Stochastic control problems for pension schemes

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Abstract

The last decades have witnessed unexpected changes in life expectancy, low financial market returns and surging inflation. Pension schemes all over the world are facing a period of extreme changes and challenges. Risk management in pension schemes is becoming highly complex and tends to be a major economic and financial topic. At the same time, stochastic optimization methods have become important tools used in fields of economics, finance and life insurance. This Ph.D. thesis is devoted to focusing on risk management and asset allocation for pension schemes in a dynamic way. We intend to develop continuous-time stochastic optimization models to tackle pension issues. Chapter 1 overviews pension schemes and risks and briefly discusses the stochastic optimization methods.

Chapter 2 studies the longevity risk management in a defined contribution pension scheme that promises a minimum guarantee such that members are able to purchase lifetime annuities upon retirement. To hedge the longevity risk, the scheme manager decides to invest in a mortality-linked security that is available on the financial market, typically a longevity bond. The manager's compensation depends on the surplus between the scheme's final wealth and the minimum guarantee. The manager maximizes his expected utility from terminal compensation by controlling the investment strategy. We transform the corresponding constrained optimal investment problem into a single investment portfolio optimization problem by replicating future contributions from members and the minimum guarantee provided by the scheme. We solve the resulting optimization problem using the dynamic programming principle. Through a series of numerical studies, we show that longevity risk has an important impact on investment performance. Our results add to the growing evidence supporting the use of mortality-linked securities for efficient hedging of longevity risk.

Chapter 3 investigates the hedging performance of the longevity bond and the role of the risk-sharing rule in a pension scheme. The scheme manager invests in a longevity bond whose coupon payment is linked to a survival index to hedge the longevity risk. We use stochastic affine processes to model the force of mortality and investigate longevity basis risk, which arises when the mortality behavior of the members and the longevity bond's reference population are not perfectly correlated. The problem is to maximize both member's and manager's utilities by controlling the investment strategy and benefit withdrawals. By applying the dynamic programming principle, we derive optimal solutions for the single- and sub-population cases. Our numerical results show that the longevity bond acts as an effective hedging instrument, even in the presence of longevity basis risk. Also, we find that the risk-sharing rule is beneficial to both the member and the manager.

Chapter 4 turns to the situation where a DB scheme sponsor plans to wind up the scheme via an insurance buy-out. The sponsor's objective is to minimize the expected quadratic deviation of the terminal scheme wealth from the buy-out cost by deciding the investment strategy and winding up time. We derive the explicit solution to the combined stochastic control and optimal stopping problem by solving the corresponding variational Hamilton-Jacobi-Bellman inequality. Our analyses show that if the scheme wealth is initially lower than the technical provisions, it is optimal to purchase the buy-out when the funding level touches a threshold under specific financial and insurance markets conditions.

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Declaration

I declare that, except where explicit reference is made to the contribution of others, that this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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Chapter 1

Introduction

A pension is a retirement arrangement that provides an individual with retirement income. The essence of pensions is to save while working such that individuals will have supplemental income to maintain the living standard or to cover unforeseen expenses in retirement. Pension schemes provide an important economic function in society. Future development and management of pension systems are clearly essential to society since it is a matter that bears on the welfare of each and every pension scheme participant across the world. This thesis is devoted to developing and analyzing stochastic optimization problems for the risk management of pension schemes. More specifically, this thesis pays particular attention to the hedging of mortality risk in pension schemes and Pension Risk Transfer.

1.1 Pension scheme and risks

A pension scheme is an organized mechanism to provide retired people with regular incomes after retirement. There are two main types of pension schemes: *Defined Benefit schemes* (hereafter DB schemes) and *Defined Contribution schemes* (hereafter DC schemes). One main difference between DB and DC schemes is how risks and liabilities are treated. In a DB scheme, the benefit payments are pre-defined and usually depend on the employee's final/mean salary and on the number of years of service. Generally, the sponsor pays flexible contributions while the employee's contribution rates are fixed. After retirement, benefits are normally paid in the form of a lifetime annuity. Thus, the sponsor faces the risk of failure to cover the liabilities, whereas the employee bears no risks. In a DC scheme, the sponsor's and/or the employee's contribution rates are pre-determined, typically as a percentage of the employee's salary. The benefits are generated by the accumulated contributions and depend on the scheme's investment performance. At retirement, DC scheme members have more freedom over how they choose to take the benefits. Options available are, for example, taking the money to purchase a lifetime annuity, choosing a flexible income-drawdown option (i.e., to withdraw money periodically with the remaining money staying in the pension pot), and taking a lump sum (or lump sums). In this case, the employee bears all the risks rather than the sponsor.

The large variety of pension scheme risks could be classified into two main categories: financial risks and demographic risks. See Devolder et al. (2013). Apart from the risks in the financial market (e.g., investment risk and interest rate risk), pension schemes are also exposed to risks outside the financial market (i.e., the background risks). Typical background risks include, for example, the labor income risk (i.e., the risk caused by uncertainties of the member's future salary), credit risk and inflation risk. One of the main demographic risks is the *longevity risk*, i.e., the risk that the actual life expectancy may be longer than anticipated. For a DB scheme, the sponsor needs to decide the investment strategy and adjust contribution rates in order to maintain the fund in balance. Thus, it is clear that the sponsor faces risks such as market risk, inflation risk and contribution rate risk. If the DB scheme pays benefits in the form of annuities, then the sponsor's liabilities increase if members' life span increases and longevity risk is crucial to the sponsor. On the contrary, the sponsor bears no risks in a DC scheme since his only responsibility is to pay pre-defined contributions. However, the member faces all financial risks (e.g., bad investment returns or significant inflation). Moreover, the member faces the risk of outliving his pension and savings (i.e., longevity risk).

1.2 Research background and motivation

Pension schemes' risk management and asset allocation problems are popular research topics. How do pension schemes optimally invest in the financial market? What are the efficient ways to hedge pension risks? Are longevity securities efficient instruments to hedge some of the pension scheme risks? Whether or not and when should a DB sponsor offload the risk and wind up the scheme? This thesis seeks to contribute to the literature by applying stochastic methods to answer these questions. In what follows, we present a general overview of the related literature and outline our motivation and contribution.

Over the last decades, DC schemes have become increasingly popular among sponsors, and there has been a gradual shift from DB towards DC schemes in the occupational pension systems. The asset allocation is crucial in DC schemes as members' retirement benefits depend heavily on schemes' investment performance. The existing literature on optimal portfolio selection problems in DC schemes using stochastic optimization methods is rich. There are two broad optimization structures: utility maximization and mean-variance criteria. Papers that maximized the expected utility from terminal wealth in DC schemes include, for example, Gao (2008), Battocchio & Menoncin (2004) and Zhang & Ewald (2010). Gao (2008) considered the stochastic interest rate and used the dual approach to solve the optimal control problem. Battocchio & Menoncin (2004) studied the salary risk and the inflation risk and solved the optimization problem by stochastic dynamic programming. Zhang & Ewald (2010) applied the martingale method to solve the utility maximization problem with inflation. Inflation-indexed bonds were used to hedge inflation risk. Mean-variance criteria aim to maximize the profit while minimizing the loss/risk. Previous studies on mean-variance problems in DC schemes include, for example, Yao et al. (2013), He & Liang (2013a), He & Liang (2013b), Vigna (2014) and He & Liang (2015). Nonetheless, these works supposed that scheme members have full trust in the manager and do not take the attractiveness and effectiveness of the scheme management into consideration.

DC schemes that provide a minimum guarantee on benefits can be more attractive to employers. Boulier et al. (2001) and Deelstra et al. (2003) studied optimal asset allocation problems with stochastic interest rate and minimum guarantees protections. Deelstra et al. (2004) further studied the optimal design of minimum guarantee and used the martingale approach. Han & Hung (2012) and Guan & Liang (2014) extended the model to include stochastic inflation. However, these papers did not consider members' stochastic mortality behavior and ignored mortality risk. If scheme members' life expectancy and actual survival rate exceed expectations, then the value of the minimum guarantee will be higher than anticipated. In other words, the scheme faces longevity risk. To fill this gap, we model the members' force of mortality (also referred to as a mortality intensity) using stochastic affine processes in Chapter 2. See Luciano & Vigna (2005), Menoncin (2009), Russo et al. (2011) and Xu et al. (2020). Moreover, to hedge the longevity risk, we suppose that the scheme manager decides to invest in a longevity bond. See Blake & Burrows (2001), Menoncin (2008), De Kort & Vellekoop (2017) and Shen & Sherris (2018).

The population basis risk arises when there is a mismatch between the hedging instrument's underlying mortality experience and the hedging population's mortality behavior. A scheme manager faces the longevity basis risk when a standardized longevity security (e.g., an indexed-based longevity bond) is used for hedging. In the literature, Coughlan et al. (2011), A. J. Cairns et al. (2014), De Rosa et al. (2017) argued that the presence of the population basis risk may have a detrimental impact on the hedging effectiveness of the instrument. Drawing from econometric literature, co-integration has been incorporated into mortality modeling, see Njenga & Sherris (2011) and Darkiewicz & Hoedemakers (2004). In Wong et al. (2014), the authors applied the co-integration technique to study the mean-variance hedging of longevity risk for an insurance company with a longevity bond. They suggested that co-integration is vital in longevity risk management. However, the impact of longevity basis risk on the risk management and effectiveness of longevity bond has not been studied in the literature on pension schemes. In Chapter 3, we study an optimal investment strategy and benefit withdrawal problem in a pension scheme that provides an income-drawn policy. To manage the scheme efficiently, we suppose the manager and members agree to share the risks, and the objective is to maximize both sides' utilities. Similar to Wong et al. (2014), our stochastic mortality models are designed to allow the scheme members and the longevity bond's reference population to have different mortality behaviors.

DC schemes have become increasingly popular as they are less difficult and less costly to maintain, and many employers have opted to shift from DB to DC schemes. Nonetheless, some still retain their DB schemes, and many companies still face an important choice of whether or not to de-risk by transferring their pension fund to another party. For example, via an Insurance buy-in or buy-out, a company can transfer part or all of its risks to an insurer. A buy-out also enables a scheme to wind up. A DB sponsor, who plans to purchase an insurance buy-out and thus to wind up the scheme, has to decide when to purchase the buy-out and how to manage the scheme until the buy-out is purchased. As far as we know, this optimal buy-out purchasing problem for DB schemes has not yet been studied in the literature. Chapter 4 intends to fill this gap and provide a new direction in modeling and tackling this problem. We suppose that the scheme sponsor's objective is to minimize the terminal solvency risk by controlling the investment strategy and determining the insurance buy-out purchasing time. Solvency risk in DB schemes are considered in papers such as Haberman et al. (2000), Josa-Fombellida & Rincón-Zapatero (2004), Josa-Fombellida & Rincón-Zapatero (2010), Hainaut & Deelstra (2011), Josa-Fombellida & Rincón-Zapatero (2012) and Hainaut (2014). Drawing from the existing literature on optimal annuitization time for individual retirees, we formulate our minimization problem as a combined stochastic control and optimal stopping time problem. See Stabile (2006), Farhi & Panageas (2007), Milevsky & Young (2007), Gerrard et al. (2012), Park & Jang (2014) and Park (2015).

This thesis contributes to the literature by developing new ideas and directions in tackling the risk management of pension schemes from a theoretical point of view. Specifically, Chapters 2 and 3 contribute to the literature by studying stochastic control problems for pension schemes in the presence of longevity risk and longevity basis risk. Besides, the two chapters investigate the hedging effectiveness of longevity bond. Chapter 4 extends the current literature on DB schemes by developing and tackling an optimal insurance buy-out purchasing problem.

1.3 Thesis outline

In Section 1.4, we present some methods and techniques used in the subsequent sections of this thesis. Chapters 2 to 4 are three independent but related chapters focusing on the risk management and asset allocation for pension schemes in a dynamic way. In a continuous-time framework, we formulate and analyze various stochastic optimization models arising from maximization and minimization problems.

In Chapter 2, we study an optimal control problem for a DC scheme that promises a minimum guarantee on the purchase of lifetime annuities. The longevity risk arises since the annuity price is unknown until the retirement time and depends on the members' future longevity trend. An analog Cox-Ingersoll-Ross model is introduced to model the members' mortality behavior. To hedge the longevity risk, the manager decides to invest in a mortality-linked security that is available on the financial market. The manager receives a fraction of the final surplus between the scheme wealth and the minimum guarantee as compensation and aims to maximize his expected utility from terminal wealth.

In Chapter 3, we investigate the first-best principal-agent problem for a pension scheme, where the manager and the member decide to share the underlying risk based on a risk-sharing rule. We describe the mortality behaviors of the scheme members and the longevity bond's reference population using affine class models. In the case where the scheme members are a sub-population of the longevity bond's reference population, the hedging effectiveness of the longevity bond is reduced due to the presence of longevity basis risk. At any instant of time, the manager controls the investment strategy and benefit withdrawal rate. The objective is to maximize both members' and manager's utilities, and the maximization problem is naturally formulated as a stochastic optimal control problem.

Chapter 4 is devoted to finding the optimal timing of the insurance buy-out purchase for a DB scheme. The DB scheme we considered is closed to new entrants, and all its members are pensioners. Up until the buy-out is purchased, the sponsor pays surviving members benefits and invests in the financial market. The sponsor's objective is to minimize the solvency risk, which is measured by the expected quadratic deviation of the terminal scheme wealth from the buy-out cost, by deciding the investment and buy-out purchasing strategies. We formulate the problem as a combined stochastic control and optimal stopping problem.

1.4 Preliminaries: stochastic optimization methods

When randomness is present, decision problems are often formulated as stochastic optimization problems where decision makers wish to optimize (maximize or minimize) some performance criteria in a dynamic way. Stochastic optimization methods have become essential tools used in a wide range of fields and applications, such as computer science, engineering and economics. There are discrete- and continuous-time stochastic optimization problems. This thesis only investigates stochastic optimization problems for pension schemes in a continuous-time framework.

1.4.1 Stochastic calculus

This section presents certain definitions and results in stochastic calculus that will be used in the following chapters. The concepts in this section were introduced in detail in many books; see Duffie (2001), Shreve (2004), Pascucci (2011) and Oksendal (2013).

In the sequel, we denote by $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ a filtered probability space.

Definition 1.4.1 (Stopping time)**.** *A random variable τ is a stopping time, if it takes values in $[0, +\infty)$ and satisfies*

$$\{\tau \leq t\} := \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}(t), \qquad \forall\, t \geq 0.$$

Definition 1.4.2 (Itô process)**.** *Let $W(t)$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$. An Itô process is a stochastic process valued in $\mathbb{R}$ and has the form*

$$\mathrm{d}X(t) = X(0) + \int_0^t \mu(s)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}W(s), \qquad \forall\, t \geq 0,$$

where $X(0)$ is nonrandom and $\mu(s)$ and $\sigma(s)$ are progressively measurable processes valued in $\mathbb{R}$ s.t.

$$\int_0^t |\mu(s)|\mathrm{d}s + \int_0^t |\sigma(s)|^2\mathrm{d}s < \infty, \qquad \forall\, t \geq 0. \tag{1.1}$$

Equation (1.1) is also referred to as a stochastic differential equation (SDE).

Theorem 1.4.1 (Itô formula)**.** *Let $X(t)$, $t \geq 0$, be an Itô process as defined in Definition 1.4.2, and let $f(t,x) \in C^{1,2}([0,+\infty) \times \mathbb{R})$. Then, for all $t \geq 0$, $Y(t) = f(t,x)$ is also an Itô process and we have*

$$\mathrm{d}Y(t) = f_t(t, X(t))\mathrm{d}t + f_x(t, X(t))\mathrm{d}X(t) + \frac{1}{2} f_{xx}(t, X(t))\mathrm{d}X(t)\mathrm{d}X(t),$$

where $\mathrm{d}X(t)\mathrm{d}X(t)$ is computed according to the rules

$$\mathrm{d}t\mathrm{d}t = \mathrm{d}t\mathrm{d}W(t) = \mathrm{d}W(t)\mathrm{d}t = 0, \qquad \mathrm{d}W(t)\mathrm{d}W(t) = \mathrm{d}t.$$

The Itô formula is an Itô integral version of the chain rule and is very useful in stochastic analysis. The Itô formula can also be applied to multivariate stochastic processes. For example, let $f(t, x, y) \in C^{1,2,2}([0, +\infty) \times \mathbb{R}^2)$, the two-dimensional Itô formula is

$$\mathrm{d}f(t, X, Y) = f_t \mathrm{d}t + f_x \mathrm{d}X + f_y \mathrm{d}Y + \frac{1}{2} f_{xx} \mathrm{d}X \mathrm{d}X + f_{xy} \mathrm{d}X \mathrm{d}Y + \frac{1}{2} f_{yy} \mathrm{d}Y \mathrm{d}Y.$$

1.4.2 Stochastic optimal control theory

Choosing an optimal decision among all possible ones, at any given point of time, to attain some future expectation is the general topic of stochastic control theory. The stochastic control theory has been developed and applied to deal with stochastic optimization problems emerging from mathematical finance. This section collects minimal basic results concerning the control theory that will be used in the subsequent chapters. The results and rigorous proof can be found in books such as Yong & Zhou (1999), Fleming & Soner (2006), Pham (2009) and Oksendal (2013).

We present a one-dimensional stochastic optimal control problem in a finite time horizon. Consider the state process $X(t)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \geq 0}, \mathbb{P})$ that is governed by a SDE:

$$\mathrm{d}X(t) = \mu(t, X(t), u(t))\mathrm{d}t + \sigma(t, X(t), u(t))\mathrm{d}W(t), \qquad X(0) = x \in \mathbb{R},$$

where $\{u(t)\}_{t \geq 0}$ is the control process. We introduce an objective function (performance criterion):

$$J(t, x; u) = \mathbb{E}_{t,x}\Big[\int_t^T f(s, X(s), u(s))\mathrm{d}s + g(T, X(T)) \Big],$$

where f and g are continuous functions. The problem is to maximize (or minimize) this objective function J by controlling u. The value function of the optimization problem is

$$V(t, x) = \sup_u J(t, x; u), \qquad \forall\ (t, x) \in [0, T] \times \mathbb{R}.$$

There are three main approaches to solving this problem: dynamic programming principle (DPP), Pontryagin optimality principle and convex duality martingale method. In this thesis, we adopt DPP approach. The central concept of DPP is that it is not necessary to optimize the control all at once in the entire time horizon. Instead, one can break the time horizon down into small subintervals and optimize over each interval individually.

Theorem 1.4.2 (Dynamic Programming Principle)**.** *Let* $(t,x) \in [0,T] \times \mathbb{R}$*, then we have*

$$V(t,x) = \sup_u \mathbb{E}_{t,x}\Big[\int_t^\tau f(s,X(s),u(s))\mathrm{d}s + V(\tau, X(\tau))\Big],$$

for any stopping time τ *valued in* $[0,T]$.

The DPP approach involves the following steps:

- Apply the DPP and obtain a non-linear partial differential equation, which is called the Hamilton-Jacobi-Bellman (HJB) equation.
- Obtain a (smooth) candidate solution to the value function by solving the HJB equation.
- Show that the candidate solution is the value function via a verification theorem.
- One obtains the optimal feedback control as a byproduct of the verification theorem.

Theorem 1.4.3 (Verification theorem)**.** *Let* $v(t,x)$ *be a function in* $C^{1,2}([0,T] \times \mathbb{R})$ *and* $v(T,x) = g(T,x)$ *such that*

$$v_t(t,x) + \sup_u\big[\mathcal{A}^u v(t,x) + f(t,x,u)\big] = 0, \qquad \forall\ (t,x) \in [0,T] \times \mathbb{R},$$

where

$$\mathcal{A}^u v(t,x) = \mu(t,x,u)v_x(t,x) + \frac{1}{2}\sigma^2(t,x,u)v_{xx}(t,x).$$

Then, we have

$$v(t,x) = V(t,x), \quad \textit{on} \quad [0,T] \times \mathbb{R}.$$

The optimal control u^* is a measurable function such that

$$\sup_u\big[\mathcal{A}^u v(t,x) + f(t,x,u)\big] = \mathcal{A}^{u^*} v(t,x) + f(t,x,u^*),$$

i.e.,

$$u^*(t,x) \in \arg\max_u\big[\mathcal{A}^u v(t,x) + f(t,x,u)\big].$$

1.4.3 Optimal stopping theory

The optimal stopping theory is one of the advanced aspects of the stochastic control theory. Optimal stopping problems arise when the time of a particular action is random and controlled by the decision maker in order to optimize some objective. Problems of this type were largely studied in the areas of, for instance, statistics, operations research, and mathematical finance (in particular, American options and real options). The interested readers are referred to the monograph Peskir & Shiryaev (2006) for additional materials on the optimal stopping theory. We also refer to the books Pham (2009) and Oksendal (2013) for applications in finance.

We describe a one-dimensional infinite horizon optimal stopping problem. For any $t \geq 0$, consider the state process $X(t)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ without control:

$$\mathrm{d}X(t) = \mu(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t), \qquad X(0) = x \in \mathbb{R}.$$

An optimal stopping problem is given as

$$V(x) = \sup_{\tau} \mathbb{E}\Big[\int_0^{\tau} e^{-\beta t} f(X(t))\mathrm{d}t + e^{-\beta\tau} g(X(\tau))\Big], \qquad \forall\, x \in \mathbb{R}.$$

The DPP in this case is: for a fixed $x \in \mathbb{R}$,

$$V(x) = \sup_{\tau} \mathbb{E}\Big[\int_0^{\tau\wedge\theta} e^{-\beta t} f(X(t))\mathrm{d}t + e^{-\beta\tau} g(X(\tau))\mathbb{1}_{\tau<\theta} + e^{-\beta\theta} V(X(\theta))\mathbb{1}_{\theta\leq\tau}\Big],$$

for any stopping time θ valued in $[0, \infty]$. By applying DPP and Itô formula, we expect the value function $V(x)$ satisfies the following variational HJB inequality (HJBVI):

$$\max\{\mathcal{A}v(x) + f(x) - \beta v(x), g(x) - v(x)\} = 0, \qquad \forall\, x \in \mathbb{R}, \tag{1.2}$$

where

$$\mathcal{A}v(x) = \mu(x)v_x(x) + \frac{1}{2}\sigma^2(x)v_{xx}(x).$$

We introduce the open set

$$\mathcal{C} := \{x \in \mathbb{R} : V(x) > g(x)\}.$$

$\mathcal{C}$ is called the continuation region. As long as $x \in \mathcal{C}$, it is optimal to let the diffusion continue. The HJBVI (1.2) means that $\mathcal{A}V(x) + f(x) - \beta V(x) = 0$ is satisfied within region $\mathcal{C}$. On the free boundary $\partial\mathcal{C}$ of the continuation region, the value-matching and smooth-pasting conditions are $V(x) = g(x)$ and $V'(x) = g'(x)$. The optimal stopping time τ^* is the first exist time of the continuation region $\mathcal{C}$, i.e., $\tau^* := \inf\{t \geq 0 : V(X(t)) = g(X(t))\}$. When the smoothness assumption of $v(x)$ is not satisfied on the free boundary $\partial\mathcal{C}$, other techniques must be applied, see Peskir & Shiryaev (2006).

Chapter 2

Hedging longevity risk in Defined Contribution pension schemes

2.1 Introduction

Pension schemes provide an important economic function in the society. They provide people with regular incomes after retirement from the productive labor workforce and incentivize sustainable consumption over a life time. With regards to benefit and contribution policies, there are two main categories of pension schemes: defined benefit schemes (DB schemes) and defined contribution schemes (DC schemes). In a DB scheme, pension benefits to be paid by the scheme after retirement are pre-defined. In this case, scheme members only need to pay contributions regularly and bear no investment risk. The scheme sponsor bears the risk of bad investment performance and may fail to deliver the benefits. In a DC scheme, the amount of contributions payable by scheme members is pre-determined rather than the benefit payments. The benefits depend on the size of the accumulated contributions and the scheme's investment performance, and are uncertain until the retirement time. The sponsor bears no risk as its only responsibility is to pay the contributions, while the employees face risks originating from the market. Historically, pension schemes were dominated by PAYGO DB schemes. Over the last decades, however, DC schemes have become increasingly popular, a consequence of dealing with financial sustainability issues of DB schemes, especially in the presence of population aging.

In this chapter, we focus our attention on DC schemes and study the optimal investment strategy for DC schemes from a theoretical point of view. In a DC scheme, members' benefits rely heavily on the scheme's investment performance. Therefore, it is crucial to study the optimal portfolio selection problem in order that the scheme will deliver satisfactory benefits at retirement. Gao (2008) used the dual approach to solve the optimal asset allocation problem for a DC scheme in a market with stochastic interest rates. Battocchio & Menoncin (2004) studied the optimal asset allocation problem for a DC pension plan manager who maximize expected exponential utility of final wealth considering salary and inflation risk. However, Gao (2008) and Battocchio & Menoncin (2004) supposed that scheme members have full trust in the manager and do not take the attractiveness and effectiveness of the scheme management into consideration. Classically, DC schemes do not guarantee any minimum benefits, leaving its members exposed to the risk of receiving insufficient benefits after retirement. DC schemes that do provide a minimum guarantee on the benefits however can also be more attractive to employers. Boulier et al. (2001) studied the optimal investment problem for a DC scheme in a framework with stochastic interest rates where a downside protection for the member's benefits is provided. They obtained the optimal investment strategy which maximizes the expected terminal utility from the surplus between the scheme's final wealth and the downside guarantee by applying the dynamic programming principle. Deelstra et al. (2003) extended their model to the case where the contribution process is stochastic. They modeled the manager's remuneration as an increasing concave function of the surplus between the scheme's terminal wealth and the minimum guarantee. The martingale method is used to find the optimal investment strategy that maximizes the manager's expected utility from remuneration. Han & Hung (2012) further developed the model to consider inflation and labor income risks. They introduced a minimum guarantee on the purchase of an inflation-indexed annuity at retirement. To hedge the inflation risk, they included an inflation-indexed bond in the investment portfolio. Nonetheless, these papers did not take the members' stochastic mortality behavior into consideration and ignored the mortality risk. To fill this gap, we consider a DC scheme in which the scheme manager allocates the wealth in various assets in order to achieve the level needed to buy lifetime annuities upon retirement of the members. The annuities act as the minimum guarantee and the price depends on the members' expected remaining lifetime and future risk-free interest rate. Furthermore, this work incorporates stochastic interest rate and force of mortality.

According to Cocco & Gomes (2012), the average life expectancy of a 65-year-old US (UK) male increases by 1.2 (1.5) years per decade. Consequently, a DB scheme for those populations in the US for example would have needed 29% more wealth in 2007 than in 1970. The increases in life expectancy in the UK are largely responsible for the underfunding of pay-as-you-go state pensions, defined benefit company pensions, and state-sponsored pension plans. Biffis & Blake (2014) mentioned that the estimate of the global amount of annuity- and pension-related longevity risk exposure amounts to $15 trillion. However, most articles studying optimal portfolio strategies for DC schemes focus on the financial risks (for example, interest rate risk, inflation risk) and leave longevity risk aside. Those studies that take longevity risk into account, mainly focus on optimal asset allocation problems for DB pension schemes, using time-varying but deterministic forces of mortality to implement longevity risk into their models. This ignores that the force of mortality can itself vary stochastically and be exposed to shocks. In this chapter, we study an optimal portfolio problem for a DC scheme within the framework of a stochastic force of mortality as well as stochastic interest rates. Force of mortality, or the instantaneous rate of mortality, is often used within the context of survival analysis in actuarial science. Classical work, including De Moivre (1725) and Gompertz (1825), has studied deterministic force of mortality models. However, more recent research on mortality risk modeling considers discrete-time and continuous-time models with stochastic force of mortality. It is straightforward to model the force of mortality in a discrete-time setting since the mortality data are usually reported annually. Lee & Carter (1992) were among the earliest to model and estimate the force of mortality using time series methods. Other discrete-time models include, for example, the CBD model and Renshaw-Haberman cohort model (A. J. Cairns et al. (2006); Renshaw & Haberman (2006)). Some studies, such as Milevsky & Promislow (2001) and Dahl (2004), found similarities in the methodology between interest rates and force of mortality; for example, that both are positive and have a term structure. Thus, drawing from the interest rate modeling literature, diffusion processes and jump processes are now used to study the impact of the force of mortality. In particular, affine mortality models are popular and are studied in work such as Dahl (2004), Biffis & Millossovich (2006), Luciano & Vigna (2005) and Russo et al. (2011). In this chapter, we assume that the evolution of the mortality rate of all the scheme members can be described by the same continuous-time stochastic process. We follow Menoncin (2009) and describe the force of mortality using an affine model which is an analog of the Cox-Ingersoll-Ross (CIR) process. The considered pension scheme is attractive to members as it provides a minimum guarantee on purchasing lifetime annuities upon retirement. The longevity risk arises as the value of the minimum guarantee depends on members' actual survival rate and expected remaining lifetime and is uncertain until retirement time. Besides, future contribution payments depend on members' actual survival rate. To hedge the scheme's longevity risk, the manager decides to invest in a longevity security.

Proposed by Blake & Burrows (2001), a longevity bond provides coupon payment based on the number of survivors in a chosen reference population. Therefore, investment in a longevity bond not only provides an efficient way to hedge the longevity risk, but also allows diversification of investment portfolios. Menoncin (2008) studied an optimal consumption and investment problem for an investor with a stochastic time of death. He maximized the investor's inter-temporal consumption until the time of death and used a rolling longevity bond to hedge against the investor's longevity risk. De Kort & Vellekoop (2017) modeled the force of mortality using the CIR process which guarantees the mortality rates to be non-negative. They argued that although there is no liquid market for such longevity bonds, it is not practical to put the market price of longevity risk at zero. Instead, they assumed a time-varying market price of longevity risk which is proportional to the square root of the mortality rate. Cocco & Gomes (2012) studied the optimal consumption and investment problem in a life-cycle model. By calibrating to US historical data and current projections, they showed considerable uncertainty with respect to future improvements in mortality rates. They also suggested that longevity linked securities can help in longevity risk management. Menoncin & Regis (2017) studied the optimal consumption and investment problem for an individual investor to hedge his longevity risk before retirement. They showed that the optimal proportion that should be invested in longevity bonds is higher than for other assets.

In this chapter, we consider a financial market that consists of three risky assets: a stock, a rolling bond and a rolling longevity bond. Our results show that the longevity bond provides an efficient way to hedge the longevity risk. Our main contribution is in extending the work of Boulier et al. (2001), Gao (2008) and Menoncin & Regis (2017), to investigate the optimal portfolio allocation for DC schemes while hedging the longevity risk. The pension scheme promises that the scheme's wealth level must be sufficient in order to allow members buying lifetime annuities at retirement, in this way providing a minimum guarantee. The scheme is exposed to longevity risk as the value of the minimum guarantee will be higher than anticipated if members' life expectancy and actual survival rate exceed expectations. At retirement time, the manager receives a fixed fraction of the surplus between the scheme wealth and the minimum guarantee as remuneration. The aim is to maximize his expected utility from remuneration by controlling the investment strategy. To hedge the longevity risk, a rolling longevity bond as introduced in Menoncin (2008) is added to the investment portfolio. Our results show that the longevity risk plays an important role in the pension scheme's risk management and reveals that the longevity bond can not only offer an efficient way to hedge future longevity risk, but also provide attractive risk premiums.

The rest of this chapter is organized as follows. In Section 2.2, we present the mathematical framework of the problem and introduce the risky assets considered in the financial market model. In Section 2.3.1, we first formulate the constrained optimization problem in which the manager maximize his utility of terminal wealth. We identify different components of the investment portfolio and reformulate the portfolio selection problem as a single investment portfolio optimization problem in Section 2.3.2. We derive the analytical solution for the optimal investment strategy by using the dynamic programming principle in Section 2.3.3. Section 2.4 discusses several numerical studies including sensitivity analyses with respect to different model parameters, which reveal the significance of introducing a rolling longevity bond in the investment portfolio.

2.2 Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions on an infinite time horizon $\mathcal{T} = [0, \infty)$. $\mathbb{P}$ is the physical (observable) probability measure and $\mathcal{F}(t)$ signifies the information available to the investor at time t. On this probability space, we consider a frictionless financial market consisting of a stock, a *rolling bond* and a *rolling longevity bond*. For practical pricing of zero-coupon and longevity bonds, we consider a stochastic risk-free interest rate $r(t)$ and a stochastic force of mortality $\lambda(t)$. Furthermore, we denote a three-dimensional standard Brownian motion under $\mathbb{P}$ by $\{W(t) \mid t \in \mathcal{T}\} = \{[W_1(t), W_2(t), W_3(t)]' \mid t \in \mathcal{T}\}$. We assume that $r(t)$ is described by a CIR process:

$$\mathrm{d}r(t) = (a_r - b_r r(t))\mathrm{d}t + \sigma_r \sqrt{r(t)}\mathrm{d}W_1(t), \qquad r(0) = r_0,$$

where a_r, b_r and σ_r are positive constants. We further assume that the Feller condition $2a_r > \sigma_r^2$ is satisfied so that, for any $t \in \mathcal{T}$, $r(t) > 0$ almost surely under $\mathbb{P}$.

As stated earlier in Section 2.1, affine models are popular when modeling the stochastic force of mortality. Luciano & Vigna (2005) described the force of mortality (also referred to a mortality intensity) through affine models and calibrated the models using observed and projected UK mortality tables. They claimed that affine processes with a deterministic part which increases exponentially could describe the evolution of force of mortality properly. Russo et al. (2011) calibrated three different affine stochastic mortality models using term assurance premiums of three Italian insurance companies, and proposed that such affine models can be used for pricing mortality-linked securities. Thus, in the same spirit, we assume that $\lambda(t)$ evolves as

$$\mathrm{d}\lambda(t) = (a_\lambda(t) - b_\lambda \lambda(t))\,\mathrm{d}t + \sigma_\lambda \sqrt{\lambda(t)}\mathrm{d}W_2(t), \qquad \lambda(0) = \lambda_0, \tag{2.1}$$

where $a_\lambda(t)$ is a deterministic function, b_λ and σ_λ are positive constants. We restrict the mortality model parameters to satisfy the condition $2a_\lambda(t) > \sigma_\lambda^2$, to ensure the strict positivity of $\lambda(t)$. The analytical tractability of the affine model allows us to price mortality-linked securities using the arbitrage-free pricing framework that has been developed for interest-rate derivatives.

The initial value of the mortality intensity λ_0 is calculated according to the deterministic Gompertz-Makeham law and is given by

$$\lambda_0 = \phi + \frac{1}{b} e^{\frac{t_0 - m}{b}}, \tag{2.2}$$

where t_0, m, ϕ and b are constants. As argued in Menoncin (2009), we require that the expected value of $\lambda(t)$ equals to the corresponding deterministic Gompertz-Makeham force of mortality to ensure at any time t, $\lambda(t)$ has a reasonable value. To achieve this, we require that $a_\lambda(t)$ is of the following form

$$a_\lambda(t) = b_\lambda \left(\phi + \left(\frac{1}{b_\lambda b} + 1 \right) \frac{1}{b} e^{\frac{t-m}{b}} \right).$$

The force of mortality is used as a tool to study the instantaneous survival rate of a population. If we denote by $p(t)$ the fraction of a population that survives from time 0 to t, then $p(t)$ measures the cumulative survival rate which coincides with the survival probability. Since the force of mortality measures the instantaneous rate of mortality, we can write

$$\frac{\mathrm{d}p(t)}{p(t)} = -\lambda(t)\mathrm{d}t, \qquad\qquad p(0) = 1.$$

Given information up to time $t \in \mathcal{T}$, the (conditional) expected survival probability from t to $s > t$ is given by (see Menoncin (2008, Section 2.2))

$$\mathbb{E}\left[\frac{p(s)}{p(t)} \;\middle|\; \mathcal{F}(t) \right] = \mathbb{E}\left[e^{-\int_t^s \lambda(u)\mathrm{d}u} \;\middle|\; \mathcal{F}(t) \right].$$

To discuss the prices of tradable financial risky assets in the market, we first introduce a risk-neutral pricing measure $\widetilde{\mathbb{P}}$ by the following Radon-Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = Z(t) = \exp\left(-\int_0^t \Theta(s)'\mathrm{d}W(s) - \frac{1}{2}\int_0^t |\Theta(s)|^2 \mathrm{d}s \right).$$

In the above $\{\Theta(t) \mid t \in \mathcal{T}\} = \left\{ \left[\theta_1(t), \theta_2(t), \theta_3(t)\right]' \mid t \in \mathcal{T} \right\}$ is an $\mathbb{R}^3$-valued, $\mathcal{F}$-adapted process such that $Z(t)$ is a martingale and $\mathbb{E}[Z(t)] = 1$. Here, we denote by $\mathbb{E}[\cdot]$ the expectation operator under $\mathbb{P}$. By Girsanov's theorem,

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(s)\mathrm{d}s$$

is a three-dimensional standard Brownian motion under $\widetilde{\mathbb{P}}$. We use the notation $\{\widetilde{W}(t) \mid t \in \mathcal{T}\} = \left\{ \left[\widetilde{W}_1(t), \widetilde{W}_2(t), \widetilde{W}_3(t)\right]' \mid t \in \mathcal{T} \right\}$. The introduction of a risk-neutral measure also allows us to motivate the idea of market price of risk or risk-premium through $\Theta(t)$ in our financial market framework. Since we use square root processes to model the stochastic interest rate $r(t)$ and force of mortality $\lambda(t)$, the Novikov's condition for measure change is not necessarily satisfied. It is known that Novikov's condition is sufficient to guarantee $Z(t)$ is a martingale under $\mathbb{P}$. However, it is not a necessary condition. Later, we show in Lemma 2.2.1 that $Z(t)$ is a martingale even if Novikov's condition is not imposed.

The first financial asset in the market is a representative stock. We suppose that the stock price process $S(t)$ under $\mathbb{P}$ evolves as

$$\frac{\mathrm{d}S(t)}{S(t)} = \left(r(t) + \theta_r \sigma_S^r r(t) + \theta_S \sigma_S\right) \mathrm{d}t + \sigma_S^r \sqrt{r(t)} \mathrm{d}W_1(t) + \sigma_S \mathrm{d}W_3(t), \qquad S(0) = S_0,$$

where $\sigma_S, \sigma_S^r, \theta_r, \theta_S$ are constants. Here, we have assumed that the market prices of interest rate risk and stock price risk are $\theta_1(t) = \theta_r \sqrt{r(t)}$ and $\theta_3(t) = \theta_S$, respectively. The instantaneous covariance between the stock price and risk-free interest rate is captured by $\sigma_S^r \sqrt{r(t)}$. The market price of stock price risk and different volatility coefficients could be stochastic and take many different forms. However, as we mainly focus on the interest rate risk and longevity risk rather than investment risk, it is reasonable to suppose that they are constants.

