Trigonometric $\vee$-systems and solutions of WDVV and related equations

by

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Abstract

This thesis contains three parts related to trigonometric solutions of Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations.

In the first part of the thesis we consider a class of trigonometric solutions of WDVV equations determined by collections $\mathcal{A}$ of covectors with multiplicities. This class of solutions involves an extra variable which makes them non-trivial already for planar collections. These solutions have the general form

$$F = \sum_{\alpha \in \mathcal{A}} c_\alpha f(\alpha(x)) + Q, \tag{0.1}$$

where $c_\alpha \in \mathbb{C}$ are multiplicity parameters and Q is a cubic polynomial in $x = (x_1, \ldots, x_N)$ and additional variable y, and f is the function of a single variable z satisfying $f'''(z) = \cot z$. We show that such solutions can be restricted to special subspaces to produce new solutions of the same type. We find new solutions given by restrictions of root systems, as well as examples which are not of this form. Further, we consider a closely related notion of a trigonometric $\vee$-system, and we show that its subsystems are also trigonometric $\vee$-systems. While reviewing the root system case we determine a version of generalised Coxeter number for the exterior square of the reflection representation of a Weyl group. We give a list of all the known trigonometric $\vee$-systems on the plane.

In the second part of the thesis, we consider solutions of WDVV equations in N-dimensional space (without extra variable), which are of the form (0.1) with $Q = 0$. Such class of solutions does not exist in general even for the case of root system $\mathcal{A}$ and invariant multiplicities c_α. However, it is known to exist for the root system B_N and specific choice of invariant multiplicities [33]. We generalize this solution to a multi-parameter family so that the underlying configuration $\mathcal{A}$ is the root system BC_N. These BC_N type solutions of WDVV equations are found by applying restrictions to the known solutions of the commutativity equations and by relating commutativity equations with WDVV equations for the corresponding prepotential. We apply these solutions to define $\mathcal{N} = 4$ supersymmetric mechanical systems.

In the third part of the thesis we reveal the relation between the set of WDVV equations and the set of the commutativity equations for an arbitrary function F. We reformulate it as the existence of the identity vector field for the natural associative multiplication associated with a solution F of commutativity equations. We give explicit formulas of the identity vector field corresponding to root systems for all the cases when commutativity equations are known to be satisfied. We also get new solutions of WDVV equations related to root system F_4.

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Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

1.1 Background

The Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations are a remarkable set of nonlinear third order partial differential equations for a single function $\mathcal{F}$. They were discovered originally in two-dimensional topological field theories at the end of 1980's by the physicists E. Witten, R. Dijkgraaf, E.Verlinde, and H.Verlinde [15, 55]. One of the forms of these equations for a holomorphic function $\mathcal{F}$ is

$$\mathcal{F}_i \mathcal{F}_j^{-1} \mathcal{F}_k = \mathcal{F}_k \mathcal{F}_j^{-1} \mathcal{F}_i, \quad i, j, k = 1, \ldots, N, \tag{1.1}$$

where $\mathcal{F}_i$'s are $N \times N$ matrices constructed from the third derivatives of the unique function $\mathcal{F} = \mathcal{F}(t_1, \ldots, t_N)$ of N variables with entries

$$(\mathcal{F}_i)_{pq} = \mathcal{F}_{ipq} = \frac{\partial^3 \mathcal{F}}{\partial t_i \partial t_p \partial t_q} \quad (p, q = 1, \ldots, N). \tag{1.2}$$

When $\mathcal{F}$ is a function of one or two variables ($N = 1, 2$) then equations (1.1) are always satisfied. However for more variables, regardless the simplicity of their compact matrix form these equations form a highly nontrivial overdetermined system of nonlinear partial differential equations for the function $\mathcal{F}$. These equations have been widely discussed and have found various interesting applications in connection with many areas of mathematical physics. Sometimes equations (1.1) are referred to as the generalized WDVV equations while the usual WDVV equations have the form (1.1) with a fixed index j. However, we will call the full set of equations (1.1) simply as WDVV equations. Following [15, 55], WDVV equations appeared in the core of Frobenius manifolds theory in the early '1990's, where the mathematical structure of these equations had been thoroughly studied by Boris Dubrovin as a way to provide a geometric setting of the solutions to the WDVV equations [16, 17]. Dubrovin's Frobenius manifolds give invariant coordinates free way to think about WDVV equations. In his work he gave differential geometric formulation of WDVV equations as he proved that locally any solution of the WDVV equations defines the structure of a Frobenius manifold and vice versa.

Let us briefly outline the setting of this relation before giving more details in the next chapter. The main features of a Frobenius manifold $\mathcal{M}$ are the existence of an associative commutative multiplication $\circ$ on each of its tangent space $T_t\mathcal{M}$ and a holomorphic flat metric η so that it has zero curvature. This metric is compatible with the multiplication, that is

$$\eta(x \circ y, z) = \eta(x, y \circ z), \quad x, y, z \in T_t\mathcal{M}. \tag{1.3}$$

Another characters that play roles in the structure of a Frobenius manifold are two distinguished vector fields $e, \mathcal{E} \in \Gamma(T\mathcal{M})$. The vector field e is the unity for the multiplication $\circ$ and it is required to be flat with respect to the Levi-Civita connection of the metric η, namely $\nabla e = 0$. The other vector field $\mathcal{E}$ is called Euler vector field and the metric and multiplication are assumed to be homogeneous with respect to this additional vector field. Furthermore, there exists a symmetric $(0, 3)$ tensor c related to the metric η and the multiplication $\circ$ by the formula

$$\eta(x \circ y, z) = c(x, y, z), \quad x, y, z \in \Gamma(T\mathcal{M}). \tag{1.4}$$

The multiplication $\circ$ with condition (1.3) makes $T\mathcal{M}$ into a family of commutative associative algebras with unity e, which is a family of Frobenius algebras. The axioms of the Frobenius manifold require that the tensor c is totally symmetric in all its arguments. Such a condition implies local existence of a function $\mathcal{F} = \mathcal{F}(t_1, \dots, t_N)$ on $\mathcal{M}$ called the (free energy) prepotential of the Frobenius manifold $\mathcal{M}$, and allows one to express the structure constants of the multiplication in terms of the third order derivatives of the function $\mathcal{F}$, that is

$$c_{\alpha\beta\gamma} = c(\alpha, \beta, \gamma) = \frac{\partial^3 \mathcal{F}(t)}{\partial t_\alpha \partial t_\beta \partial t_\gamma}, \tag{1.5}$$

where variables t_i, are flat coordinates of the metric η. Hence, by (1.3) and (1.5) the flat metric η can be defined in terms of the third order derivatives of $\mathcal{F}$ which allows to reformulate WDVV equations as the associativity condition of a multiplication $\circ$ in a family of Frobenius algebras defined on the tangent spaces $T_t\mathcal{M}$. By this reason sometimes WDVV equations are referred as associativity equations.

Let us recall some of the important known classes of solutions of WDVV equations they are related to finite Coxeter groups $\mathcal{W}$ which are finite groups of linear transformations of a real vector space V of dimension N generated by reflections.

There is a remarkable class of polynomial solutions of WDVV equations which corresponds to (finite) Coxeter groups $\mathcal{W}$ [17]. In this case the Frobenius manifold $\mathcal{M} = \mathcal{M}_\mathcal{W}$

is the (complexified) space of $\mathcal{W}$-orbits in the reflection representation V of $\mathcal{W}$. K. Saito proved that there exists a flat structure on $\mathcal{M}_{\mathcal{W}}$ [51], and this structure can be presented as a flat metric, which is known as the Saito metric. It has been shown by Dubrovin that the orbit space $\mathcal{M}_{\mathcal{W}}$ admits the structure of a Frobenius manifold [17, Lecture 4]. Moreover, his construction shows that the corresponding prepotential $\mathcal{F}$ of this Frobenius manifold is a polynomial in the flat coordinates of the Saito metric. Following Dubrovin's result and conjecture, Hertling proved that under some assumptions every polynomial Frobenius manifold arises in this way [32]. Thus, the orbit space construction gives all polynomial solutions to the WDVV equations under certain assumptions. Note that the polynomial prepotential corresponding to the space of orbits $\mathcal{M}_{\mathcal{W}}$ of a finite Coxeter group $\mathcal{W}$ cannot be written explicitly in a simple way.

For any Frobenius manifold there is an associated almost dual Frobenius manifold introduced by Dubrovin in [20]. In this new structure some of the axioms of a Frobenius manifold are relaxed. New multiplication is considered on the tangent spaces which is defined via the old multiplication of the Frobenius manifold and its Euler vector field. Tangent spaces remain to be Frobenius algebras with respect to new multiplication and new metric. There is also a new prepotential $\mathcal{F}^*$ satisfying WDVV equations which are associativity conditions for the new multiplication. For the orbit space $\mathcal{M}_{\mathcal{W}}$ the prepotential $\mathcal{F}^*$ can be expressed in a simple form

$$\mathcal{F}^* = \mathcal{F}^{rat} = \sum_{\alpha\in\mathcal{A}} \alpha(x)^2 \log \alpha(x), \quad x \in V, \tag{1.6}$$

where $\mathcal{A} = \mathcal{R}$ is the root system of the group $\mathcal{W}$. In this case the constant metric is the $\mathcal{W}$-invariant form on the vector space V of the reflection representation of the group $\mathcal{W}$.

A class of solutions to WDVV equations (1.1) includes some prepotentials arising from low-energy effective actions of $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions via Seiberg-Witten theory. More precisely, perturbative parts of Seiberg–Witten prepotentials take the form (1.6) and solve WDVV equations. Note that the geometric structure of the WDVV equations in Seiberg-Witten theory appears to be different from that of Frobenius manifolds as for WDVV equations in the case of Seiberg-Witten theory there may be no full structure of Frobenius manifolds associated with them. Marshakov, Mironov and Morozov found in this context solutions (1.6) of WDVV equations for classical root systems $\mathcal{R}$ in [39, 40]. More generally, it has been shown in [40] that the prepotential

$$\mathcal{F}^{rat} = \sum_{i,j} \big(\nu_-(x_i - x_j)^2 \log(x_i - x_j) + \nu_+(x_i + x_j)^2 \log(x_i + x_j)\big) + \eta \sum_i x_i^2 \log x_i \tag{1.7}$$

solves WDVV equations for any value of η if $\nu_+ = \nu_-$ or $\nu_+ = 0$. Suitable choices of scalars η, ν_+, ν_- correspond to the prepotentials for the classical groups D_N, B_N and A_N.

Note that solutions corresponding to some deformations of the root system A_N are also contained in the family (1.7) when $\nu_+ = 0$ although this point was not emphasized in [40]. Solutions (1.6) for non-classical root systems were found by Gragert and Martini in [41].

Veselov found solutions $\mathcal{F}^{rat}$ of the form (1.6) for non-crystallographic (Coxeter) root systems as well as, more importantly, for not fully symmetric configurations of covectors $\mathcal{A} \subset V^*$, where V can now be a complex vector space. He introduced the notion of a $\vee$-system [53] formulated in terms of linear algebra. The following theorem describes the connection between $\vee$-systems and WDVV equations.

Theorem 1.1.1. [26, 53]. *Under a non-degeneracy assumption configuration of vectors $\mathcal{A}$ is a $\vee$-system if and only if the corresponding logarithmic prepotential (1.6) satisfies WDVV equations.*

Another equivalent statement that relates the class of $\vee$-systems and WDVV equations is given via associativity of a multiplication defined on the tangent space of the complement $M_{\mathcal{A}} = V \setminus \cup_{\alpha \in \mathcal{A}} \Pi_\alpha$ to the union of all hyperplanes $\Pi_\alpha = \{x \in V \colon \alpha(x) = 0\}$. The multiplication is given by the formula

$$u * v = \sum_{\alpha \in \mathcal{A}} \frac{\alpha(u)\alpha(v)}{\alpha(x)} \alpha^\vee, \quad u, v \in T_x M_{\mathcal{A}}, \tag{1.8}$$

where $\alpha^\vee \in V$ is the vector corresponding to the covector $\alpha \in V^*$ with respect to a non-degenerate bilinear form defined on V. The relation then is formulated by the following statement.

Theorem 1.1.2. [26, 54] *The multiplication* (1.8) *is associative if and only if $\mathcal{A}$ is a $\vee$-system.*

This statement together with Theorem 1.1.1 leads to the fact that associativity of the multiplication (1.8) is equivalent to the WDVV equations (1.1) for prepotential (1.6).

This property can also be reformulated in terms of flatness of a connection on the tangent bundle TV [54]. The connection is given by the formula

$$\nabla_u = \partial_u - \sum_{\alpha \in \mathcal{A}} \frac{\alpha(u)}{\alpha(x)} \alpha^\vee \otimes \alpha. \tag{1.9}$$

Then the following statement holds.

Theorem 1.1.3. [54] *Connection* (1.9) *is flat if and only if $\mathcal{A}$ is a $\vee$-system.*

As a corollary of this statement together with Theorem 1.1.1 it is easy to see that the flatness condition for connection (1.9) is equivalent to the WDVV equations (1.1) for prepotential (1.6). A closely related notion of the Dunkl system was introduced and studied in [14]. That structure for complex reflection groups was investigated further in [5] in relation with Frobenius manifolds theory.

The class of $\vee$-systems is closed under the natural operations of taking subsystems [26] and under restriction of a system to the intersection W of some of the hyperplanes Π_α where $\alpha \in \mathcal{A}$ [25]. A brief formulations of these facts can be given as follows, more details are given later in Chapter 2.

By a subsystem $\mathcal{B} \subset \mathcal{A}$ we mean the intersection of $\mathcal{A}$ with a linear subspace in V.

Theorem 1.1.4. *Let $\mathcal{A}$ be a $\vee$-system. Then under some non-degeneracy conditions any subsystem $\mathcal{B} \subset \mathcal{A}$ as well as the set of non-zero restricted covectors of $\mathcal{A}$ to the subspace W are also $\vee$-systems.*

By considering these two operations of taking subsystems and restrictions one can obtain new solutions to the WDVV equations from known solutions. Note that the restriction of Coxeter root systems in general are not root sytems, so the class of $\vee$-systems can be thought of as an extension of the class of Coxeter root systems that satisfies such property. In fact, the class of $\vee$-systems contains multi-parameter deformations of the root systems A_N and B_N ([13], see also [26] for more examples). The underlying matroids were examined in [46]. The problem of classification of $\vee$-systems remains open.

Other remarkable solutions to WDVV equations that arise in theory of Frobenius manifolds are trigonometric generalisation of solutions (1.6). These solutions have the form

$$\mathcal{F}^{trig} = \sum_{\alpha \in \mathcal{A}} c_\alpha f(\alpha(x)) + Q, \tag{1.10}$$

where $c_\alpha \in \mathbb{C}$ are some multiplicity parameters and Q is a cubic polynomial in $x = (x_1, \ldots, x_N)$ and, often, in additional variable y, and f is a function of a single variable z given by

$$f(z) = \frac{1}{6} i z^3 + \frac{1}{4} Li_3(e^{-2iz}), \tag{1.11}$$

where Li_3 is the (trilogarithm) function so that $f'''(z) = \cot z$. The WDVV equations in this case have the form

$$\mathcal{F}_i^{trig} (\mathcal{F}_j^{trig})^{-1} \mathcal{F}_k^{trig} = \mathcal{F}_k^{trig} (\mathcal{F}_j^{trig})^{-1} \mathcal{F}_i^{trig}, \tag{1.12}$$

similar to equations (1.1) but they involve $N+1$ variables $x_1, \ldots, x_N, x_{N+1} = y$, where

$$(\mathcal{F}_i^{trig})_{pq} = \frac{\partial^3 \mathcal{F}^{trig}}{\partial x_i \partial x_p \partial x_q} \quad (i, p, q = 1, \ldots, N+1). \tag{1.13}$$

Such solutions (for root system of type A_N) appeared in five-dimensional Seiberg–Witten theory as perturbative parts of prepotentials [40]. Solutions of the form (1.10) corresponding to (crystallographic) irreducible reduced root systems $\mathcal{A} = \mathcal{R}$ of Weyl groups and $\mathcal{W}$-invariant multiplicities c_α were studied by Hoevenaars and Martini in [33, 42]. The polynomial Q is $\mathcal{W}$-invariant, and it was specified fully explicitly in the case of constant multiplicity.

Such solutions also appear as prepotentials for the almost dual Frobenius manifold structures on the extended affine Weyl groups orbit spaces [18, 21]– see [45] for type A_N. In some cases such solutions may be related to the rational solutions (1.6) by twisted Legendre transformations [45].

Bryan and Gholampour found another remarkable appearance of trigonometric solutions (1.10), or rather the corresponding associative product, in geometry as they studied quantum cohomology of resolutions of (simple) A, D, E singularities [11]. The associative quantum product on these cohomologies is governed by the corresponding solutions $\mathcal{F}^{trig}$ with $\mathcal{A} = A_N, D_N, E_N$ respectively and with special multiplicities. The invariant cubic polynomial Q is given in terms of the highest root of the root system. Furthermore, a family of associative products was given in [11] for any irreducible reduced root system. The corresponding prepotential can be checked to be of the form (1.10).

In Shen's work [47, 48] families of Frobenius algebras in trigonometric settings are considered. A related object is one-parameter family of torsion free and flat connections on the tangent bundle of the complement of a toric arrangement associated with a root system. This family of connections is the Dubrovin (deformed) connection ∇^a and it has the form

$$\nabla^a_{\widetilde{u}}\widetilde{v} = \partial_{\widetilde{u}}\widetilde{v} + a\widetilde{u} * \widetilde{v}, \tag{1.14}$$

where $a \in \mathbb{C}$ and the multiplication $*$ is defined so that each fiber of the tangent bundle of the complement of a toric arrangement for a root system $\mathcal{R}$ with a $\mathcal{W}$-invariant multiplicity c_α is endowed with Frobenius algebra structure. This associative multiplication is given as follows.

Let E be the vector field corresponding to the additional variable $y = x_{N+1}$, that is $E = \partial_{x_{N+1}}$. Consider two vector fields $\widetilde{X} = X + \lambda_1 E$, $\widetilde{Y} = Y + \lambda_2 E$, where $X, Y \in V$, $\lambda_1, \lambda_2 \in \mathbb{C}$. The product $*$ is given by the formula

$$\widetilde{X} * \widetilde{Y} = \frac{1}{2}\sum_{\alpha\in\mathcal{R}^+} c_\alpha \frac{1+e^\alpha}{1-e^\alpha}\alpha(X)\alpha(Y)\alpha^\vee - \mu\langle X, Y\rangle E + \lambda_2 X + \lambda_1 Y + \lambda_1\lambda_2 E. \tag{1.15}$$

In this algebra E plays the role of the identity of the product. The corresponding potential functions to these Frobenius algebras have the form (1.10). Shen finds the form of μ explicitly so that multiplication (1.15) is associative which is equivalent to the flatness of connection (1.14). This can be rephrased as explicit specification of polynomial Q in (1.10). Thus it complements results from [33, 42] to the case of arbitrary invariant multiplicities for all the reduced root systems.

Solutions of WDVV equations of the form (1.10) without full Weyl symmetry were considered by Feigin in [27] where the notion of a trigonometric $\vee$-system was introduced and its close relation with WDVV equations was established. Feigin derived geometric and algebraic conditions for a system of vectors with multiplicities so that the corresponding function of the form (1.10) for a configuration $\mathcal{A}$ satisfies the WDVV equations (see Definition 2.6.17 for precise definition of a trigonometric $\vee$-system). A key difference with the rational case is the existence of a rigid geometrical structure of a series decomposition of vectors from $\mathcal{A}$ parallel to a chosen one which generalizes the notion of strings for root systems. In more detail, for any $\alpha \in \mathcal{A}$ the set of vectors in $\mathcal{A}$ not collinear to α can be decomposed as $\mathcal{A} \setminus \langle \alpha \rangle = \bigsqcup_s \Gamma_\alpha^s$, where for any two covectors $\gamma_1, \gamma_2 \in \Gamma_\alpha^s$ one has either $\gamma_1 + \gamma_2 = m\alpha$ or $\gamma_1 - \gamma_2 = m\alpha$ for some $m \in \mathbb{Z}$. There are further algebraic conditions for each series Γ_α^s. The following example illustrates such series/strings decompositions for the root system BC_2.

Example 1.1.5. Let $\mathcal{A} = \{e^1, 2e^1, e^2, 2e^2, e^1 \pm e^2\}$ be the positive half of the root system BC_2. Then we have the following series:

$$\Gamma_{e^1}^1 = \{e^2, e^1 \pm e^2\},\ \Gamma_{e^1}^2 = \{2e^2\};\ \Gamma_{e^2}^1 = \{e^1, e^1 \pm e^2\},\ \Gamma_{e^2}^2 = \{2e^1\};$$
$$\Gamma_{2e^1}^1 = \{e^1 \pm e^2\},\ \Gamma_{2e^1}^2 = \{e^2\},\ \Gamma_{2e^1}^3 = \{2e^2\};\ \Gamma_{2e^2}^1 = \{e^1 \pm e^2\},\ \Gamma_{2e^2}^2 = \{e^1\},\ \Gamma_{2e^2}^3 = \{2e^1\};$$
$$\Gamma_{e^1+e^2}^1 = \{e^1, e^2\},\ \Gamma_{e^1+e^2}^2 = \{2e^1, 2e^2, e^1 - e^2\};\ \Gamma_{e^1-e^2}^1 = \{e^1, e^2\},\ \Gamma_{e^1-e^2}^2 = \{2e^1, 2e^2, e^1 + e^2\}.$$

The algebraic conditions for each α-series have the form

$$\sum_{\beta \in \Gamma_\alpha^s} c_\beta \beta(\alpha^\vee)\beta = b\alpha \tag{1.16}$$

for some $b \in \mathbb{C}$, where $\alpha^\vee$ is a vector corresponding to covector α under a certain identification of V and V^*. Under additional conditions it was shown in [27] that configuration of covectors $\mathcal{A}$ with multiplicities $c\colon \mathcal{A} \to \mathbb{C}$ such that (1.10) is a solution of WDVV equations has to be a trigonometric $\vee$-system, and the converse statement holds true as well. We give a version of this result below in Theorem 1.2.1.

All irreducible trigonometric $\vee$-systems with up to five vectors on the plane were determined in [27] and some more examples were given.

The study of the trigonometric and rational cases is related as the next statement illustrates.

Proposition 1.1.6. [27] *If a configuration $\mathcal{A}$ with collection of multiplicities $c_\alpha, \alpha \in \mathcal{A}$ is a trigonometric $\vee$-system then configuration $\sqrt{c_\alpha}\alpha$ is a rational $\vee$-system, that is $\mathcal{F}^{rat} = \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha(x)^2 \log \alpha(x)$ is a solution of the WDVV equations on the space V.*

However, due to the presence of the extra variable y in the trigonometric case the

WDVV equations are already nontrivial for $\dim V = 2$ while the smallest nontrivial dimension of V in the rational case is 3.

Many-parameter deformations of solutions $\mathcal{F}^{trig}$ for the classical root systems were obtained by Pavlov from reductions of Egorov hydrodynamic chains [43]. In his work Pavlov derived an effective algorithm to construct infinitely many particular solutions of the WDVV equations written in an explicit form in the flat coordinates of corresponding Egorov hydrodynamic type systems. Closely related many-parameter family of flat connections in type A_N was considered by Shen in [47, 48].

It has been proven in [41] (see also [40]) that WDVV equations (1.1) can be written equivalently in the form

$$\mathcal{F}_i B^{-1} \mathcal{F}_j = \mathcal{F}_j B^{-1} \mathcal{F}_i, \quad i, j = 1, \ldots, N, \tag{1.17}$$

where B is any non-degenerate linear combination of matrices $\mathcal{F}_k$ with functional coefficients. Solution of WDVV equations of the form (1.10) when the cubic corrections are absent, that is when $Q = 0$, does not exist in general even for the case of a root system $\mathcal{A} = \mathcal{R}$ and invariant multiplicities c_α. However, such a solution is known to exist for the root system B_N and specific choice of invariant multiplicities [33]. In fact, in their derivation Hoevenaars and Martini proved that for such a solution for the root system B_N the corresponding WDVV equations (1.17) are reduced to the commutativity equations

$$\mathcal{F}_i \mathcal{F}_j = \mathcal{F}_j \mathcal{F}_i, \quad i, j = 1, \ldots, N, \tag{1.18}$$

as they found a matrix $B = (B_{ij})_{i,j=1}^N$ given as a linear combination of the third order derivatives of the function $\mathcal{F}$ which is a multiple of the identity matrix of size $N \times N$.

Another area where WDVV equations or rather commutativity equations (1.18) emerge is $\mathcal{N} = 4$ supersymmetric mechanics. Relations of such mechanical systems with WDVV equations were known since [56] and [9]. Similarly to the case of Seiberg-Witten theory, there is no full structure of Frobenius manifolds in these settings. In order to construct an $\mathcal{N} = 4$ supersymmetric mechanical system, N (quantum) particles on a line with coordinates x_j and momenta $p_j = -i\partial_{x_j}$, $(j = 1, \ldots, N)$ are considered. For each particle four fermionic variables $\{\psi^{aj}, \bar{\psi}^j_a \,|\, a = 1, 2,\ j = 1, \ldots, N\}$ are assigned. These fermionic variables are chosen so that they obey the standard (anti)-commutation rules. The $\mathcal{N} = 4$ supersymmetry algebra then is generated by four supercharges $\{Q^a, \bar{Q}_b \,|\, a, b = 1, 2\}$. Those supercharges are differential operators in variables $x_1, \ldots, x_N$ whose coefficients depend on particle coordinates $x_1, \ldots, x_N$ and the additional (fermonic) variables ψ. The $\mathcal{N} = 4$ supersymmetry algebra has the form

$$\{Q^a, Q^b\} = \{\bar{Q}_a, \bar{Q}_b\} = 0, \quad H_{SUSY} = -\frac{1}{2}(Q^a \bar{Q}_a + \bar{Q}_a Q^a), \tag{1.19}$$

where H_{SUSY} is the supersymmetric Hamiltonian, and $\{\cdot,\cdot\}$ is the anti-commutator.

Wyllard in [56] (see also [9]) constructed a set of generators of $\mathcal{N}=4$ supersymmetric mechanics, where his ansatz for the four supercharges involves two scalar prepotentials F and W which depend on positions of particles only. The case when the second prepotentional $W=0$ was also considered in [56] (see also [30,31]). The structure of such an algebra forces functions F and W to satisfy a system of partial differential equations, which are the commutativity equations (1.18) for F [9]. Wyllard's ansatz for F has the form (1.6), where $\mathcal{A}$ is the root system A_{N-1}. Examples with root systems $\mathcal{A}=G_2, B_3$ were discussed in [56]. Wyllard's potential for $\mathcal{N}=4$ supercharges was extended to other root systems in [30, 31] where solutions for a small number of particles were considered. More generally, any rational $\vee$-system leads to $\mathcal{N}=4$ supersymmetric mechanical systems (see [4]). Trigonometric solutions of the form (1.10) with $Q=0$ to the commutativity equations (1.18) were obtained in [3]. In this case $\mathcal{A}$ is an irreducible root system with more than one orbit of the Weyl group, namely BC_N, F_4 and G_2. These solutions were used to construct $\mathcal{N}=4$ supersymmetric Hamiltonians in [3]. This gave, in particular, supersymmetric version of quantum Calogero–Moser–Sutherland system of type BC_N with two independent coupling parameters. The corresponding solution $\mathcal{F}$ generalizes B_N solution from [33]. More recently, other $\mathcal{N}=4$ supersymmetric extensions of Calogero–Moser–Sutherland type systems were obtained in [38] for the models with many fermionic variables, and in [22] for the models with extra spin variables.

There is also an important class of elliptic solutions of WDVV equations, which was considered by Strachan in [44, 49] where, in particular, certain solutions related to A_N and B_N root systems were found. The prepotentials appear as almost dual prepotential associated to Frobenius manifold structures on A_N and B_N Jacobi groups orbit spaces [7,8]. Such solutions appear also in six-dimensional Seiberg–Witten theory [10].

1.2 Present work and plan of this thesis

1.2.1 Main results I (Chapter 3)

In the first part of the thesis (Chapter 3) we study trigonometric solutions $\mathcal{F}^{trig}$ of the form (1.10) of WDVV equations. Let us explain the relation between a trigonometric $\vee$-system and WDVV equations.

Let $\mathcal{A}\subset V^*$ be a finite set of covectors. Assume it belongs to a lattice of rank N. Define the bilinear form $G_{\mathcal{A}}$ (which will be assumed to be non-degenerate) on V by

$$G_{\mathcal{A}}(x,y)=\sum_{\alpha\in\mathcal{A}} c_\alpha \alpha(x)\alpha(y), \quad x,y\in V. \tag{1.20}$$

Let $\alpha^\vee \in V$ be the vector corresponding to the covector $\alpha \in V^*$ with respect to the bilinear form $G_\mathcal{A}$. Let $U \cong \mathbb{C}$ be a one-dimensional vector space. Consider a function $F\colon V \oplus U \to \mathbb{C}$ of the form

$$F = \frac{1}{3}y^3 + \sum_{\alpha\in\mathcal{A}} c_\alpha \alpha(x)^2 y + \lambda \sum_{\alpha\in\mathcal{A}} c_\alpha f(\alpha(x)), \tag{1.21}$$

where $\lambda \in \mathbb{C}^*$ and function $f(z)$ given by (1.11).

Define the following two bilnear forms on $\Lambda^2 V$:

$$G_\mathcal{A}^{(1)}(z,w) = \sum_{\alpha,\beta\in\mathcal{A}} c_\alpha c_\beta B_{\alpha,\beta}(z) B_{\alpha,\beta}(w),$$

$$G_\mathcal{A}^{(2)}(z,w) = \sum_{\alpha,\beta\in\mathcal{A}_+} c_\alpha c_\beta G_\mathcal{A}(\alpha^\vee,\beta^\vee) B_{\alpha,\beta}(z) B_{\alpha,\beta}(w), \tag{1.22}$$

where $z, w \in \Lambda^2 V$, and for any $a, b \in V$,

$$B_{\alpha,\beta}(a\otimes b) = \alpha\wedge\beta(a\otimes b) = \alpha(a)\beta(b) - \alpha(b)\beta(a).$$

In the formula (1.22) $\mathcal{A}^+$ is obtained by replacing some vectors from $\mathcal{A}$ with their opposite ones and keeping the multiplicity unchanged so that the new configuration of vectors belongs to a half-space.

The following theorem gives the relation between WDVV equations and trigonometric $\vee$-systems.

Theorem 1.2.1. [1, 27] *Under some non-degeneracy conditions, the WDVV equations* (1.12) *for the function* (1.21) *imply the following two conditions:*

(1) $\mathcal{A}$ is a trigonometric $\vee$-system,

(2) Bilinear forms $G_\mathcal{A}^{(1)}$ and $G_\mathcal{A}^{(2)}$ are proportional: $G_\mathcal{A}^{(1)} = \frac{\lambda^2}{4} G_\mathcal{A}^{(2)}$.

Conversely, if a configuration $(\mathcal{A}, c)$ satisfies conditions (1) and (2) then the WDVV equations for the function (1.21) *hold.*

A version of Theorem 1.2.1 that relates solutions of the form (1.21) of WDVV equations with the trigonometric $\vee$-systems was also given in [27, Theorem 1], and it was obtained by analyzing WDVV equations for the function (1.21). In this thesis we derive and clarify condition (2) of Theorem 1.2.1. Solution (1.21) depends on a scalar $\lambda = \lambda_{(\mathcal{A},c)}$ which is invariant under linear transformations applied to the configuration $\mathcal{A}$. In the case of root systems $\mathcal{A} = \mathcal{R}$ the scalar $\lambda_{(\mathcal{R},c)}$ may be thought of as a version of generalized Coxeter number for the irreducible $\mathcal{W}$-module $\Lambda^2 V$ since it is given as a ratio of the two canonical $\mathcal{W}$-invariant symmetric bilinear forms $G_\mathcal{A}^{(1)}$ and $G_\mathcal{A}^{(2)}$ on $\Lambda^2 V$.

We define a multiplication $*$ on the tangent space $T_x(V\otimes U)$, where $x = (x_1, \dots, x_{N+1})$, given by the formula

$$\partial_{x_i} * \partial_{x_j} = \sum_{k,l=1}^{N+1} \eta^{kl} F_{ijk} \partial_{x_l}, \quad i, j = 1, \ldots, N+1, \tag{1.23}$$

where η^{kl} is defined by $\eta^{ij} = (F_{N+1}^{-1})_{ij}$. We arrive at Theorem 1.2.1 by making use of flatness of Dubrovin connection (1.14) and the following well-known result.

Theorem 1.2.2. *The flatness of connnection* (1.14) *for all $a \in \mathbb{C}$ where the multiplication $*$ is given by* (1.23) *is equivalent to WDVV equations for the function F.*

By working with the flatness conditions we arrive at equivalent identities for the configuration $(\mathcal{A}, c)$. These identities under some non-degeneracy conditions led to the trigonometric $\vee$-conditions (1.16) for each series. In our derivation we notice that certain conditions for collinear vectors should be satisfied in order to study possible singularities in the identities. Note that these conditions for collinear vectors were not considered in [27, Theorem 1] although they are needed for the analysis. Formulation of Theorem 1.2.1 in the thesis and [1] rectifies this. An important class of solutions of the form (1.21) of WDVV equations is given by root systems with $\mathcal{W}$-invariant multiplicities as we review above. Theorem 1.2.1 gives another way to see this since any such configuration can be easily checked to be a trigonometric $\vee$-system.

A natural problem to study is the operations of taking subsystems and restrictions in the trigonometric case. Moreover, one has to clarify whether the resulting configurations form trigonometric $\vee$-systems or not, and whether these two processes lead to new solutions of WDVV equations. These questions are motivated by the rational version of $\vee$-systems. Analogous questions for the rational case were answered positively in [26] (see also [24]) for taking subsystems and in [25] for the restriction operation, see Theorem 1.1.4 above. Before we state our results for the trigonometric case, let us review the settings.

For any $\alpha \in \mathcal{A}$ let $\delta_\alpha := \{\gamma \in \mathcal{A} \colon \gamma \sim \alpha\}$, where $\sim$ stands for proportionality. Consider a subsystem $\mathcal{B} = \mathcal{A} \cap W$ and assume that $W = \langle \mathcal{B} \rangle$. Let $W^\vee = \{\alpha^\vee \in V, \alpha \in W\}$. Define a bilinear form $G_\mathcal{B}$ on V by

$$G_\mathcal{B}(u, v) := \sum_{\beta \in \mathcal{B}} c_\beta \beta(u) \beta(v), \quad u, v \in V.$$

Define the subspace $W_\mathcal{B} = \{x \in V \colon \beta(x) = 0 \ \forall \beta \in \mathcal{B}\} \subset V$. We show that the following statements take place.

Theorem 1.2.3. *The subsystem $\mathcal{B}$ of a trigonometric $\vee$-system $\mathcal{A}$ is also a trigonometric $\vee$-system if $G_\mathcal{B}|_{W^\vee}$ is non-degenerate, where $G_\mathcal{B}|_{W^\vee}$ is the restriction of the bilinear form $G_\mathcal{B}$ to the subspace $W^\vee$.*

Theorem 1.2.4. *Let $\mathcal{B} \subset \mathcal{A}$ be a subsystem. Assume that prepotential F satisfies WDVV equations. Then under certain generic conditions on multiplicities, the function*

$$F_{\mathcal{B}} = F_{\mathcal{B}}(\xi, y) = \frac{1}{3}y^3 + \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} c_\alpha \overline{\alpha}(\xi)^2 y + \lambda \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} c_\alpha f(\overline{\alpha}(\xi)),$$

where $\xi \in W_{\mathcal{B}}, \overline{\alpha} = \alpha|_{\mathcal{B}}$, satisfies WDVV equations on $W_{\mathcal{B}} \oplus U$.

In contrast to the rational case, the collinear vectors in the trigonometric case should be taken into account and be dealt with as they are since proportional vectors cannot be replaced with a single vector which is the case in the rational version. Thus a set of collinear vectors $\{k_i\alpha \colon k_i \in \mathbb{R}, i = 1, \dots, m, m \in \mathbb{N}\}$ in the rational case can be equivalently replaced by a single covector $\widetilde{\alpha} = k\alpha$, where $k^2 = \sum_{i=1}^m k_i^2$. This is also true for the restricted systems where collinear vectors may appear. In the trigonometric case if the restricted system contains collinear vectors we keep them with their multiplicities and if a vectors α is repeated (up to a sign) with multiplicities c_α^i then we replace this collection with the single vector α with multiplicity $c_\alpha \coloneqq \sum_i c_\alpha^i$.

The following example illustrates restrictions for both the rational and trigonometric cases.

Example 1.2.5. Let $\mathcal{A} = \{e^1, e^2, e^3, e^1 \pm e^2, e^1 \pm e^3, e^2 \pm e^3,\}$ be the positive half of the root system B_3. Let multiplicity $c_\alpha = 1$ for all $\alpha \in \mathcal{A}$. Let us restrict $\mathcal{A}$ to the hyperplane $x_2 = x_3$. The rational restricted system is $\mathcal{A}^{rat} = \{e^1, \sqrt{6}e^2, \sqrt{2}(e^1 \pm e^2)\}$, and the trigonometric restricted system is $\mathcal{A}^{trig} = \{e^1, e^2, 2e^2, e^1 \pm e^2\}$ with the corresponding multiplicities $\{1, 2, 1, 2\}$.

Note that we cannot show that given a solution of the form (1.21) a subsystem $\mathcal{B}$ provides another solution. Also we cannot show that a restriction of a trigonometric $\vee$-system is a trigonometric $\vee$-system.

In order to apply Theorem 1.2.4 for classical root systems we firstly find a family of solutions of the form (1.21) of WDVV equations corresponding to the (non-reduced) root system BC_N which depends on three independent multiplicity parameters. This result generalizes the results found in [33], [42], [11] and [47] for root systems D_N, B_N and C_N. Then we apply the restriction operation to these solutions and we obtain a family of solutions depending on $(n + 3)$ parameters, where $n = \dim W_{\mathcal{B}}$, which generalizes the $(n + 1)$-parametric family from [43]. This result is given in the following statement.

Theorem 1.2.6. *Assume that parameters r, q, s and $m_1, \dots, m_n$ satisfy the relation $r + 4s + 2q(m_i - 1) \neq 0$ for any $1 \leq i \leq n$. Then for a suitable subspace $W_{\mathcal{B}}$ of dimension n*

the function

$$F_{\mathcal{B}}(\xi,y)=\frac{1}{3}y^3+\big(r+4s+2q(\sum_{i=1}^{n}m_i-1)\big)y\sum_{i=1}^{n}m_i\xi_i^2+\lambda r\sum_{i=1}^{n}m_i f(\xi_i)$$
$$+\lambda\sum_{i=1}^{n}\big(sm_i+\frac{1}{2}qm_i(m_i-1)\big)f(2\xi_i)+\lambda q\sum_{i<j}m_im_jf(\xi_i\pm\xi_j),$$

where $\xi=(\xi_1,\ldots,\xi_n)\in W_{\mathcal{B}}, y\in U\cong\mathbb{C}$, *satisfies the WDVV equations on* $W_{\mathcal{B}}\oplus U$ *provided that* $\big(r+8s+2(\sum_{i=1}^{n}m_i-2)q\big)q\neq 0$.

Similarly, the restriction process for type A_N solutions gives a multi-parameter family of solutions which can be specialized to Pavlov's $(n+1)$-parametric family from [43] (there seem to be typos in [43] for type A_n solutions). This result can be summarized as follows.

Theorem 1.2.7. *Let* $\xi=(\xi_1,\ldots,\xi_{n+1})\in\mathbb{C}^{n+1}$ *and* $y=\sum_{i=1}^{n+1}\xi_i$. *Then prepotential*

$$F=F_{\mathcal{B}}(\xi)=(\frac{1}{3}-t)y^3+ty\sum_{k=1}^{n+1}m_k\sum_{i=1}^{n+1}m_i\xi_i^2+2t^{3/2}\sum_{k=1}^{n+1}m_k\sum_{i<j}m_im_jf(\xi_i-\xi_j)$$

satisfies WDVV equations in $\mathbb{C}^{n+1}$ *for any generic* $t,m_1,\ldots,m_{n+1}\in\mathbb{C}$.

Solutions given by Theorems 1.2.6, 1.2.7 give certain deformations of BC_N and A_N root systems solutions depending on many parameters. We show that in rank two all trigonometric $\vee$-systems with small number of vectors fit into these families.

Theorem 1.2.8. *Any trigonometric* $\vee$*-system with up to five vectors on the plane is a deformation of type* A_2 *or* BC_2.

The value of the scalar λ in solution (1.21) for each root system is given in the following table.

Table 1.1: The scalar λ for root systems

$\mathcal{R}$	BC_N	G_2	E_6	E_7	E_8	F_4	A_N
λ	$\frac{\sqrt{2}\big(r+4s+2q(N-1)\big)^{3/2}}{\sqrt{q}\big(r+8s+2q(N-2)\big)^{1/2}}$	$\frac{6(p+3q)}{\sqrt{p+9q}}$	$12\sqrt{2p}$	$9\sqrt{6p}$	$30\sqrt{p}$	$\frac{6\sqrt{3}(2q+p)}{\sqrt{4q+p}}$	$2(N+1)\sqrt{p}$

Here p is the multiplicity of short roots and q is the multiplicity of long roots in the case of a reduced not simply-laced root system $\mathcal{R}$. In the case of BC_N the multiplicities $c_\alpha=c(\alpha)$ are defined as follows:

$$c(e^i)=r\,(1\leq i\leq N),\ c(2e^i)=s\,(1\leq i\leq N),\ c(e^i\pm e^j)=q\,(1\leq i<j\leq N).\quad (1.24)$$

1.2.2 Main results II (Chapter 4)

In the second part of the thesis (Capter 4), we find multiparametric family of trigonometric solutions without extra variable, that is solutions of the form (1.10) with $Q = 0$ for both the WDVV equations (1.17) and the commutativity equations (1.18). These solutions are associated with root system BC_N and they generalize solutions found in [33] for type B_N root systems for the WDVV equations and they also generalize two-parameter solutions found in [3] for the commutativity equations. The following statement takes place.

Theorem 1.2.9. *Suppose that parameters r, s, q and $m_1, \dots, m_n$ satisfy the relation*

$$r = -8s - 2q(\sum_{i=1}^{n} m_i - 2).$$

Then the prepotential

$$F = \sum_{i=1}^{n} rm_i f(x_i) + \sum_{i=1}^{n} \big(sm_i + \frac{1}{2}qm_i(m_i-1)\big) f(2x_i) + \sum_{i<j}^{n} qm_im_j f(x_i \pm x_j) \qquad (1.25)$$

satisfies both the WDVV equations $F_iF_j^{-1}F_k = F_kF_j^{-1}F_i$ $(i,j,k = 1,\dots,n)$ as well as the commutativity equations $F_iF_j = F_jF_i$ $(i,j = 1,\dots,n)$.

This result is established firstly for the case of WDVV equations using a version of restriction procedure, then it is proven for the commuttivity equations by specifying the matrix B as a linear combination of the third order derivatives of F which is a multiple of identity.

Restrictions of exceptional root systems by Theorem 1.2.4 give other examples of trigonometric solutions of WDVV equations of the form (1.21). We work out all two-dimensional cases explicitly in the Appendix. We also note that there are trigonometric solutions which are not given as a restriction of root systems. An example is given in the next statement.

Proposition 1.2.10. *Let a configuration $\mathcal{A} \subset \mathbb{C}^4$ consist of the following covectors:*

$$\begin{aligned}
&e^i, \quad \textit{with multiplicity} \quad p, \quad 1 \le i \le 3,\\
&e^4, \quad \textit{with multiplicity} \quad q,\\
&e^i \pm e^j, \quad \textit{with multiplicity} \quad r, \quad 1 \le i < j \le 3,\\
&\frac{1}{2}(e^1 \pm e^2 \pm e^3 \pm e^4), \quad \textit{with multiplicity} \quad s,
\end{aligned}$$

where $p, q, r, s \in \mathbb{C}$ are such that $4r + s \neq 0$. Then this configuration is a trigonometric $\vee$-system and the corresponding prepotential of the form (1.21) gives a solution of WDVV equations for some λ provided that $p = 2r + s$, $\quad q = \frac{s(s-2r)}{4r+s}$ and $ps \neq 0$.

In analogy with trigonometric $\vee$-systems, we define *Euclidean trigonometric $\vee$-systems*. In this case we have a Euclidean vector space V with inner product $(\cdot,\cdot)$ and a collection of vectors $\mathcal{A}\subset V$ with multiplicity function $c\colon \mathcal{A}\to\mathbb{C}$. The bilinear form $G_{\mathcal{A}}$ given by (1.20) is replaced in the considerations by the inner product $(\cdot,\cdot)$. The class of Euclidean trigonometric $\vee$-systems contains root systems with Weyl invariant multiplicity.

Let us consider the function F given by the formula

$$F=\sum_{\alpha\in\mathcal{A}} c_\alpha f((\alpha,x)). \tag{1.26}$$

We are interested in configurations $\mathcal{A}$ with a multiplicity function $c(\alpha)=c_\alpha, \alpha\in\mathcal{A}$, such that the following commutativity equations hold:

$$F_iF_j=F_jF_i,\quad i,j=1,\dots,N, \tag{1.27}$$

where F_i is the $N\times N$ matrix with entries

$$(F_i)_{pq}=F_{ipq}=\frac{\partial^3 F}{\partial x_i\partial x_p\partial x_q}.$$

We investigate the relation between the commutativity equations (1.27) for the function (1.26) and the class of Euclidean trigonometric $\vee$-systems. Analogously to Theorem 1.2.1 the following statement clarifies this relation.

Theorem 1.2.11. *Under some non-degeneracy conditions, the commutativity equations* (1.27) *for the prepotential* (1.26) *imply the following two conditions:*

(1) $\mathcal{A}$ is a Euclidean trigonometric $\vee$-system,

(2) $\sum_{\alpha,\beta\in\mathcal{A}_+} c_\alpha c_\beta(\alpha,\beta)B_{\alpha,\beta}(a,b)\alpha\wedge\beta=0$ for every $a,b\in V$.

Conversely, if a configuration $(\mathcal{A},c)$ satisfies conditions (1), (2) then commutativity equations (1.27) *hold.*

Similarly to the trigonometric $\vee$-system case we show that the class of Euclidean trigonometric $\vee$-system (under some non-degeneracy conditions) is closed under the natural operation of taking subsystems. We also show that the natural restriction process can be applied to a given solution of the commutativity equations (1.27) to get new solutions.

We apply the restriction operation to the known solutions of the form (1.26) corresponding to root systems BC_N and F_4 and we get new solutions by this procedure. These new solutions imply existence of certain $\mathcal{N}=4$ supersymmetric Hamiltonians. We find a multiparameter family of Hamiltonians which extends the supersymmetric Hamiltonians found in [3] in the case of BC_N. This result is given as follows.

Theorem 1.2.12. *Let $\widehat{\mathcal{A}} \subset \mathbb{C}^n$ be the configuration of vectors α with multiplicities c_α given by*

$$\begin{aligned}
&m_i^{-1/2}e_i, \quad \text{with multiplicity} \quad rm_i, \quad 1 \le i \le n,\\
&2m_i^{-1/2}e_i, \quad \text{with multiplicity} \quad sm_i + \frac{1}{2}qm_i(m_i-1), \quad 1 \le i \le n,\\
&m_i^{-1/2}e_i \pm m_j^{-1/2}e_j, \quad \text{with multiplicity} \quad qm_im_j, \quad 1 \le i < j \le n,
\end{aligned}$$

where $m_1, \dots, m_n \in \mathbb{C}^*$. Let the Hamiltonian H be given by

$$H = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{\alpha\in\widehat{\mathcal{A}}} \frac{c_\alpha(\alpha,\alpha)^2}{\sin^2(\alpha,x)} + \frac{1}{4}\sum_{\alpha,\beta\in\widehat{\mathcal{A}}} c_\alpha c_\beta(\alpha,\alpha)(\beta,\beta)(\alpha,\beta)\cot(\alpha,x)\cot(\beta,x) + \Phi,$$

with the fermionic term

$$\Phi = \sum_{\alpha\in\widehat{\mathcal{A}}} \frac{2c_\alpha\alpha_i\alpha_j}{\sin^2(\alpha,x)}\Big(\alpha_l\alpha_k\epsilon_{bc}\epsilon_{ad}\psi^{bi}\psi^{cj}\bar\psi_d^l\bar\psi_a^k + (\alpha,\alpha)\psi^{ai}\bar\psi_a^j\Big),$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and summation over repeated indices is assumed. Then H can be included into $\mathcal{N} = 4$ supersymmetry algebra for suitable supercharges Q^a, $\bar Q_b$, $a, b = 1, 2$.

We also find new supersymmetric Hamiltonians corresponding to restrictions of the root system F_4.

We also show that under certain assumptions Euclidean trigonometric $\vee$-system defines a trigonometric $\vee$-system.

1.2.3 Main results III (Chapter 5)

In the final part of the thesis (Chapter 5) we establish and explore the close relation between the set of commutativity equations (1.27) and the WDVV equations

$$F_iB^{-1}F_j = F_jB^{-1}F_i, \quad i, j = 1, \dots, N, \tag{1.28}$$

where B is any non-degenerate linear combination of matrices F_k with functional coefficients. Starting with a function $F = F(x_1, \dots, x_N)$ that satisfies commutativity equations (1.27), the question arises under what conditions one can obtain a linear combination B of the third order derivatives of F which is a multiple of the identity matrix so that the two sets of equations (1.27) and (1.28) become equivalent. We answer this question and reveal the sufficient non-degeneracy condition formulated as maximality of rank of a certain matrix associated with prepotential F. We give a general formula for the constant metric $B = \sum_{k=1}^N A^kF_k$ for arbitrary dimension N, where functions A^k are given as specific determinants of matrices of size $N-1$. Namely,

$$A^k = (-1)^{k+1} \det \begin{pmatrix} F_{112} & F_{212} & \cdots & F_{(k-1)12} & F_{(k+1)12} & \cdots & F_{N12} \\ F_{113} & F_{213} & \cdots & F_{(k-1)13} & F_{(k+1)13} & \cdots & F_{N13} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ F_{11N} & F_{21N} & \cdots & F_{(k-1)1N} & F_{(k+1)1N} & \cdots & F_{N1N} \end{pmatrix}. \tag{1.29}$$

The following statement relates the commutativity equations and WDVV equations for arbitrary dimension N.

Theorem 1.2.13. *Assume that a function $F = F(x_1, \dots, x_N)$ on $V \cong \mathbb{C}^N$ satisfies the commutativity equations* (1.27)*. Suppose that for a fixed i_0, $1 \le i_0 \le N$ the rank of the matrix $(F_{i_0 ij})$ where $1 \le i, j \le N$, $i \ne i_0$ is $N-1$. Suppose also that there exists a non-degenerate linear combination $\sum_{i=1}^N \eta^i F_i$ for some functions η^i. Then F is a solution of WDVV equations* (1.28)*, where the matrix B is given by $B = \sum_{k=1}^N A^k F_k$ and functions A^k are given by formula* (1.29)*. Moreover, B is proportional to the identity matrix.*

The vector field $\sum_{k=1}^N A^k \partial_{x_k}$ corresponding to the metric B is in fact proportional to the identity vector field of the natural associative multiplication on the tangent spaces $T_* V$ given by

$$\partial_u * \partial_v = u^i v^j F_{ijk} \partial_{x^k}. \tag{1.30}$$

Thus Theorem 1.2.13 implies the existence of the identity vector field for the multiplication (1.30). We also give a generalized version of Theorem 1.2.13 in which the standard metric δ_{ij} is replaced with an arbitrary constant (non-degenerate) metric g. In this case we consider equations of the form

$$F_{ij\alpha} g^{\alpha\beta} F_{\beta kl} = F_{kj\alpha} g^{\alpha\beta} F_{\beta il}, \tag{1.31}$$

where the summation over repeated indices is assumed. Under certain non-degeneracy conditions we show that $g_{\alpha\beta} = \sum_k \eta^k F_{k\alpha\beta}$ for some functions η^k.

Theorem 1.2.13 allows us to obtain new solutions of WDVV equations from known solutions of the commutativity equations. We apply these results to solutions of the form (1.26). Commutativity equations for function F of the form (1.26), where $\mathcal{A}$ is a root system, are known to be satisfied if there are more than one orbits of the Weyl group of the root system $\mathcal{A}$, and multiplicities of different orbits have to be related (see [3]). This leads to new solutions of WDVV equations for root systems BC_N and F_4 and their restrictions. The corresponding identity fields can be given explicitly as the following statement demonstrates for $\mathcal{A} = BC_N^+$, the positive half of the root system BC_N.

Proposition 1.2.14. *Assume that parameters r, s, q satisfy the condition $r = -8s -$*

$2q(N-2)$. *Then the vector filed*

$$e = h^{-1}\sum_{k=1}^{N} \sin 2x_k \partial_{x_k},$$

where $h = 2q\sum_{k=1}^{N}\cos 2x_k + r$, *is the identity vector field for the multiplication* (1.30), *where function* F *is given by*

$$F = \sum_{i=1}^{N}\Big(rf(x_i) + sf(2x_i)\Big) + q\sum_{i<j}^{N} f\big((x_i \pm x_j)\big).$$

We also find the identity vector field for the remaining cases of F_4 and G_2. The following statement reveals the explicit formula of the identity field e for root system F_4 under a specific relation between the two multiplicities.

Proposition 1.2.15. *Let* r *be the multiplicity of the short roots and* q *be the multiplicity of the long roots of root system* F_4. *The identity vector field* e *for* $\mathcal{A} = F_4^+$ *under the condition* $r = -2q$ *is given by the formula*

$$e = h^{-1}\sum_{k=1}^{4} A^k \partial_{x_k}, \tag{1.32}$$

where function h *is given by the formula*

$$h(x) = \frac{1}{2}\big(12q + \sum_{\alpha\in F_4^+} c_\alpha \cos\left(2(\alpha, x)\right)\big), \tag{1.33}$$

and functions A^k *are given by the formula*

$$A^k = \sin x_k\Big(\cos x_k(-1 + \sum_{i\neq k}\cos 2x_i) - 2\prod_{i\neq k}\cos x_i\Big).$$

The only other possibility for the multiplicities of root system F_4 for the commutativity equations to hold is $r = -4q$, and it is dealt with as well.

For the root system G_2 its multiplicities p, q have to satisfy the condition $(p+3q)(p+9q) = 0$ for the commutativity equations to hold [3]. We find the identity field explicitly in both cases.

We also deal with the restrictions of a given solution of the commutativity equations and observe that the identity vector field of a restricted system can be obtained by restricting the identity vector field of the original system. We apply this to the three-dimensional restrictions of root system F_4 (along the hyperplanes $x_4 = 0$ and $x_3 = x_4$), and we find the corresponding identity vector fields and deduce new solutions of WDVV equations of the form (1.26).

Let us now describe the structure of the thesis more specifically.

1.2.4 Structure of the thesis

Chapter 2 contains introductory material on the theory of Frobenius manifolds. We describe the foundation and background topics which are relevant to our subsequent chapters. These include the notations, basic definitions, and results one should be familiar with so that to make the thesis self-contained.

In Section 2.1 we start by giving a brief introduction to the notions of Frobenius algebra and Frobenius manifold as well as the appearance of the associativity (WDVV) equations in these contexts. In Section 2.2 we recall key notions and some general properties of Dubrovin connection which will be useful later. In Section 2.3 the concept of almost Frobenius structure and its construction is reviewed including the intersection form and Dubrovin's almost duality. Orbit spaces of finite Coxeter groups examples are discussed in Section 2.4 as an important class of Frobenius manifolds which admits polynomial prepotentials.

Section 2.5 is devoted to reviewing the rational solutions of WDVV equations corresponding to the class of (rational) $\vee$–systems. We also recall the natural operations of restriction and taking subsystems of a $\vee$-system.

In Section 2.6 we introduce the class of trigonometric solutions F of WDVV equations which involve an extra variable. These solutions are of the form (1.21) and are associated with a configuration of vectors $\mathcal{A}$ with multiplicity function $c\colon \mathcal{A} \to \mathbb{C}$. We give the corresponding associative multiplication explicitly in Propositions 2.6.4, 2.6.5. Then we work out Dubrovin's connection explicitly and relate its flatness with WDVV equations for the prepotential (1.21) in Theorem 2.6.14, which is known in general theory of Frobenius manifolds.

In Subsection 2.6.2 we present the notion of a trigonometric $\vee$-system and we derive some identities from the flatness condition of Dubrovin connections which lead to the statement that relates trigonometric $\vee$-systems and WDVV equations. We present this result in Theorem 2.6.21 (see also Theorem 1.2.1 above). Note that this result was essentially given in [27], where it was derived from the WDVV equations directly. We also note that some conditions regarding collinear vectors were missing in [27], and we clarify the conditions which trigonometric $\vee$-system should satisfy in order to give a solution of WDVV equations.

In Section 2.7 we introduce root systems of Weyl groups and prove that they give examples of trigonometric $\vee$-systems. We also review the trigonometric solutions related to root systems which can be found in [11, 33, 42, 47, 48].

We review the notion of Coxeter number and we give a version of generalized Coxeter number in Section 2.8. The additional condition for a trigonometric $\vee$-system in our Theorem 2.6.21 can be interpreted in terms of this number.

Considerations of Chapter 2 except for Sections 2.6.2, 2.7 are known from the previous literature. The main new statement which is Theorem 2.6.21 improves its earlier version from [27].

In Chapter 3 we investigate operations of taking subsystems and restrictions in Sections 3.1 and 3.2 respectively. Thus in Theorem 3.1.9 we show that a subsystem of a trigonometric $\vee$-system is also a trigonometric $\vee$-system. Then in Theorem 3.2.4 we show that one can restrict solutions of WDVV equations of the form (1.10) to the intersections of hyperplanes to get new solutions. These results are analogous to the results for the rational $\vee$-systems case.

In Section 3.3 we find solutions $\mathcal{F}^{trig}$ for the root system BC_N which depend on three parameters. By applying restrictions we obtain in Sections 3.3 and 3.4 the multi-parameter families of solutions $\mathcal{F}^{trig}$ for the classical root systems thus recovering and extending results from [43]. In the case of BC_N we get a family of solutions depending on $N+3$ parameters which can be specialized to Pavlov's $(N+1)$-parametric family from [43]. These solutions are given in Theorem 3.3.5 (see also Theorem 1.2.6 above). In Section 3.5 we consider solutions $\mathcal{F}^{trig}$ for $N = \dim V \leq 4$. We show that solutions with up to five vectors on the plane belong to deformations of classical root systems. The statement is given in Propositions 3.5.4, 3.5.5 (see also Theorem 1.2.8 above). We also get new examples of solutions $\mathcal{F}^{trig}$ of the form (1.10) some of which cannot be obtained as restrictions of solutions (1.10) for the root systems, see for example Proposition 1.2.10 above. Other examples are also given in the chapter.

In Section 3.6 we revisit solutions $\mathcal{F}^{trig}$ for the root systems studied in [11,33,42,47,48]. The polynomial Q in this case depends on a scalar $\gamma_{(\mathcal{R},c)}$ which is determined in these references for any invariant multiplicity function $c\colon \mathcal{R} \to \mathbb{C}$. We give a formula for $\gamma_{(\mathcal{R},c)}$ in terms of the highest root of $\mathcal{R}$ generalizing a statement from [11] for special multiplicities. The result corresponding to all the reduced not simply-laced root systems $\mathcal{R}$ is given in Proposition 3.7.2.

We also find a related scalar $\lambda_{(\mathcal{R},c)}$ which is invariant under linear transformations applied to the root system $\mathcal{R}$. This scalar may be thought of as a version of generalized Coxeter number (see e.g. [28]) for the irreducible $\mathcal{W}$-module $\Lambda^2 V$ since it is given as a ratio of two canonical $\mathcal{W}$-invariant symmetric bilinear forms on $\Lambda^2 V$. The main results of Chapter 3 are published in [1].

In Chapter 4 we consider trigonometric solutions of WDVV equations without extra variable. We review such solutions found in the previous literature in Section 4.1. In Section 4.2 we generalize solution of the form (1.10) with $Q = 0$ corresponding to B_N root systems found in [33] so that it is included in $(N+2)$-parametric family. The underlying configuration $\mathcal{A}$ is the positive half of BC_N root system, and multiplicities are chosen in a specific way. In order to get such a solution, we find firstly a two-parameter family of solutions where the configuration $\mathcal{A}$ is the positive half of BC_N and multiplicities are Weyl-invariant. We find a metric B given as a linear combination of the third order derivatives for these prepotentials, so that it is a multiple of the identity. The choice of the metric B is motivated by the metric B for the root system B_N which is contained in [33]. Then we generalize these considerations in Section 4.3 to obtain the family of solutions with many parameters corresponding to BC_N solutions by taking special restrictions of these solutions using procedure similar to one we applied in Section 3.3. This result is given in Theorem 4.3.7.

In Section 4.4 we recall the construction of $\mathcal{N}=4$ supersymmetric mechanics and their relations with the trigonometric solutions of WDVV equations. Then we use the multi-parameter deformation of BC_N solutions to construct $\mathcal{N}=4$ supersymmetric mechanical systems. Thus we extend Hamiltonians with two independent coupling parameters found in [3] into multiparameter family. This result is given in Theorem 4.4.7. The above considerations of Chapter 4 are joint work with G. Antoniou and M. Feigin published in [2].

In Section 4.5 we define Euclidean trigonometric $\vee$-systems for which the bilinear form (1.20) is replaced by the standard inner product. We establish a close relation between the commutativity equations with the class of Euclidean trigonometric $\vee$-systems and generalize a result given in [3] for root systems.

In Section 4.6 we discuss the natural operation of taking a subsystem and show that (under some non-degeneracy conditions) the subsystem of a Euclidean trigonometric $\vee$-system is also a Euclidean trigonometric $\vee$-system.

In Section 4.7 we discuss the relation between Euclidean trigonometric $\vee$-systems and trigonometric $\vee$-systems and show that an irreducible Euclidean trigonometric $\vee$-system is a trigonometric $\vee$-system (under a non-degeneracy assumption). We also study the relation between Euclidean trigonometric $\vee$-systems and (rational) complex Euclidean $\vee$-systems, which were introduced in [26] as a generalization of (rational) $\vee$-systems to the class when the canonical form degenerates. We prove that if $(\mathcal{A}, c)$ is an irreducible Euclidean trigonometric $\vee$-system then the set $\{\sqrt{c_\alpha}\alpha\}$ is a (rational) complex Euclidean $\vee$-system.

Finally, in Section 4.8 we apply the restriction procedure to a given solution of the commutativity equations and prove that under some assumptions one can get new solutions throughout this process. These multi-parameter solutions can be applied to construct $\mathcal{N}=4$ supersymmetric mechanical systems.

In Chapter 5 we investigate the relation between the set of commutativity equations and the set of WDVV equations in N-dimensional space. We show that under some non-degeneracy conditions the function F which satisfies the commutativity equations also satisfies the WDVV equations. This is done by showing that the commutativity equations lead to the existence of a non-degenerate linear combination (metric) B of matrices of the third order derivatives F_k with functional coefficients A^k, $(k = 1, \ldots, N)$ where the matrix $B = \sum_{k=1}^{N} A^k F_k$ is proportional to the identity matrix. Therefore WDVV equations follow from the commutativity equations. This leads to new solutions of WDVV equations.

Another way to interpret the linear combination $\sum_{k=1}^{N} A^k F_k$ is to note that the corresponding vector field $\sum_{k=1}^{N} A^k \partial_{x_k}$ is a multiple of the identity vector field for the natural product associated with solution F of the commutativity equations. In Section 5.3 we find the formula for the identity vector field in terms of F for arbitrary dimension N. In Subsection 5.5.1 we apply these results to establish that Euclidean trigonometric $\vee$-systems lead to solutions of commutativity equations provided that certain additional conditions hold. In Subsections 5.5.4–5.5.7 we give examples to illustrate these results for the root system F_4 and its restrictions as well as for the root system G_2.

In the Appendix we present explicitly all the two-dimensional trigonometric $\vee$-systems which are restrictions of root systems. We finish the appendix by listing all the known trigonometric $\vee$-systems on the plane.

Chapter 2

Frobenius manifolds structures

In this chapter we provide an overview of notations, basic definitions and results including rational and trigonometric $\vee$-systems, which one should be familiar with throughout the rest of the thesis. We give a brief introduction to the notions of Frobenius algebra and Frobenius manifold and we review the appearance of the associativity (WDVV) equations in these contexts, which feature in this work.

2.1 Frobenius manifolds and WDVV equations

In this section we start by giving a brief introduction to the notion of Frobenius Algebra and some general properties of Dubrovin connection which which will be useful later.

Definition 2.1.1. [17] Let A be some N-dimensional vector space over $\mathbb{C}$ endowed with a symmetric non-degenerate bilinear form (metric) $\langle \cdot, \cdot \rangle$ on A and a commutative multiplication of vectors

$$\circ \colon A \times A \to A.$$

Then the 4-tuple $\Big(A, \circ, e, \langle \cdot, \cdot \rangle\Big)$ is said to constitute a (commutative, associative) Frobenius algebra if the following conditions hold:

1. $(A, \circ)$ is a commutative associative algebra over $\mathbb{C}$ with unity e;
2. The symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ together with the multiplication $\circ$ satisfy the following condition (Frobenius condition)

$$\langle X \circ Y, Z \rangle = \langle X, Y \circ Z \rangle, \quad X, Y, Z \in A.$$

Remark 2.1.2. Let $\omega \in A^*$ ($\omega \colon A \to \mathbb{C}$), be the map defined by

$$\omega(v) = \langle e, v \rangle.$$

Then we have

$$\omega(u \circ v) = \langle e, u \circ v \rangle = \langle e \circ u, v \rangle = \langle u, v \rangle.$$

Hence the non-degenerate metric $\langle \cdot, \cdot \rangle$ determines the form ω and visa-versa. This linear form ω is often called a trace form (or Frobenius form).

A Frobenius manifold has such a family of Frobenius algebra structure on each of its tangent space which also possesses some additional properties. Before we state the definition of the Frobenius manifold, let us introduce two tensors as follows.

Let ∇ be the Levi-Civita connection of the metric $\langle \cdot, \cdot \rangle$, which has zero curvature, and let us define the $(0, 3)$-tensor c given by the formula

$$c(X, Y, Z) := \langle X \circ Y, Z \rangle, \tag{2.1}$$

as well as the $(0, 4)$- tensor $\Big(\nabla_W c\Big)(X, Y, Z)$, which is given by the formula

$$\Big(\nabla_W c\Big)(X, Y, Z) := \nabla_W \Big(c(X, Y, Z)\Big) - c(\nabla_W X, Y, Z) - c(X, \nabla_W Y, Z) - c(X, Y, \nabla_W Z). \tag{2.2}$$

Let $\mathcal{M}$ be an N-dimensional complex smooth manifold equipped with a (non-degenerate) metric η which does not need to be positive definite. For any $p \in \mathcal{M}$ we denote by $\langle \cdot, \cdot \rangle_p$ the restriction of η to the tangent space $T_p\mathcal{M}$. Sometimes we will omit $p \in \mathcal{M}$ in the notation $\langle \cdot, \cdot \rangle_p$. Let $\circ \colon T_p\mathcal{M} \times T_p\mathcal{M} \to T_p\mathcal{M}$ be a commutative, associative multiplication such that $T_p\mathcal{M}$ is a Frobenius algebra. Let $e \in \Gamma(T\mathcal{M})$ be the unity vector field for the multiplication $\circ$. Denote the Levi-Civita connection of the metric η by ∇. Now we give the definition of a Frobenius manifold.

Definition 2.1.3. [17] The set $(\mathcal{M}, \circ, e, \eta, \mathcal{E})$ is a Frobenius manifold, where $\mathcal{E} \in \Gamma(T\mathcal{M})$, if the following properties hold:

1. The metric η is flat,
2. The unit vector field e is flat, that is $\nabla e = 0$,
3. The two tensors $c(X, Y, Z)$ and $\big(\nabla_W c\big)(X, Y, Z)$ given by formulas (2.1), (2.2) respectively are totally symmetric,
4. The vector field $\mathcal{E}$ is linear in the flat variables, such that the corresponding group of diffeomorphisms acts by conformal transformation on the metric and by rescalings on the algebra on $T_t\mathcal{M}$, namely
    - $\nabla(\nabla\mathcal{E}) = 0$,
    - $\mathcal{E}\langle X, Y \rangle - \langle [\mathcal{E}, X], Y \rangle - \langle X, [\mathcal{E}, Y] \rangle = D\langle X, Y \rangle$,

- $[\mathcal{E}, X \circ Y] - [\mathcal{E}, X] \circ Y - X \circ [\mathcal{E}, Y] = d_1 X \circ Y$,
  for some constants D, d_1.

These axioms of the existence of the two totally symmetric tensors together with the successive application of the Poincaré Lemma imply a local existence of a prepotential F (called the free energy of the Frobenius manifold) which satisfies the WDVV equations of associativity in the flat coordinates of the metric on $\mathcal{M}$. Before we explain this fact in the following statement, let us first explain the notion of a flat coordinates by recalling some facts from the Riemannian geometry.

Let $\{x_k\}_{k=1}^N$ be a smooth coordinate system on an N-dimensional smooth manifold $\mathcal{M}$. With respect to this smooth coordinate system, let $\partial_i := \frac{\partial}{\partial x_i}$. Then locally, the Levi-Civita connection ∇ can be described by how it behaves on the basis field $\{\partial_i\}$ by

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^{N} \Gamma_{ij}^k \partial_k, \tag{2.3}$$

where the connection coefficients $\Gamma_{ij}^k : U \to \mathbb{C}$ are called the *Christoffel symbols*. Now, let us consider two vector fields $X = \sum_{i=1}^N f_i \partial_i$, $Y = \sum_{j=1}^N g_j \partial_j$, where $f, g \in C^\infty(\mathcal{M})$. Then by the properties of the connection and formula (2.3) it is easy to see that

$$\nabla_X Y = \sum_{i=1}^{N} \Big(\sum_{j=1}^{N} f_j \partial_j g_i + \sum_{j,k=1}^{N} \Gamma_{jk}^i f_j g_k \Big) \partial_i.$$

If the metric is flat then there exists a coordinate system such that components of the metric are constant and the Christoffel symbols of the Levi-Civita connection of the metric are zero.

Proposition 2.1.4. *Let $\mathcal{M}$ be an N-dimensional complex smooth manifold. Then there exists locally a single holomorphic function $F = F(t_1, \dots, t_N)$ defined on some region $\mathcal{M} \subset \mathbb{C}^N$ and unique up to a polynomial of degree two such that*

$$\frac{\partial^3 F}{\partial_{t_i} \partial_{t_j} \partial_{t_k}} = c\Big(\frac{\partial}{\partial_{t_i}}, \frac{\partial}{\partial_{t_j}}, \frac{\partial}{\partial_{t_k}}\Big),$$

where c is the tensor given by (2.1) and $\{t_k\}_{k=1}^N$ is some flat coordinates on $\mathcal{M}$.

Proof. It is known by Poincaré Lemma that one can locally solve the equation $\partial_k F = \mathcal{G}_k$ if and only if the condition $\partial_k \mathcal{G}_l = \partial_l \mathcal{G}_k$ is satisfied for all k, l. Since $\langle \cdot, \cdot \rangle$ is flat, one can choose the flat coordinates $\{t_k\}_{k=1}^N$ such that the functions $\langle \frac{\partial}{\partial t_p}, \frac{\partial}{\partial t_q} \rangle$ are constants. Let $\partial_p := \frac{\partial}{\partial t_p}$, and $c_{ijk} = c(\partial_i, \partial_j, \partial_k)$. By the axioms of the Frebinus manifold the tensor $\big(\nabla_W c\big)(X, Y, Z)$ is totally symmetric, also note that in t-coordinates the tensor (2.2) reduces to $\big(\nabla_{\partial_p} c\big)(\partial_i, \partial_j, \partial_k) = \partial_p c_{ijk}$. Then by Poincaré Lemma we can introduce a potential

H_{jk} such that

$$c_{ijk} = \partial_i(H_{jk}).$$

But again by the axioms the tensor c_{ijk} is also totally symmetric, hence we have $c_{ijk} = c_{jik}$, that is $\partial_i(H_{jk}) = \partial_j(H_{ik})$, with $H_{ik} = H_{ki}$ (since c_{ijk} is totally symmetric). Then Poincaré Lemma implies that we can introduce a potential $\mathcal{G}_k$ such that

$$H_{ik} = \partial_i \mathcal{G}_k.$$

Since $H_{ik} = H_{ki}$ then we have

$$\partial_i \mathcal{G}_k = \partial_k \mathcal{G}_i,$$

which again by Poincaré Lemma implies that there exists a prepotential F such that

$$\mathcal{G}_k = \partial_k F.$$

Hence the prepotential F exists, moreover, $c_{ijk} = \partial_i \partial_j \partial_k F$.

□

Proposition 2.1.5. *In the notations of Proposition 2.1.4, the associativity of the product $\circ$ is equivalent to the following condition*

$$c_{ijk}(t)\eta^{kn}c_{lmn}(t) = c_{ljk}(t)\eta^{kn}c_{imn}(t), \quad c_{ijm} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_m}, \tag{2.4}$$

for any $1 \leq i, j, l, m \leq N$.

Proof. By choosing the variables t_k $(1 \leq k \leq N)$ to be flat coordinates of the metric η, together with the condition $\nabla e = 0$ means this can be done in such a way that $e = \frac{\partial}{\partial t_1}$. Hence by the normalization $\partial_1 := \frac{\partial}{\partial t_1} = e$, we have

$$\eta_{ij} := \langle \partial_i, \partial_j \rangle = \langle e \circ \partial_i, \partial_j \rangle = c_{1ij}$$

is a constant non-degenerate matrix. Let $\eta^{ij} = (\eta_{ij})^{-1}$. Then the components of the product $\circ$ are given by the functions

$$c_{ij}^k = \eta^{kl} c_{ijl}, \tag{2.5}$$

which define an associative, commutative algebra

$$\partial_i \circ \partial_j|_t := c_{ij}^k(t)\partial_k \tag{2.6}$$

on each tangent space $T_t\mathcal{M}$ with unity e. By Proposition 2.1.4 the function F exists and satisfies that $c_{ijm} = \partial_i \partial_j \partial_m F$. The associativity of the multiplication $\circ$ reads in

t-coordinates

$$(\partial_i \circ \partial_j) \circ \partial_m = \partial_i \circ (\partial_j \circ \partial_m),$$

which is by formulae (2.5), (2.6) leads to following system of nonlinear partial differential equations for the function F:

$$c_{ijk}(t)\eta^{kn}c_{lmn}(t) = c_{ljk}(t)\eta^{kn}c_{imn}(t),$$

for any $1 \leq i, j, l, m \leq N$ as required. $\square$

Remark 2.1.6. (1) Equations (2.4) are known as the WDVV equations after Edward Witten, Robbert Dijkgraaf, Erik Verlinde, and Herman Verlinde who discovered the system of these equations for the first time [17].

(2) The invariant properties (4) in Definition 2.1.3 with respect to the Euler vector field lead to demanding F be a quasi-homogeneous function, namely,

$$\mathcal{E}(F) = d_F F + \text{quadratic polynomial in}\, t, \tag{2.7}$$

for some constant d_F.

The curvature form of any connection ∇ is defined by

$$R(X,Y)Z \coloneqq [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \quad X, Y, Z \in \Gamma\Big(T\mathcal{M}\Big). \tag{2.8}$$

We say that ∇ is a *flat connection* if it has zero curvature. The connection ∇ is said to be *torsion free* if it satisfies the condition:

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{2.9}$$

where $[\cdot, \cdot]$ is the Lie bracket operator and X, Y are any smooth vector fields. The connection ∇ is said to be *compatible with the metric* $\langle\cdot, \cdot\rangle$ if it satisfies

$$Z\langle X, Y\rangle = \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

2.2 Dubrovin connection

Dubrovin connection (deformed flat connection) is a one-parameter family of flat connections which is defined on a Frobenius manifold as follows. Let ∇ be a Levi-Civita connection of a metric η, and $\circ$ define a multiplication of tangent vectors on a manifold $\mathcal{M}$. Then Dubrovin Connection $\widetilde{\nabla}^{\mu}$ for any $\mu \in \mathbb{C}$ (here μ is called the spectral parameter) has the form

$$\widetilde{\nabla}^{\mu}_X Y = \nabla_X Y + \mu X \circ Y. \tag{2.10}$$

In the following two propositions we show some general properties for the Dubrovin connection (2.10).

Proposition 2.2.1. *The Dubrovin connection (2.10) is torsion free for any μ if and only if the product $\circ$ is commutative.*

Proof. By formula (2.10) we have

$$\begin{aligned}\widetilde{\nabla}^{\mu}_X Y - \widetilde{\nabla}^{\mu}_Y X - [X,Y] &= \Big(\nabla_X Y - \nabla_Y X - [X,Y]\Big) + \mu\Big(X \circ Y - Y \circ X\Big)\\ &= \mu\Big(X \circ Y - Y \circ X\Big)\end{aligned}$$

since the Levi-Civita connection ∇ is a flat connection. This implies the statement. □

Proposition 2.2.2. *Assume that the product $\circ$ is commutative and the tensor $c(X,Y,Z)$ is a totally symmetric tensor. Then Dubrovin connection (2.10) is flat for any μ if and only if the product $\circ$ is associative and the tensor $T(X,Y,Z,W) = \Big(\nabla_W c\Big)(X,Y,Z)$ given by (2.2) is totally symmetric.*

Proof. By the definition of Dubrovin connection (2.10) its curvature (2.8) takes the form

$$\begin{aligned}\widetilde{R}^{\mu}(X,Y)Z = R(X,Y)Z &+ \mu\Big(\nabla_X(Y \circ Z) - \nabla_Y(X \circ Z) + X \circ \nabla_Y Z - Y \circ \nabla_X Z - [X,Y] \circ Z\Big)\\ &+ \mu^2\Big(X \circ (Y \circ Z) - Y \circ (X \circ Z)\Big),\end{aligned} \tag{2.11}$$

where $R(X,Y)Z$ is the curvature of the Levi-Civita connection ∇ and it is equal to zero since it is a flat connection. Together with the commutativity of the product $\circ$ relation (2.11) reduces to the form

$$\begin{aligned}\widetilde{R}^{\mu}(X,Y)Z = \mu&\Big(\nabla_X(Y \circ Z) - \nabla_Y(X \circ Z) + X \circ \nabla_Y Z - Y \circ \nabla_X Z - [X,Y] \circ Z\Big)\\ &+ \mu^2\Big(X \circ (Z \circ Y) - (X \circ Z) \circ Y\Big).\end{aligned} \tag{2.12}$$

While the vanishing of μ^2-terms gives rise to the associativity of the multiplication $\circ$, we will show that the vanishing of μ-terms is equivalent to the totally symmetric of the tensor $T(X,Y,Z,W)$. To do so, let us take the product of μ-terms with any vector field $W \in T\mathcal{M}$ via the bilinear form $\langle\cdot,\cdot\rangle$, where $\langle u,v\rangle = \eta(u,v), u,v \in T_*\mathcal{M}$. Then by the compatibility and torsion freeness of the Levi-Civita connection we have

$$\begin{aligned}&\langle \nabla_X(Y \circ Z) - \nabla_Y(X \circ Z) + X \circ \nabla_Y Z - Y \circ \nabla_X Z - [X,Y] \circ Z, W\rangle\\ &= \Big(\nabla_X\langle Y \circ Z, W\rangle - \langle Y \circ Z, \nabla_X W\rangle\Big) - \Big(\nabla_Y\langle X \circ Z, W\rangle - \langle X \circ Z, \nabla_Y W\rangle\Big)\\ &+ \langle X \circ \nabla_Y Z, W\rangle - \langle Y \circ \nabla_X Z, W\rangle - \langle \nabla_X Y \circ Z, W\rangle + \langle \nabla_Y X \circ Z, W\rangle.\end{aligned} \tag{2.13}$$

Using the notation $c(X,Y,Z) = \langle X \circ Y, Z\rangle$ relation (2.13) can be rearranged as

$$\begin{aligned}
&\langle \nabla_X(Y\circ Z) - \nabla_Y(X\circ Z) + X\circ \nabla_Y Z - Y \circ \nabla_X Z - [X,Y]\circ Z, W\rangle \\
&= \Big(\nabla_X\Big(c(Y,Z,W)\Big) - c(\nabla_X Y, Z, W) - c(Y, \nabla_X Z, W) - c(Y,Z,\nabla_X W)\Big) \\
&- \Big(\nabla_Y\Big(c(X,Z,W)\Big) - c(\nabla_Y X, Z, W). - c(Y, \nabla_X Z, W) - (X,Z,\nabla_Y W)\Big) \\
&= \Big(\nabla_X c\Big)(Y,Z,W) - \Big(\nabla_Y c\Big)(X,Z,W) = T(Y,Z,W,X) - T(X,Z,W,Y). \qquad (2.14)
\end{aligned}$$

Since $c(X,Y,Z)$ is a totally symmetric tensor, then relation (2.14) tells that vanishing of μ-terms is equivalent to the symmetric of the tensor $T(X,Y,Z,W)$ in all its four arguments. Hence the statement follows by the above discussion together with relation (2.12). □

Remark 2.2.3. Proposition 2.2.2 guarantees that on a Frobenius manifold, $\widetilde{\nabla}^\mu$ is flat for all μ.

The following two propositions are general facts that hold for any connection ∇ on the tangent bundle TW, where W is a vector space.

