Essays in Industrial Organisation

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Submitted in fulfilment of the requirements for the degree of Doctor of Philosophy (PhD)

University of Glasgow

College of Social Sciences

May 2022

Abstract

This thesis is a collection of three related chapters in the field of theoretical Industrial Organisation. The overarching issue connecting all three is the impact of social interactions among consumers on firms' strategies to provide quality or effort.

In the first chapter, *Product Quality and Social Influence*, I study how opposing forces in consumer's influence have an effect on the bi-dimensional quality choices of a monopolist. The opposing forces are captured by assuming a co-existence of conformists and anti-conformists in the market, while the multiple dimensions of quality are defined according to whether they induce marginal or fixed costs of production.

In the second chapter, *Optimal Design of Ratings History*, social interactions are studied through the lens of consumer ratings. First, in a dynamic framework, I investigate how product ratings with variable history have an impact on the quality choices of a firm. Second, to maximise the steady-state welfare, I study whether a platform can intervene when there are changes in the management of a business. As the title suggests, these interventions are captured by manipulating the length of history of consumer reviews.

In the third chapter, *Impact of Consumer Ratings on Quality Choice*, I extend the initial analysis in the second chapter to a duopolistic framework. I study the impact of the average rating and the number of reviews posted on the quality choices of a firm. The ratings of both firms are considered in the analysis.

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Acknowledgments

This thesis would not have been possible without the constant guidance and support I have received from my supervisors, Prof. Hervé Moulin and Prof. Anna Bogomolnaia. I would particularly like to mention Prof. Moulin for guiding me to write this thesis and providing me with valuable skills that are essential to being a good researcher. He has taught me to ask relevant questions about the world and to be creative. I would also like to mention Prof. Bogomolnaia, who was the first teacher I had in Glasgow, for encouraging me to take up challenges that I myself was not very confident in pursuing. In the end, those challenges added up and provided me with the skills that led to the completion of this thesis. Admittedly, this is only the beginning of my research career, and I still have a mountain to climb, but thanks to them, I have been endowed with skills that should make the job easier.

During my time as a PhD student, I have had the privilege of meeting so many brilliant academic researchers whom I look up to. Some have provided valuable feedback in seminars, conferences or workshops, which has improved this research. This includes, among others, Prof. Takashi Hayashi, Dr. Francesca Flamini, Dr. John Levy, Dr. Hisayuki Yoshimoto, Prof. Francis Bloch and Dr. Andrew Clausen. I am also very grateful to my course lecturers during my MRes, where I learnt the skills necessary to write a PhD thesis in Economics. Particular, but not exhaustive, mentions include Prof. Yiannis Vailakis, Prof. Sayantan Ghosal, Prof. Richard Dennis, Prof. Jim Malley, Prof. Tatiana Kirsanova and Dr. Marco Avarucci.

I would also like to thank my colleagues and friends, who have made the journey relatively enjoyable. They include the following (not an exhaustive list nor in any particular order): Rohan Chowdhury, Damiano Turchet, Foivos Savva, Nikita Mahjabeen, Laura Kukkonen, Sara Villa, Malcolm McRobbie, Ravi Bakshi, Anh Pham, Mariia Vartuzova, Nikolas Zivanas, Max Schroder, Helena Saenz and Arthur Galichere. I am also thankful to the administrative staff at the university who ensured everything went trouble-free, with a particular mention going to the friendly and diligent Sophie Watson.

Last, but definitely far from the least, I give my utmost gratitude to my family. Particularly, I would like to mention my father, Mohammad Ataul Haque, and my mother, Mahfuza Rowshan Akhter, without whom none of this would have been remotely possible. Connecting the dots backwards, it has been their decisions and sacrifices, sometimes more so than mine, which has enabled me to write this thesis.

Author's Declaration

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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27th May 2022

Introduction

It is difficult, if not impossible, to imagine a world without markets. As Industrial Organisation (IO) is a study of the functioning of markets (Tirole, 1988, p. 1), its importance in understanding society and economic behaviour cannot be underscored enough. Different market environments can incentivise agents to make decisions that affect other market participants (Belleflamme and Peitz, 2015, p. xix). The rapid increase in the use of digital technologies in the last two decades has greatly boosted a particular market environment: social interactions[1] among various agents. The presence of these social interactions may also have an impact on the strategic choices of firms, which include, among others,[2] the provision of quality. Quality is a vertical component and the creation of value for the end consumer. Therefore, studying how firms' quality provision depends on the social interactions among buyers is essential to understanding the impact on market outcomes. With these motivations, this thesis produces three self-contained chapters in this broad topic.

In Chapter 1, I study quality provision by firms in markets with "social influence", i.e. *consumption externalities*. It is generally agreed in the literature that consumers' purchasing decisions may not only depend on the functional value of products but also the consumption choices of other individuals. For instance, some people may want to conform to others, whereas others may want to anti-conform. Assuming there is a mix of these two types of consumers in a market, I formulated a theoretical model (based on the seminal model by Spence (1975)) to analyse how this social influence can affect quality provision by firms. The model is briefly described as follows. A monopolist chooses the price and quality of a single product to sell to consumers with unit mass. Quality has two dimensions: m-quality, which increases

[1]Examples include the valuation of goods and services not only depending on the functional value but also on how many other users are using a particular product (e.g. *network* effects on platforms), or the flow of information through the widely available consumer reviews which drastically reduce the verification costs. Please see Belleflamme and Peitz (2021) for a nice comprehensive review on the economics of platforms.

[2]Other clear strategic choices include price, quantity, variety, location, etc.

the marginal cost (e.g. durable materials), and f-quality, which increases the fixed cost (e.g. innovation). A proportion of consumers are conformists, whose utility increases with more buyers, while the others are anti-conformists, whose utility decreases with more buyers.

I find that the impact of social influence on quality provision is dependent on the firm's production technology. The provision of m-quality decreases with conformity and increases with anti-conformity. On the other hand, if the conformity effect is sufficiently low, the provision of f-quality increases with conformity and decreases with anti-conformity. These findings may explain the phenomenon observed in real-world markets where relatively more m-quality (gold plating, hand-made production) is provided for luxury goods such as a Rolex watch. In contrast, relatively more Research and Development (R&D) expenditure is incurred in producing a similar accessory, the Apple Watch. I also conduct some welfare analysis and find that quality may be over-provided if its provision primarily increases variable costs of production.

In Chapter 2, I study the design of rating systems when consumer reviews can have an effect on the quality outcomes of a firm. Many popular reputation platforms, including Google Maps and Yelp, do not generally delete past reviews if there are changes in management or ownership. In this regard, I conduct a theoretical study to assess whether this policy can be improved by only keeping some recent reviews upon such a change. I formulate a dynamic model (influenced by Mailath and Samuelson (2001)) with short-lived consumers, a long-lived firm and a third-party platform. The firm is owned by different sellers (managers) changing over time and invests in a new quality level unknown to the consumers. Assuming the change of sellers or the quality level cannot be observed by the consumers, if the platform's objective is to maximise total welfare, then only a finite length of history should be aggregated for the rating. Alternatively, if the platform aims to maximise consumer welfare, reviews should be deleted upon each new seller when prior expectations are sufficiently low; otherwise, they should never be deleted.

In addition to the clear, relevant contribution to the literature on digital platforms and specifically on the design of reputation systems, the paper also contributes to other fields. First, contributing to the field of Social Learning, the paper formulated a linear weighted updating process, where the weight on ratings increases with the number of observed reviews. This intuitive and robust formulation includes Bayesian learning with commonly assumed conjugate prior distributions, such as the Normal distribution. Secondly, in the literature on Asymmetric Information, the paper contributes by formulating a scenario where the setting is a moral hazard problem but the principal (the buyer) misidentifies it as adverse selection (the buyers attribute beliefs to a static brand rather than the changing sellers).

In Chapter 3, I probe further into the effects of consumer ratings on the quality provisions of firms, this time taking into account the strategic variables of other competing firms in the industry. In this regard, I formulate a duopoly model based on Hotelling (1929), and incorporate the choice of quality as done in Economides (1989). Considering that aggregated consumer ratings consist of both the average score and the number of reviews, I analyse the impact of these variables on the quality choices of the firms when they maximise short term profits. Assuming consumers update prior beliefs using Bayes' law and that both the prior beliefs and quality outcomes are normally distributed, I find the following results. A firms' quality provision increases with its historical rating and decreases with the competitor's rating. Additionally, more reviews will only incentivise higher quality provision if the firm's rating is higher than the consumer's prior mean, but only up to a certain extent. After some point, more reviews will necessarily decrease quality provision due to the lower effect current quality will have on the aggregated rating. Alternatively, more reviews left for a competitor will increase (decrease) quality provision if the competitor's rating is lower (higher) than the consumer's prior mean. Other comparative statics concerning the variance of quality and transportation costs are also derived.

The chapters are self-contained, each including a motivating introduction, followed by a review of relevant literature. All technical proofs are contained in the Appendix of each chapter. Finally, the chapters are arranged in the chronological order in which they have been studied.

Chapter 1

Product Quality and Social Influence

1.1 Introduction

It is widely accepted in the literature that social influence plays a significant role in the choices and actions of individuals.[1] Social influence may also have an impact on goods markets, such that the consumption choices of individuals may depend not only on the functional value of products but also on the consumption choices of other individuals. According to Leibenstein (1950), three kinds of effects may exist in these markets: (i) the *bandwagon* (conformity) effect, where consumers are positively affected by increased consumption by neighbours and thus have a tendency to *conform*; (ii) the *snob* (anti-conformity) effect, where consumers value exclusivity and are negatively affected by increased consumption by neighbours, and (iii) the *Veblen* effect (named after the distinguished economist[2]), where the demand increases with price, due to those goods being a signal of social status. This chapter analyses how social influence in consumers' purchasing decisions affects multi-dimensional quality choices of a firm with market power.

Several studies established links between social influence in consumers' purchasing decisions and the behaviour of firms. Amaldoss and Jain (2005a) analysed a monopolistic market where there is a mix of conformists and anti-conformists and found

[1] Turner (1991) gives a good overview of the theories of social influence on individuals.
[2] Veblen (1899) published one of the earliest works on conspicuous consumption.

that profits increase with conformity and decrease with anti-conformity. Additionally, they found that under some conditions, the demand curve for anti-conformists is upward sloping, demonstrating the *Veblen* effect, which was further verified with an experiment in the same paper. This effect has been supported by signalling models based on the concept of *conspicuous consumption*, where consumers purchase a good not because of its functional value but to signal their wealth.[3] In a follow-up paper, Amaldoss and Jain (2005b) studied a duopolistic model and found that anti-conformity may soften the price competition among firms, leading to higher prices. Lambertini and Orsini (2001) studied the impact of positive network externalities (which is equivalent to the conformity effect) on a monopoly's quality choice and found that there may be over-provision of quality relative to the social optimum. To the best of the author's knowledge, there has not yet been any work on multi-dimensional quality choice in a market where consumers experience both positive and negative consumption externalities. This chapter aims to fill that gap.

Quality can have different dimensions, with varying impacts on the firm's production technology. For instance, one quality dimension can increase a firm's marginal cost. This can include the type of inputs in production (e.g. premium leather, gold plating), exclusive sales service, among others. In contrast, other quality dimensions can increase a firm's fixed cost. This can include, among others, product innovation, design, and automated production. To this end, the paper considers two dimensions of quality and shows that social influence affects a firm's choice of quality in these various dimensions in significantly different ways.

Let us consider two distant substitutes: the Rolex Day-Date 36 and the Apple Watch Series 5. The similarity between these products is that they fall in the class of "wristwatches". However, they cater to different markets and have significantly contrasting characteristics. The Rolex Day-Date has a price tag of around £30,000, while the Apple Watch has a lower price tag of £400. The usage features of the Rolex include telling the current time and day, while the Apple Watch has more features, like making calls, monitoring heart-beat, and telling the current time. In usage, it can be observed that the Apple Watch has significantly more features than the Rolex. Regarding materials and production, the Rolex is made of 18-carat gold and is said to be hand-made. The Apple Watch, on the other hand, is made of aluminium. The production is automated and incurs high expenditure on research and development. It can be argued that the Rolex is a conspicuous product, where

[3]Examples in this class of signalling models include, among others, Corneo and Jeanne (1997) and Bagwell and Bernheim (1996).

the brand's exclusivity is highly valued. This is aligned with the anti-conformity effect, such that the exclusiveness of the product falls when more people purchase it, leading to a decrease in value. On the other hand, it can be argued that there can be positive externalities in the purchase of an Apple Watch. This can be due to the recently growing trend in using smartwatches, or directly due to network externalities of being in the Apple "ecosystem" (using iMessage, FaceTime etc.).

This chapter aims to formulate a simple theoretical model to explain the above phenomenon and examine how social influence in markets affects firms' quality choices. With this motivation, the chapter formulates a model of vertical differentiation based on the seminal paper by Spence (1975). In a consumer market with unit mass, a profit maximising monopolist sells a product with varying quality levels. There are two types of quality dimensions: *m-quality*, which increases the marginal cost of production, and *f-quality*, which increases the fixed cost. The total quality of the product is a linear combination of the individual quality dimensions, and the monopolist chooses these components in addition to price to maximise profits. The consumer market is partitioned into types: conformists and anti-conformists. The conformists' payoff from purchasing the product increases with an increasing fraction of buyers, while the anti-conformists' payoff from buying the product decreases with an increasing fraction of buyers. Technically, conformists experience positive consumption externalities from other purchases, while anti-conformists experience negative externalities. The consumers are heterogeneous in their *taste for quality*, which is assumed to be uniformly distributed.

For analytical tractability, we focus on partially covered markets. Specifically, for most of the chapter, we ignore the possibility of a *bandwagon*, a scenario where all conformists purchase the product due to high externalities.[4] This is required due to the added complexity of incorporating consumption externalities in the presence of three choice variables (m-quality, f-quality and price) in the model.[5] This simplification enables us to define "net social influence" as the net consumption externality in the consumer market. This net influence increases with the conformity effect and decreases with the anti-conformity effect. The findings of this paper are as follows. The optimal choice of m-quality always decreases with net social influence. Alternatively, the optimal choice of f-quality decreases with net social influence only if the magnitude of the conformity effect is sufficiently high. Otherwise, the level

[4] In Section 1.6.3, we analyse the impact of this on the paper's findings.

[5] Amaldoss and Jain (2005a,b), the closest papers to this chapter with regards to consumption externalities, followed this approach as well, even though they only considered one choice variable of price.

of this quality dimension increases. The welfare criterion considered in the paper is a simple sum of consumer and producer surplus. As per this criterion, the firm under-provides f-quality compared to the socially optimal level but over-provides m-quality.

Regarding the wristwatch industry example mentioned above, these results are consistent with the market outcomes. Both Rolex and Apple can be assumed to have some degree of market power through their brand reputation, but they serve significantly different types of consumers in their respective markets. Rolex serves the luxury watches market, where there may be a demand for exclusivity, implying a higher proportion of anti-conformists than conformists in the consumer base. In line with this research, there is more expenditure along quality dimensions that increase the marginal cost of production, like gold plating and having greater attention to detail with hand-made assembly. On the other hand, in the growing and trendy smartwatches market, which Apple serves, there is likely to be more conformists than anti-conformists. As predicted by the model, it is observed that Apple invests more resources in product innovation and providing quality dimensions that incur relatively high fixed costs but low marginal costs.

The remainder of the chapter is structured as follows. Section 1.2 provides a brief review of the additional Economics literature on quality choice and consumption externalities. Section 1.3 provides the model premise, followed by the construction of the demand function. Section 1.4 analyses the profit maximising outcomes of the firm, and Section 1.5 discusses the welfare implication. Section 1.6 contains further discussions, and the conclusions follow this in Section 1.7. All proofs and technical derivations are included in the Appendix.

1.2 Additional Literature

This section provides a brief survey of the existing literature related to this research, broadly divided into two parts. First, we discuss the literature on quality choice (or *vertical differentiation*, as commonly referred to), describing the various modelling assumptions in this area concerning consumer demand as well as industry structure. This is followed by the economics literature on social influence.

Quality Choice

The seminal paper which introduced the work on endogenous quality choice is credited to Spence (1975), who analysed the provision of quality of a single good in a monopolistic market and its implications for welfare. Due to the difference in optimality conditions for the monopolist and the social planner, there may be a sub-optimal provision of quality for a given output. The monopolist considers the valuation of quality for the marginal consumer, while the social planner considers the valuation of quality for the average consumer. This may lead to over-provision of quality if the valuation of quality is increasing with quantity. Spence also suggested that rate of return regulation may have attractive features in reducing the distortion in the market. While Spence assumed the firm could only sell a single product, Mussa and Rosen (1978) extended on this by allowing the possibility of the firm to sell multiple products with varying qualities to consumers who vary in their "taste for quality". They concluded that allowing for different varieties enables firms to imperfectly price discriminate, where different price-quality menus lead consumers to self-select different varieties. Compared to a perfectly competitive market, most consumers (except those with the highest taste for quality) would get lower quality in a monopolistic market. Also, prices are always higher in a monopoly, leading to costly distortions for the consumer.

Gabszewicz and Thisse (1979) devised a duopolistic model where consumers have identical tastes but differ in their income levels and found that allowing for quality selection by firms leads to higher product differentiation. This is in stark contrast to Hotelling's (1929) location theory, where differentiation is minimal. Shaked and Sutton (1982, 1983) introduced the "finiteness property", which states that the number of firms with a positive market share in a vertically differentiated market is finite. Sutton (1986) analysed multi-product monopolistic market and found that, depending on the dispersion of consumers' willingness to pay for quality, the firm will either produce the maximum number of qualities or only sell a single product. Additionally, according to him, "what appears to matter is the extent to which the burden of product improvement falls primarily on fixed costs, or on variable costs." Although these papers have primarily assumed that consumers differ according to incomes, it was shown by Tirole (1988) that the two models are analogous, as the marginal utility of quality can be thought of as the inverse of the marginal utility of income (pp. 96). There have been few relevant extensions to these broad findings in the literature for vertical differentiation, and it is beyond the scope of this paper to review all of them.[6]

Regarding multiple dimensions in quality choice, few papers have studied duopolistic markets where firms choose between different quality attributes. Vandenbosch and Weinberg (1995) analysed the basic strategic outcomes when firms choose between two vertical dimensions. Relevant extensions to this work include Lauga and Ofek (2011) and Garella and Lambertini (2014). These papers study duopolistic markets and assume linear costs of quality improvement. Barigozzi and Ma (2018) formulated a more general model with multiple characteristics where there is an arbitrary number of characteristics that the firms can choose. In contrast to the preceding literature, they do not assume uniform distribution in the consumer's taste and separability of the cost functions. They derive necessary conditions of equilibrium and find different outcomes for quality differentiation due to non-uniform distributions and cost spillovers. Similar work by Novo-Peteiro (2020) analysed the impact of attribute dependence (whether complements or substitutes) on product differentiation.

Consumption Externalities

The present work is related to the literature on consumption externalities. Positive consumption externalities are equivalent to direct and indirect network effects, and there is a considerable portion of literature in this area. These include the classical works by Katz and Shapiro (1985, 1994), Economides (1996) and Cabral et al. (1999). The phenomenon of negative consumption externalities, or anti-conformity, is consistent with the demand for scarcity, which is the phenomenon of increased attractiveness of a product when it is more scarce. This demand for "exclusivity" has been supported in various studies.[7] Anti-conformity is also in line with the "Uniqueness Theory" by Fromkin and Snyder (1980), which proposes that people generally desire to be moderately dissimilar to others. That is, there is an inherent need for uniqueness, although there might be heterogeneity with regard to this need.

Although there is considerable literature on both positive and negative consumption externalities, studies that assume the coexistence of conformists and anti-conformists in the same market are relatively rare. Pesendorfer (1995) studied the dynamic formation of fashion cycles in a model where individuals, segmented into high and

[6] Lambertini (2006) provides a comprehensive review of the literature on vertically differentiated markets.

[7] Lynn (1991) provides a good overview of this strand of literature.

low types, are "matched" to other individuals, and a monopoly firm designs a new product. Although he did not directly formulate externalities, he showed the endogenous formation of both types of effects: when more high types buy a product, the demand increases. In comparison, the opposite holds when more low types buy a product. More recently, Grabisch et al. (2019) devised a stochastic model where conformists and anti-conformists can change their opinion based on their type and the number of individuals (anonymous) holding a certain opinion. Finally, Duffy et al. (2013) conducted an experiment to separate rational agents from "lone wolves" (anti-conformists who excessively use private information for learning) and "herd animals" (conformists who excessively use social information for learning) and found significant numbers of all three types of agents.

1.3 Model

A monopolist supplies a product with variable quality at a price $p \in \mathbb{R}_+$ to a continuum of consumers with unit mass. The quality of the product has two dimensions, $v_m \in \mathbb{R}_+$ and $v_f \in \mathbb{R}_+$, with varying impacts on the firm's production technology. Marginal costs are constant and invariant with output but assumed to be increasing and convex with v_m, with the functional form $k_m v_m^2$, where $k_m > 0$. Additionally, the firm incurs a fixed cost which increases with v_f, again assuming the specific quadratic form: $k_f v_f^2$, where $k_f > 0$. Combining the above, the firm's objective is to choose v_m, v_f and p to maximise the following profit function:

$$\Pi[v_m, v_f, p] = (p - k_m v_m^2) \cdot \bar{x}[v_m, v_f, p] - k_f v_f^2 \tag{1.1}$$

where $\bar{x}[v_m, v_f, p]$ is the demand of the product for a given menu of price and the quality components.

For the demand side, there is a continuum of consumers who are segmented into two broad types, *conformists* and *anti-conformists*. Conformists are assumed to value the product more with increasing buyers, while anti-conformists value the product more with decreasing buyers. Technically, when there is an increase in buyers, conformists experience positive externalities, while anti-conformists experience negative externalities. The proportion of conformists is exogenous and denoted by $\lambda \in [0, 1]$, with the rest of the population $(1 - \lambda)$ being anti-conformists.

The action space for consumers is binary, i.e. $A = \{0, 1\}$, where action 1 refers to buying a single unit of the good, while 0 refers to not buying. The utility from not buying (action 0) is normalised to be zero. The utility from purchasing the product is the sum of their characteristic utility from the intrinsic quality, the consumption externalities and the disutility from the price paid for the product. Consumers are heterogeneous in their taste for quality, θ, which is assumed to be uniformly distributed in $[0, 1]$.[8] The utility of the product to the conformists and the anti-conformist is denoted by u_C and u_A, respectively.

$$u_C = \theta v + \alpha x - p \tag{1.2}$$

$$u_A = \theta v - \beta x - p \tag{1.3}$$

where,

- $\theta \sim U[0, 1]$.
- $v = \omega v_m + \gamma v_f > p$,[9] where $\omega > 0$ and $\gamma > 0$ are the consumer's marginal valuation of the individual quality components respectively.
- $x \in [0, 1]$ is the fraction of buyers.
- $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}_+$ are the *social influence* parameters. They can be interpreted as the linear coefficient of conformity and anti-conformity, respectively.

The Demand Function

This section derives the demand function faced by the firm for a given set of quality components and price. Aligned with the literature, the aim is to solve for a *rational expectations equilibrium*, assuming that all consumers are fully rational and make correct expectations.

Given the formulation in the previous section, and assuming all consumers form the same expectation regarding the proportion of buyers, the fraction of conformists

[8] This can be thought along two equivalent ways: (i) a continuum of infinite consumers in the unit interval, where we are only concerned about the *measure* (length of the interval) of consumers purchasing to derive the demand, or (ii) there are two consumers, one being a conformist and the other an anti-conformist, for whom θ is an independent random draw from the uniform distribution in [0,1].

[9] Without this assumption of $v > p$, anti-conformists have no incentive to buy the product. Moreover, following the demand derivation through a rational expectations equilibrium as will be done below, having $v \leq p$ will result in multiple equilibria where only conformists may purchase. In this case, there is an existence of a *tipping point* of sales which further incentivises new buyers and the only equilibrium is a bandwagon. This is outside the scope of this paper's motivation.

opting to purchase is estimated as follows. Define x_E as the consumers' expectation of the proportion of buyers. For a given v and p, a conformist would only choose to buy if her expected utility is positive:

$$\begin{aligned} \theta v + \alpha x_E - p &> 0 \\ \implies \theta &> \frac{p - \alpha x_E}{v} \end{aligned} \tag{1.4}$$

Letting $\hat{\theta}_C = \frac{p-\alpha x_E}{v}$, any conformist with $\theta > \hat{\theta}_C$ will purchase. This can be interpreted as the marginal utility for quality for the indifferent conformist. Therefore, the proportion of conformists choosing to purchase is:

$$x_C[x, v, p] = 1 - F(\hat{\theta}_C) = 1 - F\left(\frac{p - \alpha x_E}{v}\right) \tag{1.5}$$

where $F(.)$ is the cumulative distribution function of $U[0, 1]$. Similarly, the proportion of anti-conformists choosing to purchase for a given x, v and p is:

$$x_A[x, v, p] = 1 - F(\hat{\theta}_A) = 1 - F\left(\frac{p + \beta x_E}{v}\right) \tag{1.6}$$

where $\hat{\theta}_A = \frac{p+\beta x_E}{v}$, and is likewise the marginal utility for quality for the indifferent anti-conformist. Now we solve for the equilibrium demand for a given price and quality. Combining both types of consumers, the total proportion of buyers, x, for a given v and p is:

$$\begin{aligned} x &= \lambda(x_C) + (1 - \lambda)(x_A) \\ &= \lambda\left(1 - F\left(\frac{p - \alpha x_E}{v}\right)\right) + (1 - \lambda)\left(1 - F\left(\frac{p + \beta x_E}{v}\right)\right) \end{aligned} \tag{1.7}$$

The equilibrium demand for a given menu, $\bar{x}[v, p]$, will be fixed points of the above function 1.7:

$$\bar{x}[v, p] = \{x : x = x_E\}$$

Aligned with the concept of rational expectations, all consumers can correctly form beliefs, which are confirmed in equilibrium. The full demand function, derived through computation of the fixed points, is shown below (the complete derivation is shown in the Appendix).

$$\bar{x}[v,p] = \begin{cases} x_1 : \dfrac{v-p}{v-\alpha\lambda+\beta(1-\lambda)}, & \text{if condition } c_1 \text{ holds} \\ x_2 : \dfrac{v-p(1-\lambda)}{v+(1-\lambda)\beta}, & \text{if condition } c_2 \text{ holds} \\ x_3 : \dfrac{\lambda(v-p)}{v-\alpha\lambda}, & \text{if condition } c_3 \text{ holds} \\ x_4 : \lambda, & \text{if condition } c_4 \text{ holds} \end{cases} \tag{1.8}$$

where:

- c_1: $v \geq \alpha\lambda + \beta\lambda$ and $p > \frac{v\alpha}{v+(\alpha+\beta)(1-\lambda)}$
- c_2: ($v \geq \alpha\lambda + \beta\lambda$ and $p \leq \frac{v\alpha}{v+(\alpha+\beta)(1-\lambda)}$) or ($v < \alpha\lambda + \beta\lambda$ and $v > p + \beta\lambda$)
- c_3: $p > \alpha\lambda$ and $v < \alpha\lambda + \beta\lambda$
- c_4: $p \leq \alpha\lambda$ and $v < p + \beta\lambda$

Interpretations of $x_1 - x_4$ are given as follows:

- x_1: Some conformists and some anti-conformists purchase.
- x_2: All conformists and some anti-conformists purchase.
- x_3: Some conformists purchase, while the anti-conformists refrain.
- x_4: All conformists purchase, while the anti-conformists refrain.

In the subsequent sections, in order to simplify the analysis, it is assumed that there is partial coverage of both consumer segments and the demand function takes the form of x_1. This also requires the imposition of the conditions c_1: $v \geq \alpha\lambda + \beta\lambda$ and $p > \frac{v\alpha}{v+(\alpha+\beta)(1-\lambda)}$. Assuming c_1 holds, the demand function is therefore:

$$\bar{x}[v,p] = \frac{v-p}{v-\alpha\lambda+\beta(1-\lambda)} \tag{1.9}$$

The expression in 1.9 allows us to define the parameter:

$$\phi = \alpha\lambda - \beta(1-\lambda)$$

The parameter ϕ can be interpreted as the "net social influence" in the market. If $\phi \geq 0$, then $\alpha\lambda \geq \beta(1-\lambda)$ which implies that the conformity effect weakly dominates the anti-conformity effect, and the net social influence of buyers is positive. On the other hand, if $\phi < 0$, then $\alpha\lambda < \beta(1-\lambda)$, which means that the anti-conformity effect dominates the conformity effect, and the net social influence of buyers is negative.