For the pricing of a zero-coupon bond $B(t, T_B)$ which pays one unit of currency at a fixed maturity time T_B, we first introduce a money market account $R(t)$ via

$$\frac{\mathrm{d}R(t)}{R(t)} = r(t)\mathrm{d}t, \qquad\qquad R(0) = 1.$$

The risk-neutral pricing formula then gives

$$B(t, T_B) = \widetilde{\mathbb{E}}\left[\frac{R(t)}{R(T_B)} \,\middle|\, \mathcal{F}(t)\right] = \widetilde{\mathbb{E}}\left[e^{-\int_t^{T_B} r(u)\mathrm{d}u} \,\middle|\, \mathcal{F}(t)\right],$$

where $\widetilde{\mathbb{E}}[\cdot]$ is the expectation operator under the measure $\widetilde{\mathbb{P}}$. As the interest rate $r(t)$ follows an affine model, we can solve for the bond price as

$$B(t, T_B) = e^{f_0(t,T_B) - f_1(t,T_B) r(t)}, \tag{2.3}$$

where

$$\begin{aligned}
f_0(t, T_B) &= \frac{2a_r}{\sigma_r^2} \log\left(\frac{2\eta_r e^{\frac{1}{2}(\tilde{b}_r + \eta_r)(T_B - t)}}{(\tilde{b}_r + \eta_r)(e^{\eta_r (T_B - t)} - 1) + 2\eta_r}\right), \\
f_1(t, T_B) &= \frac{2(e^{\eta_r (T_B - t)} - 1)}{(\tilde{b}_r + \eta_r)(e^{\eta_r (T_B - t)} - 1) + 2\eta_r}, \\
\eta_r &= \sqrt{\tilde{b}_r^2 + 2\sigma_r^2}, \qquad \tilde{b}_r = b_r + \theta_r \sigma_r.
\end{aligned}$$

This results can be found in several sources, for example, Brigo & Mercurio (2006, Section 3.2.3), Cuchiero (2006, Section 3.1.2). The dynamics of $B(t, T_B)$ under $\mathbb{P}$ is given as

$$\frac{\mathrm{d}B(t, T_B)}{B(t, T_B)} = \Big(r(t) + \theta_r \sqrt{r(t)} \sigma_B(t, T_B)\Big) \mathrm{d}t + \sigma_B(t, T_B) \mathrm{d}W_1(t),$$

where we denote $\sigma_B(t,T_B) = -f_1(t,T_B)\sigma_r\sqrt{r(t)}$.

As argued in Boulier et al. (2001), it is more convenient to use a single bond with a rolling maturity that can dynamically replicate any bond in the market. To do so, we introduce a rolling bond $B(t)$ with a constant time to maturity T_B by specifying its dynamics as

$$\frac{\mathrm{d}B(t)}{B(t)} = \left(r(t) + \theta_r\sqrt{r(t)}\sigma_B(t,t+T_B)\right)\mathrm{d}t + \sigma_B(t,t+T_B)\mathrm{d}W_1(t). \tag{2.4}$$

The zero-coupon bond $B(t,T_B)$ is then replicable using cash and the rolling bond $B(t)$ via

$$\frac{\mathrm{d}B(t,T_B)}{B(t,T_B)} = \left(1 - \frac{\sigma_B(t,T_B)}{\sigma_B(t,t+T_B)}\right)\frac{\mathrm{d}R(t)}{R(t)} + \frac{\sigma_B(t,T_B)}{\sigma_B(t,t+T_B)}\frac{\mathrm{d}B(t)}{B(t)}.$$

This equation also shows that the rolling bond can be dynamically replicated by using a self-financing strategy investing in the zero coupon bond and the money market account, and hence the introduction of the rolling bond via equation (2.4) does not create any arbitrage. In consequence, the use of rolling bond is equivalent to using a fixed maturity zero-coupon bond in the market.

The third and final asset in the market is a zero-coupon longevity bond, which is primarily used to hedge the longevity risk. A longevity bond is a financial security paying, at the expiration date, an amount equal to the fraction of survivors from time 0 to the maturity time in a reference population. The reference population can be a large number of similar individuals, e.g. same age, whose mortality behavior could be described by the process $\lambda(t)$. Let T_L be the fixed maturity time, the payoff of the zero-coupon longevity bond is $p(T_L)$. Suppose that the market price of longevity risk is $\theta_2(t) = \theta_\lambda\sqrt{\lambda(t)}$, then the arbitrage-free price $L(t,T_L)$ of a zero-coupon longevity bond with fixed maturity time T_L is given as

$$L(t,T_L) = \widetilde{\mathbb{E}}\left[\frac{R(t)}{R(T_L)}p(T_L)\,\middle|\,\mathcal{F}(t)\right] = e^{-\int_0^t \lambda(u)\mathrm{d}u}\widetilde{\mathbb{E}}\left[e^{-\int_t^{T_L}(r(u)+\lambda(u))\mathrm{d}u}\,\middle|\,\mathcal{F}(t)\right].$$

Due to the affine nature of $r(t)$ and $\lambda(t)$ and their independence, the longevity bond price can be expressed in the following form

$$L(t,T_L) = e^{-\int_0^t \lambda(u)\mathrm{d}u}N(t,T_L), \tag{2.5}$$

where

$$
\begin{aligned}
N(t,T_L) &= e^{f_0(t,T_L)-f_1(t,T_L)r(t)+h_0(t,T_L)-h_1(t,T_L)\lambda(t)},\\
h_1(t,T_L) &= \frac{2(e^{\eta_\lambda(T_L-t)}-1)}{(\tilde{b}_\lambda+\eta_\lambda)(e^{\eta_\lambda(T_L-t)}-1)+2\eta_\lambda},\\
h_0(t,T_L) &= -\int_t^{T_L} a_\lambda(u)h_1(u,T_L)\mathrm{d}u,\\
\eta_\lambda &= \sqrt{\tilde{b}_\lambda^2+2\sigma_\lambda^2}, \qquad \tilde{b_\lambda} = b_\lambda+\theta_\lambda\sigma_\lambda.
\end{aligned}
$$

By denoting $\sigma_L^r(t,T_L) = -f_1(t,T_L)\sigma_r\sqrt{r(t)}$ and $\sigma_L^\lambda(t,T_L) = -h_1(t,T_L)\sigma_\lambda\sqrt{\lambda(t)}$, the evolution of $L(t,T_L)$ is then described as

$$
\begin{aligned}
\frac{\mathrm{d}L(t,T_L)}{L(t,T_L)} =&\Big(r(t)+\theta_r\sqrt{r(t)}\sigma_L^r(t,T_L)+\theta_\lambda\sqrt{\lambda(t)}\sigma_L^\lambda(t,T_L)\Big)\mathrm{d}t\\
&+\sigma_L^r(t,T_L)\mathrm{d}W_1(t)+\sigma_L^\lambda(t,T_L)\mathrm{d}W_2(t).
\end{aligned}
$$

In the same manner as for the zero-coupon bond, we consider a rolling longevity bond $L(t)$ with a constant time to maturity T_L whose price process under $\mathbb{P}$ is given as

$$
\begin{aligned}
\frac{\mathrm{d}L(t)}{L(t)} =& \Big(r(t)+\theta_r\sqrt{r(t)}\sigma_L^r(t,t+T_L)+\theta_\lambda\sqrt{\lambda(t)}\sigma_L^\lambda(t,t+T_L)\Big)\,\mathrm{d}t\\
&+\sigma_L^r(t,t+T_L)\mathrm{d}W_1(t)+\sigma_L^\lambda(t,t+T_L)\mathrm{d}W_2(t).
\end{aligned}
\tag{2.6}
$$

We see that the rolling longevity bond correlates with interest rate $r(t)$ as well as force of mortality $\lambda(t)$. In fact, zero-coupon longevity bonds with any fixed maturity can be replicated using rolling bond, rolling longevity bond and cash

$$
\frac{\mathrm{d}L(t,T_L)}{L(t,T_L)} = n_0(t)\frac{\mathrm{d}R(t)}{R(t)}+n_B(t)\frac{\mathrm{d}B(t)}{B(t)}+n_L(t)\frac{\mathrm{d}L(t)}{L(t)},
$$

where

$$
\begin{aligned}
n_B(t) =&\frac{\sigma_L^r(t,T_L)}{\sigma_B(t,t+T_B)}-n_L(t)\frac{\sigma_L^r(t,t+T_L)}{\sigma_B(t,t+T_B)},\\
n_L(t) =&\frac{\sigma_L^\lambda(t,T_L)}{\sigma_L^\lambda(t,t+T_L)}, \qquad n_0(t)=1-n_B(t)-n_L(t).
\end{aligned}
$$

As before, the latter also shows that the creation of the rolling longevity bond does not lead to any arbitrage. It is common in the literature in fact to use rolling bonds: Han & Hung (2012) introduced a rolling indexed bond to hedge the inflation risk for a DC scheme.

Menoncin (2008) used a rolling longevity bond to transfer an individual's longevity risk. In principle, the problems considered in this chapter could also be solved using a fixed term maturity zero-coupon longevity bond and a fixed term zero-coupon bond. However, the use of rolling longevity bond and rolling bond simplifies the calculations in Section 2.3 significantly. Moreover, our specific choices for the market prices of risks $\theta_r\sqrt{r(t)}$ and $\theta_\lambda\sqrt{\lambda(t)}$ maintain the affine form of our models (see, for example, Duffee (2002)).

Next, we show that the Radon-Nikodym derivative $Z(t)$ is a martingale and the risk-neutral measure $\mathbb{P}$ is well-defined.

Lemma 2.2.1. *For any $t \in \mathcal{T}$, the stochastic exponential process*

$$Z(t) = \exp\left(-\int_0^t \Theta(s)'\mathrm{d}W(s) - \frac{1}{2}\int_0^t |\Theta(s)|^2\mathrm{d}s\right)$$

is a $\mathbb{P}$-martingale and $\mathbb{E}[Z(t)] = 1$.

See Appendix 2.A for the proof.

For any $t, T_B, T_L \in \mathcal{T}$, we describe the risky asset prices in the form of a vector:

$$\begin{bmatrix} \frac{\mathrm{d}B(t)}{B(t)} \\ \frac{\mathrm{d}L(t)}{L(t)} \\ \frac{\mathrm{d}S(t)}{S(t)} \end{bmatrix} = \left(r(t)\mathbf{1} + M(t)\right)\mathrm{d}t + \Sigma(t)'\mathrm{d}W(t), \tag{2.7}$$

where

$$M(t) = \begin{bmatrix} \theta_r\sqrt{r(t)}\sigma_B(t, t+T_B) \\ \theta_r\sqrt{r(t)}\sigma_L^r(t, t+T_L) + \theta_\lambda\sqrt{\lambda(t)}\sigma_L^\lambda(t, t+T_L) \\ \theta_r\sigma_S^r r(t) + \theta_S\sigma_S \end{bmatrix},$$

$$\Sigma(t)' = \begin{bmatrix} \sigma_B(t, t+T_B) & 0 & 0 \\ \sigma_L^r(t, t+T_L) & \sigma_L^\lambda(t, t+T_L) & 0 \\ \sigma_S^r\sqrt{r(t)} & 0 & \sigma_S \end{bmatrix}.$$

For ease of presentation, we also use the notation $z(t) = [r(t), \lambda(t)]'$ whose dynamics is given as

$$\mathrm{d}z(t) = \mu(t, z(t))\mathrm{d}t + \xi(t, z(t))'\mathrm{d}W(t), \qquad z(0) = [r_0, \lambda_0]', \tag{2.8}$$

where

$$\mu(t, z(t)) = \begin{bmatrix} a_r - b_r r(t) \\ a_\lambda(t) - b_\lambda \lambda(t) \end{bmatrix}, \qquad \xi(t, z(t))' = \begin{bmatrix} \sigma_r \sqrt{r(t)} & 0 & 0 \\ 0 & \sigma_\lambda \sqrt{\lambda(t)} & 0 \end{bmatrix}.$$

2.3 Main results

In this chapter, we assume the scheme members are identical individuals, i.e., same age, gender, wage, etc. Moreover, we model the members' mortality behavior using the stochastic force of mortality $\lambda(t)$ as given in (2.1). We consider a financial agent who is the manager of the scheme. The members continuously contribute to the pension scheme during the accumulation phase and delegate the scheme's management to the agent. That is to say, the manager is responsible for the investment. In return, the scheme promises a minimum guarantee at retirement in the form of lifetime annuities to its members. The scheme manager receives a fraction of the surplus between the scheme's final wealth and the minimum guarantee as his one-off remuneration. We follow Merton (1969) and suppose the manager aims to maximize his expected utility of terminal wealth.

2.3.1 The utility maximization problem

We assume that each surviving member contributes a constant fraction r_c of his instantaneous wage $w(t)$ before retirement. Previous studies, such as Han & Hung (2012) and Guan & Liang (2014), model the wage (or, contribution) as a stochastic process to study the optimal asset allocation problem for DC schemes. To simplify our calculations, the instantaneous wage in this chapter is assumed to be constant, that is, for any $t \in [0, T]$, $w(t) = w$. Thus, the contribution $c(t) = r_c w(t) = r_c w = c$ is also constant. We note that our following analysis is also applicable when $w(t)$ and $c(t)$ are treated as independent stochastic processes or deterministic functions. Suppose there are $n \geq 1$ members in the scheme at the initial time of the management. At time $t \in \mathcal{T}$, $np(t)$ members have survived. In the case where a member dies before retirement time, we assume that he stops paying contribution immediately and his heirs receive nothing. For any $t \in [0, T]$, the scheme manager invests the amounts $\alpha_S(t), \alpha_B(t)$ and $\alpha_L(t)$ of money in stock, rolling bond and rolling longevity bond, respectively. It then follows that the amount of money invested in the money market is $\alpha_0(t) = F(t) - \alpha_B(t) - \alpha_L(t) - \alpha_S(t)$, where $F(t)$ denotes the scheme's wealth level. The dynamics of $F(t)$ is given as

$$\mathrm{d}F(t) = \Big(r(t)F(t) + cnp(t) + \alpha(t)'M(t)\Big)\mathrm{d}t + \alpha(t)'\Sigma(t)'\mathrm{d}W(t), \quad F(0) = F_0, \tag{2.9}$$

where $\{\alpha(t) \mid t \in [0,T]\} = \{[\alpha_B(t), \alpha_L(t), \alpha_S(t)]' \mid t \in [0,T]\}$ denotes the investment in risky assets.

At retirement time T, the scheme manager promises that the pension wealth $F(T)$ must exceed the lifetime annuities' price, this acts as minimum guarantee $G(T)$. Moreover, the manager's remuneration will be positive only if $F(T) - G(T) > 0$. Such minimum guarantee was also previously considered in works such as Boulier et al. (2001), Deelstra et al. (2003), Han & Hung (2012) and Guan & Liang (2014). Here, we extend this work to the case where the death time is uncertain and the force of mortality is stochastic. To compute the price of the lifetime annuity, we first need to decide the level of installments that the annuity delivers. Typically, the wage replacement ratio r_w, the percentage of retirement income to pre-retirement income, is a good estimate of the income needed to maintain the living standard in retirement. We set the instantaneous installment of the annuity to be $\pi = r_w w$, so that the lifetime annuity provides sufficient retirement income for subsistence. By denoting $a(T)$ as the price of a lifetime annuity at retirement time T, we have

$$a(T) = \widetilde{\mathbb{E}}\left[\int_T^\infty \pi \frac{R(T)}{R(s)}\frac{p(s)}{p(T)}\mathrm{d}s \,\middle|\, \mathcal{F}(T)\right].$$

Here, we have assumed that the annuity provider models the mortality behavior of members using $\lambda(t)$. The minimum guarantee is met in purchasing lifetime annuities for surviving members at retirement time T. Therefore, its value at T is given as

$$G(T) = np(T)a(T) = n\widetilde{\mathbb{E}}\left[\int_T^\infty \pi \frac{R(T)}{R(s)}p(s)\mathrm{d}s \,\middle|\, \mathcal{F}(T)\right].$$

At the terminal time of the scheme management, we suppose the manager receives a fraction β of the surplus between the fund level and the minimum guarantee, i.e., $\beta\Big(F(T) - G(T)\Big)^+$, as remuneration. Here, β is a positive constant and is less than 1. Deelstra et al. (2003) used a similar assumption on the manager's remuneration. We assume that the manager aims to maximize his expected utility of terminal wealth. Thus, for a given investment strategy α, the manager's objective function for the utility maximization problem is

$$J(t, f, z; \alpha) = \mathbb{E}\left[U\left(\beta\Big(F(T) - G(T)\Big)^+\right) \,\Big|\, \mathcal{F}(t)\right], \quad 0 < t \leq T,$$

where $U : \mathbb{R}_+ \to \mathbb{R}_+$ is a utility function, $F(t) = f > 0$ and $z(t) = z$.

Hyperbolic absolute risk aversion (HARA) identifies a class of utility functions which is most commonly used. Constant relative risk aversion (CRRA), constant absolute risk aversion (CARA), and quadratic utility are all HARA type utility functions which have been used frequently in the past. For instance, Gao (2008) and Boulier et al. (2001) used CRRA utility function to study optimal asset allocation problems for DC schemes. Battocchio & Menoncin (2004) and A. Cairns (2000a) adopted CARA and quadratic utilities, respectively. In this chapter, we use CRRA in the form of

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \tag{2.10}$$

where $\gamma > 0$ and $\gamma \neq 1$. In the case when $\gamma = 1$, $U(x) = \ln x$ is the log-utility function.

Our choice of CRRA utility function is motivated by

- Pension schemes usually manage a large amount of money. With an increasing or decreasing relative risk aversion, the fraction of wealth invested in risky assets is affected by the total level of wealth. However, the CRRA utility function shows constant relative risk aversion and in conclusion the investment strategy is not affected by scale. Also, the risk-aversion coefficient γ allows us to investigate the impact of the manager's risk aversion on the optimal investment strategy.
- Our optimization problem is analytically tractable when using CRRA utility. For other types of utility functions, we would lose analytical tractability and the ability to make precise qualitative statements, even if we were to numerically solve the optimization problem using our approach.

We formulate the scheme manager's utility maximization problem as

$$\sup_{\alpha \in \mathscr{A}} \mathbb{E}\left[\frac{\left(\beta\left(F(T) - G(T)\right)\right)^{1-\gamma}}{1-\gamma} \,\Bigg|\, \mathcal{F}(t)\right] \text{ such that } F(T) \geq G(T) \text{ a.s..}$$

Since β is a positive constant, the optimization problem is equivalent to

$$\sup_{\alpha \in \mathscr{A}} \mathbb{E}\left[\frac{(F(T) - G(T))^{1-\gamma}}{1-\gamma} \,\middle|\, \mathcal{F}(t) \right] \text{ such that } F(T) \geq G(T) \text{ a.s..} \tag{2.11}$$

In the above, the set $\mathscr{A}$ denotes the set of all *admissible* strategies which are defined as below.

Definition 2.3.1. *A portfolio strategy* $\{\alpha(t) \in \mathbb{R}^3 \mid t \in [0,T]\}$ *is called admissible if* $\alpha(t)$ *is progressively measurable with respect to* $\mathcal{F}$ *and* $\mathbb{E}\left[\int_0^T |\alpha(t)|^2 \mathrm{d}t\right] < \infty$.

In our setting, scheme members continuously contribute to the scheme during the accumulation phase. The term $cnp(t)\mathrm{d}t$ in (2.9) reveals that the wealth process $F(t)$ is not self-financing. Besides, at the retirement time T, there is a minimum guarantee $G(T)$ to be met. This means that the proposed problem (3.12) is not a classical Merton type optimal investment problem. To solve this non-self-financing constrained problem, we convert it into a self-financing investment portfolio optimization problem by introducing an auxiliary surplus process. We then solve the transformed problem using the dynamic programming principle.

2.3.2 Single investment portfolio optimization problem

Inspired by Boulier et al. (2001), we split the scheme's wealth into two parts: one part is the future contributions to be paid by living members and the other part is a self-financing portfolio. For any $t \in [0,T]$, denoting by $D(t)$ the present value of future contributions until retirement time T, we can write

$$D(t) = \widetilde{\mathbb{E}}\left[\int_t^T cnp(s) \frac{R(t)}{R(s)} \mathrm{d}s \,\middle|\, \mathcal{F}(t) \right]. \tag{2.12}$$

$D(t)$ can be viewed as a coupon-paying bond that pays a continuous coupon at rate $cnp(t)$ until time T. Thus, $D(t)$ can be replicated by investing in the rolling bond, rolling longevity bond and money market.

Proposition 2.3.1. *For any* $t \in [0,T]$, $D(t)$ *in* (2.12) *can be replicated as*

$$\mathrm{d}D(t) = \big(r(t)D(t) - cnp(t) + \alpha^D(t)' M(t)\big)\,\mathrm{d}t + \alpha^D(t)'\Sigma(t)'\mathrm{d}W(t), \tag{2.13}$$

where

$$\alpha^D(t) = \begin{bmatrix} \alpha_B^D(t) \\ \alpha_L^D(t) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{cn\int_t^T L(t,s)f_1(t,s)\mathrm{d}s}{f_1(t,t+T_B)} - \frac{f_1(t,t+T_L)}{f_1(t,t+T_B)}\frac{cn\int_t^T L(t,s)h_1(t,s)\mathrm{d}s}{h_1(t,t+T_L)} \\ \frac{cn\int_t^T L(t,s)h_1(t,s)\mathrm{d}s}{h_1(t,t+T_L)} \\ 0 \end{bmatrix} \tag{2.14}$$

is the investment in risky assets. The holding in the money market is $\alpha_0^D(t) = D(t) - \alpha_B^D(t) - \alpha_L^D(t)$.

See Appendix 2.B for the proof.

Our next step is to construct a replicating portfolio for $G(T)$. At time $t \in [0, T]$, the present value of $G(T)$ is given by

$$G(t) = \widetilde{\mathbb{E}}\left[G(T)\frac{R(t)}{R(T)} \,\middle|\, \mathcal{F}(t) \right]. \tag{2.15}$$

Similar to replicating $D(t)$, we show in the following proposition that $G(t)$ can be replicated by investing in the bond, longevity bond and money market.

Proposition 2.3.2. *For any* $t \in [0, T]$, $G(t)$ *in* (2.15) *can be replicated as*

$$\mathrm{d}G(t) = \big(r(t)G(t) + \alpha^G(t)'M(t)\big)\,\mathrm{d}t + \alpha^G(t)'\Sigma(t)'\mathrm{d}W(t) \tag{2.16}$$

where

$$\alpha^G(t) = \begin{bmatrix} \alpha_B^G(t) \\ \alpha_L^G(t) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\pi n\int_T^\infty L(t,s)f_1(t,s)\mathrm{d}s}{f_1(t,t+T_B)} - \frac{f_1(t,t+T_L)}{f_1(t,t+T_B)}\frac{\pi n\int_T^\infty L(t,s)f_1(t,s)\mathrm{d}s}{h_1(t,t+T_B)} \\ \frac{\pi n\int_T^\infty L(t,s)f_1(t,s)\mathrm{d}s}{h_1(t,t+T_B)} \\ 0 \end{bmatrix} \tag{2.17}$$

is the investment in risky assets. The holding in the money market is $\alpha_0^G(t) = G(t) - \alpha_B^G(t) - \alpha_L^G(t)$.

The proof is similar to Proposition 2.3.1 and is omitted here for the sake of brevity.

Finally, we construct an auxiliary process $Y(t) = F(t) + D(t) - G(t)$. At retirement time T, from (2.12), we see that $D(T) = 0$ and we have $Y(T) = F(T) - G(T)$. That is, $Y(T)$ is the surplus of the terminal scheme wealth over the minimum guarantee. From (2.9), (2.13) and (2.16), we obtain the following equation

$$\begin{aligned} \mathrm{d}Y(t) =& \mathrm{d}F(t) + \mathrm{d}D(t) - \mathrm{d}G(t) \\ =& \big(r(t)Y(t) + \alpha^Y(t)'M(t)\big)\,\mathrm{d}t + \alpha^Y(t)'\Sigma(t)'\mathrm{d}W(t) \end{aligned} \tag{2.18}$$

where

$$\alpha^Y(t) = \alpha(t) + \alpha^D(t) - \alpha^G(t). \tag{2.19}$$

Thus, our simplified portfolio optimization problem is given as

$$\sup_{\alpha^Y \in \mathscr{A}} \mathbb{E}\left[\frac{Y(T)^{1-\gamma}}{1-\gamma} \,\middle|\, \mathcal{F}(t)\right] \text{ such that } Y(T) \geq 0 \text{ a.s..} \tag{2.20}$$

Lemma 2.3.1. *For any $t \in [0,T]$, if $\alpha^Y(t) \in \mathscr{A}$, then $\alpha(t) \in \mathscr{A}$, and the optimization problems* (3.12) *and* (2.20) *are equivalent.*

See Appendix 2.C for the proof.

We see that the wealth process (2.18) is self-financing. Once we are able to solve (2.20) and obtain the unique optimal control $\alpha^{Y\star}(t)$, we can use (2.14), (2.17) and (2.19) to obtain $\alpha^\star(t)$. We require that $Y(0) = F(0) + D(0) - G(0) > 0$ and suppose that $Y(t)$ does not become negative before time T. If $Y(t)$ hits zero, it stays there and no further investment takes place. Moreover, we assume that the manager is under the supervision of a regulator and choose the investment strategy as such that $Y(T) = F(T) - G(T) \geq 0$ almost surely.

2.3.3 The optimal solution

We define the value function of our simplified problem (2.20) as

$$V(t,y,z) := \sup_{\alpha^Y \in \mathscr{A}} \mathbb{E}\left[\frac{Y(T)^{1-\gamma}}{1-\gamma}\right],$$

with terminal condition $V(T,y,z) = \frac{Y(T)^{1-\gamma}}{1-\gamma}$. We assume that $V(t,y,z) \in C^{1,2,2,2}([0,T] \times \mathbb{R}^3_+)$. Then, by following the usual dynamic programming principle (see, for example, Pham (2009, Chapter 3)), V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$0 = V_t(t,y,z) + \sup_{\alpha^Y \in \mathbb{R}^3} \mathcal{A}^{\alpha^Y} V(t,y,z) \tag{2.21}$$

where

$$\mathcal{A}^{\alpha^Y} V = \left[V_y(ry + \alpha^{Y\prime} M) + \mu' V_z + \frac{1}{2}\text{tr}(\xi'\xi V_{zz}) + \frac{1}{2}\alpha^{Y\prime}\Sigma'\Sigma\alpha^Y V_{yy} + \alpha^{Y\prime}\Sigma'\xi V_{yz}\right].$$

$V_t,\ V_y,\ V_{yy},\ V_z,\ V_{zz}$ and V_{yz} are the first and second order partial derivatives with respect to t, y, z. In particular, $V_z = [V_r, V_\lambda]'$, $V_{yz} = [V_{yr}, V_{y\lambda}]'$ and $V_{zz} = \big[[V_{rr}, V_{\lambda r}]', [V_{r\lambda}, V_{\lambda\lambda}]'\big]$. Solving the first order condition on α^Y, we obtain the unique investment strategy as

$$\alpha^{Y\star} = -\frac{V_y}{V_{yy}}(\Sigma'\Sigma)^{-1}M - \frac{1}{V_{yy}}\Sigma^{-1}\xi V_{yz}. \tag{2.22}$$

Substituting (2.22) in (2.21), we obtain

$$\begin{aligned} 0 =& V_t + V_y r y - \frac{1}{2}\frac{{V_y}^2}{V_{yy}}M'(\Sigma'\Sigma)^{-1}M - \frac{V_y}{V_{yy}}M'\Sigma^{-1}\xi V_{yz} + \mu' V_z \\ &+ \frac{1}{2}tr(\xi'\xi V_{zz}) - \frac{1}{2}\frac{1}{V_{yy}}{V_{yz}}'\xi'\xi V_{yz}. \end{aligned} \tag{2.23}$$

Once we solve for the value function $V(t, y, z)$ in (2.23), we can obtain the optimal control $\alpha^{Y\star}(t)$. The following proposition provides the explicit optimal investment strategy for the transformed problem (2.20).

Proposition 2.3.3. *For any* $t \in [0, T]$ *and risk-aversion parameter*

$$\gamma > \max\left\{\frac{2\sigma_r^2 + \sigma_r^2\theta_r^2 + 2b_r\theta_r\sigma_r}{(b_r + \theta_r\sigma_r)^2 + 2\sigma_r^2},\ \frac{2b_\lambda\theta_\lambda\sigma_\lambda + \sigma_\lambda^2\theta_\lambda^2}{(b_\lambda + \theta_\lambda\sigma_\lambda)^2}\right\} \tag{2.24}$$

under the financial market setting (2.7)–(2.8)*, the optimal solution to* (2.20) *is given as*

$$\alpha^{Y\star}(t) = \begin{bmatrix} \alpha_B^{Y\star}(t) \\ \alpha_L^{Y\star}(t) \\ \alpha_S^{Y\star}(t) \end{bmatrix} = \begin{bmatrix} \frac{\theta_S\sigma_S^r - \theta_r\sigma_S - \sigma_S\sigma_r A_1(t,T)}{\sigma_S\sigma_r f_1(t,t+T_B)} + \frac{f_1(t,t+T_L)}{f_1(t,t+T_B)}\frac{\theta_\lambda + \sigma_\lambda A_2(t,T)}{\sigma_\lambda h_1(t,t+T_L)} \\ -\frac{\theta_\lambda + \sigma_\lambda A_2(t,T)}{\sigma_\lambda h_1(t,t+T_L)} \\ \frac{\theta_S}{\sigma_S} \end{bmatrix} \frac{Y(t)}{\gamma}$$

where

$$\begin{cases} A_1(t,T) = \frac{a_{11}a_{12}\exp\left(-\sqrt{\Delta_1}(T-t)\right) - a_{11}a_{12}}{a_{12}\exp\left(-\sqrt{\Delta_1}(T-t)\right) - a_{11}}, \\ \Delta_1 = b_r^2 + \frac{\gamma-1}{\gamma}\left(2\sigma_r^2 + \theta_r^2\sigma_r^2 + 2b_r\theta_r\sigma_r\right), \\ a_{11,12} = \frac{(\gamma-1)\theta_r\sigma_r + b_r\gamma \pm \gamma\sqrt{\Delta_1}}{\sigma_r^2}, \end{cases}$$

$$\begin{cases} A_2(t,T) = \frac{a_{21}a_{22}\exp\left(-\sqrt{\Delta_2}(T-t)\right) - a_{21}a_{22}}{a_{22}\exp\left(-\sqrt{\Delta_2}(T-t)\right) - a_{21}}, \\ \Delta_2 = b_\lambda^2 + \frac{\gamma-1}{\gamma}(2b_\lambda\theta_\lambda\sigma_\lambda + \theta_\lambda^2\sigma_\lambda^2), \\ a_{21,22} = \frac{(\gamma-1)\theta_\lambda\sigma_\lambda + b_\lambda\gamma \pm \gamma\sqrt{\Delta_2}}{\sigma_\lambda^2}, \end{cases}$$

$$A_0(t,T) = \int_t^T \left(a_r A_1(s,T) + a_\lambda(s)A_2(s,T) + \frac{1-\gamma}{2\gamma}\theta_S^2\right) \mathrm{d}s.$$

We include the proof in Appendix 2.D.

Next, we show that $Y^\star(T)$ (obtained using $\alpha^{Y^\star}(t)$) is always positive and verify the admissibility of the optimal control $\alpha^{Y^\star}(t)$.

Remark 2.3.1. For any $t \in [0, T]$, let $\tilde{\alpha}^Y(t) = \frac{\alpha^Y(t)}{Y(t)}$, we have

$$\mathrm{d}Y(t) = \big(r(t) + \tilde{\alpha}^Y(t)'M(t)\big)\, Y(t)\mathrm{d}t + \tilde{\alpha}^Y(t)'\Sigma(t)'Y(t)\mathrm{d}W(t).$$

According to Proposition 2.3.3, we see that the optimal $\tilde{\alpha}^{Y^\star}(t)$ is a vector of continuous (deterministic) functions. Thus, $\mathbb{E}\left[\int_0^T |\tilde{\alpha}^{Y^\star}(t)|^2\mathrm{d}t\right] < +\infty$, $Y^\star(t)$ admits a unique solution and is bounded over $[0, T]$. Thereafter, we have $\mathbb{E}\left[\int_0^T |\alpha^{Y^\star}(t)|^2\mathrm{d}t\right] < +\infty$ and $\alpha^{Y^\star}(t) \in \mathscr{A}$. Besides, we have

$$\begin{aligned}\frac{Y^\star(T)}{Y(0)} =& exp\Big(\int_0^T \big(r(u) + \tilde{\alpha}^{Y^\star}(u)'M(u) - \frac{1}{2}\tilde{\alpha}^{Y^\star}(u)'\Sigma(u)'\Sigma(u)\tilde{\alpha}^{Y^\star}(u)\big)\mathrm{d}u\Big)\\ &\times exp\Big(\int_0^T \tilde{\alpha}^{Y^\star}(u)'\Sigma(u)'\mathrm{d}W(u)\Big).\end{aligned}$$

From the above proposition, we observe a proportional relationship between $\alpha_S^{Y^\star}(t)$ and $Y(t)$ for constant θ_S, σ_S and γ. Namely, the optimal stock weight $\frac{\alpha_S^{Y^\star}(t)}{Y(t)}$ always stays the same throughout the investment horizon. This is similar to the classical Merton portfolio problem where the optimal portfolio weight on the risky asset is constant over time. The convention is that the constant market price of risk causes no change in the investment behavior. We also find that the optimal investment in the longevity bond $\alpha_L^{Y^\star}(t)$ is actually taken from the investment in the bond $\alpha_B^{Y^\star}(t)$ proportionally by a factor of $-\frac{f_1(t,t+T_L)}{f_1(t,t+T_B)}$. From (2.3) and (2.5), we see that $f_1(t, t+T_B)$ and $f_1(t, t+T_L)$ are the duration of the rolling bond and rolling longevity bond, respectively. Since the duration is always positive, $-\frac{f_1(t,t+T_L)}{f_1(t,t+T_B)}$ is negative. We conclude that there is a negative relationship between the optimal investments in bond and longevity bond. Besides, if the maturities of the rolling bond and the rolling longevity bond are the same (that is, $T_B = T_L$), we have $f_1(t, t+T_L) = f_1(t, t+T_B)$ and $\alpha_L^{Y^\star}(t)$ is fully deduced from $\alpha_B^{Y^\star}(t)$. For any $t < T$, $h_1(t, t+T_L)$ stays constant and $A_2(t, T)$ is negative and increases with t. Thus, it is easy to deduce that the optimal investment proportion in longevity bond $\frac{\alpha_L^{Y^\star}(t)}{Y(t)}$ is decreasing over time. While, the behavior of optimal proportions invested in bond and money market are not clear. Also, it is clear that the greater the manager's risk-aversion, the lower the portfolio weights on the longevity bond and stock.

Lemma 2.3.1 shows that Problem (3.12) and (2.20) are equivalent. According to (2.14), (2.17), (2.19) and Proposition 2.3.3, we obtain the solution to the initial problem (3.12) by straightforward calculations.

Proposition 2.3.4. *For any* $t \in [0,T]$*, under condition* (2.24) *and the financial market setting* (2.7)–(2.8)*, the optimal solution to* (3.12) *is given as*

$$
\begin{aligned}
\alpha_B^\star(t) =& \frac{\theta_S\sigma_S^r - \theta_r\sigma_S - \sigma_S\sigma_r A_1(t,T)}{\gamma\sigma_S\sigma_r f_1(t,t+T_B)} Y(t) - \frac{cn\int_t^T L(t,s) f_1(t,s)\mathrm{d}s}{f_1(t,t+T_B)} \\
&+ \frac{\pi n \int_T^\infty L(t,s) f_1(t,s)\mathrm{d}s}{f_1(t,t+T_B)} - \frac{f_1(t,t+T_L)}{f_1(t,t+T_B)}\alpha_L^\star(t), \\
\alpha_L^\star(t) =& -\frac{\theta_\lambda + \sigma_\lambda A_2(t,T)}{\gamma\sigma_\lambda h_1(t,t+T_L)} Y(t) - \frac{cn\int_t^T L(t,s) h_1(t,s)\mathrm{d}s}{h_1(t,t+T_L)} + \frac{\pi n \int_T^\infty L(t,s) h_1(t,s)\mathrm{d}s}{h_1(t,t+T_L)}, \\
\alpha_S^\star(t) =& \frac{\theta_S}{\gamma\sigma_S} Y(t), \quad \alpha_0^\star(t) = F(t) - \alpha_B^\star(t) - \alpha_L^\star(t) - \alpha_S^\star(t).
\end{aligned}
$$

We find that the optimal amount invested in the stock does not depend directly on the fund's wealth $F(t)$ but on the process $Y(t)$. Besides, the optimal stock weight $\frac{\alpha_S^\star(t)}{F(t)}$ does not keep constant any more and depends on the ratio $\frac{Y(t)}{F(t)}$. From the drift terms of $F(t)$ and $Y(t)$ in (2.9) and (2.18), we deduce that $F(t)$ is expected to increase with t faster than $Y(t)$. Thus, we conclude that the optimal stock weight falls over time. Similar to the result in Proposition 2.3.3, the optimal investments in bond and longevity bond correlate negatively. We can not easily infer from the solution how the optimal weights in bond and longevity bond change over time. Section 2.4 shows the results of numerical simulations which allows us to observe the optimal investment strategy dynamically. It is not straightforward to detect, from the optimal solution, how the risk-aversion coefficient γ, contribution rate r_c and wage replacement ratio r_w affect the optimal strategy. We provide a numerical analysis and comparative statics on these parameters in the following section.

2.4 Numerical applications

We first provide a base scenario to visualize the optimal proportions invested in risky assets and money market. Then, sensitivity analyses are provided to study the impact of model parameters on the optimal investment strategy. In what follows, we denote by $w_B(t)$, $w_L(t)$,

$w_S(t)$ and $w_0(t)$ the investment proportions in rolling bond, rolling longevity bond, stock and money market, respectively.

2.4.1 The base scenario

The values of the parameters for the base scenario are given in Table 2.1. We do not use real market data but most of these parameter values are taken from Menoncin & Regis (2017) and Han & Hung (2012) and represent a consensus of the current literature. Here, we assume that there exists a rolling bond and a rolling longevity bond with constant maturity time (in years) $T_B = 10$ and $T_L = 10$, respectively. As the longevity bond is supposed to be issued based on the mortality index of an older population, we assume that the market consists of a rolling longevity bond whose underlying is the survival index of the 40-year-old population. The longevity risk tends to be largely ignored in very early ages. Hence, we suppose that the scheme manager considers to add the longevity bond to the investment portfolio after the scheme members reach the age of 40, that is, the initial age (in years) is $t_0 = 40$. λ_0 given in (2.2) is computed by using the parameters given in Table 2.1. The retirement age is chosen as 65 years old, in other words the retirement time is $T = 25$ years. θ_λ is the parameter of the market price of longevity risk. It not easy to infer upon the value of θ_λ as longevity bonds are traded over the counter (OTC) and not through an exchange. In our base scenario, we use $\theta_\lambda = -0.10$. At time 0, the rolling bond offers a risk premium of about 0.01370 and the longevity bond provides a total risk premium of around 0.01372. The stock offers a total risk premium of around 0.01670. In this setting, the longevity risk premium is far less than the interest rate and stock risk premia. Later, we provide optimal investment strategies for different scenarios with different values of θ_λ. Without loss of generality, we normalize the size of membership of the fund to $n = 1$. Pension contribution rates differ widely among schemes and countries. According to HMRC (2018), in the UK, there is a limit on the amount of tax-free pension savings that an individual can pay into his pension account in each tax year. However, there is no cap on the contribution rate. As stated in DWP (2021), the Department for Work and Pensions requires that the minimum contribution rate for DC schemes is 8%, which is UK legislation. We first consider $r_c = 0.15$ then give a sensitivity analysis on the contribution rate. OECD (2019) shows that the net replacement rates vary across a large range from around 30% to 90% or more in OECD countries. The average net replacement rates of an average earner from mandatory schemes is 59%. Since in the proposed scheme contributions are not returned to dead members, it is natural to set a replacement ratio that is above average. In our base scenario, we adopt $r_w = 0.75$. We set the instantaneous wage as $w = 15$, thus the instantaneous contribution and annuity installment are $c = 2.25$ and $\pi = 11.25$, respectively. We suppose the initial scheme wealth is $F_0 = 20$ and the manager's risk aversion coefficient is $\gamma = 2.5$.