Proposition 2.2.4. *Let ∇ be a connection on the tangent bundle TW, where W is a vector space of dimension N. Let $x_1,\ldots,x_N$ be coordinates in W and let $\partial_i := \partial_{x_i}$, $(i = 1,\ldots,N)$, be the corresponding constant vector fields. Define $\nabla_i := \nabla_{\partial_{x_i}}$. Then $[\nabla_i,\nabla_j]\partial_k = 0$ for all k implies that $[\nabla_i,\nabla_j](Z) = 0$, where $Z = \sum_{i=1}^N z_i(x)\partial_i \in \Gamma(TW)$.*

Proof. We have

$$\nabla_j(Z) = \sum_{k=1}^N z_k(x)\nabla_j(\partial_k) + \sum_{k=1}^N(\partial_j z_k)\partial_k.$$

Hence we have

$$\nabla_i\nabla_j(Z) = \sum_{k=1}^N z_k \nabla_i\nabla_j(\partial_k) + \sum_{k=1}^N (\partial_i z_k)\nabla_j(\partial_k) + \sum_{k=1}^N(\partial_i\partial_j z_k)\partial_k + \sum_{k=1}^N(\partial_j z_k)\nabla_i(\partial_k). \qquad (2.15)$$

Similarly, we get (by exchanging $i \leftrightarrow j$) the formula of $\nabla_j\nabla_i(Z)$, then we have

$$[\nabla_i,\nabla_j](Z) = \sum_{k=1}^N z_k\Big(\nabla_i\nabla_j - \nabla_j\nabla_i\Big)\partial_k,$$

which implies the statement since $[\nabla_i,\nabla_j]\partial_k = 0$ for all k. □

Proposition 2.2.5. *In the notations of Proposition 2.2.4, let $u,v,z \in \Gamma(TW)$ be given by $u = \sum_{i=1}^N u_i(x)\partial_i$, $v = \sum_{i=1}^N v_i(x)\partial_i$, $z = \sum_{i=1}^N z_i(x)\partial_i$. If $[\nabla_i,\nabla_j]\partial_k = 0$ for all k, then*

$$[\nabla_u,\nabla_v]Z = \nabla_{[u,v]}Z.$$

Proof. Firstly, we have

$$[u,v]=\Big[\sum_{i=1}^{N}u_i\partial_i,\sum_{j=1}^{N}v_j\partial_j\Big]=\sum_{i,j=1}^{N}\Big(u_i\partial_i(v_j)\partial_j-v_j\partial_j(u_i)\partial_i\Big). \tag{2.16}$$

Since $\nabla_{f_1w_1+f_2w_2}Z=f_1\nabla_{w_1}Z+f_2\nabla_{w_2}Z$, where f_1,f_2 are functions in W, then by (2.16) we have

$$\begin{aligned}\nabla_{[u,v]}(Z)&=\sum_{i,j=1}^{N}u_i(\partial_iv_j)(\nabla_jZ)-\sum_{i,j=1}^{N}v_j(\partial_ju_i)(\nabla_iZ)\\&=\sum_{i,j,k=1}^{N}u_i(\partial_iv_j)\nabla_j(z_k\partial_k)-\sum_{i,j,k=1}^{N}v_j(\partial_ju_i)\nabla_i(z_k\partial_k)\\&=\sum_{i,j,k=1}^{N}u_i(\partial_iv_j)(\partial_jz_k)\partial_k+\sum_{i,j,k=1}^{N}u_iz_k(\partial_iv_j)(\nabla_j\partial_k)\\&\quad-\sum_{i,j,k=1}^{N}v_j(\partial_ju_i)(\partial_iz_k)\partial_k-\sum_{i,j,k=1}^{N}v_jz_k(\partial_ju_i)(\nabla_i\partial_k).\end{aligned} \tag{2.17}$$

On the other hand we have

$$\nabla_vZ=\sum_{j,k=1}^{N}v_jz_k(\nabla_j\partial_k)+\sum_{j,k=1}^{N}v_j(\partial_jz_k)\partial_k.$$

Hence we have

$$\begin{aligned}\nabla_u\nabla_vZ=&\sum_{i,j,k=1}^{N}u_iv_jz_k(\nabla_i\nabla_j\partial_k)+\sum_{i,j,k=1}^{N}u_iz_k(\partial_iv_j)(\nabla_j\partial_k)+\sum_{i,j,k=1}^{N}u_iv_j(\partial_iz_k)(\nabla_j\partial_k)\\&+\sum_{i,j,k=1}^{N}u_iv_j(\partial_jz_k)(\nabla_i\partial_k)+\sum_{i,j,k=1}^{N}u_iv_j(\partial_i\partial_jz_k)\partial_k+\sum_{i,j,k=1}^{N}u_i(\partial_iv_j)(\partial_jz_k)\partial_k.\end{aligned} \tag{2.18}$$

Similarly, (by swapping $i\leftrightarrow j, u\leftrightarrow v$) we get the formula of $\nabla_v\nabla_uZ$ and hence we have

$$\begin{aligned}[\nabla_u,\nabla_v](Z)=&\sum_{i,j,k=1}^{N}u_iv_jz_k(\nabla_i\nabla_j\partial_k-\nabla_j\nabla_i\partial_k)+\sum_{i,j,k=1}^{N}u_iz_k(\partial_iv_j)(\nabla_j\partial_k)\\&+\sum_{i,j,k=1}^{N}u_i(\partial_iv_j)(\partial_jz_k)\partial_k-\sum_{i,j,k=1}^{N}v_jz_k(\partial_ju_i)(\nabla_i\partial_k)-\sum_{i,j,k=1}^{N}v_j(\partial_ju_i)(\partial_iz_k)\partial_k.\end{aligned} \tag{2.19}$$

The statement follows from relations (2.17) and (2.19) since $[\nabla_i,\nabla_j]\partial_k=0$ for all k. $\square$

2.3 Almost dual Frobenius structure

In this section we review the concept of almost Frobenius structure and its construction, this includes the notion of the intersection form and Dubrovin's almost duality.

2.3.1 The intersection form

Another important character naturally associated to any Frobenius manifold is the existence of a second flat metric [17]. This new metric is defined as an inner product of 1-forms on the cotangent bundle of the Frobenius manifold and it is related to the original metric η, where the original metric η is used to provide an isomorphism between $T\mathcal{M}$ and $T^*\mathcal{M}$. The second metric g is called the intersection form of the Frobenius manifold.

Definition 2.3.1. [17]. Let $(\mathcal{M}, \circ, e, \eta, \mathcal{E})$ be a Frobenius manifold. Let $\omega_1, \omega_2 \in T_t^*\mathcal{M}$ be two 1-forms, and let $\omega_1 \circ \omega_2$ be the product which is induced from the product of vectors in $T_t\mathcal{M}$ by the isomorphism $\eta\colon T_t\mathcal{M} \to T_t^*\mathcal{M}$. The intersection form is a symmetric bilinear form g defined on each of the cotangent space $T_t^*\mathcal{M}$ by the formula

$$g(\omega_1, \omega_2) = \mathcal{E}(\omega_1 \circ \omega_2). \tag{2.20}$$

In the flat coordinates $\{t_k\}$ for the metric η, we have

$$dt_\alpha \circ dt_\beta = c_\gamma^{\alpha\beta}(t) dt_\gamma, \quad c_\gamma^{\alpha\beta}(t) = \eta^{\alpha\lambda}\eta^{\beta\nu}c_{\lambda\nu\gamma}. \tag{2.21}$$

Then we have the Gram matrix

$$g^{\alpha\beta}(t) := g(dt_\alpha, dt_\beta) = \mathcal{E}(dt_\alpha \circ dt_\beta) = c_\gamma^{\alpha\beta}\mathcal{E}(dt_\gamma) \tag{2.22}$$

with the basic property that $\frac{\partial g^{\alpha\beta}}{\partial_{t_1}} = \eta^{\alpha\beta}$ [17]. The following proposition gives the relation between the original metric η and the new metric g (cf.[17], note that the proof of the statement was omitted there).

Proposition 2.3.2. *The intersection metric g is related to the original metric η by the following relation:*

$$g(\mathcal{E} \circ u, v) = \eta(u, v), \quad u, v \in \Gamma(T\mathcal{M}).$$

Proof. In some flat coordinates $\{t_k\}$ of $\mathcal{M}$, let $\mathcal{E} = \mathcal{E}^\mu \partial_\mu$, $u = \partial_\rho$, $v = \partial_\lambda$. Then by formula (2.6) we have

$$\mathcal{E} \circ u = \mathcal{E}^\mu c_{\rho\mu}^\beta(t)\partial_\beta.$$

Hence we have

$$g(\mathcal{E} \circ u, v) = \mathcal{E}^\mu g_{\beta\lambda}(t) c_{\rho\mu}^\beta(t). \tag{2.23}$$

Note that from relation (2.5) we have

$$c_{\alpha\beta\gamma} = \eta_{\varepsilon\gamma} c^{\varepsilon}_{\alpha\beta}. \tag{2.24}$$

Now by formulae (2.21), (2.22) we have

$$g^{\alpha\beta}(t) = \eta^{\alpha\gamma}\eta^{\beta\varepsilon} c_{\gamma\varepsilon\mu}(t)\mathcal{E}^{\mu}.$$

By multiplying both sides by $g_{\beta\lambda}(t)$ we get

$$\delta^{\alpha}_{\lambda} = g^{\alpha\beta}(t)g_{\beta\lambda}(t) = g_{\beta\lambda}(t)\eta^{\alpha\gamma}\eta^{\beta\varepsilon} c_{\gamma\varepsilon\mu}(t)\mathcal{E}^{\mu}.$$

Then by multiplying both sides by $\eta_{\alpha\rho}$ and using formulae (2.5),(2.24) we have

$$\begin{aligned}
\eta_{\lambda\rho} &= \mathcal{E}^{\mu} g_{\beta\lambda}(t)\eta_{\lambda\rho}\eta^{\beta\varepsilon}\left(\eta^{\lambda\gamma} c_{\gamma\varepsilon\mu}(t)\right) \\
&= \mathcal{E}^{\mu} g_{\beta\lambda}(t)\eta^{\beta\varepsilon}\left(\eta_{\lambda\rho} c^{\lambda}_{\varepsilon\mu}(t)\right) \\
&= \mathcal{E}^{\mu} g_{\beta\lambda}(t)\left(\eta^{\beta\varepsilon} c_{\rho\varepsilon\mu}(t)\right) \\
&= \mathcal{E}^{\mu} g_{\beta\lambda}(t) c^{\beta}_{\rho\mu}(t).
\end{aligned} \tag{2.25}$$

The statement follows by relations (2.23), (2.25). □

Definition 2.3.3. Two contravariant metrics $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ form a flat pencil if:

(1) The metric

$$(\cdot,\cdot)_1 + \mu(\cdot,\cdot)_2 \tag{2.26}$$

is flat for all μ.

(2) The components of the Levi-Civita connection for the metric (2.26) have the form

$${}^{1}\Gamma^{\alpha\beta}_{\gamma} + \mu\, {}^{2}\Gamma^{\alpha\beta}_{\gamma},$$

where ${}^{1}\Gamma^{\alpha\beta}_{\gamma}$ and ${}^{2}\Gamma^{\alpha\beta}_{\gamma}$ are the contravariant components of corresponding Levi-Civita connections of metrics $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ respectively.

The following statement holds.

Proposition 2.3.4. [17] *Let $\mathcal{M}$ be a Frobenius manifold and let η^* be the metric induced on $T^*\mathcal{M}$ by the contravariant metric η. Assume that the Euler vector field $\mathcal{E}$ is invertible. Then the intersection form g and the metric η^* on $\mathcal{M}$ form a flat pencil.*

Remark 2.3.5. Consider points $t \in \mathcal{M}$ where there exists $\mathcal{E}^{-1} \in T_t\mathcal{M}$ such that $\mathcal{E}^{-1} \circ \mathcal{E} = e$. Then from Frobenius condition with respect to the original metric η we have

$$\eta(\mathcal{E}^{-1} \circ u, v) = \eta(\mathcal{E}^{-1}, u \circ v). \tag{2.27}$$

On the other hand from Proposition 2.3.2 we have

$$\eta(\mathcal{E}^{-1} \circ u, v) = g\big(\mathcal{E} \circ (\mathcal{E}^{-1} \circ u), v\big) = g(u, v). \tag{2.28}$$

Hence from relations (2.27), (2.28) we have

$$g(u, v) = \eta(\mathcal{E}^{-1}, u \circ v). \tag{2.29}$$

Thus the metric g is well-defined on the points of $\mathcal{M}$ where the vector field $\mathcal{E}$ is an invertible element of the algebra.

2.3.2 Dubrovin's almost duality

The concept of almost duality was introduced by Dubrovin [20]. In the construction of this structure one of the Frobenius manifold axioms is relaxed, but a solution of the WDVV equations still exists. The unity vector field is not covariantly constant, that is the axiom of the flatness of the identity vector field is relaxed.

Given a Frobenius manifold $(\mathcal{M}, \circ, e, \langle\cdot,\cdot\rangle, \mathcal{E})$, one can define a new multiplication $\star$ on the tangent bundle via the original product $\circ$ by twisting with the help of the Euler field $\mathcal{E}$ as follows:

$$X \star Y = \mathcal{E}^{-1} \circ X \circ Y, \quad X, Y \in \Gamma(T\mathcal{M}), \tag{2.30}$$

where $\mathcal{E} \circ \mathcal{E}^{-1} = \mathcal{E}^{-1} \circ \mathcal{E} = e$. Hence, the new product (2.30) is well defined whenever $\mathcal{E}$ is invertible. Now to construct new Frobenius algebra, we first define

$$\mathcal{M}^* := \mathcal{M} \setminus \{t \in \mathcal{M} : \mathcal{E} \text{ is not invertible}\}. \tag{2.31}$$

Since $\circ$ is commutative and associative, then it is clear from (2.30) that $\star$ is also commutative and associative. Also since

$$\mathcal{E} \star X = \mathcal{E}^{-1} \circ \mathcal{E} \circ X = e \circ X = X,$$

then the Euler vector field $\mathcal{E}$ plays the role of the unity for the product (2.30). Moreover, if g is the intersection form defined by formula (2.20) and $w_1, w_2, w_3 \in \Gamma(T\mathcal{M})$, then by relations (2.29) and (2.30) we have

$$g(w_1 \star w_2, w_3) = g(\mathcal{E}^{-1} \circ w_1 \circ w_2, w_3) = \eta\big(\mathcal{E}^{-1}, (\mathcal{E}^{-1} \circ w_1 \circ w_2) \circ w_3\big). \tag{2.32}$$

Since the product $\circ$ is commutative, then the right-hand side of relation (2.32) is symmetric in w_1, w_2, w_3. Hence we have $g(w_1 \star w_2, w_3) = g(w_1, w_2 \star w_3)$ which means that the intersection form g satisfies the Frobenius condition with respect to the product $\star$, and hence the the tangent bundle $T\mathcal{M}^*$ is endowed with the structure of a Frobenius algebra. Let us denote this Frobenius algebra by $A_p = (T_p\mathcal{M}, \star)$, $p \in \mathcal{M}^*$. This algebra is called a *rescaling* of the original algebra. Now we give the precise definition of almost Frobenius manifold.

Definition 2.3.6. [20] Let $\mathcal{M}^*$ be a smooth manifold equipped with a (non-degenerate) metric g. Let $\star$ be a commutative, associative multiplication such that $T_p\mathcal{M}^*$ is a Frobenius algebra. Let $\mathcal{E}, e \in \Gamma(T\mathcal{M}^*)$. An almost Frobenius structure of the charge $d \neq 1$ on the manifold $\mathcal{M}^*$ is the structure of a Frobenius algebra on the tangent spaces $T_p\mathcal{M}^* = (T_p\mathcal{M}^*, \star, g), p \in \mathcal{M}^*$, depending (smoothly, analytically etc.) on the point $p \in \mathcal{M}^*$. It must satisfy the following axioms.

(1) The metric g is flat.

(2) In the flat coordinates $p_1, ..., p_n$ for the metric g, the structure constants $\overset{*}{c}{}^i_{jk}$ of the algebra A_p can be locally represented in the form

$$\overset{*}{c}{}^i_{jk}(p) = g^{il}\frac{\partial^3 F^\star(p)}{\partial p_l \partial p_j \partial p_k}, \tag{2.33}$$

for some function $F^\star(p)$ and $g^{ij} = (dp_i, dp_j)$. The function $F^\star(p)$ must satisfy the following homogeneity condition:

$$\sum_{i=1}^{N} p_i \frac{\partial F^\star(p)}{\partial p_i} = 2F^\star(p) + \frac{g(p,p)}{1-d}.$$

(3) The vector field $\mathcal{E}$ (will be called the Euler vector field) takes the form

$$\mathcal{E} = \frac{1-d}{2}\sum_{i=1}^{N} p_i \frac{\partial}{\partial p_i}, \tag{2.34}$$

and it is the unity of the Frobenius algebra.

(4) The vector field e has the form

$$e = \sum_{i=1}^{N} e_i(p)\frac{\partial}{\partial p_i}$$

and being an invertible element of the Frobenius algebra A_p, $p \in \mathcal{M}^*$ such that it acts by shifts $\nu \mapsto \nu - 1$ on the solutions of the system of equations

$$\frac{\partial^2 \widetilde{p}}{\partial p_i \partial p_j} = \nu \overset{*}{c}{}^k_{ij}(p)\frac{\partial \widetilde{p}}{\partial p_k}, \tag{2.35}$$

for some function $\widetilde{p} = \widetilde{p}(p;\nu)$.

Such structure given by Definition 2.3.6 is also known as *dual (almost) Frobenius manifold.* This is due to the following result.

Theorem 2.3.7. [20] *Let $(\mathcal{M}, \circ, e, \langle \cdot, \cdot \rangle, \mathcal{E})$ be a Frobenius manifold. Then $(\mathcal{M}^*, \star, e, g, \mathcal{E})$, where $\mathcal{M}^*, \star, g$ are given by formulae (2.31), (2.30), (2.29) respectively, is an almost Frobenius structure. Furthermore $(\mathcal{M}^*, \star, \mathcal{E}, g, \mathcal{E})$, satisfies all the axioms of Frobenius manifold given in Definition 2.1.3 except axiom (2).*

The following statement gives the relation between the almost Frobenius structure and WDVV equations.

Theorem 2.3.8. [20] *Let $(\mathcal{M}, \circ, e, \langle \cdot, \cdot \rangle, \mathcal{E})$ be a Frobenius manifold. Let $(\mathcal{M}^*, \star, \mathcal{E}, g, \mathcal{E})$, be the corresponding almost dual Frobenius manifold where $\mathcal{M}^*, \star, g$ are given by formulae (2.31), (2.30), (2.29) respectively. Then the function $F^\star$ defined (locally) by the condition (2.33) satisfies the WDVV equations in the flat coordinates $\{p_i\}_{i=1}^N$ of the metric g.*

2.4 Orbit spaces examples

The construction of the Frobenius structure on the orbit spaces of finite Coxeter groups was established in [19]. Flat pencils of metrics can be used to provide the (complexified) orbit space of a finite Coxeter group with the structure of a Frobenius manifold. One of the flat metrics on the complexified orbit space of a finite Coxeter group is given by invariant bilinear form while the other one was found by K. Saito et al. [50]. Dubrovin used this metric to construct the structure of a Frobenius manifold on the orbit space [17, Lecture 4]. It follows from the construction that the corresponding Frobenius prepotential which solves the WDVV equations will be a polynomial in the flat coordinates $\{t_k\}_{k=1}^N$ of this metric. Here we give a brief summary for this structure.

Definition 2.4.1. Let $V = \mathbb{R}^N$ and let $\langle \cdot, \cdot \rangle$ be the standard positive definite symmetric bilinear form in V. A *reflection* is a linear operator s on V which sends some non-zero vector α to its negative while fixing point-wise the hyperplane (called a *mirror*) $\Pi_\alpha = \{x \in V : \langle \alpha, x \rangle = 0\}$ orthogonal to α.

We will write $s = s_\alpha$. This map can be given as follows:

$$s_\alpha v = v - \frac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha, \quad v \in V.$$

Indeed $V = \mathbb{R}_\alpha \oplus \Pi_\alpha$, where $\mathbb{R}_\alpha$ denotes the one-dimensional vector space spanned by α

$$\mathbb{R}_\alpha = \{u \in V : u = r\alpha, r \in \mathbb{R}\}.$$

Definition 2.4.2. Let $\mathcal{R}$ be a finite set of non-zero vectors in V such that:

1. $\mathcal{R} \cap \mathbb{R}_\alpha = \{\alpha, -\alpha\}, \quad \forall \alpha \in \mathcal{R}$,

2. For any $\alpha \in \mathcal{R}$, $s_\alpha \mathcal{R} = \mathcal{R}$.

The set $\mathcal{R}$ is called a *reduced Coxeter root system* with associated reflection group $\mathcal{W}$, where

$$\mathcal{W} = \langle s_\alpha : \alpha \in \mathcal{R} \rangle.$$

$\mathcal{W}$ is called the Coxeter group of the root system $\mathcal{R}$ and it is a subgroup of the group $O(V)$ of all orthogonal transformations of V.

Definition 2.4.3. A root system $\mathcal{R}$ is called *irreducible* if $\mathcal{R}$ cannot be written as a union $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ of two non-empty subsets $\mathcal{R}_1, \mathcal{R}_2$ of $\mathcal{R}$ orthogonal with respect to the standard bilinear form $\langle \cdot, \cdot \rangle$ on V, that is $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in \mathcal{R}_1$, $\beta \in \mathcal{R}_2$.

Definition 2.4.4. A root system $\mathcal{R}$ is called *crystallographic* root system if for all $\alpha, \beta \in \mathcal{R}$ the following condition holds:

$$\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$

The associated reflection group $\mathcal{W}$ is called *Weyl group*.

Consider a root system $\mathcal{R}$. One can find a vector $d \in V$, such that $\forall \alpha \in \mathcal{R}$, $\langle d, \alpha \rangle \neq 0$. Then we can decompose the root system into two disjoint parts: $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$, where $\mathcal{R}_+ = \mathcal{R}_+(d) = \{\alpha \in \mathcal{R} : \langle d, \alpha \rangle > 0\}$ and $\mathcal{R}_- = \mathcal{R}_-(d) = -\mathcal{R}_+(d)$.

Let $V = \mathbb{R}^N$ and let $\mathcal{W}$ be a finite irreducible Coxeter group. By definition, $\mathcal{W}$ acts on the vector space V. The action of the group $\mathcal{W}$ is extended linearly to the complexified space $V \otimes \mathbb{C} \cong \mathbb{C}^N$. The orbit space $M_\mathcal{W}$ of $\mathcal{W}$ is defined by

$$M_\mathcal{W} = V \otimes \mathbb{C}/\mathcal{W} \cong \mathbb{C}^N/\mathcal{W}.$$

Let us choose a basis in V such that $\{x_k\}_{k=1}^N$ is the corresponding coordinate system. The group $\mathcal{W}$ acts also on these coordinates as well as on the symmetric algebra $S(V) \cong \mathbb{C}[x_1, \ldots, x_N]$ of polynomials in these coordinates. A coordinate system on the orbit space $M_\mathcal{W}$ is given by choosing N homogeneous $\mathcal{W}$-invariant polynomials $y_1(x), \ldots, y_N(x) \in S(V)$ generating freely the ring $\mathbb{C}[x_1, \ldots, x_N]^\mathcal{W} \cong \mathbb{C}[y_1, \ldots, y_N]$ of $\mathcal{W}$-invariant polynomials on $\mathbb{C}^N$ [17,19]. The degrees of these invariant polynomials are uniquely determined by the Coxeter group. Let $d_\alpha = \deg(y_\alpha)$, and arrange the degrees such that

$$d_1 > d_2 \geq \cdots \geq d_{N-1} > d_N = 2.$$

The maximal degree $d_1 = h$ is called *Coxeter number* of the group $\mathcal{W}$.

Theorem 2.4.5. [17] *There exists a unique, up to an equivalence, Frobenius manifold structure on the space of orbits of a finite Coxeter group with the following properties:*

(1) the vector field $e := \frac{\partial}{\partial y_1}$ is unity field for the Frobenius manifold.

(2) The Euler vector field is given by

$$\mathcal{E} := \frac{1}{h}\sum_{i=1}^{N} d_i y_i \frac{\partial}{\partial y_i}.$$

(3) The intersection form is given by the formula

$$g_{\alpha\beta}(y) := \sum_{i=1}^{N} \frac{\partial y_\alpha}{\partial x_i}\frac{\partial y_\beta}{\partial x_i}.$$

The components $g_{\alpha\beta}(y)$ are polynomials in $y_1, \dots, y_N$. The corresponding connection one-form is given by

$$\sum_{\gamma=1}^{N} {}^{g}\Gamma_{\gamma}^{\alpha\beta}(y) dy_\gamma = \sum_{i,j,k=1}^{N} \frac{\partial^2 y_\alpha}{\partial x_i \partial x_j}\frac{\partial y_\beta}{\partial x_k} dx_k$$

and it is also a polynomial one. Another metric η (called the Saito metric) which is given by

$$\eta_{\alpha\beta}(y) := \frac{\partial}{\partial y_1}\big(g_{\alpha\beta}(y)\big)$$

and the corresponding contravariant Levi-Civita connection

$${}^{\eta}\Gamma_{\gamma}^{\alpha\beta}(y) := \frac{\partial}{\partial y_1}\big({}^{g}\Gamma_{\gamma}^{\alpha\beta}(y)\big).$$

(4) The pair of metrics g, η given in (3) form a flat pencil.

The following Theorem by Dubrovin gives the formula of the Frobenius prepotential associated to the Frobenius manifold structure on the orbit space.

Theorem 2.4.6. [20] *For any finite irreducible Coxeter group the prepotential $F^\star(z)$ for the almost dual structure has the form*

$$F^\star(z) = \frac{h}{4}\sum_{\alpha\in\mathcal{R}} \alpha(z)^2 \log \alpha(z)^2, \tag{2.36}$$

where the roots are normalised so that $\langle\alpha,\alpha\rangle = 2,\ z \in \mathbb{C}^N \setminus \cup_{\alpha\in\mathcal{R}_+}\Pi_\alpha$

Remark 2.4.7. Martini and Gragert showed in [41], by straightforward computation, that function of type (2.36) corresponding to root systems of any semisimple Lie algebra satisfies the WDVV equations. Note that the root systems of any semisimple Lie algebra are the particular examples of the Coxeter systems.

Veselov in [53] also followed a different approach to obtain other examples of the logarithmic solutions to the WDVV equations given by a formula of the form (2.36) where $\mathcal{R}$ may no longer be a root system. The fact that prepotential (2.36) satisfies WDVV equations for any Coxeter root system was established by Veselov in [53] (see Corollary 2.5.6 below).

In Veselov's work a special class of such solutions was investigated for some not fully symmetric configurations of finite set of covectors $\mathcal{A} \subset V^*$ (so-called $\vee$-systems). These kind of configurations were formulated in terms of linear algebra, and can be considered as a generalization of the class of Coxeter systems. Deformed root systems were discovered by Chalykh, Feigin and Veselov [12, 52]. These multi parameters deformed versions of root systems appeared in the theory of the generalised Calogero-Moser problems and it was shown that they also give examples of the logarithmic solutions of the WDVV equations.

2.5 Rational solutions of WDVV equations

In this section we review the rational solutions of WDVV equations corresponding to the class of (rational) $\vee$–systems. We also recall the natural operations of restriction and taking subsystems of a $\vee$-system and present the known results related to this class.

2.5.1 (Rational) $\vee$-systems

An important class of soultions of WDVV has the form

$$F = \sum_{\alpha \in \mathcal{A}} \alpha(x)^2 \log \alpha(x), \quad x \in V, \tag{2.37}$$

where $\mathcal{A}$ is a system of covectors in V. This class of solutions was given by Veselov in [53], and the corresponding configurations $\mathcal{A}$ are known as $\vee$-systems. The class of $\vee$–systems includes any two-dimensional system (trivial examples). This class also contains all Coxeter root systems. As we have seen in Section 2.4 that solutions of WDVV equations constructed from a Coxeter group define the almost dual structure of the Frobenius manifolds defined on the space of orbits of these groups. These solutions can be expressed in the form (2.37) with $\mathcal{A} = \mathcal{R}_+$ (see Theorem 2.4.6). The class of $\vee$–systems includes also the deformed versions of the Coxeter root systems related to simple Lie superalgebras coming from the theory of the generalised Calogero-Moser systems, but the full classification of the $\vee$-systems is still an open problem. It has been shown in [26] (see also references therein) that the class of $\vee$-systems is closed under the natural operations of restriction and taking subsystems. In this section we present the notion of $\vee$-system and we describe the geometric conditions ($\vee$- conditions) on such a special collection of covectors which are necessary and sufficient for the corresponding function to satisfy the generalised WDVV equations. We also give a proof that any Coxeter root system belongs to such a system. We also summarize some important results which are found in the literature which confirms that (under some mild assumptions) a subsystem of a $\vee$-system is also a $\vee$-system

[24, 26] and that the restrictions of a $\vee$-system to certain subspaces are also $\vee$-systems [25, 26].

For this section let V be an n-dimensional complex vector space and $\mathcal{A} \subset V^*$ be a finite set of covectors in the dual space V^*. To such a set one can associated the following canonical form $G_{\mathcal{A}}$ on V:

$$G_{\mathcal{A}}(x, y) = \sum_{\alpha \in \mathcal{A}} \alpha(x)\alpha(y), \quad x, y \in V. \tag{2.38}$$

We will assume that the bilinear form (2.38) is non-degenerate, which establishes an isomorphism

$$\phi_{\mathcal{A}} : V \to V^*.$$

Let us denote the inverse $\phi_{\mathcal{A}}^{-1}(\alpha)$ by $\alpha^\vee$. let us also identify $V \cong V^*$ using the canonical form $G_{\mathcal{A}}$ and for any $\alpha, \beta \in V^*$, we define $G_{\mathcal{A}}(\alpha, \beta) := G_{\mathcal{A}}(\alpha^\vee, \beta^\vee)$. We will also assume without loss of generality that $\mathcal{A}$ has no collinear vectors. Indeed, if a configuration has collinear covectors $\{k_i\alpha,\ k_i \in \mathbb{R},\ i = 1, \dots m,\ m \in \mathbb{N}\}$ then by replacing this group of covectors with a single covector $\widetilde{\alpha} = k\alpha$, where $k^2 = \sum_{i=1}^m k_i^2$, the corresponding prepotential (2.37) is unchanged up to quadratic terms. The vector $k\alpha$ is removed from the configuration if $k = 0$.

Definition 2.5.1. [53] The system $\mathcal{A}$ is called (rational) $\vee$-system if the following relations (called $\vee$-conditions) are satisfied

$$\sum_{\beta \in \pi \cap \mathcal{A}} \beta(\alpha^\vee)\beta^\vee = \lambda \alpha^\vee, \tag{2.39}$$

for any $\alpha \in \mathcal{A}$ and any two-dimensional plane $\pi \subset V^*$ containing α and some λ, which may depend on π and α.

Define an operator $A_\pi \colon \pi^\vee \to \pi^\vee$ by the formula

$$A_\pi := \sum_{\beta \in \mathcal{A} \cap \pi} \beta \otimes \beta^\vee, \tag{2.40}$$

that is $A_\pi(v) = \sum_{\beta \in \mathcal{A} \cap \pi} \beta(v)\beta^\vee$ for any $v \in \pi^\vee$. In this notation, for a fixed $\alpha \in \mathcal{A}$ the $\vee$-condition (2.39) reads

$$A_\pi(\alpha^\vee) = \lambda \alpha^\vee,$$

that is, $\alpha^\vee$ is an eigenvector of A_π with the eigenvalue λ. The following lemma is known from linear algebra.

Lemma 2.5.2. *If the operator A_π given by (2.40) has at least three non-proportional eigenvectors, then eigenvalues do not depend on the eigenvectors. Moreover, the operator*

A_π is proportional to the identity.

By Definition 2.5.1, the condition for a system $\mathcal{A}$ to be a $\vee$-system is equivalent to the following conditions:

- If $\pi \cap \mathcal{A} = \{\alpha\}$, then the $\vee$-condition is obviously satisfied.
- If $\pi \cap \mathcal{A} = \{\alpha, \beta\}$, where $\alpha \nsim \beta$, then condition (2.39) means that $\alpha^\vee, \beta^\vee$ are orthogonal with repect to the bilinear form $G_\mathcal{A}$, that is $\beta(\alpha^\vee) = G_\mathcal{A}(\beta, \alpha^\vee) = 0$.
- If $|\mathcal{A} \cap \pi| > 2$, then by Lemma 2.5.2 the scalar λ in condition (2.39) does not depend on α, that is $\lambda = \lambda(\pi)$. Moreover, the condition (2.39) means that bilinear forms $G_\mathcal{A}$ given by (2.38) and $G_\pi(x,y) = \sum_{\beta \in \mathcal{A}\cap\pi} \beta(x)\beta(y)$ restricted to the plane $\pi^\vee \subset V$ are proportional:
$$G_\pi|_{\pi^\vee \times V} = \lambda G_\mathcal{A}|_{\pi^\vee \times V}.$$
To see this, take $\alpha \in \pi \cap \mathcal{A}$ and $v \in V$, then by using condition (2.39) we have
$$G_\pi(\alpha^\vee, v) = \sum_{\beta \in \mathcal{A}\cap\pi} \beta(\alpha^\vee)\beta(v) = \sum_{\beta \in \mathcal{A}\cap\pi} \beta(\alpha^\vee) G_\mathcal{A}(\beta^\vee, v) = \lambda G_\mathcal{A}(\alpha^\vee, v)$$
which implies the result.

Originally $\vee$-system appeared in [53] as geometric reformulation of the Wittern-Dijkgraaf-Verlinde-Verlinde (WDVV) equations for the prepotential (2.37). The (generalized) WDVV equations have the form

$$F_i F_k^{-1} F_j = F_j F_k^{-1} F_i, \quad i, j, k = 1, ..., n, \tag{2.41}$$

where F_i is the matrix of third derivatives

$$(F_i)_{ab} = \frac{\partial^3 F}{\partial x_i \partial x_a \partial x_b}, \quad i, a, b = 1, \dots, n.$$

The system (2.41) is equivalent to the system

$$F_i G^{-1} F_j = F_j G^{-1} F_i, \quad i, j = 1, ..., n, \tag{2.42}$$

where G is any non-degenerate linear combination

$$G = \sum_{i=1}^n \eta_i(x) F_i, \tag{2.43}$$

for some analytic functions $\eta_i(x)$ [41] (see also [40]). The following theorem shows the equivalence of $\vee$-system conditions and the WDVV equations for the prepotential (2.37).

Theorem 2.5.3. [26, 53] *The prepotential (2.37) satisfies the WDVV equations (2.42) if and only if $\mathcal{A}$ is a $\vee$-system.*

Before we present the statement that gives the relation between root system and the (rational) $\vee$-system, let us give some properties with respect to root systems.

Let $V = \mathbb{R}^n$ and let $\langle \cdot, \cdot \rangle$ be the standard positive definite symmetric bilinear form on V. Let $\mathcal{R}$ be a Coxeter root system in a real vector space $V^{\mathbb{R}}$, where complexification is V. Let $\mathcal{A} = \mathcal{R}_+$. We recall that by identifying $V \cong V^*$ using the canonical form $G_{\mathcal{A}}$ and for any $\alpha, \beta \in V^*$ we define

$$G_{\mathcal{A}}(\alpha, \beta) := G_{\mathcal{A}}(\alpha^\vee, \beta^\vee). \tag{2.44}$$

Also, since the bilinear form (2.38) is $\mathcal{W}$-invariant then it is proportional to the standard inner product $\langle \cdot, \cdot \rangle$ on V. Let

$$G_{\mathcal{A}}(u, v) = \mu \langle u, v \rangle, \quad \mu \in \mathbb{R},\, u, v \in V. \tag{2.45}$$

Also, one can identify $V \cong V^*$ using the standard inner product $\langle \cdot, \cdot \rangle$ and for any $\alpha, \beta \in V^*$ we define

$$\langle \alpha, \beta \rangle := \langle \alpha^\vee, \beta^\vee \rangle. \tag{2.46}$$

Thus relations (2.44)–(2.46) leads to the relation

$$G_{\mathcal{A}}(\alpha, \beta) = \mu \langle \alpha, \beta \rangle, \quad \mu \in \mathbb{R},\, \alpha, \beta \in V^*.$$

Lemma 2.5.4. *Let $\mathcal{A} = \mathcal{R}_+$. Let $\alpha \in \mathcal{A}$ and let $\pi \subset V^*$ be any two-dimensional plane containing α. Then $s_\alpha \pi = \pi$.*

Proof. Let $\beta \in \pi$. Then by the definition of the reflection s_α we have

$$s_\alpha \beta = \beta - m\alpha,$$

where $m = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$. Since $s_\alpha \beta$ is a linear combination of vectors in the plane π then $s_\alpha \beta \in \pi$ for all $\beta \in \pi$. The statement follows. $\square$

The following theorem gives us the relation between root system and the (rational) $\vee$-system.

Theorem 2.5.5. [53] *Let $\mathcal{R}$ be a Coxeter root system in $\mathbb{R}^n$. Then $\mathcal{A} = \mathcal{R}_+$ is a $\vee$-system.*

Proof. In order to check that $\mathcal{A} = \mathcal{R}_+$ is a $\vee$-system we have to check that conditions (2.39) hold. Fix $\alpha \in \mathcal{A}$. Let π be any two-dimensional plane containing α. Let $\beta \in \mathcal{A} \cap \pi$. Since $\mathcal{R}$ is invariant under the action of Weyl group, then $s_\alpha \beta \in \mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_- = \mathcal{A} \cup (-\mathcal{A})$, and hence either $s_\alpha \beta \in \mathcal{A}$ or $-s_\alpha \beta \in \mathcal{A}$. If $|\mathcal{A} \cap \pi| = 2$, say $\mathcal{A} \cap \pi = \{\alpha, \beta\}$, then by

Lemma 2.5.4 $s_\alpha\beta = \beta$. Hence $G_\mathcal{A}(\alpha,\beta) = \mu\langle\alpha,\beta\rangle = 0$ and the $\vee$-condition holds in this case. Assume that $|\mathcal{A}\cap\pi| > 2$. We study two cases:

Case (1). Assume that $s_\alpha\beta \nsim \beta$. Let $s_\alpha\beta \in \mathcal{A}$. Then by Lemma 2.5.4 we have $s_\alpha\beta \in \mathcal{A}\cap\pi$. Let us now study the contribution of the pairs β and $s_\alpha\beta$ to conditions (2.39). We have (we use (2.45) thoroughly)

$$\begin{aligned}
&G_\mathcal{A}(\alpha,\beta)\beta^\vee + G_\mathcal{A}(\alpha,s_\alpha\beta)(s_\alpha\beta)^\vee\\
&= G_\mathcal{A}(\alpha,\beta)\beta^\vee + \Big(G_\mathcal{A}(\alpha,\beta) - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}G_\mathcal{A}(\alpha,\alpha)\Big)\Big(\beta^\vee - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha^\vee\Big)\\
&= G_\mathcal{A}(\alpha,\beta)\beta^\vee + \Big(G_\mathcal{A}(\alpha,\beta) - 2\mu\langle\alpha,\beta\rangle\Big)\Big(\beta^\vee - 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha^\vee\Big)\\
&= G_\mathcal{A}(\alpha,\beta)\beta^\vee + \Big(G_\mathcal{A}(\alpha,\beta) - 2G_\mathcal{A}(\alpha,\beta)\Big)\Big(\beta^\vee - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha^\vee\Big)\\
&= G_\mathcal{A}(\alpha,\beta)\beta^\vee - G_\mathcal{A}(\alpha,\beta)\beta^\vee + \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}G_\mathcal{A}(\alpha,\beta)\alpha^\vee\\
&= \frac{2\mu\langle\alpha,\beta\rangle^2}{\langle\alpha,\alpha\rangle}\alpha^\vee.
\end{aligned}$$

Hence the contribution of pairs of vectors $\beta, s_\alpha\beta \in \mathcal{A}\cap\pi$ in the left-hand side of $\vee$-condition (2.39) is proportional to $\alpha^\vee$. If $-s_\alpha\beta \in \mathcal{A}$ then similarly the contribution of pairs of vectors $\beta, -s_\alpha\beta \in \mathcal{A}\cap\pi$ in the left-hand side of $\vee$-condition (2.39) is proportional to $\alpha^\vee$.

Case (2). Assume that $s_\alpha\beta \sim \beta$. Assume firstly that $s_\alpha\beta \in \mathcal{A}$. Hence we have $\beta, s_\alpha\beta \in \mathcal{A}\cap\pi$. This implies that $s_\alpha\beta = \beta$ (since the only multiples of β are $\pm\beta$). Hence for this case the vector β is orthogonal to α and vector β has zero contribution in the left-hand side of $\vee$-condition (2.39). Now assume that $-s_\alpha\beta \in \mathcal{A}$. Hence we have $\beta, -s_\alpha\beta \in \mathcal{A}\cap\pi$. This implies that $s_\alpha\beta = -\beta$ (since the only multiples of β are $\pm\beta$). Thus we have

$$\beta - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha = -\beta.$$

That is

$$\beta = \frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha.$$

But this means that β is proportional to α and hence $\beta = \alpha$. Hence the contribution of vector β in the left-hand side of $\vee$-condition (2.39) is proportional to $\alpha^\vee$. The lemma follows by the above considerations.

□

As a corollary of Theorems 2.5.3 and 2.5.5 the following statement holds.

Corollary 2.5.6. [53] *For any Coxeter root system $\mathcal{R}$ the function*

$$F = \sum_{\alpha \in \mathcal{R}_+} \alpha(z)^2 \log \alpha(z)^2$$

satisfies WDVV equations (2.41).

Remark 2.5.7. Note that all the results of $\vee$-systems have been originally proven for real vector space in [53] then these results have been generalized to complex vector spaces where the natural complex version of the $\vee$-systems appeared firstly in [26].

2.5.2 The associative product

Theorem 2.5.3 can also be reformulated in terms of flatness of a connection on the tangent bundle TV [25, 53]. By defining $M_{\mathcal{A}} = V \setminus \cup_{\alpha\in\mathcal{A}}\Pi_\alpha$, it was shown that the considerations of WDVV equations with respect to function (2.37) leads to the following multiplication for the tangent vectors u and v on $M_{\mathcal{A}}$:

$$u * v = \sum_{\alpha\in\mathcal{A}} \frac{\alpha(u)\alpha(v)}{\alpha(x)}\alpha^\vee, \quad x \in M_{\mathcal{A}}, u, v \in T_x M_{\mathcal{A}}. \tag{2.47}$$

It is clear from formula (2.47) that the multiplication $*$ is commutative.

Before we find the identity vector field for multiplication (2.47), let us define an operator $K_{\mathcal{A}} \colon V \longrightarrow V$ by the formula

$$K_{\mathcal{A}} \coloneqq \sum_{\alpha\in\mathcal{A}} \alpha \otimes \alpha^\vee, \tag{2.48}$$

that is $K_{\mathcal{A}}(v) = \sum_{\alpha\in\mathcal{A}} \alpha(v)\alpha^\vee$ for any $v \in V$. The following property holds.

Lemma 2.5.8. *The operator $K_{\mathcal{A}}$ given by formula (2.48) is the identity operator that is $K_{\mathcal{A}}(v) = v$ for any $v \in V$.*

Proof. Let $u, v \in V$. Then we have

$$G_{\mathcal{A}}(u, K_{\mathcal{A}}(v)) = \sum_{\alpha\in\mathcal{A}} \alpha(v) G_{\mathcal{A}}(\alpha^\vee, u) = \sum_{\alpha\in\mathcal{A}} \alpha(v)\alpha(u) = G_{\mathcal{A}}(u, v).$$

Hence we have $G_{\mathcal{A}}(u, K_{\mathcal{A}}(v) - v) = 0$, for any $u \in V$. This implies the statement since the bilinear form $G_{\mathcal{A}}$ is non-degenerate. $\square$

The following proposition gives the identity vector field for the product (2.47).

Proposition 2.5.9. *The vector field*

$$\mathcal{E} = \sum_{i=1}^{n} x_i \partial_i \in \Gamma(TV) \tag{2.49}$$

is the identity vector field of the multiplication (2.47).

Proof. Let $v \in V$. Then by formulae (2.47), (2.48) and Lemma 2.5.8 we have

$$\mathcal{E} * v = \sum_{\alpha \in \mathcal{A}} \frac{\alpha(x)\alpha(v)}{\alpha(x)} \alpha^\vee = \sum_{\alpha \in \mathcal{A}} \alpha(v)\alpha^\vee = K_{\mathcal{A}}(v) = v.$$

The statement follows since by commutativity of $*$ we have $v * \mathcal{E} = \mathcal{E} * v = v$. □

Proposition 2.5.10. *The multiplication (2.47) satisfies the Frobenius algebra condition with respect to the bilinear form (2.38), that is for any $u, v, w \in T_x M_{\mathcal{A}}$ for generic $x \in M_{\mathcal{A}}$ the following condition holds:*

$$G_{\mathcal{A}}(u * v, w) = G_{\mathcal{A}}(u, v * w).$$

Proof. By the product formula (2.47) we have

$$G_{\mathcal{A}}(u * v, w) = \sum_{\alpha \in \mathcal{A}} \frac{\alpha(u)\alpha(v)}{\alpha(x)} G_{\mathcal{A}}(\alpha^\vee, w) = \sum_{\alpha \in \mathcal{A}} \frac{\alpha(u)\alpha(v)\alpha(w)}{\alpha(x)}. \tag{2.50}$$

The statement follows since the right-hand side of relation (2.50) is symmetric in u, v, w. □

The following result relates the associativity of product $*$ and WDVV equations.

Proposition 2.5.11. [25] *The associativity of the product (2.47) is equivalent to the WDVV equations (2.41) for the prepotential F given by (2.37).*

It has been shown in [25] that the associativity of multiplication (2.47) can be rewritten as

$$\sum_{\alpha \neq \beta, \alpha, \beta \in \mathcal{A}} \frac{G_{\mathcal{A}}(\alpha^\vee, \beta^\vee) B_{\alpha,\beta}(a, b)}{\alpha(x)\beta(x)} \alpha \wedge \beta \equiv 0, \tag{2.51}$$

where $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$, and $B_{\alpha,\beta}(a, b) = \alpha \wedge \beta(a, b) = \alpha(a)\beta(b) - \alpha(b)\beta(a)$.

The following statement relates the associativity of the product (2.47) with the $\vee$-system.

Theorem 2.5.12. [25] *$\mathcal{A}$ is a $\vee$-system if and only if product (2.47) is associative.*

Note that Theorem 2.5.3 follows from Proposition 2.5.11 and Theorem 2.5.12.

The following flat connection defined on tangent bundle $TM_{\mathcal{A}}$ was introduced in [54]:

$$\nabla_u v = \partial_u v - k u * v, \tag{2.52}$$

where $*$ is the product given by formula (2.47) and $k \in \mathbb{C}$ is a parameter. The following statement relates the flatness of connection (2.5.13) with the $\vee$-condition.

Theorem 2.5.13. [54] *The connection (2.52) is flat for any $k \in \mathbb{C}$ if and only if $\mathcal{A}$ is a $\vee$-system.*

Remark 2.5.14. [25] For any $\vee$-system $\mathcal{A}$ the product (2.47) defines what is called *logarithmic Frobenius structure* on $\mathcal{A}$ with prepotential (2.37). Provided that $*$ is associative, the set $(V, *, \mathcal{E}, G_{\mathcal{A}}, \mathcal{E})$ satisfies all the properties of the Frobenius manifold except the covariant constancy of the unit vector field (axiom (2) in Definition 2.1.3).

2.5.3 Subsystems of a $\vee$-system

Now we present notion of a subsystem of a $\vee$-system, then we proceed to theorems related to this notion.

Definition 2.5.15. [26] Let $\mathcal{A} \subset V^*$ be a $\vee$-system. The subset $\mathcal{B} \subset \mathcal{A}$ is called subsystem if

$$\mathcal{B} = \mathcal{A} \cap W,$$

for some linear subspace $W \subset V^*$. We will assume that $W = \langle \mathcal{B} \rangle$. A subsystem $\mathcal{B}$ is called reducible if $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a disjoint union of two non-empty subsystems orthogonal with respect to the canonical form on V, that is $G_{\mathcal{A}}(\beta_1, \beta_2) = 0, \forall \beta_1 \in \mathcal{B}_1, \beta_2 \in \mathcal{B}_2$.

Definition 2.5.16. [26] Consider the following bilinear form on V

$$G_{\mathcal{B}}(x, y) := \sum_{\beta \in \mathcal{B}} \beta(x)\beta(y), \quad x, y \in V,$$

associated with subsystem $\mathcal{B}$. The subsystem $\mathcal{B}$ is called *isotropic* if the restriction $G_{\mathcal{B}} \mid_{W^\vee}$ of the form $G_{\mathcal{B}}$ on to the subspace $W^\vee \subset V$ is degenerate and *non-isotropic* otherwise.

Theorem 2.5.17. [26] *Any non-isotropic subsystem of a $\vee$-system is also a $\vee$-system.*

Theorem 2.5.18. [26] *For any subsystem $\mathcal{B} = \mathcal{A} \cap W$ of a $\vee$-system $\mathcal{A}$, either $G_B \mid_{W^\vee \times V}$ and $G_A \mid_{W^\vee \times V}$ are proportional or $\mathcal{B}$ is reducible.*

2.5.4 Restrictions of $\vee$-systems

Let us now consider the restriction operation for $\vee$-systems.

For any subsystem $\mathcal{B} \subset \mathcal{A}$ consider the corresponding subspace $W_{\mathcal{B}} \subset V$ defined as the intersection of hyperplanes $\beta(x) = 0$, where $\beta \in \mathcal{B}$, that is

$$W_{\mathcal{B}} = \{x \in V : \beta(x) = 0, \forall \beta \in \mathcal{B}\}. \tag{2.53}$$

For any $\alpha \in \mathcal{A}$ let us denote the restriction $\alpha \mid_{W_{\mathcal{B}}}$ as $\pi_{\mathcal{B}}(\alpha)$, that is $\pi_{\mathcal{B}}(\alpha)(x) = \alpha(x)$. Let

$$\pi_{\mathcal{B}}(\mathcal{A}) = \{\pi_{\mathcal{B}}(\alpha) : \alpha \in \mathcal{A} \setminus \mathcal{B}\}$$

be the restriction of covectors $\alpha \in \mathcal{A}$ on $W_{\mathcal{B}}$. Let $M_{\mathcal{B}} = \mathcal{B} \setminus \cup_{\alpha \in \mathcal{A} \setminus \mathcal{B}} \Pi_\alpha$. The following statement shows that the class of $\vee$-systems is closed under the restriction operator.

Theorem 2.5.19. [25, 26] *Assume that the restriction $G_{\mathcal{A}} \mid_{W_{\mathcal{B}}}$ is non-degenerate. Then the restriction $\pi_{\mathcal{B}}(\mathcal{A})$ of a $\vee$-system $\mathcal{A}$ to the subspace $M_{\mathcal{B}}$ is also a $\vee$-system.*

Remark 2.5.20. Note that Theorem 2.5.19 applied to a root system $\mathcal{A}$ gives $\vee$-systems which are not root systems in general.

The following statement gives the solution of WDVV equations corresponding to the restricted system.

Theorem 2.5.21. [25, 26] *The logarithmic Frobenius structure (2.47) with its corresponding prepotential (2.37) has a natural restriction to the space $M_{\mathcal{B}}$ with the prepotential*

$$F_{\mathcal{B}} = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} \alpha(x)^2 \log \alpha(x)^2, \quad x \in M_{\mathcal{B}},$$

which also satisfies the WDVV equations.

2.6 Trigonometric case

In this section we introduce the class of trigonometric solutions F of WDVV equations which involve an extra variable. We present the notion of a trigonometric $\vee$-system and we review results associated to this class.

2.6.1 Prepotential, product and Dubrovin connection

Let V be a vector space of dimension N over $\mathbb{C}$ and let V^* be its dual space. Let $\mathcal{A}$ be a finite collection of covectors $\alpha \in V^*$ which belongs to a lattice of rank N.

Let us also consider a multiplicity function $c\colon \mathcal{A} \to \mathbb{C}$. We denote $c(\alpha)$ as c_α. We will assume throughout that the corresponding symmetric bilinear form

$$G_{(\mathcal{A},c)}(u, v) := \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha(u)\alpha(v), \quad u, v \in V \tag{2.54}$$

is non-degenerate. We will also write $G_{\mathcal{A}}$ for $G_{(\mathcal{A},c)}$ to simplify notations. The form $G_{\mathcal{A}}$ establishes an isomorphism $\phi\colon V \to V^*$, and we denote the inverse $\phi^{-1}(\alpha)$ by $\alpha^\vee$, where $G_{\mathcal{A}}(\alpha^\vee, v) = \alpha(v)$ for any $v \in V$.

Let $U \cong \mathbb{C}$ be a one-dimensional vector space. We choose a basis in $V \oplus U$ such that $e_1, \dots, e_N$ is a basis in V and e_{N+1} is the basis vector in U, and let $x_1, \dots, x_{N+1}$ be the corresponding coordinates. We represent vectors $x \in V, y \in U$ as $x = (x_1, ..., x_N)$ and $y = x_{N+1}$. Consider a function $F\colon V \oplus U \to \mathbb{C}$ of the form

$$F = \frac{1}{3}y^3 + \sum_{\alpha\in\mathcal{A}} c_\alpha \alpha(x)^2 y + \lambda \sum_{\alpha\in\mathcal{A}} c_\alpha f(\alpha(x)), \tag{2.55}$$

where $\lambda \in \mathbb{C}^*$ and function f is given by (1.11). The WDVV equations is the following system of partial differential equations

$$F_i F_{N+1}^{-1} F_j = F_j F_{N+1}^{-1} F_i, \quad i, j = 1, ..., N, \tag{2.56}$$

where F_i is $(N+1)\times(N+1)$ matrix with entries $(F_i)_{pq} = \frac{\partial^3 F}{\partial x_i \partial x_p \partial x_q}$ $(p, q = 1, \ldots, N+1)$.

Let $e^1, ..., e^N$ be the basis in V^* dual to the basis $e_1, ..., e_N \in V$. Then for any covector $\alpha \in V^*$ we have $\alpha = \sum_{i=1}^N \alpha_i e^i$ and $\alpha^\vee = \sum_{i=1}^N \alpha_i^\vee e_i$, where $\alpha_i, \alpha_i^\vee \in \mathbb{C}$. Then

$$F_{N+1} = 2\begin{pmatrix} \sum_{\alpha\in\mathcal{A}} c_\alpha \alpha\otimes\alpha & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.57}$$

where we denoted by α both column and row vectors $\alpha = (\alpha_1, ..., \alpha_N)$, and $\alpha\otimes\alpha$ is $N\times N$ matrix with matrix entries $(\alpha\otimes\alpha)_{jk} = \alpha_j\alpha_k$. Let us define

$$\eta_{ij} = (F_{N+1})_{ij}, \quad \eta^{ij} = (F_{N+1}^{-1})_{ij}, \tag{2.58}$$

where $i, j = 1, \ldots, N+1$. Now we will establish a few lemmas which will be useful later. The next statement is standard.

Lemma 2.6.1. *Let $\widetilde{G}$ be the matrix of the bilinear form $G_\mathcal{A}$, that is its matrix entry $(\widetilde{G})_{ij} = G_\mathcal{A}(e_i, e_j)$, where $i, j = 1, \ldots, N$. Then for any covector $\gamma = (\gamma_1, ..., \gamma_N) \in V^*$ and $\gamma^\vee = (\gamma_1^\vee, ..., \gamma_N^\vee) \in V$, we have $\widetilde{G}^{-1}\gamma^T = (\gamma^\vee)^T$.*

Let $M_\mathcal{A} = V \setminus \cup_{\alpha\in\mathcal{A}} \Pi_\alpha$ be the complement to the union of all the hyperplanes $\Pi_\alpha :=$ $\{x \in V : \alpha(x) = 0\}$. For any vector $\overline{a} = (a_1, \ldots, a_{N+1}) \in V \oplus U$ let us introduce the corresponding vector field $\partial_{\overline{a}} = \sum_{i=1}^{N+1} a_i \partial_{x_i} \in \Gamma(T(V\oplus U))$. For any $\overline{b} = (b_1, \ldots, b_{N+1}) \in V \oplus U$ we define the following multiplication on the tangent space $T_{(x,y)}(M_\mathcal{A} \oplus U)$:

$$\partial_{\overline{a}} * \partial_{\overline{b}} = a_i b_j \eta^{kl} F_{ijk} \partial_{x_l}, \quad i, j, k, l = 1, \ldots, N+1, \tag{2.59}$$

where η^{kl} is defined in (2.58) and the summation over repeated indices here and below is assumed. It is clear from the definition that the multiplication $*$ is commutative and distributive. The next statement follows from Proposition 2.1.5.

Proposition 2.6.2. *The associativity of multiplication $*$ is equivalent to the WDVV equation (2.56).*

Let us introduce vector field E by

$$E = \partial_{x_{N+1}} \in \Gamma(T(V \oplus U)).$$

For a fixed $(x, y) \in M_{\mathcal{A}} \oplus U$ after the identification $T_{(x,y)}(V \oplus U) \cong V \oplus U$ we have that $E \in U$.

Proposition 2.6.3. *Vector field E is the identity for the multiplication (2.59).*

Proof. For all $1 \le i \le N+1$ we have

$$\partial_{x_i} * E = \eta^{kj} F_{i,N+1,j} \partial_{x_k} = \eta^{kj} \eta_{ij} \partial_{x_k} = \partial_{x_i}.$$

$\square$

Proposition 2.6.4. *Let $a = (a_1, \dots, a_N), b = (b_1, \dots, b_N) \in V$, and let $\partial_a = \sum_{i=1}^N a_i \partial_{x_i}$, $\partial_b = \sum_{i=1}^N b_i \partial_{x_i}$. Then the product (2.59) has the following explicit form*

$$\partial_a * \partial_b = \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha(a) \alpha(b) (\frac{\lambda}{2} \cot \alpha(x) \partial_{\alpha^\vee} + E). \tag{2.60}$$

Proof. Note that $\eta^{m,N+1} = \frac{1}{2}\delta_m^{N+1}$ for any $m = 1, ..., N+1$, where δ_i^j is the Kronecker symbol. Therefore from (2.59) we have

$$\partial_a * \partial_b = a_i b_j (\sum_{k,l=1}^N \eta^{kl} F_{ijk} \partial_{x_l} + \frac{1}{2} F_{i,j,N+1} \partial_{x_{N+1}}),$$

where

$$F_{ijk} = \lambda \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha_i \alpha_j \alpha_k \cot \alpha(x), \quad 1 \le i, j, k \le N.$$

Then we have

$$\sum_{k,l=1}^N a_i b_j \eta^{kl} F_{ijk} \partial_{x_l} = \lambda \sum_{\alpha \in \mathcal{A}} \sum_{k,l=1}^N c_\alpha \alpha(a) \alpha(b) \alpha_k \eta^{kl} \cot \alpha(x) \partial_{x_l} = \frac{\lambda}{2} \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha(a) \alpha(b) \cot \alpha(x) \partial_{\alpha^\vee} \tag{2.61}$$

by Lemma 2.6.1. Also by formula (2.57) we have that

$$\frac{1}{2} \sum_{i,j=1}^N a_i b_j F_{i,j,N+1} = \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha(a) \alpha(b). \tag{2.62}$$

The statement follows from formulas (2.61) and (2.62). $\square$

If we identify vector space $V \oplus U$ with the tangent space $T_{(x,y)}(V \oplus U) \cong V \oplus U$, then multiplication (2.60) can also be written as

$$a * b = \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha(a)\alpha(b)(\frac{\lambda}{2} \cot \alpha(x)\alpha^\vee + E), \quad a, b \in V. \tag{2.63}$$

The following proposition gives the explicit formula for the multiplication (2.59) on the tangent space $T_{(x,y)}(M_\mathcal{A} \oplus U)$.

Proposition 2.6.5. *Let $\widetilde{u} = u + \rho_1 E, \widetilde{v} = v + \rho_2 E \in \Gamma(T(V \oplus U))$, where $u, v \in V$ and $\rho_1, \rho_2 \in \mathbb{C}$. Then multiplication (2.59) takes the following form on $T_{(x,y)}(V \oplus U) \cong V \oplus U$:*

$$\widetilde{u} * \widetilde{v} = \frac{\lambda}{2} \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha(u)\alpha(v) \cot \alpha(x)\alpha^\vee + \rho_2 u + \rho_1 v + \big(G_\mathcal{A}(u, v) + \rho_1\rho_2\big)E. \tag{2.64}$$

Proof. We have by Proposition 2.6.3

$$\widetilde{u} * \widetilde{v} = u * v + u * \rho_2 E + \rho_1 E * v + \rho_1 E * \rho_2 E = u * v + \rho_2 u + \rho_1 v + \rho_1\rho_2 E.$$

Then formula (2.64) follows from formula (2.63). □

Let us extend the bilinear form (2.54) to the symmetric bilinear form $\widetilde{G}_\mathcal{A}$ on $V \oplus U$ by defining

$$\begin{aligned} \widetilde{G}_\mathcal{A}(u, v) &= G_\mathcal{A}(u, v), \\ \widetilde{G}_\mathcal{A}(u, E) &= 0, \\ \widetilde{G}_\mathcal{A}(E, E) &= 1, \end{aligned} \tag{2.65}$$

for all $u, v \in V$. It is clear that the multiplication (2.64) is commutative. Let us now recall the notion of Frobenius algebra in which its prepotential appears as a solution of WDVV equations. The following result holds.

Proposition 2.6.6. *The multiplication (2.64) satisfies the Frobenius algebra condition with respect to the bilinear form (2.65), that is for any $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in T_{(x,y)}(V \oplus U)$ for generic (x, y) the following condition holds:*

$$\widetilde{G}_\mathcal{A}(\widetilde{X} * \widetilde{Y}, \widetilde{Z}) = \widetilde{G}_\mathcal{A}(\widetilde{X}, \widetilde{Y} * \widetilde{Z}).$$

Proof. Let $\widetilde{X} = X + \rho_1 E$, $\widetilde{Y} = Y + \rho_2 E$, $\widetilde{Z} = Z + \rho_3 E$, where $X, Y, Z \in V$, and $\rho_1, \rho_2, \rho_3 \in$

$\mathbb{C}$. Then by the product formula (2.64) we have

$$\begin{aligned}\widetilde{G}_{\mathcal{A}}(\widetilde{X}*\widetilde{Y},\widetilde{Z}) &= \frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha \alpha(X)\alpha(Y)\cot\alpha(x)\widetilde{G}_{\mathcal{A}}(\alpha^\vee,\widetilde{Z}) + \rho_2\widetilde{G}_{\mathcal{A}}(X,\widetilde{Z}) + \rho_1\widetilde{G}_{\mathcal{A}}(Y,\widetilde{Z}) \\ &+ \big(G_{\mathcal{A}}(X,Y)+\rho_1\rho_2\big)\widetilde{G}_{\mathcal{A}}(E,\widetilde{Z}) \\ &= \frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha\alpha(X)\alpha(Y)\alpha(Z)\cot\alpha(x) + \rho_1 G_{\mathcal{A}}(Y,Z) + \rho_2 G_{\mathcal{A}}(X,Z) \\ &+ \rho_3 G_{\mathcal{A}}(X,Y) + \rho_1\rho_2\rho_3. \end{aligned} \tag{2.66}$$

The statement follows since the right-hand side of relation (2.66) is invariant under the arbitrary permutations of $\widetilde{X},\widetilde{Y},\widetilde{Z}$.

□

Remark 2.6.7. The commutativity of the product (2.64) together with Proposition 2.6.6 imply that the 4-tuple $\Big(T(V\oplus U), *, E, \widetilde{G}_{\mathcal{A}}\Big)$ constitutes a commutative, associative Frobenius algebra provided that the product $*$ is associative.

Let us now define the (deformed) Dubrovin connection corresponding to the multiplication (2.64). Let $\nabla^0_{\widetilde{X}}\widetilde{Y} := \partial_{\widetilde{X}}\widetilde{Y}$ be the trivial connection. We know that the trivial connection is a torsion free and also a flat connection, that is

$$\nabla^0_{\widetilde{X}}\widetilde{Y} - \nabla^0_{\widetilde{Y}}\widetilde{X} = [\widetilde{X},\widetilde{Y}],$$

and

$$R^0(\widetilde{X},\widetilde{Y})\widetilde{Z} := [\nabla^0_{\widetilde{X}},\nabla^0_{\widetilde{Y}}]\widetilde{Z} - \nabla^0_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z} = 0,$$

for any smooth vector fields $\widetilde{X},\widetilde{Y},\widetilde{Z}\in\Gamma\Big(T(V\oplus U)\Big)$.

The Dubrovin connection (2.10) on the tangent bundle $T(V\oplus U)$ takes the form

$$\nabla^\mu_{\widetilde{u}}\widetilde{v} = \nabla^0_{\widetilde{u}}\widetilde{v} + \mu\widetilde{u}*\widetilde{v}, \tag{2.67}$$

where the multiplication $*$ is given by (2.64), and ∇^0 is the trivial connection, and $\mu\in\mathbb{C}$.

The $(0,3)$-tensor c given by formula (2.1) takes the form

$$c(\widetilde{X},\widetilde{Y},\widetilde{Z}) = \widetilde{G}_{\mathcal{A}}(\widetilde{X}*\widetilde{Y},\widetilde{Z}), \tag{2.68}$$

and the $(0,4)$-tensor T^0 given by formula (2.2) takes the form

$$\begin{aligned} T^0(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}) &:= \Big(\nabla^0_{\widetilde{W}}c\Big)(\widetilde{X},\widetilde{Y},\widetilde{Z}) \\ &= \nabla^0_{\widetilde{W}}\Big(c(\widetilde{X},\widetilde{Y},\widetilde{Z})\Big) - c(\nabla^0_{\widetilde{W}}\widetilde{X},\widetilde{Y},\widetilde{Z}) - c(\widetilde{X},\nabla^0_{\widetilde{W}}\widetilde{Y},\widetilde{Z}) - c(\widetilde{X},\widetilde{Y},\nabla^0_{\widetilde{W}}\widetilde{Z}) \\ &= \partial_{\widetilde{W}}\Big(c(\widetilde{X},\widetilde{Y},\widetilde{Z})\Big) - c(\partial_{\widetilde{W}}\widetilde{X},\widetilde{Y},\widetilde{Z}) - c(\widetilde{X},\partial_{\widetilde{W}}\widetilde{Y},\widetilde{Z}) - c(\widetilde{X},\widetilde{Y},\partial_{\widetilde{W}}\widetilde{Z}), \end{aligned} \tag{2.69}$$

where $\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W} \in \Gamma\Big(T(V\oplus U)\Big)$, the product $*$ is given by formula (2.64) and $\widetilde{G}_{\mathcal{A}}$ is given by formula (2.65).

The following statement takes place.

Proposition 2.6.8. *The tensor $c(\widetilde{X},\widetilde{Y},\widetilde{Z})$ given by (2.68) is totally symmetric.*

Proof. Let $\widetilde{X} = X + \rho_1 E$, $\widetilde{Y} = Y + \rho_2 E$, $\widetilde{Z} = Z + \rho_3 E$, where $X, Y, Z \in \Gamma(TV)$, and $\rho_1, \rho_2, \rho_3 \in C^\infty(U)$. Then by the product formula (2.64) we have

$$\begin{aligned} c(\widetilde{X},\widetilde{Y},\widetilde{Z}) = \widetilde{G}_{\mathcal{A}}(\widetilde{X} * \widetilde{Y},\widetilde{Z}) &= \frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha \alpha(X)\alpha(Y)\alpha(Z)\cot\alpha(x) + \rho_1 G_{\mathcal{A}}(Y,Z) \\ &+ \rho_2 G_{\mathcal{A}}(X,Z) + \rho_3 G_{\mathcal{A}}(X,Y) + \rho_1\rho_2\rho_3. \end{aligned} \tag{2.70}$$

The statement follows since the right-hand side of relation (2.66) is invariant under the arbitrary permutations of $\widetilde{X},\widetilde{Y},\widetilde{Z}$. $\square$

The following statement holds.

Proposition 2.6.9. *For any vector fields $\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W} \in \Gamma\Big(T(V\oplus U)\Big)$, the $(0,4)$-tensor $T^0(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W})$ given by formula (2.69) is totally symmetric in all its arguments.*

Proof. Let $\widetilde{X} = X+\rho_1 E$, $\widetilde{Y} = Y+\rho_2 E$, $\widetilde{Z} = Z+\rho_3 E$, $\widetilde{W} = W+\rho_4 E$, where $X, Y, Z, W \in \Gamma(TV)$, and $\rho_1, \rho_2, \rho_3, \rho_4 \in C^\infty(U)$. By formula (2.66) we have

$$\begin{aligned} c(\widetilde{X},\widetilde{Y},\widetilde{Z}) &= \frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha \alpha(X)\alpha(Y)\alpha(Z)\cot\alpha(x) + \rho_1 G_{\mathcal{A}}(Y,Z) + \rho_2 G_{\mathcal{A}}(X,Z) \\ &+ \rho_3 G_{\mathcal{A}}(X,Y) + \rho_1\rho_2\rho_3. \end{aligned}$$

Then to find the explicit formula of the tensor (2.69) we have

$$
\begin{aligned}
&\partial_{\widetilde{W}}\Big(c(\widetilde{X},\widetilde{Y},\widetilde{Z})\Big) = -\frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha \alpha(X)\alpha(Y)\alpha(Z)\frac{\alpha(W)}{\sin^2\alpha(x)} \\
&+\frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha \cot\alpha(x)\Bigg(\Big(\partial_{\widetilde{W}}\alpha(X)\Big)\alpha(Y)\alpha(Z)+\alpha(X)\Big(\partial_{\widetilde{W}}\alpha(Y)\Big)\alpha(Z)+\alpha(X)\alpha(Y)\Big(\partial_{\widetilde{W}}\alpha(Z)\Big)\Bigg) \\
&+\Big(\rho_1\partial_{\widetilde{W}}\widetilde{G}_{\mathcal{A}}(Y,Z)+\big(\partial_{\widetilde{W}}\rho_1\big)\widetilde{G}_{\mathcal{A}}(Y,Z)\Big)+\Big(\rho_2\partial_{\widetilde{W}}\widetilde{G}_{\mathcal{A}}(X,Z)+\big(\partial_{\widetilde{W}}\rho_2\big)\widetilde{G}_{\mathcal{A}}(X,Z)\Big) \\
&+\Big(\rho_3\partial_{\widetilde{W}}\widetilde{G}_{\mathcal{A}}(X,Y)+\big(\partial_{\widetilde{W}}\rho_3\big)\widetilde{G}_{\mathcal{A}}(X,Y)\Big)+\Big(\big(\partial_{\widetilde{W}}\rho_1\big)\rho_2\rho_3+\rho_1\big(\partial_{\widetilde{W}}\rho_2\big)\rho_3+\rho_1\rho_2\big(\partial_{\widetilde{W}}\rho_3\big)\Big).
\end{aligned}
\tag{2.71}
$$

Also using Frobenius condition we have

$$
\begin{aligned}
&c\big(\partial_{\widetilde{W}}\widetilde{X},\widetilde{Y},\widetilde{Z}\big)=\widetilde{G}_{\mathcal{A}}\big(\partial_{\widetilde{W}}\widetilde{X}*\widetilde{Y},\widetilde{Z}\big)=\widetilde{G}_{\mathcal{A}}\big(\partial_{\widetilde{W}}\widetilde{X},\widetilde{Y}*\widetilde{Z}\big) \\
&=\frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha\alpha(Y)\alpha(Z)\cot\alpha(x)\widetilde{G}_{\mathcal{A}}(\alpha^\vee,\partial_{\widetilde{W}}\widetilde{X})+\rho_3\widetilde{G}_{\mathcal{A}}(\partial_{\widetilde{W}}\widetilde{X},Y)+\rho_2\widetilde{G}_{\mathcal{A}}(\partial_{\widetilde{W}}\widetilde{X},Z) \\
&+\widetilde{G}_{\mathcal{A}}(Y,Z)\widetilde{G}_{\mathcal{A}}(\partial_{\widetilde{W}}\widetilde{X},E)+\rho_2\rho_3\widetilde{G}_{\mathcal{A}}(\partial_{\widetilde{W}}\widetilde{X},E).
\end{aligned}
\tag{2.72}
$$

But using the compatibility condition we have

$$
\widetilde{G}_{\mathcal{A}}(\alpha^\vee,\partial_{\widetilde{W}}\widetilde{X})=\partial_{\widetilde{W}}\widetilde{G}_{\mathcal{A}}(\alpha^\vee,\widetilde{X})-\widetilde{G}_{\mathcal{A}}(\partial_{\widetilde{W}}\alpha^\vee,\widetilde{X})=\partial_{\widetilde{W}}\alpha(X),
\tag{2.73}
$$

since $\partial_{\widetilde{W}}\alpha^\vee=0$. Also by the compatibility condition we have

$$
\widetilde{G}_{\mathcal{A}}(\partial_{\widetilde{W}}\widetilde{X},E)=\partial_{\widetilde{W}}\widetilde{G}_{\mathcal{A}}(\widetilde{X},E)-\widetilde{G}_{\mathcal{A}}(,\widetilde{X},\partial_{\widetilde{W}}E)=\partial_{\widetilde{W}}\widetilde{G}_{\mathcal{A}}(\widetilde{X},E)=\partial_{\widetilde{W}}(\rho_1),
\tag{2.74}
$$

since $\partial_{\widetilde{W}}E=0$. Hence by relations (2.73) and (2.74) relation (2.72) becomes

$$
\begin{aligned}
&c\big(\partial_{\widetilde{W}}\widetilde{X},\widetilde{Y},\widetilde{Z}\big)=\frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha\alpha(Y)\alpha(Z)\cot\alpha(x)\partial_{\widetilde{W}}\alpha(X)+\rho_3\widetilde{G}_{\mathcal{A}}(\partial_{\widetilde{W}}\widetilde{X},Y)+\rho_2\widetilde{G}_{\mathcal{A}}(\partial_{\widetilde{W}}\widetilde{X},Z) \\
&+\big(\partial_{\widetilde{W}}(\rho_1)\big)\widetilde{G}_{\mathcal{A}}(Y,Z)+\big(\partial_{\widetilde{W}}(\rho_1)\big)\rho_2\rho_3.
\end{aligned}
\tag{2.75}
$$

Similarly, we have

$$
\begin{aligned}
&c\big(\widetilde{X},\partial_{\widetilde{W}}\widetilde{Y},\widetilde{Z}\big)=\frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha\alpha(X)\alpha(Z)\cot\alpha(x)\partial_{\widetilde{W}}\alpha(Y)+\rho_3\widetilde{G}_{\mathcal{A}}(X,\partial_{\widetilde{W}}\widetilde{Y})+\rho_1\widetilde{G}_{\mathcal{A}}(\partial_{\widetilde{W}}\widetilde{Y},Z) \\
&+\big(\partial_{\widetilde{W}}(\rho_2)\big)\widetilde{G}_{\mathcal{A}}(X,Z)+\rho_1\big(\partial_{\widetilde{W}}(\rho_2)\big)\rho_3.
\end{aligned}
\tag{2.76}
$$

Also we have

$$c\big(\widetilde{X},\widetilde{Y},\partial_{\widetilde{W}}\widetilde{Z}\big)=\frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}}c_\alpha\alpha(X)\alpha(Y)\cot\alpha(x)\partial_{\widetilde{W}}\alpha(Z)+\rho_2\widetilde{G}_{\mathcal{A}}(X,\partial_{\widetilde{W}}\widetilde{Z})+\rho_1\widetilde{G}_{\mathcal{A}}(\partial_{\widetilde{W}}\widetilde{Y},Z)$$
$$+\big(\partial_{\widetilde{W}}(\rho_3)\big)\widetilde{G}_{\mathcal{A}}(X,Y)+\rho_1\rho_2\big(\partial_{\widetilde{W}}(\rho_3)\big). \tag{2.77}$$

Then by substituting relations (2.71), (2.75), (2.76) and (2.77) into the formula (2.69) and making use of the compatibility condition when required the formula of the tensor (2.69) reduces to

$$T(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W})=\Big(\nabla^0_{\widetilde{W}}c\Big)(\widetilde{X},\widetilde{Y},\widetilde{Z})=-\frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}}c_\alpha\alpha(X)\alpha(Y)\alpha(Z)\frac{\alpha(W)}{\sin^2\alpha(x)}. \tag{2.78}$$

The statement follows since formula (2.78) is symmetric in all its arguments $\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}$.

□

Propositions 2.6.8, 2.6.9 together with Poincaré Lemma confirm the existence of Frobenius prepotential as in the following proposition.

Proposition 2.6.10. *There exists a (local) prepotential $\widehat{F}=\widehat{F}(x_1,\dots,x_N,y)$ satisfies*

$$\partial_{\widetilde{X}}\partial_{\widetilde{Y}}\partial_{\widetilde{Z}}\widehat{F}=\widetilde{G}_{\mathcal{A}}(\widetilde{X}*\widetilde{Y},\widetilde{Z}),$$

where the multiplication $*$ is given by formula (2.64) and $\widetilde{X},\widetilde{Y},\widetilde{Z}$ are flat vector fields. This prepotential takes the form

$$\widehat{F}=\frac{y^3}{6}+\frac{y}{2}\sum_{\alpha\in\mathcal{A}}c_\alpha\alpha(x)^2+\frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}}c_\alpha f(\alpha(x))=\frac{1}{2}F(x,y),$$

where F is the solution given by (2.55).

Proof. By Propositions 2.6.8, 2.6.9 the two tensors (2.1), (2.2) are totally symmetric in all their arguments. Then Proposition 2.1.4 agrees of existence of the prepotential $\widehat{F}$. Let us denote by $c_{\widetilde{X}\widetilde{Y}\widetilde{Z}}:=c(\widetilde{X},\widetilde{Y},\widetilde{Z})$. Let us now find the explicit formula of the prepotential $\widehat{F}$ corresponding to the flat connection ∇^0 which can be extracted by integrating the polynomial $c_{\widetilde{X}\widetilde{Y}\widetilde{Z}}$ as follows.

By the formula of the multiplication (2.64) we have

$$\partial_{\widetilde{X}}\partial_{\widetilde{Y}}\partial_{\widetilde{Z}}\widehat{F}=c_{\widetilde{X}\widetilde{Y}\widetilde{Z}}=\widetilde{G}_{\mathcal{A}}(\widetilde{X}*\widetilde{Y},\widetilde{Z})=\frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}}c_\alpha\alpha(X)\alpha(Y)\alpha(Z)\cot\alpha(x)+\rho_1G_{\mathcal{A}}(Y,Z)$$
$$+\rho_2G_{\mathcal{A}}(X,Z)+\rho_3G_{\mathcal{A}}(X,Y)+\rho_1\rho_2\rho_3. \tag{2.79}$$

Then it is easy to see that

$$\partial_{\widetilde{X}}\partial_{\widetilde{Y}}\partial_{\widetilde{Z}}(\frac{y^3}{6}) = \rho_1\rho_2\rho_3. \tag{2.80}$$

Also we can show that

$$\partial_{\widetilde{X}}\partial_{\widetilde{Y}}\partial_{\widetilde{Z}}\Big(\frac{y}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha\alpha(x)^2\Big) = \rho_1 G_{\mathcal{A}}(Y,Z) + \rho_2 G_{\mathcal{A}}(X,Z) + \rho_3 G_{\mathcal{A}}(X,Y). \tag{2.81}$$

For the term $\frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha\alpha(X)\alpha(Y)\alpha(Z)\cot\alpha(x)$ we consider a function f to be such of that $f'''(\alpha(x)) = \cot\alpha(x)$ and hence we have

$$\partial_{\widetilde{X}}\partial_{\widetilde{Y}}\partial_{\widetilde{Z}}\Big(\sum_{\alpha\in\mathcal{A}} c_\alpha f(\alpha(x))\Big) = \sum_{\alpha\in\mathcal{A}} c_\alpha\alpha(X)\alpha(Y)\alpha(Z)\cot\alpha(x). \tag{2.82}$$

Then by relations (2.80)–(2.82) the prepotential $\widehat{F}$ in (2.79) takes the form

$$\widehat{F}(x,y) = \frac{y^3}{6} + \frac{y}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha\alpha(x)^2 + \frac{\lambda}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha f(\alpha(x)) = \frac{1}{2}F(x,y),$$

where F is the solution given by (2.55).

□

Remark 2.6.11. In the Definition 2.1.3 of the Frobenius manifold it is usually assumed that the scaling constant d_1 is nonzero. One can allow to choose $d_1 = 0$. This happens if we define $\mathcal{E} = e = E$, that is the Euler vector field coincides with the identity of the product. Then

$$\mathcal{E}(\widehat{F}) = \partial_y(\widehat{F}) = \frac{1}{2}\big(y^2 + \sum_{\alpha\in\mathcal{A}} c_\alpha\alpha(x)^2\big),$$

which agrees with the relation (2.7) where $d_F = 0$. Thus the set $\Big(V\oplus U, *, E, \widetilde{G}_{\mathcal{A}}, E\Big)$ becomes a Frobenius manifold provided that the product $*$ is associative.

The following property holds.

Proposition 2.6.12. *Dubrovin connection (2.67) is torsion free.*

Proof. The statement follows from Proposition 2.2.2 since the product $*$ is commutative.

□

The curvature of connection (2.67) is defined by

$$\widetilde{R}^\mu(X,Y)Z := [\nabla^\mu_X, \nabla^\mu_Y]Z - \nabla^\mu_{[X,Y]}Z, \quad X,Y,Z\in\Gamma\Big(T_*(V\oplus U)\Big). \tag{2.83}$$

The following result holds.

Proposition 2.6.13. *The flatness of the connection (2.67) is equivalent to the associativity of the product (2.64).*

Proof. By Proposition 2.2.2 the flatness of the connection (2.67) is equivalent to the totally symmetric of the tensor $T^0(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W})$ given by formula (2.69) together with the associativity of the product $*$ is given by (2.64). But from Proposition (2.6.9) the tensor T^0 is totally symmetric, hence the statement follows. □

As a corollary of Propositions 2.6.2, 2.6.13 the following result holds.

Theorem 2.6.14. *The flatness of Dubrovin connection (2.67) for all μ is equivalent to WDVV equations (2.56).*

Let us now consider $\Lambda^2 V^*$ as a subspace in $(V \otimes V)^*$ given by the anti-symmetric tensors, then we can define the quantity $B_{\alpha,\beta} \colon V \otimes V \to \mathbb{C}$ as follows. For any $\alpha, \beta \in \mathcal{A}$ we define $B_{\alpha,\beta} := \alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha \in \Lambda^2 V^*$, such that

$$B_{\alpha,\beta}(a \otimes b) = \alpha \wedge \beta(a \otimes b) = \alpha(a)\beta(b) - \alpha(b)\beta(a)$$

for any $a, b \in V$. The following property holds.

Lemma 2.6.15. *For any $\alpha, \beta \in V^*, a, b \in V$ we have*

$$B_{\alpha,\beta}(a \wedge b) = 2B_{\alpha,\beta}(a \otimes b).$$

Proof. Since $a \wedge b = a \otimes b - b \otimes a$, then

$$\begin{aligned} B_{\alpha,\beta}(a \wedge b) &= B_{\alpha,\beta}(a \otimes b - b \otimes a) = B_{\alpha,\beta}(a \otimes b) - B_{\alpha,\beta}(b \otimes a) \\ &= \Big(\alpha(a)\beta(b) - \alpha(b)\beta(a)\Big) - \Big(\alpha(b)\beta(a) - \alpha(a)\beta(b)\Big) = 2\Big(\alpha(a)\beta(b) - \alpha(b)\beta(a)\Big), \end{aligned}$$

which implies the statement. □

Let us introduce the following symmetric bilinear form $G^{(1)}_{\mathcal{A}} = G^{(1)}_{(\mathcal{A},c)}$, on the vector space $\Lambda^2 V \subset V \otimes V$ given by

$$G^{(1)}_{\mathcal{A}}(z, w) = \sum_{\alpha,\beta \in \mathcal{A}} c_\alpha c_\beta B_{\alpha,\beta}(z) B_{\alpha,\beta}(w), \tag{2.84}$$

where $z, w \in \Lambda^2 V$. It is easy to see that for $z = u_1 \wedge v_1, w = u_2 \wedge v_2$, where $u_1, u_2, v_1, v_2 \in V$, we have

$$G^{(1)}_{\mathcal{A}}(z, w) = 8\Big(G_{\mathcal{A}}(u_1, u_2) G_{\mathcal{A}}(v_1, v_2) - G_{\mathcal{A}}(u_1, v_2) G_{\mathcal{A}}(u_2, v_1)\Big), \tag{2.85}$$

which is a natural extension of the bilinear form $G_{\mathcal{A}}$ to the space $\Lambda^2 V$. It is also easy to see that this form $G^{(1)}_{\mathcal{A}}$ is non-degenerate and that it is $\mathcal{W}$-invariant.

