National debt, Public capital, and Welfare in Developing Economies

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Abstract

The economic effects of public debt remains one of few subjects that evoke controversy among academics and policy makers, despite being a subject of intense scrutiny for prolonged periods. Earlier literature examined national debt in neoclassical growth models, and focused on its effects on steady state equilibrium. The general finding was that debt issuance by the state reduces factor accumulation in the long run. By the fundamental welfare theorem, the competitive equilibrium is pareto-optimal, and thus, state intervention distorts market-driven outcomes and interferes with optimal allocations in equilibrium. Later research contributions established conditions under which the competitive market environment may reach equilibrium that is not optimal. In this case, the issuance of debt by the government is capable of generating pareto improvements over market equilibrium. Thus far, the theoretical neoclassical literature have limited the positive welfare effect of government debt to the perverse case in which the competitive equilibrium is suboptimal. Recent strands in the literature have explored conditions under which public debt may be invested in infrastructure and other forms of public capital in endogenous growth models. These have been much less rigorous, but have shown that contrary to the result from neoclassical theory, debt issuance for investment may increase equilibrium growth rate in a socially planned economy where the state dictates private saving and consumption decisions. But this is a very restrictive requirement. In addition, even though the recent literature addresses an important aspect of government spending, namely that governments issue debt to provide key infrastructure and public services, no existing work (to best knowledge) has examined the welfare effects of such purposeful government borrowing in a decentralized environment.

This dissertation aims to contribute to filling the gap by exploring the effects of debt issuance for investment in public capital using a production structure that is widely used in endogenous growth models, but dispenses with the usual assumptions of a socially

planned economy. In other words, we need not require the government to internalize and/or dictate private consumption and savings behaviour for equilibrium to exist. In doing so, a competitive framework is used to examine the role of government debt in a growing economy where public capital is essential to private sector production. We will see first, that the long-run equilibrium of the planning problem elaborated in endogenous growth theory can be supported as competitive equilibrium outcome with lump-sum transfers by government. However, the growth maximizing capital ratio in the planning problem may not yield efficient allocations. Secondly, we will see that in a simple analytic environment, debt-financed public investment can enhance both private wealth and welfare in general equilibrium. This is a novel finding that compares favourably with the Diamond result where debt may increase welfare in the dynamically inefficient equilibrium, but always reduces capital labour ratio in the long-run. Similarly, as opposed to Blanchard result where government expenditure does not benefit individual's utility but imposes a debt-service burden and hence reduces capital and consumption at steady state, debt-financed public investment directly affects agents utility by increasing not only the prevailing interest rate, but in general the rate of return to investment in the economy. As no known existing work has explored the welfare effects of government expenditure in public capital formation through issuance of debt, this is the crux of my contribution to the literature in this dissertation. The analytic results generally show that debt-financed public expenditure that increases the public capital stock does not only improve efficiency where the equilibrium is sub-optimal, it can increase economic efficiency and welfare of agents in the long-run. The data for a large section of developing and advanced economies seem to check out the necessary condition for debt-financed public investment to be welfare improving.

I conclude the research with policy discussions on the implications of the analytic and empirical insights on the future of developing country borrowing in the wake of the high debt levels occasioned by the COVID-19 pandemic.

Dedication

To the memory of my late father.

Declaration

I declare that except where explicit references are made to the contribution of others, this dissertation is the output of my own research work, and this has not been submitted for any other degree to the University of Glasgow or any other institution.

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Chapter 1

Introduction

1.1 Motivation

Government debt has remained a relevant subject of enquiry in both theoretical and empirical research since Adam Smith's[111] wealth of nations. Diamond's (1965) seminal contribution rigorously examined the effects of national debt on steady state wealth and consumption in a neoclassical framework. The Diamond model showed that the market equilibrium in a decentralised economy, like the Nash equilibrium in non-cooperative games, can be inefficient. In other words, there can be room for pareto improvement in utility by government distortion to the market driven outcomes. This result is at odds with the first fundamental welfare theorem in economics (Geerolf, 2013)[55]. The key to this result is the infinite existence of the economy and infinite number of agents. Barro (1974)[15] takes up this result for further scrutiny by extending the model of Diamond to an economy with bequest motive. The latter concludes that debt has no effect on net wealth - a conclusion that stands in contrast to the former, where debt is considered to decrease capital labour ratio at steady-state equilibrium. But by allowing for intergenerational linkages in the Hosehold utility structure, Barro (1974) essentially introduced finite number of agents into the overlapping generations (OLG) model. Blanchard (1985)[24] refined the original OLG model of Diamond where the economy is composed of two generations to one with many generations in a continuous time setting. This allows for the effect on steady-state wealth and consumption, of life cycle features such as mortality rates and retirement to be examined. Due to finite life of agents, a debt policy or re-allocation of taxes to finance non-utility bearing government expenditure in the Blanchard model merely improves consumption level of some generations at the expense of others. Thus, outside of the dynamically inefficient case, debt leads to non-pareto improvement in welfare.

The three contributions stand out in shaping present understanding on the effects of public debt at steady-state equilibrium in the competitive growth model with complete markets. Another segment of the literature (most notably Aiyagari and McGrattan, 1998[3]) examines the role of government debt in heterogeneous agent models with incomplete markets. They find that government debt softens the precautionary savings behaviour of households and provides an avenue for individuals to smooth consumption in the face of idiosyncratic shocks to labour income. By raising the level of debt, the government loosens the borrowing constraint on households and improves intertemporal allocations in the face of negative shocks. More recently, several authors have examined the economic effects of government debt in endogenous growth models. Unlike the neoclassical and heterogeneous agent models where public debt enhances present consumption, endogenous growth models suppose that government issues debt to invest in public capital formation (See Aschauer, 2000[7] and Checherita-Westphal, 2014[37]). In Aschauer (2000) and Checcherita-Westphal (2014), government debt to increase the public capital stock, and hence the public to private capital ratio, increases long-run growth rate of the economy if the initial capital ratio is below the optimal level. The key distinction here is that the focus is on the effect of debt on economic growth rate as opposed to welfare which is the subject of attention in the neoclassical models. In addition, the endogenous growth theory generally assume a centralized planning approach, which effectively requires government to dictate private consumption and saving decisions for long term equilibrium to exist.

The manner in which debt is analysed in the latter models leaves much to be desired. Apart from the abstraction from a decentralized market economy, the environment in which debt has a positive effect on economic growth need not arise. The government acting as a social planner that exact taxes to form public capital and dictate savings to keep the capital ratio constant in equilibrium, need not keep an inefficient ratio, to begin with. From the outset, the Social planner, maximising the economic growth rate and knowing the optimal capital ratio, should dictate saving decisions accordingly to attain optimality. This makes debt irrelevant in the planned economy in the sense that the growth rate can be maximised with respect to the tax rate (Greiner, 2012). Thus, while public capital is shown to have significant productivity effects (Aschauer, 2000) and can be more important for output growth in some states (Ascahuer, 1989A[9]), the effects of government debt for public investment needs to be examined in a decentralized environment where private agents freely make their savings and consumption decisions. The shift from the Social Planner's problem to a decentralized equilibrium is motivated by two factors. First, growth maximization by the Social Planner may not yield efficient allocations. This point is clearer by comparing the public to private capital ratio at optimum growth in the planning problem to the decentralized equilibrium solution. In other words, the Planner maximises growth by keeping a relatively higher level of public capital along the balanced growth path. This requires suppressing the marginal productivity of private capital via taxes to meet the competitive loan market condition. This ultimately lowers consumption level for the given amount of aggregate capital at various states along the balanced growth path. Secondly, a decentralised solution allows us to observe how households saving and consumption behaviour reacts to changes in the debt level. Thus, it is possible to analytically trace the welfare effects of changes in debt level via its productivity effects, tax effects and incentives for inter-temporal re-allocations. Thus, in general, while the Planning problem reflects repressionary policies in developing countries (such as China) that induces higher aggregate savings and may enhance growth, it does preclude explicit analysis on savings behaviour driven by the incentive to maintain a balanced marginal utility of consumption between periods. This offers important policy perspectives on government borrowing for public capital formation, since governments play the predominant role of providing and operating infrastructure facilities (e.g., schools, hospitals, railways, ports, roads, bridges, telecommunication networks, water and electricity supply facilities) in developing countries and deficit financing remains a key medium for exercising this role (Chan et al, 2009[36]). These important points are the subject of analysis in this dissertation.

In the first instance, I make a case for the use of the Blanchard model in analysis of debt finance as it allows for one to look at various aspects of debt (debt for consumption purposes, non-utility bearing government expenditure, or with slight modification, debt for public investment), and to focus on welfare effects at steady state. The formulations in the recent literature (Barro, 1990; Aschauer, 2000; Agénor, 2010; and Checherita-Westphal, 2014) suggest that the natural environment for analysis of debt-financed public investments is endogenous growth model environment. The downside of these formulations, however, is that they are generally framed as planning problems, and do not permit explicit analysis on welfare in decentralized equilibrium. To address this, I attempt to decentralize the Aschauer (2000) model in chapter three to allow for separate decision making on public and private capital formation. It will be seen that in the absence of any form of inefficiency and government consumption, equilibrium with endogenously determined growth rate arises when the burden of consumption by households is shared symmetrically between returns to public and private capital. This arises because of the requirement of competitive loan market. But the conditions characterising the equilibrium suggests optimality in the long-run with no potential for the use of debt to improve equilibrium outcomes.

Given the long-run result in the decentralized version of the endogenous growth model, I proceed to introduce the notion of public capital in the Blanchard overlapping generations model using the Aschauer (2000)[7] production function. Therefore, unlike in Blanchard (1985)[24], where government expenditure affects household consumption only through the imposition of taxes to finance it or to service debt accumulated from the expenditure, the presence of public capital as a production input relates government expenditure more directly to household utility. First, government's demand for capital in the closed economy raises the interest rate, and hence returns to private capital. Secondly, since this expenditure is only for public capital formation, it increases individual and aggregate output. In general, this may enhance steady state consumption depending on the relative proportions of public and private capital in the economy. In this case, when public capital formation is financed by debt, the net effect on steady state outcomes depend on two things; the efficiency effect of public capital formation, and the burden of debt service and public capital maintenance due to depreciation. Where the latter two effects dominate the efficiency effect, public capital formation financed by debt may reduce capital and consumption at steady state, otherwise it improves both outcome variables in the long-run. In this regard, I show broadly that for any given equilibrium, there is a limit to the instantaneous addition to debt that can enhance private wealth and welfare. This relates remotely to the notion of debt sustainability in the literature (as conceptualised in Blanchard and Das, 2017[25]; Krugman, 1988[77]; and Eggertsson and Krugman, 2012[49]; and Blanchard et al., 1991[26], for example).

Finally, I take a cursory look at the data with the view to estimating the relative marginal productivities of public and private capital in a large panel of advanced and developing economies. The objective is to check the primary condition under which debt finance for public investment is welfare enhancing. The data broadly reveals sizeable higher marginal productivity for public capital than private capital in the various categories of countries. Perhaps, the surprising finding is that despite observed deficits in the public capital stocks in low income countries relative to advanced economies, marginal productivity for capital remains comparatively lower in the low income countries. That said, the general insight from the data suggests that there is room for local improvement in capital and consumption levels, by use of debt-financed public investment. This view of the data via the lens of the model should be read with caution as it does not provide significant evidence in support of debt-financed public investments. Rather, it merely emphasizes the potential for debt finance to be used in stimulating growth and welfare improvements when invested efficiently in public capital, given the observe productivity differentials.

In summary, the research conducted in this thesis is motivated by the scarcity of research output on the economic impacts of debt-financed public investments. The well known literature, most notably the Diamond model, examined the effects of public debt when issued to raise consumption of present generations. Alternative formulations in Blanchard looks at the effects of debt and tax re-allocations in the presence of non-utility bearing government expenditure. What these models did not explored concerns government expenditure in building infrastructure and other forms of public capital. The effects of this form of expenditure when financed by publicly issued debt is the crux of my contribution in this thesis. I examine the role of government debt for public investment in private wealth creation and on equilibrium consumption level of agents in a competitive economy. In other words, a decentralized environment where private savings and consumption decisions are freely determined is used together with a production structure that potentially yields endogenous growth in the special case. A look to the data readily avails the substance in the analytic results. The obvious implications of the findings for fiscal policy in developing countries are especially important in the wake of the ongoing COVID-19 pandemic.

1.2 Contribution to the literature

The thesis contributes to existing literature on endogenous growth theory and on the welfare effects of public debt when it finances investment in public capital in a number of ways:

- First, the planning problems of Barro (1990), Aschauer (2000), and Checherita-Westphal et al. (2014) is decentralized, and it is shown that there exist a competitive solution to the endogenous growth model with transfers from government to households. This finding suggests that ownership of public capital is not as important as how returns on public capital is utilized. Whether the given stock of public capital is owned by the government or effectively owned by households through public debt, equilibrium requires that some or all of the returns on public capital is funnelled to households as interest on debt or transfers from government. In the absence of inefficiencies, this is a necessary condition for the existence of long-run equilibrium.

- Second, the thesis also shows that given a production system with public capital as an input in private sector production - in particular, given the Aschauer (2000) output function - a long run equilibrium with endogenous growth need not be a default outcome. A steady state equilibrium may arise and aggregate economic growth will be determined by the rate of population growth.

- Third, at steady state equilibrium, debt issuance for public capital formation can increase both private wealth and welfare in the long run. This result, apart from being novel, adds flavour to the Diamond (1965[44]) result, where it is impossible for government debt to increase both welfare and capital labour ratio in equilibrium. In addition, it is shown that conditional on government investment in public capital, dynamic inefficiency need not be a default requirement for debt to enhance welfare at steady state.

- Finally, the data reveals large deficits in capital stocks in low income developing countries compared to the rest of the world. While this finding would have suggested a higher productivity for capital in low income countries, the estimates show that this is only the case if the output-capital price ratios are not accounted for in the computation of marginal productivities. Due to relatively expensive nature of capital in low income countries, the marginal productivity is rather much lower in these economies than in more advanced economies. Nevertheless, the marginal productivity for public capital is larger than private capital across all estimations. Thus, as theoretical model implies, this insights from the data presents an opportunity for debt financing of public capital formation to be welfare improving. That said, whether debt finance is indeed growth or welfare improving in these countries should be subject to further empirical research.

1.3 Overview of chapters

The thesis comprises of five chapters. Chapter one introduces the research and presents its motivation and the key contributions to the literature on government debt. The motivation is explicit on the scarcity of analytic research on the effects of debt-financed public investments on decentralized equilibrium outcomes. Chapter two conducts a survey of the literature on public debt, public capital, and government debt in economic growth models. The review tracks two distinct strands in the literature; the link between public debt and economic growth as well as the relationship between public capital and economic growth. To lay the foundation for analysis in subsequent chapters, I review the classic Blanchard model and comment on its unique equilibrium features, and the role for debt in equilibrium. I elaborate on why admitting public capital to the Blanchard model makes it amenable to analysis on welfare effects of debt-financed public investment. This justifies a shift from the use of endogenous growth model, which is the usual analytic device in the literature for examining debt-financed government spending.

In chapter three, I examine two closely related papers in endogenous growth theory - namely, Aschauer (2000) and Checherita-Westphal et al. (2014), and consider the burden of consumption in equilibrium when households consumption-saving decision is unimpeded. The resulting long-run outcomes shows that the competitive equilibrium with endogenously determined growth leads to optimal allocation of output between public capital, private capital, and consumption. This leaves no obvious room for debt financing of public investment to be welfare improving. In view of this, chapter four introduces public capital to the Blanchard model. But in line with the Aschauer (2000) output function, the abstraction from labour necessitates some simplifications to allow for infinite existence of the economy. With public capital provided by government as an input in private sector production, it is shown that steady state values of output, private wealth and consumption depend on the available supply of the public capital. I then show that under these conditions, the issuance of government debt to raise the available supply of the public capital stock may increase equilibrium private capital and consumption levels. In other words, the positive welfare effect of debt may no longer be limited to the dynamically inefficient equilibrium. I take turns to examine external and domestic debt separately and numerically compute a ten percentage increase in public capital through external and domestic debt. As the focus is on the effects of debt, no attempt is made to conduct alternative fiscal policy analysis that compares debt financing to tax financing. Finally, I use data for a large sample of countries from the IMF Investment and Capital Stocks Dataset, the Penn World Tables, and the Historical Public Debt Database of the IMF to check the necessary condition for debt finance to be welfare enhancing in the long-run.

Chapter five discusses a number of policy-oriented issues related to the findings, and contextualise them in current policy sector discussions on debt distress in developing countries attributed largely to the ongoing COVID-19 pandemic. Finally, I emphasise the need to preserve productive public investments in less developed economies and make recommendations for developing country debt policy.

Chapter 2

Debt, Public capital, Dynamic inefficiency and the Blanchard model

2.1 Introduction

The fiscal and welfare effects of public debt continue to generate substantial academic research and policy interest. Blanchard's (2019)[23] presidential lecture at the American Economic Association reignited policy-oriented discussions on the potential benefits of public debt accumulation in advanced economies, especially in the context of record low interest rates prior to the COVID-19 pandemic. More recently, however, the discussion has been turned on its head following the accumulation of record high debt levels by sovereign states to stimulate recovery from the economic slowdown induced by the pandemic and to protect welfare of citizens. At the height of the pandemic, the World Bank and the International Monetary Fund (IMF) called for government financing, suspended debt service payments by the world's poorest countries, and introduced fast-track concessional loans to bolster economic recovery. At the time, proponents for the use of debt cited the ultra low interest rates in the capital markets which leave positive growth-interest rate differentials for advanced economies (Davies, 2020[42]), while critics cautioned against excessive deficit bias. Wyplosz (2019[114]) argue that dynamic inefficiency is not the norm in the cross country historical evidence, observing that excessive deficit bias often lead to delayed or misguided government reaction when the growth-interest rate differential switches sign. Also, by excluding off-balance sheet liabilities such as pensions and medical care, standard measures of debt underestimates government burden even in low interest rate environment (Rogoff, 2020[102]). But while major advanced economies - in particular the US, UK, and Japan maintained historic low interest rates prior to the pandemic (Blanchard, 2019), developing countries faced relatively high interest rate on government debt. In some countries, nominal growth rates were substantially below interest rates (Kharas and Dooley, 2020[73]). This raised serious concerns on the welfare effects of public debt in developing countries. These have been heightened more recently by many developing countries falling into debt distress. Indeed, these concerns may no longer be limited to developing countries in the wake of recent interest rate hikes by the US Federal Reserve, the Bank of England and in many other countries across the developed and developing world in the universal attempt to fight inflation. In other words, interest rates on government debt may be relatively higher in the foreseeable future than it has been in the last decades. With high interest rates, there would seem to be no room for positive welfare effect of debt in the analytic growth model.

The above suggestion arises from the way the effects of debt has been conceptualised in the literature. While in theory, it is often assumed that debt is issued to raise consumption levels of people presently alive but impose debt service or redemption burden on future populations, empirical research do not generally pay attention to the use of funds, but rather focuses on the relationship between debt-GDP ratio and economic growth rate. But making progress on the effects of public debt, under the current environment of relatively high interest rates, requires that we acknowledge the various uses to which funds are placed, and account for the effects of such uses in long-run growth and welfare analysis. For example, it is widely acknowledged that national governments in developing countries contract loans to financed infrastructure provision (see for example, Gurata et al., 2018[61]; Asquer, 2018[10]; and Estache, 2015[52]). In this case, by providing infrastructure and other forms of public capital, governments use debt finance as a tool to engineer growth and development. The mechanical application of debt finance in this form has seen little research in the literature. On the contrary, research on the effects of national debt on economic growth on the one hand, and the effects of public capital on growth on the other hand have proceeded in parallel, with little focus on the intersection of debt finance and public capital formation in the growth process.

This chapter reviews the literature on the long-run effects of public debt, the debt-growth nexus, and the effects of public capital on economic growth. The review proceeds in parallel on the effects of national debt on economic growth and welfare on the one hand, and the role of public capital in economic growth on the other hand. To stay focused on the main contributions in this thesis, I do not cover a huge section of the literature that have focused on the conditions that support sovereign lending given the unenforceability of sovereign debt as opposed to private debt and the doctrine of sovereign immunity that has characterised international law on sovereign lending and borrowing. Much of the literature in this vein are in the image of Eaton and Gesorvitz (1981)[47] and Bullow and Rogoff (1989)[30]. The earlier proposed reputational preservation for continued access to the capital markets as condition to support international lending, while the latter suggests the necessity of direct sanctions available to creditor country governments to deter repudiation as a requirement for sovereign lending. A recent paper by Bloise, Polemarchakis and Vailakis (2021)[27] places the reputational argument as basis for sustainable lending and borowing in the context of dynamic inefficiency where debt contracts are less expensive risk-sharing instruments. Eaton and Raquel (1995)[46] provides a survey of the earlier literature in this strand.

The review here focuses on the strand of the literature that examines the public finance aspect of government debt and its long-run effects. It provide a flavour of the relevant literature related to the subject of the dissertation, and both contextualises and motivates subsequent analysis. To this end, I review the research on economic effects of public debt and the impacts of public capital on economic growth. While these have been conceived separately in the literature, my intervention in this thesis lies at their intersection. As shown in chapter four, I use the Blanchard (1985)[24] model to tie debt finance to public capital and show that at their intersection arises novel theoretical results on the public finance effects of debt. Thus, as part of the literature review, I elaborate on the basic Blanchard model and its unique steady state equilibrium characteristics. In particular, I comment on why in the presence of population growth, the equilibrium interest rate always exceed the rate of time preference, while not precluding the existence of steady state equilibrium, despite the absence of technical progress in the model. Finally, I note that like in Diamond (1965), dynamic inefficiency remains the key requirements for positive welfare effect of debt in the long-run, but conjecture that under conditions of productive public investments through debt finance, the positive welfare effect of debt may no longer be limited to the dynamically inefficient case. Explicit analysis on this is deferred to chapter four.

2.2 Literature Review

"At moderate levels, debt improves welfare and enhances growth. But high levels can be damaging" (Cecchetti, Mohanty and Zampoli, 2011)[35]. This statement reflects the double-edged sword nature of public debt, but stands in stark contrast to the basic intuition from neoclassical economic theory. In the standard representative agent model, where the long-run equilibrium is pareto optimal, debt generates negative interim growth and thus reduces capital labour ratio in the long run. In this case, consumption level of the representative agent is lowered. However, in the overlapping generations model, where the long-run equilibrium may be dynamically inefficient, debt may improve welfare despite reducing the capital labour ratio. This is the cornerstone result from early theoretical contributions from Diamond (1965)[44] and Blanchard (1985)[24]. Therefore, the general theoretical finding is that public debt has a negative effect on growth at steady state, even if temporarily (Greiner, 2013[58]; and Checherita-Westphal and Rother, 2012[38]). Despite the clear-cut nature of the classic theoretical results, "the relationship between [external] debt and growth remains a subject of intense debate to both policy makers and academics alike" (Clemens, Bhattacharya, and Nguyen; 2003)[89].

The bulk of the recent literature on the economic impacts of public debt have been decidedly empirical. While most of them have identified a negative linear relationship or a non-linear relationship in which debt initially enhances economic growth rate, attains a maximum and begins to affect growth rate negatively, few have focused on identifying a precise channel through which public debt influences growth. Aside the plethora of research on the economic effects of debt, a distinct line of study have focused on estimating the impact of public capital on economic growth in both theoretical[1] and empirical[2] models. In general, the empirical literature have noted a positive effect of public capital accumulation on output, with Aschauer (1989A)[9] contending that the non-military

[1]For theoretical expositions on public capital effect on output or economic growth, see Barro (1990)[16], Aschauer (2000)[7], Checherita-Westphal (2014)[37], and Agenor (2010)[2], among others.

[2]A few of the empirical work on the effect of public capital on output and/or private capital include Kamps (2005)[71], Cavallo and Daude (2011)[34], Santiago et al (2019)[109], and Lowe et al (2019)[84].

stock of public capital was dramatically more important in determining productivity in the United States, and attributed productivity slowdown in the 1980s and 1990s to decreased spending on public infrastructure. Cavallo and Daude (2011)[34] on the other hand find that due to distortions with public capital investments, there is crowding out effect on private capital accumulation for a sample of 116 developing countries between 1980 and 2006. Santiago et al (2019)[109] offer support to this view by suggesting a short-run crowding out effect of public capital on private capital in their analysis of thirty Latin American and Caribbean countries over the period 1970 and 2014. In a related research work, Lowe et al (2019)[84] documents substantial cross-country variation in marginal product of public capital across countries arising from miss-allocations. They observed that the differences in marginal product of public capital produce a loss of about 9 percent in global GDP, and note that due to the imperfect substitutability between public and private capital, investment in public capital leads to inflow of private capital by raising the rate of return to private capital.

The rest of the review takes after the parallel lines of the academic literature. I start with the economic and growth effects of debt, and considers separately an important result of the theoretical literature from overlapping generations models on dynamic inefficiency. Next, I shift attention to the literature on public capital accumulation and economic growth, before reviewing the Blanchard (1985) OLG model.

2.2.1 Public debt and economic growth

The nexus between public debt and economic growth has generated substantial research interest, particularly in the empirical literature. A good starting point is the influential and controversial paper of Reinhart and Rogoff (2010)[99] who suggested the existence of a non-linear relationship between debt-GDP ratio and economic growth rate, with a negative correlation between growth rate and debt levels above 90 percent of GDP for advanced economies and about 60 percent for emerging economies. Replicating the work of Reinhart and Rogoff (2010), Hendern et al (2013)[65] found that the finding of a negative correlation between growth rate and debt levels above 90 percent of GDP is attenuated when data errors are corrected and the sample of countries expanded. Even though Herndon et al. (2014) failed to reject entirely the existence of a negative correlation between debt and growth at higher levels of debt, their critique of the Reinhart and Rogoff (2010) results has been influential in the empirical literature. That notwithstanding, other authors have emphasized the non-linear relationship using small samples of euro area and OECD countries. Cecchetti, Mohanty and Zampoli (2011)[35], for example, examined the effects of debt (separately considering public, corporate and household debt) on economic growth using a sample of 18 OECD economies for the period 1980 and 2006. Their results are consistent with the findings of Reinhart and Rogoff (2010) on the existence of a tipping point debt-GDP ratio where the relationship switches from positive to negative. Similarly, Checherita-Westphal and Rother (2012)[38] used a sample of twelve euro area countries to investigate the relationship, using a quadratic functional form for debt and instrumental variable estimators. They also conclude on an inverted U-shaped relationship, with debt having a negative impact on long-term growth at around 90-100 percent of GDP. In their estimates, further increase in public debt beyond the region of 85 percent decreases economic growth rate. In particular, 10 percentage point increase in debt-GDP ratio beyond the tipping point reduces growth by one-tenth of a percentage point. They identify channels through which debt affects growth to include private savings, public investments, and total factor productivity.

In a related earlier study, Schclarek (2005)[110] found that lower external debt levels is associated with higher growth rate in developing countries, with public external debt being the main determinant of this negative relationship. Similarly, Clemens, Bhattacharya, and Nguyen (2003)[89] used fixed effects and system GMM estimation methods and found that for a set of 55 heavily indebted poor countries (HIPCs) that were eligible for the IMF's Poverty Reduction and Growth Facility (PRGF), external debt burden was significantly and negatively related to per capita income growth between 1970 and 1999. The findings of Kumar and Woo (2010)[80] are very close to Checherita-Westphal and Rother (2012). The results of their analysis on the data of 38 advanced and emerging economies show that a 10 percentage point increase in debt-GDP ratio is associated with a slowdown of economic growth by 0.2 percent. In attempting to uncover non-linearity in the relationship between debt and growth, most of these studies have employed threshold debt levels, except for Checherita-Westphal and Rother (2012) who used a quadratic functional form for debt in their econometric estimation. The threshold estimates, in general, support the main conclusion of Reinhart and Rogoff (2010). Despite their contribution to the literature, the threshold debt levels are chosen arbitrarily and do not offer precise estimates of the debt-GDP level at which the effect switches to negative.

To address this, Baum, Chechrita-Westphal, and Rother (2013)[17] used a dynamic threshold method with the view to increasing precision on the non-linear debt-growth nexus. Their dynamic panel threshold model allow for the inclusion of endogenous variable (a one year backward lag of the dependent variable) and exogenous regressors. They find that the shortrun effect of debt (a one year lagged debt-GDP ratio) on growth is positive and statistically significant, but reduces to zero and statistical significance is lost gradually at around debt to GDP ratio of 67 percent. Unlike the previous threshold estimations where arbitrary debt to GDP ratios are chosen and used to estimate the tipping point, Baum, Chechrita-Westphal and Rother (2013) adopted a procedure developed by Hansen (1999)[63] to determine the threshold endogenously with enhanced precision.

It should be observed that while majority of the empirical literature point to a negative relationship between debt and economic growth, especially at high levels of debt, other have noted that causality is yet to be established. For example, Panizza and Prebistero (2013)[92] in a survey of the literature on the link between debt and economic growth observes that the literature has generally shown no more than a correlation between the two variables. They also note that the idea of a threshold (as in Reinhart and Rogoff) beyond which growth collapses is not robust. Similarly, Panizza and Prebistero (2014)[93] used an instrumental variable (i.e., exchange rate, which affects level of debt but not economic growth) approach to determine the causal link between high debt levels and economic growth using a sample of 17 countries. They conclude that their analysis is unable to establish a causal relationship running from debt to economic growth, and do not reject the null hypothesis that debt has no effect on growth. Other notable finding on this subject are found in Clemens et al. (2003[89]) and Salmon and de Rugy (2020[105]). The evidence of Schclarek (2005) and Clemens et al (2003) observes a negative relationship between public external debt and economic growth rate at higher levels of debt, with the former outlining the channel of effect to be mainly through capital accumulation growth. The latter, on the other hand, found the negative impact of debt to be influenced largely by the efficiency of resource use, rather than the effect on private investment.

In summary, results from the empirical literature is mixed on the growth effects of government debt. The majority of research shows a negative relationship at high levels of debt, while a handful of others point to the lack of causality and in some cases the absence of significant evidence on the relationship. In addition, the empirical literature have been distinct in focusing on the effect of debt on economic growth rate, as opposed to the theoretical literature where the focus is largely on effects on steady state welfare.

Theoretical work on the economic effects of national debt goes back to the work of Adam Smith, when he noted "the progress of the enormous debt which at the present oppress, and will in the long-run probably ruin all the great nations of Europe, has been pretty uniform"[3]. Diamond (1965[44]) made a seminal contribution, in which he rigorously examined national debt in a two-period overlapping generations model. In the Diamond model, national debt has negative impact on capital labour ratio through a reduction in savings rate in the long-run. However, its effect on welfare depends on the nature of the competitive equilibrium. If the steady state equilibrium is dynamically efficient (where interest rates exceed economic growth rate), debt issuance reduces welfare. On the other hand, if the equilibrium is dynamically inefficient, a relevant debt policy can increase welfare due to the positive difference between the equilibrium growth rate and interest rate on debt. Fundamentally, these results depend on a number of features of the neoclassical model. In general, steady state growth rate of the economy is exogenously given by the population growth rate. Also, the production structure does not admit public capital and funds from public debt is assumed to be distributed for consumption of present generations.

A synthesis of this argument advanced by Elmendorff and Mankiw (1999)[50] observes that debt may have long run negative effects on growth rates via the savings channel if Ricardian equivalence does not hold. This is the conventional view on the economic effects of debt held by most economist and almost all policy makers (Elmendorff and Mankiw, 1999). The intuition is that government debt stimulates aggregate demand due to factors such as sticky wages and prices. This in turn raises national income by increasing efficiency in the use of factors of production in the short-run. In the long-run, the effect on capital levels is negative due to the taxes to service or redeem the debt as illustrated in Diamond (1965). The Ricardian equivalence result, on the other hand, suggests that debt is neutral in a long run growth model that fully internalizes the consumption and saving decisions of present and future generations. Barro

[3]Adam Smith (1776). An Inquiry into the nature and the causes of wealth of nations. Book V, Chapter III

(1974)[15] rigorously established this result by introducing bequest motive into the overlapping generations model. He argues that given intergenerational linkages, the rise in disposable income due to government debt at the present, is fully compensated for by a corresponding fall in disposable income in the future when the debt is paid back. Internalising this, households adjusts bequest accordingly. Put differently, the Barro (1974) model rules out dynamic inefficiency as a long run equilibrium outcome. Hence, issuance of debt to raise consumption of the present generation is met with a corresponding rise in net bequest that accounts for the size of the debt, leaving overall household utility and long run capital level unchanged.

Other class of models have examined government debt as a tax smoothing vehicle and a safe asset provision in times of unanticipated shocks to the economy, even though the long-run trajectory of debt accumulation observed in the advanced economies can sometimes be at odds with the tax smoothing hypothesis. In addition, others have examined the welfare effects of government debt in incomplete market environment with heterogeneous agents (See for example Aiyagari and McGrattan 1998[3]). In their model, when markets are incomplete and hence do not provide insurance contracts for idiosyncratic shocks to labour productivity, individuals are driven by precautionary saving motives. In this case, individuals may hold excess amount of assets in equilibrium to cater for negative earning shocks. However, because government debt enters the individual's borrowing constraint, increasing the size of debt effectively loosens this constraint and allows households to rely on loans, rather than saving more, to insure against the shocks to earnings. Rather than using taxes which affects all agents, the absence of aggregate shocks allows increase in government debt to channel resources from those not experiencing shocks to their earnings to those most affected by negative shocks. Thus, by loosening the borrowing constraint, government debt provides a mechanism for individuals to insure against negative shocks and hence smooth out consumption. For these, and a brief overview of the various conceptualization of debt in the theoretical literature, see Yared (2019)[117] and Alesina and Passalacqua (2016[5]). The key results from the influential literature (the conventional view on economic effects of debt and the ricardian equivalence hypothesis) depend crucially on the neoclassical production technology and the requirement that government debt is used to raise present consumption. These features have been subjects of attention in a small section of recent models. Under the tagline of endogenous growth theory, Barro (1990)[16], Aschauer (2000)[7], Agenor (2010)[2] and

Checheritta-Westphal et al. (2014)[7], among others, have elaborated simple models in which the production technology allow for government spending on flow of services, or public capital provided by the government as an input in private sector production. These allow for government debt to be issued for investment in public capital as opposed to being used to raise present consumption in conventional models. As we will see later, the effect of debt on steady state capital and consumption levels differ significantly from the result of standard economic theory.

2.2.2 Dynamic inefficiency and welfare effect of debt

As seen above, government debt is generally acknowledged to reduce long run capital labour ratio in the absence of ricardian equivalence. Nonetheless, welfare is improved in the presence of dynamic inefficiency as an equilibrium outcome. This perverse nature of equilibrium implies over-accumulation of capital in the economy, leading to equilibrium interest rate falling below steady state economic growth rate. The idea of dynamic inefficiency arising in a perfectly competitive framework seem counter-intuitive (Geerolf, 2013[55]). This inefficiency arises from the independence of the Household's inter-temporal utility problem and the Firm's profit optimization problem. The Household, endowed exogenously with a subjective rate of time preference, optimises lifetime utility by discounting future consumption at the subjective discount rate. This ensures that for a small enough discount rate, equilibrium interest rate will be low. This can be low enough to be beneath the steady state growth rate. The two critical features associated with this outcome is that the steady state capital labour ratio will exceed the amount needed for optimal (golden rule) consumption given the production system, and importantly, the excess amount of capital supposes that actual consumption level in equilibrium is lower than potential. This outcome is directly the result of perfect competition, which ensures that market-given prices for capital, i.e., the interest rate, continuously adjusts to clear the market. But as interest rate falls to clear savings, households save even more as dictated by inter-temporal utility optimization. It is fairly straightforward that the long run is characterized by excess capital and lower interest rate than socially desirable - a perverse outcome of perfectly competitive structure (Diamond, 1965). This outcome leads to what is commonly referred to as liquidity trap (see for example, Krugman 1998[79], 2000[78] and Eggertsson and Krugman 2012[49]). The problem of dynamic inefficiency arises from the need for consumption to be lowered to enable the dynamic system maintain the higher capital labour ratio, given depreciation and the rate of population growth.

