ASSET PRICE BUBBLES AND MACROECONOMIC POLICIES

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ABSTRACT

Asset price bubbles have been affecting economies with ‘modern’ financial systems for at least 400 years. From the Dutch Tulip Mania to the Subprime bubble, passing by the South Sea bubble, the Japanese housing bubble or the Dot-Com bubble, the emergence of bubbles and their burst have marked financial markets and economies all around the world, and are often the cause of financial crises.

In the light of the 2008 financial crisis, there is still some ambiguity on the measures that need to be adopted to deflate or limit the impact of these bubbles. Since bursting bubbles may lead to financial crises, it is crucial to make clarifications on their behaviours and policy recommendations on how to deal with asset price bubbles. My Thesis proposes an investigation of their existences, of their effects on the economy and analyses on the utilisation of monetary policy as an instrument to deflate bubbles, or at least, to limits their impact on economies.

Chapter 1 starts with an overview on what are asset price bubbles and what do we know about them. Chapter 2 analyses the dynamic ownership of bubbles and their effects on the real activity and financial stability. Chapter 3 studies monetary policy in a New Keynesian model with an asset price bubble. Chapter 4 develops and estimates a DSGE model with stock market bubbles and nominal rigidities using Bayesian methods.

Life must be a constant education; one must learn everything,
from speaking to dying.

Gustave Flaubert

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DECLARATION OF THE AUTHOR

"I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution."

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I would like to thank my thesis committee for the great discussion we had during my viva voce, and the excellent comments they gave me, which, I hope, will help me to improve and publish my research.

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OVERVIEW

This Thesis is composed of four chapters. Chapter 1 is a review of the literature. It presents an overview on how assets are priced, what are the impacts of bubbles on the economy and possible policies that affect bubbles.

Chapter 2 studies the dynamic ownerships of risky asset price bubbles and their implications for financial stability and real activity in a heterogeneous agent model with occasionally binding borrowing constraints. The model is based on Aoki and Nikolov (2015), which is populated by entrepreneurs à la Kiyotaki (1998), bankers à la Gertler and Karadi (2011), and workers who have an hand-to-mouth behaviour. There is a government which supervises Banks, who have the incentive to invest a high fraction of their available funds in the bubble asset. The Government also provide bailout to failing banks when the bubble bursts.

Chapter 2 shows that the intensity of the banking crisis and the quantitative effects on real activity are mostly determined by both the overall contamination of the heterogeneous banking sector and the individual exposure of banks to the risky bubble. The more banks fail following the burst of the bubble, the deeper is the recession and the slower is the recovery. Importantly, the dynamics of bubble growth matters for financial stability: banks prefer to invest in the bubble at the beginning of its development, which makes this period extremely vulnerable to financial shocks. Although a banking supervision rule that dampens the impact of the bursting bubble should be very strict at the beginning of the bubble's growth, such rule weakens the financial health of the banks and makes them more vulnerable to economic shocks.

Chapter 3 studies monetary policy in a New Keynesian model with an asset price bubble. The model is based on Hirano et al. (2015) which represents a real economy. In this model, entrepreneurs à la Kiyotaki (1998), are producing capital goods that are sold to perfectly competitive firms. The model is augmented by the introduction of nominal rigidities and of investment capacity constraints. In contrast to Hirano et al. (2015), there is no government in this model, but a Central Bank that implements a monetary policy which reacts to the size of the bubble.

Chapter 3 claims that a monetary policy that targets the price of a bubble asset can significantly deflate the bubble; monetary policy is efficient to reduce the price of a bubble only if the asset is financed with debt. However, using monetary policy to deflate a bubble can be costly in term of output and a monetary policy that overreacts to asset prices can generate a recession.

Chapter 4 develops and estimates a DSGE model with stock market bubbles and nominal rigidities using Bayesian methods. The model is based on Miao et al. (2015a), where bubbles emerge through a positive feedback loop mechanism supported by self-fulfilling beliefs. Households believe that the value of some wholesale firms may not be equal to their fundamentals. These firms, which pledge their assets as collaterals in order to borrow funds, are able to relax their borrowing constraints because of the 'optimistic' beliefs of households on firms' values. Moreover, movements in the size of the aggregate bubble are driven by a sentiment shock.

Chapter 4 shows that stock market bubbles are an important factor in explaining the volatility of investment, output, and also of inflation. Moreover, a monetary policy rule that targets stock prices can help to diminish the impact of bubble sentiment shocks, and thus stabilise the economy faster than a policy rule that does not react to asset prices.

For my Family, and in memory of my mother

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Chapter 1

Introduction

This chapter provides an overview of the literature on asset price bubbles. It starts with a presentation on the nature of the existence asset price bubbles and continues by unveiling how bubbles were modelled in the literature. Then this introduction reveals the impact of bubbles on the economic growth and productivity, and different modelling assumptions. Finally, the introduction presents research that investigated the impact of policies on bubble, and concludes on the contribution of this thesis to the literature on asset price bubbles.

1.1 An Overview of Related Works in the Literature

1.1.1 Asset Price Bubbles

The theoretical literature on asset price bubbles is extensive, but its essence relates to the question *how are assets priced?* While we can return to the model of Samuelson (1958) because it explains why unbacked paper currency has value without resorting to special assumptions, a good starting point to answer this question begins with Tirole (1982).[1] Tirole (1982) shows that in a deterministic sequential market economy with a finite number of infinitely lived agents with rational expectation, an asset must be valued according to its market fundamental (i.e. the expected present discounted value of its dividends, or more generally its rents) no matter the differential information nor the presence of short sales constraints. Consequently, bubbles, which are defined as the difference between the market price and the fundamental of the asset, cannot emerge in such environment.[2]

However, this result does not preclude the existence of rational bubbles in economies with

[1]The special assumptions that have been used include the presence of money in the utility function, or imposing a cash-in-advance constraint on the purchase of some goods.

[2]The existence of rational bubble would violate the transversality condition implied by the optimizing behaviour of the traders.

growing numbers of traders. Tirole (1985) extends his analysis to overlapping generation models economies. Introducing an intrinsically useless asset (i.e. without fundamental value) in the OLG model developed by Diamond (1965), Tirole shows that the existence of bubbles is conditioned by the efficiency of the bubbleless equilibrium. Bubbles can only emerge if the growth rate of the economy is at least as great as the long-run interest rate.[3] The intuition behind this result is that when the equilibrium is inefficient, the agents have the incentive to use the bubble asset as a store of value to 'protect' their wealth.[4] Weil (1987) extends Tirole's analysis by demonstrating that in both exchange and production OLG economies, an intrinsically useless and unbacked asset which has a constant and exogenous probability of disappearing in every period, can also exist when the economy without bubble is inefficient.[5] He shows that the existence of these stochastic bubbles is conditioned by the survival probability of the bubble being greater than a minimum rate of confidence (i.e. a threshold level), which depends on the 'size' of the inefficiency. This inefficiency is measured by the excess of the growth rate over the interest rate in the bubbleless economy.

Nonetheless, the emergence of bubbles is not limited to OLG economies, they can also arise in endowment economies. Kocherlakota (1992) analyses the conditions under which bubbles can exist in sequential market economies with a finite number of infinitely lived agents facing different types of borrowing constraints. He shows that, when the agents face wealth constraints and short-sales constraints, the no-Ponzi-game condition can support bubbles. The intuition behind this result is that without constraints on debt accumulation, bubbles represent an arbitrage opportunity for the infinitely lived agent. Consequently, the agent can gain by permanently reducing their asset holdings and, therefore, will engage in a Ponzi-scheme. However, equilibria in infinite horizon sequential market economies can only exist if the agents cannot engage in Ponzi-scheme. Kocherlakota (1992) shows that no-Ponzi-game condition can support bubbles by preventing the agents to permanently reduce their asset holdings.[6] Dida and Grossman (1988) analyse the existence of rational bubbles in a model

[3] In his case, bubbles can emerge as long as the growth rate of the population is greater than or equal to the long-run rate of interest. If the interest rate (which corresponds to the rate of returns of the bubble asset because of the non-arbitrage condition) is higher than the growth rate of the population, bubbles are ruled out by wealth constraints since they end up growing faster than the resources of the economy.

[4] Tirole (1985) analyses deterministic bubbles on assets with and without market fundamental. Deterministic bubble assets with positive fundamentals need to satisfy either one of both following conditions to exist: i) the total rent grows at a slower rate than the economy, ii) rents are created over time but not capitalised ex-ante.

[5] Following the tradition of calling pure store of value asset 'money', Weil (1987) names his asset money as well, mentioning that this asset could be referred to as a zero-coupon bond. For criticism of this tradition, see section 6 in Tirole (1985).

[6] Kocherlakota (1992) shows in an example that two agents with short sales constraints and endowments growing at the same average rate, but which fluctuate over time, will trade an intrinsically useless asset to smooth consumption.

in which a risk-averse agent can invest in an asset that pays a real (exogenous) dividend. They show that a (positive[7]) bubble can only emerge on the first date of trading of the asset and if a bubble exists, then the asset must have been overvalued since the first date of trading of the asset. They also show that if the bubbles are risky, they never restart once they burst. This property corresponds to the specification of rational bubbles of Blanchard (1979), where the bubble component exists from the first trading date and does not re-emerge after bursting. However, Dida and Grossman's model has the property that the eigenvalue of the expectational difference equation is greater than a unit, implying that rational bubbles have explosive conditional expectations. In their paper, Santos and Woodford (1997) develop a general framework to study the general conditions under which rational bubbles can arise. They consider an intertemporal general equilibrium model involving sequential spot markets for goods and securities, in which the securities market and the participation of agents in the entire sequence of markets can potentially be incomplete (the incomplete participation of agents implies that their framework can treat standard OLG models). They show that a finite value for the aggregate endowment in the economy severely restricts the possibility for bubble emergences on securities in positive net supply. The reason is that the agents must accumulate large wealth as their consumption goes to zero. However, they also show that bubbles can emerge if the asset is in zero net supply, if the agents have "short-horizon" (this refers to OGL economies), with certain types of borrowing constraints as in Kocherlakota (1992) or if no agent is endowed with a positive fraction of the aggregate endowment. Finally, Hellwig and Lorenzoni (2009) showed that the resulting set of equilibrium allocations with self-enforcing private debt is equivalent to the allocations sustained with rational bubbles.

1.1.2 Asset Price Bubbles and Endogenous Growth

One area of the recent literature on asset price bubbles focuses on endogenous growth.[8] The economic environments in this part of the literature incorporate financial frictions and are composed of entrepreneurs facing idiosyncratic risk. Precisely, a fraction of entrepreneurs has

[7]They also investigate the existence of negative rational bubbles, i.e. the expected value of the rational-bubble component of a stock price would decrease into the infinite future. They show that negative bubbles cannot exist with rational bubbles that have explosive conditional expectations.

[8]As explained in Tirole (1985), bubbles can only exist when the economy is dynamically inefficient. In this situation, agents who lack means of saving tend to over-accumulate capital. Therefore, bubbles will correct the inefficiency by absorbing the savings of the economy. Consequently, bubbles crowd-out inefficient investments, which reduces capital stock and output. Carvalho et al. (2012) argue that, even if the theoretical relevance of Tirole's model is undeniable, its practical relevance is limited when we observe the recent macroeconomic events (i.e. output and capital stock raise with bubbles). The main difference between Tirole (1985) and bubbly models analysing endogenous growth is the absence of financial frictions in Tirole's model. Assume heterogeneous investments in Tirole. When the financial markets are frictionless, productive agents can absorb the entire savings of the unproductive entrepreneurs and invest on their behalf. This feature permits to correct the dynamic inefficiency because the entire investment becomes efficient.

good investment opportunities while the other fraction has poor investment opportunities. Consequently, the reallocation of investment funds from unproductive entrepreneurs to productive entrepreneurs is required in order to obtain an efficient production. The reallocation is achieved using loans, but the presence of borrowing constraints generates credit frictions that cause a shortage of means of saving and inefficient use of production technology. This situation creates adequate conditions for bubbles to circulate. In Kocherlakota (2009), infinitely lived entrepreneurs cannot borrow more than the value of their collateral which is an intrinsically worthless asset (defined as lands). He shows that the scarcity of sources of collaterals causes a bubble in the value of the collateral asset (i.e. the lands). The bubble leads to a better reallocation of the resources in the economy by relaxing the borrowing constraint, i.e. by expanding the borrowing capacity of the entrepreneurs. Hence, bubbles have an expansionary effect on the economy. Moreover, Kocherlakota analyses the bursting effect of the bubble, shows that it has a considerable persistent distributional and aggregate effects, and discusses policy interventions such as providing additional sources of collateral.[9]

Martin and Ventura (2012) analyse the growth of bubbles to investigate if bubbles are expansionary. While their economy is still composed of entrepreneurs facing idiosyncratic risk, they use a model with OLG instead of infinitely lived agents. The model differs from the literature for two main reasons (i) the financial markets are completely shut down due to the absence of future pledgeable income from returns on investments and (ii) there is a "random creation and destruction of bubbles" in the economy.[10][11] They found that bubbles are expansionary when the reallocation effect of bubbles is greater than their crowding-out effect.[12]

Hirano et al. (2015) also examine the crowds-in and crowds-out effects of asset bubbles. While their paper mainly focuses on how expected bailout can affect production efficiency, they show that the size of the bubble has a non-monotonic effect on output. As long as the

[9] The main policy recommendation of Kocherlakota (2009) is to let the government supply bonds to the entrepreneurs once the bubble burst. Unproductive entrepreneurs would lend to the government instead of productive entrepreneurs and accumulate higher wealth. The government could finance the debt simply by rolling it over. This recommendation is based on an insight of Caballero and Krishnamurthy (2006) where the government compensates the owners of the bubble after the burst by providing government debt that can be used as collateral in order to replace the bubble asset (land).

[10] In their paper, Martin and Ventura also developed a stylised model where the credit market is open. However, the only source of collateral that productive entrepreneurs can pledge to obtain additional funds is the bubble asset as in Kocherlakota (2009).

[11] There is no technology or preference shock in their economy, but the equilibrium is not deterministic. The economy displays stochastic equilibria with bubbles, where the stochastic process for bubbles is exogenously given. To summarise the mechanism, in every period, some young agents are lucky enough to create new bubbles. They will either immediately sell it to other young entrepreneurs or they will keep the bubble and sell it when they will be old.

[12] To summarise, the intensity of the reallocation effect is determined by the identity of the bubble creators. The creation of bubbles by productive entrepreneurs has to be high enough to balance the reduction of unproductive investment to ensure that bubbles have an expansionary effect on the economy.

bubble size is relatively small, the existence of a bubble in the economy increases output, but once the bubble becomes too large, it reduces output. Since expected bailout increases the size of the bubble because it reduces the potential loss of net worth when the bubble bursts, a "partial bailout" policy is optimal to achieve production efficiency. Hirano and Yanagawa (2017) show that the quality of the financial market is an important factor to determine the effect of bubbles on economic growth. They show that under-developed financial markets (i.e. where the degree of pledgeability is low), bubbles increase long-run growth, while when the financial market is well developed, bubbles lower growth.

Kunieda and Shibata (2016) also examine the economic growth in the presence of bubbles. Their model is based on Angeletos (2007) where the heterogeneous productivity is continuously distributed, which contrasts with the previous papers where only two types of productivity are usually assumed.[13] They show that bubbles correct the inefficiency in the resource allocation and promote economic growth when financial markets are imperfect. The two main effects of bubbles that generate this results are (i) the net worth effect (as in Kocherlakota (2009) or Hirano and Yanagawa (2017), the introduction of a bubble in the economy directly or indirectly increases the net worth of every agent) and (ii) the allocative efficiency effect (the presence of a bubble in the economy reduces the fraction of productive entrepreneurs, but increases the productive investments of remaining productive entrepreneurs). They found that the allocative efficiency effect is the most important for promoting economic growth while the net worth effect is instead important to correct the constrained dynamic inefficiency.

1.1.3 Dynamics of Asset Price Bubbles

While many papers focus on the long-run growth with bubbles, few investigate the growth of the bubble and the dynamics of the economy from its emergence. Martin and Ventura (2012) analyses the growth of bubbles in order to investigate if bubbles are expansionary. In their case, Hirano et al. (2015) and Hirano and Yanagawa (2017), who developed models of rational risky bubbles based on Kiyotaki (1998), investigate analytically the growth of the bubble asset. Their models do not analyse the implications of different bubble ownership with explicit financial intermediation. Kunieda and Shibata (2016), who examine the economic growth in the presence of bubbles, only look at the growth in the bubbly and bubbleless steady-states. Their model incorporates a representative financial intermediary that cannot gain profits from its business (i.e. there is only one interest rate) and does not have the

[13]Their model also differs from the literature by incorporating a representative financial intermediary that cannot gain profits from its business (i.e. there is only one interest rate). However, their model does not allow different bubble ownerships as in Aoki and Nikolov (2015). Only banks can invest in the bubble asset.

feature of different bubble ownerships.

1.1.4 Asset Price Bubble and Policies

While most of the previous papers investigate the emergence conditions and effects of bubbles on real activity, many of them do not analyse the financial stability of the economy with bubbles. Aoki and Nikolov (2015) developed a model based on Kiyotaki (1998) with financial intermediaries à la Gertler and Karadi (2011). They show that the macroeconomic impact of a bursting bubble depends on its ownership.[14] When banks hold the bubble, recessions are deepened while they are more muted if the bubble is held by ordinary savers. As for the long-run growth, Hirano and Yanagawa (2017) show that the effect of a bubble burst also depends on the quality of the financial market. When the bubble bursts due to a negative technology shock in economies with an under-developed financial market, the economy's growth rate decreases significantly, becomes permanently low and stagnates. When the financial market is developed, the burst of bubble immediately decreases the economic growth rate right after the productivity shock, but the economy experiences a quick recovery and high economic growth rate in the long run. Consequently, the collapse of bubbles reveals the 'true' economic conditions of an economy.

Finally, the literature on asset price bubbles and monetary policy will be presented in the Introduction 3.1.

1.2 Contribution

My Thesis proposes several contributions to the literature on asset price bubbles. First, it presents an extensive analysis of the reallocation effect of bubbles during its growth, but also in steady-state. This analysis differs from the literature because, but not only, of the possibility of different ownerships of the bubble: the bubbles can be held by ordinary savers or by banks. My analysis shows that banks' ownership of the bubble has a non-monotonic effect of financial stability, but also on the economic growth.

Second, my Thesis proposes a New-Keynesian Model with an asset price bubble and shows that a monetary policy that leans against the wind can deflate a bubble if this one is financed with leverage. However, a monetary policy that is too aggressive towards a bubble can be costly as it can generate a recession.

[14] To the best of my knowledge, I believe that Aoki and Nikolov (2015) are the first to analyse the macroeconomic effect of different explicit and endogenous ownerships. Precisely, they examine owners with different preferences and economic role.

Finally, the last contribution of my Thesis is the development and estimation of the DSGE New Keynesian model with a stock market bubbles. This chapter also contains an analysis of the effect of monetary policy on the stock market bubbles. The analysis shows that the monetary policy is not efficient to reduce the the volatility of the value of the bubble, but is efficient to stabilise the investment, output and inflation after movement in the size of the bubble.

Chapter 2

Bubbles, Endogenous Growth and Financial Stability

This chapter studies the dynamic ownership of risky asset price bubbles and its implications for financial stability and real activity in a heterogeneous agent model with occasionally binding borrowing constraints. It shows that the intensity of the banking crisis and the quantitative effects on real activity are mostly determined by both the overall contamination of the heterogeneous banking sector and the individual exposure of banks to the risky bubble. The more banks fail following the burst of the bubble, the deeper is the recession and the slower is the recovery. Importantly, the dynamics of bubble growth matters for financial stability: banks prefer to invest in the bubble at the beginning of its development, which makes this period extremely vulnerable to financial shocks. Although a banking supervision rule that dampens the impact of the bursting bubble should be very strict at the beginning of the bubble's growth, such rule weakens the financial health of the banks and makes them more vulnerable to economic shocks.

2.1 Introduction

Many countries have experienced boom-bust cycles in their asset prices, causing important fluctuations in real activity. Asset price bubbles usually increase economic activity before their bursts and contract it following their collapse. A good example of such dynamics is the economic upturn and downturn associated with the subprime mortgage bubble in the US, which led to one of the most important financial crises of all time. However, not all bubbles bust are equally harmful and trigger financial crises. Despite a sharp fall in stock prices, the burst of the early 2000s Dot-Com bubble did not cause a collapse of the financial system nor a deep recession.

Economists have different arguments to explain these various intensities. Jordà et al. (2015) argue that the financing of the bubble asset is crucial. Bubbles financed with leverage lead to more severe crises. Aoki and Nikolov (2015) advance that the economic role of the bubble asset owners matters. Because the role of banks is to provide liquidity to the real sector, banking bubbles create deepen recessions when they burst. Brunnermeier et al. (2019) claim that the allocation of risks across banks is also important and depends on the characteristics of individual financial institutions. Larger bank size, higher loan growth and other unfavourable balance sheet features increase the systemic risk associated with asset price bubbles. It is clear that the stability of the banking system is essential to determine the intensity of the recession following the burst.

However, it is not yet clear how the *development* of bubbles affects financial stability and real activity. From the emergence of a bubble to its collapse, how do financial stability and real activity evolve? And can banking supervision mitigate the impact of the bursting bubble? In this chapter, I theoretically investigate these questions by examining how the growth of risky bubbles affects the real economy and its financial stability.

I find that the emergence of bubbles leads to a significant reallocation of the resources in the economy which causes a temporary contraction of the real activity. Moreover, the emergence of bubbles has important implications for financial stability as it causes more banks to over-invest in the bubble during its early development than its late growth. Therefore the stage of development of bubbles has a direct effect on the mains factors that determine the after-effect of the burst of the bubble. I find that three key factors determine the dynamic of the economy after the collapse of a bubble. These factors are the contamination of the banking sector, the banks' exposure to the risky asset and the agents' resistance to economic shocks.

The framework is based on rational bubbles (e.g. Galí, 2014; Hirano et al., 2015; Kunieda and Shibata, 2016; Hirano and Yanagawa, 2017) where assets without fundamental values (i.e. bubbles) can exist because of financial frictions. Credit frictions, generated from borrowing constraints, cause a shortage of means of saving and inefficient use of production technology that create adequate conditions for bubbles to circulate. However, the model differs from the literature by incorporating explicit financial intermediaries à la Gertler and Karadi (2011).[1] An interesting feature of the model is that different types of agents have the opportunity to invest in the asset price bubble. Savers and bankers can invest in the bubble if they have the incentive to do so and the ownership is endogenously determined. Moreover, banks can specialise in an activity. Some banks will decide to specialise in lending activity while others

[1] Miao et al. (2015b) and Kunieda and Shibata (2016) also developed models of rational bubbles with a banking sector. However, only banks are able to invest in the bubble asset. Consequently, the implications of different ownerships cannot be analysed.

will specialise in bubble investment activity. This feature is deeply explored in this chapter by the introduction of the concept of contamination of the banking sector.

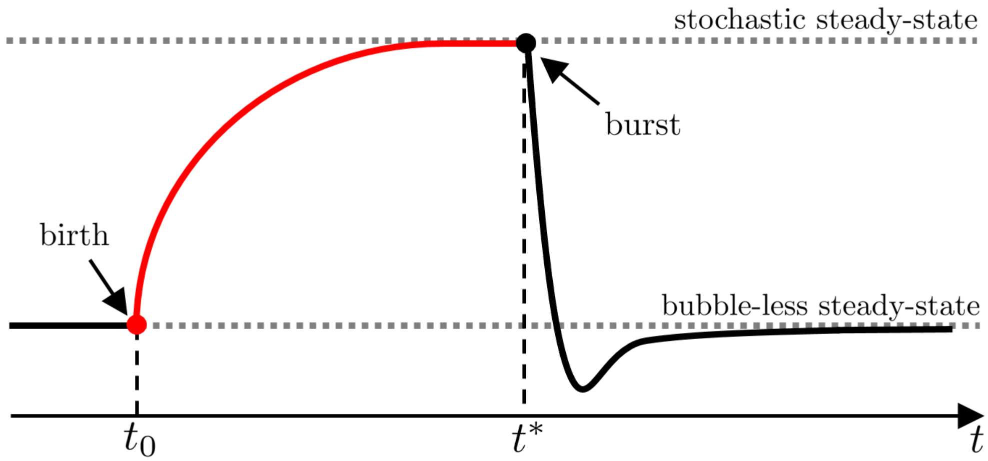

Figure 2.1: Sketch of the expected dynamics of bubble asset owners' wealth

This illustration describes the expected dynamics of the wealth of bubble asset owners. Initially, there is no bubble in the economy and the agents' wealth is at its bubbleless steady-state. At t_0, a new intrinsically useless asset (i.e. the bubble) is available and is traded by the different agents. During the growth of the asset, bubble owners see their wealth increase (red solid line) and converge to its stochastic steady-state, which is a stationary equilibrium where the bubble survives. At t^*, the bubble bursts and bubble owners experience a negative shock. Their wealth plummet (black solid line) and converge back to its bubbleless steady-state.

Rational bubbles can emerge in an economy if two conditions are satisfied, i) the bubble asset should be attractive and ii) affordable. When these conditions hold, the risky bubble positively affects the wealth of any bubble asset owners. Figure 2.1 qualitatively illustrates the dynamics of the wealth of bubble investors. Before t_0, the economy is at its bubbleless steady-state. At t_0, the bubble asset appears on the market. Agents are assumed to know the statistical process that drives the bubble, though they do not know in advance its duration. Then agents start to trade the bubble from t_0 and enjoy high returns on their investment as their wealth increases until t^*. When the bubble bursts at t^*, they see their wealth plummet and converge back to their initial level at the bubbleless steady-state.

The most similar model to the one studied in this chapter is developed by Aoki and Nikolov (2015). Their paper seeks to understand the reasons why some asset price bubble bursts lead to banking crises while others do not. They concluded that the bubble's impact on the real economy depends on its ownership. If banks hold the bubble when it bursts, recessions are deepened whereas they are more muted if the bubble is held by ordinary savers.

In contrast to their work, I focus on the dynamic equilibrium that results in very different scenarii. I investigate how the bubble affects the real activity and financial stability during the different stages of the bubble's life cycle. My analysis starts at the stochastic steady-state which is the converging point of the economy after the emergence of the bubble. I found that at the stochastic steady-state, entrepreneurs have the incentive to invest in relatively safe bubbles whereas bankers, looking for high returns, invest in risky bubbles. Furthermore, banking supervision which prevents banks to excessively invest in the bubble asset has a monotonic effect on the size of banking bubble. More relaxed supervision increases the size of the banking bubbles. However, the banking supervision has a non-monotonic effect on the economic growth and on the contamination of the banking sector: relaxing strict banking supervision increases output and the contamination of the banking sector, but once the banking supervision is too relaxed, output and the contamination of the banking sector decline.

During the growth of the bubble, I examine the dynamic ownership of the bubble asset and its impact on real activity and financial stability. I found that more banks decide to invest in the bubble at the beginning of its growth as the emergence of the bubble initially rises the interest rates and reduces the interest rate spread. However, this low spread creates important distress for the banks. Leverage becomes ineffective and their larger exposure to the risky bubble asset makes them vulnerable to economic shocks. Consequently, the emergence of the bubble asset deeply affects the structure of the economy and its transition implies a varying resistance to economic shocks.

Finally, I look at the different potential crashes of the economy along the bubble path. It turns out that recessions are significantly more intense during the first periods of the bubble growth than at the steady-state. Investors do not have time to use the bubble asset to accumulate more wealth when a bubble quickly bursts after its emergence. Consequently, the level of wealth in the economy is lower after an early burst than after a burst of a fully developed bubble. Therefore, the economy needs more time to recover because of lower initial level of investment. Moreover, the severity of these recessions is amplified by the large fraction of failing banks that makes the intermediation more difficult.

The rest of the chapter is structured as follows. Section 2.2 outlines the model and its calibration. Section 2.3 describes the main equilibrium conditions, the ownership of the bubble in the steady-state and its effects on real activity and financial stability. Section 2.4 and Section 2.5 respectively discuss the dynamics of the economy during the growth of the bubble and its financial stability along its growth path. Section 2.6 focuses on and compares the potential crashes of the economy after the burst of the bubble at different dates during its growth. Section 2.7 concludes.

2.2 The Model

The model is based on Kiyotaki (1998) as developed by Aoki and Nikolov (2015) to study bubbles. I briefly outline the model in this section, all details are given in Appendix A.1. It is a discrete-time economy with a single homogeneous good populated by three types of agents. There are entrepreneurs and workers, who have infinite lives, and bankers who live for a stochastic length of time. There is also a government whose role is limited to providing deposit insurance to failing banks by taxing entrepreneurs.

Three financial assets are available in the economy; loans, deposits and the bubble asset. Entrepreneurs and bankers have access to any of these financial assets, but the intermediation is only operated by the bankers. The production of the economy is ensured by entrepreneurs. They do not work, but hire workers and use their production technologies to produce. Workers do not have access to any financial assets, they are assumed to have a hand-to-mouth behaviour at all times.

2.2.1 Entrepreneurs

Decision Problems

There is a continuum of entrepreneurs of measure one. At each date t, some entrepreneurs are productive and the others are unproductive. Each entrepreneur is endowed with a constant returns to scale production function, which converts labour h_t to future output y_{t+1}:

$$y_{t+1} = a_t^i h_t, \tag{2.1}$$

where a_t^i is the productivity parameter of type i only known at time t. Respectively, entrepreneurs with productivity rate $a_t^i = a^H$ are called productive entrepreneurs and entrepreneurs with productivity rate $a_t^i = a^L$ are called unproductive entrepreneurs, where the productivity rates respects $a^H > a^L$.

Each agent shifts stochastically between productive and unproductive states according to a Markov chain. The productivity shifts of the entrepreneurs are exogenous and are independent across agents and over time. A productive entrepreneur during this period becomes unproductive the next one with the probability δ and an unproductive entrepreneur becomes productive next period with probability $n\delta$. The stationary fraction of productive entrepreneurs over time given by this Markov chain is equal to $n/(1+n)$.

Every entrepreneur has the same preferences and maximises the following expected discounted utility:

$$U^E = E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t, \tag{2.2}$$

where $\beta \in (0,1)$ is the subjective discount factor and c_t is the consumption of an agent at date t. Entrepreneurs purchase consumption and can invest into bubbles m_t^e at price μ_t. They can borrow funds ($b_t > 0$), make deposits ($b_t < 0$) and they pay wages w_t to the workers they hire in order to receive future revenues y_{t+1}. The government can tax their wealth at the rate τ_t after deducting debt repayments. The flow-of-funds constraint is given by:

$$c_t + w_t h_t + m_t^e \mu_t - b_t = (1-\tau_t)(y_t - R_{t-1} b_{t-1} + m_{t-1}^e \mu_t) \equiv z_t, \tag{2.3}$$

where z_t stands for entrepreneur's net worth. R_t is the interest rate which is equal to the loan rate R_t^l when the entrepreneur is a borrower and the deposit rate R_t^d when the entrepreneur is a saver.

Because of the frictions in the credit market, the borrowing entrepreneurs can pledge at most a fraction $\theta \in (0,1)$ of their future expected revenue as collateral to the banks. Hence, the borrowing constraint is given by:

$$R_t^l b_t \leq \theta E_t \left[y_{t+1} + m_t^e \tilde{\mu}_{t+1} \right], \tag{2.4}$$

where $\tilde{\mu}_{t+1}$ is the future price of the bubble asset.

In this economy, the bubble asset is an intrinsically worthless asset. Following Weil (1987), the bubble asset is risky. Its price may collapse with the constant probability $1-\pi$ in every period. Then, the stochastic realisation of the bubble value at date $t+1$ follows the process given by:

$$\tilde{\mu}_{t+1} = \begin{cases} \mu_{t+1} > 0 & \text{with probability } \pi \\ 0 & \text{with probability } 1-\pi. \end{cases} \tag{2.5}$$

Entrepreneurs maximise (2.2) subject to (2.3) and (2.4).

Optimal Behaviours

Following Sargent (1987), the decision problem of the entrepreneurs can be interpreted as a savings problem with uncertain returns. Since their instantaneous utility function is logarithmic and because there is no labour income or transfer income, entrepreneurs consume a constant fraction of their net worth z_t:

$$c_t = (1-\beta) z_t \tag{2.6}$$

and save the remaining β fraction of their wealth.

Entrepreneurs have several possibilities for accumulating net worth. They can simply deposit and obtain the rate of returns $(1-\tau_{t+1})R_t^d$, invest into bubble and earn $(1-\tau_{t+1})\mu_{t+1}/\mu_t$ when the bubble survives or use their technology to invest in unleveraged production and have a rate of returns $(1-\tau_{t+1})a_t^i/w_t$. Moreover, they can also use collateral in order to borrow extra funds to enlarge their production. If they decide to pledge only revenue from future production, the rate of return on this leveraged investment is equal to:

$$(1-\tau_{t+1})\frac{(1-\theta)a_t^i}{w_t-\frac{\theta a_t^i}{R_t^l}}.$$

In equilibrium, productive entrepreneurs will be the borrowers since they have the highest rate of returns on production.[2] Therefore productive entrepreneurs produce using leverage and obtain a rate of returns equal to:

$$(1-\tau_{t+1})\frac{(1-\theta)a^H}{w_t-\frac{\theta a^H}{R_t^l}}. \tag{2.7}$$

However, productive entrepreneurs will only borrow if their rate of returns on leveraged projects is higher or equal than the unleveraged rate of returns. This implies that the untaxed productive projects rate of returns a^H/w_t has to be greater or equal than the borrowing cost R_t^l. If $R_t^l = a^H/w_t$, the productive agents are indifferent between leveraged or unleveraged production projects. Therefore, their collateral constraint might not bind since they might not borrow or might not borrow the maximum amount possible.

The unproductive entrepreneurs have relatively poor investment opportunities in comparison to productive entrepreneurs. They can deposit, use their technology to invest in unleveraged production projects and invest in the bubble. When the productive entrepreneurs and the banking sector cannot absorb all the national saving, unproductive entrepreneurs will produce. Hence, the non-arbitrage condition implies:

$$R_t^d = \frac{a^L}{w_t} \tag{2.8}$$

whenever the unproductive entrepreneurs produce since deposits and production are risk-less investment opportunities. The unproductive entrepreneurs will only deposit if the deposit rate is higher than the rate of returns on production, $R_t^d > a^L/w_t$. When the unproductive

[2] As in Aoki and Nikolov (2015), I allow productive entrepreneurs to invest in bubble. They can use the bubble as collateral by borrowing against its value and then use this extra fund ($\theta E_t\tilde{\mu}_{t+1}/\mu_t$) as a down-payment for leveraged production. However, this investment strategy will not be realised in equilibrium because the returns of the bubble asset are too low. The arbitrage condition for this situation is given in Appendix A.1.

entrepreneurs invest into the bubble, the non-arbitrage condition is determined by the saver's state-contingent wealth valuation:

$$E_t\left[\frac{1}{c_{t+1}^L}(1-\tau_{t+1})\frac{\tilde{\mu}_{t+1}}{\mu_t}\right] = E_t\left[\frac{1}{c_{t+1}^L}(1-\tau_{t+1})\right]R_t^d, \tag{2.9}$$

where $1/c_{t+1}^L$ is the shadow value of wealth at time $t+1$ of an entrepreneur who was unproductive at time t.

2.2.2 Banks

Decision Problems

There is a continuum of bankers of measure one who are risk-neutral and operate for a stochastic length of time.[3] Once bankers receive a 'retirement' shock with probability $1-\gamma$, they liquidate all their asset holdings and consume their net worth before exiting. In order to keep the population of bankers constant, it is assumed that in each period, measure $1-\gamma$ of new bankers starts to operate with small initial endowments (net worth) received from exiting bankers.[4]

Bankers maximise the following expected discounted utility:

$$U^B = E_0\sum_{t=0}^{\infty}(\beta\gamma)^t c_t, \tag{2.10}$$

where c_t is their consumption. All lending and borrowing are intermediated by the banks since they are the only one in the economy who can enforce debt repayment. Banks receive the deposits d_t from the unproductive entrepreneurs and provide loans b_t to the productive entrepreneurs. In addition to their commercial banking activities, they can invest in the risky bubble asset m_t^b. Bankers finance their assets using their own net worth n_t and savers' deposits. Their balance sheet constraint is given by:

$$c_t + b_t + m_t^b\mu_t = d_t + n_t, \tag{2.11}$$

Bankers can default. A bank goes bankrupt when the value of its assets falls below the value of its liability. However, savers' deposits are insured by the government who provides bailout to these failing banks. Assuming that intermediation is costless and limited liability, the

[3]Following Gertler and Karadi (2011), a finite horizon for the bankers is introduced in order to ensure that over time bankers do not reach the point where they can fund all investments from their own capital.

[4]As in Aoki and Nikolov (2015), these small initial endowments are not explicitly analysed since they do not affect substantially the subsequent analysis.

evolution of the banker's net worth is given by:

$$n_{t+1} = \max[R^l_t b_t + \tilde{\mu}_{t+1} m^b_t - R^d_t d_t, 0], \tag{2.12}$$

which is non-negative.

Due to government guarantees, savers have little incentive to monitor banks. Hence, in the absence of banking regulation, two moral hazard problems arise. With positive premia, banks will have the incentive to expand their assets indefinitely by borrowing additional funds from the depositors. Furthermore, banks can excessively invest in the bubble asset and default when the bubble bursts, creating a deposit insurance cost for taxpayers. To tackle these two problems, banking regulation is introduced.

First, following Gertler and Karadi (2011), it is assumed that bankers can divert and consume $1 - \lambda$ fraction of savers' deposits at the cost of losing the ability to operate a bank forever. Consequently, bank regulators limit the amount of deposits that a bank can borrow from uninsured savers according to the following incentive compatibility constraint:

$$(1 - \lambda)d_t \leq V(n_t), \tag{2.13}$$

which is now referred to as the bank's borrowing constraint. The left hand side of (2.13) represents the fraction of deposits that a bank can divert and the right hand side is the value of an operating bank (i.e. when the banker does not divert funds). The value of the bank satisfies the following Bellman equation:

$$V(n_t) = \max_{\{c_t, b_t, m^b_t, d_t\}} \{c_t + \beta E_t[\gamma V(n_{t+1}) + (1 - \gamma)n_{t+1}]\}. \tag{2.14}$$

Second, in order to prevent banks to excessively invest in the bubble asset, bank regulators also have a supervising role. They can detect the banks that are trying to undertake large exposure to the risky bubble asset. However, banking supervision is assumed to be imperfect. Banks that hold more than ξ fraction of bubble asset on their balance sheet are caught, the others remain undetected. Then their supervision constraint is given by:

$$m^b_t \mu_t \leq \xi(b_t + m_t \mu_t). \tag{2.15}$$

Due to the different investment opportunities that can be undertaken, bankers might want to specialise in an activity, creating a heterogeneous banking sector. Depending on the quality of the banking supervision and on the intensity of the intermediation frictions, some banks, or all of them, might want to specialise in holding bubble. Hereafter, these banks that specialised in holding the bubble will be called 'bubbly banks'. The remaining ones, who decide to not hold the bubble, but to invest in safe loans will be referred to as 'lending banks'. Nonetheless, since there is no cost associated with their specialisation, bankers have the ability to change costlessly their investment strategy the following period. The investment strategy choice of a banker will be detailed in section 2.2.2.

To solve the Bellman equation (2.14), we guess and verify that V has a particular form. Let $V^l(n_t)$ be the value of a lending bank and $V^b(n_t)$ be the value of a bubbly bank at time t. Since it is costless for the banks to switch type, the continuation value of a bank $V(n_{t+1})$ at $t+1$ given by:

$$V(n_{t+1}) = \max[V^l(n_{t+1}), V^b(n_{t+1})]. \tag{2.16}$$

Since banks are risk neutral, the value function of a bank specialised in lending is guessed to be linear in n_t:

$$V^l(n_t) = \phi_t^l n_t, \tag{2.17}$$

and the value function of a bank specialised in bubble is also guessed to be linear in n_t:

$$V^b(n_t) = \phi_t^b n_t. \tag{2.18}$$

Their continuation value is given by:

$$V(n_{t+1}) = \phi_{t+1} n_{t+1}, \tag{2.19}$$

where:

$$\phi_{t+1} \equiv \max[\phi_{t+1}^l, \phi_{t+1}^b]. \tag{2.20}$$

When $\phi_t > 1$, bankers invest all their net worth and only consume when they retire.

Bankers maximise (2.10) subject to (2.11), (2.13), (2.15) and (2.12).

Optimal Behaviour of Lending Banks

If a bank decides to specialise in lending, its optimal behaviour will be characterised in the following way. When $R_t^l > R_t^d$, the bank's borrowing constraint (2.13) binds since the banker wants to issue as many loans as possible. Then, the bank's demand for deposit is given by:

$$d_t^l = \frac{\phi_t^l}{1-\lambda} n_t, \tag{2.21}$$

where $\phi_t^l/(1-\lambda)$ is the leverage ratio of the lending bank. Combining the balance sheet (2.11), the demand for deposit (2.21) and the law of motion for net worth (2.12), the lending return per unit of net worth is:

$$u_{t+1}^{l} = R_t^l + (R_t^l - R_t^d)\frac{\phi_t^l}{1-\lambda}, \tag{2.22}$$

Using (2.17), (2.19), (2.22) and the recursive form of the value function of a lending bank:

$$V^l(n_t) = \max_{\{c_t, b_t, d_t\}} \{c_t + \beta E_t[\gamma V(n_{t+1}) + (1-\gamma)n_{t+1}]\},$$

the functional equation for ϕ_t^l has the following form:

$$\phi_t^l = \beta E_t\left[(1-\gamma+\gamma\phi_{t+1})u_{t+1}^l\right], \tag{2.23}$$

where $1-\gamma+\gamma\phi_{t+1}$ is the price kernel of the banker. Substituting u_{t+1}^l out of (2.23) using (2.22), ϕ_t^l satisfies:

$$\phi_t^l = \frac{\beta E_t\left[\left(1-\gamma+\gamma\phi_{t+1}\right)R_t^l\right]}{1-\beta E_t\left[\left(1-\gamma+\gamma\phi_{t+1}\right)\frac{R_t^l - R_t^d}{1-\lambda}\right]},$$

which states that the value of a unit of net worth for a lending banker is equal to the value of the returns on its loan book (numerator), suitably boosted by leverage (denominator).

Optimal Behaviour of Bubbly Banks

The optimal behaviour of a bank that decides to hold bubble is characterised in a similar way as for a lending bank. Again, when the bank's borrowing constraint (2.13) binds, its demand for deposit is given by:

$$d_t^b = \frac{\phi_t^b}{1-\lambda}n_t, \tag{2.24}$$

where $\phi_t^b/(1-\lambda)$ is the leverage ratio of the bubbly bank. When the supervision constraint (2.15) binds, the balance sheet (2.11) combined with the demand for deposit (2.24) and the law of motion for net worth (2.12) give the stochastic rate of return $u_{t+1}^b(\tilde{\mu}_{t+1})$ of a bubbly bank:

$$u_{t+1}^b = \begin{cases} u_{t+1|s}^b = R_t^l\left(1-\xi\right) + \frac{\mu_{t+1}}{\mu_t}\xi + \left\{R_t^l\left(1-\xi\right) + \frac{\mu_{t+1}}{\mu_t}\xi - R_t^d\right\}\frac{\phi_t^b}{1-\lambda} & \text{w.p. } \pi \\ u_{t+1|c}^b = \max\left[R_t^l\left(1-\xi\right) + \left\{\left(1-\xi\right)R_t^l - R_t^d\right\}\frac{\phi_t^b}{1-\lambda}, 0\right] & \text{w.p. } 1-\pi \end{cases} \tag{2.25}$$

where $u_{t+1|s}^b$ and $u_{t+1|c}^b$ are respectively the rate of returns when the bubble survives ($\tilde{\mu}_{t+1} = \mu_{t+1}$) and when the bubble crashes ($\tilde{\mu}_{t+1} = 0$).

Using (2.18), (2.19), (2.25) and the recursive form of the value function of a bubbly bank:

$$V^b(n_t) = \max_{\{c_t, b_t, m_t^b, d_t\}} \{c_t + \beta E_t[\gamma V(n_{t+1}) + (1-\gamma)n_{t+1}]\},$$

the functional equation for ϕ_t^b has the following form:

$$\phi_t^b = \beta E_t \left[(1-\gamma+\gamma\phi_{t+1})u_{t+1}^b\right]. \tag{2.26}$$

A bank will have the incentive to invest in the bubble only if:

$$\frac{\mu_{t+1}}{\mu_t} > R_t^l,$$

or when $\mu_{t+1}/\mu_t = R_t^l$ if there is no risk that the bubble bursts. In the absence of banking supervision, banks want to only invest in bubbles. However, the regulatory constraint binds and limits their bubble holdings to a ξ fraction of total assets on their balance sheet.

Investment Strategy Choice and Banking Sector Contamination

In each period, a bank has the investment strategy of becoming either a lending bank or a bubbly bank and is able to change costlessly its specialised activity the next period. When both types of banks operate in equilibrium, the value of the two investment strategies are equal:

$$\phi_t = \phi_t^l = \phi_t^b. \tag{2.27}$$

Then, equations (2.23) and (2.26) imply that the non-arbitrage condition between investment strategies is given by:

$$E_t \left[(1-\gamma+\gamma\phi_{t+1})(u_{t+1}^l - u_{t+1}^b)\right] = 0. \tag{2.28}$$

In equilibrium, the fractions of bubbly banks and lending banks are determined to satisfy (2.28). Hereafter, the fraction of bubbly bank will be referred as 'contamination of the banking sector' and represented by the variable Ψ_t. Since the shift of specialisation is costless, endogenous and therefore untraceable, the contamination of the banking system at time t is given by the ratio of the aggregate bubbly banks' net worth at time t over the aggregate net worth of the entire banking system $\Psi_t = N_t^b/N_t$.[5]

When the banking sector is not contaminated (i.e. $\Psi_t = 0$), every banks are specialised in

[5] Because of supervision constraint, N_t^b does not represent the aggregate net worth invested in the bubble, but the aggregate net worth of banks that specialised in holding bubble because they can only hold ξ fraction of their total assets on their balance sheet. Appendix A.1.3 of the chapter presents all derivations.

lending. Hence, the value of a lending investment is strictly greater than a bubbly investment:

$$\phi_t = \phi_t^l > \phi_t^b,$$

implying that (2.28) is strictly positive. Reciprocally, when the banking sector is completely contaminated (i.e. $\Psi_t = 1$), the the value of a bubbly investment is strictly greater than a lending investment:

$$\phi_t = \phi_t^b > \phi_t^l,$$

implying that (2.28) is strictly negative.

2.2.3 Workers

Decision Problem

There is a continuum of workers of measure one who provide labour to the entrepreneurs. As common in the literature (e.g. Kocherlakota, 2009; Hirano et al., 2015), workers cannot borrow, save or invest in the bubble. That is, workers have hand-to-mouth behaviour at all times.[6]

Workers have the following utility function:

$$U^W = E_0 \sum_{t=0}^{\infty} \beta^t \left[c_t - \frac{h_t^{1+\eta}}{1+\eta} \right], \tag{2.29}$$

and the flow-of-fund constraint:

$$c_t = h_t w_t, \tag{2.30}$$

where c_t is their consumption, h_t is the quantity of labour they supply and w_t the wage they receive at time t.

Worker maximises (2.29) subject to (2.30).

Optimal Behaviour

Because it is assumed that workers have a hand-to-mouth behaviour, they will only supply labour h_t respecting:

$$h_t = w_t^{1/\eta}, \tag{2.31}$$

and consume all their labour income.

[6] In Aoki and Nikolov (2015), workers have access to the different financial assets, but it turns out that they also have a hand-to-mouth behaviour in the equilibrium they consider.

2.2.4 The Government

Role and Policy

The government endorses the role of banks regulator and is assumed to have a balanced budget rule. It provides deposit insurance to the savers only when the banks go bankrupt. In order to pay out the deposits of failing banks, the government levies taxes on entrepreneurs only when it is necessary. The rest of the time, there is no taxation.

Supervision, Deposit Insurance and Subsidy

The implicit subsidy a bubbly bank will receive from the deposit insurance fund at $t+1$ is defined as:

$$s_{t+1} = \max\left[-n^b_{t+1|c}, 0\right] \tag{2.32}$$

where $n^b_{t+1|c}$ is the future not-subsidised net worth of a banker after the collapse of the bubble at $t+1$ who specialised in bubble investment at time t. Then, $n^b_{t+1|c}$ is given by:

$$n^b_{t+1|c} = R^b_t b^b_t - R^d_t d^d_t, \tag{2.33}$$

where b^b_t is the loans issued by the bubbly bank and d^b_t is the deposits it received.

Using the supervision constraint (2.15) and the bubbly banks' demand for deposits (2.24), the aggregate subsidy S_{t+1} provided by the government to the failing banks is given by:

$$S_{t+1} = \max\left[\frac{\mu_t m^b_t}{\xi}\left(R^d_t \frac{\phi_t}{1-\lambda+\phi_t} - R^l_t(1-\xi)\right), 0\right]. \tag{2.34}$$

Intuitively, since taxes are only levied to finance the deposit insurance fund, taxation works as a lump sum tax. The tax rate τ_{t+1} will depend on the size of the aggregate subsidy S_{t+1} the bubbly banks will receive and on the future income of the entrepreneurs.

2.2.5 Equilibrium

Aggregation and Market Clearing

The aggregation is characterised as follow. The total supply of the bubble asset held by the unproductive entrepreneurs m^L_t and bankers m^b_t is normalised to 1:

$$m^L_t + m^b_t = 1. \tag{2.35}$$

Let Z^H_t and Z^L_t, respectively, denote the aggregate wealth of the productive and unproductive entrepreneurs. The expenditure side of the aggregate budget constraint at time t of the productive entrepreneurs is given by:

$$\beta Z_t^H = w_t H_t^H - B_t, \tag{2.36}$$

where B_t is the aggregate debt of the entrepreneurs and H_t^H is the aggregate labour demand of the productive entrepreneurs. The aggregate demand for entrepreneurial loans is constraint by:

$$B_t R_t^l = \theta a^H H_t^H, \tag{2.37}$$

The aggregate expenditures of the unproductive entrepreneurs are given by:

$$\beta Z_t^L = w_t H_t^L + D_t + \mu_t m_t^L, \tag{2.38}$$

where D_t is the aggregate supply of deposits of the entrepreneurs and H_t^L is the aggregate labour demand of the unproductive entrepreneurs. Aggregating Eqs (2.21) and (2.24), the expression for aggregate demand for deposits of the banking sector is:

$$D_t = \frac{\phi_t}{1-\lambda}\gamma N_t, \tag{2.39}$$

where γN_t is the aggregate net worth of active bankers and ϕ_t is equal to:

$$\phi_t = \begin{cases} \beta E_t\left[(1-\gamma+\gamma\phi_{t+1})u_{t+1}^b\right] & \text{when } \Psi_t = 1 \\ \beta E_t\left[(1-\gamma+\gamma\phi_{t+1})u_{t+1}^l\right] & \text{when } \Psi_t < 1 \end{cases} \tag{2.40}$$

When both lending and bubbly banks operate, equation (2.40) respects the non-arbitrage condition (2.27) in equilibrium. The aggregate balance sheet of the banking sector is:

$$B_t + m_t^b \mu_t = \gamma N_t + D_t. \tag{2.41}$$

Then, aggregating the production functions, aggregate output is determined from:

$$Y_t = a^H H_{t-1}^H + a^L H_{t-1}^L. \tag{2.42}$$

In this model, the aggregate state variables correspond to the wealths of the entrepreneurs and bankers. Taking into account the changes in productivity following the Markov chain described in 2.2.1, the evolution of wealth of the productive entrepreneurs is given by:

$$Z_{t+1}^H = (1-\tau_{t+1})\left[(1-\delta)\left(a^H H_t^H - R_t^l B_t\right) + n\delta\left(a^L H_t^L + R_t^d D_t + m_t^L \tilde{\mu}_{t+1}\right)\right]. \tag{2.43}$$

Similarly, the aggregate wealth of the unproductive entrepreneurs evolves as:

$$Z_{t+1}^L = (1-\tau_{t+1})\left[\delta\left(a^H H_t^H - R_t^l B_t\right) + (1-n\delta)\left(a^L H_t^L + R_t^d D_t + m_t^L \tilde{\mu}_{t+1}\right)\right]. \tag{2.44}$$

The evolution of the aggregate net worth of the banker is given by:

$$N_{t+1} = R_t^l B_t + \tilde{\mu}_{t+1} m_t^b - R_t^d D_t + S_{t+1}, \tag{2.45}$$

where S_{t+1} is the aggregate subsidy provided by the government to the banks.

The government's budget constraint implies that taxes will be levied on entrepreneurs in order to finance the subsidy to bail out the depositors in bubbly banks. Otherwise, the tax rate is zero whenever the bubble survives and no bailout is needed. For convenience, define ρ_{t+1} the total taxable aggregate revenue of the entrepreneurs when the bubble crashes at $t+1$:

$$\rho_{t+1} = a^H H_t^H + a^L H_t^L + R_t^d D_t - R_t^l B_t. \tag{2.46}$$

Consequently, the entrepreneurial tax rate will be equal to:

$$\tau_{t+1} = \begin{cases} 0 & \text{w.p. } \pi \\ S_{t+1}/\rho_{t+1} & \text{w.p. } 1-\pi \end{cases} \tag{2.47}$$

Finally, the markets for goods, labour, capital, loans and deposits must clear. From (2.31), labour market clearing implies:

$$w_t^{1/\eta} = H_t^H + H_t^L. \tag{2.48}$$

Good market clearing implies that the aggregate saving must equal aggregate investment:

$$\beta(Z_t^H + Z_t^L) + \gamma N_t = w_t(H_t^H + H_t^L) + \mu_t. \tag{2.49}$$

Equilibrium

Equations (2.8), (2.9), (2.22), (2.25), (2.28), (2.34), (2.35)-(2.48) jointly determine the 20 variables μ_t, m_t^L, m_t^b, R_t^d, R_t^l, w_t, H_t^H, H_t^L, Y_t, ϕ_t, B_t, D_t, u_{t+1}^l, u_{t+1}^b, Z_{t+1}^H, Z_{t+1}^L, N_{t+1}, τ_{t+1}, ρ_{t+1}, S_{t+1}, with three states Z_t^H, Z_t^L, N_t.

Definition 1. A competitive bubbly equilibrium is a sequence of aggregate state variables $\{Z_{t+1}^H, Z_{t+1}^L, N_{t+1}\}_{t=0}^{\infty}$, decision rules $\{H_t^H, H_t^L, D_t, B_t, m_t^L, m_t^b\}_{t=0}^{\infty}$ and price sequence $\{R_t^d$, R_t^l, $w_t, \phi_t, \mu_t\}_{t=0}^{\infty}$ such that (i) entrepreneurs, bankers and workers optimally choose decision rules $\{H_t^H, H_t^L, D_t, B_t, m_t^L, m_t^b\}_{t=0}^{\infty}$ taking the evolution of the aggregate states, prices and idiosyncratic productivity as given; (ii) the price sequence $\{R_t^d, R_t^l, w_t, \phi_t, \mu_t\}_{t=0}^{\infty}$ clears the goods, labour, capital, loan, bubble and deposit markets and (iii) government taxes τ_t and aggregate subsidy S_t satisfy the government budget constraint (2.47); (iv) the equilibrium evolution of state variables $\{Z_{t+1}^H, Z_{t+1}^L, N_{t+1}\}_{t=0}^{\infty}$ is consistent with the individual choices of entrepreneurs, bankers and workers and with exogenous evolution of productive opportunities at the individual entrepreneur level.

The whole system is presented in the Appendix A.1.

2.2.6 Calibration

The model has 11 parameters β, δ, n, a^H, a^L, η, θ, λ, γ, ξ and π. For consistency, I follow Aoki and Nikolov (2015) in calibrating the baseline scenario which is summarised in Table 2.1. The Frisch elasticity is set to 5, implying $\eta = 1/5$, which is in the range of macroeconomics estimates. The productivity ratio of the entrepreneurs $a^H/a^L = 1.1$ where a^L is normalised to 1.

The 6 parameters β, δ, n, θ, λ and γ were calibrated to match the steady-state of the model in the absence of bubbles to 7 moments in US data.[7] These are (1) the real loan rate minus the growth rate of the real GDP and minus intermediation costs; (2) the real deposit rate minus the growth rate of the real GDP; (3) the commercial bank leverage; (4) the average corporate leverage; (5) average leverage for highly leveraged corporates; (6) the rate of returns on bank equity and (7) the ratio M2 to GDP.[8]

Three intensities of banking supervision are examined: $\xi = 5\%$, 15% and 30%. With $\xi = 5\%$, banks do not fail at any time during the development of the bubble. For a higher ξ share of bubbles on their balance sheet, bubbly banks would experience important losses following a bubble collapse and might go bankrupt.

Finally, the probability of the bubble asset survival π is set to 98%. This implies an expected probability of bursting once in 49 years, the probability of the bubble bursting in the first ten periods of its growth is approximately 18.3% and the probability of bursting in the first twenty periods is around 33%.

Parameter	β	δ	n	a^H/a^L	η	θ	λ	γ	π	ξ
Value	0.958	0.167	0.011	1.100	1/5	0.622	0.788	0.907	0.980	0.05

Table 2.1: Baseline calibration

[7] Data sources used for the calibration are given in the Table A1 of Aoki and Nikolov (2015) Online Appendix.

[8] More details on the calibration of the model are presented in Apendix A.1.4.

2.3 Features of Bubbly Equilibria

Before presenting the stochastic steady-state with the benchmark calibration in Table 2.1, I describe the conditions of existence of bubbles, examine the conditions of their ownership and how they affect the economy.

2.3.1 Existence of the Bubble Equilibrium

As in Aoki and Nikolov (2015), rational bubbles can emerge when the deposit rate or both deposit and lending rates are lower than the growth rate of the economy at the bubbleless steady-state.

The interest rates are lower than the growth rate of the bubbleless economy when the degree of pledgeability of the productive entrepreneurs is in the middle range.[9] Below such degree of pledgeability, the borrowing capacity of the productive entrepreneurs is extremely limited and unproductive entrepreneurs have to support the most of the production. In these circumstances, the unproductive entrepreneurs enjoy large returns on production and possess most of the aggregate entrepreneurial wealth. At the opposite, when the degree of pledgeability is very high, productive entrepreneurs can borrow a lot of funds and support the entire production of the economy. In this situation, the unproductive entrepreneurs do not produce. They just provide funds to productive entrepreneurs and enjoy high returns on deposits. With a degree of pledgeability in the middle range, productive entrepreneurs are largely involved in the production process but are not able to absorb the entire savings of unproductive entrepreneurs. Due to their more prominent participation in the production, the cost of labour increases. Consequently, unproductive rate of returns and interest rates are low.

The rational bubble asset occurs because it attracts different agents by generating high returns. Despite being risk-averse, the unproductive entrepreneurs have the incentive to hold the bubble asset because they need to protect their wealth against bad idiosyncratic shocks.[10] Since they have poor returns on production and deposits, they will use the bubble to transfer their wealth over time until they become productive and enjoy higher returns on production. In contrast, being risk-neutral and covered by the deposit insurance in case of

[9] The middle range corresponds approximately to a value of pledgeability $\theta \in [0.55, 0.75]$. In the Appendix A.2.2, Figure A.2 maps the different degrees of financial frictions necessary to the emergence of a rational bubble and Figure A.3 shows how productive and unproductive labours are affected by the financial frictions in the bubbleless steady-state.

[10] The evolution of the wealth of an unproductive entrepreneur depends on their rate of returns. With a rate of returns lower than a unit, the unproductive entrepreneur' wealth continuously decreases over time. However the aggregate unproductive net worth remains stable because of some productive entrepreneurs becomes unproductive at each date and vice-versa.

default, bankers primarily use the bubble asset for 'speculation'. Banks have the ability to use leverage to boost their lending return on equity (ROE). Therefore they do not need the bubble to protect their wealth even if both interest rates are low. In this case, banks ride the the bubble to potentially increase their wealth and thus maximise their utility.

2.3.2 Who Holds the Bubble at the Stochastic Steady-states?

When the credit frictions are intermediate, the ownership of the bubble mostly depends on the quality of the supervision and the survival rate of the bubble (see Figure 2.2). First, weak banking supervision ξ gives the incentive to the banks to have a higher share of the bubble on their balance sheet without being detected. Therefore, bubble ownership of the banks significantly increases when the quality of the supervision decreases. Second, as the survival rate of the bubble π decreases, the size of the bubble reduces due to lower demand from all investors. Unproductive entrepreneurs will only buy the bubble if the asset is not too risky. Being risk-averse, unproductive entrepreneurs will not invest all their saving in the risky bubble, they will still have the incentive to use riskless means of saving (i.e. deposits and production).[11] When π is relatively low, unproductive entrepreneurs will barely invest in the bubble asset if they have the incentive, they will just focus on production and deposits. In this case, most of the bubble is held by the banks and it can be defined as a banking bubble. When π is high, unproductive entrepreneurs will have the incentive to invest a very large fraction of their savings in the bubble and hold most of it. The bubble can be referred to as a saver bubble in this situation.

Interestingly, keeping the survival rate of the bubble constant, the ownership of the bubble changes when the supervision rule varies, as long as the premium on the bubble is high enough to attract bankers (see Panel A of Figure 2.2). In this situation, banks 'capture' the entrepreneurs' share of the bubble as the supervision weakens (ξ rises). Banks have the incentive to expand their assets as much as possible, but they are constrained by the borrowing capacity of productive entrepreneurs, which limits their lending investments. However, the presence of bubbles allows banks to diversify their portfolio. As ξ rises, banks can undertake more bubbly investments and their share of the bubble increases. Nonetheless, the bubble share of the entrepreneurs decreases as ξ rises because the supervision rule has a direct effect on the reallocation of resources, which will be detailed in section 2.3.3. When the supervision rule is relatively tight (ξ approximately between 0% and 30%), entrepreneurs have the incentive to invest in the bubble because the deposit rate is relatively low. When the

[11] As common in the literature, entrepreneurs have a logarithmic utility function which is very helpful to simplify the model. However, the use logarithmic utility implies that the propensity to save (i.e. β in this model) is independent of asset returns, as income and substitution effects cancel each other.

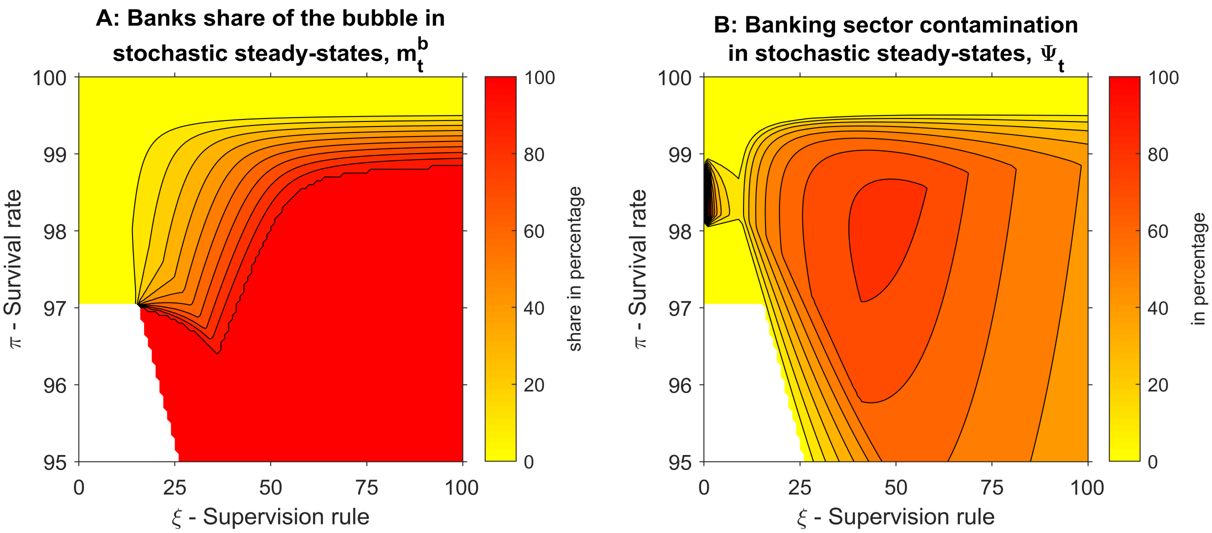

Figure 2.2: Ownership of the bubble in stochastic steady-states

This figure illustrates how the ownership of the bubble changes as a function of the intensity of the banking supervision ξ and of the bubble's survival rate π, both parameters are expressed in percentage. When ξ is low the banking supervision is strict and when ξ is high, the supervision is weak. Panel A represents the bank share of the bubble, m_t^b, and Panel B describes the percentage of banks holding the bubble, Ψ_t.

supervision becomes significantly weaker, the demand of liquidity from banks increases which rises the deposit rate. This result makes deposits more attractive than bubble investments to unproductive entrepreneurs, which leads to a reduction of entrepreneurial ownership of the bubble asset.

This chapter also investigates how the banking sector ownership of the bubble changes. Panel B in Figure 2.2 represents the fraction of banks that decided to undertake bubbly investments. Except for a small set of equilibria where the supervision rules are very tight and the bubbles relatively safe, the contamination of the banking sector is at its highest for largely relaxed supervision rules (ξ between 30% and 70%) and not when the banks are not supervised (i.e. when ξ is close to 100%).[12] When the banking sector is completely unsupervised, banks that undertake bubbly investments invest all available funds in the bubble and do not provide liquidity to the real sector. Consequently, the cost of loans R_t^l increases and a large fraction of the banks have the incentive to remain lending banks. When the supervision rule is largely relaxed (ξ between 30% and 70%), bubbly banks continue to lend a substantial quantity of funds to productive entrepreneurs. In this situation, many banks have the incentive to

[12]When ξ is close to zero and π between 98% and 99%, the banking sector is highly contaminated if not completely contaminated ($\Psi_t = 1$). There are two reasons for this result. First, the bubble is sufficiently risky for banks to enjoy a high premium, but not too risky to repel banks to invest in it. If the bubble is too risky and the supervision rule too severe, banks do not have the incentive to hold the bubble because they can enjoy higher returns on lending investments. Second, the size of bubbly investments is so restricted by the tight banking supervision that banks do not fail when the bubble bursts. Consequently, every bank has the incentive to invest some resources in the bubble.

becomes bubbly banks and enjoy high returns on both lending and bubbly activities.

2.3.3 Bubble, Growth and Reallocation Effects

One important topic in the literature on bubbles and endogenous growth concerns the effect of the bubble on the real activity. While bubbles improve the welfare of their owners, theoretical models show different results concerning bubbles effects on real activity. In dynamically inefficient economies, the capital asset is used as a productive asset and as a store of value when no other attractive assets are available in the economy, which leads to capital over-accumulation. Thus, the presence of a bubble asset allows the agents to use each asset for a specific purpose, that is to say the bubble asset will serve as a saving vehicle while capital will only be used as a productive asset.[13] Consequently, bubbles have a crowding-out effect, i.e. they reduce the capital stock and output decrases such as in Tirole (1985).

However, Tirole (1985) assumes a frictionless financial market. With credit frictions, bubbles can also crowd-in investments. Tight borrowing constraints prevent productive agents to absorb the savings of unproductive agents and invest on their behalf. While bubbles still crowd-out investments by absorbing unproductive savings, they can crowd-in investments because they have a positive net worth effect which propagates through the economy and affects the composition of the investment. However, the way this effect propagates through the economy largely differs in the literature. In Kocherlakota (2009), the bubble increases the borrowing capacity of productive entrepreneurs which directly increases the net worth of unproductive agents since they are able to supply more funds. In Martin and Ventura (2012), aggregate productive net worth increases because productive entrepreneurs sell their bubbles to unproductive agents, which allow productive agents to make larger productive investments. In Hirano et al. (2015), Kunieda and Shibata (2016) and Hirano and Yanagawa (2017), unproductive entrepreneurs who enjoy high returns on the bubble will increase the aggregate productive net worth when they become productive and thus increase aggregate productive investment. Therefore, bubbles can also crowd-in investments by replacing unproductive investments with productive ones. The final effect of bubbles on output will depend on which effect prevails. Hirano et al. (2015) shows that when bubbles are relatively small, they increase output, but once they become too large, they reduce it.

To investigate the effect of bubbles on the financial and real activities, I examine the economy where the survival rate of the bubble is 98% and under three supervision rules (i.e. $\xi = 5\%$, 15% and 30%). The results are presented in Table 2.2. Under the strictest supervision rule

[13] If the bubble asset becomes risky, risk-averse agents will continue to use a small fraction of capital as store of value (Weil, 1987).