Since the flatness of connection (2.67) is equivalent to WDVV equations (2.56) one expects that the flatness condition determines the restrictions on the scalar λ that lead to a solution of form (2.55). The following proposition gives the condition in the solution (2.55) which follows from the flatness of the corresponding connection.

Proposition 2.6.16. *The flatness of Dubrovin connection (2.67) for all μ is equivalent to the identity*

$$\sum_{\alpha,\beta\in\mathcal{A}}\left(\frac{\lambda^2}{4}c_\alpha c_\beta G_{\mathcal{A}}(\alpha^\vee,\beta^\vee)\cot\alpha(x)\cot\beta(x)+c_\alpha c_\beta\right)B_{\alpha,\beta}(u\otimes v)B_{\alpha,\beta}(z\otimes w)=0. \quad (2.86)$$

Proof. Firstly, note that by Proposition 2.2.5 and formula (2.83), it is enough to study the zero-curvature (flatness condition) by considering constant vector fields $\widetilde{u},\widetilde{v},\widetilde{z}$, and hence the flatness condition for the connection (2.67) reduces to the formula

$$[\nabla^\mu_{\widetilde{u}},\nabla^\mu_{\widetilde{v}}](\widetilde{z})=0. \quad (2.87)$$

Let us now consider constant vector fields:

$$\widetilde{u}=u+\rho_1 E,\ \widetilde{v}=v+\rho_2 E,\ \widetilde{z}=z+\rho_3 E\in\Gamma(T(V\oplus U)),$$

where $u,v,z\in V$ and $\rho_1,\rho_2,\rho_3\in\mathbb{C}$. By straightforward calculation we have

$$\begin{aligned}\nabla^\mu_{\widetilde{u}}\Big(\nabla^\mu_{\widetilde{v}}(\widetilde{z})\Big)&=-\frac{\mu\lambda}{2}\sum_{\alpha\in\mathcal{A}}c_\alpha\frac{\alpha(u)\alpha(v)\alpha(z)}{\sin^2\alpha(x)}\alpha^\vee+\frac{\mu^2\lambda\rho_1}{2}\sum_{\alpha>0}c_\alpha\alpha(v)\alpha(z)\cot\alpha(x)\alpha^\vee\\&+\frac{\mu^2\lambda\rho_2}{2}\sum_{\alpha>0}c_\alpha\alpha(u)\alpha(z)\cot\alpha(x)\alpha^\vee+\frac{\mu^2\lambda\rho_3}{2}\sum_{\alpha>0}c_\alpha\alpha(u)\alpha(v)\cot\alpha(x)\alpha^\vee\\&+\frac{\mu^2\lambda^2}{4}\sum_{\alpha,\beta>0}c_\alpha c_\beta\alpha(u)\beta(v)\cot\alpha(x)\cot\beta(x)(\alpha\otimes\alpha^\vee)(\beta\otimes\beta^\vee)(z)+\mu^2G_{\mathcal{A}}(v,z)u\\&+\mu^2\Big(\rho_1\rho_2 z+\rho_1\rho_3 v+\rho_2\rho_3 u\Big)+\mu^2\Big(\rho_1G_{\mathcal{A}}(v,z)+\rho_2G_{\mathcal{A}}(u,z)+\rho_3G_{\mathcal{A}}(u,v)+\rho_1\rho_2\rho_3\Big)E\\&+\mu^2\Big(\frac{\lambda}{2}\sum_{\alpha>0}c_\alpha\alpha(u)\alpha(v)\cot\alpha(x)\Big)E. \end{aligned}\quad (2.88)$$

Similarly, (by swapping $u\leftrightarrow v$ and $\rho_1\leftrightarrow\rho_2$) we obtain the formula of $\nabla^\mu_{\widetilde{v}}\Big(\nabla^\mu_{\widetilde{u}}(\widetilde{z})\Big)$. Hence the flatness condition (2.87) reduces to

$$\begin{aligned}0&=\frac{\mu^2\lambda^2}{4}\sum_{\alpha,\beta>0}c_\alpha c_\beta\Big(\alpha\wedge\beta(u\otimes v)\Big)\cot\alpha(x)\cot\beta(x)(\alpha\otimes\alpha^\vee)(\beta\otimes\beta^\vee)(z)\\&+\mu^2\Big(G_{\mathcal{A}}(v,z)u-G_{\mathcal{A}}(u,z)v\Big).\end{aligned}\quad (2.89)$$

Note that

$$(\alpha \otimes \alpha^\vee)(\beta \otimes \beta^\vee)(z) = (\alpha \otimes \alpha^\vee)(\beta(z)\beta^\vee) = \alpha(\beta^\vee)\beta(z)\alpha^\vee = G_{\mathcal{A}}(\alpha^\vee, \beta^\vee)\beta(z)\alpha^\vee.$$

Hence identity (2.89) can be rewritten as

$$\begin{aligned} 0 = &\frac{\mu^2\lambda^2}{4} \sum_{\alpha,\beta>0} c_\alpha c_\beta G_{\mathcal{A}}(\alpha^\vee, \beta^\vee) \cot\alpha(x) \cot\beta(x) \Big(\alpha \wedge \beta(u \otimes v)\Big)\beta(z)\alpha^\vee \\ &+ \mu^2 \Big(G_{\mathcal{A}}(v, z)u - G_{\mathcal{A}}(u, z)v\Big). \end{aligned} \tag{2.90}$$

Since the identity (2.90) valid for all vectors $\alpha, \beta > 0$, then by exchanging $\alpha \leftrightarrow \beta$ we get a similar identity in which adding them up together and multiplying the result by $\frac{4}{\mu^2}$ gives the following identity

$$\begin{aligned} &\lambda^2 \sum_{\alpha,\beta>0} c_\alpha c_\beta G_{\mathcal{A}}(\alpha^\vee, \beta^\vee) \cot\alpha(x) \cot\beta(x) \Big(\alpha \wedge \beta(u \otimes v)\Big) \Big(\alpha(z)\beta^\vee - \beta(z)\alpha^\vee\Big) \\ &+ 8\Big(G_{\mathcal{A}}(v, z)u - G_{\mathcal{A}}(u, z)v\Big) = 0. \end{aligned} \tag{2.91}$$

Let $w \in V$ and let us calculate the bilinear form $G_{\mathcal{A}}(w, \cdot)$ with respect to the identity (2.91). We have

$$\begin{aligned} &\lambda^2 \sum_{\alpha,\beta>0} c_\alpha c_\beta G_{\mathcal{A}}(\alpha^\vee, \beta^\vee) \cot\alpha(x) \cot\beta(x) \Big(\alpha \wedge \beta(u \otimes v)\Big) \Big(\alpha(z)G_{\mathcal{A}}(\beta^\vee, w) - \beta(z)G_{\mathcal{A}}(\alpha^\vee, w)\Big) \\ &+ 8\Big(G_{\mathcal{A}}(v, z)G_{\mathcal{A}}(u, w) - G_{\mathcal{A}}(u, z)G_{\mathcal{A}}(v, w)\Big) = 0, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &\lambda^2 \sum_{\alpha,\beta\in\mathcal{A}} c_\alpha c_\beta G_{\mathcal{A}}(\alpha^\vee, \beta^\vee) \cot\alpha(x) \cot\beta(x) \Big(\alpha \wedge \beta(u \otimes v)\Big) \Big(\alpha \wedge \beta(z \otimes w)\Big) \\ &+ 8\Big(G_{\mathcal{A}}(v, z)G_{\mathcal{A}}(u, w) - G_{\mathcal{A}}(u, z)G_{\mathcal{A}}(v, w)\Big) = 0. \end{aligned} \tag{2.92}$$

But from Lemma 2.6.15 and relation (2.85) we have

$$\begin{aligned} &8\Big(G_{\mathcal{A}}(v, z)G_{\mathcal{A}}(u, w) - G_{\mathcal{A}}(u, z)G_{\mathcal{A}}(v, w)\Big) = G_{\mathcal{A}}(u \wedge v, z \wedge w) \\ &= \sum_{\alpha,\beta\in\mathcal{A}} c_\alpha c_\beta B_{\alpha,\beta}(u \wedge v) B_{\alpha,\beta}(z \wedge w) = 4 \sum_{\alpha,\beta\in\mathcal{A}} c_\alpha c_\beta B_{\alpha,\beta}(u \otimes v) B_{\alpha,\beta}(z \otimes w). \end{aligned}$$

Hence the flatness condition (2.92) reduces to the required identity. $\square$

2.6.2 Trigonometric $\vee$-systems

For a fixed $\alpha \in \mathcal{A}$ identity (2.86) may contain singularities when $\tan \alpha(x) = 0$, and in order to cancel these singularities we need further investigation. To do so, we need to prove some results.

For each vector $\alpha \in \mathcal{A}$ let us introduce the set of its collinear vectors from $\mathcal{A}$:

$$\delta_\alpha := \{\gamma \in \mathcal{A} \colon \gamma \sim \alpha\}.$$

Let $\delta \subseteq \delta_\alpha$ and $\alpha_0 \in \delta_\alpha$. Then for any $\gamma \in \delta$ we have $\gamma = k_\gamma \alpha_0$ for some $k_\gamma \in \mathbb{R}$. Note that k_γ depends on the choice of α_0 and different choices of α_0 give rescaled collections of these parameters. Define $C_\delta^{\alpha_0} := \sum_{\gamma\in\delta} c_\gamma k_\gamma^2$. Note that $C_\delta^{\alpha_0}$ is non-zero if and only if $C_\delta^{\widetilde{\alpha}_0} \neq 0$ for any $\widetilde{\alpha}_0 \in \delta$.

The WDVV equations for a function F can be reformulated using geometry of the configuration $\mathcal{A}$. Such a geometric structure is embedded in the notion of a trigonometric $\vee$-system. Before defining trigonometric $\vee$-system precisely we need a notion of *series* (or *strings*) of vectors (see [27]).

For any $\alpha \in \mathcal{A}$ let us distribute all the covectors in $\mathcal{A} \setminus \delta_\alpha$ into a disjoint union of α-series

$$\mathcal{A} \setminus \delta_\alpha = \bigsqcup_{s=1}^{k} \Gamma_\alpha^s,$$

where $k \in \mathbb{N}$ depends on α. These series Γ_α^s are determined by the property that for any $s = 1, \dots, k$ and for any two covectors $\gamma_1, \gamma_2 \in \Gamma_\alpha^s$ one has either $\gamma_1 + \gamma_2 = m\alpha$ or $\gamma_1 - \gamma_2 = m\alpha$ for some $m \in \mathbb{Z}$. We assume that the series are maximal, that is if $\gamma \in \Gamma_\alpha^s$ for some $s \in \mathbb{N}$, then Γ_α^s must contain all the covectors of the form $\pm\gamma + m\alpha \in \mathcal{A}$ with $m \in \mathbb{Z}$. Note that if for some $\beta \in \mathcal{A}$ there is no $\gamma \in \mathcal{A}$ such that $\beta \pm \gamma = m\alpha$ for $m \in \mathbb{Z}$, then β itself forms a single α-series. Note also that any α-series belongs to a two-dimensional vector space.

By replacing some vectors from $\mathcal{A}$ with their opposite ones and keeping the multiplicity unchanged one can get a new configuration whose vectors belong to a half-space. We will denote such a system by $\mathcal{A}_+$. If this system contains repeated vectors α with multiplicities c_α^i then we replace them with the single vector α with multiplicity $c_\alpha := \sum_i c_\alpha^i$.

Definition 2.6.17. [27] The pair $(\mathcal{A}, c)$ is called a trigonometric $\vee$-system if for all $\alpha \in \mathcal{A}$ and for any α-series Γ_α^s, one has the relation

$$\sum_{\beta\in\Gamma_\alpha^s} c_\beta \alpha(\beta^\vee) \alpha \wedge \beta = 0. \tag{2.93}$$

Note that if $\beta_1, \beta_2 \in \Gamma_\alpha^s$ for some α, s, then $\alpha \wedge \beta_1 = \pm\alpha \wedge \beta_2$ so the identity (2.93) may be simplified by cancelling wedge products. We also note that if $\mathcal{A}$ is a trigonometric $\vee$-system then $\mathcal{A}_+$ is the one as well.

Let us also define the following bilinear form $G^{(2)}_{\mathcal{A}_+}$ on $\Lambda^2 V$:

$$G^{(2)}_{\mathcal{A}_+}(z,w) = \sum_{\alpha,\beta\in\mathcal{A}_+} c_\alpha c_\beta G_{\mathcal{A}}(\alpha^\vee,\beta^\vee) B_{\alpha,\beta}(z) B_{\alpha,\beta}(w), \tag{2.94}$$

where $z, w \in \Lambda^2 V$.

The following statement shows that the bilinear form $G^{(2)}_{\mathcal{A}_+}$ is independent of the choice of the positive system $\mathcal{A}_+$.

Lemma 2.6.18. *For any positive systems $\mathcal{A}^{(1)}_+, \mathcal{A}^{(2)}_+$ for a trigonometric $\vee$-system $(\mathcal{A}, c)$ we have $G^{(2)}_{\mathcal{A}^{(1)}_+} = G^{(2)}_{\mathcal{A}^{(2)}_+}$.*

Proof. Suppose firstly that two positive systems $\mathcal{A}^{(1)}_+, \mathcal{A}^{(2)}_+$ for a trigonometric $\vee$-system $(\mathcal{A}, c)$ satisfy the condition

$$\mathcal{A}^{(2)}_+ = \Big(\mathcal{A}^{(1)}_+ \setminus \delta_\alpha\Big) \cup \Big(-\delta_\alpha\Big)$$

for some $\alpha \in \mathcal{A}^{(1)}_+$. Notice that vector α cannot be a linear combination of vectors in $\mathcal{A}^{(1)}_+ \setminus \delta_\alpha$ with positive coefficients. Hence for each α-series Γ^s_α in $\mathcal{A}^{(1)}_+$ we have

$$\sum_{\beta\in\Gamma^s_\alpha} c_\beta \alpha(\beta^\vee) = 0 \tag{2.95}$$

since $B_{\alpha,\beta_1} = B_{\alpha,\beta_2}$ for all $\beta_1, \beta_2 \in \Gamma^s_\alpha$.

Let us consider terms in $G^{(2)}_{\mathcal{A}^{(1)}_+}(z,w)$ which contain α. They are proportional to

$$\sum_{\beta\in\mathcal{A}^{(1)}_+} c_\beta G_{\mathcal{A}}(\alpha^\vee,\beta^\vee) B_{\alpha,\beta}(z) B_{\alpha,\beta}(w) = \sum_s \sum_{\beta\in\Gamma^s_\alpha} c_\beta \alpha(\beta^\vee) B_{\alpha,\beta}(z) B_{\alpha,\beta}(w) = 0$$

by (2.95). The statement follows in this case.

In general, the system $\mathcal{A}^{(2)}_+$ can be obtained from the system $\mathcal{A}^{(1)}_+$ by a sequence of steps where in each one we replace the subset of vectors δ_α with vectors $-\delta_\alpha$ and the resulting system is still a positive one. In order to see this, one moves continuously the hyperplane defining $\mathcal{A}^{(1)}_+$ into the hyperplane defining $\mathcal{A}^{(2)}_+$ so that at each moment the hyperplane contains at most one vector from $\mathcal{A}$ up to proportionality. The statement follows from the case considered above. $\square$

As a consequence of Lemma 2.6.18 we can and will denote the form $G^{(2)}_{\mathcal{A}_+}$ as $G^{(2)}_{\mathcal{A}}$.

The following proposition holds.

Proposition 2.6.19. *Assume that prepotential (2.55) satisfies the WDVV equations (2.56). Suppose that $C_{\delta_\alpha}^{\alpha_0} \neq 0$ for any $\alpha \in \mathcal{A}, \alpha_0 \in \delta_\alpha$. Then the identity*

$$\sum_{\beta \in \mathcal{A} \setminus \delta_\alpha} c_\beta \alpha(\beta^\vee) \cot \beta(x) B_{\alpha,\beta}(a \otimes b) \alpha \wedge \beta = 0 \tag{2.96}$$

holds for all $a, b \in V$ provided that $\alpha(x) = 0$.

Proof. By Theorem 2.6.14 the WDVV equations (2.56) are equivalent to the flatness of connection (2.67), and by Proposition 2.6.16 the flatness condition of connection (2.67) is equivalent to relation (2.86).

Let us consider terms in the left-hand side of relation (2.86), where β or γ is proportional to α. The sum of these terms has to be regular at $\alpha(x) = 0$. This implies that the product

$$\Big(\lambda^2 \sum_{\gamma \in \delta_\alpha} k_\gamma^3 c_\gamma \cot \gamma(x)\Big)\Big(\sum_{\beta \in \mathcal{A} \setminus \delta_\alpha} c_\beta \alpha_0(\beta^\vee) \cot \beta(x) B_{\alpha_0,\beta}(a \otimes b) \alpha_0 \wedge \beta\Big) \tag{2.97}$$

is regular at $\alpha(x) = 0$. The first factor in the product (2.97) has the first order pole at $\alpha(x) = 0$ by the assumption that $C_{\delta_\alpha}^{\alpha_0} \neq 0$ for any $\alpha \in \mathcal{A}, \alpha_0 \in \delta_\alpha$. This implies the statement. □

Similarly to Proposition 2.6.19 the following proposition can also be established.

Proposition 2.6.20. *Assume that prepotential (2.55) satisfies the WDVV equations (2.56). Suppose that $C_\delta^{\alpha_0} \neq 0$ for any $\alpha \in \mathcal{A}, \delta \subseteq \delta_\alpha, \alpha_0 \in \delta_\alpha$. Then the identity (2.96) holds for any $a, b \in V$ provided that* $\tan \alpha(x) = 0$.

The proof is similar to the proof of Proposition 2.6.19. Indeed, we have that expression (2.97) is regular at $\alpha(x) = \pi m, m \in \mathbb{Z}$. Assumptions imply that the first factor in (2.97) has the first order pole, which implies the statement.

A close relation between trigonometric $\vee$-systems and solutions of WDVV equations is given by the following theorem.

Theorem 2.6.21. (cf.[27]) *Suppose that a configuration $(\mathcal{A}, c)$ satisfies the condition $C_\delta^{\alpha_0} \neq 0$ for all $\alpha \in \mathcal{A}$, $\delta \subseteq \delta_\alpha$, $\alpha_0 \in \delta_\alpha$. Then WDVV equations (2.56) for the function (2.55) imply the following two conditions:*

1. *$\mathcal{A}$ is a trigonometric $\vee$-system,*
2. *Bilinear forms* (2.84), (2.94) *satisfy proportionality $G_\mathcal{A}^{(1)} = \frac{\lambda^2}{4} G_\mathcal{A}^{(2)}$.*

Conversely, if a configuration $(\mathcal{A}, c)$ satisfies conditions (1) and (2) then WDVV equations (2.56) hold.

The key part of the proof is to derive trigonometric $\vee$-conditions from WDVV equations, which goes along the following lines (see [27] for details). By Proposition 2.6.20 identity (2.96) holds if $\tan\alpha(x) = 0$. The identity (2.96) is a linear combination of $\cot\beta(x)|_{\tan\alpha(x)=0}$, which can vanish only if it vanishes for each α-series. Hence identity (2.96) implies relations (2.93) so $\mathcal{A}$ is a trigonometric $\vee$-system.

Remark 2.6.22. A version of Theorem 2.6.21 is given in [27, Theorem 1] without specifying conditions $C_\delta^{\alpha_0} \neq 0$. However these assumptions seem needed in general in order to derive trigonometric $\vee$-conditions for α-series in the case when $\delta_\alpha \setminus \{\pm\alpha\} \neq \emptyset$ as above arguments and proofs of Propositions 2.6.19, 2.6.20 explain.

2.7 Root systems examples

An important class of trigonometric solutions of WDVV equations is given by (crystallographic) root systems $\mathcal{A} = \mathcal{R}$ of Weyl groups $\mathcal{W}$. Recall that a root system $\mathcal{R}$ satisfies the property

$$s_\alpha\beta = \beta - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha \in \mathcal{R} \tag{2.98}$$

for any $\alpha, \beta \in \mathcal{R}$, and one has $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$, where $\langle\cdot,\cdot\rangle$ is a $\mathcal{W}$-invariant scalar product on $V^* \cong V$. The corresponding Weyl group is generated by reflections $s_\alpha, \alpha \in \mathcal{R}$.

The following statement was established in [42] for the non-reduced root systems.

Theorem 2.7.1. (cf. [42]) *Let $\mathcal{A} = \mathcal{R}$ be an irreducible root system with the Weyl group $\mathcal{W}$ and suppose that the multiplicity function $c\colon \mathcal{R} \to \mathbb{C}$ is $\mathcal{W}$-invariant. Then prepotential (2.55) satisfies WDVV equations (2.56) for some $\lambda \in \mathbb{C}$.*

Let us explain a proof of this statement different from [42] by making use the notion of a trigonometric $\vee$-system and Theorem 2.6.21.

Proposition 2.7.2. *Root system $\mathcal{A} = \mathcal{R}$ with $\mathcal{W}$-invariant multiplicity function c is a trigonometric $\vee$-system.*

Proof. Fix $\alpha \in \mathcal{R}$. Take any $\beta \in \mathcal{R}$, and let $\gamma = s_\alpha\beta$. Then from (2.98) we have that

$$\beta - \gamma = m\alpha, \quad m \in \mathbb{Z}.$$

Hence $\beta, \gamma \in \Gamma_\alpha^s$ for some s. The bilinear form $G_\mathcal{R}$ is $\mathcal{W}$-invariant so is proportional to $\langle\cdot,\cdot\rangle$. Therefore we have

$$c_\beta = c_\gamma, \quad G_\mathcal{R}(\alpha,\beta) = -G_\mathcal{R}(\alpha,\gamma), \quad \alpha\wedge\beta = \alpha\wedge\gamma.$$

Hence,

$$c_\beta G_{\mathcal{R}}(\alpha,\beta)\alpha\wedge\beta + c_\gamma G_{\mathcal{R}}(\alpha,\gamma)\alpha\wedge\gamma = 0,$$

which implies trigonometric $\vee$-conditions (2.93). $\square$

It is easy to see that the bilinear form $G^{(1)}_{\mathcal{R}}$ is $\mathcal{W}$-invariant, and the same is true for the bilinear form $G^{(2)}_{\mathcal{R}}$ (see e.g. [3, Proposition 6.4]). Since $\mathcal{W}$-module $\Lambda^2 V$ is irreducible, the forms $G^{(1)}_{\mathcal{R}}$ and $G^{(2)}_{\mathcal{R}}$ have to be proportional. By Theorem 2.6.21 this implies Theorem 2.7.1 provided that the form $G^{(2)}_{\mathcal{R}}$ is non-zero. The latter fact is claimed in [42] where the corresponding solution of WDVV equations was explicitly stated for the constant multiplicity function. It was found for any multiplicity function for the non-reduced root systems in [47,48].

It follows that a positive half $\mathcal{A}=\mathcal{R}^+$ of a root system $\mathcal{R}$ also defines a solution of WDVV equations (2.56). We find the corresponding form $G^{(2)}_{\mathcal{R}^+}$ for the root system $\mathcal{R}=BC_N$ explicitly in Section 3.3. We also specify corresponding constants $\lambda=\lambda_{(\mathcal{R},c)}$ for (the positive halves of) reduced root systems $\mathcal{R}$ in Section 3.6. Note that λ is invariant under the linear transformations applied to $\mathcal{A}$. In the root system case the scalar $\lambda_{(\mathcal{R},c)}$ may be thought of as a version of the (generalized) Coxeter number for the case of the representation $\Lambda^2 V$, as the usual (generalized) Coxeter number can also be given as a ratio of two $\mathcal{W}$-invariant forms on V ([6,28]).

The value of the parameter λ was found in a few earlier works which we now recall.

1. Martini's and Hoevenaars' works

A prepotential $\widetilde{F}$ of $N+1$ variables $(x_1,\dots,x_N,y)$ was considered in [42] in the form

$$\widetilde{F}(x,y)=\frac{\gamma}{6}y^3+\frac{\gamma}{2}y\langle x,x\rangle+\sum_{\alpha\in\mathcal{R}^+}k_\alpha\widetilde{f}(\alpha(x)), \tag{2.99}$$

where $\mathcal{R}^+$ is a positive half of the (crystallographic) root system $\mathcal{R}$ of rank N, the multiplicity function $k(\alpha):=k_\alpha$ and the inner product $\langle\cdot,\cdot\rangle$ are invariant under the corresponding Weyl group $\mathcal{W}$ of $\mathcal{R}$, $\gamma\in\mathbb{C}$ is depending on the root system $\mathcal{R}$ and function $\widetilde{f}$ given by

$$\widetilde{f}(z)=\frac{1}{6}z^3-\frac{1}{4}Li_3(e^{-2z}) \tag{2.100}$$

satisfies $\widetilde{f}'''(z)=\coth z$. With respect to the extra variable $y=x_{N+1}$ the matrix of the third order derivatives

$$(F_{N+1})_{ij}=\frac{\partial^3 F}{\partial_{N+1}\partial_i\partial_j}$$

becomes a multiple of the identity, more precisely $(F_{N+1})_{ij}=\gamma\delta_{ij}$. Hence the WDVV systems (2.56) in this case reduces to the system

$$F_iF_j=F_jF_i.$$

It has been shown that function (2.99) satisfies WDVV equations (2.56) and the corresponding values of $\gamma_{(\mathcal{R},c)}$ were given explicitly (except for $\mathcal{R} = BC_N, G_2$) where a constant $\mathcal{W}$-invariant multiplicity function was considered as the following statement explains.

Proposition 2.7.3. [42] *The function (2.99) satisfies WDVV equations (2.56) and the corresponding values of $\gamma_{(\mathcal{R},c)}$ for a constant $\mathcal{W}$-invariant multiplicity function $k_\alpha = 1\ \forall\alpha$ are given in the following table:*

$\mathcal{R}$	B_N	C_N	D_N	A_N	E_6	E_7	E_8	F_4
$-\gamma^2$	$2(2N-3)$	$4(N+2)$	$4(N-2)$	$(N+1)$	24	48	120	15

Remark 2.7.4. (1) A function of the form (2.99) was also considered in [33] for special value of multiplicities were $k(e^i) = \eta, (1 \le i \le N),\ k(e^i \pm e^j) = 1, (1 \le i < j \le N)$, and it has been shown that function (2.99) for this case satisfies the WDVV system (2.56) if and only if

$$\eta = -2(N-2) - \frac{\gamma^2}{2}, \tag{2.101}$$

which agrees with the value of γ given above where the values $\eta = 1, 0$ in (2.101) are corresponding to the root systems of type B_N, D_N respectively.

(2) There seems to be typos in [42] in the values of γ which are given for root systems A_N, E_6, E_8 as we will clarify that later when we generalize these results to general multiplicities.

2. Bryan's and Gholampour's case

Solutions of WDVV equations of the form (2.99) can also be extracted from the considerations in [11, Section 4], where a family of (Frobenius) algebras for irreducible reduced root systems $\mathcal{R} \subset V^*$ of rank N with special multiplicities function were considered, where V is a real N-dimensional vector space. The multiplication $*$ on the tangent space $T_{(x,y)}(V \oplus U) \cong V \oplus U$, where $\dim U = 1, x \in V, y \in U$ was given by the formula

$$u * v = \langle u, v\rangle E + \widetilde{\gamma}^{-1} \sum_{\beta\in\mathcal{R}^+} \frac{c_\beta}{\langle\beta,\beta\rangle}\beta(u)\beta(v)\coth\beta(x)\beta, \quad (u, v \in V), \tag{2.102}$$

where $\widetilde{\gamma} = \widetilde{\gamma}_{(\mathcal{R},c)} \in \mathbb{C}$, and $E \in U$ is the identity of $*$. It was shown in [11] that this multiplication is associative and satisfies the Frobenius condition. This gives rise to existence of solutions of WDVV equations corresponding to each root system. The constant $\widetilde{\gamma} = \widetilde{\gamma}_{(\mathcal{R},c)}$ was expressed in [11] in terms of the highest root θ of the root system $\mathcal{R}$.

Proposition 2.7.5. [11] *The function*

$$\widetilde{F}(x,y) = \frac{\widetilde{\gamma}}{6}y^3 + \frac{\widetilde{\gamma}}{2}y\langle x,x\rangle + \sum_{\alpha\in\mathcal{R}^+} \frac{c_\alpha}{\langle\alpha,\alpha\rangle}\widetilde{f}(\alpha(x)) \tag{2.103}$$

corresponding to the multiplication (2.102) solves WDVV equations (2.56) and the value of $\widetilde{\gamma}_{(\mathcal{R},c)}$ *in the solution (2.103) in the case of constant multiplicity function* $c_\alpha = t$ *is given by*

$$\widetilde{\gamma}^2_{(\mathcal{R},c)} = -\frac{t^2}{8}\Big(\langle\theta,\theta\rangle + \sum_{i=1}^{N} n_i^2\langle\alpha_i,\alpha_i\rangle\Big).$$

3. Shen's result

Note that in [47, 48] a prepotential function for a Frobenius structure was considered, this prepotential gives a solution of WDVV equations for root systems $\mathcal{R}$ for arbitrary (not simply-laced) root system $\mathcal{R}$ with invariant multiplicity. Let us recall that solution which is given by

$$\Phi = -\frac{y^3}{6} + \frac{\mu}{2}\langle x,x\rangle y + 2\mu\sum_{\alpha\in\mathcal{R}^+}\frac{k_\alpha}{\langle\alpha,\alpha\rangle}q(\alpha(x)), \tag{2.104}$$

for some scalar $\mu\in\mathbb{R}$, where k_α is a $\mathcal{W}$-invariant multiplicity function and the function q satisfies $q'''(z) = -\frac{1}{2}\coth(\frac{z}{2})$. For each k_α a product structure was defined on the tangent bundle $T(V\oplus U)$ which endows each fiber of $T(V\oplus U)$ with Frobenius algebra structure. Let us recall this algebra structure.

Let $E = \partial_{x_{N+1}} \in \Gamma(T(V\oplus U))$. Consider two vector fields $\widetilde{X} = X + \lambda_1 E$, $\widetilde{Y} = Y + \lambda_2 E \in \Gamma(T(V\oplus U))$, where $X, Y\in V, \lambda_1, \lambda_2\in\mathbb{C}$. The product $\circ$ was defined by the formula

$$\widetilde{X}\circ\widetilde{Y} = -\sum_{\alpha\in\mathcal{R}^+}\frac{k_\alpha}{\langle\alpha,\alpha\rangle}\coth(\frac{z}{2})\alpha(X)\alpha(Y)\alpha - \mu\langle X,Y\rangle E + \lambda_2 X + \lambda_1 Y + \lambda_1\lambda_2 E.$$

In this algebra E plays the role of the identity of the product. Then a family of the corresponding Dubrovin connections $\widetilde{\nabla}^k$ was considered in which its flatness conditions determine the value of μ for each root system. Note that the root system A_N was treated differently from our configuration's form of type A_N root system.

Proposition 2.7.6. [47] *Function (2.104) solves WDVV equations (2.56) and the values of* μ *in solution (2.104) for each root system are given in the following table:*

$\mathcal{R}$	B_N	C_N	D_N	G_2	E_6	E_7	E_8	F_4
μ	$q(p+(N-2)q)$	$p(2q+(N-2)p)$	$(N-2)p^2$	$\frac{3}{4}(p+q)(p+3q)$	$6p^2$	$12p^2$	$30p^2$	$(p+q)(p+2q)$

Here p *is the multiplicity of short roots and* q *is the multiplicity of long roots in a reduced not simply-laced root system* $\mathcal{R}$.

2.8 Generalized Coxeter number

In this section let $V \cong \mathbb{R}^N$ be a real Euclidean space of dimension N. Let $\mathcal{R}$ be a reduced irreducible root system of rank N with the corresponding finite irreducible Coxeter group $\mathcal{W}$ acting on the vector space V by compositions of reflections. Consider a system of simple roots $\Delta = \{\alpha_1, \dots, \alpha_N\}$ with the corresponding simple reflections $s_1, \dots, s_N$. Then the product of all simple reflections $s_1 \dots s_N \in \mathcal{W}$ is called a *Coxeter element*. It is clear that it depends on the choice of Δ, but the following proposition holds.

Proposition 2.8.1. [34] *All Coxeter elements are conjugate in $\mathcal{W}$.*

Since all Coxeter elements are conjugate, they have the same order h, which is called the *Coxeter number* of $\mathcal{W}$.

In fact there are a few different ways to define the Coxeter number h of an irreducible root system. In addition to the previous definition of the Coxeter number we also have the following equivalent definitions:

- The Coxeter number $h = \frac{2m}{N}$, where m is the number of reflections in $\mathcal{W}$. In the crystallographic case, $2m + N$ is the dimension of the corresponding semi-simple Lie algebra.
- If the highest root is $\sum_{i=1}^N m_i \alpha_i$ for simple roots α_i, then the Coxeter number $h = 1 + \sum_{i=1}^N m_i$.
- The Coxeter number is the highest degree of a fundamental invariant of the Coxeter group acting on polynomials (see Section 2.4).

A remarkable property that the Coxeter number admits, is that it can be written as the factor of proportionality of two $\mathcal{W}$-invariant bilinear forms. The statement is given in the following proposition.

Proposition 2.8.2. [6] *Let $\langle \cdot, \cdot \rangle$ be the standard inner product on V which is assumed to be non-degenerate and $\mathcal{W}$-invariant. Then*

$$\sum_{\alpha \in \mathcal{R}} \frac{\langle \alpha, x \rangle^2}{\langle \alpha, \alpha \rangle} = h \langle x, x \rangle \tag{2.105}$$

for all $x \in V$.

In fact relation (2.105) holds for any symmetric, non-degenerate and $\mathcal{W}$-invariant bilinear form on V as such a form is unique up to proportionality for irreducible group $\mathcal{W}$. Since the scalar λ in Theorem 2.6.21 appears as the coefficient of the proportionality between two $\mathcal{W}$-invariant bilinear forms $G^{(1)}_{\mathcal{A}}$ and $G^{(2)}_{\mathcal{A}}$ on $\Lambda^2 V$ then for the case when $\mathcal{A} = \mathcal{R}$ be a root system the scalar λ can be thought as a generalized version of the Coxeter number h

in this sense for the exterior square $\Lambda^2 V$ of the reflection representation $\mathcal{W}$ of a Coxeter group.

Chapter 3

Operations with trigonometric $\vee$-systems and solutions

In this chapter we consider the operation of taking subsystems of a trigonometric $\vee$-system and the operation of restriction of a solution of WDVV equations. We prove statements analogous to the results for the rational case. We also find trigonometric solutions of WDVV equations corresponding to root systems and their restrictions as well as some other non-Coxeter examples. We also generalize some results found in the literature corresponding to trigonometric solutions of WDVV equations.

3.1 Subsystems of trigonometric $\vee$-systems

In this section we consider subsystems of trigonometric $\vee$-systems and show that they are also trigonometric $\vee$-systems. An analogous statement for the rational case was shown in [26] (see also [24]).

A subset $\mathcal{B} \subset \mathcal{A}$ is called a *subsystem* if $\mathcal{B} = \mathcal{A} \cap W$ for some linear subspace $W \subset V^*$. The subsystem $\mathcal{B}$ is called *reducible* if $\mathcal{B}$ is a disjoint union of two non-empty subsystems, and it is called *irreducible* otherwise. Consider the following bilinear form on V associated with a subsystem $\mathcal{B}$:

$$G_{\mathcal{B}}(u,v) := \sum_{\beta \in \mathcal{B}} c_\beta \beta(u)\beta(v), \quad u, v \in V.$$

The subsystem $\mathcal{B}$ is called *isotropic* if the restriction $G_{\mathcal{B}}|_{W^\vee}$ of the form $G_{\mathcal{B}}$ onto the subspace $W^\vee \subset V$, where $W = \langle \mathcal{B} \rangle$, is degenerate and $\mathcal{B}$ is called *non-isotropic* otherwise.

Remark 3.1.1. Suppose that $\mathcal{B}$ is reducible so that $W = W_1 \oplus W_2$ for some subspaces $W_1, W_2 \subset W$ where $\langle \mathcal{B} \rangle = W$ and $\mathcal{B} \subset W_1 \cup W_2$. Then one can show that $G_{\mathcal{A}}(v_1, v_2) = G_{\mathcal{B}}(v_1, v_2) = 0$ for any $v_1 \in W_1^\vee, v_2 \in W_2^\vee$ (see Corollary 3.1.7 below).

Let us prove some lemmas which will be useful for the proof of the main theorem of this section.

Lemma 3.1.2. *Let $\mathcal{A}$ be a trigonometric $\vee$-system. Let $\mathcal{B} = \mathcal{A} \cap W$ be a subsystem of $\mathcal{A}$ for some linear subspace $W \subset V^*$ such that $W = \langle \mathcal{B} \rangle$. Consider the linear operator $M \colon V \to W^\vee$ given by*

$$M := \sum_{\beta \in \mathcal{B}} c_\beta \beta \otimes \beta^\vee, \tag{3.1}$$

that is, $M(v) = \sum_{\beta \in \mathcal{B}} c_\beta \beta(v) \beta^\vee$, for any $v \in V$. Then

1. *For any $u, v \in V$ we have $G_{\mathcal{A}}(u, M(v)) = G_{\mathcal{B}}(u, v)$.*

2. *For any $\alpha \in \mathcal{B}$, $\alpha^\vee$ is an eigenvector for M.*

3. *The space $W^\vee$ can be decomposed as a direct sum*

$$W^\vee = U_{\lambda_1} \oplus U_{\lambda_2} \oplus \cdots \oplus U_{\lambda_k}, \quad k \in \mathbb{N}, \tag{3.2}$$

where $\lambda_i \in \mathbb{C}$ are distinct, and the restriction $M|_{U_{\lambda_i}} = \lambda_i I$, where I is the identity operator.

Proof. Let $u, v \in V$. We have

$$G_{\mathcal{A}}(u, M(v)) = \sum_{\beta \in \mathcal{B}} c_\beta \beta(v) G_{\mathcal{A}}(u, \beta^\vee) = \sum_{\beta \in \mathcal{B}} c_\beta \beta(u) \beta(v) = G_{\mathcal{B}}(u, v),$$

which proves the first statement.

Let us consider a two-dimensional plane $\pi \subset V^*$ such that π contains α and another covector from $\mathcal{B}$ which is not collinear with α. Let us sum up $\vee$-conditions (2.93) over α-series which belong to the plane π. We get that

$$\sum_{\beta \in \pi \cap \mathcal{A}} c_\beta \alpha(\beta^\vee) \alpha \wedge \beta = \sum_{\beta \in \pi \cap \mathcal{B}} c_\beta \beta(\alpha^\vee) \alpha \wedge \beta = 0,$$

hence

$$\sum_{\beta \in \pi \cap \mathcal{A}} c_\beta \beta(\alpha^\vee) \beta^\vee = \lambda_\pi \alpha^\vee \tag{3.3}$$

for some $\lambda_\pi \in \mathbb{C}$. Let us now sum up relation (3.3) over all such two-dimensional planes π which contain α and another non-collinear covector from $\mathcal{B}$. It follows that $M(\alpha^\vee) = \lambda \alpha^\vee$, for some $\lambda \in \mathbb{C}$, hence property (2) holds.

The set of vectors $\{\alpha^\vee \colon \alpha \in \mathcal{B}\}$ spans $W^\vee$ since $\mathcal{B}$ spans W. As $\alpha^\vee$ is an eigenvector for $M|_{W^\vee}$ for any $\alpha \in \mathcal{B}$ we get that $M|_{W^\vee}$ is diagonalizable, and $W^\vee$ has the eigenspace decomposition as stated in (3.2). $\square$

Lemma 3.1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be as stated in Lemma 3.1.2. Suppose that $\mathcal{B}$ is non-isotropic. Then*

$$G_{\mathcal{B}}|_{U_{\lambda_i}\times V} = \lambda_i G_{\mathcal{A}}|_{U_{\lambda_i}\times V}, \tag{3.4}$$

where $\lambda_i \neq 0$ for all $i = 1, \dots, k$.

Proof. Let $u \in V$ and $v \in U_{\lambda_i}$ for some i, where U_{λ_i} is given by (3.2). Then by Lemma 3.1.2 we have

$$G_{\mathcal{A}}(u, M(v)) = \lambda_i G_{\mathcal{A}}(u, v) = G_{\mathcal{B}}(u, v).$$

Hence we have the required relation (3.4). Note that $\lambda_i \neq 0$ for all i as otherwise $G_{\mathcal{B}}|_{U_{\lambda_i}\times V} = 0$ which contradicts the non-isotropicity of $\mathcal{B}$. □

Assume that the subsystem $\mathcal{B} = \mathcal{A} \cap W$, $W = \langle \mathcal{B} \rangle$, is non-isotropic so that the bilinear form $G_{\mathcal{B}}|_{W^\vee}$ is nondegenerate. Then it establishes an isomorphism $\phi_{\mathcal{B}} \colon W^\vee \to (W^\vee)^*$. For any $\beta \in \mathcal{B}$, let us denote $\phi_{\mathcal{B}}^{-1}(\beta|_{W^\vee})$ by $\beta^{\vee_{\mathcal{B}}}$. The following lemma relates vectors $\beta^{\vee_{\mathcal{B}}}$ and $\beta^\vee$.

Lemma 3.1.4. *In the assumptions and notations of Lemmas 3.1.2 and 3.1.3 let $\beta \in \mathcal{B}$. Let $i \in \mathbb{N}$ be such that $\beta^\vee \in U_{\lambda_i}$. Then $\beta^{\vee_{\mathcal{B}}} = \lambda_i^{-1}\beta^\vee$.*

Proof. Let $u \in W^\vee$. By Lemma 3.1.3 we have $G_{\mathcal{B}}(\beta^\vee, u) = \lambda_i \beta(u)$. By the definition of $\beta^{\vee_{\mathcal{B}}}$ we have $G_{\mathcal{B}}(\beta^{\vee_{\mathcal{B}}}, u) = \beta(u)$. It follows that $G_{\mathcal{B}}(\lambda_i^{-1}\beta^\vee - \beta^{\vee_{\mathcal{B}}}, u) = 0$, which implies the statement since the form $G_{\mathcal{B}}$ is non-degenerate on $W^\vee$. □

Lemma 3.1.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be as stated in Lemma 3.1.2. Let $\alpha \in \mathcal{B}$ and let $i \in \mathbb{N}$ be such that $\alpha^\vee \in U_{\lambda_i}$. Consider an α-series $\Gamma_\alpha^{\mathcal{B}}$ in $\mathcal{B}$ and let $\beta \in \Gamma_\alpha^{\mathcal{B}}$. Then $\Gamma_\alpha^{\mathcal{B}} \subset U_{\lambda_i}$ or $\Gamma_\alpha^{\mathcal{B}} \subseteq \{\pm\beta\}$.*

Proof. Suppose firstly that $\beta^\vee \in U_{\lambda_i}$. Since any covector $\gamma \in \Gamma_\alpha^{\mathcal{B}}$ is a linear combination of β and α, we get that $\gamma \in U_{\lambda_i}$ as required.

Suppose now that $\beta^\vee \notin U_{\lambda_i}$. Then $\beta^\vee \in U_{\lambda_j}$ for some $j \neq i$. Since we have a direct sum decomposition (3.2) it follows that $\Gamma_\alpha^{\mathcal{B}} \subseteq \{\pm\beta\}$. □

Lemma 3.1.6. *Let $\mathcal{A} \subset V^*$ be a finite collection of covectors, and let $\mathcal{B} \subset \mathcal{A}$ be a subsystem. Let $\alpha, \beta \in \mathcal{B}$. Let $\Gamma_\alpha^{\mathcal{A}}, \Gamma_\alpha^{\mathcal{B}}$ be the α-series in $\mathcal{A}$ and $\mathcal{B}$ respectively containing β. Then the set $\Gamma_\alpha^{\mathcal{A}}$ coincides with the set $\Gamma_\alpha^{\mathcal{B}}$.*

Proof. Let $\gamma \in \Gamma_\alpha^{\mathcal{A}}$. It follows that $\gamma \in \mathcal{B}$. By maximality of $\Gamma_\alpha^{\mathcal{B}}$, it follows that $\gamma \in \Gamma_\alpha^{\mathcal{B}}$. Hence $\Gamma_\alpha^{\mathcal{A}} \subset \Gamma_\alpha^{\mathcal{B}}$. The opposite inclusion is obvious. □

Corollary 3.1.7. *The statement of Remark 3.1.1 holds.*

Proof. Let $\beta_1 \in \mathcal{B} \cap W_1$, $\beta_2 \in \mathcal{B} \cap W_2$. Consider a β_1-series containing β_2. It is easy to see that this series contains no other elements. Hence $G_{\mathcal{A}}(\beta_1^\vee, \beta_2^\vee) = G_{\mathcal{B}}(\beta_1^\vee, \beta_2^\vee) = 0$. □

Proposition 3.1.8. *In the assumptions and notations of Lemma 3.1.2 we have* $G_{\mathcal{B}}(u,v) = 0$ *for any* $u \in U_{\lambda_i}$ *and* $v \in U_{\lambda_j}$ *such that* $i \neq j$.

Proof. From Lemma 3.1.3 we have $G_{\mathcal{B}}(u,v) = \lambda_i G_{\mathcal{A}}(u,v) = \lambda_j G_{\mathcal{A}}(u,v)$. Hence $G_{\mathcal{A}}(u,v) = 0$, which implies the statement. □

Now we present the main theorem of this section.

Theorem 3.1.9. *Any non-isotropic subsystem of a trigonometric* $\vee$*-system is also a trigonometric* $\vee$*-system.*

Proof. Let $\mathcal{A}$ be a trigonometric $\vee$-system and let $\mathcal{B}$ be its non-isotropic subsystem. Let $\alpha \in \mathcal{B}$. Then $\alpha^\vee \in U_{\lambda_i}$ in the decomposition (3.2) for some i. Consider an α-series $\Gamma_\alpha^{\mathcal{B}}$ in $\mathcal{B}$. Let $\beta \in \Gamma_\alpha^{\mathcal{B}}$. Then by Lemma 3.1.5 we have the following two cases.

(i) Suppose $\beta^\vee \in U_{\lambda_i}$. Then $\Gamma_\alpha^{\mathcal{B}} \subset U_{\lambda_i}$ and by Lemmas 3.1.3, 3.1.4 we have

$$G_{\mathcal{B}}(\alpha^{\vee_{\mathcal{B}}}, \beta^{\vee_{\mathcal{B}}}) = \lambda_i^{-2} G_{\mathcal{B}}(\alpha^\vee, \beta^\vee) = \lambda_i^{-1} G_{\mathcal{A}}(\alpha^\vee, \beta^\vee).$$

Hence we have

$$\sum_{\beta \in \Gamma_\alpha^{\mathcal{B}}} c_\beta G_{\mathcal{B}}(\alpha^{\vee_{\mathcal{B}}}, \beta^{\vee_{\mathcal{B}}}) \alpha \wedge \beta = \lambda_i^{-1} \sum_{\beta \in \Gamma_\alpha^{\mathcal{B}}} c_\beta G_{\mathcal{A}}(\alpha^\vee, \beta^\vee) \alpha \wedge \beta = 0$$

by Lemma 3.1.2 and since $\mathcal{A}$ is a trigonometric $\vee$-system. Hence the $\vee$-condition (2.93) for $\mathcal{B}$ holds.

(ii) Suppose $\beta^\vee \in U_{\lambda j}$, where $j \neq i$. Then $G_{\mathcal{B}}(\alpha^{\vee_{\mathcal{B}}}, \beta^{\vee_{\mathcal{B}}}) = \lambda_i^{-1}\lambda_j^{-1} G_{\mathcal{B}}(\alpha^\vee, \beta^\vee) = 0$, by Proposition 3.1.8, and $\Gamma_\alpha^{\mathcal{B}} \subseteq \{\pm\beta\}$ by Lemma 3.1.5. Hence the $\vee$-condition (2.93) for $\mathcal{B}$ holds. □

3.2 Restriction of trigonometric solutions of WDVV equations

In this section we consider the restriction operation for the trigonometric solutions of WDVV equations and show that this gives new solutions of WDVV equations. An analogous statement in the rational case was established in [25].

Let

$$\mathcal{B} = \mathcal{A} \cap W \tag{3.5}$$

be a subsystem of $\mathcal{A}$ for some linear subspace $W = \langle \mathcal{B} \rangle \subset V^*$. Define

$$W_{\mathcal{B}} := \{x \in V : \beta(x) = 0 \quad \forall \beta \in \mathcal{B}\}. \tag{3.6}$$

Let us denote the restriction $\alpha|_{W_\mathcal{B}}$ of a covector $\alpha \in V^*$ as $\pi_\mathcal{B}(\alpha)$, then

$$\pi_\mathcal{B}(\mathcal{A}) = \{\pi_\mathcal{B}(\alpha) \colon \pi_\mathcal{B}(\alpha) \neq 0, \quad \alpha \in \mathcal{A} \setminus \mathcal{B}\}$$

is the set of non-zero restrictions of covectors $\alpha \in \mathcal{A}$ on $W_\mathcal{B}$. Define $M_\mathcal{B} = W_\mathcal{B} \setminus \bigcup_{\alpha \in \mathcal{A} \setminus \mathcal{B}} \Pi_\alpha$.

Consider a point $x_0 \in M_\mathcal{B}$ and tangent vectors $u_0, v_0 \in T_{x_0} M_\mathcal{B}$. We extend vectors u_0 and v_0 to two local analytic vector fields $u(x), v(x)$ in the neighbourhood U of x_0 that are tangent to the subspace $W_\mathcal{B}$ at any point $x \in W_\mathcal{B} \cap U$ such that $u_0 = u(x_0)$ and $v_0 = v(x_0)$. Consider the multiplication $*$ given by (2.63). We want to study the limit of $u(x) * v(x)$ when x tends to x_0. The limit may have singularities at $x \in W_\mathcal{B}$ as $\cot \alpha(x)$ with $\alpha \in \mathcal{B}$ is not defined for such x. Also we note that outside $W_\mathcal{B}$ we have a well-defined multiplication $u(x) * v(x)$.

The proof of the next lemma is similar to the proof of [25, Lemma 1] in the rational case (see also [2]).

Lemma 3.2.1. *The limit of the product $u(x) * v(x)$ exists when vector x tends to $x_0 \in M_\mathcal{B}$ and it satisfies*

$$u_0 * v_0 = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} c_\alpha \alpha(u_0) \alpha(v_0) (\frac{\lambda}{2} \cot \alpha(x_0) \alpha^\vee + E). \tag{3.7}$$

In particular, the product $u_0 * v_0$ is determined by vectors u_0 and v_0 only.

Proof. We are going firstly to analyse the singular part of $u(x) * v(x)$ near a generic point on the hyperplane $\Pi_\beta = \{x \in V \colon \beta(x) = 0\}$, where $\beta \in \mathcal{B}$. We choose a basis $\{f_1, ..., f_{N-1}\}$ for Π_β and we extend this basis to the basis $\{f_1, ..., f_{N-1}, f_N\}$ for V such that $\beta(f_N) = 1$. Any $x \in V$ can be represented as $x = \sum_{i=1}^{N-1} s_i f_i + t f_N$, where $(s_1, ..., s_{N-1}, t) = (s, t) \in \mathbb{C}^N$ and $t = \beta(x)$. The vector fields $u(x), v(x)$ can be represented as

$$u(x) = u(s, t) = \sum_{i=1}^{N-1} \zeta_i(s, t) f_i + a(s, t) f_N,$$

$$v(x) = v(s, t) = \sum_{i=1}^{N-1} \eta_i(s, t) f_i + b(s, t) f_N,$$

where ζ_i, η_i, a, b are some analytic functions. Note that $\beta(f_i) = 0$ for all $i = 1, ..., N-1$ since $f_i \in \Pi_\beta$.

Also since $u(x_0), v(x_0)$ are assumed to be tangential to $W_\mathcal{B}$ then for x_0 with coordinates $(s^0, 0)$, where $s^0 = (s_1^0, ..., s_{N-1}^0)$, we must have $\beta(u(x_0)) = \beta(v(x_0)) = 0$. Hence we have

$$0 = \beta(u(x_0)) = \sum_{i=1}^{N-1} \zeta_i(s^0, 0) \beta(f_i) + a(s^0, 0) \beta(f_N) = a(s^0, 0) \beta(f_N) = a(s^0, 0).$$

Similarly we have that $b(s^0, 0) = 0$. Now we have

$$\lim_{t\to 0} \beta(u)\beta(v)\cot\beta(x)\beta^\vee = \lim_{t\to 0} a(s,t)b(s,t)\beta^\vee \cot t = \lim_{t\to 0} \frac{a(s,t)b(s,t)\beta^\vee}{t}. \tag{3.8}$$

As the coefficients $a(s,t)$ and $b(s,t)$ are analytic then for $s = s^0$ we have

$$a(s^0,t) = t\tilde{a}(t),$$
$$b(s^0,t) = t\tilde{b}(t),$$

where $\tilde{a}(t), \tilde{b}(t)$ are some analytic functions. Hence (3.8) implies that

$$\lim_{t\to 0} \beta(u)\beta(v)\cot\beta(x)\beta^\vee = \lim_{t\to 0} t\tilde{a}(t)\tilde{b}(t)\beta^\vee = 0.$$

This means that $u(x) * v(x)$ is non-singular at $\beta(x) = 0$ and that β term vanishes when we calculate the product at Π_β. Lemma follows as β is an arbitrary element from $\mathcal{B}$ and $W_\mathcal{B} = \bigcap_{\gamma\in\mathcal{B}} \Pi_\gamma$. □

Now for the subsystem $\mathcal{B} \subset \mathcal{A}$ given by (3.5) let

$$S = \{\alpha_1, \dots, \alpha_k\} \subset \mathcal{B}, \tag{3.9}$$

where $k = \dim W$, be a basis of W. The following lemma shows that multiplication (3.7) is closed on the tangent space $T_*(M_\mathcal{B} \oplus U)$.

Lemma 3.2.2. *Let $\mathcal{B} \subset \mathcal{A}$ be a subsystem. Assume that prepotential (2.55) corresponding to a configuration $(\mathcal{A}, c)$ satisfies WDVV equations (2.56). Suppose that $C^{\alpha_0}_{\delta_\alpha} \neq 0$ for any $\alpha \in S, \alpha_0 \in \delta_\alpha$. If $u, v \in T_{(x,y)}(M_\mathcal{B} \oplus U)$, where $x \in W_\mathcal{B}, y \in U$, then one has $u * v \in T_{(x,y)}(M_\mathcal{B} \oplus U)$, that is*

$$* : T_{(x,y)}(M_\mathcal{B} \oplus U) \times T_{(x,y)}(M_\mathcal{B} \oplus U) \to T_{(x,y)}(M_\mathcal{B} \oplus U),$$

where multiplication $*$ is given by (3.7).

Proof. Suppose that the subspace $W_\mathcal{B}$ given by (3.6) has codimension 1 in V, and let $\alpha \in S$. We have $\mathcal{B} = \delta_\alpha$. Let $x \in M_\mathcal{B} \subset \Pi_\alpha$. Let $u, v \in T_{(x,y)}(\Pi_\alpha \oplus U)$. Then u and v can be written as $u = a_u\overline{u} + b_uE$, $v = a_v\overline{v} + b_vE$, where $\overline{u}, \overline{v} \in \Pi_\alpha$, and $a_u, b_u, a_v, b_v \in \mathbb{C}$. By Proposition 2.6.19 we have

$$\sum_{\beta\in\mathcal{A}\setminus\delta_\alpha} c_\beta G_\mathcal{A}(\alpha^\vee, \beta^\vee)\cot\beta(x)\alpha(z)\beta(\overline{u})\alpha(w)\beta(\overline{v}) = 0 \tag{3.10}$$

for any $z, w \in V$. By taking $z, w \notin \Pi_\alpha$ we derive from (3.10) that

$$\sum_{\beta\in\mathcal{A}\setminus\mathcal{B}} c_\beta\alpha(\beta^\vee)\beta(\overline{u})\beta(\overline{v})\cot\beta(x)=0,$$

which implies the statement by Lemma 3.2.1.

Let us now consider $W_\mathcal{B}$ of codimension 2. Let $S=\{\alpha_1,\alpha_2\}$. By the above arguments

$$u*v\in T_{(x,y)}(\Pi_{\alpha_i}\oplus U)$$

if $x\in\Pi_{\alpha_i}$ is generic and $u,v\in T_{(x,y)}(\Pi_{\alpha_i}\oplus U),\quad (i=1,2)$. By Lemma 3.2.1, $u*v$ exists for $x\in M_\mathcal{B}$ and hence $u*v\in T_{(x,y)}\big((\Pi_{\alpha_1}\cap\Pi_{\alpha_2})\oplus U\big)$. This proves the statement for the case when $W_\mathcal{B}$ has codimension 2. General $\mathcal{B}$ is dealt with similarly. □

Let us assume that $G_\mathcal{A}|_{W_\mathcal{B}}$ is non-degenerate. Then we have the orthogonal decomposition

$$V=W_\mathcal{B}\oplus W_\mathcal{B}^\perp.$$

Vector $\alpha^\vee\in V$ can be represented as

$$\alpha^\vee=\widetilde{\alpha^\vee}+w, \tag{3.11}$$

where $\widetilde{\alpha^\vee}\in W_\mathcal{B}$ and $w\in W_\mathcal{B}^\perp$. By Lemmas 3.2.1, 3.2.2 we have associative product

$$u*v=\sum_{\alpha\in\mathcal{A}\setminus\mathcal{B}} c_\alpha\alpha(u)\alpha(v)(\frac{\lambda}{2}\cot\alpha(x_0)\widetilde{\alpha^\vee}+E),$$

where $x_0\in M_\mathcal{B}, u,v\in W_\mathcal{B}$.

For any $\gamma\in W_\mathcal{B}^*$ we define $\gamma^{\vee_{W_\mathcal{B}}}\in W_\mathcal{B}$ by $G_\mathcal{A}(\gamma^{\vee_{W_\mathcal{B}}},v)=\gamma(v),\quad\forall v\in W_\mathcal{B}$.

Lemma 3.2.3. *Suppose that the restriction $G_\mathcal{A}|_{W_\mathcal{B}}$ is non-degenerate. Then $\widetilde{\alpha^\vee}=\pi_\mathcal{B}(\alpha)^{\vee_{W_\mathcal{B}}}$ for any $\alpha\in V^*$.*

Proof. From decomposition (3.11) we have

$$\alpha(v)=G_\mathcal{A}(\alpha^\vee,v)=G_\mathcal{A}(\widetilde{\alpha^\vee}+w,v)=G_\mathcal{A}(\widetilde{\alpha^\vee},v)$$

for any $v\in W_\mathcal{B}$. It follows that $G_\mathcal{A}(\pi_\mathcal{B}(\alpha)^{\vee_{W_\mathcal{B}}}-\widetilde{\alpha^\vee},v)=0$, which implies the statement as $G_\mathcal{A}|_{W_\mathcal{B}}$ is non-degenerate. □

Let us choose a basis in the space $W_\mathcal{B}\oplus U$ such that $f_1,\ldots,f_n$ is a basis in $W_\mathcal{B}, n=\dim W_\mathcal{B}$, and f_{n+1} is the basis vector in U, and let $\xi_1,\ldots,\xi_{n+1}$ be the corresponding coordinates. We represent vectors $\xi\in W_\mathcal{B}, y\in U$ as $\xi=(\xi_1,...,\xi_n)$ and $y=\xi_{n+1}$. The WDVV equations for a function $F\colon W_\mathcal{B}\oplus U\to\mathbb{C}$ is the following system of partial differential equations:

$$F_i F_{n+1}^{-1} F_j = F_j F_{n+1}^{-1} F_i, \quad i,j = 1, ..., n, \tag{3.12}$$

where F_i is $(n+1)\times(n+1)$ matrix with entries $(F_i)_{pq} = \frac{\partial^3 F}{\partial\xi_i \partial\xi_p \partial\xi_q}$ $(p,q = 1,\dots,n+1)$. The previous considerations lead to the following theorem.

Theorem 3.2.4. *Let $\mathcal{B}\subset\mathcal{A}$ be a subsystem, and let S be as defined in (3.9). Assume that prepotential (2.55) satisfies WDVV equations (2.56). Suppose that $C_{\delta_\alpha}^{\alpha_0} \neq 0$ for any $\alpha\in S, \alpha_0\in\delta_\alpha$. Then the prepotential*

$$F_{\mathcal{B}} = F_{\mathcal{B}}(\xi, y) = \frac{1}{3}y^3 + \sum_{\alpha\in\mathcal{A}\setminus\mathcal{B}} c_\alpha \overline{\alpha}(\xi)^2 y + \lambda \sum_{\alpha\in\mathcal{A}\setminus\mathcal{B}} c_\alpha f(\overline{\alpha}(\xi)), \quad \xi\in W_{\mathcal{B}}, y\in U\cong\mathbb{C}, \tag{3.13}$$

where $\overline{\alpha} = \pi_{\mathcal{B}}(\alpha)$, satisfies the WDVV equations (3.12). The corresponding associative multiplication has the form

$$u * v = \sum_{\alpha\in\mathcal{A}\setminus\mathcal{B}} c_\alpha \overline{\alpha}(u)\overline{\alpha}(v)(\frac{\lambda}{2}\cot\overline{\alpha}(\xi)\overline{\alpha}^{\vee_{W_{\mathcal{B}}}} + E), \tag{3.14}$$

where $\xi\in M_{\mathcal{B}}, u, v\in T_{(\xi,y)}M_{\mathcal{B}}$.

Proof. It follows by Lemmas 2.6.2, 3.2.1–3.2.3, that multiplication (3.14) is associative. The corresponding prepotential has the form (4.44) and it satisfies WDVV equations (3.12) by Lemma 2.6.2. □

In general a restriction of a root system is not a root system, so we get new solutions of WDVV equations by applying Theorem 3.2.4 in this case. In Sections 3.3, 3.4 and 3.6 we consider such solutions in more details.

3.3 BC_N type configurations

In this section we discuss a family of configurations of BC_N type and show that it gives trigonometric solutions of the WDVV equations. Let the set $\mathcal{A} = BC_N^+$ consist of the following covectors:

$$e^i, 2e^i, \quad (1\le i\le N), \quad e^i\pm e^j, (1\le i<j\le N).$$

Let us define the multiplicity function $c\colon BC_N^+\to\mathbb{C}$ by $c(e^i) = r$, $c(2e^i) = s$, $c(e^i\pm e^j) = q$, where $r,s,q\in\mathbb{C}$. We will denote the configuration (BC_N^+, c) as $BC_N^+(r,s,q)$. It is easy to check that

$$G_{\mathcal{A}}(u,v) = h\langle u, v\rangle, \quad u,v\in V, \tag{3.15}$$

where
$$h = r + 4s + 2q(N-1) \tag{3.16}$$
is assumed to be non-zero, and $\langle u, v\rangle = \sum_{i=1}^N u_i v_i$ is the standard inner product for $u = (u_1, \dots, u_N)$, $v = (v_1, \dots, v_N)$. For any $\alpha, \beta \in V^*$, $(\alpha \wedge \beta)^2 \colon V \otimes V \to \mathbb{C}$ denotes the square of the covector $\alpha \wedge \beta \in (V \otimes V)^*$.

Lemma 3.3.1. *The following two identities hold:*
$$\sum_{1\le i<j<k\le N} \Big((e^i \wedge e^j)^2 + (e^i \wedge e^k)^2 + (e^j \wedge e^k)^2 \Big) = (N-2) \sum_{1 \le i < j \le N} (e^i \wedge e^j)^2 \tag{3.17}$$
and
$$\sum_{1\le i<j<k<l\le N} \Big((e^i \wedge e^j)^2 + (e^i \wedge e^k)^2 + (e^i \wedge e^l)^2 + (e^j \wedge e^k)^2 + (e^j \wedge e^l)^2 + (e^k \wedge e^l)^2 \Big)$$
$$= \frac{1}{2}(N-2)(N-3) \sum_{1\le i<j\le N} (e^i \wedge e^j)^2. \tag{3.18}$$

Proof. Note that
$$\sum_{1\le i<j<k\le N} (e^i \wedge e^j)^2 = \sum_{1\le i<j\le N} (N-j)(e^i \wedge e^j)^2, \tag{3.19}$$
$$\sum_{1\le i<j<k\le N} (e^i \wedge e^k)^2 = \sum_{1\le i<k\le N} (k-i-1)(e^i \wedge e^k)^2, \tag{3.20}$$
and
$$\sum_{1\le i<j<k\le N} (e^j \wedge e^k)^2 = \sum_{1\le j<k\le N} (j-1)(e^j \wedge e^k)^2. \tag{3.21}$$
By adding together relations (3.19)–(3.21) we get identity (3.17).

We also have
$$\sum_{1\le i<j<k<l\le N} (e^i \wedge e^j)^2 = \sum_{1\le i<j\le N} \frac{1}{2}(N-j-1)(N-j)(e^i \wedge e^j)^2, \tag{3.22}$$
$$\sum_{1\le i<j<k<l\le N} (e^i \wedge e^k)^2 = \sum_{1\le i<k\le N} (N-k)(k-i-1)(e^i \wedge e^k)^2, \tag{3.23}$$
$$\sum_{1\le i<j<k<l\le N} (e^i \wedge e^l)^2 = \sum_{1\le i<l\le N} \frac{1}{2}(l-i-2)(l-i-1)(e^i \wedge e^l)^2, \tag{3.24}$$
$$\sum_{1\le i<j<k<l\le N} (e^j \wedge e^k)^2 = \sum_{1\le i<j\le N} (N-j)(i-1)(e^i \wedge e^j)^2, \tag{3.25}$$
$$\sum_{1\le i<j<k<l\le N} (e^j \wedge e^l)^2 = \sum_{1\le j<l\le N} (l-j-1)(j-1)(e^j \wedge e^l)^2, \tag{3.26}$$

and

$$\sum_{1\le i<j<k<l\le N} (e^k \wedge e^l)^2 = \sum_{1\le k<l\le N} \frac{1}{2}(k-2)(k-1)(e^k \wedge e^l)^2. \tag{3.27}$$

Then by adding together identities (3.22)–(3.27) we obtain identity (3.18).

□

Proposition 3.3.2. *The quadratic forms $G^{(1)}_{\mathcal{A}}, G^{(2)}_{\mathcal{A}}$ corresponding to the bilinear forms $G^{(1)}_{\mathcal{A}}(\cdot,\cdot)$, $G^{(2)}_{\mathcal{A}}(\cdot,\cdot)$ given by formulas* (2.84), (2.94) *respectively have the following forms:*

$$G^{(1)}_{\mathcal{A}} = 2h^2 \sum_{1\le i<j\le N} (e^i \wedge e^j)^2, \tag{3.28}$$

and

$$G^{(2)}_{\mathcal{A}} = 4q\big(r+8s+2(N-2)q\big)h^{-1} \sum_{1\le i<j\le N} (e^i \wedge e^j)^2, \tag{3.29}$$

where h is given by (3.16).

Proof. Let us first prove identity (3.28). Note that $G^{(1)}_{\mathcal{A}}$ is a quadratic polynomial in r, s and q. The terms containing r^2 add up to

$$2r^2 \sum_{1\le i<j\le N} (e^i \wedge e^j)^2. \tag{3.30}$$

Similarly, the terms containing s^2 add up to

$$32s^2 \sum_{1\le i<j\le N} (e^i \wedge e^j)^2. \tag{3.31}$$

The terms containing rs add up to

$$2rs \sum_{1\le i<j\le N} \Big((e^i \wedge 2e^j)^2 + (2e^i \wedge e^j)^2\Big) = 16rs \sum_{1\le i<j\le N} (e^i \wedge e^j)^2. \tag{3.32}$$

Now the terms containing rq have the form

$$\begin{aligned}
&2rq \sum_{1\leq i<j\leq N} \Big(\big(e^i \wedge (e^i + e^j)\big)^2 + \big(e^i \wedge (e^i - e^j)\big)^2 + \big(e^j \wedge (e^i + e^j)\big)^2 + \big(e^j \wedge (e^i - e^j)\big)^2 \Big) \\
&+ 2rq \sum_{1\leq i<j<k\leq N} \Big(\big(e^i \wedge (e^j + e^k)\big)^2 + \big(e^i \wedge (e^j - e^k)\big)^2 + \big(e^k \wedge (e^i + e^j)\big)^2 + \big(e^k \wedge (e^i - e^j)\big)^2 \\
&+ \big(e^j \wedge (e^i + e^k)\big)^2 + \big(e^j \wedge (e^i - e^k)\big)^2 \Big) \\
&= 8rq \sum_{1\leq i<j\leq N} (e^i \wedge e^j)^2 + 2rq \sum_{1\leq i<j<k\leq N} \Bigg(\Big(e^i \wedge e^j + e^i \wedge e^k\Big)^2 + \Big(e^i \wedge e^j - e^i \wedge e^k\Big)^2 \\
&+ \Big(e^i \wedge e^k + e^j \wedge e^k\Big)^2 + \Big(e^i \wedge e^k - e^j \wedge e^k\Big)^2 + \Big(e^i \wedge e^j - e^j \wedge e^k\Big)^2 + \Big(e^i \wedge e^j + e^j \wedge e^k\Big)^2 \Bigg) \\
&= 8rq \sum_{1\leq i<j\leq N} (e^i \wedge e^j)^2 + 8rq \sum_{1\leq i<j<k\leq N} \Big((e^i \wedge e^j)^2 + (e^i \wedge e^k)^2 + (e^j \wedge e^k)^2 \Big) \\
&= 8rq(N-1) \sum_{1\leq i<j\leq N} (e^i \wedge e^j)^2 \qquad (3.33)
\end{aligned}$$

by Lemma 3.3.1. Similarly, the terms containing sq add up to

$$32sq(N-1) \sum_{1\leq i<j\leq N} (e^i \wedge e^j)^2. \qquad (3.34)$$

The terms containing q^2 have the form

$$\begin{aligned}
&2q^2 \sum_{1\leq i<j\leq N} \big((e^i + e^j) \wedge (e^i - e^j)^2\big) \\
&+ 2q^2 \sum_{1\leq i<j<k\leq N} \Big(\big((e^i + e^j) \wedge (e^i + e^k)\big)^2 + \big((e^i + e^j) \wedge (e^i - e^k)\big)^2 + \big((e^i - e^j) \wedge (e^i + e^k)\big)^2 \\
&+ \big((e^i - e^j) \wedge (e^i - e^k)\big)^2 + \big((e^i + e^j) \wedge (e^j + e^k)\big)^2 + \big((e^i + e^j) \wedge (e^j - e^k)\big)^2 \\
&+ \big((e^i - e^j) \wedge (e^j + e^k)\big)^2 + \big((e^i - e^j) \wedge (e^j - e^k)\big)^2 + \big((e^i + e^k) \wedge (e^j + e^k)\big)^2 \\
&+ \big((e^i + e^k) \wedge (e^j - e^k)\big)^2 + \big((e^i - e^k) \wedge (e^j + e^k\big)^2 + \big((e^i - e^k) \wedge (e^j - e^k\big)^2 \Big) \\
&+ 2q^2 \sum_{1\leq i<j<k<l\leq N} \Big(\big((e^i + e^j) \wedge (e^k + e^l)\big)^2 + \big((e^i + e^j) \wedge (e^k - e^l)\big)^2 \\
&+ \big((e^i - e^j) \wedge (e^k + e^l)\big)^2 + \big((e^i - e^j) \wedge (e^k - e^l)\big)^2 + \big((e^i + e^k) \wedge (e^j + e^l)\big)^2 \\
&+ \big((e^i + e^k) \wedge (e^j - e^l)\big)^2 + \big((e^i - e^k) \wedge (e^j + e^l)\big)^2 + \big((e^i - e^k) \wedge (e^j - e^l)\big)^2 \\
&+ \big((e^i + e^l) \wedge (e^j + e^k)\big)^2 + \big((e^i + e^l) \wedge (e^j - e^k)\big)^2 + \big((e^i - e^l) \wedge (e^j + e^k)\big)^2 \\
&+ \big((e^i - e^l) \wedge (e^j - e^k)\big)^2 \Big). \qquad (3.35)
\end{aligned}$$

Expression (3.35) is equal to

$$\begin{aligned}
&8q^2 \sum_{1\le i<j\le N} (e^i \wedge e^j)^2 + 24q^2 \sum_{1\le i<j<k\le N} \Big((e^i \wedge e^j)^2 + (e^i \wedge e^k)^2 + (e^j \wedge e^k)^2\Big) \\
&+ 16q^2 \sum_{1\le i<j<k<l\le N} \Big((e^i \wedge e^j)^2 + (e^i \wedge e^k)^2 + (e^i \wedge e^l)^2 + (e^j \wedge e^k)^2 + (e^j \wedge e^l)^2 + (e^k \wedge e^l)^2\Big) \\
&= 8q^2(N-1)^2 \sum_{1\le i<j\le N} (e^i \wedge e^j)^2 \qquad (3.36)
\end{aligned}$$

by Lemma 3.3.1. By adding together expressions (3.30)–(3.34) and (3.36) we get identity (3.28).

Let us now prove identity (3.29). Note that $hG^{(2)}_{\mathcal{A}}$ is a quadratic polynomial in r, s and q and that terms containing r^2, rs and s^2 all vanish. Terms containing rq in $hG^{(2)}_{\mathcal{A}}$ are given by

$$\begin{aligned}
&2rq \sum_{1\le i<j\le N} \Big(e^i(e^i+e^j)^\vee\big(e^i \wedge (e^i+e^j)\big)^2 + e^i(e^i-e^j)^\vee\big(e^i \wedge (e^i-e^j)\big)^2 \\
&+ e^j(e^i+e^j)^\vee\big(e^j \wedge (e^i+e^j)\big)^2 + e^j(e^i-e^j)^\vee\big(e^j \wedge (e^i-e^j)\big)^2\Big) = 4rq \sum_{1\le i<j\le N} (e^i \wedge e^j)^2. \\
&\qquad (3.37)
\end{aligned}$$

Similarly, the terms containing sq in $hG^{(2)}_{\mathcal{A}}$ add up to

$$32sq \sum_{1\le i<j\le N} (e^i \wedge e^j)^2. \qquad (3.38)$$

Finally, the terms containing q^2 in $hG^{(2)}_{\mathcal{A}}$ are given by

$$\begin{aligned}
&2q^2 \sum_{1\le i<j<k\le N} \Big((e^i+e^j)\big((e^i+e^k)^\vee\big)\big((e^i+e^j) \wedge (e^i+e^k)\big)^2 \\
&+ (e^i+e^j)\big((e^i-e^k)^\vee\big)\big((e^i+e^j) \wedge (e^i-e^k)\big)^2 + (e^i-e^j)\big((e^i+e^k)^\vee\big)\big((e^i-e^j) \wedge (e^i+e^k)\big)^2 \\
&+ (e^i-e^j)\big((e^i-e^k)^\vee\big)\big((e^i-e^j) \wedge (e^i-e^k)\big)^2 + (e^i+e^j)\big((e^j+e^k)^\vee\big)\big((e^i+e^j) \wedge (e^j+e^k)\big)^2 \\
&+ (e^i+e^j)\big((e^j-e^k)^\vee\big)\big((e^i+e^j) \wedge (e^j-e^k)\big)^2 + (e^i-e^j)\big((e^j+e^k)^\vee\big)\big((e^i-e^j) \wedge (e^j+e^k)\big)^2 \\
&+ (e^i-e^j)\big((e^j-e^k)^\vee\big)\big((e^i-e^j) \wedge (e^j-e^k)\big)^2 + (e^i+e^k)\big((e^j+e^k)^\vee\big)\big((e^i+e^k) \wedge (e^j+e^k)\big)^2 \\
&+ (e^i+e^k)\big((e^j-e^k)^\vee\big)\big((e^i+e^k) \wedge (e^j-e^k)\big)^2 + (e^i-e^k)\big((e^j+e^k)^\vee\big)\big((e^i-e^k) \wedge (e^j+e^k)\big)^2 \\
&+ (e^i-e^k)\big((e^j-e^k)^\vee\big)\big((e^i-e^k) \wedge (e^j-e^k)\big)^2\Big). \qquad (3.39)
\end{aligned}$$

Expression (3.39) is equal to

$$2q^2 \sum_{1\le i<j<k\le N} \Big(\big(e^i\wedge e^k - e^i\wedge e^j + e^j\wedge e^k\big)^2 + \big(e^i\wedge e^k + e^i\wedge e^j - e^j\wedge e^k\big)^2$$
$$+ \big(e^i\wedge e^k - e^i\wedge e^j - e^j\wedge e^k\big)^2 + \big(e^i\wedge e^j + e^i\wedge e^k + e^j\wedge e^k\big)^2 \Big)$$
$$= 8q^2 \sum_{1\le i<j<k\le N} \Big((e^i\wedge e^j)^2 + (e^i\wedge e^k)^2 + (e^j\wedge e^k)^2 \Big) = 8q^2(N-2) \sum_{1\le i<j\le N} (e^i\wedge e^j)^2 \tag{3.40}$$

by Lemma 3.3.1. By adding together expressions (3.37), (3.38) and (3.40) we get identity (3.29). □

The previous proposition allows us to prove the following theorem.

Theorem 3.3.3. *Prepotential (2.55) for the configuration* $(\mathcal{A}, c) = BC_N^+(r,s,q)$ *satisfies WDVV equations (2.56) with*

$$\lambda = \Big(\frac{2h^3}{q\big(r+8s+2(N-2)q\big)} \Big)^{1/2}, \tag{3.41}$$

where h *is given by (3.16), provided that* $q(r+8s+2(N-2)q) \neq 0$.

Proof. Firstly, $BC_N^+(r,s,q)$ is a trigonometric $\vee$-system by Proposition 2.7.2. Secondly, by Proposition 3.3.2 we have that $G_{\mathcal{A}}^{(1)} - \frac{\lambda^2}{4} G_{\mathcal{A}}^{(2)} = 0$ if λ is given by (3.41). The statement follows by Theorem 2.6.21. □

Theorem 3.3.3 gives a generalization of the results in [33], [42], [11] and [47], where, in particular, solutions of the WDVV equations for the root systems D_N, B_N and C_N were obtained (see Section 2.7). Following [33], [42] consider the function $\widetilde{F}$ of $N+1$ variables $(x_1, \ldots, x_N, y)$ of the form

$$\widetilde{F}(x,y) = \frac{\gamma}{6} y^3 + \frac{\gamma}{2} y \langle x, x\rangle + \sum_{\alpha\in\mathcal{R}^+} c_\alpha \widetilde{f}(\alpha(x)), \tag{3.42}$$

where $\mathcal{R}^+$ is a positive half of the root system $\mathcal{R}$, multiplicities c_α are invariant under the Weyl group, $\gamma \in \mathbb{C}$ and function $\widetilde{f}$ given by (2.100). Note that $\widetilde{f}(z) = -f(-iz)$.

Let us explain that our solution (2.55) for the configuration $BC_N^+(r,s,q)$ leads to a solution of the form (3.42).

Proposition 3.3.4. *Function* $\widetilde{F}$ *given by (3.42) with* $\mathcal{R}^+ = BC_N^+$ *satisfies WDVV equations (2.56) if*

$$\gamma^2 = -2q(r+8s+2(N-2)q). \tag{3.43}$$

Proof. By formula (3.15) solution F given by (2.55) for $\mathcal{A} = BC_N^+$ has the form

$$F(\widetilde{x}, \widetilde{y}) = \frac{1}{3}\widetilde{y}^3 + h\widetilde{y}\sum_{i=1}^{N} \widetilde{x}_i^2 + \lambda \sum_{\alpha \in BC_N^+} c_\alpha f(\alpha(\widetilde{x})), \tag{3.44}$$

where we redenoted variables (x, y) by $(\widetilde{x}, \widetilde{y})$. By changing variables $\widetilde{x} = -ix$, $\widetilde{y} = \frac{\gamma\lambda}{2h}y$ and dividing F by $-\lambda$ solution (3.44) takes the form (3.42) provided that $\gamma^2\lambda^2 = -4h^3$ which is satisfied for γ given by (3.43). $\square$

Let $n \in \mathbb{N}$ and let $\underline{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$ be such that

$$\sum_{i=1}^{n} m_i = N. \tag{3.45}$$

Let us consider the subsystem $\mathcal{B} \subset \mathcal{A} = BC_N^+$ given by

$$\mathcal{B} = \{e^{\sum_{j=1}^{i-1} m_j + k} - e^{\sum_{j=1}^{i-1} m_j + l}, \quad 1 \leq k < l \leq m_i, \quad i = 1, \dots, n\}.$$

Let us also consider the corresponding subspace $W_\mathcal{B} = \{x \in V \colon \beta(x) = 0, \forall \beta \in \mathcal{B}\}$. It can be given explicitly by the equations

$$\begin{cases} x_1 = \dots = x_{m_1} = \xi_1, \\ x_{m_1+1} = \dots = x_{m_1+m_2} = \xi_2, \\ \vdots \\ x_{\sum_{i=1}^{n-1} m_i + 1} = \dots = x_N = \xi_n, \end{cases}$$

where $\xi_1, \dots, \xi_n$ are coordinates on $W_\mathcal{B}$. Let us now restrict the configuration $BC_N^+(r, s, q)$ to the subspace $W_\mathcal{B}$. That is we consider non-zero restricted covectors $\overline{\alpha} = \pi_\mathcal{B}(\alpha), \alpha \in BC_N^+$ with multiplicities c_α, and we add up multiplicities if the same covector on $W_\mathcal{B}$ is obtained a few times. Let us denote the resulting configuration as $BC_n(q, r, s; \underline{m})$. It is easy to see that it consists of covectors

$$\begin{aligned} &f^i, \quad \text{with multiplicity} \quad rm_i, \quad 1 \leq i \leq n, \\ &2f^i, \quad \text{with multiplicity} \quad sm_i + \frac{1}{2}qm_i(m_i - 1), \quad 1 \leq i \leq n, \\ &f^i \pm f^j, \quad \text{with multiplicity} \quad qm_im_j, \quad 1 \leq i < j \leq n, \end{aligned}$$

where $f^1, \dots, f^n$ is the basis in $W_\mathcal{B}^*$ corresponding to coordinates $\xi_1, \dots, \xi_n$.

As a corollary of Theorem 3.2.4 and Theorem 3.3.3 we get the following result on $(n + 3)$-parametric family of solutions of WDVV equations, which can be specialized to

$(n+1)$-parametric family of solutions from [43].

Theorem 3.3.5. *Let $\xi = (\xi_1, \ldots, \xi_n) \in W_{\mathcal{B}}, y \in U \cong \mathbb{C}$. Assume that parameters r, q, s and $\underline{m}$ satisfy the relation $r + 4s + 2q(m_i - 1) \neq 0$ for any $1 \leq i \leq n$. Then function*

$$F_{\mathcal{B}}(\xi, y) = \frac{1}{3}y^3 + \big(r + 4s + 2q(N-1)\big)y\sum_{i=1}^{n} m_i\xi_i^2 + \lambda r\sum_{i=1}^{n} m_i f(\xi_i)$$
$$+ \lambda\sum_{i=1}^{n}\big(sm_i + \frac{1}{2}qm_i(m_i-1)\big)f(2\xi_i) + \lambda q\sum_{i<j} m_i m_j f(\xi_i \pm \xi_j), \tag{3.46}$$

where N is given by (3.45), satisfies the WDVV equations (3.12) if $\lambda = \left(\frac{2h^3}{q\left(r+8s+2(N-2)q\right)}\right)^{1/2}$, where h is given by (3.16)*, and* $\big(r + 8s + 2(N-2)q\big)q \neq 0$.

