Associated sheaf functors in tensor triangular geometry

by

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Abstract

Given a tensor triangulated category we investigate the geometry of the Balmer spectrum as a locally ringed space. Specifically we construct functors assigning to every object in the category a corresponding sheaf and a notion of support based upon these sheaves. We compare this support to the usual support in tt-geometry and show that under reasonable conditions they agree on compact objects. We show that when tt-categories satisfy a scheme-like property then the sheaf associated to an object is quasi-coherent, and that in the presence of an appropriate t-structure and affine assumption, this sheaf is in fact the sheaf associated to the object's zeroth cohomology. When the tensor triangulated structure is replaced with a *monoidal triangulated* structure we show that one can form localising bimodules and central idempotents given particular localisation sequences. Finally, we provide a computation of the spectrum for the enveloping algebra of the A_2 quiver and determine that spectrum consists of a single point.

Contents

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Author’s declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

§ 1.1 | Associated sheaf functors

Given an essentially small tensor triangulated category, one can construct a topological space, called the Balmer spectrum, from the collection of prime tensor ideals. Important collections of subcategories such as localising or thick tensor ideals can often be classified by subsets of this space. Moreover, just as the spectrum of a commutative ring can be equipped with the usual structure sheaf, the Balmer spectrum also naturally admits the structure of a locally ringed space. In this work we investigate sheaves over the Balmer spectrum equipped with this structure sheaf, emphasising the geometry of the space and its influence on ideas of support and the cohomology of objects. More precisely, given a compactly generated tt-category T generated by the tensor unit $\mathbf{1}$ we construct an *associated sheaf functor*

$$[\mathbf{1}, -]^{\bullet} : \mathsf{T} \longrightarrow \operatorname{Shv}(\operatorname{Spc}(\mathsf{T}^c))$$

Using the geometric information of this sheaf and its stalks we can define a support for an object $X \in \mathsf{T}$ by setting $\operatorname{supp}^{\bullet}(X, \mathbf{1}) = \{\mathcal{P} \mid [\mathbf{1}, X]^{\bullet}_{\mathcal{P}} \neq 0\}$. This notion of support satisfies many of the desirable properties usually satisfied by support theories and it is natural to ask how this geometric flavour of support compares to the usual notions. We have the following comparison:

Theorem. *Suppose* $\mathsf{T}^c = \operatorname{thick}(\mathbf{1})$ *and the Balmer spectrum of* T^c *is a noetherian topological space. Then for all objects* $X \in \mathsf{T}$ *there is containment*

$$\operatorname{supp} X \subseteq \operatorname{supp}^{\bullet}(X, \mathbf{1}),$$

where the support on the left is the support in the sense of Balmer-Favi. Moreover if X *is compact then there is an equality*

$$\operatorname{supp} X = \operatorname{supp}^{\bullet}(X, \mathbf{1}).$$

In algebraic geometry, the notion of affine spaces and schemes are both powerful and prolific. By using the spectrum as an avatar for our tt-category we can pull these geometric properties into our setting. Specifically we consider Balmer's comparison map of locally ringed spaces $\rho : (\mathrm{Spc}(\mathsf{T}^c), \mathcal{O}_{\mathsf{T}}) \longrightarrow (\mathrm{Spec}(R_{\mathsf{T}}), \mathcal{O}_{R_{\mathsf{T}}})$ between the Balmer spectrum and the spectrum of the endomorphism ring of the tensor unit. We call categories where ρ is an isomorphism *affine*, and call a category *schematic* if it is locally affine with respect to some open cover of the spectrum. With this we obtain the following theorem:

Theorem. *Let* T *be schematic. Then for every* $X \in \mathsf{T}$*, the sheaf* $[\mathbf{1}, X]^{\#}$ *is quasi-coherent on* $(\mathrm{Spc}(\mathsf{T}^c), \mathcal{O}_{\mathsf{T}})$.

If a triangulated category admits a generator satisfying certain conditions, then the generator equips the category with a natural t-structure. This allows one to ask questions about the cohomology of objects. Our construction of the support $\mathrm{supp}^{\bullet}(X, \mathbf{1})$ includes information about the suspensions of the object X. We can construct a slightly different support $\mathrm{supp}(X, \mathbf{1})$ without this suspension data, which we call the untwisted support. Combining these ideas with the machinery of t-structures allows us to obtain the following theorem:

Theorem. *Let* T *be an affine category generated by the tensor unit* $\mathbf{1}$*. Assume that* $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \Sigma^n \mathbf{1}) = 0$ *for all* $n > 0$*. Then for all objects* $X \in \mathsf{T}$ *we have*

$$\mathrm{supp}^{\bullet}(X, \mathbf{1}) = \bigcup_{i \in \mathbb{Z}} \mathrm{supp}(H^i(X), \mathbf{1}).$$

The material on associated sheaf functors is laid out as follows. In §2 we give the general preliminaries on triangulated categories, tensor triangular geometry and the various elements of algebraic geometry we will use. §3.1 contains our construction of relative sheaf functors, in the generality of a triangulated category acted on by a tt-category, and including twisting by an invertible object. In §3.2 we compare the support produced by these sheaf functors to the usual notion of support. §3.3 details the conditions under which properties such as (quasi-)coherence can lead to the formation of thick subcategories. Affine and schematic categories are analysed in §3.4, leading us to show that in such categories the sheaves associated to objects are always quasi-coherent. We also introduce the notion of a quasi-affine tt-category and investigate the associated comparison maps. §3.5 compares the usual gluing of sheaves with the tensor-triangular notion of gluing over a Mayer-Vietoris cover and the significant differences between the operations. In §3.6 we show how the sheaf functors interact with naturally occurring t-structures and work towards the theorem showing that when our category is affine, the twisted support of an object can be obtained by considering the untwisted supports of its cohomologies. The author first presented the content of these sections in [Row21]. We round off the discussion in §3.7 by investigating some particular examples.

§ 1.2 | Monoidal geometry

There has been a growing interest in loosening the conditions on tensor triangulated categories in order to capture a wider range of examples. Such examples occur in the study of Hopf algebras and Borel subalgebras. The particular tenet under examination is the requirement that the tensor product be symmetric. By loosening this requirement one can now analyse so called *monoidal triangulated categories*, which we refer to as *mt-categories*. Just like tt-categories, these mt-categories admit a notion of prime ideals, Balmer spectra and support data. Various familiar pieces of the symmetric setting now have analogues in the monoidal framework, including various classification results for different types of ideals, such as in [NVY19]. We work on lifting further techinical results into the monoidal setting. In particular, we build towards proving the following theorem regarding central idemopotents in the monoidal case:

Theorem. *Let* T *be a rigidly-compactly generated mt-category,* $\mathcal{S} \subseteq \mathsf{T}^c$ *a set of compact objects, and* $\mathsf{S} = \mathrm{Biloc}(\mathcal{S})$ *i.e* S *is the smallest two-sided ideal containing* $\mathcal{S}$*. Consider the corresponding smashing localisation sequence*

$$\mathsf{S} \underset{i^!}{\overset{i_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{T} \underset{j_*}{\overset{j^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{S}^{\perp}$$

Then

1. $\mathsf{S}^{\perp}$ *is a localising two-sided ideal of* T.
2. *there are isomorphisms of functors* $i_*i^!\mathbf{1} \otimes (-) \cong i_*i^!$ *and* $j_*j^*\mathbf{1} \otimes (-) \cong j_*j^*$.
3. *the objects* $i_*i^!\mathbf{1}$ *and* $j_*j^*\mathbf{1}$ *satisfy*

$$\begin{aligned} i_*i^!\mathbf{1} \otimes i_*i^!\mathbf{1} &\cong i_*i^!\mathbf{1}, \\ j_*j^*\mathbf{1} \otimes j_*j^*\mathbf{1} &\cong j_*j^*\mathbf{1}, \\ i_*i^!\mathbf{1} \otimes j_*j^*\mathbf{1} &\cong 0, \\ j_*j^*\mathbf{1} \otimes i_*i^!\mathbf{1} &\cong 0. \end{aligned}$$

We also compute a new example of the spectrum for a mt-category. We compute the spectrum of two-sided prime ideals for the enveloping algebra of the A_2 quiver and show that it consists of a single point. Specifically

$$\mathrm{Spc}(\mathsf{D}^b(A_2^e)) \cong \{*\}.$$

The material on mt-catgeories is laid out as follows. In §4.1 we recall some facts from the literature about these categories and provide proofs of a few technical lemmas. In §4.2 we extend the notion of actions to cover mt-categories acting on a triangulated category, with potentially different mt-categories and actions acting on the left and right. In particular we verify that the formation of localising bimodules is well behaved and prove the above theorem regarding central idempotents. We finish in §4.3 with our computation of the spectrum for the enveloping algebra of the A_2 quiver.

Chapter 2

Preliminaries

We lay out the core building blocks of triangulated categories, and the essential elements of tensor triangular geometry. Although our work will take us far from some of these fundamentals, it is important to keep these notions ready at hand.

§ 2.1 | Triangulated categories

We will take a quick tour through the basics of triangulated categories. We will cover the axioms of triangulated categories and some basic definitions, Verider localisations, thick and localising subcategories, and finally discuss compact objects and the generation of subcategories.

§ 2.1.1 | First axioms and definitions

Definition 2.1.1. Let T be an additive category equipped with an invertible endofunctor $\Sigma : \mathsf{T} \longrightarrow \mathsf{T}$, which we call the *suspension functor*. A *candidate triangle* in T is a diagram of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

where each of the composites $v \circ u$, $w \circ v$ and $\Sigma u \circ w$ are equal to the zero morphism. A *morphism* of candidate triangles is a commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{u_1} & Y_1 & \xrightarrow{v_1} & Z_1 & \xrightarrow{w_1} & \Sigma X_1 \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & {\scriptstyle h}\big\downarrow & & {\scriptstyle \Sigma f}\big\downarrow \\ X_2 & \xrightarrow{u_2} & Y_2 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & \Sigma X_2 \end{array}$$

where each row $X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Z_i \xrightarrow{w_i} \Sigma X_i$ is a candidate triangle. Such a morphism is an *isomorphism* if each of the arrows f, g, h is an isomorphism.

Remark 2.1.2. Various authors will write triangle diagrams in the following form:

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & Y \\ & {\scriptstyle w}\nwarrow \quad \swarrow{\scriptstyle v} & \\ & Z & \end{array}$$

This form of diagram often suppresses the presence of the suspension functor. In exchange, this make the triangles actually look like triangles! We will refrain from using this form for triangles, and keep the suspension functor present throughout.

Also note that some authors denote the suspension of an object X by $X[1]$, or refer to it as a *translation functor*. More generally for an integer i, one can denote $\Sigma^i X = X[i]$. Again, we will refrain from this and keep the suspension functor Σ obvious throughout.

Definition 2.1.3. Let T be an additive category with suspension functor Σ. Let $\mathcal{T}$ be some collection of candidate triangles in T, called *distinguished triangles.* Then T is a *pre-triangulated category* if it satisfies the following four axioms:

TR0: The candidate triangle

$$X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow \Sigma X$$

is a distinguished triangle. Every candidate triangle isomorphic to a distinguished triangle is itself distinguished.

TR1: For every morphism $X \xrightarrow{f} Y$ in T there exists a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X.$$

The object Z is referred to as the *cone* of f.

TR2: Consider a candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

and its *rotation*

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y.$$

If one of these candidate triangles is distinguished, then so is the other.

TR3: For any diagram of the form

$$\begin{array}{ccccccc} X_1 & \xrightarrow{u_1} & Y_1 & \xrightarrow{v_1} & Z_1 & \xrightarrow{w_1} & \Sigma X_1 \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & \\ X_2 & \xrightarrow{u_2} & Y_2 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & \Sigma X_2 \end{array}$$

where the rows are distinguished triangles, there exists a (not necessarily unique) morphism $h : Z_1 \longrightarrow Z_2$ such that the diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{u_1} & Y_1 & \xrightarrow{v_1} & Z_1 & \xrightarrow{w_1} & \Sigma X_1 \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \Sigma f}\downarrow \\ X_2 & \xrightarrow{u_2} & Y_2 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & \Sigma X_2 \end{array}$$

is commutative.

A pre-triangulated category $(\mathsf{T}, \Sigma, \mathcal{T})$ is *triangulated* if it satisfies the additional axiom

TR4: (Octahedral Axiom) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Y'$ be morphisms in T. Consider triangles

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

$$X \xrightarrow{gf} Y' \longrightarrow Z' \longrightarrow \Sigma X$$

$$Y \xrightarrow{g} Y' \longrightarrow Y'' \longrightarrow \Sigma Y$$

Then this completes to a diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
{\scriptstyle 1}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow & & {\scriptstyle 1}\downarrow \\
X & \xrightarrow{gf} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Y'' & \xrightarrow{1} & Y'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma X & \xrightarrow{\Sigma f} & \Sigma Y & \longrightarrow & \Sigma Z & \longrightarrow & \Sigma^2 X
\end{array}$$

in which every row and column is a distinguished triangle in T.

Remark 2.1.4. Note that the axiom **TR4** appears in may different guises and forms throughout the literature. We use the version given above as it ensures that all of the suspensions are obvious. The price is that the diagram does not look particularly octahedral. It is currently an open problem whether or not every pre-triangulated category is automatically triangulated.

Equipped with the notion of a triangulated category, we can now consider some particular types of functors. The first are the *homological* functors.

Definition 2.1.5. Let T be a triangulated category and A an abelian category. A functor $H : \mathsf{T} \longrightarrow \mathsf{A}$ is *homological* if for every distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

the sequence

$$H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z)$$

is exact. Similarly, a contravariant functor $H : \mathsf{T} \longrightarrow \mathsf{A}$ is *cohomological* if for every distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

the sequence

$$H(Z) \xrightarrow{H(v)} H(Y) \xrightarrow{H(u)} H(X)$$

is exact.

Remark 2.1.6. Given a homological functor $H : \mathsf{T} \longrightarrow \mathsf{A}$ and applying the rotation axiom **TR2**, one can extend the exact sequence infinitely in either direction. More specifically, given a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

the infinite sequence

$$\cdots \longrightarrow H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}w)} H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z) \xrightarrow{H(w)} H(\Sigma X) \longrightarrow \cdots$$

is everywhere exact.

We end this subsection with a few more definitions of interest.

Definition 2.1.7. Let T be a triangulated category admitting all countable coproducts. Consider a sequence

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} \cdots$$

of objects and morphisms in T. The *homotopy colimit* of this sequence, denoted $\operatorname{hocolim} X_i$ is by definiton given by the triangle

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{1-shift} \coprod_{i=0}^{\infty} X_i \longrightarrow \operatorname{hocolim} X_i \longrightarrow \Sigma(\coprod_{i=0}^{\infty} X_i).$$

where the shift map is the direct sum of the j_i. Note that the homotopy colimit is defined up to non-canonical isomorphism.

Definition 2.1.8. A commutative square

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow[f']{} & Z' \end{array}$$

is called *homotopy cartesian* if there exists a distinguished triangle

$$Y \xrightarrow{\begin{pmatrix} g \\ -f \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} f' & g' \end{pmatrix}} Z' \longrightarrow \Sigma Y.$$

If

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow[f']{} & Z' \end{array}$$

is a homotopy cartesian square, then we say that Y is the *homotopy pullback* of the diagram

$$\begin{array}{ccc} & & Z \\ & & \big\downarrow{\scriptstyle g'} \\ Y' & \xrightarrow[f']{} & Z' \end{array}.$$

Similarly, we say that Y' is the *homotopy pushout* of the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle g}\big\downarrow & & \\ Y' & & \end{array}$$

Remark 2.1.9. It follows immediately from the axioms of a triangulated category that every diagram of the form

$$\begin{array}{ccc} & & Z \\ & & \big\downarrow{\scriptstyle g'} \\ Y' & \xrightarrow[f']{} & Z' \end{array}.$$

admits a homotopy pullback. Similarly every diagram of the form

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle g}\big\downarrow & & \\ Y' & & \end{array}$$

admits a homotopy pushout.

§ 2.1.2 | Verdier quotients, thick subcategories and localising subcategories.

In this subsection, we will recall the notion of Verdier quotients. Given a triangulated category, taking a Verdier quotient is one of the few known recipes from which we can make a new category and will appear throughout many of the constructions in this work.

Related to Verdier quotients, but also incredibly important in their own right are the notions of thick subcategories and localising subcategories. Classifying these subcategories provides powerful structural information about the original triangulated category, and is a deep area of research within tensor-triangular geometry (perhaps with a few more adjectives depending on the category in question).

Definition 2.1.10. Let $(\mathsf{T}, \Sigma_{\mathsf{T}}, \mathcal{T})$ and $(\mathsf{S}, \Sigma_{\mathsf{S}}, \mathcal{S})$ be triangulated categories. A *triangulated functor* (or *exact functor*) is a functor $F : \mathsf{T} \longrightarrow \mathsf{S}$, together with a natural isomorphism $\psi : F \circ \Sigma_{\mathsf{T}} \cong \Sigma_{\mathsf{S}} \circ F$ such that for every distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in T, the image

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow \Sigma_{\mathsf{S}} F(X)$$

is a distinguished triangle in S.

Definition 2.1.11. Let T be a triangulated category. A subcategory $\mathsf{S} \subseteq \mathsf{T}$ is a *triangulated subcategory* if it is a full additive subcategory of T such that every object isomorphic to an object in S is itself in S, $\Sigma\mathsf{S} = \mathsf{S}$ and for any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

such that $X, Y \in \mathsf{S}$, we have $Z \in \mathsf{S}$.

Remark 2.1.12. If S is a triangulated subcategory of a triangulated category T, then the inclusion functor $\iota : \mathsf{S} \hookrightarrow \mathsf{T}$ is a triangulated functor.

Definition 2.1.13. Consider a triangulated functor $F : \mathsf{T} \longrightarrow \mathsf{S}$. Define the *kernel* of F to be the full subcategory

$$\ker(F) = \{X \in \mathsf{T} \mid F(X) \cong 0\}.$$

Lemma 2.1.14. *The kernel of a triangulated functor is itself a triangulated subcategory.*

Definition 2.1.15. Given a triangulated category T, a subcategory is said to be *thick* if it is triangulated and closed under direct summands. Explicitly, a full subcategory is thick if it is closed under suspensions, cones, and direct summands. Given a collection of objects X in T, we denote by $\mathrm{thick}(X)$ the smallest thick subcategory of T containing X. It is referred to as the *thick subcategory generated by X.*

Definition 2.1.16. A subcategory of T is said to be *localising* if it is triangulated and closed under arbitrary coproducts. Given a collection of objects X in T, we denote by $\mathrm{loc}(X)$ the smallest localising subcategory of T containing X. It is referred to as the *localising subcategory generated by X.*

Lemma 2.1.17. *Let T be a triangulated category admitting countable coproducts. Then every localising subcategory is thick.*

Proof. Consider a localising subcategory $\mathcal{S} \subseteq \mathsf{T}$ containing an object of the form $Z = X \oplus Y$. As T admits countable coproducts and $\mathcal{S}$ is localising, $\mathcal{S}$ is closed under arbitrary coproducts and so contains the coproduct $\coprod_{n\in\mathbb{N}}(X \oplus Y) \cong X \oplus \coprod_{n\in\mathbb{N}}(Y \oplus X)$. Therefore $\mathcal{S}$ contains the triangle

$$\coprod_{n\in\mathbb{N}}(Y \oplus X) \longrightarrow X \oplus \coprod_{n\in\mathbb{N}}(Y \oplus X) \longrightarrow X \longrightarrow \Sigma\left(\coprod_{n\in\mathbb{N}}(Y \oplus X)\right)$$

where the first map is the inclusion $\coprod_{n\in\mathbb{N}}(Y \oplus X) \hookrightarrow X \oplus \coprod_{n\in\mathbb{N}}(Y \oplus X)$ and the second map is the projection onto X. Therefore $\mathcal{S}$ contains X and and we conclude that $\mathcal{S}$ is thick. □

We have the following useful lemma from [Ste13, 3.8]:

Lemma 2.1.18. *Suppose* R *and* S *are triangulated categories having enough coproducts, and* M *is a localizing subcategory of* S*. Condiser a family* $(F_\lambda)_{\lambda\in\Lambda}$ *of coproduct preserving exact functors* $\mathsf{R} \longrightarrow \mathsf{S}$*. Then the full subcategory*

$$L = \{X \in \mathsf{R} \mid F_\lambda(X) \in M \textit{ for all } \lambda \in \Lambda\}$$

is a localizing subcategory of R*. In particular, if* C *is a collection of objects of* R *such that for all* $\lambda \in \Lambda$ *we have* $F_\lambda(C) \subseteq M$ *then every object* $X \in \mathrm{loc}(C)$ *satisfies* $F_\lambda(X) \in M$ *for all* $\lambda \in \Lambda$*.*

So far we have kernels in one hand, and thick subcategories in the other. The following results from the literature illuminate the connections between the two and introduce us to Verdier quotients.

Lemma 2.1.19. *The kernel of a triangulated functor is a thick subcategory.*

The following theorem is essentially due to Verdier, although the form we give below is most similar to [Nee01].

Theorem 2.1.20. *Let* S *be a triangulated subcategory of a triangulated category* T*. Then there exists a triangulated category* T/S*, called the* Verdier quotient *of* T *by* S*, and a functor* $F : \mathsf{T} \longrightarrow \mathsf{T}/\mathsf{S}$ *such that* $\mathsf{S} \in \ker(F)$ *and* F *is universal with this property. Explicitly, if* $G : \mathsf{T} \longrightarrow \mathsf{R}$ *is a triangulated functor such that* $\mathsf{S} \in ker(G)$*, then there exists a unique functor* $H : \mathsf{T}/\mathsf{S} \longrightarrow \mathsf{R}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathsf{T} & \xrightarrow{F} & \mathsf{T}/\mathsf{S} \\ {\scriptstyle G}\downarrow & \swarrow {\scriptstyle H} & \\ \mathsf{R} & & \end{array}$$

Moreover, the kernel $\ker(F)$ *is the smallest thick subcategory of* T *containing* S*.*

Corollary 2.1.21. *Every thick subcategory is the kernel of a triangulated functor.*

The above theorem therefore gives one of the motivations for classifying different types of thick subcategories, as it is equivalent to classifying the possible Verdier quotients of a triangulated category, up to whatever additional conditions are attached to the thick subcategories.

We present the definition of the quotient category as given in the proof of the theorem, but we will omit the numerous lemmas required to fully prove the theorem.

Definition 2.1.22. Let T be a triangulated category with triangulated subcategory S. Denote by $\mathrm{Mor}_{\mathsf{S}}$ the collection of those morphisms f such that the cone of f lies in S. That is, a morphism f is in $\mathrm{Mor}_{\mathsf{S}}$ if it fits into a triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

with $Z \in \mathsf{S}$.

Definition 2.1.23. For objects X, Y in T define $\alpha(X, Y)$ to be the collection of all diagrams of the form

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & & Y \end{array}$$

where $f \in \mathrm{Mor}_{\mathsf{S}}$. These diagrams are represented by the triples (Z, f, g). Define a relation $R(X, Y)$ on $\alpha(X, Y)$ by declaring that a pair of diagrams $((Z, f, g), (Z', f', g'))$ is in $R(X, Y)$ if there exists a third diagram (Z'', f'', g'') and morphisms

$$u : Z'' \longrightarrow Z$$
$$v : Z'' \longrightarrow Z'$$

such that the diagram

$$\begin{array}{ccccc} & & Z & & \\ & {}^{f}\swarrow & \uparrow{\scriptstyle u} & \searrow^{g} & \\ X & \xleftarrow{f''} & Z'' & \xrightarrow{g''} & Y \\ & {}_{f'}\nwarrow & \downarrow{\scriptstyle v} & \nearrow_{g'} & \\ & & Z' & & \end{array}$$

commutes.

It turns out that the relation defined above is actually an equivalence relation. This leads to the following construction.

Construction 2.1.24. Let T be a triangulated category with triangulated subcategory S. Define a new category T/S with the same objects as T with morphisms given by

$$\mathrm{Hom}_{\mathsf{T}/\mathsf{S}}(X, Y) = \frac{\alpha(X, Y)}{R(X, Y)}.$$

Composition of representatives (W_1, f_1, g_1) and (W_2, f_2, g_2) is induced by the diagram

$$\begin{array}{ccccc} W_3 & \xrightarrow{u} & W_2 & \xrightarrow{g_2} & Z \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle f_2} & & \\ W_1 & \xrightarrow{g_1} & Y & & \\ {\scriptstyle f_1}\downarrow & & & & \\ X & & & & \end{array}$$

where the commutative square

$$\begin{array}{ccc} W_3 & \xrightarrow{u} & W_2 \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle f_2} \\ W_1 & \xrightarrow{g_1} & Y \end{array}$$

is induced by homotopy pullback.

The proof of the theorem consists of showing that the above construction is valid, and that the resulting category does indeed satisfy the universal property of the Verdier quotient.

Warning 2.1.25. Here be dragons. In general the homs in the Verdier quotient T/S may not be sets, even if all of the homs in T are sets. Fortunately, throughout this work the quotients we will take are well behaved and there is no risk of accidentally producing some monstrous class of morphisms.

We now have in our minds an idea of what the Verdier quotient looks like, and some motivation for thick and localising subcategories. For the rest of this work we will refrain from using this explicit description of morphisms unless absolutely necessary, although the quotients will appear constantly from now on.

§ 2.1.3 | Compact objects

A common theme in mathematics is to take a large structure, and then try and investigate the different ways the structure can be constructed. One flavour of this is to consider ideas of *generation*. For example, groups and rings are often studied by considering which objects *generate* the algebraic structure via the group or ring operations. For the categories we are interested in, the type of generation that fits best is *compact generation*. In this short subsection we will set out the very basics on compact objects. Beyond this subsection, such objects will appear constantly and play a key role in many ideas and proofs.

Definition 2.1.26. Let T be a triangulated category admitting all set-indexed coproducts. An object $t \in \mathsf{T}$ is *compact* if $\mathrm{Hom}_{\mathsf{T}}(t, -)$ preserves arbitrary coproducts. That is, for any family $\{x_\lambda \mid \lambda \in \Lambda\}$ we have $\mathrm{Hom}_{\mathsf{T}}(t, \coprod_{\lambda\in\Lambda} x_\lambda) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathsf{T}}(t, x_\lambda)$.

Definition 2.1.27. We say T is *compactly generated* if there is a set of compact objects $\mathcal{G}$ such that an object $t \in \mathsf{T}$ is zero if and only if $\mathrm{Hom}_{\mathsf{T}}(g, \Sigma^i t) = 0$ for all $g \in \mathcal{G}$ and all $i \in \mathbb{Z}$. We call $\mathcal{G}$ a *(compact) generating set* for T. We denote by T^c the full subcategory of compact objects in T. Note that T^c is an essentially small thick subcategory of T.

The following lemma shows the connection between thick subcategories and the idea of a generating set.

Lemma 2.1.28. *Let T be a triangulated category and $g \in \mathsf{T}$ a compact object such that $\mathsf{T} = \mathrm{thick}(g)$. For an object $x \in \mathsf{T}$, if for all $j \in \mathbb{Z}$ we have $\mathrm{Hom}_{\mathsf{T}}(g, \Sigma^j x) = 0$, then $x = 0$. That is, if $\mathsf{T} = \mathrm{thick}(g)$ then $\{g\}$ is a compact generating set for T.*

Proof. Consider the collection

$$^{\perp}x = \{y \in \mathsf{T} \mid \forall j \in \mathbb{Z} : \mathrm{Hom}_{\mathsf{T}}(y, \Sigma^j x) = 0\}.$$

Observe that ${}^{\perp}x$ is thick. By assumption $g \in {}^{\perp}x$ and so ${}^{\perp}x = \mathsf{T}$. Therefore $\mathrm{Hom}_{\mathsf{T}}(x,x) = 0$. □

Remark 2.1.29. The above lemma holds when $\mathsf{T} = \mathrm{loc}(g)$ for a compact object g. Note also that the lemma holds when $\mathsf{T} = \mathrm{thick}(\mathcal{G})$ for some set of compact objects $\mathcal{G}$.

We will see more about compact generation in the discussion of Bousfield localisation.

As we have seen, compact objects are defined by their good behaviour with arbitrary coproducts. The following lemma shows that such objects interact well with homotopy colimits.

Lemma 2.1.30. [Nee92, 1.5] *Let t be a compact object of a triangulated category T. Consider a sequence of objects and maps in T given by:*

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} \cdots$$

Then there is an isomorphism of abelian groups

$$\operatorname{colim} \mathrm{Hom}_{\mathsf{T}}(t, X_i) \cong \mathrm{Hom}_{\mathsf{T}}(t, \operatorname{hocolim} X_i).$$

Proof. Consider the triangle defining the homotopy colimit $\operatorname{hocolim} X_i$:

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{1-shift} \coprod_{i=0}^{\infty} X_i \longrightarrow \operatorname{hocolim} X_i \longrightarrow \Sigma(\coprod_{i=0}^{\infty} X_i).$$

Applying the homological functor $\mathrm{Hom}(t,-)$ and rotating via **TR1**, we obtain an exact sequence

$$\mathrm{Hom}(t, \operatorname{hocolim} X_i) \longrightarrow \mathrm{Hom}(t, \coprod_{i=0}^{\infty} \Sigma X_i) \xrightarrow{\Sigma(1-shift)} \mathrm{Hom}(t, \coprod_{i=0}^{\infty} \Sigma X_i)$$

As t is compact we obtain a commutative diagram of abelian groups

$$\begin{array}{ccccc} \mathrm{Hom}(t, \operatorname{hocolim} X_i) & \longrightarrow & \mathrm{Hom}(t, \coprod_{i=0}^{\infty} \Sigma X_i) & \xrightarrow{1-shift} & \mathrm{Hom}(t, \coprod_{i=0}^{\infty} \Sigma X_i) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \mathrm{Hom}(t, \operatorname{hocolim} X_i) & \longrightarrow & \bigoplus_{i=0}^{\infty} \mathrm{Hom}(t, \Sigma X_i) & \xrightarrow{1-shift} & \bigoplus_{i=0}^{\infty} \mathrm{Hom}(t, \Sigma X_i) \end{array}$$

with each row exact, noting that in the above the morphism $1-shift$ is the morphism obtained by applying $\mathrm{Hom}(t, \Sigma -)$ to the original $1-shift$ map. The morphism

$$\bigoplus_{i=0}^{\infty} \mathrm{Hom}(t, \Sigma X_i) \xrightarrow{1-shift} \bigoplus_{i=0}^{\infty} \mathrm{Hom}(t, \Sigma X_i)$$

is clearly injective, and therefore the morphism

$$\mathrm{Hom}(t, \operatorname{hocolim} X_i) \longrightarrow \bigoplus_{i=0}^{\infty} \mathrm{Hom}(t, \Sigma X_i)$$

is the zero morphism. Now consider the following diagram

$$\begin{array}{ccccccc}
\bigoplus_{i=0}^{\infty} \mathrm{Hom}(t, X_i) & \xrightarrow{1-shift} & \bigoplus_{i=0}^{\infty} \mathrm{Hom}(t, X_i) & \longrightarrow & \operatorname{colim} \mathrm{Hom}(t, X_i) & \longrightarrow & 0 \\
\cong\downarrow & & \cong\downarrow & & \downarrow & & \\
\mathrm{Hom}(t, \coprod_{i=0}^{\infty} X_i) & \xrightarrow{1-shift} & \mathrm{Hom}(t, \coprod_{i=0}^{\infty} X_i) & \longrightarrow & \mathrm{Hom}(t, \operatorname{hocolim} X_i) & \longrightarrow & 0
\end{array}$$

where the arrow $\operatorname{colim} \mathrm{Hom}(t, X_i) \longrightarrow \mathrm{Hom}(t, \operatorname{hocolim} X_i)$ is the canonical map. The bottom row is exact by the previous argument. The top row is exact by the definition of the colimit. We conclude by the five lemma that the canonical map

$$\operatorname{colim} \mathrm{Hom}(t, X_i) \longrightarrow \mathrm{Hom}(t, \operatorname{hocolim} X_i)$$

is indeed an isomorphism. □

§ 2.2 | Primer for algebraic geometry

The bulk of this work is concerned with the properties of certain sheaves constructed out of data found within triangulated categories. In this section we will provide a lightning tour of the basic concepts used throughout. Note that these concepts are indeed very basic, and this material is included to reinforce these ideas and set the scene for comparisons with the constructions in the section on tensor triangular geometry.