Given the perverse and socially undesirable nature of dynamic inefficiency as a long run equilibrium outcome of the perfectly competitive market, external intervention in the form of intergenerational transfers or public debt can be welfare enhancing. The positive welfare effect of debt or intergenerational transfers work through two channels. First, the household utility function is monotonic in consumption. Thus, debt-financed lump-sum transfers to an older generation for consumption increases life-time utility. Second, by reducing the capital labour ratio (a particularly strong effect in the case of domestically issued debt), the burden imposed by depreciation and population growth in maintaining a high capital labour ratio is attenuated thereby enabling increased consumption. In the absence of dynamic inefficiency, public debt is a nuisance in the neoclassical growth model, as it reduces both capital levels available to firms and household consumption.

2.2.3 Public capital and economic growth

The role of public capital in economic growth is a subject of fairly recent research. Much of the earlier research on economic growth were neoclassical in nature. In general, these did not incorporate public capital in long run analysis of economic growth. The influential models of Samuelson (1958[106], 1962[107]), Phelps (1961[95], 1965[96]), Solow (1956)[112], and Diamond (1965)[44], among others, either take public capital as granted or abstract from it for simplicity in the analysis of long run growth. This arises out of the difficulty of treating public capital as a trade-able commodity in a perfectly competitive framework. Later research such as Romer (1990)[103] have introduced the effect of varying returns to scale, a feature contemplated in Solow (1956), due to technical change (attributable to research and development). Similarly, Azariadis and Drazen (1990)[11] explored the externalities arising out of threshold effects in which economies associated with a given level of capital experience increasing returns to scale while economies with lower levels experience decreasing returns. The threshold effect is extended to human capital levels (aggregate knowledge in an economy influencing agents decisions on what amount of training to acquire), which in turn influence the steady state level of capital per worker in the economy. Even though the threshold effects in Azariadis and Drazen (1990) does not reference public capital, it is straightforward to extend the argument informally. More explicitly, Barro (1990)[16] elaborates a growth model in which long-run equilibrium with positive per capita growth is supported by non-technical change when the production function exhibits constant elasticity to broad capital.

The more recent literature have explored the role of public capital in the economic growth and development process. Agenor (2010)[2] develops a deterministic growth model in which public capital in infrastructure is the engine of growth. Agenor's (2010) formulation incorporates stylized facts in developing countries where the government is the major provider of health services, and due to relatively low levels of infrastructure, public investment in it reduces production costs, and increases the rate of return to capital. In Agenor (2010), the rate of time preference is assumed to depend positively on private capital and negatively on health services. This dependence is necessary to ensure that private savings, and hence equilibrium consumption, is elastic to government investment in production of health services. Agenor's (2010) framework relates to Azariadis and Drazen (1990) in terms of their incorporation of human capital through training and investment in health services respectively. Infrastructure serves as the engine of growth in Agenor (2010) in part because it is used in the production of health services, and together with health services, enter as input in production of final goods. Aschauer (2000) presents a much simpler model in which the rate of time preference is exogenous, as in conventional theory. Checherita-Westphal et al. (2014)[37] relies on the optimal capital ratio result from Aschauer (2000) to estimate growth-maximizing fiscal rules using data for euro area countries. The general insight from this section of the theoretical literature is that public capital have significant positive effects on economic growth. They generally acknowledge the pivotal role of public goods and services such as infrastructure, health and education services provided by state governments in the growth process. Public goods such as highways and streets, bridges, railways, airports, electricity, water and sewage systems, schools and hospitals constitute the stock of public capital, which is argued to induce an increase in the rate of return to private capital (Aschauer, 1989A[9]; Agenor, 2010[2]).

The body of research on the effects of public capital on private capital formation and economic growth have become important in better understanding the drivers of growth and development in developing countries. Unlike developed nations, less developed economies are facing large infrastructure gaps (Berg et al, 1992[19]) and this continue to serve as key obstacle to growth and development (Agenor, 2010[2]). Thus, the centrality of public capital to growth, which is generally ignored or implicitly assumed away in the neoclassical literature, is the motivation for the analysis conducted in chapter four. On the empirical findings on the effects of public capital on growth, I start with Aschauer (1989A)[9] who made a seminal contribution using data on the United States economy. He found that a 1 percent increase in the ratio of public to private capital stocks is associated with 0.39 percent increase in total factor productivity. He concluded that the non-military stock of public capital was dramatically more important for productivity growth in the United States in the 1980s. Similar conclusion is reached in Munell (1992)[87], who find that 1 percent increase in stock of public capital increases output by 0.34 percent, with public capital having a marginal productivity in the order of 60 percent compared to 30 percent for private capital. Using a sample of 57 countries across the globe and a distinct sample of 19 Latin American countries, Sanchez-Robles (1998)[108] found that public infrastructure is positive and significantly related to output and economic growth. Similar results are found by Button (1998)[32] and particularly in Esfahani et al. (2003)[51], who highlighted the role of institutions in economic growth, in addition to infrastructure. Also, Dambala-Norris et al. (2012)[40] constructs a public investment efficiency index to capture the institutional environment characterising public investment particularly in low income countries. Gupta et al. (2014)[60], using dataset for 52 countries and adjusting for public investment management index (PIMI), examined the relationship between public capital and economic growth using a reduced form specification of a Cobb-Douglas production with three inputs; private capital stock, public capital and skill adjusted labour supply. They find that previous studies grossly underestimated the effect of public capital on economic growth. Controlling for efficiency of public investment processes, they find the productivity of public capital to significantly exceed the marginal cost of funds under normal conditions. In addition, Dreger and Reimers (2014)[45] find evidence using data for the Euro-area to show that private investments reacts positively to the stock and flow of public investment.

In addition, Arslanalp et al (2010)[6] estimated the effect of public capital on growth using the standard production function to estimate the empirical model. For the sample of forty-eight OECD and non-OECD economies, they find that public capital is positively related to growth, when initial levels are controlled for. This finding is similar to the result of Gupta et al (2014)[60]. Romp and De Haan (2007)[104] provides a good survey of earlier literature in this line of research. Despite the documented positive impacts of public capital on output levels and economic growth rate in developing and advanced countries, large deficits of infrastructure remain in developing countries in particular. For example, about 1.4 billion people have no access to electricity while 880 million people drink from unsafe sources, with more than 50 percent being in Sub Sahara Africa (MDB Working Group on Infrastructure, 2011[68]). In assessing the ways in which this challenge can be met, Bhattacharya et al (2012)[21] considers government borrowing to finance economically productive infrastructure while remaining within prudent debt levels as a viable alternative. This conundrum has seen little to no attention in the academic literature on the impacts of public debt.

In sum, the precise role of public capital in factor accumulation remains inconclusive in the economics literature, although the vast majority of empirical studies conclude on a significant positive effect. While the stock of public capital in infrastructure for example, is viewed to facilitate economic activity by enhancing efficiency of private investments, an increase in pubic capital investments may be done at the expense of private investment. This ex-ante crowding out effect may have negative consequences for private investment growth. However, by complimenting and enhancing efficiency of private investments, the positive effects of public capital investment may outweigh the crowding out effect (Aschauer, 1989B[8]). But this depends on the efficiency of the state machinery in converting resources into public capital.

2.3 The Blanchard (1985) OLG model

2.3.1 Introduction

In this section, I introduce the Blanchard overlapping generations model, which will be the workhorse for the simple model developed in chapter four. Blanchard (1985)[24] developed his model as a solution to the aggregation problem encountered in simple growth models with heterogeneous agents. Unlike the representative agent model of Ramsey (1928)[98], for example, where households are treated as dynasties with infinite horizon, making irrelevant the question of finite life of individuals, and the problem of aggregation due to mixed generations, overlapping generation models bring to the fore the need for approximate aggregation of wealth and consumption levels to derive equilibrium. In the latter class of models, it is not simply the multiplication of an individual's asset and consumption levels by the size of the population at any point in time to obtain their aggregate values. Individuals differ by generations and hence asset levels, which in turn occasion different consumption levels. These differences led Diamond (1965)[44] to develop the two period OLG model, where agents live for two periods working and earning wages in the first, and retiring and consuming out of savings in the second. With only two generations at each time, aggregate consumption becomes simply the sum over two generations, given the level of consumption being the same for members of each generation, but different between generations. The coarse population structure, while solving the problem of aggregation in the presence of mixed generations, assumes a constant survival period for all individuals. In reality, period of life and equivalently, the probability of survival and time of death are entirely random. It is this uncertain aspect of life, among other things, that the Blanchard model was designed to solve.

In his classic, “Debt, deficits, and finite horizons”, Blanchard (1965)[24] allowed for infinite existence of the economy as in previous models, but assumes that individuals live for many periods and are subject to a constant instantaneous probability of death. Thus, while an individual is capable of having infinite horizon, a positive probability of death means that people live finitely and may die at any time. This uncertainty about time of death supposes that individual may die and leave behind assets, revealing the obvious challenge about inter-temporal allocation of consumption. To address this, Blanchard relies on the technique of modelling uncertain life by Yaari (1961)[116], by assuming that there are actuarial life insurance companies offering an actuarial interest rate over and above the market interest rate on capital. An individual receives this rate on assets as long as (s)he lives. However, upon his death, the insurance company takes over ownership of the individual's assets. These contracts are also called negative life insurance contracts (Groth, 2011[59]). It turns out that for these companies to operate competitively and make zero profit, the actuarial interest rate must exceed the regular interest rate by an amount equal to the mortality rate (the probability of death). To see this, let the aggregate assets in the economy at time, t, be $A(t)$, and the capital labour ratio be given by $\bar{a}(t) = A(t)/L(t)$, where $L(t)$ is the total population at present time. By holding negative life insurance contracts, individuals deposit their assets with the Insurance company and receive interest amounting to $\phi = r+\hat{m}$. In other words, the insurance company pays out to all households a return on assets amounting to $\phi A(t) = (r + \hat{m})A(t)$. The insurance company invest the deposits of the household with the manufacturing firms, and receive the market interest rate. Thus, it receives $r * A(t)$ in revenue from assets, but pays an extra premium of $\hat{m} * A(t)$ to households.

If the probability of death faced by each individual is constant and the same for each person as assumed, then over the total population, the expected number of people that die at a given time is given by $m * L(t)$, where $0 < m < 1$ is the instantaneous probability of death (i.e., mortality rate). Since the total assets left behind by the fraction of the population that die shifts into the ownership of the insurance company, if follows that for each period, the insurance company takes ownership of $m * L(t) * \bar{a}(t) = m * A(t)$ in assets of the dead.

In summary, the revenue of the insurance company equals receipts from manufacturing firms and assets it assumes ownership of upon death of some agents. Let the total revenue be $R = r * A(t) + m * A(t)$. Since it pays out $\phi A(t)$ to households and expected to break even, its profit,

$$\pi = -[\phi A(t) - (r + m)A(t)] = -(r + \hat{m})A(t) + (r + m)A(t) = 0.$$

It follows that the insurance premium equals the mortality rate, $m = \hat{m}$. This result is crucial in allowing inter-temporal allocation of consumption in the face of uncertainty about lifespan.

2.3.2 The model framework

In this subsection, I present the primitives of the Blanchard continuous time OLG model, starting with the household utility structure, and then the production technology of the representative firm.

Households

In line with usual practice, let an infinitely lived Household maximize a standard CRRA utility function of the form,

$$U_t = \int_t^\infty \frac{c^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t} dt.$$

For simplicity let $\sigma = 1$. Since agents are subject to a probability of death in the Blanchard economy, future consumption is discounted by not only the rate of time preference but also by the exogenous mortality rate. Thus, let the utility function be restated as:

$$U_t = \int_t^\infty \ln c(v, s) e^{-(\rho+m)s} ds, \tag{2.1}$$

where $c(v,s) > 0$ is the consumption at time s of a person born at time v, $0 < \rho < 1$ is the consumer's rate of time preference, and m is the mortality rate (the same as the premium on negative life insurance contracts). Like the Diamond model, Blanchard (1985) does not factor bequest motive in the utility function. The Household problem becomes one of maximising U_t subject to $c(v,s) > 0$, and the rate of change of household wealth given by:

$$\frac{\partial a(v,s)}{\partial s} = \dot{a}(v,s) = (r(s)+m)a(v,s) - c(v,s) + w(v,s) \tag{2.2}$$

where $w(v,s)$ is labour income. As would be seen later, labour earns it marginal product and hence the generation of a person has no effect on the labour income. Therefore, v becomes redundant and $w(v,s) = w(s)$. The household problem can be solved using the current value Hamiltonian:

$$H[a,c,\lambda,s] = \ln c(v,s) + \lambda(s)[(r(s)+m)a(v,s) - c(v,s) + w(s)]. \tag{2.3}$$

The first order conditions are:

$$\partial H/\partial a(v,s) = (r(s)+m)\lambda = (\rho+m)\lambda(s) - \dot{\lambda}(s).......(A1)$$

$$\partial H/\partial c(v,s) = c(v,s)^{-1} - \lambda(s) = 0.......(A2)$$

$$\partial H/\partial \lambda(s) = (r(s)+m)a(v,s) - c(v,s) + w(s) = \dot{k}(v,s).......(A3)$$

Given the first order conditions, the sufficient condition for a solution to exist is a transversality condition given by:

$$\lim_{t\to\infty} a(v,t)e^{-\int_0^t (r(t)+m)dt} = 0...............(A4),$$

Combining (A1) and (A2) gives the growth rate of consumption in dynamic equilibrium as $\dot{c}(v,s) = (r(s)-\rho)c(v,s)$. At $s = t = 0$, this solves to $c(v,s) = c(v,t)e^{\int_t^s [r(s)-\rho]ds}$. Clearly, to determine $c(v,s)$ for any time s, one needs to know $c(v,t)$. This requires an inter-temporal budget constraint (IBC) which states the present discounted value of an individual's consumption as a function of endowment in physical and human capital. Let the relevant IBC be:

$$\int_t^\infty c(v,s)e^{-\int_t^s (r(s)+m)ds} ds \leq a(v,t) + \omega(v,t) \tag{2.4}$$

where $\omega(v,t)$ is the present discounted value of of future streams of wages. This is given as $\omega(v,t) = \int_t^\infty w(s)e^{-\int_t^s (r(s)+m)ds} ds$. Unlike the Diamond (1965) model where individuals work for one period and retire the next, in this basic framework, individuals work from birth until they die, and wages is the same for all labour. Thus the total human capital in the economy is simply the human capital endowment of an individual multiplied by the size of the population. In other words, $\omega(v,t) = \bar{\omega}(t) = \Omega(t)/L(t)$, where $L(t)$ is the total population of the economy at time, t. Substituting $c(v,s) = c(v,t)e^{\int_t^s [r(s)-\rho]ds}$ into equation (2.4) and suppose it holds with equality, one obtains

$$\int_t^{\infty} c(v,t)e^{\int_t^s [r(s)-\rho]ds} e^{-\int_t^s (r(s)+m)ds} ds = a(v,t) + \omega(v,t)$$

This gives the present consumption of a person at time, t, as a function of assets and human capital as $c(v,t) = (\rho+m)[a(v,t) + \bar{\omega}(t)]$. One of the key contributions of the Blanchard (1985)[24] paper is the development of the aggregate consumption function. The difference in age between existing members of the economy, and consequently wealth levels leads naturally to different levels of consumption between members of different generations. This difference in wealth arises solely from different levels of physical capital. The human wealth on the other hand is the same for each existing member of the economy as the expression above shows. Hence, $\forall v, \omega(v,t) = \bar{\omega}(t)$, and aggregate human wealth is determined as $\Omega(t) = \bar{\omega}(t) * L(t)$. Given aggregate physical capital as $A(t)$, aggregate consumption as a function of total wealth (physical and human capital) follows the per capita counterpart and is given as:

$$C(t) = (\rho+m)[A(t) + \Omega(t)] \tag{2.5}$$

But $A(t)$ is yet to be determined. Take the economy's population given at some time in the past as $L(0)$. With a birth rate of b, it follows that the population grows at the rate of $n = b - m$. Clearly, the size of a cohort born at time v in present time, t, is given by $L(v,t) = L(0)be^{-m(t-v)}$, if $v = 0$, otherwise $L(v,t) = L(0)e^{nv}be^{-m(t-v)}$ where $L(v) = L(0)e^{nv}$. Thus, the total population at time, t, can be written as:

$$L(t) = L(0)e^{nt} = \int_{-\infty}^{t} L(v,t)dv = \int_{-\infty}^{t} L(0)e^{nv}be^{-m(t-v)}dv \tag{2.6}$$

Therefore, aggregate physical capital may be obtained by integrating across generations as follows:

$$A(t) = \int_{-\infty}^{t} L(0)e^{nv}be^{-m(t-v)}a(v,t)dv \tag{2.7}$$

and equation (2.5) can be written as

$$C(t) = (\rho+m)[\int_{-\infty}^{t} L(0)e^{nv}be^{-m(t-v)}a(v,t)dv + \int_{-\infty}^{t}\int_t^{\infty} w(s)e^{-\int_t^s (r(s)+m)ds}dsL(v,t)dv] \tag{2.8}$$

From these aggregate equations, the dynamic evolution of the economy can be derived by differentiating with respect to time. Using the rule for differentiation under the integral (i.e., the Leibniz rule), which says that for a function given by $F(x) = \int_{u(x)}^{v(x)} f(x,t)dt$, its derivative is given by

$$F'(x) = f(x, v(x))v'(x) - f(x, u(x))u'(x) + \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x} dt.$$

Starting with equation (2.7) and using the condition $u'(t), v'(t) = 0$ if $u(t), v(t) = \pm\infty$ and $u'(t), v'(t) = 1$ if $u(t), v(t) = t$, it follows that

$$\dot{A}(t) = a(t,t)L(0)e^{nt}b - 0 + \int_{-\infty}^{t} L(0)b\frac{\partial}{\partial t}[a(v,t)e^{nv}e^{-m(t-v)}]dv$$

$$= 0 - 0 + L(0)be^{nv}\int_{-\infty}^{t} a(v,t)e^{-m(t-v)}(-m)dv + L(0)be^{nv}\int_{-\infty}^{t} \frac{\partial a(v,t)}{\partial t}e^{-m(t-v)}dv$$

From the individual's budget constraint (equation 2.2),

$$\frac{\partial a(v,t)}{\partial t} = (r(t) + m)a(v,t) - c(v,t) + w(v,t).$$

Substituting obtains

$$\dot{A}(t) = -mA(t) + L(0)be^{nv}\int_{-\infty}^{t} [(r(t) + m)a(v,t) - c(v,t) + w(v,t)]e^{-m(t-v)}dv$$

$$= -mA(t) + (r(t) + m)A(t) - C(t) + w(t)L(t) = r(t)A(t) - C(t) + w(t)L(t)$$

Similarly, from equation (2.8),

$$C(t) = (\rho + m)[\int_{-\infty}^{t} L(0)e^{nv}be^{-m(t-v)}a(v,t)dv + \int_{-\infty}^{t} \omega(\bar{t})L(v,t)dv]$$

$$= (\rho + m)[a(v,t) + \bar{\omega}(t)]\int_{-\infty}^{t} L(v,t)dv$$

Replacing $(\rho + m)[a(v,t) + \bar{\omega}(t)]$ by $c(v,t)$, and differentiating under the integral, this becomes:

$$\dot{C}(t) = c(t,t)L(0)e^{nt}b - 0 + \int_{-\infty}^{t} L(0)b\frac{\partial}{\partial t}[c(v,t)e^{nv}e^{-m(t-v)}]dv$$

Observe that for $v = t$, $a(t,t) = 0$ due to the fact that people are born with no physical capital (this is due to the absence of bequest motive). This will not be the same in the Barro (1974)[15] model, for example. However, given that people are endowed with their human capital at time of birth, the aggregate change in consumption becomes:

$$\dot{C}(t) = (\rho+m)\omega(t)L(0)e^{nt}b + \int_{-\infty}^{t} L(0)be^{nv}[(-m)c(v,t)e^{-m(t-v)} + \frac{\partial c(v,t)}{\partial t}e^{-m(t-v)}]dv$$

By the optimality condition for consumption,

$$\dot{C}(t) = (\rho+m)b\Omega(t) - m\int_{-\infty}^{t} L(0)be^{nv}c(v,t)e^{-m(t-v)}$$

$$+ \int_{-\infty}^{t} L(0)be^{nv}[r(t)-\rho]c(v,t)e^{-m(t-v)}dv$$

$$= (\rho+m)b\Omega(t) - mC(t) + [r(t)-\rho]C(t)$$

Substituting equation (2.5) and $-m = n - b$ into $\dot{C(t)}$ yields:

$$\dot{C(t)} = b(\rho+m)\Omega(t) - b(\rho+m)[A(t)+\Omega(t)] + nC(t) + [r(t)-\rho]C(t)$$

This gives the two aggregate dynamic equations of the Blanchard model as:

$$\dot{A}(t) = r(t)A(t) - C(t) + w(t)L(t) \tag{2.9}$$

$$\dot{C}(t) = -b(\rho+m)A(t) + [r(t)-\rho+n]C(t) \tag{2.10}$$

With equations (2.9) and (2.10), the steady state equilibrium of the Blanchard model is straightforward to derive, given a representative Firm in a perfectly competitive market. Our focus here is on the closed economy version, considering that in an open economy, Blanchard assumed that wages and interest rate on assets, which is invested externally, are fixed. This gives partial equilibrium and relatively uninteresting dynamics in the aggregate economy, with total consumption increasing indefinitely if global interest rate exceeds the exogenously given rate of time preference, and decreasing forever if otherwise.

The closed economy - the representative firm

Given the household decision problem elaborated above, the economy is closed by a relevant market structure for production. A typical Cobb Douglas production function with or without technology augmentation may be used. In line with the original model of Blanchard (1985), I abstract from technological progress and use a basic output function of the form:

$$Y = F(K,L) = K^{\alpha}L^{1-\alpha} \tag{2.11}$$

Standard features of neoclassical production applies (i.e., F is characterised by constant returns to scale, CRS, diminishing marginal product to each input, and bound by inada conditions). The profit maximising Firm chooses capital and labour to optimise profits. This implies that at the optimum, the Firm pays the marginal product to each input. These are:

$$\hat{r} = \partial Y/\partial K = \alpha K^{\alpha-1} L^{1-\alpha}$$

$$w = \partial Y/\partial L = (1-\alpha)(K/L)^{\alpha}$$

where $\hat{r}$ is the gross interest rate, w is wages, K is aggregate capital stock, and L denotes the labour force. Clearly, these imply that the firm's profit, $\Pi = Y - \hat{r}K - wL = 0$. Supposing that physical capital depreciates at a constant rate, δ, we can define the net interest rate as $r = \hat{r} - \delta$. Due to CRS, equation (2.11) may be written in per capita form as $y = f(k)$. It follows that $r = f'(k) - \delta$ and $w = f(k) - [r + \delta]k$. From the perspective of the Firm, aggregate change in physical capital takes the form $\dot{K} = Y - \delta K - C$. In equilibrium, markets clear at all times so that aggregate assets held by the households equal aggregate capital used by the Firm (i.e., $K = A(t)$). In other words, $\dot{K}(t) = \dot{A}(t)$. This equality is easily checked by substituting for r and w into equation (2.9).

2.3.3 Steady state equilibrium

The market conditions together with the aggregate dynamic equations are sufficient for deriving the steady state equilibrium of the Blanchard closed economy. Using the conditions that $a = k = A(t)/L(t) = K/L$, and $c = C(t)/L(t)$, and substituting for equations (2.9) and (2.10) after log differentiating both sides with respect to time obtains:

$$\dot{k} = f(k) - (n + \delta)k - c \tag{2.12}$$

$$\dot{c} = -b(\rho + m)k + [f'(k) - \delta - \rho]c \tag{2.13}$$

These two coupled differential equations characterise the dynamic equilibrium path, and move together towards steady state. It is straightforward that at $\dot{k} = 0$,

$$c = f(k) - (n + \delta)k$$

and for $\dot{c} = 0$,

$$c = \frac{b(\rho + m)}{f'(k) - \delta - \rho} k$$

Since stationary equilibrium requires both $\dot{c}, \dot{k} = 0$, it follows that steady state is given by:

$$\frac{b(\rho + m)}{f'(k) - \delta - \rho} k = f(k) - (n + \delta)k.$$

The crucial point about steady state equilibrium in the Blanchard model is that the optimality condition for inter-temporal allocation of consumption by the household is satisfied, given market conditions. Despite this, the equilibrium may be dynamically inefficient, and consumption can be improved by simply disposing off some amount of private capital or committing to a fiscal policy rule. In other words, the market equilibrium may not be pareto optimal. In this case, it is possible for government to intervene in the market economy in a way that improves welfare for present generations without hurting that of subsequent generations. If the equilibrium is efficient, however, any such intervention will be welfare reducing. We illustrate the two cases below.

Starting with the dynamically efficient equilibrium, where $r^*(k^*) \geq n$, observe that optimizing consumption with respect to capital, subject to $\dot{k} = 0$, gives a first order condition of $f'(k) = n + \delta$. Let k satisfying this be k^{**}. Similarly, the first order condition for optimum consumption subject to $\dot{c} = 0$, gives

$$\frac{\partial c}{\partial k} = \frac{[f'(k) - \delta - \rho]b(\rho + m) - b(\rho + m)kf''(k)}{[f'(k) - \delta - \rho]^2} = 0$$

But clearly, $\partial c/\partial k > 0$, as long as $r = f'(k) - \delta > \rho$, and consumption can be increased by some addition to physical capital. But since $f''(k) < 0$ and r is falling in k, it implies that there is a $\bar{k}$ that satisfies $f'(\bar{k}) = \rho + \delta$, at which there is no incentive for increased capital accumulation, given $\partial c/\partial k$ may be negative for $k > \bar{k}$. Thus, $\bar{k}$ is such that $\dot{c} = 0$. Observe that $\lim_{k \to \bar{k}},\ c|_{\dot{c}=0} = \infty$. In this case, $\bar{k}$ can be thought of as a limiting value, the vicinity of which contains the equilibrium capital labour ratio. Comparing k^{**} and $\bar{k}$ readily shows the nature of the steady state equilibrium for $\rho \neq n$. For $\rho > n$ such that $\bar{r}(\bar{k}) = f'(\bar{k}) - \delta > f'(k^{**}) - \delta = n$, then by $f''(k) < 0$, $\bar{k} < k^{**}$. Note that the steady state equilibrium capital level, k^*, is in the neighbourhood of $\bar{k}$. More specifically, for steady state consumption to be well defined, we must rule out the possibility of infinite consumption in equilibrium (i.e., $c^*(k^*) \neq \infty$). Therefore, the equilibrium consumption as a function of capital is such that $0 < c^*(k^*) < \bar{c}(\bar{k}) = \infty$. With consumption a positive function of capital along the dynamic equilibrium path, it follows directly that $k^* < \bar{k} < k^{**}$. In other words, an increase in k along the transition path will always increase utility for the individual, starting from below the equilibrium capital labour ratio. But by definition, k^* cannot exceed $\bar{k}$. Any attempt to increase capital labour ratio beyond $\bar{k}$ will lead to $f'(k) - \delta < \rho$, given $f''(k) < 0$ and by the optimality condition for inter-temporal consumption, the consumption level has to be lowered.

Similarly, if government intervenes to lower the level of capital labour ratio, it is straightforward that by $k^* < \bar{k}$ and given consumption as a positive function of capital when $\dot{c} = 0$, consumption will be lowered. A third option is a fiscal policy rule. Suppose the government issues a given amount of debt, D, at time, t_0. Let the debt labour ratio be d and further suppose that the government intends to maintain this debt labour ratio constant indefinitely. Since the population grows at rate, n, it implies that $\partial D/\partial t = n$. In other words, for all $t > t_0$, the government simultaneously retires old debt, denote as $P = (1 + r)D(t)$, and issues new debt, $N = (1 + n)D(t)$. For a dynamically efficient equilibrium where $r^* > n$, the debt policy of maintaining $\partial d/\partial t = 0$ indefinitely imply that $\forall t > t_0$, net government expenditure due to debt equals $P - N = (r - n)D(t) > 0$. In this case, the government spends more in debt service than it accumulates in new aggregate debt. This will necessitate the introduction of a tax to finance the difference in expenditure.

In summary, if the steady state equilibrium is characterised by $r^* > n$, external intervention in the market system to increase or decrease the capital labour ratio will have a net negative effect on welfare. Also a fiscal policy rule will reduce consumption level of agents. In other words, the market equilibrium is pareto optimal and it is impossible to improve outcomes by the intervention of a Social Planner.

2.3.4 A note on the Blanchard equilibrium

The requirement above that $k^* < \bar{k}$ implies that the steady state interest rate exceeds the rate of time preference, $(r^* = f'(k^*) - \delta > \rho)$. This requirement may seem to be at odds with the notion of steady state equilibrium given the optimality condition for consumption. The first order conditions for maximizing equation (2.3) gives $\dot{c}(v, t) = (r(t) - \rho)c(v, t)$, replacing s by t. Clearly, for $\dot{c}(v, t) = 0$, either $c(v, t) = 0$, or $(r(t) - \rho) = 0$ or both. Since we rule out $c(v, t) = 0$, it follows that $\dot{c}(v, t) = 0$ if and only if $(r(t) - \rho) = 0$. To see why $r^* > \rho$ at steady state equilibrium in the Blanchard model, observe that the condition for steady state is $\dot{c}(t) = 0$, not $\dot{c}(v, t) = 0$. In other words, $\dot{c}(v, t)|_{c(t)=c^*} =$

$(r^* - \rho)c(v,t) > 0$, while $\dot{c}(t) = 0$. Thus, the consumption of each household is increasing until they die, while average consumption is constant at steady state. Why this seeming contradiction? It arises from the fact that at every period, new agents who are born enter the economy with no physical assets. Therefore, while prior existing households have incentive to save and increase assets due to $r^* > \rho$, leading to positive growth in household consumption, the new agents who are born without assets ensure that on the average the per capita asset level remains constant, and hence consumption. At steady state, the amount of capital required to keep the capital labour ratio constant given the birth rate, exactly equals the new savings induced by $r^* > \rho$, plus the amount of assets left behind by those who die. This mechanism is called the generations replacement effect. Note that by equation (2.13), were $r^* = \rho$ at steady state, we will have

$$\dot{c} = -b(\rho + m)k^* + [f'(k^*) - \delta - \rho]c \neq 0,$$

since $r^* = f'(k^*) - \delta = \rho$. In other words, at $r^*(k^*) = \rho$, $\dot{c} = -b(\rho + m)k^* < 0$, and existence of steady state equilibrium will be ruled out. The implication is the same if we assume that steady state has $r^* < \rho$. Thus, by contradiction, we have seen that steady state equilibrium in the Blanchard model cannot be supported by $r^* \leq \rho$. Therefore, $r^* > \rho$ at $k(t) = k^*$.

2.3.5 Dynamic inefficiency as a general equilibrium outcome and the unique role for debt

The key result of the Diamond (1965)[44] model that also arises in the Blanchard (1985)[24] model is that the market economy may reach a dynamically inefficient equilibrium without bequest motive in the utility structure. In other words, the equilibrium capital labour ratio may exceed what is required for optimal consumption, and it is possible to improve utility by reducing the amount of capital per person. This finding might seem paradoxical considering the requirements of market driven outcomes with the steady state equilibrium satisfying market clearing conditions, profit maximization by the Firm, and the household optimality condition for inter-temporal allocation of consumption. How will the market-driven equilibrium lead to sub-optimal outcomes? The answer lies in the exogenous parameter values for population growth and rate of time preference. Unlike the efficient equilibrium where $r^* > \rho \geq n$, if the exogenous parameters are such that $\rho < n$ by a margin large enough, $r^* < n$ may arise at steady state. Under this condition, there exist the possibility of a negative equilibrium interest rate arising in the Blanchard model if population growth rate is zero ($n = 0$). By the definition for $\bar{k}$ and k^{**}, if $f'(\bar{k}) - \delta = \rho < n = f'(k^{**}) - \delta$, then $\bar{k} > k^{**}$. Similarly by $r^* < n$ it follows that $k^* > k^{**}$. Given the requirement of $k^* < \bar{k}$, we have $k^{**} < k^* < \bar{k}$. Using equation (2.12), maximizing consumption with respect to capital under $\dot{k} = 0$ satisfy the first order condition $f'(k) - n - \delta = 0$. Since k^{**} satisfies this condition, any capital level to the right of of k^{**} imply $f'(k) < n + \delta$, and $\partial c/\partial k|_{\dot{k}=0} < 0$. Therefore, $c(k^*) < c(k^{**})$, given $\dot{k} = 0$. In this case, it is possible to increase steady state consumption by reducing the capital labour ratio. But how can this be achieved in a market equilibrium? This question arises because any temporary intervention in the market system will have no effect on consumption level in the presence of bequest motive. Where agents have no bequest motive, different effects would be imposed on consumption levels of present and future generations. Thus, permanently increasing consumption level in the dynamically inefficient equilibrium requires a permanent intervention in the form of a relevant debt policy.

The unique role for debt in the dynamically inefficient equilibrium is a well-known result dating back to Diamond (1965)[44] . This case is not examined in Blanchard (1985)[24]. It can be shown that a permanent debt policy can increase consumption by existing generations without reducing that of future generations. In fact, if the relevant debt policy takes the form of perpetual inter-generational transfers from the young to the older generation, consumption level will be increased for all generations as shown by Diamond (1965)[44] . Given $r^* < n$, it follows that a relevant debt policy that maintain a constant debt labour ratio will generate net government expenditure amounting to $P - N = (r^* - n)D(t) < 0$. In other words, the debt policy leaves behind excess resources every period which can be distributed to living generations for consumption. In the Diamond two period model, the excess resources can be transferred to the older generation for consumption. Conducting intergenerational transfers in this manner will increase life time consumption of all agents.

Since population growth is given by n, it implies that $\forall t, d(t) = d$, given $\partial D/\partial t = n$. Similarly, because aggregate output, consumption and capital are growing at the rate of population growth at steady state, it follows that $\tilde{D}(t) = \tilde{D}$, where $\tilde{D} = D(t)/Y(t)$, is the constant debt-GDP ratio at all times. Apart from the uncertain life span (and time of death), the Blanchard model abstracts from all forms of uncertainty. Therefore, for the given population growth rate, maintaining a constant debt labour ratio requires increasing the aggregate debt level by rate, n, hence a constant debt-GDP ratio. A relevant question concerns the size of the initial debt, and hence the debt-GDP ratio that should be maintained. In the closed economy where the interest rate is dependent on capital demand, there is a limit to debt-GDP ratio and debt labour ratio since interest rate rises with the debt ratio. In the open economy where the interest rate is independent of the size of debt, there would appear to be expansive space for debt labour ratio. In reality, many factors including the debt-GDP ratio determine the rate of interest on external debt.

In Blanchard (1985), the effects of tax re-allocations and fiscal policy in the presence of government expenditure depends on the horizon of agents. These are beyond the scope of this dissertation. To summarise briefly, tax reallocation in the form of a decrease in taxes at the present followed by an increase in taxes in the future, given a government expenditure path, will increase present consumption while reducing that of future generations. This result arises from the imposition of finite horizon and absence of bequest. If the probability of death equal zero, agents have infinite horizons and such re-allocations with have no net effect on overall utility. Similarly, the issuance and distribution of debt, and thereafter servicing the debt indefinitely creates initial wealth effect and hence raises present consumption. However, due to the imposition of taxes to service debt, steady state capital level and consumption are reduced. Thus, in general, the issuance of debt and refinancing it indefinitely, with or without government expenditure, reduces steady state capital and consumption levels when agents have finite horizons.

What is not explored in Blanchard (1985) is where the government expenditure, financed by issuance of debt, is invested in public capital formation. Given the Cobb Douglas production function, and households being the owners of capital in the Blanchard economy, any such investment will be unnecessary, and potentially distortionary, given market conditions. It is likely that the effect will differ from the Blanchard results if government expenditure raises public capital stock, conditional on this stock being an input in the economy's production function. This is the starting point of my contribution in this dissertation. I adopt a production function that incorporates public capital in the next chapters and examine the effects of debt-financed investment in this broad category of capital on consumption levels of households in the infinitely lived economy with uncertain life-span.