Table 2.1: Values of parameters in the base scenario

Interest rate	Mortality	Stock
$r_0 = 0.0621328$	$b = 12.9374$	$\sigma_S = 0.14926$
$a_r = 0.0056210$	$\phi = 0.0009944$	$\sigma_S^r = -0.0046306$
$b_r = 0.0904668$	$m = 86.4515$	$\theta_S = 0.1108301$
$\sigma_r = 0.0543625$	$b_\lambda = 0.5610000$	
$\theta_r = -0.5590635$	$\sigma_\lambda = 0.0352$	

We obtain 1000 simulated paths and present the average paths of optimal investment proportions in the left plot in Figure 2.1. The right plot in Figure 2.1 displays the average path of $\frac{Y(t)}{F(t)}$, $\frac{G(t)}{F(t)}$ and $\frac{D(t)}{F(t)}$. As discussed in Section 2.3.3, we see that the ratio $\frac{Y(t)}{F(t)}$ and the optimal stock weight $w_S^\star(t)$ decline over time. The optimal weight in cash is initially negative and is increasing over time. In the beginning, the short position in the money market reveals that the scheme manager borrows money and invests in risky assets to obtain risk premia. It implies that the manager takes an aggressive approach to quickly increase the pension scheme's wealth to a high level in the early stage. The reduction in the total proportion invested in risky assets shows that, when closer to retirement, the manager becomes more conservative and shifts the scheme's wealth to safer assets. Throughout the whole management period, the optimal bond weight is declining. The optimal weight in the longevity bond rises first and drops slightly in the last few years. Besides, we see that the sum of bond and longevity bond weights declines over time. Furthermore, the right plot of Figure 2.1 shows that the ratios of future contributions $D(t)$ and minimum guarantee $G(t)$ to the scheme wealth $F(t)$ are both decreasing, and $\frac{D(t)}{F(t)}$ drops faster than $\frac{G(t)}{F(t)}$. These observations reveal that, when approaching retirement, the need for interest rate hedging becomes lower while the need to hedge against longevity risk is still significant. Consequently, the manager cuts down the total proportion of wealth invested in the bond and longevity bond while increasing the longevity bond's weight. Moreover, the longevity bond dominates the portfolio in the late period of management. Generally, our base scenario implies that the longevity bond is an important element in the investment portfolio and provides an efficient way to hedge against both interest rate and longevity risks.

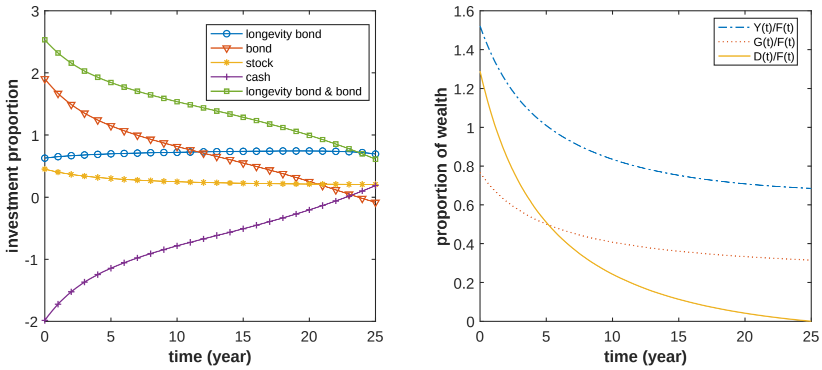

Figure 2.1: Average paths of optimal investment proportions; $Y(t)/F(t)$, $G(t)/F(t)$ and $D(t)/F(t)$

2.4.2 Sensitivity analysis

In Section 2.3.3, we give some comments on the impact of model parameters on the optimal investment strategy. This section provides various scenarios to investigate the impact of model parameters numerically. We are interested in the following parameters: risk aversion coefficient (γ), market price of longevity risk parameter (θ_λ), maturity of rolling longevity bond (T_L), contribution rate (r_c) and wage replacement ratio (r_w). Other factors such as the market prices of interest rate risk and stock risk may also affect the optimal investment strategy sufficiently. Nonetheless, we do not look into those factors as the focus of our study is on hedging longevity risk.

Risk-aversion

In the context of the CRRA utility function (2.10), γ measures the manager's relative risk-aversion. The higher γ, the more risk-averse the scheme's manager. The four plots in Figure

2.2 show the average paths of optimal investment proportions with risk-aversion coefficient γ equal to 2, 3, 4 and 5, respectively. Comparing the plots in Figure 2.1 and 2.2, we observe that the higher (or lower) the risk-aversion coefficient γ, the higher (or lower) the investment proportions in bond and cash. Whereas, the investment proportions in longevity bond and stock decrease with γ. The bond is used to hedge the interest rate risk only while the longevity bond provides a hedge against interest rate risk as well as longevity risk. A risk-averse investor tends to avoid relatively higher risk and prefers investments with lower risk but higher guaranteed returns. Although stock and longevity bond provide higher risk premiums, a scheme manager with a high risk-aversion coefficient prefers to invest more in safer assets, i.e., bond and cash. The different strategies in these cases can also be explained by the optimal solution in Proposition 2.3.4. The optimal investment proportions in bond and longevity bond are negatively correlated. In particular, if we set $T_B = T_L = 10$ we have $-\frac{f_1(t,t+T_L)}{f_1(t,t+T_B)} = -1$. This implies that the investment in longevity bond is fully taken from the investment in the bond. Since the longevity bond weight decreases with γ, the bond weight increases accordingly. This behavior occurs as the bond is a safer asset compared to the longevity bond. It is surprising that the optimal bond weight can become negative in the late period of investment, meanwhile the holding in the money market is positive. This indicates that the manager chooses to short sell bonds when approaching the retirement time. Gao (2008) and Han & Hung (2012) also had similar findings. They showed that the pension portfolio would shift from investments into risky assets to the money market. The convention is that the bond guarantees a fixed amount of money at maturity. In the beginning, the weight put on the bond is relatively high and is declining when moving closer to T (the retirement time). It also implies that the need to hedge interest rate risk becomes lower when approaching T. The bottom right plot in Figure 2.2 reveals that the longevity bond suppresses other assets in the late management period even for a highly risk-averse investor. Moreover, the longevity bond weight rises to around 50% by the end. Further, we find that the optimal weight for the longevity bond is almost always higher than for the stock, in all scenarios. Overall, we conclude that, even though highly risk-averse managers invest less in the longevity bond, the latter is always an important element in the scheme's investment portfolio.

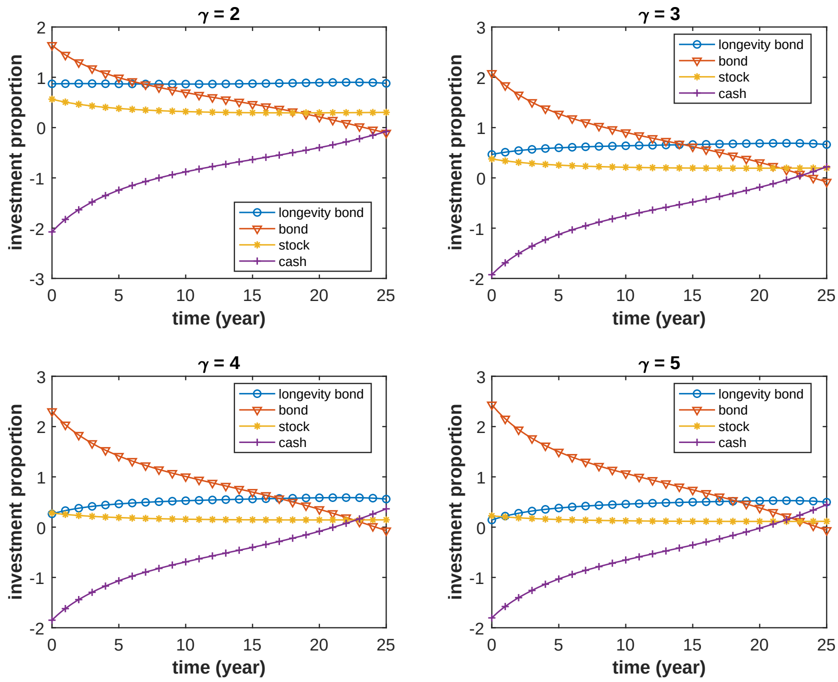

Figure 2.2: Average paths of optimal investment proportions with $\gamma = 2$, 3, 4 and 5

Market price of longevity risk

The rolling bond provides an interest rate risk premium of $-f_1(t, t + T_B)\sigma_r\theta_r r(t)$. As the rolling longevity bond has the same maturity time as the rolling bond, it offers the same premium on interest rate risk. Besides, the longevity bond provides a longevity risk premium of $-h_1(t, t + T_L)\sigma_\lambda\theta_\lambda\lambda(t)$. The stock offers an interest rate risk premium of $\sigma_S^r\theta_r r(t)$ and a stock risk premium of $\theta_S\sigma_S$. Clearly, θ_r and θ_S affect the optimal strategy. However, we are more interested in the impact of the longevity risk premium and therefore focus on the parameter θ_λ. From (2.6), we learn that the longevity risk premium offered by the rolling longevity bond increases with $-\theta_\lambda$. Figure 2.3 shows the average optimal strategies for $\theta_\lambda = -0.06, -0.08, -0.12$ and -0.14. At initial time, the corresponding longevity risk premium for these choices are 1.1749×10^{-5}, 1.5684×10^{-5}, 2.3584×10^{-5} and 2.7549×10^{-5}, respectively. In all these cases, the longevity risk premium is much smaller than the interest rate risk and stock risk premia. In general, we observe that the optimal weight for the longevity bond increases with $-\theta_\lambda$ while the optimal bond weight decreases with $-\theta_\lambda$. The optimal proportions invested in stock and money market do not change much when θ_λ changes. Besides, the higher $-\theta_\lambda$ the more the longevity bond suppresses other assets. This behavior is consistent with conventional thinking that the higher the risk premium the more attractive the asset is. The top left plot in Figure 2.3 indicates that even in the case where the longevity risk premium is very low, the longevity bond dominates the portfolio over the last 8 years. It reveals that the longevity bond is an efficient instrument to hedge the longevity risk. When $\theta_\lambda = -0.08, -0.12$ and -0.14, we observe short positions in the rolling bond in the late management period. It indicates that the need to hedge against interest rate risk becomes lower when approaching the retirement time, while, the need to hedge against uncertain changes in mortality rate is still very significant.

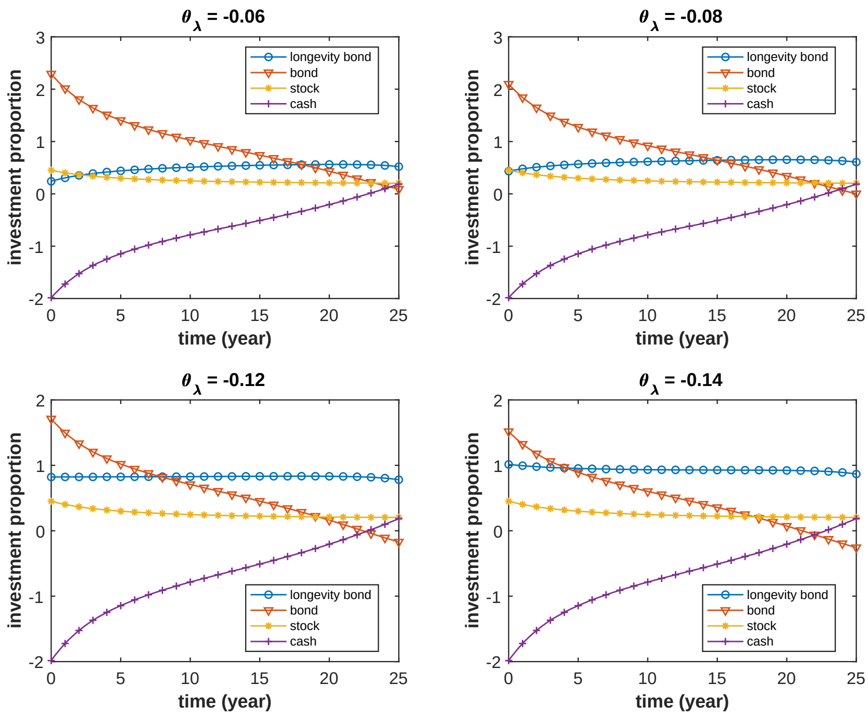

Figure 2.3: Average paths of optimal investment proportions with $\theta_\lambda = -0.06, -0.08, -0.12$ and -0.14

We provide Figure 2.4 to take a closer look at the impact of θ_λ on the optimal investment strategies. Again, we find that the higher the longevity risk premium (that is, the lower the θ_λ), the higher the portfolio weight for the longevity bond. The interpretation is that with other parameters unchanged, a lower θ_λ increases the longevity risk premium but does not increase the uncertainty in the longevity bond value. Thus, it makes the longevity bond more attractive for investment. The opposite reaction of bond weight and longevity bond weight against θ_λ is explained in the discussion of Proposition 2.3.4. It is not a surprise that the optimal weight for the stock is barely affected by θ_λ. The reason can be inferred from the optimal solution and the fact that the portfolio's weight for the stock is $\frac{\theta_S}{\sigma_S\gamma}\frac{Y(t)}{F(t)}$. $\frac{\theta_S}{\sigma_S\gamma}$ does not depend on θ_λ and $\frac{Y(t)}{F(t)}$ is only slightly affected by θ_λ. Besides, the weight for cash is not sensitive to θ_λ. This is because the changes in bond and longevity bond weights offset each other, and the total proportion in risky assets is not sensitive to θ_λ. In summary, we conclude that the longevity bond is an important element in the pension scheme's investment portfolio and hedges the scheme's longevity risk efficiently. Even when the longevity bond provides a relatively low longevity risk premium, it is optimal to invest a large proportion of the scheme's wealth into the latter, especially during the late management period.

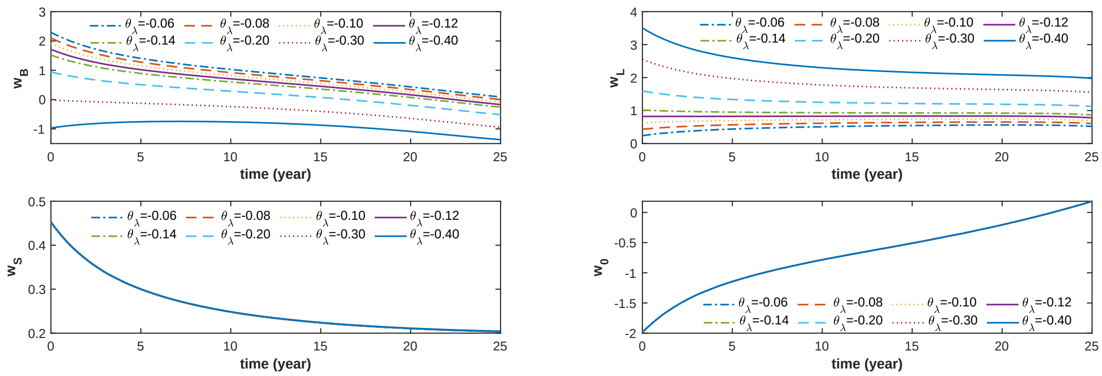

Figure 2.4: Average paths of optimal investment proportions varying θ_λ

Maturity of the rolling longevity bond

For the previous numerical scenarios, we assumed that $T_B = T_L = 10$ and therefore $f_1(t, t + T_B) = f_1(t, t + T_L)$. Now, we investigate the optimal strategy when T_L equals to 5, 15, 20 and 25. We provide Figure 2.5 to look into the impact of T_L on the optimal weights for the risky asset separately. Generally, we observe that the longer the rolling longevity bond's maturity, the lower the investment proportions in bond and longevity bond, while more weight is shifted into cash. The optimal stock weight does not change much when T_L increases. We conclude from Proposition 2.3.4, that at any time t the optimal stock weight depends on $\frac{Y(t)}{F(t)}$. Accordingly, there is numerical evidence that $\frac{Y(t)}{F(t)}$ is not very sensitive to T_L. From the top right plot in Figure 2.5, we observe that w_L shows an obvious decline when T_L rises from 5 to 10. No distinguishable change in w_L is observed when T_L takes the values 15, 20 and 25. Intuitively, longer maturity times result in more uncertainty in the rolling longevity bonds. For a risk-averse investor, it is therefore better to have less portfolio weight attached to a longevity bond with a longer maturity. Figure 2.6 shows that $\frac{1}{h_1(t,t+T_L)}$ is decreasing with T_L and $\frac{1}{F(t)}$ does not change much when T_L changes. According to Proposition 2.3.4, this results in the decline of longevity bond's weights. At first, $\frac{1}{h_1(t,t+T_L)}$ drops dramatically when T_L increases and later on changes only slightly. Thereafter, we observe that the optimal longevity bond weight decreases with T_L but is not sensitive to longer maturities.

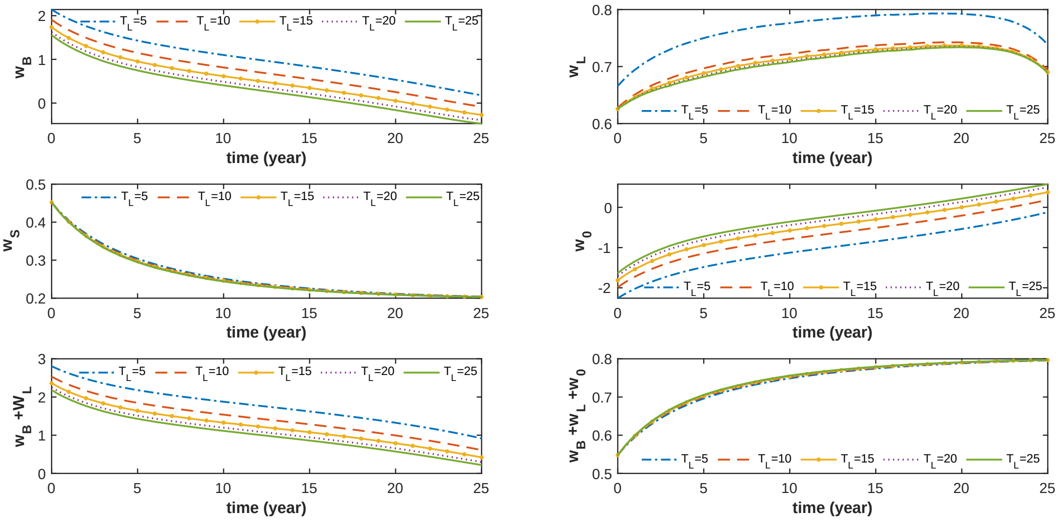

Figure 2.5: Average paths of optimal investment proportions varying T_L

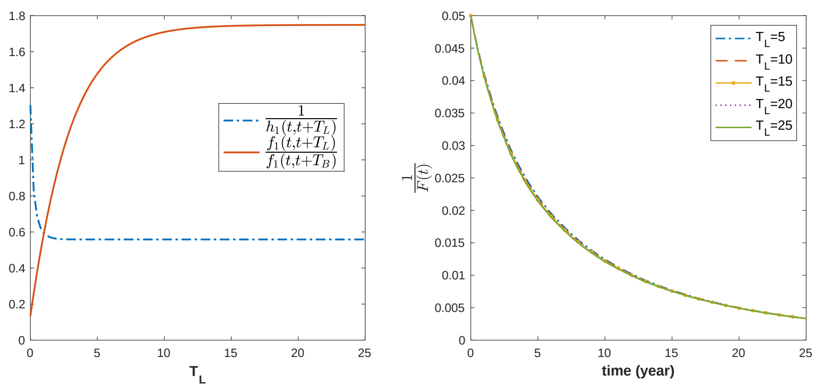

Figure 2.6: $\frac{1}{h_1(t,t+T_L)}$ and $\frac{f_1(t,t+T_L)}{f_1(t,t+T_B)}$ versus T_L ; Average path of $\frac{1}{F(t)}$ varying T_L

Compared to w_L, w_B reacts stronger to changes in T_L. This is apparent from the optimal solution as the changes in w_L are scaled up by the factor $-\frac{f_1(t,t+T_L)}{f_1(t,t+T_B)}$. The left plot in Figure 2.6 shows that $\frac{f_1(t,t+T_L)}{f_1(t,t+T_B)}$ increases with T_L. Therefore, the optimal weight for the bond decreases significantly with T_L due to decreasing $-\frac{f_1(t,t+T_L)}{f_1(t,t+T_B)}w_L(t)$. The optimal investment proportion in the bond becomes negative during the later years except for the case when $T_L = 5$. This is because the longevity bond hedges longevity risk as well as interest rate risk. When nearing retirement, the need to hedge interest rate risk reduces while the longevity risk is still high. The negative position in the bond offsets the interest rate risk hedge provided by the longevity bond. Even though a negative position in the bond can be observed, the sum of the weights for longevity bond and bond are always positive as shown in the bottom left plot in Figure 2.5. In addition, we observe that the longer the longevity bond's maturity, the sooner the longevity bond dominates the portfolio. Overall, longer maturity times may lessen the attractiveness of longevity bonds. Even though, longer maturity times might be detrimental to investment appeal, longevity bond's always play an important role in the DC scheme's risk management, dominating the portfolio during late management period.

Contribution rate

The minimum requirement for contribution rates for DC schemes in the UK is 8%, see DWP (2013) and OECD (2019), but keeping aside tax consideration there is in principle no upper limit on contribution rates. In this section we studie the four cases when the contribution rate is 10%, 20%, 30% and 40% respectively. In Figure 2.7, we show the optimal proportions with contribution rate r_c equals to 0.10, 0.20, 0.30 and 0.40. That is, the instantaneous contribution c equals to 1.5, 3, 4.5 and 6. In general, we observe that the higher the contribution rate, the higher the total weight in risky assets while the lower the weight in cash. The intuition behind this observation is that the higher the contribution rate, the higher the present value of the scheme's future income. A high present value of the future income incentives the manager to increase the total investment into risky assets. Since there is guaranteed future income, the manager would like to take more risk and earn a higher risk premiums. Nonetheless, the manager invests less into the longevity bond when the contribution rate is higher. Although the manager reduces the investment into the longevity bond, it remains an important element in his investment portfolio.

From Proposition 2.3.4, the optimal stock weight depends on $\frac{Y(t)}{F(t)}$. At any time t, we have $Y(t) = F(t) + D(t) - G(t)$. A higher contribution paid into the scheme results in a greater present value of future contributions $D(t)$. Besides, the wealth level $F(t)$ and the present value of the minimum guarantee $G(t)$ does not depend on the contribution rate. Consequently, $\frac{Y(t)}{F(t)}$ and the optimal investment proportion in the stock increase with r_c. When closer to

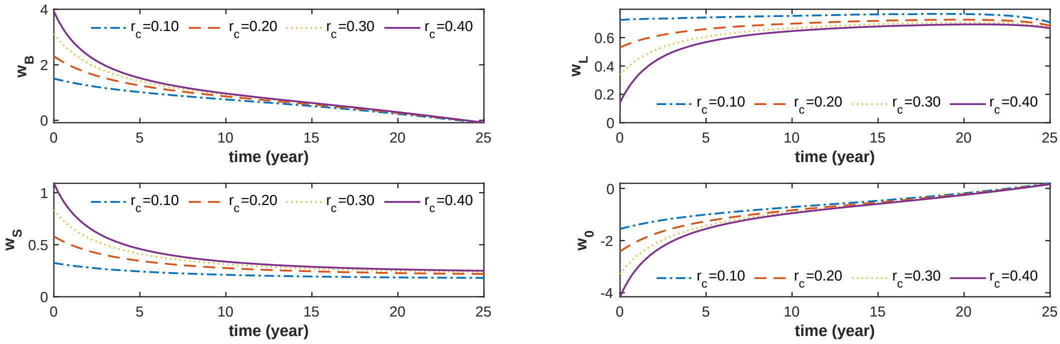

Figure 2.7: Average paths of optimal investment proportions varying r_c

the retirement time, the optimal stock weight is less sensitive to r_c. The interpretation is that the wealth process $F(t)$ increases over time as there is continuous contribution paid into the scheme. Besides, the scheme receives investment returns. The expected value of future contributions $D(t)$ decreases over time and eventually reaches zero at T. Therefore, $\frac{Y(t)}{F(t)}$ drops over time and the impact of r_c on optimal stock weight reduces. From the optimal solution, the optimal investment proportion in the longevity bond at any time t is

$$w_L^\star(t) = w_L^{Y\star}(t)\frac{Y(t)}{F(t)} - w_L^D(t)\frac{D(t)}{F(t)} + w_L^G(t)\frac{G(t)}{F(t)} \tag{2.25}$$

where

$$w_L^{Y\star}(t) = -\frac{\theta_\lambda + \sigma_\lambda A_2(t,T)}{\gamma\sigma_\lambda h_1(t,t+T_L)}, \quad w_L^D(t) = \frac{\int_t^T L(t,s)h_1(t,s)\mathrm{d}s}{h_1(t,t+T_L)\int_t^T L(t,s)\mathrm{d}s},$$
$$w_L^G(t) = \frac{\int_T^\infty L(t,s)h_1(t,s)\mathrm{d}s}{h_1(t,t+T_L)\int_T^\infty L(t,s)\mathrm{d}s}.$$

For a fixed t, it is clear that $Y(t)$ and $D(t)$ increase when r_c rises. However, $w_L^{Y\star}(t)$ and $w_L^D(t)$ do not depend on the contribution rate. It is difficult to see from (2.25) how $w_L^\star(t)$ changes with r_c. From the numerical results, we observe that a higher r_c leads to a lower weight in longevity bond and $w_L^\star(t)$ becomes less sensitive to r_c in later periods. We infer that the second term in (2.25) decreases faster and the first term increases slower when r_c rises. Thus, $w_L^\star(t)$ declines with r_c. When approaching T, the movements in the two terms offset each other gradually. Consequently, insensitivity is observed. Analyzing $w_B^\star(t)$ in the same way, we find that the optimal bond weight increases with the contribution rate and is less sensitive to r_c in later periods. We also find that, when approaching T, both the proportions of the wealth invested into risky asset and cash are less sensitive to the contribution rate.

Generally, our numerical results imply that the longevity bond always dominates other assets during the late period, the higher the contribution rate the lower its portfolio weight.

Wage replacement ratio

The wage replacement ratio is a good tool when estimating retirement income needs. A high wage replacement ratio implies that a high fraction of the pre-retirement income is needed to maintain living standard in retirement. OECD (2019) reveals that the net replacement ratio varies from 30% to 90% among OECD countries. Accordingly, we set r_w equal to 0.30, 0.50, 0.70 and 0.90 to test the impact of wage replacement ratio.

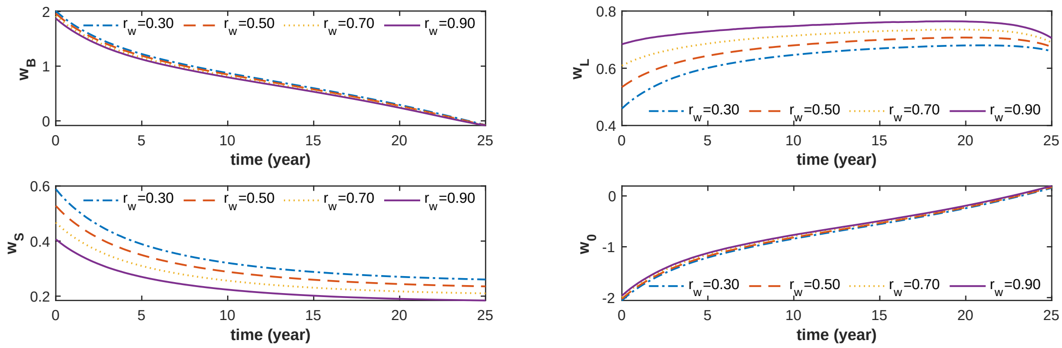

Figure 2.8: Average paths of optimal investment proportions varying r_w

In Figure 2.8, we look into the effect of r_w on the optimal weights in bond, longevity bond, stock and cash separately. It is clear that the higher the wage replacement ratio, the higher the annuity installments π and the higher the guarantee $G(t)$. While the wealth process $F(t)$ and the discounted future contributions $D(t)$ do not depend on the wage replacement ratio. Thus, $\frac{Y(t)}{F(t)} = \frac{F(t)+D(t)-G(t)}{F(t)}$ declines with r_w. From Proposition 2.3.4, we see that the optimal stock weight increases with $\frac{Y(t)}{F(t)}$, thus the optimal stock weight decreases with the wage replacement ratio. Similar to the analyses in Section 2.4.2, it is not clear from the optimal solution how the weights in bond, longevity bond and cash react when the wage replacement ratio changes. Our numerical results show that the optimal weights in bond and cash are less responsive to changes in the wage replacement ratio. It is optimal to increase the fraction of wealth invested into the longevity bond when the wage replacement ratio is high. Intuitively, a high wage replacement ratio implies that the members require high annuity installments (or, the minimum guarantee) and the scheme is exposed to greater longevity risk. As a consequence, the scheme manager invests a large proportion of the scheme's wealth into the longevity bond to hedge the longevity risk.

2.5 Conclusion

We studied the optimal investment problem underlying the management of a DC pension scheme in a framework where both interest rate risk and longevity risk are considered. Our theoretical results and subsequent numerical studies showed evidence that the longevity bond plays an important role in DC scheme's risk management. We observed that the more risk-averse the scheme manager, the lower the proportion of investment into the longevity bond. However, even for a highly risk-averse manager, we showed that it is still optimal to invest a significant proportion of the scheme's wealth into the longevity bond. Also, compared with the investment ratio's of the other risky assets, the investment proportion for the longevity bond is shown to be relatively high even in the case where the longevity risk premium is relatively low. Moreover, we observed that longer maturity times could reduce attractiveness of longevity bonds, however even longevity bonds with longer maturity dominate the other assets in the investment portfolio during the later periods of the scheme. Although the manager reduces investment into longevity bonds when the contribution rate is high, the longevity bond remains an important element of the investment portfolio. Further, we observed that high wage replacement ratios incentivize the scheme manager to invest a higher proportion of wealth into the longevity bond. We conclude that longevity bonds play an important role for DC pension schemes, in particular at times when mortality risk is increased. They are very attractive to pension schemes and there is genuine potential in the development of mortality-linked derivatives and exchanges for longevity bonds.

Further research: in this work, we have assumed that the longevity bond's reference population and the scheme members have the same mortality behavior. However, population basis risk arises when there is a mismatch between the hedging instrument's underlying mortality experience and the hedging population's mortality behavior. We investigate the impact of longevity basis risk on the asset allocation and longevity risk hedge in Chapter 3.

Appendices

Appendix 2.A: Proof of Lemma 2.2.1

Proof. First, we show that there exists $h_0 > 0$ such that for any $\nu \in \mathcal{T}$ the statement holds for $t \in [\nu, \nu + h_0]$. For $t \in [\nu, \nu + h_0]$, suppose $\sqrt{C_0} = (\theta_r, \theta_\lambda, \theta_S)^+$, denote by

$$\varphi_1(t) = e^{-(C_0+\frac{1}{2}\sigma_r^2)(t-\nu)}, \qquad \varphi_2(t) = e^{-(C_0+\frac{1}{2}\sigma_\lambda^2)(t-\nu)}, \qquad \varphi_3(t) = e^{-C_0(t-\nu)},$$

and let

$$\begin{aligned}
\Phi_1(t) &= \theta_r^2 + \varphi_1'(t) - b_r\varphi_1(t) + \frac{1}{2}\varphi_1^2(t)\sigma_r^2, \\
\Phi_2(t) &= \theta_\lambda^2 + \varphi_2'(t) - b_\lambda\varphi_2(t) + \frac{1}{2}\varphi_2^2(t)\sigma_r^2, \\
\Phi_3(t) &= \varphi_3'(t).
\end{aligned}$$

Then,

$$\begin{aligned}
\Phi_1(t) &\le C_0 - \big(C_0 + \frac{1}{2}\sigma_r^2\big)\varphi_1(t) - b_r\varphi_1(t) + \frac{1}{2}\sigma_r^2\varphi_1^2(t), \\
\Phi_2(t) &\le C_0 - \big(C_0 + \frac{1}{2}\sigma_\lambda^2\big)\varphi_2(t) - b_\lambda\varphi_2(t) + \frac{1}{2}\sigma_\lambda^2\varphi_2^2(t), \\
\Phi_3(t) &\le -C_0\varphi_3(t).
\end{aligned}$$

As $t \to \nu$, the right-hand sides of the above inequalities go to $-b_r < 0$, $-b_\lambda < 0$ and $-C_0 < 0$, respectively. Thus, for all $t \in [\nu, \nu + h_0]$, there exists $h_0 > 0$ such that $\Phi_1(t) < 0$, $\Phi_2(t) < 0$ and $\Phi_3(t) < 0$.

Next, denote by

$$\Gamma(t) = exp\left\{\int_\nu^t (\theta_r^2 r(u) + \theta_\lambda^2\lambda(u) + \theta_S^2)\mathrm{d}u + \Phi_1(t)r(t) + \Phi_2(t)\lambda(t) + \Phi_3(t)\right\}.$$

Applying Itô's formula, we have

$$\begin{aligned}
\frac{\mathrm{d}\Gamma(s)}{\Gamma(s)} =& a_r\varphi_1(s)\mathrm{d}s + a_\lambda(s)\varphi_1(s)\mathrm{d}s + \Phi_1(s)r(s)\mathrm{d}s + \Phi_2(s)\lambda(s)\mathrm{d}s + \Phi_3(s)\mathrm{d}s \\
&+ \sigma_r\sqrt{r(s)}\varphi_1(s)\mathrm{d}W_1(s) + \sigma_\lambda\sqrt{\lambda(s)}\varphi_2(s)\mathrm{d}W_2(s).
\end{aligned}$$

Then, we obtain

$$
\begin{aligned}
\mathbb{E}[\Gamma(t) \mid \mathcal{F}(\nu)] =& \mathbb{E}\Bigg[exp\Big\{ \int_{\nu}^{t} \big(a_r\varphi_1(s) + a_\lambda(s)\varphi_1(s) + \Phi_1(s)r(s) + \Phi_2(s)\lambda(s) + \Phi_3(s)\big)\mathrm{d}s \\
&- \frac{1}{2}\int_{\nu}^{t} \big(\sigma_r^2 r(s)\varphi_1^2(s) + \sigma_\lambda^2\lambda(s)\varphi_2^2(s)\big)\mathrm{d}s + \int_{\nu}^{t} \sigma_r\sqrt{r(s)}\varphi_1(s)\mathrm{d}W_1(s) \\
&+ \int_{\nu}^{t} \sigma_\lambda\sqrt{\lambda(s)}\varphi_2(s)\mathrm{d}W_2(s)\Big\} \,\Big|\, \mathcal{F}(\nu)\Bigg] \\
\leq& \mathbb{E}\left[exp\Big\{ \int_{\nu}^{t} \big(a_r\varphi_1(s) + a_\lambda(s)\varphi_1(s)\big)\mathrm{d}s\Big\} \,\Big|\, \mathcal{F}(\nu)\right] < \infty.
\end{aligned}
$$

Therefore, for $t \in [\nu, \nu + h_0]$, the Novikov condition

$$
\mathbb{E}\left[exp\left\{\int_{\nu}^{t} (\theta_r^2 r(u) + \theta_\lambda^2\lambda(u) + \theta_S^2)\mathrm{d}u\right\} \,\Big|\, \mathcal{F}(\nu)\right] \leq \mathbb{E}[\Gamma(t) \mid \mathcal{F}(\nu)] < \infty
$$

is satisfied and

$$
\frac{Z(t)}{Z(\nu)} = \exp\left(-\int_{\nu}^{t} \Theta(s)'\mathrm{d}W(s) - \frac{1}{2}\int_{\nu}^{t} |\Theta(s)|^2\mathrm{d}s\right)
$$

is a martingale.

Next, consider a partition $0 = t_0 < t_1 < \cdots < t_i < \cdots < t_N = t \in \mathcal{T}$, such that $\sup_i |t_{i+1} - t_i| < h_0$. Then, for $s \in [t_i, t_{i+1}]$, $\frac{Z(s)}{Z(t_i)}$ is a martingale and $\mathbb{E}[Z(t_{i+1})/Z(t_i) \mid \mathcal{F}(t_i)] = 1$. Hence,

$$
\mathbb{E}[Z(t)] = \mathbb{E}\left[\prod_{i=0}^{N-1} \frac{Z(t_{i+1})}{Z(t_i)}\right] = \mathbb{E}\left[\frac{Z(t_1)}{Z(t_0)}\frac{Z(t_2)}{Z(t_1)}\cdots\mathbb{E}\Big[\frac{Z(t_N)}{Z(t_{N-1})}\Big|\mathcal{F}(t_{N-1})\Big]\right] = 1.
$$

Thus, $Z(t)$ is a martingale and $\mathbb{E}[Z(t)] = 1$. See also Shirakawa (2002, Theorem 3.2)). □

Appendix 2.B: Proof of Proposition 2.3.1

Proof. At ant time $t \in [0, T]$, by interchanging the order of integration, we can rewrite (2.12) as

$$
D(t) = \int_t^T cne^{-\int_0^t \lambda(u)\mathrm{d}u}\widetilde{\mathbb{E}}\left[e^{-\int_t^s \lambda(u)\mathrm{d}u}e^{-\int_t^s r(u)\mathrm{d}u} \,\Big|\, \mathcal{F}(t)\right]\mathrm{d}s = cn\int_t^T L(t,s)\mathrm{d}s.
$$

From Leibniz' integral rule, we get

$$
\frac{\mathrm{d}D(t)}{\mathrm{d}t} = cn\left(0 - L(t,t) + \int_t^T \frac{\mathrm{d}L(t,s)}{\mathrm{d}t}\mathrm{d}s\right).
$$

Then, we obtain

$$\begin{aligned}
\mathrm{d}D(t) =& - cnp(t)\mathrm{d}t + cn\int_t^T L(t,s)\frac{\mathrm{d}L(t,s)}{L(t,s)}\mathrm{d}s \\
=& - cnp(t)\mathrm{d}t + r(t)D(t)\mathrm{d}t + cn\int_t^T L(t,s)\sigma_L^r(t,s)\mathrm{d}s\left(\mathrm{d}W_1(t) + \theta_r\sqrt{r(t)}\mathrm{d}t\right) \\
&+ cn\int_t^T L(t,s)\sigma_L^\lambda(t,s)\mathrm{d}s\left(\mathrm{d}W_2(t) + \theta_\lambda\sqrt{\lambda(t)}\mathrm{d}t\right).
\end{aligned} \tag{2.26}$$

Comparing the coefficients in (2.4) and (2.26), we obtain the holdings in rolling longevity bond, rolling bond and money market:

$$\begin{aligned}
\alpha_L^D(t) &= \frac{cn\int_t^T L(t,s)\sigma_L^\lambda(t,s)\mathrm{d}s}{\sigma_L^\lambda(t,t+T_L)} = \frac{cn\int_t^T L(t,s)h_1(t,s)\mathrm{d}s}{h_1(t,t+T_L)}, \\
\alpha_B^D(t) &= \frac{cn\int_t^T L(t,s)\sigma_L^r(t,s)\mathrm{d}s - \alpha_L^D(t)\sigma_L^r(t,t+T_L)}{\sigma_L^r(t,t+T_B)} \\
&= \frac{cn\int_t^T L(t,s)f_1(t,s)\mathrm{d}s}{f_1(t,t+T_B)} - \alpha_L^D(t)\frac{f_1(t,t+T_L)}{f_1(t,t+T_B)}, \\
\alpha_0^D(t) &= D(t) - \alpha_B^D(t) - \alpha_L^D(t).
\end{aligned}$$

□

Appendix 2.C: Proof of Lemma 2.3.1

Proof. It is clear from (2.19) that we need to verify the admissibility condition for the deterministic functions $\alpha^D(t)$ and $\alpha^G(t)$. Now for any fixed $t \in [0,T]$ and any $s \in [T,\infty)$, $f_1(t,s)$, $h_1(t,s)$ and $L(t,s)$ are continuous functions. It is easy to see that $\mathbb{E}\left[\int_0^T |\alpha^D(t)|^2\mathrm{d}t\right] < +\infty$.

