Weighted Projective Planes and Threefold Singularities

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Abstract

This thesis studies weighted projective planes and their connection to threefold singularities. In particular, we study the Veronese subring $S^{\vec{x}}$ of the ring S associated with the weighted projective plane $\mathbb{X}$ for choices of $\vec{x}$ in the grading group $\mathbb{L}$. We show that there exists a projective, birational map $T^{\vec{x}} \to \operatorname{Spec} S^{\vec{x}}$ under mild restrictions on $\vec{x}$. We then show that when $\vec{x} = -\vec{\omega}$, the dualising element, this map is a blow-up. In the toric setting, we show that in certain situations the singularities of $S^{-\vec{\omega}}$ can be identified with the familiar cyclic quotient singularities and the map $T^{-\vec{\omega}} \to \operatorname{Spec} S^{-\vec{\omega}}$ is a weighted blow-up. In particular, it is a crepant map. We also construct a tilting object on $T^{-\vec{\omega}}$ in this setting. Away from the toric setting, we are able to construct tilting objects in some instances and we study some examples in depth to construct a full resolution and identify noncommutative resolutions of these singularities.

Contents

Abstract **i**

Acknowledgements **v**

Declaration **vii**

1 Introduction **1**

1.1 Weighted Projective Lines 1

1.2 A Generalisation of Weighted Projective Lines 3

1.2.1 The Toric Case 3

1.2.2 The Non-Toric Setting 4

1.3 The Main Results 5

1.3.1 The Morphism 5

1.3.2 It's a Blow-Up 6

1.3.3 Tilting and Full Resolutions 6

2 Preliminaries **9**

2.1 Noncommutative Crepant Resolutions 9

2.1.1 Cohen–Macaulay Modules 10

2.1.2 Crepant Resolutions 11

2.2 Geigle–Lenzing Weighted Projective Space 12

2.2.1 Geigle–Lenzing Complete Intersections 13

2.2.2 Weighted Projective Space 16

2.2.3 $\mathbb{L}$-Graded Cohen–Macaulay Modules 16

2.2.4 Noncommutative Crepant Resolutions of $S^{-\vec{\omega}}$ . . . 18

3 The Toric Case **20**

3.1 An Introduction to Toric Varieties . . . 20

3.2 Cyclic Quotient Singularities . . . 21

3.2.1 Hirzebruch–Jung Continued Fractions . . . 22

3.2.2 Three Dimensional Cyclic Quotient Singularities . . . 23

3.3 Toric Weighted Projective Planes . . . 26

3.4 Weighted Blow-up . . . 28

3.4.1 The Non-coprime Case . . . 28

4 A Partial Resolution of Singularities **31**

4.1 General Results about the Ring $S^{\vec{x}}$ . . . 31

4.2 The Total Space . . . 32

4.3 Constructing the Morphism . . . 34

4.4 $T^{\vec{x}}$ is a Blow-up of $S^{\vec{x}}$ . . . 47

5 Tilting **54**

5.1 Tilting for $\vec{x} \in \mathbb{L}_+$ . . . 54

5.2 Tilting with Four Weights . . . 57

5.3 Another Natural Bundle . . . 65

5.4 A Crepant Map . . . 66

6 Towards Full Resolutions **69**

6.1 A Resolution of $S^{-\vec{\omega}}$ when $\mathbf{p} = (2,2,2,2)$ . . . 69

6.2 The Family $(2,2,2,2n)$ . . . 76

6.3 When We Do Not Have Toda-Uehera Tilting . . . 78

A Blow-Ups and Local NCCRs **82**

A.1 A Blow-up . . . 85

A.2 Local NCCRs on $T^{-\vec{\omega}}$ . . . 87

List of Figures

3.1 The singularity of type $\frac{1}{31}(1, 11)$ 22

3.2 The resolution of $\frac{1}{31}(1, 11)$ 24

3.3 The fan of $\frac{1}{31}(15, 10, 6)$ after initial analysis 25

3.4 The fan of $\frac{1}{31}(15, 10, 6)$-Hilb $\mathbb{C}^3$ 26

3.5 Weighted blow-up of $\frac{1}{31}(15, 10, 6)$ 29

3.6 The weighted blow-up for $\mathbb{Z}_2 \times \mathbb{Z}_6$ 30

4.1 The subdivision of $\mathbb{C}[x_1, x_2, x_3]^G$ by the weighted blow-up at $\frac{1}{m}(a_1, a_2, a_3)$. 52

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Declaration

With the exception of Chapter 2, which contains introductory material, all work in this thesis was carried out by the author unless otherwise explicitly stated.

Chapter 1

Introduction

The study of algebraic surfaces has a long and rich history. Throughout the 19th century surfaces were extensively studied and their basic properties described with an early highlight being the discovery of the cubic and its 27 lines by Cayley and Salmon [Rei88]. A more precise, and powerful, method of studying surfaces was then developed by the Italian school of geometry. The works of the likes of Segre, M. Noether and Veronese paved the way, with Segre suggesting the best way to study surfaces was birationally. It was Castelnuovo, a student of Segre, together with Enriques, who in 1914 finally gave us the complete birational classification of algebraic surfaces [Gra99].

This thesis studies *weighted* projective surfaces, as defined by [HIMO]. These are a modern-day variant on the above, which enshrine projective spaces with an additional group structure. Whilst weighted projective lines are relatively well understood, the same cannot be said for their higher dimensional cousins. This thesis develops a body of results on the behaviour of weighted projective surfaces, and applies these results to 3-dimensional algebraic geometry.

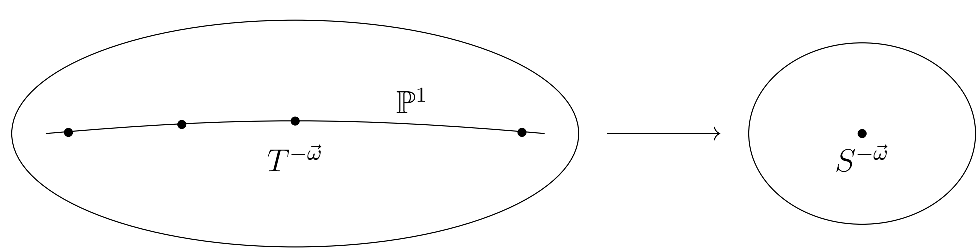
1.1 Weighted Projective Lines

In 1987 Geigle and Lenzing [GL87] introduced a new class of, what are now called, weighted projective lines which are encoded by a commutative ring with defining equations of the form $\pm x^a \pm y^b \pm z^c = 0$. These equations also have a long and remarkable history, going back to the work of Klein [Kle93] and Poincaré [Poi82]. Because of their long history, they also appear frequently in other areas of study such as representation theory, invariant theory, the McKay correspondence, singularities and more.

On one hand, weighted projective lines are interesting because they can be approached entirely using representation theory, in particular via the canonical algebra of Ringel [Rin84]. On the other hand, as we will see in this thesis, studying them from a geometric viewpoint produces fresh perspectives, and fruitful results. For example, the Serre functor [HIMO, Thm 3.4(b)] plays a crucial role, and its properties subdivide weighted projective lines into families, akin to standard geometric trichotomies of Fano, Calabi-Yau or anti-Fano.

In the simplest instances, known as the domestic case, weighted projective lines can be used to produce the familiar, and well understood, ADE-surface singularities. The standard Veronese subring trick of Lenzing is what moves from lines (dimension one) to surfaces (dimension two) [Len11, §1.4]. This elementary, but powerful, construction leads to the natural question: can more general singularities be produced and understood in this way?

Iyama and Wemyss [IW19] realised that the trick of Lenzing could be extended to more general surface singularities. Not only can more general singularities $S^{-\vec{\omega}}$ be produced, but under very mild assumptions, a partial resolution of singularites $T^{-\vec{\omega}} \to \operatorname{Spec} S^{-\vec{\omega}}$ can also be naturally constructed. The beauty of this method is that all the singularities of $T^{-\vec{\omega}}$ lie on a single copy of $\mathbb{P}^1$ inside the partial resolution, as illustrated in the following diagram.



(1.1.A)

Chapter 4 of this thesis extends this partial resolution to higher dimensions. In turn, this raises the question: what is the higher dimensional analogue of ADE-singularities?

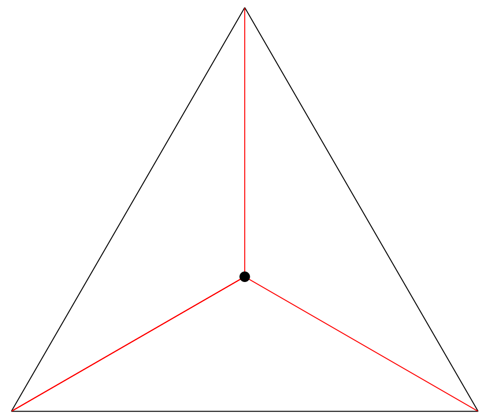
1.2 A Generalisation of Weighted Projective Lines

Much more recently, the classical construction of Geigle-Lenzing weighted projective lines was extended to dimension $d \geq 2$ by Herschend, Iyama, Minamoto and Opperman [HIMO]. The details of these weighted projective spaces, namely how they are defined by a ring S and grading group $\mathbb{L}$ with n weights $(p_1, \ldots, p_n)$, can be found in §2.2. As one would expect, increasing dimension brings added complications, subtleties and many interesting questions.

We first probe how the Iyama–Wemyss partial resolution $T^{-\vec{\omega}} \to \operatorname{Spec} S^{-\vec{\omega}}$ generalises as we raise the dimension. Conceptually, this is still constructed via the total space of the line bundle $\mathcal{O}(\vec{\omega})$, but now controlling its properties is much harder. For any fixed dimension d, the complexity largely goes with the number of weights (namely, n), and we outline our main results in this direction in the next section (§1.3). Before doing this, we briefly summarise two key cases.

1.2.1 The Toric Case

The easiest case is when $n = 3$, in which case the ring $S^{-\vec{\omega}}$ turns out to yield a threefold abelian quotient singularity (see §3.3). These are toric varieties, and so the methods of toric geometry apply. Remarkably, we are able to not only identify the toric polytope, we are also able to show that our partial resolution is, in fact, the Kawamata weighted blow-up. Pictorially, the weighted blow-up can be viewed as the following toric subdivision



where more details can be found in §4.4. The point is that, applied to the easiest case ($n = 3$), our construction recovers the Kawamata weighted blow-up, which is a fundamental construction in threefold birational geometry. The power of our method is that it also applies to many other cases, where toric methods do not apply.

1.2.2 The Non-Toric Setting

When the number of weights $n \geq 4$, the singularities $\operatorname{Spec} S^{-\vec{\omega}}$ are no longer toric, so we instead search for tilting objects to better understand the partial resolution. Tilting objects allow us to fully control the homological algebra of $\operatorname{Spec} S^{-\vec{\omega}}$. In particular, they make it possible to study the geometry of $T^{-\vec{\omega}}$ via the representation theory of noncommutative resolutions (and their variants). This information is crucial when constructing minimal models of the singularities $\operatorname{Spec} S^{-\vec{\omega}}$.

We will construct tilting objects on $T^{-\vec{\omega}}$ using the method of Toda-Uehara. In Chapter 5 we prove that the following list contains precisely all cases in which these tilting objects occur.

- $n = 4$ and $(p_1, p_2, p_3, p_4) = (2, 2, p_3, p_4)$.
- $n = 4$ and $(p_1, p_2, p_3, p_4) = (2, 3, p_3, p_4)$ where $p_3, p_4 \in \{3, 4, 5\}$.

The same computation in dimension two gives precisely the ADE classification. Therefore our result can be viewed as an ADE classification in dimension three. This suggests that there is a higher dimensional analogue of ADE singularities, from the viewpoint of weighted projective spaces.

1.3 The Main Results

We now briefly summarise the results of this thesis, in more technical detail. Given the weighted projective surface $\mathbb{X}$ and associated ring S from §2.2, consider the graded ring $S^{-\vec{\omega}} = \bigoplus_{i\in\mathbb{Z}} S_{-i\vec{\omega}}$.

1.3.1 The Morphism

The main results of this thesis are concerned with the 3-dimensional singularities $S^{-\vec{\omega}}$, although many of the results hold in arbitrary dimension. The following Theorem (shown in Chapter 4), which is one of the main results, constructs a partial resolution γ analogous to that of the one in [IW19] for these threefold singularities.

Theorem 1.3.1. (4.3.21) *If $(S,\mathbb{L})$ is Fano then the morphism $\gamma\colon T^{-\vec{\omega}} \to \operatorname{Spec} S^{-\vec{\omega}}$ is a projective, birational morphism.*

The Theorem is, in fact, much more general, but we restrict to the case $S^{-\vec{\omega}}$ for this introduction. We now highlight some of the keys steps in the proof of Theorem 1.3.1.

Using Čech cohomology, we prove the following strong vanishing results hold for weighted projective surfaces, namely

$$H^i(\mathcal{O}_{T^{-\vec{\omega}}}) = \begin{cases} \bigoplus_{j\geq 0} S_{-j\vec{\omega}} & i=0, \\ \bigoplus_{j\geq 0}(S_{\vec{\omega}+j\vec{\omega}})^* & i=2, \\ 0 & \text{else.} \end{cases}$$

These imply that $\gamma\colon T^{-\vec{\omega}} \to \operatorname{Spec} S^{-\vec{\omega}}$ satisfies $\mathbf{R}\gamma_*\mathcal{O}_{T^{-\vec{\omega}}} = \mathcal{O}_{S^{-\vec{\omega}}}$ provided $(S,\mathbb{L})$ is Fano. To prove that γ is a projective, birational morphism takes more work. In Lemma 4.3.15 we find a particular ample bundle $\mathcal{L}$ on $T^{-\vec{\omega}}$, and a similar Cech cohomology calculation gives

$$H^i(\mathcal{L}) = \begin{cases} \bigoplus_{j\geq 0} S_{j\vec{\omega}+\vec{c}} & i=0, \\ \bigoplus_{j\geq 0} (S_{\vec{\omega}-j\vec{\omega}-\vec{c}})^* & i=2, \\ 0 & \text{else.} \end{cases}$$

It follows that $\mathbf{R}^1\gamma_*\mathcal{L} = 0$. Although a technical result, this is what allows us to check that the right derived functor $\mathbf{R}^1\gamma_*$ preserves coherent sheaves, and thus that γ is proper. In fact, leveraging [Gro67, II.5.5.3] we are able to then prove Theorem 1.3.1.

1.3.2 It's a Blow-Up

As was mentioned earlier, in the toric setting we will prove that the map γ is a weighted blow up; these weighted blow-ups are recalled in §3.4. To achieve this we establish the following result which is shown to hold for any weighted projective space. The result is new even for weighted projective lines studied in [IW19].

Theorem 1.3.2. (4.4.1) *There is an isomorphism* $T^{-\vec{\omega}} \cong \operatorname{Proj}(\mathcal{I})$ *where*

$$\mathcal{I} = \mathcal{O}_{\operatorname{Spec} S^{-\vec{\omega}}} \oplus \bigoplus_{n\geq 1} \gamma_*\mathcal{L}^n.$$

Returning to the toric case, using the fact that $S^{-\vec{\omega}} \cong S^G$ for some cyclic group G (by 3.3.1), a comparison between the open charts of $\operatorname{Proj}(\mathcal{I})$ and the open charts of the weighted blow-up yields our next result.

Corollary 1.3.3. (4.4.4) *For pairwise coprime p_i, $T^{-\vec{\omega}}$ is a weighted blow-up of* $\operatorname{Spec} S^G$.

1.3.3 Tilting and Full Resolutions

Chapter 5 deals with the existence of tilting bundles on $T^{-\vec{\omega}}$. The first method of constructing a tilting bundle is via pulling back the well known tilting bundle $\mathcal{O}\oplus\mathcal{O}(1)\oplus\mathcal{O}(2)$ from $\mathbb{P}^2$ to a bundle $\mathcal{V}$ on $T^{-\vec{\omega}}$. In the toric setting ($n=3$) we show in Corollary 5.1.5 that this bundle is indeed tilting on $T^{-\vec{\omega}}$. However, outside of the toric case this does not hold.

Corollary 1.3.4. (5.2.1) *Let $p_i \geq 2$ for all i and $n \geq 4$. The pullback $\mathcal{V}$ is not a tilting object on $T^{-\vec{\omega}}$.*

In this case, we must work harder to find a tilting bundle. We next investigate the powerful method of Tuda–Uehara [TU10], and show that the only instances in which tilting bundles of this kind exist are the following.

Lemma 1.3.5. (5.2.2, 5.2.4) *When $n = 4$ there exists a tilting bundle constructed by the method of Toda–Uehara [TU10] if and only if the following conditions are satisfied:*

- $(p_1, p_2, p_3, p_4) = (2, 2, p_3, p_4)$.
- $(p_1, p_2, p_3, p_4) = (2, 3, p_3, p_4)$ *where* $p_3, p_4 \in \{3, 4, 5\}$.

This result is particularly interesting due to the fact that in [HIMO, Thm 3.64] a list is given for the existence of noncommutative crepant resolutions of $S^{-\vec{\omega}}$ and when $n = 4$ the above list is precisely six cases short of the list in [HIMO].

We remark that it is sometimes possible to produce tilting bundles on $T^{-\vec{\omega}}$ by pulling back other natural tilting bundles, such as $\mathcal{O} \oplus \Omega^1(1) \oplus \Omega^2(2)$, from $\mathbb{P}^2$. However, and interestingly, we are able to prove in Theorem 5.3.2 that this pullback produces a tilting bundle only when a Toda–Uehara tilting bundle exists. This exhausts all known methods of producing tilting objects on $T^{-\vec{\omega}}$.

We lastly focus on the problem of producing full resolutions of $\operatorname{Spec} S^{-\vec{\omega}}$. In the case of weighted projective lines, this is easy, since by (1.1.A) all singularities of $T^{-\vec{\omega}}$ are isolated and locally toric, so can be resolved by blowing up points. The higher dimensional case is *significantly* more complicated: the singularities of $T^{-\vec{\omega}}$ are no longer isolated, and birational models are not unique. We illustrate this in two cases, namely weights $(2, 2, 2, 2n)$ and $(3, 3, 3, 3)$, leaving details to Chapter 6.

In the case of $(2, 2, 2, 2)$ we are able to recognise the singularities on $T^{-\vec{\omega}}$ locally at each point of the singular locus. In this instance the singularities are locally $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ except at the three 'worst' points where the singularities are locally $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotients. Since each of these can be easily resolved locally, we can extend this result to construct a full resolution of $T^{-\vec{\omega}}$ as in Theorem 6.1.2.

However, the extension of this result even to the family $(2, 2, 2, 2n)$ for $n \in \mathbb{Z}_{\geq 0}$ is far from straightforward. We use the case $(2, 2, 2, 6)$ as an example (see §6.2). We again analyse the singularities on $T^{-\vec{\omega}}$ locally and in this instance, while we can recognise the singularities, the next 'best' steps towards a full resolution are not clear. Locally, the blow-up at points of these singularities are not all smooth and it is not obvious what should be blown first to construct a resolution.

The final example of interest is the case $\mathbf{p} = (3, 3, 3, 3)$. This case is particularly interesting due to the existence of an NCCR for $S^{-\vec{\omega}}$ [HIMO, Theorem 3.64] and yet, we have shown that we have exhausted all obvious methods of constructing a tilting bundle on $T^{-\vec{\omega}}$ in this instance. Similarly to the $(2, 2, 2, 2n)$ case, we can analyse the singularities locally on the open charts covering $T^{-\vec{\omega}}$ and recognise them as either $\mathbb{C}/\mathbb{Z}_3 \times \mathbb{C}$ or, at the three 'worst' points, $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotients. Again we are met with the problem of choice and cannot conclusively say one resolution is a better choice than another.

Chapter 2

Preliminaries

This chapter introduces some well-known results which will be important in the work of this thesis. Section 2.1 provides an overview of noncommutative crepant resolutions (NCCRs) for general Gorenstein rings, by first introducing Cohen–Macaulay modules and crepant resolutions, and then adapting them to the noncommutative setting. Then Section 2.2 introduces and illustrates the higher dimensional generalisation given in [HIMO] of Geigle–Lenzing weighted projective lines. Noncommutative resolutions are then translated into the language of weighted projective spaces in Section 2.2.4 and we give an overview of what is known about their existence in this context. Much is known about the existence of NCCRs for weighted projective lines, however much less is known for higher dimensions.

2.1 Noncommutative Crepant Resolutions

Traditionally, the geometry of a singular variety X is studied by resolving its singularities; that is, finding a non-singular variety Y equipped with a proper, birational morphism $Y \to X$ which is an isomorphism away from the singular locus. Noncommutative Crepant Resolutions (NCCRs) were first introduced by Van den Bergh [VdB04] in 2004, in his search for a noncommutative algebra from which the geometry of a singularity $X = \operatorname{Spec} R$ can be extracted.

2.1.1 Cohen–Macaulay Modules

Cohen–Macaulay modules are required for the purpose of introducing NCCRs, and so we recall the main concepts here. The material in this section, and a more in-depth study of Cohen–Macaulay rings, can be found in [BH93]. Details of their role in the setting of weighted projective space can be found later, in §2.2.3.

If R is a commutative Noetherian ring, then the category of finitely generated right R-modules will be denoted by $\operatorname{mod} R$. If $(R, \mathfrak{m})$ is further local, then the depth of a module $X \in \operatorname{mod} R$ is

$$\operatorname{depth} X = \min\{i \mid \operatorname{Ext}^i_R(R/\mathfrak{m}, X) \neq 0\},$$

and the objects in the full subcategory

$$\operatorname{CM} R = \{X \in \operatorname{mod} R \mid \operatorname{depth} X = \dim R\} \cup \{0\}$$

are called (maximal) Cohen–Macaulay R-modules. Furthermore, if R is Gorenstein then the above full subcategory can alternatively be defined as

$$\operatorname{CM} R = \{X \in \operatorname{mod} R \mid \operatorname{Ext}^i_R(X, R) = 0 \text{ for all } i > 0\}.$$

If R is a commutative Noetherian ring which is non-local, then $X \in \operatorname{mod} R$ is Cohen–Macaulay if, for all prime ideals $\mathfrak{p}$ in R, $X_{\mathfrak{p}} \in \operatorname{CM} R_{\mathfrak{p}}$. Further, the ring R is said to be Cohen–Macaulay if it is Cohen–Macaulay as an R-module.

In some instances below the stable category of Cohen–Macaulay modules, $\underline{\operatorname{CM}} R$, will be required. The objects are the same as in $\operatorname{CM} R$, but the morphism spaces are

$$\operatorname{Hom}_{\underline{\operatorname{CM}} R}(X, Y) = \operatorname{Hom}_R(X, Y)/\mathcal{P}(X, Y)$$

where $X, Y \in \operatorname{CM} R$ and $\mathcal{P}(X, Y)$ is the subspace of morphisms factoring through $\operatorname{proj} R$, the full subcategory of $\operatorname{mod} R$ containing the finitely generated projective right R-modules.

2.1.2 Crepant Resolutions

The algebraic varieties in this thesis will always be over $\mathbb{C}$, and can be assumed to be normal with only Cohen–Macaulay singularities. If X is such a variety, it automatically comes equipped with a canonical sheaf ω_X. A proper morphism $f\colon Y \to X = \operatorname{Spec} R$ is called crepant if $f^*\omega_X = \omega_Y$. The term crepant plays on the fact that the discrepancy is zero, and was first coined by Reid in [Rei83].

A scheme or variety X has rational singularities if there exists a proper, birational morphism $f\colon Y \to X$ with Y regular such that $\mathbf{R}f_*\mathcal{O}_Y = \mathcal{O}_X$. It is known that if a Gorenstein variety has a crepant resolution then the singularities are rational [KM98, 5.10]. However, the existence of rational singularities does not imply that there exists a crepant resolution, as the following example illustrates.

Example 2.1.1. Quotient singularities over $\mathbb{C}^d/G$ always have rational singularities [Vie77, Prop. 1]. When $G \subset \mathrm{SL}(d, \mathbb{C})$ and $d = 2$, these are precisely the Kleinian singularities, and a crepant resolution always exists. For $d = 3$, the existence of a crepant resolution was shown case-by-case in [IR96], [Ito95a], [Ito95b], [Mar97], [Roa94] and [Roa96], then spectacularly in all cases via derived category methods in [BKR01]. However, when $d \geq 4$ a crepant resolution does not necessarily exist. For example, the singularities $\mathbb{C}^4/G$, where $G \subset \mathrm{SL}(4, \mathbb{C})$ is generated by the diagonal matrix

$$g = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon^{r-1} & 0 & 0 \\ 0 & 0 & \varepsilon^{i} & 0 \\ 0 & 0 & 0 & \varepsilon^{r-i} \end{bmatrix}$$

and $i > 0$, $r \geq 2$ ε is an r-th primitive root of unity, are $\mathbb{Q}$-factorial terminal singularities and therefore do not admit a crepant resolution [Rei02, 5.4].

To introduce noncommutative crepant resolutions requires the notion of reflexive modules.

Definition 2.1.2. *Let R be a commutative ring, and $M \in \operatorname{mod} R$. Then M is called a* *reflexive module if the natural map* $M \to M^{**} = \operatorname{Hom}_R(\operatorname{Hom}_R(M,R),R)$ *is an isomorphism.*

Definition 2.1.3. [VdB04] *Let* R *be a normal Gorenstein domain. A noncommutative crepant resolution of* R *is an algebra* $\Lambda = \operatorname{End}_R(M)$*, where* M *is a reflexive module, such that* Λ *is a maximal Cohen-Macaulay* R*-module and* Λ *has finite global dimension.*

In analogy with the existence of crepant resolutions, Stafford and Van den Bergh [SVdB08] showed that the existence of an NCCR for a Gorenstein, normal, affine domain R implies that R has rational singularities. The term crepancy in NCCR refers to the property $\Lambda \in \operatorname{CM} R$, and is justified by the following results.

Theorem 2.1.4. [IW14b, 4.14] *Let* $f\colon X \to Y = \operatorname{Spec} R$ *be a projective, birational map between normal Gorenstein domains of dimension* d*. If* X *is derived equivalent to some ring* Λ*, then the morphism* f *is crepant if and only if* $\Lambda \in \operatorname{CM} R$*. In this case* $\Lambda \cong \operatorname{End}_R(M)$ *for some reflexive* R*-module* M*.*

Corollary 2.1.5. [IW14b, 4.15] *Consider* $f\colon X \to Y$ *under the same conditions as Theorem 2.1.4, and assume that* X *is smooth. If* X *is derived equivalent to some ring* Λ*, then* f *is crepant if and only if* Λ *is an NCCR of* R*.*

When $\dim Y \geq 4$, having a crepant resolution does not imply the existence of an NCCR as a derived equivalence between X and some Λ does not necessarily occur (see e.g. [IW14b, Theorem 4.20]). Likewise, the existence of an NCCR does not imply the existence of a crepant resolution, since the quotient singularity $\frac{1}{2}(1,1,1,1)$ admits an NCCR [VdB04, 1.1], however by Example 2.1.1, it does not admit a crepant resolution.

2.2 Geigle–Lenzing Weighted Projective Space

This section introduces Geigle–Lenzing weighted projective lines and also their extension to higher dimensions. Weighted projective lines were first introduced in [GL87], and the substance of this section is a summary of [HIMO].

2.2.1 Geigle–Lenzing Complete Intersections

The following definitions hold for an arbitrary field k. For any choice of positive integers $p_1, \dots, p_n$ and any choice of linear forms $\ell_1, \dots, \ell_n \in k[t_0, \dots, t_d]$ that define pairwise distinct points $\lambda_1, \dots, \lambda_n$ in $\mathbb{P}^d$, set $\mathbf{p} := (p_1, \dots, p_n)$, $\lambda := (\lambda_1, \dots, \lambda_n)$ and consider

$$S_{\mathbf{p},\lambda} = S := \frac{k[t_0, \dots, t_d, x_1, \dots, x_n]}{(x_i^{p_i} - \ell_i(t_0, \dots, t_d) \mid 1 \leq i \leq n)}.$$

The algebra S depends on $\mathbf{p}$ and λ as well as the linear forms ℓ_i but we usually suppress this from the notation. By construction, S is graded by the abelian group defined by generators and relations

$$\mathbb{L} = \langle \vec{x}_1, \dots, \vec{x}_n, \vec{c} \rangle / \langle p_i \vec{x}_i - \vec{c} \mid 1 \leq i \leq n \rangle$$

via $\deg x_i = \vec{x}_i$ and $\deg t_i = \vec{c}$. Observe that if $p_i = 1$, then $\vec{x}_i = \vec{c}$ in $\mathbb{L}$ and $x_i = \ell_i$ in S. Therefore x_i can be eliminated in the quotient, and so it makes sense to assume that all $p_i \geq 2$, which we do from now on.