Bubbleless and Stochastic Steady-states where $\pi = 98\%$					
Variables		No bubble	$\xi = 5\%$	$\xi = 15\%$	$\xi = 30\%$
Bubble's price	μ_t	-	0.3862	0.5000	0.5203
Bankers' bubble share	m_t^b	-	0.0000	0.1216	0.4471
Banking sector contamination	Ψ_t	-	0.0000	0.4615	0.7454
Deposit rate	R_t^d	0.9703	0.9700	0.9690	0.9739
Lending rate	R_t^l	0.9813	0.9812	0.9796	0.9824
Deposits	D_t	0.5835	0.6983	0.7999	0.9394
Loans	B_t	0.6364	0.7628	0.8179	0.8076
Wage	w_t	1.0306	1.0309	1.0321	1.0301
Productive labour	H_t^H	0.9127	1.0939	1.1710	1.1596
Unproductive labour	H_t^L	0.2501	0.0707	0.0000	0.0000
Total output	Y_t	1.2541	1.2740	1.2881	1.2756
Productive wealth	Z_t^H	0.3176	0.3809	0.4078	0.4038
Unproductive wealth	Z_t^L	0.8782	1.2082	1.2934	1.2808
Banks' net worth	N_t	0.0583	0.0711	0.0869	0.1112
Workers' wealth	Z_t^W	1.1984	1.2006	1.2086	1.1945

Table 2.2: Bubbleless and stochastic steady-states with a survival rate of 98%

The second column reports the values of the main variables of the economy without bubble, i.e. the bubbleless steady-state. The third, fourth and fifth columns report the values of the main variables in their stationary equilibrium with bubble, i.e. the stochastic steady-state, under three different supervision rules ξ (i.e. $\xi = 5\%, 15\%$ and 30%).

(i.e. $\xi = 5\%$), banks do not have the incentive to invest in the bubble due to a too low rate of returns. Consequently, only entrepreneurs hold the bubbly asset and enjoy its high returns. When the supervision rule is relaxed, banks can enjoy a higher rate of returns and therefore they largely invest in the bubble. Their share of the bubble increases from 0% under $\xi = 5\%$ to 12% under $\xi = 15\%$ and to 45% under $\xi = 15\%$. Correspondingly, the contamination of the banking sector increases from 0% under $\xi = 5\%$ to 46% under $\xi = 15\%$ and to 75% under $\xi = 30\%$.

As in the literature, the bubble crowds-out investment. The labour demand of unproductive entrepreneurs falls from 0.25 at the bubbleless steady-state to 0.07 under $\xi = 5\%$ and to 0 under both $\xi = 15\%$ and 30%. Nonetheless, the bubble also has a crowding-in effect. Productive investment increases from 0.91 in the bubbleless steady-state to 1.09 under $\xi = 5\%$, 1.17 under $\xi = 15\%$ and 1.15 under $\xi = 30\%$. Overall, the bubble has an expansionary effect on output under the three supervisions. The output level is 1.25 in the bubbleless steady-state and peaks to 1.28 with the bubble under $\xi = 15\%$. Interestingly, banks ownership of the bubble can also crowd-out investments as the banking supervision weakens (ξ rises). While output peaks to 1.28 when $\xi = 15\%$, it decreases to 1.27 as ξ increases from 15% to 30%.

In Hirano et al. (2015), the output is maximised when all unproductive investments are reallocated to productive ones, that is to say when there is zero unproductive investment, and when the interest rate is still 'determined' by the non-arbitrage condition between lending and producing. This model displays the same result. In Figure 2.3, Panel B presents the different types of investments and bubble ownerships equilibria of the economy as a function of the riskiness of the bubble and of the intensity of the supervision constraint. Unproductive entrepreneurs produce in the areas 1, 3 and 5, but they do not produce in areas 2, 4 and 6. Aside, Panel A illustrates the levels of output in stochastic steady-states. Superposing Panel B on Panel A reveals that output is at its highest at the frontier that separates the states where the unproductive agents produce from the states where they do not produce.

However, there is an additional dimension in this model. In addition to reallocating all unproductive funds to productive investments, the output can be maximised in this economy when unproductive entrepreneurs do not hold the bubble and when the interest rate is still given by the non-arbitrage condition between holding the bubble and making deposits. In this situation, the role of unproductive entrepreneurs is to only provide funds to the banks. Panel B summarises the different activities of the unproductive entrepreneurs and ownerships of the bubble. The bubble is only held by unproductive entrepreneurs in areas 1 and 2. In areas 3 and 4, both banks and entrepreneurs hold the bubble. In areas 5 and 6, only banks invest in the bubble. The stochastic steady-state where output is maximised is represented by the red triangle on both panels.

The determination of the deposit rate is important to understand what characterises these situations. The deposit rate is determined by three conditions, i) the unproductive entrepreneurs' non-arbitrage condition between bubble investments and making deposits, ii) the unproductive entrepreneurs' non-arbitrage condition between producing and making deposits and iii) the market clearing conditions for the 'deposit' market. In areas 1 and 3, all these conditions hold: the unproductive entrepreneurs hold the bubble and produce. In areas 2 and 4, the unproductive entrepreneurs hold the bubble but do not produce. In these cases, the deposit rate is given by i) and iii). In area 5, unproductive entrepreneurs produce but do not hold the bubble. Consequently, the deposit rate is determined by ii) and iii). Finally, in area 6, unproductive entrepreneurs neither produce nor hold the bubble. Here, the deposit rate is only determined by the market-clearing condition iii).

As it is shown in Panel B, the supervision rule is an important factor that determines the

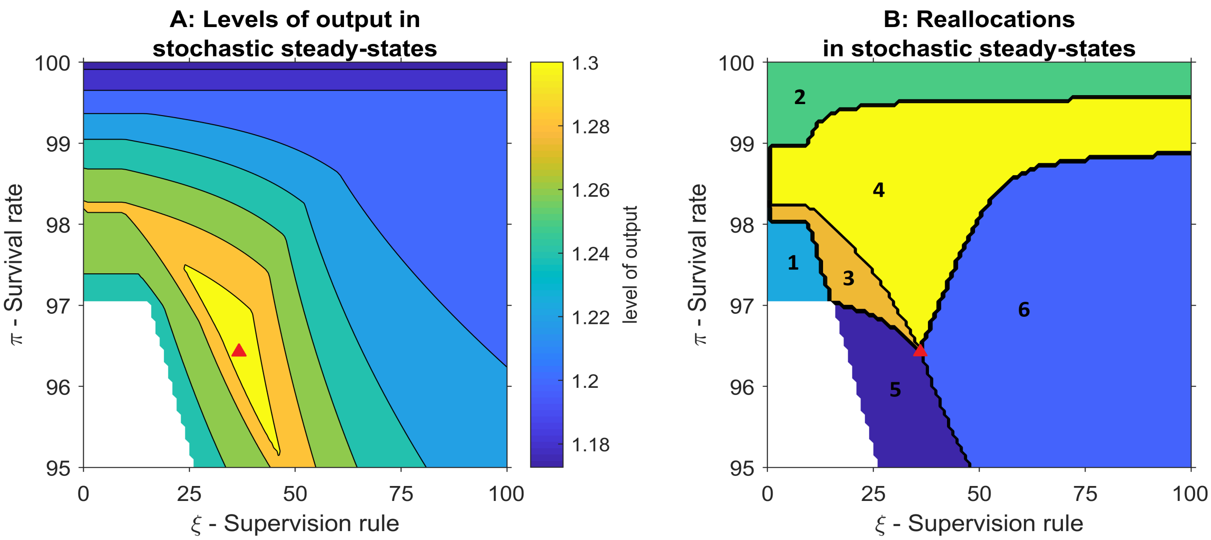

Figure 2.3: Reallocations effects of the bubble in stochastic steady-states

This figure illustrates how the bubble affects output and reallocates resources in the economy as a function of the intensity of the banking supervision ξ and of the bubble's survival rate π, both parameters are expressed in percentage. Panel A represents the levels of output in stochastic steady-states. In comparison, the bubbleless level of output is given by Table 2.2 is 1.25. Panel B depicts the combinations of different ownerships and unproductive investment decisions. Unproductive entrepreneurs produce in areas with odd numbers and do not produce in areas with even numbers. In areas 1 and 2, only entrepreneurs hold the bubble. In areas 3 and 4, both entrepreneurs and bankers hold the bubble. In areas 5 and 6, only bankers hold the bubble. The red triangle on both panels depicts the steady-state with the highest output level, where output reaches 1.319.

repartition of resources in the economy. When $\xi = 0$, the model is comparable to Hirano et al. (2015). Only the entrepreneurs hold the bubble and the size of the bubble, determined by its survival rate, affects the level of output.[14] When the bubble is relatively small, i.e. when the survival rate is low, unproductive entrepreneurs produce (area 1). When the bubble becomes larger as the survival rate increases, unproductive entrepreneurs produce less, invest more in the bubble and enjoy higher bubbly returns. The net worth effect of the bubble indirectly increases aggregate productive net worth and thus crowds-in productive investments which rise output.[15] However, the increase in labour demand raises the wage and consequently lowers the deposit rate since it is determined by the conditions i), ii) and ii). Once the bubble becomes too large, the unproductive entrepreneurs do not produce anymore and largely invest in the bubble (area 2). From this point, the deposit rate is only determined by the conditions i) and iii). Since the risk of the bubble asset decreases as π increases, the

[14] In Hirano et al. (2015), the survival rate is constant. The size of the bubble in their model is determined by the size of the bailout provided to the entrepreneurs when they fail. The mechanisms are the same since they affect the riskiness of bubble investments and so the size of the bubble.

[15] To be precise, the increase in output is not due to an equal substitution of the unproductive investments by productive ones. The reduction of the unproductive labour demand is smaller than the increase in productive labour which leads to a higher aggregate labour demand and so to a higher wage.

premium required by the unproductive entrepreneurs in order to invest in the bubble reduces and the deposit rate increases. While the net worth of unproductive entrepreneurs continue to increase as the bubble increases, the rise of the deposit rate R_t^d increases the cost of loans R_t^l. Consequently, the bubble crowds-out investments more than it crowds-in and output decreases.

However, financial intermediaries that invest in the bubble also affect the reallocation of resources in the economy. When the banking supervision is relaxed, some banks might have the incentive to invest in the bubble. When they do, the increase of aggregate demand for the bubble asset makes the bubble bigger, assuming that the survival rate remains constant. This explains why the frontier, which separates the states where the unproductive agents still produce (areas 1, 3 and 5) from the states where they do not produce (areas 2, 4 and 6), goes down as the supervision rule increases. Examining only on areas 5 and 6, i.e. areas where only banks hold the bubble, it occurs that small banking bubbles increase output, but once they become too large they also reduce output. Consequently, the survival rate and the supervision rule are both parameters that positively affect the size of the bubble and have a similar effect on the reallocation of resources in the economy.

So why is output maximised at the intersection of theses frontiers? It appears that banking bubbles are better at correcting the inefficiency in the economy than saver bubbles for two reasons. First, bubbles have a larger net worth effect on banks than on the entrepreneurs due to their ability to use leverage. Second, banking bubbles are an indicator of low lending rates since banks only have the incentive to invest in the bubble when $\mu_{t+1}/\mu_t > R_t^l$. In this situation, productive entrepreneurs can borrow at a lower cost and invest more in their production. Consequently, a banking supervision that maximises the efficiency of the investments looks for reallocating the savings of unproductive entrepreneurs to the banks while keeping the lending rate low.

2.4 Bubble Ownership and Dynamics of the Ecomomy

This section focuses on the dynamics of the economy after the birth of the bubble until its convergence to its stochastic steady-state. Suppose that at date 0, the economy is in the bubbleless steady-state. At date 1, the bubble emerges and agents make their decisions. Along the growth path, the agents expect the bubble will collapse at each date with the constant probability $1 - \pi$. Once the bubble has collapsed, it is not available any more and it disappears forever. Otherwise, the bubble continues to grow with probability π and converges to its stochastic steady-state. Figure 2.4 depicts the growth path of the bubble and its effects on the interest rates and Figure 2.5 illustrates the dynamics of the bubble ownership under three different intensities of banking supervisions ($\xi = 5\%, 15\%$ and 30%).

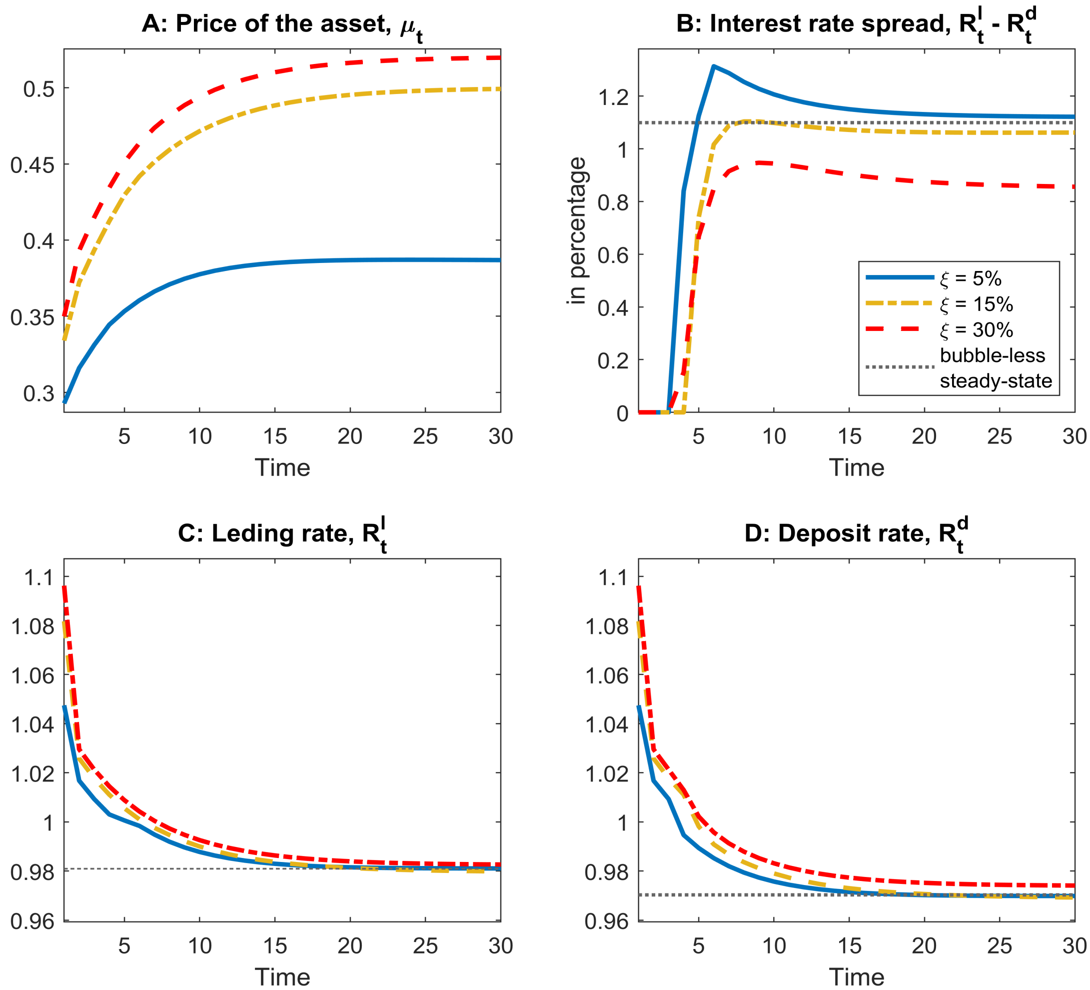

Figure 2.4: Growth of the bubble and effects on the interest rates

This figure illustrates the dynamics of main variables after the emergence of the bubble under three different intensities of banking supervision ξ, which are represented by the solid line ($\xi = 5\%$), the dash-dotted line ($\xi = 15\%$) and the dashed line ($\xi = 30\%$). The dotted line depicts the bubbleless steady-states of the variables. Panel A represents the growth path of the price of the bubble asset, μ_t. Panel B depicts the dynamics of the interest rate spread, $R^l_t - R^d_t$. Panel C and D respectively illustrate the dynamics of the lending and deposit rates, R^l_t and R^d_t.

2.4.1 Who Holds the Bubble and When?

When the bubble emerges at date 1, unproductive entrepreneurs in need of a new investment opportunity to protect their wealth know that the bubble will yield a high rate of returns if it survives. Consequently, unproductive entrepreneurs massively reallocate their savings in bubbly investments. They stop producing and reduce the deposits they supply to the banks, causing an increase of the deposit rate (see Panel D of Figure 2.4). With high pressure on the deposit rate in the first periods, banks have to reduce their demand for deposits until the deposit rate is equal to the lending rate (see Panel B of Figure 2.4). That is to say, the borrowing constraints of the banks are not binding during these periods. Thus, banks' leverage is ineffective to boost the ROE on lending investments since the interest rate spread is nil. In this situation, the ROE of lending banks falls, $u^l_{t+1} = R^l_t$, making bubbly investments more attractive because the positive premium of the bubble allows bubbly banks to use leverage to boost their wealth. Consequently, more banks decide to undertake bubbly investments and the contamination of the banking sector increases.

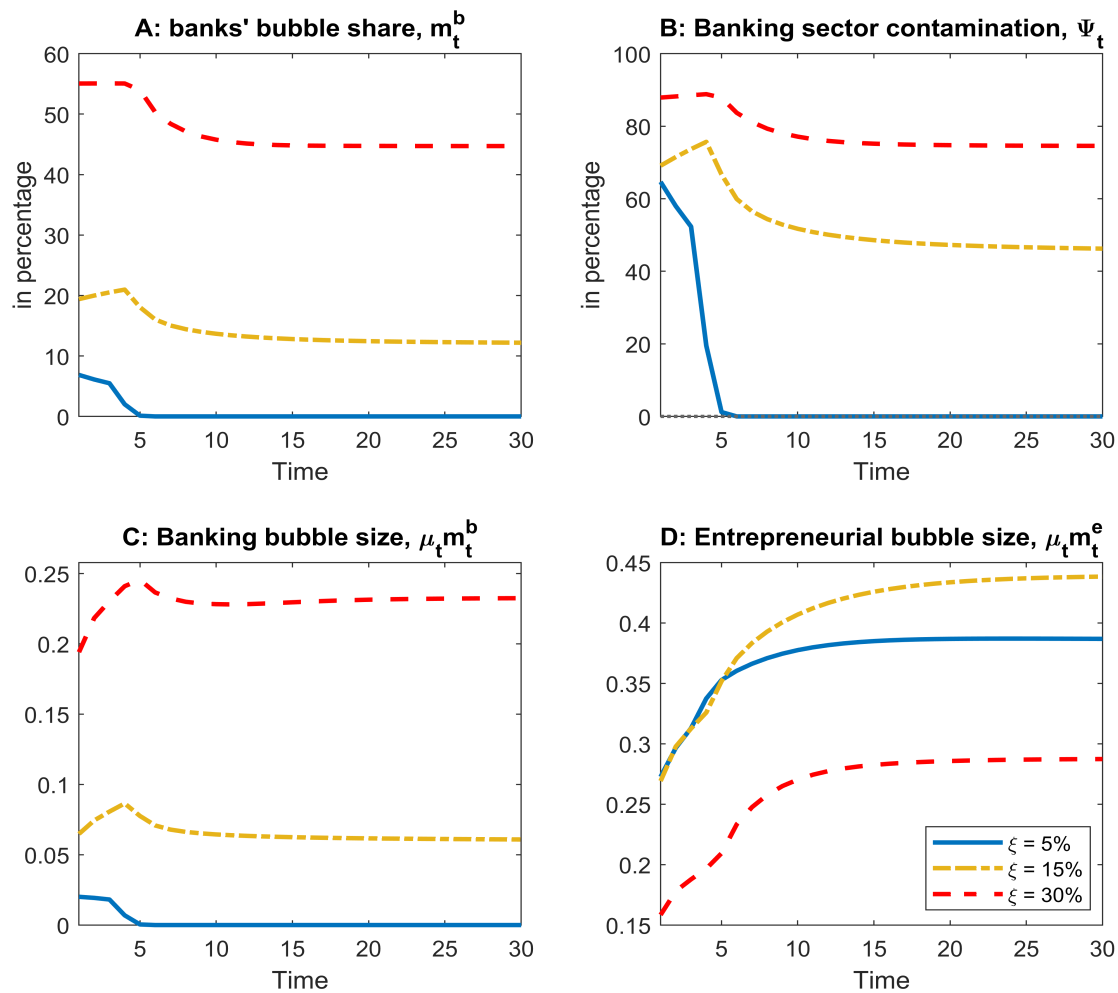

Figure 2.5: Dynamics of the ownership

This figure illustrates the growth paths of each agent's share of the bubble after its emergence under three different intensities of banking supervision ξ, which are represented by the solid line ($\xi = 5\%$), the dash-dotted line ($\xi = 15\%$) and the dashed line ($\xi = 30\%$). Panel A shows how the bubble ownership of the banks (expressed in share), m^b_t, evolves. Panel B describes the dynamics of contamination of the banking sector, Ψ_t. The banking sector contamination refers to the fraction of banks that hold the bubble asset. Panel C represents the growth of the banking bubble, $\mu_t m^b_t$. Panel D depicts the growth of the entrepreneurial bubble, $\mu_t m^e_t$.

Panel A in Figure 2.5 clearly shows that the bubble share of the banks is higher in the first periods of the growth of the bubble asset because most of the banks decided to undertake bubbly investments. For example, approximately 65% of the banks undertake bubble investments at date 1 under $\xi = 5\%$ due to the higher expected rate of returns on the bubble while they do not invest at all in the bubble at the steady-state (see in Panel B of Figure 2.5). When the supervision is weaker, banks will still invest more at the beginning of the growth of the bubble, but the difference between the beginning of the growth and the steady-state level is less substantial.

As in the steady-state, banks capture the entrepreneurs' share of the bubble when the supervision is relaxed. While banks' ownership of the bubble is a linear function of the supervision, entrepreneurial ownership of the bubble is not. Panel D of Figure 2.5 shows that when ξ increases from 5% to 15%, the size of the entrepreneurial bubble is larger even if their dynamics initially evolve in a similar way. However, when the supervision is relaxed from 15% to 30%, the entrepreneurial bubble is significantly smaller. While the sizes and the growths of the bubble are comparable under 15% to 30%, the ability of banks to invest a larger quantity of funds in the bubble increases the deposit rate when the unproductive entrepreneurs do not produce. This small increase in the deposit rate gives the incentive to unproductive entrepreneurs to provide more deposits to banks and invest less in the risky asset (see Panel C of 2.6).

2.4.2 Reallocation Effects during the Growth

As common in the recent literature, the expansionary effect of bubble shifts resources from unproductive investments to productive investments, leading to higher level of production in the long run. However, this reallocation of resources that leads to a more efficient production is not instantaneous in this economy because it first implies a drop in the short run production (see panel A in Figure 2.6). In order to invest in the bubble, unproductive entrepreneurs stop their production and reduce their deposits right after the emergence of the bubble. Consequently, there is a reduction of both unproductive and productive productions and total output sharply falls. Productive production is also reduced because there are less lending investments from the banks and the lending rate is therefore higher. Nonetheless, the output starts to recover after few periods and exceeds its initial level only once the wealth effect of the bubble has significantly increased the agents' wealth.

It was shown that the effect of the supervision rule on output is not monotonic in steady-state (output increases when ξ increases from 5% to 15%, but decreases when ξ increases from 15% to 30%). During the dynamics, the drop in output at the beginning of the growth

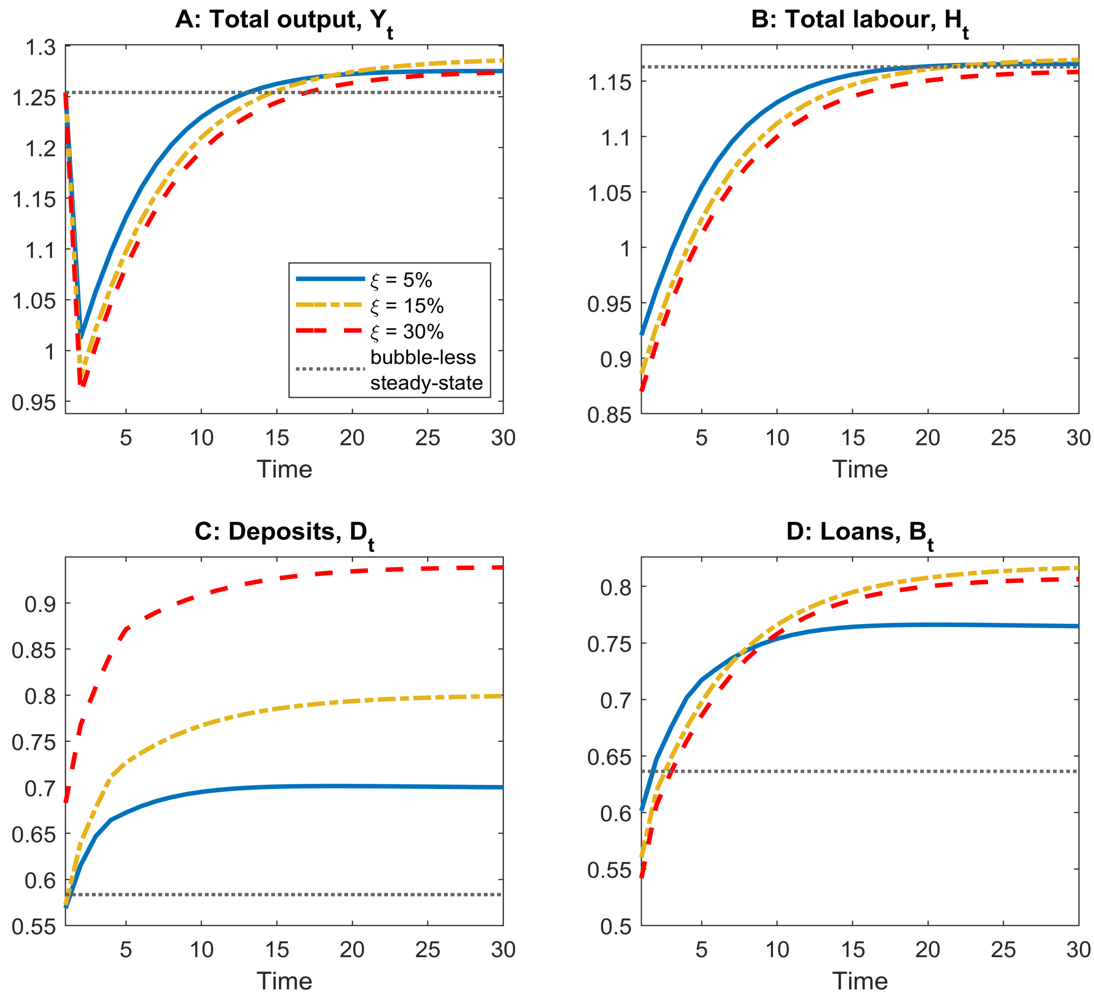

Figure 2.6: Dynamics and reallocation of the resources

This figure illustrates how the bubble shifts resources in the economy under three different intensities of banking supervision ξ, which are represented by the solid line ($\xi = 5\%$), the dash-dotted line ($\xi = 15\%$) and the dashed line ($\xi = 30\%$). The dotted line depicts the bubbleless steady-state of the variables. Panel A depicts the dynamics of the aggregate output, Y_t. Panel B shows the dynamics of the labour market, $H_t = H_t^H + H_t^L$. Panel C and D respectively represent the evolutions of the aggregate deposits and loans in the economy, D_t and B_t.

is larger when the supervision is weaker. When the supervision is weak ($\xi = 15$ or 30%), the bubble asset holding of the banks is greater. Thus, banks will allocate more funds to bubbly investments instead of lending funds to productive entrepreneurs. Panel C of Figure 2.6 clearly shows that weaker supervisions, which have a positive effect on the deposit rate, increase the supply of deposit. However, these funds are not immediately provided to the productive entrepreneurs but are used for bubbly investments. Consequently, the bubble has a contractionary effect on output in the short run which is larger when the supervision is relaxed before having an expansionary effect in the long run.

2.4.3 Wealth Effect and Transition of the Economy

The bubble positively affects the wealth of the asset owners when it survives. Unproductive entrepreneurs' wealth and the net worth of bubbly banks increase after the emergence of the bubble until its collapse. However, the bubble has a different effect on the agents that do not hold the bubble asset.[16]

While most banks decided to specialise in bubbly activities at date 1 in order to enjoy high returns on the bubble, Panel A of Figure 2.7 shows that the aggregate banks' net worth initially decreases under $\xi = 10\%$ and 15%. Because of the zero interest rate spread and a decreasing rate of returns on the bubble, the ROEs on banking activities decrease.[17] Despite the ROEs still being positive, the banking returns are not high enough to compensate the net worth consumed by the retiring bankers. Consequently, the aggregate banking net worth reduces until the spread is large enough to generate returns that can balance the wealth consumed by exiting bankers. As soon as the interest rate spread increases, ROEs soar and the aggregate net worth of the banks converges back to its stochastic steady-state. Nonetheless, the aggregate banks' net worth directly increases only under $\xi = 30\%$. The reason is that the supervision rule is weak enough to let banks boost their ROE on bubble investment with leverage.

Productive entrepreneurs' net worth also initially decreases during the growth of the bubble, though this drop lasts only one period. Productive wealth initially decreases because there are fewer available funds to borrow at a higher cost, therefore productive returns decreases. However, the negative effect of the bubble on the rate of returns of the productive entrepreneurs is balanced by a significant fall in the wages since unproductive entrepreneurs stop producing. Once the lending rate decreases because of the lower rate of return on the bubble, the productive rate of return increases again. Then, productive wealth smoothly grows and converges to its stochastic steady-state.

While the wealth of banks and productive entrepreneurs temporarily decrease during the growth of the bubble asset because of the reallocation effects of the bubble, they still reach a higher level in the stochastic steady-state than in the bubbleless steady-state. In contrast, the effect of the bubble on the aggregate wealth of the workers follows a similar path to

[16]In Hirano et al. (2015), the aggregate wealth of both types of entrepreneurs rises together with the increase in bubble price (what they call the wealth effect of the bubble). Here, it is not the case (see the growth paths of the wealth of both types of the entrepreneurs under $\xi = 15\%$ in Figure 2.9 Panel C and D and the growth path of bankers' net worth in Figure 2.7). The main difference resides in the assumption of the initial condition. In Hirano et al. (2015), entrepreneurs own the bubble asset before its emergence whereas, in this chapter, they do not own the bubble asset before its emergence. Consequently, the bubble does not positively affect the wealth of the entrepreneurs at the date of its emergence.

[17]Figure A.5 in the Appendix A.3.2 describes the dynamics of the different ROEs of the banks.

the one of output. In steady-state, the bubble increases the wealth of the workers when the bubble is relatively small because the total labour demand and the wage increase, but once it becomes too large, it reduces the aggregate wealth of the workers since they work less and receive a lower wage. During the dynamics, workers' wealth also drops because savings are initially reallocated to the trading of the bubble instead of producing at the beginning of the growth. While output recovers and even exceeds its initial level in the long run due to a better technology rate held by the productive entrepreneurs, total labour is slightly above to its bubbleless steady-state ($\xi = 5\%$ and 15%) or below steady-state when the bubble is too large ($\xi = 30\%$). To summarize, the bubble has a negative effect on workers' wealth in the short run whereas its effect in the long run depends on the severity of the supervision rule.[18]

2.4.4 Discussion: Expansionary and Contractionary Effects of the Bubble.

In this model, it is assumed that the bubble asset does not exist before date 1 and therefore is initially not owned by any agents.[19] When the bubble emerges at date 1, agents who have the incentive to invest in the bubble reallocate some funds they used for their usual activities to invest in the risky asset. Consequently, the bubble has a contractionary effect on output in the short run due to the immediate reallocation of resources to a non-productive asset. However, the bubble has an expansionary effect on the output in the long run because it indirectly rises the net worth of productive entrepreneurs. This result is important to understand the effect of bubbles in term of reallocation.

Some models only examine on the expansionary effect of bubbles, but their contractionary effect in the short run can also be found. For example, the bubble only has a expansionary effect on output in Martin and Ventura (2012). They obtain this result not because they assume a random creation and destruction of bubbles, but because only productive entrepreneurs can create the bubble asset. As a result, productive entrepreneurs sell their bubbles to the unproductive agents and all unproductive investments are replaced by productive ones, which increase output. This assumption about the ownership is relatively strong and as soon as it is relaxed, the expansionary effect of the bubble is reduced in the short

[18] In Kocherlakota (2009), workers are always better off when a bubble exists. Since the supply of labour is exogenous, the reallocation of resources due to the bubble increases output and therefore the wage. Consequently, workers receive a higher labour income and are better off with a bubble.

[19] While it is assumed in this chapter that the bubble asset does not exist before its emergence, this assumption can be changed. Assume, for example, that workers own the bubble asset before it emerges. With this particular assumption, the results in this situation will be very similar to the results presented here, at the exception of the path for the wealth of the workers. This situations is represented in Figure A.3.3 in Appendix A.3.3.

run. In Hirano et al. (2015), both types of entrepreneurs initially hold the bubble.[20] When the bubble emerges, productive entrepreneurs sell their share of the bubble asset to the unproductive agents and use these funds for productive investments. In that case, output also increases as long as the reduction of the capital produced by the unproductive entrepreneurs is smaller than the increase in capital produced by the productive entrepreneurs. That is to say, the initial bubble share of the productive entrepreneur, which determines the change in wealth of the productive agent, is important to determine the dynamics of aggregate output.

To summarise, the initial ownership of the bubble, i.e. the initial condition, is a crucial factor to determine the short run effect of the bubble on the economy. The dynamics of the economy would be very different, for example, if banks were the only initial owners of the bubble. This is beyond the scope of this chapter, but deserve future attention.

2.5 Financial Stability along the Bubble Path

As it was shown in the previous section, banks hold a higher fraction of the bubble among their assets during the growth of the bubble than in the stochastic steady-state. Consequently, the macroeconomic effect of the bubble's burst on the economy would be deepened. In this section, I evaluate the financial stability along the growth path of the bubble.

To evaluate the financial stability during the growth of the bubble, several measures of banks vulnerabilities are examined under the three different supervisions ($\xi = 5\%, 15\%$ and 30%). In addition to the evolution of the aggregate net worth of the banks, I mainly focus on the changes in i) their debt to equity ratio, ii) the size of the banking bubble, iii) the relative size of potential bailout needed if the bubble bursts and iv) the contamination of the banking sector. The indicators i) and iii) are depicted in Figure 2.7, ii) and iv) in Figure 2.5.

The debt to equity ratio (DER) is a common measure that shows how a company finances its assets. It may also indicate the level of risk a company faces in case of an economic slowdown. A low DER usually indicates less risk and a more financially stable business, whereas a high ratio indicates a higher fraction of its asset financed by debt and may reflect a high exposure to the risk of bankruptcy. In this economy, both types of banks finance their assets in the same way, meaning that the DER is the same for both types.[21] Under $\xi = 5\%$ and 15%, the DERs initially decrease during the first period because of the reallocation effect. Since

[20] In Hirano et al. (2015), entrepreneurs produce capital that they supply to competitive firms which use capital and labour for production. Workers inelastically supply labour to firms, meaning that the wage is only a function of capital.

[21] When both types of banks operate, the non-arbitrage condition implies that $\phi_t^l = \phi_t^b = \phi_t$. Thus, their borrowing is proportionally the same, $D_t^l = \gamma N_t^l \phi_t/(1-\lambda)$ and $D_t^b = \gamma N_t^b \phi_t/(1-\lambda)$. Consequently, the DER is the same for both types $D_t^l/(\gamma N_t^l) = D_t^b/(\gamma N_t^b) = \phi_t/(1-\lambda)$.

unproductive entrepreneurs reduce their supply of deposits in order to invest in the bubble, banks' liabilities decrease at date 1. However, banks net worth initially remains the same since it is realised by the investments undertook at date 0.[22] Under $\xi = 30\%$, unproductive entrepreneurs have the incentive to make more deposit in period 1 because the deposit rate is higher than under both stronger supervision rules. Therefore, the DER increases from the emergence of the bubble in period 1. Afterwards, the DERs under the three supervision rules significantly increase up to period 5 (they reach up to 37% in term of deviation from the bubbleless steady-state when $\xi = 5\%$ and up to 28% when $\xi = 15\%$).

While the DERs increase for every banks, leverage is ineffective for lending banks. Both ROEs fall and banks have a lower performance during this period. Nonetheless, deposits from the unproductive entrepreneurs quickly increase. Consequently, banks liabilities grow faster than their net worth and banks are more vulnerable to shocks during this period. Once the pressure on the interest rates falls, the spread grows and the DERs converge to their stochastic steady-state. Interestingly, the DERs of the banks are lower in the stochastic steady-states than in the bubbleless steady state, indicating a better stability of the banking sector in the long run with bubbles. Moreover, weaker supervision rules lower the DERs. Since more banks enjoy higher returns because of the premium of the bubble, the aggregate wealth of the banks is greater and less debt is needed to finance the assets. Hence, a weaker banking supervision allows banks to have a relatively better resistance to shocks in the short run while making the banking sector more stable in the long run. To summarise, banks are less resistant to financial shocks in the short run than in the long run. With higher DERs in the first periods, new bubbly investments will amplify profits or losses and so are riskier. However, with a lower DER in the long run, investments are relatively 'less risky'.

The emergence of the bubble reduces the interest rate spread and incentivises to banks to specialise in bubbly investments at the beginning of its growth. The contamination of the banking sector is then higher and the size of the banking bubble is bigger (see Panel C of Figure 2.5). Consequently, if bubbly banks default during this period, the deposit subsidy that the government will have to provide will be more substantial. Panel D of Figure 2.7 represents the potential size of the bubbly banks' bailouts as a fraction of their net worth. When $\xi = 5\%$, no banks fail at any time if the bubble bursts (banks stop trading the bubble from period 6). However, when $\xi = 15\%$ and 30%, bubbly banks will require deposit subsidy at any time. Moreover, the potential losses of the bubbly banks remain larger than their steady-state levels for a long time while the banking bubble size is relatively larger only

[22] The asset does not exist before date 0, so the wealth of the bankers is predetermined; $N_1 = R_0^l B_0 - R_0^d D_0$. In contrast, aggregate wealth at date 2 is not completely predetermined since it also depends on the price of the asset only given at date 2; $N_{2|s} = R_1^l B_1 + m_1^b \mu_{2|s} - R_1^d D_1$.

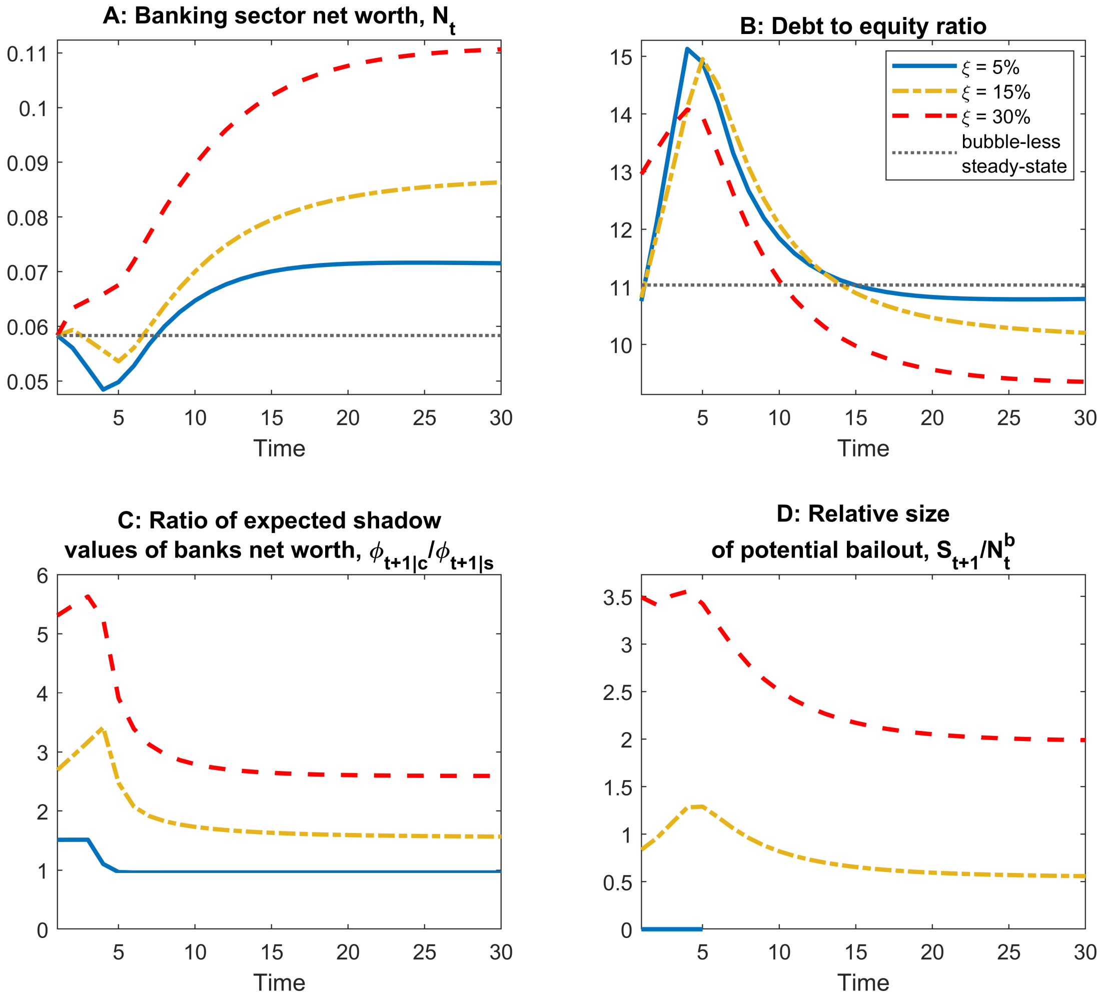

Figure 2.7: Dynamics of financial stability

This figure illustrates how several indicators of financial stability evolve under three different intensities of banking supervision ξ, which are represented by the solid line ($\xi = 5\%$), the dash-dotted line ($\xi = 15\%$) and the dashed line ($\xi = 30\%$). The dotted line depicts the bubbleless steady-state of these indicators. Panel A illustrates the dynamics of the aggregate net worth of the banking sector, N_t. Panel B depicts the debt to equity ratio, $D_t/(\gamma N_t)$. This ratio mainly indicates how a company finances its assets. Panel C depicts the ratio of expected state-contingent values of investments, $\phi_{t+1|c}/\phi_{t+1|s}$, where $\phi_{t+1|c}$ ($\phi_{t+1|s}$) stands for the shadow value of the bank net worth when the bubble collapses (survives) at time $t+1$. Panel D represents the relative size of the potential bailout that bubbly banks could require if the bubble bursts. The size of the potential bailout is expressed in term of bubbly banks' the net worth, S_{t+1}/N_t^b.

during the first four periods of the growth. This is the consequence of two related factors. First, banks are less resistant when the interest rate spread is zero, and second, bubbly investments are financed with more debt. Therefore, if the bubble bursts during this period, a higher bailout would be required to save the bubbly banks.

While the macroeconomic response to the bubble burst primarily depends on the degree of contamination of the banking sector, the size of the bailout is also an important indicator to evaluate the macroeconomic impact because it determines whether the banks fail and in what magnitude. For example, the banking sector can be highly contaminated, but if the losses are small and no bailout is required, then there will be no insolvencies and the macroeconomic effect of the burst is limited (e.g when $\xi = 5\%$, no banks fail, but the banking sector is significantly contaminated in the first periods). On the contrary, the losses can be high and require a bailout, but if the banking sector is barely contaminated, the number of insolvencies will be marginal and the macroeconomic effect is limited as well (these equilibria exist in the stochastic steady-state for very weak banking supervisions and risky bubbles). The parameter that impacts the contamination and the potential losses in this economy is the supervision rule. When $\xi = 5\%$, there are no insolvencies because banks resist well to the shock due to their small exposure to the bubble. However, when $\xi = 15\%$ and 30%, the magnitude of insolvencies can be colossal. Under $\xi = 30\%$, 88% of the banks fail in the first periods and need a bailout up to 3.5 times their wealth to be rescued. The relaxation of the banking supervision gives the incentive to the bubbly banks to invest a huge quantity of funds in the bubble. The main reason is that once the banks know that they could fail, they are indifferent about the size of their losses since they know that the government will subsidise them. Here, the magnitude of insolvencies is at its highest at date 5 but remains high during the first ten to fifteen periods of the growth of the bubble due to very high contamination of the banking sector.

To summarise, banks are relatively less resistant during the first periods of the growth of the bubble because their net worth contracts due to the pressure on the interest rates. However, in term of potential losses, the results are mixed. When banks balance sheet are significantly exposed to the bubble, their potential losses are relatively high and the macroeconomic effect of the burst of the bubble asset would be amplified by the high contamination of the banking sector.

2.5.1 The Impact of Bubble Holding of Entrepreneurs

When entrepreneurs have the incentive to hold the bubble during its growth, there is a strong pressure on the deposit rate, which weakens the banking sector vulnerability to economic shocks. Moreover, if the banking supervision is relatively weak, banks have the incentive to invest in the bubble asset. Thus, the banking sector will be even more vulnerable to financial shocks and bubble burst.

In contrast, when entrepreneurs do not have the incentive to hold the bubble asset during its growth, but banks still do, the bubble will affect the lending rate because of non-arbitrage (there could be arbitrage and the banking sector could be completely contaminated, but it can only happen if banks do not fail when the bubble bursts, i.e. only if the supervision constraint is severe). In this case, the banking sector is not as vulnerable as in a situation where entrepreneurs also hold the bubble, because the larger interest rate spread keeps the contamination of the banking sector relatively constant.

2.6 Recessions and Banking Crises

In this section, I examine the effects of the bubble burst along the growth path of the economy. The dynamics of the economy after the burst crucially depends on bubble ownership. A burst at the stochastic steady-state will serve as a benchmark to analyse the effects of the bubble's collapse during its growth.

2.6.1 Bursts at the Stochastic Steady-state

When the bubble bursts in the steady-state, bubble owners are directly affected by the shock and see their wealth plummet.

For the unproductive entrepreneurs, the magnitude of the losses is relatively similar under the different banking supervisions (wealth falls by 24%-27% after the burst). After a large drop in their wealth, it slowly converges back to the bubbleless level (see Panel C of Figure 2.8). However, the investment decisions of the unproductive entrepreneurs will depend on the financial state of the banking sector. For the banks, the magnitude of their losses is significantly determined by the share of the bubble that they hold and the contamination of the banking sector during the burst. When $\xi = 5\%$, banks are not negatively impacted by the burst of the bubble (see Panel D of Figure 2.8). On the contrary, their net worth initially increases after the burst. After the burst, the unproductive entrepreneurs have to start to produce again and unproductive labour demand slowly increases.[23] The higher labour demand rises the wage and thus decreases the deposit rate. While the deposit rate decreases, the lending rate increases due to a lower supply of loans from the banks. Consequently, the interest rate spread slightly increases and banks net worth increases as well. In this case, the reallocation of resources initially increases output before returning to its bubbleless level. When $\xi = 15\%$, the macroeconomic impact of the burst is more mitigated. When the bubble bursts, the aggregate net worth of the banking sector falls by 47%. Despite this large drop, 54% of the banks survive the burst with enough net worth to continue to absorb unproductive funds. Consequently, output also increases before returning to its bubbleless level such as the dynamics under $\xi = 5\%$.

However, when $\xi = 30\%$, 75% of the banks fail and the aggregate net worth of the banking

[23]Figure A.8 in the Appendix section A.3.4 illustrates the dynamics of resources after the burst.

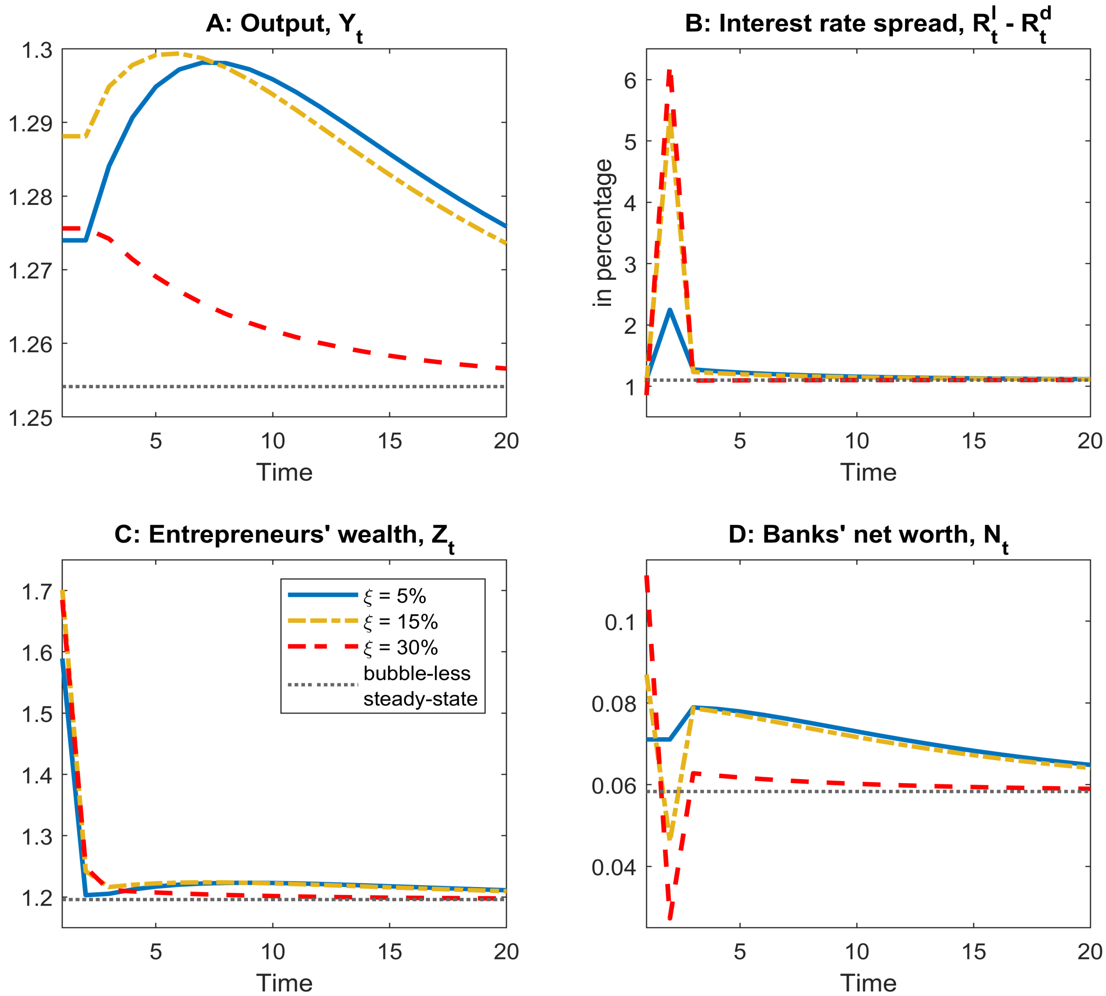

Figure 2.8: Dynamics of the economy after the burst

This figure illustrates the dynamics of the economy under three different intensities of banking supervision ξ, which are represented by the solid line ($\xi = 5\%$), the dash-dotted line ($\xi = 15\%$) and the dashed line ($\xi = 30\%$). The dotted line depicts the bubbleless steady-state of these variables. Panel A illustrates the dynamics of the aggregate output, Y_t. Panel B depicts interest rate spread, $R^l_t - R^d_t$. Panel C describes the aggregate wealth of the entrepreneurs, $Z_t = Z^H_t + Z^L_t$. Panel D represents the aggregate net worth of banks, N_t.

sector drops by almost 75%. In this case, the contraction of the aggregate net worth of the banks is so large that it is difficult for 'surviving' banks to absorb all the unproductive funds due to their borrowing constraint. As a result, unproductive entrepreneurs allocate all their fund to unproductive production and the deposits directly plummets to its bubbleless steady-state. While the wage increases due to the high unproductive demand for labour, the deposit rate falls and the large drop of deposits increases the lending rate. In this instance, the interest rate spread soars and the turmoil of the banking sector blocks the financial intermediation. Therefore, unproductive investments directly replace the productive investments and output decreases and converges back to its bubbleless level.

2.6.2 Burst during the Growth

This subsection investigates the effects of the burst at different dates during the growth of the bubble. At $t = 0$, the economy is at the bubbleless steady-state. At $t = 1$, the bubble asset appears on the market and the agents trade the asset. Figure 2.9 illustrates a sample of the potential crashes of the economy after the emergence of the bubble at period one when the supervision constraint is $\xi = 15\%$.[24] In addition, Tables 2.3 and 2.4 compare the responses of the economy after the burst of the bubble using different reference frames, i) the bubbleless steady-state and ii) the bubble path.

The results of this subsection are essentially based on the conclusion of section 2.5. Figure 2.9 clearly shows that the economy severely contracts after the burst of the bubble during the first periods of its growth and needs time to recover. For example, banks net worth falls below the bubbleless steady-state by 79% at date 5 and needs at least 5 periods to recover. This dynamic is the result of a highly contaminated banking sector during this period due to high pressure on the interest rates and of a lack of time to sufficiently increase their wealth in order to be more resistant to shocks. Consequently, many banks fail when the bubble bursts and the aggregate net worth of the banking sector plummets. In comparison, the wealth of the agents barely falls under their bubbleless level if the bubble bursts in the late periods of its growth. However, if their wealth drop below the bubbleless steady-state, their net worth quickly recover. For instance, banks net worth falls below the bubbleless steady-state by 26% at date 20 but needs only one period to recover. In both early and late bursts situations, the lending rate and the wage increase following the failure of banks, making productive entrepreneurs worse off whereas they were not owning the bubble. Table 2.3 shows that the drop in productive wealth as the bubble matures becomes smaller in term of deviations from the bubbleless steady-state, but the losses are larger in term of deviations from the bubble path.

Unproductive entrepreneurs who were owning the bubble are also deeply impacted by the burst. If the bubble bursts in the early periods of its growth, unproductive entrepreneurs, like the banks, barely had time to use the bubble to accumulate wealth. Therefore, they are relatively vulnerable to economic shocks during this period. However, even if the intensity of the shock is more severe at the beginning of the growth of the asset, the size of the drop appears to be more important in the late period of the growth (see Table 2.4). To illustrate, when the bubble bursts at date 5, the wealth of the unproductive entrepreneurs drops by 0.33, which represents a deviation from the bubble path of -30%. In comparison, a burst at date 20 causes a drop of 0.45 in the wealth of the unproductive entrepreneurs, but only

[24]In Appendix A.3.4, Figures A.9 and A.10 describe some potential paths that the economy can follow after the emergence of the bubble under the supervision constraint $\xi = 5\%$ and $\xi = 30\%$.

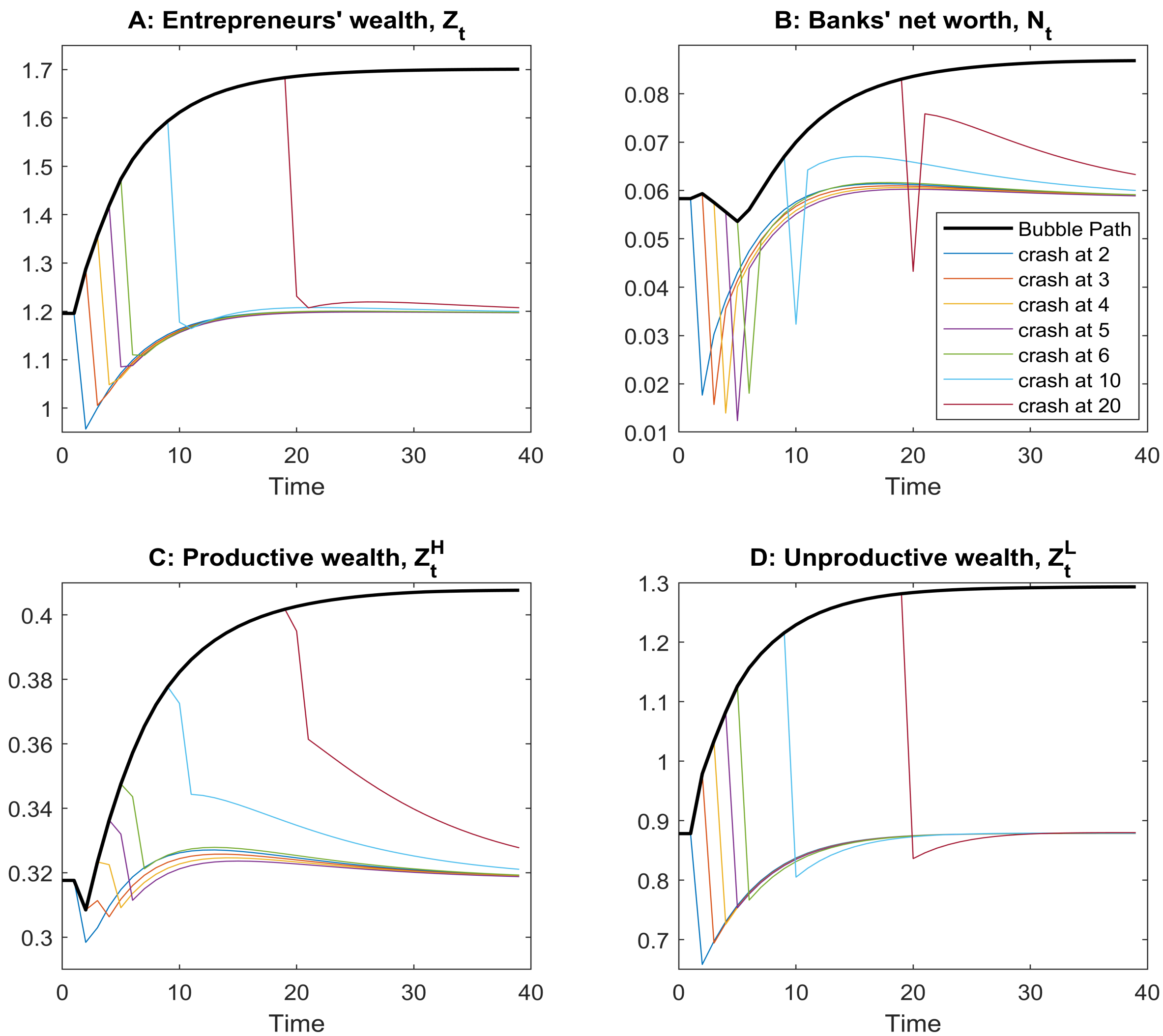

Figure 2.9: Bubble path and potential crashes

This figure illustrates a sample of the potential paths that the economy can follow after the emergence of the bubble at period one under the supervision constraint $\xi = 15\%$ (Figures A.9 and A.10 which show the potential dynamics under $\xi = 5\%$ and $\xi = 30\%$ are available in Appendix A.3.4). The bold black line represents the surviving path of the wealth of the different agents. Departing from this bubble path at different periods, the thin solid lines depict the dynamics of the agents' wealth after a potential burst of the bubble that converge to the bubbleless steady-state. Panel A represents the potential dynamics of the aggregate entrepreneurial wealth, $Z_t = Z_t^H + Z_t^L$. Panel B depicts the potential dynamics of the aggregate net worth of the banks, N_t. Panel C and D respectively represent the potential dynamics of the aggregate wealth of the productive and unproductive entrepreneurs, Z_t^H and Z_t^L.

represents a deviation from the bubble path of -35%. This means that even if agents losses are larger when the bubble is more mature, their relative losses in term of their wealth is similar.

To summarise, investors have more funds to allocate in bubbly investments as the bubble grows because of the wealth effect of the bubble. Thus, bubble owners will experience larger losses as the bubble matures. However, an early burst of the bubble is more intense for the

Percentage deviation from bubbleless steady-state where $\xi = 15\%$						
Variables		period 2	period 5	period 10	period 20	bubble Ss
Productive wealth	Z_t^H	-6.06%	4.52%	17.29%	24.35%	26.11%
Unproductive wealth	Z_t^L	-25.05%	-14.23%	-8.33%	-4.78%	-4.19%
Banks' net worth	N_t	-69.68%	-78.75%	-44.55%	-25.80%	-20.85%
Workers' wealth	Z_t^W	-22.20%	-12.31%	-3.42%	1.69%	2.76%
Total output*	Y_{t+1}	-17.82%	-9.76%	-2.04%	2.31%	3.25%

Table 2.3: Percentage deviations of the wealth and output from the bubbleless steady-state the period after the burst

This table report the value of the different variables after the burst of the bubble at different periods during the growth of the bubble where $\xi = 15\%$ and $\pi = 98\%$. The bubble emerges at period 1. If the bubble bursts at period 5, the unproductive wealth deviates from the bubbleless steady-state by -14.23%. $*$ Output is not affected by the burst of the bubble during the same period because it is completely pre-determined by state variables of the period that precedes the burst. Thus, the reported values of output in this table correspond to the output level one period after the burst of the bubble.

Percentage deviation from bubble path where $\xi = 15\%$						
Variables		period 2	period 5	period 10	period 20	bubble Ss
Productive wealth	Z_t^H	-6.06%	-1.29%	-1.36%	-1.69%	-1.78%
Unproductive wealth	Z_t^L	-25.05%	-30.43%	-33.79%	-34.75%	-34.94%
Banks' net worth	N_t	-69.68%	-77.7%	-51.75%	-47.88%	-46.92%
Workers' wealth	Z_t^W	7.86%	5.36%	3.22%	2.11%	1.91%
Total output*	Y_{t+1}	-17.82%	6.55%	2.85%	0.88%	0.52%

Table 2.4: Percentage deviations of wealth and output from the bubble path the period after the burst

This table reports the values of the different variables after the burst of the bubble at different periods during the growth of the bubble where $\xi = 15\%$ and $\pi = 98\%$. The bubble emerges at period 1. If the bubble bursts at period 5, the unproductive wealth will drop by -30.43%. $*$ Output is not affected by the burst of the bubble during the same period because it is completely pre-determined by state variables of the period that precedes the burst. Thus, the reported values of output in this table correspond to the output level the period after the burst of the bubble.

agents because of the vulnerability of their financial situations.

2.7 Conclusion

In this chapter, I analysed how the ownership of stochastic bubbles affects the financial stability and the real activity during the different stages of the development of the bubble. Based on the presented analysis, I can draw the following conclusions.

First, the intensity of the banking crisis and its quantitative effects on real activity are mostly determined by both the overall contamination of the heterogeneous banking sector and the individual exposure of banks to the risky bubble. The more banks fail following the burst of the bubble, the deeper is the recession and the slower is the recovery. Importantly, the dynamics of bubble growth matters for financial stability. I show that banks prefer to invest in the bubble at the beginning of its development as the emergence of the bubble initially rises the interest rates and reduces the spread. A low interest rate spread creates important distress for the banks as leverage becomes less effective. Consequently, the unusual larger exposure of banks to the risky bubble asset makes this period extremely vulnerable to financial shocks. In order to dampen the impact of the bursting bubble, a banking supervision rule that limits the exposure of the banks to the risky bubble should be very strict at the beginning of the bubble's growth. However, such strict banking rule weakens the financial health of the banks and makes them more vulnerable to economic shocks.

Second, the model is particularly interesting to investigate how bubbles affect production efficiency in the economy during the different stages of their development. This chapter finds that entrepreneurial bubbles can crowd-in and crowd-out investments and the final effect on output depends on the size of the bubble: while relatively small entrepreneurial bubbles substantially increase production level, large bubbles reduce it. A similar result was found for banking bubbles at the exception that banking bubbles reduce relative less output when they become too large. Because banking bubbles are an indicator of low lending rates, productive entrepreneurs can borrow at a lower cost and invest more in their production which helps to maintain a relatively higher level of output.

Finally, this chapter emphasises the importance of the initial ownership of the bubble asset to determine the dynamics of the economy after its emergence. When a bubble emerges, there is a reallocation of resources that drives the economy towards a new (stochastic) equilibrium. In this chapter, the reallocation effect of the bubble causes a temporally fall in output because the propagation of the wealth effect of the bubble in the economy is slow. This dynamics differs from the results found in the recent literature which assumes that the bubble asset initially belongs to productive agents. The direct effect of the bubble on the wealth of productive agents compensates the reduction of unproductive investments by productive ones and so maintains or even increases the output right after the bubble's emergence.

Chapter 3

Asset Price Bubbles and Monetary Policy: Deflate the Bubble?

This chapter studies monetary policy in a New Keynesian model with an asset price bubble. It shows that monetary policy that targets asset prices can significantly deflate an asset price bubble; monetary policy is efficient to reduce the price of a bubble only if the asset is financed with debt. However, using monetary policy to deflate a bubble can be costly in term of output and a monetary policy that overreacts to asset prices can generate a recession.

3.1 Introduction

The economic upturn and downturn associated with the subprime mortgage bubble in the US revived a long debate on how monetary policy should react to asset price developments. The current consensus about monetary policy is that the main objective of central banks is to maintain price stability, that is to say keeping a low and stable inflation. However, price stability generally concerns the stabilisation of the consumer price index, which covers only a segment of prices in the economy. While the omission of asset prices for monetary policy is normally not considered a problem, large movements in asset prices and bubbles' bursts led many economists to reconsider if the focus of monetary policy on consumer prices alone is still pertinent.

Before the 2008 financial crisis, the conventional strategy often named the "Jackson Hole Consensus", called for central banks to focus on maintaining price stability and stabilizing the output gap. Thus, central banks were recommended to ignore asset price fluctuations unless they were a threat to price and output stability (e.g. Bernanke and Gertler, 1999, 2001; Kohn, 2006). One of the main reasons for this consensus is that instruments of monetary policy were judged 'too blunt' to successfully target asset prices. However, this strategy prescribed that central banks should take the necessary actions (e.g. via interest rate cuts) once a bubble collapsed in order to protect the economy against the harmful effects of the bubble's burst.

This asymmetric strategy for reacting to asset price developments was challenged by many economists arguing that price stability would not guaranty financial stability (e.g. Cecchetti et al., 2000; Borio and Lowe, 2002). The opposing prominent strategy for central banks is 'leaning against the wind', which advocates that central banks should try to mitigate the risk associated to the developments and bursts of bubbles. In this case, central banks are required to tighten their monetary policy stance in the face of an inflating asset market, even if it creates a temporary deviation from their price stability objective.

The depression of the US economy following the 2008 banking crisis led many economists to agree that monetary policy should also focus on financial stability. A pure passive 'cleaning up the mess' policy is likely to be more costly than a 'leaning against the wind' policy. Ikeda (2017) concluded that the optimal monetary policy should be tightened to control the output boom caused by the bubble at the expense of inflation stabilisation. Miao et al. (2019) argue that, under adaptive learning, monetary policy that leans against the wind can reduce the volatility of bubbles. Galichère (2022) found that monetary policy that targets asset price bubbles is inefficient in reducing the volatility of the bubble size. However, using monetary policy that leans against the wind can stabilise faster investment, output and inflation than a policy rule that does not react to asset prices. In contrast, Galí and Gambetti (2015) argued central banks should not lean against the wind. They found that tightening monetary policy would persistently increase stock prices. While there is no agreement on how central banks should react to bubbles, there is a consensus that monetary policy may have a role in addressing bubbles.

This chapter revisits this debate and investigates if monetary policy has a role to play; Can we deflate a bubble? How and at what cost? To answer this questions, I develop a New Keynesian model with an asset price bubble and investment capacity constraints. I find that monetary policy can deflate a bubble (i.e. significantly bring back the market price of an asset to its fundamental value) if the bubble asset is financed with debt. Using a Taylor rule, monetary policy can deflate the bubble in two different ways i) by targeting the price of the bubble asset and ii) by lowering the inflation target. Both methods have similar effects on the bubble, but can also be costly. Reacting too much to the asset price bubble can strongly lower output and create a recession. This result supports the idea that interest rates are too blunt instruments to react to asset prices.

Nonetheless, reacting to the price of bubble using the interest rate may be a proactive solution to mitigate a recession of the economy associated to a potential bust; bubbles financed with leverage are considered the most costly for the economy if they burst (e.g. Jordà et al., 2015; Aoki and Nikolov, 2015; Galichère, 2020). Consequently, it is desirable to have the least leveraged bubbles as possible to avoid financial crisis risks. This chapter shows that a bubble financed using leverage can be 'unleveraged' using monetary policy. Precisely, a monetary policy that reacts to the price of the bubble will reduce the bubble ownership of investors who use debt to finance their bubble holdings, but will not affect the bubble investments from investors who financed their asset holdings with their own funds.