2.4 Conclusion

In this chapter, I have reviewed key literature on the economic effects of debt and public capital as conceived in the bulk of the literature. The review pays attention to three issues in the literature. First is on the growth effects of debt viewed largely as a public finance problem as opposed to a sovereign debt issue. While in theory, debt is viewed to either negatively affects growth (in the absence of Ricardian equivalence) or have no effect (with Ricardian equivalence), the majority of empirical findings suggest a negative effect of debt on economic growth at high debt-GDP ratios. Second is on dynamic inefficiency as a special case on the economic effects of debt. With dynamic inefficiency as a long-run equilibrium outcome, debt may have a positive effect on welfare despite reducing capital labour ratio. Third is on the effects of public capital on economic growth. As elaborated above, a growing amount of theoretical literature shows that under some conditions, government investment in public capital enhances economic growth. Similarly, the empirical literature have broadly revealed that due to significant gaps in public capital, especially in less advanced economies, not only do investment in public capital enhances growth, it improves the rate of return to private capital. Finally, I have reviewed the basic Blanchard (1985) model and examined its steady state features and the positive welfare effects of debt when equilibrium is dynamically inefficient. Blanchard (1985) has shown that government expenditure financed by debt or re-allocation of taxes has a negative effect on steady state capital and consumption levels, in general. These are not directly relevant for the focus of this dissertation, and hence not reviewed here. However, since government expenditure may take the form of investment in the formation of public capital as is generally the case in developing countries, there is clearly a gap that requires filling in the literature. As the focus of the next chapters, I examine the role of government expenditure in public capital formation, financed through issuance of debt, on long-run equilibrium, first with a decentralized endogenous growth model in chapter three, and secondly with an overlapping generations model with stationary steady state outcomes in chapter four.

Chapter 3

The burden of consumption in endogenous growth model

3.1 Introduction

At the center of any growth model is equilibrium consumption as the key measure of long-run outcomes. In the neoclassical growth models of Phelps (1956[95], 1965[96]), Diamond (1965)[44], Barro (1974)[15], and Blanchard (1985[24], 2019[23]), analysis on the effects of various government policies, and debt in particular, center around its effects on long-run consumption level. Similarly, endogenous growth models such as Barro (1990)[16], Aschauer (2000)[7], and Checherita-Westphal (2014)[37] have focused on the effects of any government intervention on the equilibrium growth rate of consumption. Thus, consumption is the single most important outcome variable in the analysis of long-run growth. Nevertheless, the recent models of endogenous growth have not adequately examined the question of returns to various forms of investment and how the burden of consumption is disposed off in equilibrium. Take the Aschauer (2000) and Checheritta-Westphal et al. (2014) models for example, the planning nature of their formulation effectively imposes a dictatorial restriction on household capital accumulation. In other words the central planner decides the shares of output that goes into consumption, investment in private capital, and investment in public capital stock, and hence equilibrium growth rate. This approach shelves the problem of capital accumulation as separate decisions between households who focus on inter-temporal consumption-savings problem, and hence the rate of private capital accumulation versus the government that focuses on public capital formation. But if these are considered separate decision problems, a central question that emerges is whether government consumes. If this is the case, then the government faces a dynamic problem like the households. This complicates equilibrium outcomes. On the other hand, if government does not consume out of returns to public capital, then the problem reduces to investing public returns in renewable public capital. In the context of a consuming household, whose returns on private capital is split between consumption and private capital formation, there may well arise a case where the rate of growth of consumption and private capital determined by the household does not coincide with the government's rate of public capital formation. This outcome in itself will be inconsistent with the notion of balanced growth path in endogenous growth models.

The above problem is at the heart of the analysis conducted in this chapter. I consider the rate of public capital formation as a government decision, and private capital formation as a household problem. In this case, I have abstracted from the assumption of state dictated behaviour - the central assumption shaping equilibrium outcomes in Aschauer (2000) and Checherita-Westphal et al. (2014). Thus, I examine a decentralized system in which long-run equilibrium is endogenously determined, and explore the conditions under which government public capital formation decisions are compatible with the households dynamic inter-temporal choice problem. The central result of this analysis is that the Social Planner's problem of Barro (1990), Aschauer (2000), and Checherita-Westphal et al. (2014) can be supported as a decentralized equilibrium outcome with lump-sum transfers to households. This is a novel result in this section of the literature, and broadly emphasizes the wisdom in the second fundamental welfare theorem in Economics.

3.2 Related Literature

Recent model of endogenous growth theory such as Barro (1990) and Agenor (2010)[2] have proposed models in which public infrastructure or government services is central to long-run growth. Along similar lines, Aschauer (2000) and Checherita-Westphal et al. (2014) have elaborated models in which the centrality of public capital to long-run growth is exploited to analyse the effect of government debt on equilibrium growth rate. Implicitly, these models suggest that much like debt finance, taxes for public investments can have positive long-run growth effects. Other contributions in the literature have focused on optimal taxation and the effect of consumption taxes on long-run economic growth (See for example, Jones, Manuelli, and Rossi; 1993[69], and Petrucci, 2002[94]). In Jones, Manuelli and Rossi (1993), a switch from one tax regime to an optimal tax policy induces large growth and welfare effects irrespective of whether labour is supplied elastically or inelastically and/or whether government expenditure is exogenous or endogenous.

The general approach of the models above is to formulate a Social Planner's problem and impose competitive conditions to characterise equilibrium. In the specific case of Aschauer (2000), for example, the competitive condition is in essence a competitive loan market where government trades bonds to finance public capital formation and private agents trade in private capital endowments. The necessary condition for government debt to exist in a closed economy in that environment is for the interest on government debt to equal the interest rate on private capital. In other words, the tax rate for public capital formation is set such that public capital earns the prevailing interest rate. Like Aschauer (2000)[7], the analysis of Checherita-Westphal et al. (2014)[37] broadly categorises capital into two forms; public capital and private capital. The public capital stock is provided by the government and used as input in private sector production. They derive a long-run equilibrium in which growth is endogenously determined, and in which the constant public to private capital ratio determines the efficiency level of the production system.

Their models aim ultimately to derive a so called "optimal public to private capital ratio" and by extension (in Checherita-Westphal et al., 2014) to derive debt-output ratios that are consistent with growth maximization in OECD, EU and euro area countries. Both papers go beyond the basic formulation to empirically estimate productivity effects of the various capital stocks. For the purpose of the ensuing analysis, the focus will be on the theoretical sections. Starting with the Aschauer (2000) model, which expresses a production function in per capita form as a Cobb-Douglas function of public capital, k_p, and private capital, k (i.e., $y = k^{\alpha} k_p^{1-\alpha}$). The returns to private capital in this context is deemed to broadly encompass returns to labour and private capital. In the basic framework, Aschauer imposes steady state conditions requiring output per worker, public capital and private capital per worker, as well as consumption labour ratio to grow at a common rate. This common rate is determined through optimising the sum of discounted lifetime consumption of the household using a relevant utility function. While both public and private capital grow at the same rate in the long-run, their exact ratio is taken as given. But maintaining any ratio of public to private capital in the long run will require the government to dictate private savings and consumption decisions.

Checheritta-Westphal et al. (2014) state the output function in aggregate form as $Y = [L^{\beta} K^{1-\beta}]^{\alpha} K_p^{1-\alpha}$, where K_p is aggregate public capital, L denotes total supply of labour and K is the private capital stock. This production function converges to Aschauer (2000), when labour supply is normalized to one. With the same utility function, long-run equilibrium in Checheritta-westphal et al. (2014) has the same dynamics as the former. Thus, both papers show that while the Social Planner can maintain any level of public-private capital ratio in equilibrium, there exist an optimal capital ratio that maximizes economic growth rate in the long-run. At the optimal level, "the government chooses a ratio of public to private capital so as to equate the after tax marginal product of private capital to the marginal product of public capital" (Aschauer, 2000). This condition has the effect of requiring both public and private capital to earn their respective marginal products, which are equated by virtue of competitive money market. But these conditions suggests that one needs not maintain the inconvenience of social planning problem to characterise equilibrium and/or show the effect of debt in the endogenous growth model. To properly examine the effect of government policies such as debt, for example, in this economy, the assumption of state-dictated household consumption behaviour appears too strong for a real world economy. Thus, it is important to decentralise the Planner's problem to evaluate the effect of government public capital accumulation efforts on household inter-temporal decisions and long-run equilibrium outcomes. This is the subject of the next section.

3.3 The model environment

I start with the planning problem of Achauer (2000), where the objective of the Social Planner is to optimise consumption of the representative household subject to a budget constraint. This objective requires the planner to allocate output between public capital formation, private capital accumulation and consumption at every point in time. The utility function of the household is taken to be of the CRRA (constant relative risk aversion) form;

$$U_t = \int_t^{\infty} \frac{c^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt. \tag{3.1}$$

Time is continuous and households discount consumption at the subjective rate of time preference, ρ. Note that σ is a positive constant for the degree of relative risk aversion (or the inverse of elasticity of inter-temporal substitution). Suppose that the initial stock of private capital is given, while public capital is procured by the issuance of government bonds. The government commits to service the debt indefinitely into the future but also maintain the public to private capital ratio. This entails growing public capital stock at the same rate as private capital. Thus, a tax rate on output is set for the purpose of debt service and public capital formation. In this case, maximising household utility is subject to the evolution of private capital given as,

$$\dot{k} = (1-\theta) k^{\alpha} k_p^{1-\alpha} - c,$$

where θ is the tax rate on output for investment in public capital and debt service, α is the output elasticity with respect to private capital, and c is consumption. The household inter-temporal choice problem solves to

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}[(1-\theta)\alpha\phi^{1-\alpha} - \rho],$$

where ϕ denotes the ratio of public to private capital (k_p/k). A relevant question concerns the the size of the tax rate to achieve equilibrium where all variables grow at a constant rate (debt, public capital, private capital, output, and consumption). Given no arbitrage condition, public capital is expected to earn its marginal product. But competitive money market imply that the marginal productivities are equal for public and private capital. The tax rate on output is set to achieve this equality, hence $\theta y = r.k_p$. From the consumption rule, $r = (1-\theta)\alpha\phi^{1-\alpha}$. Substituting for θ and the equilibrium interest rate is given as;

$$r^* = \frac{\alpha\phi^{1-\alpha}}{1+\alpha\phi}.$$

Thus,

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}[\frac{\alpha\phi^{1-\alpha}}{1+\alpha\phi} - \rho]$$

Optimizing the resulting growth rate of consumption with respect to the public-private capital ratio gives the maximum growth rate of the economy, and a corresponding optimal public to private capital ratio as;

$$\phi^{max} = \frac{\beta}{(1-\beta)^2}, \quad for \;\; \beta = 1-\alpha.$$

The optimal capital ratio serve as a reference economic efficiency criteria, with which the government can increase the long-run growth rate of the economy if the given ratio is below the optimal ratio[1]. The optimal ratio "maximises economic growth rate" and "equates the after-tax marginal product of private capital to the marginal product of public capital" (Aschauer, 2000). Checherita-Westphal et al. (2014) describes it as "maximising..[the] steady state growth of consumption and output". But this outcome has generally been shown as the product of a planning problem in Aschauer (2000) and Checheritta-Westphal et al. (2014). Thus, like Barro (1990)[16], the long-run equilibrium in which growth is endogenously determined (i.e., balanced growth path in which consumption, output and capital stock in per capita form grow at a constant rate) is not conceived under perfect competition where the Firm's problem is independent of the household. In the present formulation, where the requirement of competitive loan market necessitates that marginal productivity of private capital equate that of public capital, a government committed to maintaining the capital ratio in the long-run, effectively ensures that the production function becomes one with constant elasticity to "broad" reproducible capital. This is not significantly different to the canonical AK model under current assumptions. Therefore, given a bounded utility function, it amounts to positive long-term equilibrium growth rate in all variables for the representative agent under some conditions as shown by Acemoglu (2009)[41].

[1] Aschauer (2000) arrived at this result by making a number of simplifying assumptions. First, the government procures the initial stock of public capital by debt, but faces a lifetime budget constraint of the form;

$$k_{p0} + \int_0^\infty \dot{k_p}\ e^{-rt} dt = \int_0^\infty \theta.y\ e^{-rt},$$

where k_{p0} is the initial public capital stock, and θ is the tax rate. Secondly, for government debt to be feasible in a competitive money market, the interest rate on debt must equal the after tax returns to private capital. The tax rate on output, θ is shown to comprise two components. Part of it goes to service interest payments on debt, and another part goes into public capital formation at a rate consistent with the endogenous growth rate.

With the result that at the optimum capital ratio, the government chooses a ratio of public to private capital so as to equate the marginal product of public capital to the after tax marginal product of private capital, and the steady state budget constraint requiring total interest payments on debt (equivalently total returns on public capital) to equal total tax on output ($r.k_p = \theta.y$), the accumulation process of the capital stocks can be subjected to conventional perfect market. In other words, the initial public capital need not be procured by issuance of debt, rather a lumpsum capital tax on private capital can be used to provide the stock of public capital, with a tax on output amounting to $r.k_p = \theta.y$ used subsequently for continuous investment in the stock.

My intervention in this chapter is show that one needs not impose a dictatorial planning regime to to show the optimal equilibrium in the endogenous growth model. Indeed, allowing a competitive market environment allows us to make an important point regarding government decisions that facilitate optimal allocation of resources in the long-run. In particular, I can explicitly address the burden of consumption without having to assume that government dictates consumption for households.

3.3.1 Decentralized Aschauer economy

To decentralize the Aschauer (2000) economy, is to explicitly introduce the Firm's behavior and have equilibrium outcomes depend on conditions of profit maximization by the competitive Firm. This is deemed unimportant in the Planner's problem. Given the production function, and the absence of externalities such as new independent agents arising in the economy, the equilibrium capital ratio is required to be optimal in view of the First Fundamental welfare theorem in Economics.

Thus, for the representative agent framework, we need only replace the socially planned environment with a decentralized market structure and the ensuing equilibrium will be optimal, given the requirement of equal marginal productivities due to competitive loan market. But this evokes relevant question of the burden of consumption in equilibrium. This will be addressed subsequently. To make comparison with the original result of Aschauer (2000) straightforward, it is necessary to abstract from population growth and depreciation of both capital stocks.

The Firm

As above, the representative Firm is owned by the households and produces one unit of output by combining public capital and private capital. The government supplies public capital, while households supply private capital. As in Aschauer (2000) and Checherita-Westphal et al. (2014), I abstract from labour. The output function is given as;

$$y = k^{\alpha} k_p^{\beta} \text{ s.t } \alpha + \beta = 1 \tag{3.2}$$

The production function is Cobb Douglas technology, subject to constant returns to scale and diminishing returns to each input. Concretely, for any given k_p, $y = f(k)$ is such that $f'(k) > 0$ and $f''(k) < 0$. Given r_k as the rental price of private capital and r_p as the price of public capital, the Firm chooses k and k_p to maximise profits. The first order conditions imply that private capital and public capital are paid their marginal productivities. Hence, rent to private capital is given by;

$$r_k = \alpha k^{\alpha-1} k_p^{\beta} \tag{3.3}$$

and rent to public capital is given as;

$$r_p = \beta k^{\alpha} k_p^{\beta-1} \tag{3.4}$$

It is straightforward from the above that for the values of k and k_p demanded by the Firm to maximize profits, it operates at the margin, yielding profits as;

$$\pi = y - r_p k_p - r_k k \tag{3.5}$$

Substituting equations (3.3 and 3.4) into equation (3.5), one sees that $y = r_p k_p + r_k k$. This implies that;

$$y - \alpha k^{\alpha-1} k_p^{\beta}(k) = \beta k^{\alpha} k_p^{\beta-1}(k_p) \tag{3.6}$$

Dividing through by k^{β} gives

$$(\frac{k_p}{k})^{\beta} = \frac{(1-\alpha)y}{\beta k} \tag{3.7}$$

The ratio of public to private capital, $\frac{k_p}{k}$, satisfying equation (3.7) represents the Firm's chosen values of public and private capital given market prices. This result derives solely from the Firm's profit optimisation objective, without recourse to the household savings behaviour or the government's capital accumulation effort. The capital choices of the Firm reduces to;

$$\phi_* = (\frac{f(k)}{k})^{\frac{1}{1-\alpha}}. \tag{3.8}$$

As before, if the government can buy and sell capital in the capital market and given no arbitrage condition, the returns on private capital will be expected to equal the returns on public capital, hence $r_k = r_p$. Observe that this balance is ensured by inada conditions on the output function. Using this, equation (3.8) modifies to:

$$\phi^d = \frac{k_p(t)}{k(t)} = \frac{\beta}{1-\beta} \tag{3.9}$$

[2] Thus, the capital ratio that is consistent with the amount of public and private capital demanded by the Firm is given by ϕ^d.

[2]Note that this result relies largely on specific features of the Cobb-Douglas technology. First the

The Household

Unlike the planning problem where the household budget constraint is dependent on the government's choice of tax rate, here the household makes consumption and savings decisions based on the returns to private investments. Thus capital evolution,

$$\dot{k} = (1-\theta)k^{\alpha}k_p^{1-\alpha} - c,$$

modifies to;

$$\dot{k} = r_k k - c$$

The Household problem can be represented by the current value Hamiltonian;

$$H[\lambda, c, k] = \frac{c^{1-\sigma} - 1}{1-\sigma} + \lambda[r_k k - c] \tag{3.10}$$

The first order conditions for optimum are:

$$\frac{\partial H}{\partial c} = c^{-\sigma} - \lambda = 0$$

$$\frac{\partial H}{\partial k} = r_k \lambda = \rho\lambda - \dot{\lambda}$$

$$\frac{\partial H}{\partial \lambda} = r_k k - c = \dot{k}$$

---

imperfect substitutability of public and private capital implies that the Firm operates at a certain marginal rate of technical substitution to ensure that marginal productivities meet the competitive loan market condition. If the production function is of the general CES type with the substitution parameter equalling one (i.e., the elasticity of substitution between public and private capital approaching infinity), then public capital will be a perfect substitute for private capital. In this case, there will be no justification for the Firm to demand public and private capital in the fixed proportion shown in equation (3.9). The reason being that the linear isoquant and constant technical rate of substitution (TRS) will co-exist with the competitive loan market condition irrespective of the amount of private capital relative to public capital. However, a Cobb-Douglas specification allows some degree of flexibility in the substitution of public for private capital. But the assumption of a competitive loan market where the government competes for funds to provide public capital effectively imposes a constraint and provides the flavour of a Leontief production function. If the Firm increases its use of public capital, it will need to increase private capital by a similar proportion to keep the capital ratio constant and hence maintain balanced marginal productivities to meet the loan market condition. This effectively underlies some degree of complementarity between public and private capital. Second, the constant returns to scale condition is required to ensure that the Firm operates competitively. Were $\alpha + \beta < 1$, for example, there will be no reason why the Firm will not make positive profits if it pays the marginal products for public and private capital. The same argument goes for the Firm having to make losses if $\alpha + \beta > 1$.

Using the first two F.O.Cs, the household inter-temporal problem solves to:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}[r_k - \rho].$$

In this formulation, the only condition for positive growth in the long-run is for $r_k > \rho$. But how is this possible when $f''(k) < 0$? If public capital is exogenously given and constant across time, then there will exist a $\tilde{k}$ such that $r_k = \rho$, for $\rho > 0$. However, if public capital is continuously adjusted at the same rate as private capital, then r_k may be constant.

The Government

So far, I have made no assumption regarding the formation process of public capital. Suppose the government does not consume and engages only in productive public investment to produce public capital. In this case, the change in public capital stock at any point in time is given by $\dot{k_p} = r_p k_p$. Abstracting from population growth and depreciation, the rate of growth of public capital compared with private capital is:

$$\frac{\dot{k_p}}{k_p} = r_p, \quad \textit{while} \;\; \frac{\dot{k}}{k} = r_k - \frac{c}{k}$$

Suppose also that the government enters the domestic loan market for finance to invest in public capital formation. If the government pays a lower interest rate than private capital, people will choose to not invest in government bonds. Conversely, if government pays a higher interest rate, private agents will invest only in government bonds. Since both investment options are risk-less, public and private capital will co-exist in the economy if government pays the prevailing interest rate, hence $r_k = r_p$. The equilibrium requires $\dot{k}/k = \dot{k_p}/k_p$, hence, $r_k - \frac{c}{k} = r_p$. It follows from the foregoing that for $c > 0$, $r_k \neq r_p$. In particular, $c > 0$ if $r_k > r_p$ given $c = (r_k - r_p)k$. But this is incompatible with competitive money market. As long as the burden of consumption is borne only by returns to private capital, the marginal returns must differ for positive consumption along the balanced growth path.

3.3.2 Endogenous growth in equilibrium and the burden of consumption

The potential for different rates of accumulation for public capital and private capital arises here because the accumulation processes are considered independent. This need not arise in Aschauer (2000) and Checherita-Westphal et al. (2014) with the decision processes centralized. In the present environment, the requirement of competitive loan market, and hence equal marginal productivities imply different growth rates of public and private capital so long as government does not consume and there exist no inefficiencies in public investment. For public and private capital to grow at a common rate, the burden of consumption must be shared symmetrically. To see this, note that $\frac{\dot{k}}{k} = \frac{\dot{kp}}{kp}$ is an equilibrium requirement. When this is fulfilled, then $r_k = r_p = r$ if and only if

$$r_k - \frac{c}{2k} = r_p - \frac{c}{2k},$$

otherwise $r_k \neq r_p$. This is shown by substituting $c = (r_k - r_p)k$ into the above equation.

To finally characterise the endogenously determined equilibrium, it is important to note that equilibrium must solve not only the Firm's problem, but also the household inter-temporal choice problem. At the same time, the solution to the household problem must ensure that the rate of private capital accumulation is consistent with the rate of public capital accumulation. But if the government enters the domestic capital market to raise funds for public investments so that at any point in time, the value of public capital equals the amount of government debt, then the public capital stock is effectively owned by households. In this case, the amount of debt (and hence public capital) can be treated as a state variable in the household budget constraint. Here, the government need not exact a tax per capita to form public capital. Since households own private capital and public capital through their holding of government debt, the household budget constraint can be restated as;

$$\dot{a} = \dot{k} + \dot{kp} = r_k k + r_p kp - c \tag{3.11}$$

This is the same as the household budget constraint in the planning problem, where $\dot{k} = (1-\theta)k^{\alpha}k_p^{1-\alpha} - c$, if $\dot{kp} = \theta y$. Again, the rate of growth of consumption is $\frac{\dot{c}}{c} = \frac{1}{\sigma}[(1-\theta)\alpha\phi^{1-\alpha} - \rho]$. With rent to public capital equaling rent to private capital, $r_k = r_p = r$, the long-run equilibrium with positive growth is given by $\frac{\dot{k}}{k} = \frac{\dot{kp}}{kp} = \frac{\dot{c}}{c}$, and hence, for $\sigma = 1$, equilibrium imply:

$$r - \frac{c}{2k} = (1-\theta)\alpha\phi^{1-\alpha} - \rho,$$

where $\phi = \phi^d$, given by conditions of profit maximization by the Firm within a competitive loan market. The exact value of θ that meets the above condition remains to be determined. If we suppose, as in Aschauer (2000), that $\theta y = r.kp$, then it is now immaterial whether public capital is given and owned by the government, or whether it is owned by households through holding government debt in their asset portfolio. In either case, there must be some transfers from government (either in the form of interest on debt or transfers on returns to public capital) to households each period to ensure that public capital grow at the same rate as private capital as seen above. Therefore equilibrium with endogenous growth is characterised by a constant rate of return to capital, given by:

$$r^* = \frac{\alpha\phi^{*1-\alpha} - \rho + \frac{c}{2k}}{1 + \alpha\phi^*} \tag{3.12}$$

The equation for interest rate in equilibrium suggests that all that is required for a competitive equilibrium in the endogenous growth model is the presence of competitive loan market. For any level of private capital, there is a corresponding value of public capital that equalises the marginal productivities. This gives a unique public to private capital ratio in equilibrium. Thus, instead of issuing debt to form public capital, the initial supply of public capital by the government can be procured by a lump-sum tax on private capital and must be such that marginal productivities are equalised, given the constant output elasticities.

A notable feature of equilibrium in any growth model is the requirement to solve the control variable in terms of the state variable. Technically, there are two state variables growing at the same rate, i.e., public and private capital. But in terms of the household consumption-saving problem, the relevant state variable is the private capital. Thus I characterise consumption along the equilibrium path as a function of the private capital stock. This is given as:

$$c(t) = ([1 + \alpha\phi^*]r^* + \rho - \alpha\phi^{*1-\alpha}) * 2k(t) \tag{3.13}$$

Clearly, by equation for $c(t)$, the consumption value at any given time in equilibrium can be determined by the relevant value of private capital. As observed above, competitive loan market and no arbitrage condition requires the marginal products of public and private capital to equalise. At this point, the economy will grow positively in the long-run as long as the constant returns to capital in equilibrium exceeds the rate of time preference[3]. In addition, given the market rental prices of public and private

[3]It should be noted that this result depends on the abstraction from labour supply in the production function. Suppose the aggregate production function is Cobb-Douglas across three inputs (i.e., $Y = K^{\alpha}K_p^{\beta}L^{1-\alpha-\beta}$ s.t. $\alpha + \beta < 1$), it is clear that output is subject to diminishing returns to each input.

capital, the Firm demands $\phi^d = k_p(t)/k(t) = \beta/(1-\beta)$. For this ratio to stay constant, investments in public and private capital must proceed in similar proportions. In other words, for each period, $\dot{k_p} = \beta(r_k k + r_p k_p - c)$, while $\dot{k} = (1-\beta)[r_k k + r_p k_p - c]$. Where r_k, r_p given by equations (3.3. and 3.4) respectively are such that $r_k = r_p = r^*$ given competitive loan market, and $c(t)$ given by (3.13). Also note that for any initial endowments, k_0 and k_{p0}, ϕ_0 will adjust to $\phi^* = \phi^d$ given by the Firm's demand schedule. Therefore, if the given values of k, k_p are not consistent with the ratio required for competitive equilibrium, redistribution of capital may be required. This is necessary only if government own the initial stock of public capital, so that for $k_p > \phi^d k$, a conversion of $k_p - \phi^d k$ into private capital will be required for productive efficiency.

3.3.3 Allocative inefficiency of the growth-maximizing capital ratio in the Planning problem

The idea that the growth-maximizing capital ratio in the Social Planner's problem can yield inefficient allocations may not seem apparent at first glance. To see why it is associated with inefficiency, observe that maximizing the growth rate with respect to the capital ratio, as in Aschauer (2000) and Chechetita-Westphal et al. (2014), leads to the public-private capital ratio being $\phi^{max} = (1-\alpha)/\alpha^2$ at the optimum. Compared with the competitive equilibrium where the capital ratio is given by $\phi^* = k_p/k = (1-\alpha)/\alpha$, which is dictated by the Firm's demand schedule and no arbitrage condition, this becomes clearer. Note that $\phi^{max} = (1-\alpha)/\alpha^2 > \phi^d = \phi^* = k_p/k^* = (1-\alpha)/\alpha$, for $\alpha < 1$. In words, the growth maximizing ratio of public to private capital maintains a higher capital ratio than that realised in competitive equilibrium. Since at ϕ^d given by equation (3.9), $r_k = r_p$. It follows that at ϕ^{max}, $r_k > r_p$. As shown by Aschauer (2000, pg 7 of paper and 349 of journal issue), the Social Planner maintains this ratio of public to private capital by imposing a tax on marginal productivity of private capital (i.e., θ is set such that $(1-\theta)r_k = r_p$). This is necessary to ensure than government debt pays the same rate of return as the after-tax rate on private capital, but importantly it is this feature that keeps the economy growing at higher rate than it would in competitive equilibrium.

Note that if the tax on private capital productivity is invested in public capital forma-

Thus, if labour is supplied inelastically, output will be diminishing across public and private capital altogether, for any given L. In this case, it will be difficult to justify positive growth in the long-run.

tion, it raises the relative size of public capital, and consequently the relative productivity of private capital. Thus, households anticipating this are compelled to raise their savings in private capital to take advantage of a higher capital productivity. Clearly, household are "forced" to save at a higher rate than they would without the tax. In this procedure, the government ensures a higher aggregate capital savings rate and hence higher economic growth rate. But this is done at the expense of consumption each period. Note that in the absence of taxes, the government can only issue debt to provide public capital if $r_k = r_p$. Thus, at ϕ^{max} where $r_k > r_p$, the government will be compelled to sell off public capital to raise its marginal productivity, and hence revert back to the competitive equilibrium ratio, ϕ^*. The fact that consumption along the balanced growth path may be relatively higher in the competitive case arises from the fact that $r_k = r_p$. This suggests that the relative marginal productivity of private capital is lower in competitive equilibrium compared to the growth-maximizing path. With a lower relative productivity for private capital, households would save less and consume more at the present than if the productivity was higher.

In summary, the analysis here shows that a government can keep a higher economic growth rate than may be realised in competitive equilibrium. However, this is done by using taxes to raise the savings rate and keep present consumption levels low. Thus, in general, if the objective is to raise the growth rate of an economy irrespective of the temporal cost to utility optimization, the planning solution may deliver a superior outcome. This would not be realised in a competitive solution due to the desire to maintain a balanced consumption plan dictated by the inter-temporal utility problem, and a balanced investment portfolios in private capital and government debt dictated by competitive loan market.

3.3.4 Numerical Illustration

In this subsection, I numerically compute the endogenous growth rate in equilibrium, given an initial level of private capital and the exogenous output elasticities. In addition, I show that not only does this equilibrium satisfy the market clearing and break even conditions, it also satisfy the requirement of competitive money market. Thus, the equilibrium interest rate characterised above coincides with the marginal productivities of public and private capital. Similarly, we would see that for the given private capital and the corresponding value of public capital required for equilibrium, consumption is a linear function of the level of private capital, and is also satisfied. Finally, the growth rate of consumption, private capital, public capital, and output are all shown to be equal in equilibrium. The exogenous parameter values and starting level of capital in the economy are presented on table 3.1.

Table 3.1: Chosen Parameter values for exogenous variables

$\alpha = 0.78$	$\beta = 1 - \alpha = 0.22$
private capital, $k = 454.545456$	public capital, $k_p =?$
discount rate, $\rho = 0.05$	

The output elasticities are chosen to satisfy the Cobb-Douglas specification.

Given the private capital stock on table 3.1, and the exogenous output elasticities, the market clearing and competitive money market conditions imply that the corresponding public capital required for equilibrium is given as $k_p = \phi^* k = \frac{(1-\alpha)}{\alpha} k$. Thus for $k = 454.545456$, the level of public capital that must be provided by government to satisfy equilibrium conditions must be $k_p = 128.205128615385$. But since this will be taxed from the private capital stock in the closed economy, the task of government is to determine the proportion of private capital that must be converted to public capital to equate marginal products. Starting at time, t, and using the output elasticities, $k_t = \alpha k = 354.545456$, and $k_{p_t} = (1 - \alpha)k = 100.0$. Given these, the balanced growth path and corresponding values for five periods is presented on table 3.2.

As seen in the table, all conditions for equilibrium are satisfied with the competitive money market condition ensured. Comparing figures 3.1 and 3.2 reveals the differences between the growth maximizing solution in the planning problem and the competitive equilibrium. Starting at time 1, with the same level of aggregate capital (sum of public and private capital) and exogenous constants being equal, one observes that the planning solution on figure 3.2 achieves a redistribution with higher amount of public capital relative to private capital when compared with competitive solution on figure 3.1. By imposing a higher ratio of public to private capital, the Social Planner obtains an economic growth rate of 43.6% compared with a rate of 31.0% in the competitive solution.

Nevertheless, for the five periods for which the equilibrium values are computed, it is seen that consumption is higher in every period in the competitive case than the planning solution. At time 1, consumption is 127.65 in the competitive equilibrium compared

Table 3.2: Competitive equilibrium path for five periods in the endogenous growth model

Time	Parameter	value
t	Private capital, k	354.545456
	Public capital, kp	100.0
	Output, y	268.3762625
	Consumption, $c(t)$	127.5612785
$t+1$	Private capital, k	464.3811435
	Public capital, kp	130.9792965
	Output, y	351.5173403
	Consumption, $c(t+1)$	167.0788651
$t+2$	Private capital, k	608.2431541
	Public capital, kp	171.555761
	Output, y	460.4149388
	Consumption, $c(t+1)$	218.838722
$t+3$	Private capital, k	796.6726032
	Public capital, kp	224.7025287
	Output, y	603.0482472
	Consumption, $c(t+3)$	286.6334184
$t+4$	Private capital, k	1043.47617
	Public capital, kp	294.313791
	Output, y	789.8683507
	Consumption, $c(t+4)$	375.4304345

Notes: The constant equilibrium Interest rate, $r^* = \partial y/\partial k = \partial y/\partial k_p = 0.590427775$, the endogenous growth rate, $\frac{\dot{k}}{k} = \frac{\dot{k_p}}{k_p} = \frac{\dot{y}}{y} = \frac{\dot{c}}{c} = 0.309792964$, and finally, the public-private capital ratio, $\phi^* = k_p(t)/k(t) = 0.282051281$.

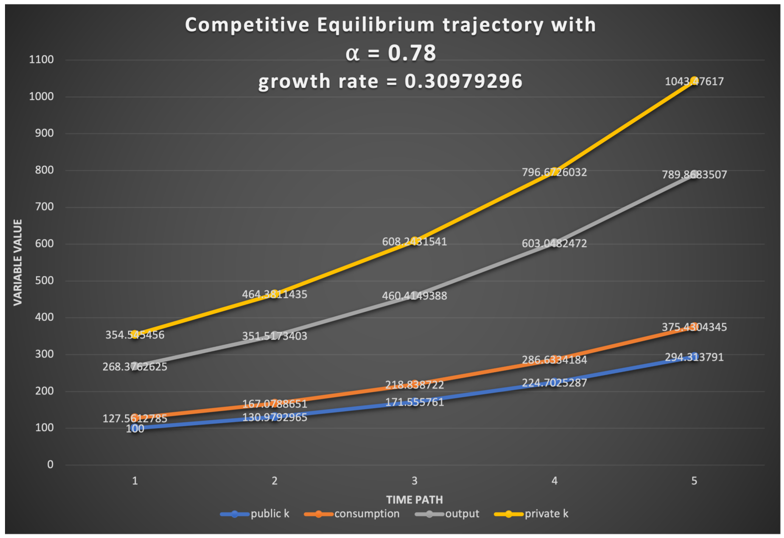

Figure 3.1: Endogenous growth in competitive equilibrium

Notes: For the given initial endowment, k, on table 3.1, the balanced growth path illustrated on the figure starts out with a redistribution into public and private capital to achieve the ratio $\phi^* = k_p(t)/k(t) = \beta/1-\beta$. Then, following production with the starting capital stocks, consumption is determined by equation (3.13) as a function of $k(t)$, given $\phi^* = \phi^d$.

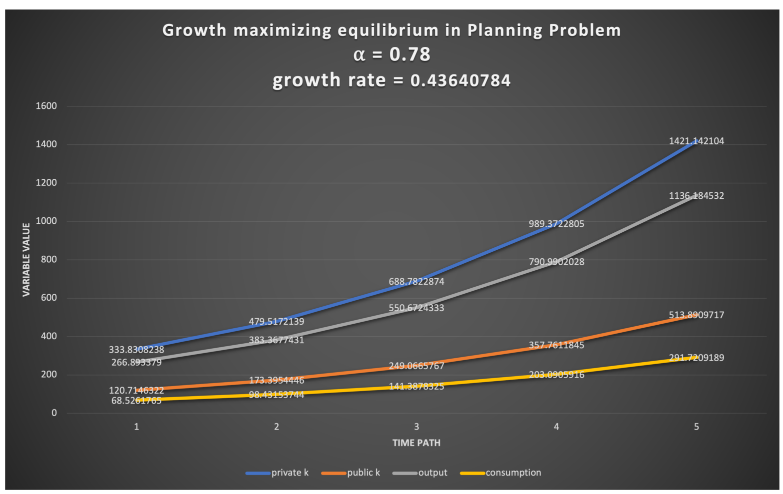

Figure 3.2: Growth maximizing equilibrium in Planning solution

Notes: Unlike the competitive case, here the initial endowment, k, on table 3.1, is redistributed into public and private capital to achieve a ratio approximating $\phi^{max} = k_p(t)/k(t) = \beta/(1-\beta)^2$. Then, following production with the starting capital stocks, consumption for the period is determined by first estimating the shares of output that goes into investment in the respective capital stock to achieve the maximum growth rate. The maximum growth rate is determined in Aschauer (2000) and adapted here for the computations. Using this growth rate automatically yields the required investments in public and private capital and hence the feasible amount of consumption for the period.