By [HIMO, 2.1(a)], the elements of $\mathbb{L}$ can be written in normal form as follows

$$\vec{x} = \sum_{i=1}^{n} a_i \vec{x}_i + a\vec{c} \tag{2.2.A}$$

where $0 \leq a_i < p_i$ and $a \in \mathbb{Z}$. It will often be the case that the choice of $\vec{x}$ is restricted to the subset $\mathbb{L}_+$, defined as follows.

Definition 2.2.1. *$\vec{x} \in \mathbb{L}$ is in $\mathbb{L}_+$ if, when written in normal form (2.2.A), $a \geq 0$.*

Remark 2.2.2. For elements $\vec{x}, \vec{y} \in \mathbb{L}$, we will write $\vec{x} \leq \vec{y}$ if $0 \leq \vec{y} - \vec{x}$, or equivalently if $\vec{y} - \vec{x} \in \mathbb{L}_+$.

Remark 2.2.3. [HIMO, 2.1.(b)] The group $\mathbb{L}$ is a rank one abelian group which is torsion free if and only if $p_1, \dots, p_n$ are pairwise coprime.

The pair $(S, \mathbb{L})$ is called a Geigle-Lenzing complete intersection if the linear forms are in general position, that is, any set of at most $d + 1$ of the linear forms ℓ_i is linearly independent. In this case, as explained in [HIMO, Observation 2.3] it can be assumed that

$$\ell_i(t_0, \dots, t_d) = \begin{cases} t_{i-1} & 1 \leq i \leq \min\{d+1, n\}, \\ \sum_{j=0}^{d} \lambda_{ij} t_j & \min\{d+1, n\} < i \leq n. \end{cases}$$

We then obtain the relations $t_j = x_{j+1}^{p_{j+1}}$ and S has the form

$$S = \begin{cases} k[t_n, \dots, t_d, x_1, \dots, x_n] & 1 \leq n \leq d+1, \\ k[x_1, \dots, x_n]/(x_i^{p_i} - \sum_{j=0}^{d} \lambda_{ij} x_{j+1}^{p_{j+1}} \mid d+2 \leq i \leq n) & d+2 \leq n. \end{cases}$$

When $n = d+2$, after rescaling if necessary,

$$S \cong \frac{k[x_1, \dots, x_n]}{x_n^{p_n} - (x_1^{p_1} + x_2^{p_2} + \dots + x_{d+1}^{p_{d+1}})}.$$

Veronese subrings of S will have a key role in the work of this thesis. Given any $\vec{y} \in \mathbb{L}$, the corresponding Veronese subalgebra is defined to be

$$S^{\vec{y}} = \bigoplus_{i \in \mathbb{Z}} S_{i\vec{y}},$$

where $S_{i\vec{y}}$ is the $i\vec{y}$-graded piece of S. The following results, that describe the graded pieces of S, will be pivotal in understanding Veronese subrings.

Lemma 2.2.4. [HIMO, 2.5, 2.1(c)] *For $\vec{x} = \sum_{i=1}^{n} a_i \vec{x}_i + a\vec{c} \in \mathbb{L}_+$ the following hold*

1. $S_{\vec{x}} = (\prod_{i=1}^{n} x_i^{a_i}) S_{a\vec{c}}$,
2. $S_{\vec{x}+m\vec{c}} = S_{\vec{x}} \cdot S_{m\vec{c}}$ *for all* $m \geq 0$.
3. *The graded piece $S_{\vec{x}}$ is non-zero if and only if $\vec{x} \in \mathbb{L}_+$.*

Write $\operatorname{mod}^{\mathbb{L}} S$ for the category of $\mathbb{L}$-graded finitely generated S-modules. For the purpose of understanding certain calculations later, such as those involving Čech cohomology, we require the notion of regular sequences. This is a general concept, given here in the context of Geigle–Lenzing complete intersections.

Definition 2.2.5. [HIMO, Definition 2.6] *Let* $(a_1, \dots, a_l)$ *be a sequence of homogeneous elements in* S*, whose degrees are in* $\mathbb{L}_+ \setminus \{0\}$*. For* $M \in \operatorname{mod}^{\mathbb{L}} S$*,* $(a_1, \dots, a_l)$ *is an* M*-regular sequence if the multiplication map*

$$a_i \colon M/M(a_1, \dots, a_{i-1}) \to M/M(a_1, \dots, a_{i-1})$$

is injective for all $1 \leq i \leq l$*.*

The following regular sequence will be vital in Chapter 4.

Lemma 2.2.6. [HIMO, Lemma 2.7] *The sequence* $(t_0, \dots, t_d)$ *is an* S*-regular sequence.*

Proposition 2.2.7. [HIMO, Prop. 2.8(a)] *The ring* S *is a complete intersection and* $\dim S = d + 1$.

Equipped with this regular sequence the following implication is well known.

Proposition 2.2.8. [HIMO, Prop. 2.8] S *is a Gorenstein ring.*

It is then easy to extract the dualising element of $(S, \mathbb{L})$.

Definition 2.2.9. *The dualising element* $\vec{\omega}$ *of* $(S, \mathbb{L})$ *is defined to be*

$$\vec{\omega} = (n - d - 1)\vec{c} - \sum_{i=1}^{n} \vec{x}_i.$$

Geigle–Lenzing complete intersections are divided into three natural classes: Fano, anti-Fano and Calabi-Yau. For this, consider the unique homomorphism $\delta \colon \mathbb{L} \to \mathbb{Q}$ such that $\delta(\vec{x}_i) = \frac{1}{p_i}$ and $\delta(\vec{c}) = 1$.

Definition 2.2.10. *A Geigle–Lenzing complete intersection* $(S, \mathbb{L})$ *is Fano (respectively, Calabi-Yau, anti-Fano) when* $\delta(\vec{\omega}) < 0$ *(respectively,* $\delta(\vec{\omega}) = 0$*,* $\delta(\vec{\omega}) > 0$*).*

This trichotomy is motivated by the ampleness [HIMO, Cor. 4.13] of the automorphisms $(\vec{\omega})$ and $(-\vec{\omega})$ on the coherent sheaves of the stack $\mathbb{X}$ defined in the next subsection. In the case of weighted projective lines, $d = 1$, the classes Fano, Calabi-Yau and anti-Fano correspond to the domestic, tubular and wild representation types respectively [GL87, 5.4.1-5.4.3].

2.2.2 Weighted Projective Space

The d-dimensional weighted projective space associated to a Geigle–Lenzing complete intersection $(S, \mathbb{L})$ is defined as

$$\mathbb{X} := [(\operatorname{Spec} S \setminus 0)/ \operatorname{Spec} \mathbb{C}\mathbb{L}],$$

where $\mathbb{C}\mathbb{L}$ is the group algebra of $\mathbb{L}$ and 0 is the (maximal) augmentation ideal of S. The above definition is made using stack notation. However, the language is not so important in this instance, since the work of this thesis focuses entirely on coherent sheaves on $\mathbb{X}$, which are defined to be the quotient category

$$\operatorname{coh}\mathbb{X} = \operatorname{qgr}^{\mathbb{L}} S = \operatorname{mod}^{\mathbb{L}} S/\operatorname{mod}^{\mathbb{L}}_0 S,$$

where $\operatorname{mod}^{\mathbb{L}}_0 S$ is the full (Serre) subcategory of finite dimensional $\mathbb{L}$-graded S-modules.

The coarse moduli space of $\mathbb{X}$ is

$$X := (\operatorname{Spec} S \setminus 0)/ \operatorname{Spec} \mathbb{C}\mathbb{L}.$$

It is well known that $X \cong \mathbb{P}^d$. Indeed, for the case $d = 2$, it is clear that the open cover $U_0 \cup U_1 \cup U_2 = (\operatorname{Spec} S) \setminus \{0\}$, where $U_i = \operatorname{Spec} S_{t_i}$, induces an open cover $X = X_0 \cup X_1 \cup X_2$ with $X_i := \operatorname{Spec}(S_{t_i})_0$, where $(S_{t_i})_0$ is the degree zero part of S_{t_i}. It can be checked that $(S_{t_0})_0 = \mathbb{C}[t_1/t_0, t_2/t_0]$, and similarly for the other charts. From this, it is clear that $X \cong \mathbb{P}^2$. Similar things happen for $d \geq 2$.

2.2.3 $\mathbb{L}$-Graded Cohen–Macaulay Modules

This section considers Cohen–Macaulay modules, as defined in §2.1, in the context of Geigle–Lenzing complete intersections.

Let $(S, \mathbb{L})$ be a Geigle–Lenzing complete intersection The category of i-dimensional $\mathbb{L}$-graded Cohen–Macaulay modules is defined to be

$$\mathrm{CM}_i^{\mathbb{L}} S = \{X \in \mathrm{mod}^{\mathbb{L}} S \mid \mathrm{Ext}_S^j(X, S) = 0 \text{ for all } j \neq d+1-i\}.$$

The maximal $\mathbb{L}$-graded Cohen–Macaulay modules are the case $i = d+1$, and are denoted

$$\mathrm{CM}^{\mathbb{L}} S := \mathrm{CM}_{d+1}^{\mathbb{L}} S.$$

The stable category $\underline{\mathrm{CM}}^{\mathbb{L}} S$ of $\mathbb{L}$-graded CM-modules has the same objects as $\mathrm{CM}^{\mathbb{L}} S$, and the morphism spaces are

$$\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{L}} S}(X, Y) = \mathrm{Hom}_S^{\mathbb{L}}(X, Y)/\mathcal{P}(X, Y)$$

where $\mathcal{P}(X, Y)$ is the subspace of morphisms $X \to Y$ that factor through $\mathrm{proj}^{\mathbb{L}} S$.

Since the Veronese $S^{-\vec{\omega}}$, which is $\mathbb{Z}$-graded, is largely the focus of this thesis we will also require some of its basic properties. The following is an $\mathbb{L}$-graded version of Goto–Watanabe's results on Veronese subrings [GW78]. Assuming that $(S, \mathbb{L})$ is not Calabi-Yau (which is true of all examples in this thesis) we have the following results.

Theorem 2.2.11. [HIMO, 3.56, 3.57] *The ring $S^{-\vec{\omega}}$ is $\mathbb{Z}$-graded of dimension $d+1$.*

As such, the category of $\mathbb{Z}$-graded maximal CM modules is

$$\mathrm{CM}^{\mathbb{Z}} S^{-\vec{\omega}} := \{X \in \mathrm{mod}^{\mathbb{Z}} S^{-\vec{\omega}} \mid \mathrm{Ext}_{S^{-\vec{\omega}}}^i(X, S^{-\vec{\omega}}) = 0 \text{ for all } i \neq 0\}.$$

Remark 2.2.12. As observed in [HIMO, above 3.57], see also [BH93, Ex. 1.2.26], the functor

$$(-)^{-\vec{\omega}} : \mathrm{mod}^{\mathbb{L}} S \to \mathrm{mod}^{\mathbb{Z}} S^{-\vec{\omega}}$$
$$M \mapsto M^{-\vec{\omega}} = \bigoplus_{i \in \mathbb{Z}} M_{-i\vec{\omega}}$$

restricts to a functor $\mathrm{CM}^{\mathbb{L}} S \to \mathrm{CM}^{\mathbb{Z}} S^{-\vec{\omega}}$.

2.2.4 Noncommutative Crepant Resolutions of $S^{-\vec{\omega}}$

The general concept of a noncommutative crepant resolution was outlined in Section 2.1. This section contains a review of what is known about the existence of NCCRs for weighted projective spaces, their relationship to cluster tilting and other various properties.

Definition 2.2.13. *A full subcategory $\mathcal{C}$ of an abelian or exact category $\mathcal{A}$ is d-cluster tilting if it is functorially finite (see [HIMO, 1.12]) and furthermore*

$$\begin{aligned}\mathcal{C} &= \{a \in \mathcal{A} \mid \operatorname{Ext}^i(a, \mathcal{C}) = 0 \textit{ for } 0 < i < d\} \\ &= \{a \in \mathcal{A} \mid \operatorname{Ext}^i(\mathcal{C}, a) = 0 \textit{ for } 0 < i < d\}.\end{aligned}$$

It is known that d-cluster tilting subcategories are strongly related to d-tilting objects. Tilting objects will be seen in action in Chapter 5.

Definition 2.2.14. *An object U in $\underline{\mathrm{CM}}^{\mathbb{L}} S$ is tilting if $\operatorname{Hom}(U, U[i]) = 0$ for all $i \neq 0$ and $\underline{\mathrm{CM}}^{\mathbb{L}} S = \operatorname{thick}(U)$. A tilting object $U \in \underline{\mathrm{CM}}^{\mathbb{L}} S$ is d-tilting if $\underline{\mathrm{End}}^{\mathbb{L}}_S(U)$ has global dimension at most d.*

By [HIMO, 6.11] the latter condition is equivalent to the global dimension being precisely d. The link between d-tilting and d-cluster tilting is provided by d-Cohen-Macaulay finiteness.

Definition 2.2.15. [HIMO, 3.49] *A complete intersection $(S, \mathbb{L})$ is d-Cohen–Macaulay finite if there exists a d-cluster tilting subcategory of $\mathrm{CM}^{\mathbb{L}} S$ such that there are only finitely many isomorphism classes of indecomposable objects up to degree shift.*

Theorem 2.2.16. [HIMO, Theorem 3.53] *If $\underline{\mathrm{CM}}^{\mathbb{L}} S$ has a d-tilting object, then $(S, \mathbb{L})$ is d-CM finite.*

Furthermore, d-CM finiteness is linked to the existence of NCCRs of the Veronese subring by the following result.

Theorem 2.2.17. [HIMO, Theorem 3.59] *If $(S, \mathbb{L})$ is d-CM finite, then $S^{-\vec{\omega}}$ has an NCCR.*

These results lead to the following open question.

Question 2.2.18. How are the following conditions related to each other?

1. $(S, \mathbb{L})$ is Fano in the sense of 2.2.10.

2. $(S, \mathbb{L})$ is d-CM finite.

3. $S^{-\vec{\omega}}$ has an NCCR.

4. $\underline{\mathrm{CM}}^{\mathbb{L}} S$ has a d-tilting object.

The follow connections are known through the results referenced above and others found in [HIMO].

$$\text{NCCR} \Leftarrow d\text{-CM finite} \Leftarrow d\text{-tilting object} \Rightarrow \text{Fano}$$

When $d = 2$, the example in [HIMO, Example 3.40] demonstrates that the above implications are not all if and only if statements. It is possible for $(S, \mathbb{L})$ to be Fano without the existence of a 2-tilting object. The precise restrictions required to obtain if and only if statements are unknown. There are families which are known to have d-tilting objects listed in [HIMO, 3.64], but the exclusivity of this list is not proven.

For the remainder of this thesis we will assume that $d = 2$, unless stated otherwise.

Chapter 3

The Toric Case

This chapter considers the simplest class of weighted projective planes $(S, \mathbb{L})$, where $n = 3$ and p_1, p_2 and p_3 are pairwise coprime. We show that the associated 3-dimensional Veronese subrings $S^{-\vec{\omega}}$ are toric varieties. In fact, they are cyclic quotient singularities. In this instance we can use toric geometry to understand and resolve $\operatorname{Spec} S^{-\vec{\omega}}$.

Before this, a brief overview of toric varieties is given, which includes 2-dimensional examples, and we also describe two very different methods of constructing partial resolutions of 3-dimensional cyclic singularities.

3.1 An Introduction to Toric Varieties

A detailed introduction to toric varieties can be found in [Ful93]. Here we provide a brief summary of the results needed in order to understand toric weighted projective planes.

To define a toric variety requires a lattice $L \cong \mathbb{Z}^n$ and dual lattice $M = \operatorname{Hom}(L, \mathbb{Z})$. Let σ be a cone in L. Then the dual cone $\sigma^\vee$ is the set of vectors in $M \otimes \mathbb{R}$ which are nonnegative on σ. A cone σ in $L \otimes \mathbb{R}$ is said to be a strongly convex polyhedral cone if it has a vertex at the origin and is generated by finitely many vectors.

A fan Σ is a collection of strongly convex polyhedral cones in L satisfying the following conditions:

- every face of a cone in Σ is also a cone in Σ,
- the intersection of two cones in Σ is a face of each cone.

A toric variety is constructed from a lattice L and a fan Σ. First, consider the commutative semigroup

$$S_\sigma := \sigma^\vee \cap M = \{u \in M : (u, v) \geq 0 \text{ for all } v \in \sigma\},$$

with corresponding finitely generated commutative $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$. Then the affine toric variety U_σ is defined to be

$$U_\sigma := \mathrm{Spec}(\mathbb{C}[S_\sigma]).$$

The toric variety U_Σ associated to the fan $\Sigma = \{\sigma_i\}_{i \in I}$ is given by the disjoint union of varieties U_{σ_i} and gluing along the intersection $U_{\sigma_i \cap \sigma_j}$ for each $i, j \in I$.

Many interesting varieties are toric, and as we will see in the next section, this includes the well-known cyclic and abelian quotient singularities.

3.2 Cyclic Quotient Singularities

We observe that cyclic quotient singularities are toric varieties by following the set up and notation described in [Rei87]. Let $r > 1$ and choose integers $0 \leq a_i < r$ for $i = 1, \ldots, n$. Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be generated by the diagonal matrix

$$g = \begin{bmatrix} \varepsilon^{a_1} & 0 & \cdots & 0 \\ 0 & \varepsilon^{a_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon^{a_n} \end{bmatrix},$$

where ε is an r-th root of unity. Let $\bar{M} \cong \mathbb{Z}^n$ be the lattice of Laurent monomials in $x_1, \ldots, x_n$ and $\bar{L}$ the dual lattice with generators $e_1, \ldots, e_n$. Take the overlattice

$$L = \bar{L} + \mathbb{Z} \cdot \frac{1}{r}(a_1, \ldots, a_n),$$

where $r \in \mathbb{N}$ and $0 \leq a_i < r$ for each $i = 1, \dots, n$. Let $M \subset \bar{M}$ be the sublattice dual to L. Then a monomial $x^{\mathbf{m}} \in M$ if and only if it is invariant under the action of the group G. We assume that $\gcd(r, a_1, \dots, \hat{a}_j, \dots, a_n) = 1$, where a_j is removed for any choice of j. Then, for $\sigma = \langle e_1, \dots, e_n \rangle \in L_{\mathbb{R}}$,

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]) = \mathrm{Spec}\, \mathbb{C}[x_1, \dots, x_n]^G = \mathbb{C}^n/G,$$

is a cyclic quotient singularity of type $\frac{1}{r}(a_1, \dots, a_n)$. We define the unit box as

$$\square = \{(r_1, \dots, r_n) \in \mathbb{R}^n \mid 0 \leq r_i < 1\}.$$

Since every $g \in G$ has a representative in $\square$, when drawing the toric picture for σ we can simply focus on $L \cap \square$ (see Figure 3.1).

3.2.1 Hirzebruch–Jung Continued Fractions

Since cyclic quotient singularities are toric, this section unpacks how to draw these when $G \subset \mathrm{GL}(2, \mathbb{C})$, which can be reduced to studying the case $\frac{1}{r}(1, a)$ where r and a are coprime. The cone $\langle e_1, e_2 \rangle$ is then the positive quadrant, and this can be drawn easily. The cyclic quotient singularity of type $\frac{1}{31}(1, 11)$ will be the running example in this section and the unit box of the lattice $\mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{31}(1, 11)$ is drawn in Figure 3.1.

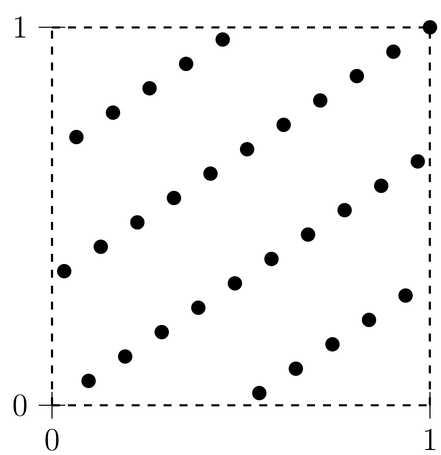


Figure 3.1: The singularity of type $\frac{1}{31}(1, 11)$

In order to calculate the resolution we need to look at the convex hull of the lattice. The points forming the convex hull of lattice points in Figure 3.1 are determined by expanding the fraction r/a as a Hirzebruch–Jung continued fraction, namely

$$\frac{r}{a} = \alpha_1 - \frac{1}{\alpha_2 - \frac{1}{\dots - \frac{1}{\alpha_m}}} = [\alpha_1, \dots, \alpha_m],$$

with all $\alpha_i \geq 2$. To calculate the points lying on the convex hull, set $v_0 = (0,1)$, $v_1 = \frac{1}{r}(1,a)$ and

$$v_{i+1} = \alpha_i v_i - v_{i-1}$$

for $i = 1, \dots, m$. The resolution is the subdivision of the quadrant generated by the rays $v_0, \dots, v_{m+1}$.

Returning to the running example, the Hirzebruch-Jung continued fraction for $31/11$ is

$$\frac{31}{11} = 3 - \frac{2}{11} = 3 - \frac{1}{6 - \frac{1}{2}} = [3, 6, 2].$$

Therefore the points generating the convex hull are $v_0 = (0,1)$, $v_1 = \frac{1}{31}(1,11)$ and

$$\begin{aligned} v_2 &= 3\frac{1}{31}(1,11) - (0,1) = \frac{1}{31}(3,2), \\ v_3 &= 6\frac{1}{31}(3,2) - \frac{1}{31}(1,11) = \frac{1}{31}(17,1), \\ v_4 &= 2\frac{1}{31} - \frac{1}{31}(3,2) = (1,0). \end{aligned}$$

Plotting the rays from the origin through these points gives the resolution depicted in Figure 3.2.

3.2.2 Three Dimensional Cyclic Quotient Singularities

The above method of resolving two dimensional cycle quotient singularities can be extended to three dimensions. It was first shown by Nakamura [Nak01] that a crepant resolution of all cyclic quotient singularities $\mathbb{C}^3/G$ exists, and can be realised as the G-Hilbert scheme G-Hilb $\mathbb{C}^3$. Craw and Reid [CR02] then constructed a visually pleasing

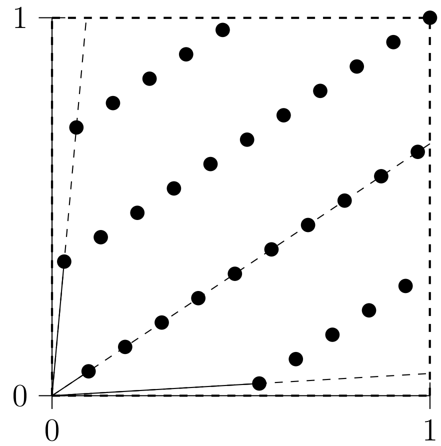


Figure 3.2: The resolution of $\frac{1}{31}(1,11)$

method, using toric geometry, to easily compute G-Hilb $\mathbb{C}^3$ for cyclic quotient singularities. In this section we give a brief overview of their construction.

Let $A \subset \mathrm{SL}(3,\mathbb{C})$ be the cyclic group generated by an element of the form $\frac{1}{r}(a_1,a_2,a_3)$, and define L as before. To study the singularities pictorially we take a slice Δ of the unit box in $\mathbb{R}^3$ where

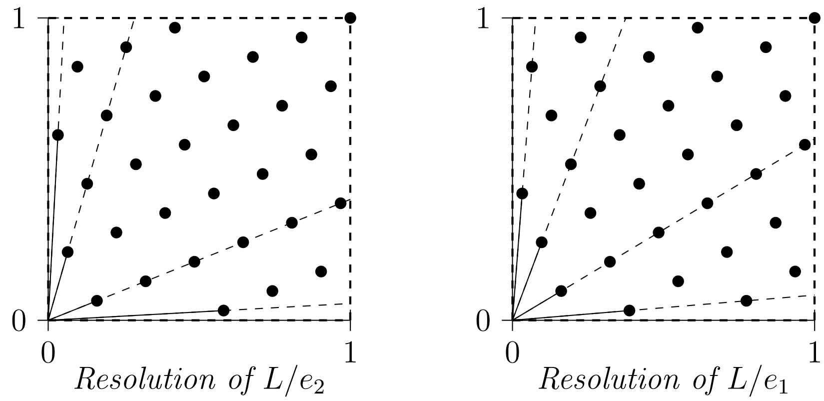
$$\Delta := \left\{(r_1,r_2,r_3) \in \square \mid \sum r_i = 1\right\}.$$

This contains all the lattice points corresponding to the generating elements of G since they take the form $\frac{1}{m}(a_1,a_2,a_3)$ where $a_1+a_2+a_3=m$. The slice Δ has three vertices $e_1=(1,0,0), e_2=(0,1,0)$ and $e_3=(0,0,1)$. When generating the toric picture of the resolution G-Hilb $\mathbb{C}^3$, we first think of it as three surface singularities taking a different e_i as the origin each time. We construct the Newton polygons of each $\Delta \setminus e_i$ as the convex hull as in §3.2.1 and concatenate these to form the triangle illustrated in Figure 3.3 (see [CR02, §2.5] for more details).

Consider the group $G = \frac{1}{31}(15,10,16)$. Then the three surface singularities to consider are the following, listed with their with Hirzebruch-Jung continued fractions.

$$\frac{1}{31}(15,10) \sim \frac{1}{31}(1,11) = [3,6,2],$$
$$\frac{1}{31}(15,6) \sim \frac{1}{31}(1,19) = [2,3,4,2],$$
$$\frac{1}{31}(10,6) \sim \frac{1}{31}(1,13) = [3,2,3,3].$$

The resolution of $L \setminus e_3$ has already been constructed in the previous section. Following the same procedure the resolutions for $L \setminus e_1$ and $L \setminus e_2$ are pictured below.



Concatinating the three resolutions gives the following picture

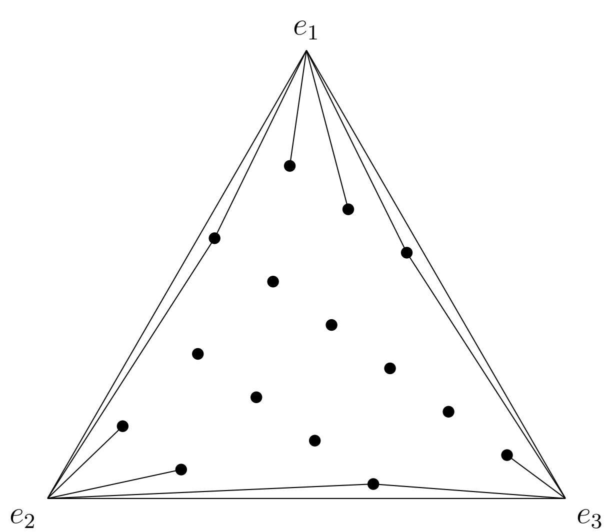


Figure 3.3: The fan of $\frac{1}{31}(15,10,6)$ after initial analysis

Continuing with Craw and Reid's method [CR02, §2] of constructing G-Hilb $\mathbb{C}^3$, step 2 gives the solid lines in Figure 3.4. Each of the rays from each corner is labelled with the associated integer α_i from the Hirzebruch-Jung continued fractions. These rays are extended, and when two rays meet, whichever has a higher number attached to it continues on, losing one in strength and the other ray stops. Step 3 produces the dotted lines, which are the regular tessellation of the remaining regular triangles. This results in the fan of G-Hilb $\mathbb{C}^3$.