Proof. We only have to check that cubic terms in (3.46) have the required form. For any $\xi \in W_{\mathcal{B}}$ we have

$$\sum_{\alpha \in BC_N^+} c_\alpha \overline{\alpha}(\xi)^2 = r\sum_{i=1}^{n} m_i\xi_i^2 + 4\sum_{i=1}^{n}\big(sm_i + \frac{1}{2}qm_i(m_i-1)\big)\xi_i^2$$
$$+ 2q\sum_{1\leq i<j\leq n} m_i m_j(\xi_i^2 + \xi_j^2). \tag{3.47}$$

Note that

$$\sum_{1\leq i<j\leq n} m_i m_j(\xi_i^2+\xi_j^2) = \frac{1}{2}\sum_{i,j=1}^{n} m_i m_j(\xi_i^2+\xi_j^2) - \sum_{i=1}^{n} m_i^2\xi_i^2 = \sum_{i=1}^{n}(N-m_i)m_i\xi_i^2$$

by formula (3.45). Hence (3.47) becomes

$$\sum_{\alpha \in BC_N^+} c_\alpha \overline{\alpha}(\xi)^2 = \big(r + 4s + 2q(N-1)\big)\sum_{i=1}^{n} m_i\xi_i^2$$

as required. □

Solution (3.46) gives a generalization of the results in [43]. Let us recall the following solution of WDVV equations from that paper. Consider the function Φ_{BC_n} of $n+1$

variables $(x_1, \dots, x_n, y)$ of the form

$$\begin{aligned}\Phi_{BC_n} &= \frac{\mu^2}{6}y^3 + \frac{\mu}{2}y\sum_{i=1}^n \varepsilon_i x_i^2 + \sum_{i=1}^n \frac{\varepsilon_i(1+\varepsilon_i)}{32}\Big(Li_3(e^{4x_i}) + Li_3(e^{-4x_i})\Big)\\ &- \frac{2+\mu+\sum_{k=1}^n \varepsilon_k}{8}\sum_{i=1}^n \varepsilon_i\Big(Li_3(e^{2x_i}) + Li_3(e^{-2x_i})\Big)\\ &+ \frac{1}{16}\sum_{i<j}\varepsilon_i\varepsilon_j\Big(Li_3(e^{2(x_i+x_j)}) + Li_3(e^{-2(x_i+x_j)}) + Li_3(e^{2(x_i-x_j)}) + Li_3(e^{-2(x_i-x_j)})\Big),\end{aligned} \tag{3.48}$$

where $\mu \neq 0$, $\mu, \varepsilon_i \in \mathbb{C}, i = 1, \dots, n$. It is stated in [43] that Φ_{BC_n} satisfies WDVV equations.

Proposition 3.3.6. *Assume that $\mu \neq -1$. Then solution (3.46) for the configuration $BC_n(q,r,s;\underline{m})$ reduces to solution (3.48) if one specifies parameters*

$$\begin{aligned}m_i &= \frac{2s-q}{q}\varepsilon_i, \quad r = \frac{q^{1/2}\Big(\sqrt{2}(2+\mu) - 2N(q\mu)^{1/2}(1+\mu)^{1/2}\Big)}{\big(\mu(1+\mu)\big)^{1/2}},\\ s &= \frac{2q\mu(1+\mu) + \Big(2q\mu(1+\mu)\Big)^{1/2}}{4\mu(1+\mu)}, \quad q \in \mathbb{C}.\end{aligned} \tag{3.49}$$

Proof. First, solution (3.46) has the general formula (where we redenoted the varible ξ by $\widetilde{x}$)

$$F = \frac{1}{3}y^3 + \big(r + 4s + 2q(N-1)\big)y\sum_{i=1}^n m_i\widetilde{x}_i^2 + \frac{\lambda}{2}\sum_{\alpha\in BC_n(q,r,s;\underline{m})} c_\alpha f(\alpha(\widetilde{x})),$$

which takes the following form after making the change of variable $\widetilde{x} = -ix$

$$F = \frac{1}{3}y^3 - \big(r + 4s + 2q(N-1)\big)y\sum_{i=1}^n m_i x_i^2 - \frac{\lambda}{2}\sum_{\alpha\in BC_n(q,r,s;\underline{m})} c_\alpha \widetilde{f}(\alpha(x)), \tag{3.50}$$

where $\widetilde{f}'''(z) = \coth z$. Note that

$$\widetilde{f}(\alpha(x)) + \widetilde{f}(-\alpha(x)) = -\frac{1}{4}\Big(Li_3(e^{2\alpha(x)}) + Li_3(e^{-2\alpha(x)})\Big).$$

Hence solution (3.50) takes the form

$$\begin{aligned}F = &\frac{1}{3}y^3 - \big(r+4s+2q(N-1)\big)y\sum_{i=1}^{n} m_i x_i^2 + \frac{\lambda}{8}\sum_{i=1}^{n} rm_i\Big(Li_3(e^{2x_i})+Li_3(e^{-2x_i})\Big)\\ &+\frac{\lambda}{8}\sum_{i=1}^{n}\Big(sm_i+\frac{1}{2}qm_i(m_i-1)\Big)\Big(Li_3(e^{4x_i})+Li_3(e^{-4x_i})\Big)\\ &+\frac{\lambda}{8}\sum_{i=1}^{n} qm_im_j\Big(Li_3(e^{2(x_i+x_j)})+Li_3(e^{-2(x_i+x_j)})+Li_3(e^{2(x_i-x_j)})+Li_3(e^{-2(x_i-x_j)})\Big).\end{aligned} \tag{3.51}$$

By dividing solution (3.51) by 2 and dividing solution (3.48) by μ^2 and comparing the corresponding coefficients in both solutions, we get the following set of equations

$$\begin{aligned}\big(r+4s+2q(N-1)\big)m_i &= -\frac{1}{\mu}\varepsilon_i, \quad & \lambda m_i\big(2s-q+qm_i\big) &= \frac{\varepsilon_i(1+\varepsilon_i)}{\mu^2}\\ \lambda rm_i &= -2\Big(\frac{2+\mu+\sum_{k=1}^n\varepsilon_k}{\mu^2}\Big)\varepsilon_i, \quad & \lambda qm_im_j &= \frac{\varepsilon_i\varepsilon_j}{\mu^2},\end{aligned}$$

which are satisfied for the given values (3.49). □

Note that the case $\mu = -1$ is not covered by the generalization given in the Proposition 3.3.6 since parameters r, s, λ are not defined. Let us also recall another solution from [43] which has the form

$$\begin{aligned}\Phi_{C_n} = &\frac{\mu^2}{6}y^3 + \frac{\mu}{2}y\sum_{i=1}^{n}\varepsilon_i x_i^2 + \sum_{i=1}^{n}\frac{\varepsilon_i(\sum_{k=1}^n\varepsilon_k - 2\varepsilon_i)}{8}\Big(Li_3(e^{2x_i})+Li_3(e^{-2x_i})\Big)\\ &-\frac{1}{2}\sum_{i<j}\varepsilon_i\varepsilon_j\Big(Li_3(e^{(x_i+x_j)})+Li_3(e^{-(x_i+x_j)})+Li_3(e^{(x_i-x_j)})+Li_3(e^{-(x_i-x_j)})\Big),\end{aligned} \tag{3.52}$$

where $\mu \neq 0$, $\mu, \varepsilon_i \in \mathbb{C}, i = 1, \ldots, n$.

Proposition 3.3.7. *Solution (3.46) for the configuration $BC_n(q, r, s; \underline{m})$ reduces to solution (3.52) if one specialized parameters*

$$m_i = \varepsilon_i, \quad r = 0, \quad q = \frac{4}{\mu(\mu-N)}, \quad s = \frac{\mu+N-2}{\mu(\mu-N)}, \tag{3.53}$$

where $N = \sum_{i=1}^n m_i$.

Proof. First, solution (3.46) with $r = 0$ has the general form

$$F = \frac{1}{3}y^3 + 2\big(2s+q(N-1)\big)y\sum_{i=1}^{n} m_i\widetilde{x}_i^2 + \frac{\lambda}{2}\sum_{\alpha\in C_n} c_\alpha f(\alpha(\widetilde{x})),$$

where we redenoted the variables ξ_i by $\widetilde{x}_i$. It takes the following form after making the change of variable $\widetilde{x} = -i\frac{x}{2}$:

$$F = \frac{1}{3}y^3 - \frac{1}{2}\big(2s + q(N-1)\big)y\sum_{i=1}^{n} m_i x_i^2 - \frac{\lambda}{2}\sum_{\alpha\in C_n} c_\alpha \widetilde{f}(\alpha(\frac{x}{2})), \tag{3.54}$$

where $\widetilde{f}'''(z) = \coth z$. Similarly to the proof of Proposition 3.3.6, solution (3.54) then takes the form

$$\begin{aligned} F &= \frac{1}{3}y^3 - \frac{1}{2}\big(2s + q(N-1)\big)y\sum_{i=1}^{n} m_i x_i^2 \\ &+ \frac{\lambda}{8}\sum_{i=1}^{n}\Big(sm_i + \frac{1}{2}qm_i(m_i-1)\Big)\Big(Li_3(e^{2x_i}) + Li_3(e^{-2x_i})\Big) \\ &+ \frac{\lambda}{8}\sum_{i=1}^{n} qm_i m_j\Big(Li_3(e^{(x_i+x_j)}) + Li_3(e^{-(x_i+x_j)}) + Li_3(e^{(x_i-x_j)}) + Li_3(e^{-(x_i-x_j)})\Big). \end{aligned} \tag{3.55}$$

By dividing solution (3.55) by 2 and dividing solution (3.52) by μ^2 and comparing the corresponding coefficients in both solutions, we get the following set of equations

$$\begin{aligned} \big(2s + q(N-1)\big)m_i &= -\frac{2}{\mu}\varepsilon_i, \quad \lambda q m_i m_j = -\frac{8\varepsilon_i\varepsilon_j}{\mu^2}, \\ \lambda m_i\big(2s - q + qm_i\big) &= 4\Big(\frac{\mu - 2\varepsilon_i + \sum_{k=1}^{n}\varepsilon_k}{\mu^2}\Big)\varepsilon_i, \end{aligned}$$

which are satisfied for the given values (3.53). □

3.4 A_N type configurations

In this section we discuss a family of configurations of type A_N and show that it gives trigonometric solutions of the WDVV equations.

Let $V \subset \mathbb{C}^{N+1}$ be the hyperplane $V = \{(x_1, \dots, x_{n+1})\colon \sum_{i=1}^{N+1} x_i = 0\}$. Let $\mathcal{A} = A_N^+$ be the positive half of the root system A_N given by

$$\mathcal{A} = \{e^i - e^j, \quad 1 \le i < j \le N+1\}.$$

Let $t = c(e^i - e^j) \in \mathbb{C}$ be the constant multiplicity. The following lemma gives the relation between covectors in $\mathcal{A}$ and their dual vectors in V.

Lemma 3.4.1. *We have*

$$(e^i - e^j)^\vee = \frac{1}{t(N+1)}(e_i - e_j), \quad 1 \le i, j \le N+1.$$

Proof. Let $x = (x_1, \dots, x_{N+1}), y = (y_1, \dots, y_{N+1}) \in V$. Then the bilinear form $G_{\mathcal{A}}$ takes the form

$$G_{\mathcal{A}}(x, y) = t \sum_{i<j}^{N+1} (x_i - x_j)(y_i - y_j) = t(N+1) \sum_{i=1}^{N+1} x_i y_i,$$

which implies the statement. □

Now we can find the forms $G_{\mathcal{A}}^{(i)}, i = 1, 2$.

Proposition 3.4.2. *The quadratic forms $G_{\mathcal{A}}^{(1)}, G_{\mathcal{A}}^{(2)}$ corresponding to the bilinear forms $G_{\mathcal{A}}^{(1)}(\cdot, \cdot)$, $G_{\mathcal{A}}^{(2)}(\cdot, \cdot)$ respectively have the following forms:*

$$G_{\mathcal{A}}^{(1)} = (N+1)^2 t^2 \sum_{i,j=1}^{N+1} (e^i \wedge e^j)^2,$$

and

$$G_{\mathcal{A}}^{(2)} = t \sum_{i,j=1}^{N+1} (e^i \wedge e^j)^2. \tag{3.56}$$

Proof. For the first equality we have

$$\begin{aligned} G_{\mathcal{A}}^{(1)} &= t^2 \sum_{i<j}^{N+1} \sum_{k<l}^{N+1} \Big((e^i - e^j) \wedge (e^k - e^l) \Big)^2 \\ &= \frac{t^2}{4} \sum_{i,j,k,l=1}^{N+1} \Big((e^i \wedge e^k) - (e^i \wedge e^l) - (e^j \wedge e^k) + (e^j \wedge e^l) \Big)^2 \\ &= t^2 (N+1)^2 \sum_{i,j=1}^{N+1} (e^i \wedge e^j)^2 \end{aligned}$$

since $\sum_{i=1}^{N+1} e^i|_V = 0$. For equality (3.56) we have by Lemma 3.4.1 that

$$\begin{aligned} G_{\mathcal{A}}^{(2)} = 2t^2 \sum_{1 \le i<j<k \le N+1} \Bigg(& (e^i - e^j)\big((e^i - e^k)^\vee\big)\big((e^i - e^j) \wedge (e^i - e^k)\big)^2 \\ &+ (e^i - e^j)\big((e^j - e^k)^\vee\big)\big((e^i - e^j) \wedge (e^j - e^k)\big)^2 + (e^i - e^k)\big((e^j - e^k)^\vee\big)\big((e^i - e^k) \wedge (e^j - e^k)\big)^2 \Bigg) \\ = \frac{2t}{N+1} & \sum_{1 \le i<j<k \le N+1} (e^i \wedge e^j - e^i \wedge e^k + e^j \wedge e^k)^2. \end{aligned} \tag{3.57}$$

Note that

$$\sum_{i,j,k=1}^{N+1} (e^i \wedge e^j - e^i \wedge e^k + e^j \wedge e^k)^2 = 3(N+1) \sum_{i,j=1}^{N+1} (e^i \wedge e^j)^2 \tag{3.58}$$

since $\sum_{i=1}^{N+1} e^i|_V = 0$. Also it is easy to see that

$$\sum_{1\le i<j<k\le N+1} (e^i\wedge e^j - e^i\wedge e^k + e^j\wedge e^k)^2 = \frac{1}{6}\sum_{i,j,k=1}^{N+1} (e^i\wedge e^j - e^i\wedge e^k + e^j\wedge e^k)^2. \quad (3.59)$$

Equality (3.56) follows from formulas (3.57)–(3.59). □

This leads us to the following result which can also be extracted from [42].

Theorem 3.4.3. (cf. [42]) *Prepotential (2.55), where $y = \sum_{i=1}^{N+1} x_i$, for the configuration $(\mathcal{A}, c) = (A_N^+, t)$ satisfies WDVV equations*

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i, \quad i,j,k = 1,\dots,N+1,$$

where $(F_i)_{pq} = \frac{\partial^3 F}{\partial x_i \partial x_p \partial x_q}$, $(p,q = 1,\dots,N+1)$, with

$$\lambda = 2(N+1)\sqrt{t}. \quad (3.60)$$

Proof. Firstly, $\mathcal{A}$ is a trigonometric $\vee$-system by Proposition 2.7.2. Secondly, by Proposition 3.4.2 we have that

$$G_{\mathcal{A}}^{(1)} - \frac{\lambda^2}{4} G_{\mathcal{A}}^{(2)} = \Big((N+1)^2 t - \frac{\lambda^2}{4}\Big) t \sum_{i,j=1}^{N+1} (e^i\wedge e^j)^2,$$

which is equal to 0 for λ given by (3.60). It follows by Theorem 2.6.21 that F satisfies WDVV equations (2.56) as a function on the hyperplane $V \subset \mathbb{C}^{N+1}$ which also depends on the auxiliary variable y. Now we change variables to $(x_1,\dots,x_{N+1})$ by putting $y = \sum_{i=1}^{N+1} x_i$, which implies the statement. □

Theorem 3.4.3 gives the value of the scalar λ for the general root system of type A_N for arbitrary multiplicity of its vectors. This result matches the result from [33], [42] where, in particular, solutions for the root systems A_N (with multiplicity equal to 1 for its vectors) were obtained. Following [33], [42], consider the function $\widetilde{F}$ given by (3.42). We will explain that our solution (2.55) for the configuration (A_N^+, t) leads to a solution of the form (3.42). The following lemmas will be used.

Lemma 3.4.4. *We have*

$$\sum_{1\le i<j\le N+1} (x_i^2 + x_j^2) = N \sum_{i=1}^{N+1} x_i^2.$$

Proof.

$$\sum_{1\le i<j\le N+1}(x_i^2+x_j^2)=\frac{1}{2}\left(\sum_{i,j=1}^{N+1}(x_i^2+x_j^2)-2\sum_{i=1}^{N+1}x_i^2\right)=\frac{1}{2}\left(2(N+1)\sum_{i=1}^{N+1}(x_i^2)-2\sum_{i=1}^{N+1}x_i^2\right),$$

which implies the required relation. □

Lemma 3.4.5. *We have*

$$\sum_{\alpha\in A_N^+}\alpha(x)^2=(N+1)\sum_{i=1}^{N+1}x_i^2-y^2, \tag{3.61}$$

where $y=\sum_{i=1}^{N+1}x_i$.

Proof. Let $x\in\mathbb{C}^{N+1}$. Then we have

$$\begin{aligned}\sum_{\alpha\in A_N^+}\alpha(x)^2&=\sum_{1\le i<j\le N+1}(x_i-x_j)^2=\sum_{1\le i<j\le N+1}(x_i^2+x_j^2-2x_ix_j)\\&=N\sum_{1\le i<j\le N+1}x_i^2-2\sum_{1\le i<j\le N+1}x_ix_j\end{aligned} \tag{3.62}$$

by Lemma 3.4.4. Note that

$$\begin{aligned}2\sum_{1\le i<j\le N+1}x_ix_j&=\sum_{i,j=1}^{N+1}x_ix_j-\sum_{i=1}^{N+1}x_i^2=\sum_{i=1}^{N+1}x_i\sum_{j=1}^{N+1}x_j-\sum_{i=1}^{N+1}x_i^2\\&=\Big(\sum_{i=1}^{N+1}x_i\Big)^2-\sum_{i=1}^{N+1}x_i^2=y^2-\sum_{i=1}^{N+1}x_i^2.\end{aligned}$$

Hence, the lemma follows. □

As we have seen in Section 2.7 that solutions of WDVV equations corresponding to root systems were obtained by Hoevenaars and Martini. Let us recall their solution for type A_N root system and show that our solution leads to such a solution.

Proposition 3.4.6. (cf.[33], [42]) *Function* $\widetilde{F}$ *given by*

$$\begin{aligned}\widetilde{F}(x,y)&=\frac{\gamma}{6}y^3+\frac{\gamma}{2}y\Big(\sum_{i=1}^{N}x_i^2-\frac{y^2}{N+1}\Big)+\sum_{\alpha\in A_N^+}\widetilde{f}(\alpha(x))\\&=\frac{N-2}{6(N+1)}\gamma y^3+\frac{\gamma}{2}y\sum_{i=1}^{N}x_i^2+\sum_{\alpha\in A_N^+}\widetilde{f}(\alpha(x)),\end{aligned} \tag{3.63}$$

where the variable y *is defined by* $y=\sum_{i=1}^{N+1}x_i$, *satisfies WDVV equations (2.56) for root system* A_N *and the value of* γ *is given by* $-\gamma^2=N+1$.

Proof. Let $c_\alpha = t = 1$. Then by Lemma 3.4.5 solution (2.55) has the form

$$F(\widetilde{x},\widetilde{y}) = \frac{-2}{3}\widetilde{y}^3 + (N+1)\widetilde{y}\sum_{i=1}^{N+1}\widetilde{x}^2 + \lambda\sum_{\alpha\in A_N^+} f(\alpha(\widetilde{x})), \tag{3.64}$$

where we redenoted variables (x,y) by $(\widetilde{x},\widetilde{y})$. By changing the variable $\widetilde{x} = -i\widehat{x}$, one accordingly has $\widetilde{y} = -i\widehat{y}$ where $\widehat{y} = \sum_{i=1}^{N+1}\widehat{x}_i$. Then by dividing the resulting solution by $-\lambda$ and calling it $\widehat{F}$ we get

$$\widehat{F} = \frac{2i}{3\lambda}\widehat{y}^3 - \frac{(N+1)i}{\lambda}\widehat{y}\sum_{i=1}^{N+1}\widehat{x}_i^{\,2} + \sum_{\alpha\in A_N^+}\widetilde{f}(\alpha(\widehat{x})). \tag{3.65}$$

Now let us replace variable $\widehat{x}$ with $\widehat{x}_i = x_i + Sy$, where $S = \frac{-\lambda\gamma i - 2(N+1)}{2(N+1)^2}$. Accordingly one can show that change of variable for $\widehat{y}$ takes the form $\widehat{y} = \frac{\gamma\lambda}{2(N+1)}iy$. Hence solution (3.65) takes the form

$$\Big(\frac{\lambda^2\gamma^2 + 6(N+1)^3\big((N+1)s^2+2s\big)}{12(N+1)^3}\Big)\gamma y^3 + \frac{\gamma}{2}y\sum_{i=1}^{N+1}x_i^2 + \sum_{\alpha\in A_N^+}\widetilde{f}(\alpha(x)). \tag{3.66}$$

In order to compare functions (3.66) and (3.42), we let $\frac{\lambda^2\gamma^2+6(N+1)^3\big((N+1)s^2+2s\big)}{12(N+1)^3} = \frac{N-2}{6(N+1)}$, which implies the required value of γ since $\lambda = 2(N+1)$. ◻

Let us now apply the restriction operation to the root system A_N. Let $n\in\mathbb{N}$ and $\underline{m} = (m_1,\dots,m_{n+1})\in\mathbb{N}^{n+1}$ be such that $\sum_{i=1}^{n+1} m_i = N+1$. Let us consider the subsystem $\mathcal{B}\subset\mathcal{A}$ given as follows:

$$\mathcal{B} = \{e^{\sum_{j=1}^{i-1}m_j+k} - e^{\sum_{j=1}^{i-1}m_j+l},\quad 1\le k<l\le m_i, i=1,\dots,n+1\}.$$

The corresponding subspace $W_\mathcal{B}$ defined by (3.6) can be given explicitly by the equations

$$\begin{cases} x_1 = \dots = x_{m_1},\\ x_{m_1+1} = \dots = x_{m_1+m_2},\\ \vdots\\ x_{\sum_{i=1}^n m_i+1} = \dots = x_{N+1}.\end{cases}$$

Define covectros $f^1,\dots,f^{n+1}\in W_\mathcal{B}^*$ by restrictions $f^i = \pi_\mathcal{B}\big(e^{\sum_{j=1}^i m_j}\big)$. Let us denote by $A_n(t;\underline{m})$ the restriction of the configuration A_N^+ to the subspace $W_\mathcal{B}$. It consists of the following covectors:

$$f^i - f^j, \quad \text{with multiplicity} \quad tm_i m_j, \quad 1 \le i < j \le n+1. \tag{3.67}$$

The following result holds, which is closely related to a multi-parameter family of solutions found in [43] (see also [47]).

Theorem 3.4.7. *The prepotential*

$$F(\xi) = (\frac{1}{3} - t)y^3 + ty \sum_{k=1}^{n+1} m_k \sum_{i=1}^{n+1} m_i \xi_i^2 + 2t^{3/2} \sum_{k=1}^{n+1} m_k \sum_{i<j}^{n+1} m_i m_j f(\xi_i - \xi_j), \tag{3.68}$$

where $\xi = (\xi_1, \dots, \xi_{n+1}) \in \mathbb{C}^{n+1}$ *and* $y = \sum_{i=1}^{n+1} \xi_i$, *satisfies WDVV equations*

$$F_i F_k^{-1} F_j = F_j F_k^{-1} F_i, \quad i, j, k = 1, \dots, n+1,$$

where $(F_i)_{pq} = \frac{\partial^3 F}{\partial \xi_i \partial \xi_p \partial \xi_q}$, $(p, q = 1, \dots, n+1)$, *for any generic* $t, m_1, \dots, m_{n+1} \in \mathbb{C}$.

Proof. Let us suppose firstly that $m_i \in \mathbb{N}$ for all $i = 1, \dots, n+1$. Define $N = -1 + \sum_{i=1}^{n+1} m_i$. By Theorem 3.4.3 function (2.55) with $\mathcal{A} = A_N^+$ and λ given by (3.60) is a solution of WDVV equations (2.56). By Theorem 3.2.4 the prepotential given by

$$F(\xi, y) = \frac{1}{3} y^3 + ty \sum_{i<j}^{n+1} m_i m_j (\xi_i - \xi_j)^2 + 2(N+1) t^{3/2} \sum_{i<j}^{n+1} m_i m_j f(\xi_i - \xi_j), \quad \xi \in W_{\mathcal{B}}, \tag{3.69}$$

as a function on $W_{\mathcal{B}} \oplus \mathbb{C}$ satisfies WDVV equations. Note that

$$\sum_{i<j}^{n+1} m_i m_j (\xi_i - \xi_j)^2 = \sum_{i<j}^{n+1} m_i m_j (\xi_i^2 + \xi_j^2) - 2 \sum_{i<j}^{n+1} m_i m_j \xi_i \xi_j. \tag{3.70}$$

Note also that

$$\sum_{i<j}^{n+1} m_i m_j (\xi_i^2 + \xi_j^2) = \frac{1}{2} \sum_{i,j=1}^{n+1} m_i m_j (\xi_i^2 + \xi_j^2) - \sum_{i=1}^{n+1} m_i^2 \xi_i^2 = \sum_{i=1}^{n+1} (N + 1 - m_i) m_i \xi_i^2, \tag{3.71}$$

and that

$$\sum_{1 \le i<j \le n+1} 2 m_i m_j \xi_i \xi_j = \Big(\sum_{i=1}^{n+1} m_i \xi_i \Big)^2 - \sum_{i=1}^{n+1} m_i^2 \xi_i^2. \tag{3.72}$$

By making use of relations (3.70)–(3.72) the function (3.69) takes the form

$$F(\xi, y) = \frac{1}{3} y^3 + (N+1) t \sum_{i=1}^{n+1} m_i \xi_i^2 y - t (\sum_{i=1}^{n+1} m_i \xi_i)^2 y + 2(N+1) t^{3/2} \sum_{i<j}^{n+1} m_i m_j f(\xi_i - \xi_j). \tag{3.73}$$

By setting $y = \sum_{i=1}^{n+1} m_i \xi_i$ and moving to variables $(\xi_1, \dots \xi_{n+1}) \in \mathbb{C}^{n+1}$ solution (3.73) takes the required form (3.68). The case of complex m_i follows from the above considerations since F depends on m_i polynomially. $\square$

Remark 3.4.8. (1) We note that Theorem 3.4.3 and the solution F given by (2.55) is valid if one takes any generic linear combination of coordinates x_i to form the extra variable $y = \sum_{i=1}^{N+1} a_i x_i, a_i \in \mathbb{C}$. The corresponding solution after restriction is given by the formula

$$F(\xi) = \frac{1}{3}y^3 + ty \sum_{i<j}^{n+1} m_i m_j (\xi_i - \xi_j)^2 + 2(N+1)t^{3/2} \sum_{i<j}^{n+1} m_i m_j f(\xi_i - \xi_j),$$

where y is a linear combination of $\xi_1, \dots, \xi_{n+1}, \xi_i \in \mathbb{C}$.

(2) A multi-parameter family of solutions related to the root system of type A_n was stated in [43, Section 5.2]. We expect that our solution (3.73) is equivalent to the family of solutions in [43]. It seems that there is a typo in the formula of the solution for type A_n in [43], since that it seems to not satisfy the WDVV equations.

3.5 Further examples in small dimensions

In Section 3.2 we presented the method of obtaining new solutions of WDVV equations through restrictions of known solutions. We applied it to classical families of root systems in Sections 3.3, 3.4. Similarly, starting from any root system and the corresponding solution of WDVV equations one can obtain further solutions by restrictions. In the next proposition we deal with a family of configurations in 4-dimensional space which in general is not a restriction of a root system.

Proposition 3.5.1. *Let a configuration $\mathcal{A} \subset \mathbb{C}^4$ consist of the following covectors:*

$$\begin{aligned}
&e^i, \quad \textit{with multiplicity} \quad p, \quad 1 \le i \le 3,\\
&e^4, \quad \textit{with multiplicity} \quad q,\\
&e^i \pm e^j, \quad \textit{with multiplicity} \quad r, \quad 1 \le i < j \le 3,\\
&\frac{1}{2}(e^1 \pm e^2 \pm e^3 \pm e^4), \quad \textit{with multiplicity} \quad s,
\end{aligned}$$

where $p, q, r, s \in \mathbb{C}$ are such that $4r + s \neq 0$. Then $\mathcal{A}$ is a trigonometric $\vee$-system if

$$p = 2r + s, \tag{3.74}$$

$$q = \frac{s(s-2r)}{4r+s}, \tag{3.75}$$

and $ps \neq 0$. The corresponding prepotential (2.55) with

$$\lambda = 6\sqrt{3}(2r+s)(4r+s)^{-1/2} \tag{3.76}$$

is a solution of WDVV equations.

Proof. For $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4) \in \mathbb{C}^4$ the bilinear form $G_{\mathcal{A}}$ is given by

$$G_{\mathcal{A}}(x, y) = (p + 4r + 2s)(x_1 y_1 + x_2 y_2 + x_3 y_3) + (q + 2s) x_4 y_4.$$

To simplify notations let us introduce covectors

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}(e^1 + e^2 + e^3 + e^4), & \alpha_2 &= \frac{1}{2}(e^1 + e^2 + e^3 - e^4),\\
\alpha_3 &= \frac{1}{2}(e^1 + e^2 - e^3 + e^4), & \alpha_4 &= \frac{1}{2}(e^1 - e^2 + e^3 + e^4),\\
\alpha_5 &= \frac{1}{2}(e^1 - e^2 - e^3 + e^4), & \alpha_6 &= \frac{1}{2}(e^1 - e^2 + e^3 - e^4),\\
\alpha_7 &= \frac{1}{2}(e^1 + e^2 - e^3 - e^4), & \alpha_8 &= \frac{1}{2}(e^1 - e^2 - e^3 - e^4).
\end{aligned}$$

Because of $B_3 \times A_1$-symmetry it is enough to check the trigonometric $\vee$-conditions for the following series only:

$\Gamma_{e_1} = \{\alpha_1, \alpha_8\}$, $\Gamma_{e_4} = \{\alpha_1, \alpha_2\}$, $\Gamma_{e^1+e^2} = \{\alpha_1, \alpha_7\}$, $\Gamma^1_{\alpha_1} = \{\alpha_2, e^4\}$, $\Gamma^2_{\alpha_1} = \{\alpha_3, e^3\}$, $\Gamma^3_{\alpha_1} = \{\alpha_5, e^2 + e^3\}$.

Trigonometric $\vee$-conditions for the series $\Gamma_{e^1}, \Gamma_{e^4}, \Gamma_{e^1+e^2}$ are immediate to check. Let us consider the trigonometric $\vee$-condition for α_1-series. We have

$$\alpha_1(\alpha_2^\vee) = \frac{3q + 4s - p - 4r}{4(p + 4r + 2s)(q + 2s)}, \quad \alpha_1(e^{4\vee}) = \frac{1}{2(q + 2s)}, \quad \alpha_1 \wedge \alpha_2 = -\alpha_1 \wedge e^4,$$

which implies the $\vee$-condition (2.93) for $\Gamma^1_{\alpha_1}$ since $s(3q+4s-p-4r) - 2q(p+4r+2s) = 0$ by relations (3.74), (3.75).

Also we have

$$\alpha_1(\alpha_3^\vee) = \frac{q + p + 4r + 4s}{4(p + 4r + 2s)(q + 2s)}, \quad \alpha_1(e^{3\vee}) = \frac{1}{2(p + 4r + 2s)}, \quad \alpha_1 \wedge \alpha_3 = -\alpha_1 \wedge e^3,$$

which implies the $\vee$-condition for $\Gamma^2_{\alpha_1}$ since $s(q+p+4r+4s) - 2p(q+2s) = 0$ by relations (3.74), (3.75).

Finally, we have

$$\alpha_1(\alpha_5^\vee) = \frac{p + 4r - q}{4(p + 4r + 2s)(q + 2s)}, \ \alpha_1((e^2 + e^3)^\vee) = \frac{1}{p + 4r + 2s}, \ \alpha_1 \wedge \alpha_5 = -\alpha_1 \wedge (e^2 + e^3),$$

which implies the $\vee$-condition for $\Gamma^3_{\alpha_1}$ since $s(p+4r-q)-4r(q+2s)=0$ by relations (3.74), (3.75).

Let us now find the quadratic form $G^{(1)}_{\mathcal{A}}$. By straightforward calculations we get

$$\begin{aligned} G^{(1)}_{\mathcal{A}} &= 2\big(p^2+8pr+4ps+16rs+16r^2+4s^2\big)\sum_{i<j}^{3}\big(e^i\wedge e^j\big)^2 \\ &+ 2(pq+2ps+4qr+2qs+8rs+4s^2)\sum_{i=1}^{3}\big(e^i\wedge e^4\big)^2 \\ &= \frac{18(2r+s)^2}{4r+s}\left((4r+s)\sum_{i<j}^{3}\big(e^i\wedge e^j\big)^2+s\sum_{i=1}^{3}\big(e^i\wedge e^4\big)^2\right). \end{aligned} \tag{3.77}$$

Now let us find the quadratic form $G^{(2)}_{\mathcal{A}}$. We have

$$\begin{aligned} G^{(2)}_{\mathcal{A}} &= pr\sum_{i=1}^{3}\sum_{j<k}^{3}e^i\big((e^j\pm e^k)^\vee\big)\big(e^i\wedge(e^j\pm e^k)\big)^2+ps\sum_{i=1}^{3}\sum_{j=1}^{8}e^i(\alpha_j^\vee)\big(e^i\wedge\alpha_j\big)^2 \\ &+rs\sum_{i<j}^{3}\sum_{k=1}^{8}(e^i\pm e^j)(\alpha_k^\vee)\big((e^i\pm e^j)\wedge\alpha_k\big)^2+s^2\sum_{i,j=1}^{8}\alpha_i(\alpha_j^\vee)(\alpha_i\wedge\alpha_j)^2 \\ &+r^2\sum_{i<j}^{3}\sum_{k<l}^{3}(e^i\pm e^j)((e^k\pm e^l)^\vee)\big((e^i\pm e^j)\wedge(e^k\pm e^l)\big)^2 \\ &= \frac{4pr}{p+4r+2s}\sum_{i<j}^{3}(e^i\wedge e^j)^2+\frac{2ps}{p+4r+2s}\sum_{i=2}^{4}(e^1\wedge e^i)^2 \\ &+\frac{4rs}{p+4r+2s}\big((e^1\wedge e^2)^2+(e^1\wedge e^3)^2+2(e^2\wedge e^3)^2+2(e^1\wedge e^4)^2+(e^2\wedge e^4)^2+(e^3\wedge e^4)^2\big) \\ &+\frac{8r^2}{p+4r+2s}\sum_{i<j}^{3}(e^i\wedge e^j)^2+\frac{2s^2}{p+4r+2s}\Big((e^2\wedge e^3)^2-\frac{4r}{s}(e^1\wedge e^4)^2+(e^2\wedge e^4)^2+(e^3\wedge e^4)^2\Big). \end{aligned} \tag{3.78}$$

By making further use of relations (3.74), (3.75) the expression (3.78) can be simplified to the form

$$G^{(2)}_{\mathcal{A}} = \frac{2}{3}(4r+s)\sum_{i<j}^{3}(e^i\wedge e^j)^2+\frac{2}{3}s\sum_{i=1}^{3}(e^i\wedge e^4)^2. \tag{3.79}$$

The final statement of the proposition follows from formulas (3.77), (3.79) and Theorem 2.6.21. $\square$

Remark 3.5.2. We note that for special values of the parameters configuration $\mathcal{A}$ is a restriction of a root system (cf. [25] where the rational version of this configuration was considered). Thus if $r=0$ and $p=q=s$ then $\mathcal{A}$ reduces to the root system D_4. If

$r = 1, s = 4$, then $p = 6$ and $q = 1$ and the resulting configuration is the restriction of the root system E_7 along subsystem of type A_3. If $s = 2r$ then the resulting configuration is the restriction of the root system E_6 along subsystem of type $A_1 \times A_1$.

Further solutions of WDVV equations can be obtained from Proposition 3.5.1 by restricting the configuration $\mathcal{A}$.

Proposition 3.5.3. *Let $\mathcal{A}_1 \subset \mathbb{C}^3$ be the configuration*

$$\mathcal{A}_1 = \{2e^1, e^1, e^2, e^3, e^1 \pm e^2, \frac{1}{2}(e^2 \pm e^3), \frac{1}{2}(2e^1 \pm e^2 \pm e^3)\},$$

with the corresponding multiplicities $\{r, 2p, p, q, 2r, 2s, s\}$, where $p, q, r, s \in \mathbb{C}$. Let configuration $\mathcal{A}_2 \subset \mathbb{C}^3$ consist of the following set of covectors:

$$\begin{aligned}
&e^i, \quad \textit{with multiplicity} \quad p+s, \quad 1 \leq i \leq 3,\\
&e^i + e^j, \quad \textit{with multiplicity} \quad r+s, \quad 1 \leq i < j \leq 3,\\
&e^i - e^j, \quad \textit{with multiplicity} \quad r, \quad 1 \leq i < j \leq 3,\\
&e^1 + e^2 + e^3, \quad \textit{with multiplicity} \quad q+s.
\end{aligned}$$

Suppose that relations (3.74), (3.75) hold and that $ps(4r+s) \neq 0$. Then $\mathcal{A}_1, \mathcal{A}_2$ are trigonometric $\vee$-systems which also define solutions of WDVV equations given by formula (2.55) with λ given by (3.76).

Proof of this proposition follows from an observation that configuration $\mathcal{A}_1$ can be obtained from the configuration $\mathcal{A}$ from Proposition 3.5.1 by restricting it to the hyperplane $x_1 = x_2$ (up to renaming the vectors). Similarly, configuration $\mathcal{A}_2$ can be obtained by restricting the configuration $\mathcal{A}$ to the hyperplane $x_1 + x_2 + x_3 - x_4 = 0$ (and up to renaming the vectors). Other three-dimensional restrictions of the configuration $\mathcal{A}$ give restriction of the root system F_4 and a configuration from BC_3 family.

Rational versions of configurations $\mathcal{A}_1, \mathcal{A}_2$ were considered in [25]. Note that configuration $\mathcal{A}_1$ has collinear vectors $2e^1, e^1$, so its rational version has different size.

Two-dimensional restrictions of $\mathcal{A}$ are considered below in Proposition 3.5.6 and Proposition 3.5.9, or can belong to BC_2 family of configuration, or have the form of configuration G_2 or appear in [27, Proposition 5].

Let us now consider examples of solutions (2.55) of WDVV equations where configuration $\mathcal{A}$ contains a small number of vectors on the plane. The next two propositions confirm that trigonometric $\vee$-systems with up to five covectors belong to A_2 or BC_2 families.

Proposition 3.5.4. *Any irreducible trigonometric $\vee$-system $\mathcal{A} \subset \mathbb{C}^2$ consisting of three vectors with non-zero multiplicities has the form (3.67) where $n = 2$ for some values of parameters.*

Proof. By [27, Proposition 2] any such configuration has the form $\mathcal{A} = \{\alpha, \beta, \gamma\}$ with the corresponding multiplicities $\{c_\alpha, c_\beta, c_\gamma\}$, where vectors in $\mathcal{A}$ satisfy $\alpha \pm \beta \pm \gamma = 0$ for some choice of signs. It is easy to see that equations

$$tm_1m_2 = c_\alpha, \quad tm_1m_3 = c_\beta, \quad tm_2m_3 = c_\gamma,$$

for $m_1, m_2, m_3, t \in \mathbb{C}$ can be resolved. $\square$

Proposition 3.5.5. *Any irreducible trigonometric* $\vee$*-system* $\mathcal{A} \subset \mathbb{C}^2$ *consisting of four or five vectors with non-zero multiplicities has the form* $BC_2(r, s, q; \underline{m})$ *for some values of parameters.*

Proof. By [27, Proposition 3] any irreducible trigonometric $\vee$-system $\mathcal{A}$ consisting of four vectors has the form $\mathcal{A} = \{2e^1, 2e^2, e^1 \pm e^2\}$ in a suitable basis, and the corresponding multiplicities $\{c_1, c_2, c_0\}$ where $c_0 \neq -2c_i$ for $i = 1, 2$. Now we require parameters r, s, q, m_1, m_2 to satisfy

$$\begin{aligned} &sm_1 + \frac{1}{2}qm_1(m_1 - 1) = c_1, \\ &sm_2 + \frac{1}{2}qm_2(m_2 - 1) = c_2, \\ &qm_1m_2 = c_0, \quad r = 0, \end{aligned}$$

which can be done by taking

$$\begin{aligned} s &= \frac{1}{m_1}\Big(c_1 - \frac{c_0(m_1 - 1)(2c_1 + c_0)}{2m_1(2c_2 + c_0)}\Big), \\ q &= \frac{c_0(2c_1 + c_0)}{m_1^2(2c_2 + c_0)}, \quad m_2 = \frac{(2c_2 + c_0)m_1}{2c_1 + c_0}, \quad m_1 \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

By [27, Proposition 4] any irreducible trigonometric $\vee$-system $\mathcal{B}$ consisting of five vectors in a suitable basis has the form $\mathcal{B} = \{e^1, 2e^1, e^2, e^1 \pm e^2\}$, and the corresponding multiplicities $\{c_1, \widetilde{c_1}, c_2, c_\pm\}$ satisfy $c_+ = c_-$ and $2\widetilde{c_1}c_2 = c_+(c_1 - c_2)$, where $(c_1 + 4\widetilde{c_1} + 2c_+)(c_2 + 2c_+) \neq 0$. In order to compare the configuration $\mathcal{B}$ with the configuration $BC_2(r, s, q; \underline{m})$, we require parameters r, s, q, m_1, m_2 to satisfy

$$\begin{aligned} &rm_1 = c_1, \quad rm_2 = c_2, \quad qm_1m_2 = c_+, \\ &sm_1 + \frac{1}{2}qm_1(m_1 - 1) = \widetilde{c_1}, \quad sm_2 + \frac{1}{2}qm_2(m_2 - 1) = 0. \end{aligned}$$

These equations can be solved by taking

$$r = \frac{c_1}{m_1}, \quad s = \frac{c_+(c_1 - c_2m_1)}{2c_2m_1^2}, \quad q = \frac{c_+c_1}{c_2m_1^2}, \quad m_2 = \frac{c_2m_1}{c_1}, \quad m_1 \in \mathbb{C} \setminus \{0\}.$$

□

In the rest of this section we give more examples of trigonometric $\vee$-systems on the plane, which can be checked directly or using Theorem 2.6.21. The configuration in the following proposition can be obtained by restricting configuration $\mathcal{A}_1$ from Proposition 3.5.3 to the plane $2x_1 + x_2 - x_3 = 0$.

Proposition 3.5.6. *Let*

$$\mathcal{A} = \{e^1, 2e^1, e^2, e^1 + e^2, e^1 - e^2, 2e^1 + e^2\} \subset \mathbb{C}^2$$

be with the corresponding multiplicities $\{4a, a, 2a, 2a, 2(a-b), \frac{2ab}{4a-3b}\}$*, where* $4a - 3b \neq 0$*. Then* $\mathcal{A}$ *is a trigonometric* $\vee$*-system provided that* $a(2a-b) \neq 0$*. The corresponding solution of the WDVV equations has the form (2.55) with* $\lambda = 6\sqrt{3}(2a-b)(4a-3b)^{-1/2}$*.*

The configuration in the following proposition can be obtained by restricting configuration $\mathcal{A}_1$ from Proposition 3.5.3 to the plane $x_1 = x_2$.

Proposition 3.5.7. (cf. [27]) *Let*

$$\mathcal{A} = \{e^1, e^2, 2e^2, \frac{1}{2}(e^1 \pm e^2), \frac{1}{2}(e^1 \pm 3e^2)\} \subset \mathbb{C}^2$$

be with the corresponding multiplicities $\{\frac{a(3a-2b)}{3a+4b}, 3a+2b, b, 3a, a\}$*. Then* $\mathcal{A}$ *is a trigonometric* $\vee$*-system. The corresponding solution of the WDVV equations has the form (2.55) with* $\lambda = 6(3a+2b)(3a+4b)^{-1/2}$*.*

Remark 3.5.8. If we let $b = 0$ for the configuration in Proposition 3.5.7 then we recover the root system G_2 with special multiplicities.

The configuration in the following proposition can be obtained by restricting configuration $\mathcal{A}_1$ from Proposition 3.5.3 to the plane $x_3 = 0$.

Proposition 3.5.9. *Let*

$$\mathcal{A} = \{e^1, 2e^1, e^2, 2e^2, e^1 \pm e^2, e^1 \pm 2e^2\} \subset \mathbb{C}^2$$

be with the corresponding multiplicities $\{2a, \frac{a}{2} - \frac{b}{4}, 2b, a, b, a - \frac{b}{2}\}$*, where* $a \neq 0$*. Then* $\mathcal{A}$ *is a trigonometric* $\vee$*-system and the corresponding solution of the WDVV equations has the form (2.55) with* $\lambda = 6\sqrt{6}a\big(4a-b\big)^{-1/2}$*.*

In the next two propositions we give examples of trigonometric $\vee$-systems with nine and ten covectors on the plane.

Proposition 3.5.10. (cf. [27]) *Let*

$$\mathcal{A} = \{e^1, 2e^1, e^2, e^1 \pm e^2, \frac{1}{2}(3e^1 \pm e^2), \frac{1}{2}(e^1 \pm e^2)\} \subset \mathbb{C}^2$$

be with the corresponding multiplicities $\{a, b, \frac{a}{3}, b, \frac{a}{3}, a\}$. *Then* $\mathcal{A}$ *is a trigonometric* $\vee$-*system provided that* $a \neq -2b$. *The corresponding solution of the WDVV equations has the form (2.55) with* $\lambda = 6(a+2b)(a+4b)^{-1/2}$.

Note that if $b = 0$ then after rescaling $e^2 \to \sqrt{2}e^2$ this configuration reduces to the positive half of the root system G_2.

Proposition 3.5.11. (cf. [27]) *Let*

$$\mathcal{A} = \{e^1, 2e^1, e^2, 2e^2, e^1 \pm e^2, e^1 \pm 2e^2, 2e^1 \pm e^2\} \subset \mathbb{C}^2$$

be with the corresponding multiplicities $\{6a, \frac{3a}{2}, 6a, \frac{3a}{2}, 4a, a, a\}$. *Then* $\mathcal{A}$ *is a trigonometric* $\vee$-*system provided that* $a \neq 0$. *The corresponding solution of the WDVV equations has the form (2.55) with* $\lambda = 15a^{1/2}$.

The following configuration containing 14 vectors on the plane.

Proposition 3.5.12. *Let*

$$\mathcal{A} = \{e^1, 2e^1, 3e^1, e^2, 2e^2, 3e^2, e^1 \pm e^2, 2(e^1 \pm e^2), 2e^1 \pm e^2, e^1 \pm 2e^2\} \subset \mathbb{C}^2$$

be with the corresponding multiplicities $\{3a, \frac{3a}{2}, \frac{a}{3}, 3a, \frac{3a}{2}, \frac{a}{3}, 2a, \frac{a}{2}, a, a\}$. *Then* $\mathcal{A}$ *is a trigonometric* $\vee$-*system provided that* $a \neq 0$. *The corresponding solution of the WDVV equations has the form (2.55) with* $\lambda = 5(6a)^{1/2}$.

The following configuration containing 9 vectors on the plane which does not belong to any restrictions of Coxeter root system for general values of multiplicities.

Proposition 3.5.13. *Let*

$$\mathcal{A} = \{e^1, 2e^1, 3e^1, e^2, 2e^2, e^1 \pm e^2, 2e^1 \pm e^2\} \subset \mathbb{C}^2$$

be with the corresponding multiplicities $\{\frac{2b(3b+2a)}{b+2a}, \frac{b(3b+2a)}{b+2a}, \frac{2b(b-2a)}{3(b+2a)}, 2a+3b, a, 2b, b\}$, *where* $b + 2a \neq 0$. *Then* $\mathcal{A}$ *is a trigonometric* $\vee$-*system provided that* $b(3b+2a) \neq 0$. *The corresponding solution of the WDVV equations has the form (2.55) with* $\lambda = 3\sqrt{2}(3b + 2a)(b+2a)^{-1/2}$.

3.6 Exceptional root systems solutions revisited

We know that a function F with extra variable y given by formula (2.55) is a solution of WDVV equations (2.56) if the nonzero constant $\lambda \in \mathbb{C}$ exists. In Section 3.2 we have shown that any restricted function of a solution F also gives a solution of WDVV equations with the same value of λ. This fact can be used to find the value of λ for any configuration through out the process of restriction. This can help us to determine the value of the scalar λ for a higher dimensional configuration by making restriction of the configuration to obtain a restricted system in a small dimension which make it easier to calculate. Also if two configurations have the same restricted system then actually one can derive the value of λ for one of these configurations whenever the value of λ is known for the other configuration. In this section we will make use of this procedure to find the value of λ for root systems of type E_6, E_7, E_8, F_4 and G_2. Following [33,42], recall that WDVV equations (2.56) have solutions of the form

$$\widetilde{F}(x,y) = \frac{\gamma}{6}y^3 + \frac{\gamma}{2}y\langle x,x\rangle + \sum_{\alpha\in\mathcal{R}^+} c_\alpha \widetilde{f}(\alpha(x)), \quad (x \in V, y \in \mathbb{C}), \tag{3.80}$$

where $V \cong C^N$ and $\mathcal{R} \subset V^*$ is a root system of rank N, multiplicities c_α and the inner product $\langle\cdot,\cdot\rangle$ are invariant under the Weyl group, $\gamma = \gamma_{(\mathcal{R},c)} \in \mathbb{C}$ and function $\widetilde{f}$ is given by (2.100). By identifying $V \cong V^*$ via the standard inner product $\langle\cdot,\cdot\rangle$, we define $\langle\alpha,\beta\rangle := \langle\alpha^*,\beta^*\rangle$. The corresponding values of $\gamma_{(\mathcal{R},c)}$ were given explicitly in [33,42] for constant multiplicity functions $c_\alpha = t$ $\forall\alpha$ (except for $\mathcal{R} = BC_N, G_2$), they were found in [11] for special multiplicities and in [47,48] for arbitrary (non-reduced) root system $\mathcal{R}$ with invariant multiplicity. For type E root systems we have

$$\gamma_{(E_6,t)} = 2i\sqrt{6}t, \quad \gamma_{(E_7,t)} = 4i\sqrt{3}t, \quad \gamma_{(E_8,t)} = 2i\sqrt{30}t.$$

Similarly to analysis of the BC_N case in Section 3.3 these solutions lead to solutions F of the form (2.55) for $\mathcal{A} = \mathcal{R}^+$ and the corresponding values of $\lambda = \lambda_{(\mathcal{R},c)}$ are given by

$$\lambda_{(E_6,t)} = 12\sqrt{2t}, \quad \lambda_{(E_7,t)} = 9\sqrt{6t}, \quad \lambda_{(E_8,t)} = 30\sqrt{t}. \tag{3.81}$$

We recall that $\lambda_{(\mathcal{R},c)}$, in contrast to $\gamma_{(\mathcal{R},c)}$, is invariant under linear transformations applied to $\mathcal{R}$. An alternative way to derive values (3.81) is to apply Theorem 3.2.4 to already known solutions. Thus $\lambda_{(E_6,t)}$ can be derived, for example, by considering the four-dimensional restriction of E_6 along a subsystem of type $A_1 \times A_1$ as this restriction is equivalent to the configuration from Proposition 3.5.1 when parameter $s = 2r$. Likewise restriction of E_7 along a subsystem of type A_3 gives the same configuration from Proposition 3.5.1 with $r = 1$ and $s = 4$. Similarly, restriction of E_8 along a subsystem of type D_6

gives the configuration of type BC_2 which allows to get $\lambda_{(E_8,t)}$.

3.6.1 E_6 type configuration

Proposition 3.6.1. *Let $\mathcal{A} = E_6^+$ be the positive half of the root system E_6 consisting of the following vectors*

$$\begin{aligned} &e^i \pm e^j, \quad 1 \leq i < j \leq 5, \\ &\frac{1}{2}(e^1 \pm e^2 \pm e^3 \pm e^4 \pm e^5 \pm \sqrt{3}e^6), \quad \text{with odd number of plus sign,} \end{aligned}$$

where all vectors have the same multiplicity r. Then the corresponding solution of the WDVV equations has the form (2.55) with $\lambda = 12\sqrt{2r}$.

Proof. From Remark 3.5.2, since the restriction of the root system E_6 along subsystem of type $A_1 \times A_1$ gives the same configuration one can obtain from the configuration $\mathcal{A}$ in Proposition 3.5.1 by putting the parameter $s = 2r$ in proposition 3.5.1, then the value of the scalar λ for the root system E_6 is the same value of λ for $\mathcal{A}$ in Proposition 3.5.1 when $s = 2r$. Let us recall the value of λ for $\mathcal{A}$ in Proposition 3.5.1 which is given by

$$\lambda = \frac{6\sqrt{3}(2r+s)}{\sqrt{4r+s}}.$$

Hence for $s = 2r$ we have $\lambda = 12\sqrt{2r}$. This proves the proposition. □

In the following proposition we show that our solution F for the root system E_6 takes the form of the solution obtained in [33] for E_6.

Proposition 3.6.2. [42] *Function $\widetilde{F}$ given by (3.42) satisfies WDVV equations (2.56) for root system E_6 and the value of γ is given by $\gamma = 2i\sqrt{6}$.*

Proof. For the root system E_6 as defined above we have

$$\sum_{\alpha \in E_6^+} c_\alpha \alpha(x)^2 = 12r \sum_{i=1}^{6} x_i^2.$$

Let us take parameter $r = 1$. Then the solution F given by (2.55) takes the form

$$F(\widetilde{x}, \widetilde{y}) = \frac{1}{3}\widetilde{y}^3 + 12\widetilde{y}\sum_{i=1}^{6} \widetilde{x}_i^2 + \lambda \sum_{\alpha \in E_6^+} f(\alpha(\widetilde{x})), \tag{3.82}$$

where we redenoted variables (x, y) by $(\widetilde{x}, \widetilde{y})$. By changing variables $\widetilde{x} = -ix$ and $\widetilde{y} = \frac{\gamma\lambda}{24}y$ and dividing by $-\lambda$ solution (3.82) takes the form (3.42) provided that $\gamma^2\lambda^2 = -6912$ which is satisfied for $\gamma = 2i\sqrt{6}$.

□

Remark 3.6.3. There seems to be a typo in [42] as the value of γ stated in [42] for root system E_6 should be $2i\sqrt{6}$ instead of $i\sqrt{6}$.

3.6.2 E_7 type configuration

Here we discuss a family of E_7 type configurations and show that it gives trigonometric solutions of the WDVV equations.

Let V be the hyperplane in $\mathbb{C}^8$ consisting of vectors whose coordinates add up to 0. Let E_7^+ be the positive half of the root system E_7 consisting of the following vectors in $\mathbb{C}^8$

$$\frac{1}{2}(e^1 \pm e^2 \pm e^3 \pm e^4 \pm e^5 \pm e^6 \pm e^7 \pm e^8), \tag{3.83}$$

$$e^i - e^j, \quad 1 \leq i < j \leq 8, \tag{3.84}$$

where the sum of all eight coordinates of vectors (3.83) is zero. Let $t \in \mathbb{C}$ be the multiplicity of all vectors (3.83), (3.84).

Before we discuss solutions of WDVV equations corresponding to the root system E_7, the following lemma will be used.

Lemma 3.6.4. *For $\mathcal{A} = E_7^+$, consider the extra variable y given by $y = \sum_{i=1}^{8} x_i$. Then we have*

$$\sum_{\alpha \in E_7^+} \alpha(x)^2 = 18 \sum_{i=1}^{8} x_i^2 - \frac{9}{4} y^2. \tag{3.85}$$

Proof. Let $x = (x_1, \dots, x_8) \in \mathbb{C}^8$. Then we have

$$\begin{aligned}\sum_{\alpha \in E_7^+} \alpha(x)^2 &= \frac{1}{4}(x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 \pm x_6 \pm x_7 \pm x_8)^2 + \sum_{1 \leq i < j \leq 8} (x_i - x_j)^2 \\ &= \frac{9}{4}(7\sum_{i=1}^{8} x_i^2 - \sum_{i=1}^{8} x_i \sum_{j \neq i}^{8} x_j) = \frac{9}{4}\Big(7\sum_{i=1}^{8} x_i^2 - \sum_{i=1}^{8} x_i(y - x_i)\Big) \\ &= \frac{9}{4}(8\sum_{i=1}^{8} x_i^2 - y^2)\end{aligned}$$

which gives the required result. □

In the following proposition we show that our solution F for the root system E_7 takes the form of the solution obtained in [33] for E_7.

Proposition 3.6.5. *Let E_7^+ be the positive half of the root system E_7 as defined above. Then the corresponding solution of the WDVV equations has the form (2.55) with $\mathcal{A} = E_7^+$*

and $y = \sum_{i=1}^{8} x_i$*, satisfies WDVV equations*

$$F_i F_k^{-1} F_j = F_j F_k^{-1} F_i, \quad i, j, k = 1, ..., 8, \tag{3.86}$$

if

$$\lambda = 9\sqrt{6t}. \tag{3.87}$$

Proof. From Remark 3.5.2, since the restriction of the root system E_7 along subsystem of type A_3 gives the same configuration one can obtain from the configuration $\mathcal{A}$ in Proposition 3.5.1 by putting parameters $r = 1, s = 4, p = 6, q = 1$ in proposition 3.5.1, then the value of the scalar λ for the root system E_7 is the same value of λ for $\mathcal{A}$ in Proposition 3.5.1 for this special choices of parameters. Let us recall the value of λ for $\mathcal{A}$ in Proposition 3.5.1 which is given by

$$\lambda = \frac{6\sqrt{3}(2r + s)}{\sqrt{4r + s}}.$$

Hence for $r = 1, s = 4, p = 6, q = 1$ we have $\lambda = 9\sqrt{6}$. This proves the proposition. □

Proposition 3.6.6. [42] *Function* $\widetilde{F}$ *given by*

$$\begin{aligned}\widetilde{F}(x, y) &= \frac{\gamma}{6} y^3 + \frac{\gamma}{2} y \Big(\sum_{i=1}^{8} x_i^2 - \frac{y^2}{8} \Big) + \sum_{\alpha \in E_7{}^+} \widetilde{f}(\alpha(x)) \\ &= \frac{5}{48} \gamma y^3 + \frac{\gamma}{2} y \sum_{i=1}^{8} x_i^2 + \sum_{\alpha \in E_7{}^+} \widetilde{f}(\alpha(x)),\end{aligned} \tag{3.88}$$

where the variable y *is defined by* $y = \sum_{i=1}^{8} x_i$*, satisfies WDVV equations (3.86) and the value of* γ *is given by* $\gamma^2 = -48$.

Proof. Let $c_\alpha = 1$. Then by Lemma 3.6.4 solution (2.55) takes the form

$$F(\widetilde{x}, \widetilde{y}) = -\frac{23}{12} \widetilde{y}^3 + 18 \widetilde{y} \sum_{i=1}^{8} \widetilde{x}_i^2 + \lambda \sum_{\alpha \in E_7^+} f(\alpha(\widetilde{x})), \tag{3.89}$$

where we redenote variables (x, y) by $(\widetilde{x}, \widetilde{y})$. By changing the variable $\widetilde{x} = -i\widehat{x}$, one accordingly has $\widetilde{y} = -i\widehat{y}$ where $\widehat{y} = \sum_{i=1}^{8} \widehat{x}_i$. Then by dividing the resulting solution by $-\lambda$ and calling it $\widehat{F}$ we get

$$\widehat{F} = \frac{23}{12\lambda} i \widehat{y}^3 - \frac{18i}{\lambda} \widehat{y} \sum_{i=1}^{8} \widehat{x}_i^2 + \sum_{\alpha \in E_7^+} \widetilde{f}(\alpha(\widehat{x})). \tag{3.90}$$

Now let us replace variable $\widehat{x}$ with $\widehat{x} = x + sy$, where $s = \frac{\gamma\lambda i - 36}{288}$. Accordingly we have

$\widehat{y} = \frac{\gamma\lambda i}{36} y$. Hence solution (3.90) takes the form

$$\gamma(\frac{8s^2+2s}{2} + \frac{23\gamma^2\lambda^2}{559872})y^3 + \frac{\gamma}{2} y \sum_{i=1}^{8} x_i^2 + \sum_{\alpha \in E_7^+} \widetilde{f}(\alpha(x)). \tag{3.91}$$

In order to compare function (3.88) with (3.91), we let $\frac{8s^2+2s}{2} + \frac{23\gamma^2\lambda^2}{559872} = \frac{5}{48}$, which implies that

$$\gamma^2 = -\frac{23328}{\lambda^2}.$$

Since $\lambda = 9\sqrt{6}$, we get the required value of γ. □

We can also have a slightly different solution associated with root system E_7.

Proposition 3.6.7. *Function $\widetilde{F}$ given by*

$$\widetilde{F}(x,y) = \frac{\gamma}{6} y^3 + \frac{\gamma}{2} y \sum_{i=1}^{8} x_i^2 + \sum_{\alpha \in E_7{}^+} \widetilde{f}(\alpha(x)), \tag{3.92}$$

where the variable y is defined by $y = \sum_{i=1}^{N+1} x_i$, satisfies WDVV equations (3.86) and the value of γ is given by $-\gamma^2 = 66$.

Proof. As in the proof of Proposition 3.6.6 we get relation (3.91), and in order to compare function (3.92) with (3.91), we let $\frac{8s^2+2s}{2} + \frac{23\gamma^2\lambda^2}{559872} = \frac{1}{6}$, which implies that

$$\gamma^2 = -\frac{32076}{\lambda^2}.$$

Since $\lambda = 9\sqrt{6}$, we get the required value of γ. □

3.6.3 E_8 type configuration

In the following proposition we derived the solution of WDVV equations corresponding to the root system E_8 using the restriction operation in order to reduce the dimension.

Proposition 3.6.8. *Let $\mathcal{A} = E_8^+$ be the positive half of the root system E_8 consisting of the following vectors*

$$\begin{aligned}
&e^i \pm e^j, \textit{with multiplicity} \quad t, \quad 1 \leq i < j \leq 8, \\
&\frac{1}{2}(e^1 \pm e^2 \pm e^3 \pm e^4 \pm e^5 \pm e^6 \pm e^7 \pm e^8), \quad \textit{with multiplicity} \quad t,
\end{aligned}$$

where the sum of all eight coordinates is even. Then the corresponding solution of the WDVV equations has the form (2.55) with

$$\lambda = 30\sqrt{t}. \tag{3.93}$$

Proof. Let us restrict the root system E_8 to the subspace where

$$x_1 = x_2 = x_3 = x_4,$$
$$x_5 = x_6 = x_7 = x_8.$$

The resulting restricted system (after renaming vectors) is given by

$$\mathcal{A} = \{2e^1, 2e^2, e^1 + e^2, e^1 - e^2, 2(e^1 + e^2), 2(e^1 - e^2)\}$$

with the corresponding multiplicities $12t, 12t, 32t, 32t, t, t$. Let us make the following linear transformation

$$e^1 + e^2 \to \widetilde{e^1},$$
$$e^1 - e^2 \to \widetilde{e^2}.$$

The resulting equivalent system is given by

$$\widetilde{\mathcal{A}} = \{\widetilde{e^1}, \widetilde{e^2}, 2\widetilde{e^1}, 2\widetilde{e^2}, \widetilde{e^1} + \widetilde{e^2}, \widetilde{e^1} - \widetilde{e^2}\}$$

with the corresponding multiplicities $\{32t, 32t, t, t, 12t, 12t\}$. We note that the configuration is of type BC_2, hence the value of the scalar λ of E_8 is the same of that for the restricted configuration $\widetilde{\mathcal{A}}$. The value of λ for $\widetilde{\mathcal{A}}$ is given by (3.41) with

$$r = 32t, \quad s = t, \quad q = 12t,$$

which gives the required value of λ. □

In the following proposition we show that our solution F for the root system E_8 takes the form of the solution obtained in [33] for E_8.

Proposition 3.6.9. [42] *Function $\widetilde{F}$ given by (3.42) satisfies WDVV equations (2.56) for root system E_8 and the value of γ is given by $\gamma = 2i\sqrt{30}$.*

Proof. For the root system E_8 as defined in Proposition 3.6.8 we have

$$\sum_{\alpha \in E_8^+} c_\alpha \alpha(x)^2 = 30t \sum_{i=1}^{8} x_i^2.$$

Let us take parameter $t = 1$. Then the solution F given by (2.55) takes the form

$$F(\widetilde{x}, \widetilde{y}) = \frac{1}{3}\widetilde{y}^3 + 30\widetilde{y} \sum_{i=1}^{8} \widetilde{x}_i^2 + \lambda \sum_{\alpha \in E_8^+} f(\alpha(\widetilde{x})), \tag{3.94}$$

where we redenoted variables (x, y) by $(\widetilde{x}, \widetilde{y})$. By changing of variables $\widetilde{x} = -ix$, $\widetilde{y} = \frac{\gamma\lambda}{60}y$ and dividing by $-\lambda$ solution (3.94) takes the form (3.42) provided that $\gamma^2\lambda^2 = -108000$ which is satisfied for $\gamma = 2i\sqrt{30}$. $\square$

Remark 3.6.10. There seems to be a typo in [42] as the value of γ stated in [42] for root system E_8 should be $i2\sqrt{30}$ instead of $i\sqrt{320}$.

3.6.4 F_4 type configuration

Proposition 3.6.11. *Let $\mathcal{A} = F_4^+$ be the positive half of the root system F_4 with the multiplicity function c given by*

$$c\Big(\frac{1}{2}(e^1 \pm e^2 \pm e^3 \pm e^4)\Big) = c(e^i) = s, \quad (1 \le i \le 4),$$
$$c(e^i \pm e^j) = r, \quad (1 \le i < j \le 4), \tag{3.95}$$

where $r, s \in \mathbb{C}$. Then in the corresponding solution (2.55) of the WDVV equations (2.56) we have

$$\lambda = \lambda_{(F_4,c)} = 6\sqrt{3}(2r+s)(4r+s)^{-1/2}. \tag{3.96}$$

Proof. We note that the restriction of the configuration defined in Proposition 3.5.1 to the hyperplane $x_4 = 0$ gives the same configuration as one gets by restricting $\mathcal{A} = F_4^+$ to the hyperplane $x_4 = 0$. Hence λ is given by formula (3.76). $\square$

Proposition 3.6.11 has the following implication for the corresponding solution of the form (3.80), which is also contained in [47].

Proposition 3.6.12. [47] *For $\mathcal{R} = F_4$ with the multiplicity function (3.95) we have*

$$\gamma^2_{(F_4,c)} = -(s+2r)(s+4r).$$

Proof. We have $\sum_{\alpha\in F_4^+} c_\alpha \alpha(x)^2 = 3(s+2r)\sum_{i=1}^4 x_i^2$. Then solution F given by (2.55) for $\mathcal{A} = F_4^+$ takes the form

$$F(\widetilde{x}, \widetilde{y}) = \frac{1}{3}\widetilde{y}^3 + 3(s+2r)\widetilde{y}\sum_{i=1}^{4}\widetilde{x}_i^2 + \lambda\sum_{\alpha\in F_4^+} c_\alpha f(\alpha(\widetilde{x})), \tag{3.97}$$

where λ is given by (3.96), and we redenoted variables (x, y) by $(\widetilde{x}, \widetilde{y})$. By dividing F by $-\lambda$ and changing variables $\widetilde{x} = -ix$, $\widetilde{y} = \frac{\gamma\lambda}{6(s+2r)}y$, solution (3.97) takes the form (3.80) provided that $\gamma^2\lambda^2 = -108(s+2r)^3$, which implies the statement. $\square$

Let us now find the value of λ for $\mathcal{R} = G_2$.

3.6.5 G_2 type configuration

Proposition 3.6.13. *Let $\mathcal{A} = G_2^+$ be the positive half of the root system G_2 with the multiplicity function given by*

$$c(\sqrt{3}e^1) = c(\frac{\sqrt{3}e^1}{2} \pm \frac{3e^2}{2}) = q, \quad c(e^2) = c(\frac{\sqrt{3}e^1}{2} \pm \frac{e^2}{2}) = p, \tag{3.98}$$

where $q, p \in \mathbb{C}$. Then in the corresponding solution (2.55) of the WDVV equations (2.56) we have

$$\lambda = \lambda_{(G_2,c)} = 6(p+3q)(p+9q)^{-1/2}. \tag{3.99}$$

Proof. Note that by restricting the configuration $\mathcal{A}_2$ defined in Proposition 3.5.3 to the hyperplane $x_1 + x_2 + x_3 = 0$ we get the two-dimensional configuration

$$\widetilde{\mathcal{A}}_2 = \{e^1, e^2, e^1 + e^2, e^1 - e^2, e^1 + 2e^2, 2e^1 + e^2\}$$

which can be mapped to the configuration G_2 by a linear transformation. The corresponding multiplicities satisfy

$$p = 3(r+s), \quad q = r,$$

which implies the statement by Proposition 3.5.3 and Theorem 3.2.4. $\square$

Proposition 3.6.13 has the following implication for the corresponding solution of the form (3.80), which is also contained in [47].

Proposition 3.6.14. [47] *For $\mathcal{R} = G_2$ with multiplicity function (3.98) we have*

$$\gamma^2_{(G_2,c)} = -\frac{3}{8}(p+3q)(p+9q).$$

Proof. We have $\sum_{\alpha \in G_2^+} c_\alpha \alpha(x)^2 = \frac{3}{2}(p+3q)(x_1^2 + x_2^2)$. Then solution F given by (2.55) for $\mathcal{A} = G_2^+$ takes the form

$$F(\widetilde{x}, \widetilde{y}) = \frac{1}{3}\widetilde{y}^3 + \frac{3}{2}(p+3q)\widetilde{y}(\widetilde{x}_1^2 + \widetilde{x}_2^2) + \lambda \sum_{\alpha \in G_2^+} c_\alpha f(\alpha(\widetilde{x})), \tag{3.100}$$

where λ is given by (3.99), and we redenoted variables (x, y) by $(\widetilde{x}, \widetilde{y})$. By dividing F by $-\lambda$ and changing variables $\widetilde{x} = -ix$, $\widetilde{y} = \frac{\gamma\lambda}{3(p+3q)}y$, solution (3.100) takes the form (3.80) provided that $\gamma^2\lambda^2 = \frac{27}{2}(p+3q)^3$ which implies the statement. $\square$

3.7 Bryan-Gholampour solutions revisited

Solutions of WDVV equations of the form (3.80) were also obtained in [11]. More exactly, consider the multiplication $*$ on the tangent space $T_{(x,y)}(V \oplus U) \cong V \oplus U$, where

$\dim U = 1, x \in V, y \in U$ which is given by

$$u * v = \langle u, v\rangle E + \widetilde{\gamma}^{-1} \sum_{\beta\in\mathcal{R}^+} \frac{c_\beta}{\langle\beta,\beta\rangle}\beta(u)\beta(v)\coth\beta(x)\beta, \quad (u, v \in V), \tag{3.101}$$

$\widetilde{\gamma} = \widetilde{\gamma}_{(\mathcal{R},c)} \in \mathbb{C}$ and $E \in U$ is the identity of the product (3.101). It was shown in [11] that this multiplication is associative. It can be seen (cf. Section 2.1 above) that associativity of (3.101) is equivalent to the statement that function

$$\widetilde{F}(x, y) = \frac{\widetilde{\gamma}}{6}y^3 + \frac{\widetilde{\gamma}}{2}y\langle x, x\rangle + \sum_{\alpha\in\mathcal{R}^+} d_\alpha \widetilde{f}(\alpha(x)), \tag{3.102}$$

where $d_\alpha = \frac{c_\alpha}{\langle\alpha,\alpha\rangle}$ satisfies WDVV equations, hence $\widetilde{\gamma} = \widetilde{\gamma}_{(\mathcal{R},c)} = \gamma_{(\mathcal{R},d)}$.

Let $\{\alpha_1, \dots, \alpha_N\}$ be a basis of simple roots of $\mathcal{R}$. Recall that there exists the *highest root* $\theta = \theta_{\mathcal{R}} = \sum_{i=1}^N n_i\alpha_i \in \mathcal{R}$ such that, for every $\beta = \sum_{i=1}^N p_i\alpha_i \in \mathcal{R}$, we have $n_i \geq p_i$ for all $i = 1, \dots, N$ [6]. The constant $\widetilde{\gamma} = \widetilde{\gamma}_{(\mathcal{R},c)}$ was expressed in [11] in terms of the highest root of the root system $\mathcal{R}$.

Proposition 3.7.1. [11] *The value of $\widetilde{\gamma}_{(\mathcal{R},c)}$ in the solution (3.102) in the case of constant multiplicity function $c_\alpha = t$ is given by*

$$\widetilde{\gamma}^2_{(\mathcal{R},c)} = -\frac{t^2}{8}\Big(\langle\theta,\theta\rangle + \sum_{i=1}^N n_i^2\langle\alpha_i,\alpha_i\rangle\Big).$$

Now we give a generalization of Proposition 3.7.1 to the case of non-constant multiplicity function. Let p be the multiplicity of short roots and q be the multiplicity of long roots in a reduced not simply-laced root system $\mathcal{R}$.

Proposition 3.7.2. *We have*

$$\widetilde{\gamma}^2_{(\mathcal{R},c)} = -\frac{1}{8}\Big(a_0\langle\theta,\theta\rangle + \sum_{i=1}^N a_i n_i^2\langle\alpha_i,\alpha_i\rangle\Big), \tag{3.103}$$

where scalars a_i for all irreducible reduced not simply-laced root systems are given as follows. (1) Let $\mathcal{R} = B_N$ with the basis of simple roots

$$\alpha_1 = e^1 - e^2, \dots, \alpha_{N-1} = e^{N-1} - e^N, \alpha_N = e^N.$$

Then

$$a_0 = a_1 = a_N = pq, \quad a_i = q^2, \quad (2 \leq i \leq N-1). \tag{3.104}$$

(2) Let $\mathcal{R} = C_N$ with the basis of simple roots

$$\alpha_1 = e^1 - e^2,\ \ldots,\ \alpha_{N-1} = e^{N-1} - e^N,\ \alpha_N = 2e^N.$$

Then

$$a_0 = a_1 = a_N = pq, \quad a_i = p^2, \quad (2 \leq i \leq N-1). \tag{3.105}$$

(3) Let $\mathcal{R} = F_4$ with the basis of simple roots

$$\alpha_1 = e^2 - e^3,\ \alpha_2 = e^3 - e^4,\ \alpha_3 = e^4,\ \alpha_4 = \frac{1}{2}(e^1 - e^2 - e^3 - e^4).$$

Then

$$a_0 = a_2 = a_4 = pq, \quad a_1 = p^2, \quad a_3 = q^2. \tag{3.106}$$

(4) Let $\mathcal{R} = G_2$ with the basis of simple roots $\alpha_1 = \frac{\sqrt{3}e^1}{2} - \frac{3e^2}{2}$, $\alpha_2 = e^2$. Then

$$a_0 = p^2, \quad a_1 = pq, \quad a_2 = q^2. \tag{3.107}$$

Proof. It follows from Proposition 3.3.4 that

$$\widetilde{\gamma}^2_{(B_N,c)} = -q(p + (N-2)q).$$

Note that $\theta_{B_N} = e^1 + e^2 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_N)$. Then it is easy to see that the substitution of (3.104) into formula (3.103) gives the same value of $\widetilde{\gamma}_{(B_N,c)}$. Similarly, we have

$$\widetilde{\gamma}^2_{(C_N,c)} = -p\big(2q + (N-2)p\big),$$

which is equal to the value given by formula (3.103) after substitution a_i from (3.105) and by using $\theta_{C_N} = 2e^1 = 2(\alpha_1 + \cdots + \alpha_{N-1}) + \alpha_N$. It follows from Proposition 3.6.12 that

$$\widetilde{\gamma}^2_{(F_4,c)} = -(p+q)(p+2q).$$

Note that $\theta_{F_4} = e^1 + e^2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. Then it is easy to see that the substitution of values (3.106) into formula (3.103) gives the same value of $\widetilde{\gamma}_{(F_4,c)}$. Similarly, it follows from Proposition 3.6.14 that

$$\widetilde{\gamma}^2_{(G_2,c)} = -\frac{3}{8}(p+q)(p+3q),$$

which is equal to the expression in formula (3.103) after the substitution of (3.107) and by using $\theta_{G_2} = \sqrt{3}e^1 = 2\alpha_1 + 3\alpha_2$. □

It is not clear to us how to formulate Proposition 3.7.2 for any not simply-laced (reduced) root system in a uniform way.

Let us also give another formula for $\widetilde{\gamma}_{(\mathcal{R},c)}$ in terms of the dual root system $\mathcal{R}^\vee$ of $\mathcal{R}$ where $\mathcal{R}^\vee = \{\beta^\vee \colon \beta \in \mathcal{R}\}$, and $\beta^\vee = \frac{2\beta}{\langle\beta,\beta\rangle}$. Then we have

$$\widetilde{\gamma}^2_{(\mathcal{R},c)} = -\frac{\langle\theta,\theta\rangle^2}{32}\Big(a_0\langle\theta^\vee,\theta^\vee\rangle + \sum_{i=1}^N \overline{n}_i^2 a_i\langle\alpha_i^\vee,\alpha_i^\vee\rangle\Big), \tag{3.108}$$

where coefficients $\overline{n}_i \in \mathbb{Z}_{\geq 0}$ are determined by the expansion $\theta^\vee = \sum_{i=1}^N \overline{n}_i\alpha_i^\vee$. Formula (3.108) follows from formula (3.103) by observing the relation $\overline{n}_i = \frac{n_i\langle\alpha_i,\alpha_i\rangle}{\langle\theta,\theta\rangle}$ for $1 \leq i \leq N$.

Let us explain how solutions for root systems considered above correspond to solutions considered by Shen in [47, 48].

Note that in [47, 48] a prepotential function for a Frobenius structure was considered, this prepotential gives a solution of WDVV equations for root systems $\mathcal{R}$. Let us first recall that solution which is given by

$$\Phi = -\frac{y^3}{6} + \frac{y}{2}c\sum_{\alpha\in\mathcal{R}^+}\alpha(x)^2 + \sum_{\alpha\in\mathcal{R}^+}k_\alpha\frac{a(\alpha^\vee,\alpha^\vee)}{\alpha(\alpha^\vee)}q(\alpha(x)), \tag{3.109}$$

where k_α is a $\mathcal{W}$-invariant multiplicity function, the function q satisfies

$$q'''(z) = \frac{1}{2}(\frac{1+e^z}{1-e^z}) = -\frac{1}{2}\coth(\frac{z}{2}),$$

and $a(\cdot,\cdot)$ is a symmetric bilinear form that satisfies

$$a(u,v) = \mu\langle u,v\rangle, \tag{3.110}$$

for some scalar $\mu \in \mathbb{R}$, and c is a constant corresponding to the bilinear form a and satisfies

$$a = c\sum_{\alpha>0}\alpha\otimes\alpha. \tag{3.111}$$

Let us explain that the solution (3.102) leads to a solution of the form (3.109).

Firstly, by (3.110) and (3.111) we have

$$c\sum_{\alpha\in\mathcal{R}^+}\alpha(x)^2 = a(x,x) = \mu\langle x,x\rangle. \tag{3.112}$$

Also, since $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ then we have

$$\alpha(\alpha^\vee) = 2, \quad a(\alpha^\vee, \alpha^\vee) = \frac{4\mu}{\langle\alpha,\alpha\rangle}. \tag{3.113}$$

Hence by substituting relations (3.110) and (3.113) in (3.109) and dividing the resulting function by $\frac{1}{2\mu}$ we get

$$\widehat{\Phi}(x,y) = \frac{1}{2\mu}\Phi = -\frac{y^3}{12\mu} + \frac{y}{4}\langle x,x\rangle + \sum_{\alpha\in\mathcal{R}^+}\frac{k_\alpha}{\langle\alpha,\alpha\rangle}q(\alpha(x)). \tag{3.114}$$

On the other hand for the solution (3.102), let us make change of variable by putting $x = \frac{\widetilde{x}}{2}$. Note that $\widetilde{f}(\frac{z}{2})) = -\frac{1}{4}q(z)$. Also, let $c_\alpha = k_\alpha$. Then after multiplying the solution (3.102) by -4, we get

$$F^{(1)}(\widetilde{x},y) = -\frac{2\widetilde{\gamma}}{3}y^3 - \frac{\widetilde{\gamma}y}{2}\langle\widetilde{x},\widetilde{x}\rangle + \sum_{\alpha\in\mathcal{R}^+}\frac{k_\alpha}{\langle\alpha,\alpha\rangle}q(\alpha(\widetilde{x})). \tag{3.115}$$

Now in order to compare (3.115) with (3.114) we make a change of variable by putting $y = -\frac{\widetilde{y}}{2\widetilde{\gamma}}$ in (3.115). Hence (3.115) becomes

$$F^{(2)}(\widetilde{x},\widetilde{y}) = \frac{\widetilde{y}^3}{12\widetilde{\gamma}^2} + \frac{\widetilde{y}}{4}\langle\widetilde{x},\widetilde{x}\rangle + \sum_{\alpha\in\mathcal{R}^+}\frac{k_\alpha}{\langle\alpha,\alpha\rangle}q(\alpha(\widetilde{x})). \tag{3.116}$$

Solutions $F^{(2)}$ given by (3.116) coincides with solution $\widehat{\Phi}$ given by (3.114) if we identify $\widetilde{x} = x, \widetilde{y} = y$ and $\mu = -\widetilde{\gamma}^2$.

The values of $\widetilde{\gamma}^2$ for not simply-laced root systems B_N, C_N, F_4, G_2 are given in Proposition 3.7.2. Note also that since $\widetilde{\gamma} = \widetilde{\gamma}_{(\mathcal{R},c)} = \gamma_{(\mathcal{R},d)}$, then by Propositions 3.3.4, 3.4.6, 3.6.2, 3.6.6, 3.6.9 the values of $\widetilde{\gamma}^2$ for simply-laced root systems D_N, A_N, E_6, E_7, E_8 are given by

$\mathcal{R}$	D_N	A_N	E_6	E_7	E_8
$-\widetilde{\gamma}^2$	$(N-2)s^2$	$\frac{(N+1)}{4}s^2$	$6s^2$	$12s^2$	$30s^2$

where $c_\alpha = s$ is the multiplicity of all vectors α in the simply-laced root system $\mathcal{R}$. This matches the scalar $\mu = -\widetilde{\gamma}^2$ given in [47].