1.4 Profit Maximisation

The monopolist chooses $\{v_m, v_f, p\}$ to maximise profits. Substituting $\alpha\lambda-\beta(1-\lambda) = \phi$ and $v = \omega v_m + \gamma v_f$ in equation 1.1 yields the maximisation problem:

$$\max_{\{v_m,v_f,p\}} \Pi[v_m, v_f, p] = \left(p - k_m v_m^2\right) \cdot \left(\frac{\omega v_m + \gamma v_f - p}{\omega v_m + \gamma v_f - \phi}\right) - k_f v_f^2 \tag{1.10}$$

subject to the conditions c_1 in the previous section and the assumptions of the model parameters. In addition, the cost parameters k_m and k_f are assumed to be sufficiently high to ensure strict concavity of the profit function. Before attempting to solve the problem, it will be convenient to reduce the problem to two variables: v_m and v_f. Taking the derivative of 1.10 with respect to p and equating to zero yields:

$$p^*[v_m, v_f] = \frac{1}{2}\left(\gamma v_f + k_m v_m^2 + \omega v_m\right) \tag{1.11}$$

Equation 1.11 shows that the optimal price[10] charged by the monopolist is positively related to the quality level it provides. It should be noted that this expression may not always satisfy the conditions in c_1. For instance, plugging in $p^*[v_m, v_f] = \frac{1}{2}\left(\gamma v_f + k_m v_m^2 + \omega v_m\right)$ in $p > \frac{v\alpha}{v+(\alpha+\beta)(1-\lambda)}$ requires α to be sufficiently low.[11] If α is high, then the conformity effect is so strong partial coverage of both conformists and anti-conformists is not possible, and there will always be a bandwagon in equilibrium. In the subsequent analysis, we assume that this restriction is satisfied and there is partial coverage of both consumer types.

Subsequently, the above equation is substituted into equation 1.10 to yield a reduced

[10] Additionally, this optimal price is indeed a maximum and not only a stationary point. This can be shown by taking the second order derivative of the profit function with respect to price:

$$\frac{\partial^2\Pi}{\partial p^2} = -\frac{2}{\gamma v_f + \omega v_m - \phi} < 0$$

as the denominator is always positive for partial coverage (easily observed from equation 1.9).

[11] In particular, the following constraint is required: $\alpha < \frac{(\gamma v_f + v_m(k_m v_m + \omega))(\beta(-\lambda)+\beta+\gamma v_f+\omega v_m)}{(\lambda+1)(\gamma v_f+\omega v_m)+(\lambda-1)k_m v_m^2}$.

form for the maximisation problem:

$$\max_{\{v_m, v_f\}} \Pi_v[v_m, v_f] = \frac{(\omega v_m + \gamma v_f - k_m v_m^2)^2}{4(\omega v_m + \gamma v_f - \phi)} - k_f v_f^2 \tag{1.12}$$

subject to the usual constraints. In the following sections, we work with implicit interior solutions of the above problem, and define (v_m^*, v_f^*) as any local maximum[12] of $\Pi_v[v_m, v_f]$. A local maximum point in this case is defined as any point (v_m^*, v_f^*) such that $\frac{\partial \Pi_v[v_m^*, v_f^*]}{\partial v_m} = 0$, $\frac{\partial \Pi_v[v_m^*, v_f^*]}{\partial v_f} = 0$ and the Hessian matrix of $\Pi_v[v_m, v_f]$ is negative definite at (v_m^*, v_f^*).[13]

Some comparative static analysis is now conducted with respect to the parameter ϕ. The next proposition establishes the impact of consumption externalities on the firm's profits.

Proposition 1.1 (Profits)**.** *The equilibrium profit is always increasing with ϕ.*

The intuition behind proposition 1.1 is as follows. When there is an increase in the net social influence in the market, higher positive externalities from coverage increases the average willingness to pay for the good at every price and quality, which the firm exploits to generate higher profits. The opposite effect holds for an increase in anti-conformity.

The following proposition establishes that social influence in the market may lead to highly contrasting directional changes for the individual quality provisions, v_m^* and v_f^*.

Proposition 1.2. *If (v_m^*, v_f^*) is a local maximum point of the monopolist's profit function in (1.12), then the following holds:*

- *v_m^* is strictly decreasing in ϕ.*
- *if ϕ is positive and sufficiently high, v_f^* is decreasing in ϕ; otherwise v_f^* is strictly increasing in ϕ.*

[12] As constraints are put on k_m and k_f for the function to be strictly concave, and the domain restrictions are all open sets, any local maximum, if it exists, will also be a global maximum.

[13] The existence of a well behaved maximum (with both the necessary and sufficient conditions satisfied) is assumed. This may not be the case if the maximum is on the corner. The parameter conditions that ensure this is not explicitly constructed due to the complexity of the problem. However, similar to optimality of the price, among others, α has to be low enough to ensure that partial coverage is profit maximising. Qualitatively, relaxing this assumption may affect the resulting comparative statics if the solution is indeed on the corner, and there is a bandwagon where all conformists purchase. The impact of this is separately analysed in Section 1.6.3.

Before discussing the above proposition, it is helpful to present another result, which establishes the impact of social influence on the equilibrium market coverage.

Proposition 1.3 (Coverage)**.** *The equilibrium market coverage is always increasing in ϕ.*

In order to explain the intuition behind propositions 1.2 and 1.3, it is convenient to consider the cases when $\phi \geq 0$ and $\phi < 0$. When $\phi \geq 0$, this implies that $\alpha\lambda \geq \beta(1-\lambda)$ and the conformity effect is higher than the anti-conformity effect. In this case, the net consumption externality is positive, and sales impose a positive external effect on consumers. This allows us to consider market coverage as a third quality dimension. Suppose there is an increase in ϕ, resulting from an increase in α or λ or a decrease in β. This increases the consumer's valuation for coverage and induces the firm to trade off quality for quantity. At the same time, the increase in quantity decreases the unit costs of v_f while having no effect on the per-unit costs of v_m. If the conformity effect is sufficiently low, the firm responds by decreasing v_m and increasing v_f, as v_f is now relatively cheaper than v_m. For high conformity effects, consumers' valuation of the market coverage is so high that it is optimal for the firm to reduce the levels of both quality dimensions while increasing coverage through lower prices. This is because v_f becomes a strong enough substitute for x that the unit cost reductions of v_f due to higher quantity is overcome by the explicit costs incurred for v_f.

When $\phi < 0$, this implies that $\alpha\lambda < \beta(1-\lambda)$ and the anti-conformity effect is higher than the conformity effect. In this contrasting scenario, the net consumption externality is negative, and sales impose a negative external effect on consumers. Suppose there is a decrease in ϕ, which translates to an *increase* in anti-conformity, and can result from a decrease in α or λ or an increase in β. This would increase the dis-utility of sales to the existing consumers. The firm would respond by reducing coverage which will raise the unit costs for v_f, making v_m relatively cheaper. This would make it profitable for the firm to reduce v_f while increasing v_m. However, if the magnitude of anti-conformity is sufficiently high, then any additional decrease in ϕ may induce the firm to reduce the level of v_f enough to lead to a decrease in total quality, v^*. This will depend on the relative size of the parameters γ (consumer's valuation of v_f) and k_f (cost parameter for increasing v_f). In particular, if γ is sufficiently low or k_f is too high, then total quality will always increase with a decrease in ϕ.

Price

The impact of consumption externalities on the price depends on the relative magnitude of the changes to v_m^* and v_f^*, as well as the size of the parameters $\{\omega, \gamma, k_m, k_f\}$. For any v_m^* and v_f^* the optimal price for the firm is (equation 1.11):

$$p^*[v_m^*, v_f^*] = \frac{1}{2}\left(\gamma v_f^* + k_m(v_m^*)^2 + \omega v_m^*\right)$$

If ϕ is sufficiently low (from proposition 1.2), v_m^* is decreasing in ϕ and v_f^* is increasing in ϕ. Therefore, the optimal price may increase in ϕ if the size of the change in v_f^* is high enough. Formally, the price increases if:

$$\gamma \frac{\partial v_f^*}{\partial \phi} \geq -(\omega + 2k_m v_m^*)\left(\frac{\partial v_m^*}{\partial \phi}\right)$$

where, from proposition 1.2, $\frac{\partial v_f^*}{\partial \phi} > 0$ and $\frac{\partial v_m^*}{\partial \phi} < 0$ if ϕ is sufficiently low. If ϕ is sufficiently high, then both quality levels are decreasing, leading to a decrease in price as well.

1.5 Welfare Analysis

In this section, a simple measure of welfare (summation of the consumer surplus and producer surplus) is used to compare the decisions of the monopolist with the social optimum. There is a social planner who maximises the total surplus in the market. As defined generally in the literature, consumer surplus is measured as the integral of the differences in the prices consumers are willing to pay and the price they pay, computed as follows.

$$\begin{aligned} CS_H &= \int_p^{\omega v_m + \gamma v_f} \left(\frac{\omega v_m + \gamma v_f - a}{\omega v_m + \gamma v_f - \phi}\right) da \\ &= \frac{(\omega v_m + \gamma v_f - p)^2}{2(\omega v_m + \gamma v_f - \phi)} \end{aligned} \tag{1.13}$$

The total surplus is the sum of consumer surplus and the monopolists' profits.

$$TS[v_m, v_f, p] = CS[v_m, v_f, p] + \Pi[v_m, v_f, p] \tag{1.14}$$

It is assumed that the social planner is choosing $\{v_m, v_f, p\}$ to maximise the total surplus in the market. Taking the derivative of 1.14 with respect to p and equating to zero we find a standard result in industrial organisation:

$$p^S[v_m, v_f] = k_m v_m^2 \tag{1.15}$$

The socially optimal price for the product is equal to the marginal cost. Substituting equation 1.15 into equation 1.14 yields the planner's reduced objective function:

$$\max_{\{v_m, v_f\}} TS_v[v_m, v_f] = \frac{(\omega v_m + \gamma v_f - k_m v_m^2)^2}{2(\omega v_m + \gamma v_f - \phi)} - k_f v_f^2 \tag{1.16}$$

Similar to the previous section, the cost parameters k_m and k_f are assumed to be sufficiently high to ensure strict concavity of the above function. The following proposition compares the monopolist's optimal menu with the social optimum. For simplicity, it is further assumed that both the monopolist's and social planner's solution is interior and covers the market partially in both consumer segments.

Proposition 1.4. *The monopolist over-provides v_m and under-provides v_f compared to the social optimum.*

Let us first discuss the impact of v_m on the total welfare. The social planner aims to maximise the sum of the firm's profits and the consumer surplus. It was found from equation 1.15 that the social planner's optimal solution is when the price is equal to marginal cost. In this case, the marginal cost of production only depends quadratically on v_m. Any increase in v_m, despite creating value and surplus at a cost, also increases the price, negatively impacting consumer surplus. In comparison, the cost for the social planner on increasing v_f only includes the explicit cost of quality improvement.

For the monopolist, profit maximisation will lead to a lower coverage (relative to the planner), implying a higher price through the law of demand. This low coverage will imply a higher unit cost of v_f, while it has no impact on the per-unit cost of v_m. This makes v_m relatively more attractive for the firm and leads to a higher v_m and lower v_f compared to the planner's choice.

1.6 Further Discussions

This section has four parts. First, the results are demonstrated through some numerical examples. Second, some alternative specifications of the quality dimension are studied, followed by a discussion of the full coverage scenario where all conformists purchase. Finally, the model assumptions are discussed in greater detail.

1.6.1 Numerical Examples

Two scenarios are considered, with both being assigned the following parameters: $\{k_m = k_f = 3.0, \omega = \gamma = 10, \alpha = \beta = 1.5\}$. The only difference is in the choice of λ. In essence, the equilibrium outcomes in a market with full conformity is compared to a market with full anti-conformity.

(a) Setting $\lambda = 1$

	Monopolist	Social Planner
v_m	0.899	0.876
v_f	0.414	0.831
p	7.784	2.303
Π	1.949	-2.072
CS	1.232	7.004
TS	3.181	4.932
$\bar{x}$	0.46	0.948

(b) Setting $\lambda = 0$

	Monopolist	Social Planner
v_m	1.096	1.032
v_f	0.375	0.785
p	9.155	3.196
Π	1.481	-1.853
CS	1.168	5.704
TS	2.648	3.852
$\bar{x}$	0.420	0.761

Table 1.1: Equilibrium values for parameters $\{k_m = k_f = 3.0, \omega = \gamma = 10, \alpha = \beta = 1.5\}$

Table 1.6.1(a) provides the equilibrium solutions for some numerical assignment to the parameters, with $\lambda = 1$, implying the market is made entirely of conformists. We can see that v_m is over-provided compared to the social optimum, while v_f is under-provided. The price is very high as well compared to the socially optimal level. This is because the efficient price is equal to the marginal cost of production, while the monopolist uses its market power to charge a price above the marginal cost. We can also see that the monopolist's optimal level of output is significantly low compared to the socially optimal level (46% compared to 95%).

Table 1.6.1(b) sets the parameters $\lambda = 0$, implying the market is made entirely of anti-conformists. In contrast to the previous case, we can see that the level of v_m^* has risen compared to (a), while v_f^* has decreased (as per propositions 1.2). There is also a price increase by the monopolist, and a decrease in the optimal coverage for both the monopolist and the social planner, with a larger change for the latter.

1.6.2 Alternative Specifications of Quality

One quality dimension incurring both fixed and marginal cost

In the paper, we have assumed two separate and independent quality components of the product, which increase marginal and fixed costs. What happens if only one quality dimension increases both fixed and marginal costs? For example, if a phone manufacturer wants to provide a faster processor, this may incur both fixed costs in development and higher marginal costs due to the cost of each processor.

Let us call this quality component $\tilde{v}$ and let $\omega + \gamma = y$. In this case, the profit function becomes (after substituting $p^*[\tilde{v}]$):

$$\max_{\{\tilde{v}\}} \Pi_v[\tilde{v}] = \frac{(y\tilde{v} - k_m\tilde{v}^2)^2}{4\,(y\tilde{v} - \phi)} - k_f\tilde{v}^2 \tag{1.17}$$

Implicit solutions to this problem lead to findings consistent with the original model of this paper. When the net social influence is positive, and the conformity effect dominates the anti-conformity effect, the quality choice decreases with conformity. Similar to the discussion following proposition 1.2, coverage can be seen as another dimension of quality, and the monopolist trades off quality for higher coverage (through lower price). On the other hand, if the market is anti-conformist, then an increase in the anti-conformity effect will lead to the firm decreasing coverage, which will be compensated with higher quality. If the anti-conformity effect is sufficiently high, the firm reduces overall quality due to the higher per-unit costs.

One quality dimension incurring either fixed or marginal cost

Now we consider the cases where consumers only value one quality dimension. That is, either $\gamma = 0$ or $\omega = 0$ which will lead to monopolist's optimal quality provisions $v_f^* = 0$ or $v_m^* = 0$ respectively. Table 1.6.2 summarises the results from these special cases.

	Only marginal cost ($\gamma = 0$)	**Only fixed cost** ($\omega = 0$)
Quality provision	Decreases with ϕ $\forall \phi$	Peak at $\phi = 0$
Welfare	Socially optimal provision	Under-provided

Table 1.2: Either fixed or marginal costs of quality provision

The results are special cases of propositions 1.2 and 1.4. When consumers value only v_m ($\gamma = 0$), then v_m^* is decreasing with the size of the consumption externality. Still, the monopolist's provision of quality is equivalent to the socially optimal level (although at a higher price). On the other hand, when the consumers only value v_f ($\omega = 0$), the monopolist decreases the quality for any form of net social influence, whether conformity or anti-conformity (relative to the scenario of zero social influence). For conformity, the monopolist trades off quality for coverage. For anti-conformity, the monopolist decreases coverage, leading to higher unit costs and, therefore, a lower quality level. In this case, the quality provision is always under-provided compared to the socially optimal level.

1.6.3 Full Coverage

The chapter considered partial coverage for both types of consumers for analytical tractability. What are the implications of allowing for a bandwagon, a scenario where all conformists purchase due to high externalities? Let us assume that the size of the consumption externality α is sufficiently high that the monopolist only sells to conformists and ignores the anti-conformists. In this case, demand is no longer downward sloping concerning price:

$$\max_{\{v_m, v_f, p\}} \Pi[v_m, v_f, p] = \left(p - k_m v_m^2\right) \cdot \lambda - k_f v_f^2 \tag{1.18}$$

The marginal profit for the monopolist is always increasing with p and decreasing with v_m and v_f. Therefore, the firm will charge the maximum price and provide the minimum level of quality up until that point full coverage still entails. We require $v \geq p$ to ensure uniqueness of equilibria.[14] Therefore, minimising quality

[14] When $v < p$ there is always multiple equilibria, of which $\bar{x}[v, p] = 0$ is always a stable one.

and maximising price, the monopolist sets $p^*[v_m, v_f] = v = \omega v_m + \gamma v_f$ and the objective function becomes:

$$\max_{\{v_m,v_f\}} \Pi_v[v_m, v_f] = \left(\omega v_m + \gamma v_f - k_m v_m^2\right) \cdot \lambda - k_f v_f^2 \tag{1.19}$$

The explicit solutions to the above problem 1.19 is $v_m^* = \dfrac{\omega}{2k_m}$ and $v_f^* = \dfrac{\gamma\lambda}{2k_f}$. Therefore, the level of v_m^* does not change with the proportion of conformists, but the level of v_f^* is increasing with the proportion of conformists. Additionally, because demand is not downward sloping, these quality levels are efficient.

1.6.4 The Model Assumptions

The model has made a few simplifying assumptions, discussed below.

The Uniform Mass of Consumers

This assumption, as consistent in the literature with linear demand, drives some of the analysis of this paper. First, the uniform density aids in achieving a demand function that is linear in price and significantly increases the tractability of the analysis. Second, the simplification of two opposing masses of consumers into a single "net conformity" parameter, ϕ, was possible due to this assumption. Focusing on a single parameter greatly simplifies the analysis and avoids separately dealing with the individual parameters α, β and λ.

The quality parameters ω and γ

It has been assumed that the two different types of consumers, the conformists and anti-conformists, have the same marginal valuation of the individual quality components, ω and γ for v_m and v_f respectively. This may seem implausible considering that they have opposing views about the size of the demand. However, the goal of this paper is to focus our attention on social influence, thereby keeping the functional value of the product separate from its consumption externalities. Assuming different sets of ω and γ, which may increase plausibility, will significantly complicate the model with the high number of parameters. This can, however, be incorporated into a new research question, where the focus will be on linking social influence with the marginal valuation of the individual quality dimensions. For instance, the presence of conformists and anti-conformists may lead to the latter having a relatively higher preference for quality dimensions that incur marginal costs, which induces the firm to provide these. In contrast to this demand-side effect, the focus of this paper was wholly on the supply side.

Normalisation of zero utility when not purchasing

At first glance, it may occur to the reader that normalising utility to zero when not purchasing is not appropriate in a model with social influence. For example, a conformist may get more value from not buying if very few people buy the product. If we assume the conformity coefficient as α', the utility of not buying can be expressed as $u_C[\text{not buy}] = -\alpha' x$ and $u_C[\text{buy}] = \theta v + \alpha' x - p$. In this case, conformists will only buy the product if $\theta v + \alpha' x - p > -\alpha' x$, which implies that only those with $\theta > \frac{p - 2\alpha' x_E}{v}$ will buy. This is equivalent to setting $\alpha = 2\alpha'$ in the original model with zero utility for not purchasing. Therefore, there is no loss of generality resulting from this assumption.

1.7 Conclusion

This paper attempted to formulate a tractable model to analyse the impact of social influence in markets on the quality choices by a firm with market power. We develop a model where a monopolist sells a product with varying quality to a unit mass of consumers, who are subdivided into *conformists* and *anti-conformists*. The product's quality is a linear combination of two separate components: *m-quality*, which increases the firm's marginal cost, and *f-quality*, which increases the fixed cost.

The crucial finding of the paper is that a firm finds it optimal to decrease m-quality when conformity increases, with the effect being reversed for an increase in anti-conformity. On the other hand, an increase in conformity will decrease the provision of f-quality if the magnitude of the conformity effect is sufficiently high. In all other cases, f-quality will increase with conformity and decrease with anti-conformity. When there is an increase in the net consumption externalities, consumers put relatively more valuation on the market coverage, and the demand increases. This makes the firm increase the output level, which decreases the unit-costs of f-quality, making it relatively cheaper than m-quality.

In addition to the wristwatch example mentioned in Section 1, the above phenomenon can be observed in other markets. If we consider the market for supercars like Bentley or Lamborghini, they are advertised to be hand-made with additional attention on each output. On the other hand, in the market for electric cars, like Tesla, the production is automated, and there is a high level of expenditure on innovation, with relatively lower price tags compared to the supercars. The same phenomenon can be observed in multi-product firms as well. For instance, Volkswagen AG owns both Bentley and Audi, but the two differ in production and characteristics. Bentley is hand-assembled primarily, and the quality dimensions mainly comprise the engine power and materials. At the same time, the production of the majority of Audi brands is automated, with more focus on safety and convenience features for drivers. Similar to the example with watches, it can be argued that the supercar industry caters to consumers who value exclusivity. In contrast, for electric cars, consumers can experience positive externalities from other buyers (through indirect effects of charging stations, for example). Although the results from this paper still hold when the goods produced by multi-product firms are not close substitutes, a more rigorous analysis of multi-product choice to separately serve conformists and anti-conformists is left for future research.

Regarding welfare, it is shown that f-quality is under-provided by the firm, while m-quality is over-provided. This arises due to the differences in equilibrium coverage in the social planner's problem and the monopolist's problem. At the social optimum, the market coverage is higher than that of the monopolist's solution, which results in a lower unit cost of f-quality, making it cheaper. On the other hand, improving m-quality increases the marginal cost, increasing the socially optimal price, making it relatively less attractive for the social planner. One policy implication of this is the ineffectiveness of minimum quality standards if those are regarding components that increase the firm's marginal cost.

The present model in the paper can benefit from further theoretical extensions. Notably, we assumed that the quality dimensions are independent and the cost functions are separable. However, there can be an interaction between the quality dimensions, and we discuss a specific form of this in Section 1.6.2. This interaction between different quality dimensions can be studied with more rigour. Additionally, the findings apply to firms with market power and strategic interactions between firms were not considered. In that regard, incorporating a duopolistic market will enable us to study competition and its impact on product differentiation in this type of market. The future direction of this research would benefit from empirically testing the theoretical results established.

Finally, although this research was motivated before the Covid-19 pandemic, its practical implications are more apparent as businesses adjust to the "new normal". Particularly in the hospitality sector, a portion of the population may be relatively cautious (the anti-conformists) and only go to a restaurant or pub as long as there are not many other people. Alternatively, other consumers (the conformists) may be expected to be relatively less cautious and would be more willing to go to a crowded setting, which can be due to their inherent preference or due to a crowd being a signal of quality. There will be implications of this on the quality choices by the businesses and the subsequent welfare outcomes, as analysed in this paper.

Appendix

This Appendix consists of two parts. In part A, the proofs of the propositions will be provided. Many of the derived expressions have lengthy algebra, for which the *Mathematica* code has been provided in part B.[15]

A Proofs

Derivation of the full demand function

From equation 1.7, we know:

$$x = \lambda \left(1 - F\left(\frac{p - \alpha x_E}{v}\right)\right) + (1 - \lambda)\left(1 - F\left(\frac{p + \beta x_E}{v}\right)\right)$$

It is easy to see that for $v > p > 0$,

$$F\left(\frac{p - \alpha x_E}{v}\right) = \max\left\{0, \frac{p - \alpha x_E}{v}\right\}$$

$$F\left(\frac{p + \beta x_E}{v}\right) = \min\left\{1, \frac{p + \beta x_E}{v}\right\}$$

Therefore, for a given menu $\{v, p\}$ where $v > p > 0$, and fraction of buyers x, the new fraction of buyers $G(x)$:

$$x = \begin{cases} \lambda\left(1 - \frac{p - \alpha x_E}{v}\right) + (1 - \lambda)\left(1 - \frac{p + \beta x_E}{v}\right), & \text{if } x_E \leq \min\left\{\frac{p}{\alpha}, \frac{v - p}{\beta}\right\} \\ \lambda + (1 - \lambda)\left(1 - \frac{p + \beta x_E}{v}\right), & \text{if } \frac{p}{\alpha} \leq x_E < \frac{v - p}{\beta} \\ \lambda\left(1 - \frac{p - \alpha x_E}{v}\right), & \text{if } x_E < \frac{p}{\alpha} \text{ and } x_E \geq \frac{v - p}{\beta} \\ \lambda, & \text{if } x_E \geq \max\left\{\frac{p}{\alpha}, \frac{v - p}{\beta}\right\} \end{cases}$$

The equilibrium demand for a given menu, $\bar{x}[v, p]$, will be fixed points of the above function:

$$\bar{x}[v, p] = \{x : x_E = x\}$$

For $\bar{x} = x_E$, the following cases are possible:

[15] It should be noted that the usage of Mathematica is only used to save space. The algebra is tedious, but not overly complicated to necessitate the use of the software.

(1) $\bar{x} = \lambda\left(1 - \frac{p-\alpha\bar{x}}{v}\right) + (1-\lambda)\left(1 - \frac{p+\beta\bar{x}}{v}\right)$ for $\bar{x} \leq \min\left\{\frac{p}{\alpha}, \frac{v-p}{\beta}\right\}$

(2) $\bar{x} = \lambda + (1-\lambda)\left(1 - \frac{p+\beta\bar{x}}{v}\right)$ for $\frac{p}{\alpha} \leq \bar{x} < \frac{v-p}{\beta}$

(3) $\bar{x} = \lambda\left(1 - \frac{p-\alpha\bar{x}}{v}\right)$ for $\bar{x} < \frac{p}{\alpha}$ and $\bar{x} \geq \frac{v-p}{\beta}$

(4) $\bar{x} = \lambda$ for $\bar{x} \geq \max\left\{\frac{p}{\alpha}, \frac{v-p}{\beta}\right\}$

The respective solutions to the above are as follows:

(1) $\bar{x} = \dfrac{v-p}{v-\alpha\lambda+\beta(1-\lambda)}$ for $\bar{x} \leq \min\left\{\frac{p}{\alpha}, \frac{v-p}{\beta}\right\}$. This will hold when:

$$\frac{v-p}{v-\alpha\lambda+\beta(1-\lambda)} \leq \min\left\{\frac{p}{\alpha}, \frac{v-p}{\beta}\right\}$$
$$\implies \frac{v-p}{v-\alpha\lambda+\beta(1-\lambda)} < \frac{p}{\alpha} \quad \text{and} \quad \frac{v-p}{v-\alpha\lambda+\beta(1-\lambda)} < \frac{v-p}{\beta}$$

Solving for the above inequalities gives the condition c_1:

$$v \geq \alpha\lambda + \beta\lambda \quad \text{and} \quad p > \frac{v\alpha}{v+(\alpha+\beta)(1-\lambda)}$$

(2) $\bar{x} = \dfrac{v-p(1-\lambda)}{v+\beta(1-\lambda)}$ for $\dfrac{p}{\alpha} \leq \bar{x} < \dfrac{v-p}{\beta}$. This will hold when:

$$\frac{p}{\alpha} \leq \frac{v-p(1-\lambda)}{v+\beta(1-\lambda)} < \frac{v-p}{\beta}$$

Therefore, $\dfrac{p}{\alpha} \leq \dfrac{v-p(1-\lambda)}{v+\beta(1-\lambda)}$. Solving for this gives:

$$p \leq \frac{v\alpha}{v+(\alpha+\beta)(1-\lambda)}$$

At the same time, solving for $\dfrac{v-p(1-\lambda)}{v+\beta(1-\lambda)} < \dfrac{v-p}{\beta}$ gives:

$$p < v - \beta\lambda$$

The first inequality is only relevant when $\dfrac{v\alpha}{v+(\alpha+\beta)(1-\lambda)} \leq v - \beta\lambda$. This is only possible when $v \geq (\alpha+\beta)\lambda$ or $v \leq -\beta(1-\lambda)$. But, v cannot be negative as per our assumptions, giving us the reduced constraint c_2:

$$\left(v \geq \alpha\lambda + \beta\lambda \text{ and } p \leq \frac{v\alpha}{v+(\alpha+\beta)(1-\lambda)}\right) \text{ OR } \left(v < \alpha\lambda+\beta\lambda \text{ and } v > p+\beta\lambda\right)$$

(3) $\bar{x} = \dfrac{\lambda(v-p)}{v-\alpha\lambda}$ for $\bar{x} < \dfrac{p}{\alpha}$ and $\bar{x} \geq \dfrac{v-p}{\beta}$. This condition will hold true when:

$$\frac{\lambda(v-p)}{v-\alpha\lambda} < \frac{p}{\alpha} \quad \text{and} \quad \frac{\lambda(v-p)}{v-\alpha\lambda} \geq \frac{v-p}{\beta}$$
$$\implies p > \alpha\lambda \quad \text{and} \quad v \leq \alpha\lambda + \beta\lambda$$

which is exactly the constraint c_3.

(4) $\bar{x} = \lambda$ for $\bar{x} \geq \max\left\{\frac{p}{\alpha}, \frac{v-p}{\beta}\right\}$. For this to hold we need:

$$p \leq \alpha\lambda \quad \text{and} \quad v \leq p + \beta\lambda$$

Combining all the cases and the constraints yields the demand function in 1.8. This can be aided with the illustration in Figure 1.1.