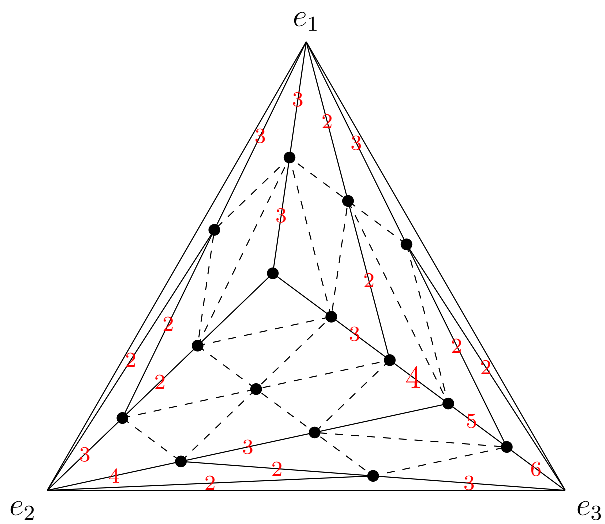


Figure 3.4: The fan of $\frac{1}{31}(15, 10, 6)$-Hilb $\mathbb{C}^3$

3.3 Toric Weighted Projective Planes

Recall, for any $(S, \mathbb{L})$ with $d = 2$, the Veronese subring $S^{-\vec{\omega}}$ from §2.2. We will show that under certain constraints ($n = 3$ and p_1, p_2, p_3 are pairwise coprime) this ring is a cyclic quotient singularity and we explicitly identify the group based on the numerical information contained within $\mathbb{L}$. We then use toric geometry to study Spec $S^{-\vec{\omega}}$.

Lemma 3.3.1. *Let* $\mathbf{p} = (p_1, p_2, p_3)$*, with* p_1, p_2, p_3 *pairwise coprime and* $-\vec{\omega} = \vec{x}_1 + \vec{x}_2 + \vec{x}_3$*. Then*

$$S^{-\vec{\omega}} \cong \mathbb{C}[x_1, x_2, x_3]^G,$$

where $G = \frac{1}{m}(a_1, a_2, a_3)$ *with* $m = p_2p_3 + p_1p_3 + p_1p_2$*,* $a_1 = p_2p_3$*,* $a_2 = p_1p_3$ *and* $a_3 = p_1p_2$*.*

Proof. Let φ be the canonical isomorphism $\varphi \colon S \to \mathbb{C}[y_1, y_2, y_3]$ given by $x_i \mapsto y_i$. We will show that this induces a map

$$\bar{\varphi}\colon S^{-\vec{\omega}} \to \mathbb{C}[y_1, y_2, y_3]^G,$$

and that it is an isomorphism. Let $x_1^{l_1}x_2^{l_2}x_3^{l_3} \in S^{-\vec{\omega}}$, and recall $-\vec{\omega} = \vec{x}_1 + \vec{x}_2 + \vec{x}_3$. Now there exists $l \in \mathbb{Z}$ such that

$$\begin{aligned}
& l_1\vec{x}_1 + l_2\vec{x}_2 + l_3\vec{x}_3 = l(\vec{x}_1 + \vec{x}_2 + \vec{x}_3) \\
\iff \quad & p_1p_2p_3(l_1\vec{x}_1 + l_2\vec{x}_2 + l_3\vec{x}_3) = p_1p_2p_3(l(\vec{x}_1 + \vec{x}_2 + \vec{x}_3)) \\
\iff \quad & (l_1p_2p_3 + l_2p_1p_3 + l_3p_1p_2)\vec{c} = l(p_2p_3 + p_1p_3 + p_1p_2)\vec{c}.
\end{aligned}$$

Since the p_i are pairwise coprime, Remark 2.2.3 implies that $\mathbb{L}$ is torsion free, and so the above is equivalent to the condition

$$(l_1p_2p_3 + l_2p_1p_3 + l_3p_1p_2) = l(p_2p_3 + p_1p_3 + p_1p_2).$$

The above shows that if $x_1^{l_1}x_2^{l_2}x_3^{l_3} \in S^{-\vec{\omega}}$ then $\varphi(x_1^{l_1}x_2^{l_2}x_3^{l_3}) \in \mathbb{C}[y_1, y_2, y_3]^G$, and so there is a well defined map

$$\bar{\varphi}\colon S^{-\vec{\omega}} \to \mathbb{C}[y_1, y_2, y_3]^G$$

which is automatically injective because it is the restriction of φ to $S^{-\vec{\omega}}$.

Let $y_1^{a_1}y_2^{a_2}y_3^{a_3} \in \mathbb{C}[x_1, x_2, x_3]^G$. Since

$$a_1p_2p_3 + a_2p_1p_3 + a_3p_1p_2 = m(p_2p_3 + p_1p_3 + p_1p_2),$$

for some $m \in \mathbb{Z}$, we have $x_1^{a_1}x_2^{a_2}x_3^{a_3} \in S^{-\vec{\omega}}$ and $\varphi(x_1^{a_1}x_2^{a_2}x_3^{a_3}) = y_1^{a_1}y_2^{a_2}y_3^{a_3}$. This implies that φ is a surjective map, and hence an isomorphism. $\square$

Remark 3.3.2. Conceptually, 3.3.1 holds because $\mathbb{L}$ is cyclic generated by $\frac{1}{p_1p_2p_3}\vec{c}$ and therefore $\mathbb{L}/\mathbb{Z}\vec{\omega} \simeq \mathbb{Z}/m\mathbb{Z}$.

Under the restriction of Lemma 3.3.1, the previous sections imply that G-Hilb $\mathbb{C}^3$ is a resolution of Spec $S^{-\vec{\omega}}$. In Chapter 4 a variety $T^{-\vec{\omega}}$ will be introduced in complete generality and it will be shown that $T^{-\vec{\omega}}$ is a partial resolution of Spec $S^{-\vec{\omega}}$. This result leads to the natural question: does the projective morphism G-Hilb $\mathbb{C}^3 \to$ Spec $S^{-\vec{\omega}}$ factor through $T^{-\vec{\omega}}$? Answering this will require weighted blow-ups, which are introduced next.

3.4 Weighted Blow-up

In this section we recall the weighted blow-up of [KM92, §10]. Let $X = \mathbb{C}^3/G$ with $G = \frac{1}{r}(a_1, a_2, a_3)$ where $r, a_1, a_2, a_3 > 0$. Recall the overlattice $L = \bar{L} + \mathbb{Z} \cdot e$ and cone $\sigma = \langle e_1, e_2, e_3 \rangle$, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ and $e = \frac{1}{r}(a_1, a_2, a_3)$, as defined in §3.2. The cone σ can be decomposed into the union of three polyhedral cones $\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3$, with $\sigma_i = \langle e_j, e_k, e \rangle$ where $j, k \neq i$. The weighted blow-up $\pi_\sigma : B_\sigma(X) \to X$ is a proper birational morphism corresponding to the cone decomposition of $C(X) = \mathbb{Q}_+ e_1 + \mathbb{Q}_+ e_2 + \mathbb{Q}_+ e_3$ by $C_i = \sum_{j \neq i} \mathbb{Q}_+ e_j + \mathbb{Q}_+ e$ for $i = 1, 2, 3$ and their intersections [KM92, 10.3]. The variety $B_\sigma(X)$ is covered by the open sets

$$U_i = \mathbb{C}^3/G_i, \tag{3.4.A}$$

where $G_1 = \frac{1}{a_1}(r, -a_2, -a_3)$, $G_2 = \frac{1}{a_2}(-a_1, r, -a_3)$ and $G_1 = \frac{1}{a_3}(-a_1, -a_2, r)$.

Referring back to the running example in this chapter, Figure 3.5 illustrates the weighted blow-up in the case $\frac{1}{31}(15, 10, 6)$. The construction described in this section will be used in §4.4 to prove that $T^{-\vec{\omega}}$ is a weighted blow-up of Spec $S^{-\vec{\omega}}$.

3.4.1 The Non-coprime Case

A final question to consider when $d = 2$ and $n = 3$ is whether there are (p_1, p_2, p_3) which are not pairwise coprime for which Spec $S^{-\vec{\omega}}$ is still toric. It is expected that Spec $S^{-\vec{\omega}}$ is still an abelian quotient singularity $\mathbb{C}^3/G$ with $G \cong \mathbb{Z}_a \times \mathbb{Z}_b$ where $ab = p_1p_2 + p_1p_3 + p_2p_3$. However, the action of the group is not obvious except in the smallest examples.

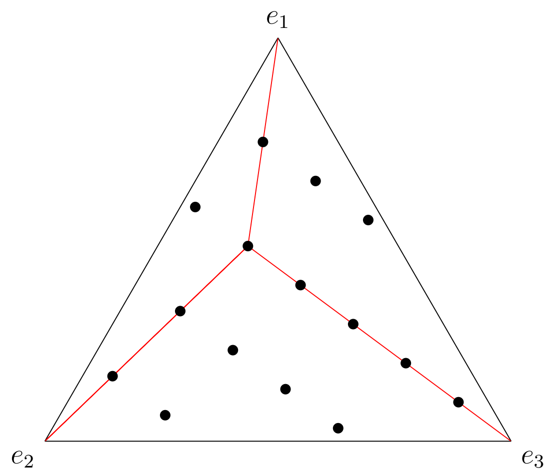


Figure 3.5: Weighted blow-up of $\frac{1}{31}(15, 10, 6)$

Example 3.4.1. Let $\mathbf{p} = (2, 2, 2)$. To understand this example we first identify the partial resolution of $\operatorname{Spec} S^{-\vec{\omega}}$ as a toric variety and, more specifically, as a weighted blow-up. Then it is easy to see what $\operatorname{Spec} S^{-\vec{\omega}}$ is. The open charts are:

$$\begin{aligned} V_0 &= \left(\mathbb{C}\left[x_1, x_2, x_3, x_1^{-1}, t\right]\right)_0 = \mathbb{C}\left[\frac{x_2^2}{x_1^2}, \frac{x_3^2}{x_1^2}, x_1x_2x_3t, x_1^6t^2\right], \\ V_1 &= \left(\mathbb{C}\left[x_1, x_2, x_3, x_2^{-1}, t\right]\right)_0 = \mathbb{C}\left[\frac{x_1^2}{x_2^2}, \frac{x_3^2}{x_2^2}, x_1x_2x_3t, x_2^6t^2\right], \\ V_2 &= \left(\mathbb{C}\left[x_1, x_2, x_3, x_3^{-1}, t\right]\right)_0 = \mathbb{C}\left[\frac{x_1^2}{x_3^2}, \frac{x_2^2}{x_3^2}, x_1x_2x_3t, x_3^6t^2\right]. \end{aligned}$$

Notice that

$$V_0 \cong V_1 \cong V_2 \cong \frac{\mathbb{C}[x, y, z, w]}{w^2 = xyz} \cong \mathbb{C}[X, Y, Z]^G$$

where $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ which is generated by

$$g = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This gives the presentation for the toric fan considered as a toric variety. It is possible, in this example, to deduce that $\operatorname{Spec} S^{-\vec{\omega}}$ is isomorphic to $\mathbb{C}^3/G$, where $G \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ is generated by

$$g' = \begin{bmatrix} \varepsilon^4 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} \text{ and } h' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

with ε a 6-th primitive root of unity. The fan in Figure 3.6 shows the weighted blow-up described as a toric variety.

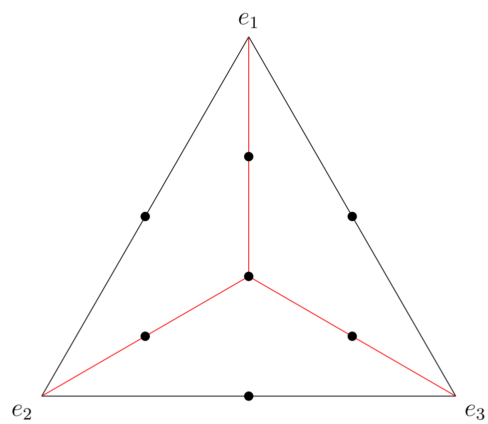


Figure 3.6: The weighted blow-up for $\mathbb{Z}_2 \times \mathbb{Z}_6$

Chapter 4

A Partial Resolution of Singularities

This section generalises results in [IW19, §3] in order to show that for certain choices of $\vec{x}$ and associated weighted projective planes, there exists a canonical partial resolution γ of the singularities of $\operatorname{Spec} S^{\vec{x}}$. This is constructed using the total space stack associated to a line bundle on the weighted projective plane. This results in the coarse moduli space of the total space giving a partial resolution of $\operatorname{Spec} S^{\vec{x}}$.

4.1 General Results about the Ring $S^{\vec{x}}$

Let $d = 2$ and $\mathbf{p} = (p_1, \dots, p_n)$ where each $p_i \geq 2$. Recall that for $\vec{x} \in \mathbb{L}$, $S^{\vec{x}} := \bigoplus_{i \in \mathbb{Z}} S_{i\vec{x}}$. The following properties of the rings $S^{\vec{x}}$ and $S^{\mathbb{N}\vec{x}} = \bigoplus_{i \in \mathbb{N}} S_{i\vec{x}}$ are needed to construct the morphism γ in §4.3.

Lemma 4.1.1. *For any $\vec{x} \in \mathbb{L}$,*

1. *If $\vec{x} \in \mathbb{L}$ is not torsion then $S^{\vec{x}}$ is a Noetherian $\mathbb{C}$-algebra with $\dim S^{\vec{x}} = 3$ and S is a finitely generated $S^{\vec{x}}$-module.*

2. *Make the ring $S[t]$ $\mathbb{L}$-graded with $\deg t = -\vec{x}$. Then the ring $(S[t])_0 \cong S^{\mathbb{N}\vec{x}}$ is a normal domain.*

3. *Suppose $-i\vec{x} \notin \mathbb{L}_+$ for all $i > 0$. Then $S^{\vec{x}} = S^{\mathbb{N}\vec{x}}$ is a Noetherian normal domain.*

Proof. 1. Recall that $\dim S = d+1$ by Proposition 2.2.7. Since $\vec{x}$ is not torsion and $\mathbb{L}$ has rank one, $\vec{x}\mathbb{Z} \subseteq \mathbb{L}$ has finite index. Therefore, S is a finitely generated $S^{\vec{x}}$-module and $\dim S^{\vec{x}} = \dim S$.

2. By definition, $(S[t])_0 = \bigoplus_{i \geq 0} S_{i\vec{x}} t^i \cong S^{\mathbb{N}\vec{x}}$. By [HIMO, Theorem 2.22] the ring S is an $\mathbb{L}$-factorial $\mathbb{L}$-domain (see [HIMO, Definition 2.19]) and $S[t]$ is an $\mathbb{L}$-factorial $\mathbb{L}$-domain by [IW19, Prop. 2.2(4)]. Therefore $S[t]$ is a normal domain and consequentially $(S[t])_0$ is also a normal domain by [IW19, Prop. 2.2(2)].

3. Since $-i\vec{x} \notin \mathbb{L}_+$ for any $i > 0$, $\vec{x}$ is not torsion by definition. Therefore, by Lemma 2.2.4(3), $S^{\vec{x}} = S^{\mathbb{N}\vec{x}}$ and the rest follows from (1) and (2).

$\square$

4.2 The Total Space

Weighted projective lines are intimately linked with two dimensional singularities. This was first realised by Geigle and Lenzing who produced the familiar ADE singularities via weighted projective lines [GL87]. Iyama and Wemyss [IW19] observed that this correspondence could be realised geometrically by using the total space of a line bundle on the weighted projective line. Therefore, if weighted projective lines are used to study surface singularities then we expect weighted projective surfaces to be useful in studying singular threefolds. In this chapter it is shown that the total space of a line bundle can also be used for weighted projective planes ($d = 2$). To construct the total space, choose $\vec{x} \in \mathbb{L}$ and let

$$\mathbb{T}^{\vec{x}} = \mathbb{T}\mathrm{ot}(\mathcal{O}(-\vec{x})) := [((\operatorname{Spec} S \setminus 0) \times \operatorname{Spec} \mathbb{C}[t]) / \operatorname{Spec} \mathbb{C}\mathbb{L}],$$

where t has weight $-\vec{x}$ in $\mathbb{L}$. Projection onto the first factor defines a morphism of stacks $q\colon \mathbb{T}^{\vec{x}} \to \mathbb{X}$. Denote the coarse moduli space of $\mathbb{T}^{\vec{x}}$ by $T^{\vec{x}}$ and the quotient map $\mathbb{T}^{\vec{x}} \to T^{\vec{x}}$

by g. The space $T^{\vec{x}}$ has an open cover

$$T^{\vec{x}} = V_0 \cup V_1 \cup V_2,$$

where $V_i := \operatorname{Spec}(S_{t_i}[t])_0$. Now, V_i maps to the plane $\operatorname{Spec}(S_{t_i})_0 = X_i \subset X \cong \mathbb{P}^2$ (the coarse moduli space of $\mathbb{X}$ §2.2.2), therefore there exists a map $p\colon T^{\vec{x}} \to X \cong \mathbb{P}^2$. This results in the following commutative diagram

$$\begin{tikzcd} \mathbb{T}^{\vec{x}} \arrow[r, "q"] \arrow[d, "g"'] & \mathbb{X} \arrow[d, "f"] \\ T^{\vec{x}} \arrow[r, "p"] & X \cong \mathbb{P}^2. \end{tikzcd} \tag{4.2.A}$$

The remainder of the chapter is devoted to constructing a projective, birational morphism $\gamma\colon T^{\vec{x}} \to \operatorname{Spec} S^{\vec{x}}$ and showing that $T^{\vec{x}}$ is a partial resolution of $\operatorname{Spec} S^{\vec{x}}$. The singularities on $T^{\vec{x}}$ are often milder. This motivates constructing a resolution of singularities for $T^{\vec{x}}$ to then obtain a full resolution of $\operatorname{Spec} S^{\vec{x}}$, by composition with γ.

To illustrate $T^{-\vec{\omega}}$ in an example, consider the toric case $d = 2$, $n = 3$ of §3, with $(p_1, p_2, p_3) = (2, 3, 5)$. Then, by the above, $T^{-\vec{\omega}}$ is covered by three open charts V_0, V_1, V_2 where $V_i = \operatorname{Spec} B_i$ and $B_i = (S_{t_i}[t])_0$. Now

$$B_2 = \mathbb{C}\left[\frac{x_1^2}{x_3^5}, \frac{x_2^3}{x_3^5}, x_1x_2x_3t, x_2^2x_3^7t^2, x_1x_3^{13}t^3, x_2x_3^{19}t^4, x_3^{31}t^6\right] \subseteq \mathbb{C}[x_1, x_2, x_3, x_3^{-1}, t].$$

It can be shown that $B_2 \cong \mathbb{C}[x, y, z]^{G_2}$ where $G_2 = \frac{1}{6}(3, 2, 1)$ and, similarly, it can be shown that $B_0 \cong \mathbb{C}[x, y, z]^{G_0}$ and $B_1 \cong \mathbb{C}[x, y, z]^{G_1}$ where $G_0 \cong \frac{1}{15}(1, 5, 9)$ and $G_1 \cong \frac{1}{10}(5, 1, 4)$. Then the subdivision of the slice Δ of the unit box centered on the point $\frac{1}{31}(15, 10, 6)$ gives the toric picture for the partial resolution $T^{-\vec{\omega}}$. Now, recall the weighted blow up of $\mathbb{C}^3/G$ where $G = \frac{1}{31}(15, 10, 6)$ from §3.4 and notice that the subdivision of $T^{-\vec{\omega}}$ is precisely the weighted blow up. We will prove that this holds in some generality in §4.4.

4.3 Constructing the Morphism

In order to construct γ, it is important to understand the global sections of certain line bundles on $T^{\vec{x}}$. This is done via Čech cohomology calculations.

We denote the twist of a module as follows.

Definition 4.3.1. *Let M be an $\mathbb{L}$-graded $S^{\vec{x}}$-module. The twist $M(\vec{y})$ of M by $\vec{y} \in \mathbb{L}$ is defined to be*

$$M(\vec{y}) = \bigoplus_{\vec{z} \in \mathbb{L}} M_{\vec{z}+\vec{y}}.$$

In particular, $S(\vec{y})^{\vec{x}} = (S^{\vec{x}})(\vec{y}) = \bigoplus_{i \in \mathbb{Z}} S_{i\vec{x}+\vec{y}}$.

Proposition 4.3.2. *For any $\vec{x} \in \mathbb{L}$,*

$$H^i(\mathcal{O}_{T^{\vec{x}}}) = \begin{cases} \bigoplus_{j \geq 0} S_{j\vec{x}} & i = 0, \\ \bigoplus_{j \geq 0} (S_{\vec{\omega}-j\vec{x}})^* & i = 2, \\ 0 & \textit{else}. \end{cases}$$

Therefore there is a canonical morphism $\gamma \colon T^{\vec{x}} \to \operatorname{Spec} S^{\mathbb{N}\vec{x}}$.

Proof. The Čech cohomology of $H^i(\mathcal{O}_{T^{\vec{x}}})$ is calculated via the open cover $\{V_0, V_1, V_2\}$. Now, $H^0(V_i, \mathcal{O}_{T^{\vec{x}}}) = (S_{t_i}[t])_0$ for $i = 0, 1, 2$; $H^0(V_i \cap V_j, \mathcal{O}_{T^{\vec{x}}}) = (S_{t_i t_j}[t])_0$ for $i \neq j$ and $i, j \in \{0, 1, 2\}$; and $H^0(V_0 \cap V_1 \cap V_2, \mathcal{O}_{T^{\vec{x}}}) = (S_{t_0 t_1 t_2}[t])_0$. Therefore the resulting complex

$$0 \to (S_{t_0}[t])_0 \oplus (S_{t_1}[t])_0 \oplus (S_{t_2}[t])_0 \xrightarrow{f} (S_{t_0 t_1}[t])_0 \oplus (S_{t_1 t_2}[t])_0 \oplus (S_{t_0 t_2}[t])_0 \xrightarrow{g} (S_{t_0 t_1 t_2}[t])_0 \to 0 \tag{4.3.A}$$

has cohomology $H^i(\mathcal{O}_{T^{\vec{x}}})$ with $i \geq 0$. Thus $H^i(\mathcal{O}_{T^{\vec{x}}}) = 0$ for any $i \geq 3$. Let $\mathfrak{a}$ be the ideal $St_0 + St_1 + St_2$ of S. Then the local cohomologies $H^i_{\mathfrak{a}}$ of S are the cohomologies of the Čech complex

$$0 \to S \to S_{t_0} \oplus S_{t_1} \oplus S_{t_2} \to S_{t_0 t_1} \oplus S_{t_1 t_2} \oplus S_{t_0 t_2} \to S_{t_0 t_1 t_2} \to 0 \tag{4.3.B}$$

by [BS13, Theorem 5.1.20]. Since t_0, t_1, t_2 is a S-sequence by Lemma 2.2.6, it follows that $H^0_{\mathfrak{a}}(S) = H^1_{\mathfrak{a}}(S) = H^2_{\mathfrak{a}}(S) = 0$ by [BS13, Theorem 6.2.7]. This results in the exact sequence

$$0 \to (S[t])_0 \to (S_{t_0}[t])_0 \oplus (S_{t_1}[t])_0 \oplus (S_{t_2}[t])_0 \xrightarrow{f} (S_{t_0t_1}[t])_0 \oplus (S_{t_1t_2}[t])_0 \oplus (S_{t_0t_2}[t])_0 \xrightarrow{g}$$
$$(S_{t_0t_1t_2}[t])_0 \to (H^3_{\mathfrak{a}}(S) \otimes_{\mathbb{C}} \mathbb{C}[t])_0 \to 0.$$

Therefore, by exactness, $H^1(\mathcal{O}_{T^{\vec{x}}}) = 0$ and comparing this with (4.3.A) shows that

$$H^0(\mathcal{O}_{T^{\vec{x}}}) \cong (S[t])_0 = \bigoplus_{j\geq 0} S_{j\vec{x}} \text{ and } H^2(\mathcal{O}_{T^{\vec{x}}}) \cong (H^3_{\mathfrak{a}}(S) \otimes_{\mathbb{C}} \mathbb{C}[t])_0 = \bigoplus_{j\geq 0} H^3_{\mathfrak{a}}(S)_{j\vec{x}}.$$

Now, $\mathfrak{m} = \sqrt{\mathfrak{a}}$ is a maximal ideal of S, and $S(\vec{\omega})$ is the $\mathbb{L}$-graded canonical module of S. Therefore, by $\mathbb{L}$-graded local duality [BS13, §14.4.1],

$$H^3_{\mathfrak{a}}(S)_{j\vec{x}} = H^3_{\mathfrak{m}}(S)_{j\vec{x}} \cong (\mathrm{Hom}_S(S, S(\vec{\omega}))_{-j\vec{x}})^* = (S_{\vec{\omega}-j\vec{x}})^*.$$

□

Classically, to contract the zero section of the total space of a line bundle, whilst maintaining projective birationality, you require negativity of the line bundle in $\mathbb{T}^{\vec{x}} = \mathbb{T}\mathrm{ot}(\mathcal{O}(-\vec{x}))$, so there must be some kind of positivity required for the choice of $\vec{x}$. Consider the following group homomorphism

$$\delta \colon \mathbb{L} \to \mathbb{Q}$$

defined by $\vec{c} \mapsto 1$ and $\vec{x}_i \mapsto \frac{1}{p_i}$. By considering the normal form on $\mathbb{L}$ (2.2.A) it is clear that $\delta(\mathbb{L}_+) \subset \mathbb{Q}_{\geq 0}$ and $\vec{x}$ is torsion if and only if $\delta(\vec{x}) = 0$. The element $\vec{z}$ in $\mathbb{L} \setminus \mathbb{L}_+$ is maximal if for any $\vec{y} \in \mathbb{L} \setminus \mathbb{L}_+$ we have $\vec{y} \leq \vec{z}$. Then it is clear that in normal form $\vec{z} = \sum_{i=1}^n (p_i - 1)\vec{x}_i - \vec{c} = \vec{\omega} + 2\vec{c}$. Using this, we deduce:

Lemma 4.3.3. *For $\vec{x} \in \mathbb{L}$ the following hold.*

1. *$-i\vec{x} \notin \mathbb{L}_+$ for all $i > 0$ if and only if $\delta(\vec{x}) > 0$.*

2. *$\vec{\omega} - i\vec{x} \notin \mathbb{L}_+$ for all $i \geq 0$ implies that $\delta(\vec{x}) \geq 0$.*

Proof. The proof is analogous to the result for weighted projective lines given in [IW19, Lemma 3.4]. $\square$

Then Lemma 4.3.3 motivates the following definition.

Definition 4.3.4. *The geometrically positive elements of* $\mathbb{L}$ *are*

$$\mathrm{GPos}(\mathbb{L}) := \{\vec{x} \in \mathbb{L} \mid \vec{x} \textit{ is not torsion, and } \vec{\omega} - j\vec{x} \notin \mathbb{L}_+ \textit{ for all } j \geq 0\}.$$

As in the case of weighted projective lines, the following result holds.

Proposition 4.3.5. *Consider* $\vec{x} \in \mathbb{L}$.

1. *If* $0 \neq \vec{x} \in \mathbb{L}_+$, *then* $\vec{x} \in \mathrm{GPos}(\mathbb{L})$.
2. *The following are equivalent:*
   (a) $\vec{x} \in \mathrm{GPos}(\mathbb{L})$,
   (b) $-i\vec{x} \notin \mathbb{L}_+$ *for all* $i > 0$, *and* $\vec{\omega} - j\vec{x} \notin \mathbb{L}_+$ *for all* $j \geq 0$,
   (c) $S^{\mathbb{N}\vec{x}} = S^{\vec{x}}$ *and* $\mathbf{R}^t\gamma_*\mathcal{O}_{T^{\vec{x}}} = 0$ *for all* $t > 0$.

Proof. The proof is analogous to the proof of [IW19, Prop. 3.6]. $\square$

Recall from Definition 2.2.10 that $(S, \mathbb{L})$ is Fano when $\delta(\vec{\omega}) < 0$.