Chapter 4

Trigonometric solutions without extra variable

In this chapter we consider trigonometric solutions of WDVV equations without extra variable. We review such solutions found in the literature. We generalize solutions of this type found in [33] corresponding to the root systems B_N. Our initial solution corresponds to the root system BC_N with Weyl-invariant multiplicities. Then we obtain a family of solutions with $N+2$ multiplicity parameters corresponding to the root system BC_N solutions by the restriction procedure. We also use these solutions to construct $\mathcal{N}=4$ supersymmetric mechanical systems.

4.1 Solutions of WDVV and related equations

Let $F = F(x_1, \ldots, x_n)$ be a function in $V \cong \mathbb{C}^n$. Consider a vector field

$$e = \sum_{i=1}^{n} A_i(x)\partial_{x_i},$$

where $A_i(x) = A_i(x_1, \ldots, x_n)$ are some functions. Define $n \times n$ matrix $B = (B_{ij})_{i,j=1}^{n}$ by

$$B_{ij} = e(F_{ij}) = \sum_{k=1}^{n} A_k(x)F_{ijk}, \quad i, j = 1, \ldots, n, \tag{4.1}$$

where

$$F_{ijk} = \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}.$$

Recall that WDVV equations (2.56) can be written equivalently in the form [41] (see also [40])

$$F_i B^{-1} F_j = F_j B^{-1} F_i, \quad i, j = 1, \dots, N, \tag{4.2}$$

where B is any linear combination of matrices F_k. We have in Chapter 2 the class of trigonometric solutions of WDVV equations (4.2) which have the form

$$F = \sum_{\alpha \in \mathcal{A}} c_\alpha f((\alpha, x)) + Q, \tag{4.3}$$

where $\mathcal{A}$ is a finite set of vectors in $V \cong \mathbb{C}^N$, $c_\alpha \in \mathbb{C}$, are some multiplicity parameters, the function f is given by (1.11), and $Q = Q(x, y)$ is a cubic polynomial which depends on $(x_1, \dots, x_N)$ and the auxiliary variable $y \in \mathbb{C}$. We have presented solutions of the form (4.3) for all root systems $\mathcal{A} = \mathcal{R}_+$. Also, solutions of the form (4.3) without full Weyl symmetry were considered where $\mathcal{A}$ is a trigonometric $\vee$-system. We also have shown that the class of trigonometric $\vee$-systems includes all root systems with $\mathcal{W}$-invariant multiplicities (Proposition 2.7.2).

In general, in the case when the cubic corrections are absent, that is $Q = 0$ the corresponding solution of the form (4.3) does not exist even for the case of root system $\mathcal{A}$ with invariant multiplicities c_α. However, in [33] Hoevenaars and Martini found such a solution for the root system of type B_N and specific choice of invariant multiplicities. Their result with respect to B_N root system is given by the following theorem ([33, Theorem 2.3]).

Theorem 4.1.1. [33] *The function*

$$F(x_1, \dots, x_N) = \sum_{1 \le i < j \le N} \left(\widetilde{f}(x_i - x_j) + \widetilde{f}(x_i + x_j) \right) + \eta \sum_{i=1}^{N} \widetilde{f}(x_i),$$

where function $\widetilde{f}$ *is given by* (2.100), *satisfies WDVV equations (4.2) if and only if* $\eta = -2(N-2)$.

The main idea of the proof of Theorem 4.1.1 in [33] is to find an appropriate invertable metric B such that WDVV equations (4.2) holds. In their proof of Theorem 4.1.1 the corresponding metric B was chosen in a specific way to be a multiple of identity, and hence the WDVV equations (4.2) reduce to the commutativity formula

$$F_i F_j = F_j F_i, \quad i, j = 1, \dots, N. \tag{4.4}$$

Solutions of the form (4.3) for equations (4.4) for root systems BC_N, F_4, G_2 with special collections of invariant multiplicities were found in [3]. Before we summarize these results, let us prove the following statement.

Proposition 4.1.2. *Let $\mathcal{A}$ be a finite set of vectors in $V \cong \mathbb{C}^N$ and $c_\alpha \in \mathbb{C}$, be some multiplicity parameters. Let*

$$F = \sum_{\alpha \in \mathcal{A}} c_\alpha f((\alpha, x)), \tag{4.5}$$

where the function f is given by (1.11)*. Let*

$$\widehat{F} = \sum_{\alpha \in \mathcal{A}} c_\alpha \widetilde{f}((\alpha, \widetilde{x})),$$

where function $\widetilde{f}$ is given by (2.100)*. Then we have*

(1) F satisfies WDVV equations (4.2) *if and only if $\widehat{F}$ satisfies WDVV equations* (4.2)*,*

(2) F satisfies the commutativity equations (4.4) *if and only if $\widehat{F}$ satisfies the commutativity equations* (4.4)*.*

Proof. By making the change of variable $x = -i\widetilde{x}$ we have

$$f(x) = f(-i\widetilde{x}) = \frac{1}{6}i(-i\widetilde{x})^3 + \frac{1}{4}Li_3(e^{-2i(-i\widetilde{x})}) = -(\frac{1}{6}\widetilde{x}^3 - \frac{1}{4}Li_3(e^{-2\widetilde{x}})) = -\widetilde{f}(\widetilde{x}).$$

The result follows since $\frac{\partial^3 F}{\partial x_j \partial x_k \partial x_l} = \frac{i\partial^3 F}{\partial \widetilde{x}_j \partial \widetilde{x}_k \partial \widetilde{x}_l}$. □

The following theorem gives the solution of equations (4.4) corresponding to the root system BC_N (see [3, Theorem 6.6], see also [4]).

Theorem 4.1.3. [3] *The function*

$$F = \sum_{i=1}^{N} \Big(rf(x_i) + sf(2x_i) \Big) + q \sum_{i<j}^{N} f(x_i \pm x_j)$$

satisfies conditions (4.4) if and only if $r = -8s - 2(N-2)q$.

The following theorem gives the solution of equations (4.4) corresponding to the root system F_4 (see [3, Theorem 6.8], see also [4]).

Theorem 4.1.4. [3] *The function*

$$F = r\sum_{i=1}^{4} f(x_i) + r \sum_{\epsilon_i \in \{1,-1\}} f\Big(\frac{1}{2}(\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + \epsilon_4 x_4)\Big) + q\sum_{i<j}^{4} f(x_i \pm x_j)$$

satisfies conditions (4.4) if and only if $r = -2q$ or $r = -4q$.

The following theorem gives the solution of equations (4.4) corresponding to the root system G_2 considered in three-dimensional space (see [3, Theorem 6.9], see also [4]).

Theorem 4.1.5. [3] *The function*

$$F = s\sum_{i<j}^{3} f\big(x_i - x_j\big) + \frac{r}{2}\sum_{\sigma\in S_3} f\big(2x_{\sigma(1)} - x_{\sigma(2)} - x_{\sigma(3)}\big)$$

satisfies conditions (4.4) if and only if $s = -3r$ *or* $s = -9r$.

Remark 4.1.6. (1) It was shown in [4, Remark 4.5.7] (see also [3]) that if $\mathcal{R} = A_{N-1}$ with constant multiplicity then equations (4.4) do not hold.

(2) In [3] solutions of the commutativity equations of the form (4.5) were considered with f replaced by $\widetilde{f}$. Proposition 4.1.2 allows us to deal with the stated trigonometric version of these results.

A question arises as to whether usual WDVV equations hold for these prepotentials considered in Theorems 4.1.3, 4.1.4, 4.1.5. Since WDVV equations (4.2) are equivalent to the system (4.4) when the metric B is proportional to the standard metric δ_i^j, the question reduces to the question whether the identity metric can be obtained as a linear combination of the third order derivatives for these prepotentials. We answer this question positively for the case of the root system BC_N. In fact we find $(N+2)$-parametric family of solutions whose underlying configuration $\mathcal{A}$ is the positive half of BC_N root system. The choice of the metric B is motivated by the metric B for the root system B_N which was chosen in [33]. Thus the $(N+2)$-parametric family of solutions reduces to the two-parameter family of prepotentials given in Theorem 4.1.3 after specialization of parameters. In fact we show firstly that a two-parameter family of prepotentilas given in Theorem 4.1.3 satisfies WDVV equations (4.2) for a suitable combination of the third order derivatives of the prepotential which gives metric B proportional to the identity. This also generalized Theorem 4.1.1. Then we generalize these considerations to obtain the family of solutions with many parameters by taking special restrictions of these solutions using procedure similar to one we applied in Section 3.3.

4.2 Metric for a family of BC_n type configurations

We are going to present solution F to the equations (4.2) for a suitable vector field e. This solution is related to BC_n root system with prescribed multiplicities of the root vectors.

$$\begin{aligned}
&e_i, \quad \text{with multiplicity} \quad rm_i, \quad 1\le i\le n,\\
&2e_i, \quad \text{with multiplicity} \quad sm_i + \frac{1}{2}qm_i(m_i-1), \quad 1\le i\le n,\\
&e_i\pm e_j, \quad \text{with multiplicity} \quad qm_im_j, \quad 1\le i<j\le n,
\end{aligned}$$

where $e_1, \ldots, e_n$ is the standard basis in $\mathbb{C}^n$. Note that if all the multiplicities $m_i = 1$ then the configuration reduces to the configuration $BC_n(r, s, q)$ which is a positive half of the root system BC_n with an invariant collection of multiplicities r, s, q.

Let us consider the function F given by (4.3) with $Q = 0$, that is

$$F = \sum_{\alpha \in BC_n(r,s,q;\underline{m})} c_\alpha f((\alpha, x)). \tag{4.6}$$

More explicitly the function F can be written as follows:

$$F = \sum_{i=1}^{n} rm_i f(x_i) + \sum_{i=1}^{n} \big(sm_i + \frac{1}{2} qm_i(m_i - 1)\big) f(2x_i) + \sum_{i<j}^{n} qm_i m_j f(x_i \pm x_j). \tag{4.7}$$

Let us now define the matrix (4.1) by taking

$$A_k = \sin 2x_k, \tag{4.8}$$

where $k = 1, ..., n$. This choice is motivated by [33] where a solution of WDVV equation (4.2) for the root system B_n was obtained.

Let us also define the following functions:

$$b_{ij} = \begin{cases} \cot(x_i + x_j) + \cot(x_i - x_j), & 1 \le i \ne j \le n, \\ 0, \quad i = j, \end{cases}$$

and

$$b_i = \cot x_i, \qquad \widetilde{b_i} = \cot 2x_i, \quad i = 1, \ldots, n.$$

Lemma 4.2.1. *We have the following expression for the third order derivatives of F:*

$$F_{klt} = rm_k b_k \delta_{kl} \delta_{lt} + 4(2sm_k + qm_k(m_k - 1))\widetilde{b}_k \delta_{kl}\delta_{lt} + q\delta_{kl}\delta_{lt} \sum_{\substack{j=1 \\ j \ne k}}^{n} m_j m_k b_{kj} \tag{4.9}$$
$$+ qm_t m_k b_{tk} \delta_{kl} + qm_l m_k b_{lk} \delta_{kt} + qm_k m_l b_{kl} \delta_{lt},$$

where $k, l, t = 1, ..., n$, and δ is the Kronecker symbol.

Proof. We note that the first two terms in (4.9) are obtained from the first two terms in formula (4.7). The last term in (4.7) contributes the following sum in F_{klt}:

$$q \sum_{i<j}^{n} m_i m_j \Big((\delta_{ki} + \delta_{kj})(\delta_{li} + \delta_{lj})(\delta_{ti} + \delta_{tj}) \cot(x_i + x_j) \tag{4.10}$$
$$+ (\delta_{ki} - \delta_{kj})(\delta_{li} - \delta_{lj})(\delta_{ti} - \delta_{tj}) \cot(x_i - x_j) \Big).$$

We rearrange some of the terms in (4.10) as follows:

$$q\sum_{i<j}^{n} m_i m_j(\delta_{ki}\delta_{li}\delta_{ti}+\delta_{kj}\delta_{lj}\delta_{tj})\cot(x_i+x_j) = q\sum_{\substack{j=1\\ j\neq k}}^{n} m_j m_k \delta_{kl}\delta_{lt}\cot(x_k+x_j), \tag{4.11}$$

and

$$q\sum_{i<j}^{n} m_i m_j(\delta_{ki}\delta_{li}\delta_{ti}-\delta_{kj}\delta_{lj}\delta_{tj})\cot(x_i-x_j) = q\sum_{\substack{j=1\\ j\neq k}}^{n} m_j m_k \delta_{kl}\delta_{lt}\cot(x_k-x_j). \tag{4.12}$$

The sum of expressions (4.11) and (4.12) equals the third term in (4.9). Further on, let us collect the following terms from (4.10):

$$q\sum_{i<j}^{n} m_i m_j\big(\delta_{ki}\delta_{li}\delta_{tj}+\delta_{kj}\delta_{lj}\delta_{ti}\big)\cot(x_i+x_j) = \begin{cases} q m_t m_k \delta_{kl}\cot(x_t+x_k), & t\neq k,\\ 0, & t=k, \end{cases} \tag{4.13}$$

and

$$q\sum_{i<j}^{n} m_i m_j\big(\delta_{ki}\delta_{li}\delta_{tj}-\delta_{kj}\delta_{lj}\delta_{ti}\big)\cot(x_j-x_i) = \begin{cases} q m_t m_k \delta_{kl}\cot(x_t-x_k), & t\neq k,\\ 0, & t=k. \end{cases} \tag{4.14}$$

The sum of the terms (4.13) and (4.14) equals to $q m_t m_k b_{tk}\delta_{kl}$. Similarly, the sum of the terms

$$q\sum_{i<j}^{n} m_i m_j\big(\delta_{ki}\delta_{lj}\delta_{tj}+\delta_{kj}\delta_{li}\delta_{ti}\big)\cot(x_i+x_j) = \begin{cases} q m_k m_l \delta_{lt}\cot(x_k+x_l), & k\neq l,\\ 0, & k=l, \end{cases}$$

and

$$q\sum_{i<j}^{n} m_i m_j\big(\delta_{ki}\delta_{lj}\delta_{tj}-\delta_{kj}\delta_{li}\delta_{ti}\big)\cot(x_i-x_j) = \begin{cases} q m_k m_l \delta_{lt}\cot(x_k-x_l), & k\neq l,\\ 0, & k=l \end{cases}$$

equals $q m_k m_l b_{kl}\delta_{lt}$. Finally, the sum of the following terms

$$q\sum_{i<j}^{n} m_i m_j\big(\delta_{ki}\delta_{lj}\delta_{ti}+\delta_{kj}\delta_{li}\delta_{tj}\big)\cot(x_i+x_j) = \begin{cases} q m_l m_k \delta_{kt}\cot(x_l+x_k), & l\neq k,\\ 0, & l=k, \end{cases}$$

and

$$q\sum_{i<j}^{n} m_i m_j\big(\delta_{ki}\delta_{lj}\delta_{ti}-\delta_{kj}\delta_{li}\delta_{tj}\big)\cot(x_j-x_i) = \begin{cases} q m_l m_k \delta_{kt}\cot(x_l-x_k), & k\neq l,\\ 0, & l=k \end{cases}$$

equals $qm_l m_k b_{lk}\delta_{kt}$. The statement follows. □

Lemma 4.2.2. *We have the following identities:*

$$A_k b_{kj} + A_j b_{jk} = 2(\cos 2x_k + \cos 2x_j), \quad 1 \le j \ne k \le n, \tag{4.15}$$

and

$$A_k b_{jk} + A_j b_{kj} = 0, \quad j, k = 1, \dots, n, \tag{4.16}$$

where A_k is given by (4.8).

Proof. We have

$$A_k b_{kj} + A_j b_{jk} = \frac{A_k \sin 2x_k - A_j \sin 2x_j}{\sin(x_k + x_j)\sin(x_k - x_j)} = \frac{\cos 4x_k - \cos 4x_j}{\cos 2x_k - \cos 2x_j},$$

which implies the first formula (4.15). Identity (4.16) follows similarly. □

Now we show that the matrix B is diagonal. Moreover, it is proportional to a constant diagonal matrix under a particular restriction on the parameters r, q, s.

Proposition 4.2.3. *The matrix $B = B(x)$ with the matrix entries*

$$B_{lt} = \sum_{k=1}^{n} A_k F_{klt}, \quad l, t = 1, \dots, n$$

is diagonal. Furthermore, if the multiplicities r, q, s and $\underline{m}$ satisfy the relation

$$r = -8s - 2q(N - 2), \tag{4.17}$$

where $N = \sum_{k=1}^{n} m_k$, then the matrix B takes the form

$$B_{lt} = m_l h(x)\delta_{lt}, \tag{4.18}$$

where $h(x) = 2q\sum_{k=1}^{n} m_k \cos 2x_k + r$.

Proof. It follows by Lemma 4.2.1 that for $l \ne t$

$$B_{lt} = qm_l m_t(A_l b_{tl} + A_t b_{lt}),$$

which is equal to zero by Lemma 4.2.2.

Let us now consider the diagonal entries of B. We have by Lemmas 4.2.1, 4.2.2

$$\begin{aligned}B_{ll} &= rm_l A_l b_l + 4\big(2sm_l + qm_l(m_l-1)\big)A_l\tilde{b}_l + q\sum_{\substack{k=1\\k\neq l}}^{n} m_k m_l (A_l b_{lk} + A_k b_{kl})\\ &= 2rm_l\cos^2 x_l + 4\big(2sm_l + qm_l(m_l-1)\big)\cos 2x_l + 2q\sum_{\substack{k=1\\k\neq l}}^{n} m_k m_l(\cos 2x_k + \cos 2x_l).\end{aligned}$$

Then

$$\begin{aligned}B_{ll} &= 2rm_l\cos^2 x_l + 4\big(2sm_l + qm_l(m_l-1)\big)\cos 2x_l + 2q(N-2m_l)m_l\cos 2x_l\\ &\quad + 2qm_l\sum_{k=1}^{n} m_k\cos 2x_k\\ &= m_l\Big(\big(r+8s+2q(N-2)\big)\cos 2x_l + 2q\sum_{k=1}^{n} m_k\cos 2x_k + r\Big),\end{aligned}$$

which implies the statement. □

Since now the matrix B is constructed to be proportional to the identity matrix, then WDVV equations (4.2) are equivalent to the system of equations (4.4) in this case. Now we define a commutative algebra on T_xV as follows. Below summation over repeated indices will be assumed. Let us now assume that multiplicities $m_i = 1$ for all $i = 1, ..., n$. For any vector $v = (v_1, \dots, v_n) \in V$ let us introduce the vector field $\partial_v = v_i\partial_{x_i} \in \Gamma(TV)$. For any $u = (u_1, \dots, u_n) \in V$ we define the following multiplication on the tangent space T_xV for generic $x \in V$:

$$\partial_u * \partial_v = F_{ijk}u_i v_j \partial_{x_k}. \tag{4.19}$$

Note that multiplication (4.19) defines a commutative algebra on T_xV. The following theorem takes place.

Theorem 4.2.4. *Suppose that parameters r, s and q satisfy the linear relation (4.17). Then function*

$$F = r\sum_{i=1}^{n} f(x_i) + s\sum_{i=1}^{n} f(2x_i) + q\sum_{1\le i<j\le n}\big(f(x_i+x_j) + f(x_i-x_j)\big) \tag{4.20}$$

satisfies WDVV equations (4.2) *where B is determined by (4.1) and (4.8). Also, multiplication* (4.19) *is associative.*

Proof. It has been shown in [3] that the function (4.20) satisfies the following system of equations if the linear relation (4.17) holds:

$$F_iF_j = F_jF_i, \tag{4.21}$$

for all $i, j = 1, \ldots, n$. It then follows from Proposition 4.2.3 that conditions (4.21) are equivalent to WDVV equations (4.2) since the matirx B is proportional to the identity matrix. Also it is easy to see that associativity of the multiplication (4.19) is equivalent to the relation (4.21). $\square$

Thus, Theorem 4.2.4 leads to Theorem 4.1.3 by the given metric B. Moreover, by letting parameters $s = 0$, $q = 1$, $r = \eta$, Theorem 4.2.4 coincides with Theorem 4.1.1, and in this way a generalisation of Theorem 4.1.1 is obtained. In fact, one can consider a generalisation of the configuration BC_N and show that the corresponding function also satisfies WDVV equations. By this we aim to generalize Theorem 4.2.4 to the configuration $BC_n(q, r, s; \underline{m})$, that is to the case of arbitrary multiplicities m_i. This generalization can be formulated as follows. In the remaining part of the chapter we prove generalization of Theorem 4.2.4 to the configuration $BC_n(q, r, s; \underline{m})$, that is to the case of arbitrary multiplicities m_i. This generalization can be formulated as follows.

Theorem 4.2.5. *Suppose parameters r, s, q and $\underline{m}$ satisfy the relation*

$$r = -8s - 2q(N - 2), \tag{4.22}$$

where $N = \sum_{i=1}^{n} m_i$. Then prepotential (4.7) satisfies WDVV equations (4.2) where $B = \sum_{i=1}^{n} \sin 2x_i F_i$.

Remark 4.2.6. Theorem 4.2.5 generalizes Theorem 2.3 from [33]. In this case we have all $m_i = 1$ and $s = 0$. Then putting $q = 1$ we get the standard B_N root system and the condition (4.22) reduces to $r = -2(N - 2)$ which is the multiplicity of the short root of B_N root system considered in [33].

Remark 4.2.7. In the rational limit solutions (4.7) of WDVV equations reduce to B_n family of $\vee$-systems found in [13].

4.3 Proof through restrictions

Let $\mathcal{A}$ be the configuration $\mathcal{A} = BC_N(r, s, q) \subset W \cong \mathbb{C}^N, N \in \mathbb{N}$. Let $e_1, \ldots, e_N$ be the standard basis of W. Let $(\cdot, \cdot)$ be the standard inner product which is defined by

$$(x, y) = \sum_{i=1}^{N} x_i y_i, \tag{4.23}$$

where $x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in W$. Let $n \in \mathbb{N}$ and $\underline{m} = (m_1, \ldots, m_n)$ with $m_i \in \mathbb{N}$ such that $\sum_{i=1}^{n} m_i = N$. Let us consider subsystem $\mathcal{B} \subset \mathcal{A}$ as follows:

$$\mathcal{B} = \{e_{\sum_{j=1}^{i-1} m_j + k} - e_{\sum_{j=1}^{i-1} m_j + l}, \quad 1 \leq k < l \leq m_i, i = 1, \ldots, n\}.$$

Now let us consider the corresponding subspace of W of dimension n given by

$$W_{\mathcal{B}} = \{x \in W : (\beta, x) = 0, \forall \beta \in \mathcal{B}\}.$$

More explicitly, vectors $x = (x_1, \ldots, x_N) \in W_{\mathcal{B}}$ satisfy conditions:

$$\begin{cases} x_1 = \cdots = x_{m_1}, \\ x_{m_1+1} = \cdots = x_{m_1+m_2}, \\ \vdots \\ x_{\sum_{i=1}^{n-1} m_i + 1} = \cdots = x_N. \end{cases}$$

For any vector $v = (v_1, \ldots, v_N) \in W$ let us define the vector field $\partial_v = v_i \partial_{x_i} \in TW$. For any $u = (u_1, \ldots, u_N) \in W$ we define the following multiplication on the tangent plane $T_x W$ for generic $x \in W$:

$$\partial_u * \partial_v = u_i v_j F_{ijk} \partial_{x_k}, \tag{4.24}$$

where the function F is given by

$$F = \sum_{\alpha \in \mathcal{A}} c_\alpha f((\alpha, x)). \tag{4.25}$$

Assume that parameters r, s, q and $\underline{m}$ satisfy the relation $r = -8s - 2q(N-2)$. Then multiplication (4.24) is associative by Theorem 4.2.4 (applied with $n = N$). Note that function (4.25) satisfies

$$F_{ijk} = \sum_{\alpha \in \mathcal{A}} c_\alpha \alpha_i \alpha_j \alpha_k \cot(\alpha, x),$$

hence multiplication (4.24) can be expressed as follows:

$$\partial_u * \partial_v = \sum_{\alpha \in \mathcal{A}} c_\alpha (\alpha, u)(\alpha, v) \cot(\alpha, x) \partial_\alpha. \tag{4.26}$$

If we identify W with $T_x W \cong W$, then multiplication (4.26) takes the form

$$u * v = \sum_{\alpha \in \mathcal{A}} c_\alpha (\alpha, u)(\alpha, v) \cot(\alpha, x) \alpha. \tag{4.27}$$

Define $M_{\mathcal{B}} = W_{\mathcal{B}} \setminus \bigcup_{\alpha \in \mathcal{A} \setminus \mathcal{B}} \Pi_\alpha$, where $\Pi_\alpha = \{x \in W \colon (\alpha, x) = 0\}$. Consider now a point $x_0 \in M_{\mathcal{B}}$ and two tangent vectors $u_0, v_0 \in T_{x_0} M_{\mathcal{B}}$. We extend vectors u_0 and v_0 to two local analytic vector fields $u(x), v(x)$ in the neighbourhood U of x_0 that are tangent to the subspace $W_{\mathcal{B}}$ at any point $x \in M_{\mathcal{B}} \cap U$ such that $u_0 = u(x_0)$ and $v_0 = v(x_0)$. Now we want to study the limit of $u(x) * v(x)$ when x tends to x_0. The limit may have singularities at $x \in W_{\mathcal{B}}$ as $\cot(\alpha, x)$ with $\alpha \in \mathcal{B}$ is not defined for such x. Also we note that outside $W_{\mathcal{B}}$

we have a well-defined multiplication $u(x) * v(x)$. Similarly to the rational case considered in [25] and trigonometric case with extra variable [1] the following lemma holds.

Lemma 4.3.1. *The product $u(x) * v(x)$ has a limit when x tends to $x_0 \in M_{\mathcal{B}}$ given by*

$$u_0 * v_0 = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} c_\alpha (\alpha, u_0)(\alpha, v_0) \cot(\alpha, x_0)\alpha. \tag{4.28}$$

In particular $u_0 * v_0$ is determined by u_0 and v_0 only.

Now we are going to show that the product $u_0 * v_0$ belongs to $T_{x_0} M_{\mathcal{B}}$. We will need the following lemma (cf. [25], [1]).

Lemma 4.3.2. *Let $\alpha \in \mathcal{A}$. Let $x \in \Pi_\alpha$ be generic. Then the identity*

$$\sum_{\substack{\beta \in \mathcal{A} \\ \beta \nsim \alpha}} c_\beta(\alpha, \beta) \cot(\beta, x) B_{\alpha,\beta}(a, b)\alpha \wedge \beta = 0 \tag{4.29}$$

holds for all $a, b \in V$ provided that $(\alpha, x) = 0$, where $B_{\alpha,\beta}(a, b) = (\alpha, a)(\beta, b) - (\alpha, b)(\beta, a)$ and $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$.

Proof. For any $\beta \in \mathcal{A}$ such that $\beta \nsim \alpha$ let $\gamma = s_\alpha \beta$, where s_α is the orthogonal reflection about Π_α. Note that $\cot(\gamma, x) = \cot(\beta, x)$ at $(\alpha, x) = 0$. Also note that

$$(\alpha, \gamma) = -(\alpha, \beta), \quad B_{\alpha,\gamma}(a, b) = B_{\alpha,\beta}(a, b), \quad \alpha \wedge \gamma = \alpha \wedge \beta.$$

We have that either γ or $-\gamma$ is an element of $\mathcal{A}$. Suppose firstly that $\gamma \in \mathcal{A}$. Then

$$c_\beta(\alpha, \beta) \cot(\beta, x) B_{\alpha,\beta}(a, b)\alpha \wedge \beta + c_\gamma(\alpha, \gamma) \cot(\gamma, x) B_{\alpha,\gamma}(a, b)\alpha \wedge \gamma = 0 \tag{4.30}$$

at $(\alpha, x) = 0$ since multiplicities are B_N-invariant. If one replaces γ with $-\gamma$ then (4.30) holds as well. □

Proposition 4.3.3. *Let $u, v \in T_x M_{\mathcal{B}}$ where $x \in M_{\mathcal{B}}$. Then $u * v \in T_x M_{\mathcal{B}}$.*

Proof. By Lemma 4.3.1 it is enough to show that

$$\sum_{\beta \in \mathcal{A} \setminus \mathcal{B}} c_\beta(\beta, u)(\beta, v)(\alpha, \beta) \cot(\beta, x) = 0 \tag{4.31}$$

for all $\alpha \in \mathcal{B}$. Assume firstly that $W_{\mathcal{B}}$ has codimension 1. By Lemma 4.3.2 we get

$$\sum_{\substack{\beta \in \mathcal{A} \\ \beta \nsim \alpha}} c_\beta(\alpha, \beta) \cot(\beta, x) \Big((\alpha, a)(\beta, b) - (\alpha, b)(\beta, a) \Big) \Big((\alpha, y)(\beta, z) - (\alpha, z)(\beta, y) \Big) = 0 \tag{4.32}$$

for any $a, b, y, z \in V$ and generic $x \in \Pi_\alpha$. Assume that $a, y \notin \Pi_\alpha$ and let $b = u \in \Pi_\alpha$ and $z = v \in \Pi_\alpha$. Then $(\alpha, b) = (\alpha, z) = 0$ and relation (4.32) implies that

$$\sum_{\substack{\beta \in \mathcal{A} \\ \beta \nsim \alpha}} c_\beta (\alpha, \beta)(\beta, u)(\beta, v) \cot(\beta, x) = 0.$$

As $W_\mathcal{B}$ has codimension 1 the relation $\beta \nsim \alpha$ is equivalent to $\beta \in \mathcal{A} \setminus \mathcal{B}$ and lemma follows.

Let us now suppose that $W_\mathcal{B}$ has codimension 2. Let $\alpha, \gamma \in \mathcal{B}$ be non-proportional to each other. By the above arguments for generic $x \in \Pi_\alpha$ and $u, v \in T_x\Pi_\alpha$, we have $u * v \in T_x\Pi_\alpha$. Similarly, if $x \in \Pi_\gamma$ is generic and $u, v \in T_x\Pi_\gamma$, then $u * v \in T_x\Pi_\gamma$. By Lemma 4.3.1, $u * v$ exists for $x \in M_\mathcal{B}$ and $u, v \in T_x M_\mathcal{B}$. It follows that for any $x \in M_\mathcal{B}$ we have $u * v \in T_x M_\mathcal{B}$, which proves the statement for the case when $W_\mathcal{B}$ has codimension 2. General $W_\mathcal{B}$ is dealt with similarly. □

Consider now the orthogonal decomposition

$$W = W_\mathcal{B} \oplus W_\mathcal{B}^\perp \tag{4.33}$$

with respect to the standard inner product. Any $\alpha \in W$ can be written as

$$\alpha = \widetilde{\alpha} + w, \tag{4.34}$$

where $\widetilde{\alpha} \in W_\mathcal{B}$ is the orthogonal projection of vector α to $W_\mathcal{B}$ and $w \in W_\mathcal{B}^\perp$. For any $x_0 \in M_\mathcal{B}$ and $u, v \in T_{x_0} M_\mathcal{B}$ one can represent product $u * v$ as

$$u * v = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} c_\alpha (\alpha, u)(\alpha, v) \cot(\alpha, x_0) \widetilde{\alpha} \tag{4.35}$$

by Proposition 4.3.3. Hence, we have

$$\partial_u * \partial_v = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} c_\alpha (\alpha, u)(\alpha, v) \cot(\alpha, x) \partial_{\widetilde{\alpha}}. \tag{4.36}$$

Let us define vectors $f_i, 1 \le i \le n$, by

$$f_i = \sum_{j=1}^{m_i} e_{\sum_{s=1}^{i-1} m_s + j}. \tag{4.37}$$

These vectors form a basis for $W_\mathcal{B}$.

The following lemma gives the general formula for the orthogonal projection of any vector $u \in W$ to $W_\mathcal{B}$.

Lemma 4.3.4. *Let $u = \sum_{i=1}^{N} u_i e_i \in W$. Then the projection $\widetilde{u}$ has the form*

$$\widetilde{u} = \Bigg(\underbrace{\frac{1}{m_1}\sum_{i=1}^{m_1} u_i, \dots, \frac{1}{m_1}\sum_{i=1}^{m_1} u_i}_{m_1}, \dots, \underbrace{\frac{1}{m_n}\sum_{i=1}^{m_n} u_{\sum_{s=1}^{n-1} m_s + i}, \dots, \frac{1}{m_n}\sum_{i=1}^{m_n} u_{\sum_{s=1}^{n-1} m_s + i}}_{m_n}\Bigg). \tag{4.38}$$

Let us now project $\mathcal{A}$ to the subspace $W_{\mathcal{B}}$. Notice that by Lemma 4.3.4 we have

$$\widetilde{e}_{\sum_{s=1}^{i-1} m_s + 1} = \dots = \widetilde{e}_{\sum_{s=1}^{i} m_s} = m_i^{-1} f_i, \quad i = 1, \dots, n.$$

Let us denote the projected system as $\widetilde{\mathcal{A}} = BC_n(q, r, s; \underline{m}) \subset W_{\mathcal{B}} \cong \mathbb{C}^n$. It consists of vectors α with the corresponding multiplicities c_α given as follows:

$$\begin{aligned}
&\widehat{f}_i = m_i^{-1} f_i, \quad \text{with multiplicity} \quad r m_i, \quad 1 \le i \le n,\\
&2\widehat{f}_i = 2 m_i^{-1} f_i, \quad \text{with multiplicity} \quad s m_i + \frac{1}{2} q m_i (m_i - 1), \quad 1 \le i \le n,\\
&\widehat{f}_i \pm \widehat{f}_j = m_i^{-1} f_i \pm m_j^{-1} f_j, \quad \text{with multiplicity} \quad q m_i m_j, \quad 1 \le i < j \le n.
\end{aligned}$$

By Lemma 4.3.4, for any $\alpha \in W$, its orthogonal projection has the form

$$\widetilde{\alpha} = \sum_{k=1}^{n} \widetilde{\alpha}_k f_k,$$

where the basis f_k is given by (4.37) and

$$\widetilde{\alpha}_k = \frac{(\widetilde{\alpha}, f_k)}{(f_k, f_k)} = \frac{(\alpha, f_k)}{m_k}. \tag{4.39}$$

Let us define

$$\widetilde{F}(\widetilde{x}) = \sum_{\gamma \in \widetilde{\mathcal{A}}} c_\gamma f((\gamma, \widetilde{x})), \tag{4.40}$$

where

$$\widetilde{x} = \sum_{i=1}^{n} \widetilde{x}_i f_i \in W_{\mathcal{B}}. \tag{4.41}$$

Note that function (4.40) can also be represented as

$$\widetilde{F}(\widetilde{x}) = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} c_\alpha f((\alpha, \widetilde{x})).$$

Let $\widetilde{F}_i$ be the $n \times n$ matrix constructed from the third-order partial derivatives of the function $\widetilde{F}$, that is

$$(\widetilde{F}_i)_{jk} = \widetilde{F}_{ijk} = \frac{\partial^3 \widetilde{F}}{\partial \widetilde{x}_i \partial \widetilde{x}_j \partial \widetilde{x}_k},$$

$i, j, k = 1, \ldots, n.$

The following lemma gives another way to represent multiplication (4.36) on $W_{\mathcal{B}}$.

Lemma 4.3.5. *The multiplication (4.36) takes the form*

$$\partial_{f_i} * \partial_{f_j} = \sum_{k=1}^{n} m_k^{-1} \widetilde{F}_{ijk} \partial_{f_k}, \quad i, j = 1, ..., n.$$

Proof. We rearrange $\partial_{\widetilde{\alpha}}$ in the right-hand side of (4.36) as

$$\partial_{\widetilde{\alpha}} = \sum_{k=1}^{n} \widetilde{\alpha}_k \partial_{f_k} = \sum_{k=1}^{n} m_k^{-1} (\alpha, f_k) \partial_{f_k}$$

by (4.39). Therefore the multiplication (4.36) can be rewritten as

$$\begin{aligned} \partial_{f_i} * \partial_{f_j} &= \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} \sum_{k=1}^{n} c_\alpha m_k^{-1} (\alpha, f_i)(\alpha, f_j)(\alpha, f_k) \cot(\alpha, \widetilde{x}) \partial_{f_k} \\ &= \sum_{k=1}^{n} m_k^{-1} \widetilde{F}_{ijk} \partial_{f_k}, \quad i, j = 1, ..., n, \end{aligned}$$

as required. □

Let $H_{\mathcal{B}}$ be the matrix of the restriction of the standard inner product on $W_{\mathcal{B}}$ in the basis $f_1, \ldots, f_n$. That is

$$(H_{\mathcal{B}})_{lt} = (f_l, f_t) = m_l \delta_{lt}, \quad l, t = 1, \ldots, n. \tag{4.42}$$

Lemma 4.3.6. *The matrix $H_{\mathcal{B}}$ can be written as a linear combination*

$$H_{\mathcal{B}} = \sum_{i=1}^{n} a_i \widetilde{F}_i,$$

where functions a_i are given by $a_i = h(\widetilde{x})^{-1} \sin 2\widetilde{x}_i$, and $h(\widetilde{x}) = 2q \sum_{k=1}^{n} m_k \cos 2\widetilde{x}_k + r$.

Proof. By Proposition 4.2.3 and (4.42), we have $H_{\mathcal{B}} = h(\widetilde{x})^{-1} B(\widetilde{x})$, where $B(\widetilde{x}) = \sum_{i=1}^{n} (\sin 2\widetilde{x}_i) \widetilde{F}_i$. This implies the statement. □

The previous considerations allow us to prove the following theorem, which is a version of stated earlier Theorem 4.2.5 where multiplicities m_i do not have to be integer.

Theorem 4.3.7. *Suppose parameters* r, s, q *and* $\underline{m}$ *satisfy the relation*

$$r = -8s - 2q(N-2), \tag{4.43}$$

where $N = \sum_{i=1}^{n} m_i$. *Then the prepotential*

$$\widetilde{F} = \sum_{\alpha \in \widetilde{\mathcal{A}}} c_\alpha f((\alpha, \widetilde{x})), \quad \widetilde{x} \in W_{\mathcal{B}}, \tag{4.44}$$

satisfies the WDVV equations

$$\widetilde{F}_i B^{-1} \widetilde{F}_j = \widetilde{F}_j B^{-1} \widetilde{F}_i, \quad i, j = 1, ..., n, \tag{4.45}$$

where $B = \sum_{i=1}^{n} \sin 2\widetilde{x}_i \widetilde{F}_i$.

The corresponding associative multiplication has the form

$$u * v = \sum_{\alpha \in \widetilde{\mathcal{A}}} c_\alpha (\alpha, u)(\alpha, v) \cot(\alpha, \widetilde{x}) \alpha$$

for any $u, v \in T_{\widetilde{x}} M_{\mathcal{B}}, \widetilde{x} \in M_{\mathcal{B}}$.

Proof. Let us assume firstly that $m_i \in \mathbb{N}$ for any i. Consider the multiplication

$$u * v = \sum_{\alpha \in BC_N(r,s,q)} c_\alpha (\alpha, u)(\alpha, v) \cot(\alpha, x) \alpha \tag{4.46}$$

on the tangent space $T_x W$ for $x \in W$. By Theorem 4.2.4 the multiplication (4.46) is associative. Now as x tends to $\widetilde{x} \in M_{\mathcal{B}}$, by Lemmas 4.3.1, 4.3.5 and Proposition 4.3.3 this product restricts to an associative product on the tangent space $T_{\widetilde{x}} M_{\mathcal{B}}$ which has the form

$$\partial_{f_i} * \partial_{f_j} = \sum_{l=1}^{n} m_l^{-1} \widetilde{F}_{ijl} \partial_{f_l}, \quad 1 \le i, j \le n. \tag{4.47}$$

The associativity condition

$$(\partial_{f_i} * \partial_{f_j}) * \partial_{f_k} = \partial_{f_i} * (\partial_{f_j} * \partial_{f_k}),$$

for any $i, j, k = 1, \ldots, n$, can be rearranged as

$$\sum_{l=1}^{n} m_l^{-1} \widetilde{F}_{ijl} \partial_{f_l} * \partial_{f_k} = \sum_{l=1}^{n} m_l^{-1} \widetilde{F}_{jkl} \partial_{f_i} * \partial_{f_l}.$$

Hence, we have

$$\sum_{l=1}^{n} m_l^{-1} \widetilde{F}_{ijl} \widetilde{F}_{lkp} = \sum_{l=1}^{n} m_l^{-1} \widetilde{F}_{jkl} \widetilde{F}_{ilp}, \tag{4.48}$$

for any $i, j, k, p = 1, \ldots, n$. In the matrix form we have

$$\widetilde{F}_i H_{\mathcal{B}}^{-1} \widetilde{F}_k = \widetilde{F}_k H_{\mathcal{B}}^{-1} \widetilde{F}_i.$$

By Lemma 4.3.6 we obtain relation (4.45) where $B = \sum_{i=1}^{n} \sinh 2\widetilde{x}_i \widetilde{F}_i$ as required. This proves the theorem for the case when $m_i \in \mathbb{N}$. Since m_i can take arbitrary integer values the statement follows for general m_i as well. $\square$

4.4 Application to supersymmetric mechanical systems

One of the topics where (generalised) WDVV equations emerge include $\mathcal{N} = 4$ supersymmetric mechanics. These corresponding differential equations are similar to what we have discussed in Chapter 2 but they rather have the form of commutativity

$$F_i F_j = F_j F_i, \quad i, j = 1, \ldots, N, \tag{4.49}$$

where F_i is the matrix of the third order derivative of a prepotential F given by

$$F_{ijk} = \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}, \quad i, j, k = 1, \ldots, N.$$

Wyllard in [56] (see also [9]) constructed a set of generators of $\mathcal{N} = 4$ supersymmetric mechanics, where his ansatz for the four supercharges involves two scalar prepotentials F and W which depend on position of particles only. The case when the second prepotentional $W = 0$ was also considered in [56] (see also [30, 31]), and we will be interested in this case only. The structure of such an algebra forces functions F and W to satisfy a system of partial differential equations. The setting of the structure leads to the result that the function F obeys the comutativity equations [9].

There are two approaches to deal with the arising system of partial differential equations [9]. The first way is to start with a particular known solution of the commutativity equations (4.49) for the prepotential F and then construct an $\mathcal{N} = 4$ superconformal multi–particle model associated to it. Alternatively, one can start with a bosonic conformally invariant multi–particle mechanics and then seek a solution of equations (4.49) that will provide an $\mathcal{N} = 4$ superconformal extension. We follow the first approach. Motivated by the family of multi-parameters solutions (4.7) we obtained in the previous section for BC_N type root system, we apply these solutions in order to construct $\mathcal{N} = 4$ supersymmetric mechanical systems. Note that trigonometric solutions of WDVV equations were used to construct $\mathcal{N}=4$ supersymmetric Hamiltonians in [3]. This gave, in particular, supersymmetric version of quantum Calogero–Moser–Sutherland system of type BC_N with two independent coupling parameters. We extend this Hamiltonian into multiparameter family. Note also that any rational $\vee$-system leads to $\mathcal{N}=4$ supersymmetric mechanical systems (see [4] and references therein).

4.4.1 Notations and ansatz for supercharges (trigonometric settings)

Particles can be classified in two general classes: *bosons* and *fermions*. Bosons and fermions have very different physical behaviour. One of the main differences is that bosonic fields obey canonical commutation relations, that is, involve the commutators $[a,b]=ab-ba$, while fermionic fields obey canonical anti-commutation relations, that is, involve the anticommutators $\{a,b\}=ab+ba$. In order to implement an $\mathcal{N}=4$ supersymmetric mechanical system, consider N (quantum) particles on a line with coordinates x_j and momenta $p_j=-i\partial_{x_j}$, $(j=1,\ldots,N)$, to each of which we assign four fermionic variables. Let us denote them by $\{\psi^{aj},\bar{\psi}^j_a\,|\,a=1,2,\ j=1,\ldots,N\}$. They obey the standard (anti)-commutation rules $(a,b=1,2,\ j,k=1,\ldots,N)$:

$$[x_j,p_k]=i\delta_{jk},\quad \{\psi^{aj},\bar{\psi}^k_b\}=-\frac{1}{2}\delta_{jk}\delta_{ab},\quad \{\psi^{aj},\psi^{bk}\}=\{\bar{\psi}^j_a,\bar{\psi}^k_b\}=0.$$

Then there are four supercharges $\{Q^a,\bar{Q}_b\,|\,a,b=1,2\}$ which generate the $\mathcal{N}=4$ supersymmetry algebra. The $\mathcal{N}=4$ supersymmetry algebra has the form

$$\{Q^a,Q^b\}=\{\bar{Q}_a,\bar{Q}_b\}=0,\quad H_{SUSY}=-\frac{1}{2}(Q^a\bar{Q}_a+\bar{Q}_aQ^a),\tag{4.50}$$

where H_{SUSY} is the supersymmetric Hamiltonian.

Two different representations of $\mathcal{N}=4$ supersymmetry algebra were constructed in [3] (see also references therein for one of the representations). The corresponding supercharges depend on a prepotential of the form (4.3) (with $Q=0$). This prepotential is assumed to satisfy equations of the form (4.4). In our application we follow the considerations of [3] of $\mathcal{N}=4$ supersymmetry algebra, so let us recall these settings.

Let ϵ_{ab}, ϵ^{ab} be the fully anti-symmetric tensors in two dimensions such that $\epsilon_{12}=\epsilon^{21}=1$. These tensors are used to lower and raise indices, for example $Q^a=\epsilon^{ab}Q_b$, $\bar{Q}^a=\epsilon^{ab}\bar{Q}_b$.

Further fermionic variables are introduced by

$$\psi^j_a=\epsilon_{ab}\psi^{bj},\quad \bar{\psi}^{aj}=\epsilon^{ab}\bar{\psi}^j_b.\tag{4.51}$$

They satisfy the following useful relations:

$$\{\psi_a^j, \bar{\psi}^{bk}\} = \frac{1}{2}\delta^{jk}\delta_a^b, \quad \{\psi^{aj}, \bar{\psi}^{bk}\} = \frac{1}{2}\delta^{jk}\epsilon^{ab}, \quad \{\psi_a^j, \bar{\psi}_b^k\} = \frac{1}{2}\delta^{jk}\epsilon_{ba}. \tag{4.52}$$

Throughout it is assumed that summation over repeated indices takes place (even when both indices are either low or upper indices) unless it is indicated that no summation is applied.

A function $F = F(x_1, \ldots, x_N)$ of the form (4.3) with $Q = 0$ is considered, that is

$$F = \sum_{\alpha \in \mathcal{A}} c_\alpha f((\alpha, x)), \tag{4.53}$$

where $\mathcal{A}$ is a finite set of vectors in $V \cong \mathbb{C}^N$, $c_\alpha \in \mathbb{C}$, are some multiplicity parameters, the function f is given by (1.11).

The function F is assumed to satisfy the commutativity equations

$$F_{rjk}F_{kmn} = F_{rmk}F_{kjn}, \tag{4.54}$$

where $F_{rjk} = \frac{\partial^3 F}{\partial x_r \partial x_j \partial x_k}$ for any $r, j, k, m, n = 1, \ldots, N$.

For the first representation the supercharges are of the form

$$\begin{aligned} Q^a &= p_r\psi^{ar} + iF_{rjk}\langle \psi^{br}\psi_b^j\bar{\psi}^{ak}\rangle, \\ \bar{Q}_c &= p_l\bar{\psi}_c^l + iF_{lmn}\langle \bar{\psi}_d^l\bar{\psi}^{dm}\psi_c^n\rangle, \end{aligned}$$

$a, c = 1, 2$, where the symbol $\langle \ldots \rangle$ stands for the anti-symmetrisation. That is given N operators A_i, $(i = 1, \ldots, N)$ we define

$$\langle A_1 \ldots A_N \rangle = \frac{1}{N!} \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) A_{\sigma(1)} \ldots A_{\sigma(N)}. \tag{4.55}$$

The following statement takes place.

Theorem 4.4.1. [3] *Let us assume that F satisfies conditions (4.54). Then for all $a, b = 1, 2$ we have*

$$\{Q^a, Q^b\} = \{\bar{Q}_a, \bar{Q}_b\} = 0 \quad \textit{and} \quad \{Q^a, \bar{Q}_b\} = -2H\delta_b^a,$$

where the Hamiltonian H is given by

$$H = \frac{p^2}{4} - \frac{\partial_i F_{jlk}}{2}(\psi^{bi}\psi_b^j\bar{\psi}_d^l\bar{\psi}^{dk} - \psi_b^i\bar{\psi}^{bj}\delta^{lk} + \frac{1}{4}\delta^{ij}\delta^{lk}) + \frac{1}{16}F_{ijk}F_{lmn}\delta^{nm}\delta^{jl}\delta^{ik}.$$

Furthermore, the rescaled Hamiltonian $H_1 = 4H$ has the form

$$H_1 = -\Delta + \frac{1}{2}\sum_{\alpha\in\mathcal{A}} \frac{c_\alpha(\alpha,\alpha)^2}{\sin^2(\alpha,x)} + \frac{1}{4}\sum_{\alpha,\beta\in\mathcal{A}} c_\alpha c_\beta(\alpha,\alpha)(\beta,\beta)(\alpha,\beta)\cot(\alpha,x)\cot(\beta,x) + \Phi, \quad (4.56)$$

where $\Delta = -p^2$ is the Laplacian in V and the fermionic term

$$\Phi = \sum_{\alpha\in\mathcal{A}} \frac{2c_\alpha\alpha_i\alpha_j}{\sin^2(\alpha,x)}\big(\alpha_l\alpha_k\psi^{bi}\psi^j_b\bar\psi^l_d\bar\psi^{dk} - (\alpha,\alpha)\psi^i_b\bar\psi^{bj}\big). \quad (4.57)$$

For the second representation the supercharges are of the form

$$Q^a = p_r\psi^{ar} + iF_{rjk}\psi^{br}\psi^j_b\bar\psi^{ak},$$
$$\bar Q_c = p_l\bar\psi^l_c + iF_{lmn}\bar\psi^l_d\bar\psi^{dm}\psi^n_c,$$

$a, c = 1, 2$. Then the following statement on supersymmetry algebra takes place [3].

Theorem 4.4.2. [3]. *Let us assume that F satisfies conditions (4.54). Then for all $a, b = 1, 2$ we have*

$$\{Q^a, Q^b\} = \{\bar Q_a, \bar Q_b\} = 0 \quad \textit{and} \quad \{Q^a, \bar Q_b\} = -2H\delta^a_b,$$

where the Hamiltonian H is given by

$$H = \frac{p^2}{4} - \frac{\partial_r F_{jlk}}{2}(\psi^{br}\psi^j_b\bar\psi^l_d\bar\psi^{dk} - \psi^r_b\bar\psi^{bj}\delta^{lk}) + \frac{i}{4}\delta^{nm}F_{rmn}p_r. \quad (4.58)$$

Furthermore, the rescaled Hamiltonian $H_2 = 4H$, has the form

$$H_2 = -\Delta + \sum_{\alpha\in\mathcal{A}} c_\alpha(\alpha,\alpha)\cot(\alpha,x)\partial_\alpha + \Phi, \quad (4.59)$$

where Φ is the fermionic term defined by (4.57).

The following result holds.

Proposition 4.4.3. [3]. *Hamiltonians H_1, H_2 from Theorems 4.4.1, 4.4.2 respectively satisfy gauge relation*

$$\delta^{-1}\circ H_2 \circ \delta = H_1,$$

where $\delta = \prod_{\alpha\in\mathcal{A}} \sin^{c_\alpha(\alpha,\alpha)/2}(\alpha,x)$.

The following theorem deals with BC_N root system and has been proven in [3].

Theorem 4.4.4. [3] *Let $\mathcal{R} = BC_N$. Under the assumptions of Theorem 4.1.3 where the positive half of the root system BC_N is defined by*

$$\eta e_i,\ 2\eta e_i,\ 1 \le i \le N; \quad \eta(e_i \pm e_j),\ 1 \le i < j \le N,$$

where $\eta \in \mathbb{C}^\times$ is a parameter, the corresponding supersymmetric Hamiltonians given by (4.56), (4.59)*, take the form*

$$H_1 = -\Delta + \eta^4 \sum_{i=1}^{N} \Big(\frac{(8s + 2(N-2)q)(2(N-2)q\eta^2 - 1)}{\sin^2 \eta x_i} + \frac{16s(4s\eta^2 + 1)}{\sin^2 2\eta x_i} \Big) \tag{4.60}$$
$$+ \eta^4 \sum_{i<j}^{N} \frac{4q(2q\eta^2 + 1)}{\sin^2(\eta(x_i \pm x_j))} + \widetilde{\Phi},$$

and

$$H_2 = -\Delta + 2\eta^3 \sum_{i=1}^{N} \big(8s \cot 2\eta x_i - (8s + 2(N-2)q) \cot \eta x_i\big)\partial_i \tag{4.61}$$
$$+ 4q\eta^3 \sum_{i<j}^{N} \cot(\eta(x_i \pm x_j))(\partial_i \pm \partial_j) + \Phi,$$

with Φ given by

$$\Phi = 4\eta^4 \sum_{i=1}^{N} \Big(\frac{-(8s + 2(N-2)q)}{\sin^2 \eta x_i} + \frac{16s}{\sin^2 2\eta x_i} \Big) \Big(\psi^{bi} \psi_b^i \bar{\psi}_d^i \bar{\psi}^{di} - \psi_b^i \bar{\psi}^{bi} \Big)$$
$$+ 4\eta^4 \sum_{\epsilon \in \{1,-1\}} \sum_{m<t}^{N} \sum_{i,j,l,k} \frac{q d_{mti} d_{mtj}}{\sin^2(\eta(x_m + \epsilon x_t))} \Big(d_{mtl} d_{mtk} \psi^{bi} \psi_b^j \bar{\psi}_d^l \bar{\psi}^{dk} - 2\psi_b^i \bar{\psi}^{bj} \Big),$$

where $d_{mtk} = d_{mtk}(\epsilon) = \delta_{mk} + \epsilon \delta_{tk}$, and $\widetilde{\Phi} = \Phi + const.$

Remark 4.4.5. The supersymmetric Hamiltonians corresponding to root systems of type F_4 and G_2 are given in details in [3] (see also [4]).

4.4.2 Multi-parametr generalization

In this section we generalize Theorem 4.4.4 by introducing extra parameters in the Hamiltonians following [2] .

Let us define coordinates $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_n) \in \mathbb{C}^n$ by $\widehat{x}_i = m_i^{1/2} x_i$ $(1 \le i \le n)$. Let Λ be the $n \times n$ diagonal matrix $\Lambda = (m_i^{1/2}\delta_{ij})_{i,j=1}^n$. Let F be given by formula (4.7) with relation (4.17) on parameters r, q, s. By Proposition 4.2.3 the matrix B can be represented as $B = h(x)\Lambda^2$. Let us define a function $\widehat{F}(\widehat{x})$ such that $\widehat{F}(\widehat{x}) = F(x(\widehat{x}))$. Consider the

$n \times n$ matrices $\widehat{F}_k$ with entries

$$(\widehat{F}_k)_{lt} = \widehat{F}_{klt} = \frac{\partial^3 \widehat{F}}{\partial \widehat{x}_k \partial \widehat{x}_l \partial \widehat{x}_t}, \tag{4.62}$$

$k, l, t = 1, \ldots, n$. Note that $\widehat{F}_k = m_k^{-1/2} \Lambda^{-1} F_k \Lambda^{-1}$, where F_k is the matrix with entries $(F_k)_{lt} = F_{klt}$. Let $\widehat{B}$ be the $n \times n$ matrix with entries $\widehat{B}_{ij} = h\delta_{ij}$, where $h = h(x(\widehat{x}))$ is given in Proposition 4.2.3.

Proposition 4.4.6. *The metric $\widehat{B}$ can be represented as*

$$\widehat{B} = \sum_{k=1}^{n} m_k^{1/2} \sin(2m_k^{-1/2}\widehat{x}_k)\widehat{F}_k,$$

and the function $\widehat{F}$ satisfies generalised WDVV equations of the form

$$\widehat{F}_i \widehat{F}_j = \widehat{F}_j \widehat{F}_i, \tag{4.63}$$

for all $i, j = 1, \ldots, n$.

Proof. The first part of the statement is immediate by Proposition 4.2.3. Consider the left-hand side of (4.63). We have

$$\widehat{F}_i \widehat{F}_j = (m_i m_j)^{-1/2} \Lambda^{-1} F_i \Lambda^{-2} F_j \Lambda^{-1} = h(m_i m_j)^{-1/2} \Lambda^{-1} F_i B^{-1} F_j \Lambda^{-1}. \tag{4.64}$$

It follows by Theorem 4.2.5 that the rightmost expression in (4.64) is unchanged if one swaps i and j, hence the statement follows. □

It follows by Proposition 4.4.6 that the function $\widehat{F}$ satisfies such type of equations, hence we obtain in this way two supersymmetric Hamiltonians H_i, $i = 1, 2$ for a family of BC_n type configurations. We give these Hamiltonians in detail. Let us consider the following configuration $\widehat{\mathcal{A}} \subset \mathbb{C}^n$ of vectors α with multiplicities c_α:

$$\begin{aligned}
&m_i^{-1/2} e_i, \quad \text{with multiplicity} \quad r m_i, \quad 1 \le i \le n, \\
&2m_i^{-1/2} e_i, \quad \text{with multiplicity} \quad s m_i + \frac{1}{2} q m_i (m_i - 1), \quad 1 \le i \le n, \\
&m_i^{-1/2} e_i \pm m_j^{-1/2} e_j, \quad \text{with multiplicity} \quad q m_i m_j, \quad 1 \le i < j \le n,
\end{aligned}$$

where $m_1, \ldots, m_n \in \mathbb{C}^*$. Consider n (quantum) particles on a line with coordinates $\widehat{x}_j$ and momenta $p_j = -i\partial_{\widehat{x}_j}$, $(j = 1, \ldots, n)$, to each of which we associate four fermionic variables $\psi^{aj}, \bar{\psi}_a^j$, $a = 1, 2$. These variables may be thought of as operators acting on wavefunctions which depend on bosonic and fermionic variables. Let ϵ_{ab} be the fully anti-symmetric tensors in two dimensions such that $\epsilon_{12} = -\epsilon_{21} = 1$.

Fermionic variables are assumed to satisfy the following (anti)-commutation relations $(a,b=1,2;j,k=1,\dots,n)$:

$$\{\psi^{aj},\bar\psi^k_b\}=-\frac{1}{2}\delta_{jk}\delta_{ab},\quad \{\psi^{aj},\psi^{bk}\}=\{\bar\psi^j_a,\bar\psi^k_b\}=0.$$

We consider supercharges of the form

$$Q^a=-i\frac{\partial}{\partial\widehat{x}_r}\psi^{ar}+i\widehat{F}_{rjk}\big(\epsilon_{bc}\epsilon_{da}\psi^{br}\psi^{cj}\bar\psi^k_d-\frac{1}{2}\psi^{ar}\delta_{jk}\big),$$
$$\bar Q_a=-i\frac{\partial}{\partial\widehat{x}_l}\bar\psi^l_a+i\widehat{F}_{lmn}\big(\epsilon_{bd}\epsilon_{ac}\bar\psi^l_d\bar\psi^m_b\psi^{cn}-\frac{1}{2}\bar\psi^l_a\delta_{nm}\big),$$

where $a=1,2$, $\widehat{F}_{ijk}$ is defined in (4.62), and we assume summation over repeated indices. Let $\Delta=\sum_{j=1}^n\partial^2_{\widehat{x}_j}$ be the Laplacian in $\mathbb{C}^n$. We have the following statement on the supersymmetry algebra which follows from [3] (see Theorems 4.4.1, 4.4.2 above).

Theorem 4.4.7. *For all $a,b=1,2$ the supercharges Q^a, $\bar Q_b$ satisfy $\mathcal{N}=4$ supersymmetry algebra relations*

$$\{Q^a,Q^b\}=\{\bar Q_a,\bar Q_b\}=0,\quad \textit{and}\quad \{Q^a,\bar Q_b\}=-\frac{1}{2}H_1\delta_{ab},$$

where the Hamiltonian H_1 is given by

$$H_1=-\Delta+\frac{1}{2}\sum_{\alpha\in\widehat{\mathcal{A}}}\frac{c_\alpha(\alpha,\alpha)^2}{\sin^2(\alpha,\widehat{x})}+\frac{1}{4}\sum_{\alpha,\beta\in\widehat{\mathcal{A}}}c_\alpha c_\beta(\alpha,\alpha)(\beta,\beta)(\alpha,\beta)\cot(\alpha,\widehat{x})\cot(\beta,\widehat{x})+\Phi, \tag{4.65}$$

with the fermionic term

$$\Phi=\sum_{\alpha\in\widehat{\mathcal{A}}}\frac{2c_\alpha\alpha_i\alpha_j}{\sin^2(\alpha,\widehat{x})}\Big(\alpha_l\alpha_k\epsilon_{bc}\epsilon_{ad}\psi^{bi}\psi^{cj}\bar\psi^l_d\bar\psi^k_a+(\alpha,\alpha)\psi^{ai}\bar\psi^j_a\Big). \tag{4.66}$$

The Hamiltonian H_1 is formally self-adjoint. Similar considerations (see [3]) yield not self-adjoint Hamiltonian of the form

$$H_2=-\Delta+\sum_{\alpha\in\widehat{\mathcal{A}}}c_\alpha(\alpha,\alpha)\cot(\alpha,\widehat{x})\partial_\alpha+\Phi \tag{4.67}$$

with the same fermionic term Φ. In fact Hamiltonians H_1, H_2 satisfy gauge relation $H_2=gH_1g^{-1}$, where $g=\prod_{\alpha\in\widehat{\mathcal{A}}}\sin^{c_\alpha(\alpha,\alpha)/2}(\alpha,\widehat{x})$ (see Proposition 4.4.3 above). Supersymmetric Hamiltonians H_1,H_2 given by formulas (4.65), (4.67) give multi-parameter generalization of the two-parameter supersymmetric Hamiltonians related to the root system BC_n considered in [3].

4.5 Commutativity equations and Euclidean trigonometric $\vee$-systems

In this section let $\mathcal{A}$ be a finite set of vectors in an N-dimensional Euclidean space V with the bilinear inner product $(\cdot,\cdot)$. Let $c_\alpha \in \mathbb{C}, \alpha \in \mathcal{A}$ be some multiplicity parameters. Assume that $\mathcal{A}$ belongs to a lattice of rank N. For each vector $\alpha \in \mathcal{A}$ we recall the set of its collinear vectors from $\mathcal{A}$:

$$\delta_\alpha := \{\gamma \in \mathcal{A} \colon \gamma \sim \alpha\}.$$

Let $\delta \subseteq \delta_\alpha$ and $\alpha_0 \in \delta_\alpha$. Then for any $\gamma \in \delta$ we have $\gamma = k_\gamma \alpha_0$ for some $k_\gamma \in \mathbb{R}$. Note that k_γ depends on the choice of α_0 and different choices of α_0 give rescaled collections of these parameters. Define $C_\delta^{\alpha_0} := \sum\limits_{\gamma\in\delta} c_\gamma k_\gamma^2$. Note that $C_\delta^{\alpha_0}$ is non-zero if and only if $C_\delta^{\widetilde{\alpha}_0} \neq 0$ for any $\widetilde{\alpha}_0 \in \delta$. Let us also recall the definition of series of vectors.

For any $\alpha \in \mathcal{A}$ let us distribute all the vectors in $\mathcal{A} \setminus \delta_\alpha$ into a disjoint union of α-series

$$\mathcal{A} \setminus \delta_\alpha = \bigsqcup_{s=1}^{k} \Gamma_\alpha^s$$

where $k \in \mathbb{N}$ depends on α. These series Γ_α^s are determined by the property that for any $s = 1, \ldots, k$ and for any two covectors $\gamma_1, \gamma_2 \in \Gamma_\alpha^s$ one has either $\gamma_1 + \gamma_2 = m\alpha$ or $\gamma_1 - \gamma_2 = m\alpha$ for some $m \in \mathbb{Z}$. We assume that the series are maximal, that is if $\gamma \in \Gamma_\alpha^s$ for some $s \in \mathbb{N}$, then Γ_α^s must contain all the covectors of the form $\pm\gamma + m\alpha \in \mathcal{A}$ with $m \in \mathbb{Z}$.

Let us now define *Euclidean trigonometric $\vee$-system* in analogy with trigonometric $\vee$-system with the bilinear form $G_{\mathcal{A}}$ replaced by the inner product $(\cdot,\cdot)$.

Definition 4.5.1. The pair $(\mathcal{A}, c)$ is called a *Euclidean trigonometric $\vee$-system* if for all $\alpha \in \mathcal{A}$ and for any α-series Γ_α^s, one has the relation

$$\sum_{\beta \in \Gamma_\alpha^s} c_\beta (\alpha, \beta) \alpha \wedge \beta = 0. \tag{4.68}$$

Now we give two examples which illustrate this notion. The corresponding canonical form $G_{\mathcal{A}}$ is identically zero in both cases.

Proposition 4.5.2. *Let $\mathcal{A} = \{e_1, \frac{2}{3}e_2, \frac{1}{2}e_1 \pm \frac{1}{6}e_2, \frac{1}{2}(e_1 \pm e_2)\} \subset \mathbb{C}^2$ with the corresponding multiplicities $\{-2, -\frac{3}{2}, 3, 1\}$. Then $\mathcal{A}$ is a Euclidean trigonometric $\vee$-system with the bilinear form defined by $(x, y) = x_1y_1 - 3x_2y_2$, where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{C}^2$.*

Proof. To simplify notations let us introduce the following vectors:

$$\alpha_1 = e_1,\ \alpha_2 = \frac{2}{3}e_2,\ \alpha_3 = \frac{1}{2}e_1 + \frac{1}{6}e_2,\ \alpha_4 = \frac{1}{2}e_1 - \frac{1}{6}e_2,\ \alpha_5 = \frac{1}{2}(e_1 + e_2),\ \alpha_6 = \frac{1}{2}(e_1 - e_2).$$

Then we have the following α-series:

$$\begin{gathered}
\Gamma^1_{\alpha_1} = \{\alpha_2\}, \quad \Gamma^2_{\alpha_1} = \{\alpha_3, \alpha_4\}, \quad \Gamma^3_{\alpha_1} = \{\alpha_5, \alpha_6\}, \\
\Gamma^1_{\alpha_2} = \{\alpha_1\}, \quad \Gamma^2_{\alpha_2} = \{\alpha_3, \alpha_6\}, \quad \Gamma^3_{\alpha_2} = \{\alpha_4, \alpha_5\}, \\
\Gamma^1_{\alpha_3} = \{\alpha_1, \alpha_4\}, \quad \Gamma^2_{\alpha_3} = \{\alpha_2, \alpha_6\}, \quad \Gamma^3_{\alpha_3} = \{\alpha_5\}, \\
\Gamma^1_{\alpha_4} = \{\alpha_1, \alpha_3\}, \quad \Gamma^2_{\alpha_4} = \{\alpha_2, \alpha_5\}, \quad \Gamma^3_{\alpha_4} = \{\alpha_6\}, \\
\Gamma^1_{\alpha_5} = \{\alpha_1, \alpha_6\}, \quad \Gamma^2_{\alpha_5} = \{\alpha_2, \alpha_4\}, \quad \Gamma^3_{\alpha_5} = \{\alpha_3\}, \\
\Gamma^1_{\alpha_6} = \{\alpha_1, \alpha_5\}, \quad \Gamma^2_{\alpha_6} = \{\alpha_2, \alpha_3\}, \quad \Gamma^3_{\alpha_6} = \{\alpha_4\}.
\end{gathered}$$

We also have the following inner products:

$$\begin{gathered}
(\alpha_1, \alpha_2) = 0, \quad (\alpha_1, \alpha_3) = \frac{1}{2}, \quad (\alpha_1, \alpha_4) = \frac{1}{2}, \quad (\alpha_1, \alpha_5) = \frac{1}{2}, \quad (\alpha_1, \alpha_6) = \frac{1}{2}, \\
(\alpha_2, \alpha_3) = -\frac{1}{3}, \quad (\alpha_2, \alpha_4) = \frac{1}{3}, \quad (\alpha_2, \alpha_5) = -1, \quad (\alpha_2, \alpha_6) = 1, \quad (\alpha_3, \alpha_4) = \frac{1}{3}, \\
(\alpha_3, \alpha_5) = 0, \quad (\alpha_3, \alpha_6) = \frac{1}{2}, \quad (\alpha_4, \alpha_5) = \frac{1}{2}, \quad (\alpha_4, \alpha_6) = 0, \quad (\alpha_5, \alpha_6) = 1.
\end{gathered}$$

Then it is easy to check that condition (4.68) holds for all α-series, $\alpha \in \mathcal{A}$. □

Remark 4.5.3. Let $\widetilde{\mathcal{A}}$ be the configuration of covectors on the plane given in Proposition 3.5.7. That is $\widetilde{\mathcal{A}} = \{e^1, e^2, 2e^2, \frac{1}{2}(e^1 \pm e^2), \frac{1}{2}(e^1 \pm 3e^2)\} \subset \mathbb{C}^2$ with the corresponding multiplicities $\{\frac{a(3a-2b)}{3a+4b}, 3a+2b, b, 3a, a\}$. Then for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{C}^2$ we have the canonical form

$$G_{\widetilde{\mathcal{A}}}(x, y) = \frac{3(3a+2b)}{3a+4b}\big(ax_1y_1 + (3a+4b)x_2y_2\big).$$

By letting multiplicity $b = -\frac{3}{2}a$ we get the following configuration $\widetilde{\mathcal{A}}_1 = \{e^1, 2e^2, \frac{1}{2}(e^1 \pm e^2), \frac{1}{2}(e^1 \pm 3e^2)\}$ with the corresponding multiplicities $\{-2a, -\frac{3}{2}a, 3a, a\}$. For this configuration the canonical form $G_{\widetilde{\mathcal{A}}_1} \equiv 0$. We regularise $G_{\widetilde{\mathcal{A}}_1}$ by defining a new non-degenerate bilinear form as follows

$$(x, y) = -\lim_{b \to -\frac{3a}{2}} \frac{G_{\widetilde{\mathcal{A}}}(x, y)}{3a+2b} = x_1y_1 - 3x_2y_2. \tag{4.69}$$

Note that configuration $\mathcal{A}$ given in Proposition 4.5.2 contains vectors dual to the covectors in system $\widetilde{\mathcal{A}}_1$ with respect to the form (4.69).

Proposition 4.5.4. *Let $\mathcal{A} = \{2e_1, \frac{1}{2}e_2, e_1 \pm \frac{1}{2}e_2, e_1 \pm e_2\} \subset \mathbb{C}^2$ with the corresponding multiplicities $\{-\frac{1}{4}, 2, 1, -\frac{1}{2}\}$. Then $\mathcal{A}$ is a Euclidean trigonometric $\vee$-system with the bilinear form defined by $(x, y) = x_1y_1 + 2x_2y_2$, where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{C}^2$.*

Proof. To simplify notations let us introduce the following vectors:

$$\alpha_1 = 2e_1,\ \alpha_2 = \frac{1}{2}e_2,\ \alpha_3 = e_1 + \frac{1}{2}e_2,\ \alpha_4 = e_1 - \frac{1}{2}e_2,\ \alpha_5 = e_1 + e_2,\ \alpha_6 = e_1 - e_2.$$

Then we have the following α-series:

$$\begin{aligned}
&\Gamma^1_{\alpha_1} = \{\alpha_2\}, \quad \Gamma^2_{\alpha_1} = \{\alpha_3, \alpha_4\}, \quad \Gamma^3_{\alpha_1} = \{\alpha_5, \alpha_6\},\\
&\Gamma^1_{\alpha_2} = \{\alpha_1\}, \quad \Gamma^2_{\alpha_2} = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6\},\\
&\Gamma^1_{\alpha_3} = \{\alpha_1, \alpha_4\}, \quad \Gamma^2_{\alpha_3} = \{\alpha_2, \alpha_5\}, \quad \Gamma^3_{\alpha_3} = \{\alpha_6\},\\
&\Gamma^1_{\alpha_4} = \{\alpha_1, \alpha_3\}, \quad \Gamma^2_{\alpha_4} = \{\alpha_2, \alpha_6\}, \quad \Gamma^3_{\alpha_4} = \{\alpha_5\},\\
&\Gamma^1_{\alpha_5} = \{\alpha_1, \alpha_6\}, \quad \Gamma^2_{\alpha_5} = \{\alpha_2, \alpha_3\}, \quad \Gamma^3_{\alpha_5} = \{\alpha_4\},\\
&\Gamma^1_{\alpha_6} = \{\alpha_1, \alpha_5\}, \quad \Gamma^2_{\alpha_6} = \{\alpha_2, \alpha_4\}, \quad \Gamma^3_{\alpha_6} = \{\alpha_3\}.
\end{aligned}$$

We also have the following inner products:

$$\begin{aligned}
&(\alpha_1, \alpha_2) = 0, \quad (\alpha_1, \alpha_3) = 2, \quad (\alpha_1, \alpha_4) = 2, \quad (\alpha_1, \alpha_5) = 2, \quad (\alpha_1, \alpha_6) = 2,\\
&(\alpha_2, \alpha_3) = \frac{1}{2}, \quad (\alpha_2, \alpha_4) = -\frac{1}{2}, \quad (\alpha_2, \alpha_5) = 1, \quad (\alpha_2, \alpha_6) = -1, \quad (\alpha_3, \alpha_4) = \frac{1}{2},\\
&(\alpha_3, \alpha_5) = 2, \quad (\alpha_3, \alpha_6) = 0, \quad (\alpha_4, \alpha_5) = 0, \quad (\alpha_4, \alpha_6) = 2, \quad (\alpha_5, \alpha_6) = -1.
\end{aligned}$$

Then it is easy to check that condition (4.68) holds for all α-series, $\alpha \in \mathcal{A}$. □

Remark 4.5.5. Let $\widetilde{\mathcal{A}}$ be the configuration of covectors on the plane given in Proposition 3.5.9. That is $\widetilde{\mathcal{A}} = \{e^1, 2e^1, e^2, 2e^2, e^1 \pm e^2, e^1 \pm 2e^2\} \subset \mathbb{C}^2$ with the corresponding multiplicities $\{2a, \frac{a}{2} - \frac{b}{4}, 2b, a, b, a - \frac{b}{2}\}$. Then for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{C}^2$ we have the canonical form

$$G_{\widetilde{\mathcal{A}}}(x, y) = 6a\big(x_1 y_1 + 2x_2 y_2\big).$$

By letting multiplicity $a = 0$ we get the following configuration $\widetilde{\mathcal{A}}_1 = \{2e^1, e^2, e^1 \pm e^2, e^1 \pm 2e^2\}$ with the corresponding multiplicities $\{-\frac{b}{4}, 2b, b, -\frac{b}{2}\}$. For this configuration the canonical form $G_{\widetilde{\mathcal{A}}_1} \equiv 0$. We regularise $G_{\widetilde{\mathcal{A}}_1}$ by defining a new non-degenerate bilinear form as follows

$$(x, y) = \lim_{a \to 0} \frac{G_{\widetilde{\mathcal{A}}}(x, y)}{6a} = x_1 y_1 + 2x_2 y_2. \tag{4.70}$$

Note that configuration $\mathcal{A}$ given in Proposition 4.5.4 contains vectors dual to the covectors in system $\widetilde{\mathcal{A}}_1$ with respect to the form (4.70).

Let us consider the function F given by the formula (4.53). We are interested in configurations $\mathcal{A}$ with a multiplicity function $c(\alpha) = c_\alpha, \alpha \in \mathcal{A}$, such that the commutativity equations (4.49) hold.

In what follows we investigate the relation between the commutativity equations (4.49) for the function (4.53) and Euclidean trigonometric $\vee$-systems.

Lemma 4.5.6. *The commutativity equations* (4.49) *for the function* (4.53) *are equivalent to the identity*

$$\sum_{\alpha,\beta\in\mathcal{A}} c_\alpha c_\beta(\alpha,\beta)\cot(\alpha,x)\cot(\beta,x)B_{\alpha,\beta}(a,b)\alpha\wedge\beta=0, \tag{4.71}$$

for all $a,b\in V$, *where* $B_{\alpha,\beta}(a,b)=\alpha\wedge\beta(a,b)=(\alpha,a)(\beta,b)-(\alpha,b)(\beta,a)$.

Proof. Let F_a be the $N\times N$ matrix of the third order derivatives of the function (4.53) with entries

$$(F_a)_{ij}=\frac{\partial^3 F}{\partial x_a\partial x_i\partial x_j}.$$

Then from the formula (4.53) we have

$$(F_a)_{ij}=\sum_{\alpha\in\mathcal{A}} c_\alpha(\alpha,a)\alpha_i\alpha_j\cot(\alpha,x).$$

Hence we have

$$F_aF_b=\sum_{\alpha,\beta\in\mathcal{A}} c_\alpha c_\beta(\alpha,a)(\beta,b)(\alpha,\beta)\cot(\alpha,x)\cot(\beta,x)\alpha\otimes\beta=0.$$

Therefore the commutativity equations (4.49) which are equivalent to the condition $[F_a,F_b]=0$ can be written as

$$\sum_{\alpha,\beta\in\mathcal{A}} c_\alpha c_\beta(\alpha,\beta)\cot(\alpha,x)\cot(\beta,x)B_{\alpha,\beta}(a,b)\alpha\otimes\beta=0. \tag{4.72}$$

By exchanging α and β one gets a similar identity to (4.72) and by subtracting the resulting identity from identity (4.72) we get the required identity. $\square$

The following result takes place.

Lemma 4.5.7. *Assume that prepotential* (4.53) *satisfies the commutativity equations* (4.49). *Suppose that* $C_\delta^{\alpha_0}\neq 0$ *for any* $\alpha\in\mathcal{A}$, $\delta\subseteq\delta_\alpha$, $\alpha_0\in\delta_\alpha$. *Assume that the identity*

$$\sum_{\alpha,\beta\in\mathcal{A}} c_\alpha c_\beta(\alpha,\beta)\cot(\alpha,x)\cot(\beta,x)B_{\alpha,\beta}(a,b)\alpha\wedge\beta=0 \tag{4.73}$$

holds for any $a,b\in V$ *and* $\alpha\in\mathcal{A}$ *provided that* $\tan(\alpha,x)=0$. *Then* $\mathcal{A}$ *is a Euclidean trigonometric* $\vee$*-system.*

Proof. Fix $\alpha\in\mathcal{A}$ such that $\tan(\alpha,x)=0$. Let us consider terms in the left-hand side of identity (4.73), where $\gamma=k_\gamma\alpha$, $k_\gamma\in\mathbb{R}$, that is γ is proportional to α. The sum of these terms has to be regular at $\tan(\alpha, x) = 0$. This implies that the product

$$\Big(\sum_{\gamma\in\delta_\alpha} k_\gamma^3 c_\gamma \cot(\gamma, x)\Big)\Big(\sum_{\beta\in\mathcal{A}\setminus\delta_\alpha} c_\beta(\alpha_0, \beta)\cot(\beta, x)B_{\alpha_0,\beta}(a, b)\alpha_0\wedge\beta\Big) \tag{4.74}$$

is regular at $\tan(\alpha, x) = 0$. The first factor in the product (4.74) has the first order pole at $\tan(\alpha, x) = 0$ by the assumption that $C_{\delta_\alpha}^{\alpha_0} \neq 0$ for any $\alpha \in \mathcal{A}, \alpha_0 \in \delta_\alpha$. This implies that

$$\sum_{\beta\in\mathcal{A}\setminus\delta_\alpha} c_\beta(\alpha_0, \beta)\cot(\beta, x)B_{\alpha_0,\beta}(a, b)\alpha_0\wedge\beta = 0. \tag{4.75}$$

Since all the vectors $\beta \in \mathcal{A}\setminus\delta_\alpha$ can be found in one of the disjoint α-series, then (4.75) is equivalent to the identity

$$\sum_s\sum_{\beta\in\Gamma_\alpha^s} c_\beta(\alpha_0, \beta)\cot(\beta, x)B_{\alpha_0,\beta}(a, b)\alpha_0\wedge\beta = 0. \tag{4.76}$$

But the sum in (4.76) is a linear combination of functions $\cot(\beta, x)|_{\tan(\alpha,x)=0}$ which can vanish only if it vanishes for each α-series (see [23, 27]) , hence we have

$$\sum_{\beta\in\Gamma_\alpha^s} c_\beta(\alpha_0, \beta)\cot(\beta, x)B_{\alpha_0,\beta}(a, b)\alpha_0\wedge\beta = 0 \tag{4.77}$$

for all α-series Γ_α^s.

Now we will show that identities (4.77) imply that $\mathcal{A}$ is a Euclidean trigonometric $\vee$-system. Let us fix a particular element $\beta_0 \in \Gamma_{\alpha_0}^s$. Then for any $\gamma \in \Gamma_{\alpha_0}^s$ we have $\gamma + \varepsilon\beta_0 = m\alpha_0$, where $m \in \mathbb{Z}$ and $\varepsilon = \pm 1$. We have $\cot(\gamma, x) = \varepsilon\cot(\beta_0, x)$ and $B_{\alpha_0,\gamma}(a, b) = \varepsilon B_{\alpha_0,\beta_0}(a, b)$ provided that $\tan(\alpha_0, x) = 0$. Hence relation (4.77) implies the Euclidean trigonometric $\vee$-condition $\sum_{\beta\in\Gamma_{\alpha_0}^s} c_\beta(\alpha_0, \beta)\alpha_0\wedge\beta = 0$. □

Note that if $\mathcal{A}$ is a Euclidean trigonometric $\vee$-system then this guaranties that the left-hand side of identity (4.71) is non-singular. Since all vectors from $\mathcal{A}$ belong to an N-dimensional lattice then the left-hand side of identity (4.71) is a rational function in suitable exponential variables which has degree zero and therefore is a constant. In order to find this constant, by changing some of the vectors from $\mathcal{A}$ to their opposite ones we can assume that all vectors from $\mathcal{A}$ belong to a half space, hence form a positive system $\mathcal{A}_+$. Then in an appropriate limit in a cone $\cot(\alpha, x) \to i$ for all $\alpha \in \mathcal{A}_+$ identity (4.71) reduces to

$$\sum_{\alpha,\beta\in\mathcal{A}_+} c_\alpha c_\beta(\alpha, \beta)B_{\alpha,\beta}(a, b)\alpha\wedge\beta = 0.$$

From these considerations together with Lemmas 4.5.6, 4.5.7 we have the following result.

Theorem 4.5.8. *Suppose that a configuration* $(\mathcal{A}, c)$ *satisfies the condition* $C_\delta^{\alpha_0} \neq 0$ *for* *any* $\alpha \in \mathcal{A}$, $\delta \subseteq \delta_\alpha$, $\alpha_0 \in \delta_\alpha$. *Then the commutativity equations* (4.49) *for the prepotential* (4.53) *imply the following two conditions:*

(1) $\mathcal{A}$ *is a Euclidean trigonometric* $\vee$*-system,*

(2) $\sum_{\alpha,\beta\in\mathcal{A}_+} c_\alpha c_\beta (\alpha,\beta) B_{\alpha,\beta}(a,b)\alpha\wedge\beta = 0$ *for any* $a, b \in V$.

Conversely, if a configuration $(\mathcal{A}, c)$ *satisfies conditions (1), (2) then commutativity equations* (4.49) *hold.*

The following statement confirms that the class of Euclidean trigonometric $\vee$-systems contains root systems with $\mathcal{W}$-invariant multiplicity.