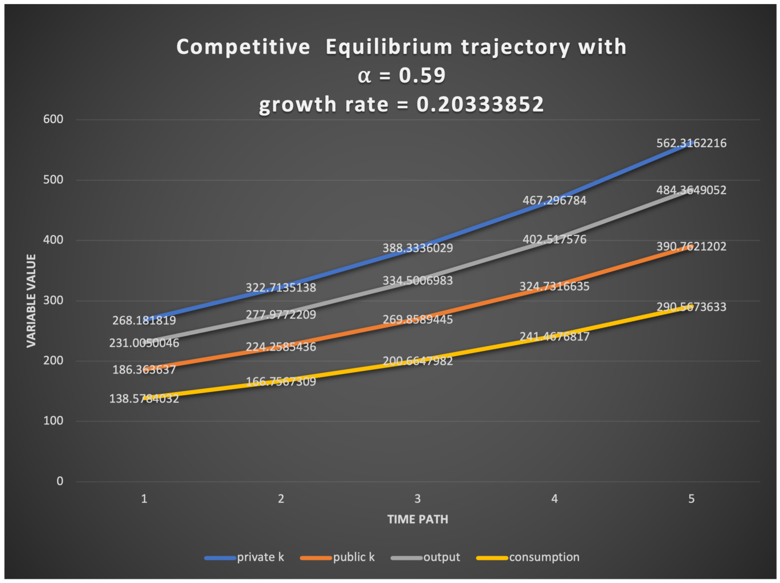

Figure 3.3: Endogenous growth in equilibrium with $\alpha = 0.59$

Notes: The balanced growth path here is computed the same way as in figure 3.1, except that α is now 0.59. The difference in growth rate compared to fig 3.1 arises solely from the differences in output elasticities, and hence ϕ^*.

with 68.53 in the planning solution. Similarly, at time 5, consumption values are 375.43 and 291.72 in the competitive and planning solutions respectively. Clearly, the planning solution may "force" a higher economic growth rate at the expense of periodic consumption. Thus, when one views this result in the context of finite life, allocations of output between various investment options and consumption may be inefficient.

Aside the observed differences between the the two frameworks, a notable feature is the role of the constant output elasticities in shaping the equilibrium growth rate. In other words, despite the notion of endogenously determined equilibrium, the exogenous output elasticities are central to the equilibrium growth rate in the competitive case. To see this, observe that when α is changed from 0.78 to 0.59, figure 3.1 modifies to figure 3.3 with the equilibrium growth rate falling from 31.0% to 20.3%. Thus, unlike the planning solution where the Government can use the tax rate to adjust the long-run economic growth rate, the competitive case arrives at a unique equilibrium growth rate for any given stock of capital. This is ensured by the requirement of competitive loan market and no arbitrage condition.

3.4 Debt in the competitive equilibrium

As we have seen from the competitive model, the equilibrium with endogenously determined growth is unique in much a similar fashion as the classic neoclassical growth model. Thus, the public to private capital ratio is unique and optimally determined. This is due in large part to the presence of competitive loan market, and hence equalled marginal productivities. In the panning problem, the Social Planner may well maintain any level of public to private capital ratio. Thus equilibrium may not be unique in that environment.

The relevant question here is the role of debt. As stated above, public debt may serve as the source of public capital in the economy. In this case, the level of debt equals the public capital stock and grows at the common rate with output, consumption and public capital. But since marginal productivities are equalled in the competitive equilibrium, it is difficult to see a positive welfare effect of a change in debt trajectory. In other words, from the competitive equilibrium, issuance of debt to change the size of public capital relative to private capital is unlikely to have a positive welfare effects as it distorts the marginal product of public capital relative the private capital. This result notwithstanding, a Social Planner may find it optimal to intervene in the decentralized economy and distort outcomes along the balanced growth path depending on the objective being pursued. Recall that the growth-maximizing capital ratio from the planning problem is greater than the capital ratio that arises in decentralized equilibrium. As a result, if the government aims to maximise economic growth irrespective of the cost to inter-temporal allocations, the existence of an interval between ϕ_{max} and ϕ^d offers an opportunity for debt and tax on private capital productivity to be used to increase the economic growth rate. The necessity of a tax on private capital productivity reflects the need to reduce the private returns to investments to match the marginal product of public capital, if government increases the relative size of public capital from the decentralized equilibrium value. This is intended to meet the loan market condition. But the welfare effect of this distortion to the market equilibrium may be negative, especially in the context of finite life. Therefore, analysis on the welfare effects of government debt in competitive equilibrium requires a different model environment than the endogenous growth model.

3.5 Conclusion

In this chapter, I have decentralized the Social Planner's problem of Barro (1990), Aschauer (2000) and Checherita-Westphal et al. (2014), and have shown that endogenous growth can be supported as a competitive equilibrium outcome without the imposition of dictatorial decision making regime. Fundamentally, this outcome suggests that given an initial level of private wealth, the endogenously determined equilibrium growth rate is optimal and entails a constant interest rate for broad capital. However, the capital ratio in competitive equilibrium is different from the one that may be obtained by the Social Planner in the quest to maximise economic growth. A competitive loan market, and no arbitrage condition, presupposes that both public and private capital pay the same rate of return. However, the existence of equilibrium requires government to make transfers from the returns on public capital to households in each period to support consumption. This transfer is necessary to allow for a common growth rate of public and private capital. But the consequence of competitive equilibrium outcome is that debt to raise the level of public capital relative to private capital does not appear to be welfare improving. It lowers the marginal product of public capital. In this case investment in public capital will pay lower than corresponding investment in private capital. This will be inconsistent with the existence of competitive equilibrium. Thus a different model environment is required to examine the welfare effects debt-financed public investments in equilibrium.

Chapter 4

National debt, Public capital and welfare in overlapping generations model

4.1 Introduction

Public debt has remained a subject of theoretical enquiry since Adam Smith's[111] wealth of nations. Early literature on the effect of national debt in an economy examined it in a neoclassical framework where funds from publicly issued debt is distributed to all or part of the population for consumption. In Diamond's (1965) seminal contribution, such a government activity will increase or decrease longrun welfare depending on the nature of the equilibrium. Given a market interest rate beneath the equilibrium growth rate of the economy, maintaining a constant debt-GDP ratio - equivalently a constant debt labour ratio - imply that in aggregate, newly issued debt will exceed overall value of outstanding debt every period. The excess amount of resources may be transferred to the older generation for consumption. Implementing a policy of intergenerational transfers indefinitely in this form enhances overall utility. But this feature is peculiar to the overlapping generations model where existing agents do not internalise the emergence of new agents in their inter-temporal utility problem. The extension of the Diamond model by Barro (1974)[15], where bequest motive is incorporated into the overlapping generations setup leave different equilibrium effect of debt. In the latter, debt has no effect on net wealth as the Household utility structure accounts for intergenerational linkages (akin to the dynasties framework of Ramsey, 1928), making equilibrium saving decisions optimal. Blanchard (1985)[24] extended the overlapping generations framework to a continuous time setting and examined the effects of life cycle features such as mortality rates and retirement on steady-state wealth and consumption levels. These have furthered understanding on the equilibrium effects of debt in a closed economy. In general, both external and domestic debt reduce utility in the long run in the normal case where equilibrium is pareto optimal.

Another segment of the literature, the endogenous growth models, have focused on an environment in which public capital or infrastructure, otherwise called public services, enter as an input in the private sector production function[1]. Building on these, Checheritta-Westphal et al (2014)[37] used the result on optimal public to private capital ratio in Aschauer (2000)[7] to derive the so called growth maximizing ratio of public debt to GDP. In the endogenous growth model, issuance of public debt for investment in public capital enhances the long-run growth rate if the initial capital ratio maintained by the government is inefficient. As we have seen in the previous chapter, when the Planner's problem is decentralized, the competitive equilibrium is optimal and the capital ratio is efficiently determined, thereby leaving no room for positive welfare effect of debt. In addition, if the Social Planner who dictates households consumption and saving decisions aims to optimise utility for households, then there is no reason a priori to maintain an inefficient ratio of public to private capital in equilibrium. Thus, as noted by Greiner (2012)[58], debt can be replaced by a corresponding tax rate in the endogenous growth model of Checheritta-Westphal et al (2014), and the balanced growth rate maximized with respect to the tax rate. In addition, if the Social Planner aims to optimise utility for the representative household, then the planning problem, like the competitive equilibrium outcome, leads to optimal allocations and debt accumulation will be sub-optimal thereafter. Given this outcome, the important question is why debt-financed public investments is deemed to be welfare or growth improving in policy discussions on government debt (See Ceccheti et al., 2011[35], for example). In what ways does debt financing of public investment improve long-run outcomes in developing economies?

[1] For example, see Barro (1990)[16] where the government is unrealistically assumed to dictate household consumption and hence can maintain a given ratio of government services to private inputs in production. Aschauer (2000)[7] and Agenor (2010)[2] consider this public input as comprising of the public infrastructure provided by the state and to a lesser extent health services in the latter.

In this chapter, I employ the Aschauer (2000) production function and the utility structure of Blanchard (1985) to examine the effect of debt in a competitive equilibrium where public capital provided by the state is an input in private sector production. As a standard result of the overlapping generations model, the competitive equilibrium interest rate may be beneath the growth rate of the economy. This is a requirement in Diamond (1965), for example, for debt to play an optimal role in equilibrium. In contrast, the population growth rate exceeding equilibrium interest rate need not be a requirement for debt accumulation to be welfare enhancing in the simple model presented here. Debt issued externally for investment in public capital enhances the rate of return to private investments and increases steady state private capital and consumption. Domestic debt has similar effect on the rate of return to private investment, but has the added effect of reducing the short-term supply of private capital, thereby inducing a greater effect on interest rate. The relative equilibrium effect of external debt as opposed to domestic debt will depend on the interest rate in the international capital market compared to the domestic economy. Another important result of the model is that unlike in Barro (1974)[15] where an operative intergenerational transfer in the form of bequest is key in showing the non-existence of real effects of debt, infinite existence of the economy in the environment here requires present generation to endow new agents with assets. This requirement is implemented via a positive tax on returns to private investments by the government. Thus, government tax has a twin objective of maintaining public capital in the face of depreciation and providing new agents with starting capital.

In the model, there are important implications for the amount of debt that can be supported from any equilibrium. We would see that at steady state, public debt issuance for investment in public capital, subject to government efficiency, can be supported if it enhances equilibrium private wealth and consumption. It is shown, however, that beyond a threshold, debt accumulation for investment in public capital ceases to improve private wealth and consumption. In particular, from an equilibrium point, issuance of debt domestically is limited by the available amount of private wealth. In other words, there is a threshold beyond which the debt labour ratio cannot be supported and the government may have to sell capital and reduce the amount of government debt. The amount of debt that can be supported will be deemed sustainable in the sense of Blanchard et al (1991)[26], where the government can sustain a given debt ratio to GDP (equivalently debt labour ratio) indefinitely with a stable level of taxes. The model abstracts from all forms of uncertainty except the uncertainty of lifetime. Incorporating other important uncertain features may require a probabilistic formulation as in Blanchard and Das (2017)[25]. This is beyond the scope of this thesis, however. In the present context, the limit to debt-financed investment in public capital (and hence debt sustainability) concerns whether or not the debt level enhances returns to overall investment in the production system than it increases interest rate in the capital market. It should be stressed that the analysis here does not compare the relative costs and benefits of debt or tax financed investment. Rather, it focuses on debt as a specific tool for financing public investment and explores conditions under which such state activity is utility enhancing.

4.2 The basic model

Consider a small closed economy with utility structure and demographic features of the Blanchard (1985)[24] overlapping generations model. Time is continuous and the population grows at the rate of $n = b - m$, the difference between birth rate and mortality rate. Thus, the economy is infinitely lived while agents have finite lives and composed of many generations, with $L(t) = L(0)e^{nt}$, and $L(0)$ being a historically given population of the economy. The economy comprises of a representative Firm that employs inputs to produce output, a government that provides public capital and coordinate taxes and transfers, and households who maximise life-time expected utility.

4.2.1 The Firm

Following Aschauer (2000)[7], the production function is stated in per worker terms as follows;

$$y = f(k, k_p) = k^{\alpha} k_p^{\beta} \; ; \alpha + \beta = 1. \tag{4.1}$$

The output function is characterised by constant returns to scale in per worker terms, but is subject to increasing returns across raw labour and capital (Aschauer, 2000). In terms of the exact inputs, it is easy to think of private capital purely as private assets and rule out labour from the production function, or consider it in terms of Checheritta-Westphal et al (2014), where private capital constitutes comprehensive inputs from the private sector (comprising of labour and private capital). For the purpose of the analysis here, private capital refers to physical capital held by private agents. As in the previous chapter, r_k and r_p are the market given rental prices of private and public capital. Profit maximization by the Firm imply:

$$r_k = \alpha k^{\alpha-1} k_p^{1-\alpha} \;\; and \;\; r_p = (1-\alpha) k^{\alpha} k_p^{1-\alpha-1} \tag{4.2}$$

Observe that $r_k = \partial y/\partial k$. For any k_p, let $\partial y/\partial k = f'(k)$. Since private agents supply the private capital stock, the returns to private investments net of depreciation can be written as, $RtP = r_k k - \delta k$, where $\delta > 0$ is depreciation rate for private capital. Note that given $r_k = f'(k)$, the total returns to public capital in per capita form can be written as $r_p k_p = f(k) - k * f'(k)$.

4.2.2 The Government

The government plays two roles in the economy. Unlike in the previous chapter, here I allow for depreciation and population growth. Therefore, the first role of the government involves supplying the stock of public capital, k_p, and exacting taxes on private agents to maintain the public capital stock in the face of depreciation. Secondly, as will be seen in the household utility function, existing agents do not "care" about new agents in their inter-temporal choice problem. In other words, there is no bequest motive in the household utility function. But, by the Aschauer (2000) output function, labour is not a production input, and hence do not earn wages. This suggests that consumption for newly born agents will be zero, since agents consume out of returns to private investments. Thus, the economy may not be viable beyond generations with private capital endowment. To ensure infinite existence, the government is assumed to provide newly born agents with the average private capital in the economy by imposing a capital gain tax on existing agents. Since at each point in time, the number of newly born agents is given by the birth rate of the economy, the capital provision in aggregate equals $b * K(t)$, where $K(t) = k(t) * L(t)$. In per capita terms, this is simply $b * k(t)$. The net returns to private investments at time, t, modifies to $RtP_n = r_k k(t) - dk(t) - bk(t)$. Define the net returns per unit of private capital (the interest rate, net of capital gain tax and depreciation) as;

$$r = \frac{RtP_n}{k} = r_k - d - b = f'(k) - d - b \tag{4.3}$$

Given the rental price of public capital, total returns to the Government from the Firm equals $r_p k_p$. The relevant question here is what the government does with the returns to public capital. There are atleast two ways to think about this. The first is to assume that government engages in unproductive expenditures (e.g., consumption), and the second is to assume that all returns to public capital is invested back into its formation. Neither of these is desirable for the problem under consideration in this chapter. The first introduces inefficiencies into the system, while the second potentially yields endogenous growth in equilibrium. As examined in the previous chapter, the equilibrium with endogenous growth is optimal given the requirement of competitive loan market. This leaves no room for a positive welfare effect of debt-financed investment in equilibrium. Thus, it is convenient to rule out government consumption and endogenous growth in equilibrium by supposing that the government aims to maintain the per capita value of public capital. Given depreciation rate as $\delta_p > 0$, it implies public investment of $\delta_p k_p$ per capita each period. If $r_p > \delta_p$, the net returns to public capital $(r_p - \delta_p)k_p$ is transferred to the household for consumption. Where $r_p < \delta_p$, the difference is financed by a tax on the household. From the above, we can write the aggregate evolution of private capital as;

$$\dot{K} = [r*k(t)+(r_p-\delta_p)k_p]*L(t)-C(t)+b*K(t) = (y*L)-\delta_p K_p(t)-\delta K(t)-C(t) \quad (4.4)$$

Note that the term $b * K(t)$ is the sum total of private capital provided to newly born agents, which is rented out to the Firm.

4.2.3 The Household

Following Blanchard (1985), I use the words “labour” and “population” interchangeably for expositional convenience, even though labour supply is ruled out. As standard of overlapping generations, an individual constitutes a household and faces an instantaneous probability of death (generalized as the mortality rate), m. Let $u(c) = \ln c$. The household has a life time expected utility (expectation defined by the probability of death) function of the form;

$$U_t = \int_t^{\infty} \ln c(v,s) e^{-(\rho+m)s} ds, \quad (4.5)$$

where $c(v,s) > 0$ and $0 < \rho < 1$ is the consumer's rate of time preference. The strict inequality imposed on $c(v,s)$ needs only be non-negative if $u(c)$ is of the general CRRA form with $\sigma \neq 1$. The wealth of the household sector is rented out to the Firm for production, but instead of receiving $r(t)$ they receive $r(t) + m$ due to the negative life insurance contracts that exist in the economy (see Blanchard, 1985). Note also that each household pays/receives $(r_p - \delta_p)k_p$ as tax/transfer from the government. Since we rule out labour from the output function, wages as a separate return does not arise. Therefore, the rate of change of household wealth is given by:

$$\frac{\partial a(v,s)}{\partial s} = \dot{a}(v,s) = (r(s)+m)a(v,s) + (r_p - \delta_p)k_p - c(v,s)$$

Define $\phi(t) = k_P/k(t)$ as the ratio of public to private capital for the average household at any given time, t. The returns to public capital net of depreciation, when positive, can be treated as an exogenous transfer from government. When negative, however, the tax imposed to finance the difference may be viewed as a tax on returns to private capital or a direct capital tax. In other words, households may perceive the depreciation on public capital in terms of their private capital, so that the household budget constraint can be restated as;

$$\frac{\partial a(v,s)}{\partial s} = \dot{a}(v,s) = (r(s)+m)a(v,s) + r_P k_P - \delta_P \phi(s) k(v,s) - c(v,s), \tag{4.6}$$

where $\dot{a}(v,s)$ is the rate of change of private capital for a generation v individual at a given time, s. The households maximise the sum of discounted consumption for all times. Given the budget constraint (equation 4.6), the Household problem can be represented by the current value Hamiltonian;

$$H = \ln c(v,s) + \lambda[(r(s)+m)a(v,s) + r_P k_P - c(v,s) - \delta_P \phi(s) a(v,s)]. \tag{4.7}$$

Note that $a(v,s)$ and k_P are given. Substituting for $a = k$ in the budget constraint (equation 4.6) and maximizing the Hamiltonian, the first order condition for optimality is the rate of change of consumption. This is given by; $\frac{\dot{c}(v,s)}{c(v,s)} = -\frac{\dot{\lambda}}{\lambda} = (r(s) - \rho - \delta_P \phi(s))$. The rate of change of consumption will be devoid of the $\delta_P \phi(s)$ term if the household perceived the tax for public investment as inconsequential to savings decisions. So far, I have captured the effect of government's tax to form public capital on the savings behaviour of households by the depreciation term. But in general, any tax rate internalised in this form will lead to a higher equilibrium interest rate, and hence a lower level of private capital and consumption at steady state. Thus, the change in consumption of an agent at each time is given by;

$$\dot{c}(v,s) = \frac{\partial c(v,s)}{\partial s} = [r(s) - \rho - \delta_P \phi(s)]c(v,s). \tag{4.8}$$

The differential equation integrates to $c(v,s) = c(v,t)e^{\int_t^s [r(s)-\rho-\delta_P\phi(s)]ds}$. As typical of infinite horizon problems, the sufficient condition for a solution to exist is a transversality condition given by:

$$\lim_{t\to\infty} a(v,t)e^{-\int_0^t (r(t)+m)} = 0 ....(A1),$$

The transversality condition also serves as a necessary condition together with the consumption euler if the objective function is finite for all admissible paths (Kamihigashi, 2001[70]). For the single state variable (i.e., a(t)), it is necessary to determine the control variable in terms of the state. In other words, consumption of an individual may be written as a function of private assets. This requires an inter-temporal budget constraint (IBC) together with the euler. Let the relevant IBC be:

$$\int_t^{\infty} c(v,s)e^{-\int_t^s (r(s)+m)ds} ds \leq a(v,t) \tag{4.9}$$

The IBC requires the discounted lifetime (planned) consumption of a person to his birth date to be no more than his endowments in private assets. The absence of wages explains the necessity of capital provision for new agents by the state. Without this, consumption is ruled out for new generations. But the provision catered for in equation (4.3) imply that new agents are endowed with the average capital stock in the economy. In this case, the generation of an individual no longer matter, as $a(v,t) = a(t)$. The solution to (4.8) when combined with (4.9) yields:

$$c(v,t)\int_t^{\infty} e^{\int_t^s (r(s)-\rho-\delta_P\phi(s))ds} \; e^{-\int_t^s (r(s)+m)ds} ds = a(v,t) \tag{4.10}$$

This shows that optimal consumption profile of an agent is a function of his endowment in private capital. Solving the improper integral, reduces equation (4.10) to $c(v,t) = (\rho + m + \delta_P\phi(t))[a(v,t)]$. This gives the consumption of a person born at time v, that is still alive at present time t. Since aggregate consumption is a sum of individual consumption of all living members of the population, aggregate consumption can be written as $C(t) = (\rho + m + \delta_P\phi(t))A(t)$, given the homogeneity across generations. Nonetheless, the trivial reason of differences in size of existing cohorts makes it convenient to keep to the standard notation in OLG models with heterogeneous capital and consumption levels. Denoting the number of people born at time v that are alive at the present time, t, as $L(v,t) = L(0)e^{nv}be^{-m(t-v)}$; where $L(v) = L(0)e^{nv}$, and $L(0)$ is historically given. We know that the total population alive is given by $L(t) = \int_{-\infty}^{t} L(v,t)dv = L(0)e^{nt}$. Hence, the aggregate values are given as:

$$C(t) = \int_{-\infty}^{t} c(v,t)L(v,t)dv = \int_{-\infty}^{t} c(v,t)L(0)e^{nv}be^{-m(t-v)}dv \tag{4.11}$$

$$A(t) = \int_{-\infty}^{t} a(v,t)L(v,t)dv = \int_{-\infty}^{t} a(v,t)L(0)e^{nv}be^{-m(t-v)}dv \tag{4.12}$$

The dynamic economy in aggregate form is derived from equations (4.11 and 3.12). Differentiating with respect to time, yields the evolution of aggregate variables for the household sector as follows[2]:

$$\dot{C}(t) = (r(t) - \rho + b + n - \delta_p\phi)C(t) - b(\rho + m + \delta_p\phi)A(t). \tag{4.13}$$

$$\dot{A}(t) = (b + r(t))A(t) + r_pK_p - C(t) - \delta_p\phi A(t). \tag{4.14}$$

Two points are worthy of note from the aggregate change in consumption and private capital. First, if one notes that $bC(t) = b(\rho + m + \delta_p\phi)A(t)$, then the aggregate dynamic consumption equation is exactly the same as the individual consumption rule, except the presence of $nC(t)$ which reflects population growth. Clearly aggregate consumption must grow at a rate higher than individual consumption growth rate to compensate for population growth. Secondly, the aggregate change in private capital reflects the fact that premium on negative life insurance contracts received by individuals is inconsequential in aggregate since the insurance companies operate competitively and make zero profit. Of particular importance is the term $bA(t)$, which reflects the capital provision for new agents.

4.3 Dynamic Equilibrium

As a condition of long-run equilibrium, the aggregate assets accumulation by households is consistent with capital demand by the Firm. The interest rate adjusts to clear the market. With consumption being a jump variable and forward-looking in the dynamic equilibrium path, the next period's capital supply depends on the expected interest rate which is fulfilled in equilibrium.

4.3.1 The dynamics of the set-up

An important question that naturally arise in this simple framework is whether a decentralized equilibrium can be supported by the given production system. The well known literature with this output set-up often assume that government keeps the inputs ratio constant by dictating private consumption, thereby ruling out equilibrium in a decentralized market structure. Secondly, the fact that agents internalise government tax in the savings decisions makes debt-financed public investment an interesting policy issue

[2]The dynamic aggregate equations for consumption and private capital are derived in Appendix A, using the rule for differentiation under the integral.

to examine in this model. While public investments enhances productivity and may positively impact earnings, we have seen that by the consumption-savings rule, a higher tax increases the equilibrium interest rate and lowers the amount of private capital. A third effects arises when government issues debt to provide public capital and services the debt indefinitely. Issuing the debt domestically reduces the available supply of private capital. These three effects are important in better understanding the effects of government intervention in public capital formation in economic growth. For endogenous growth models, where the public and private capital stocks are assume to grow continuously, these effects are not readily apparent in the competitive equilibrium. In particular, the assumption of continuous growth obscures important obstacles to economic growth in less developed economies where the public capital stock has remained relatively constant since 1960[3]. In economies where public capital (broad infrastructure) is not keeping up with the pace of population growth, congestion effects may lead to negative economic growth.

In what follows I check the condition for the existence of steady state equilibrium given population growth rate, n, and the required public capital investment rate, determined largely by the depreciation rate, δ_p. From (4.13 and 4.14), the dynamic system in per capita terms ($\dot{a} = \dot{k}$, and $\dot{c}$,) is derived as follows;

For $k = \frac{K}{L}$, log differentiating both sides with respect to time, yields $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$. The aggregate wealth of households equals the total amount of private capital held by the Firm. Hence, substituting equation (4.14) for $\dot{K} = \dot{A}(t)$, yields $\frac{\dot{k}}{k} = \frac{(b+r(t))A(t)+r_pK_p-C(t)-\delta_p\phi A(t)}{K} - n$. Markets clear and the household wealth is equivalent to private capital available to the Firm. Substitute for the interest rate, r, and the equilibrium change in private capital is given by;

$$\dot{k} = f(k) - c - (\delta + n + \delta_p\phi)k. \tag{4.15}$$

Similarly, for $c = \frac{C}{L}$, the change with respect to time of consumption is derived from the aggregate dynamic equation as follows;

$$\frac{\dot{c}}{c} = \frac{(r(t) - \rho + b + n - \delta_p\phi)C(t) - b(\rho + m + \delta_p\phi)A(t)}{C(t)} - n.$$

[3]Please see section 4.6 for discussion on the public capital stocks and its productivity relative to private capital in various categories of countries.

Again, using equation (4.3), the rate of change of consumption per capita with respect to time is given as;

$$\dot{c} = (f'(k) - \delta - \rho - \delta_p \phi)c - b(\rho + m + \delta_p \phi)k. \tag{4.16}$$

The dynamic system in per worker terms is given by the differential equations (4.15) and (4.16). The steady state equilibrium of the model occurs at where consumption and capital per labour is constant. This arises because public capital is not held as a reproducible input, rather, it is supplied by the state in fixed proportions to private agents. Allowing public capital to grow at a constant rate will entail endogenous growth in equilibrium. Thus, the equilibrium in the previous chapter is a special case in this model when public capital grows at the same rate as private capital. The value of per worker consumption at the nullcline for private capital is given as;

$$c\,|_{\dot{k}=0} = f(k) - (\delta + n + \delta_p \phi)k \tag{4.17}$$

The value of private capital per worker (denote this as $\hat{k}$), and the corresponding public to private capital ratio (denote as ϕ_{max}), that optimises consumption under equation (4.17) for the given k_p constitutes what is termed the "golden-rule" capital labour ratio in the characterization of Phelps (1965)[96]. The first order condition, $\frac{\partial c}{\partial \hat{k}}|_{\dot{k}=0} = 0$, satisfies $f'(\hat{k}) - \delta = n + \delta_p \phi_{max}$. The $k = \hat{k}$ is considered the golden rule capital-labour ratio because at this level consumption is fully optimised and it is impossible to increase household consumption by further accumulation of assets in private capital. The corresponding ratio of public to private capital, $\phi = \phi_{max}$, is unique for the given k_p. Note, however, that the equilibrium value of private capital may or may not coincide with the golden rule.

On the consumption side, the per capita value at which growth of consumption is zero is obtained by replacing $\dot{c}$ with zero, hence;

$$b(\rho + m + \delta_p \phi)k = (f'(k) - \delta - \rho - \delta_p \phi)c$$

The value of private capital per worker that satisfies the above, constitutes a stationary per capita consumption path, giving the consumption per capita as;

$$c\,|_{\dot{c}=0} = \frac{b(\rho + m + \delta_p \phi)k}{f'(k) - \delta - \rho - \delta_p \phi} \tag{4.18}$$

Any value of private capital labour ratio may satisfy equation (4.18) and hence needs not be unique. In other words, other conditions in addition to (4.18) will be required to determine the equilibrium private capital labour ratio. Notice also that the solution to the inter-temporal budget constraint (4.10) using the optimality condition from the household utility problem gives consumption of a living household as a positive function of private assets. This suggests that along the $\dot{c} = 0$ locus, consumption can always be increased by additional asset accumulation as the utility function dictates. But as equilibrium is subject to constraints of the production technology, it follows that

$$\frac{\partial c}{\partial k}|_{\dot{c}=0} = \frac{[(f'(k) - \delta - \rho - \delta p\phi)b(\rho + m + \delta p\phi)] - b(\rho + m + \delta p\phi)k(f''(k)}{[f'(k) - \delta - \rho - \delta p\phi]^2} > 0 \tag{4.19}$$

if $f'(\tilde{k}) - \delta p\phi > \rho + \delta$. This would be the case given that $f''(k) < 0$ and $k > 0$ for positive stationary consumption. Note that the term on the left side of the inequality is endogenous and expected to be a sufficiently high positive value given a small value of k. That on the right is made up of exogenous constants. Starting from a small k, $f'(k)$ is falling as k rises, so that by inada conditions the left side term is tended towards zero. Thus, there exist a positive reference value, $\tilde{k}$ satisfying;

$$f'(\tilde{k}) - \delta = \rho + \delta p\phi \quad s.t \quad lim_{k \to \tilde{k}} \; c = \infty.$$

It is straightforward that the exact value of $\tilde{k}$ makes $c|_{\dot{c}=0}$ given by equation (4.18) not well defined. However, since $\tilde{k}$ can be determined with exactitude from the condition above, it can serve as a reference value in determining the nature of steady state equilibrium as illustrated by the dotted red vertical line on figure 4.1. Obviously, $\tilde{k}$ is a vertical asymptote to the $\dot{c} = 0$ curve.

From the foregoing, the steady state equilibrium value of private capital (denote as k^*) must satisfy $\dot{c} = 0$ subject to the production technology, and hence $\dot{k} = 0$. Thus, k^* is unique and lies in between $\tilde{k}$ and $\hat{k}$ since we know that $\tilde{k} \lesseqgtr \hat{k}$. This is true when $\tilde{k} > \hat{k}$, otherwise it is to the left of both variables. The condition for the existence of private wealth per capita satisfying equations (4.17) and (4.18) suggest that in equilibrium private capital may be over-accumulated relative to public capital. The nature of the equilibrium depends on whether $\rho + \delta p\phi \lesseqgtr n + \delta p\phi_{max}$. Clearly, $\rho = n$, for $\phi = \phi_{max}$. The private capital per person that equates the right side of equations

(4.17) and (4.18) is the equilibrium private capital labour ratio. This may be such that in equilibrium, the public to private capital ratio is below the optimal value. The consumption maximizing ratio may be elusive in this model as is the case in Blanchard (1985) where the household utility structure does not cater for emergence of new agents. Thus, there can be over-accumulation of private wealth relative to the public capital stock.

4.3.2 Steady state equilibrium given public capital

The existence of steady state equilibrium is guaranteed in the model given the assumption of exogenous public good provided by the government. There exist private capital per person that equates the consumption equation given by (4.17) to that given by (4.18). In equilibrium, therefore, the public capital to private capital ratio is constant. This constancy is assured by the condition for optimality given by the solution to the Hamiltonian. Consumption increases as long as the marginal product of private capital exceed the rate of time preference and the burden depreciation of public capital imposes on a unit of private capital. But since the marginal product is diminishing, and even though total depreciation for public capital is falling in ϕ relative to, and as private capital increases, it remains positive. Together with a positive constant for consumer's rate of time preference, zero consumption growth will be obtained and hence k given by (4.18) exist. Similarly, although relative value of public capital depreciation is falling in the amount of private capital, the total value of depreciation for private capital is linear in itself. And since output per unit of private capital is falling in k due to diminishing returns, the output per labour increases at a decreasing rate. Taking the linear effect of depreciation and population growth altogether on maintaining the private capital labour ratio, and a non-linear output function on private capital in the transition to equilibrium, $\dot{k}$ given by equation (4.15) cannot be positive indefinitely. Hence, $\dot{k} = 0$ will be attained and k satisfying (4.17) is guaranteed to exist. The k satisfying (4.17) need not be the same as that satisfying (4.18). As figure 4.1 shows, any value of k along the $\dot{c} = 0$ curve satisfies equation (4.18). Similarly, any chosen value of k along the $\dot{k} = 0$ curve satisfies equation (3.17). However, there is a unique value of k along the $\dot{c}$ curve that is also found on the $\dot{k}$ curve. This satisfies both (4.17) and (4.18) and is given by:

$$f(k) - (\delta + n + \delta_p \phi)k = \frac{b(\rho + m + \delta_p \phi)k}{f'(k) - \delta - \rho - \delta_p \phi} \tag{4.20}$$

The value of k (private capital per labour) that satisfies this equation constitutes the equilibrium private capital to labour ratio (denoted as k^*) of the model. This corresponds to a ratio of public capital to private capital denoted as ϕ^* (which may or may not equal ϕ_{max}), given the public capital per labour. Since the public capital per labour is assumed to remain constant across time, we simply denote it as k_p and proceed to examine equilibrium based on the relative size of private capital. As the condition for equilibrium dictates, private capital is likely to be over-accumulated in equilibrium relative to the available public capital stock if $\rho < n$. To see this, suppose that the private capital stock is equal to the reference value (i.e., $k = \tilde{k}$) and $\dot{c} = 0$. This require $\rho = f'(\tilde{k}) - \delta - \delta_p \phi$. Observe that market clearing condition on equilibrium ensures that private capital from the households is fully used up by the Firm, and hence $r = f'(k) - \delta - b$. Assuming for a moment that $b = \delta_p \phi$, and $r = \rho < n$, the equilibrium will be suboptimal involving excess accumulation of private capital. Issuing debt for consumption purposes (as in Diamond) is pareto-optimal in this context. Note that issuing a debt of size d, and maintaining this per capita value of debt indefinitely entails retirement of debt with interest amounting to $(1+r)d$ and new issuance of $(1+n)d$ every period. As long as $n > r$, the debt policy leaves behind excess resources for consumption. Note that this debt issuance, by reducing the supply of private capital to the Firm, increases $\delta_p \phi$, and as an equilibrium requirement, raises $f'(k)$, and hence r. Clearly this will have a negative effect if in equilibrium $r > n$. As discussed in the previous chapter, if government enters the domestic loan market to issue bonds for financing public investments, then both public and private capital can be treated as trade-able goods. In this case both inputs to the production function will grow at an endogenously determined rate in equilibrium. For the economy with public capital endowment described above, dynamic inefficiency of the equilibrium (where $r < n$) presupposes that debt issuance for consumption may be welfare enhancing.