Lemma 4.3.6. $-\vec{\omega} \in \mathrm{GPos}(\mathbb{L})$ *if and only if* $(S, \mathbb{L})$ *is Fano.*

Proof. By definition, if $-\vec{\omega} \in \mathrm{GPos}(\mathbb{L})$ then $\vec{\omega} + j\vec{\omega} \notin \mathbb{L}_+$ for all $j \geq 0$. Therefore $i\vec{\omega} \notin \mathbb{L}_+$ for all $i \geq 1$. By Lemma 4.3.3(1) this holds if and only if $\delta(-\vec{\omega}) > 0$. Since $(S, \mathbb{L})$ is Fano if and only if $\delta(-\vec{\omega}) > 0$, the result follows. $\square$

A full list of when $(S, \mathbb{L})$ is Fano for $d = 2$ can be found in [HIMO, Ex. 2.15].

Corollary 4.3.7. *For* $\vec{x} \in \mathrm{GPos}(\mathbb{L})$, *there exists a canonical morphism*

$$\gamma\colon T^{\vec{x}} \to \operatorname{Spec} S^{\vec{x}}$$

such that $\mathbf{R}\gamma_*\mathcal{O}_{T^{\vec{x}}} = \mathcal{O}_{S^{\vec{x}}}$.

It can be checked that the appropriate analogue of Proposition 4.3.2 holds for arbitrary d. Specifically,

Corollary 4.3.8. *For any d and any $\vec{x} \in \mathbb{L}$,*

$$H^i(\mathcal{O}_{T^{\vec{x}}}) = \begin{cases} \bigoplus_{j\geq 0} S_{j\vec{x}} & i = 0, \\ \bigoplus_{j\geq 0} (S_{\vec{\omega}-j\vec{x}})^* & i = d, \\ 0 & \textit{else.} \end{cases}$$

Therefore there is a canonical morphism $\gamma\colon T^{\vec{x}} \to \operatorname{Spec} S^{\mathbb{N}\vec{x}}$.

The goal of the remainder of this chapter is to show that γ is projective and birational. For this we require an ample bundle on $T^{\vec{x}}$.

Definition 4.3.9. *An invertible sheaf $\mathcal{G}$ on a Noetherian scheme X is ample if for every coherent sheaf $\mathcal{F}$ on X, there exists a positive integer n (depending on $\mathcal{F}$) such that for every $N \geq n$, the sheaf $\mathcal{F} \otimes \mathcal{G}^N$ is generated by its global sections.*

The following result about ampleness will be used throughout.

Proposition 4.3.10. [Har77, Prop. II.7.5] *Let $\mathcal{G}$ be an invertible sheaf on a Noetherian scheme X. Then the following are equivalent:*

1. *$\mathcal{G}$ is ample,*
2. *$\mathcal{G}^m$ is ample for all $m > 0$,*
3. *$\mathcal{G}^m$ is ample for some $m > 0$.*

Over the course of proving the main result it will be shown that γ is of finite type.

Definition 4.3.11. *A morphism of schemes $f\colon X \to Y$ is locally of finite type if there exists a covering of Y by affine open subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}(V_i)$ can be covered by affine open subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra. The morphism f is of finite type if in addition each $f^{-1}(V_i)$ can be covered by a finite number of U_{ij}.*

In order to show that γ is of finite type, one first establishes that γ is quasi-compact.

Definition 4.3.12. *A morphism $f\colon X \to Y$ is quasi-compact if $f^{-1}(V)$ is quasi-compact for every quasi-compact open $V \subseteq Y$.*

Then, by using the following result, f can be shown to be of finite type.

Proposition 4.3.13. [Gro67, I.6.6.3] *A morphism f is of finite type if f is locally of finite type and quasi-compact.*

Finally, recall the following definition.

Definition 4.3.14. *Let $f\colon X \to Y$ be a quasi-compact morphism, $\mathcal{G}$ an invertible $\mathcal{O}_X$-module. We say $\mathcal{G}$ is f-relatively ample if there exists an open affine cover (U_α) of Y such that for every α, $\mathcal{G}|_{f^{-1}(U_\alpha)}$ is ample.*

With these tools in hand, we can show that the first key result holds.

Proposition 4.3.15. *For $\vec{x} \in \mathrm{GPos}(\mathbb{L})$ the following hold.*

1. *γ is a finite type morphism between Noetherian schemes.*
2. *$\mathcal{L} := p^*\mathcal{O}(1)$ is an ample bundle on $T^{\vec{x}}$.*
3. *$\mathcal{L}$ is γ-relatively ample.*

Proof.

1. $T^{\vec{x}}$ is covered by three affine charts given by Noetherian rings. Therefore $T^{\vec{x}}$ is Noetherian. By Lemma 4.1.1(1), $S^{\vec{x}}$ is Noetherian and hence so too is $\operatorname{Spec} S^{\vec{x}}$. The morphism γ is quasi-compact since $T^{\vec{x}}$ is quasi-compact and $\operatorname{Spec} S^{\vec{x}}$ is affine. Let $s\colon \operatorname{Spec} S^{\vec{x}} \to \operatorname{Spec} \mathbb{C}$ be the structure morphism. Then the composition $s \circ \gamma$ is of finite type and hence, by [Har77, II.Ex.3.13(f)], γ is of finite type.

2. $\mathcal{O}(1)$ is ample on $\mathbb{P}^2$ and hence relatively ample with respect to the structure morphism $\mathbb{P}^2 \to \operatorname{Spec} \mathbb{C}$. The morphism p is affine, therefore the pullback $p^*\mathcal{O}(1)$ is ample relative to the composition $T^{\vec{x}} \to \mathbb{P} \to \operatorname{Spec} \mathbb{C}$ by [Gro67, Prop. II.5.1.12]. Since this is the structure morphism for $T^{\vec{x}}$, $p^*\mathcal{O}(1)$ is ample on $T^{\vec{x}}$.

3. Since Spec $S^{\vec{x}}$ is affine this follows from 2.

□

Using a similar method to the proof of Proposition 4.3.2, the cohomology of $\mathcal{L}$ can be calculated as follows.

Lemma 4.3.16. *For any $\vec{x} \in \mathbb{L}$,*

$$H^i(\mathcal{L}) = \begin{cases} \bigoplus_{j\geq 0} S_{j\vec{x}+\vec{c}} & i = 0, \\ \bigoplus_{j\geq 0} (S_{\vec{\omega}-j\vec{x}-\vec{c}})^* & i = 2, \\ 0 & \text{else.} \end{cases}$$

In particular, $\mathbf{R}^1\gamma_*\mathcal{L} = 0$.

Proof. Use Čech cohomology and proceed as in 4.3.2. Since $H^0(V_i, \mathcal{L}) = (S_{t_i}[t])_{\vec{c}}$ for $i = 0, 1, 2$, we have $H^0(V_i \cap V_j, \mathcal{L}) = (S_{t_i t_j}[t])_{\vec{c}}$ for $i \neq j$ and $i, j \in \{0, 1, 2\}$ and $H^0(V_0 \cap V_1 \cap V_2, \mathcal{L})$ equals $(S_{t_0 t_1 t_2}[t])_{\vec{c}}$. Then the complex

$$0 \to (S_{t_0}[t])_{\vec{c}} \oplus (S_{t_1}[t])_{\vec{c}} \oplus (S_{t_2}[t])_{\vec{c}} \xrightarrow{f} (S_{t_0 t_1}[t])_{\vec{c}} \oplus (S_{t_1 t_2}[t])_{\vec{c}} \oplus (S_{t_0 t_2}[t])_{\vec{c}} \xrightarrow{g} (S_{t_0 t_1 t_2}[t])_{\vec{c}} \to 0 \tag{4.3.C}$$

has cohomology $H^i(\mathcal{L})$ for $i \geq 0$. Thus $H^i(\mathcal{L}) = 0$ for any $i \geq 3$. Once again, let $\mathfrak{a} = St_0 + St_1 + St_2$. Then the local cohomologies $H^i_{\mathfrak{a}}$ of S are the cohomologies of the Čech complex (4.3.B) as in Proposition 4.3.2. Then we have the following exact sequence

$$0 \to (S[t])_{\vec{c}} \to (S_{t_0}[t])_{\vec{c}} \oplus (S_{t_1}[t])_{\vec{c}} \oplus (S_{t_2}[t])_{\vec{c}} \xrightarrow{f} (S_{t_0 t_1}[t])_{\vec{c}} \oplus (S_{t_1 t_2}[t])_{\vec{c}} \oplus (S_{t_0 t_2}[t])_{\vec{c}} \xrightarrow{g}$$
$$(S_{t_0 t_1 t_2}[t])_{\vec{c}} \to (H^3_{\mathfrak{a}}(S) \otimes_{\mathbb{C}} \mathbb{C}[t])_{\vec{c}} \to 0.$$

Therefore, by exactness, $H^1(\mathcal{L}) = \mathbf{R}^1\gamma_*\mathcal{L} = 0$ and by comparing this with (4.3.C) the following isomorphisms are obtained:

$$H^0(\mathcal{L}) \cong (S[t])_{\vec{c}} = \bigoplus_{j\geq 0} S_{j\vec{x}+\vec{c}} \text{ and } H^2(\mathcal{L}) \cong (H^3_{\mathfrak{a}}(S) \otimes_{\mathbb{C}} \mathbb{C}[t])_{\vec{c}} = \bigoplus_{j\geq 0} H^3_{\mathfrak{a}}(S)_{j\vec{x}+\vec{c}}.$$

Recall that $\sqrt{\mathfrak{a}} = \mathfrak{m}$ is a maximal ideal of S and $S(\vec{\omega})$ is the $\mathbb{L}$-graded canonical module of S. Therefore the $\mathbb{L}$-graded local duality theorem holds and

$$H^3_{\mathfrak{a}}(S)_{j\vec{x}+\vec{c}} = H^3_{\mathfrak{m}}(S)_{j\vec{x}+\vec{c}} \cong (\mathrm{Hom}_S(S, S(\vec{\omega}))_{-j\vec{x}-\vec{c}})^* = (S_{\vec{\omega}-j\vec{x}-\vec{c}})^*.$$

$\square$

Lemma 4.3.17. *Suppose* $\vec{x} \in \mathrm{GPos}(\mathbb{L})$. *Then*

1. $\gamma_* g_* q^* \mathcal{O}_{\mathbb{X}}(\vec{y}) = \bigoplus_{i\geq 0} S_{i\vec{x}+\vec{y}}$ *for all* $\vec{y} \in \mathbb{L}$.

2. $\gamma_* \mathcal{L}^n = \bigoplus_{i\geq 0} S_{i\vec{x}+n\vec{c}}$ *where* $n \in \mathbb{Z}$.

Proof. 1. We have that

$$\gamma_* g_* q^* \mathcal{O}_{\mathbb{X}}(\vec{y}) = \mathrm{H}^0(\mathbb{P}^2, p_* g_* q^* \mathcal{O}_{\mathbb{X}}(\vec{y})) = \mathrm{H}^0(\mathbb{P}^2, f_* q_* q^* \mathcal{O}_{\mathbb{X}}(\vec{y})).$$

Then by the projection formula (see [AU15, Theorem 1.2] and [IW19, Theorem 3.9(1)]) $q_* q^*(\mathcal{O}_{\mathbb{X}}(\vec{y})) = \bigoplus_{i\geq 0} \mathcal{O}_{\mathbb{X}}(i\vec{x}+\vec{y})$, and so the above equals

$$\bigoplus_{i\geq 0} \mathrm{H}^0(\mathbb{P}^2, f_* \mathcal{O}_{\mathbb{X}}(i\vec{x}+\vec{y})) = \bigoplus_{i\geq 0} \mathrm{H}^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(i\vec{x}+\vec{y})) = \bigoplus_{i\geq 0} S_{i\vec{x}+\vec{y}}.$$

2. Recall that $\mathcal{L} = p^* \mathcal{O}(1)$. By induction assume that $\mathcal{L}^n = p^* \mathcal{O}(n)$ for $n \geq 1$. Then

$$\begin{aligned} \mathcal{L}^{n+1} &= \mathcal{L} \otimes \mathcal{L}^n \\ &= p^* \mathcal{O}_X(1) \otimes p^* \mathcal{O}_X(n) \\ &= p^*(\mathcal{O}_X(1) \otimes \mathcal{O}_X(n)) \\ &= p^* \mathcal{O}_X(n+1). \end{aligned}$$

Hence for any $n > 0$, $\mathcal{L}^n = p^*\mathcal{O}_X(n)$. Now, consider the negative case

$$\begin{aligned}\mathcal{L}^{-1} &= \mathcal{H}om_{T^{\vec{x}}}(\mathcal{L}, \mathcal{O}_{T^{\vec{x}}})\\ &= \mathcal{H}om_{T^{\vec{x}}}(p^*\mathcal{O}_X(1), p^*\mathcal{O}_X)\\ &= p^*\,\mathcal{H}om_X(\mathcal{O}_X(1), \mathcal{O}_X)\\ &= p^*\mathcal{O}_X(-1).\end{aligned}$$

By induction, assume that $\mathcal{L}^{-n} = p^*\mathcal{O}_X(-n)$. Then

$$\begin{aligned}\mathcal{L}^{-(n+1)} &= \mathcal{H}om_{T^{\vec{x}}}(\mathcal{L} \otimes \mathcal{L}^n, \mathcal{O}_{T^{\vec{x}}})\\ &= \mathcal{H}om_{T^{\vec{x}}}(\mathcal{L}, \mathcal{H}om(\mathcal{L}^n, \mathcal{O}_{T^{\vec{x}}}))\\ &= \mathcal{H}om_{T^{\vec{x}}}(\mathcal{L}, \mathcal{L}^{-n})\\ &= \mathcal{H}om_{T^{\vec{x}}}(p^*\mathcal{O}_X(1), p^*\mathcal{O}_X(-n))\\ &= p^*\,\mathcal{H}om_X(\mathcal{O}_X(1), \mathcal{O}_X(-n))\\ &= p^*\mathcal{O}_X(-(n-1)).\end{aligned}$$

Thus for any $n > 0$, $\mathcal{L}^{-n} = p^*\mathcal{O}_X(-n)$. Then, by the projection formula $g_*g^*\mathcal{L} = \mathcal{L}$, we have that

$$\gamma_*\mathcal{L}^n = \gamma_*g_*g^*\mathcal{L}^n = \gamma_*g_*g^*p^*\mathcal{O}_{\mathbb{P}^2}(n) = \gamma_*g_*q^*f^*\mathcal{O}_{\mathbb{P}^2}(n) = \gamma_*g_*q^*\mathcal{O}_{\mathbb{X}}(n\vec{c}).$$

Then, part (1) implies that $\gamma_*\mathcal{L}^n = \bigoplus_{i\geq 0} S_{i\vec{x}+n\vec{c}}$ for all $n \in \mathbb{Z} \setminus \{0\}$. This is finitely generated over $S^{\vec{x}}$ by [IW19, Theorem 4.2]. Finally, the case $n = 0$ follows from the fact that $\mathbf{R}\gamma_*\mathcal{O}_{T^{\vec{x}}} = \mathcal{O}_{S^{\vec{x}}}$ by Corollary 4.3.7.

□

The key to showing that γ is proper is the fact that γ_* and $\mathbf{R}^1\gamma_*$ preserve coherent sheaves. The following lemmas will provide the tools needed to show this.

Lemma 4.3.18. *For all $n \geq 0$ and $\vec{x} \in \mathrm{GPos}(\mathbb{L})$, $\mathbf{R}^2\gamma_*\mathcal{L}^n = 0$.*

Proof. By definition, if $\vec{x} \in \mathrm{GPos}(\mathbb{L})$ then $S_{\vec{\omega}-j\vec{x}} = 0$ for all $j \geq 0$. Therefore it follows from Lemma 4.3.5(2c) that $\mathbf{R}^2\gamma_*\mathcal{O}_{T^{-\vec{\omega}}} = 0$. Pulling back via p the well-known Koszul complex [Bae88, Section 2.5] on $\mathbb{P}^2$ and twisting by $\mathcal{L}^{-1}$ gives the exact sequence

$$0 \to \mathcal{L}^{-2} \to (\mathcal{L}^{-1})^{\oplus 3} \to \mathcal{O}^{\oplus 3} \to \mathcal{L} \to 0. \tag{4.3.D}$$

Splicing provides the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus 3} \to \mathcal{L} \to 0.$$

This results in the following long exact sequence by the push forward of γ,

$$\cdots \to \mathbf{R}^2\gamma_*\mathcal{K} \to (\mathbf{R}^2\gamma_*\mathcal{O}_{T^{-\vec{\omega}}})^{\oplus 3} \to \mathbf{R}^2\gamma_*\mathcal{L} \to 0.$$

Since $\mathbf{R}^2\gamma_*\mathcal{O}_{T^{-\vec{\omega}}} = 0$, we deduce that $\mathbf{R}^2\gamma_*\mathcal{L} = 0$. By induction, assume that for $n-1$ the sheaf $\mathbf{R}^2\gamma_*\mathcal{L}^{n-1}$ equals 0. Twisting (4.3.D) produces the long exact sequence

$$\cdots \to \mathbf{R}^2\gamma_*\mathcal{K}' \to \mathbf{R}^2\gamma_*\mathcal{L}^{n-1} \to \mathbf{R}^2\gamma_*\mathcal{L}^{n} \to 0.$$

The result then follows since $\mathbf{R}^2\gamma_*\mathcal{L}^{n-1} = 0$ gives $\mathbf{R}^2\gamma_*\mathcal{L}^{n} = 0$.

□

Now we consider negative powers of $\mathcal{L}$.

Lemma 4.3.19. *For $\vec{x} \in \mathrm{GPos}(\mathbb{L})$, $\mathbf{R}^1\gamma_*\mathcal{L}^{-n}$ and $\mathbf{R}^2\gamma_*\mathcal{L}^{-n}$ are finitely generated $S^{\vec{x}}$-modules for all $n \geq 0$.*

Proof. Corollary 4.3.7 shows that $\mathbf{R}\gamma_*\mathcal{O}_{T^{\vec{x}}} = \mathcal{O}_{S^{\vec{x}}}$. Therefore $\mathbf{R}^1\gamma_*\mathcal{L}^{-n}$ and $\mathbf{R}^2\gamma_*\mathcal{L}^{-n}$ are finitely generated $S^{\vec{x}}$-modules if $n = 0$. Twisting (4.3.D) by $\mathcal{L}$ gives an exact sequence

$$0 \to \mathcal{L}^{-1} \to \mathcal{O}^{\oplus 3} \to \mathcal{L}^{\oplus 3} \to \mathcal{L}^{2} \to 0.$$

Splicing the sequence produces the following short exact sequences

$$0 \to \mathcal{L}^{-1} \to \mathcal{O}^{\oplus 3} \to \mathcal{K}_1 \to 0 \tag{4.3.E}$$

and

$$0 \to \mathcal{K}_1 \to \mathcal{L}^{\oplus 3} \to \mathcal{L}^2 \to 0. \tag{4.3.F}$$

Then pushing down (4.3.E) gives the exact sequence

$$0 \to \gamma_* \mathcal{L}^{-1} \to (S^{\vec{x}})^{\oplus 3} \to \gamma_*(\mathcal{K}_1) \to \mathbf{R}^1\gamma_*\mathcal{L}^{-1} \to 0 \to$$

$$\mathbf{R}^1\gamma_*\mathcal{K}_1 \to \mathbf{R}^2\gamma_*\mathcal{L}^{-1} \to 0 \to \mathbf{R}^2\gamma_*\mathcal{K}_1 \to 0.$$

Therefore $\mathbf{R}^2\gamma_*\mathcal{K}_1 = 0$ and $\mathbf{R}^1\gamma_*\mathcal{K}_1 \cong \mathbf{R}^2\gamma_*\mathcal{L}^{-1}$. Now, push forward (4.3.F) and use Lemma 4.3.16 to obtain the exact sequence

$$0 \to \gamma_*\mathcal{K}_1 \to \gamma_*\mathcal{L}^{\oplus 3} \to \gamma_*\mathcal{L}^2 \to \mathbf{R}^1\gamma_*\mathcal{K}_1 \to 0.$$

Since both $\gamma_*\mathcal{L}^{\oplus 3}$ and $\gamma_*\mathcal{L}^2$ are finitely generated by Lemma 4.3.17, it follows that both $\gamma_*\mathcal{K}_1$ and $\mathbf{R}^1\gamma_*\mathcal{K}_1 \cong \mathbf{R}^2\gamma_*\mathcal{L}^{-1}$ are finitely generated. Thus, in the first sequence the first three terms are finitely generated and therefore $\mathbf{R}^1\gamma_*\mathcal{L}^{-1}$ is also finitely generated. Thus, both $\mathbf{R}^1\gamma_*\mathcal{L}^{-1}$ and $\mathbf{R}^2\gamma_*\mathcal{L}^{-1}$ are finitely generated. Twisting (4.3.E) and (4.3.F) gives the following short exact sequences

$$0 \to \mathcal{L}^{-2} \to (\mathcal{L}^{-1})^{\oplus 3} \to \mathcal{K}_2 \to 0 \tag{4.3.G}$$

and

$$0 \to \mathcal{K}_2 \to \mathcal{O}^{\oplus 3} \to \mathcal{L} \to 0. \tag{4.3.H}$$

Pushing forward (4.3.G) gives

$$0 \to \gamma_*\mathcal{L}^{-2} \to \gamma_*(\mathcal{L}^{-1})^{\oplus 3} \to \gamma_*\mathcal{K}_2 \to \mathbf{R}^1\gamma_*\mathcal{L}^{-2} \to \mathbf{R}^1\gamma_*(\mathcal{L}^{-1})^{\oplus 3} \to$$

$$\mathbf{R}^1\gamma_*\mathcal{K}_2 \to \mathbf{R}^2\gamma_*\mathcal{L}^{-2} \to \mathbf{R}^2\gamma_*(\mathcal{L}^{-1})^{\oplus 3} \to \mathbf{R}^2\gamma_*\mathcal{K}_2 \to 0$$

and pushing forward (4.3.H) gives

$$0 \to \gamma_*\mathcal{K}_2 \to (S^{\vec{x}})^{\oplus 3} \to \gamma_*\mathcal{L} \to \mathbf{R}^1\gamma_*\mathcal{K}_2 \to 0.$$

The second sequence shows that $\gamma_*\mathcal{K}_2$ and $\mathbf{R}^1\gamma_*\mathcal{K}_2$ are finitely generated. Thus, in the first sequence all but possibly the fourth and seventh term are finitely generated, hence $\mathbf{R}^1\gamma_*\mathcal{L}^{-2}$ and $\mathbf{R}^2\gamma_*\mathcal{L}^{-2}$ are also finitely generated. Now, proceeding by induction, we assume that $\mathbf{R}^1\gamma_*\mathcal{L}^{-i}$ and $\mathbf{R}^2\gamma_*\mathcal{L}^{-i}$ are finitely generated for $i = n-1, n-2, n-3$. Then consider the twisted sequence

$$0 \to \mathcal{L}^{-n} \to (\mathcal{L}^{-(n-1)})^{\oplus 3} \to (\mathcal{L}^{-(n-2)})^{\oplus 3} \to \mathcal{L}^{-(n-3)} \to 0.$$

Splicing gives

$$0 \to \mathcal{L}^{-n} \to (\mathcal{L}^{-(n-1)})^{\oplus 3} \to \mathcal{K}_3 \to 0 \tag{4.3.I}$$

and

$$0 \to \mathcal{K}_3 \to (\mathcal{L}^{-(n-2)})^{\oplus 3} \to \mathcal{L}^{-(n-3)} \to 0. \tag{4.3.J}$$

The pushforward of (4.3.I) is

$$0 \to \gamma_*\mathcal{L}^{-n} \to (\gamma_*\mathcal{L}^{-(n-1)})^{\oplus 3} \to \gamma_*\mathcal{K}_3 \to \mathbf{R}^1\gamma_*\mathcal{L}^{-n} \to (\mathbf{R}^1\gamma_*\mathcal{L}^{-(n-1)})^{\oplus 3} \to$$

$$\mathbf{R}^1\gamma_*\mathcal{K}_3 \to \mathbf{R}^2\gamma_*\mathcal{L}^{-n} \to (\mathbf{R}^2\gamma_*\mathcal{L}^{-(n-1)})^{\oplus 3} \to \mathbf{R}^2\gamma_*\mathcal{K}_3 \to 0$$

and the pushforward of (4.3.J) is

$$0 \to \gamma_*\mathcal{K}_3 \to (\gamma_*\mathcal{L}^{-(n-2)})^{\oplus 3} \to \gamma_*\mathcal{L}^{-(n-3)} \to \mathbf{R}^1\gamma_*\mathcal{K}_3 \to (\mathbf{R}^1\gamma_*\mathcal{L}^{-(n-2)})^{\oplus 3} \to$$

$$\mathbf{R}^1\gamma_*\mathcal{L}^{-(n-3)} \to \mathbf{R}^2\gamma_*\mathcal{K}_3 \to (\mathbf{R}^2\gamma_*\mathcal{L}^{-(n-2)})^{\oplus 3} \to \mathbf{R}^2\gamma_*\mathcal{L}^{-(n-3)} \to 0.$$

The second sequence shows that $\gamma_*\mathcal{K}_3$ and $\mathbf{R}^1\gamma_*\mathcal{K}_3$ are finitely generated. In the first sequence the first, second, third, fifth, sixth and eighth are finitely generated. Thus, $\mathbf{R}^1\gamma_*\mathcal{L}^{-n}$ and $\mathbf{R}^2\gamma_*\mathcal{L}^{-n}$ are also finitely generated. □

Finally, equipped with the finite generation properties shown above it can be determined that γ is projective.

Theorem 4.3.20. *For $\vec{x} \in \mathrm{GPos}(\mathbb{L})$, $\gamma\colon T^{\vec{x}} \to \operatorname{Spec} S^{\vec{x}}$ is a projective morphism.*

Proof. If γ_* and $\mathbf{R}^1\gamma_*$ preserve coherent sheaves, then γ is proper [Ryd]. Let $\mathcal{F} \in \operatorname{coh} T^{\vec{x}}$. Since $\mathcal{L}$ is ample, by Proposition 4.3.15(2), there exists some $n \geq 0$ such that $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections. Then for some $N > 0$ there exists a surjection $\mathcal{O}^{\oplus N} \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^n$ and hence a surjection $(\mathcal{L}^{-n})^{\oplus N} \twoheadrightarrow \mathcal{F}$. Pushing down gives the following exact sequence, where $\mathcal{K}$ is the kernel,

$$0 \to \gamma_*\mathcal{K} \to \gamma_*(\mathcal{L}^{-n})^{\oplus N} \to \gamma_*\mathcal{F} \to \mathbf{R}^1\gamma_*\mathcal{K} \to \mathbf{R}^1\gamma_*(\mathcal{L}^{-n})^{\oplus N}$$

$$\to \mathbf{R}^1\gamma_*\mathcal{F} \to \mathbf{R}^2\gamma_*\mathcal{K} \to \mathbf{R}^2\gamma_*(\mathcal{L}^{-n})^{\oplus N} \to \mathbf{R}^2\gamma_*\mathcal{F} \to 0.$$

Since $\gamma_*\mathcal{L}^{-n}$, $\mathbf{R}^1\gamma_*\mathcal{L}^{-n}$ and $\mathbf{R}^2\gamma_*\mathcal{L}^{-n}$ are finitely generated by Lemmas 4.3.17(2) and 4.3.19, they are also coherent by [Har77, II.5.4]. Then, from exactness, $\mathbf{R}^2\gamma_*\mathcal{F}$ must be coherent and therefore $\mathbf{R}^2\gamma_*\mathcal{K}$ is also coherent since $\mathcal{F}$ is arbitrary. Now, $\mathbf{R}^1\gamma_*(\mathcal{L}^{-n})^{\oplus N}$ is also coherent and therefore $\mathbf{R}^1\gamma_*\mathcal{F}$ is also coherent. Again, since $\mathcal{F}$ is arbitrary, it follows that $\mathbf{R}^1\gamma_*\mathcal{K}$ is also coherent. Finally, since $\gamma_*\mathcal{L}^{-n}$ is coherent, $\gamma_*\mathcal{F}$ must also be coherent. Thus $\gamma_*\mathcal{F}$ and $\mathbf{R}^1\gamma_*\mathcal{F}$ are coherent, and γ is proper.