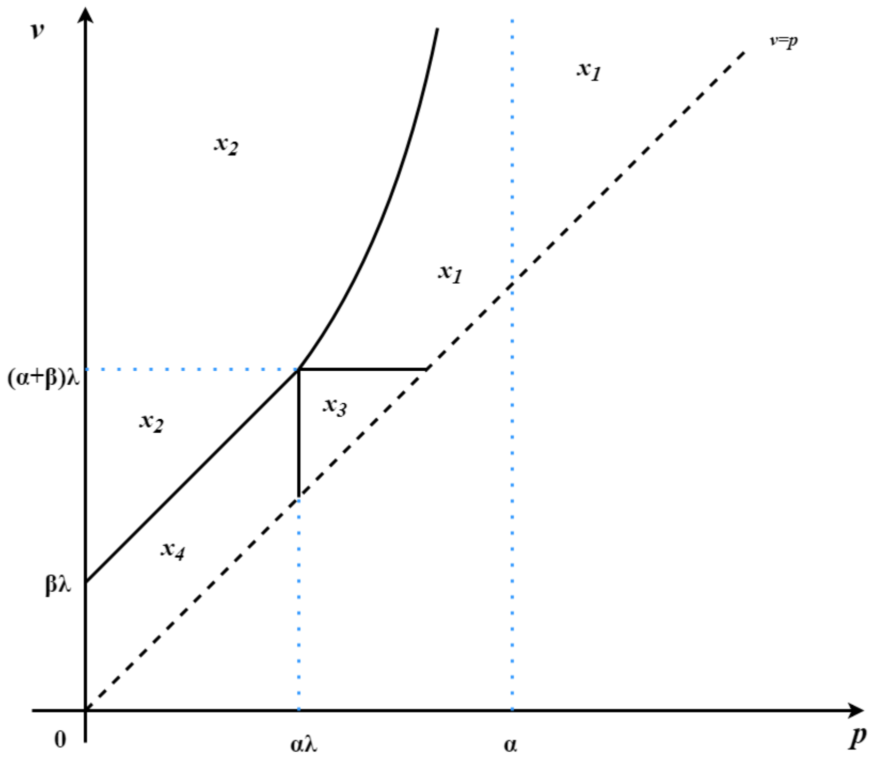

Figure 1.1: Demand Function in the domain $v > p$

Optimality Conditions

Let (v_m^*, v_f^*) be an interior point in the domain of $\mathbb{R}^2$ where $\Pi_v[v_m, v_f]$ is defined (we assume $\Pi_v[v_m, v_f] > 0$, that is, the firm will only produce if it earns a positive profit). The necessary conditions for (v_m^*, v_f^*) to be a maximum point is for the following system of equations to be satisfied at (v_m^*, v_f^*).

$$
\begin{aligned}
f_1[v_m, v_f; \phi] :=& \frac{\partial \Pi_v[v_m, v_f]}{\partial v_m} = 0 \\
& \implies \omega\left(\gamma v_f + \omega v_m - 2\phi\right) - k_m v_m \left(4\gamma v_f + 3\omega v_m - 4\phi\right) = 0 \\
f_2[v_m, v_f; \phi] :=& \frac{\partial \Pi_v[v_m, v_f]}{\partial v_m} = 0 \\
& \implies \frac{\gamma\left(\gamma v_f + v_m\left(\omega - k_m v_m\right)\right)\left(\gamma v_f + v_m\left(k_m v_m + \omega\right) - 2\phi\right)}{4\left(\gamma v_f + \omega v_m - \phi\right)^2} - 2k_f v_f = 0
\end{aligned}
\tag{1.20}
$$

where we define functions $f_1[v_m, v_f; \phi]$ and $f_2[v_m, v_f; \phi]$ as the marginal profit with respect to v_m and v_f respectively. The first order conditions in 1.20 can be rearranged to yield the following system of equations.

$$
\begin{aligned}
k_m &= \frac{\omega\left(\gamma v_f^* + \omega v_m^* - 2\phi\right)}{v_m^*\left(4\gamma v_f^* + 3\omega v_m^* - 4\phi\right)} \\
k_f &= \frac{\gamma\left(2\gamma v_f^* + \omega v_m^*\right)\left(\gamma v_f^* + \omega v_m^* - 2\phi\right)}{v_f^*\left(4\gamma v_f^* + 3\omega v_m^* - 4\phi\right)^2}
\end{aligned}
\tag{1.21}
$$

In addition, a sufficient condition for (v_m^*, v_f^*) to be a local maximum point for $\Pi_v[v_m, v_f]$ is that the Hessian matrix of $\Pi_v[v_m, v_f]$ is negative definite at (v_m^*, v_f^*), that is:

$$
\mathbf{H} = \begin{pmatrix} \frac{\partial^2 \Pi_v[v_m^*, v_f^*]}{\partial v_m^2} & \frac{\partial^2 \Pi_v[v_m^*, v_f^*]}{\partial v_m \partial v_f} \\ \frac{\partial^2 \Pi_v[v_m^*, v_f^*]}{\partial v_m \partial v_f} & \frac{\partial^2 \Pi_v[v_m^*, v_f^*]}{\partial v_f^2} \end{pmatrix} \text{ and } \mathrm{Det}[\mathbf{H}] > 0 \text{ and } \frac{\partial^2 \Pi_v[v_m^*, v_f^*]}{\partial v_m^2} < 0 \tag{1.22}
$$

where $\mathrm{Det}[\mathbf{H}]$ is the determinant of $\mathbf{H}$.

To save space and keep long equations within the page margins, define A as the following expression for the rest of the Appendix.

$$
\begin{aligned}
A :=\ & 8\gamma^2 v_f^{*2}\left(2\omega v_m^* - 3\phi\right) + \gamma\omega v_f^* v_m^*\left(11\omega v_m^* - 28\phi\right) \\
& + 8\gamma^3 v_f^{*3} + \omega v_m^*\left(3\omega^2 {v_m^*}^2 - 12\omega\phi v_m^* + 8\phi^2\right)
\end{aligned}
\tag{1.23}
$$

The next lemma will be useful for the proof of propositions 1.2 and 1.3.

Lemma 1.1. *At a local maximum point* (v_m^*, v_f^*)*,* $A > 0$.

Proof. After substituting the equations in 1.21, $\mathrm{Det}[\mathbf{H}]$ at (v_m^*, v_f^*) is computed as follows.

$$\text{Det}[\mathbf{H}] = A \cdot B \tag{1.24}$$

where $B = \dfrac{\gamma\omega\left(2\gamma v_f^* + \omega v_m^*\right)\left(\gamma v_f^* + \omega v_m^* - 2\phi\right)}{v_f^* v_m^*\left(\gamma v_f^* + \omega v_m^* - \phi\right)\left(4\gamma v_f^* + 3\omega v_m^* - 4\phi\right)^4}$. We need to show that the expression $A > 0$. At a local maximum, we know $\text{Det}[\mathbf{H}] > 0$, which means that both A and B are either strictly positive or strictly negative. We will show that $B > 0$, which will imply that $A > 0$.

First, let us consider the denominator of B. For partial market coverage, $0 < \bar{x}[v_m^*, v_f^*] < 1$, which implies:

$$\bar{x}[v_m^*, v_f^*] = \frac{\gamma v_f^* + \omega v_m^* - p}{\gamma v_f^* + \omega v_m^* - \phi} > 0 \tag{1.25}$$

As we assume $v = \gamma v_f^* + \omega v_m^* > p$, this also implies that $\gamma v_f^* + \omega v_m^* > \phi \Leftrightarrow (\gamma v_f^* + \omega v_m^* - \phi) > 0$. Using this, combined with the assumption that $v_m^* > 0$ and $v_f^* > 0$, it is easy to see that

$$v_f^* v_m^*\left(\gamma v_f^* + \omega v_m^* - \phi\right)\left(4\gamma v_f^* + 3\omega v_m^* - 4\phi\right)^4 > 0$$

Therefore, the denominator of B is always positive.

Now let us consider the numerator. From our assumptions of the parameters, it is easy to see that $\gamma\omega\left(2\gamma v_f^* + \omega v_m^*\right) > 0$. We will now show that $(\gamma v_f^* + \omega v_m^* - 2\phi) > 0$ as well.

We know the optimal price $p^*[v_m^*, v_f^*]$ is given as follows:

$$p^*[v_m^*, v_f^*] = \frac{1}{2}\left(\gamma v_f^* + k_m(v_m^*)^2 + \omega v_m^*\right)$$

Since $v = \gamma v_f^* + \omega v_m^* > p$, this implies:

$$\begin{aligned}
\gamma v_f^* + \omega v_m^* &> \frac{1}{2}\left(\gamma v_f^* + k_m(v_m^*)^2 + \omega v_m^*\right) \\
\Longrightarrow \frac{1}{2}\left(\gamma v_f^* + \omega v_m^*\right) &> \frac{1}{2}k_m(v_m^*)^2 \\
\Longrightarrow \gamma v_f^* + \omega v_m^* &> k_m(v_m^*)^2 \\
\Longrightarrow \gamma v_f^* + \omega v_m^* - k_m(v_m^*)^2 &> 0
\end{aligned} \tag{1.26}$$

Incorporating 1.21 to the above inequality, we obtain:

$$\begin{aligned}
&\gamma v_f^* + \omega v_m^* - k_m (v_m^*)^2 > 0 \\
\implies &\gamma v_f^* + \omega v_m^* - \frac{\omega\left(\gamma v_f^* + \omega v_m^* - 2\phi\right)}{v_m^*\left(4\gamma v_f^* + 3\omega v_m^* - 4\phi\right)}(v_m^*)^2 > 0
\end{aligned}$$

Simplifying the above expression yields:

$$\frac{2\left(2\gamma v_f^* + \omega v_m^*\right)\left(\gamma v_f^* + \omega v_m^* - \phi\right)}{4\gamma v_f^* + 3\omega v_m^* - 4\phi} > 0 \tag{1.27}$$

As $\gamma > 0, \omega > 0, v_f^* > 0, v_m^* > 0$, it can be easily observed that the numerator of the expression in the left hand side of 1.27 is always positive. This implies that $(4\gamma v_f^* + 3\omega v_m^* - 4\phi) > 0$. Now, from the first order condition in 1.21, we know:

$$k_m = \frac{\omega\left(\gamma v_f^* + \omega v_m^* - 2\phi\right)}{v_m^*\left(4\gamma v_f^* + 3\omega v_m^* - 4\phi\right)}$$

Because $(4\gamma v_f^* + 3\omega v_m^* - 4\phi) > 0$, we can now deduce that $\left(\gamma v_f^* + \omega v_m^* - 2\phi\right) > 0$ as well (otherwise $k_m < 0$, a contradiction). Therefore, $B > 0$ which implies that $A > 0$. ∎

Proof of Proposition 1.1. Let (v_m^*, v_f^*) be an interior solution of the problem in 1.12. Using the Envelope Theorem, we can write:

$$\begin{aligned}
\frac{\partial \Pi_v[v_m^*, v_f^*; \phi]}{\partial \phi} &= \left.\frac{\partial \Pi_v[v_m, v_f; \phi]}{\partial \phi}\right|_{(v_m = v_m^*, v_f = v_f^*)} \\
&= \frac{\left(\omega v_m^* + \gamma v_f^* - k_m (v_m^*)^2\right)^2}{4\left(\omega v_m^* + \gamma v_f^* - \phi\right)^2} > 0
\end{aligned}$$ ∎

Proof of Proposition 1.2. The Jacobian matrix of the system of equations in 1.20 is defined as:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f_1[v_m^*, v_f^*; \phi]}{\partial v_m} & \frac{\partial f_1[v_m^*, v_f^*; \phi]}{\partial v_f} \\ \frac{\partial f_2[v_m^*, v_f^*; \phi]}{\partial v_m} & \frac{\partial f_2[v_m^*, v_f^*; \phi]}{\partial v_f} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \Pi_v[v_m^*, v_f^*]}{\partial v_m^2} & \frac{\partial^2 \Pi_v[v_m^*, v_f^*]}{\partial v_m \partial v_f} \\ \frac{\partial^2 \Pi_v[v_m^*, v_f^*]}{\partial v_m \partial v_f} & \frac{\partial^2 \Pi_v[v_m^*, v_f^*]}{\partial v_f^2} \end{pmatrix} = \mathbf{H} \tag{1.28}$$

As $\text{Det}[\mathbf{H}] > 0$, $\mathbf{H}$ is invertible at (v_m^*, v_f^*), which implies $\mathbf{J}$ is also invertible. Using the Implicit Function Theorem we can state that there exists a set of differentiable functions $g_1[\phi]$ and $g_2[\phi]$ around an open neighbourhood of (v_m^*, v_f^*) such that $v_m^* =$

$g_1[\phi]$ and $v_f^* = g_1[\phi]$. The derivatives of the functions are as follows.

$$\begin{aligned}\begin{pmatrix} g_1'[\phi] \\ g_2'[\phi] \end{pmatrix} &= -\mathbf{J}^{-1} \begin{pmatrix} \frac{\partial f_1[v_m^*, v_f^*; \phi]}{\partial \phi} \\ \frac{\partial f_2[v_m^*, v_f^*; \phi]}{\partial \phi} \end{pmatrix} \\ &= - \begin{pmatrix} \frac{\partial f_1[v_m^*, v_f^*; \phi]}{\partial v_m} & \frac{\partial f_1[v_m^*, v_f^*; \phi]}{\partial v_f} \\ \frac{\partial f_2[v_m^*, v_f^*; \phi]}{\partial v_m} & \frac{\partial f_2[v_m^*, v_f^*; \phi]}{\partial v_f} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{\partial f_1[v_m^*, v_f^*; \phi]}{\partial \phi} \\ \frac{\partial f_2[v_m^*, v_f^*; \phi]}{\partial \phi} \end{pmatrix} \end{aligned} \tag{1.29}$$

After substituting for k_m and k_f from the first order conditions in 1.21, the above functions are computed as follows.

$$\begin{aligned} g_1'[\phi] &= \frac{1}{A}\left(-2v_m^*(\omega v_m^* + 2\gamma v_f^*)^2\right) \\ g_2'[\phi] &= \frac{1}{A}\left(2v_f^*(\omega v_m^* + 2\gamma v_f^*)(\omega v_m^* - 4\phi)\right) \end{aligned} \tag{1.30}$$

From Lemma 1.1, the denominator of both the functions $g_1'[\phi]$ and $g_2'[\phi]$ is strictly positive. In addition, because $v_m^* > 0$ the numerator of $g_1'[\phi]$ is always negative. Using the same assumptions, the numerator of $g_2'[\phi]$ is strictly positive when $\phi < \frac{\omega v_m^*}{4}$ and negative when $\phi \geq \frac{\omega v_m^*}{4}$. To summarise:

- $g_1'[\phi] < 0 \ \forall \phi$
- $g_2'[\phi] > 0 \ \forall \phi < \frac{\omega v_m^*}{4}$; $g_2'[\phi] = 0$ if $\phi = \frac{\omega v_m^*}{4}$ and $g_2'[\phi] < 0 \ \forall \phi \geq \frac{\omega v_m^*}{4}$

We know $v_m^* = g_1[\phi]$ and $g_1'[\phi] < 0 \ \forall \phi$. As $g_1[\phi]$ is strictly decreasing, and using the fact that $g_1[0] = v_m^* > 0$, there can exist at most one root $\phi^\tau > 0$ such that:

$$\phi^\tau = \frac{\omega g_1[\phi^\tau]}{4}$$

Therefore, $g_2'[\phi] > 0 \ \forall \phi < \phi^\tau$; $g_2'[\phi] = 0$ if $\phi = \phi^\tau$ and $g_2'[\phi] < 0 \ \forall \phi > \phi^\tau$ ■

Proof of Proposition 1.3. For a local maximum point (v_m^*, v_f^*), the equilibrium market coverage (after plugging in the optimal price in 1.11) is as follows:

$$\bar{x}[v_m^*, v_f^*; \phi] = \frac{\omega v_m^* + \gamma v_f^* - k_m (v_m^*)^2}{2\left(\omega v_m^* + \gamma v_f^* - \phi\right)}$$

Using the chain rule, the total derivative of $\bar{x}[v_m^*, v_f^*; \phi]$ with respect to ϕ is computed below.

$$\frac{d\bar{x}[v_m^*, v_f^*; \phi]}{d\phi} = \frac{\partial \bar{x}[v_m^*, v_f^*; \phi]}{\partial \phi} + \frac{\partial \bar{x}[v_m^*, v_f^*; \phi]}{\partial v_m^*} \cdot \frac{\partial v_m^*}{\partial \phi} + \frac{\partial \bar{x}[v_m^*, v_f^*; \phi]}{\partial v_f^*} \cdot \frac{\partial v_f^*}{\partial \phi} \tag{1.31}$$

Plugging in the solutions from proposition 1.2 for $\frac{\partial v_m^*}{\partial \phi}$ and $\frac{\partial v_f^*}{\partial \phi}$, we obtain:

$$\begin{aligned}\frac{d\bar{x}[v_m^*, v_f^*; \phi]}{d\phi} &= \frac{4(\omega v_m^* + 2\gamma v_f^*)^2(\gamma v_f^* + \omega v_m^* - 2\phi)}{\left(4\gamma v_f^* + 3\omega v_m^* - 4\phi\right)} \cdot \frac{1}{A} \\ &= \frac{4k_m v_m^*(\omega v_m^* + 2\gamma v_f^*)^2}{\omega} \cdot \frac{1}{A}\end{aligned} \tag{1.32}$$

Where the second equality follows by substituting $\frac{(\gamma v_f^* + \omega v_m^* - 2\phi)}{\left(4\gamma v_f^* + 3\omega v_m^* - 4\phi\right)}$ from 1.21. By Lemma 1.1 and the assumption of the parameters, $\dfrac{d\bar{x}[v_m^*, v_f^*; \phi]}{d\phi} > 0$. ■

Welfare Analysis

This section contains the proof of proposition 1.4. The next two lemmas will assist in the proof.

Lemma 1.2. *If $\phi \geq \frac{\omega^2}{16k_m}$, there is no feasible interior solution for the social planner's problem, and if $\phi < \frac{\omega^2}{16k_m}$, the only candidate interior solution for v_m is given as follows.*

$$v_m^S = \frac{\omega^2 + 4\phi k_m - 4\gamma v_f k_m + \sqrt{12\omega^2 k_m \left(\gamma v_f - 2\phi\right) + \left(-4\gamma v_f k_m + 4\phi k_m + \omega^2\right)^2}}{6\omega k_m}$$

Proof. Let us consider the optimal choice of v_m for the social planner, taking v_f as given. The first order condition is given as follows.

$$\begin{aligned}&\frac{\partial TS_v[v_m, v_f]}{\partial v_m} \\ &= \frac{\left(-\gamma v_f + k_m v_m^2 - \omega v_m\right)\left(k_m v_m \left(4\gamma v_f + 3\omega v_m - 4\phi\right) - \omega\left(\gamma v_f + \omega v_m - 2\phi\right)\right)}{2\left(\gamma v_f + \omega v_m - \phi\right)^2} = 0 \\ &\Rightarrow \left(-\gamma v_f + k_m v_m^2 - \omega v_m\right)\left(k_m v_m \left(4\gamma v_f + 3\omega v_m - 4\phi\right) - \omega\left(\gamma v_f + \omega v_m - 2\phi\right)\right) = 0 \\ &\Rightarrow \left(k_m v_m \left(4\gamma v_f + 3\omega v_m - 4\phi\right) - \omega\left(\gamma v_f + \omega v_m - 2\phi\right)\right) = 0\end{aligned} \tag{1.33}$$

where the last line follows from our assumption that $v = \omega v_m + \gamma v_f > p = k_m v_m^2$. Solving for the above equation we get two candidate solutions for v_m:

$$\text{(i)}\ \ v_m^S = \frac{\omega^2 + 4\phi k_m - 4\gamma v_f k_m - \sqrt{12\omega^2 k_m \left(\gamma v_f - 2\phi\right) + \left(-4\gamma v_f k_m + 4\phi k_m + \omega^2\right)^2}}{6\omega k_m}$$

$$\text{(ii)}\ \ v_m^S = \frac{\omega^2 + 4\phi k_m - 4\gamma v_f k_m + \sqrt{12\omega^2 k_m \left(\gamma v_f - 2\phi\right) + \left(-4\gamma v_f k_m + 4\phi k_m + \omega^2\right)^2}}{6\omega k_m}$$

We will show that the first candidate is never feasible, while the second candidate is only feasible when $\phi < \frac{\omega^2}{16k_m}$ (for feasibility we require $0 < \bar{x}[v_m, v_f] < 1$).

Let us first consider (i). Plugging in the expression of v_m to the demand function and simplifying, we obtain:

$$\bar{x}[v_m, v_f] = \frac{2}{3}\left(\frac{4\gamma v_f k_m - 4\phi k_m + 2\omega^2}{\omega^2}\right) + \frac{2}{3}\left(\sqrt{\left(\frac{4\gamma v_f k_m - 4\phi k_m + 2\omega^2}{\omega^2}\right)^2 - (3 + \frac{12\gamma v_f k_m}{\omega^2})}\right)$$

The above equation can be written in the form:

$$\bar{x}[v_m, v_f] = \frac{2}{3}\left(a + \sqrt{a^2 - b}\right) \tag{1.34}$$

where $a = \dfrac{4\gamma v_f k_m - 4\phi k_m + 2\omega^2}{\omega^2}$ and $b = 3 + \dfrac{12\gamma v_f k_m}{\omega^2}$. We require $0 < \bar{x}[v_m, v_f] < 1$. Because $b > 0$, for $\bar{x}[v_m, v_f] > 0$, we require $a > 0$ (if $a \leq 0$, then $a + \sqrt{a^2 - b} \leq 0$ due to $|a| > |\sqrt{a^2 - b}|$ when $b > 0$). Additionally, $\frac{2}{3}\left(a + \sqrt{a^2 - b}\right) < 1 \Rightarrow a + \sqrt{a^2 - b} < \frac{3}{2} \Rightarrow a < \frac{3}{2}$. At the same time, for $\sqrt{a^2 - b} \in \mathbb{R}$ we require $0 < b < a^2 \Rightarrow 0 < b < (\frac{3}{2})^2 \Rightarrow 0 < b < (\frac{9}{4})$. But $b = 3 + \dfrac{12\gamma v_f k_m}{\omega^2} > 3$, and therefore $\bar{x}[v_m, v_f] > 1$, violating feasibility.

Now let us consider (ii). Plugging in the expression to the demand function and simplifying, we obtain:

$$\bar{x}[v_m, v_f] = \frac{2}{3}\left(a - \sqrt{a^2 - b}\right) \tag{1.35}$$

where $a = \dfrac{4\gamma v_f k_m - 4\phi k_m + 2\omega^2}{\omega^2}$ and $b = 3 + \dfrac{12\gamma v_f k_m}{\omega^2}$. Again, we require $0 < \bar{x}[v_m, v_f] < 1$. Similar to above, for $\bar{x}[v_m, v_f] > 0$, we need $a > 0$. For $\bar{x}[v_m, v_f] < 1$, $\dfrac{2}{3}\left(a - \sqrt{a^2 - b}\right) < 1$. Because $b > 3$, this inequality can be easily reduced to $12a - 9 > 4b$. Substituting $a = \dfrac{4\gamma v_f k_m - 4\phi k_m + 2\omega^2}{\omega^2}$ and $b = 3 + \dfrac{12\gamma v_f k_m}{\omega^2}$ into the above inequality and simplifying, we obtain:

$$\begin{aligned}
&\frac{1}{4}\left(12\left(2-\frac{4k_m\left(\phi-\gamma v_f\right)}{\omega^2}\right)-9\right)>\frac{12\gamma v_f k_m}{\omega^2}+3\\
\implies &\frac{12\gamma v_f k_m}{\omega^2}-\frac{12\phi k_m}{\omega^2}+\frac{15}{4}>\frac{12\gamma v_f k_m}{\omega^2}+3\\
\implies &\frac{12\phi k_m}{\omega^2}<\frac{3}{4}\\
\implies &\phi<\frac{\omega^2}{16k_m}
\end{aligned}$$

■

Lemma 1.3. *For an exogenous change in v_f, the social planner's optimal choice of v_m strictly decreases with v_f.*

Proof. From lemma 1.2, the solution for v_m for a given v_f is as follows.

$$v_m^S=\frac{\omega^2+4\phi k_m-4\gamma v_f k_m+\sqrt{12\omega^2k_m\left(\gamma v_f-2\phi\right)+\left(-4\gamma v_f k_m+4\phi k_m+\omega^2\right)^2}}{6\omega k_m} \tag{1.36}$$

Taking the derivative with respect to v_f, we obtain the following:

$$\frac{\partial v_m^S}{\partial v_f}=\frac{\frac{4\gamma k_m\left(\omega^2-8k_m\left(\phi-\gamma v_f\right)\right)}{2\sqrt{12\omega^2k_m\left(\gamma v_f-2\phi\right)+\left(-4\gamma v_f k_m+4\phi k_m+\omega^2\right)^2}}-4\gamma k_m}{6\omega k_m} \tag{1.37}$$

We will show that the above expression is strictly negative. We proceed by contradiction. Suppose $\frac{\partial v_m^S}{\partial v_f}\geq 0$ for some (v_m^S, v_f). Then,

$$\begin{aligned}
&\frac{\frac{4\gamma k_m\left(\omega^2-8k_m\left(\phi-\gamma v_f\right)\right)}{2\sqrt{12\omega^2k_m\left(\gamma v_f-2\phi\right)+\left(-4\gamma v_f k_m+4\phi k_m+\omega^2\right)^2}}-4\gamma k_m}{6\omega k_m}\geq 0\\
\implies &\frac{4\gamma k_m\left(\omega^2-8k_m\left(\phi-\gamma v_f\right)\right)}{2\sqrt{12\omega^2k_m\left(\gamma v_f-2\phi\right)+\left(-4\gamma v_f k_m+4\phi k_m+\omega^2\right)^2}}-4\gamma k_m\geq 0\\
\implies &\frac{4\gamma k_m\left(\omega^2-8k_m\left(\phi-\gamma v_f\right)\right)}{2\sqrt{12\omega^2k_m\left(\gamma v_f-2\phi\right)+\left(-4\gamma v_f k_m+4\phi k_m+\omega^2\right)^2}}\geq 4\gamma k_m
\end{aligned}$$

$$\implies 4\gamma k_m\left(\omega^2-8k_m\left(\phi-\gamma v_f\right)\right)\geq 8\gamma k_m\sqrt{12\omega^2k_m\left(\gamma v_f-2\phi\right)+\left(-4\gamma v_f k_m+4\phi k_m+\omega^2\right)^2}$$

$$\implies \left(4\gamma k_m\left(\omega^2-8k_m\left(\phi-\gamma v_f\right)\right)\right)^2\geq 64\gamma^2k_m^2\left(12\omega^2k_m\left(\gamma v_f-2\phi\right)+\left(-4\gamma v_f k_m+4\phi k_m+\omega^2\right)^2\right)$$

The left hand and right hand side of the above inequality can expanded to:

$$\begin{aligned}\text{LHS} = 1024\gamma^4 v_f^2 k_m^4 + 256\gamma^3\omega^2 v_f k_m^3 - 2048\gamma^3\phi v_f k_m^4 \\ + 16\gamma^2\omega^4 k_m^2 + 1024\gamma^2\phi^2 k_m^4 - 256\gamma^2\omega^2\phi k_m^3\end{aligned}$$

$$\begin{aligned}\text{RHS} = 1024\gamma^4 v_f^2 k_m^4 + 256\gamma^3\omega^2 v_f k_m^3 - 2048\gamma^3\phi v_f k_m^4 \\ + 64\gamma^2\omega^4 k_m^2 + 1024\gamma^2\phi^2 k_m^4 - 1024\gamma^2\omega^2\phi k_m^3\end{aligned}$$

Subtracting RHS from LHS we obtain:

$$\begin{aligned} & -48\gamma^2\omega^4 k_m^2 + 768\gamma^2\omega^2\phi k_m^3 \geq 0 \\ \implies & 768\gamma^2\omega^2\phi k_m^3 \geq 48\gamma^2\omega^4 k_m^2 \\ \implies & \phi \geq \frac{\omega^2}{16k_m}\end{aligned}$$

However, we know from lemma 1.2 that there is no feasible interior solution when $\phi \geq \frac{\omega^2}{16k_m}$. ■

Proof of Proposition 1.4. We analyse the marginal surplus of the planner with respect to the v_m and v_f, and compare it with the monopolist's marginal profit. For any exogenously given v_f, the monopolist's solution of v_m solves:

$$\frac{\partial \Pi_v[v_m, v_f]}{\partial v_m} = 0 \implies k_m v_m \left(4\gamma v_f + 3\omega v_m - 4\phi\right) - \omega\left(\gamma v_f + \omega v_m - 2\phi\right) = 0 \tag{1.38}$$

In comparison, the social planner's solution of v_m solves:

$$\frac{\partial TS_v[v_m, v_f]}{\partial v_m} = 0 \implies k_m v_m \left(4\gamma v_f + 3\omega v_m - 4\phi\right) - \omega\left(\gamma v_f + \omega v_m - 2\phi\right) = 0 \tag{1.39}$$

Therefore, for any given v_f, the solution of v_m for the social planner and the monopolist is identical. Additionally, the monopolist's marginal profit from v_f is given as follows.

$$\frac{\partial \Pi_v[v_m, v_f]}{\partial v_f} = \frac{\gamma}{4}\left(1 - \frac{\left(\phi - k_m (v_m)^2\right)^2}{\left(\gamma v_f + \omega v_m - \phi\right)^2}\right) - 2k_f v_f \tag{1.40}$$

Alternatively, the marginal surplus with respect to v_f is:

$$\begin{aligned}\frac{\partial TS_v[v_m, v_f]}{\partial v_f} &= \frac{\gamma}{2}\left(1 - \frac{(\phi - k_m v_m^2)^2}{(\gamma v_f + \omega v_m - \phi)^2}\right) - 2k_f v_f \\ &= \frac{\gamma}{4}\left(1 - \frac{(\phi - k_m v_m^2)^2}{(\gamma v_f + \omega v_m - \phi)^2}\right) + \frac{\gamma}{4}\left(1 - \frac{(\phi - k_m v_m^2)^2}{(\gamma v_f + \omega v_m - \phi)^2}\right) - 2k_f v_f \\ &= \frac{\gamma}{4}\left(1 - \frac{(\phi - k_m v_m^2)^2}{(\gamma v_f + \omega v_m - \phi)^2}\right) + \frac{\partial \Pi_v[v_m, v_f]}{\partial v_f} \qquad (1.41)\end{aligned}$$

We will show that the expression $\frac{\gamma}{4}\left(1 - \frac{(\phi - k_m v_m^2)^2}{(\gamma v_f + \omega v_m - \phi)^2}\right)$ is positive. First, we can substitute $k_m v_m^2 = p^S$. We need to show that $\dfrac{(p^S - \phi)^2}{(\gamma v_f + \omega v_m - \phi)^2} < 1$. This follows from our assumption of $v > p$ and the conditions for partial market coverage. For partial coverage, we require $0 < \bar{x}[v_m, v_f, p^S] < 1 \Rightarrow 0 < \frac{v - p^S}{v - \phi} < 1 \Rightarrow 0 < v - p^S < v - \phi \Rightarrow p^S - \phi > 0$ and $v^* - \phi > 0$. Additionally, $v > p^S \Rightarrow \gamma v_f + \omega v_m > p^S \Rightarrow \gamma v_f + \omega v_m - \phi > p^S - \phi \Rightarrow (\gamma v_f + \omega v_m - \phi)^2 > (p^S - \phi)^2$ (squaring is possible because $p^S - \phi > 0$ when the market is partially covered).