Proposition 4.5.9. *Root system* $\mathcal{A} = \mathcal{R}$ *with* $\mathcal{W}$*-invariant multiplicity function* c *is a Euclidean trigonometric* $\vee$*-system.*

Proof. Fix $\alpha \in \mathcal{R}$. Take any $\beta \in \mathcal{R}$, and let $\gamma = s_\alpha\beta = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$. Since $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ we get that $\beta, \gamma \in \Gamma_\alpha^s$ for some s. We have

$$c_\beta = c_\gamma, \quad (\alpha,\beta) = -(\alpha,\gamma), \quad \alpha\wedge\beta = \alpha\wedge\gamma.$$

Hence the contribution of β and γ to the sum in (4.68) gives

$$c_\beta(\alpha,\beta)\alpha\wedge\beta + c_\gamma(\alpha,\gamma)\alpha\wedge\gamma = 0,$$

which implies that $\mathcal{R}$ is a Euclidean trigonometric $\vee$-system. $\square$

As a corollary of Proposition 4.5.9 and Theorem 4.5.8 the following result takes place.

Corollary 4.5.10. [3] *Let* $\mathcal{A} = \mathcal{R}^+$ *be a positive half of the root system* $\mathcal{R}$. *Then the commutativity equations* (4.49) *are satisfied for the function* (4.53) *if and only if*

$$\sum_{\alpha,\beta\in\mathcal{R}^+} c_\alpha c_\beta(\alpha,\beta)B_{\alpha,\beta}(a,b)\alpha\wedge\beta = 0 \tag{4.78}$$

holds for any $a, b \in V$.

The result [3, Theorem 6.5] states that if a root system with invariant multiplicity satisfies property (2) of Theorem 4.5.8 then commutativity relations (4.49) hold. It has been shown in [3] that relation (4.78) is satisfied for root systems $\mathcal{R} = BC_N$, F_4, G_2 with special invariant multiplicities, see Theorems 4.1.3, 4.1.4, 4.1.5 above. Thus Theorem 4.5.8 may be viewed as a generalization of this statement for the class of Euclidean trigonometric $\vee$-systems.

Solutions of commutativity equations can be applied to construct $\mathcal{N} = 4$ supersymmetric mechanical systems, see Theorems 4.4.1, 4.4.2 above. Hamiltonians corresponding to root systems $\mathcal{R} = BC_N$, F_4, G_2 were given explicitly in [3] (see also [4]) for more details.

4.6 Subsytems of a Euclidean trigonometric $\vee$-system

In this section we study the process of taking of a subsystem of a Euclidean trigonometric $\vee$-system and we prove some results analogous to those of trigonometric $\vee$-systems in Chapter 3. The section ends up with the main theorem which shows the fact that under some non-degeneracy conditions subsystems of a Euclidean trigonometric $\vee$-system $\mathcal{A}$ is also a Euclidean trigonometric $\vee$-system. Let us first recall the definition of a subsystem.

Definition 4.6.1. Let $\mathcal{A} \subset V$ be a Euclidean trigonometric $\vee$-system. A subset $\mathcal{B} \subset \mathcal{A}$ is called a subsystem if

$$\mathcal{B} = \mathcal{A} \cap W,$$

for some linear subspace $W \subset V$. The subsystem $\mathcal{B}$ is called *reducible* if $\mathcal{B}$ is a disjoint union of two non-empty subsystems $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. The subsystem $\mathcal{B}$ is called *irreducible* if it is not reducible. We will also equip subsystem $\mathcal{B} \subset \mathcal{A}$ with the multiplicity function which is the restriction of the multiplicity function $c\colon \mathcal{A} \to \mathbb{C}$ on $\mathcal{B}$.

Let $\mathcal{A}$ be a finite set of vectors in $V \cong \mathbb{C}^N$, and let $c_\alpha \in \mathbb{C}, \alpha \in \mathcal{A}$ be some multiplicity parameters. Assume that $\mathcal{A}$ belongs to a lattice of rank N. Let $\mathcal{B} = \mathcal{A} \cap W$ be a subsystem of $\mathcal{A}$ where $W \subset V$ be some n-dimensional linear space. Assume also that $W = \langle \mathcal{B} \rangle$. Assume now for this section that $\mathcal{A}$ is a Euclidean trigonometric $\vee$-system, that is for all $\alpha \in \mathcal{A}$ and any α-series Γ_α^s the following condition holds.

$$\sum_{\beta \in \Gamma_\alpha^s} c_\beta (\alpha, \beta) \alpha \wedge \beta = 0. \tag{4.79}$$

We say that the subsystem $\mathcal{B}$ is *non-isotropic* if the inner product $(\cdot,\cdot)_W := (\cdot,\cdot)|_W$, is non-degenerate on W.

Define the linear operator $M\colon W \to W$ given by

$$M = \sum_{\beta \in \mathcal{B}} c_\beta \beta \otimes \beta, \tag{4.80}$$

that is, $M(u) = \sum_{\beta \in \mathcal{B}} c_\beta (\beta, u) \beta$, for any $u \in W$. In what follows we prove some lemmas which are useful to prove the main theorem of this section later.

Lemma 4.6.2. *Let $\mathcal{A}$ be a Euclidean trigonometric $\vee$-system and $\mathcal{B} = \mathcal{A} \cap W$ be its subsystem. Then*

1. *Any $\alpha \in \mathcal{B}$ is an eigenvector of M, that is $M\alpha = \lambda \alpha$ for some $\lambda \in \mathbb{C}$.*

2. *The vector space W can be decomposed as*

$$W = U_{\lambda_1} \oplus U_{\lambda_2} \oplus \cdots \oplus U_{\lambda_k}, \quad k \in \mathbb{N}, \tag{4.81}$$

where $\lambda_i \in \mathbb{C}$ and $M\ |_{U_{\lambda_i}} = \lambda_i I$, where I is the identity operator.

Proof. Firstly, let $\alpha \in \mathcal{A}$. Consider a two-dimensional plane $\pi \subset V$ containing α and another vector from $\mathcal{A}$ non-collinear to α. Let us sum up the Euclidean trigonometric $\vee$-condition (4.79) over α-series which belong to the plane π. Then

$$\sum_{\beta \in \pi \cap \mathcal{A}} c_\beta (\beta, \alpha) \beta = \lambda \alpha, \tag{4.82}$$

holds for some $\lambda = \lambda(\alpha, \pi)$.

Secondly, let $\alpha \in \mathcal{B}$. Consider a collection of two-dimensional planes $\pi_1, \dots, \pi_k$ such that $\alpha \in \pi_i \subset W$ for any $i = 1, \dots, k$ and for any $\beta \in \mathcal{B}$ there exist $i, (1 \le i \le k)$ such that $\beta \in \pi_i$. Define linear operator $M_{\pi_i} \colon W \to W$, by

$$M_{\pi_i} := \sum_{\beta \in \pi_i \cap \mathcal{A}} c_\beta \beta \otimes \beta.$$

It follows from (4.82) that

$$M_{\pi_i}(\alpha) = \lambda_i \alpha, \tag{4.83}$$

for some $\lambda_i \in \mathbb{C}$. Now sum the equation (4.83) over all two-dimensional planes $\pi_1, \dots \pi_k$, gives

$$\begin{aligned} \sum_{i=1}^{k} \lambda_i \alpha = \sum_{i=1}^{k} M_{\pi_i}(\alpha) &= \sum_{i=1}^{k} \sum_{\beta \in \pi_i \cap \mathcal{A}} c_\beta (\beta, \alpha) \beta \\ &= \sum_{\beta \in W \cap \mathcal{A}} c_\beta (\beta, \alpha) \beta + (k-1) c_\alpha (\alpha, \alpha) \alpha \\ &= M(\alpha) + (k-1) c_\alpha (\alpha, \alpha) \alpha. \end{aligned}$$

Hence $M(\alpha) = \lambda \alpha$, where $\lambda = \sum_{i=1}^{k} \lambda_i + (1-k) c_\alpha (\alpha, \alpha)$. Thus α is an eigenvector for M for any $\alpha \in \mathcal{B}$ which proves the first statement.

Let us now prove the second statement. As α is an eigenvector for the operator M for any $\alpha \in \mathcal{B}$, we have that M is diagonalizable since $\langle \mathcal{B} \rangle = W$, and we have an eigenspace decomposition

$$W = U_{\lambda_1} \oplus U_{\lambda_2} \oplus \cdots \oplus U_{\lambda_k}, \quad k \in \mathbb{N}$$

where $\lambda_i \in \mathbb{C}$ and $M\ |_{U_{\lambda_i}} = \lambda_i I$, where I is the identity operator. This proves the second statement. ∎

The following two lemmas relate the series of vectors in $\mathcal{A}$ and $\mathcal{B}$.

Lemma 4.6.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be as stated in Lemma 4.6.2. Let $\alpha \in \mathcal{B}$ be such that $\alpha \in U_{\lambda_i}$ for some $i \in \mathbb{N}$. Consider an α-series $\Gamma_\alpha^{\mathcal{B}}$ in $\mathcal{B}$ and let $\beta \in \Gamma_\alpha^{\mathcal{B}}$. Then either $\Gamma_\alpha^{\mathcal{B}} \subset U_{\lambda_i}$ or $\Gamma_\alpha^{\mathcal{B}} \subset \{\pm\beta\}$.*

Proof. For $\beta \in \Gamma_\alpha^{\mathcal{B}}$ we have the following tow possible cases.

Case (i) $\beta \in U_{\lambda_i}$. Then for any $\gamma \in \Gamma_\alpha^{\mathcal{B}}$ we have for some $m \in \mathbb{Z}$ that $\gamma = m\alpha + \beta \in U_{\lambda_i}$ or $\gamma = m\alpha - \beta \in U_{\lambda_i}$. Hence $\Gamma_\alpha^{\mathcal{B}} \subset U_{\lambda_i}$.

Case (ii) $\beta \notin U_{\lambda_i}$. Hence $\beta \in U_{\lambda_j}$ for some $j \neq i$. Then for any $\gamma \in \Gamma_\alpha^{\mathcal{B}}$ we have that $\gamma \in U_{\lambda i}$ or $\gamma \in U_{\lambda j}$ since decomposition (4.81) is the direct sum. Note that $\gamma \notin U_{\lambda i}$ as otherwise for some $m \in \mathbb{Z}$ we will have $\beta = m\alpha - \gamma \in U_{\lambda i}$, or $\beta = m\alpha + \gamma \in U_{\lambda i}$, which is a contradiction. Note also that $\gamma \notin U_{\lambda_j}$ unless $\gamma = \pm\beta$ as otherwise for some $m \in \mathbb{Z}$ we have $m\alpha = \beta + \gamma \in U_{\lambda_j}$ or $m\alpha = \beta - \gamma \in U_{\lambda_j}$ which is a contradiction. Hence $\Gamma_\alpha^{\mathcal{B}} \subset \{\pm\beta\}$. □

Lemma 4.6.4. *Let $\mathcal{A}$ and $\mathcal{B}$ as stated in Lemma 4.6.2. Let $\alpha, \beta \in \mathcal{B}$. Let $\Gamma_\alpha^{\mathcal{A}}, \Gamma_\alpha^{\mathcal{B}}$ be the α-series in $\mathcal{A}$ and $\mathcal{B}$ respectively containing β. Then the set $\Gamma_\alpha^{\mathcal{A}}$ is equal to the set $\Gamma_\alpha^{\mathcal{B}}$.*

Proof. Let $\gamma \in \Gamma_\alpha^{\mathcal{A}}$. Then $\gamma = m\alpha + \beta \in \mathcal{B}$, (or $\gamma = m\alpha - \beta \in \mathcal{B}$) for some $m \in \mathbb{Z}$. Thus $\gamma \in \Gamma_\alpha^{\mathcal{B}}$ by the maximality of $\Gamma_\alpha^{\mathcal{B}}$. Hence $\Gamma_\alpha^{\mathcal{A}} \subset \Gamma_\alpha^{\mathcal{B}}$. The opposite conclusion is obvious. Therefore $\Gamma_\alpha^{\mathcal{B}} = \Gamma_\alpha^{\mathcal{A}}$. □

Proposition 4.6.5. *In the assumptions and notations of Lemma 4.6.2 we have $(u, v) = 0$ for any $u \in U_{\lambda_i}$ and $v \in U_{\lambda_j}$ such that $i \neq j$.*

Proof. It is clear that $(M(u), v) = (u, M(v))$ for any $u, v \in W$. Now for any $u \in U_{\lambda_i}$ and $v \in W$ we have

$$(M(u), v) = \lambda_i(u, v).$$

On the other hand for any $u \in W$ and $v \in U_{\lambda_j}$ we have

$$(u, M(v)) = \lambda_j(u, v).$$

Hence for $u \in U_{\lambda_i}$, $v \in U_{\lambda_j}$, $i \neq j$ we have

$$(\lambda_i - \lambda_j)(u, v) = 0.$$

Thus $(u, v) = 0$ since $\lambda_i \neq \lambda_j$. □

Now we present the main theorem in this section.

Theorem 4.6.6. *Any non-isotropic subsystem of a Euclidean trigonometric $\vee$-system is also a Euclidean trigonometric $\vee$-system.*

Proof. Let $\mathcal{A}$ be a Euclidean trigonometric $\vee$-system and let $\mathcal{B}$ be a non-isotropic subsystem of $\mathcal{A}$. We will show that the Euclidean trigonometric $\vee$-conditions hold in $\mathcal{B}$.

Let $\alpha \in \mathcal{B}$ be such that $\alpha \in U_{\lambda_i}$, $i \in \mathbb{N}$. Consider an α-series $\Gamma_\alpha^{\mathcal{B}}$ in $\mathcal{B}$. Let $\beta \in \Gamma_\alpha^{\mathcal{B}}$. By Lemma 4.6.3 we have the following two cases.

Case (i) $\beta \in U_{\lambda_i}$. Then $\Gamma_\alpha^{\mathcal{B}} \subset U_{\lambda_i}$. In this case we have

$$\sum_{\beta \in \Gamma_\alpha^{\mathcal{B}}} c_\beta (\alpha, \beta) \alpha \wedge \beta = \sum_{\beta \in \Gamma_\alpha^{\mathcal{A}}} c_\beta (\alpha, \beta) \alpha \wedge \beta = 0$$

since $\Gamma_\alpha^{\mathcal{B}} = \Gamma_\alpha^{\mathcal{A}}$ by Lemma 4.6.4 and that $\mathcal{A}$ is Euclidean trigonometric $\vee$-system.

Case (ii) $\beta \notin U_{\lambda_i}$. Then $\beta \in U_{\lambda_j}$ for some $j \neq i$. Hence $(\alpha, \beta)_W = 0$ by Proposition 4.6.5 and moreover $\Gamma_\alpha^{\mathcal{B}} \subset \{\pm\beta\}$ by Lemma 4.6.3. Hence Euclidean trigonometric $\vee$-conditions hold for $\mathcal{B}$ in this case as well. □

4.7 Euclidean trigonometric $\vee$-systems and complex Euclidean $\vee$-systems

Now let us discuss the relation between Euclidean trigonometric $\vee$-systems and complex Euclidean $\vee$-systems. Let us first recall some definitions and relations related to the complex Euclidean $\vee$-systems following [26]. Let V be a complex Euclidean space, which is a complex vector space with a non-degenerate bilinear form $(\cdot, \cdot)$. We will identify V with the dual space V^* using this form.

Definition 4.7.1. [26] Let $\mathcal{A}$ be a finite set of vectors in V. We say that the set $\mathcal{A}$ is *well-distributed* if the canonical form

$$G_{\mathcal{A}}^{rat}(x, y) = \sum_{\alpha \in \mathcal{A}} (\alpha, x)(\alpha, y), \quad x, y \in V,$$

is proportional to the form $(\cdot, \cdot)$.

Definition 4.7.2. [26] The set $\mathcal{A} \subset V$ is called *a (rational) complex Euclidean $\vee$-system* if it is well-distributed in V and any of its two-dimensional subsystems is either reducible or well-distributed in the corresponding plane.

Note in this definition we allow the canonical form $G_{\mathcal{A}}^{rat}$ to be degenerate. In analogy to Definition 4.7.1 let us introduce the trigonometric version.

For a subset $\mathcal{A} \subset V$ with multiplicity function $c \colon \mathcal{A} \to \mathbb{C}$, where $c_\alpha := c(\alpha)$, consider the bilinear form $G_{\mathcal{A}}$ on V given by

$$G_{\mathcal{A}}(x, y) = \sum_{\alpha \in \mathcal{A}} c_\alpha (\alpha, x)(\alpha, y), \quad x, y \in V. \tag{4.84}$$

Definition 4.7.3. We say that the pair $(\mathcal{A}, c)$ is *well-distributed* in V if the bilinear form (4.84) is proportional to the form $(\cdot, \cdot)$.

As we identify the vector space V with its dual space V^* and $\mathcal{A} \subset V$, we will understand the trigonometric $\vee$-system $(\mathcal{A}, c)$ as a collection of vectors satisfying relations

$$\sum_{\beta \in \Gamma_\alpha^s} c_\beta G_{\mathcal{A}}(\alpha, \beta) \alpha \wedge \beta = 0 \tag{4.85}$$

for all $\alpha \in \mathcal{A}$ and all α-series Γ_α^s, where $G_{\mathcal{A}}$ is given by formula (4.84).

Now let $(\mathcal{A}, c)$ be a Euclidean trigonometric $\vee$-system, that is conditions (4.68) hold. Define the linear operator $M \colon V \to V$ given by

$$M = \sum_{\beta \in \mathcal{A}} c_\beta \beta \otimes \beta, \tag{4.86}$$

that is, $M(u) = \sum_{\beta \in \mathcal{B}} c_\beta (\beta, u) \beta$, for any $v \in V$. Assume also that the linear span $\langle \mathcal{A} \rangle = V$. The following statement takes place.

Lemma 4.7.4. *Let $(\mathcal{A}, c)$ be a Euclidean trigonometric $\vee$-system. Then*

1. *Any $\alpha \in \mathcal{A}$ is an eigenvector of M, that is $M\alpha = \lambda\alpha$ for some $\lambda \in \mathbb{C}$.*

2. *The vector space V can be decomposed as*

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k}, \quad k \in \mathbb{N},\ \lambda_i \in \mathbb{C}, \tag{4.87}$$

where $M \mid_{V_{\lambda_i}} = \lambda_i I$, and I is the identity operator, and $\lambda_i \neq \lambda_j$ for $i \neq j$.

Proof. Firstly, let $\alpha \in \mathcal{A}$. Since $\mathcal{A}$ is a Euclidean trigonometric $\vee$-system then for each α-series Γ_α^s condition (4.68) is equivalent to the relation

$$\sum_{\beta \in \Gamma_\alpha^s} c_\beta (\beta, \alpha) \beta = \nu_s \alpha \tag{4.88}$$

for some $\nu_s = \nu_s(\alpha, \pi)$.

Now we have

$$\begin{aligned}
M(\alpha) &= \sum_{\beta \in \mathcal{A}} c_\beta (\beta, \alpha) \beta \\
&= \sum_{\substack{\beta \in \mathcal{A} \\ \beta \nsim \alpha}} c_\beta (\beta, \alpha) \beta + \sum_{\substack{\gamma \in \mathcal{A} \\ \gamma \sim \alpha}} c_\gamma (\gamma, \alpha) \gamma \\
&= \sum_s \sum_{\beta \in \Gamma_\alpha^s} c_\beta (\beta, \alpha) \beta + \sum_{k \colon k\alpha \in \mathcal{A}} c_{k\alpha} (k\alpha, \alpha) k\alpha \\
&= \sum_s \nu_s \alpha + \sum_{k \colon k\alpha \in \mathcal{A}} k^2 c_{k\alpha} (\alpha, \alpha) \alpha \\
&= \lambda \alpha,
\end{aligned}$$

where $\lambda = \sum_s \nu_s + \sum_{k:\, k\alpha\in\mathcal{A}} k^2 c_{k\alpha}(\alpha,\alpha)$. This proves the first statement.

Secondly, the space V is spanned by $\mathcal{A}$. As any vector $\alpha \in \mathcal{A}$ is an eigenvector for the operator M, the operator M is diagonalizable, and we have an eigenspace decomposition

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k}, \quad k \in \mathbb{N},\ \lambda_i \in \mathbb{C}$$

where $M\mid_{V_{\lambda_i}} = \lambda_i I$, and I is the identity operator. This proves the second statement. $\square$

Since $\mathcal{A} \subset V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}$, then $\mathcal{A}$ can be represented as

$$\mathcal{A} = \mathcal{A}_1 \sqcup \cdots \sqcup \mathcal{A}_k, \tag{4.89}$$

where $\mathcal{A}_i \coloneqq \mathcal{A} \cap V_{\lambda_i} \subset V_{\lambda_i}$, $i = 1,\dots,k$, and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$. The following two lemmas relate the series of vectors in $\mathcal{A}$ and its components $\mathcal{A}_i$.

Lemma 4.7.5. *Let $\mathcal{A}$ be a Euclidean trigonometric $\vee$-system. Let $\alpha \in \mathcal{A}$ be such that $\alpha \in V_{\lambda_i}$ for some $i \in \mathbb{N}$. Consider an α-series Γ^s_α in $\mathcal{A}_i$ and let $\beta \in \Gamma^s_\alpha$. Then either $\Gamma^s_\alpha \subset V_{\lambda_i}$ or $\Gamma^s_\alpha \subset \{\pm\beta\}$.*

Proof. For $\beta \in \Gamma^s_\alpha$ we have the two possible cases.

Case (i) $\beta \in V_{\lambda_i}$. Then for any $\gamma \in \Gamma^s_\alpha$ we have that $\gamma = m\alpha + \varepsilon\beta \in V_{\lambda_i}$ for some $m \in \mathbb{Z}$ and $\varepsilon = \pm 1$. Hence $\Gamma^s_\alpha \subset V_{\lambda_i}$.

Case (ii) $\beta \notin V_{\lambda_i}$. Hence $\beta \in V_{\lambda_j}$ for some $j \neq i$. Then for any $\gamma \in \Gamma^s_\alpha$ we have that $\gamma \in V_{\lambda_i}$ or $\gamma \in V_{\lambda_j}$ since decomposition (4.87) is the direct sum. Note that $\gamma \notin V_{\lambda_i}$ as otherwise we will have $\beta = m\alpha + \varepsilon\gamma \in V_{\lambda_i}$, for some $m \in \mathbb{Z}$ and $\varepsilon = \pm 1$, which is a contradiction. Note also that $\gamma \notin V_{\lambda_j}$ unless $\gamma = \pm\beta$ as otherwise we have $m\alpha = \beta + \varepsilon\gamma \in V_{\lambda_j}$ for some $m \in \mathbb{Z}$ and $\varepsilon = \pm 1$, which is a contradiction. Hence $\Gamma^s_\alpha \subset \{\pm\beta\}$. $\square$

Lemma 4.7.6. *Let $\alpha, \beta \in \mathcal{A}_i$. Let ${}^{\mathcal{A}}\Gamma^s_\alpha$, ${}^{\mathcal{A}_i}\Gamma^t_\alpha$ be the α-series in $\mathcal{A}$ and $\mathcal{A}_i$ respectively containing β. Then the set ${}^{\mathcal{A}}\Gamma^s_\alpha$ is equal to the set ${}^{\mathcal{A}_i}\Gamma^t_\alpha$.*

Proof. Let $\gamma \in {}^{\mathcal{A}}\Gamma^s_\alpha$. Then $\gamma = m\alpha + \varepsilon\beta \in \mathcal{A}_i$, for some $m \in \mathbb{Z}$ and $\varepsilon = \pm 1$. Thus $\gamma \in {}^{\mathcal{A}_i}\Gamma^t_\alpha$ by the maximality of ${}^{\mathcal{A}_i}\Gamma^t_\alpha$. Hence ${}^{\mathcal{A}}\Gamma^s_\alpha \subset {}^{\mathcal{A}_i}\Gamma^t_\alpha$. The opposite inclusion is obvious. Therefore ${}^{\mathcal{A}_i}\Gamma^t_\alpha = {}^{\mathcal{A}}\Gamma^s_\alpha$. $\square$

We will denote V_{λ_i} by V_i. Thus we have $M|_{V_i} = \lambda_i I$.

Proposition 4.7.7. *We have $(u,v) = 0$ for any $u \in V_i$ and $v \in V_j$ such that $i \neq j$.*

Proof. It is clear that $(M(u),v) = (u, M(v))$ for any $u, v \in V$. Now for any $u \in V_i$ and $v \in V$ we have

$$(M(u),v) = \lambda_i(u,v).$$

On the other hand for any $u \in V$ and $v \in V_j$ we have

$$(u, M(v)) = \lambda_j (u, v).$$

Hence for $u \in V_i$, $v \in V_j$, $i \neq j$ we have

$$(\lambda_i - \lambda_j)(u, v) = 0.$$

Thus $(u, v) = 0$ since $\lambda_i \neq \lambda_j$. □

The following statement takes place.

Lemma 4.7.8. *Restriction $(\cdot,\cdot)_i$ of the bilinear form $(\cdot,\cdot)$ onto the subspace V_i is non-degenerate.*

Proof. Suppose that $v \in V_i$ satisfies $(v, u)_i = 0$ for all $u \in V_i$. By Proposition 4.7.7 we have $(v, u) = 0$ for all $u \in V$. Hence $v = 0$ since $(\cdot,\cdot)$ is non-degenerate. □

The following statements relate the Euclidean trigonometric $\vee$-systems and the trigonometric $\vee$-systems.

Theorem 4.7.9. *If $\mathcal{A}$ is a Euclidean trigonometric $\vee$-system then subsystem $\mathcal{A}_i = \mathcal{A} \cap V_i$ is well-distributed in the subspace V_i with the bilinear form $(\cdot,\cdot)_i$ for all i. Furthermore, if the bilinear form*

$$G_{\mathcal{A}_i}(u, v) = \sum_{\alpha \in \mathcal{A}_i} c_\alpha (\alpha, u)(\alpha, v), \quad u, v \in V_i$$

is non-degenerate on V_i (equivalently, $G_{\mathcal{A}_i}$ is non-zero), then $\mathcal{A}_i$ is a trigonometric $\vee$-system.

Proof. By Lemma 4.7.4 we have $M|_{V_i} = \lambda_i I$. Hence for any $u, v \in V_i$ we have

$$G_{\mathcal{A}}(u, v) = (M(u), v) = \lambda_i (u, v). \tag{4.90}$$

Note also that by Proposition 4.7.7 and formula (4.90) we have for any $u, v \in V_i$ that

$$G_{\mathcal{A}_i}(u, v) = \sum_{\alpha \in \mathcal{A}_i} c_\alpha (\alpha, u)(\alpha, v) = \sum_{\alpha \in \mathcal{A}} c_\alpha (\alpha, u)(\alpha, v) = G_{\mathcal{A}}(u, v) = \lambda_i (u, v). \tag{4.91}$$

Thus subsystem $\mathcal{A}_i$ is well-distributed in the subspace V_i for all i.

Let us now assume that $G_{\mathcal{A}_i}$ is non-degenerate on V_i, that is $\lambda_i \neq 0$. Let $\alpha \in \mathcal{A}_i$. Consider an α-series Γ_α^s in $\mathcal{A}_i$. Then by Lemmas 4.7.5, 4.7.6 and formulas (4.90), (4.91) we have

$$\sum_{\beta \in \Gamma_\alpha^s} c_\beta G_{\mathcal{A}_i}(\alpha, \beta) \alpha \wedge \beta = \lambda_i \sum_{\beta \in \Gamma_\alpha^s} c_\beta (\alpha, \beta) \alpha \wedge \beta = 0$$

since $\mathcal{A}$ is a Euclidean trigonometric $\vee$-system. This proves the statement. $\square$

The following statement takes place as a corollary of Theorem 4.7.9.

Corollary 4.7.10. *If $(\mathcal{A}, c)$ is an irreducible Euclidean trigonometric $\vee$-system and $G_{\mathcal{A}}$ is non-degenerate then $(\mathcal{A}, c)$ is a trigonometric $\vee$-system.*

Proof. Since $\mathcal{A}$ is irreducible then we have $V = V_1$ and $M|_{V_1} = \lambda_1 I$. Then by Theorem 4.7.9 $\mathcal{A} \cap V_1 = \mathcal{A}$ is a trigonometric $\vee$-system since the bilinear form $G_{\mathcal{A}}$ is non-degenerate. This proves the statement. $\square$

Let π be a two-dimensional plane in V which contains two non-proportional vectors from $\mathcal{A}$.

Proposition 4.7.11. *Let $\mathcal{A}$ be a Euclidean trigonometric $\vee$-system. Then the set of vectors $\mathcal{A} \cap \pi$ is well-distributed in π or the system $\mathcal{A} \cap \pi$ is reducible.*

Proof. Let $\alpha \in \mathcal{A} \cap \pi$. Define a linear operator $M_\pi \colon \pi \to \pi$, by

$$M_\pi := \sum_{\beta \in \mathcal{A} \cap \pi} c_\beta \beta \otimes \beta.$$

Let us sum up the Euclidean trigonometric $\vee$-condition (4.68) over α-series which belong to the plane π. Then

$$\sum_{\beta \in \mathcal{A} \cap \pi} c_\beta (\beta, \alpha) \beta = \lambda \alpha, \tag{4.92}$$

holds for some $\lambda = \lambda(\alpha)$. Suppose that $\mathcal{A} \cap \pi$ is irreducible. Then λ does not depend on α and $M_\pi = \lambda I$. Therefore

$$\sum_{\beta \in \mathcal{A} \cap \pi} c_\beta (\beta, u)(\beta, v) = (M_\pi(u), v) = \lambda(u, v),$$

and the set $\mathcal{A} \cap \pi$ is well-distributed. This proves the statement. $\square$

Let us also note that if the subsystem $\mathcal{A} \cap \pi$ is reducible then it is contained in a pair of orthogonal lines, which is easy to see from the Euclidean trigonometric $\vee$-conditions.

The following statement relates the Euclidean trigonometric $\vee$-system and the (rational) complex Euclidean $\vee$-system.

Proposition 4.7.12. *Let $(\mathcal{A}, c)$ be an irreducible Euclidean trigonometric $\vee$-system. Then the set of vectors $\sqrt{c_\alpha}\alpha$ is a (rational) complex Euclidean $\vee$-system.*

Proof. Firstly, since $\mathcal{A}$ is irreducible then by Lemma 4.7.4 we have $\mathcal{A} = \mathcal{A}_1 \subset V = V_1$ and $M|_{V_1} = \lambda_1 I$. Then by Theorem 4.7.9 we have that $\mathcal{A}_1 = \mathcal{A}$ is well-distributed.

Secondly, by Proposition 4.7.11 we have that any two-dimensional subsystem $\mathcal{A} \cap \pi$ is well-distributed or reducible. Since the form $G_{\mathcal{A}}$ coincides with the canonical form for the system $\{\sqrt{c_\alpha}\alpha\}$ in the rational case, the statement follows. $\square$

4.8 Restricted solutions of commutativity equations

In this section we apply the restriction procedure to a given solution to the commutativity equations in analogy to what we have done in Chapter 3 with the solution of WDVV equations.

Let $\mathcal{B} = \mathcal{A} \cap W$ be a subsystem of $\mathcal{A}$ for some n-dimensional linear subspace $W = \langle \mathcal{B} \rangle \subset V$. Define

$$W_{\mathcal{B}} \coloneqq \{x \in V \colon (\beta, x) = 0 \quad \forall \beta \in \mathcal{B}\}. \tag{4.93}$$

Let $(\cdot,\cdot)_{\mathcal{B}}$ be the restriction of $(\cdot,\cdot)$ on $W_{\mathcal{B}}$, and assume that it is non-degenerate. Let us denote by $\pi_{\mathcal{B}}(\alpha)$ the orthogonal projection of $\alpha \in V$ to the subspace $W_{\mathcal{B}}$ with respect to the inner product $(\cdot,\cdot)$. Let $\pi_{\mathcal{B}}(\mathcal{A}) = \{\pi_{\mathcal{B}}(\alpha) \colon \pi_{\mathcal{B}}(\alpha) \neq 0,\, \alpha \in \mathcal{A}\}$. Note that we include each vector once even if it can be obtained as different projections. We define the multiplicity $c(\pi_{\mathcal{B}}(\alpha)) \coloneqq \sum_{\gamma \in \mathcal{A}} c(\gamma)$ where $\pi_{\mathcal{B}}(\gamma) = \pi_{\mathcal{B}}(\alpha)$. Let

$$S = \{\alpha_1, \dots, \alpha_n\} \subset \mathcal{B} \tag{4.94}$$

be a basis of W. Let $f_1, \dots, f_n$ be an orthonormal basis of the space $W_{\mathcal{B}}$, and let $\xi_1, \dots, \xi_n$ be the corresponding orthonormal coordinates in $W_{\mathcal{B}}$. Define $M_{\mathcal{A}} = V \setminus \bigcup_{\alpha \in \mathcal{A}} \Pi_\alpha$, and $M_{\mathcal{B}} = W_{\mathcal{B}} \setminus \bigcup_{\alpha \in \mathcal{A} \setminus \mathcal{B}} \Pi_\alpha$, where $\Pi_\alpha = \{x \in V \colon (\alpha, x) = 0\}$. The following statement shows that the class of solutions of commutativity equations corresponding to Euclidean $\vee$-systems is closed under the restrictions.

Theorem 4.8.1. *Assume that prepotential* (4.53) *satisfies commutativity equations* (4.49)*. Let $\mathcal{B} \subset \mathcal{A}$ be a subsystem, and let S be as defined in* (4.94)*. Suppose that $C_\delta^{\alpha_0} \neq 0$ for any $\alpha \in S$, $\alpha_0 \in \delta_\alpha$. Then the prepotential*

$$F_{\mathcal{B}} = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} c_\alpha f((\widetilde{\alpha}, \xi)), \quad \xi \in M_{\mathcal{B}}, \tag{4.95}$$

where $\widetilde{\alpha} = \pi_{\mathcal{B}}(\alpha)$, satisfies the commutativity equations

$$(F_{\mathcal{B}})_i (F_{\mathcal{B}})_j = (F_{\mathcal{B}})_j (F_{\mathcal{B}})_i, \quad i, j = 1, \dots, n, \tag{4.96}$$

where $(F_{\mathcal{B}})_i$ is the $n \times n$ matrix with entries

$$((F_{\mathcal{B}})_i)_{pq} = (F_{\mathcal{B}})_{ipq} = \frac{\partial^3 F_{\mathcal{B}}}{\partial \xi_i \partial \xi_p \partial \xi_q}.$$

Proof. First for any $u = (u_1, \dots, u_N)$, $v = (v_1, \dots, v_N) \in V$ let us consider the vector fields $\partial_u = \sum_{i=1}^N u_i \partial_{x_i}$, $\partial_v = \sum_{i=1}^N v_i \partial_{x_i} \in T_x M_{\mathcal{A}}$. We define the following multiplication on the tangent space $T_xM_{\mathcal{A}}$:

$$\partial_u * \partial_v = u_i v_j F_{ijk}\partial_{x_k}, \tag{4.97}$$

where the summation over repeated indices here and below is assumed. It is easy to check that the associativity of the multiplication $*$ is equivalent to the commutativity equations (4.49). From the formula (4.53) we have

$$F_{ijk} = \sum_{\alpha\in\mathcal{A}} c_\alpha \alpha_i\alpha_j\alpha_k \cot{(\alpha, x)}.$$

Hence multiplication (4.97) takes the form

$$\partial_u * \partial_v = \sum_{\alpha\in\mathcal{A}} c_\alpha(\alpha, u)(\alpha, v)\cot{(\alpha, x)}\partial_\alpha. \tag{4.98}$$

By identifying $V \cong T_xV$, we have

$$u * v = \sum_{\alpha\in\mathcal{A}} c_\alpha(\alpha, u)(\alpha, v)\cot{(\alpha, x)}\alpha. \tag{4.99}$$

Consider now a point $x_0 \in M_{\mathcal{B}}$ and two tangent vectors $u_0, v_0 \in T_{x_0}M_{\mathcal{B}}$. We extend vectors u_0 and v_0 to two local analytic vector fields $u(x), v(x)$ in the neighbourhood U of x_0 that are tangent to the subspace $W_{\mathcal{B}}$ at any point $x \in M_{\mathcal{B}} \cap U$ such that $u_0 = u(x_0)$ and $v_0 = v(x_0)$. The following lemma holds and its proof is similar to the proof of Lemma 3.2.1

Lemma 4.8.2. *The limit of the product $u(x) * v(x)$ exists when vector x tends to $x_0 \in M_{\mathcal{B}}$ and it satisfies*

$$u_0 * v_0 = \sum_{\alpha\in\mathcal{A}\setminus\mathcal{B}} c_\alpha(\alpha, u_0)(\alpha, v_0)\cot(\alpha, x_0)\alpha. \tag{4.100}$$

In particular, the product $u_0 * v_0$ is determined by vectors u_0 and v_0 only.

The following lemma holds and it shows that multiplication (4.98) is closed on the tangent space $T_{x_0}M_{\mathcal{B}}$.

Lemma 4.8.3. *Let $u, v \in T_{x_0}M_{\mathcal{B}}$ where $x_0 \in M_{\mathcal{B}}$. Then $u * v \in T_{x_0}M_{\mathcal{B}}$.*

The proof of Lemma 4.8.3 is similar to the proof of Lemma 3.2.2. It uses analogue of Proposition 2.6.20 which claims that the following identity holds for any $a, b \in V$ if $\tan(\alpha, x) = 0$:

$$\sum_{\beta\in\mathcal{A}\setminus\delta_\alpha} c_\beta(\alpha, \beta)\cot(\beta, x)B_{\alpha,\beta}(a\otimes b)\alpha\wedge\beta = 0.$$

Then for $u, v \in T_{x_0}M_{\mathcal{B}}$, $x_0 \in M_{\mathcal{B}}$, the product (4.98) takes the form

$$\partial_u * \partial_v = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} c_\alpha (\widetilde{\alpha}, u)(\widetilde{\alpha}, v) \cot(\widetilde{\alpha}, x_0) \partial_{\widetilde{\alpha}}. \tag{4.101}$$

By using the orthonormal basis $f_1, \ldots, f_n$ of $W_{\mathcal{B}}$ we rearrange $\partial_{\widetilde{\alpha}}$ as

$$\partial_{\widetilde{\alpha}} = \sum_{k=1}^{n} (\widetilde{\alpha}, f_k) \partial_{f_k}.$$

Hence for $x_0 = \xi = \sum_{i=1}^{n} \xi_i f_i$ we have

$$\begin{aligned} \partial_{f_i} * \partial_{f_j} &= \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} \sum_{k=1}^{n} c_\alpha (\widetilde{\alpha}, f_i)(\widetilde{\alpha}, f_j)(\widetilde{\alpha}, f_k) \cot(\widetilde{\alpha}, \xi) \partial_{f_k} \\ &= \sum_{k=1}^{n} \widetilde{F}_{ijk} \partial_{f_k}, \quad i, j = 1, \ldots, n, \end{aligned} \tag{4.102}$$

where $\widetilde{F}(\xi) = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{B}} c_\alpha f(\alpha, \xi) = F_{\mathcal{B}}$. Now multiplication (4.102) is associative and it is easy to check that its associativity is equivalent to the commutativity equations

$$\widetilde{F}_i \widetilde{F}_j = \widetilde{F}_j \widetilde{F}_i, \quad i, j = 1, \ldots, n.$$

Hence the restricted prepotential on $W_{\mathcal{B}}$ corresponding to the restricted system satisfies the commutativity equations, which proves the theorem. □

As a corollary of Theorems 4.8.1 and 4.1.3 the following result takes place.

Corollary 4.8.4. *Under the assumption of Theorem 4.1.3 on multiplicities, all the functions of the form* (4.53) *corresponding to restricted systems of the root system BC_N satisfy the commutativity equations.*

Note that in Section 4.2 we proved Corollary 4.8.4 following a different approach, by relating WDVV equations (4.2) with the commutativity equations for the restricted systems of the root system BC_N. See Theorem 4.2.5.

Similarly, as a corollary of Theorems 4.8.1 and 4.1.4 the following result takes place.

Corollary 4.8.5. *Under the assumptions of Theorem 4.1.4 on multiplicities, all the functions of the form* (4.53) *corresponding to the three-dimensional and two-dimensional restricted systems of the root system F_4 satisfy the commutativity equations.*

Solutions of the commutativity equations corresponding to the projected systems of root systems BC_N and F_4 imply existence of certain $\mathcal{N} = 4$ supersymmetric Hamiltonians given by formulas (4.56), (4.57) and (4.59) in Theorems 4.4.1 and 4.4.2.

In the case of BC_N root system the Hamiltonians of its restricted systems are given in Theorem 4.4.7 by formulas (4.65), (4.66) and (4.67).

In the case of F_4 root system there are two projected systems in dimension three and four projected system in dimension two. The projected systems on the plane are given in Appendix A.1, where these configuration are given explicitly and denoted by $(F_4, A_2)_1, (F_4, A_1^2)$, see Table A.1 below. The configurations $(F_4, A_2)_2$ and (F_4, B_2) are equivalent to root systems G_2 and BC_2, these cases were considered in [3]. The other four cases in dimension two and three are new. Let us give details of the three-dimensional projected systems of F_4.

Consider the positive half of root system F_4 consisting of vectors

$$F_4^+ = \{e_i\,(1 \le i \le 4),\ e_i \pm e_j\,(1 \le i < j \le 4),\ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

Let r be the multiplicity of short roots and let q be the multiplicity of long roots. Then we have the following three-dimensional projected systems.

- The projected system $(F_4, A_1)_1$ of F_4 to the hyperplane $x_4 = 0$ consists of the following set of vectors:

$$\begin{aligned}
&e_i, \quad \text{with multiplicity} \quad r + 2q, \quad 1 \le i \le 3,\\
&e_i \pm e_j, \quad \text{with multiplicity} \quad q, \quad 1 \le i < j \le 3,\\
&\frac{1}{2}(e_1 \pm e_2 \pm e_3), \quad \text{with multiplicity} \quad 2r.
\end{aligned}$$

- The projected system $(F_4, A_1)_2$ of F_4 to the hyperplane $x_3 = x_4$ consists of the following set of vectors (after making change of variables and renaming vectors):

$$\begin{aligned}
&e_1, e_2, \quad \text{with multiplicity} \quad r,\\
&\sqrt{2}e_3, \quad \text{with multiplicity} \quad q,\\
&\frac{\sqrt{2}}{2}e_3, \quad \text{with multiplicity} \quad 2r,\\
&e_1 \pm e_2, \quad \text{with multiplicity} \quad q,\\
&\frac{1}{2}(e_1 \pm e_2), \quad \text{with multiplicity} \quad 2r,\\
&e_1 \pm \frac{\sqrt{2}}{2}e_3,\ e_2 \pm \frac{\sqrt{2}}{2}e_3, \quad \text{with multiplicity} \quad 2q,\\
&\frac{1}{2}(e_1 \pm e_2 \pm \sqrt{2}e_3), \quad \text{with multiplicity} \quad r.
\end{aligned}$$

Theorem 4.8.6. *Let $(\widehat{\mathcal{A}}, c)$ be one of the configurations $(F_4, A_1)_1$, $(F_4, A_1)_2$, $(F_4, A_2)_1$, (F_4, A_1^2). Then the function $F = \sum_{\alpha\in\widehat{\mathcal{A}}} c_\alpha f((\alpha, x))$, where $f'''(z) = \cot z$, satisfies the commutativity* *equations, where* $x \in \mathbb{C}^3$ *for the first two configurations and* $x \in \mathbb{C}^2$ *for the last two configurations and parameters* r, q *satisfy the condition* $r = -2q$ *or* $r = -4q$*. Furthermore, the corresponding Hamiltonians* (4.56), (4.59) *satisfy the supersymmetry algebra relations stated in Theorems 4.4.1, 4.4.2*

Chapter 5

Commutativity equations and WDVV equations

In this chapter we investigate the relation between the set of commutativity equations and the set of WDVV equations in N-dimensional space. This leads to new solutions for WDVV equations from known solutions of the commutativity equations.

Let $V \cong \mathbb{C}^N$. Let $F = F(x_1, \dots, x_N)$ be a function on V. We recall that it has been proven in [41] (see also [40]) that the (generalized) WDVV equations

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i, \quad i, j, k = 1, \dots, N \tag{5.1}$$

can be written equivalently in the form

$$F_i B^{-1} F_j = F_j B^{-1} F_i, \quad i, j = 1, \dots, N, \tag{5.2}$$

where B is any non-degenerate linear combination of matrices F_k with functional coefficients A^k, $(k = 1, \dots, N)$. If the matrix B happens to be a multiple of identity for some functions A^k, then WDVV equations (5.2) reduce to the commutativity equations

$$F_i F_j = F_j F_i, \quad i, j = 1, \dots, N. \tag{5.3}$$

The natural question to investigate is when there exists such a linear combination B which is proportional to the standard metric δ^{ij}. We do this in this section.

Let us assume that the function $F = F(x_1, \dots, x_N)$ satisfies the commutativity equations (5.3). Let us denote by $[F_i, F_j]_{(a,b)}$ the (a, b)-entry of the commutator $[F_i, F_j]$, that is we have the explicit form

$$[F_i, F_j]_{(a,b)} = \sum_{m=1}^{N} (F_{iam} F_{jbm} - F_{ibm} F_{jam}).$$

Since the function F satisfies the commutativity equations then we have

$$[F_i, F_j]_{(a,b)} = \sum_{m=1}^{N} (F_{iam}F_{jbm} - F_{ibm}F_{jam}) = 0, \quad (1 \leq i, j, a, b \leq N).$$

These equations imply the following identity for any $1 \leq i, j, k \leq N$:

$$F_{ijk}F_{iii} = \sum_{m=1}^{N} F_{ijm}F_{ikm} - \sum_{m \neq i} F_{iim}F_{jkm}, \tag{5.4}$$

which is obtained from $[F_i, F_k]_{(i,j)} = 0$, $k \neq i, j$. We also have

$$F_{ijj}F_{iii} + F_{iij}F_{jjj} = \sum_{m=1}^{N} F_{ijm}^2 - \sum_{m \neq i,j} F_{iim}F_{jjm}, \tag{5.5}$$

for $i \neq j$, which is obtained from the identity (5.4) for $k = j$.

Observe also the equality of matrix entries $[F_a, F_b]_{(i,j)} = [F_i, F_j]_{(a,b)}$. Let us also introduce the following notation

$$[F_i, F_j]_{(a,b)}^{\{m\}} = F_{iam}F_{jbm} - F_{ibm}F_{jam}, \tag{5.6}$$

where there is no summation over m in the right-hand side. Let us define a matrix $B = (B_{ij})_{i,j=1}^{N}$ with the entries given as a linear combination of the third order derivatives of F:

$$B_{ij} = \sum_{k=1}^{N} A^k F_{kij}, \tag{5.7}$$

for some functions $A^k = A^k(x_1, \ldots, x_N)$. Now we will investigate when there exists such a combination B so that equations (5.3) imply the equations (5.2). For that it is sufficient to deduce that the matrix B is proportional to the identity.

Before we start our investigation, let us recall the formula of Laplace expansion in multiple rows of a determinant from linear algebra which will be used later.

Let Q be a given $N \times N$ matrix. Let $[N] = \{1, 2, \ldots, N\}$ and let I be a fixed subset of $[N]$. Let J be another subset of $[N]$ where subsets J and I has the same size, that is $|J| = |I|$. Let I', J' be the complements of I, J respectively in $[N]$. We want to calculate the determinant of Q by expanding along all rows which belong to the subset I. This leads to the formula

$$\det Q = \sum_{J \subset [N]} \sigma_J \det Q_{IJ} \det Q_{I'J'}, \tag{5.8}$$

where Q_{IJ} is a matrix composed of rows I and columns J of the matrix Q, and $\sigma_J = (-1)^s$ with $s = \sum_{i \in I} i + \sum_{j \in J} j$.

5.1 Constant metric in dimension two

The commutativity equations for dimension two is the equation $F_1F_2 = F_2F_1$ which is equivalent to the single relation

$$F_{122}F_{111} + F_{112}F_{222} - F_{112}^2 - F_{122}^2 = 0. \tag{5.9}$$

Note that WDVV equations hold for any function F on the plane. In the next proposition we give the formulas for coefficients A^1, A^2.

Proposition 5.1.1. *Assume that the function $F = F(x_1, x_2)$ satisfies the commutativity equation* (5.9)*. Let*

$$A^1 = F_{122}, \quad A^2 = -F_{112}. \tag{5.10}$$

Then the matrix $B = A^1F_1 + A^2F_2$ is a multiple of the identity matrix.

Proof. Firstly, we have

$$B_{12} = B_{21} = A^1F_{112} + A^2F_{122} = 0$$

by (5.10). Secondly, we have

$$B_{11} = A^1F_{111} + A^2F_{112} = F_{122}F_{111} - F_{112}^2,$$
$$B_{22} = A^1F_{122} + A^2F_{222} = F_{122}^2 - F_{112}F_{222}.$$

By relation (5.9) we have $B_{11} = B_{22}$. This proves the proposition. □

5.2 Constant metric in dimension three

In this section we assume that $N = 3$.

Proposition 5.2.1. *Assume that the function F satisfies the commutativity equations* (5.3)*. Then the matrix Q given by*

$$Q = \begin{pmatrix} F_{112} & F_{122} & F_{123} \\ F_{113} & F_{123} & F_{133} \\ F_{123} & F_{223} & F_{233} \end{pmatrix} \tag{5.11}$$

is singular.

Proof. Let D be the determinant of the matrix Q. We have

$$D = \det Q = F_{112}(F_{233}F_{123} - F_{133}F_{223}) - F_{122}(F_{113}F_{233} - F_{133}F_{123}) + F_{123}(F_{113}F_{223} - F_{123}^2). \tag{5.12}$$

Firstly, from relation (5.4) we have

$$F_{123}F_{111} = F_{112}F_{113} + F_{122}F_{123} + F_{123}F_{133} - F_{112}F_{223} - F_{113}F_{233}, \tag{5.13}$$
$$F_{123}F_{222} = F_{112}F_{123} + F_{122}F_{223} + F_{123}F_{233} - F_{122}F_{113} - F_{223}F_{133}. \tag{5.14}$$

Secondly, by multiplying relation (5.5) for $i = 1, j = 2$ by F_{123} we have

$$F_{123}(F_{122}F_{111} + F_{112}F_{222}) = F_{123}(F_{112}^2 + F_{122}^2 + F_{123}^2 - F_{113}F_{223}). \tag{5.15}$$

Now by substituting relations (5.13), (5.14) into the left-hand side of relation (5.15) we get $D = 0$. This proves the proposition. $\square$

As a corollary of Proposition 5.2.1 the following statement takes place.

Corollary 5.2.2. *Assume that the function F satisfies the commutativity equations* (5.3). *Then there exists a non-zero solution (A^1, A^2, A^3) for the following system of equations:*

$$B_{12} = \sum_{m=1}^{3} A^m F_{12m} = 0, \quad B_{13} = \sum_{m=1}^{3} A^m F_{13m} = 0, \quad B_{23} = \sum_{m=1}^{3} A^m F_{23m} = 0. \tag{5.16}$$

Proof. To prove the statement, it is enough to show that commutativity equations imply that the coefficient matrix corresponding to the homogeneous system of equations (5.16) is singular. Since this coefficient matrix is given by formula (5.11) the statement follows from Propositin 5.2.1. $\square$

Define

$$A^1 = \det\begin{pmatrix} F_{122} & F_{123} \\ F_{123} & F_{133} \end{pmatrix} = F_{122}F_{133} - F_{123}^2, \quad A^2 = -\det\begin{pmatrix} F_{112} & F_{123} \\ F_{113} & F_{133} \end{pmatrix} = F_{113}F_{123} - F_{112}F_{133},$$
$$A^3 = \det\begin{pmatrix} F_{112} & F_{122} \\ F_{113} & F_{123} \end{pmatrix} = F_{112}F_{123} - F_{113}F_{122}. \tag{5.17}$$

Proposition 5.2.3. *Assume that the function F satisfies the commutativity equations* (5.3). *Then functions A^i, $(i = 1, 2, 3)$ given by* (5.17) *solve the system of equations*

$$B_{12} = \sum_{m=1}^{3} A^m F_{12m} = 0, \quad B_{13} = \sum_{m=1}^{3} A^m F_{13m} = 0, \quad B_{23} = \sum_{m=1}^{3} A^m F_{23m} = 0. \tag{5.18}$$

Proof. After substituting of functions A^i, $(i = 1, 2, 3)$ into the formula of B_{12} we get a determinant with repeated rows, which is equal to zero. Similarly, the equality $B_{13} = 0$ is also satisfied. Let us now check that $B_{23} = 0$. We have

$$B_{23} = A^1 F_{123} + A^2 F_{223} + A^3 F_{233}.$$

After substituting the values of A^1, A^2, A^3 we get that $B_{23} = \det Q$, where Q is the matrix given by (5.11). The statement follows from Proposition (5.2.1). □

Let us define the coefficient matrix

$$Q = \begin{pmatrix} F_{112} & F_{212} & F_{312} \\ F_{113} & F_{213} & F_{313} \end{pmatrix} \tag{5.19}$$

corresponding to the system of equations $B_{12} = 0$, $B_{13} = 0$.

The following statement takes place.

Proposition 5.2.4. *Assume that the rank of of the matrix Q given by formula* (5.19) *is equal to two. Then the diagonal entries of matrix $B = \sum_{m=1}^{3} A^m F_m$, where A^m are given by formula* (5.17)*, are equal; $B_{11} = B_{ii}$ for $i = 2, 3$.*

This proposition is a particular case of Proposition 5.3.4 below which is valid for any dimension $N \geq 3$, so we postpone the proof until Section 5.3.

As a corollary of Propositions 5.2.3 and 5.2.4 the following statement takes place.

Proposition 5.2.5. *Assume that the matrix Q given by formula* (5.19) *has rank two. Then the matrix $B = \sum_{m=1}^{3} A^m F_m$, where A^m are given by relations* (5.17)*, is proportional to the identity matrix.*

5.3 Constant metric in general

Let us now assume $N \geq 4$. Let V be an N-dimensional space. Let $F = F(x_1, \ldots, x_N)$ be a function such that commutativity equations (5.3) hold. Let us also define functions A^k, $(k = 1, \ldots, N)$ by the formula

$$A^k = (-1)^{k+1} \det \begin{pmatrix} F_{112} & F_{212} & \cdots & F_{(k-1)12} & F_{(k+1)12} & \cdots & F_{N12} \\ F_{113} & F_{213} & \cdots & F_{(k-1)13} & F_{(k+1)13} & \cdots & F_{N13} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ F_{11N} & F_{21N} & \cdots & F_{(k-1)1N} & F_{(k+1)1N} & \cdots & F_{N1N} \end{pmatrix}. \tag{5.20}$$

That is A^k are given by the determinant of the $(N-1) \times (N-1)$ matrix corresponding to equations $B_{12} = B_{13} = \cdots = B_{1N} = 0$ after removing the k^{th} column. In this section the summation over repeated indices will be assumed provided that one of indices is subscript and the other is superscript.

Lemma 5.3.1. *For any function $F = F(x_1, \dots, x_N)$ which satisfies the commutativity equations* (5.3) *the following relation holds*

$$\det \begin{pmatrix} F_{a1r} & F_{b1r} & F_{c1r} \\ F_{a1t} & F_{b1t} & F_{c1t} \\ F_{art} & F_{brt} & F_{crt} \end{pmatrix} = -\sum_{m \neq t} \det \begin{pmatrix} F_{a1r} & F_{b1r} & F_{c1r} \\ F_{a1m} & F_{b1m} & F_{c1m} \\ F_{arm} & F_{brm} & F_{crm} \end{pmatrix}, \tag{5.21}$$

where $1 \leq a, b, c \leq N$ and $2 \leq r < t \leq N$.

Proof. We have by the first row expansion and commutativity equations

$$\begin{aligned} \det \begin{pmatrix} F_{a1r} & F_{b1r} & F_{c1r} \\ F_{a1t} & F_{b1t} & F_{c1t} \\ F_{art} & F_{brt} & F_{crt} \end{pmatrix} &= F_{a1r} \det \begin{pmatrix} F_{b1t} & F_{c1t} \\ F_{brt} & F_{crt} \end{pmatrix} - F_{b1r} \det \begin{pmatrix} F_{a1t} & F_{c1t} \\ F_{art} & F_{crt} \end{pmatrix} \\ + F_{c1r} \det \begin{pmatrix} F_{a1t} & F_{b1t} \\ F_{art} & F_{brt} \end{pmatrix} &= F_{a1r}[F_1, F_r]^{\{t\}}_{(b,c)} - F_{b1r}[F_1, F_r]^{\{t\}}_{(a,c)} + F_{c1r}[F_1, F_r]^{\{t\}}_{(a,b)} \\ = -F_{a1r} \sum_{m \neq t} [F_1, F_r]^{\{m\}}_{(b,c)} &+ F_{b1r} \sum_{m \neq t} [F_1, F_r]^{\{m\}}_{(a,c)} - F_{c1r} \sum_{m \neq t} [F_1, F_r]^{\{m\}}_{(a,b)} \\ = -\sum_{m \neq t} \det \begin{pmatrix} F_{a1r} & F_{b1r} & F_{c1r} \\ F_{a1m} & F_{b1m} & F_{c1m} \\ F_{arm} & F_{brm} & F_{crm} \end{pmatrix}. \end{aligned}$$

This proves the statement. $\square$

The following statement takes place.

Lemma 5.3.2. *Suppose that the function $F = F(x_1, \dots, x_N)$ satisfies the commutativity equations* (5.3). *Then the matrix*

$$Q = \begin{pmatrix} F_{112} & F_{212} & \cdots & F_{N12} \\ F_{113} & F_{213} & \cdots & F_{N13} \\ \vdots & \vdots & \ddots & \vdots \\ F_{11r} & F_{21r} & \cdots & F_{N1r} \\ \vdots & \vdots & \ddots & \vdots \\ F_{11t} & F_{21t} & \cdots & F_{N1t} \\ \vdots & \vdots & \ddots & \vdots \\ F_{11N} & F_{21N} & \cdots & F_{N1N} \\ F_{1rt} & F_{2rt} & \cdots & F_{Nrt} \end{pmatrix}, \tag{5.22}$$

where $2 \leq r < t \leq N$ is singular.

Proof. Let $D = \det Q$. Let R_i denote the i^{th} row in the matrix Q. Let us now perform Laplace expansion of D along rows number $r-1$, $t-1$, N, that is along the following rows of Q:

$$\begin{aligned} R_{r-1} &= (F_{11r}, F_{21r}, \dots, F_{N1r}), \\ R_{t-1} &= (F_{11t}, F_{21t}, \dots, F_{N1t}), \\ R_N &= (F_{1rt}, F_{2rt}, \dots, F_{Nrt}), \end{aligned}$$

In the notations of Laplace expansion formula (5.8) we choose $I = \{r-1, t-1, N\}$ and $J = \{a, b, c\}$, where $1 \le a < b < c \le N$. Hence we have

$$Q_{IJ} = \begin{pmatrix} F_{a1r} & F_{b1r} & F_{c1r} \\ F_{a1t} & F_{b1t} & F_{c1t} \\ F_{art} & F_{brt} & F_{crt} \end{pmatrix}. \tag{5.23}$$

By formula (5.8) the determinant D takes the form

$$D = \sum_J \sigma_J \det Q_{I'J'} \det \begin{pmatrix} F_{a1r} & F_{b1r} & F_{c1r} \\ F_{a1t} & F_{b1t} & F_{c1t} \\ F_{art} & F_{brt} & F_{crt} \end{pmatrix}, \tag{5.24}$$

where I', J' are the complements of I, J respectively in $[N] = \{1, \dots, N\}$, $\sigma_J = (-1)^s$ and $s = \sum_{i \in I} i + \sum_{j \in J} j = N + r + t + a + b + c - 2$. Now by Lemma 5.3.1 the determinant (5.24) can be rewritten equivalently as

$$\begin{aligned} D &= -\sum_J \sigma_J \det Q_{I'J'} \Big(\sum_{m \ne t} \det \begin{pmatrix} F_{a1r} & F_{b1r} & F_{c1r} \\ F_{a1m} & F_{b1m} & F_{c1m} \\ F_{arm} & F_{brm} & F_{crm} \end{pmatrix} \Big) \\ &= \sum_{m \ne t} \det \begin{pmatrix} F_{11r} & F_{21r} & \cdots & F_{N1r} \\ F_{11m} & F_{21m} & \cdots & F_{N1m} \\ F_{112} & F_{212} & \cdots & F_{N12} \\ \vdots & \vdots & \ddots & \vdots \\ F_{11\mu} & F_{21\mu} & \cdots & F_{N1\mu} \\ \vdots & \vdots & \ddots & \vdots \\ F_{11N} & F_{21N} & \cdots & F_{N1N} \\ F_{1rm} & F_{2rm} & \cdots & F_{Nrm} \end{pmatrix}, \end{aligned}$$

where μ runs from 2 to N excluding r and t. Hence $D = 0$ since for each m the determinant contains a pair of repeated rows. Hence the matrix Q is singular as required. □

The following statement takes place.

Proposition 5.3.3. *Assume that the function $F = F(x_1, \dots, x_N)$ satisfies the commutativity equations* (5.3). *Assume also that the rank of the matrix (F_{1ij}), where $2 \le i \le N$, $1 \le j \le N$, is $N-1$. Then the coefficient matrix corresponding to the system* (5.7) *where functions A^k, $(k = 1, \dots, N)$ are given by formula* (5.20), *is diagonal.*

Proof. Let us consider the system of linear equations

$$B_{1m} = \sum_{k=1}^{N} A^k F_{1km} = 0, \tag{5.25}$$

for some functions $A^k = A^k(x_1, \dots, x_N)$. The system (5.25) represents a homogeneous system of $N-1$ linear equations in N variables A^i, $(i = 1, \dots, N)$. The assumption that rank of the matrix $P = (F_{1ij})$, where $2 \le i \le N$, $1 \le j \le N$, is $N-1$ implies that the system (5.25) has a non-trivial solution which is unique up to proportionality.

Now, fix $2 \le s \le N$. The direct substitution of the functions A^k given by formula (5.20) into the right-hand side of relation (5.25) gives a row expansion of the determinant with the repeated rows, hence the equation $B_{1s} = 0$ is satisfied. Note also that $A^k \neq 0$ for some k since the rank of the matrix P is $N-1$. Now we will check the other non-diagonal equations, namely, we will show that off-diagonal entries $B_{rt} = 0$, where $2 \le r < t \le N$. In order to do so, we add one row corresponding to the non-diagonal entry B_{rt} to the coefficient matrix of the linear system (5.25) and we will show that the matrix is singular. This will imply the existence of a non-trivial solution to the resulting system of N equations. Indeed, as the first $N-1$ equations have a unique solution given by (5.20) up to the proportionality, it also has to solve the last equation. Thus we consider the equations

$$B_{1m} = 0, \ (m = 2, \dots, N), \quad B_{rt} = 0, \quad 2 \le r < t \le N. \tag{5.26}$$

Then the coefficient matrix Q corresponding to equations (5.26) is given by formula (5.22) which is a singular matrix as required by Lemma 5.3.2. This proves the statement. □

The following statement gives further property for the matrix of the system (5.7).

Proposition 5.3.4. *Under the assumption of Proposition 5.3.3 the matrix of the system of linear equations* (5.7), *where function A^k, $(k = 1, \dots, N)$ are given by formula* (5.20), *satisfies*

$$B_{11} = B_{pp}, \tag{5.27}$$

for all p.

Proof. Let us first consider the case $p = 2$. Since the matrix (F_{1jk}),where $(2 \leq j \leq N,\ 1 \leq k \leq N)$ has the maximal rank $N-1$, this implies that there exists some q $(1 \leq q \leq N)$ such that $F_{12q} \neq 0$. Following the idea of the proof of Proposition 5.3.3, let us consider the following set of homogeneous equations:

$$\begin{aligned} &B_{1m} = 0, \quad 2 \leq m \leq N, \\ &F_{12q}(B_{11} - B_{22}) = 0. \end{aligned} \tag{5.28}$$

It is sufficient to show that the coefficient matrix corresponding to equations (5.28) is singular. Let Q be the coefficient matrix corresponding to equations (5.28), That is

$$Q = \begin{pmatrix} F_{112} & F_{212} & \cdots & F_{N12} \\ F_{113} & F_{213} & \cdots & F_{N13} \\ \vdots & \vdots & \ddots & \vdots \\ F_{11N} & F_{21N} & \cdots & F_{N1N} \\ F_{12q}(F_{111} - F_{122}) & F_{12q}(F_{211} - F_{222}) & \cdots & F_{12q}(F_{N11} - F_{N22}) \end{pmatrix}. \tag{5.29}$$

Let $D = \det Q$. Now from the identity (5.4) for $i = 1, j = 2, k = q$, we have

$$F_{12q}F_{111} = \sum_{m=1}^{N} F_{12m}F_{1qm} - \sum_{m=2}^{N} F_{11m}F_{2qm}.$$

Similarly, from the identity (5.4) for $i = 2, j = 1, k = q$, we have

$$F_{12q}F_{222} = \sum_{m=1}^{N} F_{12m}F_{2qm} - \sum_{m \neq 2} F_{1qm}F_{22m}.$$

Let R_i denote the i^{th} row in the matrix Q. Then we have

$$\begin{aligned} R_k &= (F_{11(k+1)}, F_{21(k+1)}, \ldots, F_{N1(k+1)}), \quad (1 \leq k \leq N-1), \\ R_N &= (r_{N1}, r_{N2}, \ldots, r_{NN}), \end{aligned} \tag{5.30}$$

where

$$\begin{aligned} r_{N1} &= \sum_{m \neq 2} F_{1qm}F_{12m} - \sum_{m \neq 1} F_{11m}F_{2qm}, \\ r_{N2} &= \sum_{m \neq 2} F_{1qm}F_{22m} - \sum_{m \neq 1} F_{12m}F_{2qm}, \\ r_{Nk} &= F_{12q}(F_{k11} - F_{k22}), \quad (3 \leq k \leq N). \end{aligned}$$

Now let us perform the following row operation on the matrix Q and let $\widetilde{Q}$ be the resulting matrix:

$$R_N \to \widetilde{R}_N = R_N - F_{11q}R_1 + \sum_{k=2}^{N} F_{2qk}R_{k-1}.$$

Let $\widetilde{r}_{Nk}$ be the k^{th} element in the row $\widetilde{R}_N$ of the matrix $\widetilde{Q}$. We have

$$\begin{aligned}
\widetilde{r}_{N1} &= \sum_{m\neq 1,2} F_{1qm}F_{12m}, \\
\widetilde{r}_{N2} &= \sum_{m\neq 1,2} F_{1qm}F_{22m}, \\
\widetilde{r}_{Nk} &= \sum_{m=1}^{N} F_{2qm}F_{1km} - F_{12q}F_{22k} - F_{11q}F_{12k}, \quad (3 \leq k \leq N).
\end{aligned} \tag{5.31}$$

Note that

$$\det \begin{pmatrix} F_{112} & F_{212} & \cdots & F_{N12} \\ F_{113} & F_{213} & \cdots & F_{N13} \\ \vdots & \vdots & \ddots & \vdots \\ F_{11N} & F_{21N} & \cdots & F_{N1N} \\ F_{12m} & F_{22m} & \cdots & F_{N2m} \end{pmatrix} = 0$$

for any $3 \leq m \leq N$ by Lemma 5.3.2. Therefore one can add the row corresponding to equation $B_{2m} = 0$, $(3 \leq m \leq N)$ to the last row of the matrix $\widetilde{Q}$ without changing its determinant D. Let $S_{2m} = (F_{12m}, F_{22m}, \ldots F_{N2m})$. Let us now add the rows $-F_{1qm}S_{2m}$, $(m = 3, \ldots, N)$ consecutively to the last row of $\widetilde{Q}$. The last row $(\widehat{r}_{N1}, \widehat{r}_{N2}, \ldots, \widehat{r}_{NN})$ of the resulting matrix has the form

$$\begin{aligned}
&\widehat{r}_{N1} = 0, \quad \widehat{r}_{N2} = 0, \\
&\widehat{r}_{Nk} = \sum_{m=1}^{N} F_{2qm}F_{1km} - \sum_{m=1}^{N} F_{1qm}F_{2km} = -[F_1, F_2]_{(q,k)}, \quad (3 \leq k \leq N).
\end{aligned}$$

Since $[F_1, F_2]_{(q,k)} = 0$ by the commutativity equations, we get $\widehat{r}_{Nk} = 0$ for all $1 \leq k \leq N$. Therefore $D = 0$. This proves that $B_{11} = B_{22}$. Similarly, one can prove that $B_{11} = B_{pp}$ for all p. □

As a corollary of Propositions 5.3.3 and 5.3.4 the following statement takes place.

Theorem 5.3.5. *Under the assumptions of Proposition 5.3.3 the matrix* (5.7), *where functions* A^k, $(k = 1, \ldots, N)$ *are given by formula* (5.20), *is proportional to the identity.*

We also have the following result.

Proposition 5.3.6. *Under the assumptions of Proposition 5.3.3 suppose also that there exists a non-degenerate linear combination* $G = \eta^k F_k$ *for some functions* η^k, $(k = 1, \ldots, N)$.

Let $B = A^k F_k$, where functions A^k are given by formula (5.20) *and $A^k \neq 0$ for some $k = 1, \ldots, N$. Then B is a non-zero multiple of the identity matrix.*

Proof. From Theorem 5.3.5 we know that the matrix B is proportional to the identity. It remains to show that B is not the zero matrix. Let $B_{ij} = \sum_{k=1}^{N} A^k F_{ijk} = h\delta_{ij}$ for some function $h = h(x)$. We will show that $h \neq 0$. Assume that $h = 0$. Then $A^k F_{ijk} = 0$. Hence $\eta^l A^k F_{ljk} = 0$ which means that the non-zero vector $(A^1, \ldots, A^N)$ belongs to the kernel of the form G (cf. a similar argument in [26]). Therefore G is degenerate, which contradicts the assumption of G. Hence $h \neq 0$ and the statement follows. □

The following theorem is a corollary of Theorem 5.3.5 and Proposition 5.3.6, and it confirms that function F which satisfies the commutativity equations also solves WDVV equations under some non-degeneracy conditions.

Theorem 5.3.7. *Assume that the function $F = F(x_1, \ldots, x_N)$ on $V \cong \mathbb{C}^N$ satisfies the commutativity equations* (5.3)*. Suppose that for a fixed i_0, $1 \leq i_0 \leq N$ the rank of the matrix $(F_{i_0 ij})$ where $1 \leq i, j \leq N$, $i \neq i_0$ is $N-1$. Suppose also that there exists a non-degenerate linear combination $G = \eta^i F_i$ for some functions $\eta^i, (i = 1, \ldots, N)$. Then F is a solution of WDVV equations* (5.2) *where the matrix B is given by $B = A^k F_k$ where functions A^k are given by formula* (5.20)*.*

Proof. From Theorem 5.3.5 we know that for $i_0 = 1$ the matrix (5.7), where functions A^k, $(k = 1, \ldots, N)$ are given by formula (5.20), is proportional to the identity. Note that the same arguments can be applied for any i_0 and the matrix (5.7) is proportional to the identity in this case as well. Moreover, from Proposition 5.3.6 we know that the matrix (5.7) is a non-zero matrix, therefore the system of WDVV equations (5.2) is equivalent to the system of commutativity equations (5.3) and the statement follows since F solves the commutativity equations. □

Remark 5.3.8. Note that under the assumptions of Theorem 5.3.7, function F also satisfies WDVV equations

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i, \quad i, j, k = 1, \ldots, N$$

provided that matrices F_j are non-degenerate. Indeed these equations follow from equations (5.2) for any particular non-degenerate combination $B = A^i F_i$ by the result from [41] (see also [40]). It also follows that F satisfies the WDVV equations

$$F_i G^{-1} F_j = F_j G^{-1} F_i, \quad i, j = 1, \ldots, N$$

for any non-degenerate linear combination G of matrices F_i.

5.4 The identity field

In this section we define a natural multiplication on the tangent planes T_*V associated with a solution F of the commutativity equations. We find the identity vector field of this multiplication and establish that it is proportional to the vector field $\sum_{k=1}^N A^k \partial_{x_k}$, where functions A^k were defined in Section 5.3. Thus we will express the identity vector field in terms of F for arbitrary dimension N.

For any functions $u = (u^1, \ldots, u^N)$, $v = (v^1, \ldots, v^N)\colon V \to V$, consider vector fields $\partial_u = u^i \partial_{x^i}$, $\partial_v = v^i \partial_{x^i} \in \Gamma(TV)$. Let us define the following multiplication on the tangent space $T_x V$ for generic $x \in V$:

$$\partial_u * \partial_v = u^i v^j \delta^{kl} F_{ijk} \partial_{x^l}. \tag{5.32}$$

Note that multiplication (5.32) defines a commutative algebra on $T_x V$.

Consider a vector field

$$e = e^k \partial_{x^k}, \tag{5.33}$$

where $e^k = e^k(x^1, \ldots, x^N)$ are some functions. Consider the $N \times N$ matrix $B = (B_{ij})_{i,j=1}^N$ given by

$$B_{ij} = e(F_{ij}) = e^k F_{ijk}, \quad i, j = 1, \ldots, N. \tag{5.34}$$

The following statement takes place.

Proposition 5.4.1. *The following statements are equivalent:*

(1) The matrix B with entries given by (5.34) *is equal to the identity,*

(2) The vector field e given by formula (5.33) *is the identity vector field of the multiplication* (5.32).

Proof. From relations (5.32), (5.33) and (5.34) we have

$$e * \partial_v = e^i v^j \partial_{x^i} * \partial_{x^j} = e^i v^j F_{ijk} \partial_{x^k} = B_{jk} v^j \partial_{x^k}. \tag{5.35}$$

Let us firstly assume that $B_{jk} = \delta_{jk}$. Then relation (5.35) reduces to

$$e * \partial_v = \delta_{jk} v^j \partial_{x^k} = v^j \partial_{x^j} = \partial_v.$$

That is statement (2) follows from (1).

Secondly, assume that e is the identity vector field of the multiplication (5.32). Then from relation (5.35) we have

$$e * \partial_v = B_{jk} v^j \partial_{x^k} = \partial_v = v^j \partial_{x^j}.$$

This implies that $B_{jk} = \delta_{jk}$, that is statement (1) holds. This proves the statement. $\square$

Proposition 5.4.1 allows us to reformulate Theorem 5.3.5 as follows.

Theorem 5.4.2. *Assume that the function $F = F(x_1, \dots, x_N)$ on $V \cong \mathbb{C}^N$ satisfies the commutativity equations* (5.3)*. Suppose that for a fixed i_0, $1 \leq i_0 \leq N$ the rank of the matrix $(F_{i_0 ij})$ where $1 \leq i, j \leq N$, $i \neq i_0$ is $N-1$. Suppose also that there exists a non-degenerate linear combination $G = \eta^i F_i$ for some functions $\eta^i, (i = 1, \dots, N)$. Then there exists a unique vector field $e = e^k \partial_{x^k}$, where $e^k = e^k(x^1, \dots, x^N)$ are some functions, such that $e(F_{ij}) = \delta_{ij}$. Moreover, the vector field e is the identity vector field of the multiplication* (5.32)*, and it has the form $e^k = h^{-1} A^k$, where functions A^k are given by formula* (5.20) *and $h = A^k F_{kii}$ (for any $i = 1, \dots, N$).*

Proof. From Theorem 5.3.5 we know that the matrix B with its entries given by formula (5.7), and functions A^k, $(k = 1, \dots, N)$ are given by formula (5.20), is proportional to the identity matrix. That is we have

$$B = A^k F_k = h I_N, \tag{5.36}$$

where I_N is the $N \times N$ identity matrix and $h = A^K F_{kii}$ for any $i = 1, \dots, N$. Let $\widetilde{B} = h^{-1} B$. Thus $\widetilde{B}$ is the identity matrix and its entries have the form

$$\widetilde{B}_{ij} = h^{-1} A^k F_{ijk} = \delta_{ij}. \tag{5.37}$$

Now let $e = h^{-1} A^k \partial_{x^k}$. Then by (5.37) we have $e(F_{ij}) = h^{-1} A^k F_{ijk} = \delta_{ij}$. Therefore, by Proposition 5.4.1 the vector field $e = h^{-1} A^k \partial_{x^k}$ is the identity vector field of the multiplication (5.32) since the matrix (5.37) is the identity matrix. This proves the theorem. $\square$

Now we are going to generalize Theorem 5.3.7 to the case of arbitrary constant metric g in place of the standard metric δ_{ij}. Thus we start with equations of the form

$$F_{ij\alpha} g^{\alpha\beta} F_{\beta kl} = F_{kj\alpha} g^{\alpha\beta} F_{\beta il}, \tag{5.38}$$

where the summation over repeated indices is assumed, and we will show that metric g can be represented as a linear combination of the third order derivatives of the function F under some non-degeneracy assumptions.

Theorem 5.4.3. *Let $F = F(x^1, \dots, x^N)$ be a function on $\mathbb{C}^N$ that satisfies equations of the form* (5.38) *for some constant non-degenerate matrix $(g^{\alpha\beta})$, where $i, j, k, l, \alpha, \beta = 1, \dots, N$. Let $\widehat{C}$ be the constant matrix of change of variables such that*

$$y^i = \widehat{C}^i_j x^j, \tag{5.39}$$

where $y^1, \dots, y^N$ is a new coordinate system and the matrix g satisfies the relation $\widehat{C}^\alpha_i \widehat{C}^\beta_j g^{ij} = \delta^{\alpha\beta}$. Let $\widetilde{F}(y) = F(x)$. Suppose that there exists $i_0, (1 \leq i_0 \leq N)$ such that the matrix

$(\widetilde{F}_{i_0 ij}(y))$ has rank $N-1$, where $1 \le i,j \le N$, $i \ne i_0$. Then there exists a unique vector field $e = e^k(x)\partial_k$ for some functions e^k, $(k = 1,\dots,N)$ such that

$$e(F_{lm}) = e^k(x)F_{klm} = g_{lm}, \tag{5.40}$$

where (g_{lm}) is the inverse matrix for $(g^{\alpha\beta})$.

Before we prove the theorem, we give firstly a prove of the following lemma which is needed.

Lemma 5.4.4. *Relation $\widehat{C}_i^\alpha \widehat{C}_j^\beta g^{ij} = \delta^{\alpha\beta}$ implies that $\sum_{\alpha\beta} \widehat{C}_c^\alpha \widehat{C}_d^\beta \delta^{\alpha\beta} = g_{cd}$, where the matrix $\widehat{C}$ is defined by formula* (5.39).

Proof. Firstly, note that for any matrices G and H, if $GH = I_d$, where I_d is the identity matrix, then $HG = I_d$. This is easy to show as $GH = I_d$ implies that $H = G^{-1}$ and we know that $GG^{-1} = G^{-1}G = I_d$. Now, let us multiply relation $\widehat{C}_i^\alpha \widehat{C}_j^\beta g^{ij} = \delta^{\alpha\beta}$ by C_α^a, we get $\widehat{C}_j^\beta g^{aj} = C_\beta^a$. By multiplying this relation by C_β^b we get $g^{ab} = \sum_\beta C_\beta^a C_\beta^b$. This relation after multiplying it by g_{ac} becomes $\delta_c^b = \sum_\beta C_\beta^a C_\beta^b g_{ac}$. Let us also multiply this relation by $\widehat{C}_b^\alpha$ so we get $\widehat{C}_c^\alpha = C_\alpha^a g_{ac}$. Finally, by multiplying this relation by $\widehat{C}_d^\alpha$ we get $\sum_\alpha \widehat{C}_c^\alpha C_d^\alpha = g_{cd}$ which is equivalent to the relation $\sum_{\alpha\beta} \widehat{C}_c^\beta C_d^\alpha \delta^{\alpha\beta} = g_{cd}$. This proves the lemma. □

Now we give the proof of Theorem 5.4.3.

Proof. Let C be the matrix such that $C_k^i \widehat{C}_j^k = \delta_j^i$. Thus we have $x^i = C_j^i y^j$. We also have

$$\partial_{x^j} = \widehat{C}_j^i \partial_{y^i}, \quad \partial_{y^j} = C_j^i \partial_{x^i}. \tag{5.41}$$

Since $\widehat{C}_i^\alpha \widehat{C}_j^\beta g^{ij} = \delta^{\alpha\beta}$, then this implies that

$$\widehat{C}_a^l \widehat{C}_b^m \delta^{lm} = g_{ab} \tag{5.42}$$

by Lemma 5.4.4. From (5.41) we have the following relations:

$$F_j(x) = \partial_{x^j} F(x) = \widehat{C}_j^i \partial_{y^i} \widetilde{F}(y) = \widehat{C}_j^i \widetilde{F}_i(y). \tag{5.43}$$

Similarly, we have

$$\widetilde{F}_m(y) = C_m^j F_j(x). \tag{5.44}$$

Hence, we have the following relation:

$$F_{pjk}(x) = \widehat{C}_p^m \widehat{C}_j^i \widehat{C}_k^l \widetilde{F}_{mil}(y). \tag{5.45}$$

By multiplying relation (5.45) by $C_a^p C_b^j C_c^k$ we get

$$\widetilde{F}_{abc}(y) = C_a^p C_b^j C_c^k F_{pjk}(x). \tag{5.46}$$

Now let us consider the system of equations (5.38), that is,

$$F_{ij\alpha} g^{\alpha\beta} F_{\beta kl} = F_{kj\alpha} g^{\alpha\beta} F_{\beta il}. \tag{5.47}$$

Then from relation (5.45) we see that relations (5.47) are equivalent to

$$\widehat{C}_i^p \widehat{C}_j^q \widehat{C}_\alpha^r \widetilde{F}_{pqr} g^{\alpha\beta} \widehat{C}_\beta^a \widehat{C}_k^b \widehat{C}_l^d \widetilde{F}_{abd} = \widehat{C}_k^s \widehat{C}_j^q \widehat{C}_\alpha^r \widetilde{F}_{sqr} g^{\alpha\beta} \widehat{C}_\beta^a \widehat{C}_i^b \widehat{C}_l^d \widetilde{F}_{abd}. \tag{5.48}$$

By relation (5.42) we reduce relation (5.48) to

$$\widehat{C}_i^p \widehat{C}_k^b \widetilde{F}_{pqr} \delta^{ar} \widetilde{F}_{abd} = \widehat{C}_k^s \widehat{C}_i^b \widetilde{F}_{sqr} \delta^{ar} \widetilde{F}_{abd}. \tag{5.49}$$

By multiplying equation (5.49) by $C_n^i C_m^k$ this equation reduces to

$$\widetilde{F}_{nqr} \delta^{ar} \widetilde{F}_{amd} = \widetilde{F}_{mqr} \delta^{ar} \widetilde{F}_{and}, \tag{5.50}$$

that is $\widetilde{F}_m$ and $\widetilde{F}_n$ commutes. Now since $rank(\widetilde{F}_{i_0 ij}) = N - 1$, then by Theorem 5.3.5 there exists a unique vector field e given in the coordinate system $y^1, \ldots, y^N$ by

$$e(y) = e^j(y) \partial_{y^j}$$

such that for the function $\widetilde{F}$ we have

$$e(\widetilde{F}_{\alpha\beta}(y)) = e^j(y) \widetilde{F}_{j\alpha\beta}(y) = \delta_{\alpha\beta}. \tag{5.51}$$

Now we will show that $e(F_{ab}(x)) = g_{ab}$. From relation (5.45) we have

$$\partial_{x^i} F_{ab}(x) = F_{iab}(x) = \widehat{C}_i^k \widehat{C}_a^l \widehat{C}_b^m \widetilde{F}_{klm}(y).$$

Hence we have

$$C_j^i \partial_{x^i} F_{ab}(x) = \widehat{C}_a^l \widehat{C}_b^m \widetilde{F}_{jlm}(y).$$

This equation implies that

$$e^j(y(x)) C_j^i \partial_{x_i} F_{ab}(x) = e^j(y(x)) \widetilde{F}_{jlm}(y) \widehat{C}_a^l \widehat{C}_b^m = \delta_{lm} \widehat{C}_a^l \widehat{C}_b^m = g_{ab}$$

by relation (5.42) as required. This proves the theorem. □

To conclude this section we illustrate Theorem 5.4.3 by considering prepotential F

coming from a Frobenius manifold.