Since $\operatorname{Spec} S^{\vec{x}}$ is separated, $\mathcal{L}$ is relatively ample by Proposition 4.3.15(3), and γ is quasi-compact, it follows that γ is projective by [Gro67, Cor. II.5.5.3]. □

Theorem 4.3.21. *For $\vec{x} \in \mathrm{GPos}(\mathbb{L})$, $\gamma\colon T^{\vec{x}} \to \operatorname{Spec} S^{\vec{x}}$ is a projective, birational morphism.*

Proof. Since $S^{\vec{x}}$ is a domain by 4.1.1(3), $\operatorname{Spec} S^{\vec{x}}$ is an integral scheme. Each $V_i \subset T^{\vec{x}}$ is integral and by inspection, $T^{\vec{x}}$ is connected and therefore integral. Then, γ is birational if it induces an isomorphism of local rings over the generic points of $T^{\vec{x}}$ and $\operatorname{Spec} S^{\vec{x}}$ [GW10, 9.33]. Let $R = (S_{t_0}[t])_0$ and recall that $S^{\vec{x}} \cong (S[t])_0$, with $(S[t])_0 \subseteq R$. Therefore, if the inclusion map

$$f\colon (S[t])_0 \to R,$$

induces an isomorphism on fields of fractions then γ is birational. Certainly, f induces a map $f'\colon F \to G$ between the field of fractions F of $(S[t])_0$ and the field of fractions G of R. Since f' is a map between fields, $\ker(f') = \{0\}$ as the map is clearly non-zero. It suffices to show that all generators in $\mathbb{C}[V_0]$ are in the image of f'. Let $z \in \mathbb{C}[V_0]$. Then

$$z = \frac{x_1^{l_1} s t^l}{x_1^k},$$

where $s = x_2^{l_2} \dots x_n^{l_n} \in S$ for some $l_1, \dots, l_n, l, k \in \mathbb{N}_{\geq 0}$ such that $l_1\vec{x}_1 + \dots + l_n\vec{x}_n = l\vec{x} + k\vec{x}_1$. Now, $x_1^p t^q \in (S[t])_0$ for some $p, q \in \mathbb{N}_{>0}$, therefore if $k \leq p$

$$z = \frac{x_1^{l_1} s t^l}{x_1^k} = \frac{x_1^{l_1} s t^l}{x_1^k} \frac{x_1^{p-k} t^q}{x_1^{p-k} t^q} = \frac{x_1^{l_1+p-k} s t^{l+q}}{x_1^p t^q}.$$

Since $x_1^p t^q \in (S[t])_0$, it follows that $x_1^{l_1+p-k} s t^{l+q} \in (S[t])_0$. Likewise, if $k > p$ then, there exists $N > 0$ such that $k < Np$ and we have that

$$z = \frac{x_1^{l_1} s t^l}{x_1^k} = \frac{x_1^{l_1} s t^l}{x_1^k} \frac{x_1^{Np-k} t^{Nq}}{x_1^{Np-k} t^{Nq}} = \frac{x_1^{l_1+Np-k} s t^{l+Nq}}{x_1^{Np} t^{Nq}}.$$

Clearly $(x_1^p t^q)^N \in (S[t])_0$. Therefore, it follows that $x_1^{l_1+Np-k} s t^{l+Nq} \in (S[t])_0$. Thus, $G \subseteq \mathrm{Im}(f') = F$ and we have an isomorphism. □

Remark 4.3.22. Another approach to obtain the result in 4.3.21 would be to consider the $\mathbb{N}\vec{c}$-Veronese $S' := S[t]^{\mathbb{N}\vec{c}}$ and try to identify $T^{\vec{x}}$ with $\operatorname{Proj} S'$ by using the natural inclusion

$S' \hookrightarrow S[t]$. Since $(S'_{t_i})_0 = (S[t]^{\vec{c}}_{t_i})_0$ there would be a canoical morphism $q : \operatorname{Proj} S' \to T^{\vec{x}}$ if S' were a finitely generated $S[t]_0$-algebra. However, we have not been able to verify this. If, moreover, we were able to show that $S'_{m\vec{c}} \subseteq S' S_+^{\mathbb{N}\vec{c}} \subset S'_+ \mathrm{S}$ for $m \gg 0$ then this would further show that q is an isomorphism.

4.4 $T^{\vec{x}}$ is a Blow-up of $S^{\vec{x}}$

It has been shown that γ is a projective, birational morphism so, by [Gro67, II.8.8], γ is a blow-up. The next result makes this statement explicit.

Theorem 4.4.1. *For all $\vec{x} \in \mathrm{GPos}(\mathbb{L})$, $T^{\vec{x}} \cong \mathrm{Proj}(\mathcal{I})$ where*

$$\mathcal{I} = \mathcal{O}_{\operatorname{Spec} S^{\vec{x}}} \oplus \bigoplus_{n \geq 1} \gamma_* \mathcal{L}^n$$

and $\mathcal{L} = p^* \mathcal{O}(1)$ as defined in Proposition 4.3.15(2).

Proof. Since γ is separated and quasi-compact, $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_{S^{\vec{x}}}$-module by [Gro67, Cor I.9.2.2.a]. Then there exists a canonical homomorphism [Gro67, II.8.8.1] of graded $\mathcal{O}_{T^{\vec{x}}}$-algebras

$$\tau \colon \gamma^*(\mathcal{I}) \to \bigoplus_{n \geq 0} \mathcal{L}^n,$$

such that in degree ≥ 1 this agrees with the adjunction

$$\sigma \colon \gamma^*(\gamma_*(\mathcal{L}^n)) \to \mathcal{L}^n,$$

and in degree zero is the identity. Then, since $\mathcal{L}$ is γ-ample by Proposition 4.3.15(3), there exists a $\operatorname{Spec} S^{\vec{x}}$-morphism

$$r = r_{\mathcal{L},\tau} \colon T^{\vec{x}} \to P = \mathrm{Proj}(\mathcal{I})$$

which is everywhere defined (i.e. regular). It is a dominant open immersion such that

$$r^*(\mathcal{O}_P(n)) = \mathcal{L}^n \text{ for all } n \in \mathbb{Z}.$$

Since $T^{\vec{x}}$ is proper over Spec $S^{\vec{x}}$ and $\mathcal{L}$ is quasi-coherent, r is necessarily proper by [Gro67, II.5.4.4]. Recall that $T^{\vec{x}}$ is Noetherian by Proposition 4.3.15(1) and γ is projective by Theorem 4.3.20. Then, since r is a proper dominant closed immersion which is necessarily closed, $T^{\vec{x}} \cong P$ by [Gro67, II.8.8.3]. □

Now we return to the setting of Chapter 3. Recall that $S = \mathbb{C}[x_1, x_2, x_3]$, where the weights p_1, p_2, p_3 are chosen to be pairwise coprime. The dualising element $\vec{\omega}$ equals $-\vec{x}_1 - \vec{x}_2 - \vec{x}_3$, and the $\mathbb{L}$-grading is a $\mathbb{Z}$-grading since $\mathbb{L} \cong \mathbb{Z}$. Recall from Lemma 3.3.1 the isomorphism

$$S^{-\vec{\omega}} \cong \mathbb{C}[y_1, y_2, y_3]^G,$$

where $G = \frac{1}{m}(a_1, a_2, a_3)$, $a_1 = p_2p_3$, $a_2 = p_1p_3$, $a_3 = p_1p_2$ and $m = a_1 + a_2 + a_3$. Here monomials of weight $-n\vec{\omega}$ are exactly the weight nm monomials.

We make $\mathbb{C}[z_1, z_2, z_3, t]$ into a $\mathbb{Z}$-graded polynomial ring by defining $\deg z_i = a_i$ and $\deg t = -m$. Then the following map $z_i \mapsto y_i$ and $t \mapsto 1$ is an isomorphism

$$\mathbb{C}[z_1, z_2, z_3, t]^{\mathbb{C}^*} \xrightarrow{\sim} \mathbb{C}[y_1, y_2, y_3]^G.$$

The ring $\mathbb{C}[z_1, z_2, z_3, t]$ is the direct sum of its $\mathbb{Z}$-graded pieces $\mathbb{C}[z_1, z_2, z_3, t]_k$ for $k \in \mathbb{Z}$.

Theorem 4.4.2. *Let $\mathbf{p} := (p_1, p_2, p_3)$ where p_1, p_2, p_3 are pairwise coprime. The following $\mathbb{N}$-graded isomorphism of algebras holds*

$$\mathcal{I} = \mathcal{O}_{\operatorname{Spec} S^{\vec{\omega}}} \bigoplus \bigoplus_{n \geq 1} \gamma_* \mathcal{L}^n \cong \bigoplus_{n \geq 0} S(n\vec{c})^{-\vec{\omega}} \cong \oplus_{n \geq 0} \mathbb{C}[z_1, z_2, z_3, t]_{nb},$$

where $b = p_1p_2p_3$.

Proof. Recall that $\gamma_* \mathcal{L}^n = \bigoplus_{i \geq 0} S_{i\vec{\omega} + n\vec{c}} = S(n\vec{c})^{\vec{\omega}}$ by Lemma 4.3.17(2), and $\mathcal{O}_{\operatorname{Spec} S^{-\vec{\omega}}} =$

$S^{-\vec{\omega}} = S(0\vec{c})^{-\vec{\omega}}$ by Proposition 4.3.2. Therefore,

$$\mathcal{J} \cong \bigoplus_{n \geq 0} S(n\vec{c})^{-\vec{\omega}}.$$

Consider the canonical isomorphism $\varphi\colon S \to \mathbb{C}[z_1, z_2, z_3]$, given by $x_i \mapsto z_i$. We show that this induces a map

$$\bar{\varphi}\colon S(n\vec{c})^{-\vec{\omega}} \to \mathbb{C}[z_1, z_2, z_3, t]_{nb}$$

for any $n \in \mathbb{Z}_{\geq 0}$. Consider $x_1^{l_1} x_2^{l_2} x_3^{l_3} \in S(n\vec{c})^{-\vec{\omega}}$. Then there exists $l \geq 0$ such that $x_1^{l_1} x_2^{l_2} x_3^{l_3} \in S_{-l\vec{\omega}+n\vec{c}}$. Thus

$$\begin{aligned}
& & l_1\vec{x}_1 + l_2\vec{x}_2 + l_3\vec{x}_3 &= l(\vec{x}_1 + \vec{x}_2 + \vec{x}_3) + n\vec{c} \\
&\iff & b(l_1\vec{x}_1 + l_2\vec{x}_2 + l_3\vec{x}_3) &= b(l(\vec{x}_1 + \vec{x}_2 + \vec{x}_3) + n\vec{c}) \\
&\iff & (l_1p_2p_3 + l_2p_1p_3 + l_3p_1p_2)\vec{c} &= (l(p_2p_3 + p_1p_3 + p_1p_2)\vec{c} + nb\vec{c}) \\
&\iff & l_1p_2p_3 + l_2p_1p_3 + l_3p_1p_2 &= l(p_2p_3 + p_1p_3 + p_1p_2) + nb \\
&\iff & l_1a_1 + l_2a_2 + l_3a_3 &= lm + nb.
\end{aligned}$$

Now, consider $z_1^{j_1} z_2^{j_2} z_3^{j_3} t^j \in \mathbb{C}[z_1, z_2, z_3, t]_{nb}$. Then

$$j_1a_1 + j_2a_2 + j_3a_3 - jm = nb.$$

Therefore, define the map $\bar{\varphi}_n\colon S(n\vec{c})^{-\vec{\omega}} \to \mathbb{C}[z_1, z_2, z_3, t]_{nb}$ given by $x_1^{l_1} x_2^{l_2} x_3^{l_3} \mapsto z_1^{l_1} z_2^{l_2} z_3^{l_3} t^l$ where

$$l = \frac{l_1a_1 + l_2a_2 + l_3a_3 - nb}{m}.$$

Note that $\bar{\varphi}$ is surjective since, for any $z_1^{j_1} z_2^{j_2} z_3^{j_3} t^j \in \mathbb{C}[z_1, z_2, z_3, t]_{nb}$, we have

$$j_1a_1 + j_2a_2 + j_3a_3 = jm + nb,$$

and hence $x_1^{j_1} x_2^{j_2} x_3^{j_3} \in S_{-j\vec{\omega}+n\vec{c}}$, with $\bar{\varphi}_n(x_1^{j_1} x_2^{j_2} x_3^{j_3}) = z_1^{j_1} z_2^{j_2} z_3^{j_3} t^j$.

The inverse can be explicitly constructed. The map $\bar{\varphi}_n^{-1}$ sends $z_1^{l_1}z_2^{l_2}z_3^{l_3}t^l$ to $x_1^{l_1}x_2^{l_2}x_3^{l_3}$. This shows that $\bar{\varphi}_n$ is a bijection. We have constructed an isomorphism between each individual summand. Now put this all together to get the full isomorphism: $\bar{\varphi} = \oplus_{n\geq 0}\bar{\varphi}_n$. This is clearly linear. Let $x = x_1^{l_1}x_2^{l_2}x_3^{l_3} \in S(i\vec{c})^{-\vec{\omega}}$, $y = x_1^{k_1}x_2^{k_2}x_3^{k_3} \in S(j\vec{c})^{-\vec{\omega}}$ and define

$$l = \frac{l_1a_1 + l_2a_2 + l_3a_3 - ib}{m} \text{ and } k = \frac{k_1a_1 + k_2a_2 + k_3a_3 - jb}{m}.$$

Then

$$\begin{aligned}\bar{\varphi}_i(x)\bar{\varphi}_j(y) &= z_1^{l_1}z_2^{l_2}z_3^{l_3}t^l z_1^{k_1}z_2^{k_2}z_3^{k_3}t^k \\ &= z_1^{l_1+k_1}z_2^{l_2+k_2}z_3^{l_3+k_3}t^{l+k}.\end{aligned}$$

Note that

$$l + k = \frac{(l_1+k_1)a_1 + (l_2+k_2)a_2 + (l_3+k_3)a_3 - (i+j)b}{m}$$

therefore $z_1^{l_1+k_1}z_2^{l_2+k_2}z_3^{l_3+k_3}t^{l+k} \in \mathbb{C}[z_1, z_2, z_3, t]_{(i+j)b}$ and hence

$$\begin{aligned}\bar{\varphi}_i(x)\bar{\varphi}_j(y) &= \bar{\varphi}_{i+j}(x_1^{l_1+k_1}x_2^{l_2+k_2}x_3^{l_3+k_3}) \\ &= \bar{\varphi}_{i+j}(x_1^{l_1}x_2^{l_2}x_3^{l_3}x_1^{k_1}x_2^{k_2}x_3^{k_3}) \\ &= \bar{\varphi}_{i+j}(xy).\end{aligned}$$

Therefore $\bar{\varphi}$ is an algebra map. □

Corollary 4.4.3. *Let* $\mathbf{p} := (p_1, p_2, p_3)$ *where* p_1, p_2, p_3 *are pairwise coprime. Then*

$$\operatorname{Proj}(\mathcal{I}) \cong \operatorname{Proj}\big(\oplus_{n\geq 0}\mathbb{C}[z_1, z_2, z_3, t]_{nb}\big) \cong \operatorname{Proj}\big(\oplus_{n\geq 0}\mathbb{C}[z_1, z_2, z_3, t]_n\big).$$

Proof. The first isomorphism is immediate from Theorem 4.4.2. The second follows from the result of Grothendieck [Gro67, II.2.4.7] which states that for a graded ring $R = \oplus_{i\geq 0}R_i$,

$$\operatorname{Proj} R \cong \operatorname{Proj} R^{(d)},$$

where $R^{(d)} = \oplus_{i\geq 0} R_{id}$. □

Finally, recall from §3.2 that abelian quotient singularities are toric varieties. Via this interpretation of $\operatorname{Spec} S^{-\vec{\omega}}$, we will now prove that γ is a weighted blow-up as defined in §3.4. Let $\bar{M} \cong \mathbb{Z}^3$ be the lattice of Laurent monomials in x_1, x_2, x_3 and $\bar{L}$ the dual lattice with generators e_1, e_2, e_3. Then, take the overlattice

$$L = \bar{L} + \mathbb{Z} \cdot \frac{1}{m}(a_1, a_2, a_3).$$

Let $M \subset \bar{M}$ be the sublattice of invariant monomials as defined in Section 3.2.

Corollary 4.4.4. *Let $\mathbf{p} := (p_1, p_2, p_3)$ where p_1, p_2, p_3 are pairwise coprime. Then $T^{-\vec{\omega}}$ is the weighted blow-up of $\mathbb{C}[x_1, x_2, x_3]^G$ at $\frac{1}{m}(a_1, a_2, a_3)$.*

Proof. By Theorem 4.4.1, $T^{-\vec{\omega}} \cong \operatorname{Proj}(\mathfrak{I})$. Recall that the weighted blow-up of $\operatorname{Spec} \mathbb{C}[x_1, x_2, x_3]^G$ at $\frac{1}{m}(a_1, a_2, a_3)$ is covered by three open sets (3.4.A)

$$U_i = \mathbb{C}^3 / G_i,$$

where $G_i = \frac{1}{a_i}(-a_1, \ldots, m, \ldots, -a_n)$ and m is in the i-th position. In toric language, the weighted blow up is the subdivision of the fan through $\frac{1}{m}(a_1, a_2, a_3)$. This can be seen in Figure 4.1.

The triangle without e_i as a corner corresponds to U_i. Therefore, let $\sigma_1 = \langle e_2, e_3, e \rangle$ where $e = \frac{1}{m}(a_1, a_2, a_3)$. Then $U_1 = \operatorname{Spec} \mathbb{C}[\sigma_1^\vee \cap M]$. A monomial $x_1^\alpha x_2^\beta x_3^\gamma$ belongs to $\sigma_1^\vee \cap M$ if and only if

$$(\alpha, \beta, \gamma) \cdot (0, 1, 0) \geq 0, \tag{4.4.A}$$

$$(\alpha, \beta, \gamma) \cdot (0, 0, 1) \geq 0, \tag{4.4.B}$$

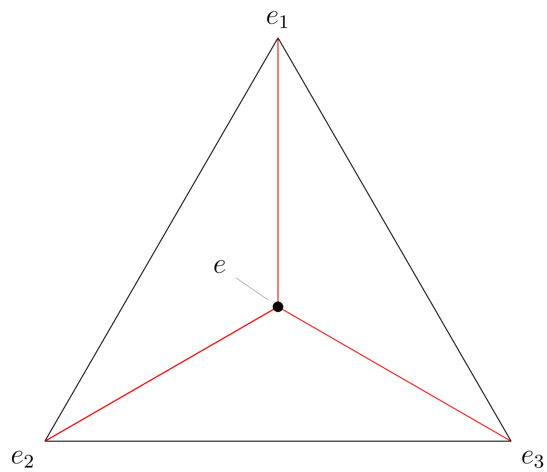


Figure 4.1: The subdivision of $\mathbb{C}[x_1, x_2, x_3]^G$ by the weighted blow-up at $\frac{1}{m}(a_1, a_2, a_3)$

$$(\alpha, \beta, \gamma) \cdot \frac{1}{m}(a_1, a_2, a_3) \geq 0. \tag{4.4.C}$$

Now, the open charts on $T^{-\vec{\omega}}$ are as follows

$$\operatorname{Spec} \bar{U}_i = \mathbb{C}[z_1, z_2, z_3, t, z_i^{-1}]^{\mathbb{C}^*}$$

for each $i \in \{1, 2, 3\}$. A monomial $z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} t^l$ is in $\mathbb{C}[\bar{U}_i]$ if and only if

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 - lm = 0, \tag{4.4.D}$$

$$\iff \frac{a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3}{m} = l \geq 0, \tag{4.4.E}$$

and $\alpha_j \geq 0$ for $j \neq i$. This inequality corresponds on the nose to the inequalities (4.4.A), (4.4.B) and (4.4.C) under the morphism $z_i \mapsto x_i$ and $t \mapsto 1$. Finally, we check that $U_i \cap U_j \cong \bar{U}_i \cap \bar{U}_j$. By symmetry, we need only check if this holds for $i = 1$ and $j = 2$. Then,

$$U_1 \cap U_2 = \operatorname{Spec} \mathbb{C}[(\sigma_1 \cap \sigma_2)^\vee \cap M].$$

Therefore the monomial $x_1^\alpha x_2^\beta x_3^\gamma$ belongs to $(\sigma_1 \cap \sigma_2)^\vee \cap M$ if and only if

$$(\alpha, \beta, \gamma) \cdot \frac{1}{m}(a_1, a_2, a_3) \geq 0 \tag{4.4.F}$$

and

$$(\alpha, \beta, \gamma) \cdot (0, 0, 1) \geq 0. \tag{4.4.G}$$

Now,

$$\mathbb{C}[\bar{U}_1 \cap \bar{U}_2] = \mathbb{C}[z_1, z_2, z_3, t, z_1^{-1}, z_2^{-1}]^{\mathbb{C}^*},$$

and a monomial $z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} t^l$ is in $\mathbb{C}[\bar{U}_1 \cap \bar{U}_2]$ if and only if

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 - lm = 0 \tag{4.4.H}$$

and

$$\alpha_3 \geq 0. \tag{4.4.I}$$

Thus, again by the usual map, an isomorphism

$$U_1 \cap U_2 \xrightarrow{\sim} \bar{U}_1 \cap \bar{U}_2$$

can be constructed since inequalities (4.4.F) and (4.4.H); and (4.4.I) and (4.4.G) are equivalent. $\square$

Chapter 5

Tilting

This chapter investigates the existence of tilting objects on $\mathbb{T}^{\vec{x}}$ and $T^{\vec{x}}$. The construction of such tilting objects will allow us to show that $\boldsymbol{\gamma}$ is a crepant map under certain restrictions.

5.1 Tilting for $\vec{x} \in \mathbb{L}_+$

We recall the following definition.

Definition 5.1.1. [TU10, Def. 3.1] *Let $\mathcal{V}$ be a vector bundle on a variety (or stack) X:*

1. *$\mathcal{V}$ is said to be pretilting if $\operatorname{Hom}^i_X(\mathcal{V}, \mathcal{V}) = 0$ for any $i \neq 0$.*

2. *$\mathcal{V}$ is called a generator of $\mathrm{D}^-(X)$ if the vanishing $\mathbf{R}\operatorname{Hom}_X(\mathcal{V}, \mathcal{K}) = 0$ for $\mathcal{K} \in \mathrm{D}^-(X)$ implies $\mathcal{K} = 0$.*

We say that $\mathcal{V}$ is a tilting bundle if it is pre-tilting and a generator.

When $\vec{x} \in \mathbb{L}_+$ we will construct a tilting bundle on $T^{\vec{x}}$ by pulling back the following known tilting objects from $\mathbb{P}^2$ and $\mathbb{X}$. As notation, set $[0, 2\vec{c}] = \{\vec{x} \in \mathbb{L} \mid 0 \leq \vec{x} \leq 2\vec{c}\}$.

Theorem 5.1.2. *Set $\mathcal{E} := \bigoplus_{\vec{y} \in [0,2\vec{c}]} \mathcal{O}(\vec{y}) \in \operatorname{coh} \mathbb{X}$ and $\mathcal{V} := \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2) \in \operatorname{coh} \mathbb{P}^2$. Then the following statements hold.*

1. *$\mathcal{V}$ is a tilting bundle on $\mathbb{P}^2$.*

2. *$\mathcal{E}$ is a tilting bundle on $\mathbb{X}$.*

Proof. We have (1) from [Bei78] and (2) from [HIMO, 5.2]. □

The following proposition will be required to control the pullbacks of the bundle in Theorem 5.1.4.

Proposition 5.1.3. [HIMO, 4.5] *For all $\vec{x}, \vec{y} \in \mathbb{L}$ and $i \in \mathbb{Z}$, we have that*

$$\operatorname{Ext}^i_{\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = \begin{cases} S_{\vec{y}-\vec{x}} & i = 0, \\ (S_{\vec{x}-\vec{y}+\vec{\omega}})^* & i = 2, \\ 0 & \textit{else.} \end{cases}$$

With this in hand, it can be shown that the pullback of $\mathcal{E}$ is a tilting object on $\mathbb{T}^{\vec{x}}$ under certain restrictions. In what follows, we use the alternative characterisation of a tilting bundle T, where naive generation is replaced by the condition that whenever $x \in \mathrm{D}(\operatorname{Qcoh} \mathbb{T}^{\vec{x}})$ satisfies $\operatorname{Hom}(T, x[i]) = 0$ for all i, then $x = 0$ (see e.g. [TU10, Def. 3.1]).

Theorem 5.1.4. *Let $0 \neq \vec{x} \in \mathbb{L}_+$, then $q^*\mathcal{E}$ is a tilting object on $\mathbb{T}^{\vec{x}}$.*

Proof. We first check that $q^*\mathcal{E}$ is a generator, following the argument of [AU15, Lemma 4.1]. Let $\operatorname{Hom}(q^*\mathcal{E}, M[i]) = 0$ for some $M \in \mathrm{D}(\operatorname{Qcoh} \mathbb{T}^{\vec{x}})$ and all $i \in \mathbb{Z}$. Then, by adjunction, $\operatorname{Hom}(\mathcal{E}, q_*M[i]) = 0$ for all $i \in \mathbb{Z}$. Since $\mathcal{E}$ is a generator in $\mathbb{X}$, this implies that $q_*M = 0$. Exactly as is stated in the proof of [AU15, Lemma 4.1], the fact that q is affine then implies that $M = 0$. Hence $q^*\mathcal{E}$ is a generator of $\mathrm{D}(\operatorname{coh} \mathbb{T}^{\vec{x}})$. Now for Ext vanishing, note that $q_*\mathcal{O}_{T^{\vec{x}}} = \bigoplus_{k \geq 0} \mathcal{O}_{\mathbb{X}}(k\vec{x})$ see e.g. [IW19, Thm 3.13] and [AU15, Lemma 4.1].

$$\begin{aligned}
\operatorname{Ext}^i_{\mathbb{T}^{\vec{x}}}(q^*\mathcal{E}, q^*\mathcal{E}) &\cong \operatorname{Ext}^i_{\mathbb{X}}(\mathcal{E}, q_*q^*\mathcal{E}) && \text{(by adjunction)} \\
&\cong \operatorname{Ext}^i_{\mathbb{X}}(\mathcal{E}, \mathcal{E} \otimes q_*\mathcal{O}_{\mathbb{T}^{\vec{x}}}) && \text{(by projection formula [Har77, Ex. II.5.1(d)])} \\
&\cong \operatorname{Ext}^i_{\mathbb{X}}(\mathcal{E}, \bigoplus_{k\geq 0} \mathcal{E} \otimes \mathcal{O}_{\mathbb{X}}(k\vec{x})) && \text{(since } q_*\mathcal{O}_{\mathbb{T}^{\vec{x}}} = \textstyle\bigoplus_{k\geq 0} \mathcal{O}_{\mathbb{X}}(k\vec{x})) \\
&\cong \bigoplus_{k\geq 0} \operatorname{Ext}^i_{\mathbb{X}}(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathbb{X}}(k\vec{x})) \\
&\cong \bigoplus_{k\geq 0} \bigoplus_{\vec{y}\in[0,2\vec{c}]} \bigoplus_{\vec{z}\in[0,2\vec{c}]} \operatorname{Ext}^i_{\mathbb{X}}(\mathcal{O}_{\mathbb{X}}(\vec{y}), \mathcal{O}_{\mathbb{X}}(\vec{z} + k\vec{x}))
\end{aligned}$$

In the first line we have again used the fact that q is affine. Now using Proposition 5.1.3,

$$\bigoplus_{k\geq 0}\bigoplus_{\vec{y}\in[0,2\vec{c}]}\bigoplus_{\vec{z}\in[0,2\vec{c}]}\mathrm{Ext}^i_{\mathbb{X}}(\mathcal{O}(\vec{y}),\mathcal{O}(\vec{z}+k\vec{x})) = \begin{cases} \bigoplus_{k\geq 0}\bigoplus_{\vec{y}\in[0,2\vec{c}]}\bigoplus_{\vec{z}\in[0,2\vec{c}]} S_{\vec{z}+k\vec{x}-\vec{y}} & i=0,\\ \bigoplus_{k\geq 0}\bigoplus_{\vec{y}\in[0,2\vec{c}]}\bigoplus_{\vec{z}\in[0,2\vec{c}]}(S_{\vec{y}-\vec{z}-k\vec{x}+\vec{\omega}})^* & i=2,\\ 0 & \text{else.}\end{cases}$$

By Remark 2.2.4(3), it is enough to consider the choice of $k,\vec{y},\vec{z}$ resulting in the most positive choice for $\vec{y}-\vec{z}-k\vec{x}+\vec{\omega}$ and check if this is in $\mathbb{L}_+$. This occurs when $k=0=\vec{z}$ and $\vec{y}=2\vec{c}$, so we have $\vec{\omega}+2\vec{c}=(n-3)\vec{c}-\sum_{i=1}^n x_i+2\vec{c}=(n-1)\vec{c}-\sum_{i=1}^n x_i\notin\mathbb{L}_+$. Therefore, the Ext groups vanish for all $i\geq 1$. □

Using the above, a tilting object on the partial resolution $T^{\vec{x}}$ exists whenever $\vec{x}\in\mathbb{L}_+$.