Since $r(t) > 0$, $\lambda(t) > 0$, $f_1(t,s) \geq 0$, $h_1(t,s) \geq 0$ and $h_0(t,s) \leq 0$, we have

$$L(t,s) \leq e^{f_0(t,s)} = \left(\frac{2\eta_r e^{\frac{1}{2}(\tilde{b}_r+\eta_r)(s-t)}}{(\tilde{b}_r+\eta_r)e^{\eta_r(s-t)} + (\eta_r - \tilde{b}_r)}\right)^{\frac{2a_r}{\sigma_r^2}}.$$

Let

$$\tilde{f}(t,s) = \left(\frac{2\eta_r e^{\frac{1}{2}(\tilde{b}_r+\eta_r)(s-t)}}{(\tilde{b}_r+\eta_r)e^{\eta_r(s-t)}}\right)^{\frac{2a_r}{\sigma_r^2}} = \left(\frac{2\eta_r e^{-\frac{1}{2}(\tilde{b}_r-\eta_r)t}}{\tilde{b}_r+\eta_r}\right)^{\frac{2a_r}{\sigma_r^2}} e^{\frac{a_r(\tilde{b}_r-\eta_r)}{\sigma_r^2}s},$$

we have

$$\tilde{f}(t,s) > e^{f_0(t,s)} \geq L(t,s) > 0 \quad on \quad [T,\infty).$$

Given $\eta_r > 0$, $\tilde{b}_r > 0$, $\tilde{b}_r - \eta_r < 0$ and $\frac{2a_r}{\sigma_r^2} > 1$, it is easy to see that $\int_T^\infty \tilde{f}(t,s)\mathrm{d}s$ is a constant. Moreover, $\int_T^\infty L(t,s)\mathrm{d}s$ is convergent. Since $f_1(t,s)$ is a monotonically increasing function and is bounded on $[T,\infty)$, Abel's test shows that $\int_T^\infty f_1(t,s)L(t,s)\mathrm{d}s$ is convergent. Similarly, $\int_T^\infty h_1(t,s)L(t,s)\mathrm{d}s$ also converges. Thus, $\mathbb{E}\left[\int_0^T |\alpha^G(t)|^2\mathrm{d}t\right] < +\infty$.

Therefore, if $\mathbb{E}\left[\int_0^T |\alpha^Y(t)|^2\mathrm{d}t\right] < +\infty$, then $\mathbb{E}\left[\int_0^T |\alpha(t)|^2\mathrm{d}t\right] < +\infty$. This means that if $\alpha^Y(t) \in \mathscr{A}$, then $\alpha(t) \in \mathscr{A}$. We have $D(T) = 0$, thus if $Y(T) \geq 0$ a.s., then $F(T) - G(T) \geq 0$ a.s.. Furthermore, since $F(t) = Y(t) - D(t) + G(t)$ and (2.19) holds, $\alpha^{Y\star}(t)$ leads to the optimal strategy $\alpha^\star(t)$ which concludes the argument. □

Appendix 2.D: Proof of Proposition 2.3.3

Proof. For any $t \in [0,T]$, let $g(t,z)$ be a function of t and $z(t)$. We make a sophisticated guess that the solution of the second order non-linear partial differential equation (2.23) is of the following form

$$V(t,y,z) = \frac{y^{1-\gamma}}{1-\gamma}g(t,z), \tag{2.27}$$

with terminal condition $g(T,z) = 1$. Substituting (2.27) in (2.23) leads to

$$\begin{aligned} 0 =& g_t + (1-\gamma)rg + \frac{1-\gamma}{2\gamma}M'(\Sigma'\Sigma)^{-1}Mg + \frac{1-\gamma}{\gamma}M'\Sigma^{-1}\xi g_z + \mu' g_z \\ &+ \frac{1}{2}tr(\xi'\xi g_{zz}) + \frac{1-\gamma}{2\gamma g}{g_z}'\xi'\xi g_z. \end{aligned} \tag{2.28}$$

We further guess that $g(t,z)$ is of the following form

$$g(t,z) = e^{A_0(t,T)+A(t,T)z(t)} = e^{A_0(t,T)+A_1(t,T)r(t)+A_2(t,T)\lambda(t)} \tag{2.29}$$

with terminal conditions $A_0(T,T) = 0$, $A_1(T,T) = 0$ and $A_2(T,T) = 0$. Substituting (2.29) in (2.28), we have

$$\begin{aligned} 0 =& (A_0' + A_1'r + A_2'\lambda) + (1-\gamma)r + \frac{1-\gamma}{2\gamma}\left(\theta_r^2 r + \theta_\lambda^2\lambda + \theta_S^2\right) + (a_r - b_r r)A_1 \\ &+ \frac{1-\gamma}{\gamma}(\theta_r\sigma_r r A_1 + \theta_\lambda\sigma_\lambda\lambda A_2) + (a_\lambda - b_\lambda\lambda)A_2 + \frac{1}{2\gamma}\left(\sigma_r^2 r A_1^2 + \sigma_\lambda^2\lambda A_2^2\right). \end{aligned}$$

By collecting the $r(t)$ and $\lambda(t)$ terms above, we obtain the following three ODEs:

$$
\begin{aligned}
0 &= A_1'(s,T) + \frac{(1-\gamma)(2\gamma+\theta_r^2)}{2\gamma} + \frac{(1-\gamma)\theta_r\sigma_r - b_r\gamma}{\gamma}A_1(s,T) + \frac{\sigma_r^2}{2\gamma}A_1^2(s,T),\\
0 &= A_2'(s,T) + \frac{(1-\gamma)\theta_\lambda^2}{2\gamma} + \frac{(1-\gamma)\theta_\lambda\sigma_\lambda - b_\lambda\gamma}{\gamma}A_2(s,T) + \frac{\sigma_\lambda^2}{2\gamma}A_2(s,T)^2,\\
0 &= A_0'(s,T) + \frac{1-\gamma}{2\gamma}\theta_S^2 + a_r A_1(s,T) + a_\lambda(s)A_2.
\end{aligned}
$$

Under the conditions $\Delta_1 > 0$ and $\Delta_2 > 0$, the solutions $A_0(t,T)$, $A_1(t,T)$ and $A_2(t,T)$ are given in Proposition 2.3.3. The first order condition (2.22) then becomes

$$
\begin{aligned}
\alpha^{Y\star} =& \frac{1}{\gamma}(\Sigma'\Sigma)^{-1}My + \frac{1}{\gamma}\Sigma^{-1}\xi Ay\\
=& \begin{bmatrix} \frac{\sigma_L^\lambda\sigma_S\theta_r\sqrt{r} - \sigma_L^r\sigma_S\theta_\lambda\sqrt{\lambda} - \sigma_L^\lambda\sigma_S^r\theta_S\sqrt{r}}{\sigma_B\sigma_S\sigma_L^\lambda} + \frac{\sigma_r\sqrt{r}}{\sigma_B}A_1 - \frac{\sigma_L^r\sigma_\lambda\sqrt{\lambda}}{\sigma_B\sigma_L^\lambda}A_2 \\ \frac{\theta_\lambda\sqrt{\lambda}}{\sigma_L^\lambda} + \frac{\sigma_\lambda\sqrt{\lambda}}{\sigma_L^\lambda}A_2 \\ \frac{\theta_S}{\sigma_S} \end{bmatrix}\frac{y}{\gamma}.
\end{aligned}
$$

□

Chapter 3

Sharing of longevity basis risk in pension schemes with income-drawdown guarantees

3.1 Introduction

Pension schemes are exposed to a wide range of risks such as investment risk, interest rate risk, inflation risk and longevity risk. Among these, longevity risk is becoming a significant challenge to pension schemes. Longevity risk is the risk that the actual life expectancy may be higher than anticipated. It is a positive trend for society that people's average life expectancy is increasing over the last decades. However, pension schemes are suffering from losses caused by the unexpected increasing benefit outgo accompanied by the longevity trend. Cocco & Gomes (2012) shows that the average life expectancy of 65-year-old US (UK) males increases by 1.2 (1.5) years per decade. As a consequence, a defined benefit scheme (DB scheme) for those populations would have needed 29% more wealth in 2007 than in 1970.

Instead of a classical DB or DC scheme, this chapter considers a pension scheme which provides an income-drawdown option to its members in the decumulation phase.[1] The decumulation phase refers to the period after the retirement of pension scheme members. The drawdown option gives the member the right to withdraw money periodically from the scheme until death time, while keeping the remaining amount in the pension scheme. The management of the scheme starts from the member's retirement time and ends until the the member passes away. Thus, the investment period can last for decades. Due to the long investment horizon, the scheme may face significant interest rate risk and inflation risk.

[1]The pension scheme considered in this chapter is similar to a classical tontine pension plan.

However, the main focus of this chapter is on hedging the longevity risk. We assume the risk-free interest rate to be constant and do not consider inflation risk. In other words, we only consider investment risk and longevity risk. The scheme manager invests in a representative stock to gain investment returns, and he is exposed to investment risk. The length of the decumulation period is increasing due to the member's growing life expectancy. One way to reduce the pension scheme's unexpected loss caused by its exposure to longevity risk is to gain a better understanding of the future mortality evolution. Another way is to seek some financial instruments to transfer the longevity risk to the financial market.

The force of mortality, which represents the instantaneous rate of mortality at a certain time, is useful when studying mortality behavior. The literature on the stochastic force of mortality models is rich. In the papers of, for example, Lee & Carter (1992), A. J. Cairns et al. (2006) and Renshaw & Haberman (2006), the authors modeled the stochastic force of mortality in discrete-time settings. In a continuous-time framework, Luciano & Vigna (2005) modeled the force of mortality by affine processes and calibrated the models to the observed and projected UK mortality tables. They claimed that the affine process with a deterministic part that increases exponentially could describe the force of mortality properly. Affine mortality models are also used in Menoncin (2009), Blackburn & Sherris (2013), Gudkov et al. (2019) and Xu et al. (2020). In this chapter, we apply analog Ornstein-Uhlenbeck (OU) and Cox-Ingersoll-Ross (CIR) processes to model the force of mortality of certain populations.

The market for longevity risk transfers has been developing, and mortality-linked financial instruments have been proposed in the literature. Blake & Burrows (2001) introduced a survivor bond which provides coupon payments based on the number of survivors in a chosen reference population. Under a continuous-time stochastic force of mortality framework, Menoncin (2008) studied the optimal consumption and investment strategy for an individual investor using the longevity bond to hedge the investor's longevity risk. He argued that the longevity bond plays a crucial role in the individual's longevity risk management. Other papers that used the longevity bond include, for instance, De Kort & Vellekoop (2017), Menoncin & Regis (2017) and Shen & Sherris (2018). On the one hand, longevity derivatives may attract capital market investors since they provide diversification benefits. On the other hand, investors may require standardized products such as index-based longevity derivatives. The reason is that longevity risk hedgers are likely to have more information on the mortality experience of the hedging population than capital market investors, which could result in information asymmetry. However, the population basis risk, which refers to the mismatch between the hedging instrument's underlying mortality experience and the hedging population's mortality behavior, arises when an index-based longevity derivative is used for hedging. Besides, the presence of the population basis risk may have a detrimental impact on the hedging effectiveness of the instrument. See Coughlan et al. (2011), A. J. Cairns et al. (2014), De Rosa et al. (2017). To deal with the longevity basis risk, Wong et al. (2014) applied the co-integration technique to study the mean-variance hedging of longevity risk for an insurance company with a longevity bond. They suggested that co-integration is vital in longevity risk management. In this chapter, we introduce a rolling zero-coupon longevity bond to hedge the scheme member's longevity exposure. We first consider the case where the reference population of the longevity bond coincides with the scheme members. Then, we assume that the scheme members are a sub-population of the reference population. In this sub-population case, the force of mortality of the longevity bond reference population correlates imperfectly with the pension members' force of mortality.

Within a continuous-time framework, this chapter aims to determine the optimal investment and benefit withdrawal strategy in a pension scheme that provides an income-drawdown policy in the decumulation phase. After retirement, the member continuously withdraws money from the pension scheme until death. Upon death, a deterministic proportion of the member's pension account balance is given to the manager as compensation. By incorporating a risk-sharing rule, the manager considers both the scheme member's benefit and his own profit. The problem is naturally formulated as a stochastic optimal control problem. In the literature, the stochastic control problems in pension schemes (or problems for individual retirees) are normally based on two broad optimization structures. One type of optimization problem uses Mean-Variance criteria. See Gerrard et al. (2012), He & Liang (2013a), Vigna (2014), He & Liang (2015) and Wong et al. (2017). The other type of optimal control problem maximizes the expected utility. See Gao (2008), Menoncin (2008), Zhang & Ewald (2010), Han & Hung (2012), Z. Liang & Ma (2015) and Shen & Sherris (2018). In this chapter, we suppose that the objective is to maximize both the member's utility from benefit withdrawals and the manager's utility from compensation.

By applying the dynamic programming principle, we first solve the optimization problem under general assumptions. Then, we derive explicit solutions for the single- and sub-population cases. Numerical studies on single-population and sub-population OU models are conducted to investigate the optimal portfolio strategy and optimal benefit withdrawal rate dynamically. Comparative studies are provided to analyze the performance of the longevity risk hedge. We also implement a sensitivity analysis to look into the impact of the market price of risk and the risk-sharing rule parameter. Our results show that with the absence of longevity basis risk, the longevity bond provides an efficient longevity risk hedge when the membership shows a longevity trend. When the membership is a sub-population of the longevity bond reference population, the presence of the longevity basis risk may weaken the longevity bond's hedging performance. However, it still provides a way to hedge the longevity risk and offers a risk premium. We also establish that an equal-risk-sharing rule is beneficial to both the member and the manager in the long run. This chapter contributes to the literature by studying a stochastic optimal control problem for pension schemes in the presence of longevity risk and longevity basis risk. Our optimal control problem also incorporates a risk-sharing rule parameter which defines the agreement on how to share the risk between the members and the manager such that both parties benefit in the long run.

The rest of this chapter is organized as follows. Section 3.2 introduces the mathematical framework of the problem and derives the optimal solution in the general case. Explicit solutions under single-population and sub-population models are given in Section 3.3. In Section 3.4, numerical simulations are carried out to investigate the optimal portfolio strategy and benefit withdrawal. A comparison and sensitivity analysis is conducted to discuss the role of the longevity bond and the impact of model parameters on the optimal solutions. Section 3.5 concludes this chapter.

3.2 Mathematical Framework

There is a rich literature on optimal control problems for pension schemes with deterministic mortality behavior of the populations. However, given the fluctuations in the mortality behavior over time, it is more practical to use stochastic mortality rates. In Section 3.2.1, we model the forces of mortality using continuous-time stochastic processes. Then, in Section 3.2.2, we introduce risky assets that define the financial market, including a mortality-linked security, i.e., a zero-coupon longevity bond, and a stock. Section 3.2.3 studies the individual member's wealth process and describes the optimization problem. The general optimal solution is derived by applying the dynamic programming principle.

3.2.1 The stochastic force of mortality

We consider an infinite time horizon $\mathcal{T} = [0, \infty)$ where time 0 represents the retirement time for all populations. Let $\{W(t) \mid t \in \mathcal{T}\} = \{(W_1(t), ..., W_n(t), W_S(t))' \mid t \in \mathcal{T}\}$ denote an $(n+1)$-dimensional standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$. Here, n denotes the number of populations and $\mathbb{P}$ denotes the physical measure where we observe the longevity behaviors of the populations, and the financial market.

For $i = 1, \dots n$, let $\lambda_i(t)$ denote the force of mortality (also called mortality intensity) of the i-th population at time t. For notational simplicity, we denote by $\{\lambda(t) \mid t \in \mathcal{T}\} = \{(\lambda_1(t), ..., \lambda_n(t))' \mid t \in \mathcal{T}\}$, the vector of forces of mortalities and assume that it evolves as

$$\mathrm{d}\lambda(t) = \mathcal{B}(t, \lambda)\mathrm{d}t + \Sigma(t, \lambda)'\mathrm{d}W(t), \tag{3.1}$$

where

$$\mathcal{B}(t, \lambda) = \begin{bmatrix} \kappa_1(t, \lambda) \\ \vdots \\ \kappa_n(t, \lambda) \end{bmatrix}, \qquad \Sigma(t, \lambda)' = \begin{bmatrix} \sigma_{11}(t, \lambda_1) & \cdots & \sigma_{1n}(t, \lambda_n) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \sigma_{n1}(t, \lambda_1) & \cdots & \sigma_{nn}(t, \lambda_n) & 0 \end{bmatrix}.$$

For any $i, j = 1, \dots, n$, $\kappa_i(t, \lambda)$ and $\sigma_{ij}(t, \lambda_j)$ are assumed to be continuous functions. In Section 3.3, we will specify the affine class models we use for $\lambda(t)$ by defining $\mathcal{B}(t, \lambda)$ and $\Sigma(t, \lambda)'$.

We can use the force of mortality to study the instantaneous survival rate of a population. Denote by $p_i(t)$ the fraction of the ith population that survives from time 0 to t, it measures the cumulative survival rate which coincides with the survival probability for a member of population i. $p_i(t)$ and $\lambda_i(t)$ are related through the following relation:

$$\frac{\mathrm{d}p_i(t)}{p_i(t)} = -\lambda_i(t)\mathrm{d}t, \qquad\qquad p_i(0) = 1.$$

Given alive at time t, the probability for a member of population i to be still alive at time s is given by

$$\mathbb{E}_t\left[\frac{p_i(s)}{p_i(t)}\right] = \mathbb{E}_t\left[e^{-\int_t^s \lambda_i(u)\mathrm{d}u}\right].$$

3.2.2 The financial market

We consider a frictionless financial market consisting of a stock and a *rolling zero-coupon longevity bond*. The money market account is denoted by $R(t)$,

$$\frac{\mathrm{d}R(t)}{R(t)} = r\mathrm{d}t, \qquad\qquad R(0) = 1,$$

where r denotes the constant risk-free interest rate. In this chapter we consider a constant risk-free rate of interest as our focus is to understand the impact of longevity and investment risk in a pension scheme. Our analysis can also be performed in the presence of a stochastic interest rate.

The price of a financial derivative, under a risk-neutral pricing measure $\mathbb{Q}$, is the discounted expected value of its future payoff. We thus introduce an equivalent risk-neutral probability measure $\mathbb{Q}$ by the following Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = Z(T) = \exp\left(-\int_0^T \theta(t,\lambda)'\mathrm{d}W(t) - \frac{1}{2}\int_0^T \|\theta(t,\lambda)\|^2\mathrm{d}t\right),$$

where $\{\theta(t,\lambda) \mid t \in \mathcal{T}\} = \{(\theta_1(t,\lambda_1), \cdots, \theta_n(t,\lambda_n), \theta_S)' \mid t \in \mathcal{T}\}$ is an $\mathbb{R}^n$-valued, $\mathcal{F}$-adapted process such that $Z(t)$ is a martingale and $\mathbb{E}[Z] = 1$. $\theta(t,\lambda)$ is called the vector of market prices of risks and measures the additional amount of investment return when risk increases by one unit. By the Girsanov's theorem, $\{W^{\mathbb{Q}}(t) \mid t \in \mathcal{T}\} = \{(W_1^{\mathbb{Q}}(t), ..., W_n^{\mathbb{Q}}(t), W_S^{\mathbb{Q}}(t))' \mid t \in \mathcal{T}\}$ is an $(n+1)$-dimensional standard Brownian motion under $\mathbb{Q}$ such that

$$W^{\mathbb{Q}}(t) = W(t) + \int_0^t \theta(s,\lambda)\mathrm{d}s. \tag{3.2}$$

The stock price process $\{S(t) \mid t \in \mathcal{T}\}$ is given as

$$\frac{\mathrm{d}S(t)}{S(t)} = (r + \sigma_S\theta_S)\,\mathrm{d}t + \sigma_S \mathrm{d}W_S(t), \qquad S(0) = S_0,$$

where σ_s denotes the constant stock price volatility. The risk premium of the stock is $\theta_S\sigma_S$.

In the literature, several types of mortality-linked securities are proposed to hedge the longevity risk. The values of these securities depend on the mortality index for some given populations: the higher the survival rate, the more valuable the securities. We suppose there is a *zero-coupon longevity bond* traded on the financial market, i.e., a financial security paying, at the expiration date T, a face amount which is equal to the fraction of survivors from time 0 to T within a reference population. There may be multiple longevity bonds based on different reference populations in the market. However, as an illustration of the use of longevity bond, we only consider one longevity bond in this chapter.

Let the ith population be the reference population of the zero-coupon longevity bond which pays $p_i(T)$ at maturity. The arbitrage-free price of the longevity bond at time t is given as

$$L(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{R(T)}{R(t)}p_i(T)\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r(T-t)-\int_0^T \lambda_i(u)\mathrm{d}u}\right],$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ is the conditional expectation operator under the probability measure $\mathbb{Q}$. Using (3.2), the dynamics of $\lambda(t)$ under $\mathbb{Q}$ is given as:

$$\mathrm{d}\lambda(t) = \big(\mathcal{B}(t,\lambda) - \Sigma(t,\lambda)'\theta(t,\lambda)\big)\,\mathrm{d}t + \Sigma(t,\lambda)'\mathrm{d}W^{\mathbb{Q}}(t).$$

Let $h(t,T,\lambda_i) = \mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T \lambda_i(u)\mathrm{d}u}]$ and $D(t) = e^{-\int_0^t \lambda_i(u)\mathrm{d}u}$, we note that $D(t)h(t,s,\lambda_i)$ is a martingale under $\mathbb{Q}$. Applying Itô's formula and setting the $\mathrm{d}t$ term equal to zero, we obtain

$$\begin{aligned}\lambda_i(t)h(t,s,\lambda_i) =& h_t(t,s,\lambda_i) + h_{\lambda_i}(t,s,\lambda_i)\big(\mathcal{B}(t,\lambda) - \Sigma(t,\lambda)'\theta(t,\lambda)\big) \\ &+ \frac{1}{2}tr\left(\Sigma'(t,\lambda)\Sigma(t,\lambda)h_{\lambda_i\lambda_i}(t,s,\lambda_i)\right),\end{aligned} \tag{3.3}$$

where we denote the corresponding partial derivatives of $h(t,s,\lambda_i)$ as $h_t(t,s,\lambda_i)$, $h_{\lambda_i}(t,s,\lambda_i)$ and $h_{\lambda_i\lambda_i}(t,s,\lambda_i)$.

Using (3.2) and (3.3) and applying Itô's formula to $L(t,T) = e^{-r(T-t)}D(t)h(t,T,\lambda_i)$ gives

$$\frac{\mathrm{d}L(t,T)}{L(t,T)} = \Big(r + \sigma_L(t,T)\theta(t,\lambda)\Big)\mathrm{d}t + \sigma_L(t,T)\mathrm{d}W(t),$$

where $\sigma_L(t,T) = \frac{e^{-r(T-t)}D(t)h_{\lambda_i}(t,T,\lambda_i)\Sigma(t,\lambda)'}{L(t,T)}$.

By taking inspiration from the arguments in Boulier et al. (2001) on rolling zero-coupon bonds, we introduce a *rolling zero-coupon longevity bond* $L(t)$ (with a little abuse of notation) with constant time to maturity T. The use of a rolling zero-coupon longevity bond in our set-up simplifies the calculations. The dynamics of $L(t)$ under $\mathbb{P}$ is given as

$$\frac{\mathrm{d}L(t)}{L(t)} = \Big(r + \sigma_L(t,t+T)\theta(t,\lambda)\Big)\mathrm{d}t + \sigma_L(t,t+T)\mathrm{d}W(t).$$

From the above, we can see that $L(t)$ provides a longevity risk premium of $\sigma_L(t,t+T)\theta(t,\lambda)$. Any zero-coupon longevity bond $L(t,T)$ can be replicated by using the rolling longevity bond $L(t)$ and cash. The following equation shows the relationship between $L(t,T)$ and $L(t)$

$$\frac{\mathrm{d}L(t,T)}{L(t,T)} = \left(1 - \frac{\sigma_L(t,T)}{\sigma_L(t,t+T)}\right)\frac{\mathrm{d}R(t)}{R(t)} + \frac{\sigma_L(t,T)}{\sigma_L(t,t+T)}\frac{\mathrm{d}L(t)}{L(t)}.$$

The latter also shows that the introduction of the rolling bond does not change the market. The original zero-coupon longevity bond can be recreated from a self-financing trading strategy that involves the rolling bond and vice versa the rolling bond can be created from a self-financing trading strategy that involves the original zero-coupon longevity bond, which means the market remains arbitrage-free.

3.2.3 The optimization problem

In this chapter, we study the optimal benefit withdrawal rate and investment strategy in the decumulation phase for a pension scheme that provides an *income-drawdown* option. The option allows the scheme member to withdraw money periodically from his pension account until death. While, the balance of the accumulated pension is invested in the financial market to gain investment returns. We suppose that the member does not have a bequest motive. Upon death, a fraction $\pi \in (0,1]$ of the member's pension balance is delivered to the scheme manager as compensation. [2] While the remaining fraction $1-\pi$ of the balance stays in the scheme's fund pool. Suppose the j-th population represents the scheme members, then we use $\lambda_j(t)$ to describe the mortality behavior of a representative scheme member. Let $Y(t)$ denote the member's accumulated pension and $\beta(t)$ denote the amount of the member's benefit withdrawal. In addition, $\alpha_S(t)$, $\alpha_L(t)$ and $\alpha_0(t)$ denote the investments in stock, rolling longevity bond and money market, respectively.

To study the dynamics of the wealth process $Y(t)$, we employ a similar method as used in He & Liang (2013a) and He & Liang (2013b) by first looking at the discrete-time changes in $Y(t)$. For any $t \in \mathcal{T}$ and small positive number Δ, let $\mu(t, t+\Delta)$ denote the rate of investment return. Then, we have

$$\begin{aligned}\mu(t, t+\Delta)Y(t) = &\frac{S(t+\Delta)-S(t)}{S(t)}\alpha_S(t) + \frac{L(t+\Delta)-L(t)}{L(t)}\alpha_L(t) \\ &+ \frac{R(t+\Delta)-R(t)}{R(t)}\alpha_0(t).\end{aligned}$$

Similar to He & Liang (2013a), the change of the individual member's pension level $Y(t)$ in the time interval $(t, t+\Delta)$ is affected by three factors: the investment return, the benefit withdrawal and the survival credit. Hence, we have

$$\begin{aligned}Y(t+\Delta) = &\Big[Y(t) + \mu(t, t+\Delta)Y(t) - \beta(t)\Delta - q(t, t+\Delta)\pi Y(t)\Big] \\ &\times \frac{1}{1-q(t, t+\Delta)},\end{aligned} \tag{3.4}$$

where $q(t, t+\Delta)$ is the probability that the scheme member who is alive at time t will die in the following Δ time period. The last coefficient in (3.4) means that the total fund assets, at time $t+\Delta$, will be equally distributed into each surviving member's pension account. In other words, the surviving member receives a survival credit.

Taylor series approximation gives

$$\begin{aligned}q(t, t+\Delta) &= 1 - \frac{p_j(t+\Delta)}{p_j(t)} = 1 - e^{-\int_t^{t+\Delta}\lambda_j(u)\mathrm{d}u} = \lambda_j(t)\Delta + o(\Delta), \\ \frac{1}{1-q(t, t+\Delta)} &= e^{\int_t^{t+\Delta}\lambda_j(u)\mathrm{d}u} = 1 + \lambda_j(t)\Delta + o(\Delta), \qquad \text{as} \quad \Delta \to 0.\end{aligned}$$

Moreover, it is easy to see that

$$q(t, t+\Delta)\Delta = o(\Delta), \qquad \mu(t, t+\Delta)\Delta = o(\Delta), \qquad \Delta^2 = o(\Delta),$$

[2] This is similar to Yaari (1964) and Yaari (1965) and not uncommon today, e.g. reverse mortgages.

as $\Delta \to 0$. Thereafter, we have

$$Y(t+\Delta) = Y(t) + \mu(t, t+\Delta)Y(t) - \beta(t)\Delta + (1-\pi)\lambda_j(t)Y(t)\Delta + o(\Delta) \quad (3.5)$$

Next, we deduce the continuous version of the dynamics of $Y(t)$. Taking $\Delta \to 0$, we have

$$\mu(t, t+\Delta)Y(t) \to \frac{\mathrm{d}S(t)}{S(t)}\alpha_S(t) + \frac{\mathrm{d}L(t)}{L(t)}\alpha_L(t) + \frac{\mathrm{d}R(t)}{R(t)}\alpha_0(t).$$

It then follows from (3.5) that

$$\begin{aligned}\mathrm{d}Y(t) =& \Big[rY(t) + \alpha_S(t)\sigma_S\theta_S + \alpha_L(t)\sigma_L(t, t+T)\theta(t, \lambda) + (1-\pi)\lambda_j(t)Y(t) - \beta(t)\Big]\mathrm{d}t \\ &+ \alpha_S(t)\sigma_S\mathrm{d}W_S(t) + \alpha_L(t)\sigma_L(t, t+T)\mathrm{d}W(t),\end{aligned}$$

where we use the fact that $\alpha_0(t) = Y(t) - \alpha_S(t) - \alpha_L(t)$.

At any time, the pension scheme manager decides the benefits withdrawal rate $\beta(t)$ and the investment strategy $(\alpha_S(t), \alpha_L(t))$. A fraction of the member's pension balance is paid to the manager as compensation, at the stochastic time of death. We suppose that the member and the manager decide to share the investment risk and longevity risk based on a *risk-sharing rule*. Moreover, the manager works not only for his own benefit but also for the benefit of the scheme member. This is also known as *first-best principal-agent problem*. More specifically, we consider an optimization problem which combines the manager's and the member's utilities. Denote by τ the member's stochastic time of death, the objective function is given as

$$\begin{aligned}J(t, Y, \lambda; \alpha_S, \alpha_L, \beta) &= \mathbb{E}_t\Big[\int_t^\tau e^{-r(s-t)}U_P(\beta(s))\mathrm{d}s + \phi e^{-r(\tau-t)}U_A(\pi Y(\tau))\Big] \\ &= \mathbb{E}_t\Big[\int_t^\infty e^{-\int_t^s(r+\lambda_j(u))\mathrm{d}u}\Big(U_P(\beta(s)) + \phi\lambda_j(s)U_A(\pi Y(s))\Big)\mathrm{d}s\Big],\end{aligned}$$

where $U_P(\cdot)$ and $U_A(\cdot)$ denote the utility functions of the principal (member) and the agent (manager). The non-negative constant ϕ can be viewed as a parameter that determines the risk-sharing rule between the principal and the agent. The case $\phi = 0$ corresponds to the situation when the manager works only for the sake of the member. In this case, the objective is to maximize the member's running utility from benefit withdrawals while the manager pays no attention to his own utility. The case $0 < \phi < 1$ prioritizes the member's utility. When $\phi = 1$, the objective function puts equal importance on member's and manager's utility.

To specify the optimization problem and to facilitate analytic solutions, we set $U_P(\cdot)$ and $U_A(\cdot)$ as log utility functions:

$$U_A(x) = \ln x, \qquad U_P(x) = \ln x, \qquad \forall x > 0.$$

Remark 3.2.1. The log utility function belongs to the Constant relative risk aversion (CRRA) utility functions. Pension schemes are, in general, large companies that determine investment strategies more or less in a scaling way. The feature of CRRA function makes the investment strategies unaffected by scale. Moreover, past papers show that a log utility function outperforms other utility functions in the long run. Since the management period of pension schemes is long, we consider log utility functions. Also, log utility function has the advantage of leading to closed-form solutions.

Besides, upon death, we assume that the total amount of the member's remaining pension is paid to the manager, i.e., $\pi = 1$. The wealth process is now

$$\begin{aligned} \mathrm{d}Y(t) =& \Big[rY(t) + \alpha_S(t)\sigma_S\theta_S + \alpha_L(t)\sigma_L(t,t+T)\theta(t,\lambda) - \beta(t)\Big]\mathrm{d}t \\ &+ \alpha_S(t)\sigma_S \mathrm{d}W_S(t) + \alpha_L(t)\sigma_L(t,t+T)\mathrm{d}W(t). \end{aligned} \tag{3.6}$$

Then, the optimization problem is defined as

$$\begin{cases} \sup\limits_{\alpha_S,\alpha_L,\beta} \mathbb{E}_t\Big[\int_t^\infty e^{-\int_t^s (r+\lambda_j(u))\mathrm{d}u}\Big(\ln(\beta(s)) + \phi\lambda_j(s)\ln(Y(s))\Big)\mathrm{d}s\Big] \\ \qquad\qquad \text{s.t.} \quad (3.1) \text{ and } (3.6) \text{ hold.} \end{cases} \tag{3.7}$$

This optimal control problem can be solved by applying the dynamic programming principle, and the general solution is provided by the following proposition.

Proposition 3.2.1. *The solution to the optimization problem* (3.7) *is*

$$\begin{aligned} \frac{\beta^*(t)}{Y(t)} &= \frac{1}{G(t,\lambda)}, \qquad \frac{\alpha_S^*(t)}{Y(t)} = \frac{\theta_S}{\sigma_S}, \\ \frac{\alpha_L^*(t)}{Y(t)} &= \frac{\sigma_L(t,t+T)\theta(t,\lambda)}{\sigma_L(t,t+T)\sigma_L(t,t+T)'} + \frac{G_\lambda(t,\lambda)\Sigma(t,\lambda)'\sigma_L(t,t+T)'}{\sigma_L(t,t+T)\sigma_L(t,t+T)'}\frac{1}{G(t,\lambda)}, \end{aligned}$$

where

$$G(t,\lambda) = \int_t^\infty \left(\phi\mathbb{E}_t\left[\lambda_j(s)e^{-\int_t^s \lambda_j(u)\mathrm{d}u}\right] + \mathbb{E}_t\left[e^{-\int_t^s \lambda_j(u)\mathrm{d}u}\right]\right)e^{r(s-t)}\mathrm{d}s.$$

Proof. The proof is given in Appendix 3.A. □

We learn from the optimal solution that the optimal portfolio weight in the stock equals $\frac{\theta_s}{\sigma_s}$ and keeps constant over time. This is similar to the classical Merton portfolio problem where the optimal weight on the risky asset is constant. The intuition behind this is that the constant value of the market price of stock risk causes no change in the investment strategy. However, it is not clear from the solution how the longevity bond investment and benefit withdrawal changes over time.

3.3 Explicit solutions

In the literature, for example Luciano & Vigna (2005) and Wong et al. (2014), several continuous-time stochastic models for force of mortality have been studied, including OU processes, CIR processes and Feller processes. In this section, we follow Menoncin (2009) and use affine models which are analogs to OU and CIR processes to model the stochastic force of mortality. In Section 3.2, population i refers to the individuals constituting the reference population of the longevity bond's survival index. Population j refers to the scheme members. In this section, we provide explicit solutions to the optimal control problem proposed in Section 3.2 for the single-population and sub-population cases. For the single-population case, we assume that the reference population and the members' population are the same. That is, we set $n = 1$ and $\lambda_i(t) = \lambda_j(t) = \lambda_1(t)$. For the sub-population case, we suppose the two populations are different and set $n = 2$. We assume that the longevity bond's underlying index is a large public survival index and the scheme members are a proper sub-population of the longevity bond reference population with different force of mortality. The latter case is very common. For example, we could regard the total public workforce in a country as a reference population, and teachers at public schools as a sub-population of the reference population who have access to membership in a dedicated pension fund for public school teachers. A report by the Society of Actuaries (SOA) reveals that teachers show a significantly different longevity pattern when compared with public sector employees who have other types of jobs. See SOA (2019).

3.3.1 Single-population model

Suppose that the reference population of the longevity bond happens to be the scheme members' population. We expect that the investment in the longevity bond can hedge the scheme's longevity risk effectively, since the uncertainty in the longevity bond value correlates with the member's longevity risk perfectly. Now, equation (3.1) becomes

$$\mathrm{d}\lambda_1(t) = \kappa(t,\lambda_1)\mathrm{d}t + \sigma_1(t,\lambda_1)\mathrm{d}W_1(t). \tag{3.8}$$

Following Menoncin (2009), we require that the expected value of $\lambda_1(t)$ equals to the Gompertz-Makeham force of mortality:

$$\mathbb{E}[\lambda_1(t)] = \nu_1 + \frac{1}{\Delta_1} e^{\frac{t-m_1}{\Delta_1}}$$

where ν_1, Δ_1 and m_1 are constants. Specifically, m_1 is the modal value of remaining life span after retirement. Suppose the retirement age is t_0, then $t_0 + m_1$ is the mode of life expectancy. The condition above is met if

$$\kappa_1(t,\lambda_1) = a_1(t) - b_1\lambda_1(t),$$

where

$$a_1(t) = b_1\nu_1 + \frac{1 + b_1\Delta_1}{\Delta_1^2} e^{\frac{t-m_1}{\Delta_1}}.$$

To construct analog OU and CIR processes for $\lambda_1(t)$, we make some assumptions on the diffusion term $\sigma_1(t,\lambda_1)$ and market price of longevity risk $\theta_1(t,\lambda_1)$.