Throughout this section we will be only defining sheaves on topological spaces. At no point in this work will we require the greater generality of sheaves defined on general categories.

§ 2.2.1 | Presheaves, sheaves and stalks

Let X be a topological space.

Definition 2.2.1. Define a category $\Omega(X)$ where:

1. $\mathrm{Ob}(\Omega(X))$ is the collection of all open subsets of X.
2. For each pair of open subsets $V, U \subseteq X$ there is a unique arrow $V \longrightarrow U$ iff $V \subseteq U$, and no arrows otherwise.

In other words, $\Omega(X)$ is the data of all opens of X ordered by inclusion. That is, it is the category generated by the poset of opens.

Definition 2.2.2. Let C be a category. A *presheaf F on X with values in* C is a functor

$$F : \Omega(X)^{\mathrm{op}} \longrightarrow \mathsf{C}.$$

Explcitly, a presheaf is a rule which to each open subset $U \subseteq X$ assigns an object $F(U) \in \mathsf{C}$, and for every inclusion $V \subseteq U$ of open subsets assigns a morphism $\rho^U_V : F(U) \longrightarrow F(V)$ such that whenever we have inclusions $W \subseteq V \subseteq U$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.

Definition 2.2.3. A *morphism* $\varphi : F \longrightarrow G$ *of presheaves valued in* C is a natural transformation of functors. Specifically, for each open subset $U \in \Omega(X)$ there is a morphism $\varphi_U : F(U) \longrightarrow G(U)$ in C, and this assignment is compatible with the restriction maps of the presheaves. This compatibility can be represented by requiring the following square to commute for all inclusions of open subsets $V \subseteq U$:

$$\begin{array}{ccc} U & & F(U) \xrightarrow{\varphi_U} G(U) \\ \uparrow & & {}_F\rho^U_V \downarrow \qquad \downarrow {}_G\rho^U_V \\ V & & F(V) \xrightarrow[\varphi_V]{} G(V) \end{array}$$

Notation 2.2.4. We denote by $\mathsf{Psh}_{\mathsf{C}}(X)$ the category of all presheaves on X with values in C, with morphisms given by morphisms of presheaves. When the category C is obvious, we will supress the notation and simply write $\mathsf{Psh}(X)$. If F is a presheaf in $\mathsf{Psh}_{\mathsf{C}}(X)$ and each object of C has an underlying set, then the elements of each $F(U)$ are referred to as *sections*.

From now on we will assume that the value category C consists of objects which have underlying sets. While this is not strictly necessary for the formalisms to make sense, it is the setting we find ourselves in for the remainder of this work.

Definition 2.2.5. A *sheaf on* X *with values in* C is a presheaf $F \in \mathsf{Psh}_{\mathsf{C}}(X)$ satsifying the following condition: Given any open covering $U = \bigcup_{i\in I} U_i$ indexed by a set I, and any collection of sections $s_i \in F(U_i)$ such that for all $i, j \in I$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

then there exists a unique section $s \in F(U)$ such that for all $i \in I$ we have $s|_{U_i} = s_i$. A *morphism of sheaves* is simply a morphism of the underlying presheaves. The category of all sheaves on X with values in C and all sheaf morphisms is denoted by $\mathrm{Shv}_{\mathsf{C}}(X)$, or by $\mathrm{Shv}(X)$ if the context is clear.

Definition 2.2.6. The *stalk* of a (pre)sheaf F at a point $x \in X$ is given by

$$F_x = \mathrm{colim}_{x\in U} F(U)$$

where the colimit is taken over all open subsets U containing x.

Warning 2.2.7. A presheaf $F : \Omega(X)^{\mathrm{op}} \longrightarrow \mathsf{C}$ may fail to have a stalk at point x if the colimit $\mathrm{colim}_{x\in U} F(U)$ does not exist in C. From now on we will assume that C admits all such colimits. This is the case in many categories which appear in practice, particular those with an algebraic flavour.

The elements of F_x are *germs* which are pairs (f, U) where U is an open subset containing x, and f is a section of $F(U)$, modulo the relation that $(f, U) \sim (g, V)$ if and only if there exists an open subset $W \subseteq U \cap V$ containing x, such that $f|_W = g|_W$.

Definition 2.2.8. Let F be a presheaf. For an open subset $U \subseteq X$ we say $(s_x) \in \prod_{x\in U} F_x$ satisfies the *local condition* if, for every $x \in U$ there exists an open neighbourhood $x \in V \subset U$ and a section $r \in F(V)$ such that for all $v \in V$ we have $s_v = (V, r)$ in F_v. For each open subset $U \subseteq X$, define

$$F^{\#}(U) = \{(s_x) \in \prod_{x\in U} F_x \mid (s_x) \text{ satisfies the local condition}\}$$

For open subsets $V \subseteq U$ there are canonical projection maps

$$\prod_{u\in U} F_u \longrightarrow \prod_{v\in V} F_v.$$

We denote by $F^{\#}$ the data of the $F^{\#}(U)$ and the projection maps. We call $F^{\#}$ the *sheafification* of the presheaf F.

The following theorem confirms that the rather suspiciously named sheafification is indeed a sheaf.

Theorem 2.2.9. *Let F be a presheaf on X and let $F^{\#}$ be the sheafification of F.*

1. *$F^{\#}$ is a sheaf.*
2. *For all $x \in X$, $F_x = F^{\#}_x$.*
3. *For any morphism $F \longrightarrow G$ where G is a sheaf, the morphism factors uniquely as $F \longrightarrow F^{\#} \longrightarrow G$.*

It is often the case that we wish to work with a particular basis for a topological space, rather than just using arbitrary open subsets of the space. Let X be a topological space with basis $\mathcal{B}$. One can define (pre)sheaves on $\mathcal{B}$ by adapting the above definitions, simply changing every instance of "open subset" to "*basic* open subset". We will sometimes refer to (pre)sheaves on a basis $\mathcal{B}$ as *partially-defined.*

Proposition 2.2.10. *Let F be a sheaf on $\mathcal{B}$. Then there exists a sheaf F^{ext} on X such that for all basic open subsets $U \in \mathcal{B}$, we have $F(U) = F^{\text{ext}}(U)$.*

Proof. The key ingredient is applying the so called *local condition* to the appropriate open subsets. Specifically, for *any open subset* $U \subseteq X$ we say $(s_x) \in \prod_{x\in U}$ satisfies the *local condition* if, for every $x \in U$ there exists a *basic open neighbourhood* $x \in V \subset U$ and a section $r \in F(V)$ such that for all $v \in V$ we have $s_v = (V, r)$ in F_v. For each open subset $U \subseteq X$, define

$$F^{\text{ext}}(U) = \{(s_x) \in \prod_{x\in U} F_x \mid (s_x) \text{ satisfies the local condition}\}$$

For open subsets $V \subseteq U$ there are canonical projection maps

$$\prod_{u\in U} F_u \longrightarrow \prod_{v\in V} F_v.$$

We denote by F^{ext} the data of the $F^{\text{ext}}(U)$ and the projection maps. We call F^{ext} the *extension* of the sheaf F.

The remainder of the proof is essentially Theorem 2.2.9. □

Remark 2.2.11. As a consequence of the proposition, each partially defined presheaf F on $\mathcal{B}$ extends uniquely to a sheaf on X via

$$F \overset{\text{sheafification}}{\longrightarrow} F' \longrightarrow (F')^{\text{ext}}.$$

From now on we will denote this extension of F by F' and freely call this the sheafification of F.

In Chapter 3, we will be constructing particular sheaves from the data of a triangulated category, and then investigating their properties. The next two subsections will lay out the extra structure present on the sheaves, as well as some of the properties of interest.

§ 2.2.2 | Locally ringed spaces and affine schemes

We start this subsection with a quick recap of $\mathcal{O}$-modules and locally ringed spaces. Throughout we fix a topological space X.

Definition 2.2.12. Let $\mathcal{O}$ be a sheaf of rings. An *$\mathcal{O}$-module* is a sheaf of abelian groups F together with a morphism of sheaves

$$\mathcal{O} \times F \longrightarrow F$$

such that for every open subset $U \subseteq X$, the induced map $\mathcal{O}(U) \times F(U) \longrightarrow F(U)$ defines an $\mathcal{O}(U)$-module structure on $F(U)$, compatible with the restriction maps. That is, for all open subsets $V \subseteq U$ the diagram

$$\begin{array}{ccc} \mathcal{O}(U) \times F(U) & \xrightarrow{\text{action}} & F(U) \\ {\scriptstyle {}_{\mathcal{O}}\rho^U_V \times {}_F\rho^U_V}\downarrow & & \downarrow{\scriptstyle {}_F\rho^U_V} \\ \mathcal{O}(V) \times F(V) & \xrightarrow[\text{action}]{} & F(V) \end{array}$$

commutes. A *morphism* of $\mathcal{O}$-modules is a map $\varphi : F \longrightarrow G$ of sheaves compatible with the actions of the $\mathcal{O}$-module structures of F and G. More specifically, we require that the diagram

$$\begin{array}{ccc} \mathcal{O} \times F & \xrightarrow{\text{action}} & F \\ {\scriptstyle \text{id} \times \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathcal{O} \times G & \xrightarrow[\text{action}]{} & G \end{array}$$

commutes. We denote by $\mathsf{Mod}\mathcal{O}$ the category of all $\mathcal{O}$-modules and $\mathcal{O}$-module morphisms between them.

Definition 2.2.13. A *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. The sheaf $\mathcal{O}_X$ is called the *structure sheaf*. Given a continuous map $f : X \longrightarrow Y$, the *pushforward* sheaf $f_*\mathcal{O}_X$ is a sheaf on Y where for each open subset $U \subseteq Y$ we have $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \longrightarrow Y$ and a collection of ring homomorphisms $\varphi_U : \mathcal{O}_Y(U) \longrightarrow f_*\mathcal{O}_X(U)$ for each open subset $U \subseteq X$, compatible with the restriction maps. That is, for all open subsets $V \subseteq U$ the diagram

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{\varphi_U} & f_*\mathcal{O}_X(U) \\ {\scriptstyle \mathcal{O}_Y\rho^U_V}\big\downarrow & & \big\downarrow{\scriptstyle f_*\mathcal{O}_X\rho^U_V} \\ \mathcal{O}_Y(V) & \xrightarrow[\varphi_V]{} & f_*\mathcal{O}_X(V) \end{array}$$

commutes. An *isomorphism of ringed spaces* is a morphism of ringed spaces $(f, (\varphi_U)_{U\in\Omega(X)})$ such that the continuous function $f : X \longrightarrow Y$ is a homeomorphism, and the data of φ is an isomorphism of sheaves $\varphi : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$.

We shall focus our attention on those ringed spaces with well-behaved *local* properties.

Definition 2.2.14. Let R be a *graded ring.* A *homogeneous ideal* of R is an ideal $I \subseteq R$ such that I is a graded module over R.

Definition 2.2.15. A ring R is *local* if it possesses a unique maximal right ideal I. If R is local then I is also the unique maximal left ideal, and the unique maximal two-sided ideal. A graded ring R is a *graded local ring* if the two-sided ideal M generated by noninvertible homogeneous elements is a proper maximal two-sided ideal.

Definition 2.2.16. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that all stalks of $\mathcal{O}_X$ are local rings. A *morphism of locally ringed spaces* $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces satisfying the additional property that the induced homomorphisms between the stalks of $\mathcal{O}_Y$ and $\mathcal{O}_X$ are local homomorphisms. More specifically, for every $x \in X$, the maximum ideal of the stalk $\mathcal{O}_{Y,f(x)}$ is mapped into the maximum ideal of the stalk $\mathcal{O}_{X,x}$.

Remark 2.2.17. If instead of a sheaf of rings, we equip the space X with a sheaf of *graded* rings, then we can repeat all of the above definitions but with the adjective *graded*, with the caveat that morphisms and restriction maps must also be compatible with the grading.

Notation 2.2.18. If a topological space X is equipped with the structure of a *graded (locally) ringed space* we will denote the structure sheaf by $\mathcal{O}^\bullet_X$. Similarly we will denote the category of graded $\mathcal{O}^\bullet_X$-modules by $\mathsf{grMod}\mathcal{O}^\bullet_X$.

Definition 2.2.19. Let R be a $\mathbb{Z}$-graded ring, let M be a $\mathbb{Z}$-graded R-module and let i be an integer. The module M *twisted by* i, or the i-twist of M, denoted $M(i)$ is defined by

$$M(i)_n = M_{n+i}$$

for all $n \in \mathbb{Z}$.

We will now introduce the building block of many key concepts in algebraic geometry: the spectrum of a commutative ring.

Definition 2.2.20. Let R be a commutative ring. The *spectrum* of R, denoted $\mathrm{Spec}(R)$, is the set of all prime ideals of R. The set $\mathrm{Spec}(R)$ can be equipped with a topology, called the *Zariski topology*, by taking all sets of the form

$$D(f) = \{P \mid f \notin P\}$$

where f is any element of R, as a basis of open subsets. We define a sheaf of rings on $\mathrm{Spec}(R)$ by defining $\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) = R_f$ for each basic open subset $D(f)$, which by Remark 2.2.10 extends to a sheaf defined on all open subsets of $\mathrm{Spec}(R)$. The pair $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is a locally ringed space.

Remark 2.2.21. For an equivalent construction of $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$, see [Har77, 2.2.1] and the discussion that follows.

We can now define schemes, affine or otherwise.

Definition 2.2.22. A locally ringed space $(X, \mathcal{O}_X)$ is an *affine scheme* if it is isomorphic to $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for some commutative ring R. A locally ringed space $(X, \mathcal{O}_X)$ is a *scheme* if for every point $x \in X$ there is an open neighbourhood U such that U equipped with the restriction of $\mathcal{O}_X$ to U is an affine scheme.

Incorporating the property of being an (affine) scheme into *tensor triangular geometry* will be a key theme throughout this work. With rings fresh in our mind, we will quickly introduce a graded version of the spectrum.

Definition 2.2.23. Let R be a graded ring. A homogeneous ideal P is *prime* in the usual sense if for all homogeneous elements $a, b \in R$, if $ab \in P$ then $a \in P$ or $b \in P$.

Definition 2.2.24. Let R be a graded ring. The *homogeneous spectrum of* R, denoted $\mathrm{Spec}^h(R)$, is the collection of *all* prime homogeneous ideals of R.

Just as with the usual spectrum, we can equip the homogeneous spectrum with a topology and structure sheaf $\mathcal{O}_{\mathrm{Spec}^{\mathrm{h}}(R)}$ in the same way. The pair $(\mathrm{Spec}^{\mathrm{h}}(R), \mathcal{O}_{\mathrm{Spec}^{\mathrm{h}}(R)})$ is a *graded locally ringed space.*

§ 2.2.3 | Quasi-coherent and coherent sheaves

In this short subsection we will lay out some important properties of sheaves which we will later investigate in the tensor triangular setting.

Definition 2.2.25. A sheaf of modules $\mathcal{F}$ on a ringed space $(X, \mathcal{O}_X)$ is of *finite type* if for every $x \in X$ there exists some open neighbourhood U such that $\mathcal{F}|_U$ is generated by finitely many sections.

Definition 2.2.26. An $\mathcal{O}_X$-module $\mathcal{F}$ on a ringed space $(X, \mathcal{O}_X)$ is *quasi-coherent* if it is locally the cokernel of a map of free modules. That is, there is an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X such that for every λ there exist index sets I_λ and J_λ and an exact sequence of sheaves of $\mathcal{O}_X$-modules of the form

$$\mathcal{O}_{U_\lambda}^{\oplus I_\lambda} \longrightarrow \mathcal{O}_{U_\lambda}^{\oplus J_\lambda} \longrightarrow \mathcal{F}|_{U_\lambda} \longrightarrow 0.$$

An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be *coherent* if

- $\mathcal{F}$ is of finite type and,
- for every open $U \subseteq X$ and every finite collection $s_i \in \mathcal{F}(U), i = 1, \ldots, n$ the kernel of the associated map $\mathcal{O}_U^{\oplus n} \longrightarrow \mathcal{F}|_U$ is of finite type.

Note that every coherent sheaf is quasi-coherent.

Notation 2.2.27. Given a ringed space $(X, \mathcal{O}_X)$ we denote by $\mathsf{QCoh}(X)$ the category of all quasi-coherent sheaves, and we write $\mathsf{Coh}(X)$ for the category of all coherent sheaves.

Coherent and quasi-coherent sheaves naturally form well-behaved categories.

Proposition 2.2.28. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

1. [Ser55, I §2] *The category $\mathsf{Coh}(X)$ of all coherent sheaves is an abelian category.*
2. *If $(X, \mathcal{O}_X)$ is a scheme, then the category $\mathsf{QCoh}(X)$ of all quasi-coherent sheaves is an abelian category.*

Remark 2.2.29. The above holds true in the graded case. We denote the category of coherent graded $\mathcal{O}_X^\bullet$-modules by $\mathsf{Coh}^\bullet(X)$ and the category of quasi-coherent graded $\mathcal{O}_X^\bullet$-modules by $\mathsf{QCoh}^\bullet(X)$.

Definition 2.2.30. Let $(X, \mathcal{O}_X)$ be a ringed space. The structure sheaf $\mathcal{O}_X$ is called *coherent* if it is coherent as a module over itself.

Definition 2.2.31. An *invertible* sheaf on a locally ringed space $(X, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ which is locally free of rank 1. That is, for each point $x \in X$ there is an open neighbourhood U of x such that $\mathcal{F}|_U \cong \mathcal{O}_U$.

Throughout this work we will frequently use the construction of a sheaf *associated to a module*. We recall the construction and some elementary properties of these sheaves.

Definition 2.2.32. Let R be a commutative ring and let M be an R-module. For each open subset $U \subseteq \operatorname{Spec}(R)$ define the group $\widetilde{M}(U)$ to be the collection of functions $s : U \longrightarrow \coprod_{\mathcal{P}\in U} M_\mathcal{P}$ (where $M_\mathcal{P}$ is the localisation of M at $\mathcal{P}$) such that

1. For all $\mathcal{P} \in U$, $s(\mathcal{P}) \in M_\mathcal{P}$.

2. For each $\mathcal{P} \in U$, there is an open neighbourhood V of $\mathcal{P}$ and elements $m \in M$ and $r \in R$, such that for all $\mathcal{Q} \in V$, $r \notin \mathcal{Q}$ and $s(\mathcal{Q}) = \frac{m}{r}$ in $M_{\mathcal{Q}}$.

Using the obvious restriction maps, these groups assemble $\widetilde{M}$ into a sheaf.

Proposition 2.2.33. [Har77, II 5.1] *Let R be a commutative ring, let M be an R-module and let $\widetilde{M}$ be the sheaf on $X = \mathrm{Spec}(R)$ associated to M. Then*

1. *$\widetilde{M}$ is an $\mathcal{O}_X$-module;*
2. *for each $\mathcal{P} \in X$ the stalk $(\widetilde{M})_{\mathcal{P}}$ is isomorphic to $M_{\mathcal{P}}$, the localisation of M at $\mathcal{P}$;*
3. *for any $f \in R$, the R_f-module $\widetilde{M}(D(f))$ is isomorphic to the localised module M_f.*

§ 2.3 | Tensor triangular geometry

We start this section with the definition of a *tensor triangulated category*, the main object of study in this work.

Definition 2.3.1. [Bal05] A *tensor triangulated category* is a triple $(\mathsf{T}, \otimes, \mathbf{1})$ consisting of a triangulated category T, and a symmetric monoidal product $\otimes : \mathsf{T} \times \mathsf{T} \longrightarrow \mathsf{T}$ which is exact in each variable, with unit $\mathbf{1}$.

We will freely abbreviate tensor triangulated category to *tt-category*. Essentially, an existing categorical structure has been equipped with an additional "multiplication". Naturally this draws comparisons to familiar algebraic objects like groups being equipped with a type of multiplication to become rings. As we go further into the building blocks of tensor triangular geometry this comparison with rings will be increasingly apparent.

This collection of preliminaries will take us through the main ideas of tt-geometry. We will start with the definitions and basic results around prime ideals, spectra and support theories. We will detail Balmer's construction of a locally ringed space structure, and recall some proofs of a few technical results. Finally we will embrace the framework of categorical actions and examine some localisations and supports in this context.

§ 2.3.1 | Rigidly-compactly generated tensor triangulated categories

Definition 2.3.2. Assume that T is closed symmetric monoidal, i.e. for each $k \in \mathsf{T}$ the functor $k \otimes -$ has a right adjoint $\hom(k, -)$. These functors can then be assembled into a bifunctor $\hom(-,-)$ which we call the *internal hom functor.* The *dual* of an object k is given by $k^\vee := \hom(k, \mathbf{1})$. An object k is *rigid* if for all other objects t we have that the natural evaluation map $k^\vee \otimes t \longrightarrow \hom(k,t)$ is an isomorphism. For a rigid object k we have $k \cong (k^\vee)^\vee$.

We can now combine the structure of a tensor triangulated category with the smallness conditions of compactness and rigidity to obtain a good structure to analyse.

Definition 2.3.3. A *rigidly-compactly generated tensor triangulated category* is a triple $(\mathsf{T}, \otimes, \mathbf{1})$ where T is a compactly generated tensor triangulated category, and $(\otimes, \mathbf{1})$ is a symmetric monoidal structure on T such that the tensor product $\otimes$ is a coproduct preserving exact functor in each variable, and the compact objects T^c form a rigid tensor subcategory. In particular we require $\mathbf{1}$ to be compact. We will refer to such a category T as a *big tt-category.*

Remark 2.3.4. Note that if T is a rigidly-compactly generated tt-category, then a rigid object must necessarily be compact. To see this observe that given some coproduct $\bigoplus_{i \in I} t_i$ in T, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{T}}(u, \bigoplus_{i \in I} t_i) &\cong \mathrm{Hom}_{\mathsf{T}}(\mathbf{1} \otimes u, \bigoplus_{i \in I} t_i) \\
&\cong \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \mathrm{hom}(u, \bigoplus_{i \in I} t_i)) \text{ as the tensor product is closed} \\
&\cong \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, u^{\vee} \otimes \bigoplus_{i \in I} t_i) \text{ as } u \text{ is rigid} \\
&\cong \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \bigoplus_{i \in I} (u^{\vee} \otimes t_i)) \text{ as the tensor product preserves coproducts} \\
&\cong \bigoplus_{i \in I} \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, u^{\vee} \otimes t_i) \text{ as the tensor unit is compact} \\
&\cong \bigoplus_{i \in I} \mathrm{Hom}_{\mathsf{T}}(u, t_i) \text{ by adjunction.}
\end{aligned}$$

Therefore u is compact.

From now on we will suppress notation and take T to be a big tt-category and denote by T^c the full subcategory of compact objects.

§ 2.3.2 | The spectrum of a tensor triangulated category

We now present the Balmer spectrum of a tensor triangulated category and the first notion of support as given in [Bal05]. Restricting to the compact objects, this will let us view the collection of prime ideals of a tensor triangulated category as a topological space, equipped with a universal support theory satisfying many desirable properties.

Definition 2.3.5. [Bal05, Def 1.2] A *thick tensor-ideal* A of T is a thick subcategory such that for all $a \in \mathsf{A}$ and $t \in \mathsf{T}$ the tensor product $a \otimes t$ also belongs to A. Just as with thick subcategories, we will denote by $\mathrm{thick}^{\otimes}(X)$ the smallest thick tensor-ideal containing X.

Following the similarity with ideals in the usual theory of commutative rings we will also be interested in those ideals which have the additional property of being *prime*.

Definition 2.3.6. [Bal05, Def 2.1] A *prime ideal* of T is a proper thick tensor-ideal $\mathcal{P} \subsetneq \mathsf{T}$ such that if $a \otimes b \in \mathcal{P}$ then $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

It is useful to understand how being prime affects the product of ideals. The following lemma is similar to [NVY19, 3.2.2].

Lemma 2.3.7. *If $\mathcal{P}$ is a prime ideal of T then for all ideals $\mathcal{I}, \mathcal{J} \subseteq \mathsf{T}$ such that $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P}$, either $\mathcal{I} \subseteq \mathcal{P}$ or $\mathcal{J} \subseteq \mathcal{P}$.*

Proof. Let $\mathcal{I}, \mathcal{J} \subseteq \mathsf{T}$ be ideals such that $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P}$ and suppose that $\mathcal{I} \nsubseteq \mathcal{P}$ and $\mathcal{J} \nsubseteq \mathcal{P}$. Then there exists $i \in \mathcal{I}$ and $j \in \mathcal{J}$ such that $i \notin \mathcal{P}$ and $j \notin \mathcal{P}$. Then

$$i \otimes j \in \mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P}.$$

As $\mathcal{P}$ is prime, $i \otimes j \in \mathcal{P}$ implies either $i \in \mathcal{P}$ or $j \in \mathcal{P}$, a contradiction. Therefore $\mathcal{I} \subseteq \mathcal{P}$ or $\mathcal{J} \subseteq \mathcal{P}$ as required. □

Definition 2.3.8. Let T be a big tt-category, with compacts T^c. The *Balmer spectrum* of T^c is given by

$$\mathrm{Spc}(\mathsf{T}^c) = \{\mathcal{P} \mid \mathcal{P} \text{ prime ideal of } \mathsf{T}^c\}.$$

For all $a \in \mathsf{T}^c$ we define the open subsets $U(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c) \mid a \in \mathcal{P}\}$. This forms a basis of quasi-compact open sets for the topology on $\mathrm{Spc}(\mathsf{T}^c)$ [Bal05, 2.7, 2.14].

Definition 2.3.9. For an object $t \in \mathsf{T}^c$ the (small) tt-support $\mathrm{supp}_{\mathsf{T}^c} t$ is defined as

$$\mathrm{supp}_{\mathsf{T}^c} t = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c) \mid t \notin \mathcal{P}\}.$$

By definition $\mathrm{supp}_{\mathsf{T}^c} t$ is the complement of the basic open subset $U(t)$. Given a subset of objects $\mathcal{J} \subset \mathsf{T}^c$ we define the support of the subset as

$$\mathrm{supp}_{\mathsf{T}^c}(\mathcal{J}) := \bigcup_{j \in \mathcal{J}} \mathrm{supp}_{\mathsf{T}^c}(j).$$

This notion of support has many desirable properties and is in fact universal amongst such constructions.

Theorem 2.3.10. (Universal property of the spectrum [Bal05, 3.2]) *We have*

1. $\mathrm{supp}_{\mathsf{T}^c}(0) = \varnothing$ *and* $\mathrm{supp}_{\mathsf{T}^c}(\mathbf{1}) = \mathrm{Spc}(\mathsf{T}^c)$.
2. $\mathrm{supp}_{\mathsf{T}^c}(a \oplus b) = \mathrm{supp}_{\mathsf{T}^c}(a) \cup \mathrm{supp}_{\mathsf{T}^c}(b)$.
3. $\mathrm{supp}_{\mathsf{T}^c}(\Sigma a) = \mathrm{supp}_{\mathsf{T}^c}(a)$ *where Σ is the suspension functor for T.*
4. $\mathrm{supp}_{\mathsf{T}^c}(a) \subseteq \mathrm{supp}_{\mathsf{T}^c}(b) \cup \mathrm{supp}_{\mathsf{T}^c}(c)$ *for any exact triangle $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma a$.*
5. $\mathrm{supp}_{\mathsf{T}^c}(a \otimes b) = \mathrm{supp}_{\mathsf{T}^c}(a) \cap \mathrm{supp}_{\mathsf{T}^c}(b)$.

Moreover, for any pair (X, σ), where X is a topological space and σ an assignment of closed subsets $\sigma(t) \subseteq X$ to objects $t \in \mathsf{T}^c$ which satisfy properties (1)-(5) above, there exists a unique continuous map $f : X \longrightarrow \mathrm{Spc}(\mathsf{T}^c)$ such that $\sigma(t) = f^{-1}(\mathrm{supp}_{\mathsf{T}^c}(t))$.

Proposition 2.3.11. [Bal05, 2.14] *Let* T *be a big tt-category. An open subset* $U \subseteq \mathrm{Spc}(\mathsf{T}^c)$ *is quasi-compact if and only if there exists an object* $t \in \mathsf{T}^c$ *such that* $U = U(t)$*. If* $\mathrm{Spc}(\mathsf{T}^c)$ *is a* noetherian *topological space, then every open subset is of the form* $U(t)$ *for some* $t \in \mathsf{T}^c$*.*

In [Bal10] Balmer constructs a locally ringed space structure on the spectrum $\mathrm{Spc}(\mathsf{T}^c)$. We will study certain collections of sheaves of modules on this space. In order to do so we will provide Balmer's construction and adjusted notation.

Throughout we assume T is a big tt-category with compact objects T^c.

Definition 2.3.12. An object u is said to be *invertible* if there exists an object v such that $u \otimes v \cong \mathbf{1}$.