The model has a standard New Keynesian structure: it includes households, wholesale firms and retailers, augmented by the 'classical' entrepreneurs' set-up in the literature on rational asset price bubbles (e.g. Kocherlakota, 2009; Hirano et al., 2015; Aoki and Nikolov, 2015; Hirano and Yanagawa, 2017). Entrepreneurs in this model provide capital to wholesale firms and face investment capacity constraint if they adjust or reallocate funds from their potential level.[1] Finally, the rational asset price bubble considered in this chapter is a pure bubble, in other words, an asset without fundamental value. The bubble can emerge in this economy because of the lack of means of saving; credit frictions create an inefficient environment with low rates of return which incentivise investors to hold an intrinsically useless asset. Additionally, the rates of return of the entrepreneurs are worsen because of an investment capacity constraint faced by the entrepreneurs and a monetary policy that targets an inflation rate of 2%.

The rest of the chapter is structured as follows. Section 3.2 outlines the model and its calibration. Section 3.3 analyses how targeting asset prices or reducing inflation target can deflate the bubble and describe the role of investment frictions in these results. Finally, section 3.4 concludes.

3.2 The Model

The model is based on New Keynesian framework augmented by the classical entrepreneurs set-up used in the recent literature to study rational bubble (e.g. Kocherlakota, 2009; Aoki et al., 2014; Hirano et al., 2015; Aoki and Nikolov, 2015; Hirano and Yanagawa, 2017). It is a discrete-time economy populated by households and entrepreneurs who have infinite lives. Households supply labour to wholesale firms. Entrepreneurs, who face idiosyncratic productivity shocks à la Kiyotaki (1998), supply capital goods to these wholesale firms. The wholesale firms are perfectly competitive and sell their production to retailers who face a

[1] The potential level of the economy in this chapter is defined as the bubbleless steady-state.

quadratic cost to change their price à la Rotemberg (1982). Finally, bubbles emerge in this economy because of credit frictions that generate a shortage of means of saving.

The model has two equilibria; one without bubble called bubbleless equilibrium and one with bubble which is called bubbly equilibrium. I briefly outline the model in this section, all details are given in Appendix B.1.

3.2.1 Entrepreneurs

Decision Problems

There is a continuum of entrepreneurs of measure one. At each date t, some entrepreneurs are productive with a probability p and the others are unproductive with probability $(1-p)$. Each entrepreneur is endowed with a production technology, which converts investment x_t to capital k_{t+1}:

$$k_{t+1} = a_t^i x_t - \frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^i} - 1\right)^2 x_t,$$

where a_t^i is the productivity parameter of type i only known at time t and λ_x^i is the investment target associated to this technology.[2] Respectively, capital goods producers with productivity rate $a_t^i = a^H$ are called productive entrepreneurs and capital goods producers with productivity rate $a_t^i = a^L$ are called unproductive entrepreneurs, where the productivity rates respects $a^H > a^L$. The specific investment target parameter λ_x^i is set such that there is no cost in the bubbleless steady-state.

Every entrepreneur has the same preferences and maximises the following expected discounted utility:

$$U^E = E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t, \tag{3.1}$$

Entrepreneurs purchase consumption and can invest into a bubble asset s_t at price μ_t. They

[2]This investment target differs from the conventional adjustment cost mechanism, which can be found for example in Miao et al. (2015b). The conventional adjustment cost includes past investment x_{t-1} instead of a constant level λ_x^i, which results in optimisable frictions. In other words, capital good producers are able to adapt their production system over time. In this chapter, it is assumed that entrepreneurs cannot adapt their production system and any deviations in capital investments from their potential level results in misproduction. Entrepreneur do not have the ability to overproduce and any additional investment above the target generates unusable capital goods. Alternatively, investing less than the target results in small but marginal misproduction. Therefore, this non-optimisable mechanism can be interpreted as a investment constraint capacity which has related function to the traditional mechanism, but allow for a simplified solution without denaturing the analysis.

can borrow funds ($b_t > 0$) or lend funds ($b_t < 0$). Their source of incomes comes from the sell of capital goods k_{t+1} at price q_{t+1} to perfectly competitive firms. The flow-of-funds constraint is given by:

$$c_t + x_t + s_t\mu_t - b_{t+1}\frac{(1+\pi_{t+1})}{R_t} = q_t k_t - b_t + s_{t-1}\mu_t = z_t, \tag{3.2}$$

where z_t stands for entrepreneur's net worth and R_t is the interest rate.

Because of the frictions in the credit market, the borrowing entrepreneurs can pledge at most a fraction $\theta \in (0,1)$ of their future expected revenue as collateral to the banks. Hence, the borrowing constraint is given by:

$$b_{t+1} \leq \theta E_t\left[q_{t+1}k_{t+1} + s_t\mu_{t+1}\right], \tag{3.3}$$

where μ_{t+1} is the future price of the bubble asset.

Entrepreneurs maximise (3.1) subject to (3.2) and (3.3).

Optimal Behaviours

Following Sargent (1987), the decision problem of the entrepreneurs can be interpreted as a savings problem with uncertain returns. Since their instantaneous utility function is logarithmic and because there is no labour income or transfer income, entrepreneurs consume a constant fraction of their net worth z_t:

$$c_t = (1-\beta)z_t \tag{3.4}$$

and save the remaining β fraction of their wealth.

Entrepreneurs have several possibilities for accumulating net worth. They can lend and obtain the expected real rate of return $E_t\left[R_t/\left(1+\pi_{t+1}\right)\right]$, invest into bubble and earn $E_t\left[\mu_{t+1}/\mu_t\right]$ or use their technology to invest in unleveraged capital good production and have a rate of return:

$$E_t\left[q_{t+1}\left(a_t^i - \frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^i}-1\right)\left(3\frac{x_t}{\lambda_x^i}-1\right)\right)\right].$$

Moreover, they can also use collateral in order to borrow extra funds to enlarge their capital good production or undertake leveraged bubble investment. If they decide to pledge only revenue from future production, the rate of return on this leveraged investment is equal to:

$$E_t\left[\frac{q_{t+1}\left(1-\theta\right)\left(a_t^i-\frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^i}-1\right)\left(3\frac{x_t}{\lambda_x^i}-1\right)\right)}{1-\theta\left(a_t^i-\frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^i}-1\right)\left(3\frac{x_t}{\lambda_x^i}-1\right)\right)\frac{q_{t+1}(1+\pi_{t+1})}{R_t}}\right].$$

Alternatively, they can pledge a part of their bubble assets and obtain the leveraged rate of return on bubble investment:

$$E_t\left[\frac{\left(1-\theta\right)\mu_{t+1}/\mu_t}{1-\theta\frac{(1+\pi_{t+1})}{R_t}\frac{\mu_{t+1}}{\mu_t}}\right]$$

In equilibrium, productive entrepreneurs will be the borrowers since they have the greatest rate of returns on capital good production. Therefore productive entrepreneurs produce using leverage and obtain a rate of returns equal to:

$$E_t\left[\frac{q_{t+1}\left(1-\theta\right)\left(a^H-\frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^H}-1\right)\left(3\frac{x_t}{\lambda_x^H}-1\right)\right)}{1-\theta\left(a^H-\frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^H}-1\right)\left(3\frac{x_t}{\lambda_x^H}-1\right)\right)\frac{q_{t+1}(1+\pi_{t+1})}{R_t}}\right].$$

However, productive entrepreneurs will only borrow if their rate of returns on leveraged projects is higher or equal than the unleveraged rate of returns. This implies that the rate of return on productive projects has to be greater than or equal to the borrowing cost. The productive agents are indifferent between leveraged or unleveraged production projects if:

$$E_t\left[\frac{R_t}{(1+\pi_{t+1})}\right]=E_t\left[q_{t+1}\left(a^H-\frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^H}-1\right)\left(3\frac{x_t}{\lambda_x^H}-1\right)\right)\right].$$

In this situation, their collateral constraint might not bind since they might not borrow or might not borrow the maximum amount possible. Alternatively, they can also undertake leveraged bubble investment as long as:

$$E_t\left[\frac{\tilde{\mu}_{t+1}/\mu_t\left(1-\theta\right)}{1-\theta\frac{(1+\pi_{t+1})}{R_t}\frac{\tilde{\mu}_{t+1}}{\mu_t}}\right]\geqslant E_t\left[\frac{q_{t+1}\left(1-\theta\right)\left(a^H-\frac{\Omega}{2}\left(\frac{x_t}{\lambda_x}-1\right)\left(3\frac{x_t}{\lambda_x}-1\right)\right)}{1-\theta\left(a^H-\frac{\Omega}{2}\left(\frac{x_t}{\lambda_x}-1\right)\left(3\frac{x_t}{\lambda_x}-1\right)\right)\frac{q_{t+1}(1+\pi_{t+1})}{R_t}}\right]$$

The unproductive entrepreneurs have relatively poor investment opportunities in comparison to productive entrepreneurs. They can lend, use their technology to invest in unleveraged production projects and invest in the bubble. When the productive entrepreneurs cannot absorb all the national saving, unproductive entrepreneurs will produce. Hence, the unproductive non-arbitrage condition implies:

$$E_t\left[\frac{R_t}{(1+\pi_{t+1})}\right]=E_t\left[q_{t+1}\left(a^L-\frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^L}-1\right)\left(3\frac{x_t}{\lambda_x^L}-1\right)\right)\right]$$

whenever the unproductive entrepreneurs produce. The unproductive entrepreneurs will only lend if the real interest rate is greater than the rate of returns on production, $E_t\left[\frac{R_t}{(1+\pi_{t+1})}\right]>E_t\left[q_{t+1}\left(a^L-\frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^L}-1\right)\left(3\frac{x_t}{\lambda_x^L}-1\right)\right)\right]$. When the unproductive entrepreneurs invest into the bubble, the non-arbitrage condition between lending funds to the productive entrepreneurs and investing in the bubble is:

$$E_t\left[\frac{\mu_{t+1}}{\mu_t}\right]=E_t\left[\frac{R_t}{(1+\pi_{t+1})}\right]. \tag{3.5}$$

3.2.2 Households

Decision Problem

There are households with a unit measure. Each household is endowed with one unit of labour in each period, n_t, which is inelastically supplied in the labour market, and earns real wage rate w_t. The lifetime utility of the representative household is:

$$U^w=E_0\sum_{t=0}^{\infty}\beta^t\ln c_t, \tag{3.6}$$

where $\beta\in(0,1)$ is the subjective discount factor and c_t is the consumption of an agent at date t. In addition of their labour income, they have the ability to save ($b<0$), borrow ($b>0$) and purchase bubble assets, s_t, at the relative price μ_t. Finally, households own the retailers and receive their profit Π_t^F each period. The flow-of-funds constraint is given by:

$$c_t-b_{t+1}\frac{(1+\pi_{t+1})}{R_t}+s_t\mu_t=w_tn_t-b_t+s_{t-1}\mu_t+\Pi_t^F=z_t, \tag{3.7}$$

where z_t the wealth of the household at time t.

Unlike the entrepreneurs, households do not have access to the production technology in order to borrow. The only collateralizable asset that households could use to borrow is the bubble asset. However, only a fraction θ of the value of households' asset holding can be seized by creditors. Hence the collateral constraint is given by:

$$b_{t+1}=\theta E_t\left[s_t\mu_{t+1}\right] \tag{3.8}$$

Households maximize (3.6) subject to (3.7) and (3.8).

Optimal Behaviours

The FOCs for consumption, saving and unleveraged bubble investments are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &: \ \Lambda_t = \frac{1}{c_t} \\
\frac{\partial \mathcal{L}}{\partial b_{t+1}} &: \ \Lambda_t = \beta E_t \left[\Lambda_{t+1} \frac{R_t}{(1+\pi_{t+1})} \right] \\
\frac{\partial \mathcal{L}}{\partial s_t} &: \ \Lambda_t = \beta E_t \left[\Lambda_{t+1} \frac{\mu_{t+1}}{\mu_t} \right]
\end{aligned}$$

If households decide to undertake leveraged bubble investments, the FOCs for bubble investments and saving, $\frac{\partial \mathcal{L}}{\partial b_{t+1}}$ and $\frac{\partial \mathcal{L}}{\partial s_t}$, imply:

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \frac{(1-\theta)\,\mu_{t+1}/\mu_t}{1-\theta \frac{(1+\pi_{t+1})}{R_t} \frac{\mu_{t+1}}{\mu_t}} \right]$$

In equilibrium, the interest rate will be low because of the credit frictions coming from the entrepreneurs' side. Consequently, households will not have the incentive to lend money. Moreover, the rate of return on the asset price bubble will be low as well because of non-arbitrage condition and so does not incentivise households to hold the bubble asset. In fact, households will have the incentive to borrow because of the low interest rate, but will not be able to do so because of their lack of collateralizable assets. Consequently, households are hand-to-mouth at all times and consume:

$$c_t = w_t n_t + \Pi_t^F = z_t$$

3.2.3 Wholesale Firms

The wholesale firms are competitive and produce intermediate goods using capital and labour. The production technology of each firm is:

$$y_t^w = k_t^\alpha n_t^{1-\alpha}$$

where k_t and n_t are capital input and labour input at date t. The real profit of the wholesale firms is:

$$\Pi_t^w = p_t^w y_t^w - (q_t k_t + w_t n_t)$$

and the factors of production are paid their marginal product. In equilibrium, the costs of inputs are given by:

$$
\begin{aligned}
q_t &= p_t^w \alpha K_t^{\alpha-1} \\
w_t &= p_t^w (1-\alpha) K_t^{\alpha}
\end{aligned}
$$

where K_t is the aggregate capital level and N_t is the aggregate labour supply which is equal to one ($N_t = \sum n_t = 1$).

3.2.4 Retailers

There is a continuum of retail firms of measure one, each indexed by i. Each retail firm i buys a wholesale good Y_t^w at the nominal price P_t^w and repackaged into a specialized retail good $Y_t(i)$. Retailers sell their specialized retail good $Y_t(i)$ to competitive final firms at price P_t^i. Retailers face a quadratic cost à la Rotemberg (1982) to change their price.

Decision Problem

The problem of a retail firm i is:

$$
V(i) = \max_{P_t^i} E_t \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left[\frac{(P_t^i - P_t^w)}{P_t} Y_t(i) - \frac{\Upsilon}{2} \left(\frac{P_t^i}{P_{t-1}^i} - 1 \right)^2 Y_t \right],
$$

subject to the competitive demand for good $Y_t(i)$:

$$
Y_t(i) = \left[\frac{P_t^i}{P_t} \right]^{-\varkappa} Y_t,
$$

where the term $\frac{\Upsilon}{2} \left(\frac{P_t^i}{P_{t-1}^i} - 1 \right)^2 Y_t$ in the objective function represents the cost of adjusting prices and where $\varkappa > 1$ governs the elasticity of substitution between any two specialized retail goods.

Optimal Behaviour

The first order condition of the retail firms yields the expression for aggregate inflation:

$$
(1+\pi_t)\pi_t = \frac{1 - \varkappa_t + \varkappa_t p_t^w}{\Upsilon} + \beta E_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \frac{Y_{t+1}}{Y_t} (1+\pi_{t+1}) \pi_{t+1} \right], \tag{3.9}
$$

where $p_t^w = P_t^w / P_t$ is the real marginal cost.

Finally, households receive the real profit Π_t^F of the retail firms, which can be found from the aggregation of firms' budget constraints:

$$\Pi_t^F = \left(1 - p_t^w - \frac{\Upsilon}{2}\pi_{t+1}^2\right) Y_t. \tag{3.10}$$

3.2.5 Monetary Policy

In this model, the monetary policy is given by the following Taylor's rule:

$$R_t = R_n \left(\frac{1+\pi_t}{1+\pi^T}\right)^{\phi_\pi} \left(\frac{Y_t}{Y_n}\right)^{\phi_Y} (1+\mu_t)^{\phi_\mu} \tag{3.11}$$

The nominal interest rate R_t reacts to the inflation rate when it deviates from its inflation target π^T, to the output gap where Y_n is the potential level of output and to the price of the bubble asset. The responsiveness is described by, respectively, ϕ_π , ϕ_Y and ϕ_μ. Finally, when the economy is in a bubbleless steady-state, the nominal interest rate is equal to its stabilising rate R_n.

3.2.6 Calibration

There are 17 parameters in this model. The bubbleless steady-state can be computed with only 9 parameters; β, p, a^H, a^L, θ, α, $\varkappa$, Υ and π^T. The other parameters are required to compute any dynamics and the bubbly steady-state.

The annual discount rate β is calibrated to 0.98, the share of capital α equals 0.3, and the adjustment cost parameter for retailers Υ is set to 80. The probability to be in a productive state p, the collateral constraint parameter θ, productivity rates a^H and a^L are calibrated to yield an average corporate leverage close to 0.5 as in Aoki and Nikolov (2015) or Galichère (2020). Thus, p is set to 0.35, θ to 0.5994 and the productivity parameters a^H and a^L respectively to 1.988 and 1.9. The elasticity of substitution between any two specialized retail goods $\varkappa$ is set to 11, which yields a net markup of 10%.

To generate an inefficient environment, the net real interest rate needs to be negative, therefore the model is calibrated such that the nominal interest rate is zero in the bubbleless steady-state ($R_n = 1$) and the inflation rate is at its inflation target π^T.[3] Then inflation target is calibrated to 2%, which generates a real interest rate of $r = 0.98$.[4] The inflation

[3]Note it is not required that the nominal interest rate is equal to zero. To generate an adequate environment for bubbles to circulate, the interest rate needs to be such that $R_t \leqslant (1+\pi^T)$. In this chapter, the interest rate is set to zero to generate the largest inefficiency which maximises the size of the bubble (Weil, 1987).

[4]Taking the average of the Federal Funds Rate adjusted by the expected inflation rate (from the University of Michigan's survey of consumer) from 1990:Q1−2018:Q1, the annual steady-state gross rate r is 0.979.

target is set to 2% in the baseline calibration of the model, but I will investigate the effect of different values of π^T on the size of the bubble and the response of the economy.

This calibration produces the following results in the bubbleless steady-state: a consumption to output ratio of 0.715, an investment to output ratio of 0.268, and an average corporate leverage of 0.494 in the bubbleless steady-state. In the bubbleless steady-state, X_t^H and X_t^L are respectively equal to 0.1800 and 0.0243 which permit to calibrate λ_x^H and λ_x^L, and the potential level of output Y_n is equal to 0.762.

Parameter	Value	Name
β	0.98	Discount factor
p	0.35	probability to be in a productive state
a^H	1.988	Productive technology
a^L	1.900	Unproductive technology
λ_x^H	0.1800	Specific adjustment cost parameter for productive entrepreneurs
λ_x^L	0.0243	Specific adjustment cost parameter for unproductive entrepreneurs
Ω	0.3500	Common adjustment cost parameter for capital good producers
θ	0.5994	Collateral constraint parameter
α	0.30	Share of capital
$\varkappa$	11	Elasticity of substitution between any two specialized retail goods
Υ	80	Adjustment cost parameter for retailers
ϕ_π	1.5	Taylor rule's inflation coefficient
ϕ_y	0.01	Taylor rule's output gap coefficient
ϕ_μ	0	Taylor rule's asset price bubble coefficient
π^T	0.02	Inflation target
R_n	1	Stabilising nominal interest rate (bubbless steady-state value R)
Y_n	0.762	Potential level of output (bubbless steady-state value Y)

Table 3.1: Baseline calibration of the model

In addition to X_t^H, X_t^L and Y_n, the values of last parameters Ω, ϕ_π, ϕ_y and ϕ_μ are needed to compute the bubbly steady-state or any dynamics of the economy. The common adjustment cost parameter for both type of capital good producers Ω is set to 0.35, and the policy feedback coefficients ϕ_π and ϕ_y are respectively calibrated to 1.5, 0.01. Finally, I calibrate ϕ_μ to 0 in the baseline calibration of the model and will investigate the effect of different values of ϕ_μ on the bubble's size and the response of the economy.

3.3 Results

This sections investigates how a policy rule, which targets asset prices or that reduces inflation target, can deflate the bubble. Moreover, I describe the role of investment frictions in the obtention of these results.

3.3.1 Emergence of Bubbles and Failure of Monetary Policy

Consider the bubbleless version of this economy as presented in the second column of Table 3.2. At the steady-state of this economy, the nominal interest rate is at the zero lower bound while the inflation is equal to its target, 2%. Consequently, the real gross rate of interest is lower than 1 ($r = 0.98$). Unproductive capital good producers, who have poor investment opportunities, see their wealth decreasing over time due to low rates of return on lending or capital good production. Because of the lack of profitable means of savings, these agents have the incentive to invest in an asset without fundamental value in order to protect their wealth until they receive a good idiosyncratic productivity shock. This mechanism generates the adequate conditions for the emergence of a pure bubble in this economy.

The existence of a bubble in this economy has a positive effect of the wealth of the unproductive capital good producers who completely stop unprofitable investments and hold the bubble asset instead. With a higher level of wealth due to the higher rate of return of the bubble, unproductive entrepreneurs who turns productive have more funds to invest in capital good production. Consequently, output increases above its bubbleless steady-state level and inflation increases above its target (inflation reaches 6% when $\phi_\mu = 0$).

The central bank, which reacts to inflation that deviates from its target $\pi^T = 2\%$ and reacts to deviation of output from its bubbleless level, increases its nominal rate. However, the central bank is unsuccessfully to bring back inflation on target the economy or output to its potential level.[5] The reason for the failure of the central bank to satisfy its targeting objectives is that the stabilising nominal interest rate is determined by the bubbleless steady-state (i.e. $R_n = 1$).[6]

[5] In an environment without investment adjustment cost and with a policy rule that only react to deviation of inflation from its target π^T, $R_t = R_n \left(\frac{1+\pi_t}{1+\pi^T}\right)^{\phi_\pi}$, I still find that the central bank is unsuccessful bringing back inflation and output to their target after the emergence of the bubble.

[6] In the bubbleless steady-state, the gross real interest rate is less than 1, which is consistent with the literature on rational bubble (e.g. Tirole, 1985). In this chapter, I decided to illustrate an economy at the zero lower bond where inflation equal to its target. Consequently, $R_n = 1$ in the bubbleless steady-state. However, in the bubbly steady-state, the real interest rate is 1 because it is pinned down by the steady-state rate of return on bubbly investments. WLOG, consider the case where $\phi_y = 0$ and $\phi_\mu = 0$, equation (3.11) shows that if inflation is on target, the nominal interest rate must be equal to $R_n = 1$. However, this yields a lower real rate which is not possible because of non-arbitrage condition. Thus inflation has to increase above

Bubbleless and Bubbly Steady-states with $\Omega = 0$, $\phi_\pi = 1.5$ and $\phi_y = 0.01$					
Variables		No bubble	$\phi_\mu = 0$	$\phi_\mu = 0.65$	$\phi_\mu = 1.3$
Bubble's price	μ_t	-	0.0218	0.0217	0.0215
Unproductives' bubble share	s_t^L	-	1.0000	1.0000	1.0000
Nominal interest	R_t	1.0000	1.0599	1.0318	1.0040
Real interest rate	r_t	0.9804	1.0000	1.0000	1.0000
Inflation	π_t	0.0200	0.0599	0.0318	0.0040
Loans	B_t	0.1107	0.1264	0.1255	0.1247
Wage	w_t	0.4865	0.4919	0.4885	0.4853
Relative price of capital	q_t	0.5160	0.5144	0.5144	0.5144
Relative price of intermediate goods	p_t^w	0.9121	0.9183	0.9139	0.9097
Productive investment	X_t^H	0.1800	0.2062	0.2047	0.2034
Unproductive investment	X_t^L	0.0243	0.0000	0.0000	0.0000
Total capital	K_t	0.4040	0.4099	0.4070	0.4043
Total output	Y_t	0.7620	0.7652	0.7636	0.7621
Productive wealth	Z_t^H	0.0730	0.0814	0.0809	0.0803
Unproductive wealth	Z_t^L	0.1355	0.1512	0.1502	0.1492
Households' wealth	Z_t^W	0.5413	0.4446	0.5235	0.5537

Table 3.2: Bubbleless and bubbly steady-states with no investment friction

The second column reports the values of the main variables of the economy without bubble, i.e. the bubble-less steady-state. The third, fourth and fifth columns report the values of the main variables in their stationary equilibrium with bubble, i.e. the bubbly steady-state, under three different Taylor rules bubble coefficient ϕ_μ (i.e. $\phi_\mu = 0, 0.65$ and 1.3).

In addition to be the source of the failure of the monetary policy in satisfying its objective, bubbles can be the cause of financial instability.[7] I will evaluate the efficiency of a policy rule that reacts to the asset price bubble. I will show that monetary policy can be partially successful in deflating the bubble when the rule targets the asset price bubble. Monetary policy will be successful in reducing the size of a bubble which is financed with debt, but will be unable to deflate a bubble fully financed without leverage. Thus, I call the fraction of the bubble financed without leverage the 'incompressible part of the bubble' because it is insensitive to monetary policy targeting.

its target.

[7]In this model, the bubble asset is assumed to be riskless (i.e. it does not burst) for simplicity. If the bubble was risky, the results would be consistent with the presented results in this chapter.

3.3.2 Reacting to the Asset Price Bubble without Investment Frictions

Without investment frictions ($\Omega = 0$), the bubble asset is only held by unproductive entrepreneurs ($s_t^L = 1$ in Table 3.2) because the rate of return on productive investment is high enough. In this situation, the bubble is fully financed without leverage, and monetary policy is inefficient in deflating the bubble. Table 3.2 shows that when monetary policy reacts to the price of the bubble, the price of the bubble remains relatively stable even when the feedback coefficient on asset price bubble is 1.3. By targeting the price of the bubble, the central bank affects the borrowing cost for the productive entrepreneurs, but not the incentive of the unproductive investors because the rate of return on the bubble for the unproductive entrepreneurs $E_t\left[\mu_{t+1}/\mu_t\right]$ is pinned down by the real interest rate $r = 1$. Consequently, a rule that targets asset prices will only affects inflation such that the unleverage rate of return on the bubble remains equal to the real interest rate, i.e. $r = 1$. This is the 'incompressible part of the bubble' by the monetary policy.

Even if the monetary policy is inefficient at significantly deflating the bubble, the increase of ϕ_μ from 0 to 1.3 marginally reduces the price of the bubble by 0.0003. While the price of the bubble marginally changes, the cost of borrowing increases and the productive rate of return decreases. Consequently, aggregate wealth of the entrepreneurs falls and unproductive entrepreneurs have less available funds to invest in the bubble. This explain the small reduction in the price of the bubble when the feedback coefficient on the price of the bubble asset increases.

In summary, a policy rule that reacts strongly to the bubble will not deflate the bubble, but will reduce inflation and close the output gap (Passing from $\phi_\mu = 0$ to $\phi_\mu = 1.3$, inflation falls from 6% to 0% and output decreases from 0.7652 to 0.7621, close to the bubbleless level equal to 0.7620).

3.3.3 Reacting to the Asset Price Bubble with Investment Frictions

With investment frictions ($\Omega = 0.35$), the bubble is not only held by the unproductive entrepreneurs, but also by the productive capital good producers. We can see in Table 3.3 that when the central bank does not react to the asset price bubble, unproductive capital good producers only hold 35% of the bubble, which means that the rest of the bubble is financed with leverage.

Why do productive entrepreneurs want to hold the bubble? Because of the investment

Bubbleless and Bubbly Steady-states with $\Omega = 0.35$, $\phi_\pi = 1.5$ and $\phi_y = 0.01$					
Variables		No bubble	$\phi_\mu = 0$	$\phi_\mu = 0.3$	$\phi_\mu = 0.6$
Bubble's price	μ_t	-	0.0756	0.0585	0.0482
Unproductives' bubble share	s_t^L	-	0.3548	0.4283	0.4978
Nominal interest	R_t	1.0000	1.0600	1.0249	1.0022
Real interest rate	r_t	0.9804	1.0000	1.0000	1.0000
Inflation	π_t	0.0200	0.0600	0.0249	0.0022
Loans	B_t	0.1107	0.1552	0.1451	0.1389
Wage	w_t	0.4865	0.4905	0.4866	0.4843
Relative price of capital	q_t	0.5160	0.5179	0.5170	0.5164
Relative price of intermediate goods	p_t^w	0.9121	0.9183	0.9128	0.9094
Productive investment	X_t^H	0.1800	0.2045	0.2032	0.2024
Unproductive investment	X_t^L	0.0243	0.0000	0.0000	0.0000
Total capital	K_t	0.4040	0.4059	0.4034	0.4019
Total output	Y_t	0.7620	0.7630	0.7616	0.7607
Productive wealth	Z_t^H	0.0730	0.1000	0.0935	0.0895
Unproductive wealth	Z_t^L	0.1355	0.1858	0.1736	0.1662
Households' wealth	Z_t^W	0.5413	0.4430	0.5342	0.5530

Table 3.3: Bubbleless and bubbly steady-states with investment frictions

The second column reports the values of the main variables of the economy without bubble, i.e. the bubble-less steady-state. The third, fourth and fifth columns report the values of the main variables in their stationary equilibrium with bubble, i.e. the bubbly steady-state, under three different Taylor rules bubble coefficient ϕ_μ (i.e. $\phi_\mu = 0, 0.3$ and 0.6).

frictions, the reallocation of resources after the emergence of the bubble generates a misproduction and decreases the rate of return on production of the productive entrepreneurs. To 'escape' this situation, productive capital good producers have the incentive to invest in the bubble asset to avoid misproducing.

In contrast to the economy without investment frictions, we can see that the inflation also reached 6% when $\phi_\mu = 0$, but the output only increases to 0.7630 because of misproduction. Can monetary policy deflate the bubble in this environment? Yes, but partially. The monetary policy will only reduce the part of the bubble financed with debt. We can see that the price of the bubble falls from 0.0756 to 0.0482 when ϕ_μ increases to 0.6, and that the share of the bubble financed without leverage increases from 35% to 50%. This result is particularly interesting when thinking about financial stability, because bubbles financed with leverage are often more harmful to the economy when they burst (e.g. Jordà et al., 2015; Aoki and Nikolov, 2015; Galichère, 2020).

As in the economy without investment friction, Table 3.3 shows that inflation also falls once the central bank reacts to the bubbles. However, we can see in this economy that output may even fall below its steady-state value. In other words, a policy that is too aggressive toward the bubble can create a recession while trying to deflate the bubble.

3.3.4 Deflating the Bubble by Lowering the Inflation Target

Is there another alternative for monetary policy to deflate the bubble instead of targeting the price of the bubble? In this subsection I show that reducing the inflation target has a very similar effect as targeting the price of the bubble.

Table 3.4 presents the steady-states under three different inflation targets post-bubble emergence and where the policy rule does not react to the price of the asset price bubble (i.e. $\phi_\mu = 0$). We can see that lowering the inflation target from 2% to 0% decreases the price of the bubble from 0.0756 to 0.0473. By lowering the inflation target π^T, the central bank reduces the real cost of borrowing for the productive entrepreneurs. In this situation, productive entrepreneurs have lower incentive to hold the bubble. However, despite increasing the rate of return of the productive entrepreneurs, a smaller bubble implies a lower expansionary effect and the aggregate economy contracts; inflation falls as the target is lowered, as well as output, which can also fall below its steady-state level.

3.3.5 Targeting Asset Prices or Lowering Inflation Target?

In the two previous subsections, I showed that in an economy where the bubble is financed with debt, monetary policy could partially deflate the bubble. I presented two alternatives to deflate the bubble, i) implementing a policy rule that targets the price of the asset price bubble, or ii) lowering the inflation target. In this subsection, I show the effect of the combination of different inflation targets and different feedback coefficients of the asset prices on output, nominal interest rate, inflation and the size of the bubble.

Figure 3.1 presents the effects of different policy mixes on the steady-states of the price of the bubble, nominal interest rate, inflation and output. We can observe that the two policy instrument, π^T and ϕ_μ, are substitutes, if not perfect substitutes. The marginal rate of substitution between the two different instruments for the same steady-state level of a variable appears to be constant. This gives a lot of flexibility to the central bank in

Bubbleless and Bubbly Steady-states with $\Omega = 0.35$, $\phi_\pi = 1.5$, $\phi_y = 0.01$ and $\phi_\mu = 0$					
Variables		No bubble	$\pi^T = 2\%$	$\pi^T = 1\%$	$\pi^T = 0\%$
Bubble's price	μ_t	-	0.0756	0.0609	0.0473
Unproductives' bubble share	s_t^L	-	0.3548	0.4155	0.5056
Nominal interest	R_t	1.0000	1.0600	1.0300	1.0000
Real interest rate	r_t	0.9804	1.0000	1.0000	1.0000
Inflation	π_t	0.0200	0.0600	0.0300	0.0000
Loans	B_t	0.1107	0.1552	0.1465	0.1389
Wage	w_t	0.4865	0.4905	0.4872	0.4840
Relative price of capital	q_t	0.5160	0.5179	0.5171	0.5164
Relative price of intermediate goods	p_t^w	0.9121	0.9183	0.9136	0.9091
Productive investment	X_t^H	0.1800	0.2045	0.2034	0.2024
Unproductive investment	X_t^L	0.0243	0.0000	0.0000	0.0000
Total capital	K_t	0.4040	0.4059	0.4037	0.4017
Total output	Y_t	0.7620	0.7630	0.7618	0.7606
Productive wealth	Z_t^H	0.0730	0.1000	0.0944	0.0892
Unproductive wealth	Z_t^L	0.1355	0.1858	0.1753	0.1656
Households' wealth	Z_t^W	0.5413	0.4430	0.5256	0.5532

Table 3.4: Bubbleless and bubbly steady-states with different inflation targets

The second column reports the values of the main variables of the economy without bubble, i.e. the bubble-less steady-state. The third, fourth and fifth columns report the values of the main variables in their stationary equilibrium with bubble, i.e. the bubbly steady-state, under three different inflation targets π^T (i.e. $\pi^T = 2\%, 1\%$ and 0%).

conducting its monetary policy.

In addition to the substitutability of these two instruments, we can observe that a higher inflation target will increase the price of the bubble asset, output, inflation and the nominal interest rate. The policy rule (3.11) shows that a higher inflation target should lower the nominal interest rate, but the inflation increases proportionally more than the target, which raises the nominal interest rate instead of decreasing it.

Finally, Figure 3.2 presents the ratio of the deviation of output deviation from its bubbleless steady-state to the size of the bubble. This ratio gives a measure of the relative expansionary effect of a bubble, which have been used in Hirano et al. (2015) and Galichère (2020) to evaluate the different effects of policies. Hirano et al. (2015) showed that the relative expan-

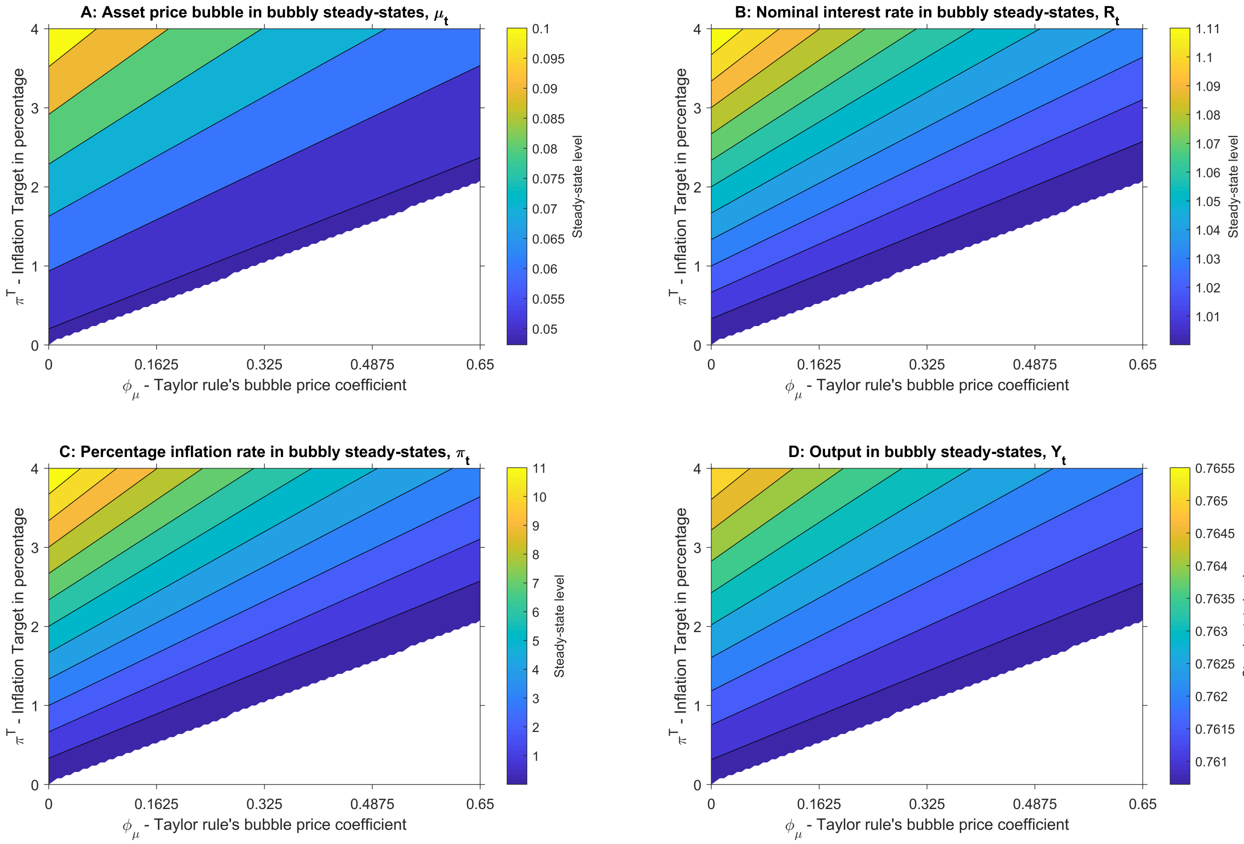

Figure 3.1: Bubbly steady-state analysis of the economy

This figure illustrates the effect of different policy mix on the steady-states values of four main variables: Panel A, the price of the bubble, Panel B, the nominal interest rate, Panel C, the inflation rate, and finally, Panel D, the output level.

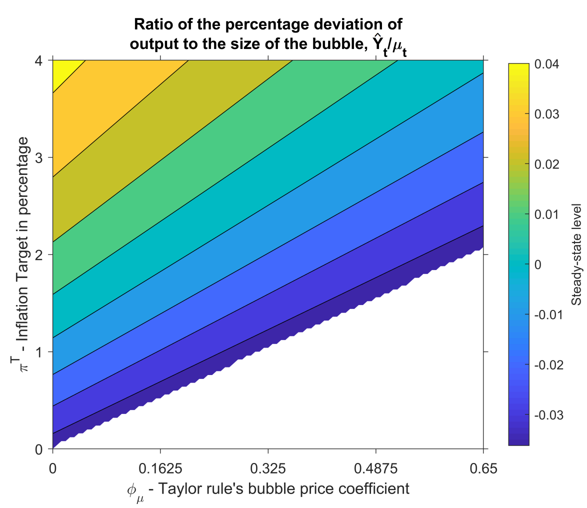

Figure 3.2: Relative expansionary effect of the bubble

This figures illustrates the percentage deviation of output in stochastic steady-states Y_t to its bubble-less steady-state Y_n, $\hat{Y}_t = (Y_t - Y_n)/Y_n$, to the price of the bubble, μ_t.

sionary effect of the bubble is a non-monotonic function of bailouts for entrepreneurs and Galichère (2020) established that this measure is also a non-monotonic function of banking supervision.[8] In this chapter, Figure 3.2 reveals that both policies have a monotonic effect on the relative expansionary effect of the bubble. The lower the feedback coefficient on the price of bubble is, the larger is the expansionary effect of the bubble relative to the size of the bubble. The higher is the inflation target, the larger is the expansionary effect of the bubble relative to the size of the bubble.

3.3.6 Discussion: Investment Capacity Constraint

This chapter introduces a simple mechanism which has important implications on the results. In this section, I want to discuss the utility for this mechanism.

The mechanism is not 'essential' to obtain the results but it allows to simplify the model enough to get these results. The goal of this chapter is to show than when agents finance their bubble investments using debt, then monetary policy can deflate the bubble because it can directly affect the leveraged rate of return of the bubble investment and therefore the arbitrage conditions. The capacity constraint mechanism was designed for this purpose.

The investment capacity constraint generates misproduction as soon as entrepreneurs deviate their level of investment from their potential levels. Without the investment capacity constraint, the leveraged rate of return of the productive entrepreneurs is sufficently high to not incite productive entrepreneurs to invest in the bubble. In this situation, only unproductive entrepreneurs invest in the bubble without using leverage. Monetary policy is therefore inefficient to affect the size of the bubble in steady-state; the non-arbitrage condition between lending and investing in the bubble (equation 3.5) cannot be affected by monetary policy because the real interest rate is pinned down by the rate of return of the bubble.

In contrast, when the entrepreneurs face an investment capacity constraint, the leveraged rate of return of the productive entrepreneurs is lower because of the expansionary effect of the bubble. The decrease in leveraged rate of return is large enough for productive entrepreneur to have the incentive to invest in the bubble with leverage instead of misproducing. In this case, the monetary policy can deflate the bubble. By leaning against the wind, the central bank increases the cost of capital good leveraged production. Despite a higher borrowing rate, the reduction in capital good misproduction has a positive effect on the leveraged rate of return, and productive entrepreneurs do not have to invest as much in the bubble anymore.

[8] Banking supervision in Galichère (2020) prevents banks to over-invest in a risky bubble asset. When the banking supervision is very tight, easing the supervision constraint increases the level of output in the economy. However, when the supervision constraint is relaxed, relaxing the constraint even more decreases the level of output in the economy.

In other words, the central bank reallocates funds from bubble investment to capital good production investments. Nonetheless, the overall capital good production is lower because of lower level of wealth in the economy.

3.4 Conclusion

In this chapter, I developed a New Keynesian model with an asset price bubble and I analysed how monetary policy which reacts to the price of the bubble can affect its size and the way it is financed. Based on the presented analysis, I can draw the following conclusions.

First, the strength of investment frictions can affect the ownership of the asset price bubble and might incentivise the investors to use debt to buy the bubble asset. The emergence of the bubble leads to a reallocation of the resources from unproductive capital producers to productive capital producers. However, this reallocation generates new inefficiencies due to the overproduction of capital goods of the productive entrepreneurs. Consequently, productive capital good producers who can use leverage to boost their rate of return might have the incentive to divert funds from their production to leveraged investments in the asset price bubble.

Second, monetary policy can deflate the bubble in an environment with investment frictions. Monetary policy can deflate the bubble in two different ways i) by targeting the price of the bubble asset and ii) by lowering the inflation target. This chapter shows that both methods have similar effects on the size of the bubble, and that a combination of these two methods can also be considered to reduce the size of the bubble. However, trying to deflate the asset price bubble can also be costly and reacting too much to the asset price bubble can strongly lower output and create a recession. This result supports the idea that interest rates may be policy instruments that are too blunt to react to asset prices.

Third, reacting to an asset price bubble can affect the way the bubble is financed. If a part of the asset price bubble is financed using debt, the leveraged rate of return on the bubble will be affected by the nominal interest rate. In this case, investment frictions are in favour of the monetary authority as increasing the nominal interest will decrease more actively the leveraged rate of return on the bubble than the leveraged rate of return on capital good production. Therefore, productive entrepreneurs will reduce their bubble investments and refocus on capital good production. This result is important because it shows that there is a proactive way to mitigate a potential recession of the economy if the bubble bursts.

Finally, without investment frictions, monetary policy has little power over the size of the bubble because the real interest rate is 'anchored' due to the non-arbitrage condition. In this case, reacting to the asset price bubble will barely affect its size, but will close the positive output gap and reduce inflation to zero.

Chapter 4

Stock Market Bubbles and Monetary policy: a Bayesian Analysis

This chapter develops and estimates a DSGE model with stock market bubbles and nominal rigidities using Bayesian methods. Bubbles emerge through a positive feedback loop mechanism supported by self-fulfilling beliefs, and their movements are driven by a sentiment shock. This chapter shows that stock market bubbles are an important factor to explain volatility in investment, output, and also in inflation. Moreover, a monetary policy rule that targets stock prices can help to diminish the impact of bubble sentiment shocks, and thus stabilise the economy faster than a policy rule that does not react to asset prices.

4.1 Introduction

This chapter investigates how monetary policy interacts with stock market bubbles and asks; Can monetary policy lessen the impact of bubbles, How and at what cost. To answer these questions, I develop a New Keynesian model with rational bubbles, where bubbles can exist because of financial friction (e.g. Miao and Wang, 2018). In contrast to the literature that mainly focuses on pure bubbles (e.g. Martin and Ventura, 2012; Galí, 2014; Hirano and Yanagawa, 2017), the proposed model is based on Miao et al. (2015a) where the stock market price of wholesale firms contains a bubble component in addition to the fundamental value.[1] Unlike pure bubbles, stock market bubbles are attached to productive firms with positive dividends and are not separately tradable from firm stocks. The stock price bubbles

[1]The value of an asset is equal to its market fundamental, that is to say, the expected and discounted present value of its dividends (or more generally its rents), plus a bubble component. Pure bubbles are defined as intrinsically useless assets, that it to say that have no fundamental value. However, these assets have a positive price. Such assets are often interpreted in the literature as money, gold or lands (e.g. Weil, 1987; Kocherlakota, 2009).

can emerge in different firms or in different sectors, and their emergence or collapse may be unrelated to the emergence or collapse of pure bubbles.

Bubbles emerge through a positive feedback loop mechanism supported by self-fulfilling beliefs. Precisely, households believe that the value of some wholesale firms may not be equal to their fundamentals. These firms, which pledge their assets as collateral in order to borrow funds, are able to relax their borrowing constraints because of the 'optimistic' beliefs of households on firms' values. Consequently, firms are able to borrow more and increase profit, which in turn raise the value of these firms. In this sense, bubbles can exist because of self-fulfilling beliefs. Without the presence of bubbles, these firms would be unable to borrow extra funds and deliver higher profits. Finally, as in Miao et al. (2015a), the beliefs of the households about the movement of the bubble are modelled with the introduction of a sentiment shock. I estimate this sentiment shock and evaluate its importance in explaining changes in real and nominal variables.

I find that bubbles can cause large fluctuations in aggregate variables such as investment or output, and can also be the cause of high inflation. Moreover, as in Miao et al. (2019) or Galichère (2021), I found that leaning against the wind can reduce the impact of the sentiment shock which drives bubbles. The latter finding is based on evaluation of two alternative policy rules that target stock prices: the first rule reacts to changes in stock prices and the second reacts to deviations of stock price from its trend. I find that these rules can reduce the impact of bubble sentiment shock, but not the volatility of the bubble size itself. Nonetheless, these alternative policies can stabilise quicker aggregate output, investment and the stock price than a policy rule that does not react to the stock price.

Finally, while these alternative policy rules can quickly stabilise the economy after a sentiment shock, their specification matters for their reaction to inflation. My analysis shows that a policy that reacts to changes in stock prices is not successful in promptly bringing back inflation to steady-state. In contrast, the policy rule that reacts to deviation from steady-state of the stock price will be more aggressive towards inflation. This type of rule will stabilise quicker inflation than the traditional rule or the first alternative rule mentioned above.

The rest of the chapter is structured as follows. Section 4.2 outlines the model. Section 4.3 presents the calibration of structural parameters, the estimation procedure and the estimated parameter. Section 4.4, which presents the main findings, is composed of four parts: i) an evaluation of the model in explaining historical bubble episodes, ii) a counterfactual experiment, iii) a analysis the transmission mechanism of monetary policy in a bubbly economy, and iv) an examination of alternate policy rules that react to stock prices. Finally, section

4.5 concludes.

4.2 The Model

The model is based on the real model presented in Miao et al. (2015a) and incorporates nominal rigidities à la Calvo (1983) to study the interaction between monetary policy and stock market bubbles. The main structure of proposed model is presented in this section, and all the details are given in Appendix C.1. The model represents a discrete time economy populated by households, capital good firms, wholesale firms, retail firms and a Central Bank. Households, capital goods firms and retail firms have infinite lives while wholesale firms operate on the market for a stochastic length of time.

Three financial assets are available in the economy: loans, deposits and stocks of wholesale firms. Households can deposit and invest in wholesale firms stocks but cannot borrow. Wholesale firms can borrow funds for their production or save unused funds. Retail and capital goods firms can neither borrow nor save. Finally, the stock price of the wholesale firms can contain a rational asset price bubble because of financial friction. The size of these bubbles is stochastic.

4.2.1 Households

Decision Problem

Each household derives utility from consumption and leisure according to the expected utility function:

$$U = E_t \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(C_t/A_t - \theta C_{t-1}/A_{t-1}\right)^{1-\sigma}}{1-\sigma} - \psi_t \frac{N_t^{1+\eta}}{1+\eta} \right] \xi_t, \tag{4.1}$$

where C_t denotes consumption and N_t is the household's labour supply. The household consumes C_t and provides labour N_t to wholesale firms for the nominal wage W_t. The household can accumulate wealth by purchasing shares of wholesale firms s_{t+1} at aggregate price P_t^s and by saving in deposits D_t (where $D_t \geq 0$) at the deposit rate R_t^d.

The representative household's budget constraint is given by:

$$C_t + p_t^s s_{t+1} + d_{t+1} \frac{(1+\pi_{t+1})}{R_t^d} = w_t N_t + \Pi_t^I + \Pi_t^F + d_t + \left(d_t^s + p_t^s\right) s_t, \tag{4.2}$$

where p_t^s is the relative aggregate price of the stock, d_t is the real quantity of deposits, π_{t+1} is the inflation rate, w_t is the real wage and d_t^s is the real aggregate dividend on their stock investment. Household receives real profits from capital goods firms Π_t^I, profit from retailers Π_t^F. Moreover, the gross inflation rate $(1+\pi_{t+1}) = P_{t+1}/P_t$, where P_t is price level of consumption.

The representative agent maximises (4.1) subject to (4.2).

Optimal Behaviour

The remainder of the household's problem is standard. The first order conditions for habit-adjusted consumption, labour and deposits are given by:

$$\Lambda_t = \frac{\xi_t}{(C_t/A_t - \theta C_{t-1}/A_{t-1})^\sigma} - \theta\beta E_t\left[\frac{\xi_{t+1}}{(C_{t+1}/A_{t+1} - \theta C_t/A_t)^\sigma}\right], \tag{4.3}$$

$$\frac{N_t^\eta}{\Lambda_t} = \frac{w_t}{\xi_t\psi_t}(1-\tau_t), \tag{4.4}$$

$$\Lambda_t = \beta E_t\left[\frac{\Lambda_{t+1}}{(1+\pi_{t+1})}\right] R_t^d \qquad \text{if } d_t > 0, \tag{4.5}$$

where Λ_t represents the marginal utility of consumption. The first order condition for share of aggregate stock investment is:

$$\Lambda_t = \beta E_t\left[\Lambda_{t+1} r_t^s\right] \qquad \text{if } s_{t+1} > 0, \tag{4.6}$$

where the expected real rate of stock return is defined as $r_t^s \equiv E_t\left[\left(d_{t+1}^s + p_{t+1}^s\right)/p_t^s\right]$. Equations (4.6) and (4.5) set the non-arbitrage condition between stock investments and deposits, which is given by:

$$E_t\left[\frac{\Lambda_{t+1}}{(1+\pi_{t+1})}\right] R_t^d = E_t\left[\Lambda_{t+1} r_t^s\right].$$

4.2.2 Capital producers

Decision Problem

The households own capital producers and receive the profit Π_t^I. A representative capital goods firm produces new capital using input of final output and subject to some adjustment costs. It sells new capital I_t to wholesales firms at price P_t^I. The objective of a capital producer is to choose I_t to maximise:

$$V^I = \max_{I_t} E_t \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0}\left[p_t^I I_t - \left(1 + \frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}} - \lambda^I\right]^2\right)\frac{I_t}{Z_t}\right],$$

where p_t^I is the relative price of capital goods, λ^I is the growth rate of aggregate investment, $\Omega > 0$ is the adjustment cost parameter and Z_t represents an investment cost shock that follow the exogenous process:

$$\ln Zt = \rho_Z \ln Z_{t-1} + \epsilon_t^Z$$

where ϵ_t^Z is an independent and identically distributed shock (IID) over time.

Optimal behaviour

The optimal level of investment goods satisfies the first order condition with respect to I_t:

$$\begin{aligned} Z_t p_t^I &= 1 + \frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}} - \lambda^I\right]^2 + \Omega \frac{I_t}{I_{t-1}}\left[\frac{I_t}{I_{t-1}} - \lambda^I\right] \\ &\quad - \beta \frac{\Lambda_{t+1}}{\Lambda_t}\left(\frac{I_{t+1}}{I_t}\right)^2 \Omega \left[\frac{I_{t+1}}{I_t} - \lambda^I\right] \frac{Z_t}{Z_{t+1}} \end{aligned} \tag{4.7}$$

and household receive the real profit:

$$\Pi_t^I = p_t^I I_t - \left(1 + \frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}} - \lambda^I\right]^2\right) \frac{I_t}{Z_t} \tag{4.8}$$

4.2.3 Wholesale Firms

Following Carlstrom and Fuerst (1997), Bernanke, Gertler, and Gilchrist (1999), and Gertler and Kiyotaki (2010), and Miao et al. (2015a), exogenous entry and exit of firms is assumed because of non-arbitrage condition. To understand the necessity of this mechanism, suppose that households believe that each wholesale firm's stock may contain a bubble and that this bubble may burst with some probability. Because of rational expectations, a bubble cannot re-emerge in the same firm after bursting. Otherwise there would be an arbitrage opportunity. This means that none of the firms would contain any bubble once all bubbles have burst if no new firms enter the economy.

A firm may exit with an exogenously given probability δ_e each period. After exiting the economy, its value is zero and a new firm enters the economy without costs so that the total measure of firms is fixed at unity in each period. A new firm entering at date t starts with an initial capital stock K_{0t} and then operates in the same way as older firms. Moreover, each new firm may bring a new bubble into the economy with probability ω.

Wholesale firms make investment decisions that maximize their stock market values. They can purchase investment goods I_t from capital producers at price P_t^I and they sell their good

Y_t^j to retail firms at price P_t^w.

Decision Problem

A wholesale firm $j \in [0,1]$ combines capital K_t^j and labour L_t^j to produce intermediate goods Y_t^j using the production function:

$$Y_t^j = \left(u_t^j K_t^j\right)^{\alpha} \left(A_t N_t^j\right)^{1-\alpha}, \tag{4.9}$$

where $\alpha \in (0,1)$, u_t^j denotes the capacity of utilisation rate and A_t denotes the labour-augmenting technology shock (or total factor productivity (TFP) shock given the Cobb-Douglas production function). For a new firm entering at date t, I set $K_t^j = K_{0t}$.

Assume that the capital depreciation rate between period t and period $t+1$ is given by $\delta_t^j = \delta(u_t^j)$, where δ is a twice continuously differentiable convex function that maps a positive number into $[0,1]$. The function $\delta(\cdot)$ does not need to be parametrised because the model will be solved using the log-linearisation solution method, where the steady-state capacity utilisation rate will be normalized to 1.

The capital stock evolves according to:

$$K_{t+1}^j = \left(1 - \delta_t^j\right) K_t^j + \varepsilon_t^j I_t^j, \tag{4.10}$$

where I_t^j denotes investment and ε_t^j is an idiosyncratic shock that measures the efficiency of the investment. Investment is assumed to be irreversible at the firm level so that $I_t^j \geq 0$. Moreover, ε_t^j is an IID shock across firms and over time, and is drawn from the fixed cumulative distribution Φ over $[\varepsilon_{min}, \varepsilon_{max}] \subset (0, \infty)$ with mean 1 and probability density function ϕ. This shock induces firm heterogeneity in the model. For tractability, assume that the capacity utilisation decision is made before the observation of investment efficiency shock ε_t^j. Consequently, the optimal capacity utilisation does not depend on the idiosyncratic shock ε_t^j.

In each period t, firm j can make investment I_t^j by purchasing investment goods from capital producers at the price P_t^I. Its real flow-of-funds constraint is given by:

$$d_t^{sj} + p_t^I I_t^j - l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} = p_t^w Y_t^j - w_t A_t N_t^j - l_t^j, \tag{4.11}$$

where $l_{t+1}^j > 0$ (< 0) represents the real quatity of borrowing (savings) at time t, R_t^l represents the lending rate, p_t^w is the relative price of wholesale firms' goods and $d_t^{sj} > 0$ (< 0) represents dividends (new equity issuance). Assume that external financial markets are imperfect so that firms are subject to the constraint on new equity issuance:

$$d_t^{sj} \geqslant -\varphi_t K_t^j, \tag{4.12}$$

where φ_t is an exogenous stochastic shock to equity issuance. The demand for capital good is constrained such that:

$$0 \leqslant p_t^I I_t^j \leqslant p_t^w Y_t^j - w_t A_t N_t^j + \varphi_t K_t^j - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l}. \tag{4.13}$$

In addition, external borrowing is subject to the credit constraint:

$$\begin{aligned} \underbrace{\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(K_{t+1}^j, l_{t+1}^j\right)}_{\text{continuation value of the firm}} \geqslant & \underbrace{\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(K_{t+1}^j, 0\right)}_{\text{value of the firm if it defaults}} \\ & - \underbrace{\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(\gamma_t K_t^j, 0\right)}_{\text{the threat value to the lender}}, \end{aligned} \tag{4.14}$$

where $\bar{V}_{t,a}\left(K_t^j, l_t^j\right) \equiv \int V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) d\Phi(\varepsilon)$ represents the ex-ante value after integrating out ε_t^j and $V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right)$ represents the cum-dividends stock market value of the firm of age a with with assets K_t^j, debt l_t^j and idiosyncratic investment shock ε at time t . In equation (4.14), γ_t represents the collateral shock that reflects the frictions in the credit market. Note that a represents the age of the firm. The equity value depends on the age of the firm because it contains a bubbles component that is age dependent.

Following Miao and Wang (2018), equation (4.14) is an incentive constraint in a contracting problem between the firm and the lender which ensures firm j has no incentive to default in equilibrium. The firm has limited commitment and can default on debt l_{t+1}^j at the beginning of period $t+1$. If the firm does not default, its continuation value is given by $\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(K_{t+1}^j, l_t^j\right)$. If the firm defaults, the debt is renegotiated, the repayment is relieved and the value of the firm is $\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(K_{t+1}^j, 0\right)$. The lender can seize the collateralised asset $\gamma_t K_t^j$ and keep the firm running with these assets by reorganizing the firm. Thus the threat value to the lender is $\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(\gamma_t K_t^j, 0\right)$.[2] Then the RHS of equation (4.14) is the value of the firm if it chooses to default.

An intermediate goods producer j with age a chooses labour, $N_t^j \geqslant 0$, investment, $I_t^j \geqslant 0$,

[2]The variable γ_t may be interpreted as an efficiency parameter in the sense that lender may not be able to efficiently use the firm's assets K_{t+1} (Miao and Wang, 2018).

and debt, $l_{t+1}^{j} \geqslant 0$, to maximize its value:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= \max_{I_t^j, N_t^j, l_{t+1}^j, u_t^j} p_t^w Y_t^j - \left(w_t A_t N_t^j + I_t^j p_t^I\right) - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} \\ &\quad + (1-\delta_e)\, \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1,a+1}\left(K_{t+1}^j, l_{t+1}^j, \varepsilon_{t+1}^j\right), \end{aligned}$$

subject to the production function (4.9), the law of motion of capital (4.10), the constraint on new equity issuance (4.12), the borrowing (4.14) and the flow of funds (4.11).

As in Miao et al. (2015a), I conjecture and verify that the value function takes the form:

$$V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) = Q_t\left(\varepsilon_t^j\right) K_t^j + B_{t,a}\left(\varepsilon_t^j\right) - Q_t^L\left(\varepsilon_t^j\right) l_t^j. \tag{4.15}$$

Optimal Behaviour

The first order condition of a wholesale firm's problem with respect to labour, given the wage rate W_t and the capacity utilisation rate u_t^j, yields the labour demand of the wholesale firm j:

$$N_t^j = \frac{u_t^j}{A_t} \left[\frac{(1-\alpha)\, p_t^w}{w_t}\right]^{\frac{1}{\alpha}} K_t^j. \tag{4.16}$$

Given the wage rate w_t and the capacity utilisation rate u_t^j, the production problem can be simplified such that:

$$\max_{N_t^j} p_t^w Y_t^j - w_t A_t N_t^j = u_t^j \Psi_t K_t^j$$

where Ψ_t is given by:

$$\Psi_t = \alpha \left[\frac{1-\alpha}{w_t}\right]^{\frac{1-\alpha}{\alpha}} \left(p_t^w\right)^{\frac{1}{\alpha}} \tag{4.17}$$

after substituting out the labour decision (4.16) of the production problem.

Using (4.6), the date-t ex-dividend stock relative price of the firm j of age a can be rewritten as:

$$p_t^{s,j} = (1-\delta_e)\, \beta E_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1,a+1}\left(K_{t+1}^j, l_{t+1}^j, \varepsilon_{t+1}^j\right)\right].$$

Given the above conjecture (4.15), stock relative price can be rewritten in the form:

$$p_{t,a}^{sj} = q_t K_{t+1}^j + b_{t,a} - q_t^L l_{t+1}^j,$$

where q_t, $b_{t,a}$ and q_t^L define such that:

$$\begin{aligned} q_t &= (1-\delta_e)\,\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} Q_{t+1}\left(\varepsilon_{t+1}^j\right), & (4.18)\\ b_{t,a} &= (1-\delta_e)\,\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} B_{t+1,a+1}\left(\varepsilon_{t+1}^j\right), & (4.19)\\ q_t^L &= (1-\delta_e)\,\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} Q_{t+1}^L\left(\varepsilon_{t+1}^j\right). \end{aligned}$$

Note that q_t, $b_{t,a}$ and q_t^L do not depend on future idiosyncratic shocks ε_{t+1}^j because they are integrated out.

The first order condition for l_{t+1}^j using the guess of the value function (4.15) gives:

$$q_t^L = \frac{(1+\pi_{t+1})}{R_t^l},$$

and the credit constraint can be rewritten such that:

$$q_t \gamma_t K_t^j + b_{t,a} \geqslant \frac{(1+\pi_{t+1})}{R_t^l} l_{t+1}^j.$$

The investment level I_t^j of a wholesale firm j with a bubble depends on the efficiency shock of the investment ε_t^j being greater that the threshold ε_t^*. Consequently, the optimal investment level I_t^j of a wholesale firm j respects that:

$$I_t^j p_t^I = \begin{cases} u_t^j \Psi_t K_t^j + \varphi_t K_t^j - l_t^j + q_t \gamma_t K_t^j + b_{t,a}, & \text{if } \varepsilon_t^j \geqslant \varepsilon_t^*, \\ 0, & \text{otherwise.} \end{cases} \tag{4.20}$$

where the investment threshold, $\varepsilon_t^* \equiv \frac{p_t^I}{q_t}$, is given by the first order condition for I_t^j using the guess of the value function (4.15).

Each firm chooses the same capacity utilisation rate u_t satisfying:

$$\Psi_t (1+G_t) = q_t \delta^{'}(u_t), \tag{4.21}$$

where G_t satisfies:

$$G_t = \int\limits_{\varepsilon \geqslant \varepsilon_t^*} \left(\frac{\varepsilon}{\varepsilon_t^*} - 1 \right) d\Phi(\varepsilon). \tag{4.22}$$

The the price of installed capital, the bubble, and the lending rate satisfy:

$$q_t = (1-\delta_e)\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left[\begin{array}{c} u_{t+1}\Psi_{t+1} + q_{t+1}(1-\delta_{t+1}) \\ +G_{t+1}\left(u_{t+1}\Psi_{t+1} + q_{t+1}\gamma_{t+1} + \varphi_{t+1}\right) \end{array} \right], \tag{4.23}$$

$$b_{t,a} = (1-\delta_e)\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}(1+G_{t+1})\, b_{t+1,a+1}, \tag{4.24}$$

$$\frac{(1+\pi_{t+1})}{R_t^l} = (1-\delta_e)\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}(1+G_{t+1}). \tag{4.25}$$

where $\delta_t = \delta(u_t)$.

Sentiment shock

The household beliefs on the movement of bubbles are modelled with the introduction of a sentiment shock κ_t. Denote $b_{t,a}$ the real value of the bubble attached to a wholesale firms with age a at time t. Households believe that a new firm in period t may contain a bubble of real size $b_{t,0} = b_t^*$ with probability ω. Then the total value of emerging bubble in the economy at date t is given by $\omega\delta_e b_t^*$. Moreover, they believe that the relative size of the bubbles at date $t+a$ for any two firms born at date t and $t+1$ is given by κ_t such that:

$$\kappa_t = \frac{b_{t+a,a}}{b_{t+a,a-1}}, \quad t \geqslant 0, \ a \geqslant 1.$$

The relative size of bubbles κ_t follows an exogenously given process:

$$\ln \kappa_t = (1-\rho_\kappa)\ln \kappa + \rho_\kappa \ln \kappa_{t-1} + \epsilon_t^\kappa \tag{4.26}$$

where ρ_κ is the persistence parameter and ϵ_t^κ is an IID normal random variable with mean zero and variance σ_κ^2. This process reflects the beliefs of households about the fluctuations of bubbles and is interpreted as the sentiment shock.

4.2.4 Retailers

There is a continuum of differentiated retail firms of measure one, each indexed by i. Each retailer i buys a wholesale good Y_t^j at price P_t^w and repackage into a specialized retail good $Y_t(i)$. Retailers sell their specialized retail good $Y_t(i)$ to competitive final firms at price $P_t(i)$.

Decision Problem

The optimisation problem of a retailer is standard. A firm i chooses prices to maximise its profit:

$$V_t(i) = \max_{\{P_t(i)\}_{s=t}^{\infty}} E_t \sum_{s=t}^{\infty} (\vartheta\beta)^s \frac{\Lambda_{t+s}}{\Lambda_t} \left[\left(\frac{P_t(i)}{P_{t+s}} - p_{t+s}^w \right) Y_{t+s}(i) \right],$$

subject to the competitive demand constraint for good $Y_t(i)$:

$$Y_t(i) = \left[\frac{P_t^i}{P_t} \right]^{-\varkappa_t} Y_t,$$

where p_t^w is the real marginal cost of the retailer and $\varkappa_t > 1$ governs the elasticity of substitution between any two specialized retail goods. Retailers are subject to cost push shocks that affect the elasticity of substitution between any two retail goods, where $\varkappa_t$ follows an exogenously given process:

$$\ln \varkappa_t = (1 - \rho_\varkappa) \ln \varkappa + \rho_\varkappa \ln \varkappa_{t-1} + \epsilon_t^\varkappa$$

Price rigidity and price indexation are introduced as following: i) like in Calvo (1983), a firm i at time t has the opportunity to reset its price $P_t(i)$ with probability ϑ ; ii) when it has the chance of resetting its price, it chooses price optimally, $P_t^*(i)$, with probability $1 - \varpi$, or chooses its new price with probability ϖ according to the simple rule of thumb $P_t^b = P_{t-1}^R \pi_{t-1}$, where P_t^R is given by:

$$P_t^R = \left[(1 - \varpi) P_t^*(i)^{1-\varkappa_t} + \varpi (P_t^b)^{1-\varkappa_t} \right]^{\frac{1}{1-\varkappa_t}}$$

Optimal behaviour

The first order condition of the retail firms with respect to $P_t(i)$ yields the following system for aggregate inflation:

$$H_t = \Lambda_t p_t^w Y_t \frac{\varkappa_t}{(\varkappa_t - 1)} + \vartheta\beta E_t (1 + \pi_{t+1})^{\varkappa_t} H_{t+1} \tag{4.27}$$

$$F_t = \Lambda_t Y_t + \vartheta\beta E_t (1 + \pi_{t+1})^{\varepsilon - 1} F_{t+1} \tag{4.28}$$

$$\begin{aligned} \frac{1 - \vartheta (1 + \pi_t)^{\varkappa_t - 1}}{1 - \vartheta} &= (1 - \varpi) \left(\frac{H_t}{F_t} \right)^{1-\varkappa_t} \\ &\quad + \varpi \frac{\left[1 - \vartheta (1 + \pi_{t-1})^{\varkappa_t - 1} \right]}{1 - \vartheta} \left(\frac{1 + \pi_{t-1}}{1 + \pi_t} \right)^{1-\varkappa_t} \end{aligned} \tag{4.29}$$

The system (4.27)-(4.29), once log-linearised, yields the log-linearise Phillips curve:

$$\pi_t = \kappa_c \hat{p}_t^w + \hat{\varkappa}_t + \chi_f \beta \pi_{t+1} + \chi_b \pi_{t-1} \tag{4.30}$$

where $\chi_f = \frac{\vartheta}{\Upsilon}$, $\chi_b = \frac{\varpi}{\Upsilon}$, $\kappa_c = \frac{(1-\varpi)(1-\vartheta)(1-\vartheta\beta)}{\Upsilon}$, $\Upsilon = \vartheta + \varpi\,(1 - \vartheta + \vartheta\beta)$, and where the cost push shock has been normalised.