Corollary 5.1.5. *Let $0\neq\vec{x}\in\mathbb{L}_+$, then $p^*\mathcal{V}$ is a tilting object on $T^{\vec{x}}$.*

Proof. The generation argument follows the same argument as in Theorem 5.1.4. For Ext vanishing a similar method as above is also used. Let $\mathcal{F}:=\mathcal{O}_{\mathbb{X}}\oplus\mathcal{O}_{\mathbb{X}}(\vec{c})\oplus\mathcal{O}_{\mathbb{X}}(2\vec{c})$.

$$\begin{aligned}
\mathrm{Ext}^i_{T^{\vec{x}}}(p^*\mathcal{V},p^*\mathcal{V}) &\cong \mathrm{Ext}^i_{T^{\vec{x}}}(p^*\mathcal{V},g_*\mathcal{O}_{\mathbb{T}^{\vec{x}}}\otimes_{T^{\vec{x}}}p^*\mathcal{V}) && (\text{since } g_*\mathcal{O}_{\mathbb{T}^{\vec{x}}}=\mathcal{O}_{T^{\vec{x}}})\\
&\cong \mathrm{Ext}^i_{T^{\vec{x}}}(p^*\mathcal{V},g_*(\mathcal{O}_{\mathbb{T}^{\vec{x}}}\otimes_{T^{\vec{x}}}g^*p^*\mathcal{V}) && (\text{by the projection formula})\\
&\cong \mathrm{Ext}^i_{T^{\vec{x}}}(p^*\mathcal{V},g_*g^*p^*\mathcal{V})\\
&\cong \mathrm{Ext}^i_{\mathbb{T}^{\vec{x}}}(g^*p^*\mathcal{V},g^*p^*\mathcal{V}) && (\text{by adjunction})\\
&\cong \mathrm{Ext}^i_{\mathbb{T}^{\vec{x}}}(q^*f^*\mathcal{V},q^*f^*\mathcal{V}) && (\text{by commutativity of 4.2.A})\\
&\cong \mathrm{Ext}^i_{\mathbb{T}^{\vec{x}}}(q^*\mathcal{F},q^*\mathcal{F})
\end{aligned}$$

Now, $\mathcal{F}$ is a summand of $\mathcal{E}$, therefore the Ext groups vanish for $i\geq 1$. □

Thus, we see that in certain cases, which includes the toric setting of Chapter 3, there always exists a tilting bundle on $T^{\vec{x}}$.

5.2 Tilting with Four Weights

Much of our interest is in the case $T^{-\vec{\omega}} \to \operatorname{Spec} S^{-\vec{\omega}}$. But when $n \geq 4$ we know that $-\vec{\omega} \notin \mathbb{L}_+$ and and so the above section does not apply. In fact, the results of §5.1 cannot be extended to this setting.

Proposition 5.2.1. *Let $n \geq 4$ and $p_i \geq 2$ for all i. Then $q^*\mathcal{E}$ is not a tilting object on $\mathbb{T}^{-\vec{\omega}}$ and $p^*\mathcal{V}$ is not a tilting object on $T^{-\vec{\omega}}$.*

Proof. We do a check for Ext vanishing as in Theorem 5.1.4, where it was shown that

$$\operatorname{Ext}^2_{\mathbb{T}^{\vec{x}}}(q^*\mathcal{E}, q^*\mathcal{E}) = \bigoplus_{k\geq 0} \bigoplus_{\vec{y}\in[0,2\vec{c}]} \bigoplus_{\vec{z}\in[0,2\vec{c}]} \operatorname{Ext}^2_{\mathbb{X}}(\mathcal{O}(\vec{y}), \mathcal{O}(\vec{z}+k\vec{x})) = \bigoplus_{k\geq 0} \bigoplus_{\vec{y}\in[0,2\vec{c}]} \bigoplus_{\vec{z}\in[0,2\vec{c}]} (S_{\vec{y}-\vec{z}-k\vec{x}+\vec{\omega}})^*.$$

Consider the summand $(S_{2\vec{c}+2\vec{\omega}})^*$ which is identified by $\vec{y} = 2\vec{c}$, $\vec{z} = 0$ and $k = 1$ for $\vec{x} = -\vec{\omega}$. Then

$$2\vec{c} + 2\vec{\omega} = -2\sum_{i=1}^{n} \vec{x}_i + 2(n-3)\vec{c} + 2\vec{c} = \sum_{i=1}^{n}(p_i - 2)\vec{x}_i + (n-4)\vec{c} \in \mathbb{L}_+.$$

By Remark 2.2.4(3), it follows that the summand $(S_{2\vec{c}+2\vec{\omega}})^* \neq 0$, and so $q^*\mathcal{E}$ is not tilting on $\mathbb{T}^{-\vec{\omega}}$. By the proof of Theorem 5.1.4 and Corollary 5.1.5, it is also known that

$$\operatorname{Ext}^2_{T^{-\vec{\omega}}}(p^*\mathcal{V}, p^*\mathcal{V}) \cong \bigoplus_{k\geq 0} \bigoplus_{\vec{y}\in\{0,\vec{c},2\vec{c}\}} \bigoplus_{\vec{z}\in\{0,\vec{c},2\vec{c}\}} \operatorname{Ext}^i_{\mathbb{X}}(\mathcal{O}(\vec{y}), \mathcal{O}(\vec{z} - k\vec{\omega})).$$

Again, $(S_{2\vec{c}+2\vec{\omega}})^*$ is a summand of $\operatorname{Ext}^2_{T^{-\vec{\omega}}}(p^*\mathcal{V}, p^*\mathcal{V})$ and so $p^*\mathcal{V}$ is not tilting on $T^{-\vec{\omega}}$. □

Of course, the above does not imply that there does not exist *some* tilting bundle on $T^{-\vec{\omega}}$, just that the obvious candidate is not tilting. In fact, the existence of a tilting object for some cases can be seen in the following result.

Proposition 5.2.2. *If the following conditions are satisfied then there exists a tilting bundle on $T^{-\vec{\omega}}$.*

- *$n = 4$ and $(p_1, p_2, p_3, p_4) = (2, 2, p_3, p_4)$.*

- $n = 4$ *and* $(p_1, p_2, p_3, p_4) = (2, 3, p_3, p_4)$ *where* $p_3, p_4 \in \{3, 4, 5\}$.

Proof. The result of Toda and Uehara [TU10, Theorem 6.1] states that if $\mathbf{R}^2\gamma_*\mathcal{L}^{-1} = 0$ for some ample globally generated line bundle $\mathcal{L}$ then there exists a tilting bundle $T^{-\vec{\omega}}$. By Proposition 4.3.15(2), $\mathcal{L} = p^*\mathcal{O}(1)$ is ample and, by construction, is globally generated. We now proceed by checking $\mathbf{R}^2\gamma_*\mathcal{L}^{-1} = 0$.

$$\begin{aligned}
\mathbf{R}^2\gamma_*\mathcal{L}^{-1} &= \mathbf{R}^2\gamma_*\,\mathcal{H}\mathrm{om}(\mathcal{L}, \mathcal{O}_{T^{-\vec{\omega}}}) \\
&\cong \mathrm{Ext}^2_{T^{-\vec{\omega}}}(\mathcal{L}, \mathcal{O}) \\
&= \mathrm{Ext}^2_{T^{-\vec{\omega}}}(p^*\mathcal{O}(1), p^*\mathcal{O}) \\
&\cong \mathrm{Ext}^2_{\mathbb{T}^{-\vec{\omega}}}(q^*\mathcal{O}(\vec{c}), q^*\mathcal{O}) && \text{(by the same logic as 5.1.5.)} \\
&\cong \mathrm{Ext}^2_{\mathbb{X}}(\mathcal{O}(\vec{c}), q_*q^*\mathcal{O}) && \text{(by adjunction)} \\
&\cong \bigoplus_{k\geq 0} \mathrm{Ext}^2_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(-k\vec{\omega})) \\
&= \bigoplus_{k\geq 0} (S_{\vec{c}+k\vec{\omega}+\vec{\omega}})^*. && \text{(by 5.1.3)}
\end{aligned}$$

Let $(p_1, p_2, p_3, p_4) = (2, 2, p_3, p_4)$. We check for non-zero summands by looking at the even and odd cases for k. Suppose $k = 2m$, then

$$\begin{aligned}
(k+1)\vec{\omega} + \vec{c} &= (2m+1)(-\sum_{i=1}^{4} \vec{x}_i + \vec{c}) + \vec{c} \\
&= -2m\vec{c} - \vec{x}_1 - \vec{x}_2 - (2m+1)(\vec{x}_3 + \vec{x}_4) + (2m+2)\vec{c} \\
&= \vec{x}_1 + \vec{x}_2 - 2\vec{c} - (2m+1)(\vec{x}_3 + \vec{x}_4) + 2\vec{c} \\
&= \vec{x}_1 + \vec{x}_2 - (2m+1)(\vec{x}_3 + \vec{x}_4) \notin \mathbb{L}_+.
\end{aligned}$$

Now, suppose $k = 2m + 1$, then

$$\begin{aligned}
(k+1)\vec{\omega} + \vec{c} &= (2m+2)(-\sum_{i=1}^{4} \vec{x}_i + \vec{c}) + \vec{c} \\
&= -(2m+2)\vec{c} - (2m+2)(\vec{x}_3 + \vec{x}_4) + (2m+2)\vec{c} \\
&= -(2m+2)(\vec{x}_3 + \vec{x}_4) \notin \mathbb{L}_+.
\end{aligned}$$

Therefore $\mathbf{R}^2\gamma_*\mathcal{L}^{-1} = 0$.

Next, let $(p_1, p_2, p_3, p_4) = (2, 3, 3, 3)$. Consider $k = (6m + i)$, for $i \in \{0, 1, 2, 3, 4, 5\}$, and $m \geq 0$. Then,

$$
\begin{aligned}
k = 6m \quad & (6m+1)\vec{\omega} + \vec{c} = (6m+1)(-\sum_{i=1}^{4}\vec{x}_i + \vec{c}) + \vec{c} \\
&= -3m\vec{c} - 2m(3\vec{c}) - \sum_{i=1}^{4}\vec{x}_i + (6m+2)\vec{c} \\
&= -\sum_{i=1}^{4}\vec{x}_i + (2-3m)\vec{c} \notin \mathbb{L}_+ \\
k = 6m+1 \quad & (6m+2)\vec{\omega} + \vec{c} = (6m+2)(-\sum_{i=1}^{4}\vec{x}_i + \vec{c}) + \vec{c} \\
&= -(3m+1)\vec{c} - 2m(3\vec{c}) - \sum_{i=2}^{4}\vec{x}_i + (6m+3)\vec{c} \\
&= -\sum_{i=2}^{4}\vec{x}_i + (2-3m)\vec{c} \notin \mathbb{L}_+ \\
k = 6m+2 \quad & (6m+3)\vec{\omega} + \vec{c} = (6m+3)(-\sum_{i=1}^{4}\vec{x}_i + \vec{c}) + \vec{c} \\
&= -(3m+1)\vec{c} - (2m+1)(3\vec{c}) - \vec{x}_1 + (6m+4)\vec{c} \\
&= -\vec{x}_1 - 3m\vec{c} \notin \mathbb{L}_+ \\
k = 6m+3 \quad & (6m+4)\vec{\omega} + \vec{c} = (6m+4)(-\sum_{i=1}^{4}\vec{x}_i + \vec{c}) + \vec{c} \\
&= -(3m+2)\vec{c} -) 2m+1)(3\vec{c}) - \sum_{i=2}^{4}\vec{x}_i + (6m+5)\vec{c} \\
&= -\sum_{i=2}^{4}\vec{x}_i - 3m\vec{c} \notin \mathbb{L}_+ \\
k = 6m+4 \quad & (6m+5)\vec{\omega} + \vec{c} = (6m+5)(-\sum_{i=1}^{4}\vec{x}_i + \vec{c}) + \vec{c} \\
&= -(3m+2)\vec{c} - (2m+1)(3\vec{c}) - \vec{x}_1 - 2\sum_{i=2}^{4}\vec{x}_i + (6m+6)\vec{c} \\
&= -\vec{x}_1 - 2\sum_{i=2}^{4}\vec{x}_i + (1-3m)\vec{c} \notin \mathbb{L}_+
\end{aligned}
$$

$$k = 6m+5 \qquad (6m+6)\vec{\omega}+\vec{c} = (6m+6)(-\sum_{i=1}^{4}\vec{x}_i+\vec{c})+\vec{c}$$
$$= -3(m+1)\vec{c}-2(m+1)(3\vec{c})+(6m+7)\vec{c}$$
$$= -(3m+2)\vec{c} \notin \mathbb{L}_+$$

Therefore $\mathbf{R}^2\gamma_*\mathcal{L}^{-1} = 0$ in this case too. The other cases can be calculated in a similar fashion. □

Unfortunately, as we show below, using $\mathcal{L}$ and [TU10] to construct tilting bundles on $T^{-\vec{\omega}}$ does not work in many cases.

Proposition 5.2.3. *In the following cases* $\mathbf{R}^2\gamma_*\mathcal{L}^{-1} \neq 0$.

1. (p_1, p_2, p_3, p_4) *where* $p_i \geq 3$ *for all* i,
2. $(2, p_2, p_3, p_4)$ *and* $p_2, p_3, p_4 \geq 4$,
3. $(2, 3, p_3, p_4)$ *and* $p_3, p_4 \geq 6$.

Proof. From the proof of Theorem 5.2.2, we have $\mathbf{R}^2\gamma_*\mathcal{L}^{-1} = \bigoplus_{k\geq 0}(S_{\vec{c}+k\vec{\omega}+\vec{\omega}})^*$. We now show that there exists a nonzero summand $\mathbf{R}^2\gamma_*\mathcal{L}^{-1}$ in each of the above cases by using the fact that if $\vec{x} \in \mathbb{L}_+$ then $S_{\vec{x}} \neq 0$ by Lemma 2.2.4(3).

1. We consider the summand $(S_{\vec{c}+3\vec{\omega}})^*$ and check if $\vec{c}+3\vec{\omega} \in \mathbb{L}_+$. We have

$$\vec{c}+3\vec{\omega} = 4\vec{c} - 3\sum_{i=1}^{4}\vec{x}_i$$
$$= \sum_{i=1}^{4}(p_i-3)\vec{x}_i$$

If $p_i \geq 3$ for all i, then $\vec{c}+3\vec{\omega} \in \mathbb{L}_+$.

2. We consider the summand $(S_{\vec{c}+4\vec{\omega}})^*$ and check if $\vec{c}+4\vec{\omega} \in \mathbb{L}_+$. We have

$$\begin{aligned}\vec{c}+4\vec{\omega} &= 5\vec{c} - 4\sum_{i=1}^{4}\vec{x}_i \\ &= 5\vec{c} - (2\vec{c} + 4\vec{x}_2 + 4\vec{x}_3 + 4\vec{x}_4) \\ &= -(4\vec{x}_2 + 4\vec{x}_3 + 4\vec{x}_4) + 3\vec{c} \\ &= (p_2-4)\vec{x}_2 + (p_3-4)\vec{x}_3 + (p_4-4)\vec{x}_4.\end{aligned}$$

   Clearly, if $p_2, p_3, p_4 \geq 4$, then $\vec{c}+6\vec{\omega} \in \mathbb{L}_+$.

3. We consider the summand $(S_{\vec{c}+6\vec{\omega}})^*$ and check if $\vec{c}+6\vec{\omega} \in \mathbb{L}_+$. We have

$$\begin{aligned}\vec{c}+6\vec{\omega} &= 7\vec{c} - 6\sum_{i=1}^{4}\vec{x}_i \\ &= 7\vec{c} - (3\vec{c} + 2\vec{c} + 6\vec{x}_3 + 6\vec{x}_4) \\ &= -(6\vec{x}_3 + 6\vec{x}_4) + 2\vec{c} \\ &= (p_3-6)\vec{x}_3 + (p_4-6)\vec{x}_4.\end{aligned}$$

   Clearly, if $p_3, p_4 \geq 6$, then $\vec{c}+6\vec{\omega} \in \mathbb{L}_+$.

□

The following summarises Proposition 5.2.2 and 5.2.3.

Theorem 5.2.4. *For $\mathcal{L} = p^*\mathcal{O}(1)$ and $n = 4$, $\mathbf{R}^2\gamma_*\mathcal{L}^{-1} = 0$ if and only if (p_1, p_2, p_3, p_4) is one of the choices in Proposition 5.2.2.*

To use the method of [TU10, Section 4] to explicitly construct a tilting bundle also requires knowledge of $\mathbf{R}^2\gamma_*\mathcal{L}^{-2}$, which is considered in the following result.

Lemma 5.2.5. *For the cases where a Toda-Uehara tilting object exists we have*

Weights	$\mathbf{R}^2\gamma_*\mathcal{L}^{-2}$
$[2,2,p_3,p_4]$ *where* $p_3,p_4\in\mathbb{Z}_{>0}$	$\bigoplus_{i=1}^{\lfloor\frac{p_3}{2}\rfloor}(S_{2\vec{c}+2i\vec{\omega}})^*$
$[2,3,3,p_4]$ *where* $p_4\in\{3,4,5\}$	$(S_{2\vec{\omega}+2\vec{c}})^*\oplus(S_{3\vec{\omega}+2\vec{c}})^*$
$[2,3,4,p_4]$ *where* $p_4\in\{4,5\}$	$(S_{2\vec{\omega}+2\vec{c}})^*\oplus(S_{3\vec{\omega}+2\vec{c}})^*\oplus(S_{4\vec{\omega}+2\vec{c}})^*$
$[2,3,5,5]$	$(S_{2\vec{\omega}+2\vec{c}})^*\oplus(S_{3\vec{\omega}+2\vec{c}})^*\oplus(S_{4\vec{\omega}+2\vec{c}})^*\oplus(S_{5\vec{\omega}+2\vec{c}})^*$

Proof. By the same method as in the proof of Theorem 5.2.2 $\mathbf{R}^2\gamma_*\mathcal{L}^{-2}=\bigoplus_{k\geq 0}(S_{2\vec{c}+(k+1)\vec{\omega}})^*$. We claim that most of the summands are zero since, $(S_{2\vec{c}+(k+1)\vec{\omega}})^*=0$ if and only if $2\vec{c}+(k+1)\vec{\omega}\notin\mathbb{L}_+$.

Consider the first case $[2,2,2,2]$. We check for $k=2m$ and $k=2m+1$ where $m\geq 0$:

$$
\begin{aligned}
k=2m \qquad\qquad 2\vec{c}+(2m+1)\vec{\omega} &= 2\vec{c}+(2m+1)(-\sum_{i=1}^{4}\vec{x}_i+\vec{c})\\
&= 2\vec{c}-(2m+1)\sum_{i=1}^{4}\vec{x}_i+(2m+1)\vec{c}\\
&= (2m+3)\vec{c}-4m\vec{c}-\sum_{i=1}^{4}\vec{x}_i\\
&= (3-2m)\vec{c}-\sum_{i=1}^{4}\vec{x}_i\\
&= -(2m-1)\vec{c}+\sum_{i=1}^{4}\vec{x}_i\notin\mathbb{L}_+
\end{aligned}
$$

and

$$
\begin{aligned}
k=2m+1 \qquad\qquad 2\vec{c}+(2m+2)\vec{\omega} &= 2\vec{c}+(2m+2)(-\sum_{i=1}^{4}\vec{x}_i+\vec{c})\\
&= (2m+4)\vec{c}-(2m+2)\sum_{i=1}^{4}\vec{x}_i\\
&= (2m+4)\vec{c}-4(m+1)\vec{c}\\
&= -(2m)\vec{c}.
\end{aligned}
$$

Therefore the only summand in $\mathbb{L}_+$ is when $k=1$ i.e. $\mathbf{R}^2\gamma_*\mathcal{L}^{-2}=(S_{2\vec{c}+2\vec{\omega}})^*$.

The other cases are calculated in a similar manner. □

Theorem 5.2.6. *In the following cases*

- $n = 4$ *and* $(p_1, p_2, p_3, p_4) = (2, 2, p_3, p_4)$,
- $n = 4$ *and* $(p_1, p_2, p_3, p_4) = (2, 3, p_3, p_4)$ *where* $p_3, p_4 \in \{3, 4, 5\}$,

$\mathbf{R}^i\gamma_*\mathcal{L}^{-j} = 0$ *holds for* $j \geq 0$ *if* $i = 1$ *or* $i \geq 3$.

Proof. Consider $i > 2$ and any j. Using the same logic as the proof of Proposition 5.2.2,

$$\mathbf{R}^i\gamma_*\mathcal{L}^{-j} \cong \bigoplus_{k\geq 0} \operatorname{Ext}^i_{\mathbb{X}}(\mathcal{O}(j\vec{c}), \mathcal{O}(-k\vec{\omega})) = 0. \qquad \text{(by 5.1.3)}$$

Similarly, for $i = 1$ and any j,

$$\mathbf{R}^1\gamma_*\mathcal{L}^{-j} \cong \bigoplus_{k\geq 0} \operatorname{Ext}^1_{\mathbb{X}}(\mathcal{O}(j\vec{c}), \mathcal{O}(-k\vec{\omega})) = 0. \qquad \text{(by 5.1.3)}$$

□

We now explain, following [TU10], how to construct the tilting bundle on $T^{-\vec{\omega}}$ in the cases considered in Theorem 5.2.6. The construction is iterative, beginning at 0, and constructs vector bundles $\mathcal{E}_k$ for $k \in \{0, 1, 2\}$. Set $\mathcal{E}_0 = \mathcal{O}$. Denote $\operatorname{End}_{T^{-\vec{\omega}}}(\mathcal{E}_{k-1}) = A_{k-1}$ and define the following functors

$$\Phi_{k-1}(-) = \mathbf{R}\operatorname{Hom}_X(\mathcal{E}_{k-1}, -) \colon \mathrm{D}(X) \longrightarrow \mathrm{D}(A_{k-1}),$$
$$\Psi_{k-1}(-) = - \otimes_{A_{k-1}} \mathcal{E}_{k-1} \colon \mathrm{D}^-(A_{k-1}) \longrightarrow \mathrm{D}^-(X).$$

For a complex $\mathcal{K}$ of coherent sheaves, define $\tau_{\leq p}\mathcal{K}(= \tau_{<p+1}\mathcal{K})$ and $\tau_{>p}\mathcal{K}(= \tau_{\geq p+1}\mathcal{K})$ to be the good truncations, namely

$$(\tau_{\leq p}\mathcal{K})^n = \begin{cases} \mathcal{K}^n & n < p \\ \operatorname{Ker} d^p & n = p \\ 0 & n > p \end{cases}$$

and

$$(\tau_{>p}\mathcal{K})^n = \begin{cases} 0 & n < p \\ \operatorname{Im} d^p & n = p \\ \mathcal{K}^n & n > p, \end{cases}$$

where $d^p\colon \mathcal{K}^p \to \mathcal{K}^{p+1}$ is the differential. Similarly, denote by $\sigma_{\leq p}\mathcal{K}(= \sigma_{<p+1}\mathcal{K})$ and $\sigma_{>p}\mathcal{K}(= \sigma_{\geq p+1}\mathcal{K})$ to be the brutal truncations, namely

$$(\sigma_{\leq p}\mathcal{K})^n = \begin{cases} \mathcal{K}^n & n \leq p \\ 0 & n > p \end{cases}$$

and

$$(\sigma_{>p}\mathcal{K})^n = \begin{cases} 0 & n \leq p \\ \mathcal{K}^n & n > p. \end{cases}$$

Then for any $j \in \mathbb{Z}$ the following distinguished triangles exist:

$$\tau_{\leq p}\mathcal{K} \to \mathcal{K} \to \tau_{>p}\mathcal{K} \to \tau_{\leq p}\mathcal{K}[1]$$

$$\sigma_{>p}\mathcal{K} \to \mathcal{K} \to \sigma_{\leq p}\mathcal{K} \to \sigma_{>p}\mathcal{K}[1].$$

Construct a A_{k-1} free resolution of $\Phi_{k-1}(\mathcal{L}^{-k})$ and denote it by P_{k-1}. This results in the triangle of [TU10, (8)], namely

$$\Psi_{k-1}(\sigma_{\geq 1}(P_{k-1})) \to \mathcal{L}^{-k} \to \mathcal{N}_{k-1} \to \Psi_{k-1}(\sigma_{\geq 1}(P_{k-1})[1]).$$

where $\Psi_{k-1}(\mathcal{N}_{k-1}) \cong \sigma_{<1}(P_{k-1})$. For the cases listed in Theorem 5.2.6, since by Theorem 5.2.4, $\Phi_0(\mathcal{L}^{-1}) = \mathbf{R}\operatorname{Hom}(\mathcal{O}, \mathcal{L}^{-1}) = \mathbf{R}\gamma_*\mathcal{L}^{-1}$ has cohomology only in deg 0, $\sigma_{\geq 1}(P_0) = 0$ so $\Psi_0(\sigma_{\geq 1}(P_0)) = 0$ and hence $\mathcal{N}_0 = \mathcal{L}^{-1}$. We set $\mathcal{E}_1 = \mathcal{E}_0 \oplus \mathcal{N}_0 = \mathcal{O} \oplus \mathcal{L}^{-1}$. We interate one more time to obtain the tilting bundle $\mathcal{E}_2 = \mathcal{E}_0 \oplus \mathcal{N}_0 \oplus N_1 = \mathcal{O} \oplus \mathcal{L}^{-1} \oplus \mathcal{N}_1$ where $\mathcal{N}_1 \in \mathrm{D}^{\mathrm{b}}(T^{-\vec{\omega}})$ sits in a triangle

$$\Psi_1(\sigma_{\geq 1}(P_1)) \to \mathcal{L}^{-2} \to \mathcal{N}_1 \to \Psi_1(\sigma_{\geq 1}(P_1)[1]).$$

Since $\Psi_1(\sigma_{\geq 1}(P_1))$ is a perfect complex [TU10, Below (13)], so is $\mathcal{N}_1$. It follows from [TU10, 4.3] that $\mathcal{N}_1$ is a vector bundle and $\mathcal{E}_2$ is a tilting bundle on $T^{-\vec{\omega}}$.

5.3 Another Natural Bundle

In this chapter we have seen that to find a tilting bundle for the cases in Proposition 5.2.3 we cannot use the pullback of $\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$ from $\mathbb{P}^2$. We have also seen that the method used by Toda-Uehara [TU10] is insufficient in this case. It is well known that $\mathbb{P}^2$ has another tilting bundle, namely $\mathcal{O} \oplus \Omega^1(1) \oplus \Omega^2(2)$ and so following [BLVdB10] another approach to constructing a tilting bundle on $T^{-\vec{\omega}}$ would be to pull this back. However, as we show in this section, this does not provide any new examples.

Lemma 5.3.1. [Har77, II.8.13] *There is a short exact sequence of sheaves on* $\mathbb{P}^2$

$$0 \to \Omega_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2} \to 0.$$

Recall that $\Omega^2 = \mathcal{O}_{\mathbb{P}^2}(-3)$ [Har77, III.7.1] and hence $\Omega^2(2) = \mathcal{O}(-1)$. Therefore, for $\mathcal{O} \oplus \Omega^1(1) \oplus \Omega^2(2)$ to pullback to give a tilting bundle on $T^{-\vec{\omega}}$ requires $p^*\Omega^1(1)$ to exhibit certain vanishing properties. We will manipulate exact sequences to show that the relevant Ext groups do not vanish.