Proposition 2.3.13. *If* $X \in \mathsf{T}$ *is an invertible object, then the inverse* X^{-1} *is isomorphic to the dual* $X^{\vee}$*. Moreover,* X *is rigid and compact.*

Proof. Suppose X is invertible with inverse X^{-1}. Then the functor $X \otimes - : \mathsf{T} \longrightarrow \mathsf{T}$ is an equivalence of categories with inverse $X^{-1} \otimes -$. These functors are therefore both left and right adjoint to one another. Therefore there is an equivalence of representable functors

$$\mathrm{Hom}(X \otimes -, \mathbf{1}) \cong \mathrm{Hom}(-, X^{-1})$$

via the adjunction. From the properties of the internal hom functor we have

$$\mathrm{Hom}(X \otimes -, \mathbf{1}) \cong \mathrm{Hom}(-, \mathrm{hom}(X, \mathbf{1}))$$

and so we conclude that $X^{-1} \cong \mathrm{hom}(X, \mathbf{1}) \cong X^{\vee}$. The rigidity of X is then immediate, as the corresponding counit of adjunction in the evaluation map is just the isomorphism realising $X \otimes X^{-1} \cong \mathbf{1}$. The fact that X is compact then follows by Remark 2.3.4. $\square$

Definition 2.3.14. For any two objects a, b in a big tt-category T, given an invertible object u we define the *twisted homomorphism group*

$$\mathrm{Hom}^{\bullet}_{\mathsf{T}}(a, b) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{T}}(a, u^{\otimes i} \otimes b).$$

In the special case $u = \Sigma \mathbf{1}$ we define the *graded homomorphism group* by

$$\mathrm{Hom}^{*}_{\mathsf{T}}(a, b) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{T}}(a, \Sigma^i b).$$

Remark 2.3.15. For completeness we should include the invertible object u in the notation, but we suppress it here. We rarely deal with more than one invertible object at a time in this setup. Also note that in the case $u = \mathbf{1}$, the twisted homomorphism group is given by

$$\mathrm{Hom}^{\bullet}_{\mathsf{T}}(a, b) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{T}}(a, \mathbf{1}^{\otimes i} \otimes b) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{T}}(a, b),$$

which is a coproduct of countably many copies of the usual homomorphism group.

Definition 2.3.16. [Bal10, 2.1] We define the *central ring* R_{T} to be the endomorphism ring of the tensor unit

$$R_{\mathsf{T}} := \mathrm{End}_{\mathsf{T}}(\mathbf{1}).$$

Given a compact invertible object u we define the *twisted central ring* $R_{\mathsf{T}}^{\bullet}$ to be

$$R_{\mathsf{T}}^{\bullet} := \mathrm{Hom}_{\mathsf{T}}^{\bullet}(\mathbf{1}, \mathbf{1}).$$

Note that this is a special case of Definition 2.3.14 where $a = b = \mathbf{1}$.

Proposition 2.3.17. [Bal10, 3.3] *Let $u \in \mathsf{T}^c$ be invertible. Then there exists $\epsilon_u \in R_{\mathsf{T}}$ such that $R_{\mathsf{T}}^{\bullet}$ is ϵ_u-commutative. That is, given two homogenous elements f and g of orders i and j respectively, we have $fg = \epsilon_u^{ij} gf$.*

Recall from Definition 2.3.8 that the space $\mathrm{Spc}(\mathsf{T}^c)$ has a basis of quasi-compact open subsets.

Construction 2.3.18. [Bal10, 6.1] Let $U \subset \mathrm{Spc}(\mathsf{T}^c)$ be a quasi-compact open subset with closed complement Z. Define $\mathsf{T}_Z^c := \{t \in \mathsf{T}^c \mid \mathrm{supp}(t) \subset Z\}$ to be the thick $\otimes$-ideal of T^c supported outside of U. Define the tensor-triangulated category T^c on U

$$\mathsf{T}^c(U) := (\mathsf{T}^c/\mathsf{T}_Z^c)^{\natural}$$

as the idempotent completion of the Verdier quotient $\mathsf{T}^c/\mathsf{T}_Z^c$. This quotient is the localisation $S^{-1}\mathsf{T}^c$ with respect to $S = \{s : a \longrightarrow b \mid \mathrm{supp}(\mathrm{cone}(s)) \subset Z\}$.

Remark 2.3.19. Note that for every quasi-compact open $U \subset \mathrm{Spc}(\mathsf{T}^c)$ we have $U \cong \mathrm{Spc}(\mathsf{T}^c(U))$ and moreover if V is a quasi-compact subset of U then $(\mathsf{T}^c(U))(V) \cong \mathsf{T}^c(V)$.

By construction we have a natural monoidal functor

$$q_U : \mathsf{T}^c \longrightarrow \mathsf{T}^c(U).$$

Notation 2.3.20. Let X be a topological space with basis $\mathcal{B}$. Recall from Remark 2.2.11 that given a (partially defined) presheaf F on $\mathcal{B}$ we denote the associated sheaf on the whole of X by $F^{\#}$.

Definition 2.3.21. [Bal10, 6.4] For each quasi-compact open $U \subset \mathrm{Spc}(\mathsf{T}^c)$, define the commutative ring ${}_{\mathrm{p}}\mathcal{O}_{\mathsf{T}}(U)$ by

$${}_{\mathrm{p}}\mathcal{O}_{\mathsf{T}}(U) := R_{\mathsf{T}(U)} = \mathrm{Hom}_{\mathsf{T}(U)}(\mathbf{1}_U, \mathbf{1}_U).$$

For an invertible object $u \in \mathsf{T}^c$ define the ϵ_u-commutative graded ring

$${}_{\mathrm{p}}\mathcal{O}_{\mathsf{T}}^{\bullet}(U) := R_{\mathsf{T}(U)}^{\bullet} = \mathrm{Hom}_{\mathsf{T}(U)}^{\bullet}(\mathbf{1}_U, \mathbf{1}_U)$$

where

$$\mathrm{Hom}_{\mathsf{T}(U)}^{\bullet}(\mathbf{1}_U, \mathbf{1}_U) \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{T}(U)}(\mathbf{1}_U, u_U^{\otimes i} \otimes \mathbf{1}_U).$$

These form partially defined presheaves on $\mathrm{Spc}(\mathsf{T}^c)$, only defined on the basis of quasi-compact open sets. The associated sheaves on the space $\mathrm{Spc}(\mathsf{T}^c)$ are denoted

$$\mathcal{O}_{\mathsf{T}} := {}_{\mathrm{p}}\mathcal{O}_{\mathsf{T}}^{\#},$$
$$\mathcal{O}_{\mathsf{T}}^{\bullet} := ({}_{\mathrm{p}}\mathcal{O}_{\mathsf{T}}^{\bullet})^{\#}.$$

We denote the locally ringed space obtained by

$$\mathrm{Spec}(\mathsf{T}) = (\mathrm{Spc}(\mathsf{T}^c), \mathcal{O}_{\mathsf{T}}),$$

and the graded locally ringed space by

$$\mathrm{Spec}^{\bullet}(\mathsf{T}) = (\mathrm{Spc}(\mathsf{T}^c), \mathcal{O}_{\mathsf{T}}^{\bullet}).$$

There is a natural map of locally ringed spaces between the Balmer spectrum of T^c and the spectrum of the commutative ring R_{T}.

Construction 2.3.22. [Bal10, 5.6, 6.10] Let $\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c)$ and define

$$\rho_{\mathsf{T}}^{\bullet}(\mathcal{P}) := \{f \in (R_{\mathsf{T}}^{\bullet})^{\mathrm{hom}} \mid \mathrm{cone}(f) \notin \mathcal{P}\}.$$

By [Bal10, 5.6] $\rho_{\mathsf{T}}^{\bullet}(\mathcal{P})$ is a homogenous prime ideal of $R_{\mathsf{T}}^{\bullet}$. Moreover, $\rho_{\mathsf{T}}^{\bullet} : \mathrm{Spc}(\mathsf{T}^c) \longrightarrow \mathrm{Spec}^{\mathrm{h}}(R_{\mathsf{T}}^{\bullet})$ is continuous and natural in T^c. For every $s \in (R_{\mathsf{T}}^{\bullet})^{\mathrm{even}}$ we define

$$U(s) := U(\mathrm{cone}(s)) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c) \mid \mathrm{cone}(s) \in \mathcal{P}\}.$$

For each distinguished open $D(s)$ in $\mathrm{Spec}(R_{\mathsf{T}})$ we have $U(s) = (\rho_{\mathsf{T}}^{\bullet})^{-1}(D(s))$. By [Bal10, Lemma 6.9] we have

$$\mathcal{O}_{\mathrm{Spec}^{\mathrm{h}}(R_{\mathsf{T}}^{\bullet})}^{\bullet}(D(s)) \cong {}_{\mathrm{p}}\mathcal{O}_{\mathsf{T}}^{\bullet}(U(s))$$

and both are naturally isomorphic to $R_{\mathsf{T}}^{\bullet}[s^{-1}]$. This allows us to construct a ring homomorphism

$$r_{D(s)} : \mathcal{O}_{\mathrm{Spec}^{\mathrm{h}}(R_{\mathsf{T}}^{\bullet})}^{\bullet}(D(s)) \longrightarrow \mathcal{O}_{\mathsf{T}}^{\bullet}(U(s))$$

as the composition of the isomorphism $\mathcal{O}_{\mathrm{Spec}^{\mathrm{h}}(R_{\mathsf{T}}^{\bullet})}^{\bullet}(D(s)) \cong {}_{\mathrm{p}}\mathcal{O}_{\mathsf{T}}^{\bullet}(U(s))$ followed by the sheafification morphism. This construction is compatible with restriction and defines a morphism of ringed spaces

$$(\rho_{\mathsf{T}}^{\bullet}, r) : \mathrm{Spec}^{\bullet}(\mathsf{T}) \longrightarrow \mathrm{Spec}^{\mathrm{h}}(R_{\mathsf{T}}^{\bullet}).$$

Proposition 2.3.23. [Bal10, 6.11] *The map*

$$(\rho_{\mathsf{T}}^{\bullet}, r) : \mathrm{Spec}^{\bullet}(\mathsf{T}) \longrightarrow \mathrm{Spec}^{\mathrm{h}}(R_{\mathsf{T}}^{\bullet})$$

is a map of graded locally ringed spaces. Moreover, if $\rho_{\mathsf{T}}^{\bullet}$ *is a homeomorphism, then* $(\rho_{\mathsf{T}}^{\bullet}, r)$ *is an isomorphism of graded locally ringed spaces. The degree* 0 *restriction of* *this map defines a map of locally ringed spaces*

$$(\rho_{\mathsf{T}}, r) : \mathrm{Spec}(\mathsf{T}) \longrightarrow \mathrm{Spec}(R_{\mathsf{T}})$$

such that if ρ_{T} *is a homeomorphism then* (ρ_{T}, r) *is an isomorphism of locally ringed spaces.*

The proofs for the above require the following useful lemma. We will use more general versions at various points in this work.

Lemma 2.3.24. [Bal10, 6.3] *Let* $\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c)$*, and let* $a, b \in \mathsf{T}^c$*. Let* $\mathcal{U}$ *be the collection of all those quasi-compact open subsets of* $\mathrm{Spc}(\mathsf{T}^c)$ *containing* $\mathcal{P}$*. Then there is a natural isomorphism*

$$\mathrm{colim}_{U \in \mathcal{U}} \mathrm{Hom}_{\mathsf{T}^c(U)}(a, b) \cong \mathrm{Hom}_{\mathsf{T}^c/\mathcal{P}}(a, b).$$

Definition 2.3.25. Given a prime $\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c)$ we denote the closure of $\mathcal{P}$ by $\mathcal{V}(\mathcal{P})$. We define a set $\mathcal{Z}(\mathcal{P})$ by

$$\mathcal{Z}(\mathcal{P}) = \{\mathcal{Q} \in \mathrm{Spc}(\mathsf{T}^c) \mid \mathcal{P} \notin \mathcal{V}(\mathcal{Q})\}.$$

Lemma 2.3.26. *Given a prime* $\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c)$ *there is an equality of sets*

1. $\mathcal{V}(\mathcal{P}) = \{\mathcal{Q} \in \mathrm{Spc}(\mathsf{T}^c) \mid \mathcal{Q} \subseteq \mathcal{P}\}$.
2. $\mathcal{Z}(\mathcal{P}) = \{\mathcal{Q} \in \mathrm{Spc}(\mathsf{T}^c) \mid \mathcal{P} \nsubseteq \mathcal{Q}\}$.

Proof. The proof of the first statement can be found in [Bal05, 2.9]. The second statement follows from the first:

$$\begin{aligned} \mathcal{Q} \in \mathcal{Z}(\mathcal{P}) &\iff \mathcal{P} \notin \mathcal{V}(\mathcal{Q}) \\ &\iff \mathcal{P} \notin \{\mathcal{R} \in \mathrm{Spc}(\mathsf{T}^c) \mid \mathcal{R} \subseteq \mathcal{Q}\} \\ &\iff \mathcal{P} \nsubseteq \mathcal{Q}. \end{aligned}$$

From this we conclude $\mathcal{Z}(\mathcal{P}) = \{\mathcal{Q} \in \mathrm{Spc}(\mathsf{T}^c) \mid \mathcal{P} \nsubseteq \mathcal{Q}\}$, completing the proof. □

Definition 2.3.27. A subset $\mathcal{V} \subset \mathrm{Spc}(\mathsf{T}^c)$ is *Thomason* if $\mathcal{V}$ is a (possibly infinite) union of closed subsets with quasi-compact open complements.

Lemma 2.3.28. *For an essentially small tensor triangulated category* T^c *the following hold for all* $\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c)$.

1. $\mathrm{supp}(\mathcal{P}) = \mathcal{Z}(\mathcal{P})$.
2. $\mathrm{supp}(\mathcal{P}) = \bigcup_{\mathcal{P} \in U} \mathrm{Spc}(\mathsf{T}^c) \setminus U$ *taken over all quasi-compact opens* U *containing* $\mathcal{P}$.
3. $\mathsf{T}^c_{\mathrm{supp}(\mathcal{P})} = \mathcal{P}$.

4. $\mathcal{P} = \bigcup_{\mathcal{P}\in U} \mathsf{T}^c_{\mathrm{Spc}(\mathsf{T}^c)\setminus U}$.

5. *For a Thomason subset* $Z \subseteq \mathrm{Spc}(\mathsf{T}^c)$ *we have* $\mathrm{supp}\,\mathsf{T}^c_Z = Z$.

Proof. 1. We have $\mathcal{S} \in \mathrm{supp}(\mathcal{P})$ iff there exists $p \in \mathcal{P}$ such that $\mathcal{S} \in \mathrm{supp}(p)$, iff $p \notin \mathcal{S}$, i.e $\mathcal{P} \nsubseteq \mathcal{S}$.

2. First note that by Proposition 2.3.11 every quasi-compact open subset of $\mathrm{Spc}(\mathsf{T}^c)$ is of the form $U(t)$ for some object $t \in \mathsf{T}^c$. We obtain the following chain of implications

$$\begin{aligned}\mathcal{Q} \in \mathrm{supp}(\mathcal{P}) &\iff \mathcal{Q} \in \mathrm{supp}(t) \text{ for some } t \in \mathcal{P}\\ &\iff \mathcal{Q} \in \mathrm{Spc}(\mathsf{T}^c) \setminus U(t) \text{ for some } t \in \mathcal{P}\\ &\iff \mathcal{Q} \in \bigcup_{\mathcal{P}\in U} \mathrm{Spc}(\mathsf{T}^c) \setminus U\end{aligned}$$

where the union is over all quasi-compact open subsets containing $\mathcal{P}$.

3. If $t \in \mathcal{P}$ then by definition $\mathrm{supp}(t) \subseteq \mathrm{supp}(\mathcal{P})$ and so $t \in \mathsf{T}^c_{\mathrm{supp}(\mathcal{P})}$. If $t \in \mathsf{T}^c_{\mathrm{supp}(\mathcal{P})}$ then as $\mathrm{supp}(t) \subseteq \mathrm{supp}(\mathcal{P})$ we have $\mathcal{P} \notin \mathrm{supp}(t)$ and so $t \in \mathcal{P}$.

4. Let $t \in \mathcal{P}$ and let $Z = \mathrm{supp}(t)$, which is closed with open complement U. Note that $\mathcal{P} \in U$. It is immediate that $t \in \mathsf{T}^c_Z$ and so $\mathcal{P} \subseteq \bigcup_{\mathcal{P}\in U} \mathsf{T}^c_Z$. For the other inclusion let $t \in \bigcup_{\mathcal{P}\in U} \mathsf{T}^c_Z$. Then there exists an open subset U with closed complement Z such that $\mathcal{P} \in U$ and $t \in \mathsf{T}^c_Z$. That is $\mathrm{supp}(t) \subseteq Z$. Then $\mathcal{P} \notin \mathrm{supp}(t)$ and so $t \in \mathcal{P}$.

5. As Z is Thomason, we have $Z = \bigcup_{\lambda\in\Lambda} \mathrm{Spc}(\mathsf{T}^c) \setminus U_\lambda$ where each open subset U_λ is quasi-compact. By Proposition 2.3.11 there exist objects $t_\lambda \in \mathsf{T}^c$ such that $U_\lambda = U(t_\lambda)$ and so $Z = \bigcup_{\lambda\in\Lambda} \mathrm{Spc}(\mathsf{T}^c) \setminus U(t_\lambda) = \bigcup_{\lambda\in\Lambda} \mathrm{supp}_{\mathsf{T}^c}(t_\lambda)$. Therefore if $\mathcal{P} \in Z$, there exists t_λ such that $\mathcal{P} \in \mathrm{supp}_{\mathsf{T}^c}(t_\lambda) \subseteq Z$. Hence $t_\lambda \in \mathsf{T}^c_Z$ and $\mathcal{P} \in \mathrm{supp}(\mathsf{T}^c_Z)$ and so $Z \subseteq \mathrm{supp}(\mathsf{T}^c_Z)$. Now fix $\mathcal{P} \in \mathrm{supp}\,\mathsf{T}^c_Z$. Then there exists $t \in \mathsf{T}^c_Z$ such that $\mathcal{P} \in \mathrm{supp}(t)$ from which it is immediate that $\mathcal{P} \in Z$. Therefore $\mathsf{T}^c_Z \subseteq Z$, concluding the proof.

□

Definition 2.3.29. The *radical* $\sqrt{\mathcal{J}}$ of a thick $\otimes$-ideal $\mathcal{J}$ is defined to be

$$\sqrt{\mathcal{J}} := \{r \in \mathsf{T} \mid \exists n \geq 1 \text{ such that } r^{\otimes n} \in \mathcal{J}\}.$$

A thick subcategory $\mathcal{J}$ is called *radical* if $\mathcal{J} = \sqrt{\mathcal{J}}$.

Remark 2.3.30. If the category T^c is rigid then every thick $\otimes$-ideal is radical. All thick $\otimes$-ideals are radical if and only if $r \in \mathrm{thick}^{\otimes}(r \otimes r)$ for every object $r \in \mathsf{T}^c$ [Bal05, 4.4].

Proposition 2.3.31. [Bal05, 4.9] *Let $\mathcal{J} \subset \mathsf{T}^c$ be a thick $\otimes$-ideal. Then*

$$\mathsf{T}^c_{\operatorname{supp}(\mathcal{J})} = \sqrt{\mathcal{J}}.$$

§ 2.3.3 | Categorical actions

The definitions so far suggest that one can think of a big tt-category as some kind of bizarre commutative ring. The work of Stevenson in [Ste13] takes this comparison further by introducing the action of a big tt-category on another triangulated category. In the comparison with rings, this is the introduction of modules.

Remark 2.3.32. From now on we will assume that the suspension functor of the big tt-category T is compatible with the tensor product. This is guaranteed when T can be realised as the homotopy category of a finitely presentable stable $\mathbb{E}_2$-monoidal ∞-category.

Definition 2.3.33. [Ste13, Def 3.2] Let T be a big tt-category and K a triangulated category. A *left action* of T on K is a functor

$$* : \mathsf{T} \times \mathsf{K} \longrightarrow \mathsf{K}$$

which is exact in each variable, together with natural isomorphisms

$$a_{X,Y,A} : (X \otimes Y) * A \xrightarrow{\sim} X * (Y * A)$$

and

$$l_A : \mathbf{1} * A \xrightarrow{\sim} A$$

for all $X, Y \in \mathsf{T}$, $A \in \mathsf{K}$, compatible with the biexactness of $(-) * (-)$ and satisfying the following conditions:

1. The associator a satisfies the pentagon condition which asserts that the following diagram commutes for all $X, Y, Z \in \mathsf{T}$ and $A \in \mathsf{K}$

$$\begin{array}{ccccc} & & X * (Y * (Z * A)) & & \\ & \nearrow^{X * a_{Y,Z,A}} & & \nwarrow^{a_{X,Y,Z*A}} & \\ X * ((Y \otimes Z) * A) & & & & (X \otimes Y) * (Z * A) \\ \uparrow{\scriptstyle a_{X,Y\otimes Z,A}} & & & & \uparrow{\scriptstyle a_{X\otimes Y,Z,A}} \\ (X \otimes (Y \otimes Z)) * A & & \longleftarrow & & ((X \otimes Y) \otimes Z) * A \end{array}$$

   where the bottom arrow is the associator of T.

2. The unitor l makes the following squares commute for every $X \in \mathsf{T}$ and $A \in \mathsf{K}$

$$\begin{array}{ccc} X * (\mathbf{1} * A) & \xrightarrow{X * l_A} & X * A \\ {\scriptstyle a_{X,\mathbf{1},A}}\uparrow & & \downarrow{\scriptstyle 1_{X*A}} \\ (X \otimes \mathbf{1}) * A & \longrightarrow & X * A \end{array} \qquad \begin{array}{ccc} \mathbf{1} * (X * A) & \xrightarrow{l_{X*A}} & X * A \\ {\scriptstyle a_{\mathbf{1},X,A}}\uparrow & & \downarrow{\scriptstyle 1_{X*A}} \\ (\mathbf{1} \otimes X) * A & \longrightarrow & X * A \end{array}$$

where the bottom arrows are the right and left unitors of T.

3. For every $A \in \mathsf{K}$ and $r, s \in \mathbb{Z}$ the diagram

$$\begin{array}{ccc} \Sigma^r \mathbf{1} * \Sigma^s A & \xrightarrow{\sim} & \Sigma^{r+s} A \\ \big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle (-1)^{rs}} \\ \Sigma^r(\mathbf{1} * \Sigma^s A) & \xrightarrow{\sim} & \Sigma^{r+s} A \end{array}$$

is commutative, where the left vertical map comes from exactness in the first varianble of the action, the bottom horzontal map is the unitor, and the top map is given by the composite

$$\Sigma^r \mathbf{1} * \Sigma^s A \longrightarrow \Sigma^s(\Sigma^r \mathbf{1} * A) \longrightarrow \Sigma^{r+s}(\mathbf{1} * A) \xrightarrow{l} \Sigma^{r+s} A$$

whose first two maps use exactness in both variables of the action.

4. The functor $*$ distributes over coproducts whenever they exist. That is, for families $\{X_i\}_{i \in I}$ in T and $\{A_j\}_{j \in J}$ in K, and X in T, A in K the canonical maps

$$\coprod_i (X_i * A) \xrightarrow{\sim} (\coprod_i X_i) * A$$

and

$$\coprod_j (X * A_j) \xrightarrow{\sim} X * (\coprod_j A_j)$$

are isomorphisms whenever the coproducts concerned exist.

If T acts on K we will say that K is a T-module.

Definition 2.3.34. Let $\mathsf{L} \subseteq \mathsf{K}$ be a localising (thick) subcategory. We say L is a localising T-*submodule* of K if the functor

$$\mathsf{T} \times \mathsf{L} \xrightarrow{*} \mathsf{K}$$

factors via L. That is, L is closed under the action of T. Given a collection of objects $\mathcal{A}$ in K, we denote by $\mathrm{loc}^*(\mathcal{A})$ (resp. $\mathrm{thick}^*(\mathcal{A})$) the smallest localising (resp. thick) submodule containing $\mathcal{A}$.

We have the following useful lemma.

Lemma 2.3.35. [Ste13, 3.13] *If* T *is generated as a localising subcategory by the tensor unit* $\mathbf{1}$*, then every localising subcategory of* K *is a* T*-submodule.*

§ 2.3.4 | Localisations, residues and big supports

The notion of support first introduced by Balmer is only defined for *compact* objects. The problem now is how to extend the concept of a support to be applicable to *all* objects, while keeping as many of the properties enjoyed by the original support theory as possible. A solution to this problem was defined by Balmer-Favi in [BF11], which was then extended by Stevenson in [Ste13] to the case of actions. As the techniques used in the definition will be used in various future sections, we will present them in some detail. We focus on the three main components:

- Localisations with respect to the topology on $\mathrm{Spc}(\mathsf{T}^c)$.
- Tensor idempotents associated to points in the spectrum.
- Support defined in terms of tensor idempotents.

We begin with the notion of a localisation sequence.

Definition 2.3.36. A *localisation sequence* is a diagram

$$\mathsf{R} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathsf{T} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathsf{S}$$

where both i_* and j_* are fully faithful, and we have equalities $(i_*\mathsf{R})^\perp = j_*\mathsf{S}$ and ${}^\perp(j_*\mathsf{S}) = i_*\mathsf{R}$ where

$$(i_*\mathsf{R})^\perp = \{t \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(i_*r, t) = 0 \text{ for all } r \in \mathsf{R}\}$$

and

$${}^\perp(j_*\mathsf{S}) = \{t \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(t, j_*s) = 0 \text{ for all } s \in \mathsf{S}\}.$$

We call $i_*i^!$ the *acyclisation* functor and j_*j^* as the *localisation* functor.

A localisation sequence provides an abundance of additional information. References for the following statements and their proofs can be found in [Nee01], [BN93], and [Kra09]. Further references can be found in [BF11]. The form of the proposition below can be found in [Ste18a, 2.15].

Proposition 2.3.37. *Given a localisation sequence as in the definition, the following hold:*

1. *The composites j^*i_* and $i^!j_*$ are zero. Moreover the kernel of j^* is* R.
2. *The composite*
$$\mathsf{S} \xrightarrow{j_*} \mathsf{T} \longrightarrow \mathsf{T}/\mathsf{R}$$
is an equivalence. In particular, the Verdier quotient T/R *is locally small and the canonical projection* $\mathsf{T} \longrightarrow \mathsf{T}/\mathsf{R}$ *has a right adjoint.*
3. *For every $X \in \mathsf{T}$ there is a distinguished triangle*
$$i_*i^!X \longrightarrow X \longrightarrow j_*j^*X \longrightarrow \Sigma i_*i^!X.$$

These triangles are functorial and unique in the sense that given any distinguished triangle

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X'$$

with $X' \in \mathsf{R}$ *and* $X'' \in \mathsf{S}$ *there are unique isomorphisms* $X' \cong i_* i^! X$ *and* $X'' \cong j_* j^* X$.

4. *A localisation sequence is completely determined by either of the pairs of adjoint functors* $(i_*, i^!)$ *or* (j_*, j^*).

Lemma 2.3.38. [Kra09, 5.5.1] *Given a localisation sequence as above, the functor* $i^!$ *is coproduct preserving if and only if* j_* *is coproduct preserving.*

Definition 2.3.39. A localisation sequence is called *smashing* if $i^!$ (or equivalently j_*) preserves coproducts. In such a sequence we call R a *smashing subcategory* of T. If R is a tensor ideal then we say it is a *smashing tensor-ideal.*

In particular we will use the following theorem, combining Proposition 2.3.37 with the definition of smashing localisations. This theorem is the work of Miller [Mil92] (for part 1) and Neeman [Nee01] (for parts 2 and 3). This particular form of the theorem is from [BF11, 4.1]

Theorem 2.3.40. [Miller, Neeman] *Let* T *be a big tt-category and let* C *be a thick* $\otimes$*-ideal of* T^c. *Then we have*

$$\begin{array}{ccccc}
\mathsf{C} & \hookrightarrow & \mathsf{T}^c & \longrightarrow & \mathsf{T}^c/\mathsf{C} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{loc}(\mathsf{C}) & \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} & \mathsf{T} & \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} & \mathsf{T}/\operatorname{loc}(\mathsf{C})
\end{array}$$

1. $\operatorname{loc}(\mathsf{C})$ *is a smashing tensor ideal, the bottom row of the diagram is a smashing localisation sequence and* $\operatorname{loc}(\mathsf{C})^c = \operatorname{loc}(\mathsf{C}) \cap \mathsf{T}^c = \mathsf{C}$.

2. $\mathsf{T}/\operatorname{loc}(\mathsf{C})$ *has small hom-sets and is a compactly generated tensor-triangulated category.*

3. T^c/C *fully faithfully embeds into the compact objects of* $\mathsf{T}/\operatorname{loc}(\mathsf{C})$ *and the additive closure of* T^c/C *is exactly* $(\mathsf{T}/\operatorname{loc}(\mathsf{C}))^c$. *That is, if* t *is a compact object in* $\mathsf{T}/\operatorname{loc}(\mathsf{C})$ *then* t *is a summand of an object in* T^c/C.

We can choose C in the theorem to connect it to the topology on $\operatorname{Spc}(\mathsf{T}^c)$ via Thomason subsets.

Given such a Thomason subset $\mathcal{V}$ define the associated thick-tensor ideal $\mathsf{T}^c_{\mathcal{V}} = \{t \in \operatorname{Spc}(\mathsf{T}^c) \mid \operatorname{supp}_{\mathsf{T}^c} t \subseteq \mathcal{V}\}$. Letting $\mathsf{C} = \mathsf{T}^c_{\mathcal{V}}$ and $\Gamma_{\mathcal{V}}\mathsf{T} = \operatorname{loc}(\mathsf{C})$ the above theorem gives us a smashing localisation sequence

$$\Gamma_{\mathcal{V}}\mathsf{T} \underset{i^!}{\overset{i_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{T} \underset{j_*}{\overset{j^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} L_{\mathcal{V}}\mathsf{T}$$

with corresponding acyclisation and localisation functors

$$\Gamma_{\mathcal{V}}(-) = i_* i^!(-) \text{ and } L_{\mathcal{V}}(-) = j_* j^*(-)$$

respectively.

With such a localisation we can now move onto the definition of tensor idempotents associated to points.