Thus far, the analysis in this chapter take the public capital stock per capita as beyond influence. Assuming however that debt may be issued to raise the public capital stock permanently, it will increase ϕ and enhance productive efficiency. If the debt is issued domestically, it reduces the relative size of private capital and increases its marginal product. Such a system of reallocation enhances efficiency in a similar fashion as threshold externalities do, in Azariadis and Drazen (1990)[11] for example. This increases output per unit of capital, and given normality of savings,[4] increases steady-

[4]The normality assumption is shown in Diamomnd (1965)[44] to entail $0 < \partial s/\partial w < 1$. Note also that in Diamond's, $\partial s/\partial r$ may be positive or negative depending on whether the capital market is

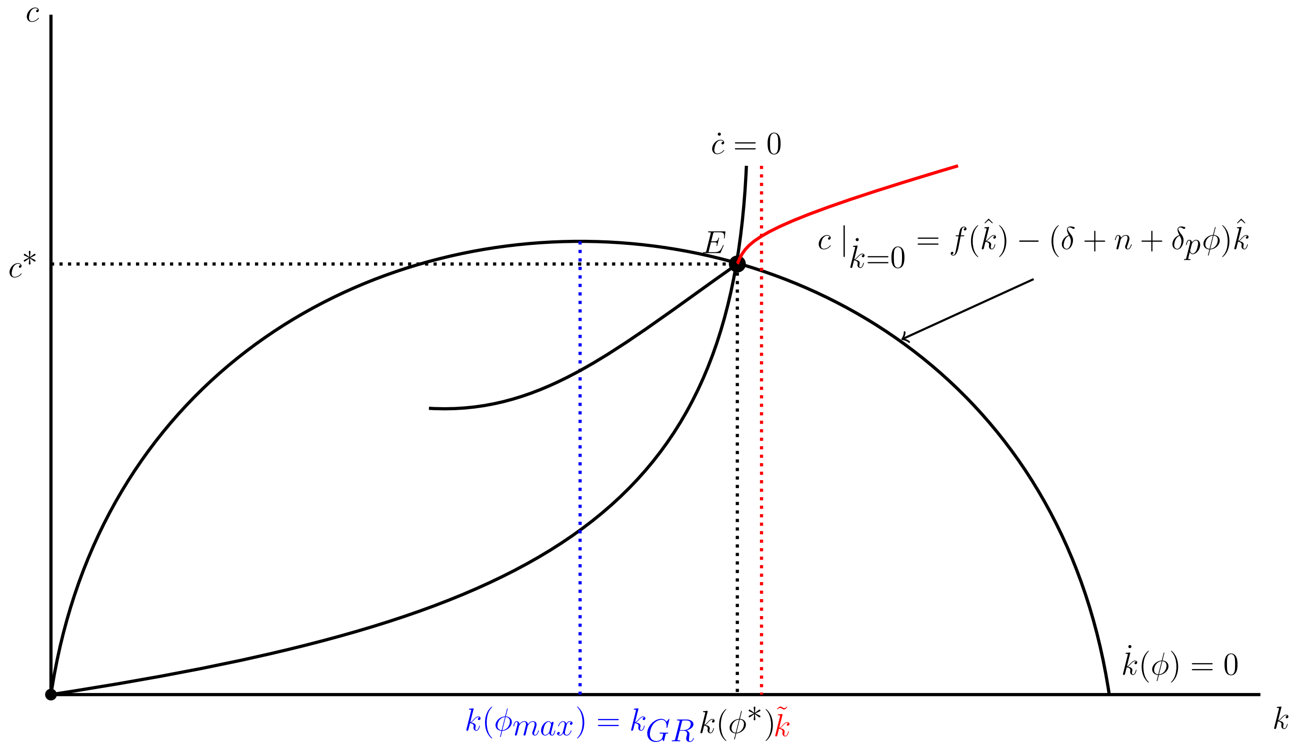

Figure 4.1: Phase diagram of the model with constant k_p and varying values of k

Notes: The phase diagram is constructed to reflect the dynamically inefficient case for the given k_p. In this case the steady state private capital per labour, k^* falls in between the golden rule value, $k(\phi_{max})$, and the reference value, $\tilde{k}$.

state amount of private capital per labour. By increasing interest rate, this enhances efficiency in the economy.

Features of the dynamic equilibrium

As the phase diagram below shows, an economy characterised by the production function here will grow at a rate higher than population growth if the relative size of private capital is small so that the marginal returns to private investment is higher than the consumer's rate of time preference and the burden imposed by public capital depreciation. The relative size of private capital increases with consumption until equilibrium is attained. On the other hand, if at the outset the relative size of private capital is very large, the public to private capital ratio (ϕ) will be very low and the efficiency of private

characterised by the normal case or the perverse case. In this environment where wages, w, as a factor earning is non-existent, the normality requirement involves $0 < \partial s/\partial r$, conditional on $\partial y/\partial k < \partial y/\partial k_p$. Clearly, $\partial s/\partial r$ may be negative if $\partial y/\partial k > \partial y/\partial k_p$.

investments low as a results. Marginal returns to private investments will be below the consumer's rate of time preference and the burden imposed by depreciation of public capital. Here, private capital will fall with consumption until marginal returns rises to fully compensate for depreciation and population growth and steady state realized. Since equilibrium interest rate is largely determined by the exogenously given rate of time preference and depreciation of public capital, a lower discount rate will necessarily yield a lower equilibrium interest rate relative to an environment with higher discount rate, ceteris paribus. Equilibrium in the model will exhibit the following features ;

- For any starting value of $\phi = \frac{kp}{k}$, there exist an equilibrium in the dynamic economy characterised by the Aschauer (2000) production function and subject to fixed value of public capital labour ratio.

- For the given kp, the equilibrium private capital and consumption pair (k^*, c^*) is unique.

- The equilibrium may be dynamically inefficient involving a relatively larger amount of private capital than is required for golden rule consumption, where $\phi_{max} > \phi^*$.

- Subject to public capital available in fixed proportions, equilibrium will require ϕ and private capital per person (k) to become constant. Private capital labour ratio and the public capital ratio to private capital are however evolving from any starting point, until equilibrium is attained.

Transition dynamics

Steady state equilibrium in the model requires $\dot{c}$, $\dot{k} = 0$, and its existence is independent of the given value of ϕ. However, transition to equilibrium will depend on the relative size of public capital in comparison to private wealth. Observe that for any given value of $\phi > \phi^*$ such that private capital is relatively under-supplied, marginal product of private investment will exceed rate of time preference and the amount needed to maintain the public capital level $(r(s) > \rho + \delta_p\phi)$. In this case, consumption is increased by the accumulation of private wealth. As interest rate is decreasing in the supply of assets, this will continue until $r(s)$ falls enough to equate $\rho + \delta_p\phi$. In this case, the relative amount of private capital in the economy starts below the steady state and grows till equilibrium is attained. Note that ϕ will be decreasing in the supply of private capital. But it stays positive as long as the public capital level remains a positive constant.

Hence, the condition $r(s) = \rho + \delta_p \phi$ is bound to be attained. Similarly, for $\phi < \phi^*$, private capital is relatively over-supplied and marginal product of private wealth is below what is required to maintain the public capital level given the subjective rate of time preference (i.e., $r(s) < \rho + \delta_p \phi$). This lowers household consumption. Here, the economy starts above steady state and grows negatively until steady state equilibrium is attained. The existence of the equilibrium is guaranteed by the joint concavity of the Hamiltonian in consumption and capital.

Non-absoluteness of optimality result

As illustrated, the steady state equilibrium capital ratio, ϕ^* may or may not be optimal for the constant k_p. In aggregate, consumption and private capital will growth at a common rate exceeding population growth in transition if the economy starts below the steady state, but consistent with population growth at steady state. The optimality of equilibrium argument, where $\phi^* \geq \phi_{max}$, takes k_p as beyond influence. Relaxing this assumption, and allowing a lump-sum tax or a debt-financed investment in public capital may have positive effects on steady state wealth and consumption for $\phi^* \lesseqgtr \phi_{max}$. By exogenously increasing the size of public capital available to a person, the relative amount of private capital falls and marginal product of private capital increases and consumption will grow until steady state is re-established.

4.3.3 The generations replacement effect

One of the key features of the standard Blanchard model is the generations replacement effect which captures the continuous replacement of people dying with wealth by people being born without physical capital. This effect ensures that in equilibrium $r^* > \rho$ and hence agents are continuously accumulating private capital until they die. In other words, $\forall t,\ c(\dot{v}, t) > 0$ in Blanchard. This feature does not arise here because of the simplification that allows new agents to be provided with private capital. Apart from the obvious introduction of the term, $\delta_p \phi$, the steady state interest rate, r^* is such that $\dot{c}(v, t) = 0$. In other words, at steady state, $r^* = \rho + \delta_p \phi$, implying that by the optimality condition (given by equation 4.7), consumption is constant for any agent at steady state. This is ruled out in the original Blanchard model due to the requirement that $r^* > \rho$ and hence $\dot{c}(v, t) > 0$. In Blanchard, the aggregate growth rate of consumption is given by

$$\dot{C}(t) = -b(\rho + m)A(t) + [r(t) - \rho + n]C(t).$$

Since steady state is given by $\dot{c}(t) = 0$, it implies that aggregate consumption grows at the rate of population growth given by, n. Thus, $\dot{C}(t) = nC(t)$. Note, however, that if steady state interest rate was given by $r^* = \rho$, $\dot{C}(t) = -b(\rho + m)A(t) + nC(t)$ hence $\dot{c}(t) < 0$ ruling out steady state equilibrium. In other words, Blanchard's steady state requires $r^* > \rho$ so that while average consumption is constant, every agent's consumption grows at a positive rate from time of birth to death.

In the economy with public and private capital as the productive inputs, the equilibrium explicitly accounts for the generations replacement effect. Note in equation (4.3) that net returns to private capital accounts for asset provision that must be made for new agents, hence $r^* = f'(k^*) - \delta - b$. The aggregate growth rate of consumption in this case is given by

$$\dot{C}(t) = -b(\rho + m + \delta_p\phi)A(t) + [r(t) - \rho + b + n - \delta_p\phi]C(t).$$

Substituting for r yields

$$\dot{C}(t) = -b(\rho + m + \delta_p\phi)A(t) + [f'(k^*) - \delta - \rho - \delta_p\phi + n]C(t)$$

Observe that if $f'(k^*) - \delta = \rho + \delta_p\phi$, then $\dot{C}(t) < nC(t)$, and hence $\dot{c}(t) < 0$. Clearly, at steady state equilibrium, $f'(k^*) - \delta \neq \rho + \delta_p\phi$. However, since $\dot{c}(t) = 0$ at steady state equilibrium, we require $f'(k^*) - \delta > \rho + \delta_p\phi$ for positive stationary consumption. It implies that $\dot{C}(t) = nC(t)$ if

$$[f'(k^*) - \delta - \rho - \delta_p\phi]C(t) = b(\rho + m + \delta_p\phi)A(t).$$

This condition can easily be shown to be met. Suppose that the equilibrium requirement of $f'(k^*) - \delta > \rho + \delta_p\phi$ is such that $f'(k^*) - \delta - \rho - \delta_p\phi = \epsilon > 0$, then

$$[f'(k^*) - \delta - \rho - \delta_p\phi]C(t) = \epsilon C(t).$$

It is straight forward that given $C(t) = (\rho + m + \delta_p\phi)A(t)$ by equation (4.10), if $\epsilon \neq b$, then $\dot{C}(t) \lesseqgtr nC(t)$. Therefore, apart from the trivial term for depreciation of public capital, the birth rate being explicitly accounted for in the interest rate is what erases the generations replacement effect in the elaboration presented here. To summarize, while individuals have incentive to save and increase consumption given equation (4.19), this is diminished by the fact that their savings are taxed indirectly through the capital gain tax aimed at providing assets for new generations. If we define the gross interest rate net of depreciation by $\bar{r} = f'(k^*) - \delta > \rho + \delta p \phi$. It follows that given $\bar{r}$, individuals have the incentive to increase consumption due to the optimality condition. However, the tax amounting to b reduces take home returns (the net interest rate) to $r^* = f'(k^*) - \delta - b$ thereby eliminating the propensity to save and increase consumption. In other words, savings by generation already alive, and hence their capital level, is constantly diminished by an amount equal to the birth rate due to the decrease in net returns. This is met by the normality condition, which requires $\partial s / \partial r > 0$.

4.3.4 A comment on debt for consumption

For a steady state equilibrium with public capital stock per capita given, the continuous issuance of debt for consumption purposes as in Diamond (1965), increases utility if $r < n$. If the equilibrium interest rate exceed the population growth rate, there is no justification, a priori, for debt issuance to finance consumption expenditure under the circumstance.

On the other hand, if the given stock of public capital per labour can be increased at some points in time, which may be taken to be lump-sum investment, through debt accumulation, the role for debt in enhancing utility is no longer limited to the specific equilibrium where $r < n$. By being invested in public capital and increasing ϕ, debt avails a different channels for increasing welfare in the economy. The analysis that follow in the next section focuses on the latter case where government debt is issued to finance investment in public capital.

4.4 National Debt

Theoretically, there are at least two relevant ways to conceptualise the effect of debt in an economy. Diamond (1965)[44] and Blanchard (1985)[24] examined this by looking at the effect of debt on consumption in a dynamic economy in equilibrium. Their analysis assume that debt finances consumption for all or part of the present generation (Blanchard examined debt for unproductive public expenditure), thereby imposing tax obligation on future generations for interest payments on the debt in equilibrium. As a result, public debt is considered to have a negative effect on the economy by reducing savings rate and steady-state capital labour ratio. The negative effect on capital, nonetheless, is welfare enhancing in the dynamically inefficient equilibrium. The approach by Barro (1974) treats debt as a zero-sum between generations. This approach, also examined in

Blanchard (1985), binds the government to a relevant transversality condition. Thus, any issuance of (or increase in) debt today must be accompanied by surpluses in the future to enable full repayment of the debt as opposed to indefinite refinancing of it (as is the case in Diamond, 1965). In this case, debt issued to enable consumption of present generation merely induces a one-on-one increase in net bequest, given the intergenerational linkages in the utility structure. This ultimately leaves steady state capital and consumption unchanged.

Aschauer (2000)[7] and Checherita-Westphal et al (2014)[37] on the other hand consider debt to be issued for investment in public capital rather than for consumption. In their endogenous growth models, the worthiness of debt depends on its effect on growth rate. In this case, the debt-GDP ratio that is consistent with the growth maximising public to private capital ratio is utility enhancing. Essentially, a fiscal policy rule that aims to maximize the equilibrium growth rate will be beneficial despite the imposition of taxes to service the debt. There are two relevant conceptual points worthy of note here. First is whether government transversality condition is binding (i.e., debt refinanced indefinitely or paid back in full in finite time), and second is whether debt finances consumption or investment. On the first point, if debt finances procurement of public capital, it is more natural to assume that it is refinanced indefinitely, and the government's budget constraint need not bind. In this case the effect of debt on equilibrium consumption is a straightforward one; it is positive (in the case where the debt level is below the growth maximizing fiscal rule) in the formulation of Checherita-Westphal et al (2014)[37]. A key problem, however, is that this avenue for debt to improve welfare will not exist in a competitive equilibrium, nor will an optimising social planner maintain an inefficient public to private capital ratio in equilibrium. In the overlapping generations model, however, the size of population growth rate compared with equilibrium interest rate determines the optimality (or otherwise) of debt for consumption in equilibrium. As we will see below, irrespective of which side of the population growth rate the equilibrium interest rate lies, debt issued for public investment can be welfare improving under some conditions.

4.4.1 Effects of External Debt

I start the analysis with external debt, which is perhaps the simplest case. It is easier to assume an economy that is small enough to have negligible effect on global interest rate.

Hence, the interest rate in the international capital market, taken as given, is considered to be beneath the rate of return to private investment in the domestic economy. This is necessary since it will not make sense for external debt issuance if domestic interest rates are lower than in the international capital market. In this partial equilibrium treatment, it is easy to see that external debt accumulation for investment increases the stock of public capital available to private agents, and hence enhances the rate of return to private investment without a corresponding decrease in private capital, as would be the case with domestic debt. Despite an increase in tax for interest payments on the debt, private wealth and consumption may rise in equilibrium.

To clarify this result, I proceed by assuming (following Aschauer, 2000; and Checherita-Westphal et al, 2014) that the overall stock of public capital in the economy is procured by the issuance of government bonds so that at every point in time, the nominal stock of debt exactly equals the aggregate public capital, $D(t) = K_P(t)$. Here, the state no longer taxes agents the per capita equivalence of $\delta_P k_P$ to keep the public capital per labour constant, rather this ratio is maintained by the continuous issuance of new public debt. The government, however, exact taxes to pay interest on the debt. The dynamic aggregate debt equation is given by;

$$\frac{\partial D(t)}{\partial t} = \dot{D}(t) = (i + n + \delta_P)D(t) - T(t). \tag{4.21}$$

Where i is the interest rate on debt, and $T(t)$ is the aggregate amount of taxes, so that equation (4.14) becomes;

$$\dot{A}(t) = (b + r(t))A(t) + r_P K_P(t) - C(t) - T(t). \tag{4.22}$$

The public debt stock is increasing in the nominal interest rate, the rate of population growth and the depreciation of public capital, but decreasing in taxes. The constant debt labour ratio, $d = \frac{D(t)}{L(t)}$. Log differentiating with respect to time gives, $\frac{\dot{d}}{d} = \frac{(i+n+\delta_P)D(t)-T(t)}{D(t)} - n$. Let the tax per person, $\tau(t) = \frac{T(t)}{L(t)}$, so that the dynamic debt equation reduces to;

$$\dot{d} = (i + \delta_P)d - \tau(t). \tag{4.23}$$

At steady state, $\dot{k}, \dot{c} = 0$, in addition to k_P which is assumed constant indefinitely ($\dot{k_P} = 0$) therefore, $\dot{d} = 0$. With a constant debt labour ratio, tax per labour is given by $\tau = (i + \delta_P)d$. Also observe from equation (4.22) that the per capital evolution of private wealth is $\frac{\partial a(t)}{\partial t} = \dot{a}(t) = [r(t) + m]a(t) + r_p k_p - c(t) - \tau(t)$. Substituting for $\tau(t) = (i + \delta_p)\phi a(t)$ and using this as the individual's budget constraint, the first order conditions for optimality yields a consumption rule of $\dot{c}(t) = (r - \rho - (i + \delta_p)\phi)c(t)$. Note that $a(t) = k(t)$ and $k_p = \phi k(t)$. Clearly, at $\dot{c}(t) = 0$, where $\phi(t) = \phi^*$, we have $r^* = \rho + (i + \delta_p)\phi^*$. Let the private capital to labour ratio be k^*.

Now consider that from this steady state equilibrium, government issues a lump-sum amount of extra debt to raise the public capital stock per labour permanently. This entails a permanent addition to aggregate debt at a given point in time. Let the new debt equation be;

$$\dot{D} = (i + n + \delta_p)(D + \epsilon D) - T_n, \tag{4.24}$$

where ϵD is the extra debt accumulated to finance the permanent increase in k_p. It is straightforward that the tax per person at the new steady state equilibrium, where $\dot{d} = 0$, becomes;

$$\tau_n = \frac{T_n}{L} = [(i + \delta_p + n)(1 + \epsilon) - n]d.$$

The time notation is dropped for convenience. From τ_n, it follows that the increase in the public capital labour ratio results in an increased tax on future generations. But this may increase their consumption as well. The investment increases the value of ϕ and hence private capital will be relatively under-supplied, raising the rate of return to private investment $r(t)$. To reflect the extra debt in capital accumulation, equation (4.22) can be restated as:

$$\dot{A}(t) = (b + r(t))A(t) + r_p K_{p_n}(t) - C(t) - T_n(t). \tag{4.25}$$

Where $T_n > T$ denotes an increase in aggregate taxes to reflect interest payments on the increased debt stock. This does not affect capital demand from the households perspectives since debt is issued externally. Thus, for households, the effect of the extra debt is seen in the increase in taxes to service it and to maintain a higher public capital stock. From equation (4.25), per capital evolution of private wealth is

$$\frac{\partial a(t)}{\partial t} = \dot{a}(t) = [r(t) + m]a(t) + r_{p_n} k_{p_n} - c(t) - \tau_n(t).$$

where $k_{p_n} > k_p$ denotes the increased supply of public capital and $r_{p_n} \neq r_p$ denotes the change in marginal returns to public capital, and hence the transfers by government. Again, the optimality condition for consumption is $\dot{c} = (r(t) - \rho - [(i + \delta_p + n)(1 + \epsilon) -$

$n]\phi_n)c$, for $\tau_n = [(i + \delta_p + n)(1 + \epsilon) - n]k_p$. The new steady state equilibrium, when $\dot{c}(t) = 0$, imply a new equilibrium interest rate $r_n^* = \rho + [(i + \delta_p + n)(1 + \epsilon) - n]\phi_n$, and the private capital to labour ratio being k_n^*.

Compared with the initial equilibrium, the debt-financed investment in public capital raises equilibrium rate of return to private investment (i.e., $r_n^* > r^*$). Had the extra debt been used for consumption, the rise in equilibrium interest would have involved only the increase in tax to service a now larger debt stock, and this will entail a lower equilibrium private capital labour ratio. However, given the increased level of public capital per capita, the total transfers from government may increase in addition to a higher return on private capital. If the increase in these returns, collectively, outweigh the rise in taxes to service the debt stock and maintain the higher public capital per person, equilibrium consumption will increase. This arises from the propensity to save and increase the private capital labour ratio. The net effect on consumption depends on the relative change in disposable income compared with the change in taxes. The change in equilibrium consumption will depend on;

$$(r_n^* + m)k_n^* - (r^* + m)k^* + (r_{p_n}k_{p_n} - r_pk_p) \lesseqgtr (i + \delta_p + n)\epsilon k_p. \tag{4.26}$$

Note that the right hand-side of the inequality above is the change in tax per capita $(\tau_n - \tau)$. If the left side of (4.26) exceed the right side, consumption will increase in equilibrium, which will be the case in all instances provided that; 1) the interest rate in the international capital market is below the domestic interest rate $(i < r^*)$, and 2) the marginal returns to public capital exceed the domestic interest rate $(r_p > r_n^*)$. The inequality (4.26) is derived from the steady state consumption values using the per capita forms of equations (4.22 and 4.25). Note that in this case, $r \lesseqgtr n$ is immaterial for the equilibrium effect of external debt issuance for investment in public capital. The result relies on the relative shares of public and private capital and their marginal productivities in the economy. In summary, therefore, external debt may increase disposable income and consumption in a low interest rate setting. This lends support, under some conditions, to the argument of Blanchard (2019)[23].

4.4.2 Effects of Domestic Debt

The effect of internally-issued debt is slightly different from the external counterpart in a sense that internal debt has to be analysed in a closed system. Unlike external debt where government demand is met from outside the economy, with internal debt, government enters the domestic capital market, and hence reduces the available supply of private capital. However, like external debt, domestic debt may be welfare improving in equilibrium if debt issuance is for investment in public capital stock.

The slight complication with domestically issued debt is that the interest rate on government bonds is no longer taken as given. The debt has the immediate effect of reducing the amount of private capital, while increasing the available supply of public capital to productive sectors of same economy. From the perspective of households, this has no immediate effect on assets portfolio, as assets holding in 'private capital' is being substituted for government bonds. To simply analysis, let us assume, like in the case of external debt, that the initial stock of public capital is procured from debt-financed investment. With the gross interest rate r in the economy, the aggregate dynamic debt equation remain the same as equation (4.21), except for replacement of i with r, denoting the domestic interest rate. Rewrite equation (4.21) as;

$$\frac{\partial D}{\partial t} = \dot{D} = (r(t) + n + \delta_p)D - T(t). \tag{4.27}$$

Because public debt is now owed to domestic agents who not only pay taxes to service debt, but also receive interest payments on the debt, the aggregate change in private assets must account for these. First, government debt is treated as asset in household asset portfolio. Secondly, government receives total returns on public capital from the Firm, but service debt at the prevailing interest rate. To simplify the analysis, suppose the government makes transfer of the total returns to public capital to households. But to service the debt, it exact taxes on agents. Altogether, this imply that equation (4.22) modifies to;

$$\dot{\tilde{A}}(t) = \dot{A}(t) + \dot{D}(t) = (b + r(t))[\tilde{A}(t)] + r_p K_p - C(t) - T(t), \tag{4.28}$$

where $\tilde{A} = A(t) + D(t)$ is the total assets of households comprising of $A(t)$ and $D(t)$, where $A(t) = K(t)$ is the portion of households assets held in private capital and $D(t)$ is the total debt stock. Note that from the perspectives of households, government debt is a substitute for asset holding in the capital market. Both are risk-free and pay the same rate of return. Using equation (4.28), the per capita change in private assets is derived by log differentiating both sides of $\tilde{a}(t) = \tilde{A}(t)/L(t)$ with respect to time. This becomes

$$\frac{\partial \tilde{a}(t)}{\partial t} = \dot{\tilde{a}}(t) = [r(t) + m]\tilde{a}(t) + r_p k_p - c(t) - \tau(t),$$

where $\tilde{a} = a + d$ and $\tau(t) = \dfrac{T(t)}{L(t)}$ is the tax per person. From equation (4.27), and the fact that the debt labour ratio is held constant together with the level of public capital, then the tax per capita reduces to $\tau(t) = (r(t) + \delta_p)d$. Similarly, given that $\dot{d} = 0$, it follows that $\dot{\tilde{a}}(t) = \dot{a}(t) + \dot{d} = \dot{a}(t)$. Thus, substituting for $\tau(t) = (r(t) + \delta_p)d$, $\dot{\tilde{a}}(t)$ becomes;

$$\frac{\partial a(t)}{\partial t} = \dot{a}(t) = [r(t) + m]a(t) + [m - \delta_p]d(t) + r_p k_p - c(t). \tag{4.29}$$

From the equation above, it is convenient to retain $r_p k_p$ since this is taken to be a transfer to households. Replacing $d(t)$ by $\phi a(t)$ in the budget constraint, the optimality condition for the household utility problem gives a consumption rule of $\dot{c} = [r - \rho + (m - \delta_p)\phi)]c$, all time notations dropped. Clearly, at steady state equilibrium when $\dot{c} = 0$, the interest rate is given by $r^* = \rho + \delta_p \phi - m\phi$, where $\phi = \phi^*$. The steady state level of consumption is given by $\dot{k} = 0$ becomes;

$$c^* = f(k^*) - (n + \delta)k^* + (m - \delta_p)\phi k^* \tag{4.30}$$

where k^* is the equilibrium level of private capital per labour.

To see the effect of debt-financed public investment, suppose that the government issues a one-off amount of domestic debt to raise the available supply of public capital per person. This has the immediate effect of reducing the supply of private capital to the Firm by the exact amount of the debt being issued since government enters the domestic capital market from the demand side. From the household perspective, asset holdings remain unchanged as private assets are being substituted for government bonds. Let the new dynamic debt equation be:

$$\frac{\partial D}{\partial t} = \dot{D} = (r_n(t) + n + \delta_p)(1 + \epsilon)D - T_n. \tag{4.31}$$

Where ϵD is the additional debt expressed as a percentage of initial stock of debt, and $T_n > T$ denotes the new aggregate tax to reflect the addition to debt. In equilibrium, $\dot{d} = 0$ and the new tax per person becomes $\tau_n = [(r_n(t)+n+\delta_p)(1+\epsilon)-n]d$. Compared with the initial equilibrium, the tax rate increases by $(r_n(t) + n + \delta_p)\epsilon d$ capturing the requirement for interest payments on the extra debt and the need to maintain the addition to public capital in the face of depreciation and population growth. Also, unlike the partial equilibrium environment of external debt, there is a third positive effect on the tax rate arising from the rise in rate of return to capital $(r_n - r > 0)$. The corresponding aggregate capital accumulation equation, following the extra debt accumulation, becomes:

$$\dot{\tilde{A}}(t) = (b + r_n(t))[A(t) - \epsilon D + (1+\epsilon)D] + r_{p_n}K_p(t) - C(t) - T_n(t) \tag{4.32}$$

The new dynamic aggregate private capital accumulation equation is written to capture the instantaneous addition to national debt by an amount ϵD, the simultaneous reduction in private capital by the same amount, and the change in tax rate due to this increase in debt. Even though the debt issuance reduces the supply of the private capital input and increases the public capital input to the Firm, households asset portfolio remains unchanged. Let us denote the new level of household assets held in private capital as $a_n(t)$, the new level of public capital to labour ratio as k_{p_n}. Therefore, in per capita terms, the household's dynamic asset equation is rewritten as

$$\frac{\partial \tilde{a_n}(t)}{\partial t} = \dot{\tilde{a_n}}(t) = [r_n(t) + m]\tilde{a_n}(t) + r_{p_n}k_{p_n} - c_n(t) - \tau_n(t),$$

where $\tilde{a_n}(t) = a_n(t) + d_n$. It is convenient to keep to the original notation for debt labour ratio since the addition to debt has been expressed as a percentage of the original debt stock. From this, the counterpart of equation (4.29) becomes;

$$\frac{\partial a(t)}{\partial t} = \dot{a}(t) = [r_n(t) + m]a_n(t) + [m - \delta_p]d + r_{p_n}k_{p_n} - c(t) - (r_n + n + \delta_p)\epsilon d. \tag{4.33}$$

Note that the government is assumed to maintain the new level of debt and public capital constant. As before, the optimality condition for consumption becomes

$$\dot{c}(t) = [r_n - \rho + (m - \delta_p)\phi(t) - (r_n + n + \delta_p)\epsilon\phi(t)]c(t).$$

This consumption rule gives a transition path to new steady state equilibrium following the addition to debt. Before examining the effect of the increased debt-financed public capital formation on steady state consumption level, it is important to redefine the new levels of private and public capital along the dynamic equilibrium path as $k_n(t)$ and $k_{p_n}(t)$. For analytical convenience, I preserve the public to private capital ratio in the initial equilibrium (ϕ^*) in the analysis below by expressing the addition to debt and public capital as a percentage of the previous stock of public capital. Following the addition of ϵd to the debt and public capital stocks, the new steady state equilibrium is given by $\dot{c}(t) = 0$ and yields a new stationary interest rate as;

$$r_n^* = \frac{\rho + \delta_p\phi^* - m\phi^* + (n + \delta_p)\epsilon\phi^*}{1 - \epsilon\phi^*} \tag{4.34}$$

The consumption effect of the changes in public capita, debt stock, private capital, and the equilibrium interest rate is better understood by first comparing the new stationary interest rate to the old one. The initial steady state equilibrium had interest rate as $r^* = \rho + \delta_p\phi^* - m\phi^*$. From the new steady state rate of return to capital in equation (4.34), it is clear that the interest rate is increased by ϵ as long as $\epsilon\phi^* < 1$. Note, however, that as $\epsilon\phi^* \to 1$, $r_n \to \infty$, and $r_n \to r$ as $\epsilon\phi^* \to 0$. This suggests that for normal reaction of interest rate (rate of return to capita, more generally), extra debt accumulation for investment must be such that $\epsilon\phi$ is within the vicinity of zero, rather than one. In other words, debt accumulation must be such that debt per person, equivalently the public capital per person, is not too large relative to the private capital component in the economy.

In general, these conditions imply that from any steady state equilibrium, there is always an opportunity to generate a local improvement in the returns to household wealth by the issuance of debt to finance public capital formation. Note that since $r_n^* > r^*$, by the normality of savings $(0 < \partial s/\partial f'(k) < 1)$, we expect savings per capita to rise as a result of the rise in interest rate. However, whether this result in higher equilibrium level of consumption and private capital, requires separation of the equilibrium effect of the extra debt into three; 1) the income effect, 2) the transfer effect, and 3) the tax effect.

Starting with the tax effect, the steady state tax per person after extra debt accumulation, from equation (4.31), is given as $\tau_n^* = (r_n^* + \delta_p)d + (r_n^* + n + \delta_p)\epsilon d$. This captures the fact that the interest rate is no longer changing, and hence the tax per capita is stationary. Clearly, the extra tax arises from the increase in interest rate on existing debt, and the interest rate on new debt in addition to the burden of maintaining the increased public capital labour ratio in the face of depreciation and population growth. Denoting the addition to tax as ϵt_n, and separate the tax into existing (old) tax and extra tax arising from the addition to debt. In the initial steady state, the tax for maintaining public capital and interest payment on debt was given by $\tau^* = (r^* + \delta_p)d$. Subtracting this from the new tax per person yields;

$$\epsilon\tau_n = (r_n^* - r^*)d + (r_n^* + n + \delta_p)\epsilon d$$

On the earnings from assets effect, we know that the returns per unit of capital (i.e., interest rate) increases with debt, so that even though debt issuance reduces the private capital labour ratio, from the households perspective, asset portfolio is left unchanged as shown in equation (4.32), since private capital is merely substituted for government debt. Government pays the prevailing interest rate on debt, hence for an individual with initial stock of private capital k, and holding government debt of size d, earnings on both asset holdings will be $r(k+d)$. Extra debt issuance by the state creates an immediate effect on earnings arising solely from the increase in interest rate. This will be;

$$(r_n - r^*)[(k - \epsilon d) + (d + \epsilon d)] = (r_n - r^*)(k + d)$$

At steady state equilibrium, however, due to the change in the amount of household wealth held in private capital and the convergence to a new stationary interest rate, let us rewrite the increase in earnings on household wealth as

$$\epsilon h_i = (r_n^* + m)(k_n^* + d) - (r^* + m)(k^* + d)$$

where ϵh_i is the extra household income, and m denotes the fact that individuals receive a premium on all assets from the negative life insurance contracts. Note that while the value of private capital changes from one equilibrium to the other, we have used the fact that the extra debt cancel out in the household wealth equation and $\dot{d} = 0$ to preserve d in the extra income equation. Finally, the transfers from government changes with the increase in the public capital stock. Denote the change in transfers as $\epsilon tr.$ given by;

$$\epsilon tr. = r_{p_n} k_{p_n} - r_p k_p.$$

Putting the three effects together, with all variables being the stationary equilibrium realizations, the change in consumption at steady state equilibrium depends on whether

$$(r_n^* + m)(k_n^* + d) - (r^* + m)(k^* + d) + r_{p_n} k_{p_n} - r_p k_p \lesseqgtr (r_n^* - r^*)d + (r_n^* + n + \delta_p)\epsilon d.$$

This simplifies to;

$$\begin{aligned}[][f(k_n^*) - (n+\delta)k_n^* + (r_n^* + m)d] - [f(k^*) - (n+\delta)k_{*+(r^*+m)d}] \\ \lesseqgtr (r_n^* - r^*)d + (r_n^* + n + \delta_p)\epsilon d\end{aligned} \tag{4.35}$$

Alternatively, one might derive the effect from the equations for consumption at steady state equilibrium. We can simplify equation (4.33) to

$$c_n^*|_{\dot{k}_n=0} = f(k_n^*) - (n+\delta)k_n^* + (m - \delta_p)\phi k_n^* - (r_n^* + n + \delta_p)\epsilon\phi_n^* k_n^*. \tag{4.36}$$

Comparing the right-hand side of equations (4.36) and (4.30) indicates the change in steady state consumption as a result of the increased public capital and debt stock.

Using the inequality (4.35), however, if the left-hand side exceed the right-hand side, the extra earnings and transfers will dominate the tax effect and private capital and consumption will rise in equilibrium. On the other hand, if the rise in taxes dominates the income and transfers, consumption will fall.

The latter case suggests that beyond a given threshold, debt-financed public investment can be utility decreasing. To see this, observe that the government receives the total returns to public capital. Denoting the gross returns to public capital per labour as $r_p k_p = (\partial y/\partial k_p) * k_p$. In turn, the government pays the prevailing interest rate on debt. Note that the gross interest rate, $r_k = \partial y/\partial k$. Abstracting from depreciation and new agents momentarily, one notes that at any point, private capital earns its marginal product, hence returns on private capital $r_k(t)k(t)$ is sustainable for any level of k. However, for any amount of private capital, the marginal product of public capital, and hence r_p is falling in the amount of public capital. Thus, from any steady state equilibrium with finite amount of private capital, there exist an upper bound, $\bar{k_p}$, such that for $k_p > \bar{k_p}$, $r_p < r_k$. Government is assumed to pay the prevailing interest rate on debt. Therefore, for $k_p = d > \bar{k_p}$ and the returns to public capital is not sufficient to pay the interest rate on debt, $r_p k_p < r_n d$ (note that $k_p = d$). In this case, government must set a tax rate in excess of the equilibrium tax dictated by equation (4.31). Thus, as a necessary condition for debt-financed public investment to be welfare improving, we require:

$$r_p k_p \geq r_k d$$

The implication of this condition is that government debt should pay for itself without increasing tax per person beyond the equilibrium tax rate. For ϵd such that $\epsilon\phi \to 1$, $r_n \to \infty$, and $r_p < r_n$. Hence, $r_n(k + d) > (r_n k + r_p k_p)$. Clearly, there exist a debt and public capital level $\bar{d} = \bar{k_p}$ such that $r_p = \partial y/\partial k_p = r_k = \partial y/\partial k$ and debt-financed investment in public capital beyond this level no longer pays for itself and hence may decrease utility. In this case, government will be required to reduce the debt level and public capital stock to restore equilibrium. The sufficient condition given by equation (4.35) where the left-hand side is greater than the right-hand side accounts for depreciation of both public and private capital as well as population growth at steady state. Observe that while the households treat government debt and private capital as substitutes, this result relies on the relative proportions of private and public capital in the production system. Even though from perspective of the capital market interest rate rises with debt, by inada conditions, the marginal product of public capital falls.