Therefore, for any given v_m, $\dfrac{\partial TS_v[v_m, v_f]}{\partial v_f} > \dfrac{\partial \Pi_v[v_m, v_f]}{\partial v_f}$. This implies that the marginal surplus from v_f is always higher than the marginal profit from v_f. Using the above, at the monopolist's optimal solution, $\dfrac{\partial TS_v[v_m^*, v_f^*]}{\partial v_m} = 0$ and $\dfrac{\partial TS_v[v_m^*, v_f^*]}{\partial v_f} > 0$. The planner can increase surplus by increasing v_f. From lemma 1.3, this implies a decrease in v_m. Additionally, because at v_f^* the solution for v_m is identical for both agents, increasing v_f at that point will imply a socially optimal level of v_m which is lower than v_m^*.

■

B Mathematica Code

Optimality Conditions

Profit Function:

In[∘]:= $$\Pi[vM\_, vF\_, p\_] := \left(p - k_m\, vM^2\right)\left(\frac{\omega\, vM + \gamma\, vF - p}{\omega\, vM + \gamma\, vF - \phi}\right) - k_f\, vF^2$$

Maximising with respect to price and reducing the profit function to two variables

In[∘]:= `Solve[D[Π[vM, vF, p], p] == 0, p] // Simplify`

Out[∘]= $$\left\{\left\{p \to \frac{1}{2}\left(vF\,\gamma + vM\,\omega + vM^2\, k_m\right)\right\}\right\}$$

In[∘]:= $$\Pi[vM, vF, p] \;/.\; p \to \frac{1}{2}\left(\gamma\, vF + vM\,(\omega + k_m\, vM)\right) \;//\; \text{FullSimplify}$$

Out[∘]= $$-vF^2\, k_f + \frac{\left(vF\,\gamma + vM\,\omega - vM^2\, k_m\right)^2}{4\,(vF\,\gamma - \phi + vM\,\omega)}$$

In[∘]:= $$\Pi_v[vM\_, vF\_] := -vF^2\, k_f + \frac{\left(vF\,\gamma + vM\,\omega - vM^2\, k_m\right)^2}{4\,(vF\,\gamma - \phi + vM\,\omega)}$$

Taking the derivative with respect to v_m and v_f:

In[∘]:= $$D[\Pi_v[vM, vF], vM] == 0 \;//\; \text{Simplify}$$

Out[∘]= $$\frac{\left(vF\,\gamma + vM\,\omega - vM^2\, k_m\right)\left(-\omega\,(vF\,\gamma - 2\,\phi + vM\,\omega) + vM\,(4\,vF\,\gamma - 4\,\phi + 3\,vM\,\omega)\,k_m\right)}{vF\,\gamma - \phi + vM\,\omega} == 0$$

In the above, $vF\,\gamma + vM\,\omega - vM^2\, k_m > 0$ as $vF\,\gamma + vM\,\omega > p$ and $p > vM^2\, k_m$, and $vF\,\gamma - \phi + vM\,\omega > 0$ to ensure positive coverage. Therefore,

Out[∘]= $$-\omega\,(vF\,\gamma - 2\,\phi + vM\,\omega) + vM\,(4\,vF\,\gamma - 4\,\phi + 3\,vM\,\omega)\,k_m == 0$$

In[∘]:= $$D[\Pi_v[vM, vF], vF] \;//\; \text{Simplify}$$

Out[∘]= $$-2\,vF\,k_f + \frac{\gamma\left(vF\,\gamma + vM\,\omega - vM^2\, k_m\right)\left(vF\,\gamma - 2\,\phi + vM\,\omega + vM^2\, k_m\right)}{4\,(vF\,\gamma - \phi + vM\,\omega)^2}$$

The Hessian:

In[∘]:= $$H = D[\Pi_v[vM, vF], \{\{vM, vF\}, 2\}] \;//\; \text{FullSimplify};$$

Determinant of The Hessian at $v_m{}^*$ and $v_f{}^*$:

In[∘]:= $$\text{Det}\left[H \;/.\; \left\{k_f \to \frac{\gamma\left(-k_m\, vM^2 + vF\,\gamma + vM\,\omega\right)\left(k_m\, vM^2 + vF\,\gamma - 2\,\phi + vM\,\omega\right)}{8\, vF\,(vF\,\gamma - \phi + vM\,\omega)^2}\right\} \;/.\; \left\{k_m \to \frac{\omega\,(vF\,\gamma - 2\,\phi + vM\,\omega)}{vM\,(4\,vF\,\gamma - 4\,\phi + 3\,vM\,\omega)}\right\}\right] \;//\; \text{FullSimplify}$$

Out[∘]= $$\left(\gamma\,\omega\,(2\,vF\,\gamma + vM\,\omega)\,(vF\,\gamma - 2\,\phi + vM\,\omega)\left(8\,vF^3\,\gamma^3 + 8\,vF^2\,\gamma^2\,(-3\,\phi + 2\,vM\,\omega) + vF\,vM\,\gamma\,\omega\,(-28\,\phi + 11\,vM\,\omega) + vM\,\omega\left(8\,\phi^2 - 12\,vM\,\phi\,\omega + 3\,vM^2\,\omega^2\right)\right)\right) \Big/ \left(vF\,vM\,(vF\,\gamma - \phi + vM\,\omega)\,(4\,vF\,\gamma - 4\,\phi + 3\,vM\,\omega)^4\right)$$

Computations for the Proofs of the Propositions

Proposition 1.1:

In[∘]:=
```
D[Π_v[vM, vF], ϕ] // Simplify
```

Out[∘]=
$$\frac{\left(\text{vF}\,\gamma+\text{vM}\,\omega-\text{vM}^2\,k_m\right)^2}{4\,(\text{vF}\,\gamma-\phi+\text{vM}\,\omega)^2}$$

Proposition 1.2:

In[∘]:=
```
FOCM = FullSimplify[D[Π_v[vM, vF], {{vM, vF}}]];
```

In[∘]:=
```
J = FullSimplify[D[FOCM, {{vM, vF}}]]ᵀ;
```

In[∘]:=
```
derivativeWithϕ = FullSimplify[-Inverse[J].D[FOCM, ϕ]]
```

Out[∘]=
$$\Big\{-\Big(\big(2\,(\text{vF}\,\gamma-\phi+\text{vM}\,\omega)\,k_f\,\left(\text{vF}\,\gamma+\text{vM}\,\omega-\text{vM}^2\,k_m\right)\,(\phi\,\omega+\text{vM}\,(2\,\text{vF}\,\gamma-2\,\phi+\text{vM}\,\omega)\,k_m)\big)\Big/$$
$$\Big(-\gamma^2\,k_m\,\left(\phi-\text{vM}^2\,k_m\right)^2\,\left(\text{vF}\,\gamma+\text{vM}\,\omega-\text{vM}^2\,k_m\right)+$$
$$2\,(\text{vF}\,\gamma-\phi+\text{vM}\,\omega)\,k_f\,\Big(-\phi^2\,\omega^2+k_m\,\Big(2\,\text{vF}\,\gamma\,(-\text{vF}\,\gamma+\phi)^2+6\,\text{vM}\,(-\text{vF}\,\gamma+\phi)^2\,\omega+6\,\text{vM}^2\,(\text{vF}\,\gamma-\phi)\,\omega^2+$$
$$2\,\text{vM}^3\,\omega^3-\text{vM}^2\,\left(6\,(-\text{vF}\,\gamma+\phi)^2+8\,\text{vM}\,(\text{vF}\,\gamma-\phi)\,\omega+3\,\text{vM}^2\,\omega^2\right)\,k_m\Big)\Big)\Big)\Big),$$
$$\Big(\gamma\,k_m\,\left(\text{vF}\,\gamma+\text{vM}\,\omega-\text{vM}^2\,k_m\right)^2\,\left(-\phi+\text{vM}^2\,k_m\right)\Big)\Big/\Big(-\gamma^2\,k_m\,\left(\phi-\text{vM}^2\,k_m\right)^2\,\left(\text{vF}\,\gamma+\text{vM}\,\omega-\text{vM}^2\,k_m\right)+$$
$$2\,(\text{vF}\,\gamma-\phi+\text{vM}\,\omega)\,k_f\,\Big(-\phi^2\,\omega^2+k_m\,\Big(2\,\text{vF}\,\gamma\,(-\text{vF}\,\gamma+\phi)^2+6\,\text{vM}\,(-\text{vF}\,\gamma+\phi)^2\,\omega+$$
$$6\,\text{vM}^2\,(\text{vF}\,\gamma-\phi)\,\omega^2+2\,\text{vM}^3\,\omega^3-\text{vM}^2\,\left(6\,(-\text{vF}\,\gamma+\phi)^2+8\,\text{vM}\,(\text{vF}\,\gamma-\phi)\,\omega+3\,\text{vM}^2\,\omega^2\right)\,k_m\Big)\Big)\Big)\Big\}$$

In[∘]:=
$$\text{derivativeAtMaximumWith}\phi=$$
$$\text{derivativeWith}\phi\ /.\ \left\{k_f\to\frac{\gamma\,\left(-k_m\,\text{vM}^2+\text{vF}\,\gamma+\text{vM}\,\omega\right)\,\left(k_m\,\text{vM}^2+\text{vF}\,\gamma-2\,\phi+\text{vM}\,\omega\right)}{8\,\text{vF}\,(\text{vF}\,\gamma-\phi+\text{vM}\,\omega)^2}\right\}\ /.$$
$$\left\{k_m\to\frac{\omega\,(\text{vF}\,\gamma-2\,\phi+\text{vM}\,\omega)}{\text{vM}\,(4\,\text{vF}\,\gamma-4\,\phi+3\,\text{vM}\,\omega)}\right\}\ //\ \text{FullSimplify}$$

Out[∘]=
$$\Big\{-\frac{2\,\text{vM}\,(2\,\text{vF}\,\gamma+\text{vM}\,\omega)^2}{8\,\text{vF}^3\,\gamma^3+8\,\text{vF}^2\,\gamma^2\,(-3\,\phi+2\,\text{vM}\,\omega)+\text{vF}\,\text{vM}\,\gamma\,\omega\,(-28\,\phi+11\,\text{vM}\,\omega)+\text{vM}\,\omega\,\left(8\,\phi^2-12\,\text{vM}\,\phi\,\omega+3\,\text{vM}^2\,\omega^2\right)},$$
$$\frac{2\,\text{vF}\,(2\,\text{vF}\,\gamma+\text{vM}\,\omega)\times(-4\,\phi+\text{vM}\,\omega)}{8\,\text{vF}^3\,\gamma^3+8\,\text{vF}^2\,\gamma^2\,(-3\,\phi+2\,\text{vM}\,\omega)+\text{vF}\,\text{vM}\,\gamma\,\omega\,(-28\,\phi+11\,\text{vM}\,\omega)+\text{vM}\,\omega\,\left(8\,\phi^2-12\,\text{vM}\,\phi\,\omega+3\,\text{vM}^2\,\omega^2\right)}\Big\}$$

Proposition 1.3:

In[∘]:=
$$x=\frac{\omega\,\text{vM}+\gamma\,\text{vF}-p}{\omega\,\text{vM}+\gamma\,\text{vF}-\phi}\ /.\ \left\{p\to\frac{1}{2}\,\left(\text{vF}\,\gamma+\text{vM}\,\omega+\text{vM}^2\,k_m\right)\right\}\ //\ \text{FullSimplify}$$

Out[∘]=
$$\frac{\text{vF}\,\gamma+\text{vM}\,\omega-\text{vM}^2\,k_m}{2\,\text{vF}\,\gamma-2\,\phi+2\,\text{vM}\,\omega}$$

Total Derivative of *x* with respect to ϕ:

In[∘]:=
```
TDOfX = D[x, ϕ] + D[x, vM] × derivativeAtMaximumWithϕ[[1]] +
    D[x, vF] × derivativeAtMaximumWithϕ[[2]];
```

In[∘]:=
$$\text{TDOfX}\ /.\ \left\{k_f\to\frac{\gamma\,\left(-k_m\,\text{vM}^2+\text{vF}\,\gamma+\text{vM}\,\omega\right)\,\left(k_m\,\text{vM}^2+\text{vF}\,\gamma-2\,\phi+\text{vM}\,\omega\right)}{8\,\text{vF}\,(\text{vF}\,\gamma-\phi+\text{vM}\,\omega)^2}\right\}\ /.$$
$$\left\{k_m\to\frac{\omega\,(\text{vF}\,\gamma-2\,\phi+\text{vM}\,\omega)}{\text{vM}\,(4\,\text{vF}\,\gamma-4\,\phi+3\,\text{vM}\,\omega)}\right\}\ //\ \text{FullSimplify}$$

Out[∘]=
$$\left(4\,(2\,\text{vF}\,\gamma+\text{vM}\,\omega)^2\,(\text{vF}\,\gamma-2\,\phi+\text{vM}\,\omega)\right)\Big/\Big((4\,\text{vF}\,\gamma-4\,\phi+3\,\text{vM}\,\omega)\times$$
$$\left(8\,\text{vF}^3\,\gamma^3+8\,\text{vF}^2\,\gamma^2\,(-3\,\phi+2\,\text{vM}\,\omega)+\text{vF}\,\text{vM}\,\gamma\,\omega\,(-28\,\phi+11\,\text{vM}\,\omega)+\text{vM}\,\omega\,\left(8\,\phi^2-12\,\text{vM}\,\phi\,\omega+3\,\text{vM}^2\,\omega^2\right)\right)\Big)$$

Welfare Analysis

In[∘]:= `CS = Integrate[`$\frac{\omega\,vM + \gamma\,vF - a}{\omega\,vM + \gamma\,vF - \phi}$`, {a, p, vF γ + vM ω}]`

Out[∘]= $\frac{(-p + vF\,\gamma + vM\,\omega)^2}{2\,(vF\,\gamma - \phi + vM\,\omega)}$

In[∘]:= `TS = CS + Π[vM, vF, p] // Simplify`

Out[∘]= $-vF^2\,k_f - \frac{(p - vF\,\gamma - vM\,\omega)\left(p + vF\,\gamma + vM\,\omega - 2\,vM^2\,k_m\right)}{2\,(vF\,\gamma - \phi + vM\,\omega)}$

Price equation

In[∘]:= `Solve[D[TS, p] == 0, p]`

Out[∘]= $\{\{p \to vM^2\,k_m\}\}$

In[∘]:= TS_v `= TS /. p →` $vM^2\,k_m$ `// FullSimplify`

Out[∘]= $-vF^2\,k_f + \frac{\left(vF\,\gamma + vM\,\omega - vM^2\,k_m\right)^2}{2\,(vF\,\gamma - \phi + vM\,\omega)}$

Lemma 1.2

In[∘]:= `Solve[(-ω (vF γ - 2 ϕ + vM ω) + vM (4 vF γ - 4 ϕ + 3 vM ω)` k_m`) == 0, vM] // FullSimplify`

Out[∘]= $\left\{\left\{vM \to \frac{\omega^2 + 4\,(-vF\,\gamma + \phi)\,k_m - \sqrt{\omega^4 + 4\,k_m\left((vF\,\gamma - 4\,\phi)\,\omega^2 + 4\,(-vF\,\gamma + \phi)^2\,k_m\right)}}{6\,\omega\,k_m}\right\},\right.$
$\left.\left\{vM \to \frac{\omega^2 + 4\,(-vF\,\gamma + \phi)\,k_m + \sqrt{\omega^4 + 4\,k_m\left((vF\,\gamma - 4\,\phi)\,\omega^2 + 4\,(-vF\,\gamma + \phi)^2\,k_m\right)}}{6\,\omega\,k_m}\right\}\right\}$

In[∘]:= `xW =`
$\frac{\omega\,vM + \gamma\,vF - p}{\omega\,vM + \gamma\,vF - \phi}$ `/.` $\{p \to vM^2\,k_m\}$ `/.`
$\left\{vM \to \frac{\omega^2 + 4\,(-vF\,\gamma + \phi)\,k_m + \sqrt{\omega^4 + 4\,k_m\left((vF\,\gamma - 4\,\phi)\,\omega^2 + 4\,(-vF\,\gamma + \phi)^2\,k_m\right)}}{6\,\omega\,k_m}\right\}$ `// FullSimplify`

Out[∘]= $\frac{4\,\omega^2 + 8\,(vF\,\gamma - \phi)\,k_m - 2\,\sqrt{\omega^4 + 4\,k_m\left((vF\,\gamma - 4\,\phi)\,\omega^2 + 4\,(-vF\,\gamma + \phi)^2\,k_m\right)}}{3\,\omega^2}$

Numerical Example (Table 1.1a)

Table 1.1 (a) Monopolist’s solution

```
In[•]:= {solMonopolyProfit, solMonopoly} =
         With[{ϕ = 1.5, kM = 3, kF = 3, ω = 10, γ = 10},
          Maximize[-kF vF^2 + ((p - kM vM^2) (-p + vF γ + vM ω))/(vF γ - ϕ + vM ω), {vM, vF, p}]]
Out[•]= {1.94939, {vM → 0.899871, vF → 0.41401, p → 7.78406}}

In[•]:= csMonopoly = With[{ϕ = 1.5, kM = 3, kF = 3, ω = 10, γ = 10}, (-p + vF γ + vM ω)^2/(2 (vF γ - ϕ + vM ω)) /. solMonopoly]
Out[•]= 1.2318

In[•]:= tsMonopoly = csMonopoly + solMonopolyProfit
Out[•]= 3.18119

In[•]:= xSolution = With[{ϕ = 1.5, kM = 3, kF = 3, ω = 10, γ = 10}, (ω vM + γ vF - p)/(ω vM + γ vF - ϕ) /. solMonopoly]
Out[•]= 0.460077
```

Table 1.1 (a) Social Planner’s solution

```
In[•]:= {solPlannerSurplus, solPlanner} =
         With[{ϕ = 1.5, kM = 3, kF = 3, ω = 10, γ = 10},
          Maximize[-vF^2 kF + (vF γ + vM ω - vM^2 kM)^2/(2 (vF γ - ϕ + vM ω)), {vM, vF}]]
Out[•]= {4.93185, {vM → 0.87635, vF → 0.831113}}

In[•]:= csSocial = With[{ϕ = 1.5, kM = 3, kF = 3, ω = 10, γ = 10},
          (-p + vF γ + vM ω)^2/(2 (vF γ - ϕ + vM ω)) /. p → kM vM^2 /. solPlanner]
Out[•]= 7.0041

In[•]:= profitSocial = With[{ϕ = 1.5, kM = 3, kF = 3, ω = 10, γ = 10},
          -kF vF^2 + ((p - kM vM^2) (-p + vF γ + vM ω))/(vF γ - ϕ + vM ω) /. p → kM vM^2 /. solPlanner]
Out[•]= -2.07225

In[•]:= xSocial = With[{ϕ = 1.5, kM = 3, kF = 3, ω = 10, γ = 10},
          (ω vM + γ vF - p)/(ω vM + γ vF - ϕ) /. p → kM vM^2 /. solPlanner]
Out[•]= 0.94838
```

Numerical Example (Table 1.1b)

Table 1.1 (b) Monopolist's solution

In[∘]:= {solMonopolyProfit, solMonopoly} =
With$\left[\{\phi = -1.5,\ kM = 3,\ kF = 3,\ \omega = 10,\ \gamma = 10\},\right.$
$\left.\text{Maximize}\left[-kF\, vF^2 + \frac{(p - kM\, vM^2)\,(-p + vF\,\gamma + vM\,\omega)}{vF\,\gamma - \phi + vM\,\omega},\ \{vM,\ vF,\ p\}\right]\right]$

Out[∘]= {1.48052, {vM → 1.09556, vF → 0.375409, p → 9.15523}}

In[∘]:= csMonopoly = With$\left[\{\phi = 1.5,\ kM = 3,\ kF = 3,\ \omega = 10,\ \gamma = 10\},\ \frac{(-p + vF\,\gamma + vM\,\omega)^2}{2\,(vF\,\gamma - \phi + vM\,\omega)}\ /.\ \text{solMonopoly}\right]$

Out[∘]= 1.16778

In[∘]:= tsMonopoly = csMonopoly + solMonopolyProfit

Out[∘]= 2.6483

In[∘]:= xSolution = With$\left[\{\phi = 1.5,\ kM = 3,\ kF = 3,\ \omega = 10,\ \gamma = 10\},\ \frac{\omega\, vM + \gamma\, vF - p}{\omega\, vM + \gamma\, vF - \phi}\ /.\ \text{solMonopoly}\right]$

Out[∘]= 0.420484

Table 1.1 (b) Social Planner's solution

In[∘]:= {solPlannerSurplus, solPlanner} =
With$\left[\{\phi = -1.5,\ kM = 3,\ kF = 3,\ \omega = 10,\ \gamma = 10\},\right.$
$\left.\text{Maximize}\left[-vF^2\, kF + \frac{(vF\,\gamma + vM\,\omega - vM^2\, kM)^2}{2\,(vF\,\gamma - \phi + vM\,\omega)},\ \{vM,\ vF\}\right]\right]$

Out[∘]= {3.85159, {vM → 1.03218, vF → 0.785884}}

In[∘]:= csSocial = With$\left[\{\phi = -1.5,\ kM = 3,\ kF = 3,\ \omega = 10,\ \gamma = 10\},\right.$
$\left.\frac{(-p + vF\,\gamma + vM\,\omega)^2}{2\,(vF\,\gamma - \phi + vM\,\omega)}\ /.\ p \to kM\, vM^2\ /.\ \text{solPlanner}\right]$

Out[∘]= 5.70443

In[∘]:= profitSocial = With$\left[\{\phi = -1.5,\ kM = 3,\ kF = 3,\ \omega = 10,\ \gamma = 10\},\right.$
$\left.-kF\, vF^2 + \frac{(p - kM\, vM^2)\,(-p + vF\,\gamma + vM\,\omega)}{vF\,\gamma - \phi + vM\,\omega}\ /.\ p \to kM\, vM^2\ /.\ \text{solPlanner}\right]$

Out[∘]= -1.85284

In[∘]:= xSocial = With$\left[\{\phi = -1.5,\ kM = 3,\ kF = 3,\ \omega = 10,\ \gamma = 10\},\right.$
$\left.\frac{\omega\, vM + \gamma\, vF - p}{\omega\, vM + \gamma\, vF - \phi}\ /.\ p \to kM\, vM^2\ /.\ \text{solPlanner}\right]$

Out[∘]= 0.76138

Chapter 2

Optimal Design of Ratings History

2.1 Introduction

Feedback from past consumers aid in future consumers making more informed choices when purchasing experience goods. In third-party reputation platforms, such as Google Maps and Yelp, feedback is often aggregated into a standard rating, usually comprising the average score and the number of reviews. To make these ratings more informative, significant effort has been given to identify and reduce the amount of non-genuine or "fake" reviews. However, even genuine reviews may carry misinformation, particularly if the businesses make periodic quality decisions that are not observed by consumers. An obvious example of this is when the business has new management (or, in the case of the restaurant industry, a new chef), and consumers are not aware of this change. The new management can be expected to make quality investments to maximise expected short-term profits (anticipating another new management in the future). Clearly, if ratings from the past are aggregated into the current rating, then consumers make incorrect inferences from the observed rating. As of writing, two of the popular reputation platforms, Google and Yelp, do *not* have a policy of deleting past reviews, no matter how old they are.

This paper aims to examine whether the third-party platforms should delete reviews upon new management or keep all or some of the past reviews.[1] In this regard, a

[1]Assuming, of course, that the platform knows when the management is changed. It can be reasonable when the entrepreneur has a different identity and therefore has to register the new details with the platform. In other cases, it can be made a rule to notify management changes to the platform. Whether the businesses do it or not and how to incentivise them to report truthfully would constitute a separate problem in mechanism design.

dynamic model in discrete time (influenced by Mailath and Samuelson (2001)) is formulated where there is a single business controlled by an exogenous sequence of identical[2] sellers. Each seller owns the business for k periods, after which the tenancy is terminated, and the ownership is transferred to a new identical seller. During each tenancy, each seller invests in the *average* quality of a single product (the true quality or experience in each transaction being the realisation of a continuous random variable). The seller selects this average quality and commits to it at the beginning of their tenancy.

Every period, a fixed number of short-lived buyers enter the market and buy one unit of the product from the firm. They have identical prior beliefs about the quality, and before purchase they can consult an aggregated rating of past experiences from other past buyers on a third-party platform. After consulting the rating, they update their beliefs of the average quality using a simple weighted average of the prior mean and the average rating, where the weight placed on the rating is a general increasing function of the number of reviews aggregated.

Following related literature such as Mailath and Samuelson (2001) and Holmström (1999), the buyers are assumed to be willing to pay their expectation of the interim belief after consulting the ratings. In the present formulation, buyers not only do not observe the changes but also are naive regarding the possibility of management changes.[3] Specifically, they are assumed to make beliefs about the brand rather than the existing management, which implies that from the buyer's point of view, the quality is fixed.[4] After purchase, an exogenous proportion of buyers leave a review every period, which is aggregated into the next period's rating.

The platform chooses a policy of how many of the most recent past periods' reviews to consider upon the change of management. Specifically, as the process progresses, some amount of previous reviews from the recent τ periods are kept, with the rest of the earlier reviews deleted upon the change in ownership.[5] If $\tau = \infty$, reviews are never deleted, and all historical feedback is aggregated. If τ is finite, all earlier reviews, except those posted in the most recent τ periods, are deleted upon each

[2]In order to focus on quality choices, only one type of seller is considered.

[3]This is a departure from Mailath and Samuelson (2001), where buyers cannot observe the management changes, but know that these changes are possible.

[4]In relation to the literature on information asymmetry, this constitutes a dynamic hidden action problem (seller's choice of quality), but from the buyer's point of view, the problem is misidentified as a "hidden type".

[5]A more intuitive policy may be to keep a certain fixed length of recent reviews at all times. In that case, all the theoretical results still hold, implying that the two policies are analytically equivalent. This is discussed in Section 2.6.1.

new seller (provided enough time has passed). The sellers then take the remaining reviews and the inherited rating as given and choose the average quality to maximise the expected profits for k periods, also accounting for the evolution of the rating during their tenancy due to their quality choice. Welfare is defined as the steady-state level of total surplus in the long run.

The first main result (Theorem 2.1) says that the welfare-maximising level of τ is finite, and any finite level of τ yields at least as much surplus as the case where reviews are never deleted. This is primarily driven by the fact that if reviews are never deleted, then eventually, the sellers have little incentive to provide any quality due to the shrinking marginal effect their effort will have on the aggregated rating.

The second main result (Theorem 2.2) demonstrates the significant difference in the optimal policy of the platform if the objective is to maximise consumer surplus instead of total surplus. In this case, if the prior mean is sufficiently high, the optimal length of history is higher than the welfare-maximising length. The intuition behind this result is that a sufficiently high prior mean would result in the quality provision being "lower than expected", and a higher number of reviews reduce the level of expectations prior to purchase, which would result in a lower price. This will result in the reduction of the magnitude of a negative consumer surplus. As a reasonable special case, if the weight placed on ratings goes to 1 as the number of aggregated reviews goes to infinity, then a high prior mean would imply that reviews should never be deleted (the optimal length is infinity). The intuition behind this is that never deleting reviews would lead to the sellers providing zero quality in the long run. Consumers would eventually catch up with this as they place full weight on ratings if the number of reviews is high enough. Although this creates no positive surplus for the consumer, it is better than consistently experiencing quality below their expectations.

There are a few secondary results on top of the aforementioned main results. Interestingly, additional feedback, even if being informative, may result in a lower average consumer surplus. This is the case when quality is higher than the buyer's beliefs. More information about a better-than-expected product may increase prices, reducing consumer surplus. Additionally, Bayesian inference for some conjugate prior distributions is a special case of the general weighted updating assumed as the learning process in this paper. This includes the case where the prior distribution and quality realisation are normally distributed. In this scenario, the optimal policy follows an "all-or-nothing" strategy. To maximise the total surplus, all past reviews should be deleted upon each new seller. If the objective is to maximise consumer surplus, then reviews should never be deleted as long as the prior mean is sufficiently high. Otherwise, all past reviews should be deleted.