Theorem 5.3.2. *Let* $\mathcal{K} = p^*\Omega^1(1)$. *Then* $\operatorname{Ext}^1_{T^{-\vec{\omega}}}(\mathcal{K}, \mathcal{O}) = \mathbf{R}^1\gamma_*\mathcal{K}^* = 0$ *if and only if* (p_1, p_2, p_3, p_4) *is one of those listed in Theorem 5.2.4.*

Proof. Consider the Koszul complex on $\mathbb{P}^2$, namely

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 3} \to \mathcal{O}^{\oplus 3} \to \mathcal{O}(1) \to 0.$$

Splicing gives an exact sequence

$$0 \to \mathrm{Ker} \to \mathcal{O}^{\oplus 3} \to \mathcal{O}(1) \to 0.$$

Therefore, by shifting this sequence, we see from Theorem 5.3.1 that $\mathrm{Ker} = \Omega^1(1)$. Pulling back this sequence to $T^{-\vec{\omega}}$ results in an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus 3} \to \mathcal{L} \to 0.$$

Taking the dual gives exact sequence

$$0 \to \mathcal{L}^{-1} \to \mathcal{O}^{\oplus 3} \to \mathcal{K}^* \to 0.$$

Pushing down gives the long exact sequence

$$0 \to \gamma_*\mathcal{L}^{-1} \to (S^{-\vec{\omega}})^{\oplus 3} \to \gamma_*\mathcal{K}^* \to \mathbf{R}^1\gamma_*\mathcal{L}^{-1} \to 0 \to \mathbf{R}^1\gamma_*\mathcal{K}^* \to \mathbf{R}^2\gamma_*\mathcal{L}^{-1} \to 0$$

since $\mathbf{R}^1\gamma_*S^{-\vec{\omega}} = 0$ by Proposition 4.3.2. Therefore $\mathbf{R}^2\gamma_*\mathcal{L}^{-1} \cong \mathbf{R}^1\gamma_*\mathcal{K}^*$. The statement now follows from Theorem 5.2.4. □

5.4 A Crepant Map

In the previous section we have considered the existence of tilting bundles on $T^{-\vec{\omega}}$, and implicitly, the existence of modifying algebras for $\operatorname{Spec} S^{-\vec{\omega}}$. In this final section we consider commutative crepant resolutions. It seems that constructing a crepant resolution of $\operatorname{Spec} S^{-\vec{\omega}}$ is very tricky in general. In the toric setting we are able to show that γ is a crepant map when $\vec{x} = -\vec{\omega}$ in two very different ways.

Lemma 5.4.1. *Let $0 \neq \vec{x} \in \mathbb{L}_+$, then there is an isomorphism $\mathrm{End}_{T^{\vec{x}}}(p^*\mathcal{V}) \cong \mathrm{End}_{S^{\vec{x}}}(\gamma_*p^*\mathcal{V})$.*

Proof. We note that γ is a projective birational morphism by Theorem 4.3.21 and $S^{\vec{x}}$ is normal by Lemma 4.1.1(3). The vector bundle $p^*\mathcal{V}$ is of finite rank and generated by global sections, and $\mathrm{Ext}^1(p^*\mathcal{V}, p^*\mathcal{V}) = 0$ by Corollary 5.1.5. Therefore it follows from [DW19, 4.3] that $\mathrm{End}_{T^{\vec{x}}}(p^*\mathcal{V}) \cong \mathrm{End}_{S^{\vec{x}}}(\gamma_*p^*\mathcal{V})$. □

Lemma 5.4.2. *Let* $\vec{x} = \sum_{j=1}^{3} a_j\vec{x}_j + b\vec{c}$ *where* $a_1, a_2, a_3 > 0$ *and* $b \geq 0$. *Then* $S(a\vec{c})^{\vec{x}} = \bigoplus_{i\geq 0} S_{i\vec{x}+a\vec{c}}$ *for* $a \in \{-2, -1, 0, 1, 2\}$.

Proof. In general, $S(a\vec{c})^{\vec{x}} = \bigoplus_{i\in\mathbb{Z}} S_{i\vec{x}+a\vec{c}}$. Consider $i = -1$. Then

$$-\vec{x} + a\vec{c} = -(\sum_{j=1}^{3} a_j\vec{x}_j + b\vec{c}) + a\vec{c} = \sum_{j=1}^{n}(p_j - a_j)\vec{x}_j + (a - b - 3)\vec{c}.$$

We have $-\vec{x} + a\vec{c} \in \mathbb{L}_+$ if and only if

$$a - b - 3 \geq 0$$
$$a \geq 3 + b \geq 3.$$

Therefore $-\vec{x}+a\vec{c} \notin \mathbb{L}_+$ for $a \leq 2$. Since we are simply taking away multiples of $\vec{x}$ for $i > 1$, $-i\vec{x}+a\vec{c}$ is not in $\mathbb{L}_+$ for $a \leq 2$, and so the sum $S(a\vec{c})^{\vec{x}} = \bigoplus_{i\in\mathbb{Z}} S_{i\vec{x}+a\vec{c}} = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} S_{i\vec{x}+a\vec{c}}$. □

Lemma 5.4.3. *Let* $\vec{x} = \sum_{j=1}^{3} a_j\vec{x}_j + b\vec{c}$ *where* $a_1, a_2, a_3 > 0$ *and* $b \geq 0$, *then*

$$\mathrm{End}_{S^{\vec{x}}}(\gamma_* p^* \mathcal{V}) \cong \begin{pmatrix} S^{\vec{x}} & S(\vec{c})^{\vec{x}} & S(2\vec{c})^{\vec{x}} \\ S(-\vec{c})^{\vec{x}} & S^{\vec{x}} & S(\vec{c})^{\vec{x}} \\ S(-2\vec{c})^{\vec{x}} & S(-\vec{c})^{\vec{x}} & S^{\vec{x}} \end{pmatrix}.$$

Proof. We have that

$$\gamma_* p^* \mathcal{V} = \gamma_* p^*(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)) = \gamma_*(\mathcal{O} \oplus \mathcal{L} \oplus \mathcal{L}^2),$$

and so, by Lemmas 4.3.17(2) and 5.4.2, $\gamma_* p^* \mathcal{V} = S^{\vec{x}} \oplus S(\vec{c})^{\vec{x}} \oplus S(2\vec{c})^{\vec{x}}$. Therefore

$$\mathrm{End}_{S^{\vec{x}}}(\gamma_* p^* \mathcal{V}) \cong \mathrm{End}_{S^{\vec{x}}}(S^{\vec{x}} \oplus S(\vec{c})^{\vec{x}} \oplus S(2\vec{c})^{\vec{x}}) \cong \begin{pmatrix} S^{\vec{x}} & S(\vec{c})^{\vec{x}} & S(2\vec{c})^{\vec{x}} \\ S(-\vec{c})^{\vec{x}} & S^{\vec{x}} & S(\vec{c})^{\vec{x}} \\ S(-2\vec{c})^{\vec{x}} & S(-\vec{c})^{\vec{x}} & S^{\vec{x}} \end{pmatrix}$$

since $\mathrm{Hom}_{S^{\vec{x}}}(S(\vec{y})^{\vec{x}}, S(\vec{z})^{\vec{x}}) = \mathrm{Hom}_S^{\mathbb{L}}(S(\vec{y}), S(\vec{z}))^{\vec{x}} = S(\vec{z} - \vec{y})^{\vec{x}}$ by [IW19, Lemma 4.6]. □

Corollary 5.4.4. *In the toric setting* ($n = 3$)*,* $\operatorname{End}_{S^{-\vec{\omega}}}(\gamma_* p^* \mathcal{V}) \in \operatorname{CM} S^{-\vec{\omega}}$.

Proof. By Theorem 2.2.11 $S^{-\vec{\omega}}$ is Gorenstein and, necessarily, Cohen-Macaulay. Since $n = 3$, $-\vec{\omega} = \Sigma_{i=1}^{3} x_i \in \mathbb{L}_+$, so we may use Lemma 5.4.3. Further, since $S^{-\vec{\omega}} \in \operatorname{CM} S^{-\vec{\omega}}$, the twists $S(-2\vec{c})^{-\vec{\omega}}$, $S(-\vec{c})^{-\vec{\omega}}$, $S(\vec{c})^{-\vec{\omega}}$ and $S(2\vec{c})^{-\vec{\omega}}$ are also Cohen-Macaulay, by e.g. Remark 2.2.12. □

Corollary 5.4.5. *In the toric setting* ($n = 3$)*,* $T^{-\vec{\omega}} \to \operatorname{Spec} S^{-\vec{\omega}}$ *is crepant.*

Proof. Since $p^* \mathcal{V}$ is tilting on $T^{-\vec{\omega}}$ by 5.1.5, by Theorem 2.1.4 we just need to check that $\operatorname{End}_{T^{-\vec{\omega}}}(p^* \mathcal{V}) \in \operatorname{CM} S^{-\vec{\omega}}$. But, by Lemma 5.4.1 this is equivalent to $\operatorname{End}_{S^{-\vec{\omega}}}(\gamma_* p^* \mathcal{V}) \in \operatorname{CM} S^{-\vec{\omega}}$, which holds by Corollary 5.4.4 above. □

The benefit of Corollary 5.4.5 is that it holds in the case $n = 3$ without further restrictions. If we are further to insist on the conditions of §3.4, namely (p_1, p_2, p_3) are pairwise coprime, then we know from §4.4 that $T^{-\vec{\omega}} \to \operatorname{Spec} S^{-\vec{\omega}}$ is also a weighted blow-up. In this more restrictive setting this gives an alternative method of showing the $T^{-\vec{\omega}} \to \operatorname{Spec} S^{-\vec{\omega}}$ is crepant, as then we can simply show that the weighted blow-up is crepant.

Corollary 5.4.6. *When* $n = 3$ *and* (p_1, p_2, p_3) *are pairwise coprime, then* $T^{-\vec{\omega}} \to \operatorname{Spec} S^{-\vec{\omega}}$ *is crepant.*

Proof. As observed above, $T^{-\vec{\omega}} \to \operatorname{Spec} S^{-\vec{\omega}}$ is the weighted blow-up at $\frac{1}{m}(a_1, a_2, a_3)$ and this is the subdivision of the toric fan by the simplicial ray $\frac{1}{m}(a_1, a_2, a_3)$. Therefore $\boldsymbol{\gamma}$ is crepant by [DHZ01, Prop. 2.4]. □

Chapter 6

Towards Full Resolutions

This chapter investigates some examples of the partial resolution constructed in Chapter 4, and in some instances extends this to a full resolution of $\operatorname{Spec} S^{-\vec{\omega}}$. Along the way, we illustrate the complexity of the problem of resolving the singularities of $T^{-\vec{\omega}}$ in general.

6.1 A Resolution of $S^{-\vec{\omega}}$ when $\mathbf{p} = (2, 2, 2, 2)$

In this section we will find a full resolution of $\operatorname{Spec} S^{-\vec{\omega}}$ when $\mathbf{p} = (2, 2, 2, 2)$ by blowing up the singular locus of $T^{-\vec{\omega}}$. Recall that, in this setting, S can be written in the form

$$S = \frac{\mathbb{C}[x_1, x_2, x_3, x_4]}{x_1^2 + x_2^2 + x_3^2 = x_4^2}$$

and that $S^{-\vec{\omega}} = \bigoplus_{i \in \mathbb{Z}} S_{-i\vec{\omega}}$ where $\vec{\omega} = \vec{c} - \vec{x}_1 - \vec{x}_2 - \vec{x}_3 - \vec{x}_4$ is the dualizing element. We begin by finding all generators of the ring $S^{-\vec{\omega}}$. The following table calculates the multiples of $-\vec{\omega}$, and lists all monomials of the corresponding weight:

$-\vec{\omega}$	$\vec{x}_1 + \vec{x}_2 + \vec{x}_3 + \vec{x}_4 - \vec{c}$	-
$-2\vec{\omega}$	$2(\vec{x}_1 + \vec{x}_2 + \vec{x}_3 + \vec{x}_4 - \vec{c}) = 2\vec{c}$	$x_1^4, x_2^4, x_3^4, x_4^4, x_1^2x_2^2, x_1^2x_3^2, x_1^2x_4^2, ..., x_3^2x_4^2$
$-3\vec{\omega}$	$\vec{x}_1 + \vec{x}_2 + \vec{x}_3 + \vec{x}_4 + \vec{c}$	$x_1^3x_2x_3x_4, x_1x_2^3x_3x_4, x_1x_2x_3^3x_4, x_1x_2x_3x_4^3$

We claim that these are all generators. Indeed, $S_{-2\vec{\omega}} = S_{2\vec{c}}$ and $S_{-3\vec{\omega}} = S_{\vec{x}_1+\vec{x}_2+\vec{x}_3+\vec{x}_4+\vec{c}} = x_1x_2x_3x_4S_{\vec{c}}$ by Lemma 2.2.4, so for $i \in \mathbb{N}$ it follows that

$$\begin{aligned} S_{-2i\vec{\omega}} &= S_{2\vec{c}} \cdots S_{2\vec{c}} && (i \text{ copies of } S_{2\vec{c}}) \\ S_{-(2i+1)\vec{\omega}} &= S_{\vec{x}_1+\vec{x}_2+\vec{x}_3+\vec{x}_4+(2i-1)\vec{c}} = x_1x_2x_3x_4S_{\vec{c}} \cdot S_{2\vec{c}} \cdots S_{2\vec{c}} && (i-1 \text{ copies of } S_{2\vec{c}}). \end{aligned}$$

It is thus clear that everything in $S^{-\vec{\omega}}$ is generated by elements in $S_{2\vec{c}}$ and $x_1x_2x_3x_4S_{\vec{c}}$. Furthermore, using the relation $x_1^2 + x_2^2 + x_3^2 = x_4^2$ we may ignore x_4^4, and likewise

$$x_1x_2x_3x_4^3 = x_1x_2x_3x_4(x_4^2) = x_1x_2x_3x_4(x_1^2 + x_2^2 + x_3^2) = x_1^3x_2x_3x_4 + x_1x_2^3x_3x_4 + x_1x_2x_3^3x_4,$$

so we may ignore $x_1x_2x_3x_4^3$. We conclude that $S^{-\vec{\omega}}$ is generated by the set

$$\{x_1^4, x_2^4, x_3^4, x_1^2x_2^2, x_1^2x_3^2, x_2^2x_3^2, x_1^3x_2x_3x_4, x_1x_2^3x_3x_4, x_1x_2x_3^3x_4\}.$$

Whilst it is possible to write the defining relations, we refrain from doing so as this will not be needed to understand $T^{-\vec{\omega}}$. In comparison with the later example in §6.3, the key point here is that $S^{-\vec{\omega}}$ is not a hypersurface.

Using the generators, the open charts on $T^{-\vec{\omega}}$ can easily be calculated, and are Spec of the following rings:

$$\begin{aligned} V_0 &= \left(\mathbb{C}\left[x_1, x_2, x_3, x_4, x_1^{-1}, t\right]\right)_0 = \mathbb{C}\left[\frac{x_2^2}{x_1^2}, \frac{x_3^2}{x_1^2}, \frac{x_2x_3x_4t}{x_1}, x_1^4t^2\right], \\ V_1 &= \left(\mathbb{C}\left[x_1, x_2, x_3, x_4, x_2^{-1}, t\right]\right)_0 = \mathbb{C}\left[\frac{x_1^2}{x_2^2}, \frac{x_3^2}{x_2^2}, \frac{x_1x_3x_4t}{x_2}, x_2^4t^2\right], \\ V_2 &= \left(\mathbb{C}\left[x_1, x_2, x_3, x_4, x_3^{-1}, t\right]\right)_0 = \mathbb{C}\left[\frac{x_1^2}{x_3^2}, \frac{x_2^2}{x_3^2}, \frac{x_1x_2x_4t}{x_3}, x_3^4t^2\right]. \end{aligned}$$

To present each abstractly, by symmetry we need only consider V_0. Set $x = \frac{x_2^2}{x_1^2}$, $y = \frac{x_3^2}{x_1^2}$, $z = x_1^4t^2$ and $u = x_1^{-1}x_2x_3x_4t$. Then the defining relation of S yields

$$u^2 = xyz(x + y + 1),$$

and there is an isomorphism

$$V_0 \cong \frac{\mathbb{C}[u,x,y,z]}{u^2 = xyz(x+y+1)} =: R.$$

We now blow up R along the (reduced) singular locus, which in this example is given by the ideal $I = (u, yz(2x+y+1), xz(2y+x+1), xy(x+y+1))$. It follows that the singular locus of $\operatorname{Spec} R$ has the following irreducible components

$$u = z = x+y+1 = 0, \tag{6.1.A}$$

$$u = z = y = 0, \tag{6.1.B}$$

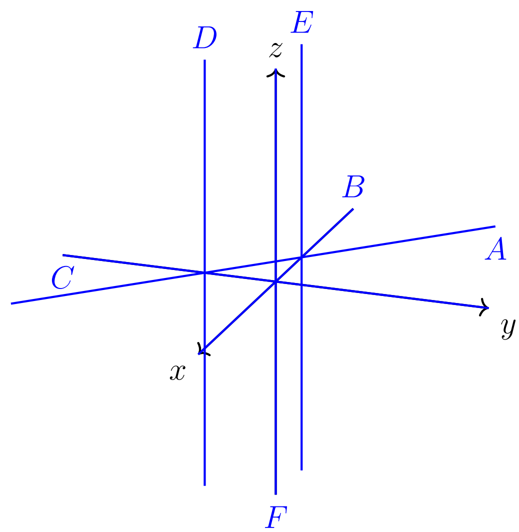
$$u = z = x = 0, \tag{6.1.C}$$

$$u = y+1 = x = 0, \tag{6.1.D}$$

$$u = y = x+1 = 0, \tag{6.1.E}$$

$$u = y = x = 0. \tag{6.1.F}$$

Ignoring the condition $u = 0$, since this always holds, we visualise the above as follows



We will next prove that the singularity transverse to the singular locus is generically an A_1 surface singularity. This will be the case at all points, except the three 'worst' points where (Zariski) locally the space is Spec of the ring $N := \frac{\mathbb{C}[U,X,Y,Z]}{U^2=XYZ}$, which is the coordinate ring of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ threefold quotient singularity.

Proposition 6.1.1. *Let* $R = \frac{\mathbb{C}[u,x,y,z]}{u^2=xyz(x+y+1)}$ *and* $N = \frac{\mathbb{C}[U,X,Y,Z]}{U^2=XYZ}$. *Then the following hold:*

1. $R_{(0,0,0,0)} \cong R_{(0,-1,0,0)} \cong R_{(0,0,-1,0)} \cong N_{(0,0,0,0)}$.

2. *At all other singular points* $\operatorname{Spec} R$ *is locally isomorphic to* $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$.

Proof. 1. We first argue that $R_{(0,0,0,0)} \cong N_{(0,0,0,0)}$. After localising R at the origin, $(0,0,0,0)$, $x+y+1$ is a unit. We first construct a map $\phi\colon N \to R$ sending

$$U \mapsto u, X \mapsto x, Y \mapsto y, Z \mapsto z(x+y+1).$$

Composing this with the canonical map $R \hookrightarrow R_{(0,0,0,0)}$ gives a map $N \to R_{(0,0,0,0)}$ given by $n \mapsto \frac{\phi(n)}{1}$. We have that $\phi^{-1}((u,x,y,z)) = (U,X,Y,Z)$ since $\phi(n)$ is invertible in $R_{(0,0,0,0)}$ if and only if $\phi(n) \in R \setminus (u,x,y,z)$, this implies that we get a map $\phi'\colon N_{(0,0,0,0)} \to R_{(0,0,0,0)}$ given by

$$\frac{r}{s} \mapsto \frac{\phi(r)}{\phi(s)}.$$

Conversely, consider the morphism $\varphi\colon R \to N_{(0,0,0,0)}$ sending

$$u \mapsto U, x \mapsto X, y \mapsto Y, z \mapsto \frac{Z}{X+Y+1}.$$

Again, this induces a map $R_{(0,0,0,0)} \to N_{(0,0,0,0)}$. Composing $\varphi' \circ \phi'$ we have

$$\varphi' \circ \phi'\left(\frac{r}{s}\right) = \frac{\varphi \circ \phi(r)}{\varphi \circ \phi(s)} \tag{6.1.G}$$

Since $\varphi \circ \phi(Z) = \varphi(z(x+y+1)) = Z$ and $\varphi \circ \phi$ is clearly the identity on U, X, Y it follows from (6.1.G) that $\varphi' \circ \phi' = \mathrm{id}$. Similarly $\phi' \circ \varphi' = \mathrm{id}$ and hence $R_{(0,0,0,0)} \cong N_{(0,0,0,0)}$.

The cases $(0,-1,0,0)$ and $(0,0,-1,0)$ are symmetric, so we only consider the former in which case x is a unit in $R_{(0,-1,0,0)}$. This is similar to before: define $\phi \colon N \to R$ by

$$U \mapsto u, X \mapsto xz, Y \mapsto y, Z \mapsto (x+y+1)$$

and proceed as before to construct an isomorphism $R_{(0,-1,0,0)} \cong N_{(0,0,0,0)}$.

2. Finally, consider a point on the singular locus away from these three 'worst' points. To illustrate this, choose $(0,1,0,0)$. Then localising R at this point, both x and $x+y+1$ are units and

$$R_{(0,1,0,0)} \cong \left(\frac{\mathbb{C}[U,X,Y,Z]}{U^2 = YZ} \right)_{(0,1,0,0)}.$$

This illustrates that at any point of the singular locus, away from the three 'worst' points, $\operatorname{Spec} R$ is Zariski locally isomorphic to $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$.

$\square$

We blow up the reduced singular locus, namely the ideal $I = (u, yz(2x+y+1), xz(2y+x+1), xy(x+y+1))$, to obtain a birational morphism

$$X \xrightarrow{f} \operatorname{Spec} R.$$

Proposition 6.1.2. *The morphism f is a resolution of singularities. In particular, X is smooth.*

The proof of Proposition 6.1.2 requires the following result. Let $\mathfrak{m} \lhd R$ be a maximal ideal, and consider the following pullback diagram

$$\begin{tikzcd} X' \arrow[r, "\phi"] \arrow[d, "g"] & X \arrow[d, "f"] \\ \operatorname{Spec} R_{\mathfrak{m}} \arrow[r, "h"] & \operatorname{Spec} R, \end{tikzcd}$$

where X' is the fiber product $X \times_{\operatorname{Spec} R} \operatorname{Spec} R_{\mathfrak{m}}$. Here g is the blow up of $I_{\mathfrak{m}}$ since blow ups are preserved under flat base change [Liu02, Lemma 3.48].

Lemma 6.1.3. *The closed points of X' are in bijection, via ϕ, with $f^{-1}(\mathfrak{m}) \subseteq X$. For each closed point $x \in X'$, the stalks $\mathcal{O}_{X',x}$ and $\mathcal{O}_{X,\phi(x)}$ are isomorphic.*

Proof. By [Sta18, Tag 01JS] X' admits an affine open cover of the form $\{W_i \times_{\operatorname{Spec} R} \operatorname{Spec} R_{\mathfrak{m}}\}$ with each W_i affine. Since the statement of the lemma is local on X', we may assume that $X' = W \times_{\operatorname{Spec} R} \operatorname{Spec} R_{\mathfrak{m}}$, where $W = \operatorname{Spec} A$ for some R-algebra A. Since W, $\operatorname{Spec} R$ and $\operatorname{Spec} R_{\mathfrak{m}}$ are affine, $X' = \operatorname{Spec}(A \otimes_R R_{\mathfrak{m}})$ by [Sta18, Tag 01JQ]. Let $\varphi\colon = R \to A$ be the (co)morphism corresponding to $f\colon W \to \operatorname{Spec} R$. Let $S = \varphi(R \setminus \mathfrak{m})$, which is a multiplicatively closed set of A (possibly containing zero). We claim that there is an isomorphism

$$A[S^{-1}] \xrightarrow{\sim} A \otimes_R R_{\mathfrak{m}}.$$

If $s = \varphi(r) \in S$ then $r^{-1} \in R_{\mathfrak{m}}$, and thus

$$(s \otimes 1)(1 \otimes r^{-1}) = s \otimes r^{-1} = 1 \otimes rr^{-1} = 1 \otimes 1.$$

This means that the map $A \to A \otimes_R R_{\mathfrak{m}}$, given by $a \mapsto a \otimes 1$, extends to a map $A[S^{-1}] \to A \otimes_R R_{\mathfrak{m}}$. This latter map is an isomorphism, since the inverse $A \otimes_R R_{\mathfrak{m}} \to A[S^{-1}]$ is given by

$$a \otimes br^{-1} \mapsto a\varphi(b)\varphi(r)^{-1}.$$

It follows that the maximal ideals of $A \otimes_R R_{\mathfrak{m}}$ are the maximal ideals $\mathfrak{n}$ of A not intersecting S. Let $\mathfrak{n}$ be one such maximal ideal. Since $S = \varphi(R \setminus \mathfrak{m})$, we have

$$\begin{aligned} S \cap \mathfrak{n} = \emptyset &\iff \varphi^{-1}(\mathfrak{n}) \subseteq \mathfrak{m} \\ &\iff \varphi^{-1}(\mathfrak{n}) = \mathfrak{m} \\ &\iff f(\mathfrak{n}) = \mathfrak{m}. \end{aligned}$$

Thus, $\mathfrak{n} \in f^{-1}(\mathfrak{m})$. Finally, if $x \in X'$ corresponds to such a maximal ideal $\mathfrak{n} \lhd A$, then

$S \cap \mathfrak{n} = \emptyset$ implies that $S \subseteq A \setminus \mathfrak{n}$. Consequently,

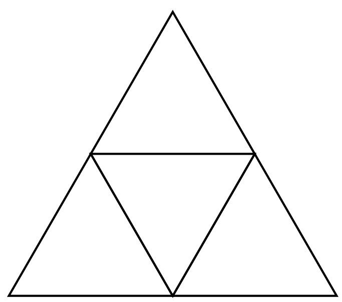
$$\mathcal{O}_{X,\phi(x)} = A_{\mathfrak{n}} = A[S^{-1}]_{\mathfrak{n}S^{-1}} \xrightarrow{\sim} (A \otimes_R R_{\mathfrak{m}})_{(\mathfrak{n}\otimes R_{\mathfrak{m}})} = \mathcal{O}_{X',x}. \qquad \square$$

Therefore, we can prove Proposition 6.1.2 if we can show that, for each maximal ideal of R, the pullback $X' \to \operatorname{Spec} R_{\mathfrak{m}}$ is a resolution of singularities.