Example 5.4.5. Consider the prepotential F of an N-dimensional Frobenius manifold given by

$$F(t) = Q + f(t^2, \dots, t^N), \tag{5.52}$$

where Q is a cubic term given by

$$Q = \frac{1}{2}\Big((t^1)^2 t^N + t^1 \sum_{i=2}^{N-1} t^i t^{N+1-i}\Big). \tag{5.53}$$

We assume that $F(t)$ satisfies the equations $F_i G F_j = F_j G F_i$ for all $i, j = 1, \dots, N$, where

$$G = G^{-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ 0 & \iddots & 0 & 0 \\ 1 & \cdots & 0 & 0 \end{pmatrix}. \tag{5.54}$$

It is immediate to check that $G = F_1$. Now we are going to derive this equality by applying Theorem 5.4.3.

Let $C, \widehat{C}$ be the constant matrices of the change of variables such that $\widehat{C} = C^{-1}$ and

$$t^i = C^i_j x^j, \quad x^i = \widehat{C}^i_j t^j, \tag{5.55}$$

where $x^1, \dots, x^N$ is a new coordinate system and the matrix $G = (g^{ij})$ satisfies the relation $\widehat{C}^\alpha_i \widehat{C}^\beta_j g^{ij} = \delta^{\alpha\beta}$. Then matrix C is given as follows:

$$C = \begin{pmatrix} -\frac{i}{2} & & & 0 & & & \frac{1}{2} \\ & \ddots & & & & \iddots & \\ & & -\frac{i}{2} & & \frac{1}{2} & & \\ 0 & & & 1 & & & 0 \\ & & i & & 1 & & \\ & \iddots & & & & \ddots & \\ i & & & 0 & & & 1 \end{pmatrix}, \quad \text{or} \quad C = \begin{pmatrix} -\frac{i}{2} & & & 0 & & & \frac{1}{2} \\ & \ddots & & & & \iddots & \\ & & -\frac{i}{2} & & \frac{1}{2} & & \\ 0 & & & & & & 0 \\ & & i & & 1 & & \\ & \iddots & & & & \ddots & \\ i & & & 0 & & & 1 \end{pmatrix},$$

for N being odd or even respectively. That is the ij-entries C^i_j of the matrix C are given by

- If N is odd let $b = \frac{N+1}{2}$, then the entries are

$$\begin{aligned}
&C_1^1 = C_2^2 = \cdots = C_{b-1}^{b-1} = -\frac{i}{2},\\
&C_b^b = C_{b+1}^{b+1} = \cdots = C_N^N = 1,\\
&C_N^1 = C_{N-1}^2 = \cdots = C_{b+1}^{b-1} = \frac{1}{2},\\
&C_{b-1}^{b+1} = C_{b-2}^{b+2} = \cdots = C_1^N = i,
\end{aligned}$$

  and all the other entries are zeros.

- If N is even let $b = \frac{N}{2}$, the entries are

$$\begin{aligned}
&C_1^1 = C_2^2 = \cdots = C_b^b = -\frac{i}{2},\\
&C_{b+1}^{b+1} = C_{b+2}^{b+2} = \cdots = C_N^N = 1,\\
&C_N^1 = C_{N-1}^2 = \cdots = C_{b+1}^b = \frac{1}{2},\\
&C_b^{b+1} = C_{b-1}^{b+2} = \cdots = C_1^N = i,
\end{aligned}$$

  and all the other entries are zeros.

The matrix $\widehat{C}$ is given as follows:

$$\widehat{C} = \begin{pmatrix}
i & & & 0 & & & -\frac{i}{2}\\
& \ddots & & & & \iddots & \\
& & i & & -\frac{i}{2} & & \\
0 & & & 1 & & & 0\\
& & 1 & & \frac{1}{2} & & \\
& \iddots & & & & \ddots & \\
1 & & & 0 & & & \frac{1}{2}
\end{pmatrix}, \quad \text{or} \quad \widehat{C} = \begin{pmatrix}
i & & & 0 & & & -\frac{i}{2}\\
& \ddots & & & & \iddots & \\
& & i & & -\frac{i}{2} & & \\
0 & & & & & & 0\\
& & 1 & & \frac{1}{2} & & \\
& \iddots & & & & \ddots & \\
1 & & & 0 & & & \frac{1}{2}
\end{pmatrix},$$

for N being odd or even respectively. Hence we have the following relations. Firstly, for N being odd number we have

$$\partial_{x^k} = \begin{cases}
-\frac{i}{2}\partial_{t^1} + i\partial_{t^N}, & k = 1,\\
-\frac{i}{2}\partial_{t^k} + i\partial_{t^{N+1-k}}, & 2 \leq k \leq \frac{N-1}{2},\\
\partial_{t^k}, & k = \frac{N+1}{2},\\
\frac{1}{2}\partial_{t^{N+1-k}} + \partial_{t^k}, & \frac{N+3}{2} \leq k \leq N-1,\\
\frac{1}{2}\partial_{t^1} + \partial_{t^N}, & k = N.
\end{cases}$$

Secondly, for N being even number we have

$$\partial_{x^k}=\begin{cases}-\frac{i}{2}\partial_{t^1}+i\partial_{t^N}, & k=1,\\ -\frac{i}{2}\partial_{t^k}+i\partial_{t^{N+1-k}}, & 2\le k\le \frac{N}{2},\\ \frac{1}{2}\partial_{t^{N+1-k}}+\partial_{t^k}, & \frac{N}{2}+1\le k\le N-1,\\ \frac{1}{2}\partial_{t^1}+\partial_{t^N}, & k=N.\end{cases}$$

Let $\widetilde{F}(x)=F(t)$. Define $\widetilde{F}_{jkl}(x)=\partial_{x_j}\partial_{x_k}\partial_{x_l}F(t)$. Since $\partial_{x^1}f(t^2,\dots,t^N)=i\partial_{x^N}f(t^2,\dots,t^N)$, we also have

$$\begin{aligned}&\partial_{x^1}\partial_{x^1}\partial_{x^r}f(t^2,\dots,t^N)=i\partial_{x^1}\partial_{x^N}\partial_{x^r}f(t^2,\dots,t^N),\quad 2\le r\le N,\\ &\partial_{x^1}\partial_{x^j}\partial_{x^k}f(t^2,\dots,t^N)=i\partial_{x^N}\partial_{x^j}\partial_{x^k}f(t^2,\dots,t^N).\end{aligned}\tag{5.56}$$

The following relations are also easy to verify:

$$\begin{aligned}&\partial_{x^1}\partial_{x^1}\partial_{x^r}Q(t^1,\dots,t^N)=i\partial_{x^1}\partial_{x^N}\partial_{x^r}Q(t^1,\dots,t^N)=0,\quad 2\le r\le N-1,\\ &\partial_{x^1}\partial_{x^1}\partial_{x^N}Q(t^1,\dots,t^N)=i\partial_{x^1}\partial_{x^N}\partial_{x^N}Q(t^1,\dots,t^N),\\ &\partial_{x^1}\partial_{x^j}\partial_{x^k}Q(t^1,\dots,t^N)=i\partial_{x^N}\partial_{x^j}\partial_{x^k}Q(t^1,\dots,t^N),\quad j\ne k,\\ &\partial_{x^1}\partial_{x^k}\partial_{x^k}Q(t^1,\dots,t^N)=i\partial_{x^N}\partial_{x^k}\partial_{x^k}Q(t^1,\dots,t^N)-i,\quad 1\le k\le N.\end{aligned}\tag{5.57}$$

Hence relations (5.56), (5.57) imply the relations

$$\widetilde{F}_{11r}=i\widetilde{F}_{1Nr},\quad 2\le r\le N,\tag{5.58}$$

$$\widetilde{F}_{1jk}=i\widetilde{F}_{Njk},\quad j\ne k,\tag{5.59}$$

$$\widetilde{F}_{1kk}=i\widetilde{F}_{Nkk}-i,\quad 1\le k\le N.\tag{5.60}$$

Assume that there exists $j_0,(1\le j_0\le N)$ such that the matrix $(\widetilde{F}_{j_0jk}(x))$ has rank $N-1$, where $1\le j,k\le N$, $j\ne j_0$. One can now define the matrix $B=A^k\widetilde{F}_k$, where A^k are given by formula (5.20), and there exist at least r, where $(1\le r\le N)$, such that $A^r\ne 0$. Now from relations (5.58), (5.59) and formula (5.20) it is easy to see that $A^1=iA^N$ and that $A^k=0$ for $2\le k\le N-1$ since the right-hand side of formula (5.20) for $2\le k\le N-1$ is a determinant of a matrix which contains two proportional columns. Thus the matrix B takes the form

$$B=A^1\widetilde{F}_1+A^N\widetilde{F}_N=A^N(i\widetilde{F}_1+\widetilde{F}_N).\tag{5.61}$$

Hence for $j\ne k$ we have $B_{jk}=A^N(i\widetilde{F}_{1jk}+\widetilde{F}_{Njk})=0$ by (5.59). Let us now find B_{kk}. We have from (5.60) that

$$B_{kk} = A^1\widetilde{F}_{1kk} + A^N\widetilde{F}_{Nkk} = iA^N(i\widetilde{F}_{Nkk} - i) + A^N\widetilde{F}_{Nkk} = A^N.$$

Hence we have shown that $B = hI_N$, where $h = A^N$. Let us now define the vector field $e = e^k\partial_{x^k}$ by $e = h^{-1}A^k\partial_{x^k}$. Then we have

$$e = h^{-1}\Big(A^1\partial_{x^1} + A^N\partial_{x^N}\Big) = h^{-1}A^N\Big(i\partial_{x^1} + \partial_{x^N}\Big) = i\partial_{x^1} + \partial_{x^N} = \partial_{t^1}.$$

As expected, this formula coincides with the well-known formula of the identity field of a Frobenius manifold.

Remark 5.4.6. We note that the maximality of rank condition is sufficient but not necessary for the existence of the identity. Indeed, in the case of Example 5.4.5 with $N = 2$ we have

$$\widetilde{F}_{112} = \frac{1}{4} - f'''(ix^1 + x^2), \quad \widetilde{F}_{122} = -\frac{i}{4} + if'''(ix^1 + x^2).$$

Then the matrix $\begin{pmatrix}\widetilde{F}_{112} & \widetilde{F}_{122}\end{pmatrix}$ has rank zero if $f(t^2) = \frac{1}{24}(t^2)^3$. Nonetheless $e = \partial_{t^1}$ is the identity field.

5.5 Applications

In this section we explore the close relation between commutativity equations and WDVV equations established in Section 5.3 through the existence of the identity field. This leads to new solutions of WDVV equations.

5.5.1 Applications to Euclidean trigonometric $\vee$-systems

Let us recall the function F given by the formula (4.53)

$$F = \sum_{\alpha\in\mathcal{A}} c_\alpha f((\alpha, x)), \tag{5.62}$$

where $\mathcal{A}$ is a finite set of vectors in $V \cong \mathbb{C}^N$, $c_\alpha \in \mathbb{C}$, are some multiplicity parameters, where the function f is given by (1.11). Let us recall some notations from Section 4.5. Let $\mathcal{A}$ be a finite set of vectors in a Euclidean space V with the bilinear inner product $(\cdot,\cdot)$. Let $c_\alpha \in \mathbb{C}, \alpha \in \mathcal{A}$ be some multiplicity parameters. Assume that $\mathcal{A}$ belongs to a lattice of rank N. For each vector $\alpha \in \mathcal{A}$ we recall the set of its collinear vectors from $\mathcal{A}$:

$$\delta_\alpha := \{\gamma \in \mathcal{A} \colon \gamma \sim \alpha\}.$$

Let $\delta \subseteq \delta_\alpha$ and $\alpha_0 \in \delta_\alpha$. Then for any $\gamma \in \delta$ we have $\gamma = k_\gamma \alpha_0$ for some $k_\gamma \in \mathbb{R}$. Note that k_γ depends on the choice of α_0 and different choices of α_0 give rescaled collections of these parameters. Define $C_\delta^{\alpha_0} := \sum\limits_{\gamma\in\delta} c_\gamma k_\gamma^2$. Note that $C_\delta^{\alpha_0}$ is non-zero if and only if $C_\delta^{\widetilde{\alpha}_0} \neq 0$ for any $\widetilde{\alpha}_0 \in \delta$.

As a corollary of Theorems 4.5.8, 5.3.7, the following statement takes place.

Theorem 5.5.1. *Suppose that a configuration $(\mathcal{A}, c)$ satisfies the following conditions*

(1) $\mathcal{A}$ is a Euclidean trigonometric $\vee$-system,

(2) $\sum_{\alpha,\beta\in\mathcal{A}_+} c_\alpha c_\beta (\alpha,\beta) B_{\alpha,\beta}(a,b)\alpha\wedge\beta = 0$,

(3) $C_\delta^{\alpha_0} \neq 0$ for any $\alpha \in \mathcal{A}$, $\delta \subseteq \delta_\alpha$, $\alpha_0 \in \delta_\alpha$,

(4) The function F given by formula (5.62) satisfies that for a fixed i_0 the rank of the matrix $(F_{i_0 ij})$, where $1 \leq i,j \leq N$, $i \neq i_0$, is $N-1$,

(5) There exists a non-degenerate linear combination $G = C^i F_i$ for some functions $C^i, (i = 1,\dots,N)$.

Then function (5.62) satisfies both the commutativity equations (5.3) and the WDVV equations (5.2), where the entries of the matrix $B = (B_{ij})_{i,j=1}^N$ are given by formula (5.7) and functions $A_k = A_k(x_1,\dots,x_N)$ are given by formula (5.20).

We have shown in Theorem 4.8.1 that the class of solutions of commutativity equations corresponding to Euclidean $\vee$-systems is closed under the restrictions. As a corollary of Theorem 4.8.1 and Theorem 5.3.7, the following statement takes place.

Theorem 5.5.2. *In the notations and under the assumptions of Theorem 4.8.1, assume that prepotential (5.62) satisfies the commutativity equations (5.3). Let $\mathcal{B} \subset \mathcal{A}$ be a subsystem of rank n. Suppose that the prepotential*

$$F_\mathcal{B} = \sum_{\alpha\in\mathcal{A}\setminus\mathcal{B}} c_\alpha f((\widetilde{\alpha},\xi)), \quad \xi \in M_\mathcal{B}, \tag{5.63}$$

satisfies that for a fixed i_0 the rank of the matrix $(F_\mathcal{B})_{i_0 ij}$, where $1 \leq i,j \leq n$, $i \neq i_0$, is $n-1$. Then prepotential (5.63) satisfies the WDVV equations of the form

$$(F_\mathcal{B})_i B^{-1} (F_\mathcal{B})_j = (F_\mathcal{B})_j B^{-1} (F_\mathcal{B})_i, \quad i,j = 1,\dots,n, \tag{5.64}$$

where $(F_\mathcal{B})_i$ is the $n\times n$ matrix with entries

$$((F_\mathcal{B})_i)_{pq} = (F_\mathcal{B})_{ipq} = \frac{\partial^3 F_\mathcal{B}}{\partial x_i \partial x_p \partial x_q}$$

and the entries of the matrix $B = (B_{ij})_{i,j=1}^n$ are given by formula (5.7) (with N=n) and functions $A_k = A_k(x_1,\dots,x_n)$ are given by formula (5.20) provided that $B \neq 0$.

5.5.2 New solutions of WDVV equations

In this subsection we check that maximal rank assumption from Theorem 5.4.2 is satisfied in the example related to root system F_4. This leads to a new solution of WDVV equations.

Consider the positive half of root system F_4 consisting of vectors

$$F_4^+ = \{e_i\,(1 \le i \le 4),\ e_i \pm e_j\,(1 \le i < j \le 4),\ \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}. \tag{5.65}$$

Let r be the multiplicity of the short roots and let q be the multiplicity of the long roots. We assume that $r, q \neq 0$.

Recall that by Theorem 4.1.4 (see [3]) function (5.62) for the collection $\mathcal{A} = F_4^+$ satisfies commutativity equations (5.3) if and only if $r = -2q$ or $r = -4q$. Now we will check that the condition of Theorem 5.3.5 is satisfied for both cases.

Consider the following 3×4 matrix

$$Q = \begin{pmatrix} F_{112} & F_{212} & F_{312} & F_{412} \\ F_{113} & F_{213} & F_{313} & F_{413} \\ F_{114} & F_{214} & F_{314} & F_{414} \end{pmatrix}. \tag{5.66}$$

Matrix Q is the coefficient matrix corresponding to the homogeneous system of equations $B_{12} = B_{13} = B_{14} = 0$ where B_{1m} is given by formula (5.25) for some functions $A_k = A_k(x_1, x_2, x_3, x_4)$, where function F has the form (5.62) and $\mathcal{A} = F_4^+$ is given by formula (5.65). The following statement takes place.

Lemma 5.5.3. *Matrix Q has rank three.*

Proof. To show that $rank(Q) = 3$, we will show that the following 3×3 sub-matrix $\widetilde{Q}$ has rank three, where

$$\widetilde{Q} = \begin{pmatrix} F_{112} & F_{212} & F_{312} \\ F_{113} & F_{213} & F_{313} \\ F_{114} & F_{214} & F_{314} \end{pmatrix}. \tag{5.67}$$

That is we will show that matrix (5.67) is not singular. Let $D = D(x) = \det \widetilde{Q}$. From (5.67) we have

$$D(x) = F_{112}(F_{123}F_{134} - F_{133}F_{124}) - F_{122}(F_{113}F_{134} - F_{114}F_{133}) + F_{123}(F_{113}F_{124} - F_{114}F_{12}). \tag{5.68}$$

We will establish that $D(x_0) \neq 0$, where $x_0 = (x_1, 0, 0, x_4)$. We have the following third order derivatives:

$$
\begin{aligned}
F_{112} = {} & q\Big(\cot(x_1+x_2)-\cot(x_1-x_2)\Big)+\frac{r}{8}\Big(\cot(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2})\\
& -\cot(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2})+\cot(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2})-\cot(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2})\\
& +\cot(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2})-\cot(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2})+\cot(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2})\\
& -\cot(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2})\Big),
\end{aligned}
$$

$$
\begin{aligned}
F_{114} = {} & q\Big(\cot(x_1+x_4)-\cot(x_1-x_4)\Big)+\frac{r}{8}\Big(\cot(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2})\\
& -\cot(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2})-\cot(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2})-\cot(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2})\\
& -\cot(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2})+\cot(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2})+\cot(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2})\\
& +\cot(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2})\Big),
\end{aligned}
$$

$$
\begin{aligned}
F_{122} = {} & q\Big(\cot(x_1+x_2)+\cot(x_1-x_2)\Big)+\frac{r}{8}\Big(\cot(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2})\\
& +\cot(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2})+\cot(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2})+\cot(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2})\\
& +\cot(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2})+\cot(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2})+\cot(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2})\\
& +\cot(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2})\Big),
\end{aligned}
$$

$$
\begin{aligned}
F_{133} = {} & q\Big(\cot(x_1+x_3)+\cot(x_1-x_3)\Big)+\frac{r}{8}\Big(\cot(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2})\\
& +\cot(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2})+\cot(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2})+\cot(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2})\\
& +\cot(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2})+\cot(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2})+\cot(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2})\\
& +\cot(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2})\Big),
\end{aligned}
$$

$$
\begin{aligned}
F_{123} = {} & \frac{r}{8}\Big(\cot(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2})+\cot(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2})-\cot(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2})\\
& -\cot(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2})+\cot(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2})+\cot(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2})\\
& -\cot(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2})-\cot(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2})\Big),
\end{aligned}
$$

$$
\begin{aligned}
F_{134} = \frac{r}{8}\Big(& \cot\big(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2}\big)+\cot\big(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2}\big)+\cot\big(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}-\frac{x_4}{2}\big) \\
& -\cot\big(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2}\big)-\cot\big(\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}-\frac{x_4}{2}\big)-\cot\big(\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2}\big) \\
& -\cot\big(\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}+\frac{x_4}{2}\big)+\cot\big(\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}+\frac{x_4}{2}\big)\Big).
\end{aligned}
$$

Now we have the following two cases to consider.

Case (1). $r=-2q$. We have

$$
\begin{aligned}
& F_{112}|_{x_0}=0, \quad F_{123}|_{x_0}=0, \quad F_{134}|_{x_0}=0, \\
& F_{114}|_{x_0}=q\Big(\cot(x_1+x_4)-\cot(x_1-x_4)-\cot\big(\frac{x_1}{2}+\frac{x_4}{2}\big)+\cot\big(\frac{x_1}{2}-\frac{x_4}{2}\big)\Big), \\
& F_{122}|_{x_0}=q\Big(2\cot x_1-\cot\big(\frac{x_1}{2}+\frac{x_4}{2}\big)-\cot\big(\frac{x_1}{2}-\frac{x_4}{2}\big)\Big), \\
& F_{133}|_{x_0}=q\Big(2\cot x_1-\cot\big(\frac{x_1}{2}+\frac{x_4}{2}\big)-\cot\big(\frac{x_1}{2}-\frac{x_4}{2}\big)\Big).
\end{aligned}
$$

Hence, the determinant (5.68) reduces to

$$
\begin{aligned}
D(x_0) &= F_{122}F_{114}F_{133} \\
&= q^3\Big(\cot(x_1+x_4)-\cot(x_1-x_4)-\cot\big(\frac{x_1}{2}+\frac{x_4}{2}\big)+\cot\big(\frac{x_1}{2}-\frac{x_4}{2}\big)\Big) \\
&\times\Big(2\cot x_1-\cot\big(\frac{x_1}{2}+\frac{x_4}{2}\big)-\cot\big(\frac{x_1}{2}-\frac{x_4}{2}\big)\Big)^2 \neq 0.
\end{aligned} \tag{5.69}
$$

Case (2). $r=-4q$. We have

$$
\begin{aligned}
& F_{112}|_{x_0}=0, \quad F_{123}|_{x_0}=0, \quad F_{134}|_{x_0}=0, \\
& F_{114}|_{x_0}=q\Big(\cot(x_1+x_4)-\cot(x_1-x_4)-2\cot\big(\frac{x_1}{2}+\frac{x_4}{2}\big)+2\cot\big(\frac{x_1}{2}-\frac{x_4}{2}\big)\Big), \\
& F_{122}|_{x_0}=2q\Big(\cot x_1-\cot\big(\frac{x_1}{2}+\frac{x_4}{2}\big)-\cot\big(\frac{x_1}{2}-\frac{x_4}{2}\big)\Big), \\
& F_{133}|_{x_0}=2q\Big(\cot x_1-\cot\big(\frac{x_1}{2}+\frac{x_4}{2}\big)-\cot\big(\frac{x_1}{2}-\frac{x_4}{2}\big)\Big).
\end{aligned}
$$

Hence, the determinant (5.68) reduces to

$$
\begin{aligned}
D(x_0) &= F_{122}F_{114}F_{133} \\
&= 4q^3\Big(\cot(x_1+x_4)-\cot(x_1-x_4)-2\cot\big(\frac{x_1}{2}+2\frac{x_4}{2}\big)+\cot\big(\frac{x_1}{2}-\frac{x_4}{2}\big)\Big) \\
&\times\Big(\cot x_1-\cot\big(\frac{x_1}{2}+\frac{x_4}{2}\big)-\cot\big(\frac{x_1}{2}-\frac{x_4}{2}\big)\Big)^2 \neq 0.
\end{aligned} \tag{5.70}
$$

This proves the statement. □

Now let us define

$$A^1 = \det\begin{pmatrix} F_{212} & F_{312} & F_{412} \\ F_{213} & F_{313} & F_{413} \\ F_{214} & F_{314} & F_{414} \end{pmatrix}, \quad A^2 = -\det\begin{pmatrix} F_{112} & F_{312} & F_{412} \\ F_{113} & F_{313} & F_{413} \\ F_{114} & F_{314} & F_{414} \end{pmatrix},$$
$$A^3 = \det\begin{pmatrix} F_{112} & F_{212} & F_{412} \\ F_{113} & F_{213} & F_{413} \\ F_{114} & F_{214} & F_{414} \end{pmatrix}, \quad A^4 = -\det\begin{pmatrix} F_{112} & F_{212} & F_{312} \\ F_{113} & F_{213} & F_{313} \\ F_{114} & F_{214} & F_{314} \end{pmatrix}. \tag{5.71}$$

As a corollary of Theorem 5.3.5 and Lemma 5.5.3 the following statement takes place.

Theorem 5.5.4. *Consider the function*

$$F = r\sum_{i=1}^{4} f(x_i) + r\sum_{\varepsilon_i\in\{1,-1\}} f\Big(\frac{1}{2}(\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4)\Big) + q\sum_{i<j}^{4} f(x_i \pm x_j), \tag{5.72}$$

Suppose that parameters r, q satisfy the conditions $r = -2q$ or $r = -4q$. Define the vector field

$$e = \sum_{i=1}^{4} A^i(x)\partial_{x^i}, \tag{5.73}$$

where functions $A^i(x)$ are given by formulas (5.71)*. Define 4×4 matrix $B = (B_{ij})_{i,j=1}^4$ by*

$$B_{ij} = e(F_{ij}) = \sum_{k=1}^{4} A^k(x) F_{ijk}, \quad i,j = 1,\ldots,4. \tag{5.74}$$

Then the matrix B is proportional to the identity matrix.

This leads us to another solution of WDVV equations.

Theorem 5.5.5. *Consider the function* (5.72) *such that parameters r, q satisfy the conditions $r = -2q$ or $r = -4q$. Then F satisfies WDVV equations* (5.2)*.*

5.5.3 The identity field for root system F_4

In this subection we consider $\mathcal{A} = F_4^+$ given by formula (5.65) and the corresponding function of the form (5.62) which satisfies commutativity equations (5.3). We give explicit simple formulas for the functions A^k $(k = 1, 2, 3, 4)$, and we show that the metric $B = \sum_{k=1}^4 A^k F_k$ is a multiple of the identity matrix I_4. By uniqueness of the identity field this implies, in particular, that these functions A^k coincides up to proportionality with corresponding functions defined in Section 5.3 by determinant formulas.

Let $[4] = \{1, 2, 3, 4\}$. Let us also introduce the following functions:

$$\begin{aligned}
b_1 &= (A^1 + A^2 + A^3 + A^4)\cot(\frac{x_1 + x_2 + x_3 + x_4}{2}),\\
b_2 &= (A^1 + A^2 - A^3 - A^4)\cot(\frac{x_1 + x_2 - x_3 - x_4}{2}),\\
b_3 &= (A^1 + A^2 + A^3 - A^4)\cot(\frac{x_1 + x_2 + x_3 - x_4}{2}),\\
b_4 &= (A^1 + A^2 - A^3 + A^4)\cot(\frac{x_1 + x_2 - x_3 + x_4}{2}),\\
b_5 &= (A^1 - A^2 + A^3 + A^4)\cot(\frac{x_1 - x_2 + x_3 + x_4}{2}),\\
b_6 &= (A^1 - A^2 - A^3 - A^4)\cot(\frac{x_1 - x_2 - x_3 - x_4}{2}),\\
b_7 &= (A^1 - A^2 + A^3 - A^4)\cot(\frac{x_1 - x_2 + x_3 - x_4}{2}),\\
b_8 &= (A^1 - A^2 - A^3 + A^4)\cot(\frac{x_1 - x_2 - x_3 + x_4}{2}),\\
b_9 &= (A^1 + A^2)\cot(x_1 + x_2), \quad b_{10} = (A^1 - A^2)\cot(x_1 - x_2),\\
b_{11} &= (A^1 + A^3)\cot(x_1 + x_3), \quad b_{12} = (A^1 - A^3)\cot(x_1 - x_3),\\
b_{13} &= (A^1 + A^4)\cot(x_1 + x_4), \quad b_{14} = (A^1 - A^4)\cot(x_1 - x_4).
\end{aligned} \tag{5.75}$$

We consider separately two cases $r = -2q$ and $r = -4q$.

5.5.4 F_4 with the condition $r = -2q$

Assume that the multiplicity parameters r, q satisfy the condition $r = -2q$. Define

$$A^k = \sin x_k\Big(\cos x_k(-1 + \sum_{i\neq k}\cos 2x_i) - 2\prod_{i\neq k}\cos x_i\Big), \quad k = 1, 2, 3, 4. \tag{5.76}$$

The following relation takes place.

Lemma 5.5.6. *Functions A^k, $(k = 1, 2, 3, 4)$ given by formula* (5.76) *satisfy the relation*

$$\sum_{i=1}^{4}\varepsilon_i A^i = \sin(\frac{\varepsilon_1x_1 + \varepsilon_2x_2 + \varepsilon_3x_3 + \varepsilon_4x_4}{2})\Big(-2\cos(\frac{\varepsilon_1x_1 + \varepsilon_2x_2 + \varepsilon_3x_3 + \varepsilon_4x_4}{2})$$
$$+\sum_{\substack{1\le i<j\le 4\\ k,l\in[4]\setminus\{i,j\}\\ k<l}}\cos(\frac{\varepsilon_ix_i + \varepsilon_jx_j - 3\varepsilon_kx_k - 3\varepsilon_lx_l}{2}) - \sum_{\substack{1\le i<j<k\le 4\\ l\in[4]\setminus\{i,j,k\}}}\cos(\frac{\varepsilon_ix_i + \varepsilon_jx_j + \varepsilon_kx_k - 3\varepsilon_lx_l}{2})\Big), \tag{5.77}$$

where $\varepsilon_i \in \{1, -1\}$ for all i.

Proof. Let us substitute functions A^k, $(k = 1, 2, 3, 4)$ from relation (5.76) into the left-hand side of the equation (5.77). Then by rearranging the terms one gets the formula

$$\sum_{i=1}^{4}\varepsilon_i A^i = \frac{1}{2}\Big(\sum_{1\le i<j\le 4}\sin\left(2\varepsilon_i x_i + 2\varepsilon_j x_j\right) - \sum_{i=1}^{4}\sin\left(2\varepsilon_i x_i\right)\Big)$$
$$-2\Big(\sin\left(\varepsilon_1 x_1+\varepsilon_2 x_2\right)\cos\left(\varepsilon_3 x_3\right)\cos\left(\varepsilon_4 x_4\right)+\sin\left(\varepsilon_3 x_3+\varepsilon_4 x_4\right)\cos\left(\varepsilon_1 x_1\right)\cos\left(\varepsilon_2 x_2\right)\Big). \tag{5.78}$$

On the other hand we have

$$\sin\Big(\frac{\varepsilon_i x_i+\varepsilon_j x_j+\varepsilon_k x_k+\varepsilon_l x_l}{2}\Big)\cos\Big(\frac{\varepsilon_i x_i+\varepsilon_j x_j-3\varepsilon_k x_k-3\varepsilon_l x_l}{2}\Big)$$
$$=\frac{1}{2}\Big(\sin\left(\varepsilon_i x_i+\varepsilon_j x_j-\varepsilon_k x_k-\varepsilon_l x_l\right)+\sin\left(2\varepsilon_k x_k+2\varepsilon_l x_l\right)\Big). \tag{5.79}$$

Also we have

$$\sin\Big(\frac{\varepsilon_i x_i+\varepsilon_j x_j+\varepsilon_k x_k+\varepsilon_l x_l}{2}\Big)\cos\Big(\frac{\varepsilon_i x_i+\varepsilon_j x_j+\varepsilon_k x_k-3\varepsilon_l x_l}{2}\Big)$$
$$=\frac{1}{2}\Big(\sin\left(\varepsilon_i x_i+\varepsilon_j x_j+\varepsilon_k x_k-\varepsilon_l x_l\right)+\sin\left(2\varepsilon_l x_l\right)\Big). \tag{5.80}$$

Now let M be the right-hand side of the equation (5.77). Then from relations (5.79) and (5.80) we have

$$M=-\sin\left(\varepsilon_1 x_1+\varepsilon_2 x_2+\varepsilon_3 x_3+\varepsilon_4 x_4\right)+\frac{1}{2}\sum_{\substack{i<j\\ k<l\\ k,l\neq i,j}}\Big(\sin\left(\varepsilon_i x_i+\varepsilon_j x_j-\varepsilon_k x_k-\varepsilon_l x_l\right)$$
$$+\sin\left(2\varepsilon_k x_k+2\varepsilon_l x_l\right)\Big)-\frac{1}{2}\sum_{\substack{i<j<k\\ l\neq i,j,k}}\Big(\sin\left(\varepsilon_i x_i+\varepsilon_j x_j+\varepsilon_k x_k-\varepsilon_l x_l\right)+\sin\left(2\varepsilon_l x_l\right)\Big). \tag{5.81}$$

Note that

$$\sin\left(\varepsilon_1 x_1+\varepsilon_2 x_2+\varepsilon_3 x_3+\varepsilon_4 x_4\right)=\sin\left(\varepsilon_1 x_1+\varepsilon_2 x_2\right)\cos\left(\varepsilon_3 x_3+\varepsilon_4 x_4\right)$$
$$+\cos\left(\varepsilon_1 x_1+\varepsilon_2 x_2\right)\sin\left(\varepsilon_3 x_3+\varepsilon_4 x_4\right).$$

Note also that

$$\sum_{\substack{i<j\\ k<l\\ k,l\neq i,j}}\sin\left(\varepsilon_i x_i+\varepsilon_j x_j-\varepsilon_k x_k-\varepsilon_l x_l\right)=0.$$

Also we have

$$\begin{aligned}
&\sin(\varepsilon_1x_1+\varepsilon_2x_2+\varepsilon_3x_3-\varepsilon_4x_4)+\sin(\varepsilon_1x_1+\varepsilon_2x_2-\varepsilon_3x_3+\varepsilon_4x_4)\\
&=2\sin(\varepsilon_1x_1+\varepsilon_2x_2)\cos(\varepsilon_3x_3-\varepsilon_4x_4),
\end{aligned}$$

and

$$\begin{aligned}
&\sin(\varepsilon_1x_1-\varepsilon_2x_2+\varepsilon_3x_3+\varepsilon_4x_4)-\sin(\varepsilon_1x_1-\varepsilon_2x_2-\varepsilon_3x_3-\varepsilon_4x_4)\\
&=2\cos(\varepsilon_1x_1-\varepsilon_2x_2)\sin(\varepsilon_3x_3+\varepsilon_4x_4).
\end{aligned}$$

Thus, formula (5.81) can be rearranged as

$$\begin{aligned}
M=&-\sin(\varepsilon_1x_1+\varepsilon_2x_2)\Big(\cos(\varepsilon_3x_3+\varepsilon_4x_4)+\cos(\varepsilon_3x_3-\varepsilon_4x_4)\Big)-\frac{1}{2}\sum_{i=1}^{4}\sin(2\varepsilon_ix_i)\\
&-\sin(\varepsilon_3x_3+\varepsilon_4x_4)\Big(\cos(\varepsilon_1x_1+\varepsilon_2x_2)+\cos(\varepsilon_1x_1-\varepsilon_2x_2)\Big)+\frac{1}{2}\sum_{1\le i<j\le 4}\sin(2\varepsilon_ix_i+2\varepsilon_jx_j).
\end{aligned}\tag{5.82}$$

It is clear that the right-hand side of the formula (5.82) is equal to the right-hand side of the relation (5.78), which proves the lemma. □

The following identity holds.

Lemma 5.5.7. *Functions A^k, $(k=1,2,3,4)$ given by formula (5.76) satisfy the identity*

$$\begin{aligned}
&\sum_{i=1}^{4}\varepsilon_iA^i\cot\Big(\frac{\varepsilon_1x_1+\varepsilon_2x_2+\varepsilon_3x_3+\varepsilon_4x_4}{2}\Big)\\
&=-1-\cos(\varepsilon_1x_1+\varepsilon_2x_2+\varepsilon_3x_3+\varepsilon_4x_4)+\cos(\varepsilon_1x_1+\varepsilon_2x_2-\varepsilon_3x_3-\varepsilon_4x_4)\\
&+\cos(\varepsilon_1x_1-\varepsilon_2x_2+\varepsilon_3x_3-\varepsilon_4x_4)+\cos(\varepsilon_1x_1-\varepsilon_2x_2-\varepsilon_3x_3+\varepsilon_4x_4)\\
&+\frac{1}{2}\sum_{1\le i<j\le 4}\cos(2\varepsilon_ix_i+2\varepsilon_jx_j)-\frac{1}{2}\sum_{\substack{i<j<k\\ l\ne i,j,k}}\cos(\varepsilon_ix_i+\varepsilon_jx_j+\varepsilon_kx_k-\varepsilon_lx_l)-\frac{1}{2}\sum_{i=1}^{4}\cos(2\varepsilon_ix_i).
\end{aligned}\tag{5.83}$$

where $\varepsilon_i\in\{1,-1\}$ for all i.

Proof. We have by Lemma 5.5.6 that

$$\sum_{i=1}^{4} \varepsilon_i A^i \cot(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4}{2})$$
$$= \cos(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4}{2})\Big(-2\cos(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4}{2})$$
$$+ \sum_{\substack{1\le i<j\le 4\\ k,l\in[4]\setminus\{i,j\}\\ k<l}} \cos(\frac{\varepsilon_i x_i + \varepsilon_j x_j - 3\varepsilon_k x_k - 3\varepsilon_l x_l}{2}) - \sum_{\substack{1\le i<j<k\le 4\\ l\in[4]\setminus\{i,j,k\}}} \cos(\frac{\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k - 3\varepsilon_l x_l}{2})\Big). \tag{5.84}$$

Firstly, we have

$$2\cos^2(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4}{2}) = 1 + \cos(\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4). \tag{5.85}$$

Secondly, we have

$$\cos(\frac{\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k + \varepsilon_l x_l}{2})\cos(\frac{\varepsilon_i x_i + \varepsilon_j x_j - 3\varepsilon_k x_k - 3\varepsilon_l x_l}{2})$$
$$= \frac{1}{2}\Big(\cos(\varepsilon_i x_i + \varepsilon_j x_j - \varepsilon_k x_k - \varepsilon_l x_l) + \cos(2\varepsilon_k x_k + 2\varepsilon_l x_l)\Big). \tag{5.86}$$

Also we have

$$\cos(\frac{\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k + \varepsilon_l x_l}{2})\cos(\frac{\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k - 3\varepsilon_l x_l}{2})$$
$$= \frac{1}{2}\Big(\cos(\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k - \varepsilon_l x_l) + \cos(2\varepsilon_l x_l)\Big). \tag{5.87}$$

Relation (5.83) follows by substituting relations (5.85)–(5.87) into the right-hand side of relation (5.84). □

We will also need the following identity.

Lemma 5.5.8. *For distinct $i, j, k, l \in [4]$ we have*

$$\varepsilon_i A^i + \varepsilon_j A^j = \sin(\varepsilon_i x_i + \varepsilon_j x_j)\Big(\cos(\varepsilon_i x_i + \varepsilon_j x_j) - 2\cos(\varepsilon_k x_k)\cos(\varepsilon_l x_l)$$
$$+ \cos(\varepsilon_i x_i - \varepsilon_j x_j)\big(-1 + \cos(2\varepsilon_k x_k) + \cos(2\varepsilon_l x_l)\big)\Big), \tag{5.88}$$

where functions A^k $(k = 1, 2, 3, 4)$ are given by formula (5.76) and $\varepsilon_i \in \{1, -1\}$ for all i.

Proof. From formula (5.76) we have

$$\begin{aligned}\varepsilon_i A^i + \varepsilon_j A^j &= \frac{1}{2}\varepsilon_i \sin(2x_i)\Big(-1+\sum_{k\neq i}\cos 2x_k\Big) - 2\varepsilon_i \sin x_i \prod_{k\neq i}\cos x_k \\ &+ \frac{1}{2}\varepsilon_j \sin 2x_j\Big(-1+\sum_{k\neq j}\cos 2x_k\Big) - 2\varepsilon_j \sin x_j \prod_{k\neq j}\cos x_k \\ &= \frac{1}{2}\Big(\sin(2\varepsilon_i x_i)+\sin(2\varepsilon_j x_j)\Big)\Big(-1+\cos(2\varepsilon_k x_k)+\cos(2\varepsilon_l x_l)\Big) \\ &+ \frac{1}{2}\sin(2\varepsilon_i x_i + 2\varepsilon_j x_j) - 2\cos(\varepsilon_k x_k)\cos(\varepsilon_l x_l)\sin(\varepsilon_i x_i+\varepsilon_j x_j). \qquad (5.89)\end{aligned}$$

By applying formulas

$$\sin(2\varepsilon_i x_i)+\sin(2\varepsilon_j x_j) = 2\sin(\varepsilon_i x_i+\varepsilon_j x_j)\cos(\varepsilon_i x_i-\varepsilon_j x_j),$$

and

$$\sin(2\varepsilon_i x_i+2\varepsilon_j x_j) = 2\sin(\varepsilon_i x_i+\varepsilon_j x_j)\cos(\varepsilon_i x_i+\varepsilon_j x_j),$$

we see that relation (5.89) takes the form (5.88). This proves the lemma. □

The following statement confirms that with the choice of the functions A_i given by formula (5.76), the metric $B = \sum_{k=1}^4 A^k x_k$ is diagonal.

Proposition 5.5.9. *The linear combination $B = \sum_{k=1}^4 A^k F_k$, where functions A^k are given by formula* (5.76) *and function F has the form* (5.62), *is a diagonal matrix.*

Proof. Let us show that the entry $B_{12} = \sum_{k=1}^4 A^k F_{12k} = 0$. In the notation of formulas (5.75) we have

$$\frac{4}{q}B_{12} = 4(A^1+A^2)\cot(x_1+x_2) - 4(A^1-A^2)\cot(x_1-x_2) - \sum_{i=1}^4 b_i + \sum_{i=5}^8 b_i. \qquad (5.90)$$

By making use of Lemmas 5.5.7 and 5.5.8 formula (5.90) reduces to

$$\begin{aligned}
\frac{4}{q}B_{12} = 4&\Big(\cos(x_1+x_2)-\cos(x_1-x_2)\Big)\Big(\cos(x_1+x_2)+\cos(x_1-x_2)\Big)\\
&-8\Big(\cos(x_1+x_2)-\cos(x_1-x_2)\Big)\cos(x_3)\cos(x_4)\\
&+2\Big(\cos(x_1+x_2+x_3+x_4)+\cos(x_1+x_2-x_3-x_4)\Big)\\
&-2\Big(\cos(x_1-x_2+x_3-x_4)+\cos(x_1-x_2-x_3+x_4)\Big)\\
&-2\Big(\cos(x_1-x_2+x_3+x_4)+\cos(x_1-x_2-x_3-x_4)\Big)\\
&+2\Big(\cos(x_1+x_2+x_3-x_4)+\cos(x_1+x_2-x_3+x_4)\Big)\\
&-2\Big(\cos(2x_1+2x_2)-\cos(2x_1-2x_2)\Big).
\end{aligned}\tag{5.91}$$

Formula (5.91) can be rearranged as follows

$$\begin{aligned}
\frac{4}{q}B_{12} &= 16\sin x_1\sin x_2\Big(\cos x_1\cos x_2-\cos x_3\cos x_4\Big)\\
&+8\Big(\cos(x_1+x_2)-\cos(x_1-x_2)\Big)\cos x_3\cos x_4-4\sin 2x_1\sin 2x_2\\
&=16\sin x_1\sin x_2\Big(\cos x_1\cos x_2-\cos x_3\cos x_4\Big)+16\sin x_1\sin x_2\cos x_3\cos x_4\\
&-16\sin x_1\sin x_2\cos x_1\cos x_2=0.
\end{aligned}\tag{5.92}$$

The other off-diagonal entries can be done by symmetry. □

The following statement gives further property to the metric B.

Proposition 5.5.10. *The linear combination $B=\sum_{k=1}^4 A^kF_k$, where functions A^k are given by formula* (5.76) *and function F has the form* (5.62)*, is proportional to the identity matrix.*

Proof. By Proposition 5.5.9 we are left to show that $B_{11}=B_{ss}$ for all $s=2,3,4$. Let us consider $s=2$. We have $B_{11}=\sum_{k=1}^4 A^kF_{11k}$. In the notation of formulas (5.75) we have

$$q^{-1}B_{11}=-2A^1\cot x_1 \quad +\sum_{i=2}^4\Big((A^1+A^i)\cot(x_1+x_i)+(A^1-A^i)\cot(x_1-x_i)\Big)-\frac{1}{4}\sum_{i=1}^8 b_i.\tag{5.93}$$

Also we have

$$q^{-1}B_{22}=-2A^2\cot x_2+\sum_{i\neq 2}\Big((A^2+A^i)\cot(x_2+x_i)+(A^2-A^i)\cot(x_2-x_i)\Big)-\frac{1}{4}\sum_{i=1}^8 b_i.\tag{5.94}$$

Hence from (5.93), (5.94) we have

$$q^{-1}(B_{11}-B_{22}) = -2A^1\cot x_1 + 2A^2\cot x_2 + \sum_{i=3}^{4}\Big((A^1+A^i)\cot(x_1+x_i) + (A^1-A^i)\cot(x_1-x_i)\Big) - \sum_{i=3}^{4}\Big((A^2+A^i)\cot(x_2+x_i)+(A^2-A^i)\cot(x_2-x_i)\Big). \tag{5.95}$$

Now by making use of Lemma 5.5.8 formula (5.95) reduces to

$$\begin{aligned} q^{-1}(B_{11}-B_{22}) &= 2(\cos^2 x_1 - \cos^2 x_2) - 2\cos^2 x_1\sum_{i\neq 1}^{4}\cos 2x_i + 2\cos^2 x_2\sum_{i\neq 2}^{4}\cos 2x_i \\ &+ 2\cos x_1 \sum_{\substack{3\le i,j\le 4\\ i\neq j}} \cos x_i\big(\cos(x_2+x_j)+\cos(x_2-x_j)\big) \\ &- 2\cos x_2 \sum_{\substack{3\le i,j\le 4\\ i\neq j}} \cos x_i\big(\cos(x_1+x_j)+\cos(x_1-x_j)\big) \\ &+ \sum_{3\le i\le 4}\big(\cos^2(x_1+x_i)+\cos^2(x_1-x_i)\big) - \sum_{3\le i\le 4}\big(\cos^2(x_2+x_i)+\cos^2(x_2-x_i)\big) \\ &+ 2\sum_{\substack{3\le i,j\le 4\\ i\neq j}} \cos(x_1+x_i)\cos(x_1-x_i)\big(-1+\cos 2x_2+\cos 2x_j\big) \\ &- 2\sum_{\substack{3\le i,j\le 4\\ i\neq j}} \cos(x_2+x_i)\cos(x_2-x_i)\big(-1+\cos 2x_1+\cos 2x_j\big). \end{aligned} \tag{5.96}$$

By applying identities

$$\begin{aligned} &2\cos(a+b)\cos(a-b) = \cos 2a + \cos 2b, \\ &\cos^2(a+b)+\cos^2(a-b) = 1+\cos 2a\cos 2b, \\ &\cos(a+b)+\cos(a-b) = 2\cos a\cos b, \\ &2\cos^2 a = \cos 2a + 1, \end{aligned}$$

it follows that $B_{11}-B_{22}=0$. Similarly one can check that $B_{11}=B_{ss}$ for $s=3,4$. □

Since the metric $B=\sum_{k=1}^{4}A^kF_k$ is a multiple of the identity matrix, we can write $B=hI_4$, where $h=h(x)$ is some function. The coefficient of proportionality $h=B_{ii}$ for any $i=1,2,3,4$. In order to find the explicit formula for the function h, let us first prove some lemmas.

Lemma 5.5.11. *In the notation of formulas* (5.75) *we have*

$$\sum_{i=1}^{8} b_i = -8 - 4\sum_{i=1}^{4} \cos 2x_i + 2 \sum_{1\le i<j\le 4} \Big(\cos(2x_i + 2x_j) + \cos(2x_i - 2x_j)\Big), \tag{5.97}$$

where functions A^k are given by formula (5.76) *and function F has the form* (5.62).

Proof. Relation (5.97) follows by direct substitution of the formulas of b_i from Lemma 5.5.7 into the left-hand side of relation (5.97). □

Lemma 5.5.12. *In the notation of formulas* (5.75) *we have*

$$\begin{aligned}\sum_{i=9}^{14} b_i = 3 &- 12\prod_{i=1}^{4} \cos x_i - 3\cos 2x_1 - \sum_{i=2}^{4} \cos 2x_i + 3\sum_{i=2}^{4} \cos 2x_1 \cos 2x_i \\ &+ 2\sum_{2\le i<j\le 4} \cos 2x_i \cos 2x_j,\end{aligned} \tag{5.98}$$

where functions A^k are given by formula (5.76) *and function F has the form* (5.62).

Proof. From Lemma 5.5.8 we have

$$\begin{aligned}\sum_{i=9}^{14} b_i = 2 &\sum_{\substack{2\le i\le 4\\ j,k\in[4]\setminus\{1,i\}\\ j\ne k}} \cos(x_1+x_i)\cos(x_1-x_i)\big(-1+\cos 2x_j + \cos 2x_k\big) \\ &- 2\sum_{\substack{2\le i\le 4\\ j,k\in[4]\setminus\{1,i\}\\ j\ne k}} \Big(\cos(x_1+x_i)+\cos(x_1-x_i)\Big)\cos x_j \cos x_k \\ &+ \sum_{i=2}^{4} \Big(\cos^2(x_1+x_i) + \cos^2(x_1-x_i)\Big).\end{aligned} \tag{5.99}$$

Relation (5.98) follows by applying the following identities to the right-hand side of relation (5.99):

$$\begin{aligned}2\cos(a+b)\cos(a-b) &= \cos 2a + \cos 2b,\\ \cos^2(a+b) + \cos^2(a-b) &= 1 + \cos 2a \cos 2b,\\ \cos(a+b) + \cos(a-b) &= 2\cos a \cos b.\end{aligned}$$

□

Also we have the following relation which is easy to check directly.

Lemma 5.5.13.

$$\sum_{\varepsilon_i\in\{-1,1\}} \cos(x_1+\varepsilon_2x_2+\varepsilon_3x_3+\varepsilon_4x_4) = 8\prod_{i=1}^{4}\cos x_i.$$

The following statement gives the explicit formula of the coefficient of the proportionality $B \sim I_4$.

Proposition 5.5.14. *The linear combination $B = \sum_{k=1}^{4} A^k F_k$, where functions A^k are given by formula (5.76) and function F has the form (5.62), has the form $B = hI_4$, where the function $h = h(x)$ is given by*

$$\begin{aligned} h(x) &= \frac{q}{2}\Big(12 - 2\sum_{i=1}^{4}\cos 2x_i - 2\sum_{\varepsilon_i\in\{-1,1\}}\cos\left(x_1+\varepsilon_2x_2+\varepsilon_3x_3+\varepsilon_4x_4\right) \\ &\quad + \sum_{\substack{1\le i<j\le 4\\ \varepsilon_j\in\{-1,1\}}}\cos\left(2x_i+2\varepsilon_jx_j\right)\Big) = \frac{1}{2}\big(12q + \sum_{\alpha\in F_4^+} c_\alpha\cos\left(2(\alpha,x)\right)\big) \\ &= \frac{1}{2}\big(-\sum_{\alpha\in F_4^+}c_\alpha + \sum_{\alpha\in F_4^+}c_\alpha\cos\left(2(\alpha,x)\right)\big) = -\sum_{\alpha\in F_4^+}c_\alpha\sin^2(\alpha,x). \end{aligned} \tag{5.100}$$

Proof. From Proposition 5.5.10 we know that $h = B_{ii}$ for any $i = 1,2,3,4$. In the notation of formulas (5.75) we have

$$\frac{4}{q}B_{11} = -8A^1\cot x_1 - \sum_{i=1}^{8} b_i + 4\sum_{i=9}^{14} b_i. \tag{5.101}$$

Note that

$$2A^1\cot x_1 = -(1+\cos 2x_1) + (1+\cos 2x_1)\sum_{i\neq 1}\cos 2x_i - 4\prod_{i=1}^{4}\cos x_i. \tag{5.102}$$

Then by relations (5.101), (5.102) and Lemmas 5.5.11–5.5.13 we get

$$\begin{aligned} \frac{4}{q}B_{11} &= 24 - 4\sum_{i=1}^{4}\cos 2x_i - 4\sum_{\varepsilon_i\in\{-1,1\}}\cos\left(x_1+\varepsilon_2x_2+\varepsilon_3x_3+\varepsilon_4x_4\right) \\ &\quad + 2\sum_{\substack{1\le i<j\le 4\\ \varepsilon_j\in\{-1,1\}}}\cos\left(2x_i+2\varepsilon_jx_j\right). \end{aligned} \tag{5.103}$$

Formula (5.100) follows from relation (5.103). □

The following statement takes place.

Proposition 5.5.15. *Functions A^k given by formula* (5.76) *have the equivalent formula*

$$A^k = \frac{1}{4q} \sum_{\alpha \in F_4^+} c_\alpha (e_k, \alpha) \sin(2(\alpha, x)), \quad k = 1, 2, 3, 4. \tag{5.104}$$

Proof. Firstly, formula (5.76) is equivalent to the formula

$$A^k = -\frac{1}{2} \sin 2x_k + \frac{1}{2} \sin 2x_k \sum_{i \neq k} \cos 2x_i - 2 \sin x_k \prod_{i \neq k} \cos x_i. \tag{5.105}$$

Secondly, it is easy to check that the following identities take place in dimension four:

$$\begin{aligned} \sin 2x_k \sum_{i \neq k} \cos 2x_i &= \frac{1}{2} \sum_{i \neq k} \sin(2x_k \pm 2x_i), \\ \sum_{\substack{\varepsilon_s \in \{-1,1\} \\ s = \{i,j,l\}}} \sin(x_k + \varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_l x_l) &= 8 \sin x_k \prod_{m \neq k} \cos x_m. \end{aligned} \tag{5.106}$$

From (5.106) formula (5.105) becomes

$$A^k = \frac{1}{4} \Big(-2 \sin 2x_k + \sum_{m \neq k} \sin(2x_k \pm 2x_m) - \sum_{\substack{\varepsilon_s \in \{-1,1\} \\ s = \{i,j,l\}}} \sin(x_k + \varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_l x_l) \Big). \tag{5.107}$$

Now it is easy to check that the right-hand side of relation (5.107) is equal to the right-hand side of formula (5.104). This proves the proposition. □

The following statement is a corollary of Theorem 5.4.2 and Proposition 5.5.15.

Proposition 5.5.16. *The identity vector field e for the collection $\mathcal{A} = F_4^+$ given by formula* (5.65) *under the condition $r = -2q$ is given by the formula*

$$e = h^{-1} A^k \partial_{x_k}, \quad k = 1, 2, 3, 4. \tag{5.108}$$

where function h is given by formula (5.100) *and A^k is given by formula* (5.104).

5.5.5 F_4 with the condition $r = -4q$

Assume that the multiplicity parameters r, q satisfy the condition $r = -4q$. Define

$$A^k = \sin x_k \Big(\cos x_k + 2 \prod_{i \neq k} \cos x_i \Big), \quad k = 1, 2, 3, 4. \tag{5.109}$$

We establish that identity field in this case is proportional to $A^k \partial_k$ by performing analysis similarly to Section 5.5.4. The following relation takes place.

Lemma 5.5.17. *Functions A^k, $(k = 1, 2, 3, 4)$ given by formula (5.109) satisfy the relation*

$$\sum_{i=1}^{4} \varepsilon_i A^i = \sin(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4}{2})\Big(2\cos(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4}{2}) + \sum_{\substack{1\leq i<j<k\leq 4\\ l\in[4]\setminus\{i,j,k\}}} \cos(\frac{\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k - 3\varepsilon_l x_l}{2})\Big), \tag{5.110}$$

where $\varepsilon_i \in \{1, -1\}$.

Proof. Let us substitute functions A^k, $(k = 1, 2, 3, 4)$ from relation (5.109) into the left-hand side of the equation (5.110). Then by rearranging the terms one can get the formula

$$\varepsilon_1 A^1 + \varepsilon_2 A^2 + \varepsilon_3 A^3 + \varepsilon_4 A^4 = \frac{1}{2}\sum_{i=1}^{4} \sin(2\varepsilon_i x_i) + 2\Big(\sin(\varepsilon_1 x_1 + \varepsilon_2 x_2)\cos(\varepsilon_3 x_3)\cos(\varepsilon_4 x_4) + \sin(\varepsilon_3 x_3 + \varepsilon_4 x_4)\cos(\varepsilon_1 x_1)\cos(\varepsilon_2 x_2)\Big). \tag{5.111}$$

On the other hand we have

$$\sin(\frac{\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k + \varepsilon_l x_l}{2})\cos(\frac{\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k - 3\varepsilon_l x_l}{2}) = \frac{1}{2}\Big(\sin(\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k - \varepsilon_l x_l) + \sin(2\varepsilon_l x_l)\Big). \tag{5.112}$$

Now let M be the right-hand side of the equation (5.110). Then by relation (5.112) we have

$$M = \sin(\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4) - \frac{1}{2}\sum_{\substack{i<j<k\\ l\neq i,j,k}} \Big(\sin(\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k - \varepsilon_l x_l) + \sin(2\varepsilon_l x_l)\Big). \tag{5.113}$$

Note that

$$\sin(\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4) = \sin(\varepsilon_1 x_1 + \varepsilon_2 x_2)\cos(\varepsilon_3 x_3 + \varepsilon_4 x_4) + \cos(\varepsilon_1 x_1 + \varepsilon_2 x_2)\sin(\varepsilon_3 x_3 + \varepsilon_4 x_4).$$

Note also that

$$\sin(\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 - \varepsilon_4 x_4) + \sin(\varepsilon_1 x_1 + \varepsilon_2 x_2 - \varepsilon_3 x_3 + \varepsilon_4 x_4) = 2\sin(\varepsilon_1 x_1 + \varepsilon_2 x_2)\cos(\varepsilon_3 x_3 - \varepsilon_4 x_4),$$

Also we have

$$\begin{aligned}&\sin(\varepsilon_1 x_1 - \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4) - \sin(\varepsilon_1 x_1 - \varepsilon_2 x_2 - \varepsilon_3 x_3 - \varepsilon_4 x_4)\\ &= 2\cos(\varepsilon_1 x_1 - \varepsilon_2 x_2)\sin(\varepsilon_3 x_3 + \varepsilon_4 x_4).\end{aligned}$$

Thus, formula (5.113) can be rearranged as

$$\begin{aligned}M &= \sin(\varepsilon_1 x_1 + \varepsilon_2 x_2)\Big(\cos(\varepsilon_3 x_3 + \varepsilon_4 x_4) + \cos(\varepsilon_3 x_3 - \varepsilon_4 x_4)\Big) + \frac{1}{2}\sum_{i=1}^{4}\sin(2\varepsilon_i x_i)\\ &+ \sin(\varepsilon_3 x_3 + \varepsilon_4 x_4)\Big(\cos(\varepsilon_1 x_1 + \varepsilon_2 x_2) + \cos(\varepsilon_1 x_1 - \varepsilon_2 x_2)\Big).\end{aligned} \tag{5.114}$$

It is clear that the right-hand side of relation (5.113) is equivalent to the right-hand side of relation (5.111) which proves the lemma. □

The following identity holds.

Lemma 5.5.18. *Functions A^k, $(k = 1, 2, 3, 4)$ given by formula* (5.109) *satisfy the identity*

$$\begin{aligned}&(\varepsilon_1 A^1 + \varepsilon_2 A^2 + \varepsilon_3 A^3 + \varepsilon_4 A^4)\cot(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4}{2})\\ &= 1 + \cos(\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4)\\ &+ \frac{1}{2}\sum_{\substack{1\le i<j<k\le 4\\ l\in[4]\setminus\{i,j,k\}}}\Big(\cos(\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k - \varepsilon_l x_l) + \cos(2\varepsilon_l x_l)\Big).\end{aligned} \tag{5.115}$$

where $\varepsilon_i \in \{1, -1\}$ for all i.

Proof. We have by Lemma 5.5.17 that

$$\begin{aligned}&(\varepsilon_1 A^1 + \varepsilon_2 A^2 + \varepsilon_3 A^3 + \varepsilon_4 A^4)\cot(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4}{2})\\ &= \cos(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4}{2})\Big(2\cos(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4}{2})\\ &+ \sum_{\substack{1\le i<j<k\le 4\\ l\in[4]\setminus\{i,j,k\}}}\cos(\frac{\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k - 3\varepsilon_l x_l}{2})\Big),\end{aligned} \tag{5.116}$$

Firstly, we have

$$2\cos^2(\frac{\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4}{2}) = 1 + \cos(\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \varepsilon_4 x_4). \tag{5.117}$$

Secondly, we have

$$\cos(\frac{\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k + \varepsilon_l x_l}{2}) \cos(\frac{\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k - 3\varepsilon_l x_l}{2})$$
$$= \frac{1}{2}\Big(\cos(\varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_k x_k - \varepsilon_l x_l) + \cos(2\varepsilon_l x_l)\Big). \tag{5.118}$$

Relation (5.115) follows by substituting relations (5.117), (5.118) into the right-hand side of relation (5.116). □

We will also need the following identity.

Lemma 5.5.19. *For distinct $i, j, k, l \in [4]$ we have*

$$(\varepsilon_i A^i + \varepsilon_j A^j) = \sin(\varepsilon_i x_i + \varepsilon_j x_j)\Big(\cos(\varepsilon_i x_i - \varepsilon_j x_j) + 2\cos(\varepsilon_k x_k)\cos(\varepsilon_l x_l)\Big), \tag{5.119}$$

where functions A^k $(k = 1, 2, 3, 4)$ are given by formula (5.109) *and $\varepsilon_i \in \{1, -1\}$.*

Proof. From formula (5.109) we have

$$\varepsilon_i A^i + \varepsilon_j A^j = \frac{1}{2}(\varepsilon_i \sin 2x_i + \varepsilon_j \sin 2x_j) + 2(\varepsilon_i \sin x_i \prod_{k\neq i} \cos x_k + \varepsilon_j \sin x_j \prod_{k\neq j} \cos x_k)$$
$$= \frac{1}{2}\Big(\sin(2\varepsilon_i x_i) + \sin(2\varepsilon_j x_j)\Big) + 2\cos(\varepsilon_k x_k)\cos(\varepsilon_l x_l)\sin(\varepsilon_i x_i + \varepsilon_j x_j). \tag{5.120}$$

By applying formula

$$\sin(2\varepsilon_i x_i) + \sin(2\varepsilon_i x_i) = 2\sin(\varepsilon_i x_i + \varepsilon_j x_j)\cos(\varepsilon_i x_i - \varepsilon_j x_j),$$

we see that relation (5.120) takes the form (5.119). This proves the lemma. □

The following statement confirms that with the choice of the functions A_i given by formula (5.109), the metric $B = \sum_{k=1}^4 A^k F_k$ is diagonal.

Proposition 5.5.20. *The linear combination $B = \sum_{k=1}^4 A^k F_k$, where functions A^k are given by formula* (5.109) *and function F has the form* (5.62)*, is a diagonal matrix.*

Proof. Let us show that the entry $B_{12} = \sum_{k=1}^4 A^k F_{12k} = 0$. In the notation of formulas (5.75) we have

$$\frac{2}{q}B_{12} = 2(A^1 + A^2)\cot(x_1 + x_2) - 2(A^1 - A^2)\cot(x_1 - x_2) - \sum_{i=1}^4 b_i + \sum_{i=5}^8 b_i. \tag{5.121}$$

By making use of Lemmas 5.5.18 and 5.5.19 formula (5.90) reduces to

$$\begin{aligned}\frac{2}{q}B_{12} = 4\Big(&\cos(x_1+x_2)-\cos(x_1-x_2)\Big)\cos x_3\cos x_4\\ &-\Big(\cos(x_1+x_2+x_3+x_4)+\cos(x_1+x_2-x_3-x_4)\Big)\\ &+\Big(\cos(x_1-x_2+x_3-x_4)+\cos(x_1-x_2-x_3+x_4)\Big)\\ &+\Big(\cos(x_1-x_2+x_3+x_4)+\cos(x_1-x_2-x_3-x_4)\Big)\\ &-\Big(\cos(x_1+x_2+x_3-x_4)+\cos(x_1+x_2-x_3+x_4)\Big).\end{aligned} \tag{5.122}$$

Formula (5.122) can be rearranged as follows

$$\begin{aligned}q^{-1}B_{12} &= 4\sin x_1\sin x_2\cos x_3\cos x_4 - \cos(x_1+x_2)\Big(\cos(x_3+x_4)+\cos(x_3-x_4)\Big)\\ &\quad+\cos(x_1-x_2)\Big(\cos(x_3+x_4)+\cos(x_3-x_4)\Big)\\ &= 4\sin x_1\sin x_2\cos x_3\cos x_4 - 2\cos x_3\cos x_4\Big(\cos(x_1+x_2)-\cos(x_1-x_2)\Big)=0.\end{aligned}$$

The other off-diagonal entries can be done by symmetry. □

The following statement gives further property to the metric B.

Proposition 5.5.21. *The linear combination $B=\sum_{k=1}^4 A^k F_k$, where functions A^k are given by formula* (5.109) *and function F has the form* (5.62)*, is proportional to the identity matrix.*

Proof. By Proposition 5.5.20 we are left to show that $B_{11}=B_{ss}$ for all $s=2,3,4$. Let us consider $s=2$. We have $B_{11}=\sum_{k=1}^4 A^k F_{11k}$. In the notation of formulas (5.75) we have

$$q^{-1}B_{11} = -4A^1\cot x_1 + \sum_{i=2}^{4}\Big((A^1+A^i)\cot(x_1+x_i)+(A^1-A^i)\cot(x_1-x_i)\Big)-\frac{1}{2}\sum_{i=1}^{8}b_i. \tag{5.123}$$

Also we have

$$q^{-1}B_{22} = -4A^2\cot x_2 + \sum_{i\neq 2}\Big((A^2+A^i)\cot(x_2+x_i)+(A^2-A^i)\cot(x_2-x_i)\Big)-\frac{1}{2}\sum_{i=1}^{8}b_i. \tag{5.124}$$

Hence from (5.123), (5.124) we have

$$q^{-1}(B_{11}-B_{22}) = -4A^1\cot x_1 + 4A^2\cot x_2 + \sum_{i=3}^{4}\Big((A^1+A^i)\cot(x_1+x_i) + (A^1-A^i)\cot(x_1-x_i)\Big) - \sum_{i=3}^{4}\Big((A^2+A^i)\cot(x_2+x_i)+(A^2-A^i)\cot(x_2-x_i)\Big). \tag{5.125}$$

Now by making use of Lemma 5.5.19 formula (5.125) reduces to

$$q^{-1}(B_{11}-B_{22}) = -2(\cos^2 x_1 - \cos^2 x_2) + 2\sum_{i=3}^{4}\cos(x_1+x_i)\cos(x_1-x_i) - 2\sum_{i=3}^{4}\cos(x_2+x_i)\cos(x_2-x_i) - 2\cos x_1 \sum_{\substack{3\le i,j\le 4\\ i\ne j}}\cos x_i\big(\cos(x_2+x_j)+\cos(x_2-x_j)\big) + 2\cos x_2 \sum_{\substack{3\le i,j\le 4\\ i\ne j}}\cos x_i\big(\cos(x_1+x_j)+\cos(x_1-x_j)\big). \tag{5.126}$$

By applying the following identities

$$2\cos(a+b)\cos(a-b) = \cos 2a + \cos 2b,$$
$$\cos(a+b)+\cos(a-b) = 2\cos a\cos b,$$
$$2\cos^2 a = \cos 2a + 1,$$

it follows that $B_{11}-B_{22}=0$. Similarly one can check that $B_{11}=Bss$ for $s=3,4$. □

Now since the metric B is a multiple of the identity matrix, let $B = hI_4$ for some function $h=h(x)$. Before we find the explicit formula for the function h, let us prove some lemmas.

Lemma 5.5.22. *In the notation of formulas* (5.75) *we have*

$$\sum_{i=1}^{8} b_i = 8 + 4\sum_{i=1}^{4}\cos 2x_i + 3\sum_{\varepsilon_i\in\{-1,1\}}\cos(x_1+\varepsilon_2x_2+\varepsilon_3x_3+\varepsilon_4x_4). \tag{5.127}$$

where functions A^k are given by formula (5.109) *and function F has the form* (5.62).

Proof. Relation (5.127) follows by direct substitution of the formulas of b_i from Lemma 5.5.18 into the left-hand side of relation (5.127). □

Lemma 5.5.23. *In the notation of formulas* (5.75) *we have*

$$\sum_{i=9}^{14} b_i = 3\cos 2x_1 + \sum_{i=2}^{4} \cos x_i + 12\prod_{i=1}^{4} \cos x_i. \tag{5.128}$$

where functions A^k are given by formula (5.109) *and function F has the form* (5.62).

Proof. From Lemma 5.5.19 we have

$$\begin{aligned}\sum_{i=9}^{14} b_i &= 2\sum_{i=2}^{4} \cos(x_1+x_i)\cos(x_1-x_i)\\ &+ 2\sum_{\substack{2\le i\le 4\\ j,k\in[4]\setminus\{1,i\}\\ j\neq k}} \Big(\cos(x_1+x_i)+\cos(x_1-x_i)\Big)\cos x_j \cos x_k.\end{aligned} \tag{5.129}$$

Relation (5.128) follows by making use of Lemma 5.5.13 and by applying the following identities on the right-hand side of relation (5.129):

$$\begin{aligned}2\cos(a+b)\cos(a-b) &= \cos 2a + \cos 2b,\\ \cos(a+b)+\cos(a-b) &= 2\cos a\cos b.\end{aligned}$$

□

The following statement gives the explicit formula of the coefficient of the proportionality $B \sim I_4$.

Proposition 5.5.24. *The linear combination $B = \sum_{k=1}^4 A^k F_k$, where functions A^k are given by formula* (5.109) *and function F has the form* (5.62), *has the form $B = hI_4$, where the function $h = h(x)$ is given by*

$$\begin{aligned}h(x) &= -q\Big(6 + \sum_{i=1}^{4}\cos 2x_i + 8\prod_{i=1}^{4}\cos x_i\Big)\\ &= -q\Big(6 + \sum_{i=1}^{4}\cos 2x_i + \sum_{\varepsilon_i\in\{-1,1\}} \cos\left(x_1+\varepsilon_2x_2+\varepsilon_3x_3+\varepsilon_4x_4\right)\Big)\\ &= \frac{1}{4}\Big(-24q + \sum_{\alpha\in F_{4,s}^+} c_\alpha \cos(2(\alpha,x))\Big)\\ &= -\frac{1}{2}\Big(36q + \sum_{\alpha\in F_{4,s}^+} c_\alpha \sin^2(\alpha,x)\Big),\end{aligned} \tag{5.130}$$

where $F_{4,s}^+$ is the subset of short roots in F_4^+.

Proof. From Proposition 5.5.21 we know that B is a multiple of the identity matrix, thus the diagonal entries B_{ii} are equal and hence $h = B_{ii}$ for any $i = 1, 2, 3, 4$. In the notation of formulas (5.75) we have

$$\frac{2}{q} B_{11} = -8A^1 \cot x_1 - \sum_{i=1}^{8} b_i + 2 \sum_{i=9}^{14} b_i. \tag{5.131}$$

Note that

$$A^1 \cot x_1 = \frac{1}{2}(1 + \cos 2x_1) + 2 \prod_{i=1}^{4} \cos x_i. \tag{5.132}$$

Then by relation (5.132) and Lemmas 5.5.22, 5.5.23 we get

$$\frac{2}{q} B_{11} = -12 - 2 \sum_{i=1}^{4} \cos 2x_i - 16 \prod_{i=1}^{4} \cos x_i. \tag{5.133}$$

Formula (5.130) follows from relation (5.133). □

The following statement takes place.

Proposition 5.5.25. *Functions A^k given by formula* (5.109) *have the equivalent formula*

$$A^k = -\frac{1}{8q} \sum_{\alpha \in F_{4,s}^+} c_\alpha(e_k, \alpha) \sin(2(\alpha, x)), \quad k = 1, 2, 3, 4. \tag{5.134}$$

Proof. Since

$$\sum_{\substack{\varepsilon_s \in \{-1,1\} \\ s = \{i,j,l\}}} \sin(x_k + \varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_l x_l) = 8 \sin x_k \prod_{m \neq k} \cos x_m,$$

then formula (5.109) can be written equivalently as

$$A^k = \frac{1}{4}\Big(2 \sin 2x_k + \sum_{\substack{\varepsilon_s \in \{-1,1\} \\ s = \{i,j,l\}}} \sin(x_k + \varepsilon_i x_i + \varepsilon_j x_j + \varepsilon_l x_l)\Big) \tag{5.135}$$

Then it is easy to check that the right-hand side of formula (5.135) is equal to the right-hand side of formula (5.134). □

The following statement is a corollary of Theorem 5.4.2 and Proposition 5.5.24.

Proposition 5.5.26. *The identity vector field e for the collection $\mathcal{A} = F_4^+$ given by* *formula* (5.65) *under the condition* $r = -4q$ *is given by the formula*

$$e = h^{-1} A^k \partial_{x_k}, \quad k = 1, 2, 3, 4, \tag{5.136}$$

where function h *is given by formula* (5.130) *and* A^k *is given by formula* (5.134).

5.5.6 Identity vector field for projections of F_4

In this subsection we deal with the restrictions of a given solution of the commutativity equations and find the corresponding identity vector field and its relation with the original one before restriction. This process allows to obtain new solutions to WDVV equations.

The following statement explains this relation.

Proposition 5.5.27. *Let* $F(x)$ *be a function and* $e = e^k \partial_{x_k}$ *be a vector field such that* $e(F_{ij}) = \delta_{ij}$. *Let* π *be the hyperplane* $\pi := \{x \in V : x_N = 0\}$ *and let* $\widetilde{F}(x_1, \dots, x_{N-1}) = F(x_1, \dots, x_N)|_\pi$. *Suppose that* $e(\widetilde{x}) \in T_{\widetilde{x}}\pi$ *for all* $\widetilde{x} \in \pi$. *Suppose also that* $(F_{Nij})|_\pi$ *is well-defined for all* $i, j = 1, \dots, N-1$. *Let* $\widetilde{e} = e|_\pi \in \Gamma(T_*\pi)$. *Then* $\widetilde{e}(\widetilde{F}_{ij}) = \delta_{ij}$.

Proof. Consider equality $e(F_{ij}) = \delta_{ij}$ for $x \in \pi$, $i, j = 1, \dots, N-1$. Note that $(e^N \partial_{x_N} F_{ij})|_\pi = 0$ since $e^N|_\pi = 0$ and F_{Nij} is well-defined for all $i, j = 1, \dots, N-1$. It follows that $\widetilde{e}(\widetilde{F}_{ij}) = \delta_{ij}$. □

Now, recall that Corollary 4.8.5 confirms that all the functions of the form (5.62) corresponding to the three-dimensional and two-dimensional projections of the root system F_4 under the conditions $r = -2q$ or $r = -4q$ satisfy the commutativity equations (5.3). As an application of Proposition 5.5.27 we find the identity vector field for the three dimensional projections of root system F_4. Recall also that for F_4 root system there are two projections in dimension 3. Let us start with the first projected system.

The projected system $(F_4, A_1)_1$ of F_4 to the hyperplane $x_4 = 0$ consists of the following set of vectors:

$$\begin{aligned}
&e_i, \quad \text{with multiplicity} \quad r + 2q, \quad 1 \le i \le 3, \\
&e_i \pm e_j, \quad \text{with multiplicity} \quad q, \quad 1 \le i < j \le 3, \\
&\frac{1}{2}(e_1 \pm e_2 \pm e_3), \quad \text{with multiplicity} \quad 2r.
\end{aligned} \tag{5.137}$$

For the projected system (5.137) where parameters r, q satisfy the condition $r = -2q$ the following statement takes place. Note that the configuration (5.137) in the case $r = -2q$ contains 10 vectors only with non-zero multiplicities.

Theorem 5.5.28. *The identity vector field* $\widetilde{e}$ *for the restricted system* (5.137)*, where parameters* r, q *satisfy the condition* $r = -2q$*, exists and has the formula*

$$\widetilde{e} = \widetilde{h}^{-1} \sum_{k=1}^{3} \widetilde{A}^k \partial_{x_k},$$

where

$$\widetilde{A}^k = \sin x_k \Big(\cos x_k \sum_{i \neq k} \cos 2x_i - 2 \prod_{i \neq k} \cos x_i \Big), \quad k = 1, 2, 3, \tag{5.138}$$

and

$$\begin{aligned} \widetilde{h} &= \frac{q}{2}\Big(10 + \sum_{\substack{1 \leq i < j \leq 3 \\ \varepsilon \in \{-1,1\}}} \cos(2x_i + 2\varepsilon x_j) - 4 \sum_{\varepsilon_2, \varepsilon_3 \in \{-1,1\}} \cos(x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3)\Big) \\ &= \frac{1}{2}\Big(10q + \sum_{\alpha \in (F_4, A_1)_1} c_\alpha \cos(2(\alpha, x))\Big) = \frac{1}{2}\Big(- \sum_{\alpha \in (F_4, A_1)_1} c_\alpha + \sum_{\alpha \in (F_4, A_1)_1} c_\alpha \cos(2(\alpha, x))\Big) \\ &= - \sum_{\alpha \in (F_4, A_1)_1} c_\alpha \sin^2(\alpha, x). \end{aligned} \tag{5.139}$$

Proof. Let π be the hyperplane $\pi := \{x \in \mathbb{C}^4 \colon x_4 = 0\}$, and let $\widetilde{F}$ be the restriction of the function $F = \sum_{\alpha \in F_4^+} c_\alpha f((\alpha, x))$ to the hyperplane π. That is $\widetilde{F}(x_1, x_2, x_3) = F(x_1, x_2, x_3, x_4)|_\pi$. Note that identity vector field (5.108) for the collection $\mathcal{A} = F_4^+$ given by formula (5.65), where parameters r, q satisfy condition $r = -2q$, satisfies $e(F_{ij}) = \delta_{ij}$. It is easy to see that $(F_{4ij})|_{x_4=0}$ is regular for all $i, j = 1, 2, 3$. Now let $\widetilde{x} \in \pi$. It is easy to see that $A^4|_{x_4=0} = 0$ from formula (5.76). Hence by Proposition 5.5.27 there exist a vector field $\widetilde{e}$ that satisfies $\widetilde{e}(\widetilde{F}_{\alpha\beta}) = \delta_{\alpha\beta}$, and it is given by $\widetilde{e} = e|_\pi = (\sum_{k=1}^3 h^{-1} A^k \partial_{x_k})|_{x_4=0}$. Now from formula (5.76) it is easy to check that $(A^k)|_{x_4=0}$ is equal to the stated form $\widetilde{A}^k$ for $k = 1, 2, 3$. Also by formula (5.100) it is easy to check that $h|_{x_4=0}$ has the stated formula of $\widetilde{h}$. Vector field $\widetilde{e}$ is the identity field by Proposition 5.4.1. This complete the proof of the theorem. ∎

The following proposition gives an equivalent formula for functions (5.138).

Proposition 5.5.29. *Function* $\widetilde{A}^k$ *given by formula* (5.138) *can be written equivalently as*

$$\widetilde{A}^k = \frac{1}{4q} \sum_{\alpha \in (F_4, A_1)_1} c_\alpha (e_k, \alpha) \sin(2(\alpha, x)), \quad k = 1, 2, 3. \tag{5.140}$$

Proof. Firstly, it is easy to check that the following identities take place in dimension three

$$\sin x_k \prod_{m\neq k} \cos x_m = \frac{1}{4} \sum_{i\neq j\neq k\neq i} \sin(x_k \pm x_i \pm x_j),$$
$$\sin 2x_k \sum_{i\neq k} \cos 2x_i = \frac{1}{2} \sum_{m\neq k} \sin(2x_k \pm 2x_m).$$

Hence the right-hand side of formula (5.138) becomes

$$\widetilde{A}^k = \frac{1}{4} \sum_{m\neq k} \sin(2x_k \pm 2x_m) - \frac{1}{2} \sum_{i\neq j\neq k\neq i} \sin(x_k \pm x_i \pm x_j). \tag{5.141}$$

Now one can check that the right-hand sides of formulas (5.140) and (5.141) are equal. □

For the projected system (5.137) where parameters r, q satisfy the condition $r = -4q$ the following statement takes place.

Theorem 5.5.30. *The identity vector field $\widetilde{e}$ for the projected system* (5.137)*, where parameters r, q satisfy the condition $r = -4q$, exists and has the formula*

$$\widetilde{e} = \widetilde{h}^{-1} \sum_{k=1}^{3} \widetilde{A}^k \partial_{x_k},$$

where

$$\widetilde{A}^k = \sin x_k (\cos x_k + 2 \prod_{i\neq k} \cos x_i), \quad k = 1, 2, 3, \tag{5.142}$$

and

$$\widetilde{h} = -q\Big(7 + \sum_{i=1}^{3} \cos 2x_i + 8 \prod_{i=1}^{3} \cos x_i\Big). \tag{5.143}$$

Proof. Let π be the hyperplane $\pi := \{x \in \mathbb{C}^4 \colon x_4 = 0\}$, and let $\widetilde{F}$ be the restriction of the function $F = \sum_{\alpha\in F_4^+} c_\alpha f((\alpha, x))$ to the hyperplane π. That is $\widetilde{F}(x_1, x_2, x_3) = F(x_1, x_2, x_3, x_4)|_\pi$. Note that identity vector field (5.136) for the collection $\mathcal{A} = F_4^+$ given by formula (5.65), where parameters r, q satisfy condition $r = -4q$, satisfies $e(F_{ij}) = \delta_{ij}$. It is easy to see that $(F_{4ij})|_{x_4=0}$ is regular for all $i, j = 1, 2, 3$. Now let $\widetilde{x} \in \pi$. It is easy to see that $A^4|_{x_4=0} = 0$ from formula (5.109). Hence by Proposition 5.5.27 there exist a vector field $\widetilde{e}$ that satisfies $\widetilde{e}(\widetilde{F}_{\alpha\beta}) = \delta_{\alpha\beta}$, and it is given by $\widetilde{e} = e|_\pi = (\sum_{k=1}^{3} h^{-1} A^k \partial_{x_k})|_{x_4=0}$. Now from formula (5.109) it is easy to check that $(A^k)|_{x_4=0}$ is equal to the stated form $\widetilde{A}^k$ for $k = 1, 2, 3$. Also by formula (5.130) it is easy to see that $h|_{x_4=0}$ is equal to the stated form of $\widetilde{h}$. Vector field $\widetilde{e}$ is the identity field by Proposition 5.4.1. This complete the proof of the theorem. □

The following statement is easy to check since the identities

$$\sin x_k \prod_{m\neq k} \cos x_m = \frac{1}{4} \sum_{i\neq j\neq k\neq i} \sin(x_k \pm x_i \pm x_j),$$

and

$$\sum_{\varepsilon_2,\varepsilon_3\in\{-1,1\}} \cos(x_1+\varepsilon_2 x_2+\varepsilon_3 x_3) = 4\prod_{m=1}^{3} \cos x_m$$

hold in dimension three.