Definition 2.3.41. We say a point $x \in \operatorname{Spc}(\mathsf{T}^c)$ is *weakly visible* if there exist Thomason subsets $\mathcal{V}$ and $\mathcal{W}$ of $\operatorname{Spc}\mathsf{T}^c$ such that

$$\mathcal{V} \setminus (\mathcal{V} \cap \mathcal{W}) = \{x\}.$$

That is, the set singleton $\{x\}$ is the intersection of a Thomason subset and the complement of a Thomason subset. We denote the collection of all weakly visible points by $\operatorname{Vis}(\mathsf{T}^c)$.

Definition 2.3.42. We say a space is *weakly noetherian* if every point in the space is weakly visible.

Remark 2.3.43. The definition of “weakly visible” aligns with [BHS21], and coincides with the definition of “visible” in [Ste13]. We will say a point $x \in \operatorname{Spc}(\mathsf{T}^c)$ is *visible* in the sense of Balmer-Favi if the closure $\mathcal{V}(x) = \overline{\{x\}}$ is a Thomason subset. It follows that every visible point in this sense is also weakly visible, as the set $\mathcal{Z}(x) = \{y \in \operatorname{Spc}(\mathsf{T}^c) \mid x \notin \mathcal{V}(y)\}$ is always Thomason, and $\{x\} = \mathcal{V}(x) \setminus (\mathcal{V}(x) \cap \mathcal{Z}(x)$.

Definition 2.3.44. For a weakly visible point $x \in \operatorname{Vis}(\mathsf{T}^c)$ we define a tensor idempotent

$$\Gamma_x \mathbf{1} = \Gamma_{\mathcal{V}}\mathbf{1} \otimes L_{\mathcal{W}}\mathbf{1}.$$

where $\mathcal{V}$, $\mathcal{W}$ are Thomason subsets such that $\{x\} = \mathcal{V} \setminus (\mathcal{V} \cap \mathcal{W})$. By [BF11, Corollary 7.5] any such pairs $(\mathcal{V}_1, \mathcal{W})$ and $(\mathcal{V}_2, \mathcal{W}_2)$ of Thomason subsets will define isomorphic tensor idempotents.

We can now introduce the definition of the support relative to an action.

Definition 2.3.45. Let T act on K. Then for $A \in \mathsf{K}$ we define the support of A to be the set

$$\operatorname{supp}_{(\mathsf{T},*)} A = \{x \in \operatorname{Vis}(\mathsf{T}^c) \mid \Gamma_x A \neq 0\}$$

where $\Gamma_x A$ denotes $\Gamma_x \mathbf{1} * A$. When the action in question is clear we will omit the subscript from the notation for support.

The below properties demonstrate the good behaviour of this support:

Proposition 2.3.46. [Ste13, 5.7] *The support assignment* $\operatorname{supp}_{(\mathsf{T},*)}$ *satisfies the following properties:*

- *given a triangle*
$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$
in K *we have* $\operatorname{supp} B \subseteq \operatorname{supp} A \cup \operatorname{supp} C$;

- *for any* $A \in \mathsf{K}$ *and* $i \in \mathbb{Z}$
$$\operatorname{supp} A = \operatorname{supp} \Sigma^i A;$$

- *given a set-indexed family* $\{A_\lambda\}_{\lambda\in\Lambda}$ *of objects of* K *there is an equality*
$$\operatorname{supp} \coprod_\lambda A_\lambda = \bigcup_\lambda \operatorname{supp} A_\lambda$$
whenever the coproduct on the left exists;

- *the support satisfies the separation axiom. That is, for every specialisation closed subset* $\mathcal{V} \subseteq \operatorname{Vis} \mathsf{T}^c$ *and every object* A *of* K
$$\operatorname{supp} \Gamma_{\mathcal{V}}\mathbf{1} * A = (\operatorname{supp} A) \cap \mathcal{V}$$
$$\operatorname{supp} L_{\mathcal{V}}\mathbf{1} * A = (\operatorname{supp} A) \cap (\operatorname{Vis} \mathsf{T}^c \setminus \mathcal{V}).$$

When considering a big tt-category acting on itself via the tensor product, the following additional properties hold. Note that in the noetherian case, this result is given in [BF11, 7.17].

Proposition 2.3.47. [BHS21, 2.12, 2.18] *Let* T *be a tt-category acting on itself via the tensor product. Suppose in addition that the spectrum* $\operatorname{Spc}(\mathsf{T}^c)$ *is a weakly noetherian topological space. Then*

- *for every compact object* $t \in \mathsf{T}^c$ *we have*
$$\operatorname{supp}_{(\mathsf{T},*)} t = \operatorname{supp}_{\mathsf{T}^c} t.$$

- $\operatorname{supp}_{(\mathsf{T},*)}(0) = \varnothing$ *and* $\operatorname{supp}_{(\mathsf{T},*)}(\mathbf{1}) = \operatorname{Spc}(\mathsf{T}^c)$.

- $\operatorname{supp}_{(\mathsf{T},*)}(t \otimes t') \subseteq \operatorname{supp}_{(\mathsf{T},*)}(t) \cap \operatorname{supp}_{(\mathsf{T},*)}(t')$.

From now on we will simply write supp for any of the support theories detailed so far, unless it is useful to specify further.

We will now introduce the local-to-global principle for a tt-category T acting on a triangulated category K.

Definition 2.3.48. Let T be a tt-category with weakly noetherian spectrum acting on K. We say that $\mathsf{T} \times \mathsf{K} \xrightarrow{*} \mathsf{K}$ satisfies the *local-to-global principle* if for each $A \in \mathsf{K}$
$$\operatorname{loc}^*(A) = \operatorname{loc}^*(\Gamma_x A \mid x \in \operatorname{Spc}(\mathsf{T}^c)).$$

Theorem 2.3.49. [BHS21, 3.21] *Let* T *be a tt-category acting on itself via the tensor product. If the spectrum* $\mathrm{Spc}(\mathsf{T}^c)$ *is noetherian, then the local-to-global principle holds.*

Note that the theorem was originally proved in [Ste13, 6.9] with the assumption of that T possessed a good theory of homotopy colimits through an enhancement. The theorem given above from [BHS21] holds without any such assumption.

Chapter 3

Geometry of associated sheaf functors

§ 3.1 | Constructing associated sheaf functors

Given a big tt-category T we have now seen the construction of locally ringed spaces $\mathrm{Spec}(\mathsf{T}^c)$ and $\mathrm{Spec}^\bullet(\mathsf{T}^c)$. Given an action of T on a triangulated category K we will define an "associated sheaf functor", which associates to each object $A \in \mathsf{K}$ an $\mathcal{O}_\mathsf{T}$-module (and in the graded case an $\mathcal{O}_\mathsf{T}^\bullet$-module). This construction extends Balmer's construction of the locally ringed space structure. We will then show how these sheaves interact with the tt-support theories and under what conditions we can determine their coherence properties.

Throughout this section the action of T on K will be denoted by $*$.

Definition 3.1.1. Consider the spectrum $\mathrm{Spc}(\mathsf{T}^c)$. We define the category $\mathsf{QBasic}(\mathsf{T}^c)$ of quasi-compact open subsets of $\mathrm{Spc}(\mathsf{T}^c)$, with morphisms given by inclusion. Given an abelian category X we define $\mathsf{QBPsh}_\mathsf{X}(\mathrm{Spc}(\mathsf{T}^c))$ to be the category of X-valued presheaves defined over the basis of quasi-compact opens of $\mathrm{Spc}(\mathsf{T}^c)$. That is, the category of contravariant functors

$$F : \mathsf{QBasic}(\mathsf{T}^c)^{\mathrm{op}} \longrightarrow \mathsf{X}.$$

Remark 3.1.2. By Remark 2.2.11, as the quasi-compact open subsets of $\mathrm{Spc}(\mathsf{T}^c)$ are a basis, every $F \in \mathsf{QBPsh}_\mathsf{X}(\mathrm{Spc}(\mathsf{T}^c))$ extends uniquely to a sheaf $F^\# \in \mathrm{Shv}_\mathsf{X}(\mathrm{Spc}(\mathsf{T}^c))$.

Construction 3.1.3. Let $U \subseteq \mathrm{Spc}(\mathsf{T}^c)$ be a quasi-compact open subset with closed complement Z. Recall the definition

$$\Gamma_Z\mathsf{T} = \mathrm{loc}(\mathsf{T}^c_Z) = \mathrm{loc}(\{t \in \mathsf{T}^c \mid \mathrm{supp}(t) \subseteq Z\}).$$

Define

$$\mathsf{T}(U) = \mathsf{T}/\Gamma_Z\mathsf{T}.$$

and

$$\mathsf{K}(U) = \mathsf{K}/(\Gamma_Z\mathsf{T} * \mathsf{K}).$$

If V is a quasi-compact open subset of U then $(\mathsf{K}(U))(V) \simeq \mathsf{K}(V)$.

For a prime ideal $\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c)$ we define

$$\mathsf{K}(\mathcal{P}) = \mathsf{K}/(\mathrm{loc}^{\otimes}(\mathcal{P}) * \mathsf{K}).$$

There is an action of $\mathsf{T}(U)$ on $\mathsf{K}(U)$ induced by the action $*$ of T on K [Ste13, 8.5]. We will also use $*$ to denote this induced action.

Remark 3.1.4. Instead of restricting to a quasi-compact open, we can instead restrict to the complement of a Thomason subset of the spectrum. That is, given a Thomason subset $Z \subseteq \mathrm{Spc}(\mathsf{T}^c)$ with complement U we can repeat the above construction to produce the corresponding categories $\mathsf{T}(U)$ and $\mathsf{K}(U)$. We focus on the case where U is a quasi-compact open, as in this case we have $\mathrm{Spc}(\mathsf{T}(U)^c) \cong U$ [BF07, 1.11].

Definition 3.1.5. Fix an invertible object $u \in \mathsf{T}$. Define a functor

$${}_{\mathrm{p}^\bullet}[-,-](-) : (\mathsf{K}^c)^{\mathrm{op}} \times \mathsf{K} \times \mathsf{QBasic}(\mathrm{Spc}(\mathsf{T}^c))^{\mathrm{op}} \longrightarrow \mathsf{gr\,Ab}$$

by

$${}_{\mathrm{p}^\bullet}[A,B](U) = \mathrm{Hom}^\bullet_{\mathsf{K}(U)}(A_U, B_U),$$

where A_U, B_U are the respective images of A and B under the localisation functor $q_U : \mathsf{K} \longrightarrow \mathsf{K}(U)$, and

$$\mathrm{Hom}^\bullet_{\mathsf{K}(U)}(A_U, B_U) \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(U)}(A_U, (u^{\otimes i} * B)_U)$$

as in Definition 2.3.14. This defines a functor

$${}_{\mathrm{p}^\bullet}[-,-] : (\mathsf{K}^c)^{\mathrm{op}} \times \mathsf{K} \longrightarrow \mathsf{QBPsh}_{\mathsf{gr\,Ab}}(\mathrm{Spc}(\mathsf{T}^c)),$$

which extends uniquely to a functor

$$[-,-]^\bullet : (\mathsf{K}^c)^{\mathrm{op}} \times \mathsf{K} \longrightarrow \mathrm{Shv}_{\mathsf{gr\,Ab}}(\mathrm{Spc}(\mathsf{T}^c)).$$

We call this the *associated sheaf functor*. We say that $[A,B]^\bullet$ is the *sheaf associated to A and B relative to u*.

The *untwisted associated sheaf functor* is constructed by taking the presheaf

$${}_{\mathrm{p}}[A,B](U) = \mathrm{Hom}_{\mathsf{K}(U)}(A_U, B_U)$$

and applying sheafification as above. We denote this untwisted sheaf by $[A,B]^\#$.

Proposition 3.1.6. *The functor* $[-,-]^\bullet : (\mathsf{K}^c)^{\mathrm{op}} \times \mathsf{K} \longrightarrow \mathrm{Shv}_{\mathsf{gr\,Ab}}(\mathrm{Spc}(\mathsf{T}^c))$ *upgrades to a functor*

$$[-,-]^\bullet : (\mathsf{K}^c)^{\mathrm{op}} \times \mathsf{K} \longrightarrow \mathsf{grMod}\mathcal{O}^\bullet_{\mathsf{T}}.$$

Proof. Fix $(A,B) \in (\mathsf{K}^c)^{\mathrm{op}} \times \mathsf{K}$ and a quasi-compact open subset $U \subseteq \mathrm{Spc}(\mathsf{T}^c)$. Consider the sections ${}_{\mathrm{p}^\bullet}[A,B](U) = \mathrm{Hom}^\bullet_{\mathsf{K}(U)}(A_U, B_U)$. We define a graded module structure via the action of T on K. As noted this action restricts to an action of $\mathsf{T}(U)$ on $\mathsf{K}(U)$. Fix $f \in {}_{\mathrm{p}^\bullet}\mathcal{O}^\bullet_{\mathsf{T}}(U)$ with $\deg(f) = i$ and $g \in {}_{\mathrm{p}^\bullet}[A, B](U)$ with $\deg(g) = j$. Explicitly $f \in \mathrm{Hom}_{\mathsf{T}(U)}(\mathbf{1}_U, (u^i \otimes \mathbf{1})_U)$ and $g \in \mathrm{Hom}_{\mathsf{K}(U)}(A_U, (u^j * B)_U)$. Define $f \cdot g$ to be the composite

$$A_U \xrightarrow{\simeq} \mathbf{1}_U * A_U \xrightarrow{f*g} u^i_U * (u^j_U * B_U) \xrightarrow{\simeq} u^{i+j}_U * B_U.$$

This multiplication is compatible with the addition on morphisms, and the grading. Together with the compatibility of the action with the restriction maps this completes the proof. □

We now compute the stalks of these sheaves. To do this we will require that $\mathrm{Spc}(\mathsf{T}^c)$ be a weakly noetherian topological space satisfying the local-to-global principle and that T has a well-behaved theory of homotopy colimits by appealing to some higher structure such as some flavour of model, derivator, or ∞-category (see [Ste13, 6.5] for more discussion).

Remark 3.1.7. By assuming our tt-category has a suitable enhancement, we can expand the notion of homotopy colimit to any diagram of objects, rather than just sequences as in Definition 2.1.7. In Lemma 2.1.30 we saw that compact objects interact well with the basic notion of homotopy colimits. In the presence of a suitable enhancement, more general homotopy colimits are themselves just filtered colimits and so it is automatic that mapping out of a compact object commutes with homotopy colimits.

Lemma 3.1.8. *Let* T *act on* K. *Suppose* $\mathrm{Spc}\,\mathsf{T}^c$ *is a weakly noetherian space and assume that the local-to-global principle holds for the action of* T *on* K. *For a prime* $\mathcal{P} \in \mathrm{Spc}\,\mathsf{T}^c$ *the stalk of the sheaf* $[A, B]^\bullet$ *at* $\mathcal{P}$ *is given by*

$$[A, B]^\bullet_{\mathcal{P}} = \mathrm{Hom}^\bullet_{\mathsf{K}(\mathcal{P})}(A_{\mathcal{P}}, B_{\mathcal{P}})$$

where

$$\mathrm{Hom}^\bullet_{\mathsf{K}(\mathcal{P})}(A_{\mathcal{P}}, B_{\mathcal{P}}) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(\mathcal{P})}(A_{\mathcal{P}}, u^i_{\mathcal{P}} * B_{\mathcal{P}}).$$

Proof. We use the identification $\Gamma_{\mathcal{V}}\mathsf{K} = \Gamma_{\mathcal{V}}\mathsf{T} * \mathsf{K} = \mathsf{T}_{\mathcal{V}} * \mathsf{K}$ from [Ste13, 4.11], where $\mathcal{V}$ is the complement of a quasi-compact open subset U. Now consider all quasi-compact opens U_i containing $\mathcal{P}$, with complements $\mathcal{V}_i$. We first note that inclusions $\mathcal{V}_i \subseteq \mathcal{V}_j$ induce natural morphisms:

$$\begin{array}{ccc} L_{\mathcal{V}_i}B & \longrightarrow & L_{\mathcal{Z}(\mathcal{P})}B \\ & \searrow \quad \nearrow & \\ & L_{\mathcal{V}_j}B & \end{array}$$

which induces a map between $\mathrm{hocolim}\, L_{\mathcal{V}_i}B$ and $L_{Z(\mathcal{P})}B$. This map completes to a triangle

$$\mathrm{hocolim}\, L_{\mathcal{V}_i}B \longrightarrow L_{Z(\mathcal{P})}B \longrightarrow Z \longrightarrow \Sigma\, \mathrm{hocolim}\, L_{\mathcal{V}_i}B.$$

By a slight generalisation of [Bou83], localising subcategories are closed under homotopy colimits and so Z is an object in $L_{\mathcal{Z}(\mathcal{P})}\mathsf{K}$. Therefore $L_{\mathcal{Z}(\mathcal{P})}Z \cong Z$ and by Proposition 2.3.46 we have

$$\operatorname{supp}(B) = \operatorname{supp}(L_{\mathcal{Z}(\mathcal{P})}B) = \operatorname{supp}(B) \cap (\operatorname{Spc}(\mathsf{T}^c) \setminus \mathcal{Z}(\mathcal{P})) \subseteq \operatorname{Spc}(\mathsf{T}^c) \setminus \mathcal{Z}(\mathcal{P}).$$

For each $\mathcal{Q} \notin \mathcal{Z}(\mathcal{P})$ we have

$$\Gamma_{\mathcal{Q}} L_{Z(\mathcal{P})} B \cong \Gamma_{\mathcal{Q}} B.$$

Applying this functor to the above gives a triangle

$$\Gamma_{\mathcal{Q}} \operatorname{hocolim} L_{\mathcal{V}_i} B \longrightarrow \Gamma_{\mathcal{Q}} B \longrightarrow \Gamma_{\mathcal{Q}} Z \longrightarrow \Gamma_{\mathcal{Q}} \Sigma \operatorname{hocolim} L_{\mathcal{V}_i} B.$$

Given that $\Gamma_{\mathcal{Q}} \operatorname{hocolim} L_{\mathcal{V}_i} B \cong \Gamma_{\mathcal{Q}} B$ the first morphism in the triangle is an isomorphism and so $\Gamma_{\mathcal{Q}} Z \cong 0$. As T has weakly noetherian spectrum and we assumed a good theory of homotopy colimits this forces $Z \cong 0$ by the local-to-global principle [Ste13, 6.8]. Therefore $\operatorname{hocolim} L_{\mathcal{V}_i} B \cong L_{Z(\mathcal{P})} B$. Now, for a quasi-compact open subset $U \subseteq \operatorname{Spc}(\mathsf{T}^c)$, recall that there is an induced localisation sequence

$$\Gamma_{\mathcal{V}}\mathsf{K} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{K} \underset{\iota_U}{\overset{q_U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{K}(U) = L_{\mathcal{V}}\mathsf{K}$$

with an equivalence of functors $\iota_U q_U(-) \cong L_{\mathcal{V}}(\mathbf{1}) * (-)$, as in [Ste13, 4.4]. By definition the stalk of $[A,B]^\bullet$ at $\mathcal{P}$ is given by

$$[A,B]^\bullet_{\mathcal{P}} = \operatorname{colim}_{\mathcal{P}\in U_i} \operatorname{Hom}^\bullet_{\mathsf{K}(U_i)}(A_{U_i}, B_{U_i})$$

where the colimit is taken over all quasi-compact open subsets $U_i \subseteq \operatorname{Spc}(\mathsf{T}^c)$ containing $\mathcal{P}$. Hence

$$\begin{aligned}
[A,B]^\bullet_{\mathcal{P}} &\cong \operatorname{colim}_{\mathcal{P}\in U_i} \operatorname{Hom}^\bullet_{\mathsf{K}(U_i)}(A_{U_i}, B_{U_i}),\\
&\cong \operatorname{colim}_{\mathcal{P}\in U_i} \operatorname{Hom}^\bullet_{\mathsf{K}(U_i)}(q_{U_i}A, q_{U_i}B),\\
&\cong \operatorname{colim}_{\mathcal{P}\in U_i} \operatorname{Hom}^\bullet_{\mathsf{K}(U_i)}(A, \iota_{U_i} q_{U_i} B) \text{ by adjunction},\\
&\cong \operatorname{colim}_{\mathcal{P}\in \mathcal{V}_i^c} \operatorname{Hom}^\bullet_{\mathsf{K}}(A, L_{\mathcal{V}_i}B),\\
&\cong \operatorname{Hom}^\bullet_{\mathsf{K}}(A, \operatorname{hocolim} L_{\mathcal{V}_i}B) \text{ by compactness},\\
&\cong \operatorname{Hom}^\bullet_{\mathsf{K}}(A, L_{Z(\mathcal{P})}B) \text{ by the isomorphism in the previous paragraph},\\
&\cong \operatorname{Hom}^\bullet_{\mathsf{K}(\mathcal{P})}(A_{\mathcal{P}}, B_{\mathcal{P}}) \text{ by adjunction.}
\end{aligned}$$

□

Remark 3.1.9. Note that if T acts on itself via the tensor product and $\operatorname{Spc}(\mathsf{T}^c)$ is a noetherian space, then by Theorem 2.3.49 the local-to-global principle immediately holds, without needing to consider an enhancement.

Lemma 3.1.10. *For all $A \in \mathsf{K}^c$ the functor $[A,-]^\bullet$ is homological and coproduct* *preserving.*

Proof. Observe that as $\mathrm{Hom}(A,-)$ is homological and the associated sheaf functor is exact, the functor $[A,-]^\bullet$ must be homological. To prove that $[A,-]^\bullet$ preserves coproducts, fix a quasi-compact open set $U \subseteq \mathrm{Spc}(\mathsf{T}^c)$ and some set-indexed collection of objects $(B_\lambda)_{\lambda\in\Lambda}$ in K. As the localisation sequence used in the construction of $\mathsf{K}(U)$ is smashing [Ste13] the canonical functor $\mathsf{K} \longrightarrow \mathsf{K}(U)$ preserves compact objects. Therefore there is an isomorphism

$$\mathrm{Hom}^\bullet_{\mathsf{K}(U)}(A, \coprod_{\lambda\in\Lambda} B_\lambda) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}^\bullet_{\mathsf{K}(U)}(A, B_\lambda)$$

noting that as $-\otimes-$ is coproduct preserving in each variable, for any invertible object u we have $u \otimes \coprod_{\lambda\in\Lambda} B_\lambda \cong \coprod_{\lambda\in\Lambda}(u\otimes B_\lambda)$. It follows that there is an isomorphism

$$_{\mathrm{p}^\bullet}[A, \coprod_{\lambda\in\Lambda} B_\lambda] \cong \bigoplus_{\lambda\in\Lambda} {}_{\mathrm{p}^\bullet}[A, B_\lambda]$$

and so

$$[A, \coprod_{\lambda\in\Lambda} B_\lambda]^\bullet \cong (\bigoplus_{\lambda\in\Lambda} {}_{\mathrm{p}^\bullet}[A, B_\lambda])^\#.$$

Sheafification of a presheaf is a left adjoint and so preserves coproducts. Therefore

$$\begin{aligned}[A, \coprod_{\lambda\in\Lambda} B_\lambda]^\bullet &\cong \bigoplus_{\lambda\in\Lambda} {}_{\mathrm{p}^\bullet}[A, B_\lambda]^\# \\ &\cong \bigoplus_{\lambda\in\Lambda} [A, B_\lambda]^\bullet.\end{aligned}$$

□

The following lemma shows the effect of twisting by the invertible object at the level of presheaf sections.

Lemma 3.1.11. *Fix an invertible object $u \in \mathsf{T}$, and objects $A \in \mathsf{K}^c$ and $B \in \mathsf{K}$. For all basic open subsets $U \subseteq \mathrm{Spc}(\mathsf{T}^c)$ and for all $i \in \mathbb{Z}$ there are isomorphisms of graded rings*

$$_{\mathrm{p}^\bullet}[u^{-i} * A, B](U) \cong \Big({}_{\mathrm{p}^\bullet}[A,B](U)\Big)(i) \cong {}_{\mathrm{p}^\bullet}[A, u^i * B](U)$$

Proof. Recall that $\Big({}_{\mathrm{p}^\bullet}[A,B](U)\Big)(i)$ denotes the i-twist of ${}_{\mathrm{p}^\bullet}[A,B](U)$ as in Definition 2.2.19. Given a basic open $U \subseteq \mathrm{Spc}(\mathsf{T}^c)$, the sections of the associated presheaves are given by

$$\begin{aligned}
{}_{\mathrm{p}^\bullet}[u^{-i} * A, B](U) &\cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(U)}(u_U^{-i} * A_U, u_U^j * B_U), \\
&\cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(U)}(A_U, u_U^i * (u_U^j * B_U)), \\
&\cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(U)}(A_U, (u_U^i \otimes u_U^j) * B_U), \\
&\cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(U)}(A_U, u_U^{i+j} * B_U), \\
&\cong \Big({}_{\mathrm{p}^\bullet}[A, B](U)\Big)(i).
\end{aligned}$$

Similarly,

$$\begin{aligned}
{}_{\mathrm{p}^\bullet}[A, u^i * B](U) &\cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(U)}(A_U, u_U^j * (u_U^i * B_U)), \\
&\cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(U)}(A_U, (u_U^j \otimes u_U^i) * B_U), \\
&\cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{K}(U)}(A_U, u_U^{i+j} * B_U), \\
&\cong \Big({}_{\mathrm{p}^\bullet}[A, B](U)\Big)(i).
\end{aligned}$$

□

§ 3.2 | Comparison of support theories

Throughout this section we assume that the Balmer spectrum of the tt-category T is a noetherian topological space.

Definition 3.2.1. Fix an action of a big tt-category T on a triangulated category K. Let u be an invertible object in T^c and A an object in K^c. For an object $B \in \mathsf{K}$ we define the *u-twisted A-support* of B by

$$\mathrm{supp}^\bullet(B, A) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c) \mid [A, B]^\bullet_{\mathcal{P}} \neq 0\}.$$

We define the *untwisted A-support of B* by

$$\mathrm{supp}(B, A) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c) \mid [A, B]^{\#}_{\mathcal{P}} \neq 0\}.$$

When considering two distinct invertible objects u_1 and u_2 simultaneously, we will expand the notation and write $\mathrm{supp}^\bullet(B, A, u_1)$ and $\mathrm{supp}^\bullet(B, A, u_2)$ but will only do so to improve readability. Note that $\mathrm{supp}(B, A) = \mathrm{supp}^\bullet(B, A, \mathbf{1})$.

We note the following basic properties:

Proposition 3.2.2. *Given the notion of support defined above we have:*

1. $\mathrm{supp}^\bullet(0, A) = \varnothing$.

2. *Given a triangle* $B \longrightarrow C \longrightarrow D \longrightarrow \Sigma B$ *in* K *we have*

$$\operatorname{supp}^\bullet(C, A) \subseteq \operatorname{supp}^\bullet(B, A) \cup \operatorname{supp}^\bullet(D, A).$$

3. *For any integer* $i \in \mathbb{Z}$ *we have*

$$\operatorname{supp}^\bullet(B, A) = \operatorname{supp}^\bullet(u^i * B, A) = \operatorname{supp}^\bullet(B, u^{-i} * A).$$

4. $\operatorname{supp}^\bullet(\bigoplus_{\lambda\in\Lambda} B_\lambda, A) = \bigcup_{\lambda\in\Lambda} \operatorname{supp}^\bullet(B, A)$,

Proof. 1. As $[A, 0]^\bullet = 0$, all of the stalks vanish at each $\mathcal{P}$ and so the support is empty as required.

2. Let $\mathcal{P} \in \operatorname{supp}^\bullet(C, A)$. Applying the homological functor $[A, -]^\bullet$ followed by the stalk functor at $\mathcal{P}$ gives a long exact sequence

$$\cdots \longrightarrow [A, B]^\bullet_{\mathcal{P}} \longrightarrow [A, C]^\bullet_{\mathcal{P}} \longrightarrow [A, D]^\bullet_{\mathcal{P}} \longrightarrow [A, \Sigma B]^\bullet_{\mathcal{P}} \longrightarrow \cdots$$

If $\mathcal{P} \notin \operatorname{supp}^\bullet(B, A) \cup \operatorname{supp}^\bullet(D, A)$ then as the sequence is exact we would have $[A, C]^\bullet_{\mathcal{P}} = 0$, a contradiction. Therefore $\mathcal{P} \in \operatorname{supp}^\bullet(B, A) \cup \operatorname{supp}^\bullet(D, A)$ and the result holds.

3. Follow immediately from Lemma 3.1.11.

4. As A is compact we have an isomorphism

$$\operatorname{Hom}^\bullet_{\mathsf{K}(\mathcal{P})}(A, \bigoplus_{\lambda\in\Lambda} B_\lambda) \cong \bigoplus_{\lambda\in\Lambda} \operatorname{Hom}^\bullet_{\mathsf{K}(\mathcal{P})}(A, B_\lambda).$$

Using this gives us another chain of implications:

$$\begin{aligned} \mathcal{P} \in \operatorname{supp}^\bullet(\bigoplus_{\lambda\in\Lambda} B_\lambda, A) &\Leftrightarrow \operatorname{Hom}^\bullet_{\mathsf{K}(\mathcal{P})}(A, \bigoplus_{\lambda\in\Lambda} B_\lambda) \neq 0 \\ &\Leftrightarrow \bigoplus_{\lambda\in\Lambda} \operatorname{Hom}^\bullet_{\mathsf{K}(\mathcal{P})}(A, B_\lambda) \neq 0 \\ &\Leftrightarrow \mathcal{P} \in \operatorname{supp}(B_\lambda, A) \text{ for some } \lambda \in \Lambda \\ &\Leftrightarrow \mathcal{P} \in \bigcup_{\lambda\in\Lambda} \operatorname{supp}^\bullet(B_\lambda, A). \end{aligned}$$

□

Remark 3.2.3. Consider the twisted support $\operatorname{supp}^\bullet(B, A)$ with respect to an invertible object u. We can decompose this twisted support into a union of untwisted supports. By definition $\operatorname{Hom}^\bullet_{\mathsf{K}(\mathcal{P})}(A, B) \neq 0$ if and only if there exists $i \in \mathbb{Z}$ such that $\operatorname{Hom}_{\mathsf{K}(\mathcal{P})}(A, u^{\otimes i} * B) \neq 0$. Therefore $[A, B]^\bullet_{\mathcal{P}} \neq 0$ if and only if there exists $i \in \mathbb{Z}$ such that $[A, u^{\otimes i} * B]^{\#}_{\mathcal{P}} \neq 0$. Taking the union over all $i \in \mathbb{Z}$ we obtain

$$\operatorname{supp}^\bullet(B, A) = \bigcup_{i\in\mathbb{Z}} \operatorname{supp}(u^{\otimes i} * B, A).$$

In order to investigate further, we consider local generators relative to each prime ideal in the spectrum.