The present formulation is a clear departure from the repeated game setting with imperfect monitoring that is utilised in related papers (examples include, among others, Mailath and Samuelson (2001), Cripps et al. (2004), and Ekmekci (2011)). This serves two purposes. First, instead of characterising particular equilibria from many possible ones (as standard in the repeated games literature due to the Folk theorem), it characterises one deterministic steady-state outcome of quality choice that the platform considers in its objective function. Second, buyers are assumed to be strategic in the repeated game setting. In this formulation, that may not be the case, and the buyers are assumed to behave *mechanically* with respect to the observed ratings. This may be more reasonable in real-life settings and allows us to consider markets where the buyers may not behave as rationally as assumed by standard theory.

The structure of the paper is as follows. First, the related literature is discussed in brief. This is followed by formulating the model premise. Next, the platform's policy and the two main results are discussed. This is followed by the comparative statics and the analysis of some special cases of the general weighted updating mentioned in the previous paragraph. Finally, the paper ends with a discussion of the model assumptions and the conclusion.

Related Literature

Reputation[6]. With regards to hidden action (in this paper's context, the choice of quality), one of the earliest and seminal works on reputation formation was conducted by Klein and Leffler (1981). Using a repeated game setting, they showed that firms in a competitive market would only be incentivised to provide high quality if there is a price premium above costs for doing so. They explained the dissipation of profits through activities such as investments in advertising or buying the goodwill of an existing name. Shapiro (1983) formulated a more elegant model considering heterogeneous consumers and showed that this reputation could be achieved through selling below costs in the initial periods. There has been a high volume of extensions

[6]The literature on reputation is quite broad, and a full overview is beyond the scope of this paper. Please see the survey by Bar-Isaac and Tadelis (2008) for a general overview of the area.

to these papers using the repeated game setting, and a comprehensive review can be found in Mailath and Samuelson (2006).

Incorporating both moral hazard and adverse selection, Holmström (1999) formulated a dynamic model where the sellers (in his context, workers) do not know their own type and found that career concerns do incentivise higher effort initially, but this diminishes over time. This is due to the lower impact future efforts have on beliefs because of the higher relative weight placed on past outcomes. Although the context and framework are quite different from the present paper,[7] the intuition behind this result still holds in this chapter for the case where reviews are never deleted (Proposition 2.1).

With regards to frequent management changes which are not observed by consumers, the motivation of this paper is aligned with Mailath and Samuelson (2001). In that paper[8], the authors formulated a dynamic repeated game model with two types of firms, *competent* and *inept*, who periodically replace the control of an existing brand name. In line with this paper, the replacements are not observed by consumers. Restricting attention to only Markov Perfect Equilibria, the authors characterised conditions under which the competent firms will provide high effort. More generally, for games with imperfect monitoring, Cripps et al. (2004) showed that without the uncertainty of types, it is impossible to maintain a permanent reputation for a strategy that does not constitute an equilibrium of the stage game. In games with perfect monitoring, Liu and Skrzypacz (2014) showed that limited past records could result in short-run players trusting the opportunistic player even after knowing the type.

Design of Reputation Systems. One of the core objectives of this paper is to analyse the optimal number of past reviews to aggregate into the current rating, thus contributing to the literature on the *design* of reputation systems. In a repeated game framework, Dellarocas (2005) studied the impact of various reputation systems on

[7]For instance, Holmström solves for rational expectations and assumes a single worker, while this paper assumes a sequence of unrelated sellers and considers the possibility of incorrect beliefs by buyers.

[8]In contrast to our objectives, Mailath and Samuelson have also conducted an analysis of the market for reputations, assuming that, upon the exit of an existing firm, new firms compete for the control of the name. They found that competent firms are more likely to purchase average reputations and build a good name. In contrast, inept firms are more likely to purchase low reputations and keep them low or purchase high reputations and deplete them. This is similar to a related paper by Tadelis (1999), where the author showed, using an overlapping generations model, that it is impossible to have an equilibrium where only good type sellers buy good existing names. To ensure analytical tractability, we assume that the replacements are exogenous, and there is no difference in types among the incoming and existing sellers.

market efficiency. He found that the attainable efficiency in an eBay like reputation system (where the last N negative reviews are summarised) is independent of N. Also utilising a repeated game setting but with uncertain types (similar to Mailath and Samuelson (2001)), Ekmekci (2011) formulated a particular finite rating system where information is censored (buyers do not get to observe the entire history). He showed that this system could result in an equilibrium where the seller provides high effort irrespective of history. Quite recently, Hörner and Lambert (2021) expanded on Holmström's (1999) model and found that *linear*[9] ratings aid in maximising the effort over time.

Several papers approach the ratings design issue with objectives and formulations different from this paper. For instance, utilising a dynamic setting with entry and exit, Vellodi (2018) analysed how fully transparent ratings systems can increase the barriers of entry for new firms. Kovbasyuk and Spagnolo (2018) formulates a dynamic model with the types of sellers changing stochastically over time and shows that infinite records will necessarily lead to a breakdown of trade. Similar to this paper (Theorem 2.1), they find that a finite history will maximise welfare in the long run. With close motivations, Shi et al. (2020) formulates a model with discrete quality choices and finds that full information may hinder investments in quality. Finally, in an adverse selection framework but with a static setting, Hopenhayn and Saeedi (2019) analyses the optimal distance between a given number of categories in a ratings system, as well as the cardinality of the categories.

2.2 The Model

2.2.1 Agents

The model is formulated in a dynamic setting with discrete and infinite time horizon, indexed by time $t \in \mathbb{N}_0$. The agents in the model are defined as follows. There is one firm having the capacity to sell a fixed number $x \in \mathbb{N}$ of goods every period, but this firm is owned by different sellers changing over time. Each period, a fixed number $m \in \mathbb{N}$ of identical, short-lived and risk-neutral buyers enter the market and decide whether or not to buy a single unit of a good from the firms. For analytical convenience, it is assumed that the total number of buyers exceeds the

[9] In their framework, linearity is in terms of both time (past observations) and the kind of information, i.e. a weighted average of ability and effort.

total available supply in the market,[10] i.e. $m \geq x$. After purchase, an exogenous number $x_r \leq x$ of buyers post a genuine review of their experience on a third-party reputation platform. This platform aggregates these reviews into a rating, which future buyers can consult at zero cost.

The Firm and the Sellers

The only asset owned by the firm is its "name". The rights to sell under this name are frequently transferred to different sellers (with identical technology), who are risk-neutral and operate the firm for $k \geq 2$ periods each. Starting at $t = 0$, the first seller operates the firm for k periods, after which the tenancy is terminated, and a new but identical seller operates for the further k periods, and so on. For every transaction, the actual quality follows a continuous random variable $\hat{Q}_t$ with mean $q_t \geq 0$, which can be termed at the *average quality*. Each new seller chooses this average quality, q_t, for the fixed number of k periods. Each seller knows their own quality, but this is not known by buyers before purchase.

The Platform

The platform can observe the change of sellers and decide how much of the past reviews to aggregate into the firm's rating upon each new seller. For instance, whenever there is a new seller, the platform can decide to keep only a finite number of the most recent reviews and delete the rest, or include all historically generated reviews in the rating aggregation. The platform's policy is defined as follows. As $t \to \infty$, the platform aggregates reviews from the last $\tau \in \mathbb{N}_0 \cup \{\infty\}$ periods in the visible rating. If $\tau = \infty$, then reviews are never deleted. If τ is finite, then earlier reviews, except for those posted in the most recent τ periods, are eventually deleted whenever there is a new seller. In other words, when τ is finite, for any $t \leq \tau$, all reviews are aggregated, while for $t > \tau$, only reviews in the most recent τ periods are aggregated.

Notably, the platform only intervenes and deletes earlier reviews at $t \in \{0, k, 2k, 3k, ...\}$, i.e. whenever there is a new seller. This implies that all reviews generated within that seller's tenancy will be aggregated. An alternative policy would be to only keep a fixed finite number of periods (or equivalently, reviews) at all times. That policy

[10] This ensures that the demand every period is constant at x.

is analytically equivalent to the results in this chapter and is discussed in Section 2.6.1.

Buyers and Reputation Dynamics

The buyers cannot observe the history of sellers. The only information visible to the buyers is the firm's rating on the platform, aggregated by past reviews posted by previous buyers. Additionally, they are assumed to be *naive*, such that they do not consider that the product ratings can include consumer experiences from different sellers. From the buyer's point of view, beliefs are made about the brand rather than the sellers in control over the brand. In other words, buyers form their beliefs about the product as if one single long-lived seller owned the firm. The utility of the product to the buyers is denoted by the following (while the utility from the outside option is zero):

$$U_t = \hat{q}_t - p_t \tag{2.1}$$

where $\hat{q}_t$ is a realisation of the random variable $\hat{Q}_t$. The buyers enter the market with an identical prior belief of q_t: a continuous random variable with mean $\mu \geq 0$. Upon entering the market, the buyer consults the rating in the platform, which comprise a mean score, y_t, and the number of aggregated reviews, n_t. Consider a mapping $\omega : \mathbb{N}_0 \to [0, 1]$, which is the weight placed by buyers upon entering the market on the mean score, y_t. The expectation of the interim belief distribution (can be deemed as the firm's *reputation* at time t), denoted $\tilde{q}_t$, is assumed to be a weighted average of the prior mean and the mean score:

$$\mathbb{E}\left[\tilde{q}_t | (y_t, n_t)\right] = \left(1 - \omega[n_t]\right) \mu + \omega[n_t] y_t$$

In essence, every period the consumers arrive with prior expectations μ and observe n_t and y_t. The function $\omega[n]$ determines how much weight the consumers put on the observed rating. In the extreme case, if $\omega[n] = 1$ for some n, consumers discard their prior completely and trust only the observed rating. On the other hand, if $\omega[n] = 0$ for some $n > 0$, consumers only trust their prior and discard the observed rating. Every period, a fixed number $x_r \leq x$ of buyers post a review on the platform after purchase. This review is a real number, being the true quality realisation for those buyers. In the next period, a new set of buyers enter the market with the same prior beliefs but now consulting a (potentially) different product rating on the platform. Additionally, buyers are assumed to buy the product simultaneously within each period, and therefore the possible sequential updating of the product rating within a particular period is ignored. In essence, for simplicity, it is considered that the rating is updated after every period t rather than after every review entered.[11] It is assumed that buyers are willing to pay their expectation of the quality.

The following three assumptions are made in respect to $\omega[n]$.

Assumption 1. $\omega[0] = 0$.

Assumption 2. $\omega[n]$ *is a non-decreasing function of* n.

Assumption 3. *There exists* n' *such that for all* $n \geq n'$, $\omega[n] > 0$

Assumption 1 says that when there are no ratings, then buyers only put weight on their prior. Assumption 2 implies that the weight placed by consumers on the rating increases as the number of reviews by previous consumers increases. This is quite intuitive as consumers would be more inclined to trust the rating if they observe a higher number of other consumers regarding it in a certain way. Assumption 3 is only to rule out the trivial case where $\omega[n] = 0 \ \forall n$, where consumers place no weight on the rating. If there exists no n' where consumers place some weight on the rating, then the ratings are of no use at all.

Evolution of Ratings

Finally, before solving for the seller's quality decisions, the evolution of the ratings (n_t, y_t) has to be discussed. The evolution of n_t is dependent on the policy τ chosen by the platform. Considering $n_0 = 0$, for $t \geq 1$:

$$n_t = \begin{cases} \tau, & t \in \{k, 2k, 3k, ...\},\ \tau < \infty,\ \text{and } t \geq \tau \\ n_{t-1} + x_r, & \text{otherwise} \end{cases} \tag{2.2}$$

The aggregated rating y_t is a simple average of past reviews and the new reviews generated during the tenancy of the current seller at time t:

$$y_t = \begin{cases} \dfrac{\sum_{j=t-\tau}^{t-1} \sum_{b=1}^{x_r} \hat{q}_\eta^{b_j}}{\tau}, & t \in \{k, 2k, 3k, ...\},\ \tau < \infty,\ \text{and } t \geq \tau \\ \dfrac{n_{t-1} y_{t-1} + \sum_{b=1}^{x_r} \hat{q}_\eta^{b_{t-1}}}{n_t}, & \text{otherwise} \end{cases} \tag{2.3}$$

[11] The sequential updating can be also be incorporated by setting $x_r = 1$, in which case x can be interpreted as the "number of products the firm needs to sell to generate one review", and each period is interpreted as "the amount of time required to sell x number of products".

where $\hat{q}_{\eta}^{b_t}$ is the realisation of quality for consumer b_t with seller η at time t. The first case demonstrates the platform's intervention whenever there is a new seller: only the reviews generated in the last τ periods (as long as t is larger than τ) are aggregated upon a seller change. If the denominator is zero (for instance if $\tau = 0$ and $t = k$), then y_t can be allowed to take any arbitrary number as the rating can be thought to be non-existent when there are zero reviews aggregated. This has no impact on consumer beliefs as buyers will just base their decisions on their prior, and put zero weight on the rating.

2.2.2 Quality Outcomes

The firm is owned by different sellers which change exogenously over time, unknown to the buyers. Let $H = \{0, 1, 2, ...\}$ be the sequence of sellers. Each new seller $\eta \in H$ operates the firm during the interval $t \in \{\eta k, ..., \eta k + k - 1\}$ and chooses q_η (the average quality) to maximise the expected profits for k periods.

$$\mathbb{E}[\Pi_\eta] = \mathbb{E}\left[\sum_{i=0}^{k-1}\left(p_t x - c q_\eta^2\right)\right] \tag{2.4}$$

where:

- $c q_\eta^2$ ($c > 0$) is the cost of quality, which is the same in all periods[12] and identical across all sellers,
- x is the market size (the number of buyers) each period, and
- p_t is the market-clearing price.

Let $\tilde{q}_t$ be the buyer's interim belief about the quality after consulting the rating and updating their prior. This interim belief can be incorrect because, from the buyer's point of view, the firm essentially only consists of a single long-lived seller. The price p_t is equal to the buyer's expectation of $\tilde{q}_t$ after consulting the rating. Formally,

$$p_t = \mathbb{E}\left[\tilde{q}_t | y_t\right] = (1 - \omega[n_t])\, \mu + \omega[n_t] y_t \tag{2.5}$$

Next, the seller's expectation of the evolution of the observed rating y_t will be discussed. Seller η is fully informed about the buyer's information and belief updating,

[12] In this framework, it can be seen as a contractual obligation made at the beginning which keeps the cost constant during the entire period of tenancy for the seller. An equivalent analytical framework would be to consider the cost as an incurred fixed cost by the new seller.

and therefore takes the price in 2.5 as given. Furthermore, as per the platform's policy, the reviews from recent τ_η periods are aggregated, where:

$$\tau_\eta = \min\{\underbrace{\eta k}_{=t}, \tau\}$$

Assuming the mean ratings of previous sellers is $\bar{s}$, the expected evolution of ratings for any q_η after the seller enters is computed as follows.

$$\mathbb{E}[y_t] = \frac{\bar{s} \cdot x_r \cdot \tau_\eta + \mathbb{E}[\hat{q}_\eta] \cdot x_r(i - \tau_\eta)}{x_r \cdot i} = \frac{\bar{s} \cdot \tau_\eta + q_\eta(i - \tau_\eta)}{i} \tag{2.6}$$

where $i \in \{\tau_\eta + 1, ..., \tau_\eta + k\}$ and $\hat{q}_\eta$ is the actual quality outcome chosen by the seller but unknown to the buyers. Incorporating equations 2.5 and 2.6 into the profit function, we obtain:

$$\mathbb{E}[\Pi_\eta] = \sum_{i=\tau_\eta}^{k+\tau_\eta-1} \left(\left((1 - \omega[x_r \cdot i])\, \mu + \omega[x_r \cdot i] \cdot \frac{\bar{s} \cdot \tau_\eta + q_\eta(i - \tau_\eta)}{i} \right) x - c q_\eta^2 \right) \tag{2.7}$$

The optimal choice of quality is obtained as follows.

$$q_\eta^*[\tau_\eta] = \sum_{i=1}^{k-1} \omega[x_r \cdot (i + \tau_\eta)] \left(\frac{i}{i + \tau_\eta} \right) \cdot \frac{x}{2c \cdot k} \tag{2.8}$$

It is noteworthy to point out that the optimal choice of quality is independent of the average rating generated by previous sellers, $\bar{s}$. Mathematically, this can be explained by observing that $\bar{s}$ only enters additively without any interactions with q_η in the profit function in 2.7. This results in $\bar{s}$ monotonically transforming the profit function and having no impact on the optimal quality. Fundamentally, this is enabled by our assumption that the quantity sold every period is fixed. If demand depends on the rating, the current choice of quality will also depend on the inherited rating. This is discussed in more detail in Section 2.6.2.

Given the quality outcomes for each seller in $H = \{0, 1, 2, ...\}$, the long-run average quality for a given τ is computed as follows.

$$\bar{q}[\tau] := \lim_{\eta \to \infty} q_\eta^*[\tau_\eta] \tag{2.9}$$

As η goes to infinity, if $\tau < \infty$, then in the steady-state only the last τ period's reviews are included in the rating. This holds for every seller $\eta > \tau k$, and the quality will be constant for each (from equation 2.8). If $\tau = \infty$, the average quality still has a limit, as shown by the next proposition.

Proposition 2.1. *The function $\bar{q}(\tau)$ is derived as follows.*

$$\bar{q}[\tau] = \begin{cases} \sum_{i=1}^{k-1} \omega[x_r \cdot (i+\tau)] \left(\frac{i}{i+\tau}\right) \cdot \frac{x}{2c \cdot k}, & 0 \leq \tau < \infty \\ 0, & \tau = \infty \end{cases} \tag{2.10}$$

In the long run, the number of aggregated reviews, n_t, for the life-cycle of a new seller η is equivalent to $x_r \cdot (i+\tau)$ for $i \in \{0, ..., k-1\}$ and $t = \eta k + i$. Additionally, the expected aggregated rating for that seller is computed as $\mathbb{E}[y_t] = \frac{\bar{s} \cdot \tau + q_\eta \cdot i}{i+\tau}$. Proposition 2.1 basically says that when reviews are never deleted ($\tau = \infty$), the average quality in the long run will converge to zero. This is due to two effects:

(i) Let $\triangle\omega[n_t] = \omega[n_t] - \omega[n_{t-1}]$, which is intuitively the "marginal weight" placed by consumers on an additional review. If the number of aggregated reviews is always increasing (i.e. $n_t \to \infty$), then $\triangle\omega[n_t]$ converges to zero eventually. This, in turn, eliminates the seller's marginal return of quality due to the zero weight consumers place on any increased quality.

(ii) An increase in τ will mean that the seller's contribution to the aggregated rating is reduced. This is because increasing τ will aggregate relatively more of the previous sellers' rating realisations and less of the current seller's quality. This reduces the incentive for the seller to provide quality. As $\tau \to \infty$, sellers' contribution to the aggregated rating reduces to zero, eliminating the incentive to provide quality.

2.3 Welfare Analysis

A simple notion of welfare as the summation of average consumer surplus and profits will be considered. The ambition is to analyse the optimal value of τ the platform should choose to maximise this measure.

Definition 1. *The expected consumer surplus and profits in the long run are defined as follows.*

$$CS[\tau] := \mathbb{E}\left[\frac{1}{k}\sum_{i=0}^{k-1}\Big(\bar{q}[\tau] - p_i[\tau]\Big) \cdot x\right] \tag{2.11}$$

$$\Pi[\tau] := \mathbb{E}\left[\frac{1}{k}\sum_{i=0}^{k-1}\left(p_i[\tau]\cdot x - c\bar{q}[\tau]^2\right)\right] \tag{2.12}$$

Both CS and Π are mappings from $\mathbb{N}_0 \cup \{\infty\}$ to $\mathbb{R}$.

The above formulation considers the average aggregate consumer surplus generated by sellers over the k periods they operate (similarly for profits). It should be noted that the consumer surplus in this case is typically different to zero due to the buyers' likelihood of making incorrect expectations. When the quality level is lower than their expectations, this can also result in the surplus being negative, as the price they will be willing to pay is higher than the provision of quality. The total surplus is computed using a simple summation of these two values.

$$\begin{aligned} W[\tau] &:= CS[\tau] + \Pi[\tau] \\ &= \frac{1}{k}\sum_{i=0}^{k-1}\left(\bar{q}[\tau]\cdot x - c\cdot\bar{q}[\tau]^2\right) \end{aligned} \tag{2.13}$$

From equation 2.13, it can be seen that the total surplus only depends on the level of quality and not the payments $p_i[\tau]\cdot x$. This results in the following useful lemma.

Lemma 2.1. *The set of maximisers of $W[\tau]$ is equivalent to the set of maximisers of $\bar{q}[\tau]$. Formally,*

$$\underset{\tau}{\operatorname{argmax}}\{W[\tau]\} \equiv \underset{\tau}{\operatorname{argmax}}\{\bar{q}[\tau]\}$$

The proof is provided from the appendix.[13] Lemma 2.1 basically says that in order to maximise the expected long-run welfare, it is sufficient to maximise the average quality. This significantly simplifies the problem mathematically, allowing us to easily derive the following theorem.

Theorem 2.1. *The average long run surplus for any finite τ is (weakly) greater than that for $\tau = \infty$. Moreover, there exists a finite τ which generates strictly higher surplus than that for $\tau = \infty$. Formally,*

$$W[\tau] \geq W[\infty] \quad \forall \tau < \infty \qquad \textit{and} \qquad \exists \tau' < \infty : W[\tau'] > W[\infty]$$

Corollary (Theorem 2.1)**.** *Any maximiser of $W[\tau]$ is finite. Formally,*

$$\tau^* \in \underset{\tau}{\operatorname{argmax}}\{W[\tau]\} \implies \tau^* < \infty$$

[13]From equation 2.13 it may appear that the welfare is maximised when $\bar{q}[\tau] = \frac{x}{2c}$, but the proof essentially shows that $\bar{q}[\tau] < \frac{x}{2c}$ for any τ.

Theorem 2.1 yields a very simple but strong result: with any policy of keeping only a finite number of historical reviews upon each seller, it is possible to achieve a level of surplus which is at least as good as that of never deleting reviews. Additionally, there exists some τ which yields strictly higher surplus than that from never deleting reviews, implying that the latter policy is not welfare-maximising. The corollary which follows says exact this. The intuition for theorem 2.1 follows closely to the intuition of proposition 2.1: if reviews are never deleted, then, in the long run, sellers have no incentive to provide quality, thereby generating no surplus.

Maximising Consumer Surplus

The above analysis assumed that the platform maximises total surplus, with equal weight on both the sellers and the consumers. If, instead, the objective is to maximise only the consumer surplus, then the optimal choice of τ can be different, as shown in the following theorem.

Theorem 2.2. *Suppose the largest maximisers of $W[\tau]$ and $CS[\tau]$ are defined as follows.*

$$\tau^* := \sup_{\tau}\{\operatorname{argmax}_{\tau}\{W[\tau]\}\} \quad \text{and} \quad \tau^C := \sup_{\tau}\{\operatorname{argmax}_{\tau}\{CS[\tau]\}\}$$

Then, the following holds:

- *If $\bar{q}[\tau^*] > \mu$, then $\tau^C \leq \tau^*$;*
- *if $\bar{q}[\tau^*] \leq \mu$, then $\tau^C \geq \tau^*$;*
- *if $\bar{q}[\tau^*] \leq \mu$ and $\lim_{n\to\infty} \omega[n] = 1$, then $\tau^C = \infty$.*

If the maximum achievable quality is higher than the prior mean,[14] then increasing the number of aggregated reviews will decrease consumer surplus due to the following effects:

1. If the number of aggregated reviews increases, consumers will have more information about the high quality. This will result in a higher price, reducing consumer surplus. In other words, consumers' ignorance about a "better-than-expected" quality product makes them better off.

[14]It should be noted that the prior belief is each buyer's belief *before* they observe the rating.

2. Any $\tau > \tau^*$ will result in a lower quality, as τ^* is the largest τ which maximises $\bar{q}[\tau]$ which, ceteris paribus, yields a lower consumer surplus than that with τ^*.

On the other hand, if the maximum quality that results from τ^* is lower than the mean prior belief of the consumer, the consumer surplus is going to be negative for the full lifetime of the seller. As the resulting quality is the maximum (lemma 2.1), no other τ can result in reducing the gap between expectations and quality from the aspect of quality outcome. However, this can be alleviated by increasing the number of reviews, resulting in consumers putting more weight on the low expected ratings generated by the relatively lower quality. Consumers adjust their expectations, and therefore price, accordingly. This results in a higher τ^C.

The intuition behind the third bullet point in theorem 2.2 is as follows. From proposition 2.1, $\bar{q}[\infty] = 0$. If $\omega[n] \to 1$ as $n \to \infty$, consumers sooner or later catch up with the zero quality and update their expectations accordingly, resulting in zero price in the long run. Zero consumer surplus for the consumers would make them better off compared to having a negative surplus. It will be optimal never to delete reviews, and therefore the negative surplus eventually shrinks to zero.

2.4 Comparative Statics

The previous sections discussed the optimality conditions for the seller and the platform. This section will analyse the impact of the model parameters on the endogenous outcomes. As a reminder, there are four exogenous parameters in this model:

- x_r, which is the number of reviews left per period and can be termed as the "feedback frequency",[15]
- k, which is the number of periods served by each seller and can be termed as the "tenancy period",
- c, which is simply a cost parameter of quality, and
- x, which can be termed as the "market size".

[15] Alternatively, it can be interpreted as the number of reviews left per x number of buyers.

All of the above parameters will be analysed in turn. Particular focus, with formal analysis, will be given to the impact of feedback frequency due to its relevance to the model's social learning framework and straightforward policy implications.

Feedback Frequency

If more buyers post reviews, then for any τ, the total surplus is weakly higher due to the higher long-run quality. However, the impact on consumer surplus is less clear. To avoid unnecessary notation, the functions are defined temporarily in terms of τ and the relevant parameter (in this case, for example, $W[\tau; x_r]$ is written instead of $W[\tau]$).

Proposition 2.2. *Suppose $x_r'' > x_r' > 0$. Then, for any $\tau \in \mathbb{N} \cup \{\infty\}$, $W[\tau; x_r''] \geq W[\tau; x_r']$ (equivalently $\bar{q}[\tau; x_r''] \geq \bar{q}[\tau; x_r']$).*

When more consumers post reviews, future buyers will put more weight on the aggregated ratings. This induces the sellers to increase their quality choice, increasing the surplus for any policy τ. This suggests that increasing the feedback frequency cannot harm total surplus.

However, the impact on consumer surplus is ambiguous. It can be seen by rearranging the $CS[\tau]$ equation, as below.

$$CS[\tau] = x \cdot \Big(\bar{q}[\tau; x_r] - \mu\Big)\Big(1 - \frac{1}{k}\sum_{i=0}^{k-1} \omega[x_r \cdot (i + \tau)]\Big) \tag{2.14}$$

The long-run consumer surplus in 2.14 consists primarily of two components. The first one, expression $\bar{q}[\tau; x_r] - \mu$, is the difference between the buyer's prior belief and chosen quality, while the second is the uncertainty of information obtained by the reviews: $1 - \frac{1}{k}\sum_{i=0}^{k-1} \omega[x_r \cdot (i + \tau)]$ (more reviews decrease the uncertainty and increase the average weights during the seller's tenancy, which therefore brings the price closer to the true quality). When x_r increases, there is one positive effect and one ambiguous effect (depending on whether the quality is lower or higher than μ).

The positive effect is through the higher choice of quality (from proposition 2.2), always resulting in the buyers being at least as well off. The ambiguous effect is from the uncertainty component. If the quality is less than μ, more precision from the higher number of reviews will improve the surplus. However, if the quality is

x_r	$CS[0]$	$CS[1]$
1	9.61	11.2018
2	8.79	11.2243
3	8.60	11.2278

Table 2.1: Consumer Surplus with parameters $\{x = 3, c = 0.1, k = 4, \mu = 0\}$

more than μ, more precision from the higher number of reviews will make the buyers worse off (through higher prices). The relative changes of these two components will determine the overall effect on consumer surplus. The numerical example in table 2.1 demonstrates this. Only the level of τ is varied with the same set of parameters. When $\tau = 0$, the level of consumer surplus is decreasing with additional reviews being posted. When $\tau = 1$, consumer surplus is increasing with more reviews.

Tenancy Period

Sellers operate for k periods, after which the control of the firm is transferred to another seller. The next proposition shows that increasing this tenancy period, k, has a positive effect on the average quality chosen by each seller.

Proposition 2.3. *Suppose $k' > k \geq 2$; then $\bar{q}[\tau, k'] \geq \bar{q}[\tau, k]$ (equivalently $W[\tau, k'] \geq W[\tau, k]$).*

Increasing the tenancy period implies that sellers value the future more, which incentivises higher quality as this will translate to higher prices for future consumers through the generation of favourable reviews. The changes to welfare will follow similarly due to lemma 2.1. Similar to the previous analysis for x_r, the impact on consumer surplus will be ambiguous due to the counteracting effects of higher quality and lower uncertainty resulting from more review generation.