Thus, domestic issuance of debt increases consumption by the rise in the efficiency of private investment in the production system rather than by the rise in interest rate in the capital market per se. In this environment, whether $r_n \lesseqgtr n$ is immaterial for the effect of debt-financed investment in public capital. As long as the marginal product of public capital in the production system exceed the equilibrium interest rate, the effect on consumption and private wealth is positive for $r_n \lesseqgtr n$.

4.5 Computational Illustrations

In this section, I use numerical techniques to compute steady state equilibrium given initial value for public capital. To show the effect of debt-financed public capital investment, I follow the exposition above in supposing that the given stock of public capital is procured by debt. In this case, the differential equations characterising equilibrium are distinct from section 4 where the public capital per labour is given. It is shown generally that subject to an increase in the level of public capital per labour through debt-financed investment, equilibrium is no longer unique, and the steady state values of private capital, output, consumption, and interest rate will depend on the size of the change in public capital per person.

4.5.1 Steady state with given public capital

I start with the the computation for steady state equilibrium given public capital. The relevant equations characterising equilibrium are the output function, the interest rate, and the two coupled differential equations. These are;

$$y = f(k, k_p) = k^{\alpha} k_p^{1-\alpha} \tag{4.37}$$

$$\dot{k} = f(k) - c - (\delta + n + \delta_p \phi)k. \tag{4.38}$$

$$\dot{c} = (f'(k) - \delta - \rho - \delta_p \phi)c - b(\rho + m + \delta_p \phi)k. \tag{4.39}$$

$$r = f'(k) - \delta - b \tag{4.40}$$

Altogether, equations (4.37, 4.38, 4.39 and 4.40) are used to numerically compute the steady state equilibrium of the model for any given value of public capital. These equilibrium conditions assume that public capital per person is exogenous and there is no government debt in the economy. Thus, computing the equilibrium involves choosing an initial value of private capital and determining the corresponding value of consumption for which the pair converges to a steady state. In table 4.1, I choose numeric values for the exogenous parameters and use that to numerically compute the steady state equilibrium of the model, and solve dynamic equilibrium paths.

Table 4.1: Chosen Parameter values for exogenous variables

$\alpha = 0.44$	discount rate, $\rho = 0.018$
birth rate, $b = 0.038$	mortality rate, $m = 0.029$
pop growth rate, $n = 0.009$	depreciation of k, $\delta = 0.04$
$k_p = 6.0, k = 12.0$	depreciation of k_p, $\delta_p = 0.03$

These values are chosen to satisfy the assumptions of the model

Given the public capital stock per capita as $k_p = 6.0$, the steady state equilibrium values of private capital and consumption are 87.58 and 15.05 respectively. As is well known of the deterministic growth model, consumption is a jump variable while initial value of (private) capital is given. Here, initial "private" capital is predetermined while public capital is exogenously fixed. With these, the dynamic equilibrium path is solved using difference equations.[5] Figure 4.2 shows various initial pairs of private capital and consumption that yield different trajectories. Only trajectory 1 ("Equil. tr") stay on the dynamic path and converges to steady state, all others diverge with time from the equilibrium path. Under a given tolerance level and chosen grid points (of time), for an initial private capital stock of 12.0, the corresponding consumption value that converges, along the dynamic path, to the steady state is 4.160979244. Minor deviations from this

[5] **A note on the computational solution:** Even though the model is a continuous time model, it is computationally convenient to use difference equations to solve for the dynamic equilibrium path towards steady state for plotting purposes. In other words, once the steady state is computed using the standard newton approach, one can then solve for the level of consumption that converges to steady state for any given amount of private capital. Along the dynamic path, consumption is linear in private capital. But since for any given value of private capital, its values and the corresponding consumption values pass through several states to converge to the steady state, it is intuitive to think about the evolution of the pair in discrete space. Thus, $\dot{k} = \delta k/\delta t$ is approximated to $\Delta k/\Delta t = (k_t - k_{t-1})/(t-(t-1))$. Similarly, $\dot{c} = \delta c/\delta t$ is approximated to $\Delta c/\Delta t = (c_t - c_{t-1})/(t-(t-1))$. From these, one can write private capital and consumption as functions of their previous realizations. Thus, computing and plotting the transition path requires a set of finite states. I implement this by using a finite set of grid points to represent the number of times the state-control pair evolve in transition.

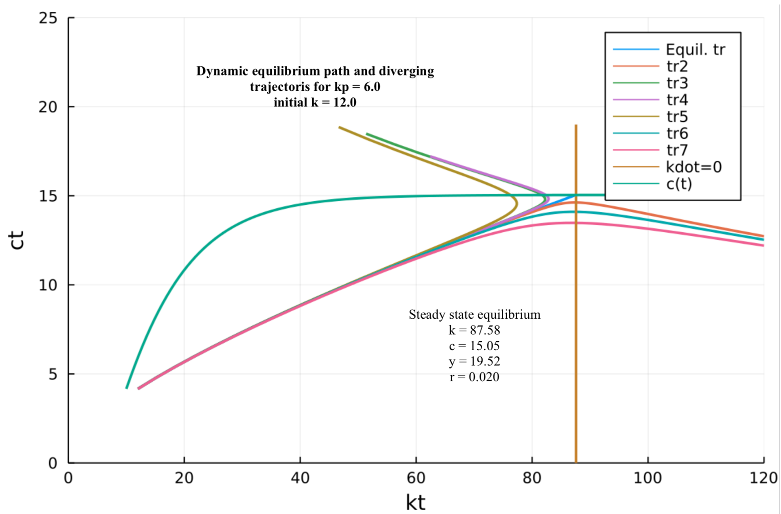

Figure 4.2: Equilibrium trajectory and diverging paths

value causes divergence from the dynamic equilibrium path.

The assumption of a given level of public capital per labour seem rather crude in the presence of population growth. For any given economy, the starting level of aggregate public capital (and hence its per capita value) may be given. However, the public capital per labour at any point in time subsequent must depend on the public investment effort by the government. This complicates the steady state equilibrium analysis. It turns out that following Aschauer (2000) and Checherita-Westphal et al. (2014) in assuming that the public capital is procured by debt and the per capita value of both public capital and debt held constant indefinitely (as seen above in the analysis of section 4) simplifies the problem and conveniently helps to show the equilibrium effect of debt in a computationally transparent manner.

4.5.2 Steady state equilibria with external debt

The analysis in section 4.4 shows that both domestic and external debt to finance public capital formation may enhance steady state outcomes. However, its effect may differ quantitatively depending on whether the debt is issued in the domestic economy or to agents outside the economy. I start with the simplest case, where the debt is issued externally, and the economy is assumed to be small enough to have no effects on global interest rates. In this case, external debt is justified only if the global interest rate is beneath the interest rate in the domestic economy. Thus interest rate on government

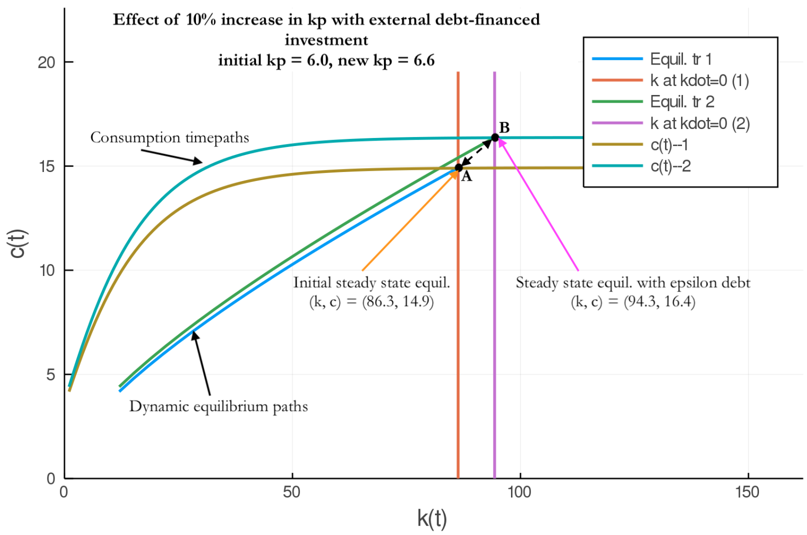

Figure 4.3: Change in equilibrium with 10 percent increase in external debt (public capital)

Notes: The diagram shows a shift in steady state equilibrium from point "A" to point "B" when the available public capitals stock per labour is increased from 6.0 to 6.6 using external debt financing.

debt is such that $(i < r)$, where i is the interest rate on government debt, and r is the domestic interest rate. Note that the interest rate on the external debt is chosen to be substantially lower than the domestic interest rate. As shown on figure 4.2, the steady state equilibrium where k_p is given, has the stable interest rate, $r = 0.020$. I therefore set the external interest rate on government debt at $i = 0.011$. Recall from section 4.4 that when government debt is issued externally the equilibrium is characterised by the following coupled differential equations:

$$\dot{k} = f(k) - c - (n + \delta)k - \tau \tag{4.41}$$

$$\dot{c} = [r - \rho - (i + \delta_p)\phi)]c \tag{4.42}$$

where $\tau = (i+\delta_p)k_p$. Note that these differential equations are from the per capita form of equation (4.22) and the corresponding solution to the household inter-temporal choice problem. Using the equations (4.37, 4.40, 4.41 and 4.42), the steady state equilibrium is graphically shown at point "A" of figure 4.3. The private capital and consumption levels are given respectively as 86.3 and 14.9, with the domestic interest rate being $r = 0.0208$. Compared to figure 4.2, the financing of the same level of public capital by external debt yields a higher equilibrium interest rate and hence a lower level of private capital and consumption. Suppose that from this steady state equilibrium, government issues a lump-sum debt amounting to 10 percent of the debt level. Investing this in public capital will increase the stock of public capital per labour as well as the debt labour ratio. Clearly, if we maintain the new level of public capital per capita and debt labour ratio, steady state equilibrium values of private capital and consumption will now be different from the initial steady state. Observe that the increase in public capital per capita changes the coupled differential equations (the counterparts of equations 4.41 and 4.42) characterising equilibrium to:

$$\dot{k} = f(k) - c - (n+\delta)k - \tau_n \tag{4.43}$$

$$\dot{c} = [r_n - \rho - [(i+n+\delta_p)(1+\epsilon) - n]\phi]c \tag{4.44}$$

where the new tax per labour is $\tau_n = [(i+n+\delta_p)(1+\epsilon) - n]k_p$. The new steady state equilibrium is illustrated at point "B" of the figure 4.3, with equilibrium interest rate being $r_n = 0.0212$, and private capital and consumption increasing to 94.3, and 16.4 respectively. Ultimately, despite the imposition of additional tax for servicing and maintaining a higher level of debt and public capital, the increase in the public capital through external debt financing enhances both private capital and consumption levels.

4.5.3 Steady state equilibria with domestic debt

The effect of domestic debt is slightly different to external debt. Unlike external debt where the interest rate on debt is given and taken to be below the domestic interest rate, with domestic debt, the interest rate is endogenously determined. Thus, the financing of public capital formation by the issuance of government debt affects not only the prevailing interest rate through the demand side effect, it also reduces the supply of private capital. The coupled differential equations for dynamic equilibrium are restated as:

$$\dot{k} = f(k) - c - (n+\delta)k + (m - \delta_p)\phi k \tag{4.45}$$

$$\dot{c} = [r - \rho + (m - \delta_p)\phi)]c \tag{4.46}$$

The interest rate on government debt is endogenously determined and is required to be equal to the prevailing interest rate in the capital market. Note that equation (4.45 and 4.46) are analogous to (4.41 and 4.42) in the case with debt issued externally. But unlike external debt where households are taxed to make interest payments to agents outside the economy, here the interest payments goes back to the households who hold government debt. Since we assumed homogeneity in household behaviour, the effect is

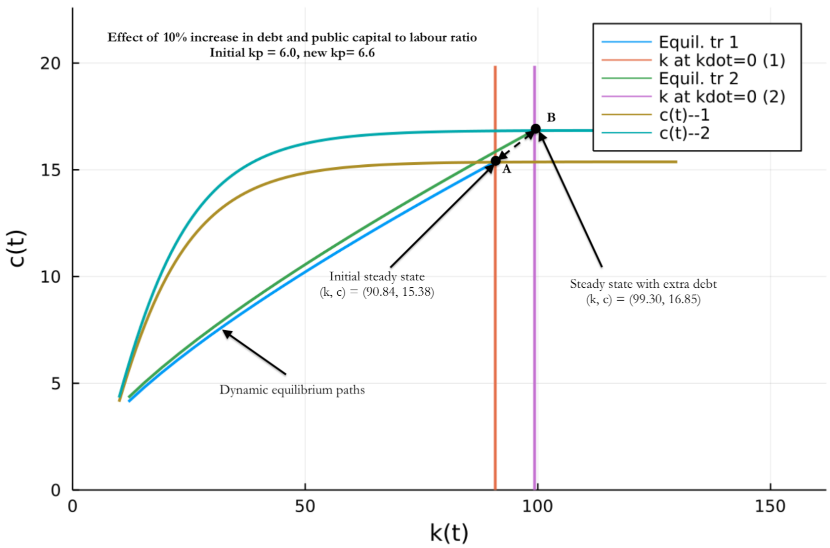

Figure 4.4: Change in equilibrium with 10 percent increase in domestic debt (public capital)

Notes: What is shown in this diagram is no different from that on figure 4.3, except that the debt financing of the increase in public capital stock is issued domestically.

the same for each existing agent.

The resulting steady state equilibrium is shown at point "A" of figure 4.4, with private capital and consumption levels being 90.84 and 15.38 respectively, and interest rate being 0.0181. Note that public capital is maintained at 6.0. For an initially predetermined value of private capital given as 12.0, the corresponding value of consumption that converges to steady state equilibrium is 4.13449208745. To illustrate the effect of a debt accumulation to increase the level of public capital on steady state equilibrium (A), suppose government issues extra bonds in the domestic capital market. This invariably entails a change in the size of debt labour ratio. For reattainment of steady state equilibrium, government must commit to maintain the new public capital (and debt) labour ratio. This changes dynamic equilibrium conditions (equations 4.45 and 4.46 respectively) to:

$$\dot{k} = f(k) - c - (n + \delta)k + (m - \delta_p)\phi k - (r_n + n + \delta_p)\epsilon\phi k \tag{4.47}$$

$$\dot{c} = [r_n - \rho + (m - \delta_p)\phi - (r_n + n + \delta_p)\epsilon\phi]c \tag{4.48}$$

As before, the instantaneous increase in the public capital stock per labour, financed by issuance of domestic bonds, equals 10 percent of its value at steady state "A". This is captured by the parameter $\epsilon = 0.1$. As figure 4.4 shows, despite an increase in the burden of debt occasioned by the rise in debt labour ratio, the increase in public capital enhances steady state private capital and consumption, and hence the shift from point "A" to point "B". From an initial steady state value of 90.84, private capital stock rises to 99.30 following the 10 percent increase in public capital and debt level. Likewise, consumption rises from 15.38 to a new steady state value of 16.85, and interest rate increases to 0.0184. While this result is based on the assumption that public debt is refinanced indefinitely, it is straightforward that a one-time capital tax may be introduced to fully defray the extra debt without reducing private capital and consumption back to its initial equilibrium level. Crucially, for the positive effect of debt-financed investment in public capital on both consumption and private capital to exist, the marginal product of public capital must exceed the prevailing interest rate ex-ante, and secondly, the issuance of debt must be within a threshold such that the net returns to households from the debt (including government transfers) is in excess of the tax burden it imposes, ex-post.

4.5.4 Comparative statics

The computations, thus far, has shown that debt financing of public investment can be welfare improving in the long-run irrespective of whether debt is issued domestically or externally. However, comparing the computations for domestic debt with external debt shows considerable quantitative differences. From the figures 4.3 and 4.4, consumption and private capital are significantly lower at steady state equilibrium when debt is issued externally. By contrast, the equilibrium rate of return to capital is higher when public capital is financed by external debt. With higher return on private capital, one would have expected consumption level to be much higher since individuals consume out of returns to capital. The reason for the relatively lower level of private capital and consumption reflects the fact that steady state interest rate accounts for the interest on debt service to external agents. Thus, even though households receive a higher return on private capital, a significant share of it goes into debt service. Since debt is serviced indefinitely, it implies the continuous withdrawal of capital from the economy, and hence the lower level of private capital in equilibrium. Domestic debt on the other hand is held by households within the economy who receive debt service payments. Therefore, total returns to household investments comprise returns to private capital and government debt. However, because the government holds a constant level of public capital labour ratio, the debt level per capita is constant. To optimise inter-temporal allocation of consumption, households assets accumulation decision is limited only to investment in private capital. Hence, in the long-run, the amount of private capital is much higher and interest rate much lower. This explains why the interest rates at steady states "A" and "B" are 0.0181 and 0.0184 respectively when public capital is financed domestically, as opposed to 0.0208 and 0.212 when financed externally.

The results on steady state interest rate in the different equilibria between domestic and external debt seem counter-intuitive. When debt is issued domestically, the government is deemed to enter the domestic capital market and hence creates a demand side effect which is expected to result in an increased interest rate compared to debt being issued externally. Clearly, this is generally a short-term effect. In the long-run, the interest rate is relatively lower because households need not account for interest payments on government debt in their consumption-savings decision as they are the recipients of this payments. In addition, there is a redistribution of private assets held in government debt by people who die to those who are alive through the negative life insurance contracts. This explains the presence of the term $m\phi$ in equations (4.46) and (4.48). Altogether, these lead to a relatively lower interest rate in the long-run. With external debt, on the other hand, households internalize the tax to pay interest on government debt in the consumption-saving decisions, and hence the presence of the term $i\phi$ in equations (4.42) and (4.44). Also, unlike domestic debt, the redistribution of assets held in government debt is unimportant here since debt is held by external agents. Therefore, interest rate is relatively higher.

Finally, it is important to note that increasing the debt and public capital level from any steady state equilibrium invariably entails an increase in long-run interest rate for both types of debt. This is largely due to the burden of maintain a marginally higher public capital stock per labour, relative to private capital. In other words, even though private capital rises following the increase in public capital, the ratio of public to private capital never settles back to its initial level in the long-run. In sum, the increase in public capital per capita raises the public to private capital ratio marginally in the long-run because individuals perceive the increased taxes to maintain the public capital. This ensures that private capital does not rise by a proportionate measure relative the public capital stock.

4.6 A look to the data

The primary occupation of this section is to check the conditions under which debt financing of public investments is welfare improving using a wide sample of developing and advanced economies. The necessary condition from section 4.4 suggests that in the face of insufficient level of public capital, debt finance for public investment may be welfare improving if public capital has a larger productivity effect than private capital. To check this, I estimate the average marginal productivities for public and private capital for different groups of countries using a dataset covering the period 1960-2015. Given the structure of the analytic model, the insights in this section should be read with caution. They indicate the presence of some statistical relationships that are consistent with the conditions of the model. The analysis here does not provide significant evidence in support of debt-financed public investment, rather the results merely show that the point estimates for public capital productivity appears larger than private capital across the different groups sampled. The dataset I use include countries at various levels of development. The primary approach I adopt is a modification of Caselli and Feyrer (2007)[33] by incorporating public capital in the aggregate Cobb Douglas production function. Thus, I run some regressions to estimate the output elasticities, after which I compute the marginal productivities using the means of the capital values for the study period. In addition, I check the likelihood of the marginal productivities of public and private capital differing by country groups in view of the huge differences in stock averages between advanced, emerging market, and low income developing countries. I do this by estimating for each country group and in separate estimations, I exclude countries with population outliers like China, India, the United States, and Japan. Note that public capital comprises government investments in infrastructure, educational institutions, and health facilities among others, while private capital comprise of physical capital holdings of the private sector. Figure 4.5 plots the yearly mean of public capital stocks (in billions of constant 2017 international dollars) for a sample of 165 IMF-member countries, categorised into country groupings defined by the World Economic Outlook (WEO); Advanced Economies (AE), Emerging Markets (EM) and Low Income Developing Countries (LIDC).

As the graph shows, the yearly average for stocks of public capital has remained largely flat for the group of 50 low income developing countries relative to the set of 36 advanced economies and 79 emerging market economies for the period 1960 and

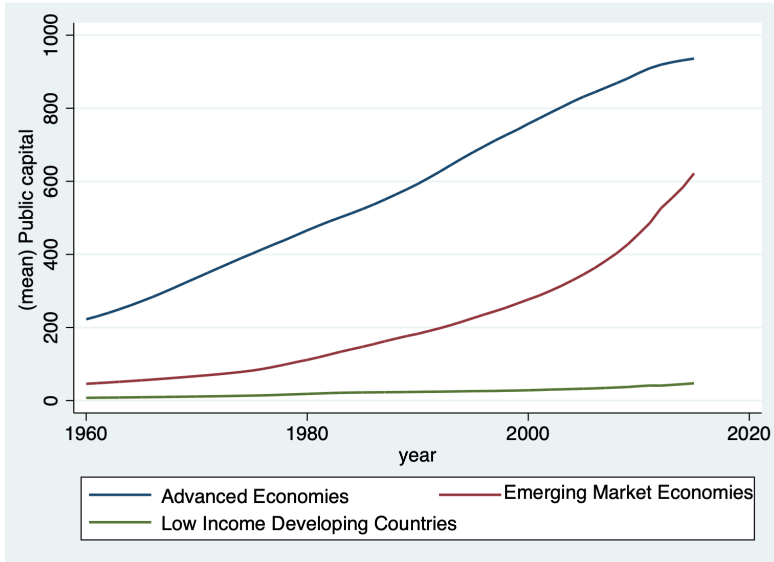

Figure 4.5: Public capital stocks for country groups between 1960 and 2015

Notes: The plots capture the mean of public capital stocks for the different country groups for the period 1960-2015. The data are in billions of constant 2017 international dollars, and adjusted for purchasing power parity.

Source: IMF Investment and Capital Stock Dataset, 2017.

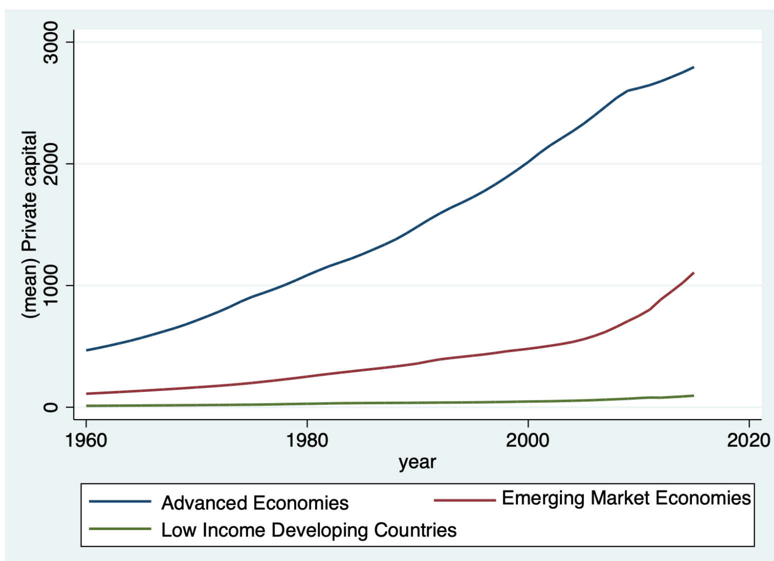

Figure 4.6: Private capital stocks for country groups between 1960 and 2015

Notes: The plots capture the mean of private capital stocks for the different country groups for the period 1960-2015. The data are in billions of constant 2017 international dollars, and adjusted for purchasing power parity.

Source: IMF Investment and Capital Stock Dataset, 2017.

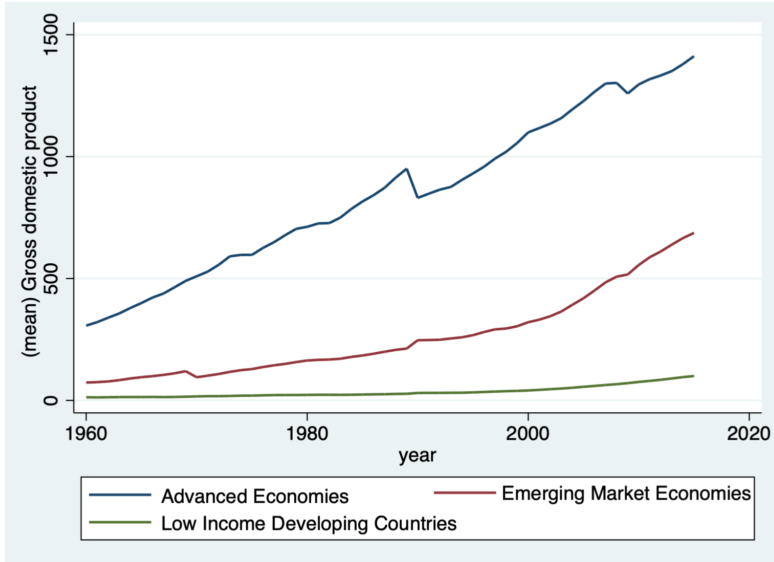

Figure 4.7: GDP for country groups between 1960 and 2015

Notes: The plots capture the mean of gross domestic product for the different country groups for the period 1960-2015. The data are in billions of constant 2017 international dollars, and adjusted for purchasing power parity.

Source: IMF Investment and Capital Stock Dataset, 2017.

2015. Private capital stocks and gross domestic product show similar patterns as shown on figure 4.6 and 4.7 respectively. Given this wide disparity, I check to see if capital productivity differs across the country groups in the ensuing analysis.

4.6.1 Data sources

The data used here is drawn from three sources. Data on output levels and capital stocks is obtained from the International Monetary Fund (IMF) Investment and Capital Stocks Dataset, 2021 (See IMF, 2015[101] for rationale and details on the dataset). The most recent publicly accessible of such data is dated May 2021 (see Xiao et al. 2021[115]). The figures are in billions of 2017 international dollars. Data for levels of employment and human capital (proxied by years of schooling) is obtained from the Penn World Tables (see Feenstra et al. 2015[53] for detail description of this dataset), while data on debt-GDP ratios are obtained from the Historical Public Debt Database (HPDD) of the IMF.

4.6.2 Estimations

I follow the approach of Caselli and Feyrer (2007)[33] to compute the marginal productivities of public and private capital. I start by log-transforming the data and testing for unit roots. As figure 4.5 through 4.7 show, the data for GDP and capital stocks look to be trending. Thus, regression in levels may yield spurious estimates. The results of the unit-roots tests are presented on Table B.1 in the appendix, and generally show that all variables of interest contain unit-roots in levels, but are stationary in first differences. To estimate the output elasticities, I use the aggregate Cobb Douglas function of the form:

$$Y = F(K, K_p, L) = K^{\alpha} K_p^{\beta} L^{\mu} \;\; s.t \;\; \alpha + \beta + \mu = 1. \tag{4.49}$$

Where K and K_p are aggregate private and public capital respectively, while L denotes employment level. Taking logs, we have $\ln Y = \alpha \ln K + \beta \ln K_p + \mu \ln L$. The coefficients in a regression of this equation can be interpreted as output elasticities. From this, the marginal productivities for private and public capital are given respectively as;

$$MPK = \alpha K^{\alpha-1} K_p^{\beta} L^{\mu} = \alpha \frac{Y}{K} \tag{4.50}$$

$$MPK_p = \beta K^{\alpha} K_p^{\beta-1} L^{\mu} = \beta \frac{Y}{K_p} \tag{4.51}$$

and the share of output per unit of labour given by

$$MPKL = \mu K^{\alpha} K_p^{\beta} L^{\mu-1} = \mu \frac{Y}{L} \tag{4.52}$$

where α is the output elasticity of private capital, and β is the elasticity with respect to public capital. Given the primary focus here being the marginal products of public and private capital, I report only the output elasticities for the capital stocks and use these to compute the marginal productivities following equations (4.50 and 4.51). The estimations of equations (4.50 and 4.51) are termed "naive" estimations by Caselli and Feyrer (2007) in their attempt to compare marginal products across different countries. The purpose, here on the other hand, is to compare marginal products of public and private capital within economies, rather than between economies. Nevertheless, to allow for inference between the different country groups, I incorporate the different price levels of output and capital and re-estimate the MPKs. It should be noted that unlike Caselli and Feyrer (2007) who also factor for natural capital such as land and natural resources, I factor only for the price levels. Thus, equations (4.50 and 4.51) modify to:

$$PMPK = \alpha \frac{p_y * Y}{p_k * K} \tag{4.53}$$

$$PMPK_p = \beta \frac{p_y * Y}{p_k * K_p} \tag{4.54}$$

where p_y captures the price level of final goods in each economy, while p_k is the price of capital stock. In other words, the marginal products are adjusted to factor for the price levels in the different categories of countries. Both the "naive" and the price level adjusted estimates are presented on table 4.3. In alternative estimations (presented on table 4.4), I control for human capital, proxied by years of schooling, and instead of price level for capital stocks, I also use price level for capital services. Unlike the price level for capital stock which is lower in LIDC, followed by EM, and then AE, there is a reversal in the price level for capital services with the average price being much higher in LIDC, compared to EM and AE. Details of the summary statistics are presented on table 4.2. Finally, it is worth noting how the marginal productivities are computed from the panel dataset. Instead of MPK/PMPK, what I compute is in fact $M\bar{P}K_{i,t}/PM\bar{P}K_{i,t}$, where the bar denotes the cross-sectional-longitudinal average. In essence, equations (4.50, 4.51, 4.53 and 4.54) may be restated as:

$$M\bar{P}K_{i,t} = \alpha \frac{\bar{Y}_{i,t}}{\bar{K}_{i,t}} \tag{4.55}$$

$$M\bar{P}Kp_{i,t} = \beta \frac{\bar{Y}_{i,t}}{\bar{K}p_{i,t}} \tag{4.56}$$

$$PM\bar{P}K_{i,t} = \alpha \frac{\bar{py}_{i,t} * \bar{Y}_{i,t}}{\bar{pk}_{i,t} * \bar{K}_{i,t}} \tag{4.57}$$

$$PM\bar{P}Kp_{i,t} = \beta \frac{\bar{py}_{i,t} * \bar{Y}_{i,t}}{\bar{pk}_{i,t} * \bar{K}p_{i,t}} \tag{4.58}$$

where $\bar{x}_{i,t}$ for example is the cross-sectional-longitudinal mean value of the respective variable estimated from the panel data, and α and β recovered from a regression equation of the form:

$$\Delta Y_{i,t} = \alpha \Delta K_{i,t} + \beta \Delta Kp_{i,t} + \gamma \Delta L_{i,t} + \lambda_t + \epsilon_{i,t} \tag{4.59}$$

where Δ represents the differencing of the data, λ_t is the time effects and $\epsilon_{i,t}$ the error term. I report only α and β from this and use them to compute the MPKs on tables 4.3 and 4.4.

4.6.3 Discussion of results

The summary statistics on Table 4.2 show the overall mean and standard deviation for all relevant variables. I estimate for the different groups of countries in the sample of 165

Table 4.2: Summary Statistics for the period 1960 - 2015

	50 LIDC		79 EM		36 AE		129 LIDC & EM	
Variable	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Public capital	22.844	72.605	208.040	1001.46	576.232	1469.853	136.326	790.371
Private capital	38.457	106.354	386.032	1130.182	1487.935	764.9804	251.440	903.148
GDP	37.045	82.238	283.481	838.936	871.975	3302.651	183.649	660.325
Debt-GDP ratio	74.196	80.697	47.593	35.5529	48.957	35.262	57.864	58.797
Employment	6.423	9.315	21.630	85.526	12.099	22.626	15.700	67.459
Years of sch.	2.661	2.126	5.722	2.751	9.370	2.487	4.4405	2.927
Capital depreciation	0.042	0.013	0.046	0.14	0.036	0.007	0.044	0.014
Pl-GDP	0.249	0.144	0.322	0.199	0.545	0.340	0.292	0.182
Pl-capital stock	0.312	0.234	0.352	0.264	0.434	0.284	0.336	0.253
Pl-capital services	1.525	2.356	1.437	2.104	1.021	0.396	1.465	2.1856

Notes: Capital stocks and GDP are in billions of constant 2017 international dollars and measured by purchasing power parity. Employment levels are in millions, and capital depreciation is expressed as a fraction of capital. Finally, Pl in bottom three rows stands for price level.

countries for which data is available for the study variables from the IMF Investment and Capital Stocks Dataset, the Penn World Tables, and the HPDD. In general, low income developing countries have large deficits in public and private capital stocks relative to emerging market and advanced economies. The average public capital stock for emerging market and advanced economies are nine and twenty-five times as large as the estimate for low income developing economies. Private capital stock on the other hand is about ten and thirty-eight times as large for EM and AE respectively relative to LIDC. Judging by standard economic theory, and from the kind complementarity between capital stocks in the theoretical sections of this thesis, one would expect that not only should public capital be more productive than private capital, but also both public and private capital ought to be substantially more productive in LIDC than EM and AE. These will be the focus of the analysis in the following section. The final point worthy of note from table 4.2 is that the average gross domestic product appears to be a convex combination of public and private capital, and thus lend support to the functional form of the output function used for the theoretical analysis.

Marginal productivities of capital

I start with the results on table 4.3 which summarises the mean for gross domestic product, private capital stock and public capital stock for LIDC, EM and AE. I then estimate the output elasticities of private capital and public capital (presented as α and β respectively) following equation (4.59). I elect to not report the output elasticity of labour

Table 4.3: Cross-country productivity effects of public and private capital

Sample	$\bar{Y}$	$\bar{K}$	$\bar{K_p}$	α	β	$M\bar{P}K$	$M\bar{P}K_p$	$PM\bar{P}K$	$PM\bar{P}K_p$
LIDC	37.045	38.457	22.844	0.116***	0.167***	0.112	0.271	0.09	0.216
LIDC & EM	183.649	251.440	136.326	0.194***	0.109***	0.142	0.147	0.123	0.128
LIDC & EM (- China)	151.617	233.072	91.788	0.190***	0.108***	0.124	0.178	0.061	0.273
LIDC & EM (- Ch., India)	134.307	216.903	79.908	0.189***	0.108***	0.117	0.182	0.102	0.158
EM	283.481	386.032	208.040	0.246***	0.066	0.181	-	0.165	-
EM (- China)	230.796	357.636	135.917	0.241***	0.066	0.155	-	0.142	-
EM (- Ch., India)	202.537	332.605	116.908	0.240***	0.064	0.146	-	0.133	-
AE	871.975	1487.935	576.232	0.186***	0.144***	0.109	0.218	0.137	0.274
AE (- USA)	580.805	1094.803	384.926	0.201***	0.128***	0.107	0.161	0.134	0.260
AE (- US, Jap.)	495.602	930.057	273.679	0.195***	0.130***	0.104	0.235	0.131	0.237

Notes: $\bar{Y}$, $\bar{K}$ and $\bar{k_p}$ denote the mean values for GDP, private and public capital for the study period. α and β are the output elasticities with respect to private and public capital, and $M\bar{P}K$, $M\bar{P}K_p$ are computed following equations (4.55) and (4.56) respectively. Similarly, $PM\bar{P}K$, $PM\bar{P}K_p$ are computed following equations (4.57) and (4.58) respectively. Observe that I do not compute $M\bar{P}K_p$ and $PM\bar{P}K_p$ for the EM groups as the reported β is insignificant for those samples. As before, statistical significance is denoted at 1% (***), 5% (**) and 10% (*).

as it is not particularly relevant for the ensuing discussion. The reported elasticities for capital is from OLS regressions with robust standard errors (where the standard errors are adjusted for country clusters, the results do not differ substantially). As shown on column eight, I do not compute the marginal product for public capital for the emerging market economies, as the output elasticity is not significant. Finally, using the estimated elasticities and mean for the capital stocks and GDP, I compute the marginal products for the different country groups. As shown on column seven and eight, the "naive" estimates show a much larger productivity for public capital relative to private capital. In LIDC and AE, MPK_p is two times or more as large as MPK. Where output elasticities are significant, the marginal product for public capital has a range of $0.16 - 0.27$, while private capital's range is $0.10 - 0.18$. Also noteworthy is that the marginal products for both private and public capital are slightly higher in low income countries compared to advanced economies. But these estimates do not account for price level differentials across country groups. As shown on table 4.2, the price level for both output and capital stocks is lower in low income developing countries compared with emerging market and advanced economies. Similarly, the output-capital stock price ratio is lower in low income countries than emerging market and advanced economies. In other words, capital is more expensive relative to the price of final goods in developing countries. Thus, when this is accounted for and the marginal products re-computed following equations (4.57 and 4.58), the outcome of the naive estimates concerning the difference between LIDC and AE is reversed. As column nine and ten show, the marginal products for both capital stocks are now larger in advanced countries than low income countries. Nevertheless, public capital maintains a larger productivity than private capital across the board.