Proposition 5.5.31. *Functions $\widetilde{A}^k$ given by formula* (5.142) *can be written equivalently as*

$$\widetilde{A}^k = \frac{1}{2}\Big(\sin 2x_k + \sum_{i\neq j\neq k\neq i} \sin(x_k \pm x_i \pm x_j)\Big), \quad k=1,2,3, \tag{5.144}$$

and function (5.143) *can be written as*

$$\widetilde{h} = -q\Big(7 + \sum_{i=1}^{3} \cos 2x_i + 2\sum_{\varepsilon_2,\varepsilon_3\in\{-1,1\}} \cos(x_1+\varepsilon_2 x_2+\varepsilon_3 x_3)\Big). \tag{5.145}$$

Now let π be the hyperplane $\pi \coloneqq \{x\in\mathbb{C}^4 \colon x_3 = x_4\}$. Let us consider the projection of the root system F_4 to π, and let us denote the resulting system by $\widetilde{\mathcal{A}} = (F_4, A_1)_2$. Let us define vectors $f_i (1\leq i\leq 4)$ by

$$f_1 = e_1, \quad f_2 = e_2, \quad f_3 = \frac{e_3+e_4}{\sqrt{2}}, \quad f_4 = \frac{e_3-e_4}{\sqrt{2}}. \tag{5.146}$$

Vectors f_1, f_2, f_3 form a basis in π, where $\{e_i\}_{i=1}^4$ is the standard basis. Then the projected system $\widetilde{\mathcal{A}}$ consists of vectors α with the corresponding multiplicities c_α given as follows:

$$\begin{aligned}
&f_1, f_2, \quad \text{with multiplicity} \quad r,\\
&\sqrt{2}f_3, \quad \text{with multiplicity} \quad q,\\
&\frac{\sqrt{2}}{2}f_3, \quad \text{with multiplicity} \quad 2r,\\
&f_1 \pm f_2, \quad \text{with multiplicity} \quad q,\\
&\frac{1}{2}(f_1 \pm f_2), \quad \text{with multiplicity} \quad 2r,\\
&f_1 \pm \frac{\sqrt{2}}{2}f_3,\ f_2 \pm \frac{\sqrt{2}}{2}f_3, \quad \text{with multiplicity} \quad 2q,\\
&\frac{1}{2}(f_1 \pm f_2 \pm \sqrt{2}f_3), \quad \text{with multiplicity} \quad r.
\end{aligned} \tag{5.147}$$

In order to make use of Proposition 5.5.27 let us perform the orthogonal change of variables

(5.146). Let C be the constant 4×4 matrix such that

$$\widetilde{x}_k = \sum_{i=1}^{4} C_i^k x_i, \tag{5.148}$$

where $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4$ is a new coordinates system given by

$$\widetilde{x}_1 = x_1, \quad \widetilde{x}_2 = x_2, \quad \widetilde{x}_3 = \frac{x_3+x_4}{\sqrt{2}}, \quad \widetilde{x}_4 = \frac{x_3-x_4}{\sqrt{2}}. \tag{5.149}$$

Thus we have

$$C = \begin{pmatrix} C_1^1 & C_2^1 & C_3^1 & C_4^1 \\ C_1^2 & C_2^2 & C_3^2 & C_4^2 \\ C_1^3 & C_2^3 & C_3^3 & C_4^3 \\ C_1^4 & C_2^4 & C_3^4 & C_4^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{5.150}$$

Note also that $\widetilde{x}_3|_\pi = \sqrt{2}x_3$, and $\widetilde{x}_4|_\pi = 0$. Hence the hyperplane π in the new coordinates reads $\pi = \{\widetilde{x} \in \mathbb{C}^4 \colon \widetilde{x}_4 = 0\}$. Before we give the formula of the identity vector field for the configuration (5.147), let us proof some general propositions.

The following statement confirms that commutativity equations are preserved under orthogonal transformations.

Proposition 5.5.32. *Let $F = F(x_1, \ldots, x_N)$ be a function on $\mathbb{C}^N$ that satisfies commutativity equations*

$$F_i F_j = F_j F_i, \quad i,j = 1,\ldots,N.$$

Let C be the constant matrix of an orthogonal change of variables such that

$$\widetilde{x}_k = C_i^k x_i, \tag{5.151}$$

where $\widetilde{x}_1, \ldots, \widetilde{x}_N$ is a new coordinates system and summation over i is assumed. Let $\widehat{C} = C^{-1}$ so we have

$$\widehat{C}_\alpha^k C_k^\beta = \delta_\alpha^\beta. \tag{5.152}$$

Then commutativity equations

$$\widetilde{F}_i \widetilde{F}_j = \widetilde{F}_j \widetilde{F}_i, \quad i,j = 1,\ldots,N,$$

where $\widetilde{F}(\widetilde{x}) = F(x)$, hold.

Proof. Since $\partial_{x_i} = C_i^k \partial_{\widetilde{x}_k}$, then we have

$$F_{ijk} = C_k^{\widetilde{k}} C_j^{\widetilde{j}} C_i^{\widetilde{i}} \widetilde{F}_{\widetilde{i}\widetilde{j}\widetilde{k}}. \tag{5.153}$$

Consider the commutativity equations

$$F_{ijk} F_{klm} = F_{mjk} F_{kli}. \tag{5.154}$$

Then by formula (5.153) equality (5.154) reads in the new coordinates as

$$C_k^{\widetilde{k}} C_j^{\widetilde{j}} C_i^{\widetilde{i}} C_k^a C_l^b C_m^d \widetilde{F}_{\widetilde{i}\widetilde{j}\widetilde{k}} \widetilde{F}_{abd} = C_m^{\widetilde{m}} C_j^{\widetilde{j}} C_k^{\widetilde{k}} C_k^a C_l^b C_i^d \widetilde{F}_{\widetilde{m}\widetilde{j}\widetilde{k}} \widetilde{F}_{abd}. \tag{5.155}$$

Let us multiply both sides of equality (5.155) by $\widehat{C}_\alpha^m \widehat{C}_\beta^j \widehat{C}_\gamma^l \widehat{C}_\epsilon^i$. We get

$$C_k^{\widetilde{k}} C_k^a \widetilde{F}_{\epsilon\beta\widetilde{k}} \widetilde{F}_{a\alpha\gamma} = C_k^{\widetilde{k}} C_k^a \widetilde{F}_{\alpha\beta\widetilde{k}} \widetilde{F}_{a\gamma\epsilon}. \tag{5.156}$$

But for the orthogonal transformation we have $C_k^{\widetilde{k}} C_k^a = \delta^{\widetilde{k}a}$. Hence equality (5.156) reduces to

$$\widetilde{F}_{\epsilon\beta a} \widetilde{F}_{a\alpha\gamma} = \widetilde{F}_{\alpha\beta a} \widetilde{F}_{a\gamma\epsilon}.$$

That is $\widetilde{F}_\epsilon \widetilde{F}_\alpha = \widetilde{F}_\alpha \widetilde{F}_\epsilon$. This proves the proposition. □

The next statement takes place.

Proposition 5.5.33. *Let $F(x)$ be a function and $e = e^k \partial_{x_k}$ be a vector field such that $e(F_{ij}) = \delta_{ij}$. Let C and $\widehat{C}$ be as given in Proposition 5.5.32. Let $\widetilde{F}(\widetilde{x}) = F(x)$ and $\widetilde{e}(\widetilde{x}) = e(x)$. Then $\widetilde{e}(\widetilde{F}_{\alpha\beta}) = \delta_{\alpha\beta}$, with $\widetilde{e} = \widetilde{e}^k C_k^{\widetilde{k}} \partial_{x_{\widetilde{k}}}$.*

Proof. We have

$$e^k F_{ijk} = \delta_{ij}. \tag{5.157}$$

By relation (5.151) we have $\partial_{x_i} = C_i^k \partial_{\widetilde{x}_k}$. Hence we have

$$F_{ijk} = C_k^{\widetilde{k}} C_j^{\widetilde{j}} C_i^{\widetilde{i}} \widetilde{F}_{\widetilde{i}\widetilde{j}\widetilde{k}}. \tag{5.158}$$

Let $e^k(x) = \widetilde{e}^k(\widetilde{x})$. Then by formula (5.158) relation (5.157) can be written as

$$\widetilde{e}^k C_k^{\widetilde{k}} C_j^{\widetilde{j}} C_i^{\widetilde{i}} \widetilde{F}_{\widetilde{i}\widetilde{j}\widetilde{k}} = \delta_{ij}. \tag{5.159}$$

Multiply equality (5.159) by $\widehat{C}_\alpha^i \widehat{C}_\beta^j$. We get

$$\widetilde{e}^k C_k^{\widetilde{k}} \widetilde{F}_{\alpha\beta\widetilde{k}} = \widehat{C}_\alpha^i \widehat{C}_\beta^i. \tag{5.160}$$

But for the orthogonal transformation we have $\widehat{C}^i_\alpha \widehat{C}^i_\beta = \delta_{\alpha\beta}$. Hence equality (5.160) becomes

$$\widetilde{e}^k C^{\widetilde{k}}_k \widetilde{F}_{\alpha\beta\widetilde{k}} = \delta_{\alpha\beta}. \tag{5.161}$$

Let $\widetilde{e} = \widetilde{e}^k C^{\widetilde{k}}_k \partial_{x_{\widetilde{k}}}$. We have by relation (5.161) that $\widetilde{e}(\widetilde{F}_{\alpha\beta}) = \delta_{\alpha\beta}$ and hence $\widetilde{e}$ is the identity vector field by Theorem 5.4.2. The new identity vector field satisfies

$$\widetilde{e}(\widetilde{x}) = \widetilde{e}^k(\widetilde{x})\partial_{x_k} = e^k(x)\partial_{x_k} = e(x). \tag{5.162}$$

This proves the proposition. □

For the restricted system (5.147) where parameters r, q satisfy the condition $r = -2q$ the following statement takes place.

Theorem 5.5.34. *The identity vector field $\widetilde{e}$ for the system* (5.147)*, where parameters r, q satisfy the condition $r = -2q$, exists and has the formula*

$$\widetilde{e} = \widetilde{h}^{-1} \sum_{k=1}^{3} \widetilde{A}^k \partial_{\widetilde{x}_k},$$

where

$$\begin{aligned}
\widetilde{A}^1 &= \sin\widetilde{x}_1\Big(\cos\widetilde{x}_1(-1+\cos 2\widetilde{x}_2 + 2\cos\sqrt{2}\widetilde{x}_3) - \cos\widetilde{x}_2(1+\cos\sqrt{2}\widetilde{x}_3)\Big),\\
\widetilde{A}^2 &= \sin\widetilde{x}_2\Big(\cos\widetilde{x}_2(-1+\cos 2\widetilde{x}_1 + 2\cos\sqrt{2}\widetilde{x}_3) - \cos\widetilde{x}_1(1+\cos\sqrt{2}\widetilde{x}_3)\Big),\\
\widetilde{A}^3 &= \frac{1}{\sqrt{2}}\sin\sqrt{2}\widetilde{x}_3\Big(-1+\cos 2\widetilde{x}_1 + \cos 2\widetilde{x}_2 + \cos\sqrt{2}\widetilde{x}_3 - 2\cos\widetilde{x}_1\cos\widetilde{x}_2\Big),
\end{aligned} \tag{5.163}$$

and

$$\begin{aligned}
\widetilde{h} = \frac{q}{2}\Big(&13 - 2\big(\cos 2\widetilde{x}_1 + \cos 2\widetilde{x}_2 + 2\cos\sqrt{2}\widetilde{x}_3\big)\\
&- 2\sum_{\varepsilon_2,\varepsilon_3\in\{-1,1\}} \cos(\widetilde{x}_1 + \varepsilon_2\widetilde{x}_2 + \varepsilon_3\sqrt{2}\widetilde{x}_3) + 2\big(\cos(2\widetilde{x}_1 + \sqrt{2}\widetilde{x}_3) + \cos(2\widetilde{x}_1 - \sqrt{2}\widetilde{x}_3)\big)\\
&+ 2\big(\cos(2\widetilde{x}_2 + \sqrt{2}\widetilde{x}_3) + \cos(2\widetilde{x}_2 - \sqrt{2}\widetilde{x}_3)\big) - 4\big(\cos(\widetilde{x}_1 + \widetilde{x}_2) + \cos(\widetilde{x}_1 - \widetilde{x}_2)\big)\\
&+ \cos 2\sqrt{2}\widetilde{x}_3 + \cos(2\widetilde{x}_1 + 2\widetilde{x}_2) + \cos(2\widetilde{x}_1 - 2\widetilde{x}_2)\Big) = \frac{1}{2}\Big(13q + \sum_{\alpha\in\widetilde{\mathcal{A}}} c_\alpha \cos(2(\alpha,\widetilde{x}))\Big)\\
= \frac{1}{2}\Big(&-\sum_{\alpha\in\widetilde{\mathcal{A}}} c_\alpha + \sum_{\alpha\in\widetilde{\mathcal{A}}} c_\alpha\cos(2(\alpha,\widetilde{x}))\Big) = -\sum_{\alpha\in\widetilde{\mathcal{A}}} c_\alpha \sin^2(\alpha,\widetilde{x}).
\end{aligned} \tag{5.164}$$

Proof. Firstly, note that identity vector field (5.136) for the collection $\mathcal{A} = F_4^+$ given by formula (5.65), where parameters r, q satisfy condition $r = -2q$, satisfies that $e(F_{ij}) = \delta_{ij}$ by

Theorem 5.4.2. Let us make the change of variables (5.149). Let $\widehat{e}(\widetilde{x}) = e(x)$ be the identity vector field in the new coordinates and let $\widehat{F}(\widetilde{x}) = F(x)$. Then by Proposition 5.5.33 the vector field $\widehat{e}$ satisfies $\widehat{e}(\widehat{F}_{\alpha\beta}) = \delta_{\alpha\beta}$. The hyperplane $x_3 = x_4$ in the new coordinates can be represented as $\widetilde{\pi} := \{\widetilde{x} \in \mathbb{C}^4 \colon \widetilde{x}_4 = 0\}$. Let $\widehat{A}^k(\widetilde{x}) = A^k(x)$, $k = 1, 2, 3, 4$, where A^k is defined by (5.76). We have

$$
\begin{aligned}
\widehat{A}^1 &= \sin\widetilde{x}_1\Big(\cos\widetilde{x}_1\big(-1+\cos 2\widetilde{x}_2+\cos(\frac{2\widetilde{x}_3+2\widetilde{x}_4}{\sqrt{2}})+\cos(\frac{2\widetilde{x}_3-2\widetilde{x}_4}{\sqrt{2}})\big)\\
&-2\cos\widetilde{x}_2\cos(\frac{\widetilde{x}_3+\widetilde{x}_4}{\sqrt{2}})\cos(\frac{\widetilde{x}_3-\widetilde{x}_4}{\sqrt{2}})\Big),\\
\widehat{A}^2 &= \sin\widetilde{x}_2\Big(\cos\widetilde{x}_2\big(-1+\cos 2\widetilde{x}_1+\cos(\frac{2\widetilde{x}_3+2\widetilde{x}_4}{\sqrt{2}})+\cos(\frac{2\widetilde{x}_3-2\widetilde{x}_4}{\sqrt{2}})\big)\\
&-2\cos\widetilde{x}_1\cos(\frac{\widetilde{x}_3+\widetilde{x}_4}{\sqrt{2}})\cos(\frac{\widetilde{x}_3-\widetilde{x}_4}{\sqrt{2}})\Big),\\
\widehat{A}^3 &= \sin(\frac{\widetilde{x}_3+\widetilde{x}_4}{\sqrt{2}})\Big(\cos(\frac{\widetilde{x}_3+\widetilde{x}_4}{\sqrt{2}})\big(-1+\cos 2\widetilde{x}_1+\cos 2\widetilde{x}_2+\cos(\frac{2\widetilde{x}_3-2\widetilde{x}_4}{\sqrt{2}})\big)\\
&-2\cos\widetilde{x}_1\cos\widetilde{x}_2\cos(\frac{\widetilde{x}_3-\widetilde{x}_4}{\sqrt{2}})\Big),\\
\widehat{A}^4 &= \sin(\frac{\widetilde{x}_3-\widetilde{x}_4}{\sqrt{2}})\Big(\cos(\frac{\widetilde{x}_3-\widetilde{x}_4}{\sqrt{2}})\big(-1+\cos 2\widetilde{x}_1+\cos 2\widetilde{x}_2+\cos(\frac{2\widetilde{x}_3+2\widetilde{x}_4}{\sqrt{2}})\big)\\
&-2\cos\widetilde{x}_1\cos\widetilde{x}_2\cos(\frac{\widetilde{x}_3+\widetilde{x}_4}{\sqrt{2}})\Big).
\end{aligned}
\tag{5.165}
$$

Let $e = e^k(x)\partial_{x_k}$, and let $\widehat{e}^k(\widetilde{x}) = e^k(x)$. Then $e = \widehat{e} = \widehat{e}^k(\widetilde{x})C_k^m\partial_{\widetilde{x}_m}$, where coefficients C_k^m are defined by (5.150).

We have that

$$
he = \sum_{k,m=1}^{4}\widehat{A}^k C_k^m\partial_{\widetilde{x}_m} = \widehat{A}^1\partial_{\widetilde{x}_1}+\widehat{A}^2\partial_{\widetilde{x}_2}+(\frac{\widehat{A}^3+\widehat{A}^4}{\sqrt{2}})\partial_{\widetilde{x}_3}+(\frac{\widehat{A}^3-\widehat{A}^4}{\sqrt{2}})\partial_{\widetilde{x}_4}, \tag{5.166}
$$

where h is defined in (5.5.14). From (5.165) it is clear that $\widehat{A}^3|_{\widetilde{x}_4=0} = \widehat{A}^4|_{\widetilde{x}_4=0}$. Thus from formula (5.166) we have $\Big(\sum_{k,m=1}^4\widehat{A}^kC_k^m\partial_{\widetilde{x}_m}\Big)|_{\widetilde{x}_4=0} \in T_*\widetilde{\pi}$. Hence $\widehat{e}|_{\widetilde{\pi}} \in T_*\widetilde{\pi}$.

It is easy also to see that $(\widehat{F}_{4ij})|_{\widetilde{x}_4=0}$ is regular for all $i, j = 1, 2, 3$. Hence by Proposition 5.5.27, the vector field $\widetilde{e}$ given by $\widetilde{e} = \widehat{e}|_{\widetilde{\pi}} = e|_{\pi} = (\sum_{k=1}^4 h^{-1}A^k\partial_{x_k})|_{x_3=x_4}$ satisfies $\widetilde{e}(\widetilde{F}_{\alpha\beta}) = \delta_{\alpha\beta}$. Vector field $\widetilde{e}$ is the identity field by Proposition 5.4.1. Now from formula (5.76) it is easy to check that $(\widehat{A}^k)|_{\widetilde{x}_4=0} = \widetilde{A}^k$ for $k = 1, 2$, and $\Big(\frac{\widehat{A}^3+\widehat{A}^4}{\sqrt{2}}\Big)|_{\widetilde{x}_4=0} = \widetilde{A}^3$. Also by formula (5.100) it is easy to check that $h|_{\widetilde{x}_4=0}$ gives the stated form of $\widetilde{h}$. This complete the proof of the theorem. □

The following proposition gives an equivalent formula for functions (5.163).

Proposition 5.5.35. *Functions $\widetilde{A}^k$ given by formula (5.163) can be written equivalently* *as*

$$\widetilde{A}^k = \frac{1}{4q} \sum_{\alpha \in (F_4, A_1)_2} c_\alpha(e_k, \alpha) \sin\left(2(\alpha, x)\right), \quad k = 1, 2, 3. \tag{5.167}$$

Proof. The straightforward calculations of the right-hand side of formula (5.167) shows that formula (5.167) leads to formulas (5.163) where the following basic identities are applied when required

$$\begin{aligned}
\sin a + \sin b &= 2\sin(\frac{a+b}{2})\cos(\frac{a-b}{2}),\\
\sin a - \sin b &= 2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2}),\\
\cos a + \cos b &= 2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2}).
\end{aligned}$$

$\square$

For the projected system (5.147) where parameters r, q satisfy the condition $r = -4q$ the following statement takes place.

Theorem 5.5.36. *The identity vector field* $\widetilde{e}$ *for the system* (5.147)*, where parameters* r, q *satisfy the condition* $r = -4q$*, exists and has the formula*

$$\widetilde{e} = \widetilde{h}^{-1} \sum_{k=1}^{3} \widetilde{A}^k \partial_{\widetilde{x}_k},$$

where

$$\begin{aligned}
\widetilde{A}^1 &= \sin\widetilde{x}_1\Big(\cos\widetilde{x}_1 + \cos\widetilde{x}_2(1 + \cos\sqrt{2}\widetilde{x}_3)\Big),\\
\widetilde{A}^2 &= \sin\widetilde{x}_2\Big(\cos\widetilde{x}_2 + \cos\widetilde{x}_1(1 + \cos\sqrt{2}\widetilde{x}_3)\Big),\\
\widetilde{A}^3 &= \frac{1}{\sqrt{2}}\sin(\sqrt{2}\widetilde{x}_3)\Big(1 + 2\cos\widetilde{x}_1\cos\widetilde{x}_2\Big),
\end{aligned} \tag{5.168}$$

and

$$\widetilde{h} = -q\Big(6 + \cos 2\widetilde{x}_1 + \cos 2\widetilde{x}_2 + 2\cos\sqrt{2}\widetilde{x}_3 + 8\cos\widetilde{x}_1\cos\widetilde{x}_2\cos^2\frac{\widetilde{x}_3}{\sqrt{2}}\Big). \tag{5.169}$$

Proof. Firstly, note that identity vector field (5.136) for the collection $\mathcal{A} = F_4^+$ given by formula (5.65), where parameters r, q satisfy condition $r = -4q$, satisfies $e(F_{ij}) = \delta_{ij}$ by Theorem 5.4.2. Let us make the change of variables (5.149). Let $\widehat{e}$ be the identity vector field in the new coordinates and let $\widehat{F}(\widetilde{x}) = F(x)$. Then by Proposition 5.5.33 the vector field $\widehat{e}$ satisfies $\widehat{e}(\widehat{F}_{\alpha\beta}) = \delta_{\alpha\beta}$. The hyperplane $x_3 = x_4$ in the new coordinates can be represented as $\widetilde{\pi} := \{\widetilde{x} \in \mathbb{C}^4 \colon \widetilde{x}_4 = 0\}$. Let $\widehat{A}^k(\widetilde{x}) = A^k(x)$, $k = 1, 2, 3, 4$, where A^k is defined by (5.109). We have

$$
\begin{aligned}
\widehat{A}^1 &= \sin\widetilde{x}_1\Big(\cos\widetilde{x}_1 + 2\cos\widetilde{x}_2\cos(\frac{\widetilde{x}_3+\widetilde{x}_4}{\sqrt{2}})\cos(\frac{\widetilde{x}_3-\widetilde{x}_4}{\sqrt{2}})\Big),\\
\widehat{A}^2 &= \sin\widetilde{x}_2\Big(\cos\widetilde{x}_2 + 2\cos\widetilde{x}_1\cos(\frac{\widetilde{x}_3+\widetilde{x}_4}{\sqrt{2}})\cos(\frac{\widetilde{x}_3-\widetilde{x}_4}{\sqrt{2}})\Big),\\
\widehat{A}^3 &= \sin(\frac{\widetilde{x}_3+\widetilde{x}_4}{\sqrt{2}})\Big(\cos(\frac{\widetilde{x}_3+\widetilde{x}_4}{\sqrt{2}}) + 2\cos\widetilde{x}_1\cos\widetilde{x}_2\cos(\frac{\widetilde{x}_3-\widetilde{x}_4}{\sqrt{2}})\Big),\\
\widehat{A}^4 &= \sin(\frac{\widetilde{x}_3-\widetilde{x}_4}{\sqrt{2}})\Big(\cos(\frac{\widetilde{x}_3-\widetilde{x}_4}{\sqrt{2}}) + 2\cos\widetilde{x}_1\cos\widetilde{x}_2\cos(\frac{\widetilde{x}_3+\widetilde{x}_4}{\sqrt{2}})\Big).
\end{aligned}
\tag{5.170}
$$

Let $e = e^k(x)\partial_{x_k}$, and let $\widehat{e}^k(\widetilde{x}) = e^k(x)$. Then $e = \widehat{e} = \widehat{e}^k(\widetilde{x})C_k^m\partial_{\widetilde{x}_m}$, where coefficients C_k^m are defined by (5.150).

We have that

$$
he = \sum_{k,m=1}^{4}\widehat{A}^k C_k^m\partial_{\widetilde{x}_m} = \widehat{A}^1\partial_{\widetilde{x}_1} + \widehat{A}^2\partial_{\widetilde{x}_2} + (\frac{\widehat{A}^3+\widehat{A}^4}{\sqrt{2}})\partial_{\widetilde{x}_3} + (\frac{\widehat{A}^3-\widehat{A}^4}{\sqrt{2}})\partial_{\widetilde{x}_4}, \tag{5.171}
$$

where function h is defined in (5.130). From (5.165) it is clear that $\widehat{A}^3|_{\widetilde{x}_4=0} = \widehat{A}^4|_{\widetilde{x}_4=0}$. Thus from formula (5.166) we have $\Big(\sum_{k,m=1}^4\widehat{A}^kC_k^m\partial_{\widetilde{x}_m}\Big)|_{\widetilde{x}_4=0}\in T_*\widetilde{\pi}$. Hence $\widehat{e}|_{\widetilde{\pi}}\in T_*\widetilde{\pi}$. It is easy also to see that $(\widehat{F}_{4ij})|_{\widetilde{x}_4=0}$ is regular for all $i,j=1,2,3$. Hence by Proposition 5.5.27 the vector field $\widetilde{e}$ given by $\widetilde{e} = \widehat{e}|_{\widetilde{\pi}} = e|_{\pi} = (\sum_{k=1}^4 h^{-1}A^k\partial_{x_k})|_{x_3=x_4}$ satisfies $\widetilde{e}(\widetilde{F}_{\alpha\beta}) = \delta_{\alpha\beta}$. Vector field $\widetilde{e}$ is the identity field by Proposition 5.4.1. Now from formula (5.109) it is easy to check that $(\widehat{A}^k)|_{\widetilde{x}_4=0} = \widetilde{A}^k$ for $k=1,2$, and $\Big(\frac{\widehat{A}^3+\widehat{A}^4}{\sqrt{2}}\Big)|_{\widetilde{x}_4=0} = \widetilde{A}^3$. Also by formula (5.130) it is easy to check that $h|_{\widetilde{x}_4=0}$ gives the stated form of $\widetilde{h}$. This complete the proof of the theorem. □

The following statement is a corollary of Corollary 4.8.5 and Theorems 5.5.28, 5.5.30, 5.5.34, and 5.5.36.

Theorem 5.5.37. *The function F given by formula* (5.62) *corresponding to configurations* (5.137) *and* (5.147) *where $r=-2q$ or $r=-4q$ satisfies WDVV equations* (5.2)*.*

Let $\mathcal{B} = \mathcal{A}\cap W$ be a subsystem of $\mathcal{A}$ for some n-dimensional linear subspace $W = \langle\mathcal{B}\rangle \subset V$. Let

$$
W_{\mathcal{B}} \coloneqq \{x\in V\colon (\beta,x)=0 \quad \forall\beta\in\mathcal{B}\}.
$$

Recall that $\pi_{\mathcal{B}}(\alpha)$ denotes the orthogonal projection of $\alpha\in V$ to the subspace $W_{\mathcal{B}}$ with respect to the inner product $(\cdot,\cdot)$ and $\pi_{\mathcal{B}}(\mathcal{A}) = \{\pi_{\mathcal{B}}(\alpha)\colon \pi_{\mathcal{B}}(\alpha)\neq 0, \alpha\in\mathcal{A}\}$. Let $f_1,\ldots,f_n$ be an orthonormal basis of the space $W_{\mathcal{B}}$, and let $\xi_1,\ldots,\xi_n$ be the corresponding orthonormal coordinates in $W_{\mathcal{B}}$. Let us extend the orthonormal basis in $W_{\mathcal{B}}$ to an orthonormal basis $f_1,\ldots,f_n,f_{n+1},\ldots,f_N$ in V and let $\xi_1,\ldots,\xi_n,\xi_{n+1},\ldots,\xi_N$ be the corresponding orthonormal coordinates in V.

Then the following statement takes place.

Theorem 5.5.38. *Let e be a vector field such that $e(F_{ij}) = \delta_{ij}$ for all $i, j = 1, \ldots, N$, where $F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$. Let $\widehat{F}(\xi_1, \ldots, \xi_N) = F(x_1, \ldots, x_N)$ and $\widehat{e}(\xi_1, \ldots, \xi_N) = e(x_1, \ldots, x_N)$. Suppose that $e(\xi) \in T_\xi W_{\mathcal{B}}$ for all $\xi \in W_{\mathcal{B}}$. Let $\widetilde{e} = \widehat{e}|_{W_{\mathcal{B}}} \in \Gamma(T_* W_{\mathcal{B}})$. Then $\widetilde{e}((F_{\mathcal{B}})_{ij}) = \delta_{ij}$, where $(F_{\mathcal{B}})_{ij} = \frac{\partial^2 F_{\mathcal{B}}}{\partial \xi_i \partial \xi_j}$ and function $F_{\mathcal{B}}$ is given by formula* (5.63).

Proof. Let C be the constant matrix of an orthogonal change of variable such that $\xi_k = \sum_{i=1}^N C_i^k x_i$. Then by Proposition 5.5.33 we have $\widehat{e}(\widehat{F}_{ij}) = \delta_{ij}$ where $\widehat{F}_{ij} = \frac{\partial^2 \widehat{F}}{\partial \xi_i \partial \xi_j}$, $(i, j = 1, \ldots, N)$. Hence $\widehat{e}(\widehat{F}_{ij}|_{W_{\mathcal{B}}}) = \delta_{ij}$, $(i, j = 1, \ldots, n)$. The statement follows since $\widetilde{e} = \widehat{e}|_{W_{\mathcal{B}}} \in \Gamma(T_* W_{\mathcal{B}})$ and $\widehat{F}|_{W_{\mathcal{B}}} = F_{\mathcal{B}}$, $\widehat{F}_{ij}|_{W_{\mathcal{B}}} = (F_{\mathcal{B}})_{ij}$. □

The following statement is a corollary of Theorem 5.5.38.

Theorem 5.5.39. *Under the assumptions of Theorem 5.5.38 let e be the vector filed given by*

$$e = c_0 H^{-1} \sum_{\alpha \in \mathcal{A}} c_\alpha \sin(2(\alpha, x)) \partial_\alpha, \tag{5.172}$$

for some constant c_0, where

$$H = H_0 + \sum_{\alpha \in \mathcal{A}} c_\alpha \sin^2(\alpha, x) \tag{5.173}$$

for some constant H_0. Let vector field $\widetilde{e}$ be given by

$$\widetilde{e} = c_0 \widetilde{H}^{-1} \sum_{\widetilde{\alpha} \in \pi_{\mathcal{B}}(\mathcal{A})} c_{\widetilde{\alpha}} \sin(2(\widetilde{\alpha}, \xi)) \partial_{\widetilde{\alpha}}, \quad \xi \in W_{\mathcal{B}}, \tag{5.174}$$

where $\widetilde{H} = H_0 + \sum_{\widetilde{\alpha} \in \pi_{\mathcal{B}}(\mathcal{A})} c_{\widetilde{\alpha}} \sin^2(\widetilde{\alpha}, \xi)$. Then $\widetilde{e}((F_{\mathcal{B}})_{ij}) = \delta_{ij}, i, j = 1, \ldots n$, where function $F_{\mathcal{B}}$ is given by formula (5.63).

Proof. For any $\alpha \in \mathcal{A}$ we have the decomposition

$$\alpha = \alpha^{(1)} + \alpha^{(2)}, \tag{5.175}$$

where $\alpha^{(1)} = \pi_{\mathcal{B}}(\alpha) \in W_{\mathcal{B}}$ and $\alpha^{(2)} \in W_{\mathcal{B}}^\perp$. Now for any $\xi \in W_{\mathcal{B}}$ we have

$$H(\xi) = H_0 + \sum_{\alpha \in \mathcal{A}} c_\alpha \sin^2(\alpha, \xi) = H_0 + \sum_{\alpha \in \mathcal{A}} c_\alpha \sin^2(\alpha^{(1)}, \xi) = H_0 + \sum_{\widetilde{\alpha} \in \pi_{\mathcal{B}}(\mathcal{A})} c_{\widetilde{\alpha}} \sin^2(\widetilde{\alpha}, \xi) = \widetilde{H}(\xi). \tag{5.176}$$

Similarly, for any $\xi \in W_{\mathcal{B}}$ we have

$$\sum_{\alpha \in \mathcal{A}} c_\alpha \sin(2(\alpha, \xi)) \partial_\alpha = \sum_{\alpha \in \mathcal{A}} c_\alpha \sin(2(\alpha^{(1)}, \xi)) \partial_{\alpha^{(1)}} = \sum_{\widetilde{\alpha} \in \pi_{\mathcal{B}}(\mathcal{A})} c_{\widetilde{\alpha}} \sin(2(\widetilde{\alpha}, \xi)) \partial_{\widetilde{\alpha}} \tag{5.177}$$

since $e(\xi) \in T_\xi W_{\mathcal{B}}$ by assumption. From (5.176) and (5.177) we have $e(\xi) = \widetilde{e}(\xi)$. It follows from Theorem 5.5.38 that $\widetilde{e}((F_{\mathcal{B}})_{ij}) = \delta_{ij}$. This completes the proof. $\square$

Note that root system F_4 has four projected system on the plane (see the Appendix for more details). Let us present these planar projections briefly.

- The projected system $(F_4, A_2)_1$ which is obtained by projecting F_4 to the subspace

$$W_{\mathcal{B}}^{(1)} = \{x \in \mathbb{C}^4, x_1 = x_2 = x_3\}, \tag{5.178}$$

- The projected system $(F_4, A_2)_2$ which is obtained by projecting F_4 to the subspace

$$W_{\mathcal{B}}^{(2)} = \{x \in \mathbb{C}^4, x_3 = x_1 + x_2,\ x_4 = 0\}, \tag{5.179}$$

- The projected system (F_4, B_2) which is obtained by projecting F_4 to the subspace

$$W_{\mathcal{B}}^{(3)} = \{x \in \mathbb{C}^4, x_3 = x_4 = 0\}, \tag{5.180}$$

- The projected system (F_4, A_1^2) which is obtained by projecting F_4 to the subspace

$$W_{\mathcal{B}}^{(4)} = \{x \in \mathbb{C}^4, x_2 = x_3,\ x_4 = 0\}. \tag{5.181}$$

In the following theorem we apply Proposition 5.5.39 to the planar projections of the root system F_4.

Theorem 5.5.40. *Let*

$$e = -\frac{1}{4q\widetilde{H}} \sum_{\alpha \in \widetilde{\mathcal{A}}} c_\alpha \sin(2(\alpha, \xi))\partial_\alpha, \tag{5.182}$$

where

$$\widetilde{H} = \sum_{\alpha \in \widetilde{\mathcal{A}}} c_\alpha \sin^2(\alpha, \xi). \tag{5.183}$$

Then $\widetilde{e}(\widetilde{F}_{ij}) = \delta_{ij}$, $i, j = 1, 2$, *where* $\widetilde{F} = \sum_{\alpha \in \widetilde{\mathcal{A}}} c_\alpha f((\alpha, \xi))$ *and* ξ_1, ξ_2 *are an orthonormal coordinates in* $\mathbb{C}^2$, *and* $\mathcal{A}$ *is one of the configurations* $(F_4, A_2)_1$, $(F_4, A_2)_2$, (F_4, B_2), (F_4, A_1^2) *with* $r = -2q$.

Proof. Firstly, for root system F_4, where parameters r, q satisfy $r = -2q$, the identity field e given by formula (5.108) satisfies that $e(F_{ij}) = \delta_{ij}$ for all $i, j = 1, 2, 3, 4$. Secondly, it is easy to check that for each of the given cases the condition $e(\widetilde{x}) \in T_{\widetilde{x}} W_{\mathcal{B}}$ for all $\widetilde{x} \in W_{\mathcal{B}}$ holds. Hence the statement follows as a corollary of Proposition 5.5.16 and Theorem 5.5.39. $\square$

For simplicity we will write $\pi_W(F_{4,s}^+) = (F_{4,s}^+, W)$ for a subspace $W \subset \mathbb{C}^4$.

Theorem 5.5.41. *Let*

$$e = \frac{1}{4q\widetilde{H}} \sum_{\alpha \in \widetilde{\mathcal{A}}} c_\alpha \sin(2(\alpha, \xi))\partial_\alpha, \tag{5.184}$$

where

$$\widetilde{H} = 36q + \sum_{\alpha \in \widetilde{\mathcal{A}}} c_\alpha \sin^2(\alpha, \xi). \tag{5.185}$$

Then $\widetilde{e}(\widetilde{F}_{ij}) = \delta_{ij}$, $i, j = 1, 2$, *where* $\widetilde{F} = \sum_{\alpha \in \widetilde{\mathcal{A}}} c_\alpha f((\alpha, \xi))$ *and* ξ_1, ξ_2 *are an orthonormal coordinates in* $\mathbb{C}^2$, *and* $\mathcal{A}$ *is one of the configurations* $(F_{4,s}^+, W_{\mathcal{B}}^{(1)})$, $(F_{4,s}^+, W_{\mathcal{B}}^{(2)})$, $(F_{4,s}^+, W_{\mathcal{B}}^{(3)})$, $(F_{4,s}^+, W_{\mathcal{B}}^{(4)})$ *with* $r = -4q$.

Proof. Firstly, for root system F_4, where parameters r, q satisfy $r = -4q$, the identity field e given by formula (5.136) satisfies that $e(F_{ij}) = \delta_{ij}$ for all $i, j = 1, 2, 3, 4$. Secondly, it is easy to check that for each of the given cases the condition $e(\widetilde{x}) \in T_{\widetilde{x}} W_{\mathcal{B}}^{(s)}$ for all $\widetilde{x} \in W_{\mathcal{B}}^{(s)}$, where $s = 1, 2, 3, 4$, holds. Hence the statement follows as a corollary of Proposition 5.5.26 and Theorem 5.5.39. $\square$

5.5.7 Identity vector field for G_2

In this subsection we give the formula of the identity vector field for root system G_2.

Let $\mathcal{A} = G_2^+$ be the positive half of the root system G_2 with the multiplicity function given by

$$c(\sqrt{3}e^1) = c(\frac{\sqrt{3}e^1}{2} \pm \frac{3e^2}{2}) = q, \quad c(e^2) = c(\frac{\sqrt{3}e^1}{2} \pm \frac{e^2}{2}) = p, \tag{5.186}$$

where $p, q \in \mathbb{C}$. Recall that by Theorem 4.1.5 function (5.62) corresponding to the collection $\mathcal{A} = G_2^+$ satisfies commutativity equations $F_1F_2 = F_2F_1$ if and only if $p = -3q$ or $p = -9q$. Define the vector field

$$e = A^1(x)\partial_{x_1} + A^2(x)\partial_{x_2}, \quad x = (x_1, x_2), \tag{5.187}$$

where functions $A^i(x)$, $i = 1, 2$, are the functions given by formulas $A^1 = F_{122}$, $A^2 = -F_{112}$. Then the matrix $B = A^1F_1 + A^2F_2$ satisfies

$$B_{ij} = e(F_{ij}) = \sum_{k=1}^{2} A^k(x)F_{ijk}, \quad i, j = 1, 2. \tag{5.188}$$

Moreover, the matrix B is proportional to the identity matrix. Let us now give the explicit formulas for functions A^1, A^2 for both cases. We have

$$A^1 = F_{122} = \frac{\sqrt{3}p}{8}\Big(\cot\big(\frac{\sqrt{3}x_1}{2}+\frac{x_2}{2}\big)+\cot\big(\frac{\sqrt{3}x_1}{2}-\frac{x_2}{2}\big)\Big)$$
$$+\frac{9\sqrt{3}q}{8}\Big(\cot\big(\frac{\sqrt{3}x_1}{2}+\frac{2x_2}{2}\big)+\cot\big(\frac{\sqrt{3}x_1}{2}-\frac{3x_2}{2}\big)\Big),$$

$$A^2 = -F_{112} = -\frac{3p}{8}\Big(\cot\big(\frac{\sqrt{3}x_1}{2}+\frac{x_2}{2}\big)-\cot\big(\frac{\sqrt{3}x_1}{2}-\frac{x_2}{2}\big)\Big)$$
$$+\frac{9q}{8}\Big(\cot\big(\frac{\sqrt{3}x_1}{2}+\frac{2x_2}{2}\big)-\cot\big(\frac{\sqrt{3}x_1}{2}-\frac{3x_2}{2}\big)\Big).$$

Now we have two cases to consider.

Case (1). $p = -3q$. We have

$$A^1 = \sqrt{3}\sin\sqrt{3}x_1\Big(\cos\sqrt{3}x_1+\cos x_2\big(\cos 2x_2 - 2\big)\Big),$$
$$A^2 = 3\sin x_2\Big(\cos\sqrt{3}x_1\cos 2x_2 - \cos x_2\Big). \tag{5.189}$$

The following proposition gives general formula for functions A^1, A^2.

Proposition 5.5.42. *Functions A^1, A^2 given by formula* (5.189) *take the form*

$$A^k = \frac{1}{2q}\sum_{\alpha\in G_2^+} c_\alpha(e_k,\alpha)\sin(2(\alpha,x)), \quad k=1,2. \tag{5.190}$$

Proof. Firstly, since $\cos 2x_2\cos x_2 = \frac{1}{2}(\cos 3x_2+\cos x_2)$, then from (5.189) we have

$$A^1 = \frac{\sqrt{3}}{2}\sin 2\sqrt{3}x_1 + \sqrt{3}\sin\sqrt{3}x_1\cos x_2\cos 2x_2 - 2\sqrt{3}\sin\sqrt{3}x_1\cos x_2$$
$$= \frac{1}{2}\Big(\sqrt{3}\sin 2\sqrt{3}x_1 + \sqrt{3}\sin\sqrt{3}x_1\cos 3x_2 - 3\sqrt{3}\sin\sqrt{3}x_1\cos x_2\Big). \tag{5.191}$$

It is easy to check that the right-hand side of formula (5.191) is equal to right-hand side of formula (5.190) for $k=1$.

Secondly, since $\cos 2x_2\sin x_2 = \frac{1}{2}(\sin 3x_2+\sin x_2)$, then from (5.189) we have

$$A^2 = 3\cos\sqrt{3}x_1\sin x_2\cos 2x_2 - \frac{3}{2}\sin 2x_2$$
$$= \frac{3}{2}\Big(\cos\sqrt{3}x_1\sin 3x_2 - \cos\sqrt{3}x_1\sin x_2 - \sin 2x_2\Big). \tag{5.192}$$

It is easy to check that the right-hand side of formula (5.192) is equal to right-hand side of formula (5.190) for $k=2$. ☐

Now we have the linear combination $B = A^iF_i = h(x)I_2$, where I_2 is the 2×2 identity matrix and function $h(x)$ has the form

$$\begin{aligned} h(x) &= \frac{9q}{4}\Big(\cos 2\sqrt{3}x_1 + (\cos 2x_2 - 2)(-3 + 4\cos\sqrt{3}x_1\cos x_2)\Big) \\ &= \frac{9}{4}\Big(6q + \sum_{\alpha\in G_2^+} c_\alpha \cos(2(\alpha, x))\Big) = \frac{9}{4}\Big(-\sum_{\alpha\in G_2^+} c_\alpha + \sum_{\alpha\in G_2^+} c_\alpha \cos(2(\alpha, x))\Big) \\ &= -\frac{9}{2}\sum_{\alpha\in G_2^+} c_\alpha \sin^2(\alpha, x). \end{aligned} \tag{5.193}$$

The following statement takes place.

Proposition 5.5.43. *The identity vector field e for the collection $\mathcal{A} = G_2^+$ defined by* (5.186) *under the condition $p = -3q$ is given by the formula*

$$e = h^{-1}(A^1\partial_{x_1} + A^2\partial_{x_2}), \tag{5.194}$$

where function h is given by formula (5.193) *and A^1, A^2 are given by formula* (5.190) *with $k = 1, 2$ respectively.*

Case (2). $p = -9q$. We have

$$\begin{aligned} A^1 &= \sqrt{3}\sin\sqrt{3}x_1\cos x_2, \\ A^2 &= \sin x_2\Big(\cos\sqrt{3}x_1 + 2\cos x_2\Big). \end{aligned} \tag{5.195}$$

The following proposition gives general formula for functions A^1, A^2.

Proposition 5.5.44. *Functions A^1, A^2 given by formula* (5.195) *take the form*

$$A^k = -\frac{1}{9q}\sum_{\alpha\in G_{2,s}^+} c_\alpha (e_k, \alpha)\sin(2(\alpha, x)), \quad k = 1, 2, \tag{5.196}$$

where $G_{2,s}^+$ is the subset of short roots in G_2^+.

Proof. It is easy to check that the right-hand side of formula (5.196) for $k = 1, 2$ gives the formula A^1, A^2 given by (5.195). □

Now we have the linear combination $B = A^iF_i = h(x)I_2$, where I_2 is the 2×2 identity matrix and function $h(x)$ has the form

$$\begin{aligned} h(x) &= -\frac{9q}{2}\Big(3 + 2\cos\sqrt{3}x_1\cos x_2 + \cos 2x_2\Big) \\ &= \frac{1}{2}\Big(-27q + \sum_{\alpha\in G_{2,s}^+} c_\alpha \cos(2(\alpha,x))\Big) \\ &= -\Big(27q + \sum_{\alpha\in G_{2,s}^+} c_\alpha \sin^2(\alpha,x)\Big). \end{aligned} \tag{5.197}$$

The following statement takes place.

Proposition 5.5.45. *The identity vector field e for the collection $\mathcal{A} = G_2^+$ defined by* (5.186) *under the condition $p = -9q$ is given by the formula*

$$e = h^{-1}(A^1\partial_{x_1} + A^2\partial_{x_2}), \tag{5.198}$$

where function h is given by formula (5.197) *and A^1, A^2 are given by formula* (5.196) *for $k = 1, 2$ respectively.*

5.5.8 Identity vector field for BC_N

In this subsection we give the formula of the identity vector field corresponding to root system BC_N based on our results from Chapter 4.

Recall that we have the configuration $\mathcal{A} = BC_N^+ \subset \mathbb{C}^N$ consisting of the following vectors and their corresponding multiplicities

$$\begin{aligned} &e_i, \quad \text{with multiplicity} \quad r, \quad 1 \le i \le N, \\ &2e_i, \quad \text{with multiplicity} \quad s, \quad 1 \le i \le N, \\ &e_i \pm e_j, \quad \text{with multiplicity} \quad q, \quad 1 \le i < j \le N, \end{aligned} \tag{5.199}$$

where $e_1, \dots, e_N$ is the standard basis in $\mathbb{C}^N$. We have shown that if the multiplicities r, s and q satisfy the relation

$$r = -8s - 2q(N-2), \tag{5.200}$$

then function

$$F = \sum_{\alpha\in\mathcal{A}} c_\alpha f((\alpha,x)) \tag{5.201}$$

satisfies both equations (5.2) and (5.3) with the matrix B given by

$$B = \sum_{k=1}^{N} \sin 2x_k F_k. \tag{5.202}$$

Moreover, the entries of the matrix B are given by the formula

$$B_{lt} = h(x)\delta_{lt}, \quad l, t = 1, \ldots, N, \tag{5.203}$$

where

$$h(x) = r + 2q \sum_{k=1}^{N} \cos 2x_k. \tag{5.204}$$

The following statement follows by Proposition 5.4.1.

Proposition 5.5.46. *The identity vector field e for BC_N^+ under the condition* (5.200) *for the multiplicity parameters is given by the formula*

$$e = h^{-1} \sum_{k=1}^{N} \sin 2x_k \partial_{x_k}, \tag{5.205}$$

where function h is given by formula (5.204).

The following proposition gives an equivalent formula for the identity field (5.205).

Proposition 5.5.47. *Let $\widehat{\mathcal{A}} = \{e_k,\ k = 1, \ldots, N\} \subset BC_N^+$. Then the identity vector field for BC_N^+ under the condition* (5.200) *for the multiplicity parameters has the formula*

$$e = -\frac{1}{4q} H^{-1} \sum_{\alpha \in \widehat{\mathcal{A}}} c_\alpha \sin(2(\alpha, x))\partial_\alpha, \tag{5.206}$$

where function H is given by

$$H = \frac{r(2s-q)}{q} + \sum_{\alpha \in \widehat{\mathcal{A}}} c_\alpha \sin^2(\alpha, x). \tag{5.207}$$

Proof. Firstly, since $c_\alpha = r$ for any $\alpha \in \widehat{\mathcal{A}}$ then we have

$$\sum_{k=1}^{N} \sin 2x_k \partial_{x_k} = \frac{1}{r} \sum_{k=1}^{N} \sum_{\alpha \in \widehat{\mathcal{A}}} c_\alpha (e_k, \alpha) \sin(2(\alpha, x))\partial_{x_k} = \frac{1}{r} \sum_{\alpha \in \widehat{\mathcal{A}}} c_\alpha \sin(2(\alpha, x))\partial_\alpha. \tag{5.208}$$

Secondly, we have

$$
\begin{aligned}
r+2q\sum_{k=1}^{N}\cos 2x_k &= r+\frac{2q}{r}\sum_{\alpha\in\widehat{\mathcal{A}}}c_\alpha\cos(2(\alpha,x))\\
&= -8s-2q(N-2)+\frac{2q}{r}\sum_{\alpha\in\widehat{\mathcal{A}}}c_\alpha\big(1-2\sin^2(\alpha,x)\big)\\
&= -\frac{4q}{r}\Big(\frac{r(2s-q)}{q}+\sum_{\alpha\in\widehat{\mathcal{A}}}c_\alpha\sin^2(\alpha,x)\Big). \qquad (5.209)
\end{aligned}
$$

Then formula (5.206) follows by substituting formulas (5.208) and (5.209) into formula (5.205). □

5.5.9 Identity vector field for restrictions of BC_N

In this subsection we give the formula of the identity vector field corresponding to root system $BC_n(q,r,s;\underline{m})$ based on our results from Chapter 4.

Recall that we have the configuration $\mathcal{A}=BC_N^+$ given by (5.199). Let $n\in\mathbb{N}$ and $\underline{m}=(m_1,\dots,m_n)$ with $m_i\in\mathbb{N}$ such that $\sum_{i=1}^n m_i=N$. Let us consider subsystem $\mathcal{B}\subset\mathcal{A}$ as follows:

$$
\mathcal{B}=\{e_{\sum_{j=1}^{i-1}m_j+k}-e_{\sum_{j=1}^{i-1}m_j+l},\quad 1\le k<l\le m_i, i=1,\dots,n\}.
$$

Let us also consider the corresponding subspace $W_{\mathcal{B}}$ of dimension n given by

$$
W_{\mathcal{B}}=\{x\in W:(\beta,x)=0,\forall\beta\in\mathcal{B}\}.
$$

More explicitly, vectors $x=(x_1,\dots,x_N)\in W_{\mathcal{B}}$ satisfy conditions:

$$
\begin{cases}
x_1=\dots=x_{m_1},\\
x_{m_1+1}=\dots=x_{m_1+m_2},\\
\vdots\\
x_{\sum_{i=1}^{n-1}m_i+1}=\dots=x_N.
\end{cases}
$$

Note that vectors $f_i, 1\le i\le n$, given by

$$
f_i=\frac{1}{\sqrt{m_i}}\sum_{j=1}^{m_i}e_{\sum_{s=1}^{i-1}m_s+j} \qquad (5.210)
$$

form an orthonormal basis for $W_{\mathcal{B}}$. Now for any vector $u=\sum_{i=1}^N u_ie_i\in\mathbb{C}^N$, let $\widetilde{u}$ be its orthogonal projection to the subspace $W_{\mathcal{B}}$. The formula of $\widetilde{u}$ is given by Lemma 4.3.4.

Now from Lemma 4.3.4 we have the following expression for the orthogonal projections of basis e_j, $(1 \leq j \leq N)$ to the subspace $W_{\mathcal{B}}$ in terms of the basis (5.210).

$$\begin{aligned}&\widetilde{e}_{m_1+\dots+m_{i-1}+1} = \widetilde{e}_{m_1+\dots+m_{i-1}+2} = \dots = \widetilde{e}_{m_1+\dots+m_i}\\ &= \frac{1}{m_i}(\underbrace{0,\dots,0}_{m_1+\dots+m_{i-1}},1,\dots,1,\underbrace{0,\dots,0}_{N-(m_1+\dots+m_i)}) = \frac{f_i}{\sqrt{m_i}}, \quad 1 \leq i \leq n.\end{aligned} \tag{5.211}$$

Let us now project BC_N^+ to the subspace $W_{\mathcal{B}}$, and denote the projected system as $\widetilde{\mathcal{A}} = BC_n(q,r,s;\underline{m}) \subset W_{\mathcal{B}} \cong \mathbb{C}^n$. Then configuration $\widetilde{\mathcal{A}}$ consists of vectors α with multiplicities c_α:

$$\begin{aligned}&m_i^{-1/2}f_i, \quad \text{with multiplicity} \quad rm_i, \quad 1 \leq i \leq n,\\ &2m_i^{-1/2}f_i, \quad \text{with multiplicity} \quad sm_i + \frac{1}{2}qm_i(m_i-1), \quad 1 \leq i \leq n,\\ &m_i^{-1/2}f_i \pm m_j^{-1/2}f_j, \quad \text{with multiplicity} \quad qm_im_j, \quad 1 \leq i < j \leq n.\end{aligned} \tag{5.212}$$

Now suppose that parameters r,s,q and $\underline{m}$ satisfy relation (5.200) with $N = \sum_{i=1}^n m_i$. Consider the function

$$\widetilde{F} = \sum_{\alpha \in \widetilde{\mathcal{A}}} c_\alpha f((\alpha,\widetilde{x})), \quad \widetilde{x} \in W_{\mathcal{B}}. \tag{5.213}$$

Note that (see Corollary 4.8.4, see also Proposition 4.4.6) function (5.213) satisfies the commutativity equations

$$\widetilde{F}_i\widetilde{F}_j = \widetilde{F}_j\widetilde{F}_i \quad i,j = 1,\dots,n.$$

It is easy to check that the identity field (5.206) satisfies that $e(\widetilde{x}) \in T_{\widetilde{x}}W_{\mathcal{B}}$ for any $\widetilde{x} \in W_{\mathcal{B}}$. Hence the following statement follows as a corollary of Theorem 5.5.39 and Proposition 5.5.47.

Theorem 5.5.48. *Let $\widehat{\mathcal{B}} = \{m_k^{-1/2}f^k,\ k = 1,\dots,n\} \subset BC_n(q,r,s;\underline{m})$. Then the identity vector field for $BC_n(q,r,s;\underline{m})$ under the condition* (5.200) *for the multiplicity parameters, where $N = \sum_{i=1}^n m_i$, has the formula*

$$\widetilde{e} = -\frac{1}{4q}\widetilde{H}^{-1}\sum_{\alpha \in \widehat{\mathcal{B}}} c_\alpha \sin(2(\alpha,\widetilde{x}))\partial_\alpha, \quad \widetilde{x} \in W_{\mathcal{B}} \tag{5.214}$$

where function $\widetilde{H}$ is given by

$$\widetilde{H} = \frac{r(2s-q)}{q} + \sum_{\alpha \in \widehat{\mathcal{B}}} c_\alpha \sin^2(\alpha,\widetilde{x}). \tag{5.215}$$

5.5.10 Uniform formulas of identity field for (not simply-laced) root systems and their projections

In this subsection we summarize the identity vector fields for all root systems with more than one orbit as well as their restrictions.

The following theorem is a corollary of Propositions 5.5.16, 5.5.43.

Theorem 5.5.49. *Let function F be given by (5.62). Consider a vector field e given by*

$$e = c_0 H^{-1} \sum_{\alpha\in\mathcal{A}} c_\alpha \sin(2(\alpha, x))\partial_\alpha, \tag{5.216}$$

for some constant c_0, where

$$H = H_0 + \sum_{\alpha\in\mathcal{A}} c_\alpha \sin^2(\alpha, x)$$

for some constant H_0. Then $e(F_{ij}) = \delta_{ij}$ if

- *$\mathcal{A} = F_4^+$ given by formula (5.65) or $\mathcal{A}$ is one of the 3-dimensional projections $(F_4, A_1)_1$, $(F_4, A_1)_2$ given by formulas (5.137), (5.147) respectively, or $\mathcal{A}$ is one of the 2-dimensional projections $(F_4, A_2)_1$, $(F_4, A_2)_2$, (F_4, B_2), (F_4, A_1^2), under the conditions*
$$r = -2q, \quad c_0 = -\frac{1}{4q}, \quad H_0 = 0,$$
- *$\mathcal{A} = F_{4,s}^+$ the subset of short roots in F_4^+ or $\mathcal{A}$ is one of the 3-dimensional projections of $F_{4,s}^+$ along the subspaces $x_4 = 0$, $x_3 = x_4$ respectively, or $\mathcal{A}$ is one of the 2-dimensional projections of $F_{4,s}^+$ to one of the subspaces $W_{\mathcal{B}}^{(1)}$, $W_{\mathcal{B}}^{(2)}$, $W_{\mathcal{B}}^{(3)}$, $W_{\mathcal{B}}^{(4)}$ given by formulas (5.178), (5.179), (5.180), (5.181) respectively, under the conditions*
$$r = -4q, \quad c_0 = \frac{1}{4q}, \quad H_0 = 36q,$$
- *$\mathcal{A} = G_2^+$ given by formula (5.186) under the conditions*
$$p = -3q, \quad c_0 = -\frac{1}{9q}, \quad H_0 = 0,$$
- *$\mathcal{A} = G_{2,s}^+$ the subset of short roots in G_2^+ under the conditions*
$$p = -9q, \quad c_0 = \frac{1}{9q}, \quad H_0 = 27q,$$
- *$\mathcal{A} = \{m_k^{-1/2} f_k,\ k = 1, \ldots, n\} \subset BC_n(q, r, s; \underline{m})$, where $BC_n(q, r, s; \underline{m})$ is given by*

(5.212), *under the conditions*

$$r = -8s - 2q(\sum_{i=1}^{n} m_i - 2), \quad c_0 = -\frac{1}{4q}, \quad H_0 = \frac{r(2s-q)}{q}$$

where in this case if all $m_i = 1\,(i = 1, \ldots, n)$, *then* $BC_n(q, r, s; \underline{m})$ *reduces to the standard* BC_N^+ *with* $N = \sum_{i=1}^{n} m_i$.

Chapter 6

Concluding remarks and open questions

6.1 Classification

In the current work we studied the trigonometric version of $\vee$-systems and trigonometric solutions to WDVV equations related to these systems. We proved that under some non-degeneracy conditions these collections are closed under the natural operations of taking subsystems and restrictions respectively, which extends the corresponding results in the rational case.

Solutions related to the trigonometric $\vee$-systems involve an extra variable which make the classification problem non-trivial already for dimension two. The classification of trigonometric $\vee$-systems remains an important open problem. We gave detailed description of all the known two-dimensional examples based on restrictions of root systems as well as some examples which are not of this form. We note that these configurations involve collinear vectors which makes the task of classification more challenging.

The existence of a rigid geometrical structure of a series decomposition of vectors in the trigonometric $\vee$-system helped to classify such systems up to five vectors on the plane (see [27]). In the current work we do further investigations and prove that these systems up to five vectors on the plane actually belong to the family of BC_2 and A_2 root systems and their deformations. We can also prove the following statement by working out the conditions of the series decomposition of vectors.

Proposition 6.1.1. *Let trigonometric $\vee$-system $\mathcal{A}$ on the plane consist of six vectors with non-zero multiplicities. Assume that $\mathcal{A}$ contains exactly two pairs of proportional vectors. Then $\mathcal{A}$ belong to the family of $BC_2(r, s, q; \underline{m})$ for some values of parameters.*

We hope that these examples as well as the strong series conditions would be instrumental in achieving classification of the trigonometric $\vee$-systems which requires further work.

6.2 Commutativity equations and WDVV equations

Commutativity equations $F_iF_j = F_jF_i$ appear in $\mathcal{N} = 4$ supersymmetric mechanics. Solutions to these equations corresponding to root systems G_2, BC_N, F_4 were given in [3]. In the current work we investigated and clarified the relation between the commutativity and WDVV equations. We have shown that under certain non-degeneracy conditions the commutativity equations imply the WDVV equations, which leads to new solutions of WDVV equations from known solutions of the commutativity equations. We gave such trigonometric solutions of WDVV equations (without extra variables) related to root systems BC_N and F_4 and their restrictions. It would be interesting to see whether there are more Frobenius manifold structures associated to solutions of WDVV equations both without and with the extra variable.

Appendix A

Trigonometric $\vee$-systems on the plane

In this appendix we present explicitly all the known trigonometric $\vee$-systems on the plane. We follow Bourbaki's work [6] for the presentation of root systems. Let us denote by $(\mathcal{R}, \mathcal{B})$ the restriction of the root system $\mathcal{R}$ along the subsystem $\mathcal{B}$.

A.1 Planar restrictions of root system F_4

Recall that $\mathcal{R} = F_4^+$ is the positive half of the root system F_4 with the multiplicity function c given by

$$\begin{aligned} &c\big(\frac{1}{2}(e^1 \pm e^2 \pm e^3 \pm e^4)\big) = c(e^i) = p, \quad (1 \leq i \leq 4), \\ &c(e^i \pm e^j) = q, \quad (1 \leq i < j \leq 4), \end{aligned} \tag{A.1}$$

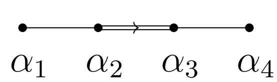
where $p, q \in \mathbb{C}$. Recall also that in the corresponding solution (2.55) of WDVV equations (2.56) we have

$$\lambda = \lambda_{(F_4,c)} = 6\sqrt{3}(2q+p)(4q+p)^{-1/2}. \tag{A.2}$$

The basis of simple roots consists of

$$\alpha_1 = e^2 - e^3, \quad \alpha_2 = e^3 - e^4, \quad \alpha_3 = e^4, \quad \alpha_4 = \frac{1}{2}(e^1 - e^2 - e^3 - e^4).$$

The Dynkin graph of F_4 is



Note that there are two different restrictions of the root system F_4 along the root system A_2. The first one $(F_4, A_2)_1$ is obtained by taking subsystem $(A_2)_1$ spanned by α_1, α_2. The second one $(F_4, A_2)_2$ is obtained taking subsystem $(A_2)_2$ spanned by α_3, α_4. The following table gives all the planar restrictions of root system F_4.

Table A.1: Restricted systems of F_4 on the plane

F_4 root system			
Subsystem	Restricted system $\mathcal{A}$	Multiplicities	$\lvert\mathcal{A}\rvert$
$(A_2)_1$	$e^1, e^2, 2e^2, e^1 \pm e^2, \frac{1}{2}(e^1 \pm e^2), \frac{1}{2}(e^1 \pm 3e^2)$	$p, 3p, 3q, 3q, 3p, p$	9
$(A_2)_2$	$e^1, e^2, \frac{1}{2}(e^1 \pm e^2), \frac{1}{2}(3e^1 \pm e^2)$	$3(p+q), q, 3(p+q), q$	6
B_2	$e^1, e^2, 2e^1, 2e^2, e^1 \pm e^2$	$4p, 4p, q, q, p+4q$	6
A_1^2	$e^1, e^2, 2e^1, 2e^2, e^1 \pm e^2, e^1 \pm 2e^2$	$2(p+2q), 4p, q, p+2q, 2p, 2q$	8

The following equivalences take place:

- Configuration $(F_4, A_2)_1$ is equivalent to the configuration given in Proposition 3.5.10.
- Configuration $(F_4, A_2)_2$ is equivalent to root system G_2 with invariant multiplicities.
- Configuration (F_4, B_2) belongs to the family of BC_2 configurations.
- Configuration (F_4, A_1^2) coincides with the configuration given in Proposition 3.5.9.

A.2 Planar restrictions of root system E_8

Recall that $\mathcal{R} = E_8^+$ is the positive half of the root system E_8 with the multiplicity function c given by

$$
\begin{aligned}
&e^i \pm e^j, \text{with multiplicity} \quad t, \quad 1 \le i < j \le 8,\\
&\frac{1}{2}(e^1 \pm e^2 \pm e^3 \pm e^4 \pm e^5 \pm e^6 \pm e^7 \pm e^8), \quad \text{with multiplicity} \quad t,
\end{aligned}
$$

where the sum of all eight coordinates is even and $t \in \mathbb{C}$.

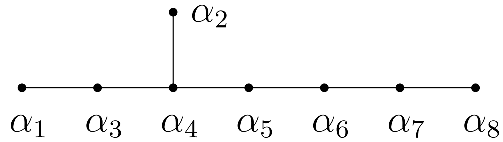
Recall also that in the corresponding solution (2.55) of WDVV equations (2.56) we have

$$\lambda = \lambda_{(E_8,t)} = 30\sqrt{t}. \tag{A.3}$$

The basis of simple roots consists of

$$
\begin{aligned}
&\alpha_1 = \frac{1}{2}e^1 + e^8 - \frac{1}{2}(e^2 + e^3 + e^4 + e^5 + e^6 + e^7), \quad \alpha_2 = e^1 + e^2, \quad \alpha_3 = e^2 - e^1,\\
&\alpha_4 = e^3 - e^2, \quad \alpha_5 = e^4 - e^3, \quad \alpha_6 = e^5 - e^4, \quad \alpha_7 = e^6 - e^5, \quad \alpha_8 = e^7 - e^6.
\end{aligned}
$$

The Dynkin graph of E_8 is



The following table gives all the planar restrictions of root system E_8.

Table A.2: Restricted systems of E_8 on the plane

E_8 root system			
Subsystem	Restricted system $\mathcal{A}$	Multiplicities	$\lvert\mathcal{A}\rvert$
E_6	$e^1, e^2, \frac{1}{2}(e^1 \pm e^2), \frac{1}{2}(3e^1 \pm e^2)$	$27, 1, 27, 1$	6
D_6	$e^1, e^2, 2e^1, 2e^2, e^1 \pm e^2$	$32, 32, 1, 1, 12$	6
A_6	$e^1, 2e^1, e^2, \frac{1}{2}(e^1 \pm e^2), \frac{1}{2}(3e^1 \pm e^2)$	$35, 7, 1, 21, 7$	7
$A_1 \times D_5$	$e^1, 2e^1, e^2, 2e^2, e^1 \pm e^2, 2e^1 \pm e^2$	$32, 10, 20, 1, 16, 2$	8
$A_1 \times A_5$	$e^1, 2e^1, 3e^1, e^2, 2e^2, e^1 \pm e^2, 2e^1 \pm e^2$	$30, 15, 2, 20, 1, 12, 6$	9
$A_2 \times D_4$	$e^1, e^2, 2e^1, e^1 \pm e^2, \frac{1}{2}(3e^1 \pm e^2), \frac{1}{2}(e^1 \pm e^2)$	$24, 8, 3, 3, 8, 24$	9
$A_2 \times A_4$	$e^1, 2e^1, 3e^1, e^2, \frac{1}{2}(e^1 \pm e^2), \frac{1}{2}(3e^1 \pm e^2),$ $\frac{1}{2}(5e^1 \pm e^2)$	$30, 15, 5, 1, 15, 10, 3$	10
A_3^2	$e^1, 2e^1, e^2, 2e^2, e^1 \pm e^2, 2e^1 \pm e^2, e^1 \pm 2e^2$	$24, 6, 24, 6, 16, 4, 4$	10
$A_1^2 \times A_4$	$e^1, 2e^1, e^2, \frac{1}{3}(e^1 + e^2), \frac{2}{3}(e^1 + e^2),$ $\frac{1}{3}(2e^1 - e^2), \frac{1}{3}(e^1 - 2e^2), \frac{2}{3}(2e^1 - e^2),$ $\frac{1}{3}(5e^1 + 2e^2), \frac{1}{3}(5e^1 - e^2), \frac{1}{3}(4e^1 + e^2)$	$20, 5, 2, 20, 10, 20, 10, 5, 2, 4, 10$	11
$A_1 \times A_2 \times A_3$	$e^1, 2e^1, 3e^1, 4e^1, e^2, 2e^2, e^1 \pm e^2,$ $2e^1 \pm e^2, 3e^1 \pm e^2$	$24, 18, 8, 3, 12, 1, 12, 6, 4$	12
$A_1^2 \times A_2^2$	$e^1, 2e^1, 3e^1, e^2, 2e^2, 3e^2, e^1 \pm e^2,$ $2(e^1 \pm e^2), 2e^1 \pm e^2, e^1 \pm 2e^2$	$18, 9, 2, 18, 9, 2, 12, 3, 6, 6$	14

The following equivalences take place:

- The restriction (E_8, E_6) is equivalent to root system G_2 with special multiplicities.
- The restriction (E_8, D_6) belongs to the family of BC_2 configurations.
- The restriction (E_8, A_6) is equivalent to a configuration from the family given in Proposition 3.5.7 with special multiplicities.
- The restriction $(E_8, A_1 \times A_5)$ belongs to the family of configuration given in Proposition 3.5.13 with special multiplicities.

- The restriction $(E_8, A_1 \times D_5)$ is equivalent to a configuration from the family given in Proposition 3.5.9 with special multiplicities.
- The restriction $(E_8, A_2 \times D_4)$ belongs to the family given in Proposition 3.5.10 with special multiplicities.

A.3 Planar restrictions of root system E_7

Recall that $\mathcal{R} = E_7^+$ is the positive half of the root system E_7 with the multiplicity function $c = t \in \mathbb{C}$ given by

$$e^i \pm e^j, \quad 1 \le i < j \le 6, \quad e^8 - e^7,$$
$$\frac{1}{2}\Big(e^8 - e^7 + \sum_{i=1}^{6}(-1)^{\nu(i)}e^i\Big), \text{with} \sum_{i=1}^{6}\nu(i) \text{ odd}.$$

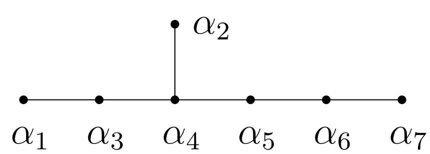
Another realization of root system E_7 is given earlier in Subsection 3.6.2. Recall also that in the corresponding solution (2.55) of WDVV equations (2.56) we have

$$\lambda = \lambda_{(E_7,t)} = 9\sqrt{6t}. \tag{A.4}$$

The basis of simple roots consists of

$$\alpha_1 = \frac{1}{2}e^1 + e^8 - \frac{1}{2}(e^2 + e^3 + e^4 + e^5 + e^6 + e^7), \quad \alpha_2 = e^1 + e^2, \quad \alpha_3 = e^2 - e^1,$$
$$\alpha_4 = e^3 - e^2, \quad \alpha_5 = e^4 - e^3, \quad \alpha_6 = e^5 - e^4, \quad \alpha_7 = e^6 - e^5.$$

The Dynkin graph of E_7 is



Note that there are two different restrictions of root system E_7 along the root system A_5. The first one $(E_7, A_5)_1$ is obtained by taking subsystem the $(A_5)_1$ spanned by the simple roots α_1, α_3, α_4, α_5, α_6. The second one $(E_7, A_5)_2$ is obtained by taking subsystem the $(A_5)_2$ spanned by the simple roots α_2, α_4, α_5, α_6, α_7. Note also that the restricted system obtaining by taking the subsystem spanned by the simple roots α_1, α_2, α_4, α_5, α_6 is equivalent to $(E_7, A_5)_1$.

The following table gives all the planar restrictions of root system E_7.

Table A.3: Restricted systems of E_7 on the plane

E_7 root system			
Subsystem	Restricted system $\mathcal{A}$	Multiplicities	$\lvert\mathcal{A}\rvert$
D_5	$e^1, e^2, e^1+e^2, e^1-e^2$	$16, 16, 10, 1$	4
$(A_5)_1$	$e^1, 2e^1, e^2, e^1 \pm e^2$	$20, 1, 15, 6$	5
$(A_5)_2$	$e^1, e^2, \frac{1}{2}(e^1 \pm e^2), \frac{1}{2}(3e^1 \pm e^2)$	$15, 1, 15, 1$	6
$A_1 \times D_4$	$e^1, e^2, 2e^1, 2e^2, e^1 \pm e^2$	$16, 16, 1, 1, 8$	6
$A_1 \times A_4$	$e^1, e^2, \frac{1}{3}(e^1+e^2), \frac{2}{3}(e^1+e^2), \frac{1}{3}(2e^1-e^2)$, $\frac{1}{3}(e^1-2e^2)$	$2, 5, 20, 5, 10, 10$	6
$A_2 \times A_3$	$2e^1, 2e^2, 4e^2, e^1 \pm e^2, e^1 \pm 3e^2$	$1, 18, 3, 12, 4$	7
$A_1^2 \times A_3$	$e^1, 2e^1, e^2, 2e^2, e^1 \pm e^2, e^1 \pm 2e^2$	$12, 1, 16, 6, 8, 2$	8
$A_1 \times A_2^2$	$e^1, 2e^1, 3e^1, e^2, e^1 \pm e^2, 2e^1 \pm e^2$	$18, 9, 2, 9, 6, 3$	8
$A_1^3 \times A_2$	$2e^1, 2e^2, 4e^2, e^1 \pm e^2, 2(e^1 \pm e^2), e^1 \pm 3e^2$	$4, 12, 3, 12, 3, 4$	9

The following equivalences take place:

- Configuration $(E_7, A_5)_1$ belongs to the family of BC_2 configurations.
- Configuration $(E_7, A_5)_2$ is equivalent to the root system G_2 with special multiplicities.
- Configurations $(E_7, D_5), (E_7, A_1 \times D_4)$ belong to the family of BC_2 configurations.
- Configuration $(E_7, A_1 \times A_4)$ is equivalent to a configuration from the family given in Proposition 3.5.6 with special multiplicities.
- Configuration $(E_7, A_2 \times A_3)$ is equivalent to the configuration given in Proposition 3.5.7 with special multiplicities.
- Configuration $(E_7, A_1^2 \times A_3)$ belongs to the family of configurations given in Proposition 3.5.9 with special multiplicities.
- Configuration $(E_7, A_1^3 \times A_2)$ is equivalent to a configuration from the family given in Proposition 3.5.10 with special multiplicities.

A.4 Planar restrictions of root system E_6

Recall that $\mathcal{R} = E_6^+$ is the positive half of the root system E_6 with the multiplicity function $c = t \in \mathbb{C}$ given by

$$\begin{aligned} &e^i \pm e^j, \quad 1 \le i < j \le 5, \\ &\frac{1}{2}\Big(e^8 - e^7 - e^6 + \sum_{i=1}^{5} (-1)^{\nu(i)} e^i\Big), \text{with} \sum_{i=1}^{5} \nu(i) \text{ even.} \end{aligned}$$

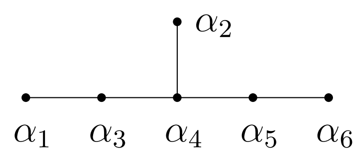
Recall also that in the corresponding solution (2.55) of WDVV equations (2.56) we have

$$\lambda = \lambda_{(E_6,t)} = 12\sqrt{2t}. \tag{A.5}$$

The basis of simple roots consists of

$$\begin{aligned} &\alpha_1 = \frac{1}{2}e^1 + e^8 - \frac{1}{2}(e^2 + e^3 + e^4 + e^5 + e^6 + e^7), \quad \alpha_2 = e^1 + e^2, \quad \alpha_3 = e^2 - e^1, \\ &\alpha_4 = e^3 - e^2, \quad \alpha_5 = e^4 - e^3, \quad \alpha_6 = e^5 - e^4. \end{aligned}$$

The Dynkin graph of E_6 is



The following table gives all the planar restrictions of root system E_7.

Table A.4: Restricted systems of E_6 on the plane

E_6 root system			
Subsystem	Restricted system $\mathcal{A}$	Multiplicities	$\lvert\mathcal{A}\rvert$
D_4	$2e^1, e^1 \pm e^2$	$8, 8$	3
A_4	$2e^1, 2e^2, e^1 \pm e^2$	$1, 5, 10$	4
$A_1 \times A_3$	$e^1, 2e^1, e^2, e^1 \pm e^2$	$12, 1, 8, 4$	5
$A_2 \times A_2$	$e^1, e^2, \frac{1}{2}(e^1 \pm e^2), \frac{1}{2}(3e^1 \pm e^2)$	$9, 1, 9, 1$	6
$A_1^2 \times A_2$	$e^1, e^2, \frac{1}{3}(e^1 + e^2), \frac{2}{3}(e^1 + e^2), \frac{1}{3}(2e^1 - e^2), \frac{1}{3}(e^1 - 2e^2)$	$2, 2, 12, 3, 6, 6$	6

The following equivalences take place:

- The restriction (E_6, D_4) is equivalent to the root system A_2.
- The restrictions $(E_6, A_4), (E_6, A_1 \times A_3)$ belong to the family of BC_2 type configurations.
- The restriction $(E_6, A_2 \times A_2)$ is equivalent to the root system G_2 with special multiplicities.
- The restriction $(E_6, A_1^2 \times A_2)$ is equivalent to a configuration from the family given in Proposition 3.5.6 with special multiplicities.

A.5 Summary

Let us summarize all the known trigonometric $\vee$-systems on the plane. In addition to configurations $BC_2(q, r, s; \underline{m})$, $A_2(t; \underline{m})$ and the root system G_2 we have the following configurations (with general multiplicities) coming from restrictions of the root systems:

Table A.5: Planar restrictions of root systems

$(\mathcal{R}, \mathcal{B})$	Restricted system $\mathcal{A}$	Multiplicities	λ	$\lvert\mathcal{A}\rvert$
(F_4, A_1^2)	$e^1, e^2, 2e^1, 2e^2,$ $e^1 \pm e^2, e^1 \pm 2e^2$	$2(p+2q), 4p, q,$ $p+2q, 2p, 2q$	$\frac{6\sqrt{3}(p+2q)}{\sqrt{p+4q}}$	8
(F_4, A_2)	$e^1, 2e^1, e^2, e^1 \pm e^2,$ $(3e^1 \pm e^2)/2, (e^1 \pm e^2)/2$	$p, q, p/3, q, p/3, p$	$\frac{6(p+2q)}{\sqrt{p+4q}}$	9
$(E_8, A_5 \times A_1)$	$e^1, 2e^1, 3e^1, e^2, 2e^2,$ $e^1 \pm e^2, 2e^1 \pm e^2$	$30t, 15t, 2t, 20t, t,$ $12t, 6t$	$30\sqrt{t}$	9
$(E_8, A_4 \times A_2)$	$e^1, 2e^1, 3e^1, e^2, (e^1 \pm e^2)/2,$ $(3e^1 \pm e^2)/2, (5e^1 \pm e^2)/2$	$30t, 15t, 5t, t, 15t,$ $10t, 3t$	$30\sqrt{t}$	10
(E_8, A_3^2)	$e^1, 2e^1, e^2, 2e^2, e^1 \pm e^2,$ $2e^1 \pm e^2, e^1 \pm 2e^2$	$24t, 6t, 24t, 6t, 16t,$ $4t, 4t$	$30\sqrt{t}$	10
$(E_8, A_4 \times A_1^2)$	$e^1, 2e^1, e^2, (e^1 + e^2)/3,$ $2(e^1 + e^2)/3, (2e^1 - e^2)/3,$ $(e^1 - 2e^2)/3, 2(2e^1 - e^2)/3,$ $(5e^1 + 2e^2)/3, (5e^1 - e^2)/3,$ $(4e^1 + e^2)/3$	$20t, 5t, 2t, 20t,$ $10t, 20t, 10t, 5t,$ $2t, 4t, 10t$	$30\sqrt{t}$	11
$(E_8, A_3 \times A_2 \times A_1)$	$e^1, 2e^1, 3e^1, 4e^1, e^2, 2e^2, e^1 \pm e^2,$ $2e^1 \pm e^2, 3e^1 \pm e^2$	$24t, 18t, 8t, 3t, 12t,$ $t, 12t, 6t, 4t$	$30\sqrt{t}$	12
$(E_8, A_2^2 \times A_1^2)$	$e^1, 2e^1, 3e^1, e^2, 2e^2, 3e^2, e^1 \pm e^2,$ $2(e^1 \pm e^2), 2e^1 \pm e^2, e^1 \pm 2e^2$	$18t, 9t, 2t, 18t, 9t,$ $2t, 12t, 3t, 6t, 6t$	$30\sqrt{t}$	14
$(E_7, A_2^2 \times A_1)$	$e^1, 2e^1, 3e^1, e^2, e^1 \pm e^2, 2e^1 \pm e^2$	$18t, 9t, 2t, 9t, 6t, 3t$	$9\sqrt{6t}$	8

In the next table we list all the known trigonometric $\vee$-systems which are not restrictions of root systems.

Table A.6: Non-Coxeter planar examples

$\mathcal{A}$	Multiplicities	Conditions	λ	$\lvert\mathcal{A}\rvert$
$e^1, 2e^1, e^2, e^1+e^2,$ $e^1-e^2, 2e^1+e^2$	$4a, a, 2a, 2a,$ $2(a-b), \frac{2ab}{4a-3b}$	$4a-3b\neq 0,$ $a(2a-b)\neq 0$	$\frac{6\sqrt{3}(2a-b)}{\sqrt{4a-3b}}$	6
$e^1, e^2, 2e^2, \frac{1}{2}(e^1\pm e^2),$ $\frac{1}{2}(e^1\pm 3e^2)$	$\frac{a(3a-2b)}{3a+4b}, 3a+2b,$ $b, 3a, a$	$3a+4b\neq 0$	$\frac{6(3a+2b)}{\sqrt{3a+4b)}}$	7
$e^1, 2e^1, 3e^1, e^2, 2e^2,$ $e^1\pm e^2, 2e^1\pm e^2$	$\frac{2b(3b+2a)}{b+2a}, \frac{b(3b+2a)}{b+2a}, \frac{2b(b-2a)}{3(b+2a)},$ $2a+3b, a, 2b, b$	$b+2a\neq 0,$ $b(3b+2a)\neq 0$	$\frac{3\sqrt{2}(3b+2a)}{\sqrt{b+2a}}$	9

Note that

- The first configuration $\mathcal{A}$ with 6 vectors can be obtained by restricting the configuration $\mathcal{A}_1$ along the plane $2x_1+x_2-x_3=0$, where $\mathcal{A}_1$ is the configuration given in Proposition 3.5.3.
- The second configuration $\mathcal{A}$ with 7 vectors can be obtained by restricting the configuration $\mathcal{A}_1$ along the plane $x_1=x_2$, where $\mathcal{A}_1$ is the configuration given in Proposition 3.5.3.
- The third configuration $\mathcal{A}$ in the special case $a=0$ reduces to the 8-vectors configuration $(E_7, A_2^2\times A_1)$.

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