Cost of Quality and Market Size

The impact of the cost parameter and market size on the endogenous outcomes is relatively trivial and therefore discussed together. A higher c decreases the incentive for quality provision and therefore decreases $\bar{q}[\tau]$ for any τ, which also decreases $W[\tau]$. For buyers, c has no impact on the uncertainty component and therefore reduces the surplus through the lower quality provision.

As for market size, x appears as a positive scalar multiplier in $W[\tau]$, $CS[\tau]$ and $\bar{q}[\tau]$; all of these variables increase (in magnitude) with x for any value of $\tau < \infty$. However, the per-unit consumer surplus is invariant, and any leftover surplus (after accounting for the higher consumers) goes to the sellers through higher profits. For $\tau = \infty$, there is no increase in surplus, as $W[\infty] = 0$.

2.5 The Weight Function: Some Discussions

The social learning mechanism that has been assumed in this chapter was a general weight function, $\omega[n]$. As shown below, this type of function covers a broad range of social learning concepts. This includes the case where consumers update using Bayes Law with the frequently used assumption of normally distributed quality and prior beliefs. A subset of Rational functions of degree 1 can be used to model this type of Bayesian learning with the said distributions. This section will first discuss the impact of faster social learning, followed by these example classes.

Higher relative weights

The following proposition compares two weight functions where one's output is always at least as high as the other (for a given number of ratings).

Proposition 2.4. *Suppose ω_1 and ω_2 are both mappings from $\mathbb{N}_0$ to $[0,1]$ and satisfy assumptions 1-3. Assume $\omega_1[n] \geq \omega_2[n]$ $\forall n$. Then $W[\tau;\omega_1] \geq W[\tau;\omega_2]$ (equivalently $\bar{q}[\tau;\omega_1] \geq \bar{q}[\tau;\omega_2]$).*

Higher weights ω_1 for any given n is equivalent to having more reviews x_r with ω_2. Therefore, the intuition follows closely to that of Proposition 2.2. Similarly, the impact on consumer surplus is ambiguous. To avoid redundancy, this will not be discussed here.

Bayesian learning with standard conjugate priors

A standard assumption made in related papers are that both the quality outcome and the consumers' prior belief are normally distributed.[16] This assumption is a

[16] Examples include, among others, Feldman et al. (2019) and Papanastasiou and Savva (2017).

special case of the framework in this chapter, with the following particular weight function:

$$\omega[n] = \frac{an}{an+b}, \text{ where } a > 0 \text{ and } b > 0$$

It is clear that $\omega[n] \to 1$ as $n \to \infty$. This functional form is consistent with Bayesian updating rules for some conjugate prior distributions,[17] as summarised in Table 2.2.

Quality Distribution	**Prior Distribution**	$\boldsymbol{\omega[n]}$
$\mathrm{B}(n, q)$	$\mathrm{Beta}(\alpha,\ \beta)$	$\frac{n}{n+\alpha+\beta}$
$N(q, \sigma^2)$ with known variance	$N(\mu, \sigma_p^2)$	$\frac{\frac{n}{\sigma^2}}{\frac{1}{\sigma_p^2}+\frac{n}{\sigma^2}}$
$N(q, \sigma^2)$ with unknown mean and unknown variance	Normal-Inverse Gamma with parameters $\{\mu, \lambda, \alpha, \beta\}$	$\frac{n}{n+\lambda}$

Table 2.2: Some conjugate priors with Bayesian learning

For this type of functional form, the results are quite extreme, as shown in the proposition below.

Proposition 2.5. *Suppose* $\omega[n] = \frac{an}{an+b}$, *where* $a > 0$ *and* $b > 0$.

(i) The welfare maximising level of τ *is always zero, i.e.* $\mathrm{argmax}_\tau\{W[\tau]\} = 0$;

(ii) To maximise consumer surplus, if the prior mean, μ, *is sufficiently low, then all reviews should be deleted upon each seller; otherwise, reviews should never be deleted. Formally:*

- $\mu < \bar{\mu} \implies \mathrm{argmax}_\tau\{CS[\tau]\} = 0$, *and*
- $\mu \geq \bar{\mu} \implies \sup_\tau\{\mathrm{argmax}_\tau\{CS[\tau]\}\} = \infty$

where $\bar{\mu} = \sum_{i=1}^{k} \frac{a(x_r \cdot i)}{a(x_r \cdot i)+b} \cdot \frac{x}{2c \cdot k}$.

In words, Proposition 2.5 says that with this form of Bayesian updating, whenever there is a new seller, all past reviews should be deleted to maximise the level of quality and total surplus. If the platform wishes to maximise consumer surplus, then the platform's policy resembles an "all or nothing" strategy. If the prior beliefs

[17] The resulting weight functions are available in standard textbooks in Bayesian Inference, but for the interested reader, the derivation of $\omega[n]$ with Beta prior and Binomial likelihood is shown in the Appendix (page 80).

are sufficiently high, consumers invariably get a negative surplus due to the quality level being lower than their expectations. In this scenario, reviews should never be deleted. On the other hand, if the prior beliefs are low, any increase in reviews decreases the quality and increases the price, and therefore all past reviews should be deleted upon each new seller.

Indicator Function

As per this model and assumptions 1-3, the weight function can also take the form $\omega[n] = \mathbb{1}_{n \geq \tilde{n}}$, where $\tilde{n} > 0$:

$$\omega[n] = \mathbb{1}_{n \geq \tilde{n}} = \begin{cases} 1, & n \geq \tilde{n} \\ 0, & \text{otherwise} \end{cases} \tag{2.15}$$

Intuitively, buyers only believe their prior unless there is a minimum number $\tilde{n}$ of reviews, after which all trust is given to the aggregated rating. For this case, the welfare-maximising choice of τ has upper and lower bounds, as shown by the following proposition.

Proposition 2.6. *Suppose* $\omega[n] = \mathbb{1}_{n \geq \tilde{n}}$*, where* $\tilde{n} \in \{1, 2, ...\}$*. Let* $\tau^* \in \operatorname{argmax}_\tau\{W[\tau]\}$*. Then:*

$$\tau^* \geq \max\left\{0, \left\lfloor \frac{\tilde{n} - x_r \cdot k}{x_r} \right\rfloor\right\} \quad \textit{and} \quad \tau^* < \left\lceil \frac{\tilde{n}}{x_r} \right\rceil$$

Intuitively, τ must be selected in such a way that total reviews generated during a seller's tenancy at least equal the buyer's threshold $\tilde{n}$. At the same time, τ should not be so high that the buyers put full trust in the ratings even before the seller starts to operate. For the more general case where $\omega[n] = 0$ for $0 \leq n < \tilde{n}$ and $\omega[n] > 0$ for all $n \geq \tilde{n}$, the lower bound still holds but the upper bound may be different.

Rate of Weight Increments

The previous sub-section discussed a scenario where the welfare-maximising level of τ may be non-zero. This sub-section will construct another weight function where $\tau^* > 0$, which results due to relatively higher marginal weights for lower time periods. Consider a weight function of the form $\omega[n] = \frac{an^2}{an^2+b}$, where $a > 0$ and $b > 0$, which is a rational function of degree 2. Setting $a = 1$ and $b = 100$, the discrete plot of the function is given below.

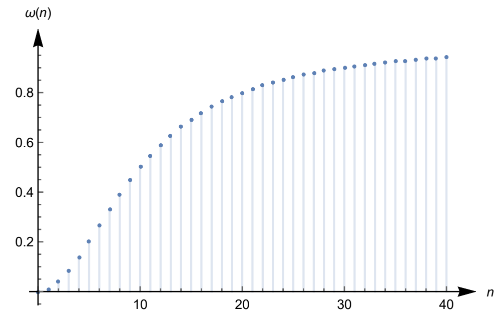

Figure 2.1: Discrete Plot of $\omega[n] = \frac{n^2}{n^2+100}$

It can be easily seen from Figure 2.1 that the marginal weight ($\triangle\omega[n_t] = \omega[n_t] - \omega[n_{t-1}]$) is increasing for some initial values of n and then decreasing after some point. This phenomenon will hold similarly for higher degree rational functions, as well as the logistic function. For this particular form, setting $x_r = 1$ and $k = 2$, a plot of $\bar{q}[\tau]$ is shown in Figure 2.2. It can be observed that for lower values of τ, where the marginal weight is increasing, the average quality $\bar{q}[\tau]$ is increasing with τ. In fact, with the aid of the programming language *Mathematica*, it is found that $\tau^* = 10$.

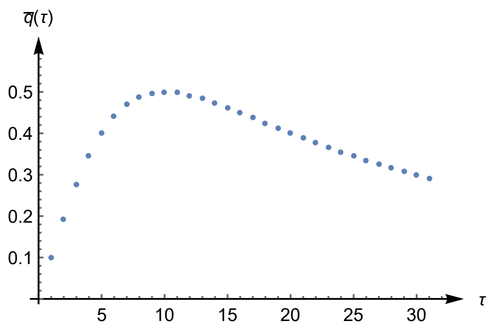

Figure 2.2: Graph of $\bar{q}[\tau]$

From the discussion following Proposition 2.1, there are two counteracting effects of increasing τ on the average quality provision: (i) the (positive) effect of the incremental weight on the product rating, and (ii) the (negative) effect due to decreased contribution to the aggregated rating. For this particular weight function, the former effect dominates the latter up until $\tau = 10$. Intuitively, it is optimal for the platform to give the new sellers a starting "boost" of existing ratings to ensure that the product rating is given sufficient weight by the buyers. However, an important point to note is that this will no longer hold if the weight increases *very* fast. In that case, solely the reviews generated by the new management will lead to the buyers placing sufficiently high weight on the rating, and aggregating past reviews is no longer necessary to induce higher quality.

2.6 Discussions

2.6.1 Alternative Policy: Fixed Interval

As per the platform's policy analysed in this chapter, past reviews are only deleted upon the entry of a new seller, and any reviews generated during the seller's tenancy are aggregated into the rating. Therefore, the length of the reviews is only adjusted at the point of replacement. This was motivated primarily by the focus on how to react to changes in management. For example, if the platform is notified of a management change on 1st September 20X1, and τ^* is one month, then all reviews entered before 1st August 20X1 are deleted. The platform intervenes again when there is another change in management, deleting everything but the last month's reviews.

However, a more intuitive approach can be to only include reviews in a fixed number of recent periods at any point in time. This would imply that the earliest x_r reviews are deleted after every period, and the latest x_r reviews are added, as long as t is sufficiently large. This policy is already used in some platforms, such as Uber Eats, where they keep only the last 90 days of user-generated feedback when aggregating the ratings for restaurants.

2.6.2 Fixed Quantity

This chapter followed similar literature by assuming a fixed quantity of products the seller produces and sells (fixed demand) every period to a competitive set of buyers (although competition among the buyers was not explicitly analysed). This greatly simplifies the problem as the seller's profit function is affected additively by past ratings and leads to a tractable solution where the seller's optimal choice of quality only depends on time (or equivalently, in this case, the number of past reviews) and not on the inherited rating. If, instead, the inherited rating also influenced demand, then the optimal quality choice will depend on this. For instance, if the seller inherits a low rating, then lower demand will also imply that fewer reviews are generated, which in turn will provide less incentives for the seller to provide high quality (Proposition 2.2). In that case, it may not be straightforward to obtain a deterministic steady state, as was the case in this model. However, this is a natural extension of the present model and is left for future research.

2.6.3 Strategic Competitors

In the model, only one firm was assumed to exist, and the analysis still holds in competitive markets where the firms are too small to have a strategic effect on other firms (as long as the buyer's side is relatively more competitive and the sellers clear whole of their capacity in each period). This holds in some practical settings, but not when the sellers can have some strategic influence on each other (for instance, when buyers have more market power). A natural extension, therefore, is to analyse an oligopolistic market. A motivating formulation may involve the setting where the sellers compete for market share, resulting in the same difficulty of variable demand as mentioned above in Section 2.6.2.

2.6.4 Naive Consumers

In the chapter, consumers are assumed to be naive with respect to expectation formations. Particularly, they are assumed to not even consider that a brand may be operated by different sellers who change quality. From their point of view, their beliefs are attributed to the brand rather than individual sellers (or management). This kind of modelling is in contrast with the literature where consumers are assumed to make rational expectations, and may be more appropriate in many markets for goods and services where buyers would not likely put much effort into search costs before purchasing. An example is when people go to a busy tourist hotspot and decide where to have lunch (and determine their willingness to pay after observing the product rating). It can be argued that in these scenarios, potential buyers are unlikely to put much cognitive energy into determining when was the last management change. This issue cannot be captured with a model with rational expectations. Moreover, with rational expectations, in a modelling sense, buyers would be able to compute exactly when was the management change, which, in addition to being quite unreasonable, would make this policy and entire research question redundant.

Additionally, in high-value markets, like that for furniture, where it can be expected that people will give more time and effort to reading actual and recent reviews rather than just observing the rating, designing an optimal length of history may not be necessary.

2.6.5 Short-lived Consumers

In the model, it has been assumed that the buyers only live for one period. It should be noted that the results may not be readily generalisable to markets where buyers are long-lived and can make repeat purchases. In that case, the buyers may make strategic choices and not behave naively when learning from ratings, as assumed in this chapter. This is due to the buyers having an extra source of information, in addition to reviews, when deciding whether to purchase from the same firm: their own past experience. Previous experiences would mean the product ratings would matter less, and consumers would trust their own experience more. However, the results should hold to a certain extent in the intermediate case: where there is a mix of short-lived and long-lived buyers. Rigorously investigating this is left for future work.[18]

2.7 Conclusions

This chapter contributes to the literature on the design of reputation systems by analysing the optimal length of history that platforms should aggregate upon changes in management (sellers), which are not observed by consumers. The results are also applicable in settings where management may not change, but the firm makes periodic quality investments to maximise medium-term profits. As the buyers are short-lived in this formulation, appropriate practical settings include markets for goods and services where the same buyers do not frequently purchase from the same firm. These include restaurants, hotels and other services in busy tourist hotspots.

[18] A mix of both short-lived and long-lived consumers is quite a realistic assumption as restaurants are visited by both tourists and locals. Intuitively, if reviews are not deleted, a high proportion of short-lived buyers (tourists) will result in the firm behaving as predicted in this chapter, reducing quality over time. Alternatively, more long-lived consumers would incentivise firms to provide higher quality due to their focus on repeat purchases. In this case, the presence of long-lived buyers can be a positive externality for the short-lived buyers, with the latter imposing a negative externality on the former.

The chapter has some clear policy recommendations. Contradicting the status quo in popular reputation platforms, reviews generated by past management should be deleted eventually if the objective is to maximise the average total surplus. If the social learning mechanism is Bayesian (with standard conjugate prior distributions), all past reviews should be deleted upon each management if the objective is to maximise total surplus. However, a never-delete policy may be applicable in specific scenarios where the objective is to maximise consumer surplus. The Bayesian case follows an all-or-nothing strategy: if the consumer's prior belief is sufficiently low, all reviews should be deleted; otherwise, reviews should never be deleted.

Inheriting past reviews will have two different effects on the new seller's choice of quality, which counteract each other: the first (positive) effect is the higher weight placed on the rating; and the second (negative) effect is the lower contribution of the seller to the aggregated rating, which is a simple average of all past ratings. Therefore, past reviews should be kept if the first effect dominates the second for some positive length of history. Two scenarios have been identified where this may be the case. The first scenario is when consumers do not place any weight on the product rating unless there is a sufficiently high number of reviews. The second scenario is when the weight on the product rating is increasing at an increasing rate for some initial reviews.

Regarding the problem of designing the optimal length of historical ratings, the focus of this chapter has been primarily on analysing how this length would affect the buyer's learning of an unobserved quality. However, in a market with few sellers, the ratings of competing firms will also impact the quality choice. There are several natural extensions to the present formulation (some of which are mentioned in the preceding section), which are left for future research. It may also be worthwhile to find a numerical value of the optimal length of past reviews by taking this model to the data.

Appendix

Derivation of the optimal quality choice. Differentiating the profit function in 2.7 with respect to q_η and equating to zero yields:

$$\begin{aligned}
\frac{\partial \mathbb{E}[\Pi_\eta]}{\partial q_\eta} &= \sum_{i=\tau_\eta}^{k+\tau_\eta-1}\left(\left(\omega[x_r \cdot i] \cdot \frac{i-\tau_\eta}{i}\right)x - 2cq_\eta\right) = 0 \\
&\Rightarrow \sum_{i=\tau_\eta+1}^{k+\tau_\eta-1}\left(\omega[x_r \cdot i] \cdot \frac{i-\tau_\eta}{i}\right) x - \sum_{i=\tau_\eta}^{k+\tau_\eta-1} 2cq_\eta = 0 \\
&\Rightarrow \sum_{i=\tau_\eta+1}^{k+\tau_\eta-1}\left(\omega[x_r \cdot i] \cdot \frac{i-\tau_\eta}{i}\right) x - q_\eta(2c \cdot k) = 0 \\
&\Rightarrow \quad q_\eta^* = \sum_{i=\tau_\eta+1}^{k+\tau_\eta-1} \omega[x_r \cdot i]\left(1 - \frac{\tau_\eta}{i}\right) \cdot \frac{x}{2c \cdot k} \qquad \blacksquare
\end{aligned}$$

Proof of Proposition 2.1. The proof for a finite τ is trivial and follows directly from equation 2.8. For $\tau = \infty$, we know that $\tau_\eta \to \infty$ as $\eta \to \infty$ (as $\tau_\eta = \min\{\eta k, \tau\}$). Moreover, as $\omega[n]$ is increasing with n and bounded above (range of ω is $[0, 1]$), using the Monotone Convergence Theorem there exists $\bar{\omega} \in (0, 1]$ such that $\lim_{n\to\infty} \omega[n] = \bar{\omega}$. Therefore,

$$\begin{aligned}
\bar{q}[\infty] &= \lim_{\eta\to\infty} q_\eta^*[\tau_\eta] \\
&= \lim_{\tau_\eta\to\infty} q_\eta^*[\tau_\eta] \\
&= \lim_{\tau_\eta\to\infty} \sum_{i=1}^{k-1} \omega[x_r \cdot (i+\tau_\eta)] \cdot \left(\frac{i}{i+\tau_\eta}\right) \cdot \frac{x}{2c \cdot k} \\
&= \frac{x}{2c \cdot k} \sum_{i=1}^{k-1} \lim_{\tau_\eta\to\infty}\left(\omega[x_r \cdot (i+\tau_\eta)]\left(\frac{i}{i+\tau_\eta}\right)\right) \\
&= \frac{x}{2c \cdot k} \sum_{i=1}^{k-1} \lim_{\tau_\eta\to\infty} \omega[x_r \cdot (i+\tau_\eta)] \cdot \lim_{\tau_\eta\to\infty}\left(\frac{i}{i+\tau_\eta}\right) \\
&= \frac{x}{2c \cdot k} \sum_{i=1}^{k-1} (\bar{\omega} \cdot 0) = 0 \qquad \blacksquare
\end{aligned}$$

Proof of Lemma 2.1. For any $\tau < \infty$,

$$\begin{aligned}
\bar{q}[\tau] &= \sum_{i=1}^{k-1} \omega[x_r \cdot (i+\tau)] \left(\frac{i}{i+\tau}\right) \cdot \frac{x}{2c \cdot k} \\
&\leq \sum_{i=1}^{k-1} \left(\frac{i}{i+\tau}\right) \cdot \frac{x}{2c \cdot k} && (\text{as } 0 \leq \omega[n] \leq 1 \ \forall n > 0) \\
&< \sum_{i=1}^{k-1} \frac{x}{2c \cdot k} && \left(\text{as } 0 < \frac{i}{i+\tau} < 1 \ \forall (i,\tau) \geq (1,0)\right) \\
&= \frac{x(k-1)}{2c \cdot k} < \frac{x}{2c} && \left(\text{as } 0 < \frac{k-1}{k} < 1 \ \forall k \geq 2\right)
\end{aligned} \tag{2.16}$$

Additionally, $\bar{q}[\infty] = 0 < \frac{x}{2c}$. Therefore, $\bar{q}[\tau] < \frac{x}{2c}$ for any τ. This implies that $x > 2c \cdot \bar{q}[\tau]$. Now, let us take the derivative of W with respect to $\bar{q}$:

$$\frac{\partial W}{\partial \bar{q}} = x - 2c \cdot \bar{q} > 0$$

As $\frac{\partial W}{\partial \bar{q}} > 0$ for any τ, under the given constraints, in order to maximise $W[\tau]$, the platform would choose τ to maximise $\bar{q}[\tau]$. Therefore, the solutions to maximising $W[\tau]$ and $\bar{q}[\tau]$ are equivalent. ■

Proof of Theorem 2.1. From Lemma 2.1, we know that the solutions to maximising $W[\tau]$ and $\bar{q}[\tau]$ are equivalent. Now, for any $\tau \in \mathbb{N}$,

$$\bar{q}[\tau] = \sum_{i=1}^{k-1} \omega[x_r \cdot (i+\tau)] \left(\frac{i}{i+\tau}\right) \cdot \frac{x}{2c \cdot k} \geq 0$$

The inequality holds because all terms in the expression are non-negative:

- $\omega[.] \in [0,1]$,
- $\left(\frac{i}{i+\tau}\right) > 0$ as $i \geq 1$ and $\tau \geq 0$,
- $x > 0$, $c > 0$ and $k \geq 2$.

Therefore, $\bar{q}[\tau] \geq 0 = \bar{q}[\infty]$, proving the first part of the theorem. Now from assumption 3, there exists n' such that $\omega[n] > 0$ for all $n \geq n'$. Let $\tau' = n'$. In this

case,
$$\bar{q}[\tau'] = \sum_{i=1}^{k-1} \omega[x_r \cdot (i+\tau')] \left(\frac{i}{i+\tau'}\right) \cdot \frac{x}{2c \cdot k} > 0 = \bar{q}[\infty]$$
because now all terms are strictly positive (including $\omega[x_r \cdot (i+\tau)]$). ■

To prove Theorem 2.2, it will be convenient to first prove a more general result:

Lemma 2.2. *Suppose $f[\tau] = g[\tau] \cdot h[\tau]$, where $f : \mathbb{N} \cup \{\infty\} \to \mathbb{R}$, $g : \mathbb{N} \cup \{\infty\} \to \mathbb{R}$ and $h : \mathbb{N} \cup \{\infty\} \to [0,1]$ and let $\tau^F := \sup_\tau\{\text{argmax}_\tau\{f[\tau]\}\}$ and $\tau^G := \sup_\tau\{\text{argmax}_\tau\{g[\tau]\}\}$. Let the following assumptions hold:*

- *$h[\tau]$ is decreasing in τ.*
- *$\lim_{\tau\to\infty} h[\tau] = \bar{h} \in [0,1]$.*
- *$\tau^G < \infty$.*
- *$h[\tau^G] > 0$.*

Then,

(i) If $g[\tau^G] > 0$, then $\tau^F \leq \tau^G$;

(ii) if $g[\tau^G] \leq 0$, then $\tau^F \geq \tau^G$;

(iii) if $g[\tau^G] \leq 0$ and $\bar{h} = 0$, then $\tau^F = \infty$.

Proof. (i) We proceed by contradiction. Suppose $\tau^F > \tau^G$. Then:
$$\begin{aligned} f[\tau^G] &= g[\tau^G] \cdot h[\tau^G] \\ &\geq g[\tau^G] \cdot h[\tau^F] \qquad \text{(because } h[\tau] \text{ is decreasing in } \tau \text{ and } g[\tau^G] > 0\text{)} \\ &\geq g[\tau^F] \cdot h[\tau^F] \qquad \text{(because } g[\tau^G] > g[\tau]\ \forall \tau > \tau^G\text{)} \\ &= f[\tau^F] \end{aligned}$$

So, $f[\tau^G] \geq f[\tau^F]$, with equality holding only when $h[\tau^G] = h[\tau^F] = 0$. But $h[\tau^G] > 0$, which implies that $f[\tau^G] > f[\tau^F]$. However, this is not possible because, by definition, $f[\tau^F] \geq f[\tau]\ \forall \tau$.

(ii) Again, we proceed by contradiction. Suppose $\tau^F < \tau^G$. Then:

$$\begin{aligned}
f[\tau^G] &= g[\tau^G] \cdot h[\tau^G] \\
&\geq g[\tau^G] \cdot h[\tau^F] && \text{(because } h[\tau] \text{ is decreasing in } \tau \text{ and } g[\tau^G] \leq 0) \\
&\geq g[\tau^F] \cdot h[\tau^F] && \text{(because } g[\tau^G] \geq g[\tau]\ \forall \tau \neq \tau^G) \\
&= f[\tau^F]
\end{aligned}$$

Please note that the second line follows because a lower value of τ will (weakly) increase h, which in turns increases the magnitude of a non-positive value. Now, $f[\tau^G] \geq f[\tau^F]$, which implies that either:

- $f[\tau^G] = f[\tau^F]$, but this is not possible as $\tau^F \neq \sup_\tau\{\text{argmax}_\tau\{f[\tau]\}\}$ as $\tau^F < \tau^G$.
- $f[\tau^G] > f[\tau^F]$, but this is not possible because, by definition, $f[\tau^F] \geq f[\tau]$ $\forall \tau$.

(iii) Because $g[\tau^G] \leq 0$ and $h[\tau] \geq 0\ \forall \tau$, this means that $f[\tau] = g[\tau] \cdot h[\tau] \leq 0\ \forall \tau$. Then, $f[\infty] = \bar{g} \cdot \bar{h} = \bar{g} \cdot 0 = 0 \implies \tau^F = \infty$. ■

Another lemma is required before the main proof.

Lemma 2.3. *If* $\tau^* \in \text{argmax}_\tau\{\bar{q}[\tau]\}$*, then* $\omega[x_r \cdot \tau^*] < 1$.

Proof. Let $\tau^* \in \text{argmax}_\tau\{\bar{q}[\tau]\}$. If $\tau^* = 0$, then trivially from assumption 1, $\omega[x_r \cdot \tau^*] = 0 < 1$. We proceed by contradiction for $\tau^* \geq 1$. Suppose $\tau^* \geq 1$ and $\omega[x_r \cdot \tau^*] = 1$. Then,

$$\begin{aligned}
\bar{q}[\tau^*] &= \sum_{i=1}^{k-1} \omega[x_r \cdot (i+\tau)] \left(\frac{i}{i+\tau^*}\right) \cdot \frac{x}{2c \cdot k} \\
&= \frac{x}{2c \cdot k} \cdot \sum_{i=1}^{k-1} \left(\frac{i}{i+\tau^*}\right)
\end{aligned}$$

because $\omega[\cdot]$ is increasing, which results in all $\omega[x_r \cdot (i+\tau)] = 1$ for $i \in \{1, ..., k-1\}$. Now, let $\tau' = \tau^* - 1$ and $i = j - 1$. Then,

$$\bar{q}[\tau'] = \sum_{j=1}^{k-1} \omega[x_r \cdot (j+\tau')] \left(\frac{j}{j+\tau'}\right) \cdot \frac{x}{2c \cdot k}$$

$$
\begin{aligned}
&= \frac{x}{2c \cdot k} \cdot \sum_{j=1}^{k-1} \omega[x_r \cdot (j+\tau^*-1)] \left(\frac{j}{j+\tau^*-1} \right) \\
&= \frac{x}{2c \cdot k} \cdot \sum_{i=0}^{k-1} \omega[x_r \cdot (i+\tau^*)] \left(\frac{i+1}{i+\tau^*} \right) \\
&= \frac{x}{2c \cdot k} \cdot \left(\frac{1}{\tau^*} + \sum_{i=1}^{k-1} \left(\frac{i+1}{i+\tau^*} \right) \right) \\
&> \frac{x}{2c \cdot k \cdot \tau^*} + \bar{q}[\tau^*] > \bar{q}[\tau^*]
\end{aligned}
$$

So, $\bar{q}[\tau^*] < \bar{q}[\tau']$, which implies $\tau^* \notin \operatorname{argmax}_\tau \{\bar{q}[\tau]\}$. ■

Proof of Theorem 2.2. We are given the following:

$$
\tau^* := \sup_\tau \{\operatorname*{argmax}_\tau \{W[\tau]\}\} \quad \text{and} \quad \tau^C := \sup_\tau \{\operatorname*{argmax}_\tau \{CS[\tau]\}\}
$$

Let $n_i = x_r(i+\tau)$. First, we rearrange $CS[\tau]$ from equation 2.11:

$$
\begin{aligned}
CS[\tau] = \mathbb{E}\left[\frac{1}{k} \sum_{i=0}^{k-1} \Big(\bar{q}[\tau] - p_i[\tau] \Big) \cdot x \right] \\
&= x \cdot \bar{q}[\tau] - \frac{x}{k} \cdot \mathbb{E}\left[\sum_{i=0}^{k-1} p_i[\tau] \right] \\
&= x \cdot \bar{q}[\tau] - \frac{x}{k} \cdot \sum_{i=0}^{k-1} \Big((1-\omega[n_i])\,\mu + \omega[n_i] \cdot \frac{\bar{s} \cdot \tau + \bar{q}[\tau](i-\tau)}{i} \Big) \\
&= x \cdot \bar{q}[\tau] - \frac{x}{k} \cdot \sum_{i=0}^{k-1} \Big((1-\omega[n_i])\,\mu + \omega[n_i] \cdot \bar{q}[\tau] \Big) \\
&= x \cdot \Big(\bar{q}[\tau] - \mu \Big) \Big(1 - \frac{1}{k} \sum_{i=0}^{k-1} \omega[n_i] \Big)
\end{aligned}
$$

where the second to last equality follows because $\bar{s} = \bar{q}$ in the limit. Now, let $f[\tau] = CS[\tau]$, $g[\tau] = x \cdot \Big(\bar{q}[\tau] - \mu\Big)$, and $h[\tau] = \Big(1 - \frac{1}{k}\sum_{i=0}^{k-1} \omega[n_i]\Big)$, where $h[\infty] = 1 - \lim_{n_i \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \omega[n_i]$.