4.2.5 Equilibrium

Aggregation and Market Clearing

The aggregation is characterised as follow. Let K_t^A denote the aggregate capital stock after the realisation of the exit shock of the wholesale firms, but before new investments and depreciation take place. Thus:

$$K_t^A = (1 - \delta_e)\, K_t + \delta_e K_{0t}, \tag{4.31}$$

where $K_t = \int_0^1 K_t^j \, dj$ is the aggregate capital stock of all firms at the end of of period $t-1$ before the realization of the exit shock, and K_{0t} is the aggregate capital stock brought by new entrants.

In equilibrium, the labour market clears so the labour demand, $\int_0^1 N_t^j \, dj$, must be equal to its supply, N_t. Then, the aggregate labour demand of wholesale firms is given by:

$$N_t = \frac{u_t}{A_t} \left[\frac{(1-\alpha)\, p_t^w}{w_t} \right]^{\frac{1}{\alpha}} K_t^A, \tag{4.32}$$

where u_t is the capacity utilisation rate which is the same across firms, because wholesale firms have the same capital-labour ratio.

Denote $Y_t^W = \int_0^1 Y_t^j \, dj$ the aggregate output of wholesale firms. The aggregation of their their production functions yields:

$$Y_t^W = \left(u_t K_t^A\right)^{\alpha} (A_t N_t)^{1-\alpha}. \tag{4.33}$$

In equilibrium, the aggregate supply of the wholesale goods Y_t^W has to be equal to the demand of the retailers $\int_0^1 Y_t\,(i)\, dj$. Thus, the final output is given by:

$$Y_t \equiv \int_0^1 \left[\frac{P_t^i}{P_t} \right]^{\varkappa} Y_t\,(i)\;\; di = \left(u_t K_t^A\right)^{\alpha} (A_t N_t)^{1-\alpha}, \tag{4.34}$$

using (4.33), where $Y_t^W = \int_0^1 Y_t(i)\, dj$.

Let b_t denotes the aggregate real bubble at time t. When adding up the bubbles of the firms of all ages, the total real value of the bubble in the economy at time t is given by:

$$\begin{aligned} b_t &= \sum_{a=0}^{t} (1-\delta_e)^a \omega \delta_e b_{t,a} \\ &= m_t b_t^*, \end{aligned} \tag{4.35}$$

where b_t^* is the size of new emerging bubbles at date t and where m_t satisfies the recursion:

$$m_t = m_{t-1}(1-\delta_e)\kappa_{t-1} + \delta_e \omega, \tag{4.36}$$

with $m_0 = \delta_e \omega$. Using the law of motion of the bubble (4.24), restriction on the size of the new bubble is given by:

$$b_t^* = (1-\delta_e)\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}(1+G_{t+1})\kappa_t b_{t+1}^*. \tag{4.37}$$

Finally, substituting total real bubble at date t (4.35) into the restriction on the size of the new bubble (4.37) yields the non arbitrage condition for the total bubble in the economy:

$$b_t = (1-\delta_e)\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\frac{m_t}{m_{t+1}}(1+G_{t+1})\kappa_t b_{t+1}. \tag{4.38}$$

The market clearing conditions for credits implies that the demand for loans is equal to the supplies of savings, $L_t = \int_0^1 L_t^j\, dj = D_t = 0$. Moreover, competitive financial intermediaries require that the deposit rate is equal to the lending rate. Therefore $R_t^d = (1-\delta_e) R_t^l$, taking into account that firms exit the market in each period with probability δ_e. However, from (4.25) and $G_{t+1} > 0$ that follow:

$$\frac{(1+\pi_{t+1})}{(1-\delta_e) R_t^l} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}(1+G_{t+1}) > \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} = \frac{(1+\pi_{t+1})}{R_t^d}.$$

where the RHS of the inequality comes from the FOC of the households (4.5). Consequently, households do not have the incentive to save, and prefer to borrow until their borrowing constraint binds (i.e. $D_t = 0$).[3] Only firms that receive low efficiency shocks save and lend funds to productive firms.

[3]Without borrowing constraints, no arbitrage implies that $G_{t+1} = 0$. In this case, (4.24) and the transversality condition would rule out bubbles.

Aggregating the value of all firms, the aggregate relative stock price is equal to:

$$p_t^s = q_t K_{t+1} + b_t, \tag{4.39}$$

where the aggregate stock holding of the household is normalised to a unit, $s_{t+1} = 1$. This equation reveals that the the relative aggregate price of the stock has two components, the fundamental $q_t K_{t+1}$ and the aggregate bubble b_t.

The total capacity of external financing is given by:

$$\varphi_t K_t + q_t \gamma_t K_t + b_t,$$

which reflects the overall financial market conditions. Following Miao et al. (2015a), a financial shock ζ_t is introduced to capture the disturbance of the overall financial constraints, which is defined as:

$$\zeta_t = \frac{\varphi_t}{q_t} + \gamma_t,$$

so that the total capacity of external financing can be rewritten such as $\zeta_t q_t K_t + b_t$.

Aggregating over the idiosyncratic shock ε_t^j, the aggregate investment is given by:

$$I_t p_t^I = \left[\left(u_t \Psi_t + \zeta_t q_t \right) K_t^A + b_t \right] \int_{\varepsilon > \varepsilon_t^*} d\Phi \left(\varepsilon \right), \tag{4.40}$$

using the financial shock ζ_t, where $\Psi_t = \alpha \left[\frac{(1-\alpha)}{w_t} \right]^{\frac{1-\alpha}{\alpha}} \left(p_t^w \right)^{\frac{1}{\alpha}}$ and $\varepsilon_t^* = \frac{p_t^I}{q_t}$. Furthermore, the law of motion of capital is given by:

$$K_{t+1} = \left(1 - \delta_t \right) K_t^A + I_t \frac{\int_{\varepsilon > \varepsilon_t^*} \varepsilon \; d\Phi \left(\varepsilon \right)}{\int_{\varepsilon > \varepsilon_t^*} d\Phi \left(\varepsilon \right)}, \tag{4.41}$$

and the aggregate price of installed capital follows:

$$q_t = \left(1 - \delta_e \right) \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left[\begin{array}{c} u_{t+1} \Psi_{t+1} + q_{t+1} \left(1 - \delta_{t+1} \right) \\ + G_{t+1} \left(u_{t+1} \Psi_{t+1} + \zeta_{t+1} q_{t+1} \right) \end{array} \right] \tag{4.42}$$

The resource constraint is given by:

$$C_t + \left(1 + \frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}} - \lambda^I\right]^2\right)\frac{I_t}{Z_t} = Y_t \tag{4.43}$$

using the budget constraint of the households (4.2), the flow-of-funds of the wholesale firms (4.11), the profit of the capital good producers (4.8) and the profit of the retailers.

Finally, the policy rule that I will consider is the following Taylor-type rule:

$$R_t^l = R\left[(1+\pi_t)^{\phi_\pi}\left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_y}\right]^{1-\phi_R}\left(\frac{R_{t-1}^l}{R}\right)^{\phi_R}\exp(\epsilon_t^R) \tag{4.44}$$

where R is the natural rate at the zero inflation steady-state, and ϕ_π, ϕ_y and ϕ_R are the three policy feedback coefficients on inflation, evolution of output and past interest rate. Empirical rules in this form are often used in empirical literature and are shown to behave well (e.g. An and Schorfheide, 2007; Chen et al., 2017).[4]

Equilibrium

Equations (4.3), (4.4), (4.7), (4.17), (4.21), (4.22), (4.25), (4.27), (4.28), (4.29), (4.31), (4.32), (4.34), (4.36), (4.38), (4.40), (4.41), (4.42), (4.43) and (4.44) jointly determine the 20 aggregate endogenous variables C_t, N_t, p_t^I, p_t^w, u_t, K_t^A, K_t, w_t, Ψ_t, I_t, q_t, G_t, R_t^l, Λ_t, b_t, m_t, Y_t, H_t, F_t and π_t where $\delta_t = \delta(u_t)$ and $\varepsilon_t^* = \frac{p_t^I}{q_t}$.

4.3 Bayesian Estimation

The model, presenting no occasionally binding condition in equilibrium, is first made stationary and then log-linearised around the bubbly non-stochastic steady-state, and fit the US data using Bayesian estimation methods.

4.3.1 Data and Shocks

The model has seven shocks: 1) a TFP shock, g_{at}, 2) an investment adjustment cost shock, Z_t, 3) an elasticity of substitution between any two specialised retail goods shock, $\varkappa_t$, 4) a financial shock, ζ_t, 5) a sentiment shock, κ_t, 6) a household taste shock ξ_t and finally, 7) a labour supply shock ψ_t.

[4]This form of policy rule has not only been found to be empirically useful, but, when adequately parameterised, can often mimic optimal policy (e.g. Schmitt-Grohé and Uribe, 2007).

These shocks are identified using seven time series to estimate the parameters of the model: 1) the Federal Funds Rate, 2) the industrial inflation rate, 3) the US real GDP, 4) the US real investment, 5) the relative price of investment, 6) the S&P 500 composite index and 7) Chicago Fed's National Financial Conditions Index (NFCI). I compute the quarterly growth rates of real GDP, real investment, relative price of investment and the relative stock price for the estimation.

The data are available quarterly and cover the period from 1975Q1 to 2019Q4. Data for the industrial inflation (2), US GDP (3) and investment (4) are from the BEA website. The Federal Funds Rate (1), the relative price of investment (5) and the Chicago Fed's National Financial Condition Index are retrieved on the FRED website. Finally, the stock price data (6) are the S&P composite index downloaded from Robert Shiller's website. Figure 4.1 presents the transformed data used for the estimation.

4.3.2 Solution and Estimation Procedure

As in Miao et al. (2015a), there is no need to to parametrise the depreciation function $\delta(\cdot)$ and the distribution function $\Phi(\cdot)$ because of the log-linearisation method. These terms will be components of estimated parameters. Yet, knowing the steady-state values of the following parameters is necessary: $\delta(1)$, $\delta'(1)$, $\delta''(1)$, $\Phi(\varepsilon^*)$ and $\mu = \frac{\phi(\varepsilon^*)\varepsilon^*}{1-\Phi(\varepsilon^*)}$ where the capacity of utilisation is equal to 1 in steady-state and ε^* is the steady-state investment threshold for the idiosyncratic shock ε_t. These parameters will be estimated, except for $\delta(1)$ which will be calibrated.

The quarterly real gross rate of interest is calibrated using the means of inflation and Federal Funds Rate times series, $R = R^l - \pi = 1.0048$. As is standard in the literature, the quarterly subjective discount rate is calibrated to 0.995 using the quarterly real gross rate of interest R. The inverse Frisch labour supply elasticity is set to 1/5 which is in the range of macroeconomics estimates. The capital share in production is also set to its traditional value 0.3. The coefficient of relative risk aversion is set at 2 and the elasticity of substitution between any two specialised retail goods is set to 11, yielding a steady-state mark-up value of 1.1 on intermediate good relative price p_t^w. Finally, I set the exit parameter δ_e to 2% as in Miao et al. (2015a).

The steady-state depreciation rate $\delta(u)$ where $u = 1$ in steady-state is calibrated to 0.025, and the steady-state investment to output ratio I/Y is set to 0.2. I use the mean of the growth rate of output to compute the steady-state quarterly gross growth rate of output g_a, which is equal to 1.0068. Finally, using the real rate R and the output growth rate g_a to compute the steady-state relative size of the old bubble to the new bubble, $\kappa = R/g_a$, which

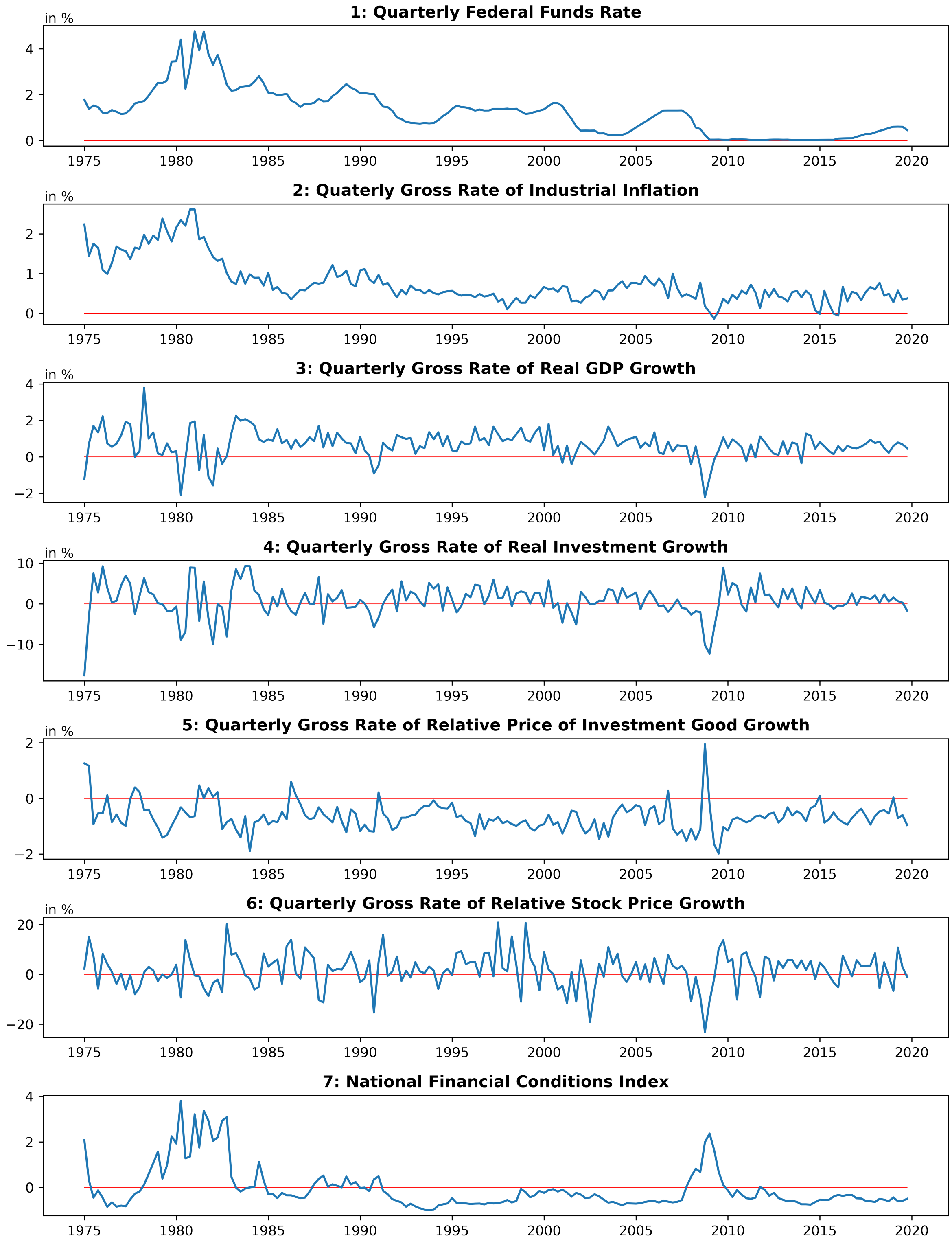

Figure 4.1: Plots of the data from 1975Q1 to 2019Q4

This figure illustrates the different time series used for the Bayesian estimation.

yields 0.9980.

The parameter $K_0/\tilde{K}$ was initially estimated through the model. The mean prior on this parameter was 0.005 with a standard deviation of 0.001. I found that it converged to zero, thus I fixed $K_0/\tilde{K}$ to 0.001.

Table 4.1 below summarises the calibrated parameters of the model.

Parameter	Value	Description
β	0.995	Quarterly subjective discount rate
σ	2	Risk aversion coefficient
η	1/5	Inverse Frisch labour supply elasticity
α	0.3	Capital share in production
δ_e	0.02	Probability of exiting the economy
$\varkappa$	11	Elasticity of substitution between any two specialised retail goods
g_a	1.0068	Steady-state quarterly gross growth rate of output
R	1.0048	Quarterly natural gross rate of interest at the zero inflation steady-state
κ	0.9980	Steady-state relative size of the old bubble to the new bubble
u	1	Steady-state capacity of utilisation rate
$\delta(1)$	0.025	Steady-state depreciation rate
I/Y	0.2	Steady-state investment-output ratio
$K_0/\tilde{K}$	0.001	Ratio of capital endowment for a entering firms to total capital stock

Table 4.1: Calibrated parameters

The estimation was first initiated using a Markov jump-linear-quadratic (MJLQ) model à la Svensson (2005), where uncertainty takes the form of different "modes" (or regimes) that follow a Markov process. The estimation was done with 4 different modes; *Dry Monetary Policy*, *Wet Monetary Policy*, *High Shock Volatility* and *Low Shock Volatility*. The analysis reflected that monetary policy was always dry and the the volatility of shocks was always low. Therefore, I continue the estimation process without using any modes: the likelihood function is combined with the prior distribution to obtain the posterior distribution. A random walk Metropolis-Hastings algorithm is then used to generate 1,000,000 draws from the posterior distribution. Four chains have been used for the simulations.

As in Miao et al. (2015a), I use the NFCI time series to better identify the financial shock ζ_t. The estimation of the model without the NFCI index produces a coherent smoothed financial shock series, yet the financial shocks are very persistent and ρ_ζ converges to 1. The introduction of the NFCI index helps to significantly reduces the persistence of the financial shock. The financial shock is identified using the following measurement equation:

$$NFCI_t = -f_\zeta \hat{\zeta}_t - f_q \hat{q}_t - f_{Kb}(\hat{b}_t - \hat{K}_t)$$

which describes movements in the financing capacity of wholesale firms. An increase in $\hat{\zeta}_t$, $\hat{q}_t$, $\hat{b}_t$ or a decreases in $\hat{K}_t$ reduces the NFCI, in turn relaxes the financial constraint of the wholesale firms. However, the coefficient of the marginal Tobin's Q, f_q, converged to zero in every estimated specifications. Thus I set it to zero ($f_q = 0$).

4.3.3 Estimated Parameters

Table 4.2 presents the prior and posterior distributions of the estimated parameters. Most prior distributions for the parameters are based on the posterior of Miao et al. (2015a) due to the structural similarities between this study and theirs. The priors for parameters related to the structural parameters of New-Keynesian models are as follow: the mean prior of the Calvo parameter ϑ is set to 0.75 with a standard deviation of 0.01, and the mean prior of the price indexation parameter ϖ is set to 0.2 with a standard deviation of 0.02. The feedback coefficient parameter on inflation ϕ_π has a prior mean of 1.5 and a standard deviation of 0.1, the feedback coefficient parameter on change in output ϕ_y has a prior mean of 0.4 and a standard deviation of 0.15, and the feedback coefficient parameter on past nominal interest rate ϕ_R has a prior mean of 0.4 and a standard deviation of 0.1.

The main differences between our estimates come from the estimation of the steady-state values for the financial constraint parameter ζ and the investment productivity distribution parameter μ. Miao et al. (2015a) had a relatively loose prior on these two parameters and explained that ζ was not particularly sensitive to the prior distribution. Using the same priors as Miao et al. (2015a) for these two parameters, the posteriors of these two were significantly different and higher. The mean posterior of the financial shock ζ peaked at 0.7 and the elasticity of the probability of undertaking investment at the steady-state cutoff μ could range between between 5 and 12. Jointly or individually, the values of these parameters yold counter-intuitive results and did not match the range in previous studies. Covas and Den Haan (2011) reported that the financial constraint parameter ranges between 0.1 and 0.4, and Miao et al. (2015a) estimated $\zeta^M = 0.3$.[5] For the elasticity μ, Miao et al. (2015a) obtained a posterior mean of $\mu^M = 2.58$ and Wang and Wen (2012) found a similar estimate equal to 2.4. Consequently, I restricted the standard deviation priors for these two parameters

[5]In the case of Miao et al. (2015a), the mean posterior of the financial shock ζ did not deviate from the mean prior.

	Prior Distribution			Posterior Distribution			
Param.	Distr.	Mean	Std. Dev.	Mean	Std. Dev.	5%	95%
θ	Beta	0.4	0.05	0.9882	0.0021	0.9846	0.9914
Ω	Normal	0.1	0.1	0.1781	0.0397	0.1174	0.2479
$\frac{\delta''}{\delta'}$	Normal	10	2	12.3344	1.4024	9.8474	14.4560
ζ	Beta	0.15	0.01	0.2970	0.0121	0.2770	0.3168
μ	Normal	2.3	0.01	2.3155	0.0098	2.2995	2.3313
ϑ	Normal	0.75	0.01	0.7671	0.0088	0.7530	0.7818
ϖ	Beta	0.2	0.02	0.2190	0.0208	0.1858	0.2543
ϕ_π	Normal	1.5	0.1	1.6374	0.0694	1.5256	1.7538
ϕ_y	Beta	0.4	0.15	0.3271	0.1175	0.1443	0.5302
ϕ_R	Beta	0.4	0.1	0.8929	0.0129	0.8722	0.9141
f_ζ	Beta	0.5	0.15	0.1465	0.0200	0.1158	0.1804
f_{Kb}	Beta	0.1	0.15	0.0034	0.0005	0.0027	0.0042
ρ_Z	Beta	0.5	0.1	0.8982	0.0204	0.8634	0.9299
$\rho_\varkappa$	Beta	0.7	0.1	0.9877	0.0036	0.9813	0.9930
ρ_ζ	Beta	0.5	0.1	0.7990	0.0329	0.7434	0.8512
ρ_κ	Beta	0.2	0.1	0.2297	0.0809	0.1225	0.3742
ρ_{g_a}	Beta	0.3	0.025	0.3450	0.0264	0.3016	0.3884
ρ_ξ	Beta	0.5	0.1	0.9966	0.0005	0.9956	0.9973
ρ_ψ	Beta	0.5	0.1	0.3390	0.0800	0.2144	0.4782
σ_Z	Inv-Gamma	0.01	0.005	0.0115	0.0009	0.0101	0.0131
$\sigma_\varkappa$	Inv-Gamma	0.005	0.005	0.0113	0.0010	0.0098	0.0129
σ_ζ	Inv-Gamma	0.02	0.005	0.0320	0.0043	0.0257	0.0400
σ_κ	Inv-Gamma	0.02	0.005	0.2937	0.1284	0.1243	0.5284
σ_{g_a}	Inv-Gamma	0.015	0.005	0.0154	0.0010	0.0139	0.0170
σ_ξ	Inv-Gamma	0.012	0.005	5.6266	0.6558	4.5567	6.7090
σ_ψ	Inv-Gamma	0.03	0.005	0.0379	0.0044	0.0313	0.0456
σ_R	Inv-Gamma	0.005	0.0015	0.0074	0.0004	0.0068	0.0081

Table 4.2: Prior and posterior distributions

to 0.01 around mean 0.15 for ζ and 2.3 for μ. It seems that these mean posteriors of these two parameters do not correspond to their priors because I use data on the interest rate in addition to data on investment and stock prices. The introduction of the interest rate as a decision variable by the Central Bank and using it as an observable variable increases the posterior means of ζ and μ. A higher ζ relaxes the borrowing constraint and makes the mean value of ζ less sensitive to changes in the interest rate.

The results of the estimation indicates that habits are very persistent, i.e. $\theta = 0.9882$, and that the adjustment cost parameter for price of investment is equal to 0.1781. These two parameters are significantly higher than in Miao et al. (2015a), who found $\theta^M = 0.54$ and $\Omega^M = 0.03$. For the 'curvature' of the depreciation function $\delta(\cdot)$, I found that $\frac{\delta''}{\delta'} = 12.33$ which is similar to Miao et al. (2015a). The posterior distributions of the feedback coefficients for the policy rule are in the usual ranges of the literature with posterior means of 1.637 for ϕ_π, 0.32 for ϕ_y, and 0.89 for ϕ_R (e.g. Chen et al., 2017). The persistence and standard deviations posteriors are conventional, except for the taste shock $\hat{\xi}_t$ which tends to 1. The implications of the finding of this estimate are discussed in Section 4.4.

4.4 Results

The Result Section is composed of four analyses: i) an evaluation of the model in explaining historical bubble episodes, ii) a counterfactual experiment, iii) a analysis the transmission mechanism of monetary policy in a bubbly economy, and iv) an examination of alternate policy rules that react to stock prices.

4.4.1 Model Evaluation: Sentiment Shock, Bubbly Firms and Aggregate Bubble

During the last 50 years, the US economy experienced two major bubble episodes, the dot-com bubble and the subprime mortgage bubble. In this subsection, I investigate the explanatory power of the presented model about these two events.

As previously established, bubbles emerge because of by self-fulfilling beliefs about the value of wholesale firms. Moreover, movements in bubbles can be driven by household sentiment shocks $\hat{\kappa_t}$ about the relative size of bubbles between two firms of different ages. A positive sentiment shock increases the bubble size of young firms and increases the total value of the bubble. Finally, the total value of the bubble depends on the aggregation of all bubbles in the economy. Due to stochastic lives of the wholesale firms, the aggregation of the bubble depends on the variable m_t whose dynamics is described by equation (4.36). As in Miao et al. (2015a), I interpret m_t as the mass of firms having bubbles. The log-linearized version of m_t is given by:

$$\hat{m}_t = (1 - \delta_e)\, \kappa \left(\hat{m}_{t-1} + \hat{\kappa}_{t-1}\right) \tag{4.45}$$

This equation (4.45) establishes that fluctuations in the mass of bubbly firms depend on past fluctuations in the mass of bubbly firms plus fluctuations in past sentiment shock. Therefore, the sentiment shock affects $\hat{m}_t$ with a lag. However, the bubble law of motion, i.e. equation (4.38), depends on both $\hat{m}_t$ and expected $\hat{m}_{t+1}$. The latter implies that fluctuations in the aggregate bubble depend on both current and lagged sentiment shocks.

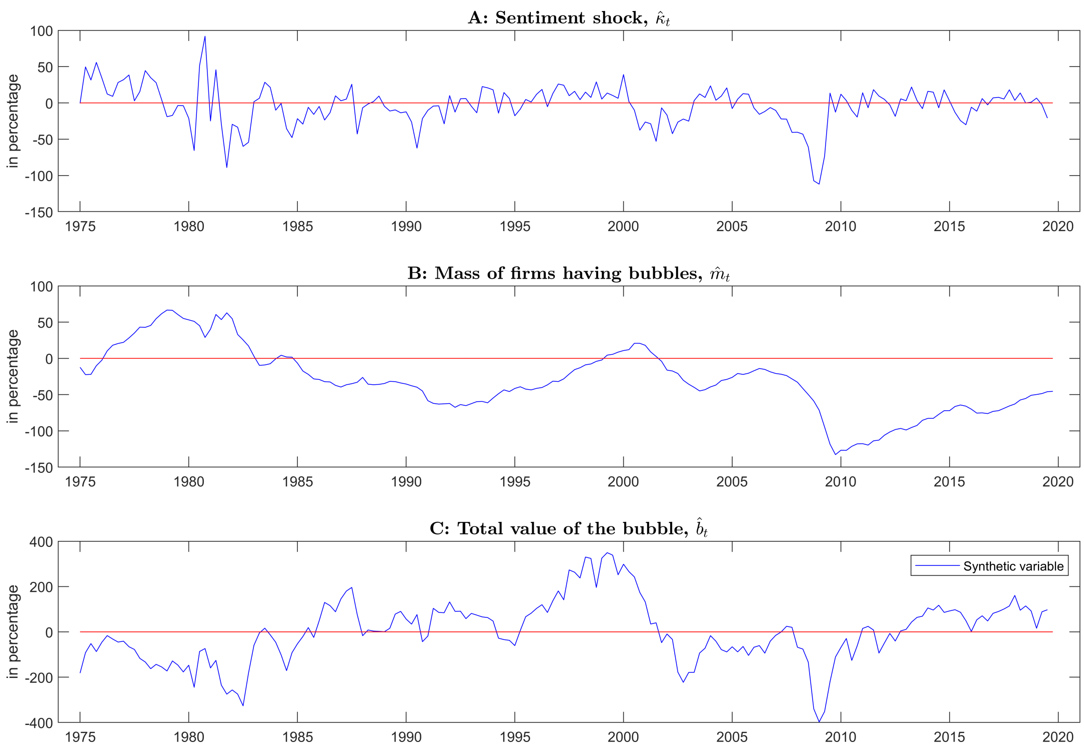

Figure 4.2: Sentiment shock, bubbly firms and aggregate bubbles

This figure illustrates the log-deviations from the steady-state of the sentiment shock $\hat{\kappa}_t$, the mass of bubbly firms $\hat{m}_t$, and the aggregate value of the bubble $\hat{b}_t$ estimated from the model.

Figure 4.2 illustrates the estimation of the sentiment shock (Panel A), the mass of bubbly firms (Panel B) and the total value of aggregate bubble (Panel C). How well does the model describe the boom and bust of the dot-com bubble? Panel B shows that the mass of bubbly firms slowly but constantly increased from the 90s until 2001 due to small positive sentiment shocks. This increase in the mass of bubbly firms seems to have a positive effect on the development of the aggregate bubble (Panel C). During Q1 2001, Panel A clearly depicts a relatively persistent, but moderate, fall in sentiment shocks during the burst of dot-com bubble. We can see that this movement in the sentiment shock implies a net sustained fall in the mass of bubbly firms. This fall also seems to pass on to the total value of the aggregate bubble with a strong fall from 2000Q1 to 2002Q4.

Concerning the subprime mortgage bubble, we can see in Panel A of Figure 4.2 that the sentiment shock was relatively stable around steady-state between 2003 and 2007. This explains why the mass of bubbly firms slightly recovered but remained significantly below steady-state during the ‘booming’ phase of the bubble episode. In other words, the model may not be able to capture well the boom of the subprime mortgage bubble. One possible explanation for this pitfall could be that this boom is relatively localised to the housing market and thus did not strongly affect the S&P 500. Therefore, the boom is not captured in the data. However, the burst of the bubble is clearly depicted by a strong fall in sentiment shock which results in a large and substantial decrease in the mass of the bubbly firms. Panel C also shows that the total value of the bubble plummets from 2007Q4 to 2009Q1.

In the next subsection, I will show that the sentiment shock is not the most significant factor to explain volatility of the bubbles which contrasts with the results of Miao et al. (2015a). The sentiment shock is able to explain the volatility in the bubble, but its effect is dominated by changes in the investment threshold $\hat{\varepsilon}_t^*$ and the taste shock $\hat{\xi}_t$.

4.4.2 Counterfactual Experiment: No Sentiment Shock pre-Financial Crisis

This section presents a counterfactual experiment in which the US economy would not be hit by a strong sentiment shock before the financial crisis.. Figure 4.3 illustrates the counterfactual experiment with $\hat{\kappa}_t = 0$ from 2007Q2.

As established, the sentiment shock $\hat{\kappa}_t$ only affects the mass of bubbly firms in the economy m_t, which in turn only affects the law of motion for the aggregate bubble b_t. Panel C shows that, while the sentiment shock is muted, the value of the aggregate bubble still plummets.

The change in the aggregate value of the firm, as small as it is, has large implications on the real variables and on some nominal variables (see Figure 4.3). A small change in the aggregate value of the bubble will have a direct effect on investment. We can see in Panel G in Figure 4.3 that investment falls far less. This change in aggregate investment has a net a positive effect on output, but this effect is relatively marginal during the crisis as exhibited in Panel F.

The change in the sentiment shock results in high inflation, and thus in a high reaction of the nominal interest rate. Because of the higher cost of labour, the price of wholesale firms' goods increases, which raises the marginal cost of retailers. Consequently, the Central Bank

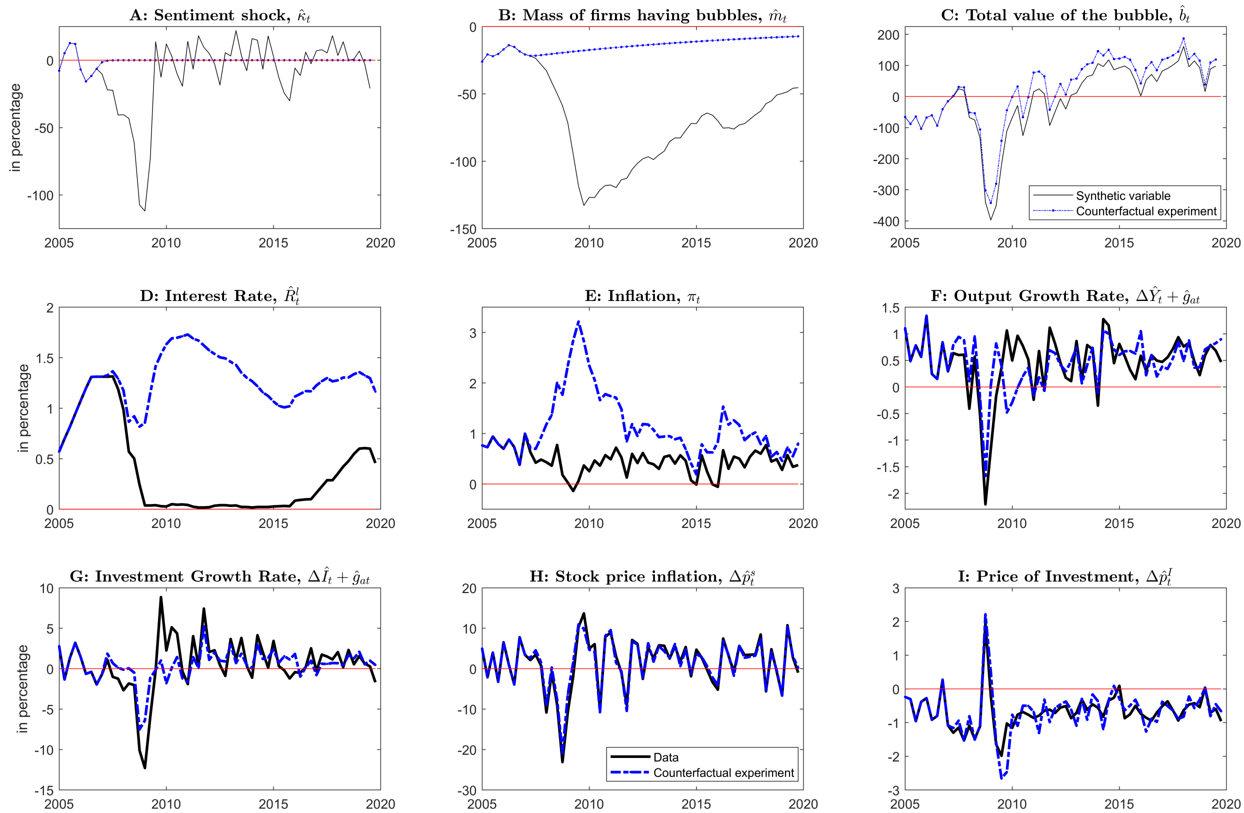

Figure 4.3: No bubble sentiment shock pre-financial crisis

This figure illustrates the response of the economy from 2007Q2 with $\hat{\kappa}_t = 0$.

reacts to the increase in inflation and raises the nominal interest rate.

Surprisingly, the growth rate of stock prices remains unaffected once the sentiment shock is muted. The log-linearised detrended stock relative price is given by:

$$\hat{p}^s_t \quad = \quad \overbrace{\frac{q\tilde{K}g_a}{\tilde{p}^s}(\hat{g}_{at+1} + \hat{q}_t + \hat{K}_{t+1})}^{\text{fundamental value, } \hat{V}^f_t} + \frac{\tilde{b}}{\tilde{p}^s}\hat{b}_t$$

While the bubble component of the stock market bubble marginally changes, we could have expected changes in the fundamental value of the stock market. It appears that the taste shock ξ_t is the main driver of fluctuations in stock market prices and the bubble component. In contrast to Miao et al. (2015a), I use data on the nominal interest rate and inflation in addition to the data on stock prices. The growth rate of stock prices appears to be too volatile while the interest rate is relatively smoother. Consequently, the taste shock $\hat{\xi}_t$ tends to capture the excess volatility of the data on the growth rate of stock price.

4.4.3 Monetary Policy Transmission in a Bubbly Economy

The monetary policy affects the entire economy by altering the borrowing cost for wholesale firms. By manipulating the credit market between wholesale firms, the Central Bank affects 1) wholesale firms' demands for investment goods and labour, 2) the supply of goods to the retailers and the real marginal cost p_t^w, and 3) the market value of the wholesale firms and in turn the households' return on the stock.

A contractionary policy of the Central Bank increases the borrowing cost due to a higher nominal rate. This policy has an intensive and extensive effect on the investment of wholesale firms. The intensive effect is a drop in the individual demand for investment goods. The extensive effect is a drop in the aggregate demand for investment goods because some wholesale firms with an efficiency shock close to the threshold will not have the incentive to borrow. This drop in demand implies a fall in the price of investment and a decrease in capital accumulation. Marginal q, price of installed capital, also falls because firms have too much capital at the Aggregate level.

This reduction in investment decreases the aggregate capital stock and leads to a lower demand for labour, a lower utilisation rate and finally, a lower production. Nonetheless, despite a fall in output, the price of wholesale firm goods decreases because of the drop in the cost of labour. Consequently, inflation falls as retailers, facing a lower marginal cost, adjust their prices.

A noteworthy feature of the effect of monetary policy is an immediate fall of the need for bubbles. Since firms have too much capital, they do not need to make new investments. Therefore, bubbles are less needed and their value falls after an increase in the nominal interest rate. A drop in the aggregate value of the bubble implies a fall in the value of the firms, reflected by the value of stock price drops. This has an immediate negative wealth effect for the households, which leads to a drop in consumption.

4.4.4 Alternative Monetary Policies

In this section, I investigate the robustness of a monetary policy that reacts to stock prices in mitigating the impact of a sentiment shock on the economy. In this intent, I specify two alternative monetary policies than the one estimated in the benchmark model presented.

The first alternative policy reacts to the change in stock prices (Model 1). In its log-linearised form, this monetary policy follows the rule:

$$\hat{R}_t^l = (1 - \phi_R)\left[\phi_\pi \pi_t + \phi_y \Delta \hat{Y}_t + \phi_{p^s} \Delta \hat{p}_t^s\right] + \phi_R \hat{R}_{t-1}^l \tag{4.46}$$

The second policy reacts to deviations of stock prices from its steady-states (Model 2). In its log-linearised form, this monetary policy follows the rule:

$$\hat{R}_t^l = (1 - \phi_R) \left[\phi_\pi \pi_t + \phi_y \Delta \hat{Y}_t + \phi_{p^s} \hat{p}_t^s \right] + \phi_R \hat{R}_{t-1}^l \tag{4.47}$$

For this experiment, I calibrate $\phi_{p^s} = 2$ for Model 1 and $\phi_{p^s} = 0.1$ for Model 2.

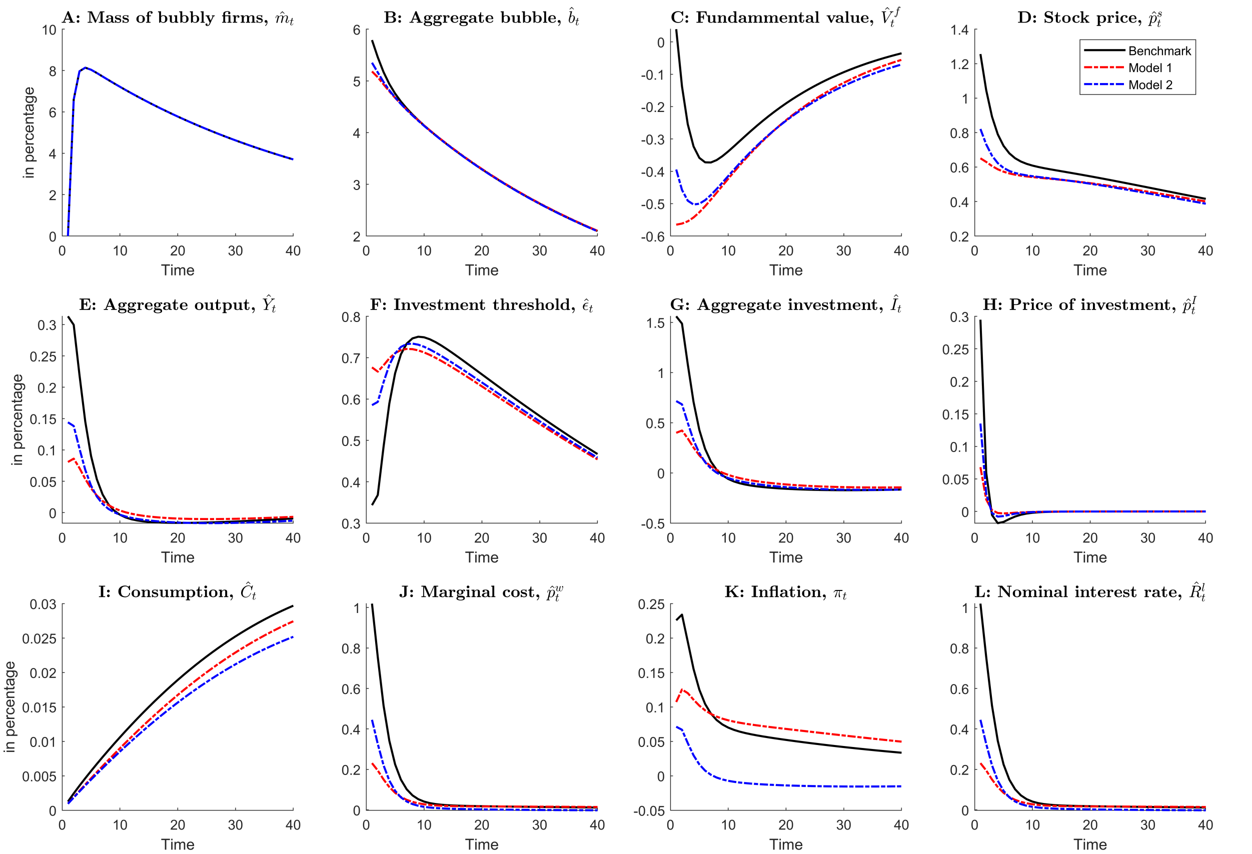

Figure 4.4: Impulse responses of alternative monetary policies after a positive sentiment shock

This figure illustrates the response of the economy after a unit sentiment shock $\hat{\kappa}_t$ under the benchmark estimation (i.e. $\phi_{p^s} = 0$) and under two alternative policy rules that reacts to changes on stock price (Model 1: $\phi_{p^s} = 2$) and to stock price deviation from its steady-state (Model 2: $\phi_{p^s} = 0.1$). The vertical axes represent the percentage deviations from the variables steady-state levels and the unit of time for the horizontal axes correspond to quarters.

Figure 4.4 presents the response of the economy after a unit sentiment shock $\hat{\kappa}_t$ under the benchmark estimation (i.e. $\phi_{p^s} = 0$) and the two alternative policy rules specified above (i.e. Model 1 and 2). Under both alternative policies, the stock price does not increase as much as under the benchmark model (Panel D). However, these smaller deviations are not due to a reduction in the value of the bubble (Panels B), which is only marginally affected by the policy reaction. The lower deviation of the stock price from its steady-state is the consequences of larger negative deviations in the fundamental value (Panels C).

The drop in the fundamental value of the stock represents a decrease in the aggregate value of the wholesale firms. This would imply a drop in investment, because of a contraction of the borrowing capacity of the firm. However, bubbles counter-balance this drop and the aggregate investment remains above its steady-state value after the sentiment shock (Panel G). A lower investment relative to the benchmark model implies a lower cost of labour and lower output.

Both alternative rules manage to stabilise investment and output faster than the benchmark model. The main difference between these two rules is that Model 2 implies a higher interest rate after the impact of the sentiment shock, which has consequently a more aggressive effect on inflation. This higher aggressiveness of the interest rate towards inflation permits to quickly stabilise inflation in contrast to Model 1 (Panel K).

Can monetary policy directly affect the total value of the bubble after a sentiment shock? The answer is mostly no. My finding shows that a rise in the interest rate has a direct negative effect on the aggregate value of the bubble. However, this effect is countered because there is a need for bubbles due to the contraction of the borrowing capacity of the wholesale firms; the fundamental value falls, so bubbles are needed to ease the borrowing constraints. Consequently, the interest rate is an adequate instrument to reduce the volatility of the value of the bubble but is still useful to quickly stabilise output (and inflation under Model 2) by reacting to stock prices.

4.5 Conclusion

In this chapter, I developed and estimate a New Keynesian model with stock market bubbles. Moreover, I analysed the effects of bubbles on real and nominal variables and the transmission mechanism of monetary policy in a bubbly economy. Finally, I investigate if a monetary policy that leans against the wind can reduce the volatility of bubbles. Based on the presented analysis, I can draw the following conclusions.

First, the volatility of value of bubbles can explain a significant fraction of movements in investment, output and inflation. Bubbles, which exist in this economy because of self-fulfilling beliefs, allow firms to increase the borrowing capacity and increase investment and production. Thus, movements in bubbles directly affect the volatility of these variables as well as inflations.

Second, the sentiment shock is not the main cause for changes in bubble size. Because of tight borrowing constraints, firms are very sensitive to changes in the cost of borrowing. Therefore, changes in the interest rates, in the price of installed capital, and in the price of capital goods can have significant intensive and extensive effects on the investment decisions of firms. Moreover, I found that the taste shock is also an important factor to explain variations in the volatility of the bubble's value. Movements in these variables, or the taste shock, affect the number of investing firms and thus the volatility of bubbles.

Finally, monetary policies that react to asset prices can mitigate the impact of the sentiment shock. In this chapter, I present two alternative policies; i) a policy rule that reacts to the changes in stock prices and ii) a policy rule that reacts to deviations of the stock price from its steady-state. Such policies can stabilise quicker output, investment and stock prices. However, these policies have different effects on inflation. Reacting to changes in stock prices is not able to stabilise fast enough inflation and thus can reduce welfare because of persistent inflation. In contrast, a policy rule that reacts to the deviations of the stock price from steady-state can stabilise faster inflation than the estimated rule and the first alternative rule.

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Appendices

Appendix A

Bubbles, Endogenous Growth and Financial Stability

A.1 The Model

The model is composed of two systems. The first system describes the growth of the bubble and comprises 13 main variables (the first 11 equations plus 2 different arbitrage conditions, 20 and 24), whereas the second system describes the crash of the economy with banking sector which is composed of 11 variables. Each systems presented below have all their constraints binding.

A.1.1 System Describing the Asset Price Bubble Growth

Reminder: ε is the Frisch elasticity which corresponds to $1/\eta$.

This first set of equations describes the main body of the system during the growth of the bubble:

$$
\begin{aligned}
1 &: \quad R_t^d = \frac{a^L}{w_t} \\
2 &: \quad w_t^{\varepsilon} = H_t^H + H_t^L \\
3 &: \quad B_t R_t^l = \theta E_t \left[a^H H_t^H + \pi \mu_{t+1} m_t^H \right] \\
4 &: \quad \beta Z_t s_t = w_t H_t^H - B_t
\end{aligned}
$$

$$5 \; : \; \beta Z_t \left(1 - s_t\right) = w_t H_t^L + D_t + \mu_t m_t^L$$

$$6 \; : \; B_t + (1 - m_t^H - m_t^L)\mu_t = \gamma N_t + D_t$$

$$7 \; : \; D_t = \frac{\phi_t}{1-\lambda}\gamma N_t$$

$$8 \; : \; Z_t = a^H H_{t-1}^H + \mu_t m_{t-1}^H - R_{t-1}^l B_{t-1} + a^L H_{t-1}^L + R_{t-1}^d D_{t-1} + m_{t-1}^L \mu_t$$

$$9 \; : \; s_t = \frac{(1-\delta)\left(a^H H_{t-1}^H + \mu_t m_{t-1}^H - R_{t-1}^l B_{t-1}\right) + n\delta\left(a^L H_{t-1}^L + R_{t-1}^d D_{t-1} + m_{t-1}^L \mu_t\right)}{Z_t}$$

$$10 \; : \; N_t = R_{t-1}^l B_{t-1} + (1 - m_{t-1}^H - m_{t-1}^L)\mu_t - R_{t-1}^d D_{t-1}$$

$$11 \; : \; \phi_t = \frac{\beta E_t \left[\pi\left(1-\gamma+\gamma\phi_{t+1|s}\right) + (1-\pi)\left(1-\gamma+\gamma\phi_{t+1|c}\right)\right] R_t^l}{1 - \beta E_t \left[\pi\left(1-\gamma+\gamma\phi_{t+1|s}\right) + (1-\pi)\left(1-\gamma+\gamma\phi_{t+1|c}\right)\right] \frac{R_t^l - R_t^d}{1-\lambda}}$$

The following set of equations describes how to model the heterogeneity of the banking sector:

$$12 \; : \; B_t^b = \frac{1-\xi}{\xi}\mu_t(1 - m_t^H - m_t^L)$$

$$13 \; : \; N_t^b = \frac{1-\lambda}{(1-\lambda+\phi_t)} \frac{\mu_t(1 - m_t^H - m_t^L)}{\gamma\xi}$$

$$14 \; : \; D_t^b = \frac{\phi_t}{1-\lambda}\gamma N_t^b$$

This set of equations describes how the government taxes entrepreneurs to provide deposit insurances to savers of the failing banks:

$$15 \; : \; N_{t+1|c}^b = R_t^l B_t^b - R_t^d D_t^b$$

$$16 \; : \; S_{t+1} = \max\left(-N_{t+1|c}^b, 0\right)$$

$$17 \; : \; \tau_{t+1} = \frac{S_{t+1}}{a^H H_t^H + a^L H_t^L + R_t^d D_t - R_t^l B_t}$$

The non-arbitrage condition for the unproductive entrepreneurs to hold the bubble is given by:

$$18 \; : \; r_{t+1|s}^L = \frac{a^L H_t^L + m_t^L \mu_{t+1} + R_t^d D_t}{w_t H_t^L + m_t^L \mu_t + D_t}$$

$$19 \; : \; r_{t+1|c}^L = (1-\tau_{t+1}) \frac{a^L H_t^L + R_t^d D_t}{w_t H_t^L + m_t^L \mu_t + D_t}$$

$$20 \; : \; E_t\left[\frac{\pi}{r_{t+1|s}^L} + \frac{(1-\pi)(1-\tau_{t+1})}{r_{t+1|c}^L}\right] R_t^d = E_t\left[\frac{\mu_{t+1}}{\mu_t}\frac{\pi}{r_{t+1|s}^L}\right]$$

The non-arbitrage condition for the bankers to hold the bubble is given by:

$$
\begin{aligned}
21 \quad &: \quad u^l_{t+1} = R^l_t + \left(R^l_t - R^d_t\right)\frac{\phi_t}{(1-\lambda)} \\
22 \quad &: \quad u^b_{t+1|s} = \frac{\phi_t}{1-\lambda}\left(R^l_t\left(1-\xi\right) + \frac{\mu_{t+1}}{\mu_t}\xi - R^d_t\right) + R^l_t\left(1-\xi\right) + \frac{\mu_{t+1}}{\mu_t}\xi \\
23 \quad &: \quad u^b_{t+1|c} = \max\left[\frac{\phi_t}{1-\lambda}\left\{\left(1-\xi\right)R^l_t - R^d_t\right\} + R^l_t\left(1-\xi\right), 0\right] \\
24 \quad &: \quad E_t\left[\pi\left(1-\gamma+\gamma\phi_{t+1|s}\right)u^b_{t+1|s} + (1-\pi)\left(1-\gamma+\gamma\phi_{t+1|c}\right)u^b_{t+1|c}\right] \\
&= \quad E_t\left[\pi\left(1-\gamma+\gamma\phi_{t+1|s}\right) + (1-\pi)\left(1-\gamma+\gamma\phi_{t+1|c}\right)\right]u^l_{t+1}
\end{aligned}
$$

The non-arbitrage condition for the productive entrepreneurs to hold the bubble is given by:

$$
\begin{aligned}
25^* \quad &: \quad r^H_{t+1|s} = \frac{a^H H^H_t + m^H_t\mu_{t+1} - R^l_t B_t}{w_t H^H_t + m^H_t\mu_t - B_t} \\
26^* \quad &: \quad r^H_{t+1|c} = (1-\tau_{t+1})\frac{a^H H^H_t - R^l_t B_t}{w_t H^H_t + m^H_t\mu_t - B_t} \\
27^* \quad &: \quad R^H_{t+1} = \frac{(1-\theta)a^H}{w_t - \theta a^H/R^l_t} \\
28^* \quad &: \quad E_t\left[\frac{\pi}{r^H_{t+1|s}} + \frac{(1-\pi)(1-\tau_{t+1})}{r^H_{t+1|c}}\right]R^H_{t+1} = E_t\left[\frac{\pi}{r^H_{t+1|s}}\left(R^H_{t+1}\frac{\theta\pi\mu_{t+1}}{R^l_t\mu_t} + \frac{\mu_{t+1}(1-\theta\pi)}{\mu_t}\right)\right. \\
&\left. + \frac{(1-\pi)(1-\tau_{t+1})}{r^H_{t+1|c}}\left(R^H_{t+1}/R^l_t - 1\right)\frac{\theta\pi\mu_{t+1}}{\mu_t}\right]
\end{aligned}
$$

* Equations (25)-(28) do not hold in the equilibrium.

A.1.2 Systems Describing the Crash from the Bubbly Equilibrium

This set of equations describes the main body of the system during the crash of economy after the bubble bursts:

$$
\begin{aligned}
1 \quad &: \quad R^d_t = \frac{a^L}{w_t} \\
2 \quad &: \quad \phi_t = \frac{\beta E_t\left[R^l_t\left(1-\gamma+\gamma\phi_{t+1}\right)\right]}{1-\beta E_t\left[\left(1-\gamma+\gamma\phi_{t+1}\right)\frac{R^l_t - R^d_t}{1-\lambda}\right]} \\
3 \quad &: \quad B_t R^l_t = \theta a^H H^H_t \\
4 \quad &: \quad \beta Z_t s_t = w_t H^H_t - B_t
\end{aligned}
$$

$$5 : \beta Z_t (1 - s_t) = w_t H_t^L + D_t$$
$$6 : H_t^H + H_t^L = w_t^{\frac{1}{\eta}}$$
$$7 : D_t = \frac{\phi_t}{(1 - \lambda_t)} \gamma N_t$$
$$8 : B_t = \gamma N_t + D_t$$
$$9 : N_t = R_{t-1}^l B_{t-1} - R_{t-1}^d D_{t-1}$$
$$10 : Z_t = (1 - \tau_t) \left(a^H H_{t-1}^H - R_{t-1}^l B_{t-1} + a^L H_{t-1}^L + R_{t-1}^d D_{t-1} \right)$$
$$11 : s_t Z_t = (1 - \tau_t) \left[(1 - \delta) \left(a^H H_{t-1}^H - R_{t-1}^l B_{t-1} \right) + n\delta \left(a^L H_{t-1}^L + R_{t-1}^d D_{t-1} \right) \right]$$

A.1.3 Banking Sector Contamination

In order to determine the heterogeneity of the banking sector, we use the set of equation (12-15). This set of equations is derived by determining the original fraction of wealth that will be used to invest into bubble:

$$N_t = N_t^b + N_t^l$$

where N_t^b is the wealth of bubbly banks and N_t^l is the wealth of the pure lending banks at time t. Let's define the contamination of the banking system by the fraction of banks that invested net worth into the bubble asset. Thus, when the deposit constraint is binding, we can compute the contamination of the banking sector Ψ_t, which is given by:

$$\Psi_t = N_t^b / N_t$$

which represent the percentage of the banks that hold the bubble. When Ψ_t equals one, the entire banking sector is contaminated. This means that the value of the bubbly investment strategy is greater than the value of the pure lending investment strategy ($\phi_t^b > \phi_t^l$). Thus, the non-arbitrage condition (24) between bubbly and pure lending investments is not holding anymore.

When the deposit constraint is not binding, we cannot compute the contamination of the banking sector directly using N_t^b. To do so, we compute the assets of the bubbly balance sheets. Since we know the quantity of loans from the bubbly banks:

$$B_t^b = \frac{1 - \xi}{\xi} \mu_t m_t^b$$

and we can determine the assets of the bubbly banks:

$$assets_t^b = B_t^b + \mu_t m_t^b$$

Thus, the contamination of the banking sector Ψ_t is given by:

$$\Psi_t = \frac{assets_t^b}{assets_t} = \frac{B_t^b + \mu_t m_t^b}{B_t + \mu_t m_t^b}$$

and the bubbly wealth N_t^b is given by:

$$N_t^b = \Psi_t N_t$$

In addition to the contamination of the banking sector, we can also to determine the fraction of wealth of the bubbly banks (which is different from the contamination of the banking sector). we need to use the returns or the law of motion.

$$frac_{t+1|s}^b = \frac{\gamma N_t^b u_{t+1|s}^b}{N_{t+1}} = \frac{R_t^l B_t^b - R_t^d D_t^b + m_t^b \mu_{t+1}}{R_t^l B_t - R_t^d D_t + m_t^b \mu_{t+1}}$$

The difference between the contamination and the share of wealth of bubbly banks is that the contamination is realised at time t and depends on the share of banks that will hold the bubbly asset, whereas the share of net worth of the bubbly banks depends on the realisation of the initial investment and therefore can only be known at $t+1$. In the case when all the banking sector is contaminated, both Ψ_t and $frac_{t+1|s}^b$ have to be equal to 1.

A.1.4 Calibration

Interpretations

Interpretations given by Aoki and Nikolov (2015) of the 3 parameters δ, n and γ which were calibrated to match the steady state of the model in the absence of bubbles to 7 moments in US data: (1) $\delta = 0.167$ implies that on average the productive entrepreneurs remains productive for six years, (2) $n = 0.011$ implies that the initial fraction of the productive entrepreneurs relative to the unproductive entrepreneurs is about 1.1% and (3) $\gamma = 0.907$ implies that the banker's expected length of staying in business is 10.7 years.

Model and Data Moments

The 6 parameters β, δ, n, θ, λ and γ were calibrated to match the steady state of the model in the absence of bubbles to 7 moments in US data. The data of 7 moments comes from Aoki and Nikolov (2015) and are presented in Table A.1 below. Their data sources used to calibrate their model are given in Table A1 of their Online Appendix D. The model moments are also presented in Table A.1.

Moment	Model Concept	Data	Model
Real deposit rate-real GDP growth	R^d	0.950	0.970
Real loan rate-real GDP growth-costs/assets	R^l	0.982	0.981
Ratio of M2 to GDP	D/Y	0.500	0.465
Bank leverage	D/N	10.00	10.00
Average corporate leverage	$B/(Z^H + Z^L)$	0.500	0.532
Leverage of indebted corporates	B/Z^H	2.000	2.004
Bank rate of returns on equity	$R^l + \frac{\phi(R^l - R^d)}{1-\lambda}$	1.100	1.103
Note: Data presented in this Table are from Aoki and Nikolov (2015).			

Table A.1: Model and data moments

A.2 Features of Bubble Stochastic Steady State

A.2.1 Stochastic Bubbly Path and Steady-State

My simulations focus on the emergence and the dynamic survival of an unique asset price bubble. Once this one has collapsed, the asset is not available anymore and disappears forever.

To illustrate the stochastic bubbly path of the economy, consider the following tree diagram (see figure A.1). At $t = 0$ and even before, the economy is in its steady-state where the asset price bubble does not exist. At $t = 1$, a risky asset appears on the market and is available. The different agents make strategic decisions about holding this new asset, knowing that the expected future asset price $\tilde{\mu}_{t+1}$ has the probability π to survive next period (i.e. $\mu_{t+1|s} > 0$) and probability $(1-\pi)$ to burst next period (i.e. $\mu_{t+1|c} = 0$). At $t = 2$, if the asset price bubble survived, the agents can decide to re-invest in the bubbly asset facing the same probability of survival of the asset price. If the asset price collapsed, the asset disappears and the economy converge to its no-bubble steady-state.

Statistically, this is a geometric distribution. Let's define the random variable X as "the number of periods needed for the bubble to burst", then $X \sim Geo(1 - \pi)$, where the mean of X is $\mu_X = \pi/(1 - \pi)$, its variance is given by $\sigma_X = \pi/(1 - \pi)^2$ and $P(X = x) = \pi^{x-1}(1 - \pi)$ where x is the number of periods.

A.2.2 Existence of Bubbly Equilibria

In this model, we consider a pure asset price bubble. That is to say the asset does not have a fundamental value. Thus the asset price only reflects the bubble component of the value

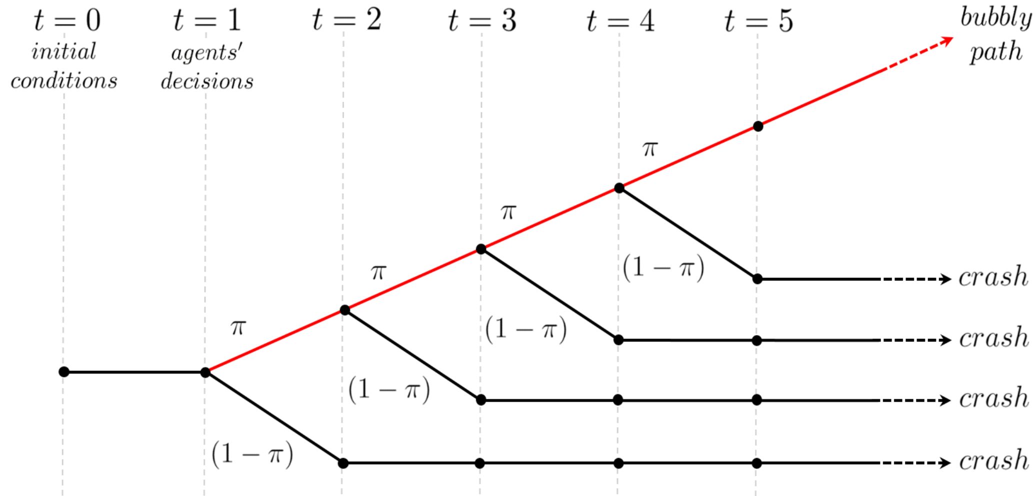

Figure A.1: Bubbly path and potential crashes

This figure represents the event tree that the economy faces. The red branch of the tree depicts the stochastic bubble path where the bubble survives forever. That is to say this path leads the economy to its stochastic bubbly steady-state. Nonetheless, the probability of staying on this path forever converges to zero. The black branches of the tree represents the potential crashes of the economy from its bubbly path that leads the economy to its deterministic no-bubble steady-states.

of the asset. Its attractiveness is a result of relatively high rate of returns. Since the agents' behaviours are driven by the returns of the different investment opportunities, they will have the incentive to invest in the bubble as soon as the rate of return on the bubble asset is equal to or greater than other investment opportunities rate of returns.

In a economical environment where the financial market is not completely developed, some agents might only have poor investment opportunities. However, an intrinsically worthless asset that they were considering useless in another economic environment can be the best investment opportunity for them to protect, transfer and increase their wealth over time.

To understand the existence of bubbles in this model, we need to focus on the long run. At the steady-state, the price of the bubbly asset will be constant, so the rate of returns on the bubble (μ_{t+1}/μ_t) at the steady-state will be a unit. Thus, if the best rate of returns for some agents is lower than or equal to a unit, they will use the bubble to protect and transfer their wealth overtime. Therefore, the two conditions needed for bubbles to circulates are: bubbles should be attractive and affordable. In our case, unproductive entrepreneurs will be attracted to the bubbly asset when their rate of returns is less than an unit, meaning when the deposit rate will be less than an unit. The banks will be attracted to the bubble when their returns on equity (ROE) from lending will be lower than or equal to their ROE from bubbly investment, implying when the lending rate will be less than an unit. Concerning the affordability of the bubble, it simply means that the agents have enough resources to be able to buy it.

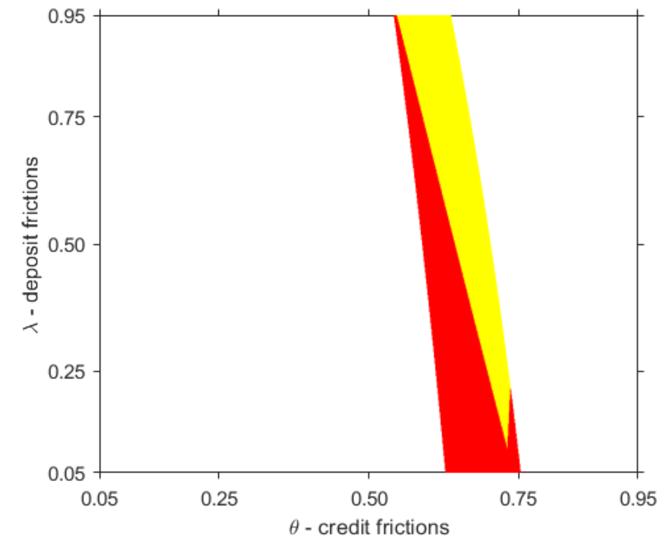

Figure A.2: Propitious areas to the existence of pure bubbly equilibria

This map reveals the levels of financial development that are necessary to make the different agents attracted to the bubbly asset. The red area represents the bubble-less steady-state equilibria where the deposit interest rate R_t^d is less than or equal to a unit. For this level of financial frictions, entrepreneurs might have the incentive to invest in the bubble asset. The yellow region depicts the bubble-less steady-state equilibria where both interest rates are less than or equal to an unit, that is to say where both banks and entrepreneurs might have the incentive to invest in the bubble asset.

I mapped the different interest rates of the no-bubble steady-states as functions of the credits frictions (see figure A.2). The purpose of this map is to find the levels of financial development that are necessary to make the different agents attracted to the bubbly asset. The red region represents the steady-states where the deposit rate R_t^d is less than or equal to an unit and the yellow region depicts the steady-state where both interest rates are less than or equal to an unit. There areas are propitious to let bubbles emerges.

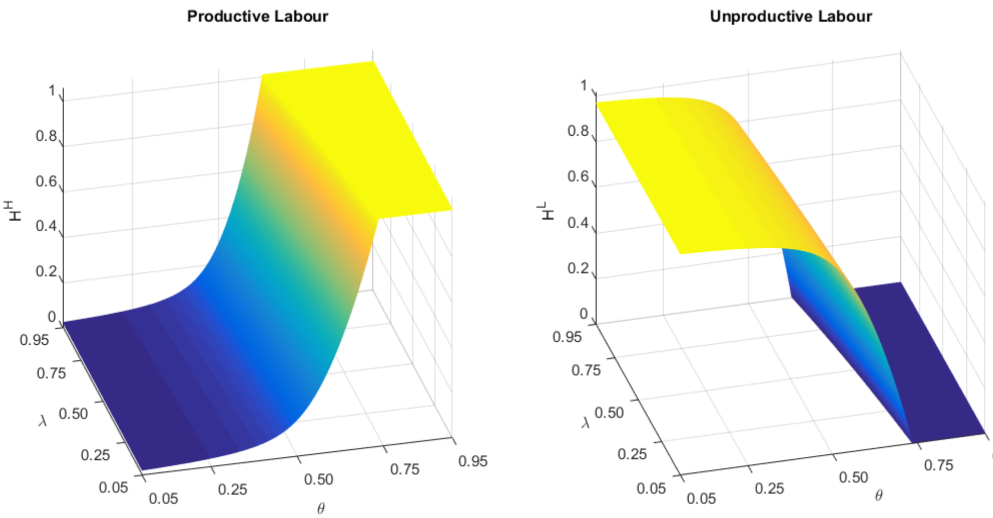

Figure A.3: Labour levels at the bubble-less steady-state

This figure depicts the level of productive and unproductive labours as a function of the financial friction. A low degree of pledgeability θ implies an extremely limited borrowing capacity of the productive entrepreneurs. Unproductive entrepreneurs have to support the most of the production. At the opposite, when θ is very high, productive entrepreneurs can borrow very important quantity of funds and support the entire production of the economy. In this situation, the unproductive entrepreneurs do not produce. When θ is in the middle range, productive entrepreneurs start to be more involved in the production, but are not yet able to support the entire production by themselves.