Definition 3.2.4. Let T act on K. Let $B \in \mathsf{K}$ be an object with non-empty support. Fix a compact object $A \in \mathsf{K}^c$ and a prime ideal $\mathcal{P} \in \operatorname{supp}(B)$. We say that B is *locally generated at* $\mathcal{P}$ *by* A if $B_{\mathcal{P}} \in \operatorname{loc}(A_{\mathcal{P}})$, realised as a full subcategory of the Verdier quotient $\mathsf{K}/\operatorname{loc}(\mathcal{P})$. Let $(A(\mathcal{P}))_{\mathcal{P}\in\operatorname{supp}(B)}$ be a collection of compact objects in K. We say B is *locally generated by* $(A(\mathcal{P}))_{\mathcal{P}\in\operatorname{supp}(B)}$ if it is locally generated at each prime $\mathcal{P} \in \operatorname{supp}(B)$ by $A(\mathcal{P})$. Finally, we say that B is *locally generated by* A if B is locally generated by the constant sequence $(A)_{\mathcal{P}\in\operatorname{supp}(B)}$.

Remark 3.2.5. Note that every compact object is locally generated by itself.

Lemma 3.2.6. *Let* T *act on* K. *Let* B *be an object in* K, *locally generated at* $\mathcal{P} \in \operatorname{supp}(B)$ *by a compact object* $A \in \mathsf{K}^c$. *Then there exists* $i \in \mathbb{Z}$ *such that*

$$\mathcal{P} \in \operatorname{supp}(\Sigma^i B, A)$$

If B *is locally generated by a collection of compact objects* $(A(\mathcal{P}))_{\mathcal{P}\in\operatorname{supp}(B)}$ *then there exists a sequence of integers* $(i(\mathcal{P}))_{\mathcal{P}\in\operatorname{supp}(B)}$ *such that*

$$\operatorname{supp}(B) \subseteq \bigcup_{\mathcal{P}\in\operatorname{supp}(B)} \operatorname{supp}(\Sigma^{i(\mathcal{P})}B, A(\mathcal{P})) \subseteq \bigcup_{\mathcal{P}\in\operatorname{supp}(B)} \operatorname{supp}^{\bullet}(\Sigma^{i(\mathcal{P})}B, A(\mathcal{P})).$$

In particular, if $u = \Sigma\mathbf{1}$ *we have*

$$\operatorname{supp}(B) \subseteq \bigcup_{\mathcal{P}\in\operatorname{supp}(B)} \operatorname{supp}^{\bullet}(B, A(\mathcal{P})).$$

Proof. Suppose B is locally generated at $\mathcal{P}$ by A. First note that $A_{\mathcal{P}}$ is indeed compact. To see this, note that as A is compact and formation of submodules is well-behaved by Lemma 3.12 in [Ste13] and the localisation functor $(-)_{\mathcal{P}}$ gives rise to a smashing localisation sequence as in Theorem 2.3.40, from which the compactness of $A_{\mathcal{P}}$ follows. As $\mathcal{P} \in \operatorname{supp}(B)$, by the adjunction in the associated localisation sequence, the localisation $B_{\mathcal{P}} \in \mathsf{K}/\operatorname{loc}^{\otimes}(\mathcal{P}) * \mathsf{K}$ is non-zero. As $B_{\mathcal{P}} \in \operatorname{loc}(A_{\mathcal{P}})$, by Lemma 2.1.28 there exists $i \in \mathbb{Z}$ such that $\operatorname{Hom}_{\operatorname{loc}(A)}(A_{\mathcal{P}}, \Sigma^i B_{\mathcal{P}}) \neq 0$. As $\operatorname{loc}(A_{\mathcal{P}})$ is full, we have $\operatorname{Hom}_{\operatorname{loc}(A)}(A_{\mathcal{P}}, \Sigma^i B_{\mathcal{P}}) = \operatorname{Hom}_{\mathsf{K}(\mathcal{P})}(A_{\mathcal{P}}, \Sigma^i B_{\mathcal{P}})$. Then $[A, \Sigma^i B]^{\#}_{\mathcal{P}} \neq 0$ and so $\mathcal{P} \in \operatorname{supp}(\Sigma^i B, A(\mathcal{P}))$, proving the first part. Now suppose B is locally generated by a collection of compact objects $(A(\mathcal{P}))_{\mathcal{P}\in\operatorname{supp}(B)}$. By the first part there exists for each prime $\mathcal{P}$ an integer $i(\mathcal{P})$ such that $\mathcal{P} \in \operatorname{supp}(\Sigma^{i(\mathcal{P})}B, A)$. Taking the union over all such supports we obtain

$$\operatorname{supp}(B) \subseteq \bigcup_{\mathcal{P}\in\operatorname{supp}(B)} \operatorname{supp}(\Sigma^{i(\mathcal{P})}B, A(\mathcal{P})) \subseteq \bigcup_{\mathcal{P}\in\operatorname{supp}(B)} \operatorname{supp}^{\bullet}(\Sigma^{i(\mathcal{P})}B, A(\mathcal{P})).$$

where the second inclusion follows from Remark 3.2.3. Finally, when the support is twisted by $u = \Sigma\mathbf{1}$ then the final statement follows from Proposition 3.2.2. □

Remark 3.2.7. If $\mathsf{K} = \mathrm{loc}(A)$ for some compact object A then every object B is locally generated by A. Then the previous lemma can be simplified by replacing all instances of $A(\mathcal{P})$ with just A. In particular when $u = \Sigma\mathbf{1}$ we have

$$\mathrm{supp}(B) \subseteq \mathrm{supp}^{\bullet}(B, A).$$

A natural place to compare support theories is for the case of a tt-category T acting on itself via the tensor product. The support theory defined by Balmer-Favi in this setting agrees with that of Balmer when restricted to compact objects of T. The following lemma pushes our theory of support in the same direction.

Lemma 3.2.8. *For any $A, B \in \mathsf{T}^c$ we have*

$$\mathrm{supp}^{\bullet}(B, A) \subseteq \mathrm{supp}\, B.$$

Proof. Fix $\mathcal{P} \in \mathrm{supp}^{\bullet}(B, A)$. By definition

$$\mathrm{Hom}^{\bullet}_{\mathsf{T}^c/\mathcal{P}}(A, B) \cong [A, B]^{\bullet}_{\mathcal{P}} \neq 0.$$

If $\mathcal{P} \notin \mathrm{supp}(B)$ i.e. $B \in \mathcal{P}$ then $B \cong 0$ in $\mathsf{T}^c/\mathcal{P}$. But then the hom-group would be 0, a contradiction. □

Theorem 3.2.9. *If $\mathsf{T}^c = \mathrm{thick}(A)$ then for all objects $B \in \mathsf{T}^c$ there is an equality*

$$\mathrm{supp}\, B = \bigcup_{i \in \mathbb{Z}} \mathrm{supp}^{\bullet}(\Sigma^i B, A).$$

When $u \cong \Sigma\mathbf{1}$ we obtain

$$\mathrm{supp}\, B = \mathrm{supp}^{\bullet}(B, A).$$

Proof. As $\mathsf{T}^c = \mathrm{thick}(A)$ by Remark 3.2.7 and Lemma 3.2.6 we have

$$\mathrm{supp}(B) \subseteq \bigcup_{\mathcal{P} \in \mathrm{supp}(B)} \mathrm{supp}^{\bullet}(\Sigma^{\psi(\mathcal{P})} B, A) \subseteq \bigcup_{i \in \mathbb{Z}} \mathrm{supp}^{\bullet}(\Sigma^i B, A),$$

which gives the first inclusion. For the reverse inclusion we have

$$\begin{aligned} \mathrm{supp}^{\bullet}(\Sigma^i B, A) &\subseteq \mathrm{supp}(\Sigma^i B) \text{ by Lemma 3.2.8,} \\ &\subseteq \mathrm{supp}(B). \end{aligned}$$

And so we obtain the first equality. The second equality follows immediately as when $u \cong \Sigma\mathbf{1}$ we have $\bigcup_{i \in \mathbb{Z}} \mathrm{supp}^{\bullet}(\Sigma^i B, A) = \mathrm{supp}^{\bullet}(B, A)$. □

We now return to the situation where T acts on K. In particular we will focus on the case where the support is twisted by $\Sigma\mathbf{1}$.

Definition 3.2.10. For an object $B \in \mathsf{K}$ we define the *localising support* by

$$\mathrm{locsupp}(B) = \{\mathcal{P} \in \mathrm{Spc}(T^c) \mid B \notin \mathrm{loc}^{\otimes}(\mathcal{P}) * \mathsf{K}\}.$$

Proposition 3.2.11. *Let B be locally generated by a compact object A. Then*

$$\operatorname{supp}^\bullet(B, A, \Sigma\mathbf{1}) = \operatorname{locsupp}(B).$$

Proof. We obtain the following

$$\begin{aligned} \mathcal{P} \in \operatorname{supp}^\bullet(B, A, \Sigma\mathbf{1}) &\iff \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{K}(\mathcal{P})}(A_\mathcal{P}, \Sigma^i B_\mathcal{P}) \neq 0 \\ &\iff B_\mathcal{P} \neq 0 \text{ in } \mathsf{K}(\mathcal{P}) \\ &\iff B \notin \operatorname{loc}^\otimes(\mathcal{P}) * \mathsf{K} \\ &\iff \mathcal{P} \in \operatorname{locsupp}(B). \end{aligned}$$

Therefore $\operatorname{supp}^\bullet(B, A, \Sigma\mathbf{1}) = \operatorname{locsupp}(B)$ as required. □

Proposition 3.2.12. *Suppose B is locally generated by a compact object A, and fix an invertible object $u \in \mathsf{T}^c$. Then*

$$\operatorname{supp}(B, A, u) \subseteq \operatorname{supp}(B, A, \Sigma\mathbf{1}).$$

Proof. Fix a prime $\mathcal{P}$ such that $\mathcal{P} \notin \operatorname{supp}(B, A, \Sigma\mathbf{1})$. By definition

$$\operatorname{Hom}_{\mathsf{K}(\mathcal{P})}(A_\mathcal{P}, \Sigma^i B_\mathcal{P}) = 0$$

for all $i \in \mathbb{Z}$. As B is locally generated by A, the hom-sets being zero implies that $B_\mathcal{P} = 0$ in the quotient $\mathsf{K}/\operatorname{loc}^\otimes(\mathcal{P}) * \mathsf{K}$. Therefore, for all $i \in \mathbb{Z}$ we have $u^{\otimes i} * B_\mathcal{P} = 0$. In particular $\operatorname{Hom}_{\mathsf{K}(\mathcal{P})}(A_\mathcal{P}, u^{\otimes i} * B_\mathcal{P}) = 0$ and so $\mathcal{P} \notin \operatorname{supp}(B, A, u)$. That is if $\mathcal{P} \notin \operatorname{supp}(B, A, \Sigma\mathbf{1})$ then $\mathcal{P} \notin \operatorname{supp}(B, A, u)$. Therefore we conclude $\operatorname{supp}(B, A, u) \subseteq \operatorname{supp}(B, A, \Sigma\mathbf{1})$ as required. □

§ 3.3 | Almost thick preimages of sheaf categories

We show that in general the collection of objects which are quasi-coherent after sheafification is almost a thick subcategory, with a clear potential obstruction. The proof only relies on the formal property of being a coproduct preserving homological functor and so the intermediate steps are simple.

Convention 3.3.1. Throughout this section $\mathbf{B} \colon \mathsf{T} \longrightarrow \mathsf{A}$ will be a homological, coproduct preserving functor from a big tt-category T to an abelian category A. X will be an additive subcategory of A. We denote the preimage of X under $\mathbf{B}$ by

$$\mathbf{B}^{-1}(\mathsf{X}) = \{C \in \mathsf{T} \mid \mathbf{B}(C) \in \mathsf{X}\}$$

as a full subcategory of T.

We note the following elementary properties of $\mathbf{B}^{-1}(\mathsf{X})$ in the following series of propositions. Proofs are included for completeness.

Proposition 3.3.2. *We have*

1. $\mathbf{B}^{-1}(\mathsf{X})$ *is an additive subcategory of* T.

2. *If* X *is closed under summands then* $\mathbf{B}^{-1}(\mathsf{X})$ *is closed under summands.*

3. *If* X *is cocomplete then* $\mathbf{B}^{-1}(\mathsf{X})$ *is cocomplete.*

Proof. 1. X is additive so contains the zero object 0_{A}. As $\mathbf{B}$ is additive we have $\mathbf{B}(0_{\mathsf{T}}) = 0_{\mathsf{A}}$ and so $0_{\mathsf{T}} \in \mathbf{B}^{-1}(\mathsf{X})$. The preimage is also closed under finite coproducts. Given $A, B \in \mathbf{B}^{-1}(X)$ we have $\mathbf{B}(A \oplus B) \cong \mathbf{B}(A) \oplus \mathbf{B}(B) \in \mathsf{X}$, from which the result follows.

2. Consider an object $A \in \mathbf{B}^{-1}(\mathsf{X})$ such that $A \cong B \oplus C$. By additivity $\mathbf{B}(A) \cong \mathbf{B}(B) \oplus \mathbf{B}(C)$. As X is closed under summands both $\mathbf{B}(B)$ and $\mathbf{B}(C)$ are in X and so $B, C \in \mathbf{B}^{-1}(\mathsf{X})$ as required.

3. Let Y_λ be a collection of objects in $\mathbf{B}^{-1}(\mathsf{X})$. As X is cocomplete it contains the coproduct $\coprod_\lambda \mathbf{B}(Y_\lambda)$. $\mathbf{B}$ is coproduct preserving so $\coprod_\lambda \mathbf{B}(Y_\lambda) \cong \mathbf{B}(\coprod_\lambda Y_\lambda)$. Then $\coprod_\lambda Y_\lambda \in \mathsf{X}$ as required.

□

Remark 3.3.3. Every abelian subcategory is closed under summands, as each summand is the kernel of a projection.

Definition 3.3.4. A full subcategory $\mathsf{W} \hookrightarrow \mathsf{A}$ of an abelian category A is *wide* if it is abelian, closed under extensions, and the kernels and cokernels in W agree with the kernels and cokernels in A.

Proposition 3.3.5. *Suppose* X *is a wide subcategory and consider an exact triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

such that all but Y *are assumed to be in* $\mathbf{B}^{-1}(\mathsf{X})$. *If* $\Sigma^{-1}Z \in \mathbf{B}^{-1}(\mathsf{X})$ *then* $Y \in \mathbf{B}^{-1}(\mathsf{X})$.

Proof. Apply $\mathbf{B}$ to the distinguished triangle given to obtain a long exact sequence

$$\cdots \longrightarrow \mathbf{B}(\Sigma^{-1}Z) \xrightarrow{e} \mathbf{B}(X) \xrightarrow{f} \mathbf{B}(Y) \xrightarrow{g} \mathbf{B}(Z) \xrightarrow{h} \mathbf{B}(\Sigma X) \longrightarrow \cdots$$

We have a short exact sequence

$$0 \longrightarrow \mathrm{coker}(e) \longrightarrow \mathbf{B}(Y) \longrightarrow \ker(h) \longrightarrow 0.$$

As X is abelian and both e and h are morphisms in X, both $\mathrm{coker}(e)$ and $\ker(h)$ are in X. As X is also closed under extensions, this forces $\mathbf{B}(Y) \in \mathsf{X}$ as required. □

Combining the above propositions gives the following corollary:

Corollary 3.3.6. *If* X *is wide and* $\mathbf{B}^{-1}(\mathsf{X})$ *is closed under suspensions, then* $\mathbf{B}^{-1}(\mathsf{X})$ *is thick.*

This continues our general theme of needing additional suspension data to obtain results. We now show that provided we have the right data on suspensions we can just focus on a collection of generators.

Construction 3.3.7. Given a collection of objects $\mathcal{X}$ we can generate the full subcategory thick($\mathcal{X}$) iteratively. Set

$$\begin{aligned} \langle \mathcal{X} \rangle_1 &= \operatorname{add}(\Sigma^k X \mid X \in \mathcal{X}, k \in \mathbb{Z}) \\ \langle \mathcal{X} \rangle_{i+1} &= \operatorname{add}(X \mid \exists\, X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow \Sigma X_1,\ X_1 \in \langle \mathcal{X} \rangle_i, X_2 \in \langle \mathcal{X} \rangle_1) \end{aligned}$$

Then thick$(\mathcal{X}) = \cup_{i=1}^{\infty} \langle \mathcal{X} \rangle_i$. Note that if $X \in \langle \mathcal{X} \rangle_i$, then so is $\Sigma^k X$ for all $k \in \mathbb{Z}$.

Lemma 3.3.8. *Suppose* X *is wide. Given a finite set of objects* $\mathcal{X} = \{x_1, \ldots, x_n\}$ *in* T*, suppose that for all* $k \in \mathbb{Z}$ *and* $j = 1, \ldots, n$ *we have* $\Sigma^k x_j \in \mathbf{B}^{-1}(\mathsf{X})$*. Then*

$$\operatorname{thick}(\mathcal{X}) \subseteq \mathbf{B}^{-1}(\mathsf{X}).$$

Proof. We show that each of the $\langle \mathcal{X} \rangle_i$ are contained in $\mathbf{B}^{-1}(\mathsf{X})$ by induction. If $Y \in \langle \mathcal{X} \rangle_1$ then it is a summand of a coproduct of objects in $\mathbf{B}^{-1}(\mathsf{X})$ and so is also in $\mathbf{B}^{-1}(\mathsf{X})$ by Proposition 3.3.2. The claim therefore holds for $i = 1$. Now suppose that the claim holds for all $i \leq m$. Fix $Y \in \langle \mathcal{X} \rangle_{m+1}$. We have a distinguished triangle

$$X \longrightarrow Y \oplus W \longrightarrow Z \longrightarrow \Sigma X$$

with $X \in \langle \mathcal{X} \rangle_m$ and $Z \in \langle \mathcal{X} \rangle_1$. By induction both X and ΣX are in $\mathbf{B}^{-1}(\mathsf{X})$ and so are both Z and $\Sigma^{-1} Z$. Therefore so is $Y \oplus W$ by Proposition 3.3.5. X is wide so Proposition 3.3.2 tells us $\mathbf{B}^{-1}(\mathsf{X})$ is closed under summands, and so $Y \in \mathbf{B}^{-1}(\mathsf{X})$. □

As an aside we have the following interaction with certain homotopy colimits.

Proposition 3.3.9. *Suppose* X *is cocomplete and wide. Let*

$$Y_0 \xrightarrow{j_1} Y_1 \xrightarrow{j_2} Y_2 \xrightarrow{j_3} \cdots$$

be a sequence in T*. If there is an increasing sequence of integers*

$$0 \leq i_0 < i_1 < i_2 < i_3 < \cdots$$

such that for all i_k *we have that* Y_{i_k} *and* ΣY_{i_k} *are objects of* $\mathbf{B}^{-1}(\mathsf{X})$*, then* $\operatorname{hocolim} Y_i$ *is an object of* $\mathbf{B}^{-1}(\mathsf{X})$*.*

Proof. By Proposition 3.3.2 the coproduct of the Y_{i_k} lies in $\mathbf{B}^{-1}(\mathsf{X})$ as is the coproduct of the ΣY_{i_k}. The homotopy colimit $\operatorname{hocolim} Y_{i_k}$ is defined by the triangle

$$\coprod_{k=0}^{\infty} Y_{i_k} \xrightarrow{1-shift} \coprod_{k=0}^{\infty} Y_{i_k} \longrightarrow \operatorname{hocolim} Y_{i_k} \longrightarrow \Sigma \coprod_{k=0}^{\infty} Y_{i_k}$$

where the shift map is the direct sum of the maps $j_{i_k+1} : X_{i_k} \longrightarrow X_{i_k+1}$. By Proposition 3.3.5 hocolim Y_{i_k} lies in $\mathbf{B}^{-1}(\mathsf{X})$. By [Nee01, 1.7.1] hocolim $Y_i \cong$ hocolim Y_{i_k}, completing the proof. □

We now apply the formalism of these wide subcategories to particular subcategories of sheaves.

Proposition 3.3.10. *Let T be a big tt-category acting on a compactly generated triangulated category K. For each $A \in \mathsf{K}^c$ consider the associated sheaf functor*

$$[A, -]^\bullet : \mathsf{K} \longrightarrow \mathsf{grMod}\mathcal{O}^\bullet_\mathsf{T}.$$

Then this functor fits into the setup of Convention 3.3.1, taking $\mathsf{A} = \mathsf{grMod}\mathcal{O}^\bullet_\mathsf{T}$ and $\mathsf{X} = \mathsf{Coh}^\bullet(\mathrm{Spec}(\mathsf{T}^c))$. Thus the results requiring X abelian hold. Moreover if $\mathrm{Spec}(\mathsf{T}^c)$ is a scheme, setting $\mathsf{X} = \mathsf{QCoh}^\bullet(\mathrm{Spec}(\mathsf{T}^c))$ will also satisfy the required conditions.

Remark 3.3.11. Note that this setup holds for both the twisted and non-twisted case as appropriate.

We can condense this section into the following result:

Corollary 3.3.12. *Suppose $\mathsf{K} = \mathrm{thick}(X)$. If for all $i \in \mathbb{Z}, [A, \Sigma^i X]^\bullet$ is coherent, then for all $B \in \mathsf{K}$ the sheaf $[A, B]^\bullet$ is coherent. If $\mathrm{Spec}(\mathsf{T}^c)$ is a scheme then if for all $i \in \mathbb{Z}, [A, \Sigma^i X]^\bullet$ is quasi-coherent, then for all $B \in \mathsf{K}$ the sheaf $[A, B]^\bullet$ is quasi-coherent.*

Corollary 3.3.13. *Suppose $\mathsf{K} = \mathrm{thick}(\mathbf{1})$ and let $u = \Sigma\mathbf{1}$. If the $\Sigma\mathbf{1}$-twisted structure sheaf $\mathcal{O}^\bullet_\mathsf{T}$ is coherent, then every sheaf of the form $[\mathbf{1}, B]^\bullet$ is coherent.*

Proof. Follows immediately from the previous corollary: if the $\Sigma\mathbf{1}$-twisted structure sheaf $\mathcal{O}^\bullet_\mathsf{T}$ is coherent then each of the sheaves $[\mathbf{1}, \Sigma^i\mathbf{1}]^\bullet$ are coherent. Therefore by the previous corollary each sheaf of the form $[\mathbf{1}, B]^\bullet$ is coherent. □

§ 3.4 | Affine and schematic categories

Our results on (quasi-)coherence so far require us to already know the nature of the suspensions of generators. We now restrict to the case of T acting on itself, and consider sheaves of the form $[\mathbf{1}, X]^\#$. We will show that when the Balmer spectrum of T is well-behaved, all such sheaves are quasi-coherent.

Definition 3.4.1. We say a big tt-category T is *affine* if the natural comparison map

$$\rho : (\mathrm{Spc}(\mathsf{T}^c), \mathcal{O}_\mathsf{T}) \longrightarrow (\mathrm{Spec}(R_\mathsf{T}), \mathcal{O}_{\mathrm{Spec}(R_\mathsf{T})})$$

is an isomorphism of locally ringed spaces. We say that T is *schematic* if there exists an open cover $\{U_i\}$ of $\mathrm{Spc}(\mathsf{T}^c)$ such that for each i the natural comparison maps

$$\rho_i : (\mathrm{Spc}(\mathsf{T}^c(U_i))\mathcal{O}_{\mathsf{T}(U_i)}) \longrightarrow (\mathrm{Spec}(R_{\mathsf{T}(U_i)}), \mathcal{O}_{\mathrm{Spec}(R_{\mathsf{T}(U_i)})})$$

are isomorphisms of locally ringed spaces. In the twisted case we instead consider the corresponding graded morphisms and say T is *twisted affine* and *twisted schematic* respectively.

Remark 3.4.2. Note that by Proposition 2.3.23 the comparison map ρ is a homeomorphism on the underlying topological spaces, then it is automatically a isomorphism of locally ringed spaces.

Immediately from the definitions we see that every affine category is schematic and that the spectrum of a schematic category is a scheme.

Proposition 3.4.3. *If X is a quasi-compact quasi-seperated scheme then $\mathcal{D}^{\mathsf{perf}}(X)$ is a schematic category.*

Proof. As X is a scheme we have an open cover of X by affines U_i. Let $E(U_i)$ be the image of U_i under the homeomorphism $X \cong \mathrm{Spc}(\mathcal{D}^{\mathsf{perf}}(X))$ of [Bal02, 7.3]. Then

$$E(U_i) \cong \mathrm{Spc}(\mathcal{D}^{\mathsf{perf}}(X))(E(U_i)) \cong \mathrm{Spc}(\mathcal{D}^{\mathsf{perf}}(U_i))$$

by [Bal02, 7.8]. By [Bal10, 8.1] each of the $\mathcal{D}^{\mathsf{perf}}(U_i)$ are affine, completing the proof. □

A category being affine is sufficient for all of the associated sheaves $[\mathbf{1}, X]^{\#}$ to be quasi-coherent. We start by considering compact objects.

Proposition 3.4.4. *For a compact object $X \in \mathsf{T}^c$ and any endomorphism $s \in R_{\mathsf{T}}$ we have*

$$_{\mathrm{p}}[\mathbf{1}, X](U(s)) \cong \mathrm{Hom}_{\mathsf{T}^c}(\mathbf{1}, X)[s^{-1}].$$

The proof of this proposition is nearly identical to [Bal10, 6.9]. We will fill in those details left to the reader in [Bal10].

Proof. By definition

$$_{\mathrm{p}}[\mathbf{1}, X](U(s)) = \mathrm{Hom}_{\mathsf{T}^c/\mathsf{T}^c_Z}(\mathbf{1}, X)$$

where Z is the closed complement of $U(s)$. In fact $Z = \mathrm{supp}(\mathrm{cone}(s))$ and clearly

$$\mathrm{supp}(\mathrm{cone}(s)) = \mathrm{supp}(\mathrm{thick}^{\otimes}(\mathrm{cone}(s))).$$

Applying Proposition 2.3.31

$$\mathsf{T}^c_Z = \sqrt{\mathrm{thick}^{\otimes}(\mathrm{cone}(s))}.$$

As T^c is rigid this gives $\mathsf{T}^c_Z = \mathrm{thick}^{\otimes}(\mathrm{cone}(s))$. By [Bal10, 2.16] we have

$$\mathrm{thick}^{\otimes}(\mathrm{cone}(s)) = \mathrm{thick}^{\otimes}(\mathrm{cone}(s^i) \mid i \geq 0).$$

This is the ideal $\mathcal{J}$ associated to the multiplicative set $S = \{s^i \mid i \geq 0\}$ as in [Bal10, 3.5]. Applying [Bal10, 3.6] gives

$$_{\mathrm{p}}[\mathbf{1}, X](U(s)) \cong \mathrm{Hom}_{\mathsf{T}^c/\mathsf{T}^c_Z}(\mathbf{1}, X) \cong \mathrm{Hom}_{\mathsf{T}^c/\mathcal{J}}(\mathbf{1}, X) \cong S^{-1}\,\mathrm{Hom}_{\mathsf{T}^c}(\mathbf{1}, X),$$

and so ${}_{\mathrm{p}}[\mathbf{1}, X](U(s)) \cong \mathrm{Hom}_{\mathsf{T}^c}(\mathbf{1}, X)[s^{-1}]$ as required. □

Proposition 3.4.5. *Suppose* T *is affine. Then for every* $X \in \mathsf{T}^c$ *the sheaf* $[\mathbf{1}, X]^{\#}$ *is quasi-coherent on* $\mathrm{Spec}(\mathsf{T})$*. Explicitly*

$$[\mathbf{1}, X]^{\#} \cong \widetilde{M}$$

where $\widetilde{M}$ *is the sheaf associated to the* R_{T}*-module* $M = \mathrm{Hom}_{\mathsf{T}^c}(\mathbf{1}, X)$*.*

Proof. By Proposition 3.4.4 we have for each $s \in R_{\mathsf{T}}$ an isomorphism

$${}_{\mathrm{p}}[\mathbf{1}, X](U(s)) \cong \mathrm{Hom}_{\mathsf{T}^c}(\mathbf{1}, X)[s^{-1}].$$

By [Har77, Ch.2, 5.1] we also have that

$$\widetilde{M}(D(s)) \cong \mathrm{Hom}_{\mathsf{T}^c}(\mathbf{1}, X)[s^{-1}].$$

Given that ρ is a homeomorphism, $D(s) \cong U(s)$ and so ${}_{\mathrm{p}}[\mathbf{1}, X]$ is isomorphic to $\widetilde{M}$ on a basis of quasi-compact opens. Therefore the associated sheaves are isomorphic which gives $[\mathbf{1}, X]^{\#} \cong \widetilde{M}$. As sheaves associated to modules are quasi-coherent, this completes the proof. □

The result can be extended to non-compact objects using the following theorem.

Theorem 3.4.6. [HPS97, 3.3.7] *Let* T *be a big tt-category. Then given a morphism* $s \in R_{\mathsf{T}}$*, the thick tensor ideal* $\mathcal{J} = \mathrm{thick}^{\otimes}(\mathrm{cone}(s))$ *fits into a localisation diagram*

$$\begin{array}{ccccc}
\mathcal{J} & \hookrightarrow & \mathsf{T}^c & \longrightarrow & \mathsf{T}^c/\mathcal{J} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{loc}(\mathcal{J}) & \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} & \mathsf{T} & \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} & \mathsf{T}/\mathrm{loc}(\mathcal{J})
\end{array}$$

such that for all objects $X \in \mathsf{T}$ *we have*

$$\mathrm{Hom}_{\mathsf{T}/\mathrm{loc}(\mathcal{J})}(\mathbf{1}, X) \cong S^{-1}\,\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X)$$

where $S = \{s^i \mid i \geq 0\}$*.*

Remark 3.4.7. The existence of the localisation diagram can actually be obtained from Theorem 2.3.40.

Immediately from the theorem we obtain:

Corollary 3.4.8. *For any object* $X \in \mathsf{T}$ *and any endomorphism* $s \in R_{\mathsf{T}}$ *we have*

$${}_{\mathrm{p}}[\mathbf{1}, X](U(s)) \cong \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X)[s^{-1}].$$

Corollary 3.4.9. *Suppose* T *is affine. Then for every* $X \in \mathsf{T}$ *the sheaf* $[\mathbf{1}, X]^{\#}$ *is quasi-coherent on* $\mathrm{Spec}(\mathsf{T})$*. Explicitly*

$$[\mathbf{1}, X]^{\#} \cong \widetilde{M},$$

where $\widetilde{M}$ *is the sheaf associated to the* R_{T}*-module* $M = \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X)$.