A shortfall of the above calculations is that the functional form of the Cobb-Douglas function meant that other factors such as human capital were excluded in the estimation of the output elasticities. Relaxing this, and controlling for human capital (using years of schooling as a proxy), I re-estimate the output elasticities, and compute the marginal products factoring in price level differentials. In addition, instead of using price level of capital stock, I alternatively compute the marginal products, using price levels of capital services. As the summary statistics show, the price level for capital services is rather much higher in low income countries than emerging market and advanced economies. With capital services being expensive in all economies, and especially very expensive in low income countries, factoring for this further dampens the estimated capital productivity. With these adjustments, the main result remains unchanged with public capital maintaining a larger marginal productivity relative to private capital across all estimates, but also advanced economies continue to have larger productivity for all capital relative to LIDC and EM. Thus, unlike the traditional economic thought of marginal product for capital being higher in low income countries compared to advanced countries, the analysis conducted here follows Caselli and Feyrer (2007) in showing that across the country groups (based on level of development), when one factors for price levels the marginal product for capital is rather higher in advanced economies. This is especially true when one uses price level of capital services to factor for price of capital.

The surprising result from this analysis is that the marginal product of both capital stocks is substantially higher in advanced economies than emerging market and low income countries. Despite the influential result of Caselli and Feyrer (2007) on equalised marginal productivities, one would have expected public capital to have much larger

Table 4.4: Cross-country productivity effects of public and private capital - adjusting for years of schooling and price levels

Adjusting for price level of capital stock					Adjusting for price level of capital services	
Country group	α	β	$P\bar{MPK}$	$P\bar{MPK}_p$	$P_s\bar{MPK}$	$P_s\bar{MPK}_p$
LIDC	0.116***	0.174***	0.089	0.225	0.018	0.046
LIDC & EM	0.130***	0.133***	0.083	0.156	0.019	0.036
LIDC & EM (- China)	0.119***	0.133***	0.067	0.191	0.015	0.044
LIDC & EM (- Ch., India)	0.117***	0.132***	0.063	0.193	0.014	0.044
EM	0.126**	0.083*	0.085	-	0.021	-
EM (- China)	0.101	0.083*	-	-	-	-
EM (- Ch., India)	0.096	0.081*	-	-	-	-
AE	0.176***	0.137***	0.130	0.260	0.055	0.111
AE (- USA)	0.190***	0.123***	0.127	0.233	0.054	0.099
AE (- US, Jap.)	0.182***	0.124***	0.122	0.283	0.052	0.120

Notes: The regressions I use to estimate α and β controls for years of schooling. The $P\bar{MPK}$, $P\bar{MPK}_p$ are computed following equations (4.57) and (4.58) respectively. Also note that the columns $P_s\bar{MPK}$ and $P_s\bar{MPK}_p$ factor for the price level of capital services, rather than capital stock. Observe that I do not compute $P\bar{MPK}/P_s\bar{MPK}$ and $P\bar{MPK}_p/P_s\bar{MPK}_p$ for the EM groups as the reported β is insignificant for those samples. As before, statistical significance is denoted at 1% (***), 5% (**) and 10% (*).

productivity level in low income countries, given the large gaps in infrastructure and other forms of public capital documented in the literature. However, the relatively expensive nature of capital in low income countries dampens marginal productivity. This is especially the case for capital services. Thus, a foreign entrepreneur looking to establish a productive enterprise in low income countries will pay a higher price to hire capital services. Ultimately, the relatively lower price for final goods effectively reduces the marginal productivity for investment in productive enterprises.

In summary, the estimations have shown that public capital has larger marginal productivity over private capital across the different groups of countries in the sample. However, in general, capital remains more productive in advanced countries than emerging market and low income countries due to capital price differentials. It does suggest that developing economies need policy measures that reduces the relative price of capital to make them competitive. That said, the larger marginal productivity for public capital suggests that efficient public investments may be exploited to improve growth and development in less developed economies.

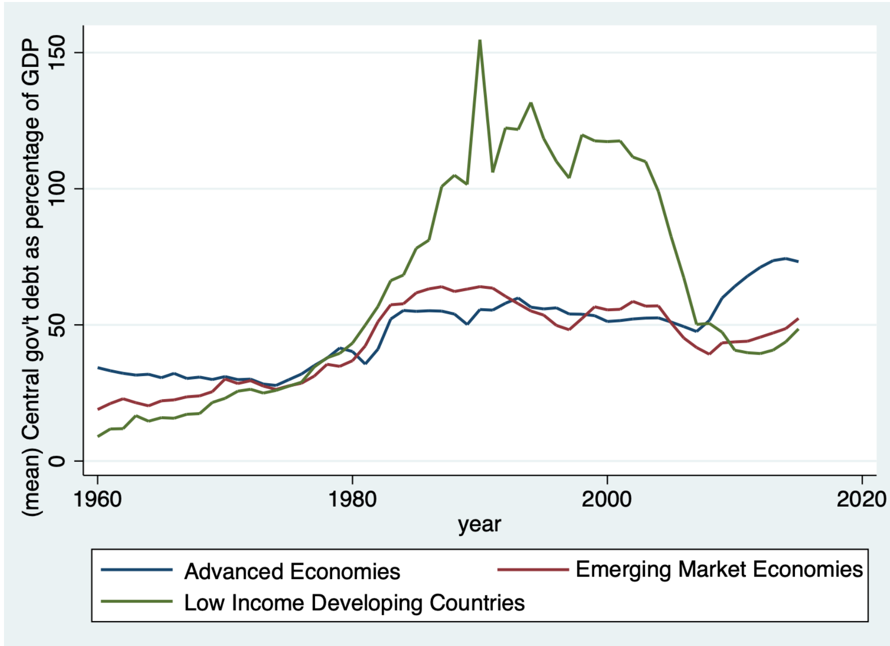

Figure 4.8: Average debt-GDP ratio for country groups between 1960 and 2015

Notes: The graphs plots the mean of debt-GDP ratio for the country groups against each year within 1960-2015.

Source: Historical Public debt database of the IMF

Implications for public debt

The empirical finding of productivity premium for public capital have implications for debt financing of public investments, especially in developing countries. Viewed in the lens of the theoretical model, it does appear that financing public capital formation through government debt may improve long-run outcomes. But this should take into account existing debt levels. From figure 4.8, which graphs the average debt-GDP ratio for the three groups of countries, it can be seen that unlike the graphs for capital stocks and output, where the average for low income developing countries have remained substantially below the average for both emerging market and advanced countries, the graph for debt level depicts a contrasting picture. For more than half of the study period (between 1980 and 2010), the debt level for LIDC on the average has been substantially higher than EM and AE, which appear to have comparable level of debt over the period. Thus, intuitively, where the debt level is low, and public capital productivity is larger, debt-financed public investments might be useful for stimulating growth.

4.7 Concluding remarks

In this chapter, I have explored the role of public debt for financing investment in public capital formation. Examining the effects of this investment on long-run equilibrium outcomes has required the use of a production function where public capital, provided by the state, is an input in private sector production. We have seen that the available supply of the public capital stock will dictate steady-state private wealth and consumption levels. Abstracting from labour and wages in the model framework, consumption is fully borne out of returns on private investments, subject to a constant public capital per person. Given initial values for public and private capital, the equilibrium may be characterised by a disproportionately larger amount of private capital relative to public capital. It is shown that under this circumstance, debt accumulation for investment in public capital can increase household wealth and consumption levels. This depends crucially on debt financed investment enhancing overall returns to private investment in the production system than it increases interest rate in the capital market. The necessary condition for this positive welfare effect requires marginal productivity of public capital to exceed the productivity level of private capital. This is only one way of enhancing utility in equilibrium, however.

As a well known result in the overlapping generations literature, dynamic inefficiency may arise in equilibrium in the absence of bequest motive. For any given public capital to labour ratio, this outcome is not ruled out in the model elaborated here. When this outcome arises at steady state, the equilibrium is characterised by $r < n$, and implementing intergenerational transfers by maintenance of a relevant debt policy may increase utility as shown by Diamond (1965). I have noted however that unlike debt issuance for consumption which requires dynamic inefficiency for public debt to enhance utility, debt issuance for public capital investment needs not require this outcome. As long as the marginal returns to public capital exceed the prevailing interest rate, debt issuance for investment may enhance utility for $r \lesseqgtr n$. This finding has significant policy implications for developing economies who use government debt to mobilize capital for growth financing.

In addition, I have used numerical computations to show that steady state equilibrium need not be unique if public capital is a given public good that can be increased by state intervention in the form of debt-financed investment. This is seen to be welfare improving irrespective of whether the debt is sovereign or issued domestically. But the computations have shown that issuing the debt domestically tend to have a much larger effect on wealth and consumption levels than external debt. Finally, I have gleaned data from various sources and for a large number of countries in an attempt to verify the most relevant condition for debt financing of public investment. The data shows that public capital is relatively under-supplied in low income developing countries relative to emerging market and advanced economies. Nevertheless, its productivity remains low relative to advanced countries, when price levels are accounted for. That said, in general, public capital appears to have larger marginal productivity relative to private capital across all country samples. This remains unchanged for all the alternative estimations of marginal productivity for the different country groups. Thus, one may generally conjecture that debt financing of public investment with high degree of efficiency can be welfare improving, a priori. However, low income countries may require policies to reduce the relative price of capital to enable them attract external financing.

Appendix A

Dynamic equations for aggregate variables

A.1 Dynamic aggregate consumption

The dynamic aggregate equations for household consumption and private capital are derived from equations (3.11 & 3.12). Starting with the aggregate consumption function given as:

$$C(t) = \int_{-\infty}^{t} c(v,t)L(v,t)dv = \int_{-\infty}^{t} c(v,t)L(0)e^{nv}be^{-m(t-v)}dv \qquad \text{(A.1)}$$

The Leibniz rule for differentiation under the integral is used to differentiate functions of the form,

$$A(t) = \int_{b(t)}^{c(t)} a(v,t)dv,$$

giving the derivative as

$$A'(t) = a[t,c(t)]c'(t) - a[t,b(t)]b'(t) + \int_{b(t)}^{c(t)} \frac{\partial a(v,t)}{\partial t}dv.$$

For $b(t), c(t) = -\infty, +\infty$, then $b'(t), c'(t) = 0$. Also, for $b(t), c(t) = t$, then $b'(t), c'(t) = 1$ (See Groth, 2011, pg. 531). Using this rule, the derivative for equation (A.1) becomes:

$$\dot{C}(t) = c(t,t)L(0)e^{nt}b - 0 \ + \ \int_{-\infty}^{t} L(0)b \ \frac{\partial}{\partial t}[c(v,t)e^{nv}e^{-m(t-v)}]dv$$

$$= c(t,t)L(t)b \ + \ L(0)b\int_{-\infty}^{t} e^{nv}[-me^{-m(t-v)}c(v,t) \ + \ e^{-m(t-v)}\frac{\partial c(v,t)}{\partial t}]dv$$

From equation (3.9), $\dot{c}(v,t) = [r(t) - \rho - \delta_p\phi]c(v,t)$. Substituting for this yields:

$$\dot{C}(t) \;=\; c(t,t)L(t)b \;+\; L(0)b\int_{-\infty}^{t} -mc(v,t)e^{-m(t-v)}e^{nv}$$

$$+\; L(0)b\int_{-\infty}^{t}[r(t) - \rho - \delta_p\phi]c(v,t)e^{-m(t-v)}e^{nv}.$$

As the government endows new agents with the average capital stock as captured by equation (3) and implied by the constraint imposed on (4), $a(v,t) = \bar{a}(t)$, at $v = t$. Therefore, $a(t,t) = \bar{a}(t)$. This implies $c(t,t) = \bar{c}(t)$. Using the fact that $\bar{c}(t)L(t) = C(t)$, we have:

$$\dot{C}(t) = bC(t) \;-\; mC(t) \;+\; [r(t) - \rho - \delta_p\phi]C(t)$$

Replace $-m$ with $n - b$ and rearrange terms. The aggregate function becomes:

$$\dot{C}(t) = -bC(t) \;+\; (r(t) - \rho - \delta_p\phi)C(t) \;+\; bC(t) \;+\; nC(t).$$

The solution to equation (3.9) gives $c(v,t) = [\rho + m + \delta_p\phi]a(v,t)$, and hence $c(t,t) = [\rho + m + \delta_p\phi]a(t,t)$. Thus, we replace $C(t) = \bar{c}(t)L(t)$, by $[\rho + m + \delta_p\phi]\bar{a}(t)L(t)$ in the first term and it gives the aggregate evolution of consumption in equation (3.13) as:

$$\dot{C}(t) = \; -b(\rho + m + \delta_p\phi)A(t) \;+\; (r(t) - \rho + b + n - \delta_p\phi)C(t). \tag{A.2}$$

A simple alternative can be used to obtain this result by noting that private assets (capital) per person, and hence individual consumption, is the same irrespective of age. Hence, $C(t) = c(t) * L(t)$. Log and differentiate both sides with respect to time, and it yields:

$$\dot{C}(t) = (\dot{c}/c + n)C(t) = (r(t) - \rho - \delta_p\phi + n)C.$$

Substituting $n = b - m$ and $C(t) = (\rho + m + \delta_p\phi)A(t)$, and rearranging, one obtains;

$$\dot{C}(t) = (r(t) - \rho - \delta_p\phi + b)C(t) + (n - b)(\rho + m + \delta_p\phi)A(t)$$

$$= (r(t) - \rho - \delta_p\phi + b + n)C(t) + -b(\rho + m + \delta_p\phi)A(t).$$

A.2 Dynamic aggregate private capital

The private capital in aggregate form is given as:

$$A(t) = \int_{-\infty}^{t} a(v,t)L(v,t)dv = \int_{-\infty}^{t} a(v,t)L(0)e^{nv}be^{-m(t-v)}dv. \tag{A.3}$$

Again, using the Leibniz rule, the derivative can be written as:

$$\dot{A}(t) = a(t,t)e^{nt}L(0)b \;-\; 0 \;+ L(0)b\int_{-\infty}^{t}\frac{\partial}{\partial t}[a(v,t)e^{nv}e^{-m(t-v)}]dv$$

$$= \; a(t,t)L(t)b \;+\; L(0)b\int_{-\infty}^{t} e^{nv}[-me^{-m(t-v)}a(v,t) \;+\; e^{-m(t-v)}\frac{\partial a(v,t)}{\partial t}]dv$$

Substituting for $a(t,t) = \bar{a}(t)$ and $\frac{\partial a(v,t)}{\partial t}$ from equation (3.6), we obtain

$$\dot{A}(t) = bA(t) - mA(t) + L(0)b\int_{-\infty}^{t} e^{nv}e^{-m(t-v)}[r(t){+}m]a(v,t){+}r_pk_p{-}c(v,t){-}\delta_pk_p \, dv$$

This yields:

$$\dot{A}(t) = bA(t) \;-\; mA(t) \;+\; (r(t)+m)A(t) \;+\; r_pK_p \;-\; C(t) \;-\; \delta_pK_p$$

Substitute for $K_p = \phi K$ and $k = A(t)$, and the aggregate evolution of private capital (household assets) is given in equation (3.14) as:

$$\dot{A}(t) = (b + r(t))A(t) \;+\; r_pK_p \;-\; C(t) \;-\; \delta_p\phi A(t). \tag{A.4}$$

As with the aggregate consumption function, taking $a = A(t)/L(t)$ and log-differentiate both sides with respect to time, we obtain $\dot{A}(t) = (\dot{a}(t) + n * a(t))L(t)$. Using the household budget constraint, we obtain;

$$\dot{A}(t) = (b + r(t))A(t) \;+\; r_pK_p \;-\; C(t) \;-\; \delta_p\phi A(t).$$

Appendix B

Computations and Statistical tests

B.1 Computational solution to steady state equilibrium

Given that the basic theoretical model is a deterministic general equilibrium model in continuous time, a number of assumptions and simplifications are made to numerically compute the equilibrium. I used the following steps to obtain numeric solutions to the model using the Julia programming language:

1. I start by setting up a system of four equations capturing the output function, the interest rate as a function of private capital, and the two coupled differential equations. The system of equations are written into a composite function and set up as a minimization problem.

2. Next, I provide initially guesses of numeric values for the variables. These guesses must be reasonable given the public capital value. For example, with output a Cobb Douglas function of public and private capital, the guess for output cannot exceed both the exogenous value of public capital and the initial guess for private capital. As it is required to be a convex combination of the two, its value must fall in between them.

3. The initially guessed values for the four parameters are used with the composite function to compute the deterministic steady state using the newton solution method and a prior written function for computing derivative of a function. Observe that the exogenous parameters take the numeric values presented on table

4.1.

4. The above steps is enough for computing the steady state equilibrium of the model. However, for plotting purposes, I solve for the dynamic equilibrium path. The deterministic solution treats the continuous time model in discrete space. This is particularly useful for the dynamic equilibrium trajectory.

5. I compute only the dynamic values of private capital and consumption, using finite differences. Thus, each period's value is a function of the previous period.

6. Since the differential equations characterising dynamic equilibrium are coupled, I set up a system of equations that determines the value of current private capital as a function of the previously determined values of private capital and consumption. This is true for determining the current value of consumption as well.

7. Finally, I take an initial given value for private capital and knowing consumption to be a jump variable, I keep changing the corresponding initial value for it until they converge to the known deterministic steady state. Note that the current value of consumption depends on its previous value and the current value of private capital but also the previous value of of private capital through the presence of the public-private capital ratio in the equation.

8. With the dynamic trajectories, necessary adjustments and plots features are applied to produce the plots presented on figures 4.1 through 4.3.

B.1.1 Statistical tests

I conduct two main statistical tests to enable computations of the marginal products. First, I check for presence of unit-roots in the panel data using the test procedures of Im, Pesaran, and Shin (2003)[66] and Choi (2001)[39]. The test statistics indicate that the log-transformed data in levels contain unit roots (see Table B.1). I limit the panel unit root tests to the two options. Other test options such as Hadri (2000)[62], Breitug (2001)[28], Levin–Lin–Chu (2002)[83], and Harris–Tzavalis (1999)[64] require strongly balanced data which is not fulfilled in this case. Given the non-stationary nature of the data in levels, I take the first-differences and as presented on table B.1, all the variables are shown to be stationary at 1% significance level. Therefore, I estimate the average output elasticities using the data in first-differences.

Table B.1: Tests for panel unit roots and Hausmann Misspecification for FE, RE

Tests for Panel unitroots			Hausman test for RE or FE	
Variable	Z-Statistic (log-t.)	Z-Statistic (log-t - FD)		
Im-Pesaran-Shin unit-root test				
GDP	7.774	-40.627***	test	value
Public capital	10.623	-8.133***	chi2(57)	67.60
Private capital	16.270	-9.448***		
Employment	14.870	-29.171***	p value	0.159
Human capital	-22.465***	-3.209***		
Fisher-type unit-root test				
GDP	4.238	-29.185***		
Public capital	9.425	-6.550***		
Private capital	12.725	-9.132***		
Employment	10.677	-22.914***		
Human capital	-0.150	-2.171**		

Note: H_0: All panels contain unit-roots. Note that I used the complete sample of 165 countries to conduct these tests. For the ADF inverse normal (Z) statistic, lag selection is set at 2. The tag "log-t-FD" in column headings stands for log transformed and first differenced. Statistical significance against the null is denoted at 1% (***), 5% (**) and 10% (*). Due to missing data for some panels in some years, other available panel unit-root tests such as Hadri, Breitung, Levin-Lin-Chu, and Harris-Tzavalis could not be applied, as they require strongly balanced data. Also note that I use years of schooling to proxy human capital.

Table B.2: List of all countries in dataset

36 AE		50 LIDCs		79 EMs		
United States	Cyprus	Bangladesh	Mauritania	Turkey	Saudi Arabia	FYR Macedonia
United Kingdom	Israel	Benin	Moldova	South Africa	Syria	Romania
Austria	Taiwan Province	Bhutan	Sudan	Argentina	UAE	Bosnia & Herz.
Belgium	Hong Kong SAR	Lesotho	Mozambique	Brazil	Egypt	Oman
Denmark	Korea	Burkina Faso	Myanmar	Chile	Sri Lanka	Armenia
France	Singapore	Burundi	Nepal	Colombia	India	Hungary
Germany	Czech Republic	Cambodia	Niger	Costa Rica	Indonesia	Poland
Italy	Slovak Republic	Cameroon	Rwanda	Dominican Rep.	Malaysia	Bolivia
Luxembourg	Estonia	Central Afr. Rep.	Yemen	Ecuador	Maldives	Montenegro
Netherlands	Latvia	Chad	Senegal	El Salvador	Pakistan	Suriname
Norway	Lithuania	Comoros	Uganda	Guatemala	Philippines	Belarus
Sweden	Slovenia	Congo Rep.	Lao P.D.R	Mexico	Thailand	Mongolia
Switzerland	New Zealand	Cote d'Ivoire	Vietnam	Panama	Algeria	Albania
Canada	Australia	DRC	Zambia	Paraguay	Angola	Bahrain
Japan		Djibouti	Zimbabwe	Peru	Botswana	Georgia
Finland		Ethiopia	Tajikistan	Uruguay	Cabo Verde	Iran
Greece		Gambia, The	Tanzania	Venezuela	Eq. Guinea	Kazakhstan
Iceland		Ghana	Togo	Antigua &Barbuda	Gabon	Iraq
Ireland		Guinea	Kenya	Bahamas, The	Mauritius	Bulgaria
Malta		Guinea-Bissau	Mali	Barbados	Morocco	Jordan
Portugal		Haiti	Nicaragua	Dominica	Seychelles	Russia
Spain		Honduras	Liberia	Grenada	Namibia	Kuwait
		Madagascar	Malawi	Belize	Eswatini	China
		Uzbekistan	Sierra Leone	St.Kitts & Nevis	Tunisia	Lebanon
		Sao Tome & Principe	Nigeria	St. Lucia	Fiji	Ukraine
				Azerbaijan	Serbia	
				St. Vincent & the Grenadines	Croatia	

Appendix C

Julia code for computations

C.1 Steady state computations given public capital

```
# Import Helper Functions (... denotes the path of the containing folder)
include("... /derivative.jl")
include("... /newton.jl")
#Exogenous parameter values
alpha = 0.44
br = 0.03800
mr = 0.02900
gr = br -mr
delta_p = 0.03
delta = 0.04
rho = 0.0180
sigma = 1.0
kp = 6.0
#Define output function
function output(x)
    y = zeros(length(x))
    y = (x[1]^(alpha))*(kp^(1.0-alpha))
    return y
end

#Define non-linear function
function steady_state(x::Array{Float64,1})
    f = zeros(length(x))

    f[1] = x[1] - (x[3] - x[2] - delta_p*kp -(delta+gr)*x[1]) - x[1]
```

```
    f[2] = x[2] - (x[4]- rho - delta_p*(kp/x[1]))*(1.0/sigma)*x[2] - x[2]
    f[3] = x[3] - (x[1]^alpha)*(kp^(1.0-alpha))
    f[4] = x[4] - (alpha*x[1]^(alpha-1.0)*kp^(1.0-alpha) - delta - br)
    return f
end

#Make a guess of the unknowns
initial_guess2 = [10,8.0,5.0,1.50]
using LinearAlgebra
(xr, fxr, iterr) = newton(steady_state, initial_guess2, 1e-8, 100)

#Verify equilibrium values using the following conditions
equi_r1 = derivative(output,[xr[1],kp])[1]- delta -br
equi_r2 = rho+delta_p*(kp/xr[1])
equi_output = output(xr[1])

#Check the generations replacement effect condition
equi_r1 = derivative(output,[xr[1],kp])[1]- delta - rho - delta_p*(kp/xr[1])

#Using finite difference method to solve differential equations.
time_nodes = [1.0:0.1:12.90;]
#remember to revert to [1.0:0.1:12.90;] after changes
dt = time_nodes[2]-time_nodes[1]
#define function for k_t_plus_1
function kt_plus1(x::Array{Float64,1})
    kt_plus1 = x[1] + (x[1]^(alpha)*kp^(1.0-alpha)) - x[2] - (delta + gr +
    delta_p*(kp/x[1]))*x[1]
    return kt_plus1
end
#define function for ct_plus1
function ct_plus1(x::Array{Float64,1})
    ct_plus1 = zeros(length(x))
    kt_plus1 = x[1] + (x[1]^(alpha)*kp^(1.0-alpha)) - x[2] - (delta + gr +
    delta_p*(kp/x[1]))*x[1]
    ct_plus1 = x[2] + (alpha*kt_plus1^(alpha-1.0)*kp^(1.0-alpha) - br - delta
    - rho - delta_p*(kp/kt_plus1))*(1/sigma)*x[2]
    return ct_plus1
end

#define function for the c-k pair in the dynamic equilibrium path
function ks_cs(x::Array{Float64,1})
    kt_nodes = zeros(length(time_nodes))
```

```
    ct_nodes = zeros(length(time_nodes))
    kt_nodes[1] = x[1]
    ct_nodes[1] = x[2]
    for i = 2:length(time_nodes)
        kt_nodes[i] = kt_plus1([kt_nodes[i-1],ct_nodes[i-1]])
        ct_nodes[i] = ct_plus1([kt_nodes[i-1],ct_nodes[i-1]])
    end
    return kt_nodes,ct_nodes
end
cdot_zero = zeros(131)
kdot_zero = zeros(20)
c_zerosx = [0.0:1.0:130;]
k_zerosx = [0.0:1.0:19;]
for i = 1:length(cdot_zero)
    cdot_zero[i] = xr[2]
end
for i = 1:length(k_zerosx)
    kdot_zero[i] = xr[1]
end

c_zeros = cdot_zero
k_zeros = kdot_zero
(kts, cts) = ks_cs([12.0, 4.160979244])
#kts and cts have converged to the equilibrium values above.
#Note that some kts and cts require time_nodes = [1.0:0.1:7.90;] to execute.
(kts2, cts2) = ks_cs([12.0, 4.1608710])
(kts6, cts6) = ks_cs([12.0, 4.16])
(kts7, cts7) = ks_cs([12.0, 4.157])
#change time_nodes to [1.0:0.1:7.90;] for kts3 and kts4
(kts3, cts3) = ks_cs([12.0, 4.1612])
(kts4, cts4) = ks_cs([12.0, 4.16114])
#change time_nodes to [1.0:0.1:6.50;] for kts5
(kts5, cts5) = ks_cs([12.0, 4.1623]) #use [1.0:0.1:6.50;] for time nodes
#create nodes for vertical line at equilibrium k
timenode_cts =[10.0:0.7:93.5;]
cts_line = zeros(0)
cts[120]
for i = 1:length(cts_line)
    cts_line[i] = cts[120]+0.0
end

cts_line = [cts;cts_line]
```

```
using Plots
plot([kts,kts2,kts3,kts4,kts5,kts6,kts7,k_zeros,timenode_cts],
[cts,cts2,cts3,cts4,cts5,cts6,cts7,k_zerosx,cts_line],
xlims = (0.0, 120),ylims = (0.0,25.0),xlabel = "kt",ylabel = "ct",
label = [:"Equil. tr" :"tr2" :"tr3" :"tr4" :"tr5" :"tr6" :"tr7" :"kdot=0" :"c(t)"]
, linewidth = 2)
#save plot as PNG, specifying the path as below
png("/Users/abdul-muminahmed/Desktop/My_Computations/plots/saddlepaths")
```

C.2 Steady state computations with external debt

```
#Import Helper Functions as above.
#Exogenous parameter values
alpha = 0.44
br = 0.03800
mr = 0.0290
gr = br -mr
delta_p = 0.03
delta = 0.04
rho = 0.0180
int_rate = 0.011
sigma = 1.0
kp = 6.0
tax = (int_rate+delta_p)*kp
#Define output function
function output(x)
    y = zeros(length(x))
    y = (x[1]^(alpha))*(kp^(1.0-alpha))
    return y
end

#Define non-linear function
function steady_state(x::Array{Float64,1})
    f = zeros(length(x))

    f[1] = x[1] - (x[3] - x[2] - tax -(delta+gr)*x[1]) - x[1]
    f[2] = x[2] - (x[4]- rho - (int_rate +delta_p)*(kp/x[1]))*(1.0/sigma)*x[2]
    - x[2]
    f[3] = x[3] - (x[1]^alpha)*(kp^(1.0-alpha))
    f[4] = x[4] - (alpha*x[1]^(alpha-1.0)*kp^(1.0-alpha) - delta - br)
    return f
```

```
end

initial_guess = [10,8.0,5.0,1.50]
using LinearAlgebra
(xr, fxr, iterr) = newton(steady_state, initial_guess, 1e-8, 100)

#check if equilibrium is correct by using ff conditions
equi_r1 = derivative(output,[xr[1],kp])[1]- delta -br
equi_r3 = rho+ (int_rate +delta_p)*(kp/xr[1])
equi_output = output(xr[1])
equiconsumption_wd = xr[3] - (gr + delta)*xr[1] - tax
##using finite difference method to solve differential eqtns.
time_nodes = [1.0:0.1:12.90;]
dt = time_nodes[2]-time_nodes[1]
#define function for k_t_plus_1
function kt_plus1(x::Array{Float64,1})
    kt_plus1 = x[1] + (x[1]^(alpha)*kp^(1.0-alpha)) - x[2] - tax -(delta+gr)*x[1]
    return kt_plus1
end
#define function for ct_plus1
function ct_plus1(x::Array{Float64,1})
    ct_plus1 = zeros(length(x))
    kt_plus1 = x[1] + (x[1]^(alpha)*kp^(1.0-alpha)) - x[2] - tax -(delta+gr)*x[1]
    ct_plus1 = x[2] + (alpha*kt_plus1^(alpha-1.0)*kp^(1.0-alpha) - br - delta
    - rho - (int_rate + delta_p)*(kp/kt_plus1))*(1.0/sigma)*x[2]
    return ct_plus1
end
# ct2 and kt2 is just for sanity check.
#ct2 = ct_plus1([5.0, 1.65])
#kt2 = kt_plus1([5.0, 1.650])
#####

function ks_cs(x::Array{Float64,1})
    kt_nodes = zeros(length(time_nodes))
    ct_nodes = zeros(length(time_nodes))
    kt_nodes[1] = x[1]
    ct_nodes[1] = x[2]
    for i = 2:length(time_nodes)
        kt_nodes[i] = kt_plus1([kt_nodes[i-1],ct_nodes[i-1]])
        ct_nodes[i] = ct_plus1([kt_nodes[i-1],ct_nodes[i-1]])
    end
    return kt_nodes,ct_nodes
```

```
end
cdot_zero = zeros(120)
kdot_zero = zeros(22)
c_zerosx = [0.0:1.0:120;]
k_zerosx = [0.0:1.0:21;]
for i = 1:length(cdot_zero)
    cdot_zero[i] = xr[2]
end
for i = 1:length(k_zerosx)
    kdot_zero[i] = xr[1]
end

c_zeros = cdot_zero #equilibrium consumption
k_zeros = kdot_zero #equilibrium capital
(kts, cts) = ks_cs([12.0, 4.172041888494649])
#kts and cts have converged to the equilibrium values above.
#set time notes for plots
timenodes_cts =[1.0:0.97:124;]
cts_line = zeros(7)
cts[120]
for i = 1:length(cts_line)
    cts_line[i] = cts[120]+0.0
end

cts_line = [cts;cts_line]
using Plots

#Functions for the values with extra debt.
kp_ini = 6.0
eps_debt = 0.1
kp_d = kp_ini + eps_debt*kp_ini
newtax = (int_rate+gr+delta_p)*(1+eps_debt)*kp_ini - gr*kp_ini
#Define output function
function output(x)
    y = zeros(length(x))
    y = (x[1]^(alpha))*(kp_d^(1.0-alpha))
    return y
end

#non-linear system
function steady_state_epsdebt(x::Array{Float64,1})
    f = zeros(length(x))
```

```
    f[1] = x[1] - (x[3] - x[2] - newtax -(delta+gr)*x[1]) - x[1]
    f[2] = x[2] - (x[4]- rho - ((int_rate +delta_p+gr)*(1+eps_debt)-gr)*
    (kp_d/x[1]))*(1.0/sigma)*x[2] - x[2]
    f[3] = x[3] - (x[1]^alpha)*(kp_d^(1.0-alpha))
    f[4] = x[4] - (alpha*x[1]^(alpha-1.0)*kp_d^(1.0-alpha) - delta - br)
    return f
end

initial_guess2 = [10.0,8.0,5.0,1.50]
(xr_d, fxr_d, iterr_d) = newton(steady_state_epsdebt, initial_guess2, 1e-8, 100)

#Verify equilibrium using the following conditions
equi_r1_d = derivative(output,[xr_d[1],kp_d])[1]- delta -br
equi_r3_d = rho - gr*(kp_d/xr_d[1])+(int_rate +delta_p+gr)*
(1+eps_debt)*(kp_d/xr_d[1])
equi_output_d = output(xr_d[1])
equiconsumption_wd2 = xr_d[3] - (gr + delta)*xr_d[1] - newtax
#Check condition for positive welfare effect of debt
income_effect = ((xr_d[4]+mr)*xr_d[1]- (xr[4]+ mr)*xr[1])+(((1-alpha)*
(xr_d[1]/kp_d)^(alpha))*kp_d - ((1-alpha)*(xr[1]/kp)^(alpha))*kp)
tax_effect = newtax-tax
consu_change = income_effect - tax_effect
verify = xr_d[2]-xr[2]
#define function for k_t_plus_1 for the values of k&c under debt conditions
function kt_plus1_d(x::Array{Float64,1})
    kt_plus1 = x[1] + (x[1]^(alpha)*kp_d^(1.0-alpha)) - x[2] - newtax -
    (delta+gr)*x[1]
    return kt_plus1
end
#define function for ct_plus1
function ct_plus1_d(x::Array{Float64,1})
    ct_plus1 = zeros(length(x))
    kt_plus1 = x[1] + (x[1]^(alpha)*kp_d^(1.0-alpha)) - x[2] - newtax -
    (delta+gr)*x[1]
    ct_plus1 = x[2] + (alpha*kt_plus1^(alpha-1.0)*kp_d^(1.0-alpha) - br
    - delta- rho - ((int_rate +delta_p+gr)*(1+eps_debt)-gr)*(kp_d/x[1]))*
    (1.0/sigma)*x[2]
    return ct_plus1
end
function ks_cs_d(x::Array{Float64,1})
    kt_nodes = zeros(length(time_nodes))
```

```
    ct_nodes = zeros(length(time_nodes))
    kt_nodes[1] = x[1]
    ct_nodes[1] = x[2]
    for i = 2:length(time_nodes)
        kt_nodes[i] = kt_plus1_d([kt_nodes[i-1],ct_nodes[i-1]])
        ct_nodes[i] = ct_plus1_d([kt_nodes[i-1],ct_nodes[i-1]])
    end
    return kt_nodes,ct_nodes
end

#####
#trajectories of (k,c) with debt
(kts_d, cts_d) = ks_cs_d([12.0, 4.405414565875061])
cdot_zero_d = zeros(120)
kdot_zero_d = zeros(22)
c_zerosx_d = [0.0:1.0:120;]
k_zerosx_d = [0.0:1.0:21;]
for i = 1:length(cdot_zero_d)
    cdot_zero_d[i] = xr_d[2]
end
for i = 1:length(k_zerosx_d)
    kdot_zero_d[i] = xr_d[1]
end

c_zeros_d = cdot_zero_d #equilibrium consumption
k_zeros_d = kdot_zero_d #equilibrium capital
timenodes_cts_d =[1.0:0.97:124;]
cts_line_d = zeros(7)
cts_d[120]
for i = 1:length(cts_line_d)
    cts_line_d[i] = cts_d[120]+0.0
end

cts_line_d = [cts_d;cts_line_d]
#All plots with equilibrium with exra debt.
plot([kts,k_zeros,kts_d,k_zeros_d,timenodes_cts,timenodes_cts_d],
[cts,k_zerosx,cts_d,k_zerosx_d,cts_line,cts_line_d],xlims = (0.0,162),
ylims = (0.0,22.6),xlabel = "k(t)",ylabel = "c(t)",label = [:"Equil. tr 1" :
"k at kdot=0 (1)" :"Equil. tr 2" :"k at kdot=0 (2)" :"c(t)--1" :"c(t)--2"],
linewidth = 2)
png("/Users/abdul-muminahmed/Desktop/Computational_Macro/plots/extdebt_eq")
```