A.3 Results and Discussions

A.3.1 Effects of Bubbles on Real Activity

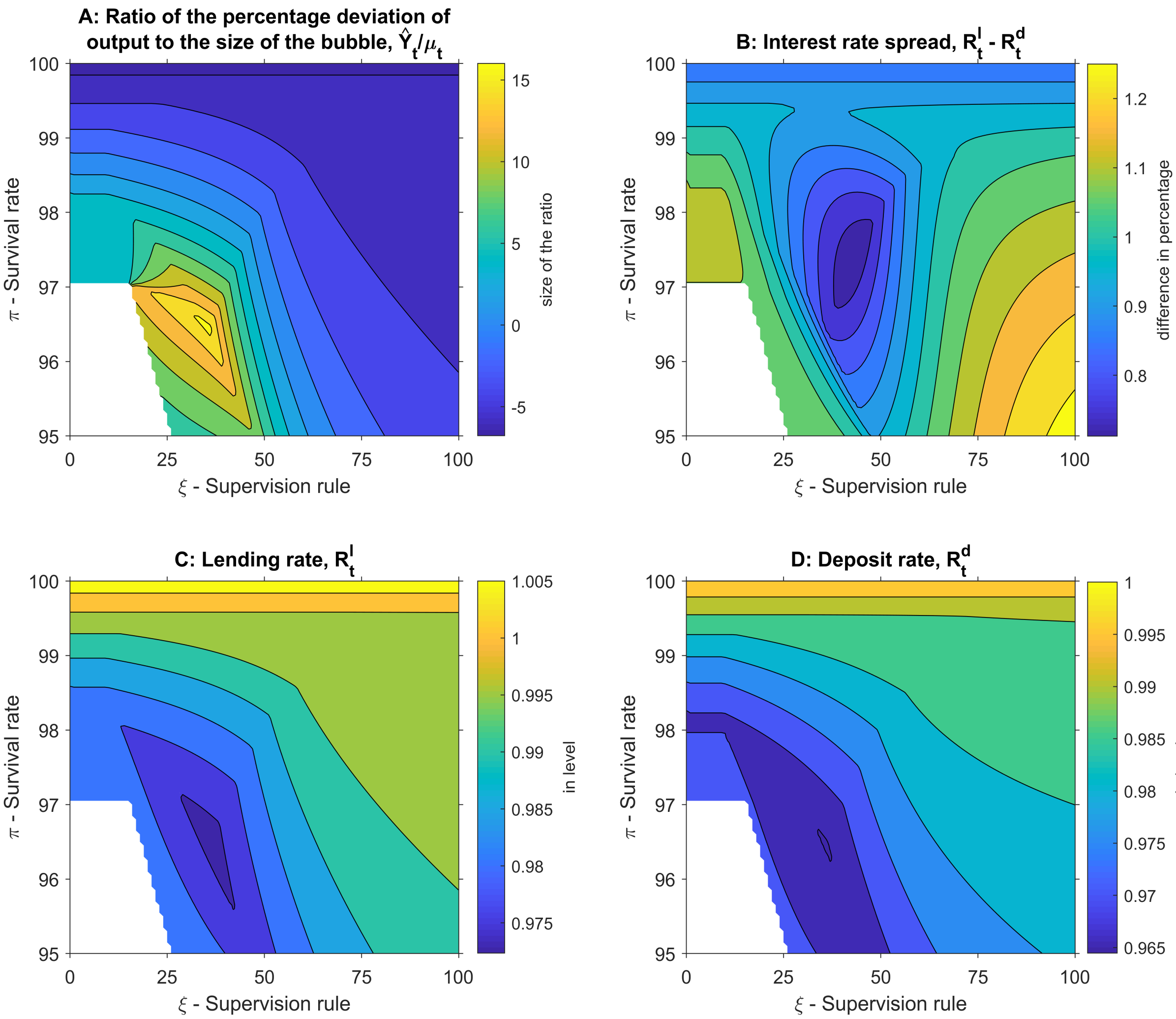

Figure A.4: Effects of the bubble on real activity and interest rates

This figure depicts how the bubble affect the real activity and the interest rates as a function of the intensity of the banking supervision ξ and of the bubble's survival rate π, both parameters are expressed in percentage. Panel A represents the percentage deviation of output in stochastic steady-states Y_t^b to its bubble-less steady-state Y_t, $\hat{Y}_t = (Y_t^b - Y_t)/Y_t$, to the price of the bubble, μ_t. Panel B depicts the stochastic steady-states of the interest rate spread, $R_t^l - R_t^d$. Panel C and D respectively illustrate the stochastic steady-states of the lending and deposit rates, R_t^l and R_t^d.

A.3.2 Bubble Ownership and Banking Sector Contamination

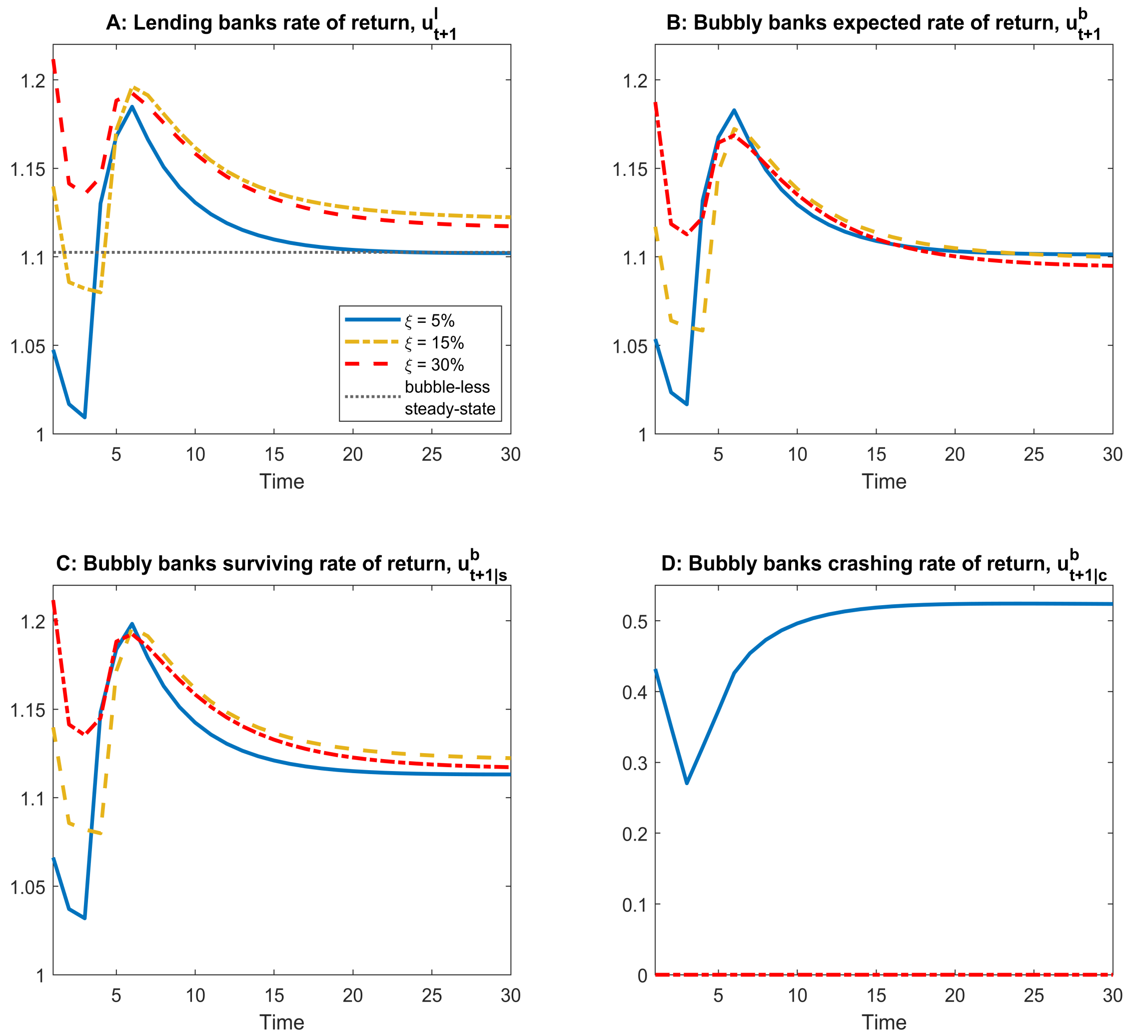

Figure A.5: Dynamics of banks' rates of return

A.3.3 Initial Ownership: Workers

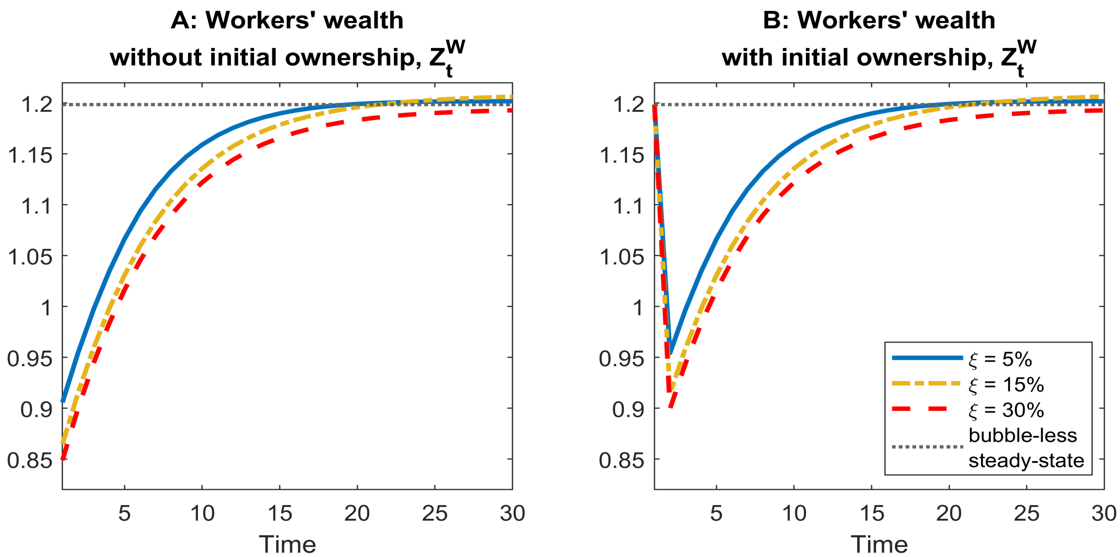

Figure A.6: Dynamics of workers' wealth under different initial ownerships

This figure illustrates the dynamics of the workers' wealth under three different intensities of banking supervision ξ, which are represented by the solid line ($\xi = 5\%$), the dash-dotted line ($\xi = 15\%$) and the dashed line ($\xi = 30\%$). The dotted line depicts the the bubble-less steady-state of this variable. Panel A represents the dynamics of the wealth of the workers when the do not own the bubble asset before its emergence. Panel B represents the dynamics of the wealth of the workers when they own the bubble asset when it emerges.

A.3.4 Recessions and Financial Crises

Bursts at the Stochastic Steady-States

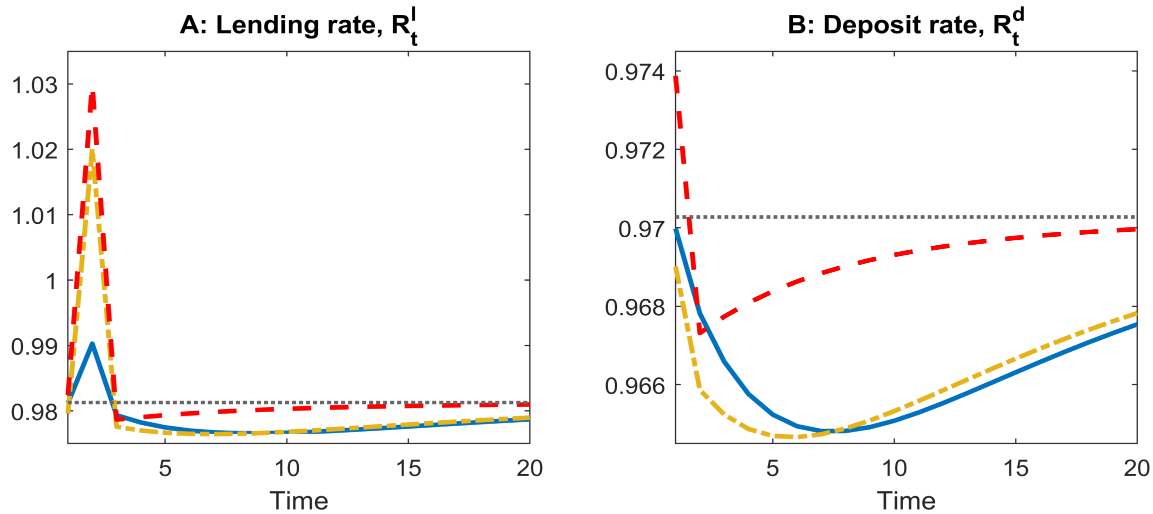

Figure A.7: Dynamics of the interest rates after the burst

This figure illustrates the dynamics of the interest rates under three different intensities of banking supervision ξ, which are represented by the solid line ($\xi = 5\%$), the dash-dotted line ($\xi = 15\%$) and the dashed line ($\xi = 30\%$). The dotted line depicts the the bubble-less steady-state of these indicators. Panel A illustrates the dynamics of the lending rate, R_t^l. Panel B depicts the deposit rate, R_t^d.

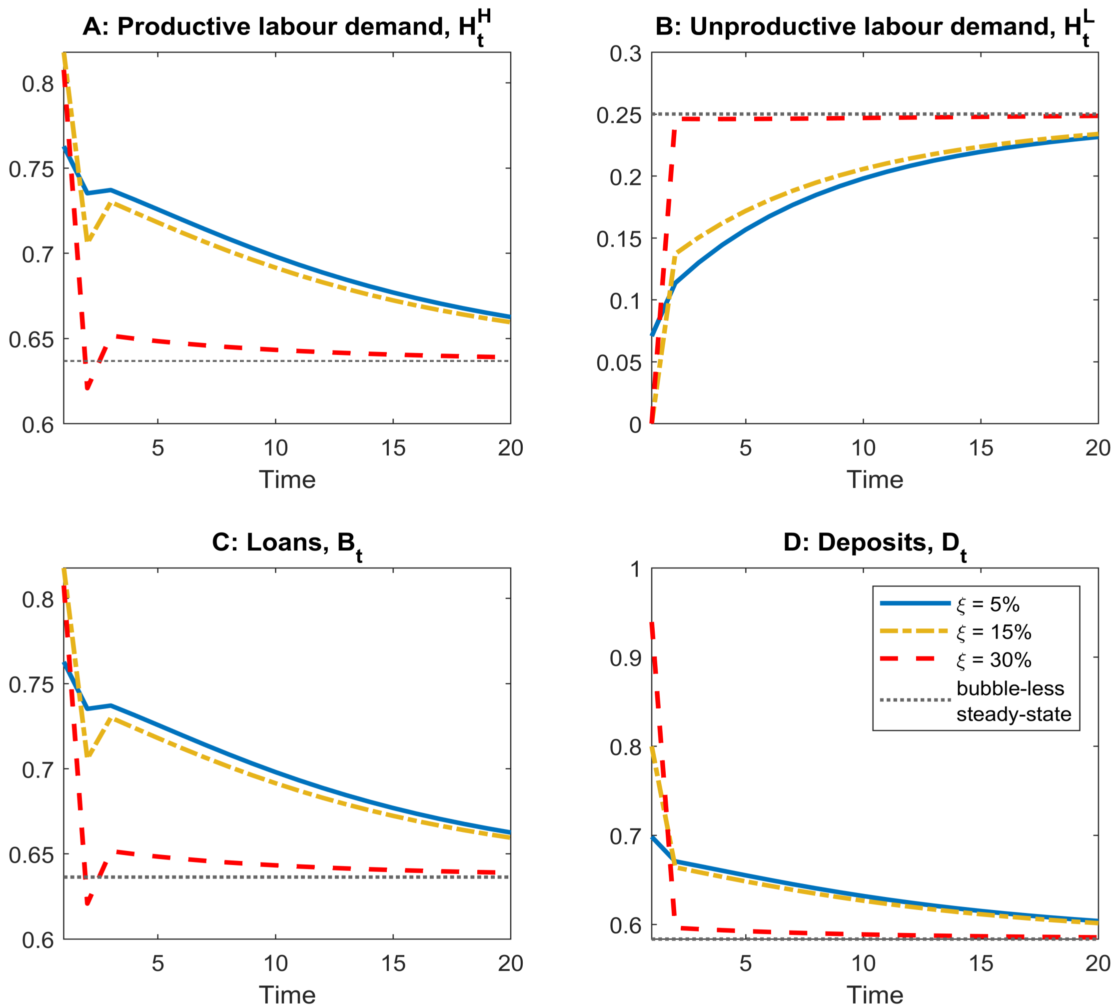

Figure A.8: Dynamics of the labour and assets after the burst

This figure illustrates the dynamics of the resources under three different intensities of banking supervision ξ, which are represented by the solid line ($\xi = 5\%$), the dash-dotted line ($\xi = 15\%$) and the dashed line ($\xi = 30\%$). The dotted line depicts the the bubble-less steady-state of these variables. Panel A depicts the dynamics of the productive demand for labour, H_t^H. Panel B shows the dynamics of unproductive demand for labour, H_t^L. Panel C and D respectively represent the evolutions of the aggregate deposits and loans in the economy, D_t and B_t.

Bubble Paths and Potential Crashes

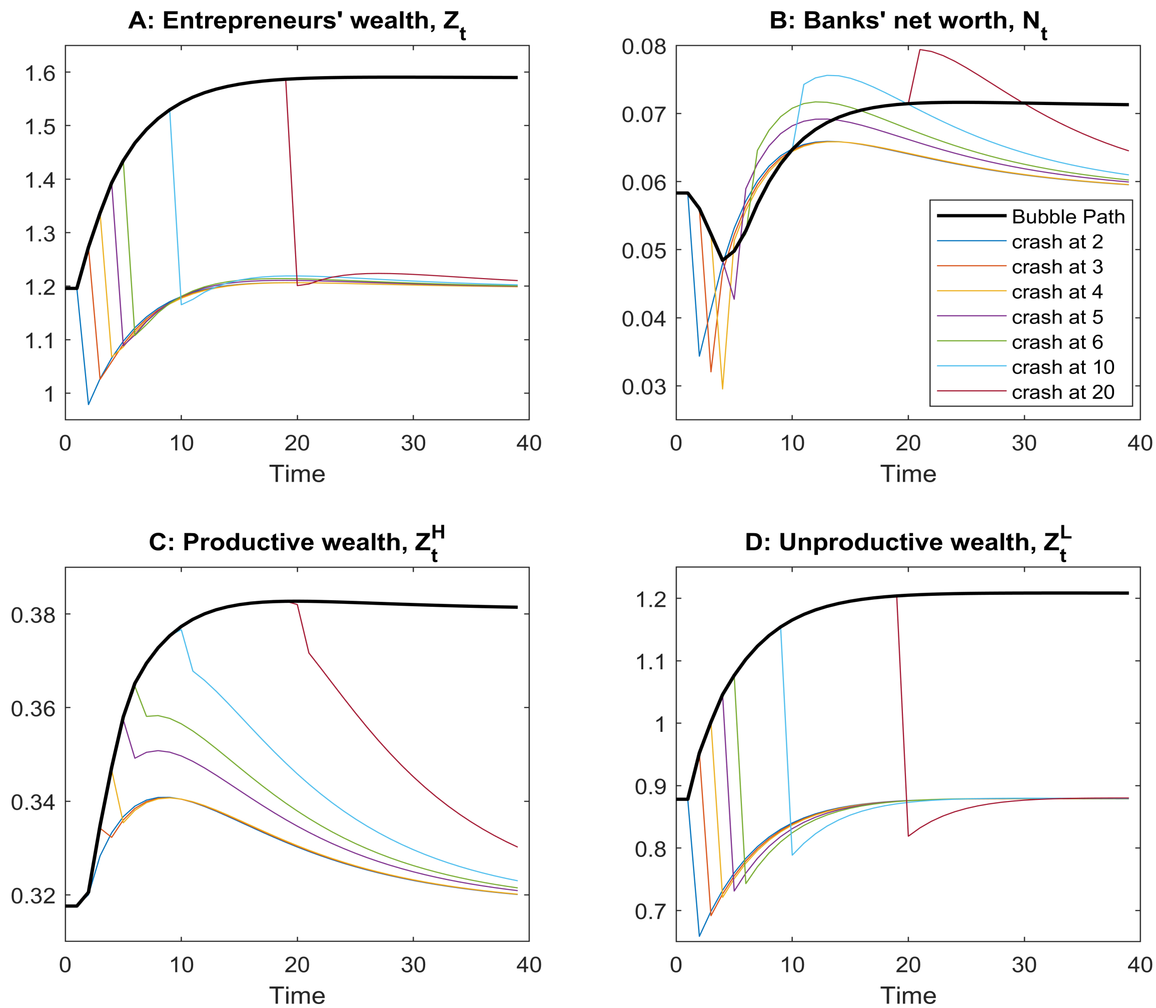

Figure A.9: Bubble path and potential crashes under a tight banking supervision

This figure illustrates a sample of the potential paths that the economy can follow after the emergence of the bubble at period one under the supervision constraint $\xi = 5\%$. The bold black line represents the surviving path of the wealth of the different agents. Departing for this bubble path at different periods, the solid lines depict the dynamics of the agents' wealth after a potential burst of the bubble that converge to their bubble-less steady-states. There are two different bubble-less steady-states, one with a banking sector and one without. Panel A represents the potential dynamics of the aggregate entrepreneurial wealth. Panel B depicts the potential dynamics of the aggregate net worth of the banks. Panel C and D respectively represent the potential dynamics of the aggregate wealth of the productive and unproductive entrepreneurs.

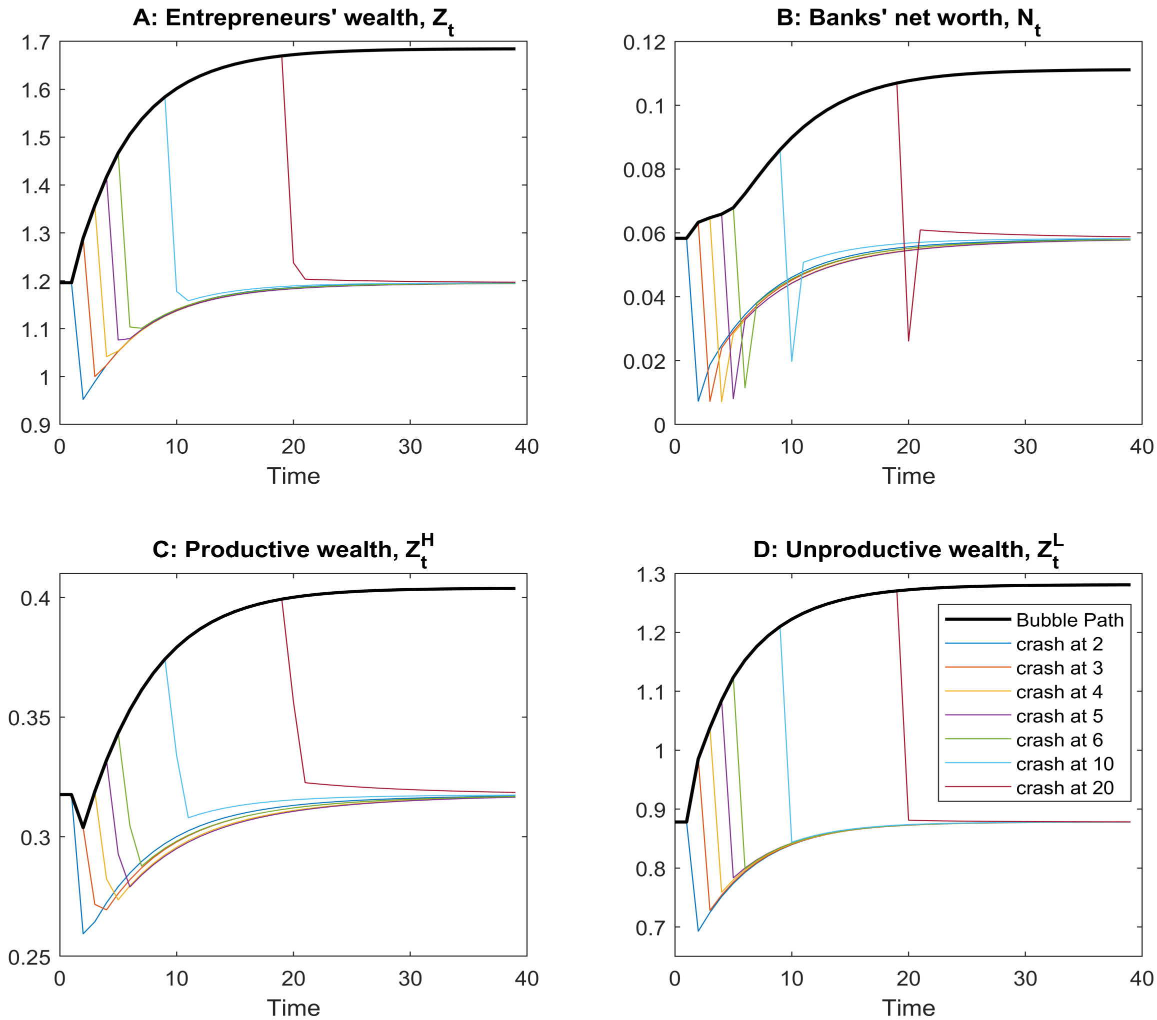

Figure A.10: Bubble path and potential crashes under a relatively relaxed banking supervision

This figure illustrates a sample of the potential paths that the economy can follow after the emergence of the bubble at period one under the supervision constraint $\xi = 30\%$. The bold black line represents the surviving path of the wealth of the different agents. Departing for this bubble path at different periods, the solid lines depict the dynamics of the agents' wealth after a potential burst of the bubble that converge to their bubble-less steady-states. There are two different bubble-less steady-states, one with a banking sector and one without. Panel A represents the potential dynamics of the aggregate entrepreneurial wealth. Panel B depicts the potential dynamics of the aggregate net worth of the banks. Panel C and D respectively represent the potential dynamics of the aggregate wealth of the productive and unproductive entrepreneurs.

Appendix B

Asset Price Bubbles and Monetary Policy

B.1 The Model

B.1.1 Maximisation Problems

Unproductive Entrepreneurs

The unproductive entrepreneurs' budget constraint in nominal terms is given by:

$$P_t c_t + P_t x_t + s_t M t - P_{t+1} \frac{b_{t+1}}{R_t} = Q_t k_t - P_t b_t + s_{t-1} M_t = P_t z_t$$

In real terms, it becomes:

$$c_t + x_t + s_t \mu t - b_{t+1} \frac{(1+\pi_{t+1})}{R_t} = q_t k_t - b_t + s_{t-1} \mu_t = z_t$$

Simplification for later. Given the capital goods production function:

$$k_{t+1} = a^L x_t - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^L} - 1 \right)^2 x_t$$

the derivative with respect to investment x_t yields:

$$\begin{aligned}
\frac{k_{t+1}}{\partial x_t} &= a^L - \left[\frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^L} - 1 \right)^2 + \frac{\Omega}{\lambda_x^L} \left(\frac{x_t}{\lambda_x^L} - 1 \right) x_t \right] \\
&= a^L - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^L} - 1 \right) \left(3 \frac{x_t}{\lambda_x^L} - 1 \right)
\end{aligned}$$

The Maximisation Problem of an unproductive entrepreneur at time t:

$$\mathcal{L} = E_s \sum_{s=t}^{\infty} \left\{ \begin{array}{c} \ln c_t \\ -\lambda_t \left(\begin{array}{c} c_t + x_t + s_t \mu_t - b_{t+1} \frac{(1+\pi_{t+1})}{R_t} \\ -q_t \left(a_{t-1}^i x_{t-1} - \frac{\Omega}{2} \left(\frac{x_{t-1}}{\lambda_x^i} - 1 \right)^2 x_{t-1} \right) + b_t - s_{t-1}\mu_t) \end{array} \right) \\ +\beta \ln c_{t+1} \\ -\beta p \lambda_{t+1}^H \left(\begin{array}{c} c_{t+1} + x_{t+1} + s_{t+1}\mu_{t+1} - b_{t+2} \frac{(1+\pi_{t+2})}{R_{t+1}} \\ -q_{t+1} \left(a_t^L x_t - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^i} - 1 \right)^2 x_t \right) + b_{t+1} - s_t \mu_{t+1}) \end{array} \right) \\ -\beta \left(1-p\right) \lambda_{t+1}^L \left(\begin{array}{c} c_{t+1} + x_{t+1} + s_{t+1}\mu_{t+1} - b_{t+2} \frac{(1+\pi_{t+2})}{R_{t+1}} \\ -q_{t+1} \left(a_t^L x_t - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^L} - 1 \right)^2 x_t \right) + b_{t+1} - s_t \mu_{t+1}) \end{array} \right) \\ +... \end{array} \right\}$$

The First Order Conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &: \frac{1}{c_t} - \lambda_t = 0 \\ \frac{\partial \mathcal{L}}{\partial x_t} &: -\beta^t \lambda_t + \beta^{t+1} E_t \left[\lambda_{t+1} q_{t+1} \left(a^L - \left[\frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^L} - 1 \right)^2 + \frac{\Omega}{\lambda_x^L} \left(\frac{x_t}{\lambda_x^L} - 1 \right) x_t \right] \right) \right] = 0 \\ \frac{\partial \mathcal{L}}{\partial b_{t+1}} &: \beta^t \lambda_t E_t \left[\frac{(1+\pi_{t+1})}{R_t} \right] - \beta^{t+1} E_t \left[\lambda_{t+1} \right] = 0 \\ \frac{\partial \mathcal{L}}{\partial s_t} &: -\beta^t \lambda_t \mu_t + \beta^{t+1} E_t \left[\lambda_{t+1} \tilde{\mu}_{t+1} \right] = 0 \end{aligned}$$

Reorganise the FOCs to obtain:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &: \frac{1}{c_t} = \lambda_t \\ \frac{\partial \mathcal{L}}{\partial x_t} &: \lambda_t = \beta E_t \left[\lambda_{t+1} q_{t+1} \left(a^L - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^L} - 1 \right) \left(3 \frac{x_t}{\lambda_x^L} - 1 \right) \right) \right] \\ \frac{\partial \mathcal{L}}{\partial b_{t+1}} &: \lambda_t = \beta E_t \left[\lambda_{t+1} \frac{R_t}{(1+\pi_{t+1})} \right] \\ \frac{\partial \mathcal{L}}{\partial s_t} &: \lambda_t = \beta E_t \left[\lambda_{t+1} \frac{\tilde{\mu}_{t+1}}{\mu_t} \right] \end{aligned}$$

and subsitute out λ_t:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &: \frac{1}{c_t} = \lambda_t \\
\frac{\partial \mathcal{L}}{\partial x_t} &: \frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} q_{t+1} \left(a^L - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^L} - 1 \right) \left(3\frac{x_t}{\lambda_x^L} - 1 \right) \right) \right] \\
\frac{\partial \mathcal{L}}{\partial b_{t+1}} &: \frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \frac{R_t}{(1+\pi_{t+1})} \right] \\
\frac{\partial \mathcal{L}}{\partial s_t} &: \frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \frac{\tilde{\mu}_{t+1}}{\mu_t} \right]
\end{aligned}$$

Productive Entrepreneur

The productive entrepreneurs' budget constraint in nominal terms is given by:

$$P_t c_t + x_t + s_t M t - P_{t+1} \frac{b_{t+1}}{R_t} = Q_t k_t - P_t b_t + s_{t-1} M_t = P_t z_t$$

and the borrowing constraint:

$$P_{t+1} b_{t+1} \leq \theta E_t \left[Q_{t+1} k_{t+1} + s_t \tilde{M}_{t+1} \right],$$

In real terms, the budget constraint becomes:

$$c_t + x_t + s_t \mu_t - b_{t+1} \frac{(1+\pi_{t+1})}{R_t} = q_t k_t - b_t + s_{t-1} \mu_t = z_t$$

and the borrowing constraint:

$$b_{t+1} \leq \theta E_t \left[q_{t+1} k_{t+1} + s_t \tilde{\mu}_{t+1} \right],$$

Simplification for later. Given the capital goods production function:

$$k_{t+1} = a^H x_t - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^H} - 1 \right)^2 x_t$$

the derivative with respect to investment x_t yields:

$$\begin{aligned}
\frac{k_{t+1}}{\partial x_t} &= a^H - \left[\frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^H} - 1 \right)^2 + \frac{\Omega}{\lambda_x^H} \left(\frac{x_t}{\lambda_x^H} - 1 \right) x_t \right] \\
&= a^H - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^H} - 1 \right) \left(3\frac{x_t}{\lambda_x^H} - 1 \right)
\end{aligned}$$

The Maximisation Problem of an productive entrepreneur at time t:

$$\mathcal{L} = E_0 \sum_{s=t}^{\infty} \left\{ \begin{array}{c} ... + \ln c_t \\ -\lambda_t \left(\begin{array}{c} c_t + x_t + s_t\mu_t - b_{t+1}\frac{(1+\pi_{t+1})}{R_t} \\ -q_t \left[a_{t-1}^i x_{t-1} - \frac{\Omega}{2}\left(\frac{x_{t-1}}{\lambda_x^i} - 1\right)^2 x_{t-1} \right] + b_t - s_{t-1}\mu_t) \end{array} \right) \\ -\varpi_t \left(b_{t+1} - \theta q_{t+1} \left[a^H x_t - \frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^H} - 1\right)^2 x_t \right] - \theta s_t \tilde{\mu}_{t+1} \right) \\ +... \end{array} \right\}$$

The First Order Conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &: \frac{1}{c_t} - \lambda_t = 0 \\
\frac{\partial \mathcal{L}}{\partial x_t} &: -\beta^t \left(\lambda_t - \varpi_t \theta q_{t+1} \left[a^H - \frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^H} - 1\right)\left(3\frac{x_t}{\lambda_x^H} - 1\right) \right] \right) \\
&\quad + \beta^{t+1} \left(E_t \left[\lambda_{t+1} q_{t+1} \left(a^H - \frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^H} - 1\right)\left(3\frac{x_t}{\lambda_x^H} - 1\right) \right) \right] \right) = 0 \\
\frac{\partial \mathcal{L}}{\partial b_{t+1}} &: \beta^t \left(\lambda_t E_t \left[\frac{(1+\pi_{t+1})}{R_t} \right] - \varpi_t \right) - \beta^{t+1} E_t \left[\lambda_{t+1} \right] = 0 \\
\frac{\partial \mathcal{L}}{\partial s_t} &: -\beta^t \left(\lambda_t \mu_t - \theta \varpi_t E_t \left[\tilde{\mu}_{t+1} \right] \right) + \beta^{t+1} E_t \left[\lambda_{t+1} \tilde{\mu}_{t+1} \right] = 0
\end{aligned}$$

Reorganise the FOCs to obtain:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &: \frac{1}{c_t} = \lambda_t \\
\frac{\partial \mathcal{L}}{\partial x_t} &: \lambda_t - \varpi_t \theta E_t \left[q_{t+1} \left(a^H - \frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^H} - 1\right)\left(3\frac{x_t}{\lambda_x} - 1\right) \right) \right] \\
&\qquad = \beta E_t \left[\lambda_{t+1} q_{t+1} \left(a^H - \frac{\Omega}{2}\left(\frac{x_t}{\lambda_x^H} - 1\right)\left(3\frac{x_t}{\lambda_x^H} - 1\right) \right) \right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial b_{t+1}} &: \lambda_t E_t \left[\frac{(1+\pi_{t+1})}{R_t} \right] - \beta E_t \left[\lambda_{t+1} \right] = \varpi_t^H \\
\frac{\partial \mathcal{L}}{\partial s_t} &: \beta E_t \left[\lambda_{t+1} \tilde{\mu}_{t+1} \right] = \lambda_t \mu_t - \theta \varpi_t^H E_t \left[\tilde{\mu}_{t+1} \right]
\end{aligned}$$

Simplify and obtain:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &: \quad \frac{1}{c_t} = \lambda_t \\
\frac{\partial \mathcal{L}}{\partial x_t} &: \quad \lambda_t \left(1 - \theta E_t \left[q_{t+1} \left(a^H - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^H} - 1 \right) \left(3 \frac{x_t}{\lambda_x^H} - 1 \right) \right) \frac{(1+\pi_{t+1})}{R_t} \right] \right) \\
&\qquad = \beta E_t \left[\lambda_{t+1} q_{t+1} \left(a^H - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^H} - 1 \right) \left(3 \frac{x_t}{\lambda_x^H} - 1 \right) \right) \right] (1-\theta) \\
&\implies \lambda_t = \frac{\beta E_t \left[\lambda_{t+1} q_{t+1} \right] (1-\theta) \left(a^H - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^H} - 1 \right) \left(3 \frac{x_t}{\lambda_x^H} - 1 \right) \right)}{1 - \theta \left(a^H - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x^H} - 1 \right) \left(3 \frac{x_t}{\lambda_x^H} - 1 \right) \right) E_t \left[\frac{q_{t+1}(1+\pi_{t+1})}{R_t} \right]}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial s_t} &: \quad \beta E_t \left[\lambda_{t+1} \tilde{\mu}_{t+1} \right] = \lambda_t \mu_t - \theta \left(\lambda_t E_t \left[\frac{(1+\pi_{t+1})}{R_t} \tilde{\mu}_{t+1} \right] - \beta E_t \left[\lambda_{t+1} \tilde{\mu}_{t+1} \right] \right) \\
&\implies \beta E_t \left[\lambda_{t+1} \tilde{\mu}_{t+1} \right] (1-\theta) = \lambda_t \left(\mu_t - \theta E_t \left[\frac{(1+\pi_{t+1})}{R_t} \tilde{\mu}_{t+1} \right] \right) \\
&\implies \lambda_t = \beta \frac{E_t \left[\lambda_{t+1} \tilde{\mu}_{t+1} / \mu_t \right] (1-\theta)}{1 - \theta E_t \left[\frac{(1+\pi_{t+1})}{R_t} \frac{\tilde{\mu}_{t+1}}{\mu_t} \right]}
\end{aligned}$$

The non-arbitrage condition implies that:

$$E_t \left[\frac{1}{c_{t+1}} \frac{\tilde{\mu}_{t+1}/\mu_t (1-\theta)}{1 - \theta E_t \left[\frac{(1+\pi_{t+1})}{R_t} \frac{\tilde{\mu}_{t+1}}{\mu_t} \right]} \right] = E_t \left[\frac{1}{c_{t+1}} \frac{q_{t+1} (1-\theta) \left(a^H - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x} - 1 \right) \left(3 \frac{x_t}{\lambda_x} - 1 \right) \right)}{1 - \theta \left(a^H - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x} - 1 \right) \left(3 \frac{x_t}{\lambda_x} - 1 \right) \right) \frac{q_{t+1}(1+\pi_{t+1})}{R_t}} \right]$$

and non-binding borrowing constraint:

$$\begin{aligned}
q_{t+1} \left(a^H - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x} - 1 \right) \left(3 \frac{x_t}{\lambda_x} - 1 \right) \right) &= \frac{q_{t+1} (1-\theta) \left(a^H - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x} - 1 \right) \left(3 \frac{x_t}{\lambda_x} - 1 \right) \right)}{1 - \theta \left(a^H - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x} - 1 \right) \left(3 \frac{x_t}{\lambda_x} - 1 \right) \right) \frac{q_{t+1}(1+\pi_{t+1})}{R_t}} \\
a^H - \frac{\Omega}{2} \left(\frac{x_t}{\lambda_x} - 1 \right) \left(3 \frac{x_t}{\lambda_x} - 1 \right) &= \frac{R_t}{q_{t+1} (1+\pi_{t+1})}
\end{aligned}$$

Intermediate Firms

The production function of the firms and the profit function:

$$\begin{aligned}
y_t^w &= k_t^\alpha n_t^{1-\alpha} \\
\Pi_t^w &= P_t^w y_t^w - (Q_t k_t + W_t n_t) \\
&= p_t^w y_t^w - (q_t k_t + w_t n_t) \\
&= p_t^w k_t^\alpha n_t^{1-\alpha} - q_t k_t - w_t n_t
\end{aligned}$$

Prices of inputs:

$$\frac{\partial \Pi_t^w}{\partial k_t} \quad : \quad q_t = p_t^w \alpha n_t^{1-\alpha} k_t^{\alpha-1}$$
$$\frac{\partial \Pi_t^j}{\partial n_t} \quad : \quad w_t = p_t^w \left(1-\alpha\right) k_t^{\alpha} n_t^{-\alpha}$$

Households inelastically supply one unit of labour unit N_t =1:

$$\begin{aligned}
\Pi_t^w &= p_t^w Y_t^w - qK_t - w_t \\
&= p_t^w K_t^{\alpha} - p_t^w \alpha K_t^{\alpha-1} K_t - p_t^w \left(1-\alpha\right) K_t^{\alpha} \\
&= p_t^w K_t^{\alpha} - p_t^w \alpha K_t^{\alpha} - p_t^w \left(1-\alpha\right) K_t^{\alpha} \\
&= p_t^w K_t^{\alpha} \left(1-\alpha-\left(1-\alpha\right)\right) \\
&= 0
\end{aligned}$$

Retailers

Decision Problem There is a continuum of retail firms of measure one, each index by i. Each retail firm i buys a wholesale good Y_t^j at price P_t^w and repackage into a specialized retail good $Y_t(i)$. Retailers sell their specialized retail good $Y_t(i)$ to competitive final firms at price P_t^i. Retailers face a quadratic cost a la Rotemberg to change their price.

The problem of a retail firm i is:

$$V\left(i\right) = \max_{P_t^i} E_t \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left[\frac{\left(P_t^i - P_t^w\right)}{P_t} - \frac{\Upsilon}{2}\left(\frac{P_t^i}{P_{t-1}^i} - 1\right)^2 Y_t\right]$$

subject to the the competitive demand for good $Y_t(i)$:

$$Y_t \ \left(i\right) = \left[\frac{P_t^i}{P_t}\right]^{-\varkappa} Y_t \tag{B.1}$$

Optimal behaviour Define $\frac{P_{t+1}}{P_t} = (1+\pi_{t+1})$, $p_t^w = \frac{P_t^w}{P_t}$ and $\tilde{p}_t = \frac{P_t^i}{P_t}$, the maximisation problem can be rewritten using (B.1) such as:

$$\begin{aligned}
V(i) &= \max_{P_t^i} E_t \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left[\frac{(P_t^i - P_t^w)}{P_t} Y_t(i) - \frac{\Upsilon}{2} \left(\frac{P_t^i}{P_{t-1}^i} - 1 \right)^2 Y_t \right] \\
&= \max_{P_t^i} E_t \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left[\frac{(P_t^i - P_t^w)}{P_t} \left[\frac{P_t^i}{P_t} \right]^{-\varkappa} Y_t - \frac{\Upsilon}{2} \left(\frac{P_t^i}{P_{t-1}^i} - 1 \right)^2 Y_t \right] \\
&= \max_{P_t^i} E_t \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left[\left[\frac{P_t^i}{P_t} \right]^{1-\varkappa} Y_t - p_t^w \left[\frac{P_t^i}{P_t} \right]^{-\varkappa} Y_t - \frac{\Upsilon}{2} \left(\frac{P_t^i}{P_{t-1}^i} - 1 \right)^2 Y_t \right] \\
&= \max_{\tilde{p}_t} E_t \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left[\tilde{p}_t^{1-\varkappa} Y_t - \tilde{p}_t^{-\varkappa} p_t^w Y_t - \frac{\Upsilon}{2} \left(\frac{\tilde{p}_t}{\tilde{p}_{t-1}} (1+\pi_t) - 1 \right)^2 Y_t \right]
\end{aligned}$$

The first order condition with respect to $\tilde{p}_t$ yields:

$$\begin{aligned}
\frac{\partial V(i)}{\partial \tilde{p}_t} &: \quad \beta^t \frac{\Lambda_t}{\Lambda_0} \left[(1-\varkappa) \tilde{p}_t^{-\varkappa} Y_t + \varkappa p_t^w \tilde{p}_t^{-(\varkappa+1)} Y_t - \Upsilon \frac{(1+\pi_t)}{\tilde{p}_{t-1}} \left(\frac{\tilde{p}_t}{\tilde{p}_{t-1}} (1+\pi_t) - 1 \right) Y_t \right] \\
&: \quad +\beta^{t+1} E_t \frac{\Lambda_{t+1}}{\Lambda_0} \left[\Upsilon \left(\frac{\tilde{p}_{t+1}}{\tilde{p}_t^2} (1+\pi_{t+1}) \right) \left(\frac{\tilde{p}_{t+1}}{\tilde{p}_t} (1+\pi_{t+1}) - 1 \right) Y_{t+1} \right] = 0 \\
&\Longrightarrow \quad (1-\varkappa) \tilde{p}_t^{-\varkappa} + \varkappa p_t^w \tilde{p}_t^{-(\varkappa+1)} - \Upsilon \frac{(1+\pi_t)}{\tilde{p}_{t-1}} \left(\frac{\tilde{p}_t}{\tilde{p}_{t-1}} (1+\pi_t) - 1 \right) \\
&: \quad = -\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left[\Upsilon \left(\frac{\tilde{p}_{t+1}}{\tilde{p}_t^2} (1+\pi_{t+1}) \right) \left(\frac{\tilde{p}_{t+1}}{\tilde{p}_t} (1+\pi_{t+1}) - 1 \right) \frac{Y_{t+1}}{Y_t} \right] \\
&\Longrightarrow \quad \frac{(1+\pi_t)}{\tilde{p}_{t-1}} \left(\frac{\tilde{p}_t}{\tilde{p}_{t-1}} (1+\pi_t) - 1 \right) = \frac{(1-\varkappa) \tilde{p}_t^{-\varkappa} + \varkappa p_t^w \tilde{p}_t^{-(\varkappa+1)}}{\Upsilon} + \\
&: \quad \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left[\left(\frac{\tilde{p}_{t+1}}{\tilde{p}_t^2} (1+\pi_{t+1}) \right) \left(\frac{\tilde{p}_{t+1}}{\tilde{p}_t} (1+\pi_{t+1}) - 1 \right) \frac{Y_{t+1}}{Y_t} \right]
\end{aligned}$$

With $\tilde{p}_t = 1$, the last expression becomes:

$$(1+\pi_t)\,\pi_t = \frac{1-\varkappa+\varkappa p_t^w}{\Upsilon} + \beta E_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \frac{Y_{t+1}}{Y_t} (1+\pi_{t+1})\,\pi_{t+1} \right]$$

which gives us an expression for aggregate inflation.

Profit is given:

$$\Pi_t^F = \left(1 - p_t^w - \frac{\Upsilon}{2} \pi_{t+1}^2 \right) Y_t$$

B.1.2 Aggregation

Resources of the entrepreneurs

$$
\begin{aligned}
Z_t^H &= p\left[q_t K_t^H - B_t + \left(1 - s_t^L\right)\mu_t\right] + p\left[q_t a^L K_t^L + B_t + s_t^L \mu_t\right] \\
&= p\left[q_t K_t + \mu_t\right] \\
Z_t^L &= (1-p)\left[q_t K_t^H - B_t + \left(1 - s_t^L\right)\mu_t\right] + (1-p)\left[q_t K_t^L + B_t + s_t^L \mu_t\right] \\
&= (1-p)\left[q_t K_t + \mu_t\right]
\end{aligned}
$$

Expenditures of the entrepreneurs:

$$
\begin{aligned}
Z_t^H &= (1-\beta) Z_t^H + X_t^H + \mu_t - \frac{R_t B_{t+1}}{(1+\pi_{t+1})} \\
Z_t^L &= (1-\beta) Z_t^L + X_t^L + \mu_t + \frac{R_t B_{t+1}}{(1+\pi_{t+1})}
\end{aligned}
$$

Consumption resources:

$$
\begin{aligned}
p\left[q_t K_t + \mu_t\right] &= (1-\beta) Z_t^H + X_t^H + \left(1 - s_t^L\right)\mu_t - \frac{R_t B_{t+1}}{(1+\pi_{t+1})} \\
(1-p)\left[q_t K_t + \mu_t\right] &= (1-\beta) Z_t^L + X_t^L + s_t^L \mu_t + \frac{R_t B_{t+1}}{(1+\pi_{t+1})} \\
w_t + \Pi_t^F + \Pi_t^w &= C_t
\end{aligned}
$$

Total output is the function of aggregate capital:

$$
Y_t^w = K_t^\alpha
$$

The resource constraint of the economy:

$$
\begin{aligned}
\begin{pmatrix} p\left(q_t K_t + \mu_t\right) \\ + (1-p)\left(q_t K_t + \mu_t\right) \end{pmatrix} + w_t + \Pi_t^F + \Pi_t^w &= \begin{pmatrix} (1-\beta) Z_t^H + X_t^H + \left(1 - s_t^L\right)\mu_t \\ -\frac{R_t B_{t+1}}{(1+\pi_{t+1})} + (1-\beta) Z_t^L \\ + X_t^L + s_t^L \mu_t + \frac{R_t B_{t+1}}{(1+\pi_{t+1})} \end{pmatrix} + C_t \\
q_t K_t + \mu_t + w_t + \Pi_t^F + 0 &= (1-\beta)\left(Z_t^H + Z_t^L\right) + X_t^H + X_t^L + \mu_t + C_t \\
p_t^w \alpha K_t^{\alpha-1} K_t + w_t + \left(1 - p_t^w - \frac{\Upsilon}{2}\pi_{t+1}^2\right) Y_t &= (1-\beta)\left(Z_t^H + Z_t^L\right) + X_t^H + X_t^L + C_t \\
p_t^w Y_t + \left(1 - p_t^w - \frac{\Upsilon}{2}\pi_{t+1}^2\right) Y_t &= (1-\beta)\left(Z_t^H + Z_t^L\right) + X_t^H + X_t^L + C_t \\
\left(1 - \frac{\Upsilon}{2}\pi_{t+1}^2\right) Y_t &= (1-\beta)\left(Z_t^H + Z_t^L\right) + X_t^H + X_t^L + C_t
\end{aligned}
$$

System: The system is given by:

$$1 \quad : \quad X_t^H : \beta Z_t^H = X_t^H + \left(1 - s_t^L\right)\mu_t - E_t \frac{R_t B_{t+1}}{(1+\pi_{t+1})}$$

$$2 \quad : \quad X_t^L : \beta Z_t^L = X_t^L + s_t^L \mu_t + E_t \frac{R_t B_{t+1}}{(1+\pi_{t+1})}$$

$$3 \quad : \quad Z_t^H : Z_t^H = p\left[q_t K_t + \mu_t\right]$$

$$4 \quad : \quad Z_t^L : Z_t^L = (1-p)\left[q_t K_t + \mu_t\right]$$

$$5 \quad : \quad B_t : B_t = \theta E_t \left[q_{t+1}\left(a^H - \frac{\Omega}{2}\left(\frac{X_t^H}{\lambda_x^H} - 1\right)^2\right) X_t^H + \left(1 - s_t^L\right)\kappa \mu_{t+1}\right]$$

$$6 \quad : \quad R_t : \frac{R_t}{(1+\pi_{t+1})} = q_{t+1}\left(a^L - \frac{\Omega}{2}\left(\frac{X_t^L}{\lambda_x^L} - 1\right)\left(3\frac{X_t^L}{\lambda_x^L} - 1\right)\right)$$

$$7 \quad : \quad \mu_t : E_t\left[\frac{\kappa}{r_{t+1|s}^L}\frac{\mu_{t+1}}{\mu_t}\right] = E_t\left[\frac{\kappa}{r_{t+1|s}^L}\frac{R_t}{(1+\pi_{t+1})} + \frac{1-\kappa}{r_{t+1|c}^L}\frac{R_t}{(1+\pi_{t+1})}\right]$$

$$8 \quad : \quad q_t : q_t = p_t^w \alpha K_t^{\alpha-1}$$

$$10 \quad : \quad p_t^w : (1+\pi_t)\pi_t = \frac{1 - \varkappa + \varkappa p_t^w}{\Upsilon} + \beta E_t\left[\frac{C_t}{C_{t+1}}\frac{Y_{t+1}}{Y_t}(1+\pi_{t+1})\pi_{t+1}\right]$$

$$11 \quad : \quad K_t : K_t = \left[a^H - \frac{\Omega}{2}\left(\frac{X_{t-1}^H}{\lambda_x^H} - 1\right)^2\right] X_{t-1}^H + \left[a^L - \frac{\Omega}{2}\left(\frac{X_{t-1}^L}{\lambda_x^L} - 1\right)^2\right] X_{t-1}^L$$

$$12 \quad : \quad C_t : C_t = \left(1 - \alpha p_t^w - \frac{\Upsilon}{2}E_t \pi_{t+1}^2\right) Y_t$$

$$13 \quad : \quad s_t^L : E_t\left[\frac{1}{c_{t+1}}\frac{\tilde{\mu}_{t+1}/\mu_t\,(1-\theta)}{1 - \theta E_t\left[\frac{(1+\pi_{t+1})}{R_t}\frac{\tilde{\mu}_{t+1}}{\mu_t}\right]}\right] =$$

$$E_t\left[\frac{1}{c_{t+1}}\frac{q_{t+1}(1-\theta)\left(a^H - \frac{\Omega}{2}\left(\frac{x_t}{\lambda_x} - 1\right)\left(3\frac{x_t}{\lambda_x} - 1\right)\right)}{1 - \theta\left(a^H - \frac{\Omega}{2}\left(\frac{x_t}{\lambda_x} - 1\right)\left(3\frac{x_t}{\lambda_x} - 1\right)\right)\frac{q_{t+1}(1+\pi_{t+1})}{R_t}}\right]$$

$$14 \quad : \quad \pi_t : R_t = R_n + \phi_\pi\left(\pi_t - \pi^T\right) + \phi_Y\left(\frac{Y_t - Y_n}{Y_n}\right) + \phi_\mu \mu_t + \nu_t$$

Retrieve the system:

$$13 \quad : \quad \left(1 - \frac{\Upsilon}{2}\pi_{t+1}^2\right) Y_t = (1-\beta)\left(Z_t^H + Z_t^L\right) + X_t^H + X_t^L + C_t$$

$$14 \quad : \quad w_t = p_t^w (1-\alpha) K_t^\alpha$$

$$15 \quad : \quad C_t = w_t + \left(1 - p_t^w - \frac{\Upsilon}{2}E_t \pi_{t+1}^2\right) Y_t$$

$$16 \quad : \quad Y_t = K_t^\alpha$$

Capital good productions

$$K_{t+1}^{H}=\left[a^{H}-\frac{\Omega}{2}\left(\frac{X_{t}^{H}}{\lambda_{x}^{H}}-1\right)^{2}\right]X_{t}^{H}$$

$$\begin{aligned}
0 &= a^{H}-\frac{\Omega}{2}\left(\frac{X_{t}^{H}}{\lambda_{x}^{H}}-1\right)^{2}\\
\left(\frac{X_{t}^{H}}{\lambda_{x}^{H}}-1\right) &= \left(a^{H}\frac{2}{\Omega}\right)^{1/2}\\
X_{t}^{H} &= \left[\left(\frac{a^{H}2}{\Omega}\right)^{1/2}+1\right]\lambda_{x}^{H}
\end{aligned}$$

Bubbleless System with Lending

$$\begin{aligned}
1 &: X_{t}^{H}:\beta Z_{t}^{H}=X_{t}^{H}\\
2 &: X_{t}^{L}:\beta Z_{t}^{L}=X_{t}^{L}\\
3 &: Z_{t}^{H}:Z_{t}^{H}=pq_{t}K_{t}\\
4 &: Z_{t}^{L}:Z_{t}^{L}=(1-p)\,q_{t}K_{t}\\
5 &: \pi_{t}:a^{H}-a^{L}=\frac{\Omega}{2}\left[\left(\frac{X_{t}^{H}}{\lambda_{x}^{H}}-1\right)\left(3\frac{X_{t}^{H}}{\lambda_{x}^{H}}-1\right)-\left(\frac{X_{t}^{L}}{\lambda_{x}^{L}}-1\right)\left(3\frac{X_{t}^{L}}{\lambda_{x}^{L}}-1\right)\right]
\end{aligned}$$

$$\begin{aligned}
6 &: q_{t}:q_{t}=p_{t}^{w}\alpha K_{t}^{\alpha-1}\\
7 &: p_{t}^{w}:(1+\pi_{t})\,\pi_{t}=\frac{1-\varkappa+\varkappa p_{t}^{w}}{\Upsilon}+\beta E_{t}\left[\frac{C_{t}}{C_{t+1}}\frac{Y_{t+1}}{Y_{t}}(1+\pi_{t+1})\,\pi_{t+1}\right]\\
8 &: K_{t}:K_{t}=\left[a^{H}-\frac{\Omega}{2}\left(\frac{X_{t-1}^{H}}{\lambda_{x}^{H}}-1\right)^{2}\right]X_{t-1}^{H}+\left[a^{L}-\frac{\Omega}{2}\left(\frac{X_{t-1}^{L}}{\lambda_{x}^{L}}-1\right)^{2}\right]X_{t-1}^{L}\\
9 &: C_{t}:C_{t}=\left(1-\alpha p_{t}^{w}-\frac{\Upsilon}{2}E_{t}\pi_{t+1}^{2}\right)Y_{t}\\
10 &: R_{t}:R_{t}=R^{T}+\phi_{\pi}\left(\pi_{t}-\pi^{T}\right)+\phi_{Y}\left(\frac{Y_{t}-Y^{T}}{Y^{T}}\right)+\phi_{\mu}\mu_{t}+\nu_{t}
\end{aligned}$$

Non-arbitrage condition between lending and capital good production implies:

$$\begin{aligned}
\frac{R_{t}}{(1+\pi_{t+1})} &= q_{t+1}\left(a^{L}-\frac{\Omega}{2}\left(\frac{X_{t}^{L}}{\lambda_{x}^{L}}-1\right)\left(3\frac{X_{t}^{L}}{\lambda_{x}^{L}}-1\right)\right)\\
\frac{R_{t}}{(1+\pi_{t+1})} &= q_{t+1}\left(a^{H}-\frac{\Omega}{2}\left(\frac{X_{t}^{H}}{\lambda_{x}^{H}}-1\right)\left(3\frac{X_{t}^{H}}{\lambda_{x}^{H}}-1\right)\right)
\end{aligned}$$

so the difference needs to be:

$$
\begin{aligned}
a^H - \frac{\Omega}{2}\left(\frac{X_t^H}{\lambda_x^H} - 1\right)\left(3\frac{X_t^H}{\lambda_x^H} - 1\right) &= a^L - \frac{\Omega}{2}\left(\frac{X_t^L}{\lambda_x^L} - 1\right)\left(3\frac{X_t^L}{\lambda_x^L} - 1\right) \\
a^H - a^L &= \frac{\Omega}{2}\left[\left(\frac{X_t^H}{\lambda_x^H} - 1\right)\left(3\frac{X_t^H}{\lambda_x^H} - 1\right) - \left(\frac{X_t^L}{\lambda_x^L} - 1\right)\left(3\frac{X_t^L}{\lambda_x^L} - 1\right)\right]
\end{aligned}
$$

B.2 Capital Goods Productions and Dynamics

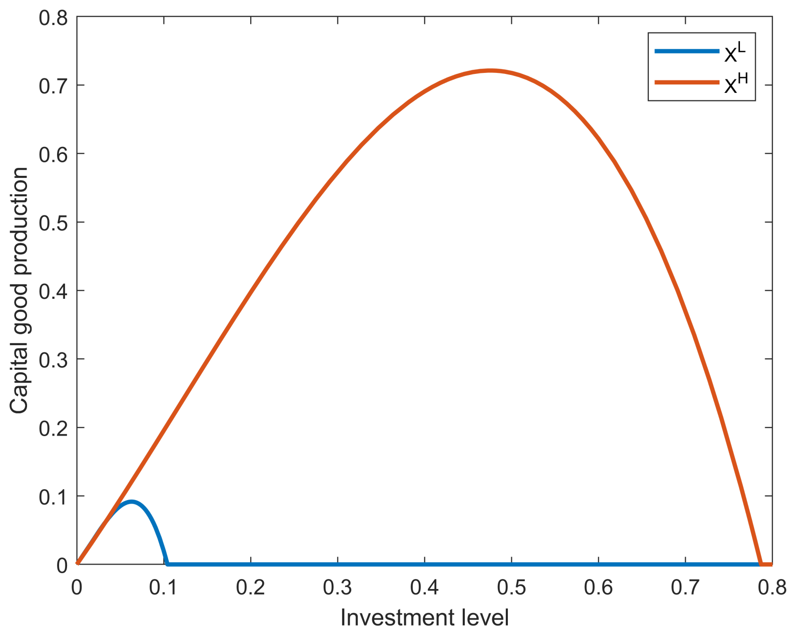

Figure B.1: Specific capital good productions

This Figure illustrates the specific investment levels.

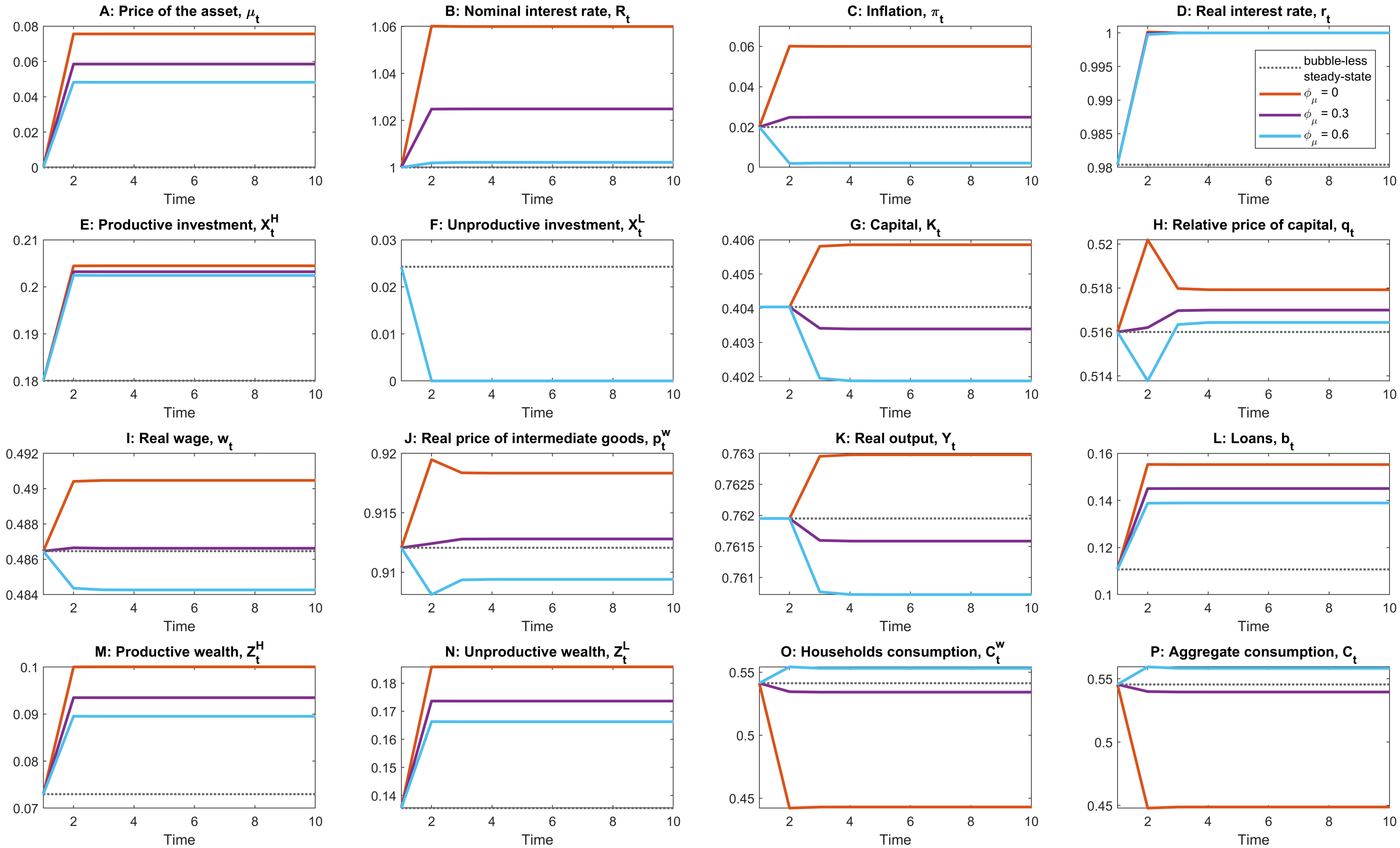

Figure B.2: Dynamics of the economy

This Figure illustrates the transitional dynamics from the bubbleless steady-state to the bubbly steady-state under three different Taylor rules bubble coefficient ϕ_μ (i.e. $\phi_\mu = 0, 0.5$ and 1) after the emergence of a bubble.

Appendix C

Stock Market Bubbles and Monetary policy

C.1 The Model

C.1.1 Households

The Lagrangian of the maximisation problem of the household is given by:

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} \xi_t \frac{(C_t/A_t - \theta C_{t-1}/A_{t-1})^{1-\sigma}}{1-\sigma} - \xi_t \psi_t \frac{N_t^{1+\eta}}{1+\eta} \\ -\Lambda_t \left(\begin{array}{c} C_t + p_t^s s_{t+1} + d_{t+1} \frac{(1+\pi_{t+1})}{R_t^d} \\ -(1-\tau_t) w_t N_t - \Pi_t^I \\ -\Pi_t^F - d_t - (d_t^s + p_t^s) s_t \end{array} \right) \end{array} \right]$$

Note that flow utility is stationary, but the period budget constraint is growing at A_t, so Λ_t behaves as $1/A_t$. The FOCs with respect to C_t, N_t, s_t and d_t yields:

$$\frac{\partial \mathcal{L}}{\partial C_t} \ : \ \beta^t \left[\frac{\xi_t}{A_t} (C_t/A_t - \theta C_{t-1}/A_{t-1})^{-\sigma} - \Lambda_t \right] \\ -\beta^{t+1} E_t \left[\theta \frac{\xi_{t+1}}{A_t} (C_{t+1}/A_{t+1} - \theta C_t/A_t)^{-\sigma} \right] = 0 \qquad \text{(C.1)}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} \quad : \quad -\beta^t \left[\xi_t \psi_t N_t^\eta - \Lambda_t (1-\tau_t) w_t\right] = 0 \tag{C.2}$$

$$\frac{\partial \mathcal{L}}{\partial s_t} \quad : \quad -\beta^t \Lambda_t p_t^s + \beta^{t+1} E_t \left[\Lambda_{t+1}\left(d_{t+1}^s + p_{t+1}^s\right)\right] = 0 \tag{C.3}$$

$$\frac{\partial \mathcal{L}}{\partial d_t} \quad : \quad -\beta^{t-1} \Lambda_{t-1} \frac{(1+\pi_t)}{R_{t-1}^d} + \beta^t \Lambda_t = 0 \tag{C.4}$$

Reorganising the FOCs, we obtain the following optimality conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} \quad : \quad \Lambda_t = {} & \frac{\xi_t}{A_t}\left(C_t/A_t - \theta C_{t-1}/A_{t-1}\right)^{-\sigma} \\ & -\theta\beta E_t\left[\frac{\xi_{t+1}}{A_t}\left(C_{t+1}/A_{t+1} - \theta C_t/A_t\right)^{-\sigma}\right] \end{aligned} \tag{C.5}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} \quad : \quad \frac{N_t^\eta}{\Lambda_t} = \frac{w_t}{\xi_t \psi_t}(1-\tau_t) \tag{C.6}$$

$$\frac{\partial \mathcal{L}}{\partial s_t} \quad : \quad \Lambda_t = \beta E_t\left[\Lambda_{t+1}\frac{\left(d_{t+1}^s + p_{t+1}^s\right)}{p_t^s}\right] \tag{C.7}$$

$$\frac{\partial \mathcal{L}}{\partial d_{t+1}} \quad : \quad \Lambda_t = \beta E_t\left[\frac{\Lambda_{t+1}}{(1+\pi_{t+1})} R_t^d\right] \tag{C.8}$$

C.1.2 Wholesale Firms

The decision problem

The maximisation problem of the firm j of age a at time t, which depends on the efficiency shock ε_t^j, is then given by:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) = {} & \max_{I_t^j, N_t^j, u_t^j, l_{t+1}^j} p_t^w Y_t^j - w_t N_t^j - I_t^j p_t^I - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} \\ & + E_t\left[\frac{(1-\delta_e)}{r_t^{sj}} V_{t+1,a+1}\left(K_{t+1}^j, l_{t+1}^j, \varepsilon_{t+1}^j\right)\right] \end{aligned} \tag{C.9}$$

where r_t^{sj} is the rate of return of stock investments of the firm j (the stochastic discount factor $\frac{1}{r_t^{sj}}$ is equal to $\beta\frac{\Lambda_{t+1}}{\Lambda_t}$ from the Euler equation for stock investement because of non-arbitrage condition). The the firm j chooses N_t^j, I_t^j, u_t^j and l_{t+1}^j to maximise (C.9) subject to their production function:

$$Y_t^j = \left(u_t^j K_t^j\right)^\alpha \left(A_t N_t^j\right)^{1-\alpha} \tag{C.10}$$

where A_t is the common trend and z_t is a stationary shock, the law of motion of capital:

$$K_{t+1}^j = \left(1-\delta_t^j\right) K_t^j + \varepsilon_t^j I_t^j \tag{C.11}$$

the constraint on new equity:

$$d_t^{sj} \geq -\varphi_t K_t^j$$

the investment good constraint:

$$0 \leq I_t^j p_t^I \leq p_t^w Y_t^j - w_t N_t^j + \varphi_t K_t^j - l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} \tag{C.12}$$

and the external borrowing constraint:

$$\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(K_{t+1}^j, l_{t+1}^j\right) \geq \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(K_{t+1}^j, 0\right) \tag{C.13}$$

$$-\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(\gamma_t K_t^j, 0\right) \tag{C.14}$$

where $\bar{V}_{t,a}\left(K_t^j, l_t^j\right) \equiv \int V_{t,a}\left(K_t^j, l_t^j, \varepsilon\right) d\Phi(\varepsilon)$ reprensents the ex-ante value after integrating out ε.