The proof of this is identical to the compact case.

We can now conclude with the schematic case.

Lemma 3.4.10. *Let* $U \subseteq \mathrm{Spc}(\mathsf{T}^c)$ *be a quasi-compact open subset. Then for objects* $A \in \mathsf{T}^c, B \in \mathsf{T}$ *there is an isomorphism of sheaves on* U*:*

$$[A, B]^{\bullet}|_U \cong [A_U, B_U]^{\bullet},$$

where the associated sheaf functor on the left is over $\mathrm{Spc}(\mathsf{T}^c)$ *and the associated sheaf functor on the right is over* $\mathrm{Spc}(\mathsf{T}^c(U)) \cong U$*. For the untwisted setting we have*

$$[A, B]^{\#}|_U \cong [A_U, B_U]^{\#}.$$

Proof. We prove the twisted case, from which the untwisted case follows. We will show that the corresponding presheaves are isomorphic from which the uniqueness of sheafification will complete the proof. We first show that the presheaves agree on all quasi-compact open subsets $V \subseteq U$. Let $V \subseteq U$ be such a quasi-compact open. Recall that $\mathsf{T}(V) \simeq (\mathsf{T}(U))(V)$. Therefore

$$\begin{aligned} {}_{\mathrm{p}^{\bullet}}[A, B](V) &\cong \mathrm{Hom}^{\bullet}_{\mathsf{T}(V)}((A_U)_V, (B_U)_V), \\ &\cong \mathrm{Hom}^{\bullet}_{\mathsf{T}(V)}(A_V, B_V), \\ &\cong {}_{\mathrm{p}^{\bullet}}[A, B](V), \\ &\cong {}_{\mathrm{p}^{\bullet}}[A, B]|_U(V). \end{aligned}$$

where the last two equalities hold as both V and U are quasi-compact basic open subsets. As these isomorphisms are all compatible with restrictions, the presheaves are isomorphic and therefore so are their sheafifications. □

Theorem 3.4.11. *Let* T *be schematic. Then for every* $X \in \mathsf{T}$*, the sheaf* $[\mathbf{1}, X]^{\#}$ *is quasi-coherent on* $\mathrm{Spec}(\mathsf{T})$.

Proof. As T is schematic we are given an open cover $\{U_i\}$ of $\mathrm{Spc}(\mathsf{T}^c)$, such that each of the $\mathrm{Spc}(\mathsf{T}^c(U_i))$ is affine. Fixing i, restricting from $\mathrm{Spc}(\mathsf{T}^c)$ to U_i and applying the affine result for the affine category $\mathsf{T}(U_i)$ and using Lemma 3.4.10 we have

$$[\mathbf{1}, X]^{\#}|_{U_i} \cong [\mathbf{1}_{U_i}, X_{U_i}]^{\#} \cong \widetilde{M}_i,$$

where $M_i = \mathrm{Hom}_{\mathsf{T}(U_i)}(\mathbf{1}, X)$. This is the definition of a quasi-coherent sheaf on a scheme, completing the proof. □

The definitions of affine and schematic provide a coarse grading through which to examine tensor triangular categories. Taking further inspiration from algebraic geometry, we can attach additional conditions to the schematic structure. In particular we translate the notion of being quasi-affine to the tensor-triangulated setting.

Definition 3.4.12. Let T be a big tt-category. We say that T is *quasi-affine* if there is an open cover $(U_i)_{i\in I}$ of $\mathrm{Spc}(\mathsf{T}^c)$ such that

1. the cover $(U_i)_{i\in I}$ realises T as a schematic category,
2. for each $i \in I$ there exists a morphism $s_i \in R_{\mathsf{T}}$ such that $U_i = U(\mathrm{cone}(s_i))$.

Example 3.4.13. To motivate the definition observe that if $(X, \mathcal{O}_X)$ is a quasi-affine scheme, then the tt-category $\mathsf{D}^{\mathsf{perf}}(X)$ is quasi-affine. Indeed, as X is quasi-affine, the structure sheaf $\mathcal{O}_X$ is ample and so for each point $x \in X$ there exists an element $s \in \Gamma(X, \mathcal{O}_X)$ such that $x \in X_s$. Now by the main result of [Bal02] there is an isomorphism $(\mathsf{D}^{\mathsf{perf}}(X), \mathcal{O}_{\mathsf{D}^{\mathsf{perf}}(X)}) \cong (X, \mathcal{O}_X)$ of schemes, and that the associated comparison map is an isomorphism when restricted to affine subsets. The collection of the X_s, each of which are affine, therefore form a cover realising the schematic structure of $\mathsf{D}^{\mathsf{perf}}(X)$. Now there is an equality

$$\Gamma(X, \mathcal{O}_X) = \mathrm{Hom}_{\mathsf{D}^{\mathsf{perf}}(X)}(\mathcal{O}_X, \mathcal{O}_X) = R_{\mathsf{D}^{\mathsf{perf}}(X)}$$

and moreover $X_s \cong U(\mathrm{cone}(s))$. Therefore the cover realising the schematic property is of the required form and we conclude that $\mathsf{D}^{\mathsf{perf}}(X)$ is quasi-affine.

Proposition 3.4.14. *Let T be quasi-affine. Then the collection*

$$\{U(\mathrm{cone}(s)) \mid s \in R_{\mathsf{T}}\}$$

is a basis for the usual topology on $\mathrm{Spc}(\mathsf{T}^c)$.

Proof. For a morphism f in T^c we will write $U(f) = U(\mathrm{cone}(f))$. As T is quasi-affine there is a collection of elements $\mathcal{S} = (s_i)_i \subseteq R_{\mathsf{T}}$ such that for each index i we have

1. $\mathrm{Spc}(\mathsf{T}^c) = \bigcup_i U(s_i)$
2. The comparison map $\mathrm{Spc}(\mathsf{T}^c(U(s_i))) \xrightarrow{\rho_{U(s_i)}} \mathrm{Spec}(R_{\mathsf{T}(U(s_i))})$ is an isomorphism.

Fix the index i. Let $S = \{\mathrm{id}, s_i, s_i^2, ...\}$ be the multiplicative set generated by s_i. By Proposition 3.4.4 we have

$$R_{\mathsf{T}(U(s_i))} \cong S^{-1}R_{\mathsf{T}}.$$

By assumption the comparison map at $U(s_i)$ is an isomorphism, and so

$$\mathrm{Spc}(\mathsf{T}^c(U(s_i))) \cong \mathrm{Spec}(S^{-1}R_{\mathsf{T}}).$$

A basis for $\mathrm{Spec}(S^{-1}R_{\mathsf{T}})$ is given by the collection

$$\{D(r/s_i^n) \mid r \in R_{\mathsf{T}}, n \in \mathbb{Z}\}.$$

Using the identification $D(s) \cong \mathrm{Spec}(S^{-1}R_{\mathsf{T}})$, one can see that for a distinguished basic open $D(r/s_i^n)$ we have

$$D(r/s_i^n) = D(r) \cap D(s_i) = D(rs_i).$$

As the comparison map is an isomorphism over $\mathsf{T}^c(U(s_i))$ we can take the preimage under $\rho_{U(s_i)}$ to obtain a basis

$$\{U(rs_i) \mid r \in R_{\mathsf{T}}\}$$

for $\mathrm{Spc}(\mathsf{T}^c(U(s_i)))$. We consider the collection

$$\mathcal{B} = \{U(rs_i) \mid r \in R_{\mathsf{T}}, s_i \in \mathcal{S}\}.$$

Unwinding the definition of a basis, one can then observe that $\mathcal{B}$ is a basis for a topology on $\mathrm{Spc}(\mathsf{T}^c)$ and that it coincides with the standard topology generated by the usual basis $\{U(x) \mid x \in \mathsf{T}^c\}$.

□

Corollary 3.4.15. *If T is quasi-affine then the natural comparison map is injective:*

$$\mathrm{Spc}(\mathsf{T}^c) \overset{\rho}{\hookrightarrow} \mathrm{Spec}(R_{\mathsf{T}}).$$

If T is quasi-affine and the unit $\mathbf{1}$ satisfies $\mathrm{Hom}(\mathbf{1}, \Sigma^i \mathbf{1}) = 0$ for all $i > 0$, then the natural comparison map is an isomorphism and T is affine.

Proof. By the previous proposition, if T is quasi-affine then the collection

$$\{U(\mathrm{cone}(s)) \mid s \in R_{\mathsf{T}}\}$$

is a basis for the usual topology on $\mathrm{Spc}(\mathsf{T}^c)$. Therefore by [DS14, Proposition 3.11], the comparison map is injective. For the second part, assume that $\mathrm{Hom}(\mathbf{1}, \Sigma^i \mathbf{1}) = 0$. Then by [Bal10, Theorem 7.13] the comparison map is surjective. Combining this with the quasi-affine assumption we conclude that the comparison map is an isomorphism.

□

We can use affine categories to give concrete examples of the bad interactions between associated sheaf functors and invertibility of objects. First we give an example of an object which is invertible but its associated sheaf is not.

Example 3.4.16. Let R be a commutative ring and consider the category $\mathcal{D}^{\mathsf{perf}}(R)$. Define

$$X = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow \cdots,$$
$$Y = \cdots \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$

with X concentrated in degree 2 and Y concentrated in degree -2. In $\mathcal{D}^{\mathsf{perf}}(R)$ the tensor unit is the complex with R concentrated in degree 0. Clearly $X \otimes Y \cong \mathbf{1}$. As $\mathcal{D}^{\mathsf{perf}}(R)$ is affine we have

$$[\mathbf{1}, X]^{\#} \cong \widetilde{H^0(X)} \cong 0,$$
$$[\mathbf{1}, Y]^{\#} \cong \widetilde{H^0(Y)} \cong 0.$$

Hence the associated sheaves are not invertible.

Now we give an example where X is not invertible but $[\mathbf{1}, X]^\#$ is invertible.

Example 3.4.17. We again consider $\mathcal{D}^{\mathsf{perf}}(R)$ and define

$$X = \cdots \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow \cdots,$$

concentrated in degrees -2 and 0. We have $[\mathbf{1}, X]^\# \cong \widetilde{H^0(X)} \cong \mathcal{O}_{\mathcal{D}^{\mathsf{perf}}(R)}$. Therefore $[\mathbf{1}, X]^\#$ is invertible. However, X is not invertible in $\mathcal{D}^{\mathsf{perf}}(R)$.

§ 3.5 | Mayer-Vietoris covers

There is a process, defined in [BF07], which allows gluing of objects inside a given triangulated category. We show that gluing behaves well with the associated sheaf functors with respect to supports and quasi-coherence. Throughout we consider T a tt-category acting on itself.

Definition 3.5.1. [BF07, 2.1] Let T be a triangulated category. A *formal Mayer-Vietoris cover* of T is a pair of thick triangulated subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathrm{Hom}_{\mathsf{T}}(X_1, X_2) = \mathrm{Hom}_{\mathsf{T}}(X_2, X_1) = 0$ for every pair of objects $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$. We denote by $\mathcal{S}_1 \oplus \mathcal{S}_2$ the thick subcategory whose objects are of the form $X = X_1 \oplus X_2$ with $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$. This setup is referred to as a *Mayer-Vietoris situation.*

Remark 3.5.2. Note that the defintion of a Mayer-Vietoris cover makes no mention of a tensor structure on T. Further applications of these covers can be found in [Rou08].

Such a situation can be presented in a square diagram:

$$\begin{array}{ccc} \mathsf{T} & \longrightarrow & \mathsf{T}/\mathcal{S}_1 \\ \downarrow & & \downarrow \\ \mathsf{T}/\mathcal{S}_2 & \longrightarrow & \mathsf{T}/(\mathcal{S}_1 \oplus \mathcal{S}_2) \end{array}$$

The content of the next theorem was first in [BF07, 4.3] but we present the form given in [BF11, 2.11]

Theorem 3.5.3. *Suppose we have a formal Mayer-Vietoris situation as in the above definition. Assume that the quotients in the associated square diagram have small hom objects. Let $X_1 \in \mathsf{T}/\mathcal{S}_1$ and $X_2 \in \mathsf{T}/\mathcal{S}_2$ be two objects and $\sigma : X_1 \xrightarrow{\sim} X_2$ an isomorphism in $\mathsf{T}/(\mathcal{S}_1 \oplus \mathcal{S}_2)$. Then there exists an object $X \in \mathsf{T}$ and isomorphisms $X \cong X_i$ in $\mathsf{T}/\mathcal{S}_i$ for $i = 1, 2$, compatible with σ in $\mathsf{T}/(\mathcal{S}_1 \oplus \mathcal{S}_2)$. The object X is unique up to possibly non-unique isomorphism and is called a* gluing *of X_1 and X_2 along σ.*

We may denote a gluing as in the theorem by $X = X_1 \cup_\sigma X_2$. One may also glue morphisms together under the same hypotheses [BF07, 3.5].

The gluing of objects can be used in the context of tt-categories:

Corollary 3.5.4. *[BF11, 5.15] If* $\mathrm{Spc}(\mathsf{T}^c) = U_1 \cup U_2$ *with each* U_i *a quasi-compact open, then*

$$\begin{array}{ccc} \mathsf{T} & \longrightarrow & \mathsf{T}(U_1) \\ \downarrow & & \downarrow \\ \mathsf{T}(U_2) & \longrightarrow & \mathsf{T}(U_1 \cap U_2) \end{array}$$

satisfies gluing of objects and morphisms as above.

In general, gluings may not be unique, or are only unique up to non-unique isomorphism. When considering more general covers gluings may not even exist. There are conditions which can be imposed to guarantee existence and uniqueness over finite covers, such as the following theorem.

Theorem 3.5.5. [BF07, 5.13] *Let* $\mathrm{Spc}(\mathsf{T}^c) = U_1 \cup U_2 \cup \cdots \cup U_n$ *be a cover by quasi-compact open subsets for* $n \geq 2$. *Consider objects* $X_i \in \mathsf{T}(U_i)$ *and isomorphisms* $\sigma_{ji} : X_i \xrightarrow{\sim} X_j$ *in* $\mathsf{T}(U_i \cap U_j)$ *satisfying the cocycle condition* $\sigma_{kj}\sigma_{ji} = \sigma_{ki}$ *in* $\mathsf{T}(U_i \cap U_j \cap U_k)$ *for* $1 \leq i, j, k \leq n$. *Assume moreover the following connectivity condition: for any* $i = 2, \ldots, n$ *and for any quasi-compact open* $V \subseteq U_i$, *we suppose that*

$$\mathrm{Hom}_{\mathsf{T}(V)}(\Sigma X_i, X_i) = 0.$$

Then there exists a gluing which is unique up to unique isomorphism.

Remark 3.5.6. As noted in [BF07] it suffices to check the connectivity condition only on those $V \subseteq U_i$ which are unions of intersections of $U_1, \ldots, U_n$.

Definition 3.5.7. We say a collection of objects admits a *connective gluing* if it satisfies the hypotheses of Theorem 3.5.5.

The next two results show that when considering a gluing, the local components contain the expected local information.

Proposition 3.5.8. *Let* $\{U_i\}_{i \in I}$ *be a collection of quasi-compact open subsets of* $\mathrm{Spc}(\mathsf{T}^c)$ *such that* $\mathrm{Spc}(\mathsf{T}^c) = \bigcup_{i \in I} U_i$, *for some index set* I. *Suppose we have a collection of objects* $X_i \in \mathsf{T}(U_i)$ *which admit some gluing* X. *Fix a compact object* A *and define* $A_i = A_{U_i}$. *If* V *is a quasi-compact open subset of* $\mathrm{Spc}(\mathsf{T}^c)$ *then*

$$_{\mathrm{p}^\bullet}[A, X](V) \cong {}_{\mathrm{p}^\bullet}[A_i, X_i](V)$$

for any i *such that* $V \subseteq U_i$.

Proof. By the definition of the gluing we have $X_{U_i} \cong X_i$. Suppose $V \subseteq U_i$. Then

$$\begin{aligned} {}_{\mathrm{p}^\bullet}[A, X](V) &\cong \mathrm{Hom}^\bullet_{\mathsf{T}(V)}(A_V, X_V), \\ &\cong \mathrm{Hom}^\bullet_{\mathsf{T}(V)}((A_{U_i})_V, (X_{U_i})_V), \text{ as } V \subseteq U_i \text{ by Lemma 3.4.10}, \\ &\cong \mathrm{Hom}^\bullet_{\mathsf{T}(V)}((A_i)_V, (X_i)_V), \text{ as } X_{U_i} \cong X_i, \\ &\cong {}_{\mathrm{p}^\bullet}[A_i, X_i](V). \end{aligned}$$

Finally one can choose any i such that $V \subseteq U_i$ because the gluing data guarantees that if $V \subseteq U_i \cap U_j$ we have

$$(X_i)_V \cong (X_{ij})_V \cong (X_j)_V,$$

and so any such i will produce the same result. □

Gluing interacts well with quasi-coherence.

Proposition 3.5.9. *Let* $\mathrm{Spc}(\mathsf{T}^c) = U_1 \cup U_2$ *with* U_i *quasi-compact open,* $i = 1, 2$, *and let* X_1, X_2 *be objects of* T *with an isomorphism* $\sigma : X_1 \xrightarrow{\sim} X_2$ *in* $\mathsf{T}(U_1 \cap U_2)$. *Let* $X = X_1 \cup_\sigma X_2$. *If* $[\mathbf{1}_1, X_1]^\#$ *and* $[\mathbf{1}_2, X_2]^\#$ *are both quasi-coherent, then* $[\mathbf{1}, X]^\#$ *is quasi-coherent.*

Proof. Fix $\mathcal{P} \in \mathrm{Spc}(\mathsf{T}^c)$. Without loss of generality suppose $\mathcal{P} \in U_1$. As $[\mathbf{1}_1, X_1]^\#$ is quasi-coherent there exists an open neighbourhood U of $\mathcal{P}$ such that there is an exact sequence

$$\mathcal{O}_{\mathsf{T}}^I|_U \longrightarrow \mathcal{O}_{\mathsf{T}}^J|_U \longrightarrow [\mathbf{1}_1, X_1]^\#|_U \longrightarrow 0.$$

Moreover we can shrink U to be contained in U_1 and quasi-compact. Now on every quasi-compact open subset V of U_1 we have $[\mathbf{1}, X]^\#(V) \cong [\mathbf{1}_1, X_1]^\#(V)$ and so the given exact sequence for X_1 also holds for X. Similarly one can obtain such a sequence using X_2 in the case where $\mathcal{P} \in U_2$. Therefore $[\mathbf{1}, X]^\#$ is quasi-coherent. □

Proposition 3.5.10. *Let* $\mathrm{Spc}(\mathsf{T}^c) = U_1 \cup U_2$ *and consider some gluing* $X = X_1 \cup_\sigma X_2$ *with* $X_1 \in \mathsf{T}^c(U_1)$ *and* $X_2 \in \mathsf{T}^c(U_2)$. *Then*

$$\mathrm{supp}^\bullet(X, A) \subseteq \mathrm{supp}(X) \subseteq \mathrm{supp}(X_1 \oplus X_2).$$

Proof. The first containment is Lemma 3.2.8. We prove the second containment. Consider a prime $\mathcal{P} \in \mathrm{supp}(X)$, so that we have $X \notin \mathcal{P}$ and so $X_{\mathcal{P}} \neq 0$. As $\mathrm{Spc}(\mathsf{T}^c) = U_1 \cup U_2$ is a cover, we may assume that $\mathcal{P} \in U_1$. By Lemma 2.3.28 we have an equality $\mathcal{P} = \mathsf{T}^c_{\mathrm{supp}(\mathcal{P})} = \mathsf{T}^c_{\bigcup_{U \ni \mathcal{P}} \mathrm{Spc}(\mathsf{T}^c) \setminus U}$ and so $\mathsf{T}^c_{\mathrm{Spc}(\mathsf{T}^c) \setminus U_1} \subseteq \mathcal{P}$. Therefore

$$(X_1)_{\mathcal{P}} \cong (X_{U_1})_{\mathcal{P}} \cong X_{\mathcal{P}} \neq 0.$$

and $X_1 \notin \mathcal{P}$ i.e. $\mathcal{P} \in \mathrm{supp}(X_1)$. We can repeat the argument for each prime, choosing U_1 or U_2 as appropriate. Therefore

$$\mathrm{supp}(X) \subseteq \mathrm{supp}(X_1) \cup \mathrm{supp}(X_2) = \mathrm{supp}(X_1 \oplus X_2).$$

□

Remark 3.5.11. Providing the gluing exists, the above propositions can be extended to finite collections of objects.

The next collection of results show that gluings can be used to determine when the presheaf functors are the same as the associated sheaf functors.

Definition 3.5.12. Let U be a quasi-compact open subset of $\mathrm{Spc}(\mathsf{T}^c)$ and let $\mathcal{U} = \{U_1, \dots, U_n\}$ be an open cover of U by quasi-compact opens. We say an object $X \in \mathsf{T}$ is *1-coconnected over* $\mathcal{U}$ if for every i and every quasi-compact open $V \subseteq U_i$ we have

$$\mathrm{Hom}^{\bullet}_{\mathsf{T}(V)}(\mathbf{1}, \Sigma^{-1}X) = 0,$$

We will say that X is *absolutely 1-coconnected over* U if it is 1-coconnected over every finite cover of U.

Theorem 3.5.13. *Let X be an object of T, U a quasi-compact open subset of $\mathrm{Spc}(\mathsf{T}^c)$ and $\mathcal{U} = \{U_1, \dots, U_n\}$ a cover of U by quasi-compact opens. If X is 1-coconnected over $\mathcal{U}$ then ${}_{\mathrm{p}^{\bullet}}[\mathbf{1}, X]$ satisfies the sheaf condition with respect to this cover. That is, the sequence*

$$0 \longrightarrow {}_{\mathrm{p}^{\bullet}}[\mathbf{1}, X](U) \longrightarrow \bigoplus_i {}_{\mathrm{p}^{\bullet}}[\mathbf{1}, X](U_i) \longrightarrow \bigoplus_{i,j} {}_{\mathrm{p}^{\bullet}}[\mathbf{1}, X](U_{ij})$$

is exact, where $U_{ij} = U_i \cap U_j$.

Proof. Consider the base case $n = 2$, where the cover $\mathcal{U}$ has two opens U_1 and U_2. For each $i \in \mathbb{Z}$ the Mayer-Vietoris sequence of [BF07, Theorem 3.5] gives an exact sequence

$$\mathrm{Hom}_{\mathsf{T}(U_{12})}(\Sigma\mathbf{1}, u^{\otimes i} \otimes X) \longrightarrow \mathrm{Hom}_{\mathsf{T}(U)}(\mathbf{1}, u^{\otimes i} \otimes X) \longrightarrow$$
$$\hookrightarrow \mathrm{Hom}_{\mathsf{T}(U_1)}(\mathbf{1}, u^{\otimes i} \otimes X) \oplus \mathrm{Hom}_{\mathsf{T}(U_2)}(\mathbf{1}, u^{\otimes i} \otimes X) \longrightarrow \mathrm{Hom}_{\mathsf{T}(U_{12})}(\mathbf{1}, u^{\otimes i} \otimes X).$$

By assumption X is 1-coconnected over $\mathcal{U}$ so for all i, $\mathrm{Hom}_{\mathsf{T}(U_{12})}(\Sigma\mathbf{1}, u^{\otimes i} \otimes X) = 0$. For each i the sequence becomes

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{T}(U)}(\mathbf{1}, u^{\otimes i} \otimes X) \longrightarrow$$
$$\hookrightarrow \mathrm{Hom}_{\mathsf{T}(U_1)}(\mathbf{1}, u^{\otimes i} \otimes X) \oplus \mathrm{Hom}_{\mathsf{T}(U_2)}(\mathbf{1}, u^{\otimes i} \otimes X) \longrightarrow \mathrm{Hom}_{\mathsf{T}(U_{12})}(\mathbf{1}, u^{\otimes i} \otimes X).$$

Therefore on the presheaf we have an exact sequence

$$0 \longrightarrow {}_{\mathrm{p}^{\bullet}}[\mathbf{1}, X](U) \longrightarrow {}_{\mathrm{p}^{\bullet}}[\mathbf{1}, X](U_1) \oplus {}_{\mathrm{p}^{\bullet}}[\mathbf{1}, X](U_2) \longrightarrow {}_{\mathrm{p}^{\bullet}}[\mathbf{1}, X](U_{12}).$$

Therefore the sheaf condition holds on the cover $\mathcal{U}$. Now suppose the result holds for all cases $n < m$ with $m > 2$. Consider the case $n = m$. Given a cover $\mathcal{U} = \{U_i\}_{i=1}^m$ let $V = \bigcup_{i=2}^n U_i$ and consider the cover $U = U_1 \cup V$. By induction the sheaf condition holds on this cover of U and the given cover of V. Therefore the condition holds for the original cover $\mathcal{U}$ and the result holds. □

Corollary 3.5.14. *Let U be a quasi-compact open subset of $\mathrm{Spc}(\mathsf{T}^c)$ and suppose $X \in \mathsf{T}$ is absolutely 1-coconnected over U. Then ${}_{\mathrm{p}^{\bullet}}[\mathbf{1}, X]$ verifies the sheaf condition at U and* *hence*

$$[\mathbf{1}, X]^\bullet(U) = {}_{\mathrm{p}^\bullet}[\mathbf{1}, X](U) = \mathrm{Hom}^\bullet_{\mathsf{T}(U)}(\mathbf{1}, X).$$

Proof. Recall that the sheaf condition is that for every finite cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of U the sequence

$$0 \longrightarrow {}_{\mathrm{p}^\bullet}[\mathbf{1}, X](U) \longrightarrow \bigoplus_i {}_{\mathrm{p}^\bullet}[\mathbf{1}, X](U_i) \longrightarrow \bigoplus_{i,j} {}_{\mathrm{p}^\bullet}[\mathbf{1}, X](U_{ij})$$

is exact, where $U_{ij} = U_i \cap U_j$. As X is assumed to be absolutely 1-coconnected over U, it is 1-coconnected with respect to each finite cover $\mathcal{U}$. Therefore by Theorem 3.5.13 the given sequence is exact for every finite cover of U, and therefore the sheaf condition is satisfied. Therefore we have an equality of sections

$$[\mathbf{1}, X]^\bullet(U) = {}_{\mathrm{p}^\bullet}[\mathbf{1}, X](U) = \mathrm{Hom}^\bullet_{\mathsf{T}(U)}(\mathbf{1}, X)$$

as required.

□

We have seen that a gluing of objects can lead to good properties in the associated sheaves. However, a gluing of the associated sheaves does not give a gluing of objects even in the simplest cases.

Example 3.5.15. Consider $\mathsf{T}^c = \mathcal{D}^{\mathsf{perf}}(R)$ for some commutative ring R. This is an affine category and moreover for each object X we have $[\mathbf{1}, X]^\# \cong \widetilde{H^0(X)}$. Consider the following two perfect complexes:

$$X = \cdots \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$
$$Y = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots,$$

where X has R in degrees -2 and 0, and Y has R in degree 0 only. Both complexes have the same zeroth cohomology and so there is an isomorphism of untwisted sheaves $[\mathbf{1}, X]^\# \cong [\mathbf{1}, Y]^\#$. However they are not isomorphic in $\mathcal{D}^{\mathsf{perf}}(R)$ as they have non-isomorphic homology in degree -2. Therefore consider the trivial cover $U_1 = U_2 = \mathrm{Spc}(\mathsf{T}^c)$. We have

$$\mathsf{T}^c(U_1 \cap U_2) \simeq \mathsf{T}^c(\mathrm{Spc}(\mathsf{T}^c)) \simeq \mathsf{T}^c,$$

and the existence of a gluing of X to Y would imply that $X \cong Y$ in $\mathcal{D}^{\mathsf{perf}}(R)$, a contradiction. Therefore the existence of a gluing of sheaves does not provide a gluing objects.

§ 3.6 | Interactions with t-structures

We now consider the case where T is equipped with a t-structure. For more details on t-structures see [BBD82].

Definition 3.6.1. Let T be a triangulated category with full subcategories $\mathsf{T}^{\leq 0}$ and $\mathsf{T}^{\geq 0}$. Denote $\mathsf{T}^{\leq n} = \Sigma^{-n}\mathsf{T}^{\leq 0}$ and $\mathsf{T}^{\geq n} = \Sigma^{-n}\mathsf{T}^{\geq 0}$. A *t-structure* on T is the data $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ such that

1. There are containments

$$\mathsf{T}^{\leq -1} \subseteq \mathsf{T}^{\leq 0} \text{ and } \mathsf{T}^{\geq 1} \subseteq \mathsf{T}^{\geq 0}.$$

2. For any $X \in \mathsf{T}^{\leq 0}, Y \in \mathsf{T}^{\geq 1}$, $\mathrm{Hom}_{\mathsf{T}}(X, Y) = 0$.

3. For any $X \in \mathsf{T}$ there exists a distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \longrightarrow \Sigma X_0$$

   such that $X_0 \in \mathsf{T}^{\leq 0}$ and $X_1 \in \mathsf{T}^{\geq 1}$.

A t-structure is *non-degenerate* if

$$\bigcap_{n\in\mathbb{Z}} \mathsf{T}^{\leq n} = \bigcap_{n\in\mathbb{Z}} \mathsf{T}^{\geq n} = 0.$$

The *heart* of the t-structure is the full subcategory

$$\mathsf{T}^{\heartsuit} = \mathsf{T}^{\leq 0} \cap \mathsf{T}^{\geq 0}.$$

The following are standard results on t-structures:

Proposition 3.6.2. *Let* $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ *be a t-structure on* T*. Then the following hold:*

1. *The heart* $\mathsf{T}^{\heartsuit}$ *is an abelian category.*

2. *Consider the natural inclusion functors* $i^{\leq n} : \mathsf{T}^{\leq n} \longrightarrow \mathsf{T}$ *and* $i^{\geq n} : \mathsf{T}^{\geq n} \longrightarrow \mathsf{T}$*. There exist functors* $\tau^{\leq n} : \mathsf{T} \longrightarrow \mathsf{T}^{\leq n}$ *and* $\tau^{\geq n} : \mathsf{T} \longrightarrow \mathsf{T}^{\geq n}$ *such that*

$$i^{\leq n} \dashv \tau^{\leq n} \text{ and } \tau^{\geq n} \dashv i^{\geq n}.$$

Explicitly

$$\mathrm{Hom}_{\mathsf{T}}(i^{\leq n}X, Y) \cong \mathrm{Hom}_{\mathsf{T}^{\leq n}}(X, \tau^{\leq n}Y),$$
$$\mathrm{Hom}_{\mathsf{T}}(X, i^{\geq n}Y) \cong \mathrm{Hom}_{\mathsf{T}^{\geq n}}(\tau^{\geq n}X, Y).$$

Definition 3.6.3. Given a t-structure $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ we define the n-th cohomology functor $H^n : \mathsf{T} \longrightarrow \mathsf{T}^{\heartsuit}$ by

$$H^n(X) = \tau^{\geq 0}\tau^{\leq 0}\Sigma^n X.$$

We give the following theorem from [HKM02] showing that a t-structure can be built from a compact object under suitable conditions.