- **OU setting:** We assume that $\sigma_1(t,\lambda_1) := \sigma_1 > 0$ and $\theta_1(t,\lambda_1) := \theta_1 \leq 0$ are constants. Then, the dynamics of $L(t)$ is given as

  $$\frac{\mathrm{d}L(t)}{L(t)} = \Big(r + \sigma_L(t,t+T)\theta_1\Big)\mathrm{d}t + \sigma_L(t,t+T)\mathrm{d}W_1(t), \tag{3.9}$$

  where

  $$\sigma_L(t,s) = -A_1(t,s)\sigma_1, \qquad A_1(t,s) = \frac{1 - e^{-b_1(s-t)}}{b_1}.$$

- **CIR setting:** We assume that $\sigma_1(t,\lambda_1) := \sigma_1\sqrt{\lambda_1(t)}$ and $\theta_1(t,\lambda_1) := \theta_1\sqrt{\lambda_1(t)}$. Suppose that $\sigma_1 > 0$ and $\theta_1 \leq 0$ are constants and $\sigma_1^2 < 2a_1(t)$ s.t. the Feller condition is satisfied. Then, the dynamics of $L(t)$ is given as

$$\frac{\mathrm{d}L(t)}{L(t)} = \Big(r + \sigma_L(t,t+T)\theta_1\sqrt{\lambda_1(t)}\Big)\mathrm{d}t + \sigma_L(t,t+T)\mathrm{d}W_1(t), \tag{3.10}$$

where

$$\begin{aligned}
\sigma_L(t,s) &= -\hat{A}_1(t,s)\sigma_1\sqrt{\lambda_1(t)},\\
\hat{A}_1(t,s) &= \frac{2(e^{\eta(s-t)}-1)}{(b_1-\sigma_1\theta_1+\eta)(e^{\eta(s-t)}-1)+2\eta},\\
\eta &= \sqrt{(b_1-\sigma_1\theta_1)^2+2\sigma_1^2}.
\end{aligned}$$

Once the process $\lambda_1(t)$ is specified, we give the wealth process as

$$\begin{aligned}
\mathrm{d}Y(t) =& \Big[rY(t) + \alpha_S(t)\sigma_S\theta_S + \alpha_L(t)\sigma_L(t,t+T)\theta_1(t,\lambda_1) - \beta(t)\Big]\mathrm{d}t \\
&+ \alpha_S(t)\sigma_S\mathrm{d}W_S(t) + \alpha_L(t)\sigma_L(t,t+T)\mathrm{d}W_1(t).
\end{aligned} \tag{3.11}$$

The optimization problem is now

$$\begin{cases}
\sup\limits_{\alpha_S,\alpha_L,\beta} \mathbb{E}_t\Big[\int_t^\infty e^{-\int_t^s(r+\lambda_1(u))\mathrm{d}u}\Big(\ln(\beta(s)) + \phi\lambda_1(s)\ln(Y(s))\Big)\mathrm{d}s\Big] \\
\qquad\qquad \text{s.t.} \quad (3.8) \text{ and } (3.11) \text{ hold.}
\end{cases} \tag{3.12}$$

Proposition 3.3.1. *Within the OU setting, the solution to the single-population optimization problem* (3.12) *is given as*

$$\frac{\beta^*(t)}{Y(t)} = \frac{1}{G(t,\lambda_1)}, \qquad \frac{\alpha_S^*(t)}{Y(t)} = \frac{\theta_S}{\sigma_S}, \qquad \frac{\alpha_L^*(t)}{Y(t)} = \frac{1}{\sigma_L(t,t+T)}\left(\theta_1 + \frac{G_{\lambda_1}(t,\lambda_1)}{G(t,\lambda_1)}\sigma_1\right),$$

where

$$\begin{aligned}
G(t,\lambda_1) &= \int_t^\infty e^{-r(s-t)}h^{\mathbb{P}}(t,s,\lambda_1)\Big(\phi M(t,s,\lambda_1)+1\Big)\mathrm{d}s,\\
h^{\mathbb{P}}(t,s,\lambda_1) &= e^{A_0^{\mathbb{P}}(t,s)-A_1(t,s)\lambda_1(t)},\\
M(t,s,\lambda_1) &= \lambda_1(t)e^{-b_1(s-t)} + \Big(b_1\nu_1 - \frac{\sigma_1^2}{b_1}\Big)(s-t) + \frac{1+b_1\Delta_1}{\Delta_1}\Big(e^{\frac{s-m_1}{\Delta_1}} - e^{\frac{t-m_1}{\Delta_1}}\Big)\\
&\quad + \frac{\sigma_1^2}{b_1}A_1(t,s),\\
A_0^{\mathbb{P}}(t,s) &= \left(\frac{1}{2}\frac{\sigma_1^2}{b_1^2} - \nu_1\right)(s-t) - \Big(e^{\frac{s-m_1}{\Delta_1}} - e^{\frac{t-m_1}{\Delta_1}}\Big) - \frac{1}{4}\frac{\sigma_1^2}{b_1}{A_1}^2(t,s)\\
&\quad + \left(\nu_1 - \frac{1}{2}\frac{\sigma_1^2}{b_1^2} + \frac{1}{\Delta_1}e^{\frac{t-m_1}{\Delta_1}}\right)A_1(t,s).
\end{aligned}$$

Proof. The proof is provided in Appendix 3.B. □

Proposition 3.3.2. *Within the CIR setting, the solution to the single-population optimization problem* (3.12) *is given as*

$$\frac{\beta^*(t)}{Y(t)} = \frac{1}{G(t,\lambda_1)}, \quad \frac{\alpha_S^*(t)}{Y(t)} = \frac{\theta_S}{\sigma_S}, \quad \frac{\alpha_L^*(t)}{Y(t)} = \frac{\sqrt{\lambda_1(t)}}{\sigma_L(t,t+T)}\left(\theta_1 + \frac{G_{\lambda_1}(t,\lambda_1)}{G(t,\lambda_1)}\sigma_1\right),$$

where

$$\begin{aligned}
G(t,\lambda_1) &= \int_t^\infty e^{-r(s-t)}\hat{h}^{\mathbb{P}}(t,s,\lambda_1)\Big(\phi\hat{M}(t,s,\lambda_1)+1\Big)\mathrm{d}s,\\
\hat{h}^{\mathbb{P}}(t,s,\lambda_1) &= e^{\hat{A}_0^{\mathbb{P}}(t,s)-\hat{A}_1^{\mathbb{P}}(t,s)\lambda_1(t)}, \qquad \eta^{\mathbb{P}} = \sqrt{b_1^2+2\sigma_1^2},\\
\hat{M}(t,s,\lambda_1) &= \lambda_1(t)m(t,s)e^{-b_1(s-t)} + \int_t^s a_1(u)m(u,s)e^{-b_1(s-u)}\mathrm{d}u,\\
m(t,s) &= (2\eta^{\mathbb{P}})^{-\frac{4b_1\sigma_1^2}{\eta^{\mathbb{P}}(\eta^{\mathbb{P}2}-b_1^2)}}\Big((\eta^{\mathbb{P}}+b_1)(e^{\eta^{\mathbb{P}}(s-t)}-1)+2\eta^{\mathbb{P}}\Big)^{-\frac{2\sigma_1^2}{\eta^{\mathbb{P}}(\eta^{\mathbb{P}}+b_1)}}\times\\
&\qquad\Big((\eta^{\mathbb{P}}-b_1)(e^{\eta^{\mathbb{P}}(s-t)}-1)+2\eta^{\mathbb{P}}\Big)^{\frac{2\sigma_1^2}{\eta^{\mathbb{P}}(\eta^{\mathbb{P}}-b_1)}},\\
\hat{A}_1^{\mathbb{P}}(t,s) &= \frac{2(e^{\eta^{\mathbb{P}}(s-t)}-1)}{(b_1+\eta^{\mathbb{P}})(e^{\eta^{\mathbb{P}}(s-t)}-1)+2\eta^{\mathbb{P}}}, \quad \hat{A}_0^{\mathbb{P}}(t,s) = -\int_t^s a_1(u)\hat{A}_1^{\mathbb{P}}(u,s)\mathrm{d}u.
\end{aligned}$$

Proof. The proof of the above result is analog to the proof of Proposition 3.3.1 and we omit it here. □

As observed in Proposition 3.2.1, the optimal portfolio weight in the stock within the OU and CIR settings are the same and constant over time, while the dynamic behavior of the longevity bond investment and benefit withdrawal are more complex. Later in Section 3.4.1, we perform numerical simulations to investigate the optimal investment strategy and benefit withdrawal rate dynamically, under the single-population OU setting. We see that the higher the market price of stock risk, the higher the investment proportion in the stock. It is easy to see that $\sigma_L(t,t+T)$ is negative, thus the lower the θ_1 (the higher the value of the market price of longevity risk) the higher the portfolio weight in the longevity bond. We learn from Proposition 3.3.1 and 3.3.2 that the functions $h(t,s,\lambda_1)$ and $M(t,s,\lambda_1)$ are always positive. Therefore, we deduce that the higher the risk-sharing rule parameter ϕ the lower the proportion of the wealth withdrawn (i.e., $\beta(t)/Y(t)$). However, it is not clear how the benefit withdrawal rate $\beta(t)$ reacts when ϕ changes. In Section 3.4.1, we provide a sensitivity analysis to study the impact of ϕ on the benefit withdrawal rate.

3.3.2 Sub-population model

Based on different mortality indices and maturity times, there may be different longevity bonds in the market. It may be interesting to study our problem in a market setting with multiple longevity bonds. However, we only consider one longevity bond in our framework as our main focus is on longevity risk hedging. In this section, we study the case where the longevity bond's reference population is different from the pension members' population. More specifically, we set $n = 2$ and denote the index population of the longevity bond by Population 1, and use Population 2 to denote the scheme members' population. In practice, the reference population of the longevity bond tends to be large and could be much larger than the pension schemes' client pools. Therefore, we assume that the pension members are a sub-population of the index population. The sub-population assumption is also used in Coughlan et al. (2011) and A. J. Cairns et al. (2014).

Under this sub-population assumption (see, Wong et al. (2014)), we suppose that (3.1) takes the following form:

$$\mathrm{d}\lambda(t) = \mathcal{B}(t,\lambda)\mathrm{d}t + \Sigma(t,\lambda)'\mathrm{d}W(t), \tag{3.13}$$

where

$$\mathcal{B}(t,\lambda) = \begin{bmatrix} a_1(t) - b_1\lambda_1(t) \\ a_2(t) - b_{21}\lambda_1(t) - b_{22}\lambda_2(t) \end{bmatrix}, \quad \Sigma(t,\lambda) = \begin{bmatrix} \sigma_1(t,\lambda_1) & \sigma_{21}(t,\lambda_1) \\ 0 & \sigma_{22}(t,\lambda_2) \\ 0 & 0 \end{bmatrix}.$$

In the above, b_1, b_{21} and b_{22} are constant numbers and

$$a_2(t) = b_{21}\nu_1 + b_{22}\nu_2 + \frac{b_{21}}{\Delta_1} e^{\frac{t-m_1}{\Delta_1}} + \frac{b_{22}}{\Delta_2}\left(1 + \frac{1}{b_{22}\Delta_2}\right) e^{\frac{t-m_2}{\Delta_2}},$$

where ν_2 are Δ_2 constants. The terms $t_0 + m_2$ and $t_0 + m_2$ are modal values of life expectancy of Population 1 and Population 2, respectively. In the following, we specify the volatility vector $\Sigma(t,\lambda)'$.

- **OU setting:** Suppose that

$$\Sigma(t,\lambda)' := \begin{bmatrix} \sigma_1 & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \end{bmatrix}, \qquad \theta(t,\lambda)' := \begin{bmatrix} \theta_1 & \theta_2 & \theta_S \end{bmatrix},$$

  where σ_1, σ_{21} and σ_{22} are positive constants, while θ_1, θ_2 and θ_S are non-positive constants. Then, the dynamics of $L(t)$ is given in (3.9).

- **CIR setting:** Suppose that

$$\Sigma(t,\lambda)' := \begin{bmatrix} \sigma_1\sqrt{\lambda_1(t)} & 0 & 0 \\ \sigma_{21}\sqrt{\lambda_1(t)} & \sigma_{22}\sqrt{\lambda_2(t)} & 0 \end{bmatrix},$$
$$\theta(t,\lambda)' := \begin{bmatrix} \theta_1\sqrt{\lambda_1(t)} & \theta_2\sqrt{\lambda_2(t)} & \theta_S \end{bmatrix},$$

  where σ_1, σ_{21} and σ_{22} are positive constants, while θ_1, θ_2 and θ_S are non-positive constants. Then, the dynamics of $L(t)$ is given in (3.10).

Once the processes $\lambda_1(t)$ and $\lambda_2(t)$ are specified, we obtain the wealth process as

$$\begin{aligned} \mathrm{d}Y(t) =& \Big[rY(t) + \alpha_S(t)\sigma_S\theta_S + \alpha_L(t)\sigma_L(t,t+T)\theta_1(t,\lambda_1) - \beta(t)\Big]\mathrm{d}t \\ &+ \alpha_S(t)\sigma_S\mathrm{d}W_S(t) + \alpha_L(t)\sigma_L(t,t+T)\mathrm{d}W(t). \end{aligned} \tag{3.14}$$

The optimization problem is now

$$\begin{cases} \sup_{\alpha_S,\alpha_L,\beta} \mathbb{E}_t\Big[\int_t^\infty e^{-\int_t^s (r+\lambda_2(u))\mathrm{d}u}\Big(\ln(\beta(s)) + \phi\lambda_2(s)\ln(Y(s))\Big)\mathrm{d}s\Big] \\ \qquad\qquad \text{s.t.} \quad \text{(3.13) and (3.14) hold.} \end{cases} \tag{3.15}$$

In this sub-population model, there are two state variables. Moreover, the state variable $\lambda_2(t)$ correlates with the state variable $\lambda_1(t)$. This increases the difficulty of solving the optimization problem. In this case, an analytical solution may not always be available. In the following, we provide Proposition 3.3.3 and 3.3.4 for the solutions to the sub-population optimization problem within the OU and CIR settings, respectively.

Proposition 3.3.3. *Within the OU setting, the solution to the sub-population optimization problem* (3.15) *is given as*

$$\frac{\beta^*(t)}{Y(t)} = \frac{1}{G(t,\lambda_1,\lambda_2)}, \qquad \frac{\alpha_S^*(t)}{Y(t)} = \frac{\theta_S}{\sigma_S},$$
$$\frac{\alpha_L^*(t)}{Y(t)} = \frac{1}{\sigma_L(t,t+T)}\left(\theta_1 + \frac{G_{\lambda_1}(t,\lambda_1,\lambda_2)}{G(t,\lambda_1,\lambda_2)}\sigma_1 + \frac{G_{\lambda_2}(t,\lambda_1,\lambda_2)}{G(t,\lambda_1,\lambda_2)}\sigma_{21}\right),$$

where

$$G(t,\lambda_1,\lambda_2) = \int_t^\infty e^{-r(s-t)} f(t,s,\lambda_1,\lambda_2)\Big(\phi M(t,s,\lambda_1,\lambda_2)+1\Big)\mathrm{d}s.$$

The functions $f(t,s,\lambda_1,\lambda_2)$ *and* $M(t,s,\lambda_1,\lambda_2)$ *are given by* (3.22) *and* (3.23)*.*

Proof. The proof is given in Appendix 3.C. □

Proposition 3.3.4. *Within the CIR setting, the solution to the sub-population optimization problem* (3.15) *is given as*

$$\frac{\beta^*(t)}{Y(t)} = \frac{1}{G(t,\lambda_1,\lambda_2)}, \qquad \frac{\alpha_S^*(t)}{Y(t)} = \frac{\theta_S}{\sigma_S},$$
$$\frac{\alpha_L^*(t)}{Y(t)} = \frac{\sqrt{\lambda_1(t)}}{\sigma_L(t,t+T)}\left(\theta_1 + \frac{G_{\lambda_1}(t,\lambda_1,\lambda_2)}{G(t,\lambda_1,\lambda_2)}\sigma_1 + \frac{G_{\lambda_1}(t,\lambda_1,\lambda_2)}{G(t,\lambda_1,\lambda_2)}\sigma_{21}\right),$$

where

$$G(t,\lambda_1,\lambda_2) = \int_t^\infty e^{-r(s-t)} \hat{f}(t,s,\lambda_1,\lambda_2)\Big(\phi \hat{M}(t,s,\lambda_1,\lambda_2)+1\Big)\mathrm{d}s,$$
$$\hat{f}(t,s,\lambda_1,\lambda_2) = e^{\hat{C}_0(t,s)-\hat{C}_1(t,s)\lambda_1(t)-\hat{C}_2(t,s)\lambda_2(t)},$$
$$\hat{C}_2(t,s) = \frac{2(e^{\xi(s-t)}-1)}{(b_1+\xi)(e^{\xi(s-t)}-1)+2\xi}, \qquad \xi = \sqrt{b_{22}^2+2\sigma_{22}^2}.$$

The functions $\hat{C}_0(t,s)$*,* $\hat{C}_1(t,s)$*,* $\hat{M}(t,s,\lambda_1,\lambda_2)$ *and* $\hat{N}(t,s,\lambda_1)$ *satisfy*

$$\begin{aligned}
0 &= -\frac{\partial \hat{C}_0}{\partial t} + a_1\hat{C}_1 + a_2\hat{C}_2, \qquad (3.16)\\
0 &= -\frac{\partial \hat{C}_1}{\partial t} + b_1\hat{C}_1 + b_{21}\hat{C}_2 + \frac{1}{2}\sigma_1^2\hat{C}_1^2 + \frac{1}{2}\sigma_{21}^2\hat{C}_2^2 + \sigma_1\sigma_{21}\hat{C}_1\hat{C}_2,\\
0 &= -\frac{\partial \hat{N}}{\partial t} + a_1 - \Big(b_1 + \sigma_1^2\hat{C}_1 + \sigma_1\sigma_{21}\hat{C}_2\Big)\hat{N},\\
0 &= -\frac{\partial \hat{M}}{\partial t} + a_2 - \Big(b_{21} + \sigma_1\sigma_{21}\hat{C}_1 + \sigma_{21}^2\hat{C}_2\Big)\hat{N} - \Big(b_{22} + \sigma_{22}^2\hat{C}_2\Big)\hat{M},
\end{aligned}$$

with boundary conditions $\hat{C}_0(s,s)=0$*,* $\hat{C}_1(s,s)=0$*,* $\hat{M}(t,t,\lambda_1,\lambda_2)=0$ *and* $\hat{N}(t,t,\lambda_1)=0$*.*

Proof. The Picard-Lindelöf theorem ensures the existence of unique solutions to the ODEs in (3.16). The rest of the proof is similar to the proof of Proposition 3.3.3 and is omitted here, except that some of the ODEs do not have explicit solutions. □

From Proposition 3.3.3 and 3.3.4, we see that the optimal portfolio weight attached to the stock remains the same as in the single-population case. The OU case has an analytical solution; however, the CIR case does not. In Section 3.4, we conduct a numerical study to assess the hedging effect of the longevity bond within the sub-population OU setting.

3.4 Numerical applications

This section provides numerical simulations for the single- and sub-population cases using the results from Proposition 3.3.1 and 3.3.3. A numerical study involving CIR models is not presented here as the results here are qualitatively not different to the OU case. We observe the dynamics of the survival probability and look into the impact of the mortality behavior on the optimal strategy. We investigate and assess the hedging performance of the longevity bond in the pension scheme's risk management and provide a sensitivity analysis on the market price of longevity risk. We also look into the effect of the risk-sharing rule between the member and the manager.

Table 3.4.1: Values of parameters for optimization problem.

Population 1	Population 2	Market	Others
$\nu_1 = 0.0009944$	$\nu_2 = 0.0009944$	$r = 0.04$	$T = 35$
$\Delta_1 = 11.4000$	$\Delta_2 = 12.9374$	$\theta_1 = -0.0005$	$Y_0 = 100$
$m_1 = 21.4515$	$m_2 = 24.18$	$\theta_S = 0.05$	$\Delta = 1/10$
$b_1 = 0.5610$	$b_{21} = 0.0028$	$\sigma_S = 0.15$	$\phi = 0.8$
$\sigma_1 = 0.0035$	$b_{22} = 0.6500$	$T_L = 20$	$t_0 = 65$
	$\sigma_{21} = 0.0040$		
	$\sigma_{22} = 0.0050$		

Table 3.4.1 shows the values of parameters in our numerical examples. We choose $t_0 = 65$ as the retirement age. The time interval $\Delta = 1/10$ means that we observe the mortality rates 10 times a year. Most of the values of the mortality model parameters are as considered in other works (e.g. Menoncin & Regis (2017) and Milevsky (2001)). The maturity of the rolling longevity bond is set as $T_L = 20$. The values of other financial market parameters are meant to be representative.

3.4.1 Single-population case

In the single-population case, we assume that the scheme members happen to be the reference population of the longevity bond. The manager invests into the longevity bond to hedge the scheme's longevity exposure. The pension scheme's management starts from the retirement time and ends until members pass away. According to the Gompertz-Makeham law of mortality, less than 5% of the population are likely to survive until 100 years old given that they are alive at 65 years old. Thus, we conduct numerical simulations with a 35-year time horizon (i.e., $T = 35$).

Figure 3.4.1 shows three simulated paths/scenarios to illustrate the dynamic mortality behavior of the member. We see that the survival probability $p_1(t)$ decreases with time. In the bottom-left plot, we observe that for all three paths, the member's survival probability is less than 5% at 100 years old. This indicates that it is reasonable to set $T = 35$. Suppose the Gompertz-Makeham mortality law describes the average trend of the member's mortality. We learn from the plots in Figure 3.4.1 that the survival probability of the simulated path 1 is always higher than average. On the contrary, the survival probability of path 2 is lower than expected. Path 3 does not show any particular trend. The bottom-right plot in Figure 3.4.1 draws the probability density function $f_\tau(t)$ of the stochastic time of death τ. We see from the plot that $f_\tau(t)$ peaks at $t = 21.5$ approximately. That is, the age of 86.5. This is consistent with our model settings and choice of parameter values: $t_0 + m_1 = 86.4515$ in our model is the modal value of life span of the member. The probability density function $f_\tau(t)$ of the stochastic time of death τ conditional on the paths of $\lambda_1(u)$ is given by

$$f_\tau(t) = \lambda_1(t) e^{-\int_0^t \lambda_1(u)\mathrm{d}u}.$$

Taking the differential we obtain

$$\mathrm{d}f_\tau(t) = e^{-\int_0^t \lambda_1(u)\mathrm{d}u}\Big(a_1(u) - b_1\lambda_1(t) - \lambda_1(t)^2\Big)\mathrm{d}t + e^{-\int_0^t \lambda_1(u)\mathrm{d}u}\sigma_1 \mathrm{d}W_1(t).$$

Since $e^{-\int_0^t \lambda_1(u)\mathrm{d}u}$ decreases with time, the diffusion term decreases with time as well. Thus, the simulated paths are more volatile at the beginning while becoming smoother towards the end.

Generally, our observations imply that:

- path 1 shows a prominent longevity trend;
- path 2 on average has a lower survival rate;
- path 3 does not show any particular trend;
- For all three paths, the mode of life span is around 86 years old.

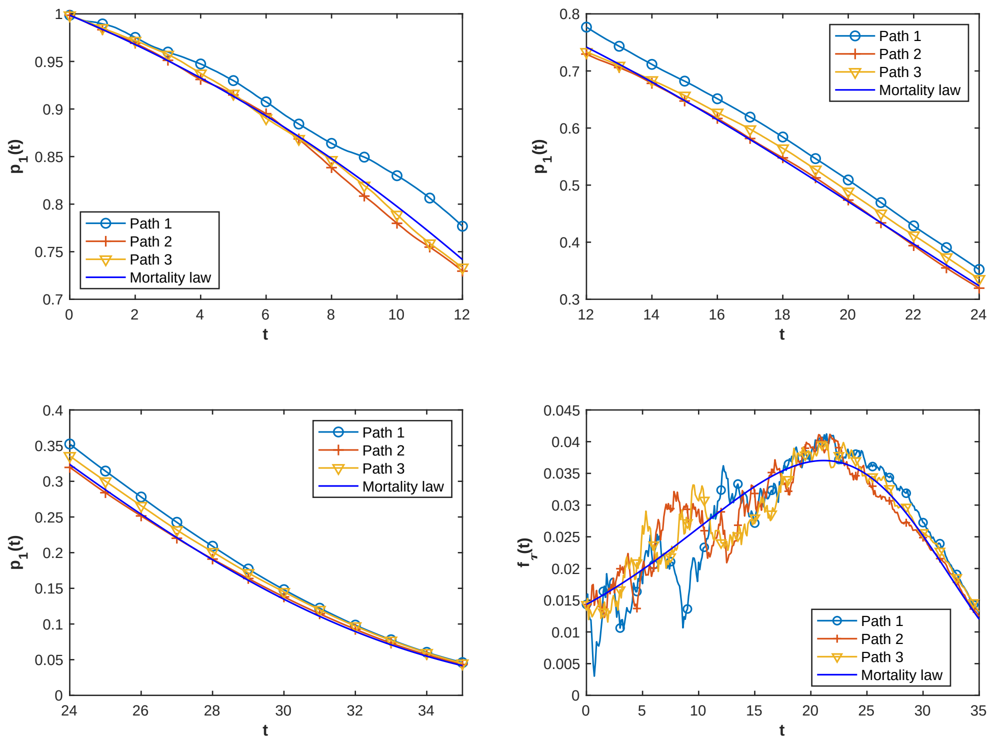

Figure 3.4.1: 3 simulation paths for the survival probability and cumulative distribution function of τ

The base scenario

In this section, we give a base scenario and investigate the optimal benefit withdrawal rate and investment strategy. Besides this, we observe the member's pension level and manager's compensation dynamically. We suppose that the manager prioritizes the member's utility and set the risk-sharing rule parameter as $\phi = 0.8$. Figure 3.4.2 shows the average investment strategy over 100 simulation paths. We observe that the portfolio weight in the longevity bond drops over time. As the member gets older, the exposure to the longevity risk reduces, and the need for longevity protection decreases. Accordingly, the manager reduces the portfolio weight in the longevity bond. The flat line which shows the investment proportion in the stock is a direct result from the optimal solutions in Proposition 3.3.1: the portfolio weight in the stock keeps constant and equals $\frac{\theta_S}{\sigma_S}$. This constant investment strategy coincides with the result in the classical Merton portfolio problem. The interpretation is that the constant value of market price of risk causes no change in the manager's investment behavior. The proportion of the portfolio in the money market is $\frac{\alpha_0(t)}{Y(t)} = 1 - \frac{\alpha_L(t)}{Y(t)} - \frac{\alpha_S(t)}{Y(t)}$. We see that the portfolio weight in the money market is negative at first and then increases over time. The negative position in the initial years indicates that the manager borrows money from the money market to invest into the risky assets to gain risk premiums and increase the wealth level. As $\frac{\alpha_S(t)}{Y(t)}$ keeps constant, and $\frac{\alpha_L(t)}{Y(t)}$ decreases over time, the manager puts more weight into the money market. With the passage of time, the manager becomes more conservative to avoid unexpected losses. Overall, the longevity bond dominates the investment portfolio throughout the investment horizon. Even when members reach the age of 100, the manager puts around 50% of the portfolio in the longevity bond, indicating that the longevity bond could provide not only longevity protection but also a considerable risk premium.

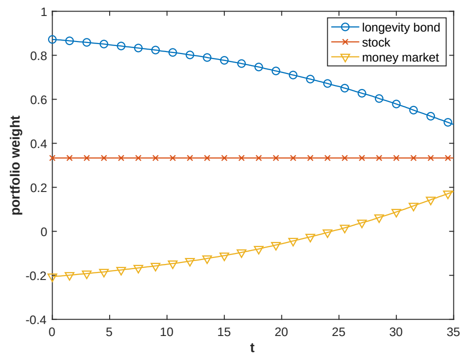

Figure 3.4.2: Average optimal investment strategies over 100 paths

To obtain general results, we present the average of 100 simulation paths of the optimal benefit withdrawal proportion $\frac{\beta^*(t)}{Y(t)}$ and rate $\beta^*(t)$, the wealth level $Y(t)$ and the manager's compensation $c(t) = \lambda_1(t)Y(t)$. From the top-left plot in Figure 3.4.3, we see that the optimal proportion of the wealth withdrawn by the member increases over time. Meanwhile, in the top-right plot, we observe that the optimal benefit withdrawal rate reduces over time. This phenomenon is explained by the declining wealth level shown in the bottom-left plot. Although the manager invests in the financial market, the average wealth level is decreasing throughout the time horizon due to the continuous benefit withdrawals and compensation payments. The wealth level declines faster while the benefit withdrawal proportion rises slower, thus the optimal benefit withdrawal rate drops over time. The average compensation received by the manager shows an interesting trend - it increases at first, peaks at around the 19th year and then drops rapidly. The reason is that, in the first 19 years, the wealth level is high while the instantaneous rate of mortality (i.e., the force of mortality $\lambda_1(t)$) rises over time. Thus, the manager's compensation increases. However, in the later years, the pension balance reduces to low levels. Although the instantaneous mortality rate increases, the manager's compensation drops.

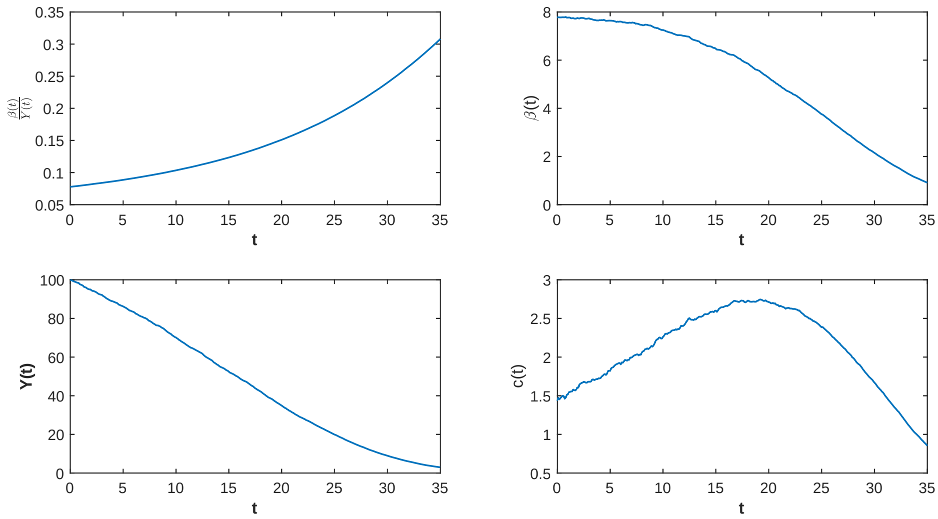

Figure 3.4.3: Average benefit withdrawal proportion, withdrawal rate, wealth, and compensation over 100 paths

Comparison analysis

To test the hedging performance of the longevity bond, we look at the results for the case when the manager does not include the longevity bond in the investment portfolio. Without the investment in the longevity bond, the optimal benefit withdrawal proportion $\frac{\beta^*(t)}{Y(t)} = \frac{1}{G(t,\lambda_1)}$, and the optimal portfolio weight in the stock $\frac{\alpha_s^*(t)}{Y(t)} = \frac{\theta_S}{\sigma_S}$, are the same as given in Proposition 3.3.1. The portfolio weight in the money market equals $1 - \frac{\alpha_S^*(t)}{Y(t)}$ and keeps constant over time. Let $\beta_1(t)$ ($c_1(t)$) and $\beta_2(t)$ ($c_1(t)$) denote the benefit withdrawal rate (compensation) without and with investment in the longevity bond, respectively. Figure 3.4.4 shows the benefit withdrawal and compensation improvement by investing in the longevity bond. It shows that, for path 1, investing in the longevity bond always results in higher benefit withdrawals and compensations. For path 3, investing in the longevity bond in general increases both the member's benefit withdrawal rate and the manager's compensation. Although, during some short period, the longevity bond investment decreases the withdrawals and compensations. However, for path 2, investment in the longevity bond seems to cut down both benefit withdrawals and compensations. As discussed earlier (see Figure 3.4.1), the survival probability on path 2 is overall lower than the Gompertz-Makeham survival probability. It implies that the member tends to live shorter than expected. Likewise, Figure 3.4.1 suggests that for path 2, the random age of death is more likely to be younger compared to the other paths - path 1 or path 3. As a result, the member does not face the longevity risk and investing in the longevity bond actually loses money rather than making gains. As it is a global trend that people's average life expectancy is increasing, we argue that pension schemes will benefit from longevity bond investment as mirrored by the situation in path 1 and 3. To support our claim, we show that the average improvements of benefit withdrawal rate and compensation over 100 simulated paths in Figure 3.4.5. As shown, there are small improvements in the first few years, but overall the improvements are significant over most of the investment period. This indicates that investing in the longevity bonds increases both the member's benefit withdrawal and the manager's compensation, and that it is beneficial to both the member and the manager.

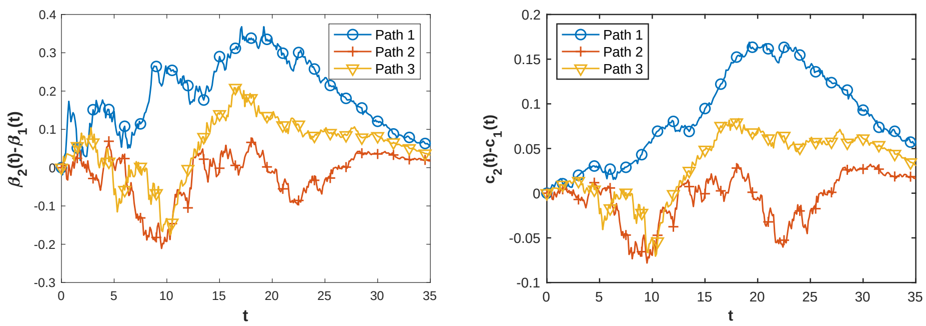

Figure 3.4.4: Improvement for the benefit withdrawal and compensation by investing into the longevity bond

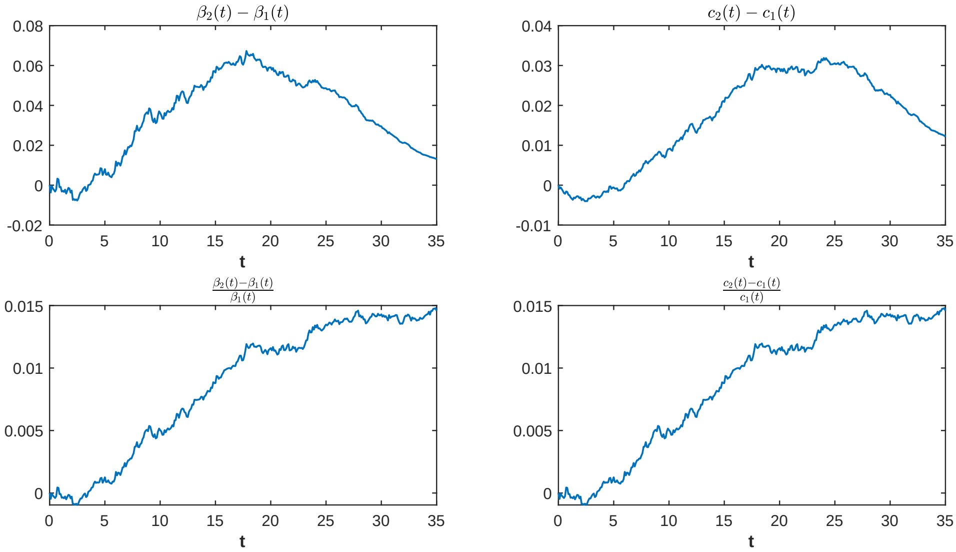

Figure 3.4.5: Average improvements for benefit withdrawal and compensation by investing into the longevity bond over 100 paths

Sensitivity analysis

We are interested in the impact of the market price of risk on the pension scheme's risk management. It is difficult to determine the market price of longevity risk due to the absence of longevity bonds in the market. We examine a number of values of θ_1 to give an illustration of the effects of the market price of longevity risk on the optimal strategy, the benefit withdrawal and the manager's compensation. In our base scenario, θ_1 is set as -5×10^{-4} and the longevity risk premium offered by the longevity bond is $\theta_1 \sigma_L = 4.4563 \times 10^{-6}$. Compared with the stock's risk premium ($\theta_S \sigma_S = 7.5 \times 10^{-3}$), the longevity risk premium is small. We show that the optimal investment strategies in the cases where $\theta_1 = 0$, -1.5×10^{-3} and -3×10^{-3}. A large absolute value of θ_1 indicates a high risk premium offered by the longevity bond. In addition, larger risk premiums should lead to more investment in the longevity bond. Figure 3.4.6 shows the investment strategies with different values of θ_1. We can see that when the manager does not add the longevity bond to his portfolio, the optimal investment proportions in the stock and money market keep constant over time. The investment in the longevity bond does not affect the portfolio weight in the stock, however it affects the investment in the money market. The top-right plot shows that even in the case when the longevity bond offers no risk premium (i.e., $\theta_1 = 0$), the optimal proportion invested in the longevity bond is always higher than 40%. It illustrates that the longevity bond provides a good way to hedge the scheme's longevity risk. As shown in Figure 3.4.6, as the longevity risk premium decreases (i.e., lower θ_1), more portfolio weight is put on the longevity bond. In the bottom-right plot, $\theta_1 = -0.0030$ and the longevity risk premium equals to 2.6738×10^{-5} which is again far less than the stock's risk premium. This implies that the manager continuously borrows money from the money market in order to invest in risky assets throughout the whole time horizon. The intuition is that the longevity bond not only provides a longevity risk hedge, but also provides an attractive risk premium.

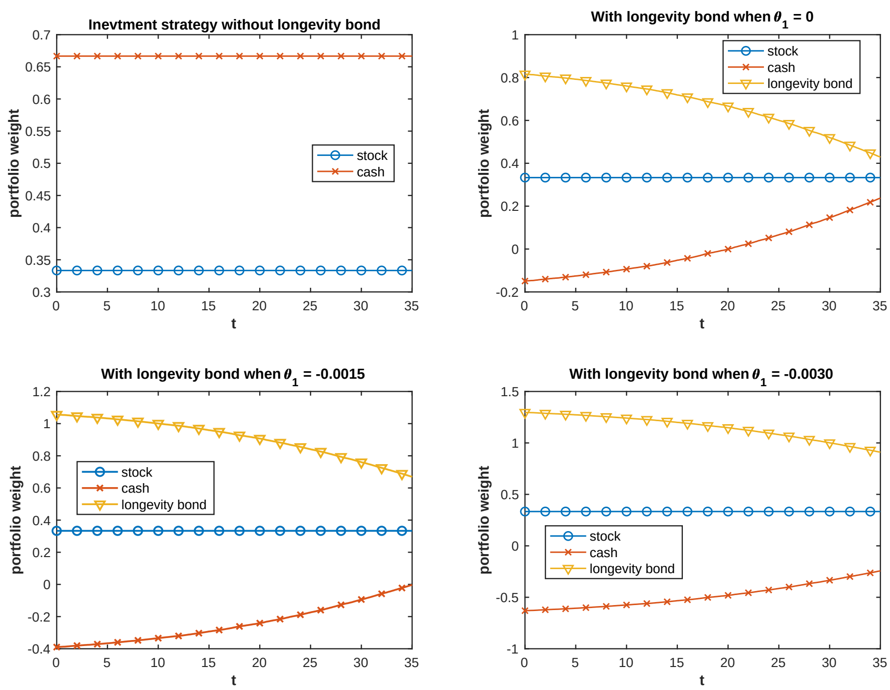

Figure 3.4.6: Impact of θ_1 on the optimal investment strategy

Figure 3.4.7 shows the improvements for benefit withdrawal and compensation when investing in longevity bonds. As shown for path 1, a high market price of longevity risk leads to high improvements in both manager's compensation and member's benefit withdrawal rate. As discussed in Section 3.4.1, investing in the longevity bond sometimes decreases the benefit withdrawal rate and compensation and thus causes a 'loss'. Here by 'loss', we mean loss in the member's benefit and the manager's compensation as the improvements by investing in the longevity bond are negative. Nevertheless, we observe from the plots in Figure 3.4.7 that a smaller θ_1 reduces this loss.

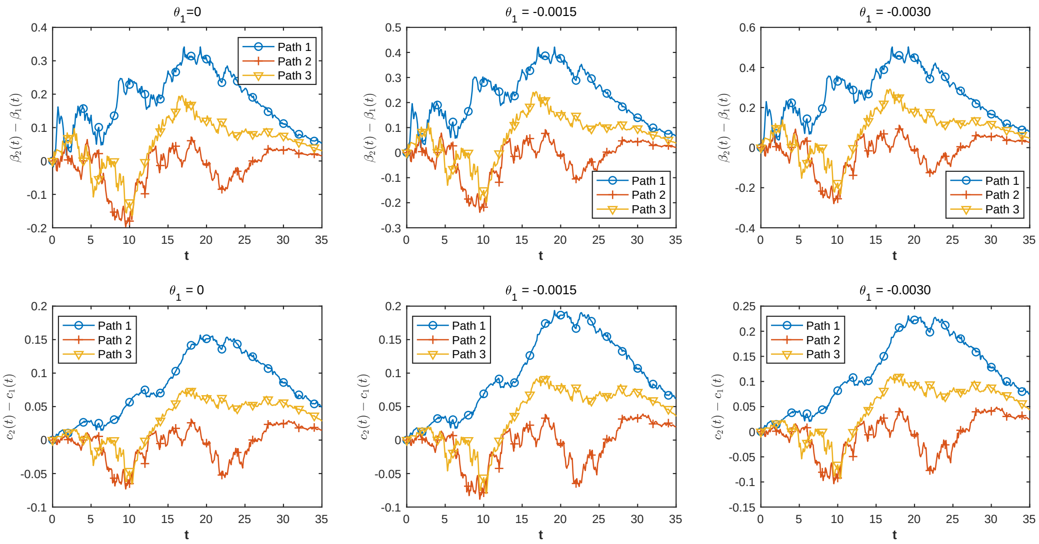

Figure 3.4.7: Impact of θ_1 on the benefit withdrawal and compensation improvements

We now investigate the impact of the risk-sharing parameter ϕ on the benefit withdrawal rate and manager compensation. As stated before, ϕ defines the agreement between the manager and the member on how to share the risk. A high value of ϕ implies that more weight is put on the manager's utility. When $\phi = 0$, the manager works on behalf of the member and only cares about the member's benefit. This case corresponds to no risk-sharing. When $0 < \phi < 1$, more emphasis is put on the member's benefit. In the case where $\phi = 1$, the manager treats his own profit and member's benefit equally which corresponds to the case of equal risk-sharing. We test the cases when ϕ takes values 0, 0.5 and 1. The case with $\phi = 0$ is chosen as the reference case. Figure 3.4.8 shows the improvement rates on benefit withdrawal and compensation (i.e., $\frac{\beta_\phi(t)-\beta_{\phi=0}(t)}{\beta_{\phi=0}(t)}$ and $\frac{c_\phi(t)-c_{\phi=0}(t)}{c_{\phi=0}(t)}$). As shown in the right plot, the higher the value of ϕ, the higher the improvement in compensation. Compared to the case $\phi = 0$, the equal risk-sharing rule agreement improves the manager's compensation by more than 20% at the end of the time horizon. For the benefit withdrawal, a higher value of ϕ leads to higher withdrawals in the last 10 years of the period, but reduces the withdrawals in the early part of the period. To illustrate the impact of ϕ, we calculate the average discounted values of benefit withdrawals and compensations over 100 simulated paths. We find that compared to the case $\phi = 0$, $\phi = 1$ increases the discounted benefit withdrawal by 4.71% and the discounted compensation by 12.82%. Thus, both the manager and the member benefit from sharing the risk equally.