- It is clear that $h[\tau]$ is decreasing with τ as $\omega[n_i]$ is increasing with n_i (which increases with τ).
- $\lim_{\tau\to\infty} h[\tau] = (1-\bar{\omega}) \in [0,1]$ (from the Monotone Convergence Theorem).[19]

[19]See proof of Proposition 2.1 above.

- In this case, $\sup_\tau\{\text{argmax}_\tau\{\bar{q}[\tau]\}\} = \sup_\tau\{\text{argmax}_\tau\{g[\tau]\}\}$ because $g[\tau]$ is a monotone scalar and linear transformation of $\bar{q}[\tau]$. From Theorem 2.1,

$$\tau^* = \sup_\tau\{\underset{\tau}{\text{argmax}}\{W[\tau]\}\} = \sup_\tau\{\underset{\tau}{\text{argmax}}\{\bar{q}[\tau]\}\} < \infty$$

- $h[\tau^*] = 1 - \frac{1}{k}\sum_{i=0}^{k-1}\omega[x_r \cdot (i+\tau^*)]$. The only way $h[\tau^*] = 0$ if $\omega[x_r \cdot (i+\tau^*)] = 1$ for all $i \in \{0, ..., k-1\}$, but $\omega[x_r \cdot \tau^*] < 1$ from Lemma 2.3. Therefore, $h[\tau^*] > 0$.

Putting it all together, using Lemma 2.2, we can conclude:

- If $\bar{q}[\tau^*] > \mu$, then $\tau^C \leq \tau^*$;
- if $\bar{q}[\tau^*] \leq \mu$, then $\tau^C \geq \tau^*$;
- if $\bar{q}[\tau^*] \leq \mu$ and $\lim_{n\to\infty}\omega[n] = 1$ (this implies $h[\infty] = 0$), then $\tau^C = \infty$. ■

Proof of Proposition 2.2. From Lemma 2.1, maximising $\bar{q}[\tau]$ and $W[\tau]$ is equivalent. Therefore, for any $\tau < \infty$ and $x_r'' > x_r' > 0$,

$$\bar{q}[\tau; x_r''] = \sum_{i=1}^{k-1}\omega[x_r'' \cdot (i+\tau)]\left(\frac{i}{i+\tau}\right)\cdot\frac{x}{2c\cdot k} \geq \sum_{i=1}^{k-1}\omega[x_r' \cdot (i+\tau)]\left(\frac{i}{i+\tau}\right)\cdot\frac{x}{2c\cdot k} = \bar{q}[\tau; x_r']$$

because $\omega[\cdot]$ is monotonically increasing. ■

Proof of Proposition 2.3. The quality expression is given is follows.

$$\bar{q}[\tau] = \frac{x}{2c}\cdot\frac{1}{k}\sum_{i=1}^{k-1}\omega[x_r \cdot (i+\tau)]\left(\frac{i}{i+\tau}\right)$$

Ignoring the constant $\frac{x}{2c}$, the expression can be seen as an average of the k terms $\omega[x_r \cdot (i+\tau)]\left(\frac{i}{i+\tau}\right)$ for $i \in \{1, ..., k-1\}$. As k increases by 1, an additional term is added, which can be expressed as $\omega[x_r \cdot (k+\tau)]\left(\frac{k}{k+\tau}\right)$. As $\omega[.]$ is assumed to monotonically increasing and because it is clear that $\left(\frac{i}{i+\tau}\right)$ is increasing with k, it is a set of increasing terms with i. Therefore, the additional term is higher than all previous terms, which implies the average is increasing. This completes the proof. ■

Proof of Proposition 2.4. The proof is quite straightforward and follows the same reasoning as that of part (i) of Proposition 2.2. For any $\tau < \infty$,

$$\bar{q}[\tau; \omega_1[\cdot]] = \sum_{i=1}^{k-1} \omega_1[x_r \cdot (i + \tau)] \left(\frac{i}{i + \tau} \right) \cdot \frac{x}{2c \cdot k} \geq \sum_{i=1}^{k-1} \omega_2[x_r \cdot (i + \tau)] \left(\frac{i}{i + \tau} \right) \cdot \frac{x}{2c \cdot k} = \bar{q}[\tau; \omega_2[\cdot]]$$

■

Derivation of the weight function for Beta prior and Binomial quality. Suppose, for seller η, the quality of the product is a random variable with a Bernoulli distribution, with the probability that $\hat{q}_\eta = 1$ (experience is good) is q_η and $\hat{q}_\eta = 0$ (experience is bad) is $1 - q_\eta$. The buyers observe (n_t, y_t) where n is the number of reviews and y is the number of positive reviews. They cannot differentiate between sellers, but they update their beliefs using Bayes' Law, assuming the reviews follow a Binomial distribution with parameters (n, q). This q is unknown to the buyers, so they have a prior belief of q, which has a Beta distribution with shape parameters α and β.

The following is based on Chapter 3 of the textbook by Johnson et al. (2022).

The pdf of the Beta distribution with parameters α and β is given as follows.

$$f[q] = \frac{\Gamma[\alpha + \beta]}{\Gamma[\alpha] \cdot \Gamma[\beta]} \cdot q^{\alpha - 1} \cdot (1 - q)^{\beta - 1}$$

where $\Gamma[.]$ is the Gamma function. Upon observing (n_t, y_t), buyers update their beliefs using Bayes' Law:

$$f[q|(n_t, y_t)] = \frac{L(y_t|q) f[q]}{\int L(y_t|q) f[q] dq} \tag{2.17}$$

where:

- $f[q|(n_t, y_t)]$ is the posterior density,
- $L(y_t|q)$ is the *likelihood* function, loosely describing the probability of observing y_t given q,
- $f[q]$ is the prior density, and
- $\int L(y_t|q) f[q] dq$ is a normalising constant.

After some algebra, the posterior distribution will be as follows.

$$f[q|(n_t, y_t)] \propto \frac{\Gamma[\alpha+\beta+n_t]}{\Gamma[\alpha+n_t y_t]\cdot\Gamma[\beta+n_t-n_t y_t]}\cdot q^{\alpha+n_t y_t-1}\cdot(1-q)^{\beta+n_t-n_t y_t-1} \tag{2.18}$$

Therefore, the posterior is distributed as Beta$(\alpha+y_t, \beta+n_t-y_t)$. The expectation of this distribution is computed as follows.

$$\begin{aligned}
\mathbb{E}[q|(n_t, y_t)] &= \frac{\alpha+n_t y_t}{\alpha+\beta+n_t} \\
&= \frac{\alpha}{\alpha+\beta+n_t}+\frac{n_t y_t}{\alpha+\beta+n_t} \\
&= \frac{\alpha+\beta}{\alpha+\beta+n_t}\cdot\frac{\alpha}{\alpha+\beta}+\frac{n_t}{\alpha+\beta+n_t}\cdot y_t \\
&= \left(1-\frac{n_t}{\alpha+\beta+n_t}\right)\cdot\mu+\frac{n_t}{\alpha+\beta+n_t}\cdot y_t
\end{aligned}$$

as the expectation of the prior Beta distribution with parameters α and β, μ, is equal to $\frac{\alpha}{\alpha+\beta}$. This shows the derivation of the weight function, $\omega[n]=\frac{n}{\alpha+\beta+n}$. ■

Proof of Proposition 2.5. Plugging $\omega[n]=\frac{an}{an+b}$ into equation $\bar{q}[\tau]$ (Proposition 2.1),

$$\bar{q}[\tau]=\begin{cases}\sum_{i=1}^{k-1}\frac{ax_r(i+\tau)}{ax_r(i+\tau)+b}\left(\frac{i}{i+\tau}\right)\cdot\frac{x}{2c\cdot k}, & 0\le\tau<\infty\\ 0, & \tau=\infty\end{cases} \tag{2.19}$$

It will be shown that for any $\tau<\infty$, increasing τ reduces $\bar{q}[\tau]$. Define $\triangle\bar{q}[\tau]=\bar{q}[\tau+1]-\bar{q}[\tau]$.

$$\begin{aligned}
&\triangle\bar{q}[\tau]\\
&=\bar{q}[\tau+1]-\bar{q}[\tau]\\
&=\frac{x}{2c\cdot k}\cdot\sum_{i=1}^{k-1}\frac{ax_r(i+\tau+1)}{ax_r(i+\tau+1)+b}\left(\frac{i}{i+\tau+1}\right)-\frac{x}{2c\cdot k}\cdot\sum_{i=1}^{k-1}\frac{ax_r(i+\tau)}{ax_r(i+\tau)+b}\left(\frac{i}{i+\tau}\right)\\
&=\frac{x}{2c\cdot k}\cdot\sum_{i=1}^{k-1}\left(\frac{ax_r\cdot i}{ax_r(i+\tau+1)+b}-\frac{ax_r\cdot i}{ax_r(i+\tau)+b}\right)\\
&=-\frac{x}{2c\cdot k}\cdot\sum_{i=1}^{k-1}\left(\frac{a^2x_r^2\cdot i}{(ax_r(i+\tau+1)+b)\,(ax_r(i+\tau)+b)}\right)<0
\end{aligned}$$

Increasing τ always decreases $\bar{q}[\tau]$ for any τ, which implies that $\text{argmax}_\tau\{\bar{q}[\tau]\}=\text{argmax}_\tau\{W[\tau]\}=0$. This proves (i), while (ii) follows trivially by applying Theorem 2.2. ■

Proof of Proposition 2.6. First, we derive the upper bound. Suppose $\exists \tau^* \geq \left\lceil \frac{\tilde{n}}{x_r} \right\rceil$. Then,

$$\omega[x_r \cdot \tau^*] \geq \omega \left[x_r \cdot \left\lceil \frac{\tilde{n}}{x_r} \right\rceil \right] \geq \omega \left[x_r \cdot \frac{\tilde{n}}{x_r} \right] = \omega[\tilde{n}] = 1$$

But this is not possible because from Lemma 2.3, we know that for any $\tau^* \in \text{argmax}_\tau\{W[\tau]\}$, $\omega[x_r \cdot \tau^*] < 1$. For the lower bound, observe that if $\tau^* < \left\lfloor \frac{\tilde{n} - x_r \cdot k}{x_r} \right\rfloor$, then

$$\begin{aligned}
\omega[x_r \cdot (k - 1 + \tau^*)] &\leq \omega \left[x_r \cdot \left(k - 1 + \left\lfloor \frac{\tilde{n} - x_r \cdot k}{x_r} \right\rfloor \right) \right] \\
&\leq \omega \left[x_r \cdot \left(k - 1 + \frac{\tilde{n} - x_r \cdot k}{x_r} \right) \right] \\
&\leq \omega \left[x_r \cdot \left(\frac{x_r \cdot k - x_r + \tilde{n} - x_r \cdot k}{x_r} \right) \right] \\
&\leq \omega \left[\tilde{n} - x_r \right] = 0
\end{aligned}$$

But if $\omega[x_r \cdot (k - 1 + \tau^*)] = 0$, then $\omega[x_r \cdot (i + \tau^*)] = 0$ for all $i \in \{1, ..., k - 1\}$ due to the monotonicity of ω. This would mean that

$$\bar{q}[\tau^*] = \sum_{i=1}^{k-1} \omega[x_r \cdot (i + \tau^*)] \left(\frac{i}{i + \tau^*} \right) \cdot \frac{x}{2c \cdot k} = 0$$

Now, set $\tau' = \tilde{n}$. In this case,

$$\bar{q}[\tau'] = \sum_{i=1}^{k-1} \omega[x_r \cdot (i + \tau')] \left(\frac{i}{i + \tau'} \right) \cdot \frac{x}{2c \cdot k} = \sum_{i=1}^{k-1} \left(\frac{i}{i + \tau'} \right) \cdot \frac{x}{2c \cdot k} > 0 = \bar{q}[\tau^*]$$

which implies that $\tau^* \notin \text{argmax}_\tau\{W[\tau]\}$. ∎

Chapter 3

Impact of Consumer Ratings on Quality Choice

3.1 Introduction

Consumer reviews and the resulting aggregated product or brand ratings play a key role in the purchasing decision of future consumers. Existing research has identified a vital relationship between product ratings and the sales and profitability of a business, implying they are pretty important strategic variables for firms. These ratings are essentially signals of a firm's quality, but is there any reverse effect of ratings affecting quality investments by those firms? Related literature has studied the implications of product ratings on the pricing strategy,[1] but studies on the impact on the resulting quality decisions and effort levels are relatively scarce. This chapter aims to analyse how product ratings of a firm and other firms in the same market affect future quality decisions when the objective is to maximise short-term profits.

Many providers of goods and services, such as restaurants and hotels, tend to make periodic quality decisions regarding their products. For the case of a restaurant, this can involve making choices on the quality of their input procurements or the relative experience or talent of the chef they will hire. For hotels, this may include the decisions on servicing and amenities, which are often short-term decisions. Alternatively, there may be an exogenous change of ownership or management in both

[1] Examples of such literature include Papanastasiou and Savva (2017), and Yu et al. (2016).

these markets who will make a new choice on the quality investments. Even though these decisions are short-term, the ratings aggregated for these markets in third-party reputation platforms such as Google Maps and Yelp include reviews from past consumers whose experiences may not be aligned with the current quality level.

In this chapter, the main focus is to answer the following questions. How do existing reputations (consumer ratings), generated from past consumers' experiences, affect the future choice of quality? Moreover, how are these affected by the reputations of competing businesses? To address these questions, a dynamic model is formulated where two firms, located on opposite ends of a Hotelling line, make quality decisions, taking into account the reputations of each other. The term *reputation* takes into account both the product rating as well the number of reviews attributed to the product rating.[2] Both the firms choose the average quality of their product, which incurs a fixed cost, for two periods. The buyers, who are short-lived and unaware of the quality changes, are uniformly drawn from the Hotelling line and decide whether to buy from one firm or the other. Before making this purchasing decision, they are endowed with an identical prior belief for both firms and update it using Bayes' Law after observing the respective ratings. After purchasing, they leave a review, which is further aggregated to the firm's rating for future consumers to observe.

A series of results are derived that study the impact of the exogenous parameters of the model on the optimal choice of quality. First, there is a potential for a "feedback loop": the present choice of quality is positively related to the current rating of the firm, ceteris paribus. So lower ratings, which decrease the profitability of businesses, may decrease the level of quality, which in turn will reduce the ratings. In a dynamic setting, this may result in a firm being driven out of the market due to bad luck. Alternatively, the quality decision is negatively related to the competitor's rating. A higher reputation of a competitor decreases the profitability and, therefore, the incentive to provide quality in this framework.

As mentioned above, product ratings typically also include the number of consumers who posted reviews. Future consumers will take this into account when they update their beliefs (more reviews will reduce the variance of the posterior and put more weight on the rating signal). If the ratings are favourable (greater than the consumer's prior), for low numbers, the firm can still have a good influence on the rating, which leads to higher incentives to provide quality if there are more reviews. If the ratings are unfavourable, then more reviews increase the precision of this

[2]Typically, both these variables are shown in aggregated ratings on many platforms.

signal, which in turn reduces the incentive to provide quality. A change in the number of reviews for the competitor will favourably affect the firm if the competitor's performance has been below the buyer's prior mean. This reinforces the buyer's belief regarding the relative worth of the present firm, increasing the relative returns of quality provision. However, the result is the exact opposite if the competitor's average rating is beyond the prior mean belief of the buyers.

Although the current model has some similarities with that of Chapter 2, there are some key technical and non-technical dissimilarities. First, both models consider ratings as being a signal of unobservable quality. However, in Chapter 2 higher ratings would lead to higher prices, while high ratings would increase the probability of sale in this model. Second, in both models, a seller makes a quality decision for short-term profits, taking into account the current reputation of the firm built on previous quality choices made by past sellers (or the same seller with short-term motivations). However, the generality of k periods in Chapter 2 is replaced by considering only 2 periods in this chapter. Finally, while chapter 2 only contained a monopoly model, this chapter also addresses new questions about how competitors' reputation affects quality choices. In short, this model extends the model in Chapter 2 in some areas, at the cost of simplification in some other areas.

Related Literature

Consumer ratings essentially can be regarded as *reputations* for businesses, and the literature on this field is quite broad.[3] Generally, the papers in this field tend to focus on sustaining reputations in the long run in a game-theoretic setting. Early works by Klein and Leffler (1981) and Shapiro (1983), using a repeated game setting, showed that firms in a competitive market would only provide high quality if the price is above costs, with the positive profits being dissipated through advertising or goodwill expenditures (e.g. initially creating exposure through supplying at a loss). Recently, the broad usage of ratings systems in product purchases has led to the literature on the optimal design of reputation systems (e.g. Hörner and Lambert (2021), Vellodi (2018), Hopenhayn and Saeedi (2019)).

Two closest research related to this chapter include Godes (2017) and Feldman et al. (2019). In the former, there is no aggregate rating per se, but information

[3]Please see Mailath and Samuelson (2006) and Bar-Isaac and Tadelis (2008) for comprehensive reviews in this field.

about the product is transmitted among the consumer base through word of mouth communications. The author found that the intensity of these communications leads to a higher choice of quality. The latter paper studied a 2-period model with a monopolist and heterogeneous consumers who can wait for more information from earlier buyers. The authors found that more uncertainty regarding the product can incentive a higher provision of quality. Both these papers have assumed a monopoly framework, but practically firms' quality provisions should depend not only on their reputations but also on the reputations of competing businesses. At the cost of making simplifying behavioural assumptions of consumers, this current research aims to contribute to the literature by analysing how these ratings can affect the future quality decisions of firms and their competitors.

Finally, the framework of the model in this chapter is related to the literature where firms compete both in terms of location and quality choice. Economides (1989) introduced quality investments in a three-stage game where firms first choose a location, then quality and then price. The present chapter focuses on the second stage, with both locations and prices being exogenously given. In this regard, there is more similarity with Brekke et al. (2006), where the authors assumed exogenous prices chosen by a regulator, while the first two stages are aligned with Economides's paper.

To the best of the author's knowledge, there is no work yet that analyses the impact of ratings components on the choice of short-term quality in a duopolistic framework, and this research aims to fill that gap.

3.2 The Model

The model is in discrete time $t \in \mathbb{N}$. Two firms, A and B, are located on opposite ends of a Hotelling line in [0,1]. They sell a product that is differentiated both vertically and horizontally. Still, only the former is chosen by a seller operating the firms, and the latter is exogenous (being on the opposite ends of the Hotelling line). Every period, one buyer[4] chooses to buy from either of the firms (binary action space, with no outside option) and then posts a review, which is a real number equal to their experience of quality, on a reputation platform. The reputation platform

[4]Having multiple buyers does not change the findings in the model, but the pivotal assumption is that there is one buyer who posts a review in a given period. If there are multiple reviewers, then the model becomes significantly more complicated.

aggregates these reviews into a rating that displays two components: (i) the average score and (ii) the number of reviews.

3.2.1 Firms and Sellers

The quality of the product for each firm $i \in \{A, B\}$ follow a normal distribution, $v_{it} \sim N(\bar{v}_i, \sigma_i^2)$ where $\bar{v}_i \in \mathbb{R}$ and $\sigma_i > 0$. At $t = t_0$, there is a new seller for each firm, but this is not observed by the buyer. Focusing on this seller and noting that the inherited history is outside the model, we normalise $t_0 = 1$. The seller chooses $\bar{v}_i$ to maximise their expected profits for two periods:

$$\mathbb{E}[\Pi_i] = px_{i1} + p\mathbb{E}[x_{i2}] - k_i\bar{v}_i^{\,2} \tag{3.1}$$

where,

- x_{i1} and x_{i2} is the probability that the buyer purchases from firm i in period 1 and 2 respectively,
- $p > 0$ is the price per unit of the product and cannot be changed by the firm (applicable in the restaurant market where competing sellers operate in a given price band),
- $k_i > 0$ is an exogenous cost parameter, and
- $\bar{v}_i$ is the mean quality chosen by the seller.

3.2.2 Buyer

Buyers are short-lived, implying that a new buyer is entering the market and making a purchasing decision every period, based on the rating aggregated by reviews from previous buyers. They perfectly know the variance of quality for each firms, σ_i^2 for $i \in \{A, B\}$, but are unaware of the mean $\bar{v}_i$. Across periods, the buyers have an identical prior belief about the mean, distributed as $N(0, \sigma_p^2)$. The utility to the buyer in period t from purchasing from firm i is given as follows:

$$u_t[i] = v_{it} - cz - p \tag{3.2}$$

where,

- v_{it} is the realised quality of firm i in period t,
- z is a independent random draw from $U[0, 1]$, expressing the horizontal taste along the Hotelling line characteristic to the buyer, and
- $c > 0$ is the transportation cost, identical across buyers.

The buyer can choose to buy either from firm A or firm B.[5] After purchasing, the buyer leaves a truthful review of her experience on the third-party platform. This review is assumed to be a real number equal to the quality experience.[6] Ratings are aggregated and displayed by the platform in the standard way: an average score of all individual reviews provided by the consumers are displayed, along with the number of consumers who have provided the reviews.

3.2.3 Period One

At $t = 1$, both firms have an existing stock of n_i reviews, which have a mean score s_i for $i \in \{A, B\}$. After observing n_i and s_i, buyers update their beliefs regarding the quality of each firm.

$$y_{i1} := \mathbb{E}_b[v_{it}|(n_i, s_i)] = \left(\frac{n_i \sigma_i^{-2}}{n_i \sigma_i^{-2} + \sigma_p^{-2}} \right) s_i \tag{3.3}$$

Any buyer z will purchase from firm A if $\mathbb{E}[u_t[A]|(n_A, s_A)] > \mathbb{E}[u_t[B]|(n_B, s_B)]$, from B otherwise. For firm A,

$$\begin{aligned} \mathbb{E}_b[u_1[A]|(n_A, s_A)] &= \mathbb{E}_b[v_{A1}|(n_A, s_A)] - cz - p \\ &= y_{A1} - cz - p \end{aligned} \tag{3.4}$$

Similarly, $\mathbb{E}[u_t[B]|(n_B, s_B)] = y_{B1} - cz - P$. The buyer will purchase from A if:

$$\begin{aligned} &\mathbb{E}_b[u_1[A]|(n_A, s_A)] > \mathbb{E}_b[u_t[B]|(n_B, s_B)] \\ \Longrightarrow\ &y_{A1} - cz - p > y_{B1} - cz - p \end{aligned}$$

[5]Eventually, a third option of not buying can be incorporated into an extension of this model.

[6]Because the model is forward-looking, only the expected rating will be relevant, and this is equal to the mean quality chosen by the firm, v_i.

$$\implies z \leq \frac{1}{2} + \frac{y_{A1} - y_{B1}}{2c} \tag{3.5}$$

As $z \sim UID[0,1]$, the probability of purchasing from firm A and B respectively are:

$$x_{A1} = \frac{1}{2} + \frac{y_{A1} - y_{B1}}{2c} \tag{3.6}$$

$$x_{B1} = \frac{1}{2} + \frac{y_{B1} - y_{A1}}{2c} \tag{3.7}$$

The above can be expressed as $x_{i1} = \frac{1}{2} + \frac{y_{i1} - y_{j1}}{2c}$ for $j \neq i$.

3.2.4 Period Two

At $t = 2$, the reputation of firm i, for $i \in \{A, B\}$, will depend on whether the reviewer purchased from firm i or for firm $j \neq i$. As an example, for firm A, with probability x_{A1} the review was left for firm A, and with probability x_{B1} the review was left for firm B. For the latter, the score will remain unchanged. Therefore,

$$\begin{aligned}\mathbb{E}[y_{A2}] &= \mathbb{E}\left[\mathbb{E}_b[u_1[A]|(n_{A(t+1)}, s_{A(t+1)})]\right] \\ &= x_{A1}\left(\frac{(n_A+1)\sigma_A^{-2}}{(n_A+1)\sigma_A^{-2} + \sigma_p^{-2}}\right)\left(\frac{n_A s_A + \bar{v}_A}{n_A+1}\right) + x_{B1}y_{A1}\end{aligned} \tag{3.8}$$

Please note the different notations for expectations for the sellers and the buyers. Seller's expectation is denoted as $\mathbb{E}$, while the buyer's expectation is denoted as $\mathbb{E}_b$. Similarly, for firm B:

$$\mathbb{E}[y_{B2}] = x_{B1}\left(\frac{(n_B+1)\sigma_B^{-2}}{(n_A+1)\sigma_B^{-2} + \sigma_p^{-2}}\right)\left(\frac{n_B s_B + \bar{v}_B}{n_B+1}\right) + x_{A1}y_{B1} \tag{3.9}$$

Using the same reasoning as in period one, the probability of purchasing from firm A and B respectively in period two are:

$$\mathbb{E}[x_{A2}] = \frac{1}{2} + \mathbb{E}\left[\frac{y_{A2} - y_{B2}}{2c}\right] \tag{3.10}$$

$$\mathbb{E}[x_{B2}] = \frac{1}{2} + \mathbb{E}\left[\frac{y_{B2} - y_{A2}}{2c}\right] \tag{3.11}$$

The two sellers choose their respective $\bar{v}_i$ to maximise the expected profits as in equation 3.1. The profit maximising level of mean quality is given as follows.

$$\bar{v}_A^* = \frac{p\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2} - \frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right) + c\right)}{8c^2 k_A \sigma_A^2\left(\frac{n_A+1}{\sigma_A^2} + \frac{1}{\sigma_P^2}\right)} \tag{3.12}$$

$$\bar{v}_B^* = \frac{p\left(\sigma_P^2\left(\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2} - \frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}\right) + c\right)}{8c^2 k_B \sigma_B^2\left(\frac{n_B+1}{\sigma_B^2} + \frac{1}{\sigma_P^2}\right)} \tag{3.13}$$

3.3 Comparative Statics

In this section, assuming the firms maximise their profits and follow the optimising behaviour derived in equations 3.12 and 3.13, a series of results will be presented that attempts to analyse the impact of the model parameters on the quality outcomes. For analytical simplicity, the analysis is restricted to interior solutions where x_{it} is in the open interval $(0, 1)$ for any firm i or any period t.

Proposition 3.1. *A firm's mean quality provision increases with its own rating and decreases with the competitor's rating.*

If the firm's inherited rating increases, then the firm inherits a higher share of the market, which drives down the relative cost of providing quality, inducing a higher level of provision. The mechanism through which this happens is due to the higher marginal sales from a unit of quality provision (while the cost per unit remains the same). A higher probability of sales would mean that future buyers would be more likely to know the level of quality with a higher degree of precision, which incentivises the firm to increase the quality level. Alternatively, if the competitor's rating increases, the firm is endowed with a relatively lower market share. This increases the relative cost of quality for every part of the demand curve, inducing a lower level of provision.

The implications of this in a dynamic setting are quite important. This is due to the potential of a "feedback loop": if a firm inherits a high rating, then the quality choice increases. If this new level of quality is higher than the inherited rating, then the new rating of the firm will go up. On the other hand, if the new level of quality is lower than the inherited rating, then the new rating will go down, which will result in an even lower level of quality. In this case, eventually, the evolution of quality would mean that quality provision would converge to zero in the long run.

Proposition 3.2. *If $s_i \leq 0$, then firm i's optimal choice of quality is strictly decreasing with each additional review. If $s_i > 0$, then there exists n^{max} such that for $n_i \leq n_{max}$, quality increases with n_i and for $n_i > n_{max}$ quality decreases with n_i.*

If firm i has a low rating (below the buyer's prior mean), more reviews, assuming the rating is unchanged, increase the precision of that rating, reducing the firm's incentive to provide quality. If the firm has a positive rating, there are two effects of increasing n_i that determine the seller's incentive to provide quality. The positive effect is the more precise signal due to a higher number of ratings. The negative effect is the decreased contribution the current seller's effort will have on the average rating. For example, if there are 2 reviews, then reviews generated from the seller's current quality will have relatively more impact on the rating than if there are 1000 reviews, for which the impact will be quite small. Until a certain finite number n^{max} (which may be zero), the former effect may dominate the latter, during which the firm has no incentive to provide quality. After that point, the latter effect dominates, and each additional review decreases the incentive for sellers to provide quality. It should be noted that in the limit as $n_i \to \infty$, $v_i^* \to 0$. This result further reinforces the finding from Chapter 2 that reviews should not be kept in perpetuity.

Proposition 3.3 (Precision of competitor's rating)**.** *A firm's mean quality provision:*

(i) decreases with the number of reviews left for the competitor, n_{-i}, if $s_{-i} > 0$ (i.e. the competitor's rating is better than the consumer's prior expectations), and increases otherwise;

(ii) increases with the competitor's variance of quality, σ^2_{-i}, if $s_{-i} > 0$, and decreases otherwise.

If the competitor's rating is higher than the consumer's prior expectations, then more reviews left for the competitor (alternatively, a lower variance of quality) increase the precision of the buyer's belief that the competitor's product is "better than expected". This decreases the incentive of a firm to provide higher quality. Alternatively, if the competitor's rating is below the prior mean, then more n_{-i}, assuming s_{-i} is unchanged, further reinforces the posterior belief that the competitor's product is "worse than expected". This increases the probability of sales for firm i, which increases the marginal returns to quality and therefore incentivises the provision of a higher level of average quality.