Definition 3.6.4. We say an object $X \in \mathsf{T}$ is *connective* if

$$\mathrm{Hom}_{\mathsf{T}}(X, \Sigma^n X) = 0 \text{ for all } n > 0.$$

If $U \subseteq \mathrm{Spc}(\mathsf{T}^c)$ is a quasi-compact open subset then we say that an object X is *locally connective over* U if $\mathrm{Hom}_{\mathsf{T}(U)}(X, \Sigma^n X) = 0$ for all $n > 0$. If $\mathcal{U} = (U_i)_{i \in I}$ is an open cover of $\mathrm{Spc}(\mathsf{T}^c)$ with each U_i quasi-compact, then we say that X is *locally connective over* $\mathcal{U}$ if X is locally connective over each of the U_i.

Theorem 3.6.5. [HKM02, 1.3] *Let* T *be a triangulated category with arbitrary coproducts,* C *a connective compact object, and* $B = \mathrm{End}_{\mathsf{T}}(C)^{\mathrm{op}}$. *Define*

$$\mathsf{T}^{\leq n} = \{X \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(C, \Sigma^i X) = 0 \textit{ for } i > n\}$$
$$\mathsf{T}^{\geq n} = \{X \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(C, \Sigma^i X) = 0 \textit{ for } i < n\}$$

Let $\mathsf{T}^{\heartsuit} = \mathsf{T}^{\leq 0} \cap \mathsf{T}^{\geq 0}$. *If* $\{\Sigma^i C \mid i \in \mathbb{Z}\}$ *is a generating set, then the following hold:*

1. $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ *is a non-degenerate t-structure on* T.
2. *The functor*
$$\mathrm{Hom}_{\mathsf{T}}(C, -) : \mathsf{T}^{\heartsuit} \longrightarrow \mathsf{Mod}\, B,$$
is an equivalence of categories.

Definition 3.6.6. Let C be a connective compact object of T. Let $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ be the t-structure of Theorem 3.6.5. We say that $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ is the t-structure *connectively generated by* C. We say that C *connectively generates* $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$.

Observe that if T is generated by the tensor unit $\mathbf{1}$ and $\mathbf{1}$ is locally connective with respect to some cover $\mathcal{U}$, then for each $U \in \mathcal{U}$ the tt-category $\mathsf{T}(U)$ can be equipped with the t-structure connectively generated by $\mathbf{1}$. We denote the heart of this structure by $\mathsf{T}(U)^{\heartsuit}$.

We now detail conditions under which these local t-structures interact well with the corresponding localisation functors.

Proposition 3.6.7. *Let* T *be a big tt-category, generated by the tensor unit* $\mathbf{1}$. *Consider a morphism* $s \in R_{\mathsf{T}} := \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \mathbf{1})$ *and the quasi-compact open subset* $U = U(\mathrm{cone}(s))$.

1. *For an object* $X \in \mathsf{T}$, *if* $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X) \cong 0$ *then* $\mathrm{Hom}_{\mathsf{T}(U)}(\mathbf{1}_U, X_U) = 0$.
2. *If* $\mathbf{1}$ *is connective in* T *then* $\mathbf{1}_U$ *is connective in* $\mathsf{T}(U)$. *As a consequence, if* $\mathbf{1}$ *connectively generates a t-structure on* T *then* $\mathbf{1}_U$ *connectively generates a t-structure on* $\mathsf{T}(U)$.

3. *If* $\mathbf{1}$ *is connective then for all* $i \in \mathbb{Z}$ *we have the following commutative diagrams:*

$$\begin{CD} \mathsf{T} @>{(-)_U}>> \mathsf{T}(U) \\ @V{\tau^{\leq i}}VV @VV{\tau_U^{\leq i}}V \\ \mathsf{T}^{\leq i} @>{(-)_U}>> \mathsf{T}(U)^{\leq i} \end{CD} \quad \textit{and} \quad \begin{CD} \mathsf{T} @>{(-)_U}>> \mathsf{T}(U) \\ @V{\tau^{\geq i}}VV @VV{\tau_U^{\geq i}}V \\ \mathsf{T}^{\geq i} @>{(-)_U}>> \mathsf{T}(U)^{\geq i} \end{CD}$$

In other words, truncation and localisation commute.

Proof. 1. By Theorem 3.4.6 we have an isomorphism

$$\mathrm{Hom}_{\mathsf{T}(U)}(\mathbf{1}_U, X_U) \cong S^{-1}\,\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X)$$

where the right term is the localisation of the module $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X)$ with respect to the multiplicative set associated to s. By assumption $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X)$ is zero, and therefore so is the localisation. Therefore $\mathrm{Hom}_{\mathsf{T}(U)}(\mathbf{1}, X) \cong 0$.

2. The unit $\mathbf{1}$ is connective and so $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \Sigma^n \mathbf{1}) \cong 0$ for all $n > 0$. Applying part (1) to each of the $\Sigma^n \mathbf{1}$ we have that for each $n > 0$, $\mathrm{Hom}_{\mathsf{T}(U)}(\mathbf{1}, \Sigma^n \mathbf{1}) \cong 0$. In other words, $\mathbf{1}$ is locally connective over U and therefore connectively generates a t-structure on $\mathsf{T}(U)$.

3. First note that by part (2), as $\mathbf{1}$ is connective it makes sense to consider the t-structure $(\mathsf{T}(U)^{\leq 0}, \mathsf{T}(U)^{\geq 0})$ connectively generated by $\mathbf{1}_U$. Fix $X \in \mathsf{T}$ and $i \in \mathbb{Z}$. We first prove that

$$(\tau^{\leq i} X)_U \in \mathsf{T}(U)^{\leq i} \text{ and } (\tau^{\geq i} X)_U \in \mathsf{T}(U)^{\geq i}.$$

Consider $\mathrm{Hom}_{\mathsf{T}(U)}(\mathbf{1}_U, \Sigma^j (\tau^{\leq i} X)_U)$. As $\tau^{\leq i} X \in \mathsf{T}^{\leq i}$ we have $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \Sigma^j \tau^{\leq i} X) = 0$ for all $j > i$. Therefore by (1) $\mathrm{Hom}_{\mathsf{T}(U)}(\mathbf{1}_U, \Sigma^j (\tau^{\leq i} X)_U) = 0$ for all $j > i$ so by definition $(\tau^{\leq i} X)_U \in \mathsf{T}(U)^{\leq i}$. A similar proof provides the second part of the claim. Now consider the diagrams

$$\begin{CD} \mathsf{T} @>{(-)_U}>> \mathsf{T}(U) \\ @V{\tau^{\leq i}}VV @VV{\tau_U^{\leq i}}V \\ \mathsf{T}^{\leq i} @>{(-)_U}>> \mathsf{T}(U)^{\leq i} \end{CD} \quad \text{and} \quad \begin{CD} \mathsf{T} @>{(-)_U}>> \mathsf{T}(U) \\ @V{\tau^{\geq i}}VV @VV{\tau_U^{\geq i}}V \\ \mathsf{T}^{\geq i} @>{(-)_U}>> \mathsf{T}(U)^{\geq i} \end{CD}$$

The proof of the first claim shows that the bottom arrows in each diagram do in fact land in $\mathsf{T}(U)^{\leq i}$ and $\mathsf{T}(U)^{\geq i}$ respectively.

Recall that the t-structure on T induces a unique triangle

$$\tau^{\leq i}X \longrightarrow X \longrightarrow \tau^{\geq i+1}X \longrightarrow \Sigma\tau^{\leq i}X.$$

Applying the localisation funtor $(-)_U$ gives a triangle

$$(\tau^{\leq i}X)_U \longrightarrow X_U \longrightarrow (\tau^{\geq i+1}X)_U \longrightarrow (\Sigma\tau^{\leq i}X)_U.$$

Now by the previous lemma $(\tau^{\leq i}X)_U \in \mathsf{T}(U)^{\leq i}$ and $(\tau^{\geq i+1}X)_U \in \mathsf{T}(U)^{\geq i+1}$. Therefore the above triangle is isomorphic to the canonical triangle induced by the t-structure on $\mathsf{T}(U)$, given by the following diagram:

$$\begin{array}{ccccccc}
(\tau^{\leq i}X)_U & \longrightarrow & X_U & \longrightarrow & (\tau^{\geq i+1}X)_U & \longrightarrow & (\Sigma\tau^{\leq i}X)_U \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tau_U^{\leq i}(X_U) & \longrightarrow & X_U & \longrightarrow & \tau_U^{\geq i+1}(X_U) & \longrightarrow & \Sigma\tau_U^{\leq i}(X_U)
\end{array}$$

As all of the vertical arrows are isomorphisms we have

$$(\tau^{\leq i}X)_U \cong \tau_U^{\leq i}(X_U) \text{ and } (\tau^{\geq i+1}X)_U \cong \tau_U^{\geq i+1}(X_U)$$

which completes the proof.

□

Remark 3.6.8. As an immediate consequence we have

$$H_U^0(X_U) \cong (H^0(X))_U$$

as the homology functor is defined by the truncation functors.

Our work on affine categories allows us to define this t-structure in terms of sheaves associated to objects. We work in the untwisted setting.

Lemma 3.6.9. *Let T be an affine category and suppose that the unit $\mathbf{1}$ connectively generates a t-structure on T. Then:*

1. *For all $n \in \mathbb{Z}$:*
   (a) $\mathsf{T}^{\leq n} = \{X \in \mathsf{T} \mid [\mathbf{1}, \Sigma^i X]^{\#} = 0 \ i > n\}$.
   (b) $\mathsf{T}^{\geq n} = \{X \in \mathsf{T} \mid [\mathbf{1}, \Sigma^i X]^{\#} = 0 \ i < n\}$.
   (c) $\mathsf{T}^{\heartsuit} = \{X \in \mathsf{T} \mid [\mathbf{1}, \Sigma^i X]^{\#} = 0 \ i \neq 0\}$.
2. *Let $X \in \mathsf{T}^c$. Then*
   (a) *If $X \in \mathsf{T}^{\leq n}$ then* $\operatorname{supp} X = \bigcup_{i\leq n} \operatorname{supp}(\Sigma^i X, \mathbf{1})$.
   (b) *If $X \in \mathsf{T}^{\geq n}$ then* $\operatorname{supp} X = \bigcup_{i\geq n} \operatorname{supp}(\Sigma^i X, \mathbf{1})$.
   (c) *If $X \in \mathsf{T}^{\heartsuit}$ then* $\operatorname{supp} X = \operatorname{supp}(X, \mathbf{1})$.

Proof. As T is affine the sheaf $[\mathbf{1}, \Sigma^i X]^\#$ is equivalent to the sheaf associated to the module $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \Sigma^i X)$. For (1), by Theorem 3.6.5 we have

$$\mathsf{T}^{\leq n} = \{X \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \Sigma^i X) = 0 \; i > n\}.$$

As T is affine, for each $i > n$, $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \Sigma^i X) = 0$ if and only if $[\mathbf{1}, \Sigma^i X]^\# = 0$. Therefore $\mathsf{T}^{\leq n} = \{X \in \mathsf{T} \mid [\mathbf{1}, \Sigma^i X]^\# = 0 \; i > n\}$, proving (1)(a). Similar proofs give us (1)(b) and (1)(c). For (2) note that by Remark 3.2.3 and Theorem 3.2.9 we have for compact X equalities

$$\operatorname{supp} X = \operatorname{supp}^\bullet(X, \mathbf{1}) = \bigcup_{i \in \mathbb{Z}} \operatorname{supp}(\Sigma^i X, \mathbf{1}),$$

where $\operatorname{supp}^\bullet(X, \mathbf{1})$ is twisted with respect to $\Sigma\mathbf{1}$. Now if $X \in \mathsf{T}^{\leq n}$, by part (1)(a) $[\mathbf{1}, \Sigma^i X]^\# = 0$ for $i < n$. Therefore by definition $\operatorname{supp}(\Sigma^i X, \mathbf{1}) = \varnothing$ for all $i < n$ and so $\bigcup_{i \in \mathbb{Z}} \operatorname{supp}(\Sigma^i x, \mathbf{1}) = \bigcup_{i \leq n} \operatorname{supp}(\Sigma^i X, \mathbf{1})$, proving (2)(a). Similar proofs provide the results in (2)(b) and (2)(c), completing the proof. □

We want to investigate when an object can be analysed via an appropriate object in the heart of the t-structure, as in the following definition.

Definition 3.6.10. Given an object $X \in \mathsf{T}$ we say that an object $X^\heartsuit \in \mathsf{T}^\heartsuit$ is a *hearty replacement* for X if

$$[\mathbf{1}, X]^\# = [\mathbf{1}, X^\heartsuit]^\#.$$

We say that X can be *heartily replaced* by $X^\heartsuit$.

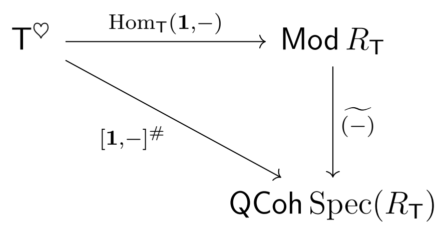
When the comparison map of spectra is an isomorphism then the heart of the t-structure encodes information about associated sheaf functors in the obvious way.

Proposition 3.6.11. *Suppose* T *is affine and that the unit object connectively generates a t-structure. Then*

- $\mathsf{T}^\heartsuit \simeq \mathsf{QCoh}\,\mathrm{Spec}(R_{\mathsf{T}})$.
- *Each object* $X \in \mathsf{T}$ *has a hearty replacement* $X^\heartsuit$. *That is, for an object* $X \in \mathsf{T}$ *there exists an object* $X^\heartsuit \in \mathsf{T}^\heartsuit$ *such that*

$$[\mathbf{1}, X]^\# \cong [\mathbf{1}, X^\heartsuit]^\#.$$

Proof. Consider the following diagram:

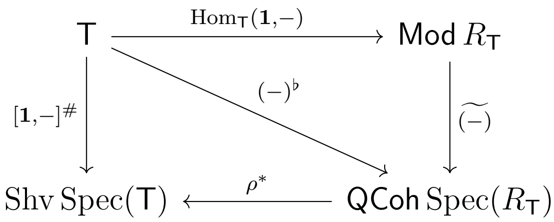


The top arrow is an equivalence by Theorem 3.6.5. The vertical arrow is an equivalence as $\widetilde{\mathrm{Spec}(R_{\mathsf{T}})}$ is an affine scheme. As T is affine, for any object $X \in \mathsf{T}$ we have $[\mathbf{1}, X]^{\#} \cong \widetilde{\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X)}$ i.e. $[\mathbf{1}, -]^{\#} \cong \widetilde{(-)} \circ \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, -)$. Therefore our diagram commutes and the final diagonal arrow is an equivalence, as required. For the second part, note that $[\mathbf{1}, X]^{\#}$ is quasi-coherent for all $X \in \mathsf{T}$ as T is affine, so via the equivalence in our diagram there must exist $X^{\heartsuit} \in \mathsf{T}^{\heartsuit}$ such that $[\mathbf{1}, X]^{\#} \cong [\mathbf{1}, X^{\heartsuit}]^{\#}$ as claimed. □

Suppose instead that T is not affine. Then the diagram of Proposition 3.6.11 may not be commutative. Define a functor

$$(-)^{\flat} = \widetilde{\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, -)}$$

and consider the diagram



By definition the upper triangle commutes. If the lower triangle commuted then $[\mathbf{1}, X]^{\#}$ would be quasi-coherent for all $X \in \mathsf{T}$. We show that under mild conditions the lower triangle fails to be commutative.

Proposition 3.6.12. *Suppose that* $\mathrm{Spc}(\mathsf{T}^c)$ *contains a non-zero prime ideal and that*

$$\mathrm{Hom}_{\mathsf{T}^c}(\mathbf{1}, \mathbf{1}) = k$$

is a field. Then

$$[\mathbf{1}, -]^{\#} \ncong \rho^*(-)^{\flat}.$$

Proof. Consider an sheaf $\mathcal{F}$ in $\mathsf{Mod}\, \mathcal{O}_k$. Such an object is in the additive closure of the structure sheaf and so the pull back $\rho^*\mathcal{F}$ must be in the additive closure of $\mathcal{O}_{\mathsf{T}}$. Every object in this closure must be supported everywhere. Now consider an object $M \in \mathcal{P}$, where $\mathcal{P}$ is a non-zero prime ideal in $\mathrm{Spc}(\mathsf{T}^c)$. The associated sheaf $[\mathbf{1}, M]^{\#}$ is not supported at $\mathcal{P}$ and so $[\mathbf{1}, M]^{\#}$ is not in the additive closure of $\mathcal{O}_{\mathsf{T}}$. Therefore $[\mathbf{1}, M]^{\#}$ is not the pullback of a $\mathcal{O}_k$-module along ρ and so $[\mathbf{1}, -]^{\#} \ncong \rho^*(-)^{\flat}$. □

In the affine case we have the following connections between the cohomology functor and the associated sheaves.

Lemma 3.6.13. *If* T *affine then for an object* X *we have*

1. $[\mathbf{1}, X]^{\#} \cong [\mathbf{1}, \tau^{\leq 0} X]^{\#}$.

2. $[\mathbf{1}, H^0 X]^{\#} \cong [\mathbf{1}, \tau^{\geq 0} X]^{\#}$.

3. *If* $X \in \mathsf{T}^{\geq 0}$ *then* $[\mathbf{1}, X]^{\#} \cong [\mathbf{1}, H^0 X]^{\#}$.

Proof. 1. First note that as the unit $\mathbf{1}$ is connective, $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \Sigma^i \mathbf{1}) = 0$ for all $i > 0$. In other words, $\mathbf{1} \in \mathsf{T}^{\leq 0}$, which is a full subcategory. Using the adjunction between the associated truncation and inclusion functors for $\mathsf{T}^{\leq 0}$ we obtain the following string of isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \tau^{\leq 0} X) &\cong \mathrm{Hom}_{\mathsf{T}^{\leq 0}}(\mathbf{1}, \tau^{\leq 0} X), \\ &\cong \mathrm{Hom}_{\mathsf{T}}(i^{\leq 0}\mathbf{1}, X), \\ &\cong \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X). \end{aligned}$$

By assumption T is affine. Combining this fact with the above isomorphism we conclude

$$[\mathbf{1}, X]^{\#} \cong \widetilde{\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X)} \cong \widetilde{\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \tau^{\leq 0} X)} \cong [\mathbf{1}, \tau^{\leq 0} X]^{\#}.$$

2. By definition $H^0 X = \tau^{\leq 0} \tau^{\geq 0} X$. Using part (1) we obtain

$$\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, H^0 X) = \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \tau^{\leq 0}\tau^{\geq 0} X) \cong \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \tau^{\geq 0} X).$$

Again, as T is affine this gives us $[\mathbf{1}, H^0 X]^{\#} \cong [\mathbf{1}, \tau^{\geq 0} X]^{\#}$ as required.

3. If $X \in \mathsf{T}^{\geq 0}$ then $X \cong \tau^{\geq 0} X$ and so

$$[\mathbf{1}, X]^{\#} \cong [\mathbf{1}, \tau^{\geq 0} X]^{\#} \cong [\mathbf{1}, H^0 X]^{\#},$$

where the second isomorphism is from (2).

□

In our hunt for hearty replacements the previous lemma states that if $X \in \mathsf{T}^{\geq 0}$ then the zeroth cohomology $H^0 X$ is a hearty replacement for X. For T affine we can extend this to all objects in T and upgrade the assignment $(-)^{\heartsuit}$ to a functor via the cohomology functor.

Proposition 3.6.14. *Suppose that* T *has the t-structure connectively generated by* $\mathbf{1}$. *Then for all* $X \in \mathsf{T}$ *we have*

$$\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X) \cong \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, H^0 X),$$

and if T *is affine we have*

$$[\mathbf{1}, X]^{\#} \cong [\mathbf{1}, H^0 X]^{\#},$$

where H^0 *is the cohomology functor induced by the t-structure connectively generated by* $\mathbf{1}$. *We can choose the functor* $H^0(-)$ *as our assignment* $(-)^{\heartsuit}$.

Proof. For $X \in \mathsf{T}$ consider the canonical triangle

$$\tau^{\leq -1} X \longrightarrow X \longrightarrow \tau^{\geq 0} X \longrightarrow \Sigma \tau^{\leq -1} X.$$

Applying the homological functor $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, -)$ produces a long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \tau^{\leq -1}X) \longrightarrow \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X) \longrightarrow \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \tau^{\geq 0}X) \longrightarrow \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \Sigma\tau^{\leq -1}X) \longrightarrow \cdots$$

As $\tau^{\leq -1}X \in \mathsf{T}^{\leq -1}$ we have $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \tau^{\leq -1}X) \cong 0$ and $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \Sigma\tau^{\leq -1}X) \cong 0$. Therefore we obtain the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X) \longrightarrow \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \tau^{\geq 0}X) \longrightarrow 0$$

which by exactness gives $\mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, X) \cong \mathrm{Hom}_{\mathsf{T}}(\mathbf{1}, \tau^{\geq 0}X)$. As we assumed T affine this gives $[\mathbf{1}, X]^{\#} \cong [\mathbf{1}, \tau^{\geq 0}X]^{\#}$. Then by Lemma 3.6.13 (2) we have

$$[\mathbf{1}, X]^{\#} \cong [\mathbf{1}, \tau^{\geq 0}X]^{\#} \cong [\mathbf{1}, H^0X]^{\#}.$$

□

Remark 3.6.15. While Proposition 3.6.14 applies to affine tt-categories, issues arise when trying to extend it to schematic categories. Indeed, Proposition 3.6.7 only guarantees that the cohomology and localisation functors are well behaved when localising over an open of the form $U(\mathrm{cone}(s))$ for some $s \in R_{\mathsf{T}}$. A quasi-affine tt-category T could therefore be a good candidate to extend the proposition. Unfortunately, requiring that the unit $\mathbf{1}$ connectively generate a t-structure on T means that the hypotheses of Corollary 3.4.15 are satisfied, and so our candidate quasi-affine category is in fact affine.

Theorem 3.6.16. *Let T be an affine category such that the tensor unit $\mathbf{1}$ generates T and connectively generates a t-structure on T. Then for all objects $X \in \mathsf{T}$ we have*

$$\mathrm{supp}^{\bullet}(X, \mathbf{1}) = \bigcup_{i \in \mathbb{Z}} \mathrm{supp}(H^i(X), \mathbf{1}),$$

where the support on the left is twisted by the invertible object $\Sigma\mathbf{1}$.

Proof. By the previous proposition, we identify the associated sheaf of an object to that of its zeroth cohomology. Applying this to the definition of the $\Sigma\mathbf{1}$-twisted support we obtain the following equalities:

$$\begin{aligned}
\mathrm{supp}^{\bullet}(X, \mathbf{1}) &= \bigcup_{i \in \mathbb{Z}} \mathrm{supp}(\Sigma^i X, \mathbf{1}), \text{ by Remark 3.2.3,} \\
&= \bigcup_{i \in \mathbb{Z}} \mathrm{supp}(H^0 \Sigma^i X, \mathbf{1}), \text{ by the previous theorem,} \\
&= \bigcup_{i \in \mathbb{Z}} \mathrm{supp}(H^i(X), \mathbf{1}),
\end{aligned}$$

completing the proof. □

§ 3.7 | Applications and examples

§ 3.7.1 | TT-categories from lattices

In this section we will produce a tt-category from a bounded lattice L.

Definition 3.7.1. A *lattice* L is a poset $(L, \leq)$ such that every pair of elements admits a *greatest lower bound/infimum/meet* $\wedge$, and a *least upper bound/supremum/join* $\vee$.

Definition 3.7.2. Let L be a lattice. We say L is *bounded* if there exists a minimal element 0 and a maximal element $\mathbf{1}$. The lattice L is *complete* if arbitrary subsets of L admit both meets and joins. We say L is *distributive* if for all $\ell_1, \ell_2, \ell_3 \in L$ we have

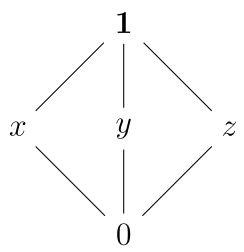
$$\ell_1 \wedge (\ell_2 \vee \ell_3) = (\ell_1 \wedge \ell_2) \vee (\ell_1 \wedge \ell_3).$$

Definition 3.7.3. A lattice L is a *frame* if it is complete and satisfies

$$\ell \wedge (\bigvee_{i \in I} m_i) = \bigvee_{i \in I} (\ell \wedge m_i)$$

for all $\ell \in L$ and any collection of elements $\{m_i\}_{\{i \in I\}}$ of L indexed by some set I.

Example 3.7.4. Consider the lattice



As it is finite, it is complete and bounded, with maximal element $\mathbf{1}$ and minimal element 0. However, it is not distributive (and so is not a frame). Indeed,

$$x \wedge (y \vee z) = x \wedge \mathbf{1} = x.$$

while on the other hand we have

$$(x \wedge y) \vee (x \wedge z) = 0 \vee 0 = 0.$$

Example 3.7.5. Let X be a topological space and consider the collection of all open subsets of X, denoted by $\Omega(X)$. Then $\Omega(X)$, ordered by inclusion, is a frame. The join is given by union $\cup$, while the meet is obtained by $\operatorname{int} \cap$, taking the (open) interior of the intersection. The minimal element is the empty set $\varnothing$ and the maximal element is the whole space X.

Definition 3.7.6. Let P be a poset. A subset $S \subseteq P$ is *downwards closed* if $s \in S$ and $\ell \in P$ such that $\ell \leq s$ then $\ell \in S$. For an element $x \in P$ define the *downwards* *closure of* x by

$$\downarrow x = \{\ell \in P \mid \ell \leq x\}.$$

Given a subset $S \subseteq P$ define the *downwards closure of* S by

$$\downarrow S = \bigcup_{x \in S} \downarrow x.$$

If S is a subset of P, then S is downwards closed if and only if $S = \downarrow S$.

Definition 3.7.7. Let $(P, \leq)$ be a poset. A subset $I \subset P$ is an *ideal* if all of the following hold:

1. $I \neq \varnothing$
2. I is downwards closed.
3. For all $x, y \in I$ there exists $z \in I$ such that $x \leq z$ and $y \leq z$

Definition 3.7.8. An ideal I of a lattice $(P, \leq)$ is *prime* if all of the following hold:

1. I is a proper ideal of P.
2. For all $x, y \in P$, if $x \wedge y \in I$ then $x \in I$ or $y \in I$.

Now that we have primes, the spectrum is not far behind.

Definition 3.7.9. Let $(P, \leq)$ be a poset. The spectrum of P is the set:

$$\mathrm{Spec}(P) = \{I \subseteq P \mid I \text{ is a prime ideal}\}$$

We define a topology on $\mathrm{Spec}(P)$ using a basis of open subsets of the form

$$D_P(x) = \{I \in \mathrm{Spec}(P) \mid x \notin I\}$$

Given a poset, downward closure produces another poset.

Definition 3.7.10. Let $(P, \leq)$ be a poset. Let $\mathrm{Dcl}(P)$ denote the collection of all non-empty downwards closed subsets of P. That is,

$$\mathrm{Dcl}(P) = \{S \subseteq P \mid S = \downarrow S\}.$$

Lemma 3.7.11. *Let L be a bounded lattice. Then the collection of all non-empty downwards closed subsets* $\mathrm{Dcl}(L)$*, ordered by inclusion, is a frame.*

Proof. The fact that $\mathrm{Dcl}(L)$ is a poset with respect to inclusion is immediate. The set P is the maximal element of $\mathrm{Dcl}(P)$, while the singleton set $\{0\}$ is the minimal element. Therefore $\mathrm{Dcl}(L)$ is bounded. The meet is intersection, and the join is union. As the arbitrary intersection and arbitrary union of downwards closed sets are again downwards closed, and intersection and union of sets satisfies the frame condition, we conclude that $\mathrm{Dcl}(L)$ is a frame. □

We will now consider a tt-category constructed from a monoid. This construction can be found in [Ste18b]. This is relevant because the data of a bounded lattice L defines a monoid, where the binary operation is given by the meet $\wedge$ and the maximal element $\mathbf{1}$ is the identity element of the operation.

Construction 3.7.12. Let $(M, *, \mathbf{1})$ be a commutative monoid with zero element 0_M and fix a field k. Consider the category

$$T_M = \prod_{m \in M \setminus \{0\}} \mathsf{D}(k_m)$$

where $k_m = k$ for all $k \in M \setminus \{0\}$, the category $\mathsf{D}(k_m)$ is the unbounded derived category of all k_m-vector spaces, and the product is taken in the category of additive categories. The objects in T_M are M-graded objects of $\mathsf{D}(k)$, and so each object can be thought of as the direct sum of each of its graded pieces. The zero object in T_M can be identified with k_{0_M} The category T_M inherits a triangulated structure from the usual triangulated structure on $\mathsf{D}(k)$ applied levelwise. T_M can then be equipped with tensor product by defining

$$k_{m_1} \otimes k_{m_2} = k_{m_1 * m_2}$$

which extends to an exact and coproduct preserving tensor product on the whole of T_M, with tensor unit $k_{\mathbf{1}}$.

Remark 3.7.13. Note that if $\mathcal{Q}$ is an ideal of the tt-category T_M^c, then $\mathcal{Q}$ is uniquely determined by the generators k_m it contains. More precisely

$$\mathcal{Q} = \text{thick}^{\otimes}(m \in M \mid k_m \in \mathcal{Q}).$$

In fact, the same is true for localising ideals. That is, if $\mathcal{Q}$ is a localising ideal, then

$$\mathcal{Q} = \text{loc}^{\otimes}(m \in M \mid k_m \in \mathcal{Q}).$$

Given a bounded lattice L, we obtain a monoid $(L, \wedge, \mathbf{1})$ and then apply the above construction to obtain the tt-category T_L.

Lemma 3.7.14. *Every thick tensor ideal of T_L is radical.*

Proof. By the construction of T_L^c it suffices to check $k_x \in \text{thick}^{\otimes}(k_x \otimes k_x)$ for each $k_x \in \mathsf{T}^c$ and we conclude by Remark 2.3.30. But $k_x \otimes k_x \cong k_{x \wedge x} \cong k_x$ and the condition holds. □

Remark 3.7.15. The category T_L^c gives an example of a tt-category which is not rigid, but all of the thick tensor ideals are radical.