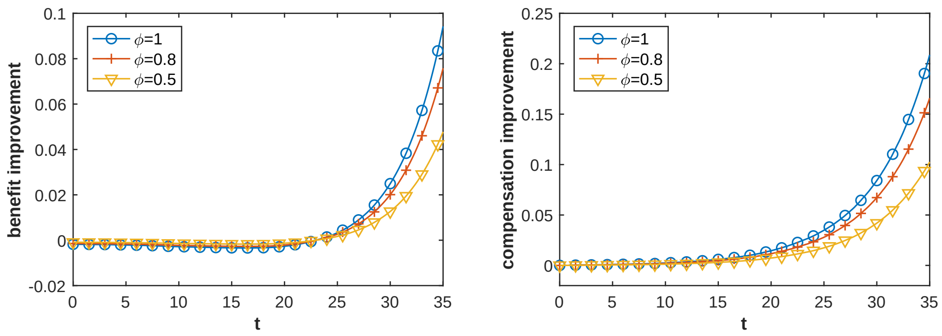

Figure 3.4.8: Impact of ϕ on the benefit withdrawal and compensation

The previous discussions can be summarized as follows. The longevity bond offers an efficient way to hedge the longevity risk. It is optimal to attach large portfolio weights to it, even in the case where it provides no risk premium, i.e., $\theta_1 = 0$. Moreover, both member and manager benefit from investment in the longevity bond. The higher the longevity risk premium, the more portfolio weight is put on the longevity bond, and the more the manager and member benefit from investing in it. Finally, an equal risk-sharing rule is the most beneficial to both member and manager.

3.4.2 Sub-population case

On the one hand, a pension scheme faces longevity risk caused by its members' extended life span. On the other hand, a pension scheme faces longevity basis risk if the mortality behavior of the scheme members is imperfectly correlated with the longevity bond's underlying mortality index. In practice, pension schemes face longevity basis risk as it is difficult to find a longevity bond in the market that is based exactly on the scheme members. Therefore, the sub-population model may be more practical compared to the single-population model. In this section, we assume that the longevity bond is based on a large population and the scheme members are a sub-population of this large population. Also, the forces of mortalities of the two populations are related. Furthermore, we assume that the expected life expectancy of the scheme members is higher than the longevity bond reference population. If it is lower, there might be no need to invest in the longevity bond. This section investigates the optimal strategy and the hedging performance of the longevity bond in the presence of longevity basis risk.

The base scenario

The values of parameters used in this section are provided in Table 3.4.1. For $i = 1, 2$, m_i is the mode of remaining life span of the i-th population. m_2 is set to be greater than m_1 implying that the expected age of death of the member in Population 2 is higher than the longevity bond reference population in Population 1. Again, we present three simulation paths in this section to observe the two populations' mortality behavior. Figure 3.4.9 shows the survival probabilities for Population 1 and Population 2. As expected, the average survival probability of Population 1 is lower than Population 2. The plots below illustrate the following:

- Population 1
    - path 1 has no particular trend;
    - path 2 has a higher survival probability than expected;
    - path 3 shows shorter life expectancy.
- Population 2:
    - path 1 displays a lower survival probability in the first half of the time horizon;
    - path 2 & 3 on average show longevity trend.

Figure 3.4.10 shows the optimal investment strategy and benefit withdrawal in the sub-population case. The optimal strategy is similar to the single-population case:

- the longevity bond dominates the portfolio;
- the portfolio weight attach to the longevity bond decreases over time while the holding in the money market increases;
- the optimal investment proportion in the stock keeps constant;
- the percentage of the wealth withdrawn by the member increases with time.

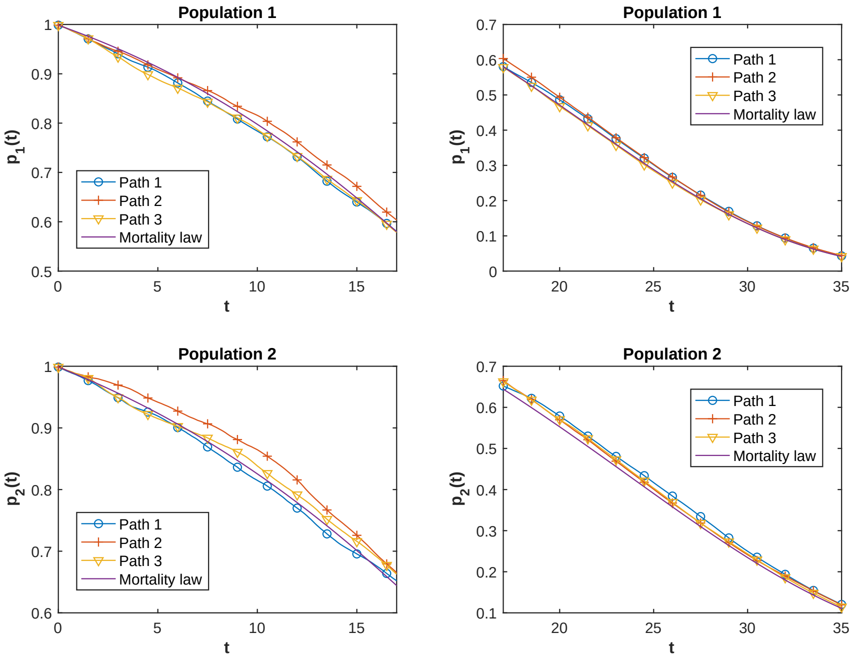

Figure 3.4.9: Survival probabilities for Population 1 and 2

We learn from the optimal strategy that the longevity bond plays an essential role in the pension scheme's risk management. It is optimal to invest large proportions of the wealth in it.

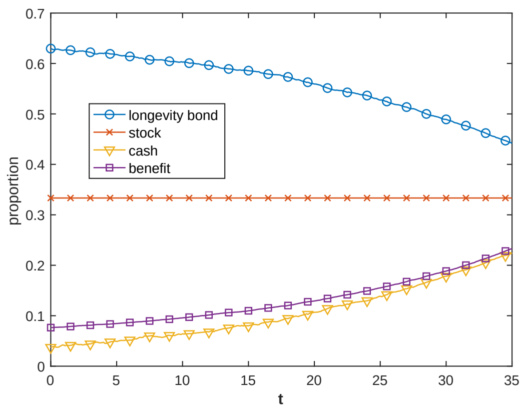

Figure 3.4.10: Average optimal portfolio strategy and benefit withdrawal in sub-population case

Comparison analysis

In Section 3.4.1, we show that in the single-population case, both the manager and the member benefit from investing in the longevity bond. Now, we conduct a comparison study in the sub-population case to see whether the use of the longevity bond is still efficient. For the simulated path 1, Figure 3.4.11 shows that investing in the longevity bond in general increases the member's benefit withdrawal and manager's compensation. For path 2, the member and the manager benefit from investing in the longevity bond in the late 20-year time period, although the member's pension suffers from some loss in the early years. In the scenario of path 3, neither the manager nor the member take advantage of longevity bond investment. To find out the reason for these phenomenon, we check the simulated survival probabilities. We learn from the plots on the bottom in Figure 3.4.9 that for path 3, the member in Population 2 lives much longer than anticipated. Whereas, the survival probability of the longevity bond reference population (Population 1) on path 3 is lower. In this situation, the member suffers from severe longevity risk. In this case, the longevity bond can not provide an efficient longevity risk hedge because the reference population displays shorter life expectancy. Since we assume the members are a sub-population of the longevity bond reference population, the members should have a similar mortality trend with the reference population, although with slightly different behavior. We believe that investing in the longevity bond can still be beneficial to both sides - manager and member - in the two population case.

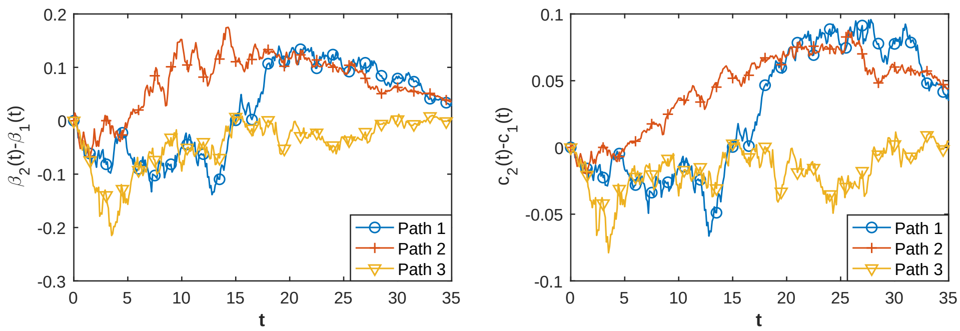

Figure 3.4.11: Benefit and compensation improvement in sub-population case

Overall, we conclude that the longevity bond provides a powerful tool to hedge the longevity risk in the sub-population case. However, due to the presence of longevity basis risk, the hedging performance is less effective than in the single-population case. A scenario with multiple longevity bonds in the market may show different results and is worthy of an independent future study. A sensitivity analysis focusing on the market price of longevity risk and the risk-sharing rule in the sub-population case deliver results which are similar to the single-population case.

3.5 Conclusion

We have studied the optimal portfolio strategy and benefit withdrawal rate for a pension scheme with an income-drawdown policy in the decumulation phase in the presence of a stochastic force of mortality, both for a reference and sub-population. The optimal solutions for the single- and sub-population cases are obtained by applying the dynamic programming principle. They are explicit solutions for for the OU case and semi-analytic solutions for the CIR case. Our numerical study shows that the longevity bond can be used efficiently to hedge the longevity risk, and both member and manager benefit from the longevity bond investment. Moreover, both member and manager benefit from an agreement on the risk-sharing rule in the long run.

Further research: One challenge that discourages hedgers from using standardized mortality-linked instruments is the population basis risk. This chapter provides mathematical evidence supporting the use of longevity bond for efficient longevity risk hedge without and with longevity basis risk. The problem with multiple longevity bonds issued with regard to different reference populations is a potential topic which we will explore in a future study. We believe that the development of a longevity market is required to provide a solution to capital markets for longevity risk hedging. Further research may also include the use of various types of mortality-linked securities, such as forward contracts and swaps. The pricing of longevity securities is also an interesting further research topic.

Appendices

Appendix 3.A: Proof of Proposition 3.2.1

Proof. The corresponding value function $V(t,y,\lambda)$ of the optimization problem (3.7) is given by

$$V(t,y,\lambda)=\sup_{\alpha_S,\alpha_L,\beta}\mathbb{E}_t\Big[\int_t^{\infty}e^{-\int_t^s(r+\lambda_j(u))\mathrm{d}u}\Big(\ln(\beta(s))+\phi\lambda_j(s)\ln(Y(s))\Big)\mathrm{d}s\Big].$$

Using the Dynamic Programming Principle (DPP), we obtain the following HJB equation

$$0=V_t(t,y,\lambda)+\phi\lambda_j\ln y-(r+\lambda_j)V(t,y,\lambda)+\sup_{\alpha_S,\alpha_L,\beta}\Big[\ln(\beta)+\mathcal{A}^{\alpha_S,\alpha_L,\beta}V(t,y,\lambda)\Big],$$

where

$$\begin{aligned}\mathcal{A}^{\alpha_S,\alpha_L,\beta}V(t,y,\lambda)=&V_y\left[ry+\alpha_S\sigma_S\theta_S+\alpha_L\sigma_L\theta-\beta\right]+V_\lambda\mathcal{B}+\frac{1}{2}tr\left(\Sigma'\Sigma V_{\lambda\lambda}\right)\\&+\frac{1}{2}\left(\alpha_S^2\sigma_S^2+\alpha_L^2\sigma_L\sigma_L'\right)V_{yy}+\alpha_L V_{y\lambda}\Sigma'\sigma_L'.\end{aligned}$$

We write $V:=V(t,y,\lambda)$ and denote the corresponding partial derivatives of V as V_t, V_y, V_λ, V_{yy}, $V_{\lambda\lambda}$ and $V_{y\lambda}$ for notational simplicity.

Solving the first order conditions on $\beta(t)$, $\alpha_S(t)$ and $\alpha_L(t)$ gives

$$\beta^*(t)=\frac{1}{V_y},\quad \alpha_S^*(t)=-\frac{\theta_S V_y}{\sigma_S V_{yy}},\quad \alpha_L^*(t)=-\frac{V_y}{V_{yy}}\frac{\sigma_L\theta}{\sigma_L\sigma_L'}-\frac{1}{V_{yy}}\frac{V_{y\lambda}\Sigma'\sigma_L'}{\sigma_L\sigma_L'}. \tag{3.17}$$

Substituting (3.17) into the HJB equation leads to

$$\begin{aligned}0=&V_t+\phi\lambda_j\ln y-(r+\lambda_j)V-\ln\left(V_y\right)+ryV_y-1+V_\lambda\mathcal{B}+\frac{1}{2}tr\left(\Sigma'\Sigma V_{\lambda\lambda}\right)\\&-\frac{1}{2}\theta_S^2\frac{V_y^2}{V_{yy}}-\frac{1}{2}\frac{V_y^2}{V_{yy}}\frac{(\sigma_L\theta)^2}{\sigma_L\sigma_L'}-\frac{V_y}{V_{yy}}\frac{V_{y\lambda}\Sigma'\sigma_L'\sigma_L\theta}{\sigma_L\sigma_L'}-\frac{1}{2}\frac{1}{V_{yy}}\frac{(V_{y\lambda}\Sigma'\sigma_L')^2}{\sigma_L\sigma_L'}.\end{aligned} \tag{3.18}$$

We make a sophisticated guess that the solution to the PDE (3.18) is of the following form

$$V(t,y,\lambda)=G(t,\lambda)\ln y+H(t,\lambda),$$

with boundary conditions

$$\lim_{t\to\infty}V(t,y,\lambda)=0,\qquad \lim_{t\to\infty}G(t,\lambda)=0,\qquad \lim_{t\to\infty}H(t,\lambda)=0.$$

The PDE (3.18) now becomes

$$\begin{aligned}0 =& G_t \ln y + H_t + \phi\lambda_j \ln y - (r+\lambda_j)(G\ln y + H) - \ln G + \ln y + rG - 1 \\ &+ (G_\lambda \ln y + H_\lambda)\mathcal{B} + \frac{1}{2}tr\left(\Sigma'\Sigma\left(G_{\lambda\lambda}\ln y + H_{\lambda\lambda}\right)\right) + \frac{1}{2}\theta_S^2 G + \frac{1}{2}\frac{(\sigma_L\theta)^2}{\sigma_L\sigma_L'}G \\ &+ \frac{G_\lambda\Sigma'\sigma_L'\sigma_L\theta}{\sigma_L\sigma_L'} + \frac{1}{2}\frac{1}{G}\frac{(G_\lambda\Sigma'\sigma_L)^2}{\sigma_L\sigma_L'},\end{aligned}$$

where we write the corresponding partial derivatives of G and H as G_t, H_t, G_λ, H_λ, $G_{\lambda\lambda}$ and $H_{\lambda\lambda}$.

Separating the $\ln y$ terms and we get two ODEs

$$\begin{aligned}0 =& G_t - (r+\lambda_j)G + G_\lambda\mathcal{B} + \frac{1}{2}tr\left(\Sigma'\Sigma G_{\lambda\lambda}\right) + \phi\lambda_j + 1, \quad (3.19)\\ 0 =& H_t - (r+\lambda_j)H - \ln G + rG - 1 + H_\lambda\mathcal{B} + \frac{1}{2}tr\left(\Sigma'\Sigma H_{\lambda\lambda}\right) + \frac{1}{2}\theta_S^2 G \\ &+ \frac{1}{2}\frac{(\sigma_L\theta)^2}{\sigma_L\sigma_L'}G + \frac{G_\lambda\Sigma'\sigma_L'\sigma_L\theta}{\sigma_L\sigma_L'} + \frac{1}{2}\frac{(G_\lambda\Sigma'\sigma_L)^2}{G\sigma_L\sigma_L'}.\end{aligned}$$

We only need $G(t,\lambda)$ to get the optimal solutions, so we solve the ODE for $G(\lambda)$ (3.19) and obtain

$$G(t,\lambda) = \mathbb{E}_t\left[\int_t^\infty (\phi\lambda_j(s)+1)e^{-\int_t^s (r+\lambda_j(u))\mathrm{d}u}\mathrm{d}s\right].$$

Substituting $G(t,\lambda)$ into (3.17) gives the optimal solutions in Proposition 3.2.1. □

Appendix 3.B: Proof of Proposition 3.3.1

Proof. We first provide the calculations for $A_0^{\mathbb{P}}(t,s)$. Within the OU setting, we denote by $h^{\mathbb{P}}(t,s,\lambda_1) = \mathbb{E}_t\left[e^{-\int_t^s \lambda_1(u)\mathrm{d}u}\right]$ and $D(t) = e^{-\int_0^t \lambda_1(u)\mathrm{d}u}$. Applying Itô's formula to $D(t)h^{\mathbb{P}}(t,s,\lambda_1)$ and setting the $\mathrm{d}t$ term equal to 0, we obtain

$$\lambda_1 h^{\mathbb{P}} = h_t^{\mathbb{P}} + h_{\lambda_1}^{\mathbb{P}}(a_1 - b_1\lambda_1) + \frac{1}{2}\sigma_1^2 h_{\lambda_1\lambda_1}^{\mathbb{P}}. \quad (3.20)$$

As $\lambda_1(t)$ follows an affine class model, we make a sophisticated guess that $h^{\mathbb{P}}(t,s,\lambda_1)$ has the following form

$$h^{\mathbb{P}}(t,s,\lambda_1) = e^{A_0^{\mathbb{P}}(t,s) - A_1^{\mathbb{P}}(t,s)\lambda_1(t)},$$

with terminal conditions $h^{\mathbb{P}}(s,s,\lambda_1)=1$, $A_0^{\mathbb{P}}(s,s)=0$ and $A_1^{\mathbb{P}}(s,s)=0$. Differentiating $h(t,\lambda_1)$ and plugging into (3.20) leads to two ODEs

$$\begin{aligned}0 &= \frac{\mathrm{d}A_0^{\mathbb{P}}(t,s)}{\mathrm{d}t} - a_1 A_1^{\mathbb{P}}(t,s) + \frac{1}{2}\sigma_1^2 A_1^{\mathbb{P}}(t,s)^2,\\ 0 &= -\frac{\mathrm{d}A_1^{\mathbb{P}}(t,s)}{\mathrm{d}t} + b_1 A_1^{\mathbb{P}}(t,s) - 1.\end{aligned}$$

Solving these ODEs gives $A_0^{\mathbb{P}}(t,s)$ and $A_1^{\mathbb{P}}(t,s)=A_1(t,s)$ in Proposition 3.3.1.

Next, we present the derivation of $G(t,\lambda_1)$. According to Proposition 3.2.1, we have

$$G(t,\lambda_1) = \int_t^\infty e^{-r(s-t)}\left(\phi\mathbb{E}_t\left[\lambda_1(s)e^{-\int_t^s \lambda_1(u)\mathrm{d}u}\right] + h^{\mathbb{P}}(t,s,\lambda_1)\right)\mathrm{d}s.$$

To solve $\mathbb{E}_t\left[\lambda_1(s)e^{-\int_t^s \lambda_1(u)\mathrm{d}u}\right]$, we denote by

$$\widetilde{Z}(t) = \frac{\mathbb{E}_t\left[e^{-\int_0^s \lambda_1(u)\mathrm{d}u}\right]}{\mathbb{E}\left[e^{-\int_0^s \lambda_1(u)\mathrm{d}u}\right]}.$$

Then, we have

$$\widetilde{Z}(t) = D(t)\frac{h^{\mathbb{P}}(t,s,\lambda_1)}{h^{\mathbb{P}}(0,s,\lambda_1)},\qquad \mathbb{E}_t\left[\lambda_1(s)e^{-\int_t^s \lambda_1(u)\mathrm{d}u}\right] = \mathbb{E}_t\left[\lambda_1(s)\frac{\widetilde{Z}(s)}{\widetilde{Z}(t)}\right]h^{\mathbb{P}}(t,s,\lambda_1).$$

Let $\mathbb{E}_t\left[\lambda_1(s)\frac{\widetilde{Z}(s)}{\widetilde{Z}(t)}\right] = \widetilde{\mathbb{E}}_t[\lambda_1(s)]$, where $\widetilde{\mathbb{E}}[\cdot]$ denotes the expectation under the measure $\widetilde{\mathbb{P}}$ which is an equivalent measure to $\mathbb{P}$ which is defined below. Applying Itô's formula to $\widetilde{Z}(t)$ gives

$$\begin{aligned}\mathrm{d}\widetilde{Z}(t) =& \frac{h^{\mathbb{P}}(t,s,\lambda_1)}{h^{\mathbb{P}}(0,s,\lambda_1)}\mathrm{d}D(t) + \frac{D(t)}{h^{\mathbb{P}}(0,s,\lambda_1)}\mathrm{d}h^{\mathbb{P}}(t,s,\lambda_1)\\ =& -\lambda_1(t)\widetilde{Z}(t)\mathrm{d}t + \frac{D(t)}{h^{\mathbb{P}}(0,s,\lambda_1)}\left(h_t^{\mathbb{P}} + h_{\lambda_1}^{\mathbb{P}}(a_1 - b_1\lambda_1) + \frac{1}{2}\sigma_1^2 h_{\lambda_1\lambda_1}^{\mathbb{P}}\right)\mathrm{d}t\\ &+ \frac{D(t)}{h^{\mathbb{P}}(0,s,\lambda_1)}h_{\lambda_1}^{\mathbb{P}}\sigma_1\mathrm{d}W_1(t)\\ =& -\sigma_1 A_1(t,s)\widetilde{Z}(t)\mathrm{d}W_1(t),\end{aligned}$$

where we use (3.20). Now, we can define $\widetilde{\mathbb{P}}$ through $\widetilde{Z}(s)$

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \widetilde{Z}(s) = \exp\left(-\int_0^s \sigma_1 A_1(u,s)\mathrm{d}W_1(u) - \frac{1}{2}\int_0^s \sigma_1^2 A_1^2(u,s)\mathrm{d}u\right),$$

and we have

$$\begin{aligned}\mathrm{d}\widetilde{W}_1(u) &= \mathrm{d}W_1(u) + \sigma_1 A_1(u,s)\mathrm{d}u,\\ \mathrm{d}\lambda_1(u) &= \left(a_1(u) - b_1\lambda_1(u) - \sigma_1^2 A_1(u,s)\right)\mathrm{d}u + \sigma_1\mathrm{d}\widetilde{W}_1(u).\end{aligned}$$

Taking expectation under $\widetilde{\mathbb{P}}$ leads to

$$\frac{\mathrm{d}\widetilde{\mathbb{E}}_t[\lambda_1(u)]}{\mathrm{d}u} = a_1(u) - \sigma_1^2 A_1(u,s) - b_1 \widetilde{\mathbb{E}}_t[\lambda_1(u)].$$

The solution of the above ODE is then obtained through

$$\begin{aligned} M(t,s,\lambda_1) =&\widetilde{\mathbb{E}}_t[\lambda_1(s)] = \lambda_1(t)e^{-b_1(s-t)} + \int_t^s \left(a_1(u) - \sigma_1^2 A_1(u,s)\right) e^{-b_1(s-u)}\mathrm{d}u \\ =&\lambda_1(t)e^{-b_1(s-t)} + \left(b_1\nu_1 - \frac{\sigma_1^2}{b_1}\right)(s-t) + \frac{1+b_1\Delta_1}{\Delta_1}\left(e^{\frac{s-m_1}{\Delta_1}} - e^{\frac{t-m_1}{\Delta_1}}\right) \\ &+ \frac{\sigma_1^2}{b_1}A_1(t,s). \end{aligned}$$

□

Appendix 3.C: Proof of Proposition 3.3.3

Proof. We denote $f(t,s,\lambda_1,\lambda_2) = \mathbb{E}_t\left[e^{-\int_t^s \lambda_2(u)\mathrm{d}u}\right]$. Applying Itô's formula to $e^{-\int_0^t \lambda_2(u)\mathrm{d}u} f(t,s,\lambda_1,\lambda_2)$ and setting the $\mathrm{d}t$ term equal to 0, we get

$$\begin{aligned} \lambda_2 f =& f_t + f_{\lambda_1}(a_1 - b_1\lambda_1) + f_{\lambda_2}(a_2 - b_{21}\lambda_1 - b_{22}\lambda_2) + \frac{1}{2}f_{\lambda_1\lambda_1}\sigma_1^2 \\ &+ \frac{1}{2}f_{\lambda_2\lambda_2}(\sigma_{21}^2 + \sigma_{22}^2) + f_{\lambda_1\lambda_2}\sigma_1\sigma_{21}. \end{aligned} \tag{3.21}$$

As $\lambda_1(t)$ and $\lambda_2(t)$ follow affine class models, we make a sophisticated guess that $f(t,s,\lambda_1,\lambda_2)$ has the following form

$$f(t,s,\lambda_1,\lambda_2) = e^{C_0(t,s) - C_1(t,s)\lambda_1(t) - C_2(t,s)\lambda_2(t)}, \tag{3.22}$$

with terminal conditions $f(s,s,\lambda_1,\lambda_2) = 1$, $C_0(s,s) = 0$, $C_1(s,s) = 0$ and $C_2(s,s) = 0$. Differentiating $f(t,s,\lambda_1,\lambda_2)$ and plugging into (3.21) leads to three ODEs

$$\begin{aligned} 0 &= \frac{\partial C_0}{\partial t} - a_1 C_1 - a_2 C_2 + \frac{1}{2}\sigma_1^2 C_1^2 + \frac{1}{2}\left(\sigma_{21}^2 + \sigma_{22}^2\right) C_2^2 + \sigma_1\sigma_{21}C_1 C_2, \\ 0 &= -\frac{\partial C_1}{\partial t} + b_1 C_1 + b_{21} C_2, \\ 1 &= -\frac{\partial C_2}{\partial t} + b_{22} C_2. \end{aligned}$$

Let $\gamma_1 = \frac{B_{21}}{b_1 - B_{22}}$ and $\gamma_2 = \sigma_{21}^2 + \sigma_{22}^2$, then we obtain

$$\begin{aligned}
C_0(t,s) =& \left(\frac{\gamma_2}{2b_{22}^2} - \frac{b_{21}\sigma_1\sigma_{21}}{b_1 b_{22}^2} + \frac{\sigma_1^2\gamma_1^2\gamma_2}{2b_1^2 b_{22}^2} + \frac{\gamma_1^2\sigma_1^2}{b_1 b_{22}} \right)(s-t) \\
&+ \frac{b_{21}(1+b_1\Delta_1) - b_1^2\Delta_1}{b_1 b_{22}\Delta_1} \left(e^{\frac{s-m_1}{\Delta_1}} - e^{\frac{t-m_1}{\Delta_1}} \right) \\
&- \frac{(1+b_1\Delta_1)\gamma_1 - b_1\Delta_1}{b_{22}(1+b_{22}\Delta_1)\Delta_1} \left(e^{\frac{s-m_1}{\Delta_1}} - e^{\frac{t-m_1}{\Delta_1} - b_{22}(s-t)} \right) \\
&- \frac{1+b_{22}\Delta_2}{b_{22}\Delta_2} \left(e^{\frac{s-m_2}{\Delta_2}} - e^{\frac{t-m_2}{\Delta_2}} \right) - \frac{1}{b_{22}\Delta_2} \left(e^{\frac{s-m_2}{\Delta_2}} - e^{\frac{t-m_2}{\Delta_2} - b_{22}(s-t)} \right) \\
&+ \left(\frac{\gamma_1(1 - \sigma_1\sigma_{21} - \gamma_1\sigma_1^2)}{b_1(b_1+b_{22})} - \frac{\sigma_1^2\gamma_1^2}{2b_1} \right) A_1(t,s) \\
&+ \left(\nu_2 + \frac{(b_{21}-1)\nu_1}{b_{22}} + \frac{\gamma_1(\sigma_1\sigma_{21} + \gamma_1\sigma_1^2 - 1)}{b_1(b_1+b_{22})} \right) C_2(t,s) \\
&+ \left(\frac{2\gamma_1\sigma_1\sigma_{21} - \sigma_1^2\gamma_1^2 - \gamma_2}{2b_{22}^2} - \frac{\gamma_1\sigma_1(\sigma_{21} + \gamma_1\sigma_1)}{b_1 b_{22}} \right) C_2(t,s) \\
&+ \frac{\gamma_1(1 - \sigma_1\sigma_{21} - \gamma_1 b_1\sigma_1^2)}{b_1 + b_{22}} A_1(t,s) C_2(t,s) \\
&+ \frac{2\gamma_1\sigma_1\sigma_{21} - \gamma_2 - \gamma_1^2\sigma_1^2}{4b_{22}} C_2^2(t,s) - \frac{\gamma_1^2\sigma_1^2}{4b_1} A_1^2(t,s), \\
C_1(t,s) =& \gamma_1 \Big(A_1(t,s) - C_2(t,s) \Big), \\
C_2(t,s) =& \frac{1 - e^{-b_{22}(s-t)}}{b_{22}}.
\end{aligned}$$

Similar to Proposition 3.3.3, in order to obtain $G(t, \lambda_1, \lambda_2)$, we first solve $\widetilde{\mathbb{E}}_t[\lambda_1(s)]$ and $\widetilde{\mathbb{E}}_t[\lambda_2(s)]$ under $\widetilde{\mathbb{P}}$ which is defined as

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \widetilde{Z}(s) = \exp\left(-\int_0^s \tilde{\theta}(u,s)' \mathrm{d}W(u) - \frac{1}{2}\int_0^s \left\| \tilde{\theta}(u,s) \right\|^2 \mathrm{d}u \right),$$

where

$$\begin{aligned}
\tilde{\theta}(u,s) =& \begin{bmatrix} \sigma_1 C_1(u,s) + \sigma_{21} C_2(u,s) \\ \sigma_{22} C_2(u,s) \\ 0 \end{bmatrix}, \\
\mathrm{d}\widetilde{W}(u) =& \mathrm{d}W(u) + \tilde{\theta}(u,s)\mathrm{d}u.
\end{aligned}$$

Then, we solve the following two ODEs

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\mathbb{E}}_t[\lambda_1(u)]}{\mathrm{d}u} &= a_1(u) - \sigma_1^2 C_1(u,s) - \sigma_1\sigma_{21} C_2(u,s) - b_1 \widetilde{\mathbb{E}}_t[\lambda_1(u)], \\
\frac{\mathrm{d}\widetilde{\mathbb{E}}_t[\lambda_2(u)]}{\mathrm{d}u} &= a_2(u) - \sigma_1\sigma_{21} C_1(u,s) - (\sigma_{21}^2 + \sigma_{22}^2) C_2(u,s) - b_{21}\widetilde{\mathbb{E}}_t[\lambda_1(u)] - b_{22}\widetilde{\mathbb{E}}_t[\lambda_2(u)],
\end{aligned}$$

and obtain

$$
\begin{aligned}
M(t,s,\lambda_1,\lambda_2) =& \widetilde{\mathbb{E}}_t[\lambda_2(s)] = \gamma_1\Big(N(t,s,\lambda_1) - \Gamma_1(t,s,\lambda_1)\Big) + \Gamma_2(t,s,\lambda_2), \qquad (3.23)\\
N(t,s,\lambda_1) =& \widetilde{\mathbb{E}}_t[\lambda_1(s)]\\
=& \lambda_1(t)e^{-b_1(s-t)} + \int_t^s \left(a_1(u) - \sigma_1^2 C_1(u,s) - \sigma_1\sigma_{21}C_2(u,s)\right) e^{-b_1(s-u)}\mathrm{d}u,\\
=& \lambda_1(t)e^{-b_1(s-t)} + \frac{1}{\Delta_1}\Big(e^{\frac{s-m_1}{\Delta_1}} - e^{\frac{t-m_1}{\Delta_1}-b_1(s-t)}\Big)\\
&+ \Big(b_1\nu_1 + \frac{\sigma_1\sigma_{21}b_{21}}{2b_1(b_1+b_{22})} - \frac{\sigma_1^2}{b_1+b_{22}}\Big)A_1(t,s)\\
&- \frac{1}{2}\sigma_1\sigma_{21}\gamma_1 A_1^2(t,s) + \frac{\sigma_1\sigma_{21}\gamma_1 b_1 - \sigma_1^2 b_1}{b_1+b_{22}}A_1(t,s)C_2(t,s)\\
&+ \frac{\sigma_1^2 - \sigma_1\sigma_{21}\gamma_1}{b_1+b_{22}}C_2(t,s),\\
\Gamma_1(t,s,\lambda_1) =& \lambda_1(t)e^{-b_{22}(s-t)} + \int_t^s \left(a_1(u) - \sigma_1^2 C_1(u,s) - \sigma_1\sigma_{21}C_2(u,s)\right) e^{-b_{22}(s-u)}\mathrm{d}u,\\
=& \lambda_1(t)e^{-b_1(s-t)} + \frac{1+b_1\Delta_1}{\Delta_1(1+b_{22}\Delta_1)}\Big(e^{\frac{s-m_1}{\Delta_1}} - e^{\frac{t-m_1}{\Delta_1}-b_{22}(s-t)}\Big)\\
&+ \frac{\gamma_1\sigma_1^2}{b_1+b_{22}}A_1(t,s) - \frac{\gamma_1\sigma_1^2 b_{22}}{b_1+b_{22}}A_1(t,s)C_2(t,s)\\
&+ \frac{1}{2}\sigma_1\Big(\gamma_1\sigma_1 - \sigma_{21}\Big)C_2^2(t,s) + \Big(b_1\nu_1 - \frac{\gamma_1 b_1\sigma_1^2}{b_1+b_{22}}\Big)C_2(t,s),\\
\Gamma_2(t,s,\lambda_2) =& \lambda_2(t)e^{-b_{22}(s-t)}\\
&+ \int_t^s \Big(a_2(u) - \sigma_1\sigma_{21}C_1(u,s) - \left(\sigma_{21}^2 + \sigma_{22}^2\right)C_2(u,s)\Big)e^{-b_{22}(s-u)}\mathrm{d}u\\
=& \lambda_2(t)e^{-b_{22}(s-t)} + \frac{b_{21}}{1+b_{22}\Delta_1}\Big(e^{\frac{s-m_1}{\Delta_1}} - e^{\frac{t-m_1}{\Delta_1}-b_{22}(s-t)}\Big)\\
&+ \frac{1}{\Delta_2}\Big(e^{\frac{s-m_2}{\Delta_2}} - e^{\frac{t-m_2}{\Delta_2}-b_{22}(s-t)}\Big) + \frac{1}{2}\Big(\gamma_1\sigma_1\sigma_{21} - \gamma_2\Big)C_2^2(t,s)\\
&- \frac{\gamma_1\sigma_1\sigma_{21}b_{22}}{b_1+b_{22}}A_1(t,s)C_2(t,s) + \frac{\gamma_1\sigma_1\sigma_{21}}{b_1+b_{22}}A_1(t,s)\\
&+ \Big(b_{21}\nu_1 + b_{22}\nu_2 - \frac{\gamma_1\sigma_1\sigma_{21}b_1}{b_1+b_{22}}\Big)C_2(t,s).
\end{aligned}
$$

□

Chapter 4

Optimal winding up time and investment strategy in a Defined Benefit scheme

4.1 Introduction

In defined benefit (DB) schemes, the benefit payments in the distribution phase are pre-determined. Thus, the scheme sponsors face the risk of failure to cover the liabilities, whereas the members bear no risks. DB scheme sponsors are increasingly looking for ways to reduce their risks, and many DB schemes have closed to new entrants. According to PPF (2020), 41% of the $5,318$ DB schemes in the UK are closed to new members (but open to new benefit accrual), and 46% of the schemes are closed to new benefit accrual in 2020. In this chapter, we study the risk management of a DB scheme that is closed to new entrants. Typically, in the accumulation phase of a DB scheme, the sponsor pays flexible contributions while the members' contributions are fixed. The sponsor needs to adjust the contribution rate and determine the investment strategy in order to maintain the fund in balance. Thus, the choice of investment strategy and contribution rate is crucial for DB scheme sponsors. The literature on risk management and asset allocation in DB schemes is rich. One main focus is to minimize contribution rate risk and solvency risk from the perspective of DB scheme sponsors. For example, Haberman et al. (2000) studied the contribution and solvency risks in DB schemes in a discrete-time setting. By controlling the spread period, the aim is to minimize a time-weighted sum of the weighted average of the quadratic deviations of the contribution rate and scheme wealth from their desired targets. In the papers of, for instance, Josa-Fombellida & Rincón-Zapatero (2004), Hainaut & Deelstra (2011) and Josa-

Fombellida & Rincón-Zapatero (2012), the normal cost and the actuarial liability in a DB scheme are defined as the ideal contribution rate and fund level, respectively. The scheme manager chooses contribution rate and investment strategy to minimize both the quadratic spread between contribution rate and normal cost and the quadratic deviation of terminal assets from liabilities. In this chapter, we suppose that the members of the DB scheme we considered are a retired cohort. In other words, the scheme is in the distribution phase, and there is no active member (i.e., members who are currently paying into the scheme) and no future contributions. Thus, we only focus on the scheme's solvency risk and do not consider the contribution rate risk. We suppose that the sponsor's objective is to minimize the quadratic deviation of the terminal assets from a desired target. See Josa-Fombellida & Rincón-Zapatero (2010) and Hainaut & Deelstra (2011).

There are several consolidation solutions for DB schemes, such as transferring all the assets and liabilities to DB master trust (or superfunds), merging with larger schemes and scheme insurance buy-in/buy-out. An *insurance buy-in* is an insurance policy covering a subset of the scheme's members. The scheme holds the insurance policy as an asset that pays benefits to the covered members. An *insurance buy-out* (or bulk annuity) is a single premium insurance policy purchased by the scheme for all its members. Members hold insurance policies individually, and the scheme can transfer all its pension liabilities and risks to the insurance company. This option also enables a scheme to wind up. PPF (2020) shows that, in the UK, the volume of buy-ins and buy-outs was £43.8bn in 2019 and £24.2bn in 2018. Since 2010, the volume of buy-ins and buy-outs has increased significantly. In this chapter, we suppose that the scheme sponsor's ultimate goal is to wind up the scheme through insurance buy-out. Moreover, we assume that the sponsor sets the buy-out cost as the scheme wealth target and measures the solvency risk as the quadratic deviation of the scheme wealth from the buy-out cost.

In this chapter, we formulate the problem as a combined stochastic control and optimal stopping time problem. The objective is to minimize the terminal solvency risk by controlling the investment strategy and determining the buy-out purchase time. As far as we know, the optimal winding up time and investment strategy problem for DB sponsors has not been studied in the literature. However, past papers on optimal annuitization time for individual retirees provide a model framework for our study. See, for example, Stabile (2006), Farhi & Panageas (2007), Milevsky & Young (2007), Gerrard et al. (2012), Park & Jang (2014) and Park (2015). We assume that the members' random death time is exponentially distributed with a positive constant. See also Stabile (2006) and Gerrard et al. (2012). There is no further building up of pension liabilities while the scheme continuously pays benefits to surviving members until the buy-out is purchased. Before the buy-out is purchased, the sponsor determines the investment in a risk-less asset and a risky asset. To solve the problem, we give a verification theorem that relates the value function of the problem with a solution to the variational HJB inequality (HJBVI). We find that the value function equals zero on one part of the continuation region. On the other part of the continuation region, we solve the HJBVI via Legendre dual. When the funding level on the technical provisions basis is less than 100%, the optimal investment in the stock is a linear function of the unfunded liability. Our main contribution is the derivation of the explicit solution to the optimal winding up and investment strategy for DB schemes. We believe this chapter could contribute to DB scheme sponsors and insurers in making buy-out purchase decisions.