Proof of Proposition 6.1.2. As explained earlier, the singular locus has 3 worst points, which locally look like

$$N = \frac{\mathbb{C}[U,X,Y,Z]}{U^2 = XYZ},$$

and the remainder of the singular locus is generically a transverse A_1 singularity. Now, as shown in Lemma A.1.1, the blow-up of N at the reduced singular locus (u, xy, yz, xz) equals the toric subdivision of the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ singularity



and this toric variety is smooth. It is well known, and easily verified, that blowing up an A_1 singularity at the singular locus also resolves it. Thus, applying Lemma 6.1.3, we can form the commutative diagram

$$\begin{tikzcd} Y \arrow[d, "g"] & Y' \cong X' \arrow[l] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} N & \operatorname{Spec} N_{\mathfrak{m}'} \cong \operatorname{Spec} R_{\mathfrak{m}} \arrow[l] \arrow[r, "h"] & \operatorname{Spec} R. \end{tikzcd}$$

Since g is a resolution of singularities, we deduce that f is a resolution in a neighbourhood of each of these points. By a similar diagram in general, we deduce that X is smooth. $\square$

6.2 The Family $(2,2,2,2n)$

Parts of the above analysis extend to the family $\mathbf{p} = (2,2,2,2n)$. However, in this more general family, singularities of the form $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_{2n})$ appear in $T^{-\vec{\omega}}$. Recall that $-\vec{\omega} = \vec{x}_1 + \vec{x}_2 + \vec{x}_3 + \vec{x}_4 - \vec{c}$ and

$$S = \frac{\mathbb{C}[x_1, x_2, x_3, x_4]}{x_1^2 + x_2^2 + x_3^2 = x_4^{2n}}.$$

Theorem 6.2.1. *For $\mathbf{p} = (2,2,2,2n)$, the 3 open charts covering $T^{-\vec{\omega}}$ are each isomorphic to the complete intersection*

$$\operatorname{Spec}\left(\frac{\mathbb{C}[u,v,x,y,z]}{(u^2 = xyv, v^n = z(x+y+1))}\right).$$

Proof. We proceed as in §6.1 by first considering all multiples of $-\vec{\omega}$ and then finding the associated generators of $S^{-\vec{\omega}}$. The following table calculates the multiples of $-\vec{\omega}$ and lists all the monomials of the corresponding weight:

$-\vec{\omega}$	$\vec{x}_1 + \vec{x}_2 + \vec{x}_3 + \vec{x}_4 - \vec{c}$	-
$-2\vec{\omega}$	$2(\vec{x}_1 + \vec{x}_2 + \vec{x}_3 + \vec{x}_4 - \vec{c}) = 2\vec{x}_4 + \vec{c}$	$x_1^2x_4^2, x_2^2x_4^2, x_3^2x_4^2, x_4^{2+2n}$
$-3\vec{\omega}$	$\vec{x}_1 + \vec{x}_2 + \vec{x}_3 + 3\vec{x}_4$	$x_1x_2x_3x_4^3$
$\vdots$	$\vdots$	$\vdots$
$-2n\vec{\omega}$	$2n(\vec{x}_1 + \vec{x}_2 + \vec{x}_3 + \vec{x}_4 + \vec{c}) = (n+1)\vec{c}$	$x_1^{2n+2}, x_2^{2n+2}, x_3^{2n+2}$
$-(2n+1)\vec{\omega}$	$x_1 + x_2 + x_3 + x_4 + n\vec{c}$	$x_1^{2a+1}x_2^{2b+1}x_3^{2c+1}x_4^{2nd+1}$

Here $a+b+c+d = n$ and $a,b,c,d \in \mathbb{Z}_{\geq 0}$. Using the generators in the third column, the open charts for $T^{-\vec{\omega}}$ are Spec of the following rings

$$V_0 = \left(\mathbb{C}\left[x_1, x_2, x_3, x_4, x_1^{-1}, t\right]\right)_0 = \mathbb{C}\left[\frac{x_2^2}{x_1^2}, \frac{x_3^2}{x_1^2}, \frac{x_2x_3x_4t}{x_1}, x_1^2x_4^2t^2, x_1^{2n+2}t^{2n}\right],$$

$$V_1 = \left(\mathbb{C}\left[x_1, x_2, x_3, x_4, x_2^{-1}, t\right]\right)_0 = \mathbb{C}\left[\frac{x_1^2}{x_2^2}, \frac{x_3^2}{x_2^2}, \frac{x_1x_3x_4t}{x_2}, x_2^2x_4^2t^2, x_2^{2n+2}t^{2n}\right],$$

$$V_2 = \left(\mathbb{C}\left[x_1, x_2, x_3, x_4, x_3^{-1}, t\right]\right)_0 = \mathbb{C}\left[\frac{x_1^2}{x_3^2}, \frac{x_2^2}{x_3^2}, \frac{x_1x_2x_4t}{x_3}, x_3^2x_4^2t^2, x_3^{2n+2}t^{2n}\right].$$

By symmetry, we need only consider V_0. Let $x = \frac{x_2^2}{x_1^2}$, $y = \frac{x_3^2}{x_1^2}$, $z = x_1^{2n+2}t^{2n}$, $u = x_1^{-1}x_2x_3x_4t$

and $v = x_1^2 x_4^2 t^2$. Then

$$v^n = z(x+y+1) \quad \text{and } u^2 = xyv$$

and further, since they are both three dimensional domains, there is an isomorphism

$$V_0 \cong \frac{\mathbb{C}[u,v,x,y,z]}{(u^2 = xyv, v^n = z(x+y+1))}.$$

□

We now restrict to the case $\mathbf{p} = (2,2,2,6)$ to illustrate the complexity of the general situation. In this case, as can be verified by Singular [DGPS22], the reduced singular locus is given by the ideal $(u, yv, xv, v^3 - z(x+y+1), yz(y+1), xyz, xz(x+1), xy(x+y+1))$, which splits into the following irreducible components

$$u = v = z = x + y + 1 = 0, \tag{6.2.A}$$

$$u = v = z = y = 0, \tag{6.2.B}$$

$$u = v = z = x = 0, \tag{6.2.C}$$

$$u = v = y + 1 = x = 0, \tag{6.2.D}$$

$$u = v = y = x + 1 = 0, \tag{6.2.E}$$

$$u = y = x = v^3 - z = 0. \tag{6.2.F}$$

We analyse a generic point on each one in turn:

1. First we have points of the form $(0,0,a,-a-1,0)$ such that $a \neq 0, -1$. Then x is a unit and y is a unit. Therefore $v = \frac{u^2}{xy}$ and $v^3 = z(x+y+1)$ so locally the space is isomorphic to the hypersurface $\frac{u^6}{x^3 y^3} = z(x+y+1)$ which we recognise as the $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_6)$.

2. Next we have points $(0,0,a,0,0)$, where $a \neq 0, -1$. Then x is a unit and $x+y+1$ is a unit. So $\frac{u^2}{x} = yv$ and $\frac{v^3}{x+y+1} = z$. This is locally a $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_2)$ singularity. At $a = -1$, $x+y+1$ is no longer a unit and we get the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_6)$ singularity acting as in 3.4.1.

3. Now consider the case $(0,0,-1,0,a)$, where $a \neq 0$. Then x and z are units. So $\frac{u^2}{x} = yv$ and $\frac{v^3}{z} = x+y+1$, which we recognise as the $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_2)$ singularity.

4. Finally we choose a point $(0,a,0,0,a^3)$ where $a \neq 0$. In this case v and z are units. Then $\frac{u^2}{v} = xy$ and $\frac{v^3}{z} = x+y+1$ which is the $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_2)$ singularity.

5. At the origin, $x+y+1$ is a unit and $\frac{v^3}{x+y+1} = z$ so this is the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ singularity singularity acting as in 3.4.1.

Thus we observe that locally the singularities are $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_2)$ or $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_6)$ along the lines of the singular locus. Then at the three 'worst' points (where the lines meet) the singularity is $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ at the origin or $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_6)$ at the other two points. Whilst we know the blow-up of $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ at the reduced singular locus is smooth, this is not the case for the blow-up of $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_6)$ at the reduced singular locus. This blow-up is covered by four open charts where three are not smooth. So we need to perform additional blow-ups in order to construct a full resolution. It is not clear what should be blown-up to further resolve the singularities.

6.3 When We Do Not Have Toda-Uehera Tilting

By Corollary 5.2.3, $\mathbf{p} = (3,3,3,3)$ is the simplest instance where the Toda-Uehara tilting bundle does not exist on $T^{-\vec{\omega}}$. Nonetheless, as we show below, the ring $S^{-\vec{\omega}}$ has a particularly nice presentation. We show that it is a hypersurface in $\mathbb{C}^4$. The graded pieces of $S^{-\vec{\omega}}$ are built as follows

$-\vec{\omega}$	$\vec{x}_1+\vec{x}_2+\vec{x}_3+\vec{x}_4-\vec{c}$	-
$-2\vec{\omega}$	$2(\vec{x}_1+\vec{x}_2+\vec{x}_3+\vec{x}_4-\vec{c})$	-
$-3\vec{\omega}$	$\vec{c}$	$x_1^3, x_2^3, x_3^3, x_4^3$
$-4\vec{\omega}$	$\vec{x}_1+\vec{x}_2+\vec{x}_3+\vec{x}_4$	$x_1x_2x_3x_4$
$\vdots$	$\vdots$	$\vdots$
$-3n\vec{\omega}$	$n\vec{c}$	$x_1^{3a''}x_2^{3b''}x_3^{3c''}x_4^{3d''}$
$-(3n+1)\vec{\omega}$	$x_1+x_2+x_3+x_4+(n-1)\vec{c}$	$x_1^{3a+1}x_2^{3b+1}x_3^{3c+1}x_4^{3d+1}$
$-(3n+2)\vec{\omega}$	$2(x_1+x_2+x_3+x_4)+(n-2)\vec{c}$	$x_1^{3a'+2}x_2^{3b'+2}x_3^{3c'+2}x_4^{3d'+2}$

where $a''+b''+c''+d''=n$, $a+b+c+d=n-1$ and $a'+b'+c'+d'=n-2$. It follows that $S^{-\vec{\omega}}$ is generated by $x_1x_2x_3x_4$, x_1^3, x_2^3, x_3^3 and x_4^3. Set $U=x_1x_2x_3x_4$, $X=x_1^3$, $Y=x_2^3$ and $Z=x_4^3$. Then the defining relation of S gives $U^3=XYZ(X+Y+1)$ and so

$$S^{-\vec{\omega}} \cong \frac{\mathbb{C}[U,X,Y,Z]}{U^3=XYZ(X+Y+Z)}.$$

Note, in comparison to the $[2,2,2,2]$ case considered earlier, it is rather surprising that now $S^{-\vec{\omega}}$ is a hypersurface. In a very similar way to before, the three open charts for $T^{-\vec{\omega}}$ are Spec of the following rings:

$$\begin{aligned}
V_0 &= \left(\mathbb{C}\left[x_1,x_2,x_3,x_4,x_1^{-1},t\right]\right)_0 = \mathbb{C}\left[\frac{x_2^3}{x_1^3},\frac{x_3^3}{x_1^3},\frac{x_2x_3x_4}{x_1^2}t, x_1^3t^3\right],\\
V_1 &= \left(\mathbb{C}\left[x_1,x_2,x_3,x_4,x_2^{-1},t\right]\right)_0 = \mathbb{C}\left[\frac{x_1^3}{x_2^3},\frac{x_3^3}{x_2^3},\frac{x_1x_3x_4}{x_2^2}t, x_2^3t^3\right],\\
V_2 &= \left(\mathbb{C}\left[x_1,x_2,x_3,x_4,x_3^{-1},t\right]\right)_0 = \mathbb{C}\left[\frac{x_2^3}{x_3^3},\frac{x_1^3}{x_3^3},\frac{x_1x_2x_4}{x_3^2}t, x_3^3t^3\right].
\end{aligned}$$

By symmetry, we need only consider V_0. Let $\frac{x_2^3}{x_1^3}=x$, $\frac{x_3^3}{x_1^3}=y$, $x_1^3t^3=z$ and $x_1^{-2}x_2x_3x_4t=u$. Then $u^3=x_1^{-6}x_2^3x_3^3x_4^3t^3=xyz(x+y+1)$ and so

$$V_0 \cong \frac{\mathbb{C}[u,x,y,z]}{u^3=xyz(x+y+1)}.$$

To understand the singularities on V_0, we proceed as in §6.2. In this case, the reduced singular locus is $(u^2, yz(2x+y+1), xz(2y+x+1), xy(x+y+1))$ which has the following irreducible components

$$u = z = x + y + 1 = 0, \tag{6.3.A}$$

$$u = z = y = 0, \tag{6.3.B}$$

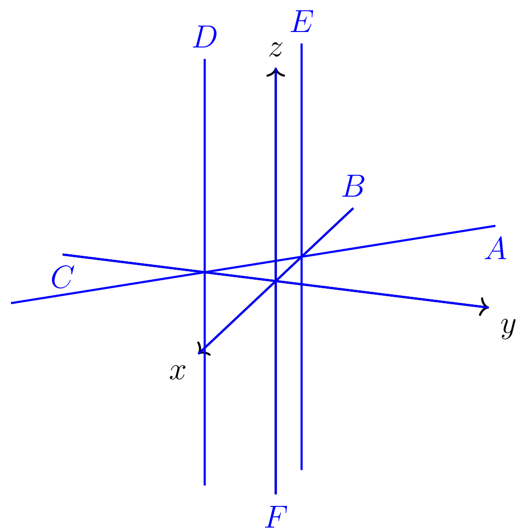
$$u = z = x = 0, \tag{6.3.C}$$

$$u = y + 1 = x = 0, \tag{6.3.D}$$

$$u = y = x + 1 = 0, \tag{6.3.E}$$

$$u = y = x = 0. \tag{6.3.F}$$

We can view this pictorially as we did in the case $(2, 2, 2, 2)$ by ignoring $u = 0$



Analysing each irreducible component in turn we observe the following,

1. For points of the form $(0, 0, -1, a)$, where $a \neq 0$, we have that y and z are units and $\frac{u^3}{yz} = x(x + y + 1)$. We recognise this as the $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_3)$ singularity. When $a = 0$ only y is a unit and the singularity is $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$.

2. By symmetry, we observe that at points of the form $(0, -1, 0, a)$ we also have the $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_3)$ singularity except where $a = 0$ then the singularity is $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$.

3. At the origin, $x + y + 1$ is a unit and $\frac{u^3}{x+y+1} = xyz$ which is the $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ singularity.

Continuing in this way, we observe that away from the three 'worst' points the singularities are locally $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_3)$ and at the 'worst' points the singularity is $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. Again, we now have a choice of possible resolutions.

Appendix A

Blow-Ups and Local NCCRs

This Appendix is a local analysis of the three-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity, its blow-up at the singular locus, and a NCCR. Parts of this analysis are then applied to find local NCCRs on $T^{-\vec{\omega}}$ for particular tuples of $\mathbf{p}$.

Consider the skew group ring $\mathbb{C}[X,Y,Z]\#G$ for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, which is by definition the vector space $\mathbb{C}[X,Y,Z] \otimes_{\mathbb{C}} \mathbb{C}G$ with multiplication $(f_1 \otimes g_1)(f_2 \otimes g_2) = (f_1 \cdot g_2(f_2)) \otimes g_1 g_2$. We begin by calculating the McKay quiver associated to this group action.

Definition A.0.1. *For a given finite group G acting on $\mathbb{C}^3 = V$, the McKay quiver is the quiver with vertices corresponding to the isomorphism classes of irreducible representations of G and the number of arrows from ρ_i to ρ_j given by*

$$\dim \operatorname{Hom}_{\mathbb{C}G}(\rho_i, \rho_j \otimes V).$$

The quiver can be constructed from the character table of the group G.

Example A.0.2. Consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ with presentation $\langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle$. Then G has four irreducible characters with corresponding character table

	e	a	b	ab
χ_0	1	1	1	1
χ_1	1	-1	1	-1
χ_2	1	-1	-1	1
χ_3	1	1	-1	-1

From above we can see that

$$\rho_0 \otimes V \cong \rho_1 \oplus \rho_2 \oplus \rho_3,$$

which gives the following arrows in the McKay quiver.

$$\begin{tikzcd}
\rho_1 \arrow[d] & \rho_2 \arrow[dl] \\
\rho_0 & \rho_3 . \arrow[l]
\end{tikzcd}$$

Similarly,

$$\rho_1 \otimes V \cong \rho_0 \oplus \rho_2 \oplus \rho_3$$

$$\rho_2 \otimes V \cong \rho_0 \oplus \rho_1 \oplus \rho_3$$

$$\rho_3 \otimes V \cong \rho_0 \oplus \rho_1 \oplus \rho_2,$$

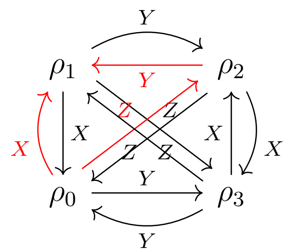
and so the McKay quiver of G is then

$$\begin{tikzcd}
\rho_1 \arrow[r, bend left, "Y"] \arrow[d, "X"] \arrow[dr, shift left, "Z"] & \rho_2 \arrow[l, "Y"] \arrow[d, bend left, "X"] \arrow[dl, shift left, "Z"] \\
\rho_0 \arrow[u, bend left, "X"] \arrow[r, "Y"] \arrow[ur, shift left, "Z"] & \rho_3 \arrow[l, bend left, "Y"] \arrow[u, "X"] \arrow[ul, shift left, "Z"]
\end{tikzcd}$$

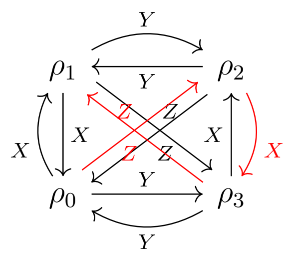
Adding in the the relations $XY = YX$, $XZ = ZX$ and $YZ = ZY$, whenever that makes sense, gives a presentation of the skew group ring [CMT07, Remark 2.7] which is known to be a NCCR of $\mathbb{C}[X, Y, Z]^G$ as explained in [VdB04, Example 1.1].

Lemma A.0.3. *The skew group ring $\mathbb{C}[X, Y, Z]\#G$, where $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, is isomorphic to $\mathrm{End}_R(M)$ where $M = R \oplus (u, x) \oplus (u, y) \oplus (u, z)$, and $R = \frac{\mathbb{C}[x,y,z,u]}{u^2=xyz}$.*

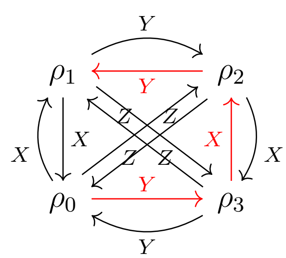
Proof. We first identify the R-module M_1 of all paths $\rho_0 \to \rho_1$ with the R-module (u, x).



Certainly, X and YZ are paths from ρ_0 and ρ_1. We claim that X and YZ generate M_1 (up to cycles). Any other path $\rho_0 \to \rho_1$ must pass through ρ_2 or ρ_3. If it passes through ρ_2 there is one path which is not X and does not obviously contain cycles, namely the red path



However $ZXZ = ZZX = Z^2X$, which is X multiplied by a cycle. Similarly, for the case where $\rho_0 \to \rho_1$ passes through ρ_3. There is only one choice which does not contain cycles, and is not $YZ = ZY$, namely



However $YXY = YYX = Y^2(X)$ and we see that this map is generated by a multiple of X. Therefore X and YZ generate M_1 as claimed. We note that M_1 is isomorphic to the ideal (u, x) since $X(X, YZ) = (X^2, XYZ) = (x, u)$. The proofs that the module of paths $\rho_0 \to \rho_2$ is isomorphic to(u, z), and the module of paths $\rho_1 \to \rho_3$ is (u, y) are similar. Now, by Auslander [Aus62]

$$\mathbb{C}[X, Y, Z]\#G \cong \mathrm{End}_{\mathbb{C}[X,Y,Z]^G}\left(\bigoplus_{\rho \in \mathrm{Irr}\, G} (\mathbb{C}[X, Y, Z] \otimes \rho)^G\right)$$

which, by our previous computations, can be identified with $\mathrm{End}_R\big(R \oplus (u,x) \oplus (u,y) \oplus (u,z)\big)$.

□

Corollary A.0.4. *$\mathrm{End}_R(M)$ is an NCCR of $R = \frac{\mathbb{C}[u,x,y,z]}{u^2=xyz}$.*

Proof. This follows from Lemma A.0.3, together with the fact that $\mathbb{C}[X,Y,Z]\#G$ is an NCCR [VdB04, Example 1.1]. □

A.1 A Blow-up

Now, we consider the blow-up of $R = \frac{\mathbb{C}[u,x,y,z]}{u^2=xyz}$ along the reduced singular locus.

Lemma A.1.1. *The blow up of $\operatorname{Spec} R$ along $I := (xy, yz, xz, u)$ is a projective birational morphism*

$$X \to \operatorname{Spec} R,$$

where X is smooth.

Proof. The blow-up X is covered by 4 open charts, the first of these is

$$U_1 = \operatorname{Spec} \mathbb{C}[x_1, y_1, y_2, y_3]/(y_1y_2y_3 - x_1) \cong \operatorname{Spec} \mathbb{C}[y_1, y_2, y_3]$$

with map to base via $(y_1, y_2, y_3) \in U_1 \mapsto (u = y_1y_2y_3, x = y_2y_3, y = y_1y_3, z = y_1y_2)$. We also have

$$U_2 = \operatorname{Spec} \mathbb{C}[x_3, x_4, y_0, y_3]/(x_3y_0^2 - y_3) \cong \operatorname{Spec} \mathbb{C}[x_3, x_4, y_0]$$

with map to the base by $(x_3, x_4, y_0) \mapsto (x_3x_4y_0, x_4x_3y_0^2, x_3, x_4)$. Charts U_3 and U_4 are symmetric to U_2.

□

We now investigate what the preimage of the singular locus on each of these four charts looks like. Note that the singular locus is the union of the three lines $\{(0, \lambda, 0, 0)\}$, $\{(0, 0, \lambda, 0)\}$ and $\{0, 0, 0, \lambda)\}$.

For the chart U_1, above the origin we have that $y_1y_2y_3 = 0$, $y_2y_3 = 0$, $y_1y_3 = 0$ and $y_1y_2 = 0$ which is the union of three lines $(y_1, 0, 0), (0, y_2, 0), (0, 0, y_3)$.

Above $(0, \lambda, 0, 0)$ we have $\lambda = y_2y_3$ and $y_1 = 0$ giving the curve $\{(0, y_2, y_3) \mid y_2y_3 = \lambda\}$. By symmetry we have $\{(y_1, 0, y_3) \mid y_1y_3 = \lambda\}$ and $\{(y_1, y_2, 0) \mid y_1y_2 = \lambda\}$ above $(0, 0, \lambda, 0)$ and $(0, 0, 0, \lambda)$ respectively.

By computing the preimage of the singular locus on U_2 we will then know what is happening on U_3 and U_4 by symmetry. Above the origin, we have $x_3 = x_4 = 0$ and there is a choice for y_0. Above $(0, \lambda, 0, 0)$ lies nothing, above $(0, 0, \lambda, 0)$ $x_4 = 0$, $x_3 = \lambda$ and y_0 is varying. Finally, above $(0, 0, 0, \lambda)$ we have $x_4 = \lambda$, $x_3 = 0$ and y_0 varying. Therefore, we have the following picture

	U_1	U_2	U_3	U_4
$(0, 0, 0, 0)$	3 lines	line	line	line
$(0, \lambda, 0, 0)$	line	-	line	line
$(0, 0, \lambda, 0)$	line	line	-	line
$(0, 0, 0, \lambda)$	line	line	line	-

We now consider what is happening above the origin and glue the charts together. First we consider $U_1 \cap U_2$. We have $y_1y_2y_3 = x_3x_4y_0$, $y_2y_3 = x_3x_4y_0^2$, $y_1y_3 = x_3$, and $y_1y_2 = x_4$. By simple manipulation we see that $x_3x_4y_0 = y_1y_2y_3 = y_2x_3$ and hence $y_2 = x_4y_0$. Similarly, $y_3 = x_3y_0$. Finally,

$$\begin{aligned} y_1y_2y_3 &= x_3x_4y_0 \\ \iff y_1(x_3x_4y_0^2) &= x_3x_4y_0 \\ \iff y_1 &= y_0^{-1}. \end{aligned}$$

Therefore we glue via

$$(y_1, 0, 0) \leftrightarrow (0, 0, y_1^{-1}).$$

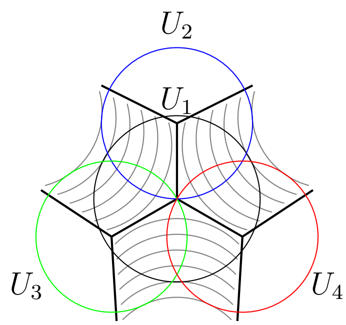
By symmetry we can also state the glue on $U_1 \cap U_3$ and $U_1 \cap U_4$ as

$$(0, y_2, 0) \leftrightarrow (0, 0, y_2^{-1})$$

and

$$(0, 0, y_3) \leftrightarrow (0, 0, y_3^{-1})$$

respectively. We can now visualise the exceptional local of the blow-up in the following picture.



A.2 Local NCCRs on $T^{-\vec{\omega}}$

In several cases considered in §6, the singularity $u^2 = xyz(x + y + 1)$ appeared as one of the open sets of an affine covering of $T^{-\vec{\omega}}$. In this subsection, we briefly justify that this singularity also has a local NCCR. The reason that this is a non-trivial statement that does not follow directly from the results of §A.1 is because of the gluing: it is not a prori clear that the local NCCRs can glue to a global NCCR of this singularity. This is similar to the problem of constructing a tilting bundle on $T^{-\vec{\omega}}$, which we considered in §5.

Lemma A.2.1. *Fix* $R = \frac{\mathbb{C}[u,x,y,z]}{u^2=xyz(x+y+1)}$. *Let* $M = R \oplus (u, x) \oplus (u, y) \oplus (u, z) \oplus (u, x+y+1)$. *Then* $\operatorname{End}_R(M)$ *is an NCCR of* R.

Proof. We consider the completion of M at every closed point in the singular locus of $\operatorname{Spec} R$; the singular locus is has been described in 6.1. First, at the origin $P = (u, x, y, z)$ we have

$$\widehat{M}_P = \widehat{R}_P \oplus \widehat{(u, x)}_P \oplus \widehat{(u, y)}_P \oplus \widehat{(u, z)}_P \oplus \widehat{(u, x + y + 1)}_P.$$

Since at the origin $x+y+1$ is a unit, this is isomorphic to $\widehat{R}_P^{\oplus 2}\oplus\widehat{(u,x)}_P\oplus\widehat{(u,y)}_P\oplus\widehat{(u,z)}_P$ which is known to give an NCCR by Lemma A.0.4.

Let $P=(0,a,0,0)$, where $a\neq -1,0$. Then x and $x+y+1$ are units and hence

$$\widehat{M}_P=\widehat{R}^{\oplus 3}\oplus\widehat{(u,y)}_P\oplus\widehat{(u,z)}_P.$$

Now, on the completion

$$\frac{z}{u}(u,y)=\frac{1}{u}(uz,yz)=\frac{1}{u}(uz,u^2)=(z,u).$$

Therefore after the completion, $(u,y)\cong(u,z)$, and

$$\widehat{M}_P=\widehat{R}^{\oplus 3}\oplus\widehat{(u,y)}_P^{\oplus 2}.$$

Therefore, $\widehat{M}_P$ gives an NCCR by Lemma A.0.4.

Suppose $a=-1$. Then x is a unit and

$$\widehat{M}_P=\widehat{R}^{\oplus 2}\oplus\widehat{(u,y)}_P\oplus\widehat{(u,z)}_P\oplus\widehat{(u,x+y+1)}_P$$

which we know gives an NCCR by Lemma A.0.4. By symmetry, it is also easy to see completing at $(0,0,a,0)$ gives an NCCR. Now consider $(0,-1,0,a)$. Then x is a unit and z is a unit. Therefore,

$$\begin{aligned}\widehat{M}_P&=\widehat{R}_P\oplus\widehat{R}_P\oplus\widehat{(u,y)}_P\oplus\widehat{R}_P\oplus\widehat{(u,x+y+1)}_P\\&=\widehat{R}_P^{\oplus 3}\oplus\widehat{(u,y)}_P^{\oplus 2},\end{aligned}$$

since

$$(u, x+y+1) = \frac{1}{u}(u^2, u(x+y+1)) = \frac{1}{u}(y(x+y+1), u(x+y+1)) = \frac{x+y+1}{u}(y, u).$$

This gives an NCCR by Lemma A.0.4. By symmetry $(0, 0, -1, a)$ also gives an NCCR.

Next, consider $(0, 0, 0, a)$ with $a \neq 0$. Then z and $x+y+1$ are units and

$$\begin{aligned}
\widehat{M}_P &= \widehat{R}_P \oplus \widehat{(u,x)}_P \oplus \widehat{(u,y)}_P \oplus \widehat{R}_P \oplus \widehat{R}_P \\
&= \widehat{R}_P^{\oplus 3} \oplus \widehat{(u,x)}_P \oplus \widehat{(u,y)}_P \\
&= \widehat{R}_P^{\oplus 3} \oplus \widehat{(u,x)}_P^{\oplus 2}.
\end{aligned}$$

Finally, at $(0, a, -1-a, 0)$ where $a \neq 0, -1$, then x and y are units and

$$\begin{aligned}
\widehat{M}_P &= \widehat{R}^{\oplus 3} \oplus \widehat{(u,z)}_P \oplus (u, \widehat{x+y}+1_P \\
&= \widehat{R}^{\oplus 3} \oplus \widehat{(u,z)^{\oplus 2}}.
\end{aligned}$$

Then since $\widehat{M}_P$ is an NCCR for all maximal ideals P, then globally M gives an NCCR by [IW14a, Cor. 5.5]. $\square$

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