Guess and New Form of the Value Function:

We conjecture and verify that the value function takes the following form:

$$V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) = Q_t\left(\varepsilon_t^j\right) K_t^j + B_{t,a}\left(\varepsilon_t^j\right) - Q_t^L\left(\varepsilon_t^j\right) l_t^j \tag{C.15}$$

Using the stock price of the firm of age a and the conjecture of the value function (C.15), we have:

$$\begin{aligned}
p_{t,a}^{sj} &= E_t\left[\frac{(1-\delta_e)\bar{V}_{t+1,a+1}\left(K_{t+1}^j, l_{t+1}^j, \varepsilon_{t+1}^j\right)}{r_t^{sj}}\right] \\
p_{t,a}^{sj} &= (1-\delta_e) E_t\left[\frac{Q_{t+1}K_{t+1}^j + B_{t+1,a+1} - Q_{t+1}^L l_{t+1}^j}{r_t^{sj}}\right] \\
p_{t,a}^{sj} &= (1-\delta_e) E_t \frac{Q_{t+1}K_{t+1}^j + B_{t+1,a+1} - Q_{t+1}^L l_{t+1}^j}{r_t^{sj}} \\
p_{t,a}^{sj} &= (1-\delta_e) E_t \frac{Q_{t+1}}{r_t^{sj}} K_{t+1}^j + (1-\delta_e) E_t \frac{B_{t+1,a+1}}{r_t^{sj}} - (1-\delta_e) E_t \frac{Q_{t+1}^L}{r_t^{sj}} l_{t+1}^j \\
p_{t,a}^{sj} &= q_t K_{t+1}^j + b_{t,a} - q_t^L l_{t+1}^j
\end{aligned}$$

where:

$$q_t = (1-\delta_e) E_t \frac{Q_{t+1}}{r_t^{sj}}$$
$$b_{t,a} = (1-\delta_e) E_t \frac{B_{t+1,a+1}}{r_t^{sj}}$$
$$q_t^L = (1-\delta_e) E_t \frac{Q_{t+1}^L}{r_t^{sj}}$$

Thus we can rewrite the continuation value of the maximisation problem as:

$$\begin{aligned} E_t\left[\frac{(1-\delta_e)}{r_t^{sj}}\bar{V}_{t+1,a+1}\left(K_{t+1}^j, l_{t+1}^j, \varepsilon_{t+1}^j\right)\right] &= \frac{(1-\delta_e)}{r_t^{sj}} E_t\left[Q_{t+1}K_{t+1}^j + B_{t+1,a+1} - Q_{t+1}^L l_{t+1}^j\right] \\ &= b_{t,a} + E_t\left[q_t K_{t+1}^j - q_t^L l_{t+1}^j\right] \end{aligned}$$

and the problem becomes:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= \max_{I_t^j, N_t^j, u_t^j, l_{t+1}^j} p_t^w Y_t^j - w_t N_t^j - I_t^j p_t^I - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} \\ &\quad + E_t\left[q_t K_{t+1}^j + b_{t,a} - q_t^L l_{t+1}^j\right] \end{aligned}$$

Substitute the law of motion of capital which depends on the idiosyncratic shock ε_t^j:

$$K_{t+1}^j = \left(1-\delta_t^j\right) K_t^j + \varepsilon_t^j I_t^j$$

in the new form of the value function, the problem becomes:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= \max_{I_t^j, N_t^j, u_t^j, l_{t+1}^j} p_t^w Y_t^j - w_t N_t^j - I_t^j p_t^I - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} \\ &\quad + E_t\left[q_t\left[\left(1-\delta_t^j\right) K_t^j + b_{t,a} + \varepsilon_t^j I_t^j\right] - q_t^L l_{t+1}^j\right] \end{aligned}$$

Finally, simplifying for I_t^j, we get the following form for $V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right)$:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= \max_{I_t^j, N_t^j, u_t^j, l_{t+1}^j} p_t^w Y_t^j - w_t N_t^j - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} \\ &\quad + E_t\left[q_t\left(1-\delta_t^j\right) K_t^j + \left(q_t \varepsilon_t^j - p_t^I\right) I_t^j + b_{t,a} - q_t^L l_{t+1}^j\right] \end{aligned}$$

FOC I_t^j: Optimal Investment

Remind that the purchase of investment goods is constrained by:

$$0 \leq I_t^j p_t^I \leq p_t^w Y_t^j - w_t N_t^j + \varphi_t K_t^j - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l}$$

First Order Condition for I_t^j yields:

$$\frac{\partial V_{t,a}}{\partial I_t^j} \quad : \quad -p_t^I + q_t \varepsilon_t^j = 0$$
$$: \quad \varepsilon_t^j = \frac{p_t^I}{q_t}$$

the investment threshold. Its means that the investment decision of the firm depends on the idiosyncratic investment efficient shock. If the shock of the firm ε_t^j is greater than $\frac{p_t^I}{q_t}$ that we can define ε_t^*, then the firm invest, otherwise the firm does not invest.

Then the optimal investment level I_t^j of a wholeslae firm j respects:

$$I_t^j p_t^I = \begin{cases} p_t^w Y_t^j - w_t N_t^j - l_t^j + \varphi_t K_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l}, & \text{if } \varepsilon_t^j \geq \varepsilon_t^*, \\ 0, & \text{otherwise,} \end{cases} \tag{C.16}$$

where $\varepsilon_t^* = \frac{p_t^I}{q_t}$. Equation (C.16) means that the constraint on new equity binds, $d_t^{sj} = \varphi_t K_t^j$.

Simplifying the problem for investing firms When $\varepsilon_t^j \geq \varepsilon_t^*$, the problem of the firm can be rewritten such that:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) \quad = \quad & \max_{N_t^j, u_t^j, l_{t+1}^j} p_t^w Y_t^j - w_t N_t^j - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} \\ & + E_t\left[q_t\left(1-\delta_t^j\right) K_t^j + \left(q_t \varepsilon_t^j - p_t^I\right) I_t^j + b_{t,a} - q_t^L l_{t+1}^j\right] \end{aligned}$$

Substituing out $I_t^j p_t^I$:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) \quad = \quad & \max_{N_t^j, u_t^j, l_{t+1}^j} p_t^w Y_t^j - w_t N_t^j - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} \\ & + E_t \left[\begin{array}{c} q_t\left(1-\delta_t^j\right) K_t^j + b_{t,a} - q_t^L l_{t+1}^j \\ + \left(\frac{\varepsilon_t^j}{\varepsilon_t^*} - 1\right)\left(p_t^w Y_t^j - w_t N_t^j - l_t^j + \varphi_t K_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l}\right) \end{array} \right] \end{aligned}$$

where $\varepsilon_t^* = \frac{p_t^I}{q_t}$

Simplifying the Problem for non-Investing Firms When $\varepsilon_t^j < \varepsilon_t^*$, the problem of the firm can be rewritten such that:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) \quad = \quad & \max_{N_t^j, u_t^j, L_{t+1}^j} p_t^w Y_t^j - w_t N_t^j - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} \\ & + E_t\left[q_t\left(1-\delta_t^j\right) K_t^j + b_{t,a} - q_t^L l_{t+1}^j\right] \end{aligned}$$

since $I_t^j p_t^I = 0$.

FOC N_t^j: Optimal Labour Demand

The labour demand N_t^j is simply derived from the maximisation problem of the firm. With respect to N_t^j, the first order conditions of the production problem is:

$$\begin{aligned}
\frac{\partial V_{t,a}}{\partial N_t^j} &: p_t^w \left(u_t^j K_t^j\right)^{\alpha} (1-\alpha) \left(A_t N_t^j\right)^{-\alpha} A_t - w_t = 0 \\
&: p_t^w \left(u_t^j K_t^j\right)^{\alpha} (1-\alpha) A_t = w_t \left(A_t N_t^j\right)^{\alpha} \\
&: u_t^j K_t^j \left[\frac{(1-\alpha) A_t p_t^w}{w_t}\right]^{\frac{1}{\alpha}} = A_t N_t^j \\
&: N_t^j = u_t^j \left[\frac{(1-\alpha) A_t p_t^w}{w_t}\right]^{\frac{1}{\alpha}} \frac{K_t^j}{A_t} \qquad \text{(C.17)}
\end{aligned}$$

Using the labour demand, the production problem becomes:

$$\begin{aligned}
p_t^w Y_t^j - w_t N_t^j &= p_t^w \left(u_t^j K_t^j\right)^{\alpha} \left(A_t N_t^j\right)^{1-\alpha} - w_t N_t^j \\
&= p_t^w \left(u_t^j K_t^j\right)^{\alpha} \left(A_t u_t^j \left[\frac{(1-\alpha) A_t p_t^w}{w_t}\right]^{\frac{1}{\alpha}} \frac{K_t^j}{A_t}\right)^{1-\alpha} - w_t \frac{u_t^j}{A_t} \left[\frac{(1-\alpha) A_t p_t^w}{w_t}\right]^{\frac{1}{\alpha}} K_t^j \\
&= p_t^w u_t^j \left[\frac{(1-\alpha) A_t p_t^w}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_t^j - w_t \frac{u_t^j}{A_t} \left[\frac{(1-\alpha) A_t p_t^w}{w_t}\right]^{\frac{1}{\alpha}} K_t^j
\end{aligned}$$

Factorise using K_t^j and simplify:

$$\begin{aligned}
p_t^w Y_t^j - w_t N_t^j &= u_t^j \left(\left[\frac{(1-\alpha) A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} - (1-\alpha) \frac{w_t}{(1-\alpha) A_t} \left[\frac{(1-\alpha) A_t}{w_t}\right]^{\frac{1}{\alpha}}\right) \left(p_t^w\right)^{\frac{1}{\alpha}} K_t^j \\
&= u_t^j \left(\left[\frac{(1-\alpha) A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} - (1-\alpha) \left[\frac{(1-\alpha) A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}}\right) \left(p_t^w\right)^{\frac{1}{\alpha}} K_t^j \\
&= u_t^j \alpha \left[\frac{(1-\alpha) A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} \left(p_t^w\right)^{\frac{1}{\alpha}} K_t^j = u_t^j K_t^j \Psi_t \qquad \text{(C.18)}
\end{aligned}$$

where $\Psi_t = \alpha \left[\frac{(1-\alpha) A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} \left(p_t^w\right)^{\frac{1}{\alpha}}$ is the gross rate of return on capital.

The maximisation problem of investing firms can be rewritten as:

$$\begin{aligned}
V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= \max_{u_t^j, l_{t+1}^j} \left[u_t^j \Psi_t + q_t \left(1-\delta_t^j\right)\right] K_t^j - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} + b_{t,a} \\
&\quad + E_t \left[\left(\frac{\varepsilon_t^j}{\varepsilon_t^*} - 1\right) \left(u_t^j \Psi_t K_t^j - l_t^j + \varphi_t K_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l}\right) - q_t^L l_{t+1}^j\right]
\end{aligned}$$

and the maximisation problem of non-investing firms can be rewritten as:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= \max_{u_t^j, l_{t+1}^j} \left[u_t^j \Psi_t + q_t\left(1-\delta_t^j\right)\right] K_t^j - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} \\ &\quad + b_{t,a} - E_t\left[q_t^L l_{t+1}^j\right] \end{aligned}$$

FOC l_{t+1}^j: Optimal Borrowing

Investing firms When $\varepsilon_t^j \geq \varepsilon_t^*$, investing firms have the following maximisation problem:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= \max_{u_t^j, l_{t+1}^j} u\left[u_t^j \Psi_t + q_t\left(1-\delta_t^j\right)\right] K_t^j - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} + b_{t,a} \\ &\quad + E_t\left[\left(\frac{\varepsilon_t^j}{\varepsilon_t^*} - 1\right)\left(u_t^j \Psi_t K_t^j - l_t^j + \varphi_t K_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l}\right) - q_t^L l_{t+1}^j\right] \end{aligned}$$

and the FOC with respect to l_{t+1}^j yields:

$$\begin{aligned} \frac{\partial V_{t,a}}{\partial l_{t+1}^j} &: E_t\left[\frac{(1+\pi_{t+1})}{R_t^l} - q_t^L + \left(\frac{\varepsilon_t^j}{\varepsilon_t^*} - 1\right)\frac{(1+\pi_{t+1})}{R_t^l}\right] = 0 \\ &: q_t^L = E_t\left[\frac{(1+\pi_{t+1})}{R_t^l}\frac{\varepsilon_t^j}{\varepsilon_t^*}\right] \end{aligned}$$

meaning that the coefficient of the value function of the firm is given by:

$$q_t^L = (1-\delta_e) E_t \frac{Q_{t+1}^L}{r_t^{sj}} = E_t\left[\frac{(1+\pi_{t+1})}{R_t^l}\frac{\varepsilon_t^j}{\varepsilon_t^*}\right]$$

Non-Investing Firms When $\varepsilon_t^j < \varepsilon_t^*$, non-investing firms have the following maximisation problem:

$$V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) = \max_{u_t^j, l_{t+1}^j} \left[u_t^j \Psi_t + q_t\left(1-\delta_t^j\right)\right] K_t^j - l_t^j + l_{t+1}^j \frac{(1+\pi_{t+1})}{R_t^l} + b_{t,a} - E_t\left[q_t^L l_{t+1}^j\right]$$

and the FOC with respect to l_{t+1}^j yields:

$$\begin{aligned} \frac{\partial V_{t,a}}{\partial l_{t+1}^j} &: \frac{(1+\pi_{t+1})}{R_t^l} - q_t^L = 0 \\ &: q_t^L = \frac{(1+\pi_{t+1})}{R_t^l} \end{aligned}$$

meaning that the coefficient of the value function of the firm is given by:

$$q_t^L = (1-\delta_e) E_t \frac{Q_{t+1}^L}{r_t^{sj}} = \frac{(1+\pi_{t+1})}{R_t^l}$$

Because of their low efficiency shock, unproductive firms do not borrow but save $\left(l_{t+1}^j < 0\right)$ at the net rate $\frac{R_t^l}{(1+\pi_{t+1})}$.

Credit Constraint

Because of the credit constraint, investing (borrowing) firms will (might) not be able to absorb the entire savings in the economy (e.g. see Kiyotaki (1998)). Consequently, the interest rate is given by the optimisation problem of the unproductive firm: $q_t^L = \frac{(1+\pi_{t+1})}{R_t^l}$. In addition, the credit constraint binds:

$$\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(K_{t+1}^j, l_{t+1}^j\right) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(K_{t+1}^j, 0\right) - \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1,a+1}\left(\gamma_t K_t^j, 0\right)$$

so that the credit constraint becomes:

$$\begin{aligned}
\frac{1}{r_t^{sj}} \bar{V}_{t+1,a+1}\left(K_{t+1}^j, l_{t+1}^j\right) &= \frac{1}{r_t^{sj}} \bar{V}_{t+1,a+1}\left(K_{t+1}^j, 0\right) - \frac{1}{r_t^{sj}} \bar{V}_{t+1,a+1}\left(\gamma_t K_t^j, 0\right) \\
\frac{\bar{Q}_{t+1} K_{t+1}^j + \bar{B}_{t+1,a+1} - \bar{Q}_{t+1}^L l_{t+1}^j}{r_t^{sj}} &= \frac{\bar{Q}_{t+1} K_{t+1}^j + \bar{B}_{t+1,a+1}}{r_t^{sj}} - \frac{\bar{Q}_{t+1} \gamma_t K_t^j + \bar{B}_{t+1,a+1}}{r_t^{sj}} \\
-\frac{\bar{Q}_{t+1}^L l_{t+1}^j}{r_t^{sj}} &= -\frac{\bar{Q}_{t+1} \gamma_t K_t^j + \bar{B}_{t+1,a+1}}{r_t^{sj}} \\
\frac{(1-\delta_e)\,\bar{Q}_{t+1}^L}{r_t^{sj}} l_{t+1}^j &= \frac{(1-\delta_e)\,\bar{Q}_{t+1}}{r_t^{sj}} \gamma_t K_t^j + \frac{(1-\delta_e)\,\bar{B}_{t+1,a+1}}{r_t^{sj}} \\
q_t^L l_{t+1}^j &= q_t \gamma_t K_t^j + b_{t,a} \\
\frac{(1+\pi_{t+1})}{R_t^l} l_{t+1}^j &= q_t \gamma_t K_t^j + b_{t,a} \qquad \text{(C.19)}
\end{aligned}$$

Substitute out $\frac{(1+\pi_{t+1})}{R_t^l} l_{t+1}^j$ and $q_t^L l_{t+1}^j$ in the maximisation problems, we can then rewrite the maximisation problems as:

◦ For investing firms:

$$\begin{aligned}
V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= \max_{u_t^j} \left[u_t^j \Psi_t + q_t\left(1-\delta_t^j\right)\right] K_t^j - l_t^j + b_{t,a} \\
&\quad + E_t \left[\left(\frac{\varepsilon_t^j}{\varepsilon_t^*} - 1\right) \left[\left(u_t^j \Psi_t + \varphi_t + q_t \gamma_t\right) K_t^j - l_t^j + b_{t,a}\right]\right]
\end{aligned}$$

◦ For non-investing firms:

$$V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) = \max_{u_t^j} \left[u_t^j \Psi_t + q_t\left(1-\delta_t^j\right)\right] K_t^j - l_t^j + b_{t,a}$$

FOC u_t^j: Optimal Capacity of Utilisation

Remind that u_t^j is determined before observing ε_t^j. Therefore each firm faces the same maximisation problem when choosing u_t^j:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= \max_{u_t^j}\left[u_t^j\Psi_t + q_t\left(1-\delta_t^j\right)\right]K_t^j - l_t^j + b_{t,a} \\ &\quad + E_t\left[\left(\frac{\varepsilon_t^j}{\varepsilon_t^*}-1\right)\left[\left(u_t^j\Psi_t+\varphi_t+q_t\gamma_t\right)K_t^j - l_t^j + b_{t,a}\right]\right] \end{aligned}$$

Thus each firm will choose the same the same capacity utilization rate u_t.

Substituing for expected investment decision, the maximisation problem in this case is:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= \max_{u_t}\left[u_t\Psi_t + q_t\left(1-\delta_t^j\right)\right]K_t^j - l_t^j + b_{t,a} \\ &\quad + G_t\left[\left(u_t^j\Psi_t+\varphi_t+q_t\gamma_t\right)K_t^j - l_t^j + b_{t,a}\right] \end{aligned}$$

where $G_t = \int\limits_{\varepsilon \geq \varepsilon_t^*} \frac{\varepsilon}{\varepsilon_t^*} - 1 \; d\Phi(\varepsilon)$. Moreover $\delta_t^j = \delta\left(u_t^j\right) = \delta(u_t)$. The FOC with respect to u_t yields:

$$\begin{aligned} \frac{\partial V_{t,a}}{\partial u_t} &: \Psi_t K_t^j - q_t\frac{d\delta_t}{du_t}K_t^j + G_t\Psi_t K_t^j = 0 \\ &: \Psi_t - \delta'(u_t) + G_t\Psi_t = 0 \\ &: \Psi_t(1+G_t) = \delta'(u_t) \end{aligned}$$

Matching the Coefficients of the Value Function

Recall that investing firms at time t have the following value function:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= \left[u_t\Psi_t + q_t(1-\delta_t)\right]K_t^j - l_t^j + b_{t,a} \\ &\quad + E_t\left[\left(\frac{\varepsilon_t^j}{\varepsilon_t^*}-1\right)\left[\left(u_t^j\Psi_t+\varphi_t+q_t\gamma_t\right)K_t^j - l_t^j + b_{t,a}\right]\right] \\ &= \left[u_t\Psi_t + q_t(1-\delta_t) + \left(\frac{\varepsilon_t^j}{\varepsilon_t^*}-1\right)(u_t\Psi_t+\varphi_t+q_t\gamma_t)\right]K_t^j \\ &\quad - \left[\frac{\varepsilon_t^j}{\varepsilon_t^*}\right]l_t^j + \left[\frac{\varepsilon_t^j}{\varepsilon_t^*}\right]b_{t,a} \end{aligned}$$

and non-investing firms at time t have the following value function:

$$\begin{aligned} V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) &= u_t\Psi_t K_t^j - l_t^j + E_t\left[q_t(1-\delta_t)K_t^j + b_{t,a}\right] \\ &= \left[u_t\Psi_t + q_t(1-\delta_t)\right]K_t^j - l_t^j + b_{t,a} \end{aligned}$$

Thus, the conjecture of the value function of the firms is:

$$V_{t,a}\left(K_t^j, l_t^j, \varepsilon_t^j\right) = Q_t\left(\varepsilon_t^j\right) K_t^j - Q_t^L\left(\varepsilon_t^j\right) l_t^j + B_{t,a}\left(\varepsilon_t^j\right)$$

where the coefficients $Q_t\left(\varepsilon_t^j\right)$, $Q_t^L\left(\varepsilon_t^j\right)$ and $B_{t,a}\left(\varepsilon_t^j\right)$ are given by:

$$Q_t\left(\varepsilon_t^j\right) = \begin{cases} \left[u_t\Psi_t + q_t\left(1-\delta_t\right) + E_t\left(\frac{\varepsilon_t^j}{\varepsilon_t^*} - 1\right)\left(u_t\Psi_t + \varphi_t + q_t\gamma_t\right)\right] & \text{if } \varepsilon_t^j \geq \varepsilon_t^* \\ u_t\Psi_t + q_t\left(1-\delta_t\right) & \text{if } \varepsilon_t^j < \varepsilon_t^* \end{cases} \tag{C.20}$$

$$Q_t^L\left(\varepsilon_t^j\right) = \begin{cases} \frac{\varepsilon_t^j}{\varepsilon_t^*} & \text{if } \varepsilon_t^j \geq \varepsilon_t^* \\ 1 & \text{if } \varepsilon_t^j < \varepsilon_t^* \end{cases} \tag{C.21}$$

$$B_{t,a}\left(\varepsilon_t^j\right) = \begin{cases} \frac{\varepsilon_t^j}{\varepsilon_t^*} b_{t,a} & \text{if } \varepsilon_t^j \geq \varepsilon_t^* \\ b_{t,a} & \text{if } \varepsilon_t^j < \varepsilon_t^* \end{cases} \tag{C.22}$$

Matching the Coefficients of the Price of the Stock

Remind that q_t, q_t^L, and $b_{t,a}$ do not directly depend on the idiosyncratic shock ε_{t+1}^j because they are integrated out. The Marginal q is given by:

$$\begin{aligned} q_t &= \left(1-\delta_e\right) E_t \frac{Q_{t+1}\left(\varepsilon_{t+1}^j\right)}{r_t^{sj}} \\ &= E_t \frac{\left(1-\delta_e\right)}{r_t^{sj}} \left[\int Q_{t+1}\left(\varepsilon\right)\ d\Phi\left(\varepsilon\right)\right] \\ &= E_t \frac{\left(1-\delta_e\right)}{r_t^{sj}} \left[\int_0^{\varepsilon_{t+1}^*} Q_{t+1}\left(\varepsilon\right)\ d\Phi\left(\varepsilon\right) + \int_{\varepsilon_{t+1}^*}^{\infty} Q_{t+1}\left(\varepsilon\right)\ d\Phi\left(\varepsilon\right)\right] \end{aligned}$$

Using the coefficient (C.20), the Marginal q becomes:

$$\begin{aligned} q_t &= E_t \frac{\left(1-\delta_e\right)}{r_t^{sj}} \left[\begin{array}{c} \int_0^{\varepsilon_{t+1}^*} u_{t+1}\Psi_{t+1} + q_{t+1}\left(1-\delta_{t+1}\right)\ d\Phi\left(\varepsilon\right) \\ + \int_{\varepsilon_{t+1}^*}^{\infty} \left(\begin{array}{c} u_{t+1}\Psi_{t+1} + q_{t+1}\left(1-\delta_{t+1}\right) \\ + \left(\frac{\varepsilon}{\varepsilon_{t+1}^*} - 1\right)\left(u_{t_{t+1}}\Psi_{t+1} + \varphi_{t+1} + q_{t+1}\gamma_{t+1}\right) \end{array}\right) d\Phi\left(\varepsilon\right) \end{array}\right] \\ &= E_t \frac{\left(1-\delta_e\right)}{r_t^{sj}} \left[\begin{array}{c} \int_0^{\infty} u_t\Psi_{t+1} + q_{t+1}\left(1-\delta_{t+1}\right)\ d\Phi\left(\varepsilon\right) \\ + \int_{\varepsilon_{t+1}^*}^{\infty} \left(\frac{\varepsilon}{\varepsilon_{t+1}^*} - 1\right)\left(u_{t+1}\Psi_{t+1} + \varphi_{t+1} + q_{t+1}\gamma_{t+1}\right)\ d\Phi\left(\varepsilon\right) \end{array}\right] \\ &= \frac{\left(1-\delta_e\right)}{r_t^{sj}} E_t\left[u_{t+1}\Psi_{t+1} + q_{t+1}\left(1-\delta_{t+1}\right) + G_{t+1}\left(u_{t+1}\Psi_{t+1} + \varphi_{t+1} + q_{t+1}\gamma_{t+1}\right)\right] \end{aligned}$$

where $G_{t+1} = \int\limits_{\varepsilon \geq \varepsilon^*_{t+1}} \frac{\varepsilon}{\varepsilon^*_{t+1}} - 1 \, d\Phi(\varepsilon)$.

The Marginal q^L is given by:

$$
\begin{aligned}
q^L_t &= (1-\delta_e) E_t \frac{Q^L_{t+1}\left(\varepsilon^j_{t+1}\right)}{r^{sj}_t} \\
&= E_t \frac{(1-\delta_e)}{r^{sj}_t} \left[\int_0^{\varepsilon^*_{t+1}} Q^L_{t+1}(\varepsilon) \; d\Phi(\varepsilon) + \int_{\varepsilon^*_{t+1}}^{\infty} Q^L_{t+1}(\varepsilon) \; d\Phi(\varepsilon) \right] \\
&= E_t \frac{(1-\delta_e)}{r^{sj}_t} \left[\int_0^{\varepsilon^*_{t+1}} 1 \; d\Phi(\varepsilon) + \int_{\varepsilon^*_{t+1}}^{\infty} \frac{\varepsilon}{\varepsilon^*_{t+1}} \; d\Phi(\varepsilon) \right] \\
&= E_t \frac{(1-\delta_e)}{r^{sj}_t} \int_{\varepsilon^*_{t+1}}^{\infty} \frac{\varepsilon}{\varepsilon^*_{t+1}} \; d\Phi(\varepsilon) \\
&= E_t \frac{(1-\delta_e)}{r^{sj}_t} (1 + G_{t+1})
\end{aligned}
$$

using the coefficient (C.21).

Finally, the real bubble of firms of age a is given by:

$$
\begin{aligned}
b_{t,a} &= E_t \frac{(1-\delta_e)}{r^{sj}_t} B_{t+1,a+1}\left(\varepsilon^j_{t+1}\right) \\
&= E_t \frac{(1-\delta_e)}{r^{sj}_t} \left[\int B_{t+1,a+1}(\varepsilon) \; d\Phi(\varepsilon) \right] \\
&= E_t \frac{(1-\delta_e)}{r^{sj}_t} \left[\int_0^{\varepsilon^*_{t+1}} B_{t+1,a+1}(\varepsilon) \; d\Phi(\varepsilon) + \int_{\varepsilon^*_{t+1}}^{\infty} B_{t+1,a+1}(\varepsilon) \; d\Phi(\varepsilon) \right]
\end{aligned}
$$

Using coefficient (C.22), the bubble becomes:

$$
\begin{aligned}
&= E_t \frac{(1-\delta_e)}{r^{sj}_t} \left[\int_0^{\varepsilon^*_{t+1}} b_{t+1,a+1} \, d\Phi(\varepsilon) + \int_{\varepsilon^*_{t+1}}^{\infty} \frac{\varepsilon^j_t}{\varepsilon^*_t} b_{t+1,a+1} \, d\Phi(\varepsilon) \right] \\
&= E_t \frac{(1-\delta_e)}{r^{sj}_t} \left[\int_0^{\varepsilon^*_{t+1}} 1 \, d\Phi(\varepsilon) + \int_{\varepsilon^*_{t+1}}^{\infty} 1 \, d\Phi(\varepsilon) + \int_{\varepsilon^*_{t+1}}^{\infty} \frac{\varepsilon^j_t}{\varepsilon^*_t} - 1 \, d\Phi(\varepsilon) \right] b_{t+1,a+1} \\
&= E_t \frac{(1-\delta_e)}{r^{sj}_t} (1 + G_{t+1}) \, b_{t+1,a+1}
\end{aligned}
$$

To summarise q_t, q_t^L, and $b_{t,a}$ are given by:

$$\begin{aligned}
q_t &= \frac{(1-\delta_e)}{r_t^{sj}} E_t \begin{bmatrix} u_{t+1}\Psi_{t+1} + q_{t+1}(1-\delta_{t+1}) \\ +G_{t+1}\left(u_{t+1}\Psi_{t+1} + \varphi_{t+1} + q_{t+1}\gamma_{t+1}\right) \end{bmatrix} \\
q_t^L &= E_t \frac{(1-\delta_e)}{r_t^{sj}} (1+G_{t+1}) = \frac{(1+\pi_{t+1})}{R_t^l} \\
b_{t,a} &= E_t \frac{(1-\delta_e)}{r_t^{sj}} (1+G_{t+1})\, b_{t+1,a+1}
\end{aligned} \tag{C.23}$$

which are the coefficients of the stock price of the following form:

$$p_{t,a}^{sj} = q_t K_{t+1}^j + b_{t,a} - q_t^L l_{t+1}^j$$

where $G_{t+1} = \int\limits_{\varepsilon \geq \varepsilon_{t+1}^*} \frac{\varepsilon}{\varepsilon_{t+1}^*} - 1 \; d\Phi(\varepsilon)$.

Finally, we can retrieve the expected rate of return on stock investment:

$$\begin{aligned}
\frac{(1+\pi_{t+1})}{R_t^l} &= E_t \frac{(1-\delta_e)}{r_t^{sj}} (1+G_{t+1}) \\
r_t^{sj} &= (1-\delta_e) E_t \left[(1+G_{t+1}) \frac{R_t^l}{(1+\pi_{t+1})} \right]
\end{aligned}$$

C.1.3 Capital Producers

Decision Problem

The households own capital producers and recive the profit Π_t^I. A representative capital goods firm produce new capital using input of final output and subject to adjustment costs. It sell new capital to intermediate goods firms at price P_t^I. The objective of a capital producer is to choose I_t to solve:

$$\Pi_t^I = \max_{I_t} E_t \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left[p_t^I I_t - \left(1 + \frac{\Omega}{2} \left[\frac{I_t}{I_{t-1}} - \lambda^I \right]^2 \right) \frac{I_t}{Z_t} \right]$$

where p_t^I is the relative price of capital goods and λ^I is the growth rate of aggregate investment, Z_t is the adjustment cost shock which is IID over time.

Optimal Behaviour

The first order condition with respect to I_t yields:

$$\begin{aligned}
\frac{\partial \Pi_t^I}{\partial I_t} &: \beta^t \frac{\Lambda_t}{\Lambda_0}\left[p_t^I-\left(1+\frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}}-\lambda^I\right]^2\right)\frac{1}{Z_t}-\frac{\Omega}{Z_t}\frac{I_t}{I_{t-1}}\left[\frac{I_t}{I_{t-1}}-\lambda^I\right]\right] \\
&: +\beta^{t+1}\frac{\Lambda_{t+1}}{\Lambda_0}\left[\left(\frac{I_{t+1}}{I_t}\right)^2\Omega\left[\frac{I_{t+1}}{I_t}-\lambda^I\right]\frac{1}{Z_{t+1}}\right]=0
\end{aligned}$$

Solving for the price of investment, we obtain:

$$\begin{aligned}
p_t^I &= \left(1+\frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}}-\lambda^I\right]^2\right)\frac{1}{Z_t}+\frac{\Omega}{Z_t}\frac{I_t}{I_{t-1}}\left[\frac{I_t}{I_{t-1}}-\lambda^I\right] \\
&\quad -\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left[\left(\frac{I_{t+1}}{I_t}\right)^2\Omega\left[\frac{I_{t+1}}{I_t}-\lambda^I\right]\frac{1}{Z_{t+1}}\right] \\
Z_t p_t^I &= 1+\frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}}-\lambda^I\right]^2+\Omega\frac{I_t}{I_{t-1}}\left[\frac{I_t}{I_{t-1}}-\lambda^I\right]-\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left(\frac{I_{t+1}}{I_t}\right)^2\Omega\left[\frac{I_{t+1}}{I_t}-\lambda^I\right]\frac{Z_t}{Z_{t+1}}
\end{aligned}$$

C.1.4 Retailers

Decision Problem: Calvo Parameterisation

There is a continuum of retail firms of measure one, each index by i. Each retail firm i buys a wholesale good Y_t^j at price P_t^w and repackage into a specialized retail good $Y_t(i)$. Retailers sell their specialized retail good $Y_t(i)$ to competitive final firms at price P_t^i.

Firm's optimisation problem is standard: a firm i chooses quantities and prices to maximise profit

$$\mathbb{E}_t \sum_{s=t}^{\infty} Q_{t,s}\left[\left(1-\tau_s\right)Y_s\left(i\right)p_s\left(i\right)-P_s^w Y_s(i)\right],$$

where $Q_{t,s}$ is household's stochastic discount factor, τ_t^{sH} is sales tax and τ^S is an employment subsidy that is used to eliminate steady-state inefficiencies associated with monopolistic competition and distornionary taxation. This optimization is subject to demand constraint:

$$Y_t\left(i\right)=Y_t\left(\frac{p_t\left(i\right)}{P_t}\right)^{-\varkappa_t}$$

Price rigidity and price indexation defined as following: i) as in Calvo's (1983) model, at time t, a firm i will not reset the price ($p_t\left(i\right)=p_{t-1}\left(i\right)$) with probability ϑ and will reset ($p_t\left(i\right)=p_t^R\left(i\right)$) with probability $1-\vartheta$; ii) when it has the chance of reseting its price, it chooses price optimally, $p_t^*(i)$, with probability $1-\varpi$, or chooses the new price, with probability ϖ, according to the simple rule of thumb $p_t^b = p_{t-1}^R \pi_{t-1}$.

Maximisation Problem

A firm i that has the chance to reset its price in period t and, with probability $1-\varpi$, adopts an optimizing behaviour chooses price $p_t^*(i)$, assuming that it still applies at s, which maximizes:

$$\begin{aligned} V_t(i) &= \max_{\{Y_t(i), p_s^*(i)\}_{s=t}^{\infty}} \mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} Q_{t,s} \left[(1-\tau_s) Y_s(i) \frac{p_t^*(i)}{P_s} - \frac{P_s^w}{P_s} Y_s(i) \right] \\ &= \max_{\{Y_t(i), p_s^*(i)\}_{s=t}^{\infty}} \mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} Q_{t,s} \left[(1-\tau_s) Y_s(i) \frac{p_t^*(i)}{P_s} - Y_s(i) \frac{MC_s}{P_s} \right] \end{aligned}$$

where the marginal cost MC_s is given by:

$$MC_s = P_t^w$$

and the competitive demand constraint for good $Y_t(i)$:

$$Y_t(i) = Y_t \left(\frac{p_t(i)}{P_t} \right)^{-\varkappa_t}$$

The maximisation problem can be rewriten such as:

$$\begin{aligned} V_t(i) &= \max_{\left\{Y_t(i), p_s^{\times}(i)\right\}_{s=t}^{\infty}} \mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} Q_{t,s} Y_s(i) \left[(1-\tau_s) \frac{p_t^*(i)}{P_s} - mc_s \right] \\ &= \max_{\{p_s^*(i)\}_{s=t}^{\infty}} \mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} Q_{t,s} Y_s \left[(1-\tau_s) \left(\frac{p_t^*(i)}{P_s} \right)^{1-\varkappa_t} - mc_s \left(\frac{p_t^*(i)}{P_s} \right)^{-\varkappa_t} \right] \end{aligned}$$

where mc_s is the real marginal cost.

First Order Condition

The FOC with respect to $p_t^*(i)$ yields:

$$\begin{aligned} 0 &= \mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} Q_{t,s} Y_s^T \left[(1-\tau_s) \left(\frac{p_t^*(i)}{P_s} \right)^{-\varkappa_t} + \frac{\varkappa_t}{(1-\varkappa_t)} mc_s \left(\frac{p_t^*(i)}{P_s} \right)^{-\varkappa_t - 1} \right] \\ 0 &= \mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} Q_{t,s} Y_s(i) \left(\frac{p_t^*(i)}{P_s} \right)^{\varkappa_t} \left[(1-\tau_s) \left(\frac{p_t^*(i)}{P_s} \right)^{-\varkappa_t} - \mu_s mc_s \left(\frac{p_t^*(i)}{P_s} \right)^{-\varkappa_t - 1} \right] \\ 0 &= \mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} Q_{t,s} Y_s(i) \left(\frac{p_t^*(i)}{P_s} \right)^{-1} \left[(1-\tau_s) \left(\frac{p_t^*(i)}{P_s} \right) - \mu_s mc_s \right] \end{aligned}$$

where the markup μ_s in steady-state is equal to:

$$\mu = -\frac{\varkappa}{1-\varkappa} = \frac{\varkappa}{\varkappa - 1}$$

Multiply by p_t^* to obtain:

$$0 = \mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} Q_{t,s} Y_s(i) P_s \left[(1-\tau_s) \left(\frac{p_t^*(i)}{P_s} \right) - \mu_s mc_s \right]$$

and rewrite such that:

$$\mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} Q_{t,s} Y_s(i) P_s (1-\tau_s) \left(\frac{p_t^*(i)}{P_s} \right) = \mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} Q_{t,s} Y_s(i) P_s \mu_s mc_s$$

Using the stochastic discount factor:

$$Q_{t,s} = \beta^{s-t} \left[\frac{\Lambda_s P_t}{\Lambda_t P_s} \right]$$

and subsituting it out and obtain:

$$\begin{aligned}
\mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} \beta^{s-t} \left[\frac{\Lambda_s P_t}{P_s \Lambda_t} \right] Y_s(i) P_s (1-\tau_s) \left(\frac{p_t^*(i)}{P_s} \right) &= \mathbb{E}_t \sum_{s=t}^{\infty} \vartheta^{s-t} \beta^{s-t} \left[\frac{\Lambda_s P_t}{P_s \Lambda_t} \right] Y_s(i) P_s \mu_s mc_s \\
\mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \Lambda_s Y_s(i) (1-\tau_s) \left(\frac{p_t^*(i)}{P_s} \right) &= \mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \Lambda_s Y_s(i) \mu_s mc_s
\end{aligned}$$

Using the competitive demand constraint for good:

$$Y_t(i) = Y_t \left(\frac{p_t(i)}{P_t} \right)^{-\varkappa_t}$$

we can substitute out $Y_t(i)$ and obtain:

$$\begin{aligned}
\mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \Lambda_s Y_s \left(\frac{p_t^*(i)}{P_s} \right)^{1-\varkappa_t} (1-\tau_s) &= \mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \Lambda_s Y_s \left(\frac{p_t^*(i)}{P_s} \right)^{-\varkappa_t} \mu_s mc_s \\
\mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \left(\frac{p_t(i)}{P_t} \frac{P_t}{P_s} \right)^{1-\varkappa_t} f_s &= \mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \left(\frac{p_t(i)}{P_t} \frac{P_t}{P_s} \right)^{-\varkappa_t} h_s \\
\left(\frac{p_t^*(i)}{P_t} \right)^{1-\varkappa_t} \mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \left(\frac{P_t}{P_s} \right)^{1-\varkappa_t} f_s &= \left(\frac{p_t(i)}{P_t} \right)^{-\varkappa_t} \mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \left(\frac{P_t}{P_s} \right)^{-\varkappa_t} h_s
\end{aligned}$$

with:

$$\begin{aligned} h_s &= \Lambda_s Y_s \mu_s mc_s \\ f_s &= \Lambda_s Y_s (1-\tau_s) \end{aligned}$$

This last expression yields the following expression:

$$\begin{aligned} \frac{p_t^*(i)}{P_t} &= \frac{\mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \left(\frac{P_t}{P_s}\right)^{-\varkappa_t} \Lambda_s Y_s \mu_s mc_s}{\mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \left(\frac{P_t}{P_s}\right)^{1-\varkappa_t} \Lambda_s Y_s (1-\tau_s)} \\ &= \frac{\mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \left(\frac{P_t}{P_s}\right)^{-\varkappa_t} h_s}{\mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} \left(\frac{P_t}{P_s}\right)^{1-\varkappa_t} f_s} = \frac{H_t}{F_t} \end{aligned}$$

where H_t and F_t satisfy:

$$\begin{aligned} H_t &= \mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} h_s \left(\frac{P_s}{P_t}\right)^{\varkappa_t} \\ F_t &= \mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} f_s \left(\frac{P_s}{P_t}\right)^{\varkappa_t - 1} \end{aligned}$$

We can now show that H_t and F_t are functional equations. Let us start with H_t:

$$\begin{aligned} H_t &= \mathbb{E}_t \sum_{s=t}^{\infty} (\vartheta\beta)^{s-t} h_s \left(\frac{P_s}{P_t}\right)^{\varkappa_t} \\ &= h_t + \mathbb{E}_t \sum_{s=t+1}^{\infty} (\vartheta\beta)^{s-t} h_s \left(\frac{P_s}{P_t}\right)^{\varkappa_t} \\ &= h_t + \vartheta\beta \mathbb{E}_t (1+\pi_{t+1})^{\varkappa_t} \sum_{s=t+1}^{\infty} (\vartheta\beta)^{s-t-1} h_s \left(\frac{P_s}{P_{t+1}}\right)^{\varkappa_t} \\ H_t &= h_t + \vartheta\beta \mathbb{E}_t (1+\pi_{t+1})^{\varkappa_t} H_{t+1} \end{aligned}$$

Similarly, we can show that the functional equation F_t has the following form:

$$F_t = f_t + \vartheta\beta \mathbb{E}_t (1+\pi_{t+1})^{\varkappa_t - 1} F_{t+1}$$

To summarise, the functional equations for H_t and F_t are given by:

$$\begin{aligned} H_t &= \Lambda_t Y_t \mu_t mc_t + \vartheta\beta\mathbb{E}_t \left(1+\pi_{t+1}\right)^{\varkappa_t} H_{t+1} \\ F_t &= \Lambda_t Y_t \left(1-\tau_t\right) + \vartheta\beta\mathbb{E}_t \left(1+\pi_{t+1}\right)^{\varkappa_t - 1} F_{t+1} \end{aligned}$$

With share ϑ of firms keeping last period's price and share $(1-\vartheta)$ of firms resetting a new price, the law of motion of aggregate price index P_{Ht} is:

$$P_t = \left[(1-\vartheta)\left(P_t^R\right)^{1-\varkappa_t} + \vartheta P_{t-1}^{1-\varkappa_t}\right]^{\frac{1}{1-\varkappa_t}}$$

where the reset price index P_t^R is given by:

$$P_t^R = \left[(1-\varpi)\left(p_t^*\right)^{1-\varkappa_t} + \varpi\left(P_t^b\right)^{1-\varkappa_t}\right]^{\frac{1}{1-\varkappa_t}}$$

with a fraction ϖ of firms choose the new price P_{Ht}^b according to a simple rule of thumb:

$$P_t^b = P_{t-1}^R \Pi_{t-1}$$

where Π is the gross rate of inflation.

Finally, the evolution of price dispersion Δ_t is given by:

$$\Delta_t = (1-\vartheta)(1-\varpi)\left(\frac{p_t^*}{P_t}\right)^{-\varkappa_t} + (1-\vartheta)\varpi\left(\frac{P_t^b}{P_t}\right)^{-\varkappa_t} + \vartheta\Pi_t^{\varkappa_t}\Delta_{t-1}$$

Linearisation of the Phillips curve

Let us start with the following set of equation:

$$\begin{aligned} \frac{p_t^*}{P_t} &= \frac{H_t}{F_t} && \text{(C.24)} \\ \left(P_t\right)^{1-\varkappa_t} &= (1-\vartheta)\left(P_t^R\right)^{1-\varkappa_t} + \vartheta P_{t-1}^{1-\varkappa_t} && \text{(C.25)} \\ \left(P_t^R\right)^{1-\varkappa_t} &= (1-\varpi)\left(p_t^*\right)^{1-\varkappa_t} + \varpi\left(P_t^b\right)^{1-\varkappa_t} && \text{(C.26)} \\ P_t^b &= P_{t-1}^R \Pi_{t-1} && \text{(C.27)} \end{aligned}$$

Equation (C.25) can be simplified to:

$$1 = (1-\vartheta)\left(\frac{P_t^R}{P_t}\right)^{1-\varkappa_t} + \vartheta\left(\frac{1}{\Pi_t}\right)^{1-\varkappa_t} \tag{C.28}$$

and equation (C.26) can be rewritten using (C.27) as:

$$\begin{aligned}
\left(\frac{P_t^R}{P_t}\right)^{1-\varkappa_t} &= (1-\varpi)\left(\frac{p_t^*}{P_t}\right)^{1-\varkappa_t} + \varpi\left(\frac{P_t^b}{P_t}\right)^{1-\varkappa_t} \\
\left(\frac{P_t^R}{P_t}\right)^{1-\varkappa_t} &= (1-\varpi)\left(\frac{p_t^*}{P_t}\right)^{1-\varkappa_t} + \varpi\left(\frac{P_{t-1}^R}{P_{t-1}}\frac{P_{t-1}}{P_t}\Pi_{t-1}\right)^{1-\varkappa_t} \\
\left(\frac{P_t^R}{P_t}\right)^{1-\varkappa_t} &= (1-\varpi)\left(\frac{p_t^*}{P_t}\right)^{1-\varkappa_t} + \varpi\left(\frac{P_{t-1}^R}{P_{t-1}}\frac{\Pi_{t-1}}{\Pi_t}\right)^{1-\varkappa_t} \qquad \text{(C.29)}
\end{aligned}$$

When $\varpi = 0$, the previous equation yields:

$$\left(\frac{P_t^R}{P_t}\right)^{1-\varkappa_t} = \left(\frac{p_t^*}{P_t}\right)^{1-\varkappa_t} = \left(\frac{H_t}{F_t}\right)^{1-\varkappa_t}$$

Substitute it in equation (C.28) and solve for $\left(\frac{H_t}{F_t}\right)^{1-\varkappa_t}$:

$$\begin{aligned}
1 &= (1-\vartheta)\left(\frac{H_t}{F_t}\right)^{1-\varkappa_t} + \vartheta\left(\frac{1}{\Pi_t}\right)^{1-\varkappa_t} \\
\left(\frac{H_t}{F_t}\right)^{1-\varkappa_t} &= \frac{1}{(1-\vartheta)} - \frac{\vartheta}{(1-\vartheta)}\left(\frac{1}{\Pi_t}\right)^{1-\varkappa_t} \\
\left(\frac{H_t}{F_t}\right)^{1-\varkappa_t} &= \frac{1-\vartheta\Pi_t^{\varkappa_t-1}}{(1-\vartheta)}
\end{aligned}$$

where H_t and F_t satisfy:

$$\begin{aligned}
H_t &= \Lambda_t Y_t \mu_t mc_t + \vartheta\beta\mathbb{E}_t\Pi_{t+1}^{\varkappa_t} H_{t+1} \\
F_t &= \Lambda_t Y_t (1-\tau_t) + \vartheta\beta\mathbb{E}_t\Pi_{t+1}^{\varkappa_t-1} F_{t+1}
\end{aligned}$$

When $\varpi \neq 0$, the equation (C.29) yields:

$$\begin{aligned}
\left(\frac{P_t^R}{P_t}\right)^{1-\varkappa_t} &= (1-\varpi)\left(\frac{p_t^*}{P_t}\right)^{1-\varkappa_t} + \varpi\left(\frac{P_{t-1}^R}{P_{t-1}}\right)^{1-\varkappa_t}\left(\frac{\Pi_{t-1}}{\Pi_t}\right)^{1-\varkappa_t} \\
\left(\frac{P_t^R}{P_t}\right)^{1-\varkappa_t} &= (1-\varpi)\left(\frac{1-\vartheta\Pi_t^{\varkappa_t-1}}{(1-\vartheta)}\right)^{1-\varkappa_t} + \varpi\left(\frac{P_{t-1}^R}{P_{t-1}}\right)^{1-\varkappa_t}\left(\frac{\Pi_{t-1}}{\Pi_t}\right)^{1-\varkappa_t} \\
\left(\frac{P_t^R}{P_t}\right)^{1-\varkappa_t} &= \frac{1}{(1-\vartheta)} - \frac{\vartheta}{(1-\vartheta)}\left(\frac{1}{\Pi_t}\right)^{1-\varkappa_t} \\
\frac{1}{(1-\vartheta)}\left(1-\vartheta\left(\frac{1}{\Pi_t}\right)^{1-\varkappa_t}\right) &= (1-\varpi)\left(\frac{p_t^*}{P_t}\right)^{1-\varkappa_t} + \frac{\varpi}{(1-\vartheta)}\left(1-\vartheta\left(\frac{1}{\Pi_{t-1}}\right)^{1-\varkappa_t}\right)\left(\frac{\Pi_{t-1}}{\Pi_t}\right)^{1-\varkappa_t}
\end{aligned}$$

and finally:

$$1-\vartheta\left(\frac{1}{\Pi_t}\right)^{1-\varkappa_t} = (1-\vartheta)(1-\varpi)\left(\frac{H_t}{F_t}\right)^{1-\varkappa_t}+\varpi\left(1-\vartheta\left(\frac{1}{\Pi_{t-1}}\right)^{1-\varkappa_t}\right)\left(\frac{\Pi_{t-1}}{\Pi_t}\right)^{1-\varkappa_t}$$

Then, we have the following system that we can log-linearised around steady-state with zero inflation:

$$\begin{aligned}
H_t &= h_t+\vartheta\beta\mathbb{E}_t\left(1+\pi_{t+1}\right)^{\varkappa_t}H_{t+1}\\
F_t &= f_t+\vartheta\beta\mathbb{E}_t\left(1+\pi_{t+1}\right)^{\varkappa_t-1}F_{t+1}\\
\frac{1}{(1-\vartheta)}\left(1-\vartheta\left(\frac{1}{\Pi_t}\right)^{1-\varkappa_t}\right) &= (1-\varpi)\left(\frac{H_t}{F_t}\right)^{1-\varkappa_t}\\
&\quad+\frac{\varpi}{(1-\vartheta)}\left(1-\vartheta\left(\frac{1}{\Pi_{t-1}}\right)^{1-\varkappa_t}\right)\left(\frac{\Pi_{t-1}}{\Pi_t}\right)^{1-\varkappa_t}
\end{aligned} \tag{C.30}$$

The functional equation H_t yields:

$$\begin{aligned}
H\left(1+\hat{H}_t^T\right) &= h\left(1+\hat{h}_t\right)+\vartheta\beta\left(1+\varkappa_t\pi_{t+1}\right)H\left(1+\hat{H}_{t+1}^T\right)\\
\hat{H}_t^T &= \frac{h}{H}\hat{h}_t+\vartheta\beta\left(\varkappa_t\pi_{t+1}+\hat{H}_{t+1}^T\right)
\end{aligned}$$

the functional equation F_t yields:

$$\begin{aligned}
F\left(1+\hat{F}_t\right) &= f\left(1+\hat{f}_t\right)+\vartheta\beta\left(1+\left(\varkappa_t-1\right)\pi_{t+1}\right)F\left(1+\hat{F}_{t+1}\right)\\
\hat{F}_t &= \frac{f}{F}\hat{f}_t+\vartheta\beta\left(\left(\varkappa_t-1\right)\pi_{t+1}+\hat{F}_{t+1}\right)
\end{aligned}$$

and equation (C.30) becomes:

$$\begin{aligned}
\frac{1-\vartheta\left(1-\left(1-\varkappa_t\right)\pi_t\right)}{\left(1-\vartheta\right)} &= \left(1-\varpi\right)\left(\frac{H\left(1+\hat{H}_t^T\right)}{F\left(1+\hat{F}_t^T\right)}\right)^{1-\varkappa_t} \\
&\quad +\frac{\varpi\left(1-\vartheta\left(1-\left(1-\varkappa_t\right)\pi_{t-1}\right)\right)}{\left(1-\vartheta\right)}\left(\begin{array}{c} 1+\left(1-\varkappa_t\right)\pi_{t-1} \\ -\left(1-\varkappa_t\right)\pi_t \end{array}\right) \\
\frac{1-\vartheta\left(1-\left(1-\varkappa_t\right)\pi_t\right)}{\left(1-\vartheta\right)} &= \left(1-\varpi\right)\left(\frac{H\left(1+\hat{H}_t^T\right)}{F\left(1+\hat{F}_t^T\right)}\right)^{1-\varkappa_t}+\varpi\left(\begin{array}{c} 1+\frac{\vartheta(1-\varkappa_t)}{(1-\vartheta)}\pi_{t-1} \\ +\left(1-\varkappa_t\right)\left[\pi_{t-1}-\pi_t\right] \end{array}\right) \\
1+\frac{\vartheta\left(1-\varkappa_t\right)\pi_t}{\left(1-\vartheta\right)} &= \left(1-\varpi\right)\left(1+\left(1-\varkappa_t\right)\left(\hat{H}_t^T-\hat{F}_t^T\right)\right)+\varpi\left(\begin{array}{c} 1+\frac{(1-\varkappa_t)}{(1-\vartheta)}\pi_{t-1} \\ -\left(1-\varkappa_t\right)\pi_t \end{array}\right) \\
\frac{\vartheta\left(1-\varkappa_t\right)}{\left(1-\vartheta\right)}\pi_t &= \left(1-\varpi\right)\left(1-\varkappa_t\right)\left(\hat{H}_t^T-\hat{F}_t^T\right)+\varpi\left(\begin{array}{c} \frac{(1-\varkappa_t)}{(1-\vartheta)}\pi_{t-1} \\ -\left(1-\varkappa_t\right)\pi_t \end{array}\right) \\
\frac{\vartheta}{\left(1-\vartheta\right)}\pi_t &= \left(1-\varpi\right)\left(\hat{H}_t^T-\hat{F}_t^T\right)+\varpi\left(\frac{\pi_{t-1}}{\left(1-\vartheta\right)}-\pi_t\right) \qquad \text{(C.31)}
\end{aligned}$$

Taking the difference of the two log-linearised functional equations:

$$\hat{H}_t^T-\hat{F}_t^T=\left(1-\vartheta\beta\right)\hat{h}_t-\left(1-\vartheta\beta\right)\hat{f}_t+\vartheta\beta\pi_{t+1}+\vartheta\beta\left(\hat{H}_{t+1}^T-\hat{F}_{t+1}^T\right)$$

$$\left(\hat{H}_t^T-\hat{F}_t^T=\frac{\vartheta+\left(1-\vartheta\right)\varpi}{\left(1-\vartheta\right)\left(1-\varpi\right)}\pi_t-\frac{\varpi}{\left(1-\varpi\right)\left(1-\vartheta\right)}\pi_{t-1}\right)$$

where:

$$H=\frac{h}{\left(1-\vartheta\beta\right)}, \quad \text{and} \quad F=\frac{f}{\left(1-\vartheta\beta\right)}$$

We can then simplify (C.31) and find:

$$\begin{aligned}
\frac{\vartheta+\left(1-\vartheta\right)\varpi}{\left(1-\vartheta\right)\left(1-\varpi\right)}\pi_t-\frac{\varpi}{\left(1-\varpi\right)\left(1-\vartheta\right)}\pi_{t-1} &= \left(1-\vartheta\beta\right)\hat{h}_t-\left(1-\vartheta\beta\right)\hat{f}_t \\
&\quad +\vartheta\beta\pi_{t+1}+\vartheta\beta\left(\begin{array}{c} \frac{(\vartheta+(1-\vartheta)\varpi)}{(1-\vartheta)(1-\varpi)}\pi_{t+1} \\ -\frac{\varpi\pi_t}{(1-\varpi)(1-\vartheta)} \end{array}\right)
\end{aligned}$$

which yields:

$$\begin{aligned}
\frac{(\vartheta+(1-\vartheta)\varpi+\vartheta\beta\varpi)\pi_t}{(1-\varpi)(1-\vartheta)} &= (1-\vartheta\beta)\left[\hat{h}_t-\hat{f}_t\right]+\begin{pmatrix}\vartheta\beta\frac{(\vartheta+(1-\vartheta)\varpi)}{(1-\vartheta)(1-\varpi)}\\ +\vartheta\beta\end{pmatrix}\pi_{t+1}+\frac{\varpi\pi_{t-1}}{(1-\varpi)(1-\vartheta)}\\
\frac{\vartheta+\varpi-\vartheta\varpi+\vartheta\beta\varpi}{(\varpi-1)(\vartheta-1)}\pi_t &= (1-\vartheta\beta)\left[\hat{h}_t-\hat{f}_t\right]+\frac{\vartheta\beta}{(\varpi-1)(\vartheta-1)}\pi_{t+1}\\
&\quad +\frac{\varpi\pi_{t-1}}{(1-\varpi)(1-\vartheta)} \qquad \text{(C.32)}
\end{aligned}$$

If $\varpi=0$, we obtain the traditional Phillips curve with the Calvo price settings:

$$\pi_t=\frac{(1-\vartheta\beta)(1-\vartheta)}{\vartheta}\left(\hat{h}_t-\hat{f}_t\right)+\beta\pi_{t+1}$$

Continuing the derivation from (C.32), we can solve for the inflation rate:

$$\begin{aligned}
\pi_t &= \frac{(1-\varpi)(1-\vartheta)(1-\vartheta\beta)}{\vartheta(1-\varpi)+(1+\vartheta\beta)\varpi}\left(\hat{h}_t-\hat{f}_t\right)+\frac{\vartheta\beta\pi_{t+1}}{\vartheta(1-\varpi)+(1+\vartheta\beta)\varpi}\\
&\quad +\frac{\varpi\pi_{t-1}}{\vartheta(1-\varpi)+(1+\vartheta\beta)\varpi}
\end{aligned}$$

where the real marginal, h_t and f_t cost:

$$\begin{aligned}
mc_t &= \frac{P_t^w}{P_t}=p_t^w\\
h_t &= \Lambda_t Y_t\mu mc_t=\Lambda_t Y_t\mu p_t^w\\
f_t &= \Lambda_t Y_t(1-\tau_t)
\end{aligned}$$

can be log-linearised such that:

$$\begin{aligned}
\hat{h}_t &= \hat{\Lambda}_t+\hat{Y}_t+\hat{p}_t^w+\hat{\mu}_t\\
\hat{f}_t &= \hat{\Lambda}_t+\hat{Y}_t
\end{aligned}$$

where $\hat{\mu}_t$ is a change in markup, perhaps because of a cost push shock. Moreover, we assume that there is no tax.

Let us log-linearised the markup:

$$\begin{aligned}
\mu(1+\hat{\mu}_t) &= \frac{\varkappa(1+\hat{\varkappa}_t)}{\varkappa(1+\hat{\varkappa}_t)-1}=\frac{\varkappa(1+\hat{\varkappa}_t)}{\varkappa-1+\varkappa\hat{\varkappa}_t}=\frac{\varkappa(1+\hat{\varkappa}_t)}{(\varkappa-1)+\frac{(\varkappa-1)}{(\varkappa-1)}\varkappa\hat{\varkappa}_t}\\
&= \frac{\varkappa(1+\hat{\varkappa}_t)}{(\varkappa-1)\left(1+\frac{\varkappa}{(\varkappa-1)}\hat{\varkappa}_t\right)}
\end{aligned}$$

Using the steady-state of μ, we get:

$$\begin{aligned}
1+\hat{\mu}_t &= \left(1+\hat{\varkappa}_t-\frac{\varkappa}{(\varkappa-1)}\hat{\varkappa}_t\right)\\
\hat{\mu}_t &= \hat{\varkappa}_t-\frac{\varkappa}{\varkappa-1}\hat{\varkappa}_t\\
\hat{\mu}_t &= (1-\mu)\,\hat{\varkappa}_t
\end{aligned}$$

So $\hat{h}_t$ becomes:

$$\hat{h}_t = \hat{\Lambda}_t + \hat{Y}_t + \hat{p}_t^w + (1-\mu)\,\hat{\varkappa}_t$$

and the difference between $\hat{h}_t$ and $\hat{f}_t$ is:

$$\hat{h}_t - \hat{f}_t = \hat{\Lambda}_t + \hat{Y}_t + \hat{p}_t^w - \hat{\Lambda}_t - \hat{Y}_t + (1-\mu)\,\hat{\varkappa}_t = \hat{p}_t^w + (1-\mu)\,\hat{\varkappa}_t$$

So Phillips curve is:

$$\begin{aligned}
\pi_t &= \frac{(1-\varpi)(1-\vartheta)(1-\vartheta\beta)}{(\vartheta(1-\varpi)+(1+\vartheta\beta)\varpi)}\left(\hat{p}_t^w+(1-\mu)\,\hat{\varkappa}_t\right)\\
&\qquad +\frac{\vartheta\beta\pi_{t+1}}{(\vartheta(1-\varpi)+(1+\vartheta\beta)\varpi)}+\frac{\varpi\pi_{t-1}}{(\vartheta(1-\varpi)+(1+\vartheta\beta)\varpi)}
\end{aligned}$$

We can introduce the coefficients:

$$\begin{aligned}
\Upsilon &= (\vartheta(1-\varpi)+(1+\vartheta\beta)\varpi) = \vartheta+\varpi(1-\vartheta+\vartheta\beta)\\
\kappa_c &= \frac{(1-\varpi)(1-\varpi)(1-\vartheta\beta)}{\Upsilon}\\
\chi_f &= \frac{\vartheta}{\Upsilon}\\
\chi_b &= \frac{\varpi}{\Upsilon}
\end{aligned}$$

and the Phillips curve simplifies to:

$$\pi_t = \kappa_c\hat{p}_t^w + \kappa_c(1-\mu)\,\hat{\varkappa}_t + \chi_f\beta\pi_{t+1} + \chi_b\pi_{t-1}$$

Finally, we can normalise the shock of the Phillips curve by its slope times $(1-\mu)$:

$$\pi_t = \kappa_c\hat{p}_t^w + \hat{\varkappa}_t^N + \chi_f\beta\pi_{t+1} + \chi_b\pi_{t-1}$$

where $\hat{\varkappa}_t^N$ is the normalised shock $(\hat{\varkappa}_t^N = \kappa_c(1-\mu)\,\hat{\varkappa}_t)$.

C.1.5 Aggregation

The Aggregate Bubble in the Economy

Then the total of emerging bubble in the economy at date t is given by $\omega\delta_e b_t^*$. Then every new period, the aggregate bubble b_t at time t is follows:

$$\begin{aligned}
t &= 0 : b_0 = \omega\delta_e b_{0,0} \\
t &= 1 : b_1 = (1-\delta_e)^1 \omega\delta_e b_{1,1} + \omega\delta_e b_{1,0} \\
t &= 2 : b_2 = (1-\delta_e)^2 \omega\delta_e b_{2,2} + (1-\delta_e)^1 \omega\delta_e b_{2,1} + \omega\delta_e b_{2,0} \\
& \quad\; t : b_t = \sum_{a=0}^{t} (1-\delta_e)^a \omega\delta_e b_{t,a}
\end{aligned}$$

The households believe that the relative size of the bubbles at date $t+a$ for any two firms born at date t and $t+1$ is given by *the sentiment shock* κ_t, such that:

$$\kappa_t = \frac{b_{t+a,a}}{b_{t+a,a-1}}.$$

It follows that, at date t, the size of the bubble for a new firm $a=0$ (i.e. born at time t), the size of the bubble for a firm that is one year old $a=1$ (i.e. born at time $t-1$), the size of the bubble for a firm that is two years old $a=2$ (i.e. born at time $t-2$), etc:

$$\begin{aligned}
b_{t,0} &= b_t^* \\
b_{t,1} &= \kappa_{t-1} b_t^* \\
b_{t,2} &= \kappa_{t-1}\kappa_{t-2} b_t^* \\
& \quad ...,
\end{aligned}$$

Then, the aggregate bubble can be rewritten as the a function of the new emerging bubble at each date:

$$\begin{aligned}
& t=0,\ b_0 = \omega\delta_e b_0^* \\
& t=1,\ b_1 = (1-\delta_e)\,\omega\delta_e\kappa_0 b_1^* + \omega\delta_e b_1^* \\
& t=2,\ b_2 = (1-\delta_e)^2 \omega\delta_e\kappa_0\kappa_1 b_2^* + (1-\delta_e)^1 \omega\delta_e\kappa_1 b_2^* + \omega\delta_e b_2^* \\
& \text{for } t,\ b_t = m_t b_t^*
\end{aligned} \tag{C.33}$$

where m_t satisfies the recursion:

$$m_t = m_{t-1}(1-\delta_e)\,\kappa_{t-1} + \delta_e\omega$$

where $m_0 = \delta_e\omega$.

Using the law of motion of the bubble (C.23):

$$b_{t,a} = (1-\delta_e)\,\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\,(1+G_{t+1})\,b_{t+1,a+1}$$

The equilibrium restriction on the size of the new bubble is given by:

$$b_t^* = (1-\delta_e)\,\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\,(1+G_{t+1})\,\kappa_t b_{t+1}^*$$

Then, using (C.33), the aggregate real bubble satisfies the non arbitrage equation:

$$\begin{aligned}
\frac{b_t}{m_t} &= (1-\delta_e)\,\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\,(1+G_{t+1})\,\kappa_t \frac{b_{t+1}}{m_{t+1}} \\
b_t &= (1-\delta_e)\,\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\frac{m_t}{m_{t+1}}\,(1+G_{t+1})\,\kappa_t b_{t+1}
\end{aligned}$$

The Resource Constraint

Finally, the resource constraint is given by the aggregation of the budget constraints and profits:

$$\begin{aligned}
C_t + p_t^s s_{t+1} &= (1-\tau_t)\,w_t N_t + \Pi_t^I + \Pi_t^F + (d_t^s + p_t^s)\,s_t + T_t \\
d_t^s + I_t p_t^I &= p_t^w Y_t - w_t N_t^j \\
\Pi_t^F &= \left(1 - p_t^w - \frac{\psi}{2}\pi_t^2\right) Y_t \\
\Pi_t^I &= p_t^I I_t - \left(1 + \frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}} - \lambda^I\right]^2\right)\frac{I_t}{Z_t}
\end{aligned}$$

Using $s_t = s_{t+1} = 1$, substitute the three last equations in the budget constraint of the household:

$$\begin{aligned}
C_t + p_t^s &= (1-\tau_t)\,w_t N_t - \left(1 + \frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}} - \lambda^I\right]^2\right)\frac{I_t}{Z_t} + \left(1 - p_t^w - \frac{\psi}{2}\pi_t^2\right) Y_t \\
&\quad + \left(p_t^w Y_t - w_t N_t^j\right) + T_t \\
\left(1 - \frac{\psi}{2}\pi_t^2\right) Y_t &= C_t - \left(1 + \frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}} - \lambda^I\right]^2\right)\frac{I_t}{Z_t}
\end{aligned}$$

C.1.6 Original System Describing the Dynamic non-Stationary Equilibrium

The entire system is composed of 18 equations solving 18 endogenous variables: C_t, N_t, p_t^I, p_t^w, u_t, K_t^A, K_t, w_t, I_t, q_t, Ψ_t, G_t, R_t^l, Λ_t, b_t, m_t, Y_t and π_t.