C.3 Steady state computations with domestic debt

```
#Import Helper Functions as above.
#Exogenous parameter values
alpha = 0.44
br = 0.03800
mr = 0.02900
gr = br -mr
delta_p = 0.03
delta = 0.04
rho = 0.0180
sigma = 1.0
kp = 6.0
#Define output function
function output(x)
    y = zeros(length(x))
    y = (x[1]^(alpha))*(kp^(1.0-alpha))
    return y
end

#non-linear system
function steady_state(x::Array{Float64,1})
    f = zeros(length(x))

    f[1] = x[1] - (x[3] - x[2] -(delta+gr)*x[1] +(kp/x[1])*(mr- delta_p)*x[1] )
    - x[1]
    f[2] = x[2] - (x[4]- rho + (mr - delta_p)*(kp/x[1]))*(1.0/sigma)*x[2] - x[2]
    f[3] = x[3] - (x[1]^alpha)*(kp^(1.0-alpha))
    f[4] = x[4] - (alpha*x[1]^(alpha-1.0)*kp^(1.0-alpha) - delta - br)
    return f
end

initial_guess = [20,12.0,8.0,1.50]
using LinearAlgebra
(xr, fxr, iterr) = newton(steady_state, initial_guess, 1e-8, 100)
#check if equilibrium is correct by using ff conditions
equi_r1 = derivative(output,[xr[1],kp])[1]- delta -br
equi_r3 = rho+ (delta_p-mr)*(kp/xr[1])
equi_output = output(xr[1])
equi_phi = kp/xr[1]
equi_consu = xr[3] -(delta+gr)*xr[1] +(kp/xr[1])*(mr- delta_p)*xr[1]
tax = (xr[4]+delta_p)*kp
```

```
marginalprodkp = (1-alpha)*xr[1]^(alpha)*kp^(1.0-alpha-1.0)
##using finite difference method to solve differential eqtns.
time_nodes = [1.0:0.1:12.90;]
dt = time_nodes[2]-time_nodes[1]
#define function for k_t_plus_1
function kt_plus1(x::Array{Float64,1})
    kt_plus1 = x[1] + (x[1]^alpha)*(kp^(1.0-alpha)) - x[2] -(delta+gr)*x[1]
    +(kp/x[1])*(mr- delta_p)*x[1]
    return kt_plus1
end
#define function for ct_plus1
function ct_plus1(x::Array{Float64,1})
    ct_plus1 = zeros(length(x))
    kt_plus1 = x[1] + (x[1]^alpha)*(kp^(1.0-alpha)) - x[2] -(delta+gr)*x[1]
    +(kp/x[1])*(mr- delta_p)*x[1]
    ct_plus1 = x[2] + (alpha*kt_plus1^(alpha-1.0)*kp^(1.0-alpha)- delta -br
    - rho + (mr - delta_p)*(kp/x[1]))*(1.0/sigma)*x[2]
    return ct_plus1
end

function ks_cs(x::Array{Float64,1})
    kt_nodes = zeros(length(time_nodes))
    ct_nodes = zeros(length(time_nodes))
    kt_nodes[1] = x[1]
    ct_nodes[1] = x[2]
    for i = 2:length(time_nodes)
        kt_nodes[i] = kt_plus1([kt_nodes[i-1],ct_nodes[i-1]])
        ct_nodes[i] = ct_plus1([kt_nodes[i-1],ct_nodes[i-1]])
    end
    return kt_nodes,ct_nodes
end
#cdot_zero = zeros(120)
kdot_zero = zeros(120)
#c_zerosx = [0.0:1.0:119;]
k_zerosy = [0.0:0.167:20;]
#for i = 1:length(cdot_zero)
#    cdot_zero[i] = xr[2]
#end
for i = 1:length(k_zerosy)
    kdot_zero[i] = xr[1]
end
```

```
#c_zeros = cdot_zero #equilibrium consumption
k_zeros = kdot_zero #equilibrium capital
(kts, cts) = ks_cs([12.0, 4.13449208745])
#kts and cts have converged to the equilibrium values above.
#timenodescts is used as x values to plot consumption time paths.
timenodes_cts =[10.0:0.93:130;]
cts_line = zeros(10)
cts[120]
for i = 1:length(cts_line)
    cts_line[i] = cts[120]+0.0
end
cts_line = [cts;cts_line]
using Plots

#Functions for the values with extra debt.
kp_ini = 6.0
eps_debt = 0.10
kp_d = kp_ini + eps_debt*kp_ini
#newtax = (int_rate+gr+delta_p)*(1+eps_debt)*kp_ini - gr*kp_ini
#Define output function
function output(x)
    y = zeros(length(x))
    y = (x[1]^(alpha))*(kp_d^(1.0-alpha))
    return y
end

#non-linear system
#edit the function below to capture extra debt.
function steady_state_epsdebt(x::Array{Float64,1})
    f = zeros(length(x))

    f[1] = x[1] - (x[3] - x[2] -(delta+gr)*x[1] + (mr-delta_p)*(kp/x[1])*x[1]
    -(x[4]+gr+delta_p)*eps_debt*(kp/x[1])*x[1]) - x[1]
    f[2] = x[2] - (x[4]- rho + (mr - delta_p)*(kp/x[1]) -(x[4]+gr+delta_p)
    *eps_debt*(kp/x[1]))*(1.0/sigma)*x[2] - x[2]
    f[3] = x[3] - (x[1]^alpha)*(kp_d^(1.0-alpha))
    f[4] = x[4] - (alpha*x[1]^(alpha-1.0)*kp_d^(1.0-alpha) - delta - br)
    return f
end

initial_guess2 = [20,15.0,8.0,1.50]
(xr_d, fxr_d, iterr_d) = newton(steady_state_epsdebt, initial_guess2, 1e-8, 100)
```

```
#check if equilibrium is correct by using ff conditions
equi_r3_d = (rho + (delta_p-mr)*(kp/xr_d[1]) + (gr+delta_p)*eps_debt*(kp/xr_d[1]))
/(1.0 - eps_debt*(kp/xr_d[1]))
equi_output_d = output(xr_d[1])
equi_phi_d = kp_d/xr_d[1]
#Verifying conditions for positive debt effects
function twov_output(x)
    y = zeros(length(x))
    y = (x[1]^(alpha))*(x[2]^(1.0-alpha))
    return y
end
marginalprods_d = derivative(twov_output,[xr_d[1],kp_d])
inte_rate = marginalprods_d[1]-delta-br
tax_d = (xr_d[4]+delta_p)*kp + (xr_d[4]+gr+delta_p)*eps_debt*kp
extra_tax = tax_d-tax
#two effects, giving necessary & sufficient conditions
tax_effect = (xr_d[4]-xr[4])*kp + (xr_d[4]+gr+delta_p)*eps_debt*kp
simp_earnings_effect = (xr_d[3]-(gr+delta)*xr_d[1]+(xr_d[4]+mr)*kp)
-(xr[3]-(gr+delta)*xr[1]+(xr[4]+mr)*kp)
chnagein_consu = simp_earnings_effect - extra_tax
verify_changein_consu = xr_d[2]-xr[2]
#Another condition
disposable_income_effect = (xr_d[4]+mr)*(xr_d[1]+kp)-(xr[4]+mr)*(xr[1]+kp)
+(marginalprods_d[2]*kp_d - marginalprodkp*kp)
changein_consu2 = disposable_income_effect-tax_effect
##using finite difference method to solve differential eqtns.
#define function for k_t_plus_1 for the values of k&c under debt conditions
function kt_plus1_d(x::Array{Float64,1})
    kt_plus1 = x[1] + ((x[1]^alpha)*(kp_d^(1.0-alpha)) - x[2] -(delta+gr)*x[1]
    + (mr-delta_p)*(kp/x[1])*x[1] - ((alpha*x[1]^(alpha-1.0)*kp_d^(1.0-alpha))
    - delta-br +gr+delta_p)*eps_debt*(kp/x[1])*x[1])
    return kt_plus1
end
#define function for ct_plus1
function ct_plus1_d(x::Array{Float64,1})
    ct_plus1 = zeros(length(x))
    kt_plus1 = x[1] + ((x[1]^alpha)*(kp_d^(1.0-alpha)) - x[2] -(delta+gr)*x[1]
    + (mr-delta_p)*(kp/x[1])*x[1] - ((alpha*x[1]^(alpha-1.0)
    *kp_d^(1.0-alpha))- delta-br +gr+delta_p)*eps_debt*(kp/x[1])*x[1])
    ct_plus1 = x[2] + (alpha*kt_plus1^(alpha-1.0)*kp_d^(1.0-alpha) - delta - br
    - rho + (mr - delta_p)*(kp_d/x[1]) - (alpha*kt_plus1^(alpha-1.0)
```

```
    *kp_d^(1.0-alpha)
    - delta - br+gr+delta_p)*eps_debt*(kp_d/x[1]))*(1.0/sigma)*x[2]
    return ct_plus1
end
function ks_cs_d(x::Array{Float64,1})
    kt_nodes = zeros(length(time_nodes))
    ct_nodes = zeros(length(time_nodes))
    kt_nodes[1] = x[1]
    ct_nodes[1] = x[2]
    for i = 2:length(time_nodes)
        kt_nodes[i] = kt_plus1_d([kt_nodes[i-1],ct_nodes[i-1]])
        ct_nodes[i] = ct_plus1_d([kt_nodes[i-1],ct_nodes[i-1]])
    end
    return kt_nodes,ct_nodes
end

(kts_d,cts_d) =ks_cs_d([12.0, 4.3304768640999])

kdot_zero = zeros(120)
kdot_zero_d = zeros(120)
k_zerosy_d = [0.0:0.167:20;]
for i = 1:length(k_zerosy_d)
    kdot_zero_d[i] = xr_d[1]
end
k_zeros_d = kdot_zero_d #equilibrium capital

timenodes_cts_d =[10.0:0.93:130;]
cts_line_d = zeros(10)
cts_d[120]
for i = 1:length(cts_line_d)
    cts_line_d[i] = cts_d[120]+0.0
end

cts_line_d = [cts_d;cts_line_d]
#All plots in equilibrium with exra debt.

plot([kts,k_zeros,kts_d,k_zeros_d,timenodes_cts,timenodes_cts_d],
[cts,k_zerosy,cts_d,k_zerosy_d,cts_line,cts_line_d],xlims = (0.0,162.0),
ylims = (0.0,22.6),xlabel = "k(t)",ylabel = "c(t)",label = [:"Equil. tr 1" :
"k at kdot=0 (1)" :"Equil. tr 2" :"k at kdot=0 (2)" :"c(t)--1" :"c(t)--2"],
linewidth = 2)
```

```
png("/Users/abdul-muminahmed/Desktop/My_Computations/plots/domesticdebt")
```

C.4 Helper Functions

```
#==================================================#
#       1. Algorithm for executing derivatives     #
#==================================================#
function derivative(f,x)

    n = length(f(x))
    m = length(x)

        h  = (eps(Float64)/2)^(1/3)*maximum(abs,[x;1.0])
    dh = Matrix(1.0I,m,m)*h

        deriv = Array{Float64}(undef,n,m)

    for i = 1:m
            f1 = f(x.+dh[:,i])
            f2 = f(x.-dh[:,i])
        deriv[:,i] .= (f1-f2)/(2.0*h)
    end

    return deriv

end
#==================================================#
# 2. Newton's method of solving non-linear systems #
#==================================================#
include(".../derivative.jl")
function newton(f,x,tol,maxiters)

    xstar = similar(x)
    len = Inf
    iters = 0

    while len > tol && iters <= maxiters

        xstar = x - vec(derivative(f,x)\f(x))
        len = maximum(abs,xstar-x)
        x = copy(xstar)
```

```
        iters += 1
    end
    return xstar, f(xstar), iters
end
```

Acknowledgements: The Helper functions (the algorithms for executing a derivative and the newton's method of solving a non-linear system were built during my participation in Professor Richard Dennis's class on Computational Macroeconomics). It is likely that after the class, he generously provided the julia code for these helper functions. Apart from these, every other code is the output of my work.

Chapter 5

Policy discussion, and Conclusion

5.1 Introduction

In this final chapter, I briefly discuss a number of policy issues concerning developing country debt especially in the unprecedented context of the COVID-19 pandemic. The pandemic has induced a sharp rise in government debt globally to record levels, averaging 97 percent of GDP, and in emerging market developing countries the average stands at 63 percent (Kose et al., 2021[85]). Before the pandemic, debt levels were generally considered safe even for heavily indebted poor countries countries (such as Benin, Ghana, Malawi, Mozambique, Niger, Sao Tome and Principe, Senegal and Uganda) that had rapidly accumulated debt post-completion of the HIPC program (Mustapha and Prizzon, 2015[88]). About 75 percent of low income developing countries were assessed to be at low or moderate risks of debt distress by 2015. Debt build-up prior to the pandemic, as well as the recession and fiscal expansion to stimulate growth due to the pandemic has increased the number of countries in debt distress, triggering concerns of imminent sovereign debt crises (Bullow et al., 2020[31]). To address this, the policy sector (in particular, the World Bank and IMF) called for debt service suspension by G20 and commercial creditors, while providing additional lending to support recovery and protect welfare. This has delivered relief to more than 40 eligible countries and helped governments address liquidity challenges during the pandemic. Nonetheless, the record debt levels has left many economies at risk of debt distress. Nearly half of all countries eligible for the debt service suspension initiative (about 35 out of 73) are either in distress or at high risk of debt distress (World Bank, 2021[13]). The unprecedented effects of the pandemic on government debt has raised policy-oriented concerns on the implications of today's high debt levels (and the increasing debt distress in developing economies) for debt-financed public investment.

While the results from previous chapters are straightforward on the role of debt-financed public investment for growth and welfare improvement in developing economies, the policy implications within the context of today's high debt levels will require nuanced examination. As emphasized by Kharas and Dooley (2021)[74], private finance and investment is not a substitute for public investment. Thus, governments will need to continue to borrow for investment, with obvious implications for debt sustainability and economic recovery. In the context of present high debt levels, where debt finance is increasingly inaccessible to developing countries, how do governments sustain productive public investments? This question takes center-stage in the ensuing discussion.

5.2 Related policy discussions

Many policy-oriented research and discussions in the wake of the pandemic have focused on fiscal policy alternatives that can facilitate accelerated recovery, promote sustainable growth and promptly address the emerging debt challenges in developing countries. To start with, de Mooij et al. (2020)[86] proposed targeted but temporary tax reliefs to support recovery, while adopting more progressive tax regimes that facilitates economic activity by shifting incomes from those with low propensity to consume to those with high propensity to consume. Beyond tax policies, concerns have raged on the need to preserve public expenditure in the face of tightening financial conditions. The OECD estimated a more than 30 percent drop in foreign direct investments (OECD, 2020[90]) with Latin America, Africa and the Middle East experiencing negative net financial flows in 2021 (Kharas and Dooley, 2021[74]). The tightening conditions will make it difficult and expensive for African government to obtain the required financing to recover from the pandemic and to refinance maturing debts (AfDB, 2021[1]). Ultimately, sovereign defaults may erupt in the future as both a consequence and inevitable way out of the post-pandemic developing country debt situation (Kose et al., 2021[85]).

To address the simmering debt challenges and preserve sustained growth, a number of alternatives for resolving developing country debt have been proposed. Key among them include:

1. Reforms in the international financial architecture for sovereign debt restructuring. Calls for reform of the international financial architecture for sovereign debt restructuring predates the current debt stress due largely to COVID-19. Krueger (2002)[76] highlighted the emerging challenges with sovereign debt restructuring in a modern world of more integrated financial markets and a shift from syndicated bank lending to trade-able securities which has expanded the creditor base of emerging sovereign states. Despite the pros of this fairly recent development, the increasing number of creditors and the diversity of claims and interests make it increasingly difficult to obtain collective action and secure prompt debt restructuring in times of debt distress. Krueger (2002) proposed an institutional based global restructuring mechanism that will offer incentives to both the sovereign debtor and its creditors to pursue predictable, orderly and efficient restructuring of debt to ensure sustainability and protect the interests of both parties. Similarly, Buchheit et al. (2013)[29] proposed an amended European Stability Mechanism that provides legal and political legitimacy for debt restructuring in cases of unsustainable debt in the Euroarea, and an IMF-based Sovereign Debt Adjustment Facility which combines lending with debt restructuring. Some of the proposed features have found expression in the IMF's Poverty Reduction and Growth Trust (PRGT) Extended Credit Facility program, even though this is not intended primarily as a debt restructuring or resolution mechanism. Renewed calls for reform have come in the wake of the pandemic. In the European Union (EU), the European Fiscal Board have proposed comprehensive reforms of the EU fiscal framework to make more predictable any change from the rules-based system in the face of exogenous shocks. Among others, this would entail provision for medium-term debt anchor, universal expenditure rules that are closely tied to economic growth rates, and a single escape clause (Thygesen et al. 2021[113]). In addition, Baarsma and Beetsma (2022[12]) suggests the replacement of the 1/20th debt reduction rule when debt level is above the Maastricht threshold of 60 percent. The 1/20th rule is a provision of the European Stability and Growth Pact (SGP) requiring countries to reduce the difference in debt by 1/20th annualy when it exceed the 60 percent cap of the Maastricht criteria. For emerging market and developing economies (EMDEs), Kose et al. (2021[85]) call for much more to be done beyond the G20's Common Framework to forestall systematic debt crises in EMDEs. Kharas (2020)[72] proposes the adoption of a new UN Security Council resolution

under Chapter VII which would call for a standstill of debt service payments for one year for any country that requests exceptional support from the IMF. Apart from allowing time for debt renegotiations without the risk of holdout litigations, such a resolution will offer legitimacy to the IMF/World Bank based procedures for sovereign debt treatment. The most recent official sector mechanism to support developing economies with unsustainable debt is the G20 Common Framework endorsed by the Paris club (MEF, 2021[48]). This is intended to be a case by case debt resolution mechanism at behest of sovereign debtors with support from the World Bank and IMF. As a sucessor to the DSSI initiative, the Common Framework has a structural improvement with signatories of bilateral creditors such as China and Saudi Arabia (Gill, 2022[67]). This will prove important for many countries in sub-Saharan Africa given the drastic shift in creditor profile with resource-backed loans from China, for example, becoming a dominant share of debt in DSSI eligible countries (Georgieva and Pazarbasioglu, 2021[56]). In general, these proposals are aimed at offering incentives to both creditors and sovereign debtors to pursue early debt restructuring when economic conditions show signs of unsustainable debt.

2. Debt restructuring. Many sovereign states in the past have used debt restructuring to cure default spells, and this is broadly categorised into two types: decisive restructuring which entails renewed capital market access and improve economic performance that enable the country avoid a major credit event for at least two years, and interim restructuring which is often followed by a relapse into default spell within two years of the restructuring. In the past, there has been atleast 279 external default spells in 113 countries between 1800 and 2020 (Von Luckner, 2021)[57]), with countries such as Poland and Nigeria undergoing at least seven restructurings to conclusively resolve their unsustainable debt (World Bank, 2022[14]). Emerging developing countries in debt distress may need to start planning for debt restructuring, and design loan guarantee programmes keeping in mind potential debt overhang problems (Becker et al., 2020[18]. As emphasized by the World Bank, developing country governments at high risk of default may need to pre-emptively initiate negotiations to restructure debt in ways that deliver haircut. This can be achieved through modification of the financial structure of liabilities to reduce their net present value. But it will require transparency and coordinated negotiation processes that obviates the risks of holdout litigation.

3. Debt relief. Historically, debt relief has been used to resolve systematic debt crisis

in developing countries. Fairly recent frameworks that offered debt relief include the Heavily Indebted Poor Countries (HIPC) initiative and the Multilateral Debt Relief initiative. The case for debt relief has been emphasized in policy sector discussions since the start of the COVID-19 pandemic. This found expression in the Debt Service Suspension Initiative (DSSI) spearheaded by the IMF and the World Bank. The relief provided by DSSI afforded developing countries some financial relief to maintain expenditure on welfare protection following revenue short-falls in the midst of covid-19. However, this does not address fundamental debt challenges. For some emerging market and developing countries, debt restructuring and market-based solutions may not be enough (Kose et al., 2021[85]). Debt relief will be required in exceptional situations to facilitate return to debt sustainability. This would call for case-by-case debt sustainability analysis conducted jointly by the IMF and Developing country governments to determine if, and by how much, debt rescheduling and write-offs will be required (Kharas, 2020[72]).

4. Innovations in financing instruments. The use of collection action clauses (CACs) continue to revolutionise the template for sovereign debt restructuring. Developing countries are increasingly incorporating collective action clauses in sovereign bond notes. These clauses ensure that all bondholders are bound by the terms of a debt restructuring agreement when a certain threshold of bondholders consent to restructuring. The use of CACs has reduced the successes and risks of holdout litigation by some creditors. For example, when a group of bond holders filed a motion of restraint to halt the Republic of Ecuador from restructuring 17.4 billion USD worth of sovereign debt, the New York court upheld Ecuador's use of CACs as the primary tool to execute the planned restructuring (Ramamurthi et al. 2020[97]). But CACs from the sovereign debtors' side alone may not solve the legal challenges that may arise occasionally. Bullow et al., (2020)[31] have called for new legislations in jurisdictions that govern international bonds such as New York, London, and Belgium to cap amount that can be reclaimed from defaulting government bonds purchased at steep discount in the market. Such legislation, among others such as the Anti-Vulture Funds Law in Belgium which prevents holdout litigation from disrupting payments mad evia Euroclear, will support orderly sovereign debt restructurings.

5. Governance reforms. The large scale fiscal stimulus that has helped many economies deal with the crisis of COVID-19 underscore the need for good governance going

forward. It often becomes known after the fact that crisis expenditure are usually not directed towards productivity and increasing export output (Kose et al. 2020[75]. Nonetheless, given the high levels of debt today in developing countries, governance reforms in the form of progressive tax policies that reduce inequality and support inclusive growth will be crucial to avoiding debt stress (de Mooij et al. 2020[86]). For example, capital income tax, taxes on wealth, measured value added taxes and taxes on unhealthy goods such as tobacco, alcohol as well as environmental taxes will support governments to generate revenues required to pursue growth generating investments and keep debt sustainable. Apart from the types of taxes that are being explored, the nature of tax administration is an important component of the needed reforms. The OECD (2021)[91] has underscored the opportunities presented by digitalisation for efficient tax delivery system. Using cutting edge techniques that exploit existing data and develop digital platforms to facilitate tax collection will ease up resources in tax administration for investment in growth interventions. This is less explored but with huge potential for growth and debt management in less developed countries. Beyond tax reforms, broader governance reforms that enhances political stability, maintain rule of law and controls corruption can be leveraged by Sub-Saharan African countries for growth and inclusive recovery post COVID-19 (Ganum and Thakoor, 2021[54]).

5.3 Policy issues on debt-financed public investment

There are a plethora of issues worthy of policy consideration in the discussion on role of debt-financed public investment in developing economies. The analysis conducted in previous chapters has been controlled to yield analytic results that do not account for various forms of market uncertainties. But even in the absence of uncertainty, a number of issues require further thought in drawing policy-oriented perspectives from the findings. They include:

1. Time lag between debt service requirement and maturity of public investments. Fundamentally, the rationale for debt finance as a tool for growth generation and welfare improvement as discussed analytically is rooted in its role in accelerating growth through investment in public capital. Yet, public capital in the form of roads, highways, railways and energy power plants are long-term investments by nature. In other words, debt-financed public investments usually impose a debt

service burden immediately the debt is incurred, but the impact on growth is delayed by the term of maturity of public projects. In the analysis of the previous chapter, the focus is on long-run effects, when the debt service burden takes effect simultaneously as the improvement in private capital, output, and consumption levels. In reality, the time lag between the opposing effect of debt and public investments may create time-varying effects of debt-financed public investments. For developing economies, an assessment of the short and long-term effects of debt and public investments will be crucial in determining the gains of debt-financed public investments.

2. Efficiency of public investments. Analytically, determining the efficiency of public investments is straightforward. It entails comparing the marginal product of public capital to the prevailing interest rate. Where there is excess marginal productivity of public over private capital, it is more efficient to accumulate public capital, and the role of debt-financed public investments immediately follows. However, estimating these marginal productivities is difficult for any given economy. Therefore, the determination of efficiency of public investment will be a judgement call for developing country governments. Nevertheless, the data in chapter four has shown that developing countries in general have large deficits in public capital, but also public capital have much larger marginal productivity than private capital. In this case, it may be growth and welfare stimulating to use debt finance for public investments provided the investment process is highly efficient. Where public investment processes is characterised by wastage in the form of rent-seeking, the welfare effects of debt finance may be questionable. Therefore, assessment of efficiency and the potential productivity of the specific public investments to be financed by debt must be country-specific.

3. Capacity for increased growth. Strengthening the nexus between public investment and economic growth is crucial to the use of debt finance as a tool for long-term growth. This involves identification of high return investments and sectors of the economy that are necessary for long-term growth. For example, investments in railways and roads and highways that facilitate transportation of agricultural produce to processing centers and between major production locations to market centers may enhance efficient production, marketing, and distribution of output. Similarly, investments in energy generation, information and technology, water processing plants, and interventions on structural transformation may directly contribute to

output and productivity growth. But using debt finance for this purpose must be preceded by rigorous assessment of the relative size and productivity contributions to national output both in the interim and over the long-term.

4. Debt sustainability threshold. The corner stone results of Reinhart et al., (2003[100]) and Reinhart and Rogoff (2010[99]) is that not only do less developed economies show substantial duress at debt levels that appear manageable for more advanced countries, debt affects growth negatively at much lower thresholds in less developed economies compared to advanced countries. This suggests the uniqueness of the debt sustainability threshold for different countries depending on their stage of development. This seem to have some theoretical basis as shown in chapter four. With the given level of public capital dictating consumption and private capital levels in the long-run, the amount of debt that can be supported from any equilibrium depends on the economy's endowment in assets. Therefore, funding public investments with debt finance is not only a question of productivity premia on investment options vis-a-vis the interest rate on debt, but must also be a question of sustainability which is concerned with the debt service burden, fiscal space and risks, and the existing level of public debt.

5. Tax capacity. By nature, tax capacities are low in less developed countries. Low income countries typically have a tax to GDP ratio of 10-20 percent compared to over 40 percent for the average advanced economy (Besley and Persson, 2014[20]). But the capacity to tax, a critical measure of state formation and effectiveness, is indispensable to revenue generation for economic development (Di John, 2006[43]). Not only does the tax capacity of an economy determine the amount of resources that can be generated by government directly, it also determine the debt sustainability threshold and consequently the amount of debt finance for public investment. The inextricable link between tax capacity and the amount of debt a sovereign state can accumulate suggests that developing economies looking to exploit debt finance must balance debt accumulation for investment with its tax capacity. A lack of careful consideration of this nexus may defeat the potential for debt finance as a growth tool. Rapid accumulation of debt can compromise sustainability in the context of weak tax capacity, and will lead to chaotic macroeconomic performance and derail economic growth.

6. Reforms and innovations to increase economic growth, tax revenue and tax capac-

ity. The tax capacity of a developing economy is closely related to its ability to execute reforms that facilitate rapid growth as well as efficient tax administration, but also its ability to pursue innovative ways of generating taxes. As found by Besley and Persson (2014), the share of income tax in revenue is inversely related to size of the informal economy. Conversely, the share of tax revenue to GDP is positively related to the GDP per capita of an economy. It is, thus, nor surprising that developing countries are generating lower tax revenue as a share of output, considering their tax capacities are hampered by the large informal sector. Being hard to tax, informality has served as a tool for tax avoidance in developing countries (La Porta and Shleifer, 2014[81]). Evidently, a tax system that offers rebates and tax incentives for informally operated establishments to formalize their operations can enhance tax capacity in the long-run. This can be achieved by developing models that use specific features of the informal enterprises to estimate their tax obligation based on similar sized enterprises in the formal sector, and apply an upward adjustment as the cost of informality. Similarly, the ongoing technological revolution can be harnessed to support tax administration and tax policy. This may entail tracking tax payers by use of unique identification numbers (Bird et al., 2008[22]) and creating a link between the tax payers and financial activity to fairly and accurately estimate tax liabilities. Importantly, developing country governments must pursue structural change policies that shift capital and labour from less productive establishments to more productive activities. This will generate economic growth and enhance tax revenue.

7. Innovations in public financing of investments, including public private partnerships. Apart from well known innovations in sovereign bond notes such as CACs and aggregation clauses and the increasing issuance of debt both in domestic currency and local covenants, new instruments of financing public investments will be required, at least in the interim, given the extremely high levels of debt in developing countries. A recent study by IMF Staff revealed the increasing use of public private-partnerships (PPP) to encourage private sector provision of infrastructure assets and infrastructure-based services (Akitoby et al., (2007[4]). This is dominated by projects on roads, ports, railways and power supply which address obvious bottlenecks, and hence are judged to be reasonably commercially viable. In general, even though the government retains contingent liability where the private sector incurs debt for projects under PPP arrangements, it allows governments

to avoid or defer expenditure on infrastructure without foregoing accumulation of it and its economic benefits. Thus, when PPPs are based on rigorous cost-benefit analysis, well structured implementation can ease the fiscal constraint and yield efficiency gains in the provision of infrastructure-based services while lowering government cost in making these services available. In the current context of COVID-19-induced debt stress in many developing countries, innovative and efficient PPP arrangements is a viable alternative to debt finance that can help maintain public investments while allowing governments the space for fiscal adjustments to reduce debt levels. But this needs to be undertaken within credible legal and institutional frameworks that ensure cost-benefit appraisals are conducted and guarantee efficiency of outcomes.

5.4 Concluding remarks

The primary objective of this dissertation was to develop an analytic model that can help explain the rationale for the accumulation of government debt observed in both developed and developing economies over the last couple of decades. Standard economic theory have shown that government debt is incapable of enhancing long-run economic conditions. It reduces capital labour ratio and decreases welfare when the equilibrium is pareto-optimal in an economy with no inter-generational linkages. Consequently, ricardian equivalence based analysis, where agents are inter-linked by a motive for bequest, have focused on optimal use of debt largely as a tax smoothing tool to smooth the deadweight losses from tax revenue over time (Yared, 2019[117]), or a countercyclical tool for economic stabilization where debt is reduced in favour of building assets in periods of strong economic growth and used to finance unanticipated shocks to expenditure (Alesina and Passalacqua, 2016[5]). The evidence however shows a broad-base build-up of debt as a share of GDP in advanced economies over the last four decades (Yared, 2019). Similarly, developing countries have seen a rapid build-up of debt following the completion of the HIPC programs in the early 2000s (Mustapha and Prizzon, 2015).

The standard motivations for debt does not adequately explain the appetite and broad-based increase in debt levels observed across advanced and the developing world. Developing countries in particular have registered remarkable economic growth rates in the years prior to the COVID-19 pandemic. Therefore, the corresponding rise in debt levels in these economies betray traditional explanations of the use of debt as countercyclical tool. The thesis has therefore explored one plausible reason for the increasing debt levels in developing countries - namely the use of debt finance for public investment purposes. In this regard, I have examined conditions under which debt finance can have positive effects on wealth and consumption levels in decentralized equilibrium. To properly motivate the study, I start by reviewing key literature and identifying the gaps, focusing mainly on the endogenous growth models of Aschauer (2000) and Checherita-Westphal et al. (2014) and the continuous time overlapping generations model of Blanchard (1985). I emphasize the difficulty of examining debt-financed public investment in the planning problem of Aschauer (2000). In particular, the decentralized Aschauer model leaves no room for positive welfare effect of debt in equilibrium. Government intervention in the decentralized economy by use of debt-financed public investment may increase the equilibrium growth rate at the cost of inter-temporal allocations. This explains why in chapter three, the growth-maximizing capital ratio exceeds the capital ratio in competitive equilibrium.

The above outcome underscores the tension, and time inconsistency, between growth maximization as may be pursued by government, and utility maximization as would often be pursued by households. In view of the difficulty of a positive welfare effect of debt in the endogenous growth model, I relax the assumptions leading to endogenous growth in the two papers, and incorporate public capital in production using the utility and demographic structure of Blanchard. The key result of the analysis on debt shows that debt finance can be used to stimulate growth in the short-term and in so doing, permanently increase wealth and consumption levels in the long-run. The flavour of this finding arises from the fact that this use of debt allows it to increase both capital and welfare in the long-run without requiring the steady state equilibrium to be dynamically inefficient. It should be noted that not only is the equilibrium required to be dynamically inefficient for debt to increase long-run welfare in Diamond (1965) and Blanchard (1985), it is also incapable of increasing the equilibrium values of private capital and consumption levels.

In the analytic model elaborated in this thesis, not only does debt improves both variables in the long-run when invested in public capital, dynamic inefficiency is no longer required for a positive welfare effect of debt to exist. In addition, when the equilibrium is dynamically inefficient in the standard Diamond sense involving a lower interest rate than equilibrium growth rate, debt-financed public investment enhances efficiency by increasing the interest rate, and generally the rate of return to capital. When the interest rate is above the equilibrium growth rate, even though debt-financed public investment does not change the long-run growth rate, it nonetheless increases the interest rate and permanently improves capital and consumption levels. This result requires the marginal returns to public capital to be significantly higher than private capital.

A look to the cross-country data shows some prospects for the existence of positive welfare effect of debt finance in a broad range of countries. While further research is required for definitive insights on this, there appears to be higher productivity for public capital than private capital in a sample of advanced economies, emerging markets, and low income developing countries. Whether or not debt finance is indeed welfare or growth enhancing in these economies is an empirical question requiring further research. But importantly, and perhaps surprising, the cross-country estimations of average marginal products for public and private capital has shown that when price levels of capital are accounted for, the marginal productivity for both capital stocks are lower in low income countries than advanced economies. This result is in line with the finding of Caselli and Feyrer (2007)[33] and suggests that even in the absence of international capital market frictions, low income countries may not be able to attract the needed capital for growth and development. Thus, policy measures that reduce the relative price of capital in less developed economies may be necessary to fill the observed gaps in infrastructure and capital stocks, improve competitiveness, and spur growth and development.

5.4.1 Future research

The thesis opens new directions of research on debt finance. The theoretical formulations abstracts from all usual forms of uncertainty and inefficiency. While this was necessary to yield simple analytic results on the potential benefits of debt finance in developing economies, it does preclude explicit analysis on the role of key factors such as institutional characteristics and the efficiency of public investment processes on the effects of debt-financed investments. Incorporating uncertainty and a parameter that captures the efficiency level of public investments will provide richer explanations on the use of debt finance and its attendant effects in developing countries. Second, the evidence from the data is nothing more than a cursory look that focused mainly on estimating marginal productivities of capital. For further empirical analysis, it may be necessary to use disaggregated data to examine the effects of debt-financed investments on household consumption and economic growth. Long run co-integration analysis will offer valuable insights. Unfortunately, data on household consumption is presently not available for a wide range of developing countries. Thus, the development of such dataset as well as its use for more focused analysis are potential future research ideas. Third, time-series analysis at the level of each developing economy will be more relevant for country-specific debt policy, as this will inherently account for idiosyncrasies in each country that may have been masked by the panel analysis conducted in this thesis. Finally, research on the causes of debt formation, and the political economy features that enhance the accumulation of debt in less developed economies even in good times, as well as the specific projects for which debt finance has been used will provide anecdotal evidence on the costs and benefits of developing country debt. This will help countries to have an improved grasp and use of debt policy for growth purposes in a way that is credible and well grounded.

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