Proposition 3.4. *A firm's mean quality provision increases with its own quality variance, σ_i^2, only if its rating, s_i, is sufficiently low, and decreases otherwise.*

A higher variance means that there is more uncertainty regarding the consumer experience. If the current rating of the firm is sufficiently low, then this higher uncertainty gives the firm more opportunity to push up the rating with higher quality provision. Otherwise, a higher variance reduces the precision of this signal, which reduces the incentive to provide quality.

Proposition 3.5. *A higher transportation cost for consumers increases the quality provision of a firm if its own rating is sufficiently low and decreases the average quality otherwise.*

Intuitively, a higher transportation cost for the buyer would mean that they would be less inclined to buy from the farthest firm, which can lead to higher or lower effort depending on the current reputation of the firm in question. If firm i has a good reputation through a sufficiently high product rating, then it is aware that the existing buyers would be less inclined to buy from the other firm, which will decrease the incentives to provide high quality. However, if the firm suffers from a poor reputation through a low product rating, then a higher transportation cost would mean losing more customers without increasing the quality level. This will incentivise the firm to provide a higher level of quality.

3.4 Conclusions and Future Work

This chapter formulated a simple model to analyse the impact of a firm's and its competitor's ratings on the future choice of quality if the objective is to maximise short term profits. It has been found that a firm's high ratings induce higher quality, while if the competitor's rating increases, this reduces the quality. This happens mainly because an increase in the rating of the firm (competitor) will increase (decrease) the probability of sale, which in turn will generate more ratings, and therefore increasing (decreasing) the incentive to provide quality.

What about the number of reviews posted? If a firm's rating is below the consumer's prior expectations, then more reviews will decrease the variance of the signal, which will hurt the firm and therefore reduce the incentive to provide quality. When the firm's rating is "better-than-expected" for the consumer, more reviews may induce higher quality up until a certain point, beyond which the firm's contribution to the rating will be minor and thus decreasing the incentive to provide high quality. Other comparative statics have been derived for some of the other model parameters.

The implications of the above are pretty important. Even if a firm's bad ratings may happen due to bad luck, these will worsen the situation as the firm's quality also decreases, reducing the chance to push up the rating. Of course, the important assumption behind these is that the firms are themselves maximising short term profits, and these may not hold if the firm's objective is for the longer horizon. This brings us to a natural extension: what if a firm's plan for the longer term is to force out the competitors by building a reputation and act as a monopoly in the market? This will require an analysis of game-theoretic strategies that may differ from the short-term profit maximisation as observed in this chapter.

The next steps with regard to this research are as follows. Considering the evolution of the rating derived with regard to short-term profit maximisation, an infinite sequence of sellers can be considered to compute the steady-state quality of the firm in the long run. However, it can be expected that if ratings are never deleted, the quality levels will converge to zero (see Proposition 3.2). In that case, following Chapter 2, a fixed *window*, n^*, of reviews can be kept. This n^* will be chosen by the platform to maximise some welfare functions (e.g. average total or consumer surplus). Additionally, another natural next step to this chapter is to empirically test the findings that have been proposed.

Appendix

Derivation of the optimal quality choice. For firm A the expected profit of the seller from equation 3.1 is as follows.

$$\Pi_A = px_{A1} + p\mathbb{E}[x_{A2}] - k_A\bar{v}_A^2 \tag{3.14}$$

We know that $x_{A1} = \frac{1}{2} + \frac{y_{A1}-y_{B1}}{2c}$ and $x_{A2} = \frac{1}{2} + \frac{y_{A2}-y_{B2}}{2c}$. Substituting $\{y_{A2}, y_{B1}, y_{A1}, y_{B2}\}$ from equations 3.3, 3.8 and 3.9 into these equations, we

$$x_{A1} = \frac{1}{2}\left(\frac{\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2} - \frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)}{c} + 1\right)$$

$$\mathbb{E}[x_{A2}] = \frac{1}{2} + \frac{\frac{n_A s_A\left(\frac{\sigma_P^2\left(\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2} - \frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}\right)}{c}+1\right)}{\frac{n_A}{\sigma_A^2}+\frac{1}{\sigma_P^2}} + \frac{(n_A s_A+\bar{v}_A)\left(\frac{\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2} - \frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)}{c}+1\right)}{\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}}}{4\cdot c\cdot\sigma_A^2}$$
$$- \frac{\frac{n_B s_B\left(\frac{\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2} - \frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)}{c}+1\right)}{\frac{n_B}{\sigma_B^2}+\frac{1}{\sigma_P^2}} + \frac{(n_B s_B+v_B)\left(\frac{\sigma_P^2\left(\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2} - \frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}\right)}{c}+1\right)}{\frac{n_B+1}{\sigma_B^2}+\frac{1}{\sigma_P^2}}}{4\cdot c\cdot\sigma_B^2}$$

Observing the above, it is easy to see that only the second term in x_{A2} contains $\bar{v}_A$. When computing for the maximum, taking the derivative of the profit function with respect to $\bar{v}_A$, we obtain:

$$\frac{\partial\Pi_A}{\bar{v}_A} = \frac{p\left(\frac{\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2} - \frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)}{c} + 1\right)}{4c\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2} + \frac{1}{\sigma_P^2}\right)} - 2k_A\bar{v}_A = 0 \tag{3.15}$$

Solving equation 3.15 we obtain:

$$\bar{v}_A^* = \frac{p\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2} - \frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right) + c\right)}{8c^2k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2} + \frac{1}{\sigma_P^2}\right)}$$

The optimal choice of mean quality for firm B, $\bar{v}_B$, can be derived similarly. ■

Proof of Proposition 3.1. We know from equation 3.12 that

$$\bar{v}_A^* = \frac{p\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2} - \frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)}{8c^2 k_A \sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}$$

Taking the derivative with respect to s_A, we obtain:

$$\frac{\partial \bar{v}_A^*}{\partial s_A} = \frac{p\cdot n_A\sigma_P^2}{8c^2 k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)\left(n_A\sigma_P^2+\sigma_A^2\right)} \tag{3.16}$$

which is positive as all the terms in both the numerator and denominator are positive (with the derivative being zero when $n_A = 0$, in which case s_A does not exist).

Taking the derivative with respect to s_B, we obtain:

$$\frac{\partial \bar{v}_A^*}{\partial s_B} = -\frac{p\cdot n_B\sigma_P^2}{8c^2 k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)\left(n_B\sigma_P^2+\sigma_B^2\right)} \tag{3.17}$$

which, similarly, can be deduced to be negative. ■

Two lemmas will be required for the proof of Proposition 3.2.

Lemma 3.1. *For the variables x_{A1} and x_{B1} to be greater than zero, the following condition needs to be satisfied.*

$$-\frac{\left(n_B\sigma_P^2+\sigma_B^2\right)\left(n_A\sigma_P^2\left(c-s_A\right)+c\sigma_A^2\right)}{n_B\sigma_P^2\left(n_A\sigma_P^2+\sigma_A^2\right)} < s_B < \frac{\left(n_B\sigma_P^2+\sigma_B^2\right)\left(n_A\sigma_P^2\left(c+s_A\right)+c\sigma_A^2\right)}{n_B\sigma_P^2\left(n_A\sigma_P^2+\sigma_A^2\right)}$$

Proof. We know that

$$x_{A1} = \frac{1}{2} + \frac{y_{A1}-y_{B1}}{2c}$$
$$x_{B1} = \frac{1}{2} + \frac{y_{B1}-y_{A1}}{2c}$$

Plugging in y_{A1} and y_{B1} from equations 3.3 into the above, we obtain:

$$x_{A1} = \frac{1}{2}\left(\frac{\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)}{c}+1\right)$$
$$x_{B1} = \frac{1}{2}\left(\frac{\sigma_P^2\left(\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}-\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}\right)}{c}+1\right)$$

Additionally, $x_{A1} > 0$ and $x_{B1} > 0$ when the solution is interior. For $x_{A1} > 0$ we require:

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)}{c}+1\right) > 0\\
\implies &\frac{1}{2}\left(\frac{\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c}{c}\right) > 0\\
\implies &\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right) > 0\\
\implies &\frac{n_B s_B \sigma_P^2}{n_B\sigma_P^2+\sigma_B^2} < \sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}\right)+c\\
\implies &\frac{n_B s_B \sigma_P^2}{n_B\sigma_P^2+\sigma_B^2} < \frac{n_A\sigma_P^2\left(s_A+c\right)+c\sigma_A^2}{n_A\sigma_P^2+\sigma_A^2}\\
\implies &s_B < \frac{\left(n_B\sigma_P^2+\sigma_B^2\right)\left(n_A\sigma_P^2\left(c+s_A\right)+c\sigma_A^2\right)}{n_B\sigma_P^2\left(n_A\sigma_P^2+\sigma_A^2\right)}
\end{aligned}
$$

The fact that $s_B > -\frac{\left(n_B\sigma_P^2+\sigma_B^2\right)\left(n_A\sigma_P^2(c-s_A)+c\sigma_A^2\right)}{n_B\sigma_P^2\left(n_A\sigma_P^2+\sigma_A^2\right)}$ can be similarly shown by solving $x_{B1} > 0$. ■

Lemma 3.2. *When $s_A > 0$, under the interior constraints, $n_B\sigma_P^4\left(s_A - s_B + c\right) + \sigma_B^2\sigma_P^2\left(s_A + c\right) > 0$.*

Proof. We proceed by contradiction. Suppose $s_A > 0$ and $n_B\sigma_P^4\left(s_A - s_B + c\right) + \sigma_B^2\sigma_P^2\left(s_A + c\right) \leq 0$. Then,

$$
\begin{aligned}
&n_B\sigma_P^4\left(s_A - s_B + c\right) + \sigma_B^2\sigma_P^2\left(s_A + c\right) \leq 0\\
\Leftrightarrow\quad &\sigma_B^2\sigma_P^2\left(s_A + c\right) + s_A n_B\sigma_P^4 + c n_B\sigma_P^4 - n_B s_B\sigma_P^4 \leq 0\\
\Leftrightarrow\quad &n_B s_B\sigma_P^4 \geq \sigma_B^2\sigma_P^2\left(s_A + c\right) + s_A n_B\sigma_P^4 + c n_B\sigma_P^4\\
\Leftrightarrow\quad &s_B \geq \frac{\sigma_B^2\left(s_A + c\right)}{n_B\sigma_P^2} + s_A + c\\
\Leftrightarrow\quad &s_B \geq \frac{\left(s_A + c\right)\left(n_B\sigma_P^2+\sigma_B^2\right)}{n_B\sigma_P^2}\\
\Leftrightarrow\quad &s_B \geq \frac{\left(s_A + c\right)\left(\sigma_A^2 + n_A\sigma_P^2\right)\left(n_B\sigma_P^2+\sigma_B^2\right)}{n_B\sigma_P^2\left(\sigma_A^2 + n_A\sigma_P^2\right)}\\
\Leftrightarrow\quad &s_B \geq \frac{\left(\left(s_A + c\right)\sigma_A^2 + \left(s_A + c\right)n_A\sigma_P^2\right)\left(n_B\sigma_P^2+\sigma_B^2\right)}{n_B\sigma_P^2\left(\sigma_A^2 + n_A\sigma_P^2\right)}\\
\Leftrightarrow\quad &s_B > \frac{\left(c\sigma_A^2 + \left(s_A + c\right)n_A\sigma_P^2\right)\left(n_B\sigma_P^2+\sigma_B^2\right)}{n_B\sigma_P^2\left(\sigma_A^2 + n_A\sigma_P^2\right)}
\end{aligned}
$$

Where $s_A > 0$ is used in the last step. However, this is not possible as from Lemma

3.1 we know that $s_B < \frac{\left(n_B\sigma_P^2+\sigma_B^2\right)\left(n_A\sigma_P^2(c+s_A)+c\sigma_A^2\right)}{n_B\sigma_P^2\left(n_A\sigma_P^2+\sigma_A^2\right)}$. ∎

Proof of Proposition 3.2. Define $\triangle\bar{v}_A^*[n_A] := \bar{v}_A^*[n_A+1] - \bar{v}[n_A]$. Then,

$$\begin{aligned}\triangle\bar{v}_A^*[n_A] &= \frac{p\left(\sigma_P^2\left(\frac{(n_A+1)s_A}{(n_A+1)\sigma_P^2+\sigma_A^2} - \frac{n_Bs_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)}{8c^2k_A\sigma_A^2\left(\frac{n_A+2}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)} - \frac{p\left(\sigma_P^2\left(\frac{n_As_A}{n_A\sigma_P^2+\sigma_A^2} - \frac{n_Bs_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)}{8c^2k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)} \\ &\vdots \\ &= \frac{H}{G} \end{aligned} \tag{3.18}$$

where:

$$\begin{aligned} H = &-p\sigma_P^4\left(\sigma_A^2\left(n_B\sigma_P^2\left(-s_A-s_B+c\right)+\sigma_B^2\left(c-s_A\right)\right)\right) \\ &- p\sigma_P^4\left(n_A\sigma_P^2\left(n_B\sigma_P^2\left(s_A-s_B+c\right)+\sigma_B^2\left(s_A+c\right)\right)\right)\end{aligned}$$

$$G = 8c^2k_A\left(n_A\sigma_P^2+\sigma_A^2\right)\left(\left(n_A+1\right)\sigma_P^2+\sigma_A^2\right)\left(\left(n_A+2\right)\sigma_P^2+\sigma_A^2\right)\left(n_B\sigma_P^2+\sigma_B^2\right)$$

It is easily observed that the denominator G is always positive under the model assumptions. What about the numerator? Let us first assume that $s_A \leq 0$. Then the numerator can be rearranged into:

$$\begin{aligned} &p\sigma_P^4\left(\sigma_A^2\left(n_B\sigma_P^2\left(-s_A-s_B+c\right)+\sigma_B^2\left(c-s_A\right)\right)\right) \\ &-p\sigma_P^4\left(n_A\sigma_P^2\left(n_B\sigma_P^2\left(s_A-s_B+c\right)+\sigma_B^2\left(s_A+c\right)\right)\right) \\ &= -p\sigma_P^4\left(n_A\sigma_B^2\sigma_P^2\left(s_A+c\right)+cn_An_B\sigma_P^4+c\sigma_A^2n_B\sigma_P^2+\sigma_A^2\sigma_B^2\left(c-s_A\right)\right) \\ &\quad -p\sigma_P^4\left(n_As_An_B\sigma_P^4-s_A\sigma_A^2n_B\sigma_P^2\right)-ps_B\sigma_P^4\left(-n_An_B\sigma_P^4-\sigma_A^2n_B\sigma_P^2\right) \\ &= -p\sigma_P^4\left(n_B\sigma_P^2+\sigma_B^2\right)\left(n_A\sigma_P^2\left(s_A+c\right)+\sigma_A^2\left(c-s_A\right)\right)+pn_Bs_B\sigma_P^6\left(n_A\sigma_P^2+\sigma_A^2\right)\end{aligned}$$

Suppose the numerator is H positive (which implies $\triangle\bar{v}_A^*[n_A] > 0$). Then

$$\begin{aligned} &-p\sigma_P^4\left(n_B\sigma_P^2+\sigma_B^2\right)\left(n_A\sigma_P^2\left(s_A+c\right)+\sigma_A^2\left(c-s_A\right)\right)+pn_Bs_B\sigma_P^6\left(n_A\sigma_P^2+\sigma_A^2\right) > 0 \\ \Leftrightarrow \quad & pn_Bs_B\sigma_P^6\left(n_A\sigma_P^2+\sigma_A^2\right) > p\sigma_P^4\left(n_B\sigma_P^2+\sigma_B^2\right)\left(n_A\sigma_P^2\left(s_A+c\right)+\sigma_A^2\left(c-s_A\right)\right) \\ \Leftrightarrow \quad & s_B > \frac{\left(n_B\sigma_P^2+\sigma_B^2\right)\left(n_A\sigma_P^2\left(s_A+c\right)+\sigma_A^2\left(c-s_A\right)\right)}{n_B\sigma_P^2\left(n_A\sigma_P^2+\sigma_A^2\right)} \\ \Leftrightarrow \quad & s_B \geq \frac{\left(n_B\sigma_P^2+\sigma_B^2\right)\left(n_A\sigma_P^2\left(c+s_A\right)+c\sigma_A^2\right)}{n_B\sigma_P^2\left(n_A\sigma_P^2+\sigma_A^2\right)}\end{aligned} \tag{3.19}$$

where the last weak inequality follows due to $s_A \leq 0$. But from Lemma 3.1 we know that $s_B < \frac{\left(n_B\sigma_P^2+\sigma_B^2\right)\left(n_A\sigma_P^2(c+s_A)+c\sigma_A^2\right)}{n_B\sigma_P^2\left(n_A\sigma_P^2+\sigma_A^2\right)}$, so this is not possible. Therefore, $s \leq 0 \implies \triangle\bar{v}_A^*[n_A] < 0$.

For $s_A > 0$, let us arrange the numerator of equation 3.18 in the following way.

$$\begin{aligned}
&-p\sigma_P^4\left(\sigma_A^2\left(n_B\sigma_P^2\left(-s_A-s_B+c\right)+\sigma_B^2\left(c-s_A\right)\right)\right)\\
&-p\sigma_P^4\left(n_A\sigma_P^2\left(n_B\sigma_P^2\left(s_A-s_B+c\right)+\sigma_B^2\left(s_A+c\right)\right)\right)\\
&=-pn_A\sigma_P^6\left(n_B\sigma_P^2\left(s_A-s_B+c\right)+\sigma_B^2\left(s_A+c\right)\right)\\
&-p\sigma_A^2\sigma_P^4\left(n_B\sigma_P^2\left(-s_A-s_B+c\right)+\sigma_B^2\left(c-s_A\right)\right)
\end{aligned} \tag{3.20}$$

This expression is non-negative when:

$$\begin{aligned}
&-pn_A\sigma_P^6\left(n_B\sigma_P^2\left(s_A-s_B+c\right)+\sigma_B^2\left(s_A+c\right)\right)\\
&-p\sigma_A^2\sigma_P^4\left(n_B\sigma_P^2\left(-s_A-s_B+c\right)+\sigma_B^2\left(c-s_A\right)\right)\geq 0\\
\Leftrightarrow\quad & pn_A\sigma_P^6\left(n_B\sigma_P^2\left(s_A-s_B+c\right)+\sigma_B^2\left(s_A+c\right)\right)\leq\\
&-p\sigma_A^2\sigma_P^4\left(n_B\sigma_P^2\left(-s_A-s_B+c\right)+\sigma_B^2\left(c-s_A\right)\right)\\
\Leftrightarrow\quad & n_A\leq-\frac{\sigma_A^2\left(n_B\sigma_P^2\left(-s_A-s_B+c\right)+\sigma_B^2\left(c-s_A\right)\right)}{n_B\sigma_P^4\left(s_A-s_B+c\right)+\sigma_B^2\sigma_P^2\left(s_A+c\right)}
\end{aligned} \tag{3.21}$$

The sign will not change due to $n_B\sigma_P^4\left(s_A-s_B+c\right)+\sigma_B^2\sigma_P^2\left(s_A+c\right) > 0$, which we know from lemma 3.2. ∎

Proof of Proposition 3.3. For (i), define $\triangle\bar{v}_A^*[n_B] := \bar{v}_A^*[n_B+1]-\bar{v}[n_B]$. Then,

$$\begin{aligned}
&\triangle\,\bar{v}_A^*[n_B]\\
&=\frac{p\left(\sigma_P^2\left(\frac{n_As_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{(n_B+1)s_B}{(n_B+1)\sigma_P^2+\sigma_B^2}\right)+c\right)}{8c^2k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}-\frac{p\left(\sigma_P^2\left(\frac{n_As_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_Bs_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)}{8c^2k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}\\
&=\frac{p\left(\sigma_P^2\left(\frac{n_As_A}{n_A\sigma_P^2+\sigma_A^2}\right)+c\right)-p\left(\sigma_P^2\left(\frac{n_As_A}{n_A\sigma_P^2+\sigma_A^2}\right)+c\right)-p\cdot\sigma_P^2\left(\frac{(n_B+1)s_B}{(n_B+1)\sigma_P^2+\sigma_B^2}-\frac{n_Bs_B}{n_B\sigma_P^2+\sigma_B^2}\right)}{8c^2k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}\\
&=\frac{-p\cdot\sigma_P^2\left(\frac{\left(n_Bs_B+s_B\right)\left(n_B\sigma_P^2+\sigma_B^2\right)-\left(n_Bs_B\right)\left((n_B+1)\sigma_P^2+\sigma_B^2\right)}{\left((n_B+1)\sigma_P^2+\sigma_B^2\right)\left(n_B\sigma_P^2+\sigma_B^2\right)}\right)}{8c^2k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}\\
&=\frac{-p\cdot\sigma_P^2\left(\left(n_Bs_B+s_B\right)\left(n_B\sigma_P^2+\sigma_B^2\right)-\left(n_Bs_B\right)\left((n_B+1)\sigma_P^2+\sigma_B^2\right)\right)}{8c^2k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)\left((n_B+1)\sigma_P^2+\sigma_B^2\right)\left(n_B\sigma_P^2+\sigma_B^2\right)}\\
&=\frac{-p\cdot\sigma_P^2\left(n_B^2s_B\sigma_P^2+s_Bn_B\sigma_P^2+n_Bs_B\sigma_B^2+s_B\sigma_B^2-n_B^2s_B\sigma_P^2-n_Bs_B\sigma_P^2-n_Bs_B\sigma_B^2\right)}{8c^2k_A\frac{1}{\sigma_P{}^2}\left(\left(n_A+1\right)\sigma_P^2+\sigma_A^2\right)\left((n_B+1)\sigma_P^2+\sigma_B^2\right)\left(n_B\sigma_P^2+\sigma_B^2\right)}
\end{aligned}$$

$$= \frac{-p\cdot\sigma_P^4 s_B\sigma_B^2}{8c^2k_A\left(\left(n_A+1\right)\sigma_P^2+\sigma_A^2\right)\left((n_B+1)\sigma_P^2+\sigma_B^2\right)\left(n_B\sigma_P^2+\sigma_B^2\right)}$$

where it is easy to observe that the denominator is always positive. The numerator, on the other hand, is negative when $s_B > 0$, positive when $s_B < 0$ and zero when $s_B = 0$. Therefore,

$$\mathrm{sign}\left(\triangle\bar{v}_A^*[n_B]\right) = -\,\mathrm{sign}(s_B)$$

For (ii), taking the derivative of $\bar{v}_A^*$ with respect to σ_B^2, we obtain:

$$\begin{aligned}\frac{\partial\bar{v}_A^*}{\partial\sigma_B^2} &= \frac{-\frac{0\cdot\left(n_B\sigma_P^2+\sigma_B^2\right)-p\cdot\sigma_p^2 n_B s_B\cdot 1}{\left(n_B\sigma_P^2+\sigma_B^2\right)^2}}{8c^2k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}\\ &= \frac{p\cdot\sigma_p^2 n_B s_B}{8c^2k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)\left(n_B\sigma_P^2+\sigma_B^2\right)^2}\end{aligned}$$

Again, the denominator is always positive, while the sign of the numerator depends on the sign of s_B. In this case, $\mathrm{sign}\left(\frac{\partial\bar{v}_A^*}{\partial\sigma_B^2}\right) = \mathrm{sign}(s_B)$. ■

Proof of Proposition 3.4. Taking the derivative of $\bar{v}_A^*$ with respect to σ_A^2, we obtain:

$$\begin{aligned}&\frac{\partial\bar{v}_A^*}{\partial\sigma_A^2}\\ &= \frac{\partial}{\partial\sigma_A^2}\left(p\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)\left(8c^2k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)\right)^{-1}\right)\\ &= \frac{p}{8c^2k_A}\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)\left(-\frac{\sigma_P^2}{\left(\left(n_A+1\right)\sigma_P^2+\sigma_A^2\right)^2}\right)\\ &\qquad\qquad +\frac{p}{8c^2k_A}\left(\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)\right)^{-1}\left(\frac{-\sigma_P^2 n_A s_A}{\left(n_A\sigma_P^2+\sigma_A^2\right)^2}\right)\\ &= -\frac{p\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)}{8c^2k_A\sigma_P^2\left(n_A+\frac{\sigma_A^2}{\sigma_P^2}+1\right)^2}-\frac{p n_A s_A\sigma_P^2}{8c^2k_A\left(n_A+\frac{\sigma_A^2}{\sigma_P^2}+1\right)\left(n_A\sigma_P^2+\sigma_A^2\right)^2}\\ &= \frac{p\left(-\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)-\frac{n_A s_A\sigma_P^4\left(n_A+\frac{\sigma_A^2}{\sigma_P^2}+1\right)}{\left(n_A\sigma_P^2+\sigma_A^2\right)^2}-c\right)}{8c^2k_A\sigma_P^2\left(n_A+\frac{\sigma_A^2}{\sigma_P^2}+1\right)^2}\end{aligned} \tag{3.22}$$

Equation 3.22 is positive when:

$$\frac{p\left(-\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)-\frac{n_A s_A\sigma_P^4\left(n_A+\frac{\sigma_A^2}{\sigma_P^2}+1\right)}{\left(n_A\sigma_P^2+\sigma_A^2\right)^2}-c\right)}{8c^2k_A\sigma_P^2\left(n_A+\frac{\sigma_A^2}{\sigma_P^2}+1\right)^2}>0$$

$$\Leftrightarrow \quad -\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)-\frac{n_A s_A\sigma_P^4\left(n_A+\frac{\sigma_A^2}{\sigma_P^2}+1\right)}{\left(n_A\sigma_P^2+\sigma_A^2\right)^2}-c>0$$

$$\Leftrightarrow \quad s_A\left(-\frac{n_A\sigma_P^4\left(n_A+\frac{\sigma_A^2}{\sigma_P^2}+1\right)}{\left(n_A\sigma_P^2+\sigma_A^2\right)^2}-\frac{n_A\sigma_P^2}{n_A\sigma_P^2+\sigma_A^2}\right)+\frac{n_B s_B\sigma_P^2}{n_B\sigma_P^2+\sigma_B^2}-c>0$$

$$\Leftrightarrow \quad s_A\left(\frac{n_A\sigma_P^4\left(n_A+\frac{\sigma_A^2}{\sigma_P^2}+1\right)}{\left(n_A\sigma_P^2+\sigma_A^2\right)^2}+\frac{n_A\sigma_P^2}{n_A\sigma_P^2+\sigma_A^2}\right)<\frac{n_B s_B\sigma_P^2}{n_B\sigma_P^2+\sigma_B^2}-c$$

$$\Leftrightarrow \quad s_A\left(\frac{n_A\left(\left(2n_A+1\right)\sigma_P^4+2\sigma_A^2\sigma_P^2\right)}{\left(n_A\sigma_P^2+\sigma_A^2\right)^2}\right)<\frac{n_B s_B\sigma_P^2}{n_B\sigma_P^2+\sigma_B^2}-c$$

$$\Leftrightarrow \quad s_A<\frac{\left(n_A\sigma_P^2+\sigma_A^2\right)^2\left(\frac{n_B s_B\sigma_P^2}{n_B\sigma_P^2+\sigma_B^2}-c\right)}{n_A\left(\left(2n_A+1\right)\sigma_P^4+2\sigma_A^2\sigma_P^2\right)} \qquad \blacksquare$$

Proof of Proposition 3.5. Taking the derivative of $\bar{v}_A^*$ with respect to c, we obtain:

$$\begin{aligned}
\frac{\partial\bar{v}_A^*}{\partial c} &= \frac{\partial}{\partial c}\left(\frac{p}{8k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)\left(\frac{1}{c^2}\right)\right)\\
&= \frac{p}{8k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)\left(\frac{-2}{c^3}\right)\\
&\qquad +\frac{p}{8k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}\left(\frac{1}{c^2}\right)\\
&= -\frac{p\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)}{4c^3k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}+\frac{p}{8c^2k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}\\
&= \frac{p\left(c-2\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)\right)}{8c^3k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}
\end{aligned} \tag{3.23}$$

Equation 3.23 is positive when:

$$\frac{p\left(c-2\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)\right)}{8c^3k_A\sigma_A^2\left(\frac{n_A+1}{\sigma_A^2}+\frac{1}{\sigma_P^2}\right)}>0$$

$$\Leftrightarrow \quad c-2\left(\sigma_P^2\left(\frac{n_A s_A}{n_A\sigma_P^2+\sigma_A^2}-\frac{n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)+c\right)>0$$

$$
\begin{aligned}
\Leftrightarrow \quad & s_A\left(-\frac{2\sigma_P^2 n_A}{n_A\sigma_P^2+\sigma_A^2}\right)-c+\left(\frac{2\sigma_P^2 n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)>0 \\
\Leftrightarrow \quad & s_A\left(\frac{2\sigma_P^2 n_A}{n_A\sigma_P^2+\sigma_A^2}\right)<\left(\frac{2\sigma_P^2 n_B s_B}{n_B\sigma_P^2+\sigma_B^2}\right)-c \\
\Leftrightarrow \quad & s_A<\frac{(n_A\sigma_P^2+\sigma_A^2)\left(\frac{2n_B s_B\sigma_P^2}{n_B\sigma_P^2+\sigma_B^2}-c\right)}{2n_A\sigma_P^2} \qquad \blacksquare
\end{aligned}
$$

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