1. Marginal utility for consumption:

$$\Lambda_t A_t = \frac{\xi_t}{\left(C_t/A_t - \theta C_{t-1}/A_{t-1}\right)^{\sigma}} - \theta\beta E_t\left[\frac{\xi_{t+1}}{\left(C_{t+1}/A_{t+1} - \theta C_t/A_t\right)^{\sigma}}\right]$$

2. Aggregate labour supply:

$$\frac{N_t^{\eta}}{\Lambda_t} = \frac{w_t}{\xi_t\psi_t}\left(1-\tau_t\right)$$

3. Euler equation for the capital producers:

$$\begin{aligned} Z_t p_t^I &= 1+\frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}}-\lambda^I\right]^2+\Omega\frac{I_t}{I_{t-1}}\left[\frac{I_t}{I_{t-1}}-\lambda^I\right] \\ & \qquad -\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left(\frac{I_{t+1}}{I_t}\right)^2\Omega\left[\frac{I_{t+1}}{I_t}-\lambda^I\right]\frac{Z_t}{Z_{t+1}} \end{aligned}$$

4. The Euler equation for the retailers:

$$1-\vartheta\left(\frac{1}{\Pi_t}\right)^{1-\varkappa_t} = \left(1-\vartheta\right)\left(1-\varpi\right)\left(\frac{H_t}{F_t}\right)^{1-\varkappa_t}+\varpi\left(1-\vartheta\left(\frac{1}{\Pi_{t-1}}\right)^{1-\varkappa_t}\right)\left(\frac{\Pi_{t-1}}{\Pi_t}\right)^{1-\varkappa_t}$$

where:

$$\begin{aligned} H_t &= \Lambda_t p_t^w Y_t\frac{\varkappa_t}{\varkappa_t-1}+\vartheta\beta\mathbb{E}_t\Pi_{t+1}^{\varkappa_t}H_{t+1} \\ F_t &= \Lambda_t Y_t+\vartheta\beta\mathbb{E}_t\Pi_{t+1}^{\varkappa_t-1}F_{t+1} \end{aligned}$$

5. Aggregate output:

$$Y_t = \left(u_t K_t^A\right)^{\alpha}\left(A_t z_t N_t\right)^{1-\alpha}$$

6. Aggregate effective capital stock:

$$K_t^A = \left(1-\delta_e\right)K_t+\delta_e K_{0t}$$

7. Law of motion for capital:

$$K_{t+1} = \left(1-\delta_t\right)K_t^A + I_t\frac{\int_{\varepsilon>\varepsilon_t^*}\varepsilon\ d\Phi\left(\varepsilon\right)}{\int_{\varepsilon>\varepsilon_t^*}d\Phi\left(\varepsilon\right)}$$

where $\varepsilon_t^* = p_t^I/q_t$ is the investment threshold.

8. Aggregate labour demand:

$$N_t = \frac{u_t}{z_t}\left[\frac{(1-\alpha) A_t z_t p_t^w}{w_t}\right]^{\frac{1}{\alpha}} \frac{K_t^A}{A_t}$$

9. Aggregate investment demand:

$$I_t p_t^I = \left[\left(u_t \Psi_t + \zeta_t q_t\right) K_t^A + b_t\right] \int_{\varepsilon > \varepsilon_t^*} d\Phi(\varepsilon)$$

10. Optimal capacity of utilisation:

$$\Psi_t (1 + G_t) = q_t \delta^{'}(u_t)$$

11. Rental rate of capital:

$$\Psi_t = \alpha \left[\frac{(1-\alpha) A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} (p_t^w)^{\frac{1}{\alpha}}$$

12. Benefits of being productive:

$$G_t = \int\limits_{\varepsilon \geq \varepsilon_t^*} \left(\frac{\varepsilon}{\varepsilon_t^*} - 1\right) d\Phi(\varepsilon) \tag{C.34}$$

13. Lending rate:

$$\frac{(1+\pi_{t+1})}{R_t^l} = (1-\delta_e) \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} (1 + G_{t+1})$$

14. Marginal Q:

$$q_t = (1-\delta_e) \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left[\begin{array}{c} u_{t+1}\Psi_{t+1} + q_{t+1}(1-\delta_{t+1}) \\ +G_{t+1}\left(u_{t+1}\Psi_{t+1} + \zeta_{t+1} q_{t+1}\right) \end{array}\right]$$

15. Law of motion of the real value of the bubble:

$$b_t = (1-\delta_e) \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \frac{m_t}{m_{t+1}} (1 + G_{t+1}) \kappa_t b_{t+1}$$

16. Evolution of the number of bubbly firms:

$$m_t = m_{t-1}(1-\delta_e) \kappa_{t-1} + \delta_e \omega$$

17. Resource constraint:

$$C_t + \left(1 + \frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}} - \lambda^I\right]^2\right)\frac{I_t}{Z_t} = Y_t$$

18. Monetary policy:

$$R^l_t = R\left[(1+\pi_t)^{\phi_\pi}\left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_y}\right]^{1-\phi_R}\left(\frac{R^l_{t-1}}{R}\right)^{\phi_R}\exp(\epsilon^R_t)$$

C.1.7 Derivations and Simplifications

Benefits of Being Productive

We rewrite $\int_{\varepsilon>\varepsilon^*_t} d\Phi(\varepsilon)$ as:

$$\int_{\varepsilon\geq\varepsilon^*_t} d\Phi(\varepsilon) = 1 - \Phi(\varepsilon^*_t)$$

and define:

$$\int_{\varepsilon\geq\varepsilon^*_t} \varepsilon\ d\Phi(\varepsilon) = \Sigma(\varepsilon^*_t)$$

With $\Sigma(x_t) = \int\limits_{\varepsilon\geq x_t} \varepsilon\ d\Phi(\varepsilon)$, so

$$\begin{aligned} G &= \int\limits_{\varepsilon\geq x_t}\left(\frac{\varepsilon}{x_t} - 1\right)d\Phi(\varepsilon) \\ &= \int\limits_{\varepsilon\geq x_t}\frac{\varepsilon}{x_t}d\Phi(\varepsilon) - \int\limits_{\varepsilon\geq x_t} 1 d\Phi(\varepsilon) \\ &= \frac{1}{x_t}\int\limits_{\varepsilon\geq x_t}\varepsilon d\Phi(\varepsilon) - [1 - \Phi(x_t)] \\ &= \frac{\Sigma(x_t)}{x_t} - [1 - \Phi(x_t)] \end{aligned}$$

therefore the investment threshold (C.34) becomes:

$$G_t = \int\limits_{\varepsilon\geq\varepsilon^*_t}\left(\frac{\varepsilon}{\varepsilon^*_t} - 1\right)d\Phi(\varepsilon) = \frac{\Sigma(\varepsilon^*_t)}{\varepsilon^*_t} + \Phi(\varepsilon^*_t) - 1$$

For later use, $\Sigma(\varepsilon_t^*)$ can be expressed as:

$$\begin{aligned} G_t &= \frac{\Sigma(\varepsilon_t^*)}{\varepsilon_t^*} - [1-\Phi(\varepsilon_t^*)] \\ \Sigma(\varepsilon_t^*) &= [G_t + 1 - \Phi(\varepsilon_t^*)]\,\varepsilon_t^* \end{aligned}$$

Substitute out the Wage of the System

We will substitue out the w_t of the system, so let's rewrite:

$$\begin{aligned} N_t &= u_t \left[\frac{(1-\alpha) A_t p_t^w}{w_t}\right]^{\frac{1}{\alpha}} \frac{K_t^A}{A_t} \\ w_t &= (u_t)^\alpha (1-\alpha) A_t p_t^w \left(\frac{K_t^A}{A_t}\right)^\alpha N_t^{-\alpha} \\ w_t &= (1-\alpha)\frac{p_t^w Y_t}{N_t} \end{aligned}$$

The retal rate of capital becomes:

$$\begin{aligned} \Psi_t &= \alpha \left[\frac{(1-\alpha) A_t}{(1-\alpha)\frac{p_t^w Y_t}{N_t}}\right]^{\frac{1-\alpha}{\alpha}} (p_t^w)^{\frac{1}{\alpha}} \\ \Psi_t^\alpha &= \alpha^\alpha (A_t N_t)^{1-\alpha} (p_t^w)^\alpha (Y_t)^{\alpha-1} \\ \Psi_t^\alpha &= \alpha^\alpha (A_t N_t)^{1-\alpha} (p_t^w)^\alpha (Y_t)^{\alpha-1} \frac{\left(u_t K_t^A\right)^\alpha}{\left(u_t K_t^A\right)^\alpha} \\ \Psi_t &= \frac{\alpha p_t^w Y_t}{u_t K_t^A} \end{aligned}$$

C.2 Stationary System

The following system presented below has a non-stochastic stationary system. It is composed of 18 equations solving 18 endogenous stationary variables: $\tilde{C}_t$, N_t, p_t^I, p_t^w, u_t, $\tilde{K}_t^A$, $\tilde{K}_t$, $\tilde{w}_t$, $\tilde{I}_t$, q_t, Ψ_t, G_t, R_t^l, $\tilde{\Lambda}_t$, $\tilde{b}_t$, m_t, $\tilde{Y}_t$ and π_t.

Shocks are as in the paper plus nominal rigidity: stationary technology shock $g_{At} = A_t/A_{t-1}$, stationary production function shock z_t, labor supply shock ψ_t, investment-specific technology shock Z_t, financial shock ζ_t, sentiment shock κ_t, cost-push shock $\varkappa_t$, taste shock ξ_t.

1. Marginal utility for consumption:

$$\tilde{\Lambda}_t = \frac{\xi_t}{\left(\tilde{C}_t - \theta \tilde{C}_{t-1}\right)^\sigma} - \theta\beta E_t \left[\frac{\xi_{t+1}}{\left(\tilde{C}_{t+1} - \theta \tilde{C}_t\right)^\sigma}\right]$$

where Λ_t is multiplied, not divided ($\tilde{\Lambda}_t = \Lambda_t A_t$).

2. Aggregate labour supply:

$$\psi_t N_t^{\eta+1} = (1-\alpha)\,\tilde{\Lambda}_t \frac{\tilde{Y}_t p_t^w}{\xi_t}\,(1-\tau_t)$$

3. Euler equation for the capital producers:

$$\begin{aligned} Z_t p_t^I &= 1 + \frac{\Omega}{2}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}} g_{At} - \lambda^I\right]^2 + \Omega \frac{\tilde{I}_t}{\tilde{I}_{t-1}} g_{At}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}} g_{At} - \lambda^I\right] \\ &\quad -\beta \mathbb{E}_t \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t} g_{At+1}\left(\frac{\tilde{I}_{t+1}}{\tilde{I}_t}\right)^2 \Omega \left[\frac{\tilde{I}_{t+1}}{\tilde{I}_t} g_{At+1} - \lambda^I\right]\frac{Z_t}{Z_{t+1}} \end{aligned}$$

4. Euler equation for the retailers:

$$1-\vartheta\left(\frac{1}{\Pi_t}\right)^{1-\varkappa_t} = (1-\vartheta)(1-\varpi)\left(\frac{H_t}{F_t}\right)^{1-\varkappa_t} + \varpi\left(1-\vartheta\left(\frac{1}{\Pi_{t-1}}\right)^{1-\varkappa_t}\right)\left(\frac{\Pi_{t-1}}{\Pi_t}\right)^{1-\varkappa_t}$$

where:

$$\begin{aligned} H_t &= \tilde{\Lambda}_t \tilde{Y}_t p_t^w \frac{\varkappa_t}{\varkappa_t - 1} + \vartheta\beta\mathbb{E}_t \Pi_{t+1}^{\varkappa_t} H_{t+1} \\ F_t &= \tilde{\Lambda}_t \tilde{Y}_t (1-\tau_t) + \vartheta\beta\mathbb{E}_t \Pi_{t+1}^{\varkappa_t - 1} F_{t+1} \end{aligned}$$

5. Aggregate output:

$$\tilde{Y}_t = \left(u_t \tilde{K}_t^A\right)^{\alpha} (N_t)^{1-\alpha}$$

6. Aggregate effective capital stock:

$$\tilde{K}_t^A = (1-\delta_e)\,\tilde{K}_t + \delta_e \tilde{K}_{0t}$$

where the normalized $\tilde{K}_{0t}$ is constant:

$$\tilde{K}_{0t} = \tilde{K}_0$$

7. Law of motion for capital:

$$g_{At+1}\tilde{K}_{t+1} = (1-\delta_t)\,\tilde{K}_t^A + \tilde{I}_t \frac{\Sigma(\varepsilon_t^*)}{1-\Phi(\varepsilon_t^*)}$$

where $\Sigma(\varepsilon_t^*) = \int_{\varepsilon>\varepsilon_t^*} \varepsilon \; d\Phi(\varepsilon)$, $1-\Phi(\varepsilon_t^*) = \int_{\varepsilon>\varepsilon_t^*} d\Phi(\varepsilon)$ and $\varepsilon_t^* = p_t^I/q_t$.

8. Real wage setting:

$$\tilde{w}_t = (1-\alpha)\, p_t^w \frac{\tilde{Y}_t}{N_t}$$

9. Aggregate investment demand:

$$\tilde{I}_t = \left[\alpha \tilde{Y}_t p_t^w + \zeta_t q_t \tilde{K}_t^A + \tilde{b}_t\right] \frac{1-\Phi\left(\varepsilon_t^*\right)}{p_t^I}$$

10. Optimal capacity of utilisation:

$$\frac{\alpha \tilde{Y}_t p_t^w}{u_t \tilde{K}_t^A}\left(\frac{\Sigma\left(\varepsilon_t^*\right)}{\varepsilon_t^*} + \Phi\left(\varepsilon_t^*\right)\right) = q_t \delta^{'}\left(u_t\right)$$

11. Rental rate of capital:

$$\Psi_t = \frac{\alpha \tilde{Y}_t p_t^w}{u_t \tilde{K}_t^A}$$

12. Benefits of being productive:

$$G_t = \frac{\Sigma\left(\varepsilon_t^*\right)}{\varepsilon_t^*} + \Phi\left(\varepsilon_t^*\right) - 1$$

13. Lending rate:

$$\frac{\left(1+\pi_{t+1}\right)}{R_t^l} = \left(1-\delta_e\right)\beta E_t \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t} \frac{\left(1+G_{t+1}\right)}{g_{At+1}}$$

14. Marginal Q:

$$q_t = \left(1-\delta_e\right)\beta E_t \frac{\tilde{\Lambda}_{t+1} q_{t+1}}{\tilde{\Lambda}_t g_{At+1}} \left[\begin{array}{c} u_{t+1}\delta^{'}\left(u_{t+1}\right) + \left(1-\delta(u_{t+1})\right) \\ +\left(\frac{\Sigma\left(\varepsilon_{t+1}^*\right)}{\varepsilon_{t+1}^*} + \Phi\left(\varepsilon_{t+1}^*\right) - 1\right)\zeta_{t+1} \end{array}\right]$$

15. Law of motion of the real value of the bubble:

$$\tilde{b}_t = \left(1-\delta_e\right)\beta E_t \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t} \frac{m_t}{m_{t+1}} \left(1+G_{t+1}\right)\kappa_t \tilde{b}_{t+1}$$

where we can see that b_t is non-stationary by design.

16. Evolution of the number of bubbly firms:

$$m_t = m_{t-1}(1-\delta_e)\kappa_{t-1} + \delta_e \omega$$

17. Resource constraint:

$$\tilde{C}_t + \left(1 + \frac{\Omega}{2}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}} g_{At} - \lambda^I\right]^2\right)\frac{\tilde{I}_t}{Z_t} = \tilde{Y}_t$$

18. Taylor rule:

$$R_t^l = R\left[(1+\pi_t)^{\phi_\pi}\left(g_{At}\frac{\tilde{Y}_t}{\tilde{Y}_{t-1}}\right)^{\phi_y}\right]^{1-\phi_R}\left(\frac{R_{t-1}^l}{R}\right)^{\phi_R}\exp(\epsilon_t^R)$$

which is linking nominal interest rate R_t^l, inflation π_t, and change in output. It will also include g_{At} and an iid shock.

Finally, we can derive two additional equations:

∘ Stock price

$$\begin{aligned} p_t^s &= q_t K_{t+1} + b_t \\ \frac{p_t^s}{A_t} &= q_t \frac{K_{t+1}}{A_{t+1}} g_{At+1} + \frac{b_t}{A_t} \\ \tilde{p}_t^s &= q_t \tilde{K}_{t+1} g_{At+1} + \tilde{b}_t \end{aligned}$$

∘ Rate of returns on stock:

$$r_t^s = (1-\delta_e) E_t\left[(1+G_{t+1})\frac{R_t^l}{(1+\pi_{t+1})}\right]$$

C.2.1 Derivations

3. Euler equation for the capital producers:

$$\begin{aligned} Z_t p_t^I &= 1 + \frac{\Omega}{2}\left[\frac{I_t}{I_{t-1}} - \lambda^I\right]^2 + \Omega \frac{I_t}{I_{t-1}}\left[\frac{I_t}{I_{t-1}} - \lambda^I\right] \\ &\quad -\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left(\frac{I_{t+1}}{I_t}\right)^2 \Omega\left[\frac{I_{t+1}}{I_t} - \lambda^I\right]\frac{Z_t}{Z_{t+1}} \end{aligned}$$

which can be log-linearised such that:

$$\begin{aligned}
Z_t p_t^I &= 1+\frac{\Omega}{2}\left[\frac{I_t}{A_t}\frac{A_{t-1}}{I_{t-1}}\frac{A_t}{A_{t-1}}-\lambda^I\right]^2+\Omega\frac{I_t}{A_t}\frac{A_{t-1}}{I_{t-1}}\frac{A_t}{A_{t-1}}\left[\frac{I_t}{A_t}\frac{A_{t-1}}{I_{t-1}}\frac{A_t}{A_{t-1}}-\lambda^I\right] \\
&\quad -\beta\frac{\Lambda_{t+1}A_{t+1}}{\Lambda_t A_t}\frac{A_t}{A_{t+1}}\left(\frac{I_{t+1}}{A_{t+1}}\frac{A_t}{I_t}\frac{A_{t+1}}{A_t}\right)^2\Omega\left[\frac{I_{t+1}}{A_{t+1}}\frac{A_t}{I_t}\frac{A_{t+1}}{A_t}-\lambda^I\right]\frac{Z_t}{Z_{t+1}} \\
Z_t p_t^I &= 1+\frac{\Omega}{2}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}}\frac{A_t}{A_{t-1}}-\lambda^I\right]^2+\Omega\frac{\tilde{I}_t}{\tilde{I}_{t-1}}\frac{A_t}{A_{t-1}}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}}\frac{A_t}{A_{t-1}}-\lambda^I\right] \\
&\quad -\beta\frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t A_t}\frac{A_t}{A_{t+1}}\left(\frac{\tilde{I}_{t+1}}{\tilde{I}_t}\frac{A_{t+1}}{A_t}\right)^2\Omega\left[\frac{\tilde{I}_{t+1}}{\tilde{I}_t}\frac{A_{t+1}}{A_t}-\lambda^I\right]\frac{Z_t}{Z_{t+1}} \\
Z_t p_t^I &= 1+\frac{\Omega}{2}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}}g_{At}-\lambda^I\right]^2+\Omega\frac{\tilde{I}_t}{\tilde{I}_{t-1}}g_{At}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}}g_{At}-\lambda^I\right] \\
&\quad -\beta\mathbb{E}_t\frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t g_{At+1}}\left(\frac{\tilde{I}_{t+1}}{\tilde{I}_t}g_{At+1}\right)^2\Omega\left[\frac{\tilde{I}_{t+1}}{\tilde{I}_t}g_{At+1}-\lambda^I\right]\frac{Z_t}{Z_{t+1}} \\
Z_t p_t^I &= 1+\frac{\Omega}{2}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}}g_{At}-\lambda^I\right]^2+\Omega\frac{\tilde{I}_t}{\tilde{I}_{t-1}}g_{At}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}}g_{At}-\lambda^I\right] \\
&\quad -\beta\mathbb{E}_t\frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t}\left(\frac{\tilde{I}_{t+1}}{\tilde{I}_t}\right)^2 g_{At+1}\Omega\left[\frac{\tilde{I}_{t+1}}{\tilde{I}_t}g_{At+1}-\lambda^I\right]\frac{Z_t}{Z_{t+1}}
\end{aligned}$$

9. Aggregate investment demand:

$$\tilde{I}_t=\left[\left(u_t\Psi_t+\zeta_t q_t\right)\tilde{K}_t^A+\tilde{b}_t\right]\frac{\left(1-\Phi\left(\varepsilon_t^*\right)\right)}{p_t^I}$$

10. Optimal capacity of utilisation:

$$\begin{aligned}
\Psi_t\left(1+G_t\right) &= q_t\delta'\left(u_t\right) \\
\alpha\frac{\tilde{Y}_t p_t^w}{u_t\tilde{K}_t^A}\left(1+G_t\right) &= q_t\delta'\left(u_t\right) \\
\alpha\frac{\tilde{Y}_t p_t^w}{u_t\tilde{K}_t^A}\left(\frac{\Sigma\left(\varepsilon_t^*\right)}{\varepsilon_t^*}+\Phi\left(\varepsilon_t^*\right)\right) &= q_t\delta'\left(u_t\right)
\end{aligned}$$

14. Marginal Q (q_t is stationary by design):

$$q_t = (1-\delta_e)\,\beta E_t \frac{\Lambda_{t+1}A_{t+1}}{\Lambda_t A_t}\frac{A_t}{A_{t+1}}\begin{bmatrix} u_{t+1}\Psi_{t+1} + q_{t+1}\left(1-\delta_{t+1}\right) \\ \\ +G_{t+1}\left(u_{t+1}\Psi_{t+1}+\zeta_{t+1}q_{t+1}\right) \end{bmatrix}$$

$$q_t = (1-\delta_e)\,\beta E_t \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t g_{At+1}}\begin{bmatrix} u_{t+1}\Psi_{t+1} + q_{t+1}\left(1-\delta_{t+1}\right) \\ \\ +G_{t+1}\left(u_{t+1}\Psi_{t+1}+\zeta_{t+1}q_{t+1}\right) \end{bmatrix}$$

$$q_t = (1-\delta_e)\,\beta E_t \frac{\tilde{\Lambda}_{t+1}q_{t+1}}{\tilde{\Lambda}_t g_{At+1}}\left[u_{t+1}\frac{\left(1+G_{t+1}\right)\Psi_{t+1}}{q_{t+1}} + \left(1-\delta_{t+1}\right) + G_{t+1}\zeta_{t+1}\right]$$

$$q_t = (1-\delta_e)\,\beta E_t \frac{\tilde{\Lambda}_{t+1}q_{t+1}}{\tilde{\Lambda}_t g_{At+1}}\left[u_{t+1}\delta^{'}\left(u_{t+1}\right) + \left(1-\delta(u_{t+1})\right) + G_{t+1}\zeta_{t+1}\right]$$

$$q_t = (1-\delta_e)\,\beta E_t \frac{\tilde{\Lambda}_{t+1}q_{t+1}}{\tilde{\Lambda}_t g_{At+1}}\begin{bmatrix} u_{t+1}\delta^{'}\left(u_{t+1}\right) + \left(1-\delta(u_{t+1})\right) \\ +\left(\frac{\Sigma\left(\varepsilon^*_{t+1}\right)}{\varepsilon^*_{t+1}} + \Phi\left(\varepsilon^*_{t+1}\right) - 1\right)\zeta_{t+1} \end{bmatrix}$$

where

$$\frac{\left(1+G_{t+1}\right)\Psi_{t+1}}{q_{t+1}} = \delta^{'}\left(u_{t+1}\right)$$

from the optimal capacity of utilisation.

15. Law of motion of the real value of the bubble (B.19):

$$b_t = (1-\delta_e)\,\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\frac{m_t}{m_{t+1}}\left(1+G_{t+1}\right)\kappa_t b_{t+1}$$

$$b_t\frac{A_t}{A_t} = (1-\delta_e)\,\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\frac{A_t}{A_t}\frac{A_{t+1}}{A_{t+1}}\frac{m_t}{m_{t+1}}\left(1+G_{t+1}\right)\kappa_t b_{t+1}\frac{A_{t+1}}{A_{t+1}}$$

$$\tilde{b}_t A_t = (1-\delta_e)\,\beta E_t \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t}\frac{A_t}{A_{t+1}}\frac{m_t}{m_{t+1}}\left(1+G_{t+1}\right)\kappa_t \tilde{b}_{t+1}A_{t+1}$$

$$\tilde{b}_t = (1-\delta_e)\,\beta E_t \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t}\frac{m_t}{m_{t+1}}\left(1+G_{t+1}\right)\kappa_t \tilde{b}_{t+1}$$

Remind that $\tilde{\Lambda}_{t+1} = \Lambda_{t+1}A_{t+1}$.

C.3 Stationary Steady-state

1. Marginal utility for consumption:

$$\tilde{\Lambda} = \frac{\xi\left(1-\theta\beta\right)}{\left(\tilde{C}-\theta\tilde{C}\right)^{\sigma}}$$

2. Aggregate labour supply:

$$\psi N^{\eta+1} = \left(1-\alpha\right)\tilde{\Lambda}\frac{\tilde{Y}p^{w}}{\xi}\left(1-\tau\right)$$

3. Euler equation for the capital producers:

$$Zp^{I} = 1$$

because $\lambda^{I} = g_A$ in steady-state.

4. Euler equation for the retailers:

$$p^{w} = \frac{\varkappa - 1}{\varkappa}$$

5. Aggregate output:

$$\tilde{Y} = \left(\tilde{K}^{A}\right)^{\alpha} N^{1-\alpha}$$

where u is equal to 1 in steady-state.

6. Aggregate effective capital stock:

$$\tilde{K}^{A} = \left(1-\delta_e\right)\tilde{K} + \delta_e\tilde{K}_0 \tag{C.35}$$

7. Law of motion for capital:

$$g_A\tilde{K} = \left(1-\delta\right)\tilde{K}^{A} + \tilde{I}\frac{\Sigma\left(\varepsilon^{*}\right)}{1-\Phi\left(\varepsilon^{*}\right)} \tag{C.36}$$

where $\Sigma\left(\varepsilon^{*}\right) = \int_{\varepsilon>\varepsilon^{*}} \varepsilon \; d\Phi\left(\varepsilon\right)$, $1-\Phi\left(\varepsilon^{*}\right) = \int_{\varepsilon>\varepsilon^{*}} d\Phi\left(\varepsilon\right)$ and $\varepsilon^{*} = p^{I}/q$.

8. Real wage setting:

$$\tilde{w} = \left(1-\alpha\right)\frac{\tilde{Y}p^{w}}{N}$$

9. Aggregate investment demand:

$$\tilde{I} = \left[\alpha \tilde{Y} p^w + \zeta q \tilde{K}^A + \tilde{b}\right] \frac{1 - \Phi\left(\varepsilon^*\right)}{p^I} \tag{C.37}$$

10. Optimal capacity of utilisation:

$$\frac{\alpha \tilde{Y} p^w}{\tilde{K}^A} (1 + G) = q \delta^{'}(1) \tag{C.38}$$

11. Rental rate of capital:

$$\Psi = \frac{\alpha \tilde{Y} p^w}{\tilde{K}^A}$$

12. Benefits of being productive:

$$G = \frac{\Sigma\left(\varepsilon^*\right)}{\varepsilon^*} + \Phi\left(\varepsilon^*\right) - 1 \tag{C.39}$$

13. Lending rate:

$$\frac{1}{R_t^l} = \left(1 - \delta_e\right) \beta \frac{(1 + G)}{g_A} \tag{C.40}$$

14. Marginal Q:

$$1 = \frac{\left(1 - \delta_e\right)}{g_A} \beta \left[\delta^{'}(1) + 1 - \delta(1) + \zeta G\right] \tag{C.41}$$

15. Law of motion of the real value of the bubble:

$$\frac{1}{\beta} = (1 + G)\left(1 - \delta_e\right) \kappa \tag{C.42}$$

16. Evolution of the number of bubbly firms:

$$m = \frac{\delta_e \omega}{\left(1 - \kappa + \kappa \delta_e\right)}$$

17. Resource constraint:

$$\tilde{C} + \frac{\tilde{I}}{Z} = \tilde{Y}$$

since $g_A = \lambda^I$.

18. Taylor rule:

$$R^l = R$$

Finally, we have the growth rate of the economy:

$$g_A = \frac{A_t}{A_{t-1}}$$

and the stock price:

$$\tilde{p}^s = q\tilde{K}g_A + \tilde{b}_t$$

C.3.1 Simplifications and Conditions:

Finding $\Sigma\left(\varepsilon^*\right)$

Equation (C.39) and (C.42) yields at steady-state:

$$G = \frac{\Sigma\left(\varepsilon^*\right)}{\varepsilon^*} - \left[1 - \Phi\left(\varepsilon^*\right)\right] = \frac{1}{\beta\left(1-\delta_e\right)\kappa} - 1 \tag{C.43}$$

Remind form C.1.7 that $\Sigma\left(\varepsilon^*\right)$ can be expressed as:

$$\Sigma\left(\varepsilon^*\right) = \left[G + 1 - \Phi\left(\varepsilon^*\right)\right]\varepsilon^*$$

This gives us steady state $\Sigma\left(\varepsilon^*\right)$ as function of G in steady state, and G is given by $\frac{1}{\beta(1-\delta_e)\kappa} - 1$. $\Phi\left(\varepsilon^*\right)$ is treated as known parameter. So:

$$\Sigma\left(\varepsilon^*\right) = \left[\frac{1}{\beta\left(1-\delta_e\right)\kappa} - \Phi\left(\varepsilon^*\right)\right]\varepsilon^*$$

Finding $\delta^{'}(1)$ **and** ϖ_δ

Reming that at the steady-state, the capacity of utilisation is equal to 1, $u = 1$. From Eq (C.38), we derive the value of $\delta^{'}(1)$:

$$\delta^{'}(1) = (1+G)\frac{\alpha\tilde{Y}p^w}{q\tilde{K}^A} \tag{C.44}$$

Eq (C.41) can be rewritten as

$$\delta^{'}(1)=\frac{g_A}{(1-\delta_e)\,\beta}-(1-\delta)-\zeta G \tag{C.45}$$

Using (C.45) and (C.44), we can derive

$$\begin{aligned}(1+G)\,\frac{\alpha\tilde{Y}p^w}{q\tilde{K}^A} &= \frac{g_A}{(1-\delta_e)\,\beta}-(1-\delta)-\zeta G\\ \frac{\alpha\tilde{Y}p^w}{q\tilde{K}^A} &= \frac{1}{(1+G)}\left[\frac{g_A}{(1-\delta_e)\,\beta}-(1-\delta)-\zeta G\right]\end{aligned} \tag{C.46}$$

then using the new equation (C.43), we get:

$$\begin{aligned}\alpha\frac{\tilde{Y}p^w}{q\tilde{K}^A} &= \beta\,(1-\delta_e)\,\kappa\left[\frac{g_A}{(1-\delta_e)\,\beta}-(1-\delta)-\zeta\left(\frac{1}{\beta\,(1-\delta_e)\,\kappa}-1\right)\right]\\ \alpha\frac{\tilde{Y}p^w}{q\tilde{K}^A} &= g_A\kappa-\beta\,(1-\delta_e)\,\kappa\,(1-\delta)-\beta\,(1-\delta_e)\,\kappa\zeta\left(\frac{1}{\beta\,(1-\delta_e)\,\kappa}-1\right)\\ \alpha\frac{\tilde{Y}p^w}{q\tilde{K}^A} &= g_A\kappa-(1-\delta)\,\beta\,(1-\delta_e)\,\kappa-\zeta\,[1-\beta\,(1-\delta_e)\,\kappa]\\ \frac{q\tilde{K}^A}{\tilde{Y}p^w} &= \frac{\alpha}{g_A\kappa-(1-\delta)\,\beta\,(1-\delta_e)\,\kappa-\zeta\,[1-\beta\,(1-\delta_e)\,\kappa]}\\ \frac{q\tilde{K}^A}{\tilde{Y}p^w} &= \varpi_\delta\end{aligned}$$

where ϖ_δ:

$$\varpi_\delta=\frac{\alpha}{g_A\kappa-(1-\delta)\,\beta\,(1-\delta_e)\,\kappa-\zeta\,[1-\beta\,(1-\delta_e)\,\kappa]} \tag{C.47}$$

Eq(C.44) becomes:

$$\begin{aligned}\delta^{'}(1) &= (1+G)\,\frac{\alpha\tilde{Y}p^w}{q\tilde{K}^A}\\ \delta^{'}(1) &= (1+G)\,\frac{\alpha}{\varpi_\delta}\\ \delta^{'}(1) &= \frac{\alpha}{\beta\,(1-\delta_e)\,\kappa\varpi_\delta}\end{aligned}$$

using the law of motion of the bubble (C.42):

$$\frac{1}{\beta\,(1-\delta_e)\,\kappa}=(1+G)$$

Find ϖ_K

Using the aggregate effective capital stock Eq (C.35):

$$\begin{aligned}
\tilde{K}^A &= (1-\delta_e)\,\tilde{K} + \delta_e \tilde{K}_0 \\
\frac{\tilde{K}^A}{\tilde{K}} &= (1-\delta_e) + \delta_e \frac{\tilde{K}_0}{\tilde{K}} \\
\frac{\tilde{K}^A}{\tilde{Y}p^w} &= \left[(1-\delta_e) + \delta_e \frac{\tilde{K}_0}{\tilde{K}}\right] \frac{\tilde{K}}{\tilde{Y}p^w} \\
\frac{\tilde{K}}{\tilde{Y}p^w} &= \varpi_K \frac{\tilde{K}^A}{\tilde{Y}p^w}
\end{aligned}$$

where ϖ_K:

$$\varpi_K = \frac{1}{\left[(1-\delta_e) + \delta_e \frac{\tilde{K}_0}{\tilde{K}}\right]}$$

Unique Bubbly Steady-state Equilibrium

Starting from the law of motion for capital, Eq (C.36):

$$\begin{aligned}
g_A \tilde{K} &= (1-\delta)\,\tilde{K}^A + \tilde{I} \frac{\Sigma(\varepsilon^*)}{1-\Phi(\varepsilon^*)} \\
\tilde{I} &= \frac{1-\Phi(\varepsilon^*)}{\Sigma(\varepsilon^*)} \left[g_A \tilde{K} - (1-\delta)\,\tilde{K}^A\right] \\
\frac{\tilde{I}}{\tilde{Y}} &= \frac{1-\Phi(\varepsilon^*)}{\Sigma(\varepsilon^*)} \left[g_A \varpi_K \frac{\tilde{K}^A}{\tilde{Y}} - (1-\delta)\,\frac{\tilde{K}^A}{\tilde{Y}}\right] \\
\frac{\tilde{I}}{\tilde{Y}} &= \frac{1-\Phi(\varepsilon^*)}{\Sigma(\varepsilon^*)} \left[g_A \varpi_K - (1-\delta)\right] \frac{\tilde{K}^A}{\tilde{Y}}
\end{aligned}$$

Using:

$$\Sigma(\varepsilon^*) = \left[G + 1 - \Phi(\varepsilon^*)\right] \varepsilon^*$$

where $\varepsilon^* = p^I/q$, we then can rewrite:

$$\begin{aligned}
\frac{\tilde{I}}{\tilde{Y}} &= \frac{1-\Phi(\varepsilon^*)}{\left[G+1-\Phi(\varepsilon^*)\right] p^I/q} \left[g_A \varpi_K - (1-\delta)\right] \frac{\tilde{K}^A}{\tilde{Y}} \\
\frac{p^I \tilde{I}}{\tilde{Y}p^w} &= \frac{1-\Phi(\varepsilon^*)}{\left[G+1-\Phi(\varepsilon^*)\right]} \left[g_A \varpi_K - (1-\delta)\right] \frac{q\tilde{K}^A}{\tilde{Y}p^w} \\
\frac{p^I \tilde{I}}{p^w \tilde{Y}} &= \frac{\left[1-\Phi(\varepsilon^*)\right]\left[g_A \varpi_K - (1-\delta)\right] \varpi_\delta}{\left[G+1-\Phi(\varepsilon^*)\right]}
\end{aligned}$$

Then, using Eq (C.42), $G+1=\frac{1}{\beta(1-\delta_e)\kappa}$, we can derive an expresion for $\Phi(\varepsilon^*)$:

$$\begin{aligned}
\frac{p^I\tilde{I}}{p^w\tilde{Y}} &= \frac{[1-\Phi(\varepsilon^*)]\,[g_A\varpi_K-(1-\delta)]\,\varpi_\delta}{[G+1-\Phi(\varepsilon^*)]} \\
G+1-\Phi(\varepsilon^*) &= [1-\Phi(\varepsilon^*)]\,[g_A\varpi_K-(1-\delta)]\,\varpi_\delta\frac{p^w\tilde{Y}}{p^I\tilde{I}} \\
-G &= [1-\Phi(\varepsilon^*)]\left(1-[g_A\varpi_K-(1-\delta)]\,\varpi_\delta\frac{p^w\tilde{Y}}{p^I\tilde{I}}\right) \\
[1-\Phi(\varepsilon^*)] &= \frac{G}{[g_A\varpi_K-(1-\delta)]\,\varpi_\delta\frac{p^w\tilde{Y}}{p^I\tilde{I}}-1} \\
\Phi(\varepsilon^*) &= 1-\frac{G}{[g_A\varpi_K-(1-\delta)]\,\varpi_\delta\frac{p^w\tilde{Y}}{p^I\tilde{I}}-1}
\end{aligned}$$

Finally, using Eq (C.37), we obtain:

$$\begin{aligned}
\tilde{I} &= \left[\alpha\tilde{Y}p^w+\zeta q\tilde{K}^A+\tilde{b}\right]\frac{1-\Phi(\varepsilon^*)}{p^I} \\
\tilde{I}p^I\frac{1}{[1-\Phi(\varepsilon^*)]} &= \left[\alpha\tilde{Y}p^w+\zeta q\tilde{K}^A+\tilde{b}\right] \\
\tilde{b} &= \tilde{I}p^I\frac{1}{[1-\Phi(\varepsilon^*)]}-\alpha\tilde{Y}p^w-\zeta q\tilde{K}^A \\
\frac{\tilde{b}}{\tilde{Y}p^w} &= \frac{\tilde{I}p^I}{\tilde{Y}p^w}\frac{1}{[1-\Phi(\varepsilon^*)]}-\alpha-\zeta\frac{q\tilde{K}^A}{\tilde{Y}p^w} \\
\frac{\tilde{b}}{p^w\tilde{Y}} &= \frac{[1-\Phi(\varepsilon^*)]\,[g_A\varpi_K-(1-\delta)]\,\varpi_\delta}{[G+1-\Phi(\varepsilon^*)]\,[1-\Phi(\varepsilon^*)]}-\alpha-\zeta\varpi_\delta \\
\frac{\tilde{b}}{p^w\tilde{Y}} &= \frac{[g_A\varpi_K-(1-\delta)]\,\varpi_\delta}{[G+1-\Phi(\varepsilon^*)]}-\alpha-\zeta\varpi_\delta>0
\end{aligned}$$

where ϖ_b is defined as:

$$\varpi_b:=\frac{\tilde{b}}{p^w\tilde{Y}}=\frac{[g_A\varpi_K-(1-\delta)]\,\varpi_\delta}{\left[\frac{1}{\beta(1-\delta_e)\kappa}-\Phi(\varepsilon^*)\right]}-\alpha-\zeta\varpi_\delta$$

The Steady-state Growth Rate of the Bubble

The bubble growth rate is given using Eq (C.40):

$$\frac{1}{R_t^l}=(1-\delta_e)\,\beta\frac{(1+G)}{g_A}$$

and Eq (C.42):

$$\frac{1}{\beta} = (1+G)(1-\delta_e)\kappa$$

which yields:

$$\begin{aligned}
\frac{g_A}{(1-\delta_e)\beta R^l} &= (1+G) \\
\frac{1}{\beta} &= \frac{g_A}{(1-\delta_e)\beta R^l}(1-\delta_e)\kappa \\
\kappa &= \frac{R^l}{g_A}
\end{aligned}$$

Relations between G, $\Sigma(\varepsilon^*)$ and φ_G

Remind that:

$$G = \frac{1}{\beta(1-\delta_e)\kappa} - 1$$

and define:

$$\varphi_G = -\frac{1-\Phi(\varepsilon_t^*)}{G} - 1$$

We can solve for G:

$$\begin{aligned}
\Sigma(\varepsilon^*) &= [G + 1 - \Phi(\varepsilon^*)]\varepsilon^* \\
G &= \frac{\Sigma(\varepsilon^*)}{\varepsilon^*} - [1 - \Phi(\varepsilon^*)]
\end{aligned}$$

Using φ_G, we get:

$$\varphi_G = \frac{\Sigma(\varepsilon^*)}{\varepsilon^*}\left(\frac{1}{(1-\Phi(\varepsilon^*)) - \frac{\Sigma(\varepsilon^*)}{\varepsilon^*}}\right) = -\frac{\Sigma(\varepsilon^*)}{G\varepsilon^*}$$

Therefore:

$$\begin{aligned}
\frac{\Sigma(\varepsilon^*)}{\varepsilon^*} &= -G\varphi_G \\
\frac{\Sigma(\varepsilon^*)}{\varepsilon^*} &= -\left(\frac{1}{\beta(1-\delta_e)\kappa} - 1\right)\varphi_G
\end{aligned}$$

and $\frac{\Sigma(\varepsilon^*)}{G\varepsilon^*}$ is:

$$\frac{\Sigma(\varepsilon^*)}{G\varepsilon^*} = -\varphi_G$$

C.3.2 Calibration

We use data to calibrate three steady-state values, 1) the growth rate of GDP, g_A, 2) the real interest rate, R^l, and 3) the ratio of investment over GDP,$\frac{\tilde{I}}{\tilde{Y}}$. Using this steady-state calibration and the calibrated parameters, we can retrive the following steady-state values:

$$p^w = \frac{\varkappa - 1}{\varkappa}$$

$$\kappa = \frac{R^l}{g_A}$$

$$\frac{\hat{C}}{\tilde{Y}} = 1 - \frac{\tilde{I}}{\tilde{Y}}$$

$$G = \frac{1}{\beta\left(1-\delta_e\right)\kappa} - 1$$

$$\delta^{'}(1) = \frac{g_A}{\left(1-\delta_e\right)\beta} - (1-\delta) - \zeta G$$

$$\varpi_\delta = \frac{q\tilde{K}^A}{\tilde{Y}p^w} = \frac{\alpha\left(1+G\right)}{\delta^{'}(1)}$$

$$\varpi_K = \frac{1}{\left[\left(1-\delta_e\right) + \delta_e\frac{\tilde{K}_0}{\tilde{K}}\right]}$$

$$\frac{q\tilde{K}}{\tilde{Y}p^w} = \varpi_K\varpi_\delta$$

$$\Phi\left(\varepsilon^*\right) = 1 - \frac{G}{\left[g_A\varpi_K - (1-\delta)\right]\varpi_\delta\frac{\tilde{Y}}{\tilde{I}} - 1}$$

$$\varpi_b = \frac{\tilde{b}}{\tilde{Y}p^w} = \frac{\left[g_A\varpi_K - (1-\delta)\right]\varpi_\delta}{\left[G + 1 - \Phi\left(\varepsilon^*\right)\right]} - \alpha - \zeta\varpi_\delta$$

$$cps = g_A\frac{q\tilde{K}}{\tilde{Y}p^w} + \frac{\tilde{b}}{\tilde{Y}p^w}$$

$$\varphi_G = -\frac{1 - \Phi\left(\varepsilon_t^*\right)}{G} - 1$$

$$\delta''(1) = \delta'(1)\,\frac{\delta''(1)}{\delta'(1)}$$

C.3.3 Alternative Method to solve the Steady-state

We can analytically reduce the probel to one non-linear equation:

$$\Xi = \Gamma \tag{C.48}$$

where:

$$\begin{aligned}
\Xi &= \xi \psi N^{\eta+1} \left[\left(1 - \frac{I}{YZ}\right)(1-\theta) N^{1-\alpha} \right]^{\sigma} \left(\left((1-\delta_e) + \delta_e \frac{\tilde{K}_0}{\tilde{K}} \right) \tilde{K} \right)^{\sigma\alpha} \\
\Gamma &= (1-\alpha)\,\xi\,(1-\theta\beta)\, p^w N^{1-\alpha} \left(\left((1-\delta_e) + \delta_e \frac{\tilde{K}_0}{\tilde{K}} \right) \tilde{K} \right)^{\alpha}
\end{aligned}$$

using:

$$\begin{aligned}
\delta^{'}(1) &= \frac{g_A}{(1-\delta_e)\,\beta} - 1 + \delta(1) - \zeta G \\
G &= \frac{1}{\beta\kappa\,(1-\delta_e)} - 1 \\
p^w &= \frac{\varkappa - 1}{\varkappa}
\end{aligned}$$

Soving equation (C.48) for $\tilde{K}$, we obtain:

$$\begin{aligned}
\tilde{K} &= \left(\frac{(1-\alpha)\,\xi\,(1-\theta\beta)\, p^w N^{1-\alpha} \left((1-\delta_e) + \delta_e \frac{\tilde{K}_0}{\tilde{K}} \right)^{\alpha}}{\xi \psi N^{\eta+1} \left[\left(1 - \frac{I}{YZ}\right)(1-\theta) N^{1-\alpha} \right]^{\sigma} \left((1-\delta_e) + \delta_e \frac{\tilde{K}_0}{\tilde{K}} \right)^{\alpha\sigma}} \right)^{\frac{1}{\sigma}} \\
\tilde{K} &= \left(\frac{(1-\alpha)\,(1-\theta\beta)\, p^w N^{1-\alpha}}{\psi N^{\eta+1} \left[\left(1 - \frac{I}{YZ}\right)(1-\theta) N^{1-\alpha} \right]^{\sigma} \left((1-\delta_e) + \delta_e \frac{\tilde{K}_0}{\tilde{K}} \right)^{\sigma}} \right)^{\frac{1}{\sigma}}
\end{aligned}$$

From this solution, we can retrieve the values of the other variables the system using the following equations:

$$\tilde{Y} = N^{1-\alpha} \left(\left((1-\delta_e) + \delta_e \frac{\tilde{K}_0}{\tilde{K}} \right) \tilde{K} \right)^{\alpha}$$

$$\tilde{K}^A = \left((1-\delta_e) + \delta_e \frac{\tilde{K}_0}{\tilde{K}} \right) \tilde{K}$$

$$\tilde{C} = \left(1 - \frac{I}{YZ}\right) \tilde{Y}$$

$$q = \frac{\alpha p^w (1+G)}{\delta^{'}(1)} \frac{\tilde{Y}}{\tilde{K}^A}$$

$$\Sigma(\varepsilon^*) = \frac{\left[g_A \frac{\tilde{K}}{\tilde{Y}} - (1-\delta)\frac{\tilde{K}^A}{\tilde{Y}}\right] G}{\left[g_A \frac{\tilde{K}}{\tilde{Y}} - (1-\delta)\frac{\tilde{K}^A}{\tilde{Y}}\right] Zq - \frac{I}{Y}}$$

$$p^I = \frac{1}{Z}$$

$$\Phi(\varepsilon^*) = 1 - \frac{q\Sigma(\varepsilon^*)}{p^I} + G$$

$$R_t^l = \frac{g_A}{(1-\delta_e)\beta(1+G)}$$

$$\tilde{\Lambda} = \frac{\xi(1-\theta\beta)}{\left(\tilde{C} - \theta\tilde{C}\right)^{\sigma}}$$

$$\Psi = \frac{\alpha\tilde{Y}p^w}{\tilde{K}^A}$$

$$\tilde{w} = (1-\alpha)\frac{\tilde{Y}p^w}{N}$$

$$\tilde{b} = \frac{\frac{I}{Y}\tilde{Y}p^I}{1-\Phi(\varepsilon^*)} - \alpha\tilde{Y}p^w - \zeta q\tilde{K}^A$$

$$m = \frac{\delta_e\omega}{(1-\kappa+\kappa\delta_e)}$$

$$\varepsilon^* = \frac{p^I}{q}$$

$$\tilde{p}^s = q\tilde{K}g_A + \tilde{b}$$

C.4 Log-linearisation

C.4.1 Log-linearisation Derivations

Preliminaries $\Sigma(\varepsilon_t^*)$:

Remind that:

$$\Sigma(\varepsilon_t^*) = \int_{\varepsilon \geq \varepsilon_t^*} \varepsilon \; d\Phi(\varepsilon)$$

Moreover, the derivative of the cdf yields the pdf:

$$
\begin{aligned}
\frac{d\Phi(\varepsilon)}{d\varepsilon} &= \phi(\varepsilon) \\
d\Phi(\varepsilon) &= \phi(\varepsilon)\,d\varepsilon \\
\Phi(x) &= \int_{-\infty}^{x} d\Phi(\varepsilon) = \int \phi(\varepsilon)\,d\varepsilon \\
\Sigma'(\varepsilon^*) &= \frac{d\Sigma(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon^*} \\
\Sigma(\varepsilon_t^*) &= \int\limits_{x>\varepsilon_t^*} x\phi(x)\,dx = \int\limits_{\varepsilon_t^*}^{\infty} x\phi(x)\,dx
\end{aligned}
$$

Consider $\Sigma(z)$:

$$
\begin{aligned}
\Sigma(z) &= \int\limits_{z}^{\infty} x\phi(x)\,dx \\
\frac{d\Sigma(z)}{dz} &= \frac{d\int\limits_{z}^{\infty} x\phi(x)\,dx}{dz} \\
\frac{dF(z)}{dz} &= \frac{d}{dz}\int\limits_{-\infty}^{z} g(x)\,dx = g(z) \\
\frac{d\Sigma(z)}{dz} &= \frac{-d\int\limits_{\infty}^{z} x\phi(x)\,dx}{dz} \\
\int\limits_{A}^{B} g(x)\,dx &= g(x)\,x|_A^B - \int\limits_{A}^{B} x dg(x)
\end{aligned}
$$

If we integrate over $f(x)$, we have:

$$
\int\limits_{A}^{B} g(x)\,df(x) = g(x)\,f(x)\,|_A^B - \int\limits_{A}^{B} f(x)\,dg(x)
$$

We can use Taylor expansion to expand $\Sigma(\varepsilon_t^*)$:

$$
\begin{aligned}
\Sigma(\varepsilon_t^*) &= \Sigma(\varepsilon^*) + \Sigma'(\varepsilon^*)(\varepsilon_t^* - \varepsilon^*) \\
&= \Sigma(\varepsilon^*) + \Sigma'(\varepsilon^*)(\varepsilon^*(1+\hat{\varepsilon}_t^*) - \varepsilon^*) \\
&= \Sigma(\varepsilon^*) + \Sigma'(\varepsilon^*)\,\varepsilon^*\hat{\varepsilon}_t^*
\end{aligned}
$$

We know that $\Phi(\varepsilon)$ is cdf over $[\varepsilon_{\min}, \varepsilon_{\max}] \subset (0, \infty)$ with mean 1. Mean 1 implies:

$$\int\limits_{-\infty}^{\infty} x d\Phi(x) = \int\limits_{-\infty}^{\infty} x\phi(x)\, dx = 1$$

We know:

$$\begin{aligned}
\Sigma(z) &= \int\limits_{z}^{\infty} x\phi(x)\, dx = \overbrace{\int\limits_{-\infty}^{\infty} x\phi(x)\, dx}^{\text{interpret as the mean}} - \int\limits_{-\infty}^{z} x\phi(x)\, dx = 1 - \int\limits_{-\infty}^{z} x\phi(x)\, dx \\
\Sigma'(z) &= -z\phi(z)
\end{aligned}$$

Therefore:

$$\Sigma(\varepsilon_t^*) = \int\limits_{\varepsilon > \varepsilon_t^*} \varepsilon d\Phi(\varepsilon) = \int\limits_{\varepsilon > \varepsilon_t^*} \varepsilon\phi(\varepsilon)\, d\varepsilon = \Sigma(\varepsilon^*) - \varepsilon^*\phi(\varepsilon^*)\,\varepsilon^*\hat{\varepsilon}_t^*$$

It remains to find the steady state $\Sigma(\varepsilon^*)$:

$$\Sigma(z) = 1 - \int\limits_{-\infty}^{z} x\phi(x)\, dx$$

such that the expression does not include integrals:

$$\int\limits_{-\infty}^{z} x\phi(x)\, dx = \int\limits_{-\infty}^{z} x d\Phi(x) = z\Phi(z) - \int\limits_{-\infty}^{z} \Phi(x)\, dx$$

Let us start with $\Sigma\left(\varepsilon_t^*\right)$:

$$\begin{aligned}
\Sigma\left(\varepsilon_t^*\right) &= \int\limits_{\varepsilon>\varepsilon_t^*} \varepsilon d\Phi\left(\varepsilon\right) \\
&= \int\limits_{\varepsilon>\varepsilon_t^*} \varepsilon\phi\left(\varepsilon\right) d\varepsilon \\
&= \Sigma\left(\varepsilon^*\right) - \varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^* \\
&= \int\limits_{\varepsilon>\varepsilon^*} \varepsilon d\Phi\left(\varepsilon\right) - \varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^* \\
&= \int\limits_{\varepsilon^*}^{\infty} \varepsilon d\Phi\left(\varepsilon\right) - \varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^* \\
&= \int\limits_{-\infty}^{\infty} \varepsilon d\Phi\left(\varepsilon\right) - \int\limits_{-\infty}^{\varepsilon^*} \varepsilon d\Phi\left(\varepsilon\right) - \varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^* \\
&= \int\limits_{-\infty}^{\infty} \varepsilon d\Phi\left(\varepsilon\right) - \left[\int\limits_{-\infty}^{\varepsilon^*} \varepsilon d\Phi\left(\varepsilon\right) = \varepsilon\Phi\left(\varepsilon\right)\Big|_{-\infty}^{\varepsilon^*} - \int\limits_{-\infty}^{\varepsilon^*} \Phi\left(\varepsilon\right) d\varepsilon\right] - \varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^*
\end{aligned}$$

The next formula is specific for normal distribution:

$$\begin{aligned}
\int\limits_{-\infty}^{\varepsilon^*} \Phi\left(\varepsilon\right) d\varepsilon &= \varepsilon^*\Phi\left(\varepsilon^*\right) + \phi\left(\varepsilon^*\right) \\
\Sigma\left(\varepsilon_t^*\right) &= \int\limits_{-\infty}^{\infty} \varepsilon d\Phi\left(\varepsilon\right) + \phi\left(\varepsilon^*\right) - \varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^* = 1 + \phi\left(\varepsilon^*\right) - \varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^*
\end{aligned}$$

and:

$$\int\limits_{-\infty}^{\infty} \varepsilon d\Phi\left(\varepsilon\right) = \int\limits_{-\infty}^{\infty} \varepsilon\phi\left(\varepsilon\right) d\varepsilon = 1$$

Finally, we can get the cdf $\Phi\left(\varepsilon_t^*\right)$:

$$\Phi\left(\varepsilon_t^*\right) = \Phi\left(\varepsilon^*\right) + \phi\left(\varepsilon^*\right)\left(\varepsilon_t^* - \varepsilon^*\right) = \Phi\left(\varepsilon^*\right) + \phi\left(\varepsilon^*\right)\left(\varepsilon^*\left(1+\hat{\varepsilon}_t^*\right) - \varepsilon^*\right) = \Phi\left(\varepsilon^*\right) + \phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^*$$

and we can thus log-linearise $\Sigma\left(\varepsilon_t^*\right)$ around its steady-state:

$$\begin{aligned}
\Sigma\left(\varepsilon_t^*\right) &= \left[G_t + 1 - \Phi\left(\varepsilon_t^*\right)\right]\varepsilon_t^* \\
\Sigma\left(\varepsilon^*\right)\left(1 + \hat{\Sigma}\left(\varepsilon_t^*\right)\right) &= \left(G\left(1+\hat{G}_t\right) + 1 - \Phi\left(\varepsilon^*\right) - \phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^*\right)\varepsilon^*\left(1+\hat{\varepsilon}_t^*\right)
\end{aligned}$$

where we need to find $\hat{G}_t$:

$$\begin{aligned}
G_t + 1 &= \frac{\Sigma\left(\varepsilon_t^*\right)}{\varepsilon_t^*} + \Phi\left(\varepsilon_t^*\right) \\
G_t &= \frac{\Sigma\left(\varepsilon_t^*\right)}{\varepsilon_t^*} + \Phi\left(\varepsilon_t^*\right) - 1
\end{aligned}$$

$$\begin{aligned}
G_t &= \frac{\Sigma\left(\varepsilon_t^*\right)}{\varepsilon_t^*} + \Phi\left(\varepsilon_t^*\right) - 1 \\
G\left(1+\hat{G}_t\right) &= \frac{\Sigma\left(\varepsilon^*\right) - \varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^*}{\varepsilon^*\left(1+\hat{\varepsilon}_t^*\right)} + \Phi\left(\varepsilon^*\right) + \phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^* - 1 \\
G\left(1+\hat{G}_t\right) &= \frac{\Sigma\left(\varepsilon^*\right)\left(1 - \frac{\varepsilon^*\phi(\varepsilon^*)\varepsilon^*}{\Sigma(\varepsilon^*)}\hat{\varepsilon}_t^*\right)}{\varepsilon^*\left(1+\hat{\varepsilon}_t^*\right)} + \Phi\left(\varepsilon^*\right) + \phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^* - 1 \\
G\left(1+\hat{G}_t\right) &= \frac{\Sigma\left(\varepsilon^*\right)}{\varepsilon^*}\left(1 - \frac{\varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*}{\Sigma\left(\varepsilon^*\right)}\hat{\varepsilon}_t^* - \hat{\varepsilon}_t^*\right) + \Phi\left(\varepsilon^*\right) + \phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^* - 1 \\
G\hat{G}_t &= \frac{\Sigma\left(\varepsilon^*\right)}{\varepsilon^*}\left(-\frac{\varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*}{\Sigma\left(\varepsilon^*\right)}\hat{\varepsilon}_t^* - \hat{\varepsilon}_t^*\right) + \phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^* \\
G\hat{G}_t &= -\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^* - \frac{\Sigma\left(\varepsilon^*\right)}{\varepsilon^*}\hat{\varepsilon}_t^* + \phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^* \\
G\hat{G}_t &= -\frac{\Sigma\left(\varepsilon^*\right)}{\varepsilon^*}\hat{\varepsilon}_t^* \qquad \text{(C.49)} \\
\hat{G}_t &= -\frac{1}{\left(1 + \frac{\varepsilon^*\left(\Phi\left(\varepsilon^*\right)-1\right)}{\Sigma\left(\varepsilon^*\right)}\right)}\hat{\varepsilon}_t^*
\end{aligned}$$

Preliminaries $\delta\left(u\right)$:

Using the Taylor expansion:

$$\begin{aligned}
\delta\left(u\right) &= \delta\left(u_0\right) + \delta'\left(u_0\right)\left(u - u_0\right) = \delta\left(u_0\right) + \delta'\left(u_0\right)\left(u_0\left(1+\hat{u}\right) - u_0\right) \\
&= \delta\left(u_0\right) + \delta'\left(u_0\right)u_0\hat{u} = \delta\left(u_0\right)\left(1 + \frac{\delta'\left(u_0\right)u_0}{\delta\left(u_0\right)}\hat{u}\right) \\
&= \delta\left(1+\hat{\delta}_t\right)
\end{aligned}$$

we can find that the log-deviation of the depreciation $\hat{\delta}_t$ is given by:

$$\begin{aligned}
\delta\left(1+\hat{\delta}_t\right) &= \delta\left(1\right)\left(1 + \frac{\delta'\left(1\right)}{\delta\left(1\right)}\hat{u}_t\right) \\
\hat{\delta}_t &= \frac{\delta'\left(1\right)}{\delta\left(1\right)}\hat{u}_t
\end{aligned}$$

As for $\delta\left(u\right)$, we can use the Taylor expansion to find the log-deviation $\widehat{\delta'}$:

$$
\begin{aligned}
\delta(u) &= \delta(u_0) + \delta'(u_0)(u-u_0) + \frac{1}{2}\delta''(u_0)(u-u_0)^2 + ... \\
\delta'(u) &= \delta'(u_0) + \delta''(u_0)(u-u_0) + ... \\
\delta'(u) &= \delta'(u_0) + \delta''(u_0)(u_0(1+\hat{u}) - u_0) \\
\overline{\delta'(u)}\left(1+\widehat{\delta'(u)}\right) &= \delta'(u_0) + \delta''(u_0)u_0\hat{u} \\
\overline{\delta'(u)}\widehat{\delta'(u)} &= \delta''(u_0)u_0\hat{u} \\
\widehat{\delta'(u)} &= \frac{\delta''(u_0)}{\delta'(u_0)}u_0\hat{u} \\
\overline{\delta'(u)}\left(1+\widehat{\delta'(u)}\right) &= \delta'(u_0)\left(1+\frac{\delta''(u_0)}{\delta'(u_0)}u_0\hat{u}\right)
\end{aligned}
$$

Simplify with $u = 1$ in steady-state, we obtain:

$$
\delta'(1)\left(1+\widehat{\delta'}\right) = \delta'(1)\left(1+\frac{\delta''(1)}{\delta'(1)}\hat{u}\right)
$$

Derivations of the System:

When I log-linearising, I substitute out $\hat{G}_t$ and $\hat{w}_t$ and $\hat{\Psi}_t$.

1. Marginal utility for consumption with steady-state $\tilde{\Lambda} = \frac{(1-\theta\beta)\xi}{\tilde{C}^{\sigma}(1-\theta)^{\sigma}}$:

$$
\begin{aligned}
\tilde{\Lambda}_t &= \frac{\xi_t}{\left(\tilde{C}_t - \theta\tilde{C}_{t-1}\right)^{\sigma}} - \theta\beta E_t\left[\frac{\xi_{t+1}}{\left(\tilde{C}_{t+1} - \theta\tilde{C}_t\right)^{\sigma}}\right] \\
\tilde{\Lambda}\left(1+\hat{\Lambda}_t\right) &= \frac{\xi\left(1+\hat{\xi}_t\right)}{\left(\tilde{C}\left(1+\hat{C}_t\right) - \theta\tilde{C}\left(1+\hat{C}_{t-1}\right)\right)^{\sigma}} \\
&\qquad - \theta\beta E_t\left[\frac{\xi\left(1+\hat{\xi}_{t+1}\right)}{\left(\tilde{C}\left(1+\hat{C}_{t+1}\right) - \theta\tilde{C}\left(1+\hat{C}_t\right)\right)^{\sigma}}\right]
\end{aligned}
$$

which yields:

$$\begin{aligned}
\tilde{\Lambda}\left(1+\hat{\Lambda}_t\right) &= \frac{\xi}{\tilde{C}^{\sigma}}\left(\begin{array}{c} \frac{\left(1+\hat{\xi}_t\right)}{\left(1+\hat{C}_t-\theta-\theta\hat{C}_{t-1}\right)^{\sigma}} \\ -\theta\beta E_t\left[\frac{\left(1+\hat{\xi}_{t+1}\right)}{\left(1+\hat{C}_{t+1}-\theta-\theta\hat{C}_t\right)^{\sigma}}\right] \end{array}\right) \\
\tilde{\Lambda}\left(1+\hat{\Lambda}_t\right) &= \frac{\xi}{\tilde{C}^{\sigma}}\left(\begin{array}{c} \frac{\left(1+\hat{\xi}_t\right)}{(1-\theta)^{\sigma}\left(1+\frac{\sigma}{(1-\theta)}\hat{C}_t-\frac{\theta\sigma}{(1-\theta)}\hat{C}_{t-1}\right)} \\ -\theta\beta E_t\left[\frac{\left(1+\hat{\xi}_{t+1}\right)}{(1-\theta)^{\sigma}\left(1+\frac{\sigma}{(1-\theta)}\hat{C}_{t+1}-\frac{\sigma\theta}{(1-\theta)}\hat{C}_t\right)}\right] \end{array}\right) \\
\tilde{\Lambda}\left(1+\hat{\Lambda}_t\right) &= \frac{\xi}{(1-\theta)^{\sigma}\tilde{C}^{\sigma}}\left(\begin{array}{c} \left(1+\hat{\xi}_t-\frac{\sigma}{(1-\theta)}\hat{C}_t+\frac{\theta\sigma}{(1-\theta)}\hat{C}_{t-1}\right) \\ -\theta\beta\left(1+\hat{\xi}_{t+1}-\frac{\sigma}{(1-\theta)}\hat{C}_{t+1}+\frac{\sigma\theta}{(1-\theta)}\hat{C}_t\right) \end{array}\right)
\end{aligned}$$

Continue and obtain:

$$\begin{aligned}
\tilde{\Lambda}(1-\theta)^{\sigma}\tilde{C}^{\sigma}\hat{\Lambda}_t &= \xi\left(\begin{array}{c} \hat{\xi}_t-\frac{\sigma}{(1-\theta)}\hat{C}_t+\frac{\theta\sigma}{(1-\theta)}\hat{C}_{t-1} \\ -\theta\beta\left(\hat{\xi}_{t+1}-\frac{\sigma}{(1-\theta)}\hat{C}_{t+1}+\frac{\sigma\theta}{(1-\theta)}\hat{C}_t\right) \end{array}\right) \\
(1-\theta\beta)\hat{\Lambda}_t &= \hat{\xi}_t-\frac{\sigma}{(1-\theta)}\hat{C}_t+\frac{\theta\sigma}{(1-\theta)}\hat{C}_{t-1} \\
&\quad -\theta\beta\left(\hat{\xi}_{t+1}-\frac{\sigma}{(1-\theta)}\hat{C}_{t+1}+\frac{\sigma\theta}{(1-\theta)}\hat{C}_t\right) \\
(1-\theta\beta)\hat{\Lambda}_t &= \hat{\xi}_t-\frac{\sigma\left(1+\beta\theta^2\right)}{(1-\theta)}\hat{C}_t+\frac{\theta\sigma}{(1-\theta)}\hat{C}_{t-1} \\
&\quad -\theta\beta\hat{\xi}_{t+1}+\frac{\theta\beta\sigma}{(1-\theta)}\hat{C}_{t+1} \\
\hat{\Lambda}_t &= \frac{1}{(1-\theta\beta)}\left(\begin{array}{c} \hat{\xi}_t-\theta\beta\hat{\xi}_{t+1}-\frac{\sigma\left(1+\beta\theta^2\right)}{(1-\theta)}\hat{C}_t \\ +\frac{\theta\sigma}{(1-\theta)}\hat{C}_{t-1}+\frac{\theta\beta\sigma}{(1-\theta)}\hat{C}_{t+1} \end{array}\right)
\end{aligned}$$

2. Aggregate labour supply with steady-state $\psi N^{\eta+1} = (1-\alpha)\,\tilde{\Lambda}\frac{\tilde{Y}p^w}{\xi}\,(1-\tau)$:

$$
\begin{aligned}
\psi_t N_t^{\eta+1} &= (1-\alpha)\,\tilde{\Lambda}_t \frac{\tilde{Y}_t p_t^w}{\xi_t}\,(1-\tau_t) \\
\psi\left(1+\hat{\psi}_t\right) N^{\eta+1}\left(1+(1+\eta)\,\hat{N}_t\right) &= (1-\alpha)\,\tilde{\Lambda}\left(1+\hat{\Lambda}_t\right)\frac{\tilde{Y}p^w\left(1+\hat{Y}_t+\hat{p}_t^w\right)}{\xi\left(1+\hat{\xi}_t\right)}\,(1-\tau\,(1+\hat{\tau}_t)) \\
\psi N^{\eta+1}\left(1+\hat{\psi}_t+(1+\eta)\,\hat{N}_t\right) &= (1-\alpha)\,\frac{\tilde{\Lambda}\tilde{Y}p^w}{\xi}\,(1-\tau)\begin{pmatrix} 1+\hat{\Lambda}_t+\hat{Y}_t+\hat{p}_t^w \\ -\hat{\xi}_t-\frac{\tau}{(1-\tau)}\hat{\tau}_t \end{pmatrix} \\
\hat{\psi}_t+(1+\eta)\,\hat{N}_t &= \hat{\Lambda}_t+\hat{Y}_t+\hat{p}_t^w-\hat{\xi}_t-\frac{\tau}{(1-\tau)}\hat{\tau}_t
\end{aligned}
$$

Alternatively using the wage:

$$
\begin{aligned}
\frac{N_t^\eta}{\tilde{\Lambda}_t} &= \frac{\tilde{w}_t}{\xi_t\psi_t}\,(1-\tau_t) \\
\frac{N^\eta\left(1+\eta\hat{N}_t\right)}{\Lambda\left(1+\hat{\Lambda}_t\right)} &= \frac{w\,(1+\hat{w}_t)}{\xi\left(1+\hat{\xi}_t\right)\psi\left(1+\hat{\psi}_t\right)}\,(1-\tau\,(1+\hat{\tau}_t)) \\
\frac{N^\eta\left(1+\eta\hat{N}_t\right)}{\Lambda\left(1+\hat{\Lambda}_t\right)} &= \frac{w\,(1+\hat{w}_t)\,(1-\tau)}{\xi\left(1+\hat{\xi}_t\right)\psi\left(1+\hat{\psi}_t\right)}\left(1-\frac{\tau}{(1-\tau)}\hat{\tau}_t\right) \\
\frac{\left(1+\eta\hat{N}_t\right)}{\left(1+\hat{\Lambda}_t\right)} &= \frac{(1+\hat{w}_t)}{\left(1+\hat{\xi}_t\right)\left(1+\hat{\psi}_t\right)}\left(1-\frac{\tau}{(1-\tau)}\hat{\tau}_t\right) \\
\eta\hat{N}_t-\hat{\Lambda}_t &= \hat{w}_t-\hat{\xi}_t-\hat{\psi}_t-\frac{\tau}{(1-\tau)}\hat{\tau}_t
\end{aligned}
$$

3. Euler equation for the capital producers:

$$\begin{aligned}
Z_t p_t^I &= 1+\frac{\Omega}{2}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}}g_{At}-\lambda^I\right]^2+\Omega\frac{\tilde{I}_t}{\tilde{I}_{t-1}}g_{At}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}}g_{At}-\lambda^I\right] \\
&\quad -\beta\mathbb{E}_t\frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t}g_{At+1}\left(\frac{\tilde{I}_{t+1}}{\tilde{I}_t}\right)^2\Omega\left[\frac{\tilde{I}_{t+1}}{\tilde{I}_t}g_{At+1}-\lambda^I\right]\frac{Z_t}{Z_{t+1}} \\
Zp^I\left(1+\hat{Z}_t+\hat{p}_t^I\right) &= 1+\frac{\Omega}{2}\left[\frac{1+\hat{I}_t}{1+\hat{I}_{t-1}}g_A\left(1+\hat{g}_{At}\right)-g_A\right]^2 \\
&\quad +\Omega\frac{1+\hat{I}_t}{1+\hat{I}_{t-1}}g_A\left(1+\hat{g}_{At}\right)\left[\frac{1+\hat{I}_t}{1+\hat{I}_{t-1}}g_A\left(1+\hat{g}_{At}\right)-g_A\right] \\
&\quad -\beta\mathbb{E}_t\frac{1+\hat{\Lambda}_{t+1}}{1+\hat{\Lambda}_t}g_A\left(1+\hat{g}_{At+1}\right)\left(\frac{1+\hat{I}_{t+1}}{1+\hat{I}_t}\right)^2 \\
&\quad \times\Omega\left[\frac{1+\hat{I}_{t+1}}{1+\hat{I}_t}g_A\left(1+\hat{g}_{At+1}\right)-g_A\right]\frac{1+\hat{Z}_t}{1+\hat{Z}_{t+1}} \\
Zp^I\left(1+\hat{Z}_t+\hat{p}_t^I\right) &= 1+\frac{\Omega}{2}g_A^2\left[\hat{I}_t-\hat{I}_{t-1}+\hat{g}_{At}\right]^2+\Omega\left(1+\hat{I}_t-\hat{I}_{t-1}+\hat{g}_{At}\right)g_A^2\left[\hat{I}_t-\hat{I}_{t-1}+\hat{g}_{At}\right] \\
&\quad -\beta\left(1+\hat{\Lambda}_{t+1}-\hat{\Lambda}_t+\hat{g}_{At+1}\right)\left(1+2\hat{I}_{t+1}-2\hat{I}_t\right) \\
&\quad \times\Omega g_A^2\left[\hat{I}_{t+1}-\hat{I}_t+\hat{g}_{At+1}\right]\left(1+\hat{Z}_t-\hat{Z}_{t+1}\right)
\end{aligned}$$

which simplifies to:

$$\begin{aligned}
\hat{Z}_t+\hat{p}_t^I &= \Omega g_A^2\left[\hat{I}_t-\hat{I}_{t-1}+\hat{g}_{At}\right]-\beta\Omega g_A^2\left[\hat{I}_{t+1}-\hat{I}_t+\hat{g}_{At+1}\right] \\
\hat{Z}_t+\hat{p}_t^I &= \Omega g_A^2\hat{I}_t-\Omega g_A^2\hat{I}_{t-1}+\Omega g_A^2\hat{g}_{At}-\beta\Omega g_A^2\hat{I}_{t+1}+\beta\Omega g_A^2\hat{I}_t-\beta\Omega g_A^2\hat{g}_{At+1} \\
\hat{Z}_t+\hat{p}_t^I &= \left(1+\beta\right)\Omega g_A^2\hat{I}_t-\Omega g_A^2\hat{I}_{t-1}-\beta\Omega g_A^2\hat{I}_{t+1}+\Omega g_A^2\hat{g}_{At}-\beta\Omega g_A^2\hat{g}_{At+1} \\
\hat{Z}_t+\hat{p}_t^I &= \Omega g_A^2\left[\hat{g}_{At}-\hat{I}_{t-1}\right]+\Omega g_A^2\left(1+\beta\right)\hat{I}_t-\Omega g_A^2\beta\mathbb{E}_t\left(\hat{I}_{t+1}+\hat{g}_{At+1}\right)
\end{aligned}$$

using $\lambda^I=g_A$ and $Zp^I=1$.