This chapter is organized as follows. In Section 4.2, we describe the financial and insurance markets and formulate the combined optimal stopping and stochastic control problem. Section 4.3 derives the explicit solution by investigating the form of the continuation region and applying the Legendre transform. In Section 4.4, we present numerical applications. Section 4.5 concludes this chapter.

4.2 Model

4.2.1 The markets and the scheme

In this section, we describe the financial and insurance markets. We assume that the scheme sponsor can trade two broad classes of assets: a bond (i.e., risk-less asset) R and a stock (i.e., a risky asset) S. The dynamics of the prices are described by the following equations

$$\mathrm{d}R_t = rR_t\mathrm{d}t,$$

$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}B_t,$$

where $r > 0$ is the constant risk-free interest rate, $\mu > r$ is the constant expected rate of stock return, $\sigma > 0$ is the stock's constant volatility and B_t is a standard Brownian motion defined on a suitable probability space.

Next, we make our insurance and mortality assumptions. We consider a DB scheme that is closed to new entrants, and all its members are pensioners. There are no future contributions, and the scheme pays benefits continuously to surviving members. We suppose that the scheme members are of the same age (i.e., a retired cohort), have a similar mortality trend and receive the same benefit payments. See Hainaut & Deelstra (2011), Cox et al. (2013) and He & Liang (2017), where the authors studied optimal control problems with a single cohort of scheme members. Denote by τ_d the members' random remaining lifetime. We assume that the scheme sponsor believes that τ_d is independent of the Brownian motion B_t and expects that τ_d is exponentially distributed with a positive constant parameter λ^S. That is, the sponsor supposes that members' force of morality is λ^S. See Stabile (2006), Gerrard et al. (2012) and He & Liang (2017). Let $n \geq 1$ be the number of scheme members at the initial time, and β be the positive constant instantaneous benefit payment to each surviving member. Then, the sponsor calculates the scheme's instantaneous benefit payments $P(t)$ and *technical provisions* $I(t)$ (i.e., the actuarial present value of future liabilities) as

$$P(t) = np(t)\beta = n\beta e^{-\lambda^S t}, \quad I(t) = \int_t^\infty P(u)e^{-r(u-t)}\mathrm{d}u = \frac{n\beta}{r+\lambda^S}e^{-\lambda^S t}, \tag{4.1}$$

where $p(t) = e^{-\lambda^S t}$ is the fraction of the members that is still alive at time t. Suppose that the scheme has an initial wealth of the size $X_0 = x$, before the termination of the scheme, the dynamics of X_t is given by

$$\mathrm{d}X_t = (rX_t + \tilde{\pi}(t)(\mu - r) - P(t))\,\mathrm{d}t + \tilde{\pi}(t)\sigma\mathrm{d}B_t, \tag{4.2}$$

where $\tilde{\pi}(t)$ is the investment in the stock at time t.

There may be some constraints on the investment strategy and scheme funding level:

- There may be no-short-selling and no-borrowing restrictions. In other words, $\tilde{\pi}(t)$ may be constrained to be non-negative, or the proportion of the wealth invested in stock may be required not to exceed 100%.

- There may be a lower limit on the funding level. For example, there may be a triggering value $l \in [0, 1)$, determined in an agreement between the sponsor and the members, that the buy-out will be purchased (and the scheme will be wound up) if the funding level on the technical provisions basis (i.e., $X_t/I(t)$) ever falls below l.
- There may be employer contributions to be paid into the scheme, though there is no further members' contribution. For example, in the UK, when a scheme does not have sufficient assets to cover its technical provisions, a recovery plan must be put in place in order to return the scheme to full funding.

In Section 4.3, we first study the situation of unconstrained $\tilde{\pi}$. Then, we assume that short-selling is not allowed, but the sponsor has the possibility of borrowing. That is, $\tilde{\pi}$ is constrained to be non-negative. In addition, we consider a non-negative wealth constraint (i.e., $l = 0$): the sponsor is forced to purchase the buy-out and thus wind up the scheme when the scheme ruins. Our analyses and results can be applied to cases where $0 < l < 1$ and where there are deficit reduction payments from the sponsor. See Appendix 4.B.

Following Stabile (2006) and Milevsky & Young (2007), we allow the scheme sponsor and the insurance company to evaluate the scheme's liability using different values of the force of mortality, and we denote by λ^O the constant force of mortality used by the insurance company. In this chapter, we suppose that $\lambda^S \geq \lambda^O$ and call λ^S and λ^O the subjective and objective force of morality, respectively. The actuarial present value of a lifetime annuity $\bar{a}$ that pays β continuously to an individual until death is

$$\bar{a} = \int_0^{\tau_d} \beta e^{-ru} \mathrm{d}u = \beta \int_0^{\infty} e^{-(r+\lambda^O)u} \mathrm{d}u = \frac{\beta}{r + \lambda^O}.$$

See Bower et al. (1997, Section 5). An *insurance buy-out* is a single premium insurance policy that passes a pension scheme's responsibilities to an insurance company. Once the buy-out is purchased, members receive individual policies. Denote by $L(t)$ the buy-out cost, we have

$$L(t) = np(t)\bar{a} = \frac{n\beta}{r + \lambda^O} e^{-\lambda^S t}.$$

If $\lambda^S > \lambda^O$, then the scheme's technical provisions $I(t)$ at any time t is always lower than the buy-out cost $L(t)$ and vice versa. The buy-out cost coincides with the technical provisions when $\lambda^S = \lambda^O$.

4.2.2 The optimization problem

The scheme sponsor plans to wind up the scheme through an insurance buy-out. The aim is to minimize the terminal solvency risk by controlling the investment strategy $\tilde{\pi}$ and deciding the buy-out purchasing time T. We suppose that the sponsor measures the solvency risk by the square of the difference between the scheme wealth and the buy-out cost. The goal is to minimize the terminal solvency risk by controlling the investment strategy and determining the buy-out purchasing time. Then, the combined optimization problem is given as

$$\phi(x) = \inf_{\tilde{\pi},T} \mathbb{E}^x \left[e^{-\rho(T\wedge\tau_0)} (X_{T\wedge\tau_0} - L(T\wedge\tau_0))^2 \right], \qquad \forall\, x \in \mathbb{R}_+,$$

where ρ is the sponsor's subjective discount rate and τ_0 denotes the time of ruin $\tau_0 := \inf\{t \geq 0 : X_t \leq 0\}$ with the convention that $\tau_0 = \infty$ if $X_t > 0$ for all t. By definition, we find that the value function $\phi(x)$ is always non-negative.

Remark 4.2.1 (Quadratic loss function)**.** Denote by $\tau = T \wedge \tau_0$ the termination time of the scheme, then $(X_\tau - L(\tau))^2$ measures the scheme's terminal solvency risk. This form of the solvency risk assessment is also used in previous works such as Josa-Fombellida & Rincón-Zapatero (2001), Josa-Fombellida & Rincón-Zapatero (2010), Hainaut (2014) and Ngwira & Gerrard (2007). $(X_\tau - L(\tau))^2$ is also known as a quadratic loss function or a disutility function. The buy-out cost $L(\tau)$ is a target level. It means that the sponsor wishes to achieve this target, and deviations of the terminal scheme wealth from this target are penalized. The use of the quadratic loss function is sometimes questioned: it also penalizes deviations above the target. However, we do not consider this to be a limiting feature for our study. Since once the target is achieved, there is no reason for further exposure to risk, and therefore the surplus becomes undesirable. Thus, our choice of this objective function is appropriate. See also Gerrard et al. (2004).

Observing the value function $\phi(x)$, we find that we can construct an auxiliary process Y_t

to rewrite ϕ in a simpler form. Let $Y_t = \frac{X_t}{L(t)}$, we have

$$\mathrm{d}Y_t = \Big((r+\lambda^S)Y_t + \pi(t)(\mu - r) - (r+\lambda^O)\Big)\mathrm{d}t + \pi(t)\sigma\mathrm{d}B_t, \tag{4.3}$$

where $\pi(t) = \frac{\tilde{\pi}(t)}{L(t)} = \frac{r+\lambda^O}{n\beta}e^{\lambda^S t}\tilde{\pi}(t)$. In the above, we used the relationship $P(t) = (r + \lambda^O)L(t)$. Y_t is the scheme's funding level on a wind up basis. On the other hand, $\frac{X_t}{I(t)}$ is the scheme's funding level on the technical provisions basis. For convenience, we hereafter call Y_t the funding level if not specified. At the initial time, the funding level is $Y_0 = y = \frac{r+\lambda^O}{n\beta}x$. The ruin time can now be written as $\tau_0 = \inf\{t \geq 0 : Y_t \leq 0\}$. Thus, denote by $\tau = T \wedge \tau_0$ the termination time (or, the winding up time), the optimization problem can now be rewritten as

$$\phi(y) = \inf_{\pi,T} \mathbb{E}^y\left[e^{-(\rho+2\lambda^S)\tau}g(Y_\tau)\right], \qquad \forall\, y \in \mathbb{R}_+, \tag{4.4}$$

where

$$g(\xi) = \left(\frac{n\beta}{r+\lambda^O}\right)^2 (\xi - 1)^2. \tag{4.5}$$

Remark 4.2.2. (i) When the scheme's funding level reaches 100%, the solvency risk becomes zero and it is optimal to wind up the scheme immediately. Moreover, when $y = 1$, we have $T^* = 0$, $\tau^* = T^* \wedge \tau_0 = 0$ and $\phi(1) = g(1) = 0$.

(ii) The compulsory wind up of the scheme when ruin occurs implies that $\tau_0 = 0$, $\tau^* = T^* \wedge \tau_0 = 0$ and $\phi(0) = g(0)$.

(iii) If the sponsor invests the total scheme wealth in the risk-less asset (i.e., $\tilde{\pi}(t) = \pi(t) = 0$), then the funding level Y_t is a deterministic function of time:

$$Y_t = \hat{y} + (y - \hat{y})e^{(r+\lambda^S)t}, \qquad \forall\, y \in \mathbb{R}_+, \tag{4.6}$$

where $\hat{y} = \frac{r+\lambda^O}{r+\lambda^S}$. For $y > \hat{y}$, we have $\lim_{t\to\infty} Y_t = +\infty$. Denote by $\tau_1 := \inf\{t \geq 0 : Y_t = 1\}$ the first time when the funding level reaches 100%. If $\lambda^S > \lambda^O$ and $y \in (\hat{y}, 1]$, then we have $\tau_0 = \infty$ and $\mathsf{P}(\tau_1 < \infty) = 1$. By definition (4.4), we find that the value function $\phi(y)$ equals zero on $(\hat{y}, 1]$. Furthermore, it implies that the sponsor can invest the total scheme wealth in the bond and wind up the scheme when the funding level reaches 100%.

(iv) When the funding level reaches $\hat{y}$, the scheme is well-funded on the technical provisions basis, i.e., the scheme wealth X_t equals the scheme's technical provisions $I(t)$. If the initial funding level is $\hat{y}$ and the sponsor invests the total scheme wealth in the bond and pays the benefits using the interest earned on the fund, then the funding level keeps constant over time, i.e., $Y_t = \hat{y}$. Moreover, we have $\tau_0 = \infty$ and

$$0 \le \phi(\hat{y}) \le \lim_{t\to\infty} \mathbb{E}^{\hat{y}} \left[e^{-(\rho+2\lambda^S)(t\wedge\tau_0)} g(Y_{t\wedge\tau_0}) \right] = 0.$$

4.2.3 The verification theorem

By applying the dynamic programming principle and the Itô formula, we expect the value function $\phi(y)$ satisfies the following HJBVI

$$\min\left\{\mathcal{L}\varphi(y), g(y) - \varphi(y)\right\} = 0, \qquad \forall\, y \in \mathbb{R}_+, \tag{4.7}$$

where

$$\begin{aligned} \mathcal{L}\varphi(y) =& -(\rho + 2\lambda^S)\varphi(y) + \inf_{\pi} \mathcal{A}^\pi \varphi(y), \\ \mathcal{A}^\pi\varphi(y) =& \Big((r+\lambda^S)y + \pi(t)(\mu - r) - (r + \lambda^O)\Big)\varphi'(y) + \frac{1}{2}\pi^2(t)\sigma^2\varphi''(y). \end{aligned} \tag{4.8}$$

See Oksendal (2013, chapter 10). The following theorem relates the value function to the solution of the HJBVI. See, e.g., Stabile (2006) and Gerrard et al. (2012).

Theorem 4.2.1 (The verification theorem)**.** *Let $\varphi(y) \in C^1(\mathbb{R}_+)$, with the second derivative continuous almost everywhere in $\mathbb{R}_+$, be a solution to* (4.7) *such that for all $y \in \mathbb{R}_+$ and $\pi(t)$,*

$$\mathbb{E}^y \left[\int_0^t e^{-(\rho+2\lambda^S)u} \sigma\pi(u)\varphi'(Y_u)\mathrm{d}B_u \right] = 0, \tag{4.9}$$

[1] *for all t. Then,*

(i) $\varphi(y) \le \phi(y)$, $\forall\, y \in \mathbb{R}_+$.

(ii) Moreover, define the sets

$$\begin{aligned} \mathcal{C} :=& \{y \in \mathbb{R}_+ : \varphi(y) < g(y)\}, \\ \mathcal{C}_1 :=& \{y \in \mathbb{R}_+ : \varphi(y) < g(y) \textit{ and } \varphi(y) \neq 0\}. \end{aligned}$$

[1] The condition (4.9) is satisfied if $\pi(u)\varphi'(Y_u)$ is bounded.

Within the region $\mathcal{C}_1$, assume that $\varphi(y)$ is strictly convex and define the control

$$\pi^*(t) := -\frac{\mu - r}{\sigma^2}\frac{\varphi'(Y_t^*)}{\varphi''(Y_t^*)}\mathbb{1}_{Y_t^* \in \mathcal{C}_1}, \tag{4.10}$$

where Y_t^* is the solution to (4.3) *with $\pi^*(t)$. Let $\tau^* := \inf\{t \geq 0 : Y_t^* \notin \mathcal{C}\}$, and assume that*

$$\lim_{t\to\infty} \mathbb{E}^y \left[e^{-(\rho+2\lambda^S)t}\varphi(Y_t^*)\mathbb{1}_{\tau^*>t}\right] = 0. \tag{4.11}$$

Then, $\pi^*(t)$ and τ^* are the optimal control and the optimal stopping time, respectively. Furthermore,

$$\varphi(y) = \phi(y), \quad \forall\, y \in \mathbb{R}_+.$$

Proof. (i) For any control $\pi(t)$ and stopping time τ, applying Ito's formula to $e^{-(\rho+2\lambda^S)t}\varphi(Y_t)$ between 0 and τ gives

$$\begin{aligned} e^{-(\rho+2\lambda^S)\tau}\varphi(Y_\tau) =& \varphi(y) + \int_0^\tau e^{-(\rho+2\lambda^S)t}\left(-(\rho+2\lambda^S)\varphi(Y_t) + \mathcal{A}^\pi\varphi(Y_t)\right)\mathrm{d}t \\ &+ \int_0^\tau e^{-(\rho+2\lambda^S)t}\pi(t)\sigma\varphi'(Y_t)\mathrm{d}B_t. \end{aligned}$$

Using (4.7) and (4.8), we have

$$\varphi(y) \leq e^{-(\rho+2\lambda^S)\tau}g(Y_\tau) - \int_0^\tau e^{-(\rho+2\lambda^S)t}\pi(t)\sigma\varphi'(Y_t)\mathrm{d}B_t.$$

Taking expectation and using (4.9), we obtain

$$\varphi(y) \leq \mathbb{E}^y\left[e^{-(\rho+2\lambda^S)\tau}g(Y_\tau)\right].$$

From the arbitrariness of $\pi(t)$ and T, we have $\varphi(y) \leq \phi(y)$ for all $y \in \mathbb{R}_+$.

(ii) Consider $\pi^*(t)$, τ^* and let Y_t^* be the corresponding trajectory. For any $0 \leq s \leq \tau^*$, we have $Y_s^* \in \mathcal{C}$ and $-(\rho+2\lambda^S)\varphi(Y_s^*) + \mathcal{A}^{\pi^*}\varphi(Y_s^*) = 0$. Thus,

$$\begin{aligned} \varphi(y) =& e^{-(\rho+2\lambda^S)(\tau^*\wedge t)}\varphi(Y_{\tau^*\wedge t}^*) - \int_0^{\tau^*\wedge t} e^{-(\rho+2\lambda^S)s}\left(-(\rho+2\lambda^S)\varphi(Y_s^*) + \mathcal{A}^{\pi^*}\varphi(Y_s^*)\right)\mathrm{d}s \\ &- \int_0^{\tau^*\wedge t} e^{-(\rho+2\lambda^S)s}\pi^*(s)\sigma\varphi'(Y_s^*)\mathrm{d}B_s \\ =& e^{-(\rho+2\lambda^S)\tau^*}\varphi(Y_{\tau^*}^*)\mathbb{1}_{\tau^*\leq t} + e^{-(\rho+2\lambda^S)t}\varphi(Y_t^*)\mathbb{1}_{\tau^*>t} - \int_0^{\tau^*\wedge t} e^{-(\rho+2\lambda^S)s}\pi^*(s)\sigma\varphi'(Y_s^*)\mathrm{d}B_s. \end{aligned}$$

Taking expectation and using the definition of τ^*, we obtain

$$\varphi(y) = \mathbb{E}^y \left[e^{-(\rho+2\lambda^S)\tau^*} g(Y^*_{\tau^*}) \mathbb{1}_{\tau^* \leq t} + e^{-(\rho+2\lambda^S)t} \varphi(Y^*_t) \mathbb{1}_{\tau^* > t} \right].$$

Setting $t \to \infty$ and using (4.11), we obtain

$$\varphi(y) = \mathbb{E}^y \left[e^{-(\rho+2\lambda^S)\tau^*} g(Y^*_{\tau^*}) \right] = \phi(y), \quad \forall\, y \in \mathbb{R}_+.$$

□

The set $\mathcal{C}$ is called the continuation region. When the funding level is in $\mathcal{C}$, it is better to keep operating the scheme. While, when the funding level is not in $\mathcal{C}$, it is optimal to wind up the scheme immediately (or, the scheme ruins and the sponsor is forced to terminate the scheme immediately). Thus, we call $\mathbb{R}_+ \backslash \mathcal{C}$ the stopping region. The optimal stopping time τ^* is the first time the funding level goes outside $\mathcal{C}$. According to the verification theorem and Remark 4.2.2, we provide the following lemmas to show some properties of $\mathcal{C}$.

Lemma 4.2.1. *The stopping region contains* $\{0, 1\}$. *If* $\lambda^S > \lambda^O$, *then the continuation region contains* $[\hat{y}, 1)$.

Proof. As discussed in Remark 4.2.2, we have $\phi(y) = 0 < g(y)$ on $[\hat{y}, 1)$ when $\lambda^S > \lambda^O$. Besides, $\varphi(1) = g(1) = 0$ and $\varphi(0) = g(0)$. □

Lemma 4.2.2. *If the set* U *is defined by* $U := \{y \in \mathbb{R}_+ : \mathcal{L}g(y) < 0\}$, *then*

(i) the continuation region $\mathcal{C}$ *contains the set* U.

(ii) Moreover, if $g(y)$ *satisfies* (4.9) *and* $U = \emptyset$, *then* $\varphi(y) = g(y)$ *for all* $y \in \mathbb{R}_+$, *the continuation region is empty and it is optimal to wind up the scheme immediately.*

Proof. (i) If $y \in \mathbb{R}_+ \backslash \mathcal{C}$, then $\varphi(y) = g(y)$ and $\mathcal{L}\varphi(y) \geq 0$, from which it follows that $\mathcal{L}g(y) \geq 0$ and $y \in \mathbb{R}_+ \backslash U$. Thus, $(\mathbb{R}_+ \backslash \mathcal{C}) \subseteq (\mathbb{R}_+ \backslash U)$ and $U \subseteq \mathcal{C}$.
(ii) If $U = \emptyset$, then $\mathcal{L}g(y) \geq 0$ for all $y \in \mathbb{R}_+$ and $\mathcal{A}^\pi g(Y_t) \geq 0$ holds for all t. Thus, $g(y) \leq \mathbb{E}^y \left[e^{-(\rho+2\lambda^S)\tau} g(Y_\tau) \right]$ for all τ. By definition, we have $\phi(y) = g(y)$ for all $y \in \mathbb{R}_+$ which implies that $\tau^* = 0$ and $\mathcal{C} = \emptyset$. □

A common way to gain information of the continuation region $\mathcal{C}$ is to analyze the set U. In Section 4.3.1, we compute the set U explicitly and use it to investigate the form of the continuation region $\mathcal{C}$.

4.3 The optimal solution

4.3.1 The analysis of the set U

In this section, we deduce the form of the continuation region $\mathcal{C}$ by analyzing U in both the situation of unconstrained $\tilde{\pi}(t)$ and the situation without short-selling.

The situation of unconstrained investment strategy

If there is no restriction on the risky investment, then the sponsor is able to short the risky asset (i.e., $\tilde{\pi}(t)$ and $\pi(t)$ can be negative) and has the possibility of borrowing (i.e., $\tilde{\pi}(t)/X_t$ can be greater than 1). From (4.5) and (4.8), we get

$$\mathcal{L}g(y) = \left(\frac{n\beta}{r+\lambda^O}\right)^2 \left\{\gamma(y-1)^2 + 2(\lambda^S - \lambda^O)(y-1)\right\}, \qquad \forall\, y \in \mathbb{R}_+,$$

where $\gamma = 2r - \rho - \left(\frac{\mu - r}{\sigma}\right)^2$. In the above, the minimizing value of π is $\pi(y) = -\frac{\mu-r}{\sigma^2}(y-1)$. We see that $\pi \leq 0$ when $y \geq 1$. When $\gamma \neq 0$, we denote

$$y_2 = 1 - 2\frac{\lambda^S - \lambda^O}{\gamma}. \tag{4.12}$$

One can show that $y_2 < \hat{y} < 1$ when $\gamma > 0$ and $y_2 > 1$ when $\gamma < 0$. We identify the following cases:

(a) equal forces of mortality $\lambda^S = \lambda^O$

$$U = \begin{cases} \emptyset, & \text{if} \quad \gamma \geq 0, \\ (0,1) \cup (1, +\infty), & \text{if} \quad \gamma < 0. \end{cases}$$

(b) higher subjective force of mortality $\lambda^S > \lambda^O$

$$U = \begin{cases} (y_2, 1), & \text{if} \quad \gamma > 2(\lambda^S - \lambda^O), \\ (0,1), & \text{if} \quad 0 \leq \gamma \leq 2(\lambda^S - \lambda^O), \\ (0,1) \cup (y_2, +\infty), & \text{if} \quad \gamma < 0. \end{cases}$$

When $\lambda^S = \lambda^O$, the buy-out cost $L(t)$ and the scheme's technical provisions $I(t)$ coincide. Using Lemma 4.2.1 and 4.2.2, one can find the explicit continuation region $\mathcal{C}$:

$$\mathcal{C} = \begin{cases} \emptyset, & \text{if} \quad \gamma \geq 0, \\ (0,1) \cup (1, +\infty), & \text{if} \quad \gamma < 0. \end{cases}$$

The situation where $\gamma < 0$ is likely to happen when the subjective discount rate ρ is relatively high or when the Sharp ratio $\frac{\mu-r}{\sigma}$ is very large. On the one hand, a large ρ means that the scheme sponsor values the present more than the future. On the other hand, a high Sharp ratio implies that the stock is very attractive to the sponsor as it has a high expected rate of return but small volatility. Therefore, the sponsor would like to invest in the financial market for a longer time and defer the buy-out purchase, though the sponsor can always wind up the scheme by paying a price that equals exactly the technical provisions and transfer all the risks to the insurance company. As a result, the continuation region is wide.

When $\gamma \geq 0$, the emptiness of the continuation region $\mathcal{C}$ implies that it is always optimal to wind up the scheme immediately. One could find a non-negative γ when the Sharp ratio $\frac{\mu-r}{\sigma}$ or the subjective discount rate ρ is relatively small. When $\frac{\mu-r}{\sigma}$ is small, the intuition behind this immediate-wind-up strategy is that the insurance market is more attractive than the financial market. Besides, the sponsor values less the present than the future (i.e., ρ is small).

Although we can find the explicit continuation region, it may be unreasonable to assume that the subjective and objective forces of mortality coincide. In the real world insurance market, the buy-out cost is normally higher than the technical provisions. For this reason, we will only focus on the situation where $\lambda^S > \lambda^O$ hereafter. To further investigate the cases where $\lambda^S > \lambda^O$, we give the following lemma.

Lemma 4.3.1. *Assume that there is no restriction on the risky investment and $\lambda^S > \lambda^O$. Then, the value function $\phi(y)$ equals zero on $y \in (1, +\infty)$ and the continuation region $\mathcal{C}$ contains* $(1, +\infty)$.

Proof. Denote by $\tau_1 := \inf\{t \geq 0 : Y_t = 1\}$ the first time that the funding level touches 100%. If there exists at least one π such that $\tau_1 < \tau_0$ and $\mathsf{P}(\tau_1 < \infty) = 1$, then

$$0 \leq \phi(y) \leq \mathbb{E}^y[e^{-(\rho+2\lambda^S)(\tau_1 \wedge \tau_0)} g(Y_{\tau_1 \wedge \tau_0})] = \mathbb{E}^y[e^{-(\rho+2\lambda^S)\tau_1} g(1)] = 0.$$

If we construct a control $\pi(t) = -\frac{r+\lambda^S}{\mu-r}(Y_t - \hat{y})$, then Y_t follows

$$\mathrm{d}Y_t = -\frac{\sigma(r+\lambda^S)}{\mu-r}\,(Y_t - \hat{y})\,\mathrm{d}B_t, \qquad \forall\, y \in \mathbb{R}_+,$$

which can be explicitly solved to give

$$Y_t = \hat{y} + (y - \hat{y})\, exp\left\{ -\frac{1}{2}\frac{\sigma^2(r+\lambda^S)^2}{(\mu - r)^2} t - \frac{\sigma(r+\lambda^S)}{\mu - r} B_t \right\}.$$

Hence, we have $\lim_{t\to\infty} Y_t = \hat{y} < 1$ almost surely. If $y \in (1, +\infty)$, then $\tau_1 < \tau_0$, $\mathsf{P}(\tau_1 < \infty) = 1$ and $\phi(y) = 0$. Besides, (4.9) and (4.11) are satisfied. □

Remark 4.3.1. Lemma 4.3.1 implies that when the investment strategy is unconstrained and the funding level is above 100%, the sponsor can short sell the stock and purchase the buy-out when the funding level falls to 100%. This paradoxical investment behavior is because the sponsor's prior objective is not to increase the scheme wealth/funding level but to reduce the solvency risk. Later in this section, we investigate the situation where short-selling is not allowed.

When $\lambda^S > \lambda^O$, using Lemma 4.2.1, 4.2.2 and 4.3.1, we infer that the continuation region is of the form:

$$\mathcal{C} = \begin{cases} (\tilde{y}, 1) \cup (1, +\infty), & \text{if} \quad \gamma > 2(\lambda^S - \lambda^O), \\ (0, 1) \cup (1, +\infty), & \text{if} \quad \gamma \leq 2(\lambda^S - \lambda^O), \end{cases}$$

where $\tilde{y}$ is a free boundary to be determined and we require that $0 \leq \tilde{y} \leq y_2$. Again, we find that the continuation region $\mathcal{C}$ is likely to be narrower when γ has a larger value. On $[\hat{y}, +\infty)$, the value function equals zero. We suppose that $\mathcal{C}_1 = (\tilde{y}, \hat{y})$ when $\gamma > 2(\lambda^S - \lambda^O)$ and $\mathcal{C}_1 = (0, \hat{y})$ otherwise. It remains to determine the free boundary $\tilde{y}$ and the value function within region $\mathcal{C}_1$.

The situation without short-selling

In this section, we consider the situation where short-selling is not allowed (i.e., $\pi(t)$ is constrained to be non-negative) and suppose that $\lambda^S > \lambda^O$. From (4.5) and (4.8), the first and second order partial derivatives of $\mathcal{A}^\pi g$ with respect to π are respectively

$$\frac{\partial \mathcal{A}^\pi g}{\partial \pi} = (\mu - r)(y - 1) + \sigma^2 \pi, \qquad \frac{\partial^2 \mathcal{A}^\pi g}{\partial \pi^2} = \sigma^2 > 0.$$

When $y > 1$, the first order partial derivative is positive for all $\pi \geq 0$. Thus, the minimizing π is

$$\pi(y) = \begin{cases} -\frac{\mu - r}{\sigma^2}(y-1), & \text{on} \quad y \in [0,1], \\ 0, & \text{on} \quad y \in (1,+\infty). \end{cases}$$

Hence,

$$\mathcal{L}g(y) = \begin{cases} \left(\frac{n\beta}{r+\lambda^O}\right)^2 \left\{\gamma(y-1)^2 + 2(\lambda^S - \lambda^O)(y-1)\right\}, & \text{on} \quad y \in [0,1], \\ \left(\frac{n\beta}{r+\lambda^O}\right)^2 \left\{(2r-\rho)(y-1)^2 + 2(\lambda^S - \lambda^O)(y-1)\right\}, & \text{on} \quad y \in (1,+\infty). \end{cases}$$

When $2r - \rho \neq 0$, we denote by $\bar{y}_2 = 1 - 2\frac{\lambda^S - \lambda^O}{2r-\rho}$ and identify the following cases:

$$U = \begin{cases} (y_2, 1), & \text{if} \quad 2r \geq \rho + 2(\lambda^S - \lambda^O) + \left(\frac{\mu-r}{\sigma^2}\right)^2, \\ (0,1), & \text{if} \quad \rho \leq 2r < \rho + 2(\lambda^S - \lambda^O) + \left(\frac{\mu-r}{\sigma^2}\right)^2, \\ (0,1) \cup (\bar{y}_2, +\infty), & \text{if} \quad 2r < \rho, \end{cases}$$

where y_2 is given in (4.12). We first study the case where $2r < \rho$ and give the following lemma.

Lemma 4.3.2. *Assume that short-selling is not allowed, $\lambda^S > \lambda^O$ and $2r < \rho$. Then, the value function $\phi(y)$ equals zero for all $y \in [\hat{y}, +\infty)$. Furthermore, the continuation region is $\mathcal{C} = (0,1) \cup (1,+\infty)$.*

Proof. We prove the statement by constructing a risk-free investment strategy. If $\pi(t) = 0$, then Y_t is deterministic and is given in (4.6). For any $y \in \mathbb{R}_+$, we have

$$\begin{aligned} \mathbb{E}^y \left[e^{-(\rho+2\lambda^S)(t\wedge\tau_0)} g(Y_{t\wedge\tau_0})\right] =& (\hat{y}-1)^2 e^{-(\rho+2\lambda^S)(t\wedge\tau_0)} + 2(\hat{y}-1)(y-\hat{y}) e^{(r-\rho-\lambda^S)(t\wedge\tau_0)} \\ &+ (y-\hat{y})^2 e^{(2r-\rho)(t\wedge\tau_0)}. \end{aligned}$$

When $y \in [\hat{y}, +\infty)$, we have $\tau_0 = \infty$. Hence,

$$0 \leq \phi(y) \leq \lim_{t\to\infty} \mathbb{E}^y \left[e^{-(\rho+2\lambda^S)(t\wedge\tau_0)} g(Y_{t\wedge\tau_0})\right] = 0, \qquad \forall\, y \in [\hat{y}, +\infty).$$

Thus, the value function equals zero for all $y \in [\hat{y}, +\infty)$ as $2r < \rho$ and $\phi(y) = g(y)$ for all $y \in (0,1) \cup (1,+\infty)$. □

Therefore, the continuation region is of the form:

$$\mathcal{C} = \begin{cases} (\tilde{y}, 1), & \text{if} \quad 2r \geq \rho + 2(\lambda^S - \lambda^O) + \left(\frac{\mu - r}{\sigma^2}\right)^2, \\ (0, 1), & \text{if} \quad \rho \leq 2r < \rho + 2(\lambda^S - \lambda^O) + \left(\frac{\mu - r}{\sigma^2}\right)^2, \\ (0,1) \cup (1, +\infty), & \text{if} \quad 2r < \rho. \end{cases}$$

Although we are able to obtain the explicit $\mathcal{C}$ when $\rho > 2r$, we will restrict our attention to the case $\rho \leq 2r$ from now on. Since it is not realistic to assume that the sponsor's subjective discount rate is more than twice the risk-free interest rate. If short-selling is not allowed, $2r \geq \rho$ and the initial funding level is above 100%, then it is optimal to purchase the buy-out immediately.

4.3.2 Legendre transform

In this section, we study the value function in region $\mathcal{C}_1$. According to the HJBVI (4.7) and using the optimal control (4.10), the value function satisfies

$$0 = -(\rho + 2\lambda^S)\varphi + (r + \lambda^S)y\varphi' - (r + \lambda^O)\varphi' - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(\varphi')^2}{\varphi''}, \tag{4.13}$$

for all $y \in \mathcal{C}_1$. Where we have assumed that $\varphi'' > 0$. If short-selling is not allowed, we also require that $\varphi' < 0$. We apply the *Legendre transform* to linearise this non-linear 2nd order ODE. See, e.g., Choulli & Hurd (2001), Jonsson & Sircar (2002), Milevsky et al. (2006) and X. Liang et al. (2014). Define

$$H(z) = \inf_{y>0}\{\varphi(y) + zy\},$$

where $z \geq 0$ is the dual variable to y. The value of y where this optimum is attained is denoted by $f(z)$, so we have

$$f(z) = \sup\{y > 0 \mid \varphi(y) \leq H(z) - zy\}.$$

The relationship between $H(z)$ and $f(z)$ is $f(z) = H'(z)$. We can choose either $f(z)$ or $H(z)$ as the dual function of $\varphi(y)$. In this chapter, we use $f(z)$. Moreover, we have

$$\varphi'(y) = -z, \qquad H(z) = \varphi(f) + zf, \qquad f(z) = y. \tag{4.14}$$

Differentiating (4.14) with respect to y, we have

$$\varphi'(y) = -z, \qquad \varphi''(y) = -\frac{1}{f'(z)} = -\frac{1}{H''(z)}.$$

Since we assume $\varphi''(y) > 0$, we require that $f'(z) < 0$. Substituting into (4.13), we obtain

$$0 = -(\rho + 2\lambda^S)(H - zy) - (r + \lambda^S)zy + (r + \lambda^O)z + \frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 z^2 H''.$$

Differentiating the above equation with respect to z and using $f(z) = H'(z)$, we get

$$\frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 z^2 f'' - \Big(r - \rho - \lambda^S - \Big(\frac{\mu - r}{\sigma}\Big)^2\Big) z f' - (r + \lambda^S) f = -(r + \lambda^O). \quad (4.15)$$

We first solve the homogeneous equation associated to (4.15) and obtain

$$f_{cf}(z) = C_1 z^{\alpha_1} + C_2 z^{\alpha_2},$$

where $\alpha_1 < -1 < 0 < \alpha_2$ are two roots of the equation

$$\frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 \alpha^2 - \left(r - \rho - \lambda^S - \frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2\right)\alpha - (r + \lambda^S) = 0.$$

It should be noticed that z does not take the value 0 when $C_1 \neq 0$, otherwise z^{α_1} is undefined. The particular solution to (4.15) is

$$f_p(z) = \frac{r + \lambda^O}{r + \lambda^S}.$$

Therefore, the solution to (4.15) is given by

$$f(z) = C_1 z^{\alpha_1} + C_2 z^{\alpha_2} + \frac{r + \lambda^O}{r + \lambda^S},$$

where C_1 and C_2 are constants to be determined. Moreover, C_1 and C_2 do not equal to zero at the same time, otherwise function $f(z)$ is not invertible. The corresponding functions H and φ are respectively

$$H(z) = \frac{1}{\alpha_1 + 1} C_1 z^{\alpha_1 + 1} + \frac{1}{\alpha_2 + 1} C_2 z^{\alpha_2 + 1} + \frac{r + \lambda^O}{r + \lambda^S} z, \quad (4.16)$$
$$\varphi(f(z)) = -\frac{\alpha_1}{\alpha_1 + 1} C_1 z^{\alpha_1 + 1} - \frac{\alpha_2}{\alpha_2 + 1} C_2 z^{\alpha_2 + 1}.$$

The optimal controls π^* and $\tilde{\pi}^*$ can now be written as

$$\pi^*(f(z)) = -\frac{\mu - r}{\sigma^2} z f'(z) = -\frac{\mu - r}{\sigma^2}\Big(C_1 \alpha_1 z^{\alpha_1} + C_2 \alpha_2 z^{\alpha_2}\Big), \quad (4.17)$$
$$\tilde{\pi}^*(t, f(z)) = \pi^*(f(z)) L(t) = -\frac{\mu - r}{\sigma^2}\Big(C_1 \alpha_1 z^{\alpha_1} + C_2 \alpha_2 z^{\alpha_2}\Big)\frac{n\beta}{r + \lambda^O} e^{-\lambda^S t}.$$

Notice that, since $z \geq 0$ and we require $f'(z) < 0$, we have $\pi^* \geq 0$ and $\tilde{\pi}^* \geq 0$. The optimal investment in stock $\tilde{\pi}^*$ is always non-negative within region $\mathcal{C}_1$ even in the situation without restriction on risky investment.

4.3.3 Explicit solution

In this section, we explicitly solve the value function $\varphi(y)$ and the free boundary $\tilde{y}$ when the buy-out cost is more expensive than the technical provisions (i.e., $\lambda^S > \lambda^O$).

Case 1

Proposition 4.3.1. *Assume that* $\lambda^S > \lambda^O$, $\gamma > 2(\lambda^S - \lambda^O)$ *and* $\lambda^O \geq \frac{(1-\alpha_2)r+(1+\alpha_2)\lambda^S}{2\alpha_2}$.

(i) If there is no restriction on the risky investment, then $\mathcal{C}_1 = (\tilde{y}, \hat{y})$ *and* $\mathcal{C} = (\tilde{y}, 1) \cup (1, +\infty)$ *with*

$$\tilde{y} = 1 - \frac{2\alpha_2}{\alpha_2 - 1}\frac{\lambda^S - \lambda^O}{r + \lambda^S}. \tag{4.18}$$

The optimal stopping time is $\tau^* = \inf\{t \geq 0 : Y_t^* \notin \mathcal{C}\}$. *The value function is*

$$\varphi(y) = \begin{cases} g(y), & \text{on} \quad y \in [0, \tilde{y}], \\ \frac{\alpha_2}{\alpha_2+1}(-C_2)^{-\frac{1}{\alpha_2}}(\hat{y} - y)^{1+\frac{1}{\alpha_2}}, & \text{on} \quad y \in (\tilde{y}, \hat{y}), \\ 0, & \text{on} \quad y \in [\hat{y}, +\infty), \end{cases}$$

where

$$C_2 = -\frac{1}{4}\frac{\alpha_2 + 1}{\alpha_2}\left(\frac{n\beta}{r + \lambda^O}\right)^{-1-\alpha_2}\left(\frac{4\alpha_2}{\alpha_2 - 1}\frac{\lambda^S - \lambda^O}{r + \lambda^S}\right)^{1-\alpha_2}. \tag{4.19}$$

Within region $\mathcal{C}_1$*, the optimal controls are*

$$\pi^*(t) = \alpha_2\frac{\mu - r}{\sigma^2}(\hat{y} - Y_t^*), \qquad \tilde{\pi}^*(t) = \alpha_2\frac{\mu - r}{\sigma^2}\left(I(t) - X_t^*\right). \tag{4.20}$$

(ii) If short-selling is not allowed, then $\mathcal{C}_1 = (\tilde{y}, \hat{y})$ *and* $\mathcal{C} = (\tilde{y}, 1)$ *with* $\tilde{y}$ *given in* (4.18). *The optimal stopping time is* $\tau^* = \inf\{t \geq 0 : Y_t^* \notin \mathcal{C}\}$. *The value function is*

$$\varphi(y) = \begin{cases} g(y), & \text{on} \quad y \in [0, \tilde{y}] \cup [1, +\infty), \\ \frac{\alpha_2}{\alpha_2+1}(-C_2)^{-\frac{1}{\alpha_2}}(\hat{y} - y)^{1+\frac{1}{\alpha_2}}, & \text{on} \quad y \in (\tilde{y}, \hat{y}), \\ 0, & \text{on} \quad y \in [\hat{y}, 1], \end{cases}$$

where C_2 *is given in* (4.19). *Within region* $\mathcal{C}_1$*, the optimal controls are given in* (4.20).