5. Aggregate output:

$$\begin{aligned}
\tilde{Y}_t &= \left(u_t\tilde{K}_t^A\right)^\alpha\left(z_tN_t\right)^{1-\alpha} \\
\hat{Y}_t &= \alpha\left(\hat{u}_t+\hat{K}_t^A\right)+\left(1-\alpha\right)\left(\hat{z}_t+\hat{N}_t\right)
\end{aligned}$$

6. Aggregate effective capital stock:

$$
\begin{aligned}
\tilde{K}_t^A &= (1-\delta_e)\,\tilde{K}_t + \delta_e \tilde{K}_0 \\
\tilde{K}^A\left(1+\hat{K}_t^A\right) &= (1-\delta_e)\,\tilde{K}\left(1+\hat{K}_t\right) + \delta_e \tilde{K}_0 \\
\tilde{K}^A \hat{K}_t^A &= (1-\delta_e)\,\tilde{K}\hat{K}_t \\
\hat{K}_t^A &= (1-\delta_e)\,\varpi_K \hat{K}_t
\end{aligned}
$$

using $\frac{\tilde{K}}{\tilde{K}^A} = \varpi_K$.

7. Law of motion for capital:

$$
\begin{aligned}
g_{At+1}\tilde{K}_{t+1} &= (1-\delta_t)\,\tilde{K}_t^A + \tilde{I}_t \frac{\Sigma\left(\varepsilon_t^*\right)}{1-\Phi\left(\varepsilon_t^*\right)} \\
g_A\left(1+\hat{g}_{At+1}\right)\tilde{K}\left(1+\hat{K}_{t+1}\right) &= \left(1-\delta\left(1+\hat{\delta}_t\right)\right)\tilde{K}^A\left(1+\hat{K}_t^A\right) \\
&\quad + \tilde{I}\left(1+\hat{I}_t\right)\frac{\Sigma\left(\varepsilon^*\right) - \varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^*}{1-\Phi\left(\varepsilon^*\right) - \phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^*} \\
g_A\tilde{K}\left(1+\hat{g}_{At+1}+\hat{K}_{t+1}\right) &= \left(1-\delta(1)\left(1+\frac{\delta'(1)}{\delta(1)}\hat{u}_t\right)\right)\tilde{K}^A\left(1+\hat{K}_t^A\right) \\
&\quad + \tilde{I}\left(1+\hat{I}_t\right)\frac{\Sigma\left(\varepsilon^*\right)}{\left(1-\Phi\left(\varepsilon^*\right)\right)}\frac{\left(1-\frac{\varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*}{\Sigma\left(\varepsilon^*\right)}\hat{\varepsilon}_t^*\right)}{\left(1-\frac{\phi\left(\varepsilon^*\right)\varepsilon^*}{1-\Phi\left(\varepsilon^*\right)}\hat{\varepsilon}_t^*\right)}
\end{aligned}
$$

Continue:

$$
\begin{aligned}
g_A\tilde{K}\left(1+\hat{g}_{At+1}+\hat{K}_{t+1}\right) &= \tilde{K}^A\left(1-\delta(1)\right)\left(1-\frac{\delta'(1)}{1-\delta(1)}\hat{u}_t\right)\left(1+\hat{K}_t^A\right) \\
&\quad + \tilde{I}\left(1+\hat{I}_t\right)\frac{\Sigma\left(\varepsilon^*\right)}{\left(1-\Phi\left(\varepsilon^*\right)\right)}\left(1-\frac{\varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*}{\Sigma\left(\varepsilon^*\right)}\hat{\varepsilon}_t^* + \frac{\phi\left(\varepsilon^*\right)\varepsilon^*}{1-\Phi\left(\varepsilon^*\right)}\hat{\varepsilon}_t^*\right) \\
g_A\left(1+\hat{g}_{At+1}+\hat{K}_{t+1}\right) &= \frac{\tilde{K}^A}{\tilde{K}}\left(1-\delta(1)\right)\left(1+\hat{K}_t^A - \frac{\delta'(1)}{1-\delta(1)}\hat{u}_t\right) \\
&\quad + \frac{\tilde{I}}{\tilde{K}}\left(1+\hat{I}_t\right)\frac{\Sigma\left(\varepsilon^*\right)}{\left(1-\Phi\left(\varepsilon^*\right)\right)}\left(1-\frac{\varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*}{\Sigma\left(\varepsilon^*\right)}\hat{\varepsilon}_t^* + \frac{\phi\left(\varepsilon^*\right)\varepsilon^*}{1-\Phi\left(\varepsilon^*\right)}\hat{\varepsilon}_t^*\right) \\
g_A\left(1+\hat{g}_{At+1}+\hat{K}_{t+1}\right) &= \frac{\tilde{K}^A}{\tilde{K}}\left(1-\delta(1)\right)\left(1+\hat{K}_t^A - \frac{\delta'(1)}{1-\delta(1)}\hat{u}_t\right) \\
&\quad + \frac{\tilde{I}}{\tilde{K}}\left(1+\hat{I}_t\right)\left(\frac{\Sigma\left(\varepsilon^*\right)}{\left(1-\Phi\left(\varepsilon^*\right)\right)} + \mu\left[\frac{\Sigma\left(\varepsilon^*\right)}{\left(1-\Phi\left(\varepsilon^*\right)\right)} - \varepsilon^*\right]\hat{\varepsilon}_t^*\right)
\end{aligned}
$$

which simplifies to:

$$
\begin{aligned}
g_A\left(1+\hat{g}_{At+1}+\hat{K}_{t+1}\right) &= \frac{\tilde{K}^A}{\tilde{K}}\left(1-\delta\left(1\right)\right)\left(1+\hat{K}_t^A-\frac{\delta'\left(1\right)}{1-\delta\left(1\right)}\hat{u}_t\right) \\
&\quad +\frac{\tilde{I}}{\tilde{K}}\left(1+\hat{I}_t\right)\left(\frac{\Sigma\left(\varepsilon^*\right)}{\left(1-\Phi\left(\varepsilon^*\right)\right)}-\frac{\Sigma\left(\varepsilon^*\right)}{\Sigma\left(\varepsilon^*\right)}\varepsilon^*\mu\left[\frac{\left(1-\Phi\left(\varepsilon^*\right)\right)-\frac{\Sigma(\varepsilon^*)}{\varepsilon^*}}{\left(1-\Phi\left(\varepsilon^*\right)\right)}\right]\hat{\varepsilon}_t^*\right) \\
g_A\left(1+\hat{g}_{At+1}+\hat{K}_{t+1}\right) &= \frac{\tilde{K}^A}{\tilde{K}}\left(1-\delta\left(1\right)\right)\left(1+\hat{K}_t^A-\frac{\delta'\left(1\right)}{1-\delta\left(1\right)}\hat{u}_t\right) \\
&\quad +\frac{\tilde{I}}{\tilde{K}}\left(1+\hat{I}_t\right)\frac{\Sigma\left(\varepsilon^*\right)}{\left(1-\Phi\left(\varepsilon^*\right)\right)}\left(1-\mu\frac{\varepsilon^*}{\Sigma\left(\varepsilon^*\right)}\left[\begin{array}{c}\left(1-\Phi\left(\varepsilon^*\right)\right) \\ -\frac{\Sigma(\varepsilon^*)}{\varepsilon^*}\end{array}\right]\hat{\varepsilon}_t^*\right) \\
g_A\left(\hat{g}_{At+1}+\hat{K}_{t+1}\right) &= \frac{\tilde{K}^A}{\tilde{K}}\left(1-\delta\left(1\right)\right)\left(\hat{K}_t^A-\frac{\delta'\left(1\right)}{1-\delta\left(1\right)}\hat{u}_t\right)+\frac{\tilde{I}}{\tilde{K}}\frac{\Sigma\left(\varepsilon^*\right)}{\left(1-\Phi\left(\varepsilon^*\right)\right)}\left(\hat{I}_t-\frac{\mu}{\varphi_G}\hat{\varepsilon}_t^*\right) \\
g_A\left(\hat{g}_{At+1}+\hat{K}_{t+1}\right) &= \frac{\left(1-\delta\left(1\right)\right)}{\varpi_K}\hat{K}_t^A-\frac{\delta'\left(1\right)}{\varpi_K}\hat{u}_t+\frac{\left[g_A\tilde{K}-\left(1-\delta\right)\tilde{K}^A\right]}{\tilde{K}}\left(\hat{I}_t-\frac{\mu}{\varphi_G}\hat{\varepsilon}_t^*\right) \\
g_A\left(\hat{g}_{At+1}+\hat{K}_{t+1}\right) &= \frac{\left(1-\delta\left(1\right)\right)}{\varpi_K}\hat{K}_t^A-\frac{\delta'\left(1\right)}{\varpi_K}\hat{u}_t+\left(g_A-\frac{\left(1-\delta\right)}{\varpi_K}\right)\left(\hat{I}_t-\frac{\mu}{\varphi_G}\hat{\varepsilon}_t^*\right)
\end{aligned}
$$

using:

$$
\begin{aligned}
\Sigma\left(\varepsilon^*\right) &= -G\varepsilon^*\varphi_G \\
\mu &= \frac{\phi\left(\varepsilon^*\right)\varepsilon^*}{\left(1-\Phi\left(\varepsilon^*\right)\right)} \\
\varpi_K &= \frac{1}{\left[\left(1-\delta_e\right)+\delta_e\frac{\tilde{K}_0}{\tilde{K}}\right]} \\
\frac{\tilde{K}^A}{\tilde{K}} &= \frac{1}{\varpi_K} \\
\tilde{I} &= \frac{1-\Phi\left(\varepsilon^*\right)}{\Sigma\left(\varepsilon^*\right)}\left[g_A\tilde{K}-\left(1-\delta\right)\tilde{K}^A\right]
\end{aligned}
$$

8. Real wage setting:

$$
\begin{aligned}
\tilde{w}_t &= \left(1-\alpha\right)p_t^w\frac{\tilde{Y}_t}{N_t} \\
\tilde{w}\left(1+\hat{w}_t\right) &= \left(1-\alpha\right)p^w\left(1+\hat{p}_t^w\right)\frac{\tilde{Y}\left(1+\hat{Y}_t\right)}{N\left(1+\hat{N}_t\right)} \\
\hat{w}_t &= \hat{p}_t^w+\hat{Y}_t-\hat{N}_t
\end{aligned}
$$

9. Aggregate investment demand:

$$
\begin{aligned}
\tilde{I}_t &= \left[\alpha \tilde{Y}_t p_t^w + \zeta_t q_t \tilde{K}_t^A + \tilde{b}_t\right] \frac{1-\Phi\left(\varepsilon_t^*\right)}{p_t^I} \\
\tilde{I}\left(1+\hat{I}_t\right) p^I\left(1+\hat{p}_t^I\right) &= \left[\begin{array}{c}
\alpha \tilde{Y}\left(1+\hat{Y}_t\right) p^w\left(1+\hat{p}_t^w\right) \\
+\zeta\left(1+\hat{\zeta}_t\right) q\left(1+\hat{q}_t\right) \tilde{K}^A\left(1+\hat{K}_t^A\right) \\
+\tilde{b}\left(1+\hat{b}_t\right)
\end{array}\right]\left(1-\left[\Phi\left(\varepsilon^*\right)+\phi\left(\varepsilon^*\right) \varepsilon^* \hat{\varepsilon}_t^*\right]\right) \\
p^I \tilde{I}\left(1+\hat{I}_t+\hat{p}_t^I\right) &= \left[\begin{array}{c}
\alpha \tilde{Y} p^w\left(1+\hat{Y}_t+\hat{p}_t^w\right) \\
+\zeta q \tilde{K}^A\left(1+\hat{\zeta}_t+\hat{q}_t+\hat{K}_t^A\right) \\
+\tilde{b}\left(1+\hat{b}_t\right)
\end{array}\right]\left(1-\Phi\left(\varepsilon^*\right)-\frac{\left(1-\Phi\left(\varepsilon^*\right)\right) \phi\left(\varepsilon^*\right) \varepsilon^*}{1-\Phi\left(\varepsilon^*\right)} \hat{\varepsilon}_t^*\right)
\end{aligned}
$$

Continue:

$$
\begin{aligned}
p^I \tilde{I}\left(1+\hat{I}_t+\hat{p}_t^I\right) &= \left[\begin{array}{c}
\alpha \tilde{Y} p^w\left(1+\hat{Y}_t+\hat{p}_t^w\right) \\
+\zeta q \tilde{K}^A\left(1+\hat{\zeta}_t+\hat{q}_t+\hat{K}_t^A\right)+\tilde{b}\left(1+\hat{b}_t\right)
\end{array}\right]\left(1-\mu \hat{\varepsilon}_t^*\right)\left(1-\Phi\left(\varepsilon^*\right)\right) \\
p^I \tilde{I}\left(\hat{I}_t+\hat{p}_t^I\right) &= \left[\begin{array}{c}
\alpha \tilde{Y} p^w\left(\hat{Y}_t+\hat{p}_t^w\right)-\alpha \tilde{Y} p^w \mu \hat{\varepsilon}_t^* \\
+\zeta q \tilde{K}^A\left(\hat{\zeta}_t+\hat{q}_t+\hat{K}_t^A\right)-\zeta q \tilde{K}^A \mu \hat{\varepsilon}_t^*+\tilde{b} \hat{b}_t-\tilde{b} \mu \hat{\varepsilon}_t^*
\end{array}\right]\left(1-\Phi\left(\varepsilon^*\right)\right)
\end{aligned}
$$

which simplifies to:

$$
\begin{aligned}
\hat{I}_t+\hat{p}_t^I = & \frac{\alpha \tilde{Y} p^w}{\left(\alpha \tilde{Y} p^w+\zeta q \tilde{K}^A+\tilde{b}\right)}\left(\hat{Y}_t+\hat{p}_t^w\right)+\frac{\zeta q \tilde{K}^A}{\left(\alpha \tilde{Y} p^w+\zeta q \tilde{K}^A+\tilde{b}\right)}\left(\hat{\zeta}_t+\hat{q}_t+\hat{K}_t^A\right) \\
& +\frac{\tilde{b}}{\left(\alpha \tilde{Y} p^w+\zeta q \tilde{K}^A+\tilde{b}\right)} \hat{b}_t-\mu \hat{\varepsilon}_t^* \\
\hat{I}_t+\hat{p}_t^I = & \frac{\alpha}{\left(\alpha+\zeta \varpi_\delta+\varpi_b\right)}\left(\hat{Y}_t+\hat{p}_t^w\right)+\frac{\zeta \varpi_\delta}{\left(\alpha+\zeta \varpi_\delta+\varpi_b\right)}\left(\hat{\zeta}_t+\hat{q}_t+\hat{K}_t^A\right) \\
& +\frac{\varpi_b}{\left(\alpha+\zeta \varpi_\delta+\varpi_b\right)} \hat{b}_t-\mu \hat{\varepsilon}_t^*
\end{aligned}
$$

where $\mu=\frac{\phi(\varepsilon^*)\varepsilon^*}{1-\Phi(\varepsilon^*)}$, $\varpi_b:=\frac{\tilde{b}}{p^w\tilde{Y}}$ and $\varpi_\delta:=\frac{q\tilde{K}^A}{\tilde{Y}p^w}$.

10. Optimal capacity of utilisation:

$$
\begin{aligned}
\frac{\alpha\tilde{Y}_t p_t^w}{u_t\tilde{K}_t^A}\left(\frac{\Sigma\left(\varepsilon_t^*\right)}{\varepsilon_t^*}+\Phi\left(\varepsilon_t^*\right)\right) &= q_t\delta^{'}\left(u_t\right)\\
\frac{\alpha\tilde{Y}\left(1+\hat{Y}_t\right)p^w\left(1+\hat{p}_t^w\right)}{\left(1+\hat{u}_t\right)\tilde{K}^A\left(1+\hat{K}_t^A\right)}\left(G\left(1+\hat{G}_t\right)+1\right) &= q\left(1+\hat{q}_t\right)\delta'\left(1\right)\left(1+\frac{\delta''\left(1\right)}{\delta'\left(1\right)}\hat{u}_t\right)\\
\frac{\alpha\tilde{Y}p^w}{\tilde{K}^A}\begin{pmatrix}1+\hat{Y}_t+\hat{p}_t^w\\ \\ -\hat{u}_t-\hat{K}_t^A\end{pmatrix}\left(\left[1+G\right]+\frac{\left[1+G\right]G}{\left[1+G\right]}\hat{G}_t\right) &= q\delta'\left(1\right)\left(1+\hat{q}_t+\frac{\delta''\left(1\right)}{\delta'\left(1\right)}\hat{u}_t\right)\\
\hat{Y}_t+\hat{p}_t^w-\hat{u}_t-\hat{K}_t^A+\frac{G}{1+G}\hat{G}_t &= \hat{q}_t+\frac{\delta''\left(1\right)}{\delta'\left(1\right)}\hat{u}_t\\
\hat{Y}_t+\hat{p}_t^w-\hat{K}_t^A-\beta\left(1-\delta_e\right)\kappa\frac{\Sigma\left(\varepsilon^*\right)}{\varepsilon^*}\hat{\varepsilon}_t^* &= \hat{q}_t+\frac{\delta''\left(1\right)}{\delta'\left(1\right)}\hat{u}_t+\hat{u}_t\\
\hat{Y}_t+\hat{p}_t^w-\hat{K}_t^A+\beta\left(1-\delta_e\right)\kappa\left(\frac{1}{\beta\left(1-\delta_e\right)\kappa}-1\right)\varphi_G\hat{\varepsilon}_t^* &= \hat{q}_t+\left(1+\frac{\delta''\left(1\right)}{\delta'\left(1\right)}\right)\hat{u}_t\\
\hat{Y}_t+\hat{p}_t^w-\hat{K}_t^A+\left(1-\beta\left(1-\delta_e\right)\kappa\right)\varphi_G\hat{\varepsilon}_t^* &= \hat{q}_t+\left(1+\frac{\delta''\left(1\right)}{\delta'\left(1\right)}\right)\hat{u}_t
\end{aligned}
$$

using (C.50) to substitute out $\hat{G}_t$.

11. Rental rate of capital:

$$
\begin{aligned}
\Psi_t &= \frac{\alpha\tilde{Y}_t p_t^w}{u_t\tilde{K}_t^A}\\
\Psi\left(1+\hat{\Psi}_t\right) &= \frac{\alpha\tilde{Y}\left(1+\hat{Y}_t\right)p^w\left(1+\hat{p}_t^w\right)}{u\left(1+u_t\right)\tilde{K}^A\left(1+\hat{K}_t^A\right)}\\
\hat{\Psi}_t &= \hat{Y}_t+\hat{p}_t^w-u_t-\hat{K}_t^A
\end{aligned}
$$

12. Benefit of being productive:

$$
\begin{aligned}
G_t &= \frac{\Sigma\left(\varepsilon_t^*\right)}{\varepsilon_t^*}+\Phi\left(\varepsilon_t^*\right)-1\\
G\left(1+\hat{G}_t\right) &= \frac{\Sigma\left(\varepsilon^*\right)-\varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^*}{\varepsilon^*\left(1+\hat{\varepsilon}_t^*\right)}+\Phi\left(\varepsilon^*\right)+\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^*-1\\
G\left(1+\hat{G}_t\right) &= \frac{\Sigma\left(\varepsilon^*\right)\left(1-\frac{\varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*}{\Sigma\left(\varepsilon^*\right)}\hat{\varepsilon}_t^*\right)}{\varepsilon^*\left(1+\hat{\varepsilon}_t^*\right)}+\Phi\left(\varepsilon^*\right)+\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^*-1\\
G\left(1+\hat{G}_t\right) &= \frac{\Sigma\left(\varepsilon^*\right)}{\varepsilon^*}\left(1-\frac{\varepsilon^*\phi\left(\varepsilon^*\right)\varepsilon^*}{\Sigma\left(\varepsilon^*\right)}\hat{\varepsilon}_t^*-\hat{\varepsilon}_t^*\right)+\Phi\left(\varepsilon^*\right)+\phi\left(\varepsilon^*\right)\varepsilon^*\hat{\varepsilon}_t^*-1
\end{aligned}
$$

which simplifies to:

$$
\begin{aligned}
G\hat{G}_t &= \frac{\Sigma(\varepsilon^*)}{\varepsilon^*}\left(-\frac{\varepsilon^*\phi(\varepsilon^*)\varepsilon^*}{\Sigma(\varepsilon^*)}-1\right)\hat{\varepsilon}_t^* + \phi(\varepsilon^*)\varepsilon^*\hat{\varepsilon}_t^* \\
G\hat{G}_t &= \left(-\frac{\Sigma(\varepsilon^*)}{\varepsilon^*}\frac{\varepsilon^*\phi(\varepsilon^*)\varepsilon^*}{\Sigma(\varepsilon^*)} - \frac{\Sigma(\varepsilon^*)}{\varepsilon^*} + \phi(\varepsilon^*)\varepsilon^*\right)\hat{\varepsilon}_t^* \\
\hat{G}_t &= -\frac{\Sigma(\varepsilon^*)}{G\varepsilon^*}\hat{\varepsilon}_t^* \\
\hat{G}_t &= \varphi_G\hat{\varepsilon}_t^* \qquad \text{(C.50)}
\end{aligned}
$$

using $\frac{\Sigma(\varepsilon^*)}{G\varepsilon^*} = -\varphi_G$.

13. Lending rate:

$$
\begin{aligned}
\frac{(1+\pi_{t+1})}{R_t^l} &= (1-\delta_e)\beta E_t\frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t}\frac{(1+G_{t+1})}{g_{At+1}} \\
\frac{(1+\pi_{t+1})}{\left(1+\hat{R}_t^l\right)R^l} &= (1-\delta_e)\beta E_t\frac{\tilde{\Lambda}\left(1+\hat{\Lambda}_{t+1}\right)}{\tilde{\Lambda}\left(1+\hat{\Lambda}_t\right)}\frac{\left(1+G\left(1+\hat{G}_{t+1}\right)\right)}{g_A(1+\hat{g}_{At+1})} \\
\frac{(1+\pi_{t+1})}{\left(1+\hat{R}_t^l\right)R^l} &= (1-\delta_e)\beta E_t\frac{\tilde{\Lambda}\left(1+\hat{\Lambda}_{t+1}\right)}{\tilde{\Lambda}\left(1+\hat{\Lambda}_t\right)}\frac{\left(1+G+\frac{(1+G)G}{(1+G)}\hat{G}_{t+1}\right)}{g_A(1+\hat{g}_{At+1})} \\
\frac{\left(1+\pi_{t+1}-\hat{R}_t^l\right)}{R^l} &= (1-\delta_e)\beta\frac{(1+G)}{g_A}E_t\begin{pmatrix} 1+\hat{\Lambda}_{t+1}-\hat{\Lambda}_t-\hat{g}_{At+1} \\ +\frac{G}{(1+G)}\hat{G}_{t+1}\end{pmatrix}
\end{aligned}
$$

which simplifies to:

$$
\begin{aligned}
\pi_{t+1}-\hat{R}_t^l &= E_t\left[\hat{\Lambda}_{t+1}-\hat{\Lambda}_t-\hat{g}_{At+1}+\frac{G}{(1+G)}\hat{G}_{t+1}\right] \\
\pi_{t+1}-\hat{R}_t^l &= E_t\left[\hat{\Lambda}_{t+1}-\hat{\Lambda}_t-\hat{g}_{At+1}-\frac{\Sigma(\varepsilon^*)}{(1+G)\varepsilon^*}\hat{\varepsilon}_{t+1}^*\right] \qquad \text{(C.51)} \\
\pi_{t+1}-\hat{R}_t^l &= E_t\left[\hat{\Lambda}_{t+1}-\hat{\Lambda}_t-\hat{g}_{At+1}-\left[-\left(\frac{1}{\beta(1-\delta_e)\kappa}-1\right)\varphi_G\right]\beta(1-\delta_e)\kappa\hat{\varepsilon}_{t+1}^*\right] \\
\pi_{t+1}-\hat{R}_t^l &= E_t\left[\hat{\Lambda}_{t+1}-\hat{\Lambda}_t-\hat{g}_{At+1}+[1-\beta(1-\delta_e)\kappa]\varphi_G\hat{\varepsilon}_{t+1}^*\right]
\end{aligned}
$$

where in the line (C.51), we used (C.50).

14. Marginal Q:

$$q_t = (1-\delta_e)\,\beta E_t \frac{\tilde{\Lambda}_{t+1} q_{t+1}}{\tilde{\Lambda}_t g_{At+1}} \left[\begin{array}{c} u_{t+1}\delta^{'}(u_{t+1}) + (1-\delta(u_{t+1})) \\ + \left(\frac{\Sigma\left(\varepsilon^*_{t+1}\right)}{\varepsilon^*_{t+1}} + \Phi\left(\varepsilon^*_{t+1}\right) - 1 \right) \zeta_{t+1} \end{array} \right]$$

$$q\,(1+\hat{q}_t) = (1-\delta_e)\,\beta E_t \frac{\tilde{\Lambda}\left(1+\hat{\Lambda}_{t+1}\right) q\,(1+\hat{q}_{t+1})}{\tilde{\Lambda}\left(1+\hat{\Lambda}_t\right) g_A\,(1+\hat{g}_{At+1})} \left[\begin{array}{c} (1+\hat{u}_{t+1})\,\delta^{'}(1)\left(1+\hat{\delta}^{'}(u_{t+1})\right) \\ + \left(1-\delta(1)\left(1+\hat{\delta}(u_{t+1})\right)\right) \\ + G\left(1+\hat{G}_{t+1}\right)\zeta\left(1+\hat{\zeta}_{t+1}\right) \end{array} \right]$$

$$(1+\hat{q}_t) = \frac{(1-\delta_e)\,\beta}{g_A} E_t \left(\begin{array}{c} 1+\hat{\Lambda}_{t+1}-\hat{\Lambda}_t \\ +\hat{q}_{t+1}-\hat{g}_{At+1} \end{array} \right) \left[\begin{array}{c} \left(1+\hat{u}_{t+1}\left(1+\frac{\delta''(1)}{\delta'(1)}\right)\right)\delta^{'}(1) \\ + \left(1-\delta(1)\left(1+\frac{\delta'(1)}{\delta(1)}\hat{u}_{t+1}\right)\right) \\ + \left(1+\hat{G}_{t+1}+\hat{\zeta}_{t+1}\right) G\zeta \end{array} \right]$$

which simplifies to:

$$(1+\hat{q}_t) = \frac{(1-\delta_e)\,\beta}{g_A} E_t \left(\begin{array}{c} 1+\hat{\Lambda}_{t+1}-\hat{\Lambda}_t \\ +\hat{q}_{t+1}-\hat{g}_{At+1} \end{array} \right) \left[\begin{array}{c} \left(1+\hat{u}_{t+1}\left(1+\frac{\delta''(1)}{\delta'(1)}\right)\right)\delta^{'}(1) \\ + \left([1-\delta(1)] - \frac{[1-\delta(1)]\delta'(1)}{[1-\delta(1)]}\hat{u}_{t+1}\right) \\ + \left(1+\hat{G}_{t+1}+\hat{\zeta}_{t+1}\right) G\zeta \end{array} \right]$$

$$(1+\hat{q}_t) = \frac{(1-\delta_e)\,\beta}{g_A} E_t \left(\begin{array}{c} 1+\hat{\Lambda}_{t+1}-\hat{\Lambda}_t \\ +\hat{q}_{t+1}-\hat{g}_{At+1} \end{array} \right) \left[\begin{array}{c} \left(\delta^{'}(1) + (1-\delta(1)) + G\zeta\right) \\ +\hat{u}_{t+1}\left(1+\frac{\delta''(1)}{\delta'(1)}\right)\delta^{'}(1) - \delta'(1)\,\hat{u}_{t+1} \\ +G\zeta\left(\hat{G}_{t+1}+\hat{\zeta}_{t+1}\right) \end{array} \right]$$

$$(1+\hat{q}_t) = \frac{(1-\delta_e)\,\beta}{g_A} E_t \left[\begin{array}{c} \left(\begin{array}{c} 1+\hat{\Lambda}_{t+1}-\hat{\Lambda}_t \\ +\hat{q}_{t+1}-\hat{g}_{At+1} \end{array} \right) \left(\delta^{'}(1) + (1-\delta(1)) + G\zeta\right) \\ +\delta''(1)\,\hat{u}_{t+1} + G\zeta\left(\hat{G}_{t+1}+\hat{\zeta}_{t+1}\right) \end{array} \right]$$

$$\hat{q}_t = E_t \left[\begin{array}{c} \left(\hat{\Lambda}_{t+1}-\hat{\Lambda}_t+\hat{q}_{t+1}-\hat{g}_{At+1}\right) + \frac{\beta(1-\delta_e)\delta'(1)}{g_A}\frac{\delta''(1)}{\delta'(1)}\hat{u}_{t+1} \\ +\frac{(1-\delta_e)\beta}{g_A} G\zeta\left(\hat{\zeta}_{t+1}+\varphi_G\hat{\varepsilon}^*_{t+1}\right) \end{array} \right]$$

using $(1-\delta_e)\,\frac{\beta}{g_A}\left[\delta^{'}(1) + (1-\delta(1)) + G\zeta\right] = 1$.

15. Law of motion of the real value of the bubble:

$$
\begin{aligned}
\tilde{b}_t &= (1-\delta_e)\,\beta E_t \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t}\frac{m_t}{m_{t+1}}\left(1+G_{t+1}\right)\kappa_t \tilde{b}_{t+1} \\
\tilde{b}\left(1+\hat{b}_t\right) &= (1-\delta_e)\,\beta E_t \frac{\tilde{\Lambda}\left(1+\hat{\Lambda}_{t+1}\right)}{\tilde{\Lambda}\left(1+\hat{\Lambda}_t\right)}\frac{m\left(1+\hat{m}_t\right)}{m\left(1+\hat{m}_{t+1}\right)}\left(1+G\left(1+G_{t+1}\right)\right)\kappa\left(1+\hat{\kappa}_t\right)\tilde{b}\left(1+\hat{b}_{t+1}\right) \\
\left(1+\hat{b}_t\right) &= (1-\delta_e)\,\beta\kappa E_t \begin{pmatrix} 1+\hat{\Lambda}_{t+1}-\hat{\Lambda}_t+\hat{m}_t \\ +\hat{\kappa}_t-\hat{m}_{t+1}+\hat{b}_{t+1} \end{pmatrix}\left([1+G]+\frac{[1+G]\,G}{[1+G]}\hat{G}_{t+1}\right) \\
\hat{b}_t &= E_t\left[\hat{\Lambda}_{t+1}-\hat{\Lambda}_t+\frac{1}{(1-\delta_e)\,\kappa}\hat{m}_{t+1}-\hat{m}_{t+1}+\hat{b}_{t+1}-\frac{1}{1+G}\frac{\Sigma\left(\varepsilon^*\right)}{\varepsilon^*}\hat{\varepsilon}^*_{t+1}\right] \\
\hat{b}_t &= E_t\left[\hat{\Lambda}_{t+1}-\hat{\Lambda}_t+\frac{1-(1-\delta_e)\,\kappa}{(1-\delta_e)\,\kappa}\hat{m}_{t+1}+\hat{b}_{t+1}+\left[1-\beta\left(1-\delta_e\right)\kappa\right]\varphi_G\hat{\varepsilon}^*_{t+1}\right]
\end{aligned}
$$

using:

$$
\hat{m}_t+\hat{\kappa}_t=\frac{1}{(1-\delta_e)\,\kappa}\hat{m}_{t+1}
$$

from 16.

16. Evolution of the number of bubbly firms:

$$
\begin{aligned}
m_t &= m_{t-1}\left(1-\delta_e\right)\kappa_{t-1}+\delta_e\omega \\
m\left(1+\hat{m}_t\right) &= m\left(1+\hat{m}_{t-1}\right)\left(1-\delta_e\right)\kappa\left(1+\hat{\kappa}_{t-1}\right)+\delta_e\omega \\
m+m\hat{m}_t &= m\left(1-\delta_e\right)\kappa+m\left(\hat{m}_{t-1}+\hat{\kappa}_{t-1}\right)\left(1-\delta_e\right)\kappa+\delta_e\omega \\
\hat{m}_t &= \left(1-\delta_e\right)\kappa\left(\hat{m}_{t-1}+\hat{\kappa}_{t-1}\right)
\end{aligned}
$$

17. Resource constraint:

$$
\begin{aligned}
\tilde{C}_t+\left(1+\frac{\Omega}{2}\left[\frac{\tilde{I}_t}{\tilde{I}_{t-1}}g_{At}-\lambda^I\right]^2\right)\frac{\tilde{I}_t}{Z_t} &= \left(1-\frac{\Upsilon}{2}\pi_t^2\right)\tilde{Y}_t \\
\tilde{C}\left(1+\hat{C}_t\right)+\left(1+\frac{\Omega}{2}\left[\frac{1+\hat{I}_t}{1+\hat{I}_{t-1}}g_A\left(1+\hat{g}_{At}\right)-g_A\right]^2\right)\frac{\tilde{I}\left(1+\hat{I}_t\right)}{Z\left(1+\hat{Z}_t\right)} &= \left(1-\frac{\Upsilon}{2}\pi_t^2\right)\tilde{Y}\left(1+\hat{Y}_t\right) \\
\tilde{C}\left(1+\hat{C}_t\right)+\left(1+\frac{\Omega}{2}g_A^2\left[\left(1+\hat{I}_t-\hat{I}_{t-1}+\hat{g}_{At}\right)-1\right]^2\right)\frac{\tilde{I}\left(1+\hat{I}_t\right)}{Z\left(1+\hat{Z}_t\right)} &= \left(1-\frac{\Upsilon}{2}\pi_t^2\right)\tilde{Y}\left(1+\hat{Y}_t\right) \\
\frac{\tilde{C}}{\tilde{Y}}\hat{C}_t+\frac{\tilde{I}}{\tilde{Y}Z}\left(\hat{I}_t-\hat{Z}_t\right) &= \hat{Y}_t
\end{aligned}
$$

18. Taylor rule

$$R_t^l = \phi_\pi \pi_t + \phi_Y \left(\hat{Y}_t - \hat{Y}_{t-1}\right) + \phi_{p^s} \hat{p}_t^s$$

Finally, we can also derive:

- Stock price:

$$\begin{aligned}
\tilde{p}_t^s &= q_t \tilde{K}_{t+1} g_{At+1} + \tilde{b}_t \\
\tilde{p}^s \left(1 + \hat{p}_t^s\right) &= q \left(1 + \hat{q}_t\right) K \left(1 + \hat{K}_{t+1}\right) g_A \left(1 + \hat{g}_{At+1}\right) + \tilde{b} \left(1 + \hat{b}_t\right) \\
\tilde{p}^s \hat{p}_t^s &= q K g_A \left(\hat{q}_t + \hat{K}_{t+1} + \hat{g}_{At+1}\right) + \tilde{b} \hat{b}_t
\end{aligned}$$

- Rate of returns on stock:

$$\begin{aligned}
r_t^s &= \left(1 - \delta_e\right) E_t \left[\left(1 + G_{t+1}\right) \frac{R_t^l}{\left(1 + \pi_{t+1}\right)}\right] \\
r^s \left(1 + \hat{r}_t^s\right) &= \left(1 - \delta_e\right) E_t \left[\left(1 + G\left(1 + G_{t+1}\right)\right) \frac{R^l \left(1 + \hat{R}_t^l\right)}{\left(1 + \pi_{t+1}\right)}\right] \\
r^s \left(1 + \hat{r}_t^s\right) &= \left(1 - \delta_e\right) E_t \left[\left([1 + G] + \frac{[1 + G]\, G}{[1 + G]} \hat{G}_{t+1}\right) R^l \left(1 + \hat{R}_t^l - \pi_{t+1}\right)\right] \\
r^s \left(1 + \hat{r}_t^s\right) &= \left(1 - \delta_e\right) E_t \left[\left(1 + G\right) R^l \left(1 + \frac{G}{1 + G} \hat{G}_{t+1} + \hat{R}_t^l - \pi_{t+1}\right)\right] \\
\hat{r}_t^s &= E_t \left[\frac{G}{1 + G} \hat{G}_{t+1} + \hat{R}_t^l - \pi_{t+1}\right] \\
\hat{r}_t^s &= E_t \left[\frac{G}{1 + G} \varphi_G \hat{\varepsilon}_t^* + \hat{R}_t^l - \pi_{t+1}\right]
\end{aligned}$$

C.4.2 Log-linearisation: Final System

1. Marginal utility for consumption:

$$\left(1 - \theta\beta\right) \hat{\Lambda}_t = \hat{\xi}_t - \theta\beta \hat{\xi}_{t+1} - \frac{\sigma \left(1 + \beta\theta^2\right)}{\left(1 - \theta\right)} \hat{C}_t + \frac{\theta\sigma}{\left(1 - \theta\right)} \hat{C}_{t-1} + \frac{\theta\beta\sigma}{\left(1 - \theta\right)} \hat{C}_{t+1}$$

2. Aggregate labour supply:

$$\left(1 + \eta\right) \hat{N}_t = \hat{\Lambda}_t + \hat{Y}_t + \hat{p}_t^w - \hat{\xi}_t - \hat{\psi}_t - \frac{\tau}{\left(1 - \tau\right)} \hat{\tau}_t \to \hat{N}_t$$

3. Euler equation for the capital producers:

$$\hat{Z}_t + \hat{p}_t^I = \Omega g_A^2 \left[\hat{g}_{At} - \hat{I}_{t-1}\right] + \Omega g_A^2 (1+\beta) \hat{I}_t - \Omega g_A^2 \beta \mathbb{E}_t \left(\hat{I}_{t+1} + \hat{g}_{At+1}\right)$$

4. Euler equation for the retailers:

$$\chi_f \beta \pi_{t+1} = \pi_t - \chi_b \pi_{t-1} - \kappa_c \left(\hat{p}_t^w + \hat{\varkappa}_t\right)$$

5. Aggregate output:

$$\hat{Y}_t = \alpha \left(\hat{u}_t + \hat{K}_t^A\right) + (1-\alpha)\left(\hat{z}_t + \hat{N}_t\right)$$

6. Aggregate effective capital stock:

$$\hat{K}_t^A = (1-\delta_e)\, \varpi_K \hat{K}_t$$

7. Law of motion for capital:

$$g_A \left(\hat{g}_{At+1} + \hat{K}_{t+1}\right) = \frac{(1-\delta(1))}{\varpi_K} \hat{K}_t^A - \frac{\delta'(1)}{\varpi_K} \hat{u}_t + \left(g_A - \frac{(1-\delta)}{\varpi_K}\right)\left(\hat{I}_t - \frac{\mu}{\varphi_G} \hat{\varepsilon}_t^*\right)$$

where:

$$\hat{\varepsilon}_t^* = \hat{p}_t^I - \hat{q}_t$$

8. Real wage setting:

$$\hat{w}_t = \hat{p}_t^w + \hat{Y}_t - \hat{N}_t$$

9. Aggregate investment demand:

$$\begin{aligned} \hat{I}_t + \hat{p}_t^I \quad = \quad & \frac{\alpha}{(\alpha + \zeta \varpi_\delta + \varpi_b)} \left(\hat{Y}_t + \hat{p}_t^w\right) + \frac{\zeta \varpi_\delta}{(\alpha + \zeta \varpi_\delta + \varpi_b)} \left(\hat{\zeta}_t + \hat{q}_t + \hat{K}_t^A\right) \\ & + \frac{\varpi_b}{(\alpha + \zeta \varpi_\delta + \varpi_b)} \hat{b}_t - \mu \hat{\varepsilon}_t^* \end{aligned}$$

10. Optimal capacity of utilisation:

$$\hat{Y}_t + \hat{p}_t^w - \hat{K}_t^A + (1 - \beta(1-\delta_e)\kappa)\, \varphi_G \hat{\varepsilon}_t^* = \hat{q}_t + \left(1 + \frac{\delta''(1)}{\delta'(1)}\right) \hat{u}_t$$

11. Rental rate of capital:

$$\hat{\Psi}_t = \hat{Y}_t + \hat{p}_t^w - \hat{u}_t - \hat{K}_t^A$$

12. Benefit of being productive:

$$\hat{G}_t = \varphi_G \hat{\varepsilon}_t^*$$

13. Lending rate:

$$\pi_{t+1} - \hat{R}_t^l = E_t \left[\hat{\Lambda}_{t+1} - \hat{\Lambda}_t - \hat{g}_{At+1} + \left[1 - \beta\left(1-\delta_e\right)\kappa\right] \varphi_G \hat{\varepsilon}_{t+1}^* \right]$$

14. Marginal Q:

$$\hat{q}_t = E_t \left[\begin{array}{c} \left(\hat{\Lambda}_{t+1} - \hat{\Lambda}_t + \hat{q}_{t+1} - \hat{g}_{At+1}\right) + \frac{\beta(1-\delta_e)\delta'(1)}{g_A} \frac{\delta''(1)}{\delta'(1)} \hat{u}_{t+1} \\ + \frac{(1-\delta_e)\beta}{g_A} G\zeta \left(\hat{\zeta}_{t+1} + \varphi_G \hat{\varepsilon}_{t+1}^*\right) \end{array} \right]$$

15. Law of motion of the real value of the bubble:

$$\hat{b}_t = E_t \left[\begin{array}{c} \hat{\Lambda}_{t+1} - \hat{\Lambda}_t + \frac{1-(1-\delta_e)\kappa}{(1-\delta_e)\kappa} \hat{m}_{t+1} + \hat{b}_{t+1} \\ + \left[1 - \beta\left(1-\delta_e\right)\kappa\right] \varphi_G \hat{\varepsilon}_{t+1}^* \end{array} \right]$$

16. Evolution of the number of bubbly firms:

$$\hat{m}_t = \left(1-\delta_e\right)\kappa\left(\hat{m}_{t-1} + \hat{\kappa}_{t-1}\right)$$

17. Resource constraint:

$$\frac{\tilde{C}}{\tilde{Y}} \hat{C}_t + \frac{\tilde{I}}{\tilde{Y}Z} \left(\hat{I}_t - \hat{Z}_t\right) = \hat{Y}_t$$

18.Taylor rule:

$$R_t^l = \phi_\pi \pi_t + \phi_Y \left(\hat{Y}_t - \hat{Y}_{t-1}\right) + \phi_{p^s} \hat{p}_t^s$$

19. Investment threshold:

$$\hat{\varepsilon}_t^* = \hat{p}_t^I - \hat{q}_t$$

and finally, stock price:

$$\tilde{p}^s \hat{p}^s_t = qKg_A \left(\hat{q}_t + \hat{K}_{t+1} + \hat{g}_{At+1}\right) + \tilde{b}\hat{b}_t$$

Shocks

Example with the sentiment shock, which has a steady-state value:

With $\hat{\kappa}_t = \rho_\kappa \hat{\kappa}_{t-1} + \epsilon_t$ and $\hat{\kappa}_{t-1} = \log \kappa_{t-1} - \log \kappa$, the sentiment shock becomes:

$$\begin{aligned}
\kappa_t &= \kappa \exp\left(\hat{\kappa}_t\right) \\
\ln \kappa_t &= \ln \kappa + \hat{\kappa}_t \\
\ln \kappa_t &= \ln \kappa + \rho_\kappa \hat{\kappa}_{t-1} + \epsilon_t \\
\ln \kappa_t &= \ln \kappa + \rho_\kappa \left(\ln \kappa_{t-1} - \ln \kappa\right) + \epsilon_t \\
\ln \kappa_t &= \left(1 - \rho_\kappa\right) \ln \kappa + \rho_\kappa \ln \kappa_{t-1} + \epsilon_t
\end{aligned}$$

If sentiment shock depends on some endogenous variables:

$$\begin{aligned}
\log \frac{\kappa_t}{\kappa} &= \log\left(\left(\frac{\kappa_{t-1}}{\kappa}\right)^\rho\right) + \log\left(1 + \epsilon_t\right) \\
\frac{\kappa_t}{\kappa} &= \left(\frac{\kappa_{t-1}}{\kappa}\right)^\rho \left(1 + \epsilon_t\right) \\
\frac{\kappa_t}{\kappa} &= \left(\frac{\kappa_{t-1}}{\kappa}\right)^\rho \left(\frac{Y_t}{Y_{t-1}}\right)^\upsilon \left(\frac{p^s_t}{p^s_{t-1}}\right)^\varpi \left(1 + \epsilon_t\right)
\end{aligned}$$

C.4.3 System reduction

Step 1: Removing Equations

We removed some equations (see at the end of this subsection). We have 18 Variables: $\hat{\Lambda}_t$, $\hat{C}_t$, $\hat{I}_t$, $\hat{N}_t$, $\hat{Y}_t$, $\hat{w}_t$, $\hat{p}^I_t$, π_t, $\hat{p}^w_t$, $\hat{K}_t$, $\hat{K}^A_t$, $\hat{G}_t$, $\hat{\varepsilon}^*_t$, $\hat{u}_t$, $\hat{\Psi}_t$, $\hat{b}_t$, $\hat{m}_t$, $\hat{R}^l_t$.

1. Resource constraint:

$$\hat{Y}_t = \frac{\tilde{C}}{\tilde{Y}}\hat{C}_t + \frac{\tilde{I}}{\tilde{Y}Z}\left(\hat{I}_t - \hat{Z}_t\right)$$

where $Z = 1$ in steady-state.

2. Aggregate investment demand:

$$\begin{aligned}
\hat{I}_t = &\frac{\alpha}{\left(\alpha + \zeta\varpi_\delta + \varpi_b\right)}\left(\hat{Y}_t + \hat{p}^w_t\right) + \frac{\zeta\varpi_\delta}{\left(\alpha + \zeta\varpi_\delta + \varpi_b\right)}\left(\hat{\zeta}_t + \left(1 - \delta_e\right)\varpi_K \hat{K}_t\right) \\
&+ \frac{\varpi_b}{\left(\alpha + \zeta\varpi_\delta + \varpi_b\right)}\hat{b}_t - \left(1 + \mu\right)\hat{\varepsilon}^*_t + \left(\frac{\zeta\varpi_\delta}{\left(\alpha + \zeta\varpi_\delta + \varpi_b\right)} - 1\right)\hat{q}_t
\end{aligned}$$

where $\mu = \frac{\phi(\varepsilon^*)\varepsilon^*}{1-\Phi(\varepsilon^*)}$ and $\hat{\varepsilon}_t^* = \hat{p}_t^I - \hat{q}_t$.

3. Aggregate output:

$$\hat{Y}_t = \alpha \hat{u}_t + \alpha (1-\delta_e) \varpi_K \hat{K}_t + \frac{(1-\alpha)}{(1+\eta)} \left(\hat{\Lambda}_t + \hat{Y}_t + \hat{p}_t^w - \hat{\xi}_t - \hat{\psi}_t \right)$$

5. Law of motion for capital (D.6):

$$\begin{aligned} \hat{g}_{At+1} + \hat{K}_{t+1} &= \frac{(1-\delta(1))}{g_A \varpi_K} (1-\delta_e) \varpi_K \hat{K}_t \\ &\quad - \frac{\delta'(1)}{g_A \varpi_K} \hat{u}_t + \left(1 - \frac{(1-\delta)}{g_A \varpi_K}\right) \left(\hat{I}_t - \frac{\mu}{\varphi_G} \hat{\varepsilon}_t^* \right) \end{aligned}$$

6. Optimal capacity of utilisation (D.8):

$$\hat{Y}_t + \hat{p}_t^w - (1-\delta_e) \varpi_K \hat{K}_t + \frac{G\varphi_G}{1+G} \hat{\varepsilon}_t^* = \hat{q}_t + \left(1 + \frac{\delta''(1)}{\delta'(1)}\right) \hat{u}_t$$

7. Marginal Q (D. 9):

$$\hat{q}_t = E_t \left[\begin{array}{c} \hat{\Lambda}_{t+1} - \hat{\Lambda}_t + \hat{q}_{t+1} - \hat{g}_{At+1} + \frac{\beta(1-\delta_e)\delta'(1)}{g_A} \frac{\delta''(1)}{\delta'(1)} \hat{u}_{t+1} \\ + \frac{(1-\delta_e)\beta}{g_A} G\zeta \left(\hat{\zeta}_{t+1} + \varphi_G \hat{\varepsilon}_{t+1}^* \right) \end{array} \right]$$

9. Euler equation for the capital producers (D.11):

$$\hat{Z}_t + \hat{\varepsilon}_t^* + \hat{q}_t = \Omega g_A^2 \left[\hat{g}_{At} - \hat{I}_{t-1} \right] + \Omega g_A^2 (1+\beta) \hat{I}_t - \Omega g_A^2 \beta \mathbb{E}_t \left(\hat{I}_{t+1} + \hat{g}_{At+1} \right)$$

10. Evolution of the number of bubbly firms (D.12):

$$\hat{m}_t = (1-\delta_e) \kappa (\hat{m}_{t-1} + \hat{\kappa}_{t-1})$$

11. Law of motion of the real value of the bubble (D.13):

$$\hat{b}_t = E_t \left[\hat{\Lambda}_{t+1} - \hat{\Lambda}_t + \frac{1-(1-\delta_e)\kappa}{(1-\delta_e)\kappa} \hat{m}_{t+1} + \hat{b}_{t+1} + \frac{G\varphi_G}{1+G} \varphi_G \hat{\varepsilon}_{t+1}^* \right]$$

12. Lending rate (D.14):

$$\pi_{t+1} - \hat{R}_t^l = E_t \left[\hat{\Lambda}_{t+1} - \hat{\Lambda}_t - \hat{g}_{At+1} + \frac{G\varphi_G}{1+G} \varphi_G \hat{\varepsilon}_{t+1}^* \right]$$

13. Marginal utility for consumption (D.15):

$$(1-\theta\beta)\,\hat{\Lambda}_t = \hat{\xi}_t - \theta\beta\hat{\xi}_{t+1} - \frac{\sigma\left(1+\beta\theta^2\right)}{(1-\theta)}\hat{C}_t + \frac{\theta\sigma}{(1-\theta)}\hat{C}_{t-1} + \frac{\theta\beta\sigma}{(1-\theta)}\hat{C}_{t+1}$$

14. Euler equation for the retailers:

$$\chi_f\beta\pi_{t+1} = \pi_t - \chi_b\pi_{t-1} - \kappa_c\left(\hat{p}_t^w + \hat{\varkappa}_t\right)$$

15.Taylor rule:

$$R_t^l = \phi_\pi\pi_t + \phi_Y\left(\hat{Y}_t - \hat{Y}_{t-1}\right) + \phi_{p^s}\hat{p}_t^s$$

REMOVED EQUATIONS:

$$\begin{aligned}
\tilde{p}^s\hat{p}_t^s &= qKg_A\left(\hat{q}_t + \hat{K}_{t+1} + \hat{g}_{At+1}\right) + \tilde{b}\hat{b}_t \\
\hat{p}_t^I &= \hat{\varepsilon}_t^* + \hat{q}_t \\
\hat{G}_t &= \varphi_G\hat{\varepsilon}_t^* \\
\hat{\Psi}_t &= \hat{Y}_t + \hat{p}_t^w - \hat{u}_t - \hat{K}_t^A \\
\hat{w}_t &= \hat{p}_t^w + \hat{Y}_t - \hat{N}_t \\
\hat{K}_t^A &= (1-\delta_e)\,\varpi_K\hat{K}_t \\
\hat{N}_t &= \frac{1}{(1+\eta)}\left(\hat{\Lambda}_t + \hat{Y}_t + \hat{p}_t^w - \hat{\xi}_t - \hat{\psi}_t\right)
\end{aligned}$$

Step 2: System Simplification

1. Resource constraint:

$$\hat{Y}_t = \frac{\tilde{C}}{\tilde{Y}}\hat{C}_t + \frac{\tilde{I}}{\tilde{Y}}\hat{I}_t - \frac{\tilde{I}}{\tilde{Y}}\hat{Z}_t$$

where $Z = 1$ in steady-state.

2. Aggregate investment demand:

$$0 = a_y\hat{Y}_t + a_y\hat{p}_t^w + a_k\hat{\zeta}_t + a_k o_k\hat{K}_t + a_b\hat{b}_t - (1+\mu)\,\hat{\varepsilon}_t^* + (a_k - 1)\,\hat{q}_t - \hat{I}_t$$

where

$$
\begin{aligned}
a_y &= \frac{\alpha}{(\alpha+\zeta\varpi_\delta+\varpi_b)} \\
a_k &= \frac{\zeta\varpi_\delta}{(\alpha+\zeta\varpi_\delta+\varpi_b)} \\
a_b &= \frac{\varpi_b}{(\alpha+\zeta\varpi_\delta+\varpi_b)} \\
o_k &= (1-\delta_e)\,\varpi_K
\end{aligned}
$$

where $\mu = \frac{\phi(\varepsilon^*)\varepsilon^*}{1-\Phi(\varepsilon^*)}$ and $\hat{\varepsilon}_t^* = \hat{p}_t^I - \hat{q}_t$.

3. Aggregate output:

$$0 = \alpha\hat{u}_t + F_K\hat{K}_t + F\hat{\Lambda}_t + F\hat{p}_t^w - F\hat{\xi}_t - F\hat{\psi}_t - F_y\hat{Y}_t$$

where

$$
\begin{aligned}
F_K &= \alpha\,(1-\delta_e)\,\varpi_K \\
F_y &= \frac{(\alpha+\eta)}{(1+\eta)} \\
F &= \frac{(1-\alpha)}{(1+\eta)}
\end{aligned}
$$

5. Law of motion for capital:

$$\hat{g}_{At+1} + \hat{K}_{t+1} = c_K\hat{K}_t - c_u\hat{u}_t + c_I\hat{I}_t - c_p\hat{\varepsilon}_t^*$$

where

$$
\begin{aligned}
c_K &= \frac{(1-\delta\,(1))}{g_A}\,(1-\delta_e)\,\varpi_K \\
c_u &= \frac{\delta'\,(1)}{g_A\varpi_K} \\
c_I &= \left(1-\frac{(1-\delta)}{g_A\varpi_K}\right) \\
c_p &= c_I\frac{\mu}{\varphi_G}
\end{aligned}
$$

6. Optimal capacity of utilisation:

$$0 = \hat{q}_t + o_u\hat{u}_t + o_k\hat{K}_t - \hat{p}_t^w - a_\varphi\hat{\varepsilon}_t^* - \hat{Y}_t$$

where:

$$
\begin{aligned}
o_k &= (1-\delta_e)\,\varpi_K \\
a_\varphi &= \frac{\varphi_G G}{1+G} \\
o_u &= \left(1+\frac{\delta''(1)}{\delta'(1)}\right)
\end{aligned}
$$

7. Marginal Q:

$$\hat{\Lambda}_{t+1}+\hat{q}_{t+1}-\hat{g}_{At+1}+d_u\hat{u}_{t+1}+d_q\hat{\zeta}_{t+1}+d_q\varphi_G\hat{\varepsilon}^*_{t+1}=\hat{q}_t+\hat{\Lambda}_t$$

where

$$
\begin{aligned}
d_q &= \frac{(1-\delta_e)\,\beta G\zeta}{g_A} \\
d_u &= d_q\frac{\delta''(1)}{G\zeta}
\end{aligned}
$$

9. Euler equation for the capital producers:

$$\beta\hat{I}_{t+1}+\beta\hat{g}_{At+1}=\hat{g}_{At}-\hat{I}_{t-1}+(1+\beta)\,\hat{I}_t-s_g\hat{Z}_t-s_g\hat{\varepsilon}^*_t-s_g\hat{q}_t$$

where

$$s_g=\frac{1}{\Omega g_A^2}$$

10. Evolution of the number of bubbly firms:

$$\hat{m}_t=(1-\delta_e)\,\kappa\,(\hat{m}_{t-1}+\hat{\kappa}_{t-1})$$

11. Law of motion of the real value of the bubble:

$$\hat{\Lambda}_{t+1}+c_{m0}\hat{m}_{t+1}+\hat{b}_{t+1}+a_\varphi\hat{\varepsilon}^*_{t+1}=\hat{b}_t+\hat{\Lambda}_t$$

where

$$
\begin{aligned}
c_{m0} &= \frac{1-(1-\delta_e)\,\kappa}{(1-\delta_e)\,\kappa} \\
a_\varphi &= \frac{\varphi_G G}{1+G}
\end{aligned}
$$

12. Lending rate:

$$\hat{\Lambda}_{t+1}-\hat{g}_{At+1}+a_\varphi\hat{\varepsilon}^*_{t+1}-\pi_{t+1}=\hat{\Lambda}_t-\hat{R}^l_t$$

13. Marginal utility for consumption:

$$\beta\hat{C}_{t+1} - m_e\hat{\xi}_{t+1} = m_l\hat{\Lambda}_t - m_s\hat{\xi}_t + m_t\hat{C}_t - \hat{C}_{t-1}$$

where

$$\begin{aligned}
m_c &= \frac{\sigma}{(1-\theta)} \\
m_t &= \frac{\left(1+\beta\theta^2\right)}{\theta} \\
m_s &= \frac{1}{m_c\theta} \\
m_e &= \frac{\beta}{m_c} \\
m_l &= \frac{(1-\theta\beta)}{m_c\theta}
\end{aligned}$$

14. Euler equation for the retailers:

$$\chi_f\beta\pi_{t+1} = \pi_t - \chi_b\pi_{t-1} - \kappa_c\left(\hat{p}_t^w + \hat{\varkappa}_t\right)$$

where:

$$\begin{aligned}
\Upsilon &= \left(\vartheta\left(1-\varpi\right) + \left(1+\vartheta\beta\right)\varpi\right) = \vartheta + \varpi\left(1-\vartheta+\vartheta\beta\right) \\
\chi_f &= \frac{\vartheta}{\Upsilon} \\
\chi_b &= \frac{\varpi}{\Upsilon} \\
\kappa_c &= \frac{\left(1-\vartheta\right)\left(1-\vartheta\beta\right)\left(1-\varpi\right)}{\Upsilon}
\end{aligned}$$

15.Taylor rule:

$$R_t^l = \phi_\pi\pi_t + \phi_Y\left(\hat{Y}_t - \hat{Y}_{t-1}\right) + \phi_{p^s}\hat{p}_t^s$$

C.5 Final System

The system can be reduced to a system of 13 equations and 13 variables: $\hat{R}^l_t$, π_t, $\hat{Y}_t$, $\hat{C}_t$, $\hat{I}_t$, $\hat{\varepsilon}^*_t$, $\hat{q}_t$, $\hat{b}_t$, $\hat{p}^w_t$, $\hat{K}_{t+1}$, $\hat{m}_{t+1}$, $\hat{u}_{t+1}$, $\hat{\Lambda}_{t+1}$

$$
\begin{aligned}
1 &: \quad 0 = \frac{\tilde{C}}{\tilde{Y}}\hat{C}_t + \frac{\tilde{I}}{\tilde{Y}}\hat{I}_t - \frac{\tilde{I}}{\tilde{Y}}\hat{Z}_t - \hat{Y}_t \\
2 &: \quad 0 = a_y\hat{Y}_t + a_y\hat{p}^w_t + a_k\hat{\zeta}_t + a_k o_k\hat{K}_t + a_b\hat{b}_t - (1+\mu)\,\hat{\varepsilon}^*_t + (a_k - 1)\,\hat{q}_t - \hat{I}_t \\
3 &: \quad 0 = \alpha\hat{u}_t + F_K\hat{K}_t + F\hat{\Lambda}_t + F\hat{p}^w_t - F\hat{\xi}_t - F\hat{\psi}_t - F_y\hat{Y}_t \\
4 &: \quad \hat{g}_{At+1} + \hat{K}_{t+1} = c_K\hat{K}_t - c_u\hat{u}_t + c_I\hat{I}_t - c_p\hat{\varepsilon}^*_t \\
5 &: \quad 0 = \hat{q}_t + o_u\hat{u}_t + o_k\hat{K}_t - \hat{p}^w_t - a_\varphi\hat{\varepsilon}^*_t - \hat{Y}_t \\
6 &: \quad \hat{\Lambda}_{t+1} + \hat{q}_{t+1} - \hat{g}_{At+1} + d_u\hat{u}_{t+1} + d_q\hat{\zeta}_{t+1} + d_q\varphi_G\hat{\varepsilon}^*_{t+1} = \hat{q}_t + \hat{\Lambda}_t \\
7 &: \quad \beta\hat{g}_{At+1} + \beta\hat{I}_{t+1} = \hat{g}_{At} - \hat{I}_{t-1} + (1+\beta)\,\hat{I}_t - s_g\hat{Z}_t - s_g\hat{\varepsilon}^*_t - s_g\hat{q}_t \\
8 &: \quad \hat{m}_{t+1} = (1-\delta_e)\,\kappa\hat{m}_t + (1-\delta_e)\,\kappa\hat{\kappa}_t \\
9 &: \quad \hat{\Lambda}_{t+1} + c_{m0}\hat{m}_{t+1} + \hat{b}_{t+1} + a_\varphi\hat{\varepsilon}^*_{t+1} = \hat{b}_t + \hat{\Lambda}_t \\
10 &: \quad \hat{\Lambda}_{t+1} - \hat{g}_{At+1} + a_\varphi\hat{\varepsilon}^*_{t+1} - \pi_{t+1} = \hat{\Lambda}_t - \hat{R}^l_t \\
11 &: \quad \beta\hat{C}_{t+1} - m_e\hat{\xi}_{t+1} = m_l\hat{\Lambda}_t - m_s\hat{\xi}_t + m_t\hat{C}_t - \hat{C}_{t-1} \\
12 &: \quad \chi_f\beta\pi_{t+1} = \pi_t - \chi_b\pi_{t-1} - \kappa_c\hat{p}^w_t - \kappa_c\hat{\varkappa}_t \\
13 &: \quad R^l_t = \phi_\pi\pi_t + \phi_Y\left(\hat{Y}_t - \hat{Y}_{t-1}\right) + \phi_{p^s}\left(\hat{p}^s_t - \hat{p}^s_{t-1}\right)
\end{aligned}
$$

with coefficients:

$$
\begin{array}{ccc}
a_y = \frac{\alpha}{(\alpha+\zeta\varpi_\delta+\varpi_b)} & c_K = \frac{(1-\delta(1))}{g_A}(1-\delta_e)\,\varpi_K & m_e = \frac{\beta}{m_c} \\
a_k = \frac{\zeta\varpi_\delta}{(\alpha+\zeta\varpi_\delta+\varpi_b)} & c_u = \frac{\delta'(1)}{g_A\varpi_K} & m_l = \frac{(1-\theta\beta)}{m_c\theta} \\
a_b = \frac{\varpi_b}{(\alpha+\zeta\varpi_\delta+\varpi_b)} & c_I = \left(1 - \frac{(1-\delta)}{g_A\varpi_K}\right) & \Upsilon = \vartheta + \varpi\,(1-\vartheta+\beta\vartheta) \\
o_k = (1-\delta_e)\,\varpi_K & c_p = c_I\frac{\mu}{\varphi_G} & \chi_f = \frac{\vartheta}{\Upsilon} \\
o_u = \left(1 + \frac{\delta''(1)}{\delta'(1)}\right) & d_q = \frac{(1-\delta_e)\beta G\zeta}{g_A} & \chi_b = \frac{\varpi}{\Upsilon} \\
a_\varphi = \frac{\varphi_G G}{1+G} & d_u = d_q\frac{\delta''(1)}{G\zeta} & \kappa_c = \frac{(1-\vartheta)(1-\beta\vartheta)(1-\varpi)}{\Upsilon} \\
F = \frac{1-\alpha}{1+\eta} & s_g = \frac{1}{\Omega g_A^2} & \\
F_y = \frac{\alpha+\eta}{1+\eta} & m_c = \frac{\sigma}{(1-\theta)} & \\
F_K = \alpha\,(1-\delta_e)\,\varpi_K & m_t = \frac{\left(1+\beta\theta^2\right)}{\theta} & \\
c_{m0} = \frac{1-(1-\delta_e)\kappa}{(1-\delta_e)\kappa} & m_s = \frac{1}{m_c\theta} &
\end{array}
$$