See Appendix 4.A for the proof.

Figure 4.3.1 describes the value function $\varphi(y)$ and function $g(y)$. We see that, on $[0, \tilde{y}]$, $\varphi(y) = g(y)$ and it is optimal to wind up the scheme immediately. Within $\mathcal{C}_1 = (\tilde{y}, \hat{y})$, $\varphi(y)$ is strictly convex and the optimal investment in stock $\tilde{\pi}^*(t)$ is proportional to the unfunded liability (i.e., the difference between the technical provisions $I(t)$ and the scheme wealth X_t). In other words, the optimal investment strategy is a linear function of the unfunded liability of the scheme. A similar investment strategy can be found in Josa-Fombellida & Rincón-Zapatero (2001), where the aim is to minimize a DB scheme's solvency and contribution rate risks on an infinite time horizon. However, one can not infer the optimal weight invested in the stock from the analytical solution (4.20). It is straightforward to see that the higher the unfunded liability, the more is invested in the stock. Since α_2 depends on r, μ and σ, the impact of the risk-free interest rate r and the Sharp ratio $\frac{\mu - r}{\sigma}$ on the optimal investment strategy $\tilde{\pi}^*$ is unclear. We will numerically investigate the impact of model parameters on the threshold value $\tilde{y}$ in Section 4.4.

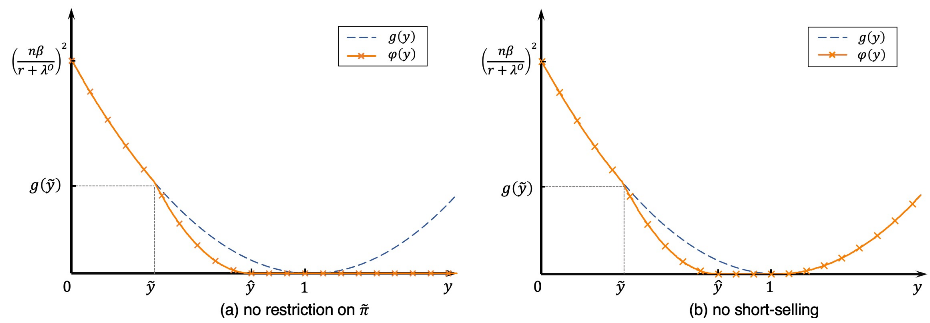

Figure 4.3.1: The value function in Case 1 when (a) there is no restriction on risky investment (b) short-selling is not allowed

Substituting $\tilde{\pi}^*(t)$ into (4.3), for all $0 \le t \le \tau^*$, we obtain

$$Y_t^* = \hat{y} + (y - \hat{y})exp\left\{\left(r + \lambda^S - \alpha_2\left(\frac{\mu - r}{\sigma}\right)^2 - \frac{1}{2}\alpha_2^2\left(\frac{\mu - r}{\sigma}\right)^2\right)t - \alpha_2\frac{\mu - r}{\sigma}B_t\right\}.$$

When $y \in (\tilde{y}, \hat{y})$, the funding level Y_t^* is always lower than $\hat{y}$ and can never reach 100%. Thus, the buy-out is purchased the first time when the funding level touches the threshold value $\tilde{y}$. Moreover, the optimal stopping time is $\tau^* = \inf\{t \ge 0 : Y_t^* = \tilde{y}\}$. The lower is $\tilde{y}$, the later the sponsor purchases the buy-out. While, the closer is $\tilde{y}$ to $\hat{y}$, the earlier the wind up happens. According to (4.25), we have $\frac{\partial \tilde{y}}{\partial \lambda^O} > 0$, $\frac{\partial^2 \tilde{y}}{\partial \lambda^{O2}} = 0$ and $\frac{\partial \tilde{y}}{\partial \lambda^S} < 0$. Thus, the higher the objective force of mortality λ^O (or the lower the subjective force of mortality λ^S), the higher the threshold value $\tilde{y}$. It implies that the cheaper the buy-out, the narrower the continuation region and the earlier the sponsor decides to wind up the scheme.

When $y \in [\hat{y}, 1)$, the sponsor can invest all the scheme wealth into the bond and wind up the scheme when the funding level increases to 100%. If the investment strategy $\tilde{\pi}^*(t)$ is unconstrained and the initial funding level is above 100%, the sponsor can short sell the stock and wind up the scheme when the funding level falls to 100%. While, if short-selling is not allowed, the sponsor winds up the scheme immediately if the initial funding level is above 100%. Since investing in the financial market is expected to increase the funding level and $(I(t) - X_t)^2$.

Case 2

Let us give the following conditions:

$$\gamma > 2(\lambda^S - \lambda^O), \quad \lambda^O < \frac{(1-\alpha_2)r + (1+\alpha_2)\lambda^S}{2\alpha_2}, \tag{4.21}$$

$$\gamma \le 2(\lambda^S - \lambda^O). \tag{4.22}$$

Proposition 4.3.2. *Assume that $\lambda^S > \lambda^O$, and either* (4.21) *or* (4.22) *is satisfied.*

(i) If there is no restriction on the risky investment, then $\mathcal{C}_1 = (0, \hat{y})$ and $\mathcal{C} = (0,1) \cup (1, +\infty)$. The optimal stopping time is $\tau^* = \inf\{t \ge 0 : Y_t^* \notin \mathcal{C}\}$. The value function is

$$\varphi(y) = \begin{cases} \frac{\alpha_2}{\alpha_2+1}(-C_2)^{-\frac{1}{\alpha_2}}(\hat{y} - y)^{1+\frac{1}{\alpha_2}}, & \text{on} \quad y \in [0, \hat{y}), \\ 0, & \text{on} \quad y \in [\hat{y}, +\infty), \end{cases}$$

where

$$C_2 = -\left(\frac{\alpha_2 + 1}{\alpha_2}\right)^{-\alpha_2} \left(\frac{n\beta}{r + \lambda^O}\right)^{-2\alpha_2} \hat{y}^{1+\alpha_2}. \tag{4.23}$$

Within region $\mathcal{C}_1$, the optimal controls are

$$\pi^*(t) = \alpha_2 \frac{\mu - r}{\sigma^2}(\hat{y} - Y_t^*), \qquad \tilde{\pi}^*(t) = \alpha_2 \frac{\mu - r}{\sigma^2}\left(I(t) - X_t^*\right). \tag{4.24}$$

(ii) If short-selling is not allowed and $2r \geq \rho$, then $\mathcal{C}_1 = (0, \hat{y})$ and $\mathcal{C} = (0, 1)$. The optimal stopping time is $\tau^* = \inf\{t \geq 0 : Y_t^* \notin \mathcal{C}\}$. The value function is

$$\varphi(y) = \begin{cases} \frac{\alpha_2}{\alpha_2+1}(-C_2)^{-\frac{1}{\alpha_2}}(\hat{y}-y)^{1+\frac{1}{\alpha_2}}, & on \quad y \in [0, \hat{y}), \\ 0, & on \quad y \in [\hat{y}, 1], \\ g(y), & on \quad y \in [1, +\infty), \end{cases}$$

where C_2 is given in (4.23)*. Within region $\mathcal{C}_1$, the optimal controls are given in* (4.24)*.*

Proof. Within region $\mathcal{C}_1$, the value function is given in (4.16). One can obtain the solution using boundary conditions $\varphi(0) = g(0)$ and $\varphi(\hat{y}) = 0$. □

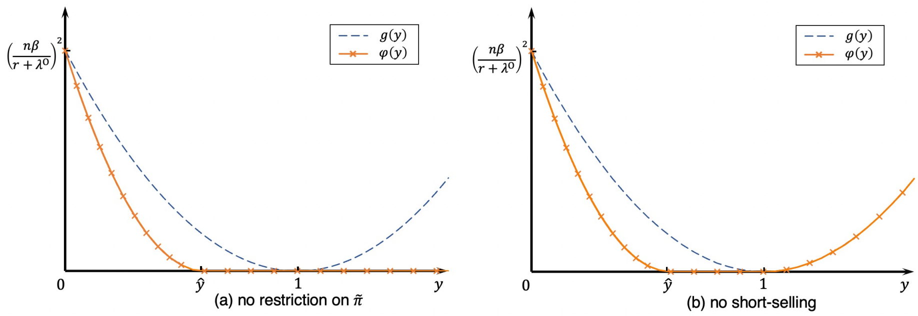

Figure 4.3.2: The value function in Case 2 when (a) there is no restriction on risky investment (b) short-selling is not allowed

In this case, the value function $\varphi(y)$ is C^2 everywhere in $\mathbb{R}_+$ and is described in Figure 4.3.2. Within $\mathcal{C}_1$, the optimal investment strategy $\tilde{\pi}^*(t)$ is again a constant proportion to the unfunded liability $I(t) - X_t$.

4.4 Numerical applications

In this section, we consider Case 1 and numerically examine the impact of the model parameters on the optimal solution. Section 4.4.1 gives a base scenario and calculates the threshold value $\tilde{y}$ of the funding level and the optimal investment strategy $\tilde{\pi}^*(t)$. Then, in Section 4.4.2, we investigate the impact of initial funding level on the winding up time and investment strategy. Section 4.4.3 and 4.4.4 provide sensitivity analyses to investigate the impact of the mortality and market assumptions on the threshold value $\tilde{y}$.

4.4.1 A base scenario

We assume that the subjective discount rate equals the risk-free interest rate, i.e., $r = \rho = 0.03$. The drift and the diffusion of the risky asset are $\mu = 0.06$ and $\sigma = 0.30$, respectively. The choice of those financial market parameters is meant to be representative. At the initial time, there are $n = 100$ members in the scheme. According to DWP (2021), the full new State Pension is £179.60 per week in the UK. Thus, we suppose that the scheme pays $\beta = 9,365$ pounds to each alive member per year continuously. The scheme actuary assumes that the members' remaining lifetime is 30 years, while the insurance company supposes that the remaining life expectancy is 32 years, i.e., $\lambda^S = \frac{1}{30}$ and $\lambda^O = \frac{1}{32}$. We obtain that $\tilde{y} = 0.7687$ and $\hat{y} = 0.9671$. We choose the initial funding level as $y = (\tilde{y} + \hat{y})/2 = 86.79\%$. Using the optimal investment strategy $\tilde{\pi}^*(t)$ presented in (4.20), we run $10,000$ Monte Carlo simulations over a 30-year time horizon. The distribution of the optimal winding up time is shown in Figure 4.4.1. In the figure, 'N' in the x-axis reports the number of cases (1,437 out of 10,000) where the wind up does not happen within 30 years. We find that there is an 85.63% chance that the sponsor winds up the scheme within 30 years. Figure 4.4.2 plots the wealth level and

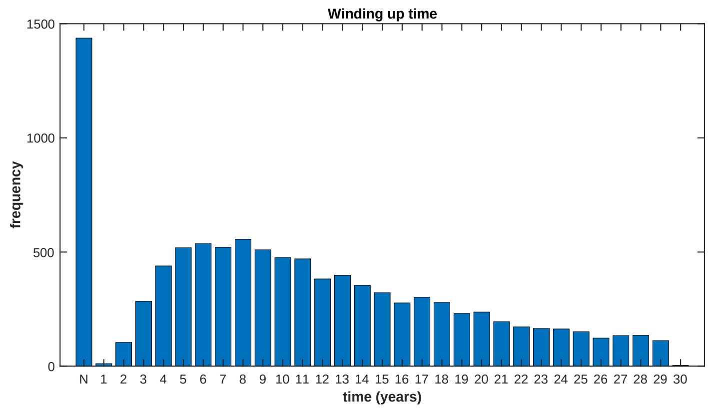

Figure 4.4.1: The distribution of the optimal winding up time

investment strategy of the 1,437 cases where the buy-out is not purchased within 30 years. In the left plot, we see that the scheme's technical provisions $I(t)$ and buy-out cost $L(t)$ are decreasing over time. It is because the number of surviving members drops continuously. Although the sponsor invests in the financial market to gain investment returns, the scheme wealth X_t declines. This is because there are continuous benefit payments to surviving members. In the right plot, we see that both the funding level on a wind up basis $\frac{X_t}{L(t)}$ and the funding level on the technical provisions basis $\frac{X_t}{I(t)}$ have similar trends: they increase slightly in the first 15 years and drop in the late 15 years.

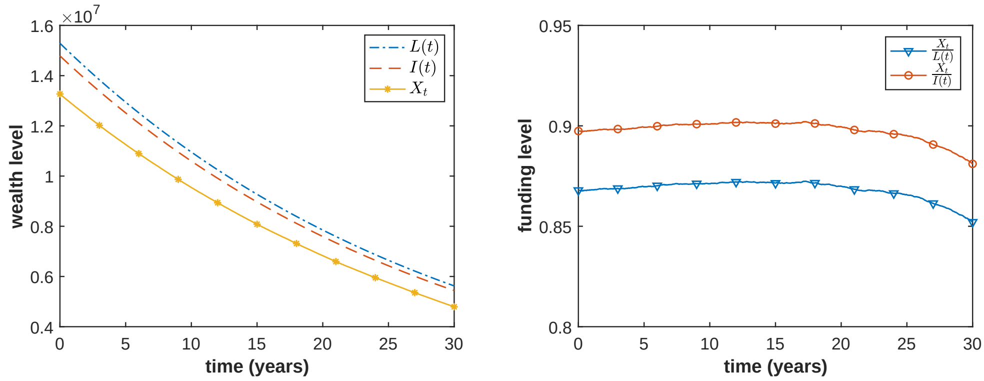

Figure 4.4.2: The dynamics of wealth level and funding level

In the left plot of Figure 4.4.3, we see that both the investment in stock $\tilde{\pi}^*(t)$ and the unfunded liability $I(t) - X_t$ decline over time, and $\tilde{\pi}^*(t)$ drops slower than $I(t) - X_t$. As shown in the right plot, the optimal proportion of the scheme wealth invested in the stock decreases in the first 17 years and increases in the late years. This is because the investment in stock $\tilde{\pi}^*(t)$ drops slower than the scheme wealth X_t in the late years. The optimal stock weight takes values between 0.05 to 0.065, implying that the sponsor invests most of the scheme wealth (over 90%) in the bond.

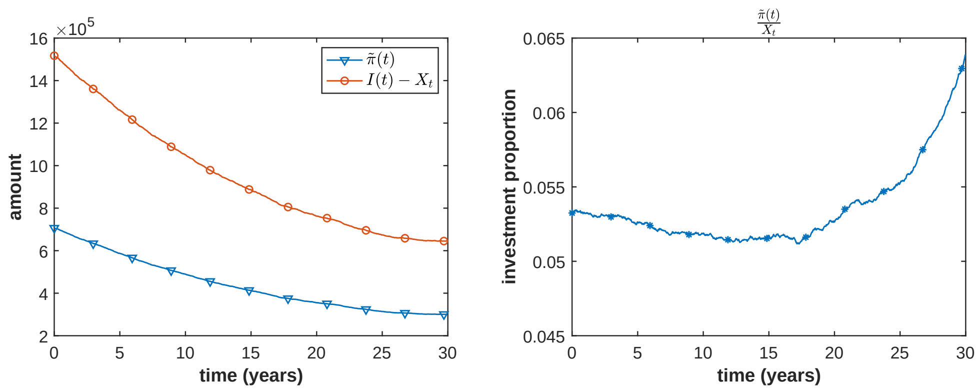

Figure 4.4.3: The dynamics of investment amount and investment proportion

4.4.2 Initial funding level

By varying the initial funding level y, we observe the impact of y on the winding up behavior. Figure 4.4.4 shows that the closer the initial funding level is to the threshold value $\tilde{y}$, the greater the chance that the buy-out is purchased within 30 years. We see that, when the initial funding level is 95%, $\tau^* \leq 30$ in only 673 out of 10,000 of the paths. When the initial funding level is 77%, there are only 11 paths where the sponsor does not wind up the scheme within 30 years. Besides, the possibility that the funding level touches $\hat{y}$ is zero.

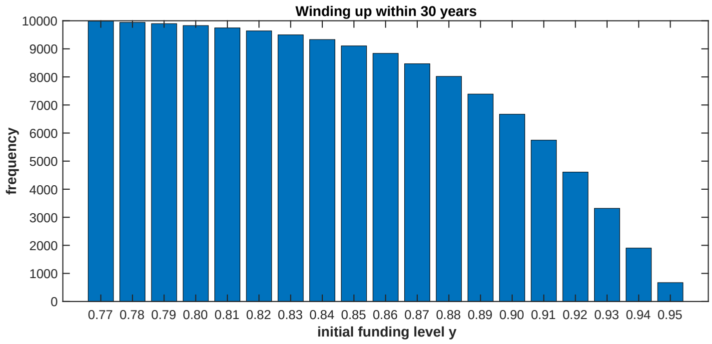

Figure 4.4.4: The impact of the initial funding level on the winding up behavior

Then, we test the impact of the initial funding level y on the optimal investment strategy $\tilde{\pi}^*(t)$ and show our results in Figure 4.4.5. We observe that the higher the initial funding level, the less is invested in the stock. It is because the lower the initial funding level, the larger the unfunded liability. Thus, the sponsor invests more in the stock to gain investment return and reduce the unfunded liability. Also, the higher the initial funding level, the smaller the proportion of the scheme wealth is invested in the stock.

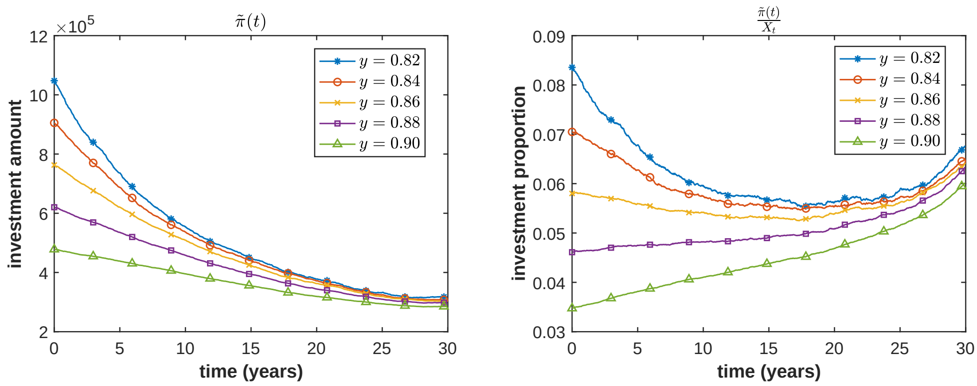

Figure 4.4.5: The impact of initial funding level on the investment strategy

4.4.3 Mortality assumption

Now, we examine the impact of mortality assumptions on the buy-out purchasing strategy by varying the values of subjective life expectancy $\frac{1}{\lambda^S}$ and objective life expectancy $\frac{1}{\lambda^O}$. In

Figure 4.4.6, we present two scenarios: (a) we suppose the subjective life expectancy goes from 15 to 50 years, and objective life expectancy is always two years more than the subjective life expectancy, i.e., $\frac{1}{\lambda^S} = \frac{1}{\lambda^O} - 2$; (b) the subjective life expectancy is fixed to 30 years, while the objective life expectancy changes from 30 to 35 years.

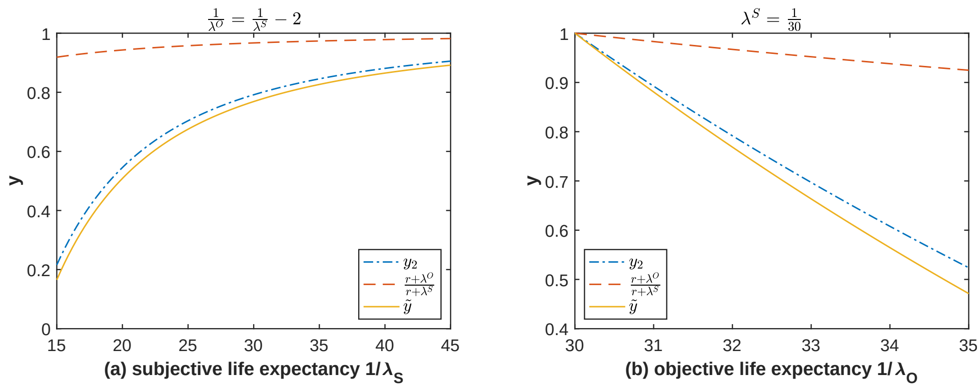

Figure 4.4.6: The impact of λ on the threshold value $\tilde{y}$

The results are consistent with our discussion in Section 4.3.3 that $\tilde{y}$ decreases when λ^S increases (λ^O decreases). Figure 4.4.6 (a) shows that $\tilde{y}$ is an increasing function of the subjective life expectancy $\left(\frac{1}{\lambda^S}\right)$. The intuition behind this is that if the scheme members are younger and expected to live longer, the sponsor is more concerned with the solvency risk and would like to purchase the buy-out earlier. As a result, $\tilde{y}$ is higher and the continuation region is narrower when the expected remaining life span is longer. For smaller values of the subjective life expectancy, we observe that the threshold value $\tilde{y}$ increases more rapidly as the expected remaining lifetime increases. It is because that when the expected remaining lifetime increases, the ratio $L(t)/I(t) = \frac{r+\lambda^S}{r+\lambda^O}$ decreases and the buy-out cost $L(t)$ is closer to the technical provisions $I(t)$. Thus, the buy-out becomes more attractive, the sponsor prefers to wind up the scheme earlier and, $\tilde{y}$ increases more sharply.

Figure 4.4.6 (b) indicates that $\tilde{y}$ declines as the objective life expectancy $\left(\frac{1}{\lambda^O}\right)$ increases. The higher the objective life expectancy, the lower the objective force of mortality λ^O and the higher the buy-out cost. Intuitively, if λ^S is much larger than λ^O, then the buy-out cost is much higher than what the scheme sponsor would expect, and the buy-out is not attractive. Meanwhile, the sponsor finds the financial market more attractive and decides to invest in the financial market for a longer time. If λ^O is slightly lower than λ^S, then the insurance market is more attractive since the buy-out cost is cheaper and closer to the price that the sponsor expects. Immediate winding up is always optimal, if the subjective and objective forces of mortality coincide. It is because that the buy-out cost equals the present value of future benefit payments, and winding up enables the sponsor to transfer all the scheme's risks to the insurance company.

4.4.4 Financial market condition

Next, we investigate the impact of the subjective discount rate ρ and market parameters r, μ and σ on the winding up strategy. Figure 4.4.7 shows how the changes in risk-free interest rate r and subjective discount rate ρ affect the threshold $\tilde{y}$. We observe that $\tilde{y}$ increases (decreases) as the risk-free interest rate r increases (the subjective discount rate ρ decreases). On the one hand, the higher the risk-free interest rate r, the cheaper the buy-out cost $L(t)$, and the more attractive is the buy-out. Besides, the Sharp ratio of the risky asset is low when r is high. This makes the stock less attractive. Thus, the sponsor would wind up the scheme earlier and $\tilde{y}$ is larger when the risk-free interest rate is higher. On the other hand, the higher the subjective discount rate ρ, the more the scheme sponsor values the present. Thus, the sponsor defers the buy-out purchasing, and $\tilde{y}$ is smaller.

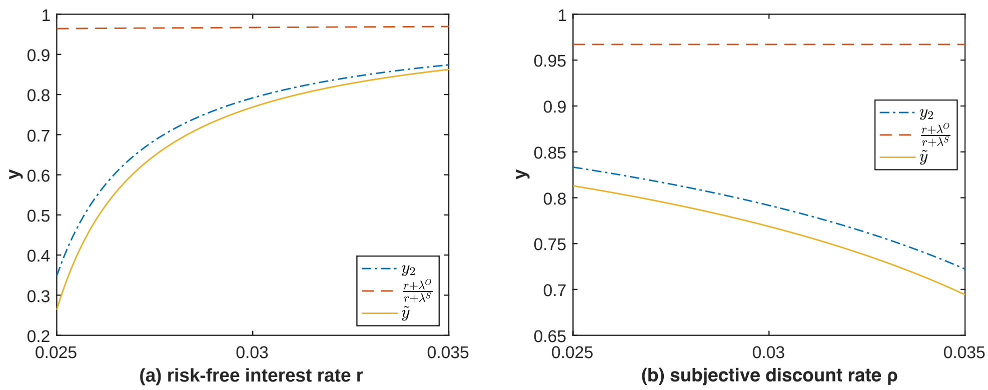

Figure 4.4.7: The impact of the risk-free interest rate r and subjective discount rate ρ on the threshold value $\tilde{y}$

In Figure 4.4.8, we plot the values of threshold $\tilde{y}$ when varying the drift μ and diffusion σ of the risky asset. We observe that the higher the μ and the lower the σ, the lower the $\tilde{y}$ and the wider the continuation region. The convention is that an increase in μ (σ) increases the Sharp ratio $\frac{\mu-r}{\sigma}$, thus making the stock more attractive. Therefore, if the Sharp ratio is high, the sponsor finds the financial market more attractive than the insurance market, and would defer the buy-out purchasing for a longer time.

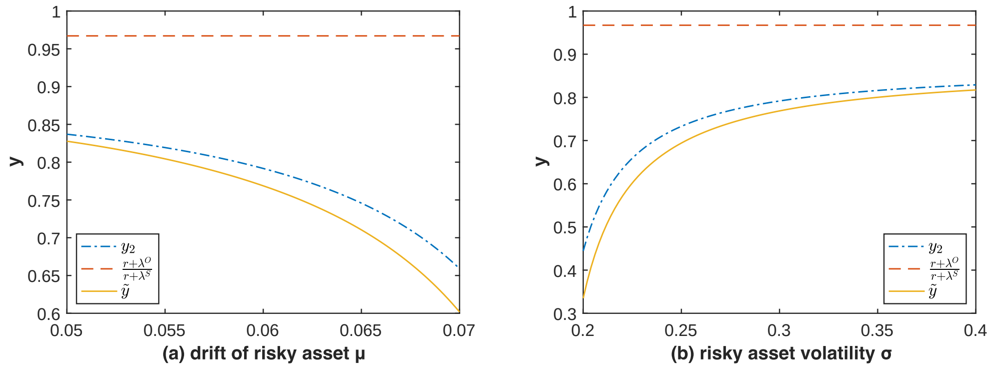

Figure 4.4.8: The impact of the risk-free interest rate r and subjective discount rate ρ on the threshold value $\tilde{y}$

4.5 Conclusion

In this chapter, we have studied the problem faced by a DB scheme sponsor who plans to wind up the scheme via insurance buy-out and has to decide the investment strategy and the buy-out purchasing time. We suppose that the scheme is closed to new entrants with all its members being pensioners. The sponsor's objective is to minimize the expected quadratic deviation of the terminal wealth from the buy-out cost. The problem is formulated as a combined stochastic control and optimal stopping time problem. We provide a verification theorem that characterizes the value function of the combined problem, and the explicit solution is obtained by solving the corresponding variational HJB inequality.

Our analyses show that an immediate-wind-up strategy is optimal when the initial funding level is above 100% if short-selling is not allowed. When the investment strategy is unconstrained, and the initial scheme wealth is sufficient to cover the technical provisions, it is optimal to wind up the scheme when the scheme wealth equals the buy-out cost. While, if the scheme wealth is initially lower than the technical provisions, it is optimal to purchase the buy-out when the funding level touches a threshold under specific financial and insurance markets conditions. Our numerical results show that the sponsor tends to wind up the scheme earlier when the members are younger. The cheaper the buy-out cost, the earlier the sponsor winds up the scheme. In addition, the sponsor would invest in the financial market for a longer time and wind up the scheme later if the Sharp ratio or the subjective discount rate is high. Moreover, before winding up the scheme, the optimal investment in the stock turns out to be a linear function of the unfunded liability.

Further research may include generalizing the model to incorporate stochastic interest rate and stochastic force of mortality: the constant risk-free interest rate and force of mortality limit the analyses to address practical issues associated with the financial and insurance markets. Another extension would be to introduce a more complicated member structure. Rather than pensioners only, one can consider a DB scheme that has both active and pensioner members.

Appendices

Appendix 4.A: Proof of Proposition 4.3.1

Proof. (i) When $\lambda^S > \lambda^O$, Remark 4.2.2 and 4.3.1 show that $\phi(y) = 0$ on $[\hat{y}, +\infty)$. When $\gamma > 2(\lambda^S - \lambda^O)$, we have deduced that $\mathcal{C}_1 = (\tilde{y}, \hat{y})$ and $\mathcal{C} = (\tilde{y}, 1) \cup (1, +\infty)$. On $[\hat{y}, +\infty)$, the value function equals zero. Within region $\mathcal{C}_1 = (\tilde{y}, \hat{y})$, the value function is given in (4.16) with the free boundary $\tilde{y}$ and constants C_1 and C_2 to be determined.

To solve for the values of $\tilde{y}$, C_1 and C_2, we apply the *smooth-fitting principle*. In addition, we require a boundary condition at $y = \hat{y}$. Two possible boundary conditions are $\varphi(\hat{y}) = 0$ and $\pi(\hat{y}) = 0$.

(a) *Boundary condition at $y = \hat{y}$.* Since the value function equals zero on $[\hat{y}, +\infty)$, one reasonable boundary condition is $\varphi(\hat{y}) = 0$. Define by $\hat{z} := \{z \geq 0 : f(z) = \hat{y}\}$, we have $\varphi(f(\hat{z})) = 0$ and $f(\hat{z}) = \hat{y}$. Using (4.16), we get

$$\begin{cases} C_1 \hat{z}^{\alpha_1} + C_2 \hat{z}^{\alpha_2} + \hat{y} = \hat{y}, \\ -\frac{\alpha_1}{\alpha_1+1} C_1 \hat{z}^{\alpha_1+1} - \frac{\alpha_2}{\alpha_2+1} C_2 \hat{z}^{\alpha_2+1} = 0. \end{cases}$$

It can be shown that $\alpha_2 > 1$ when $\gamma > 0$. Since $\hat{z} \geq 0$, $\alpha_1 < -1$, $\alpha_2 > 1$ and C_1 and C_2 do not equal to zero at the same time, we obtain that $C_1 = 0$ and $\hat{z} = 0$.

When $y \in [\hat{y}, 1)$, Remark 4.2.2 shows that the sponsor can invest the total scheme wealth in the bond and wind up the scheme at the time when the funding level increases to 100%. Thus, another possible boundary condition is $\pi(f(\hat{z})) = 0$. Using (4.17), we get

$$\begin{cases} C_1 \hat{z}^{\alpha_1} + C_2 \hat{z}^{\alpha_2} + \hat{y} = \hat{y}, \\ -\frac{\mu-r}{\sigma^2} \Big(C_1 \alpha_1 \hat{z}^{\alpha_1} + C_2 \alpha_2 \hat{z}^{\alpha_2} \Big) = 0. \end{cases}$$

Again, we obtain that $C_1 = 0$ and $\hat{z} = 0$. Now, we have

$$f(z) = C_2 z^{\alpha_2} + \frac{r + \lambda^O}{r + \lambda^S}, \qquad \varphi(f(z)) = -\frac{\alpha_2}{\alpha_2 + 1} C_2 z^{\alpha_2+1},$$

where C_2 is a negative constant such that φ is convex.

(b) *Value-matching and smooth-fitting conditions.* We require that the free boundary $\tilde{y}$ is non-negative and is smaller than y_2.

Define by $\tilde{z} := \{z \geq 0 : f(z) = \tilde{y}\}$, the value-matching and smooth-fitting conditions give

$$\begin{cases} C_2 \tilde{z}^{\alpha_2} + \hat{y} = \tilde{y}, \\ -\frac{\alpha_2}{\alpha_2+1} C_2 \tilde{z}^{\alpha_2+1} = \left(\frac{n\beta}{r+\lambda^O}\right)^2 (\tilde{y}-1)^2, \\ -\tilde{z} = 2\left(\frac{n\beta}{r+\lambda^O}\right)^2 (\tilde{y}-1). \end{cases}$$

The solution is

$$\begin{cases} \tilde{z} = \frac{4\alpha_2}{\alpha_2-1} \frac{\lambda^S-\lambda^O}{r+\lambda^S} \left(\frac{n\beta}{r+\lambda^O}\right)^2 > 0, \\ \tilde{y} = 1 - \frac{2\alpha_2}{\alpha_2-1} \frac{\lambda^S-\lambda^O}{r+\lambda^S}, \\ C_2 = -\frac{1}{4} \frac{\alpha_2+1}{\alpha_2} \left(\frac{n\beta}{r+\lambda^O}\right)^{-1-\alpha_2} \left(\frac{4\alpha_2}{\alpha_2-1} \frac{\lambda^S-\lambda^O}{r+\lambda^S}\right)^{1-\alpha_2} < 0. \end{cases} \tag{4.25}$$

After some algebra, one can prove that $\tilde{y} < y_2$. When $\lambda^O \geq \frac{(1-\alpha_2)r+(1+\alpha_2)\lambda^S}{2\alpha_2}$, we have $\tilde{y} \geq 0$. Otherwise, we set $\tilde{y} = 0$. It can be shown that $f : (0, \tilde{z}) \to (\tilde{y}, \hat{y})$ is monotonically decreasing and is invertible. In addition, we find that $\varphi''(y) > 0$ for all $y \in (\tilde{y}, \hat{y})$. Thus, the convexity of $\varphi(y)$ within the region $\mathcal{C}_1$ is verified.

(c) Within region $\mathcal{C}_1$, the optimal controls are

$$\begin{aligned} \pi^*(t) =& -\frac{\mu-r}{\sigma^2} z f'(z) = \alpha_2 \frac{\mu-r}{\sigma^2}(\hat{y} - Y_t^*), \\ \tilde{\pi}^*(t) =& \pi^*(t) L(t) = \alpha_2 \frac{\mu-r}{\sigma^2}\left(I(t) - X_t^*\right), \end{aligned}$$

where we used the relationships $I(t) = \hat{y} L(t)$ and $X_t = Y_t L(t)$. It is easy to show that $\pi^* \varphi'$ is bounded on $(\tilde{y}, \hat{y})$ and (4.9) is satisfied.

(ii) The proof is similar to statement (i) and is omitted here. □

Appendix 4.B: Extended Applications

Extension to the case with wealth constraint

Suppose that the sponsor is required to purchase the buy-out and wind up the scheme when the funding level on the technical provisions basis $\frac{X_t}{I(t)}$ touches $l \in (0, 1)$ (where $I(t)$ and X_t are given in (4.1) and (4.2)). Consider the funding level Y_t given in (4.3) and denote by $\underline{y} = \frac{r+\lambda^S}{r+\lambda^O} l$. Then, for $y \geq \underline{y}$, the optimization problem is given by (4.4) with $\tau_0 = \inf\{t \geq 0 : Y_t \leq \underline{y}\}$. In addition, we assume that $\lambda^S > \lambda^O$, and there is no restriction on the investment strategy.

When $\gamma > \frac{2(\lambda^S-\lambda^O)}{1-\underline{y}}$ and $\tilde{y} = 1-\frac{2\alpha_2}{\alpha_2-1}\frac{\lambda^S-\lambda^O}{r+\lambda^S} > \underline{y}$, one can obtain that $\mathcal{C} = (\tilde{y}, 1) \cup (1, +\infty)$ and $\mathcal{C}_1 = (\tilde{y}, \hat{y})$. Within region $\mathcal{C}_1$, the value function is

$$\varphi(y) = \frac{\alpha_2}{\alpha_2+1}(-C_2)^{-\frac{1}{\alpha_2}}(\hat{y}-y)^{1+\frac{1}{\alpha_2}},$$

where C_2 is given in (4.19). The optimal controls are given in (4.20).

Otherwise, one obtains that $\mathcal{C} = (\underline{y}, 1) \cup (1, +\infty)$, $\mathcal{C}_1 = (\underline{y}, \hat{y})$ and C_2 is given in (4.23).

Extension to the case with deficit reduction payments

When the scheme's funding level on the technical provisions basis is lower than 100%, the sponsor may be required by the regulator to pay deficit reduction payments in order to return the scheme to full funding. This section considers a special case where the sponsor continuously pays a fixed fraction of the unfunded liability into the scheme before the winding up time. That is, we suppose that the instantaneous deficit reduction payment is

$$C(t) = \delta(I(t) - X_t),$$

where $\delta > 0$. Hence, the scheme wealth evolves as

$$\begin{aligned}\mathrm{d}X_t &= (rX_t + \tilde{\pi}(t)(\mu - r) + C(t) - P(t))\,\mathrm{d}t + \tilde{\pi}(t)\sigma\mathrm{d}B_t,\\ &= \big((r-\delta)X_t + \tilde{\pi}(t)(\mu-r) - (r+\lambda^S-\delta)I(t)\big)\,\mathrm{d}t + \tilde{\pi}(t)\sigma\mathrm{d}B_t.\end{aligned}$$

The funding level follows

$$\mathrm{d}Y_t = \Big((r+\lambda^S-\delta)(Y_t-\hat{y}) + \pi(t)(\mu-r)\Big)\mathrm{d}t + \pi(t)\sigma\mathrm{d}B_t,$$

where $\tilde{\pi}(t) = \frac{\pi(t)}{L(t)}$. When $\delta > r + \lambda^S$, one can show that the value function equals zero for all $y \in \mathbb{R}_+$.

When $\delta < r + \lambda^S$, one finds that the value function equals zero on $[\hat{y}, +\infty)$. Denote by

$$\gamma = 2r - \rho - 2\delta - \big(\frac{\mu-r}{\sigma}\big)^2, \qquad \tilde{y} = 1 - \frac{2\alpha_2}{\alpha_2-1}\frac{\lambda^S-\lambda^O}{r+\lambda^S},$$

where $\alpha_2 > 1$ is the positive solution to

$$\frac{1}{2}\Big(\frac{\mu-r}{\sigma}\Big)^2\alpha^2 - \left(r-\rho-\lambda^S-\delta-\frac{1}{2}\Big(\frac{\mu-r}{\sigma}\Big)^2\right)\alpha - (r+\lambda^S-\delta) = 0.$$

If $\gamma > 2(r+\lambda^S-\delta)\frac{\lambda^S-\lambda^O}{r+\lambda^S}$ and $\tilde{y} > 0$, then $\mathcal{C} = (\tilde{y}, 1) \cup (1, +\infty)$ and $\mathcal{C}_1 = (\tilde{y}, \hat{y})$. Within region $\mathcal{C}_1$, the value function is

$$\varphi(y) = \frac{\alpha_2}{\alpha_2+1}(-C_2)^{-\frac{1}{\alpha_2}}(\hat{y}-y)^{1+\frac{1}{\alpha_2}},$$

where C_2 is given in (4.19).

Otherwise, one obtains that $\mathcal{C} = (0, 1) \cup (1, +\infty)$, $\mathcal{C}_1 = (0, \hat{y})$ and C_2 is given in (4.23).

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