Definition 3.7.16. Let L be a bounded lattice. Define a function

$$\gamma\colon\ \text{Spc}(T_L^c) \longrightarrow \mathcal{P}(\text{Dcl}\, L)$$

by

$$\gamma(\mathcal{Q}) = \{\downarrow A \mid A \subseteq \{\ell \in L \mid k_\ell \in \mathcal{Q}\}\}.$$

The fact that this function is well-defined follows immediately from the previous remark.

Lemma 3.7.17. *Let $\mathcal{Q} \in \mathrm{Spc}(T_L^c)$. Then $\gamma(\mathcal{Q})$ is a proper ideal of the frame* $\mathrm{Dcl}(L)$.

Proof. We check each of the three conditions making up the definition of an ideal.

1. Note that for every ideal $\mathcal{Q}$, we have $\{0\} \subset \{\ell \in L \mid k_\ell \in \mathcal{Q}\}$ and so $\{0\} \in \gamma(\mathcal{Q})$. In particular, $\gamma(\mathcal{Q})$ is non-empty.

2. First we observe that $\{\ell \in L \mid k_\ell \in \mathcal{Q}\}$ is downwards closed. Indeed if $f \leq \ell$ with $k_\ell \in \mathcal{Q}$, then
$$k_f = k_{f\wedge \ell} \cong k_f \otimes k_\ell \in \mathcal{Q}$$
and so $f \in \{\ell \in L \mid k_\ell \in \mathcal{Q}\}$. Now let $S \in \gamma(\mathcal{Q})$ and $R \in \mathrm{Dcl}(L)$ such that $R \subseteq S$. Then there exists a subset $A \subseteq \{\ell \in L \mid k_\ell \in \mathcal{Q}\}$ such that $R \subseteq \downarrow A$. Then for each $r \in R$ there exists $a \in A$ such that $r \leq a$, and so by the previous argument $k_r \in \mathcal{Q}$ and so $R \subseteq \{\ell \in L \mid k_\ell \in \mathcal{Q}\}$ and we conclude that $R \in \gamma(\mathcal{Q})$.

3. For $S, R \in \gamma(\mathcal{Q})$, clearly $S, R \subseteq\downarrow \{\ell \in L \mid k_\ell \in \mathcal{Q}\} \in \gamma(\mathcal{Q})$

All three conditions are satisfied and so we conclude that $\gamma(\mathcal{Q})$ is an ideal of $\mathrm{Dcl}(L)$. Finally note that as $\mathcal{Q}$ is prime in $\mathrm{Spc}(T_L)$ it is a proper ideal, the image $\gamma(\mathcal{Q})$ is also proper. □

Lemma 3.7.18. *The ideal $\gamma(\mathcal{Q})$ is prime.*

Proof. The fact that $\gamma(\mathcal{Q})$ is a proper ideal follows from the previous lemma. It remains only to show that $\gamma(\mathcal{Q})$ is prime. Let $S, R \in \mathrm{Dcl}(L)$ such that $S \cap R \in \gamma(\mathcal{Q})$. We have

$$\begin{aligned}\mathcal{Q} &\supseteq \mathrm{thick}^{\otimes}(k_x \in S \cap R)\\ &= \mathrm{thick}^{\otimes}(k_{s\wedge r} \mid s \in S, r \in R)\\ &= \mathrm{thick}^{\otimes}(k_s \otimes k_r \mid s \in S, r \in R)\\ &= \mathrm{thick}^{\otimes}(k_s \mid s \in S)\ \ \mathrm{thick}^{\otimes}(k_r \mid r \in R).\end{aligned}$$

As $\mathcal{Q}$ is prime in T_L, it is a consequence of Lemma 2.3.7 that either $\mathrm{thick}^{\otimes}(k_s \mid s \in S) \subseteq \mathcal{Q}$ or $\mathrm{thick}^{\otimes}(k_r \mid r \in R) \subseteq \mathcal{Q}$. Therefore $S \subseteq \{\ell \in L \mid k_\ell \in \mathcal{Q}\}$ or $R \subseteq \{\ell \in L \mid k_\ell \in \mathcal{Q}\}$. That is $S \in \gamma(\mathcal{Q})$ or $R \in \gamma(\mathcal{Q})$ and so we conclude that $\gamma(\mathcal{Q})$ is a prime ideal of $\mathrm{Dcl}(\mathcal{Q})$. □

Proposition 3.7.19. *There is a bijection of sets*

$$\gamma : \mathrm{Spc}(T_L^c) \longrightarrow \mathrm{Spec}(\mathrm{Dcl}(L)),$$

given by

$$\gamma(\mathcal{Q}) = \{\downarrow A \mid A \subseteq \{\ell \in L \mid k_\ell \in \mathcal{Q}\}\}.$$

Proof. The image of γ lies in $\mathrm{Spec}(\mathrm{Dcl}(L))$ by the previous lemma. The fact that γ is injective follows from the fact that each prime ideal $\mathcal{Q}$ in $\mathrm{Spc}(T_L)$ is uniquely defined by the collection $\{\ell \in L \mid k_\ell \in \mathcal{Q}\}$. It remains to show that γ is surjective. Fix a prime $\mathcal{H} \in \mathrm{Spec}(\mathrm{Dcl}(L))$ and let $H = \bigcup_{S \in \mathcal{H}} S$. Define $\beta(\mathcal{H}) = \mathrm{thick}^{\otimes}(k_h \mid h \in H)$. This is well-defined as H is non-empty. Clearly $\gamma(\beta(\mathcal{H})) = \mathcal{H}$ and so it remains to show that $\beta(\mathcal{H})$ is a prime ideal of T_L. It is clear that $\beta(\mathcal{H})$ is proper. As thick tensor ideals depend only on which objects k_ℓ they contain, this reduces to the following situation. Let $x, y \in L$ such that $k_{x\wedge y} \in \beta(\mathcal{H})$. Then $x \wedge y \in H$ so $\downarrow x \cap \downarrow y = \downarrow x \wedge y \subseteq \mathcal{H}$. As $\mathcal{H}$ is prime, $\downarrow x \in \mathcal{H}$ or $\downarrow y \in \mathcal{H}$ and so $k_x \in \beta(\mathcal{H})$ or $k_y \in \beta(\mathcal{H})$. Therefore $\beta(\mathcal{H})$ is prime. Moreover, it is clear that $\beta(\gamma(\mathcal{Q})) = \mathcal{Q}$ and so we conclude that γ is a bijection with inverse β. □

Definition 3.7.20. For a spectral space X, define the *Hochster dual* of X to be the topological space $X^\vee$ with open subbasis consisting of all those closed sets in X with quasi-compact complement. Note that $(X^\vee)^\vee = X$.

Theorem 3.7.21. *If L is a bounded lattice then there is a homeomorphism*

$$\gamma : \mathrm{Spc}(T_L)^\vee \longrightarrow \mathrm{Spec}(\mathrm{Dcl}(L)).$$

Proof. By the previous proposition, the function γ is a bijection of sets with inverse β. It remains to show that both γ and β are continuous.

1. Fix a quasi-compact open subset $D_{\mathrm{Dcl}(L)}(X) \subseteq \mathrm{Spec}(\mathrm{Dcl}(L))$. We compute its preimage:

$$\begin{aligned}
\gamma^{-1} D_{\mathrm{Dcl}(L)}(X)^c &= \gamma^{-1}\{\mathcal{H} \in \mathrm{Spec}(\mathrm{Dcl}(L)) \mid X \in \mathcal{H}\} \\
&= \{\mathcal{Q} \mid \{k_x \mid x \in X\} \subseteq \mathcal{Q}\} \\
&= \bigcap_{x \in X} \{\mathcal{Q} \mid k_x \in \mathcal{Q}\} \\
&= \bigcap_{x \in X} U(k_x) \\
&= \bigcup_{x \in X} Z(k_x).
\end{aligned}$$

   As each of all of the $Z(k_x)$ are closed in $\mathrm{Spc}(T_L)$, they are open in $\mathrm{Spc}(T_L)^\vee$ and so the union $\bigcup_{x \in X} Z(k_x)$ is open in $\mathrm{Spc}(T_L)^\vee$. Hence γ is continuous.

2. We check continuity of β on the open subbasis $Z(k_x)$ for $x \in L$:

$$\begin{aligned}
\beta^{-1} Z(k_x) &= \beta^{-1}\{\mathcal{Q} \in \mathrm{Spc}(T_L)^\vee \mid k_x \notin \mathcal{Q}\} \\
&= \{\mathcal{H} \in \mid x \notin \mathcal{H}\} \\
&= D_{\mathrm{Dcl}(L)}(\{x\}).
\end{aligned}$$

   As $\beta^{-1} Z(k_x) = D_{\mathrm{Dcl}(L)}(\{x\})$ is a basic open subset of $\mathrm{Spec}(D_{\mathrm{Dcl}(L)})$ we conclude that β is a homeomorphism.

We have shown that γ is a continuous bijection with continuous inverse β and therefore conclude that $\mathrm{Spc}(T_L)^\vee$ is homeomorphic to $\mathrm{Spec}(\mathrm{Dcl}(L))$. □

With the spectrum computed, we can now apply Balmer's classification theorem for thick tensor ideals.

Corollary 3.7.22. *The lattice of thick tensor ideals of T_L^c is dual to the lattice* $\mathrm{Dcl}(L)$.

We can also explicitly compute the structure sheaf of the spectrum.

Proposition 3.7.23. *Let L be a bounded lattice. Then the structure sheaf of* $\mathrm{Spc}(T_L^c)$ *is $\underline{k}$, the constant sheaf with value k.*

Proof. Fix an open subset $U \subseteq \mathrm{Spc}(\mathsf{T}_L^c)$ with closed complement Z. Consider the presheaf sections $\mathrm{Hom}_{\mathsf{T}_L^c(U)}(k_{\mathbf{1}}, k_{\mathbf{1}})$. Such a section is a morphism in the Verdier quotient with representative

$$k_{\mathbf{1}} \xleftarrow{\ f\ } X \xrightarrow{\ g\ } k_{\mathbf{1}}$$

for some $X \in \mathsf{T}_L^c$ such that $\mathrm{cone}(f) \in \mathsf{T}_{L,Z}^c$. Note that if either f or g are the zero morphism, then the morphism represented by the corresponding diagram is the zero morphism. Therefore we can restrict to the case where the morphisms f and g are both non-zero. By the construction of the category T_L there are no non-zero maps $h : k_m \longrightarrow k_{\mathbf{1}}$ when $m \neq \mathbf{1}$. As we assume f is non-zero, the object X must have $k_{\mathbf{1}}$ as a summand. We can rewrite our representative diagram as

$$k_{\mathbf{1}} \xleftarrow{\ f\ } k_{\mathbf{1}}^{\oplus n} \oplus Y \xrightarrow{\ g\ } k_{\mathbf{1}}$$

where n is a positive integer and the object Y does not contain $k_{\mathbf{1}}$ as a summand. By the construction of T, the component of f on Y is zero, and so the cone of f must be given by $k_{\mathbf{1}}^{\oplus n-1} \oplus Y$. As we require $\mathrm{supp}(\mathrm{cone}(f)) \subseteq Z$, we must have $n = 1$, or $Z = \mathrm{Spc}(\mathsf{T}_L^c)$ in which case every morphism in the quotient is zero. We now consider the map $g : k_{\mathbf{1}} \oplus Y \longrightarrow k_{\mathbf{1}}$. Again, the component of g on Y is again zero. Therefore our morphism is represented by the diagram

$$k_{\mathbf{1}} \xleftarrow{\ f\ } k_{\mathbf{1}} \xrightarrow{\ g\ } k_{\mathbf{1}}$$

As $\mathrm{Hom}_{\mathsf{D}(k_{\mathbf{1}})}(k_{\mathbf{1}}, k_{\mathbf{1}}) \cong k_{\mathbf{1}}$, the morphism f is already invertible. When considering the Verdier quotient, our roof diagram corresponds to the morphism gf^{-1} and the collection of all such roofs is naturally isomorphic to $k_{\mathbf{1}}$. Hence we conclude that the presheaf sections are simply given by $\mathrm{Hom}(k_{\mathbf{1}}, k_{\mathbf{1}}) \cong k_{\mathbf{1}}$, and so the associated sheaf is the constant sheaf $\underline{k}$. □

§ 3.7.2 | Group algebras of cyclic groups

Let k be a field and G a finite group. The group algebra kG provides us with two categories of interest. The first is the bounded derived category $\mathsf{D}^b(\mathsf{mod}\, kG)$, with tensor product $\otimes = \otimes_k$, and the stable module category $\underline{\mathsf{mod}}\, kG$, again with tensor product $\otimes_k$. These categories are of great interest to representation theory, and the corresponding spectra have been calculated, for instance in [BCR97], [FP07] and [BIK11]. In this section we will investigate the behaviour of associated sheaf functors in the context of these categories.

We will start with the bounded derived category. While the original computation can be found in [FP07, 7.5], the translation and proof of the theorem into the tt-geometry framework is from [Bal10, 8.5]

Theorem 3.7.24. [FP07; Bal10] *Let k be a field and G a finite group. Consider the bounded derived category $\mathsf{T} = \mathsf{D}^b(\mathsf{mod}\, kG)$ equipped with the tensor product $\otimes_k$. The natural comparison map is an isomorphism of locally ringed spaces*

$$\mathrm{Spec}^\bullet(\mathsf{T}) \cong \mathrm{Spec}^h(H^\bullet(G, k))$$

We will restrict ourselves to the case where $\mathsf{T} = \mathsf{D}^b(\mathsf{mod}\, kC_p)$ where k is a field with positive characteristic p and C_p is the cyclic group of order p.

Lemma 3.7.25. *The graded homomorphism ring of T is given by*

$$R^\bullet_{\mathsf{T}} = \mathrm{Hom}^\bullet_{\mathsf{D}^b(\mathsf{mod}\, kC_p)}(k, k) = H^\bullet(C_p, k).$$

In particular

$$H^\bullet(C_p, k) = \begin{cases} k[t], \ |t| = 1 \ \textit{if}\ p = 2 \\ \frac{k[t,s]}{(s^2)}, \ |t| = 2, \ |s| = 1 \ \textit{if}\ p \geq 3 \end{cases}$$

Proof. The central ring is isomorphic to the usual group cohomology ring as in [FP07; Bal10]. The calculation of this group cohomology ring is standard and can be found in [Eve91]. □

By Theorem 3.7.24 there is an isomorphism of locally ringed spaces

$$(\mathrm{Spc}(\mathsf{T}), \mathcal{O}^\bullet_{\mathsf{T}}) \cong (\mathrm{Spec}^h(H^\bullet(C_p, k)), \mathcal{O}^\bullet_{H^\bullet(G,k)}).$$

The spectrum $\mathrm{Spc}\, \mathsf{T}$ is the two point space

$$\begin{array}{c} (0) \\ | \\ \mathsf{D}^{\mathsf{perf}}(kC_p) \end{array}$$

with open subsets $\varnothing \subseteq \{\mathsf{D}^{\mathsf{perf}}(kC_p)\} \subseteq \mathrm{Spc}(\mathsf{T})$.

Proposition 3.7.26. *The graded structure sheaf $\mathcal{O}^\bullet_\mathsf{T}$ is isomorphic to the graded presheaf ${}_\mathrm{p}\mathcal{O}^\bullet_\mathsf{T}$. In particular, the structure sheaf of the ungraded spectrum $(\mathrm{Spc}(\mathsf{T}), \mathcal{O}_\mathsf{T})$ is the constant sheaf $\underline{k}$.*

Proof. We first compute the sections of the presheaf ${}_p\mathcal{O}^\bullet_\mathsf{T}$ over the open subsets of $\mathrm{Spc}(\mathsf{T})$.

1. For the open subset $\varnothing$, the localisation of T is $\mathsf{T}(\varnothing) = 0$. Therefore ${}_p\mathcal{O}^\bullet_\mathsf{T}(\varnothing) = 0$.
2. For the open subset $\{\mathsf{D}^{\mathsf{perf}}(kC_p)\}$, the localisation of T is $\mathsf{T}(\{\mathsf{D}^{\mathsf{perf}}(kC_p)\}) = \underline{\mathsf{mod}}(kC_p)$. Therefore ${}_p\mathcal{O}^\bullet_\mathsf{T}(\{\mathsf{D}^{\mathsf{perf}}(kC_p)\}) = (H^\bullet(C_p, k))_t$.
3. For the open subset $\mathrm{Spc}(\mathsf{T})$, the localisation of T is $\mathsf{T}(\mathrm{Spc}(\mathsf{T})) = \mathsf{T}$. Therefore ${}_p\mathcal{O}^\bullet_\mathsf{T}(\mathrm{Spc}(\mathsf{T})) = H^\bullet(C_p, k)$.

We observe that this presheaf is already a sheaf, and so ${}_p\mathcal{O}^\bullet_\mathsf{T} = \mathcal{O}^\bullet_\mathsf{T}$. Considering this sheaf in degree zero only we observe that $H^0(C_p, k) = k$ and so $\mathcal{O}_\mathsf{T}$ is the constant sheaf $\underline{k}$. □

Proposition 3.7.27. *The ungraded associated sheaf functor $[k, -]^\# : \mathsf{T} \longrightarrow \mathsf{Mod}\,\mathcal{O}_\mathsf{T}$ is not an equivalence of categories, independent of the characteristic of k.*

Proof. By considering k and kC_p and applying the ungraded associated sheaf functor $[k, -]^\#$ we obtain

$$[k, k]^\# \cong \underline{k} \text{ and } [k, kC_p]^\# \cong \mathrm{Sky}_{(0)}(k)$$

where $\mathrm{Sky}_{(0)}(k)$ is the skyscraper sheaf of value k at the closed point (0). Applying the associated sheaf functor to any perfect complex results in a skyscraper sheaf supported at (0). Therefore there exists no sheaf in the image of $[k, -]^\#$ supported only at the open point $\{\mathsf{D}^{\mathsf{perf}}(kC_p)\}$. □

Proposition 3.7.28. *When* $\mathrm{char}(k) = 2$ *the graded associated sheaf functor*

$$[k, -]^\bullet : \mathsf{D}^b(\mathsf{mod}\,kC_2) \longrightarrow \mathsf{Coh}^\bullet\,\mathrm{Spec}^\bullet(\mathsf{T})$$

is essentially surjective and conservative.

Proof. Suppose $\mathrm{char}(k) = 2$. As the cohomology ring $H^\bullet(G, k)$ is noetherian, the structure sheaf is coherent and by Corollary 3.3.13 every object $M \in \mathsf{D}^b(\mathsf{mod}\,kC_2)$ has associated sheaf $[k, M]^\bullet$ and this sheaf must be coherent. Therefore the image of $[k, -]^\bullet$ certainly lies in $\mathsf{Coh}^\bullet(\mathrm{Spec}^\bullet(\mathsf{T}))$. In order to show the graded associated sheaf functor is conservative, let us consider the indecomposable objects in $\mathsf{D}^b(\mathsf{mod}\,kC_2)$. Two of the indecomposable modules are given by the complexes with k or kC_2 in degree zero, and the zero module in all other degrees. Additionally, for each $n > 1$ consider the complex M_n given by

$$\cdots \longrightarrow 0 \longrightarrow kC_2 \xrightarrow{1-g} kC_2 \xrightarrow{1-g} \cdots \xrightarrow{1-g} kC_2 \longrightarrow 0 \longrightarrow \cdots$$

consisting of n copies of kC_2 with arrows given by multiplication by $1-g$, where g is the non-trivial element of C_2. These complexes are indecomposable objects in $\mathsf{D}^b(\mathsf{mod}\, kC_2)$. By noting that kC_2 is self-injective, we can compute the global presheaf sections by simply considering the usual hom-complex. Taking stalks at the closed point (0) we see that $[k,k]^\bullet_{(0)} \cong k[t]$, $[k,kC_2]^\bullet_{(0)} \cong k$ and for each n we have $[k,M_n]^\bullet_{(0)} \cong \frac{k[t]}{t^n}$. Each of these rings is graded, and suspending any of these indecomposables will result in the same stalk but with a new grading uniquely determined by the suspension. It is now immediate that the graded associated sheaf functor is conservative, as any isomorphism between the corresponding sheaves would force the stalks to agree. Finally let $\mathcal{F} \in \mathsf{Coh}^\bullet \operatorname{Spec}^\bullet(\mathsf{T})$ be an indecomposable graded coherent sheaf. By Proposition 3.7.26 we deduce that every such module must have global sections consisting of an indecomposable $k[t]$-module, with the sections over $\{\mathsf{D}^{\mathsf{perf}}(kC_p)\}$ being the corresponding localisation as a $k[t,t^{-1}]$-module. This can all be obtained by applying the graded associated sheaf functor to the indecomposable objects in T. In particular, every such sheaf is a finite sum of the indecomposables $k[t](j) = [k, \Sigma^j k]^\bullet$ and $\frac{k[t]}{t^n}(i) = [k, \Sigma^i M_n]^\bullet$. We conclude that the functor is essentially surjective. □

We now consider the stable module category $\mathsf{K} = \underline{\mathsf{mod}}\, kG$.

Theorem 3.7.29. [FP07, 7.5] *Let k be a field and G a finite group. Consider the stable module category $\mathsf{K} = \underline{\mathsf{mod}}\, kG$. Then there is a homeomorphism of topological spaces*

$$\operatorname{Spc}(\mathsf{K}) \cong \mathsf{Proj}(H^\bullet(G,k))$$

Just as with the derived category we will restrict our attention to the group algebra kC_p where k is a field of characteristic p and C_p is the cyclic group of order p. In this case there is an equivalence of categories

$$\mathsf{K} = \underline{\mathsf{mod}}\, kC_p \cong \underline{\mathsf{mod}}\, \frac{k[x]}{(x^p)}.$$

Lemma 3.7.30. *The graded homomorphism group of the tensor unit k of K is given by*

$$R^\bullet_{\mathsf{K}} = \operatorname{Hom}^\bullet_{\mathsf{K}}(k,k) = \begin{cases} k[t,t^{-1}], & |t| = 1 \text{ if } p = 2 \\ \frac{k[t,t^{-1},s]}{(s^2)}, & |t| = 2,\ |s| = 1,\ \text{if } p \geq 3 \end{cases}$$

Proof. The central ring $\operatorname{Hom}^\bullet_{\mathsf{K}}(k,k)$ is isomorphic to the Tate cohomology ring [FP07; Bal10]. The cohomology ring itself can be computed using standard techniques, which can be found in [AW67]. We provide a tt-flavoured proof for fields of odd characteristic, with the case $p = 2$ being similar. We identify kC_p with $A = \frac{k[x]}{(x^p)}$. Let X be the perfect complex

$$\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{x} A \longrightarrow 0 \longrightarrow \cdots$$

There is a triangle

$$\Sigma k \longrightarrow X \longrightarrow k \xrightarrow{t} \Sigma^2 k$$

where t is the same as in Lemma 3.7.25. As $\mathrm{thick}(X) = \mathsf{Perf}(A)$, the stable module category K can be obtained by taking the quotient of $\mathsf{D}^b(\mathsf{mod}\, kG)$ by $\mathrm{thick}(X)$. By [Bal10, 3.6], this is equivalent to inverting t in the corresponding graded homomorphism ring, completing the proof. □

By using the identification $\mathrm{Spc}(\mathsf{K}) \cong \mathsf{Proj}(H^\bullet(G, k))$ we can explicitly compute the corresponding structure sheaves.

Proposition 3.7.31. *In the untwisted setting, for all prime p, we have*

$$\mathrm{Spec}(\mathsf{K}) \cong \mathrm{Spec}(k).$$

In the graded setting,

$$\mathrm{Spec}^\bullet(\mathsf{K}) \cong \mathrm{Spec}^h(R^\bullet_{\mathsf{K}}).$$

Proof. First note that $\mathsf{Proj}(H^\bullet(G, k))$ consists of a single point. Therefore the structure sheaf of the spectrum is determined entirely by its global sections. In degree zero we have

$$R_{\mathsf{K}} = k$$

and so we can immediately conclude that $\mathrm{Spec}(\mathsf{K}) \cong \mathrm{Spec}(k)$ as locally ringed spaces. For the graded case, observe by our computation of $R^\bullet_{\mathsf{K}}$ in Lemma 3.7.30 that $\mathrm{Spec}^h(R^\bullet_{\mathsf{K}})$ also consists of a single point and so we conclude by the same argument as the untwisted case. □

Proposition 3.7.32. *Let k be a field with $\mathrm{char}(k) = 2$. Then the associated sheaf functor $[k, -]^\bullet$ is an equivalence of categories*

$$[k, -]^\bullet : \mathsf{K} \longrightarrow \mathsf{Coh}^\bullet(\mathrm{Spec}^\bullet(\mathsf{K})).$$

Proof. When $p = 2$, the stable module category K is equivalent to $\underline{\mathsf{mod}}\, \frac{k[x]}{(x^2)}$. This is equivalent to $\mathsf{mod}\, k$, the category of finite-dimensional k-vector spaces. As the space $\mathrm{Spec}^\bullet(\mathsf{K})$ is a point, the category $\mathsf{Coh}^\bullet(\mathrm{Spec}^\bullet(\mathsf{K}))$ is equivalent to $\mathsf{grmod}\, k[t, t^{-1}]$, the category of finitely generated graded modules over the Laurent polynomial ring $k[t, t^{-1}]$. This is also equivalent to $\mathsf{mod}\, k$. The result then follows, as

$$[k, k]^\bullet \cong k[t, t^{-1}]$$

and the equivalence $\mathsf{grmod}\, k[t, t^{-1}] \cong \mathsf{mod}\, k$ identifies $k[t, t^{-1}]$ with k. □

Proposition 3.7.33. *Let k be a field with characteristic $p = 3$. Then the associated sheaf functor $[k, -]^\bullet$ is not essentially surjective.*

Proof. If $p = 3$ then the structure sheaf of $\mathrm{Spec}^\bullet(\mathsf{K})$ is given by $\mathcal{O}^\bullet_{\mathsf{K}} = \frac{\widetilde{k[t,t^{-1},s]}}{(s^2)}$ with indecomposable sheaves given by $\mathcal{O}^\bullet_{\mathsf{K}}$, $\mathcal{O}^\bullet_{\mathsf{K}}(1), \mathcal{O}^\bullet_{\mathsf{K}}/(s)$, and $\mathcal{O}^\bullet_{\mathsf{K}}/(s)(1)$. Meanwhile $\mathsf{K} = \underline{\mathsf{mod}}\, kC_3$ only has objects k and $\Sigma k = \frac{k(x)}{(x^p-1)}$. Taking associated sheaves we obtain

$$[k,k]^\bullet = \mathcal{O}^\bullet_{\mathsf{K}} \text{ and } [k,\Sigma k]^\bullet = \mathcal{O}^\bullet_{\mathsf{K}}(1),$$

and so conclude that the associated sheaf functor $[k,-]^\bullet$ is not essentially surjective. □

Chapter 4

Monoidal triangular geometry

Many of the definitions and concepts of tensor-triangular geometry can be naturally extended to the noncommutative setting. Examples of interest relate to Hopf and Lie algebras, in which the associated monoidal product fails to be symmetric. In [NVY19] the authors define the usual machinery of prime ideals, Balmer spectra and support data in the non-commutative setting. A classification of thick two-sided ideals is also obtained, albeit with significantly more hypotheses than in the symmetric setting. In this chapter, we continue translating various tensor triangulated concepts into the noncommutative world. In §4.1 we lay out the basic definitions from [NVY19], and prove that minimal primes exist in the monoidal setting. We adapt the proof from [Bal05], adjusting the argument to use the monoidal definition of prime ideals. In §4.2 we begin translating the definition of the action of a tt-category on a triangulated category into the noncommutative setting. We lay out the formation of localising bimodules and show how the asymmetry of the tensor product affects the formation of residue objects. So far, examples of non-commutative Balmer spectra being explicitly described have been of a "quantum" or "Hopf" flavour. In §4.3 we diverge from this by instead investigating the enveloping algebra of the A_2 quiver and show that its associated noncommutative Balmer spectrum consists of a single point.

§ 4.1 | Machinery of monoidal geometry

In this section we recall the definitions of monoidal geometry, including the Balmer spectrum of prime ideals where the tensor product is no longer symmetric. We prove the existence of minimal primes in non-zero monoidal triangulated categories.

We begin with some basic definitions and results from [NVY19].

Definition 4.1.1. An *essentially small monoidal triangulated category* is a triple of the form $(\mathsf{T}, \otimes, \mathbf{1})$, where T is an essentially small triangulated category and $(\otimes, \mathbf{1})$ is a monoidal structure on T such that $\otimes$ is an exact functor in each variable. In other words, it is a tensor triangulated category but the tensor need not be symmetric.

We will abbreviate "monoidal triangulated category" to "mt-category".

The following definition contains the different notions of prime in the noncommutative setting.

Definition 4.1.2. Let T be an essentially small mt-category.

1. A *thick two-sided ideal* of T is a thick subcategory closed under left and right tensoring with arbitrary objects of T.

2. A *prime ideal* of T is a proper thick ideal $\mathcal{P}$ such that $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P}$ implies $\mathcal{I} \subseteq \mathcal{P}$ or $\mathcal{J} \subseteq \mathcal{P}$ for all thick two-sided ideals $\mathcal{I}$ and $\mathcal{J}$ or T.

3. A *semiprime* ideal of T is an intersection of prime ideals of T.

4. A *completely prime* ideal of T is a proper thick ideal $\mathcal{P}$ such that $A \otimes B \in \mathcal{P}$ implies $A \in \mathcal{P}$ or $B \in \mathcal{P}$ for all objects $A, B \in \mathsf{T}$.

Definition 4.1.3. The *noncommutative Balmer spectrum* of an essentially small mt-category T is the topological space of prime ideals of T. We denote the noncommutative spectrum by $\operatorname{Spc}(\mathsf{T})$. The topology on $\operatorname{Spc}(\mathsf{T})$ is generated by the collection of closed subsets

$$V(S) = \{\mathcal{P} \in \operatorname{Spc}(\mathsf{T}) \mid \mathcal{P} \cap S = \varnothing\}$$

for all subsets S of T.

We have the following characterisation of prime ideals in terms of objects rather than ideals:

Theorem 4.1.4. [NVY19, 1.2.1] *Let T be an essentially small mt-category. Then the following hold:*

1. *A proper thick ideal $\mathcal{P}$ of T is prime if and only if $A \otimes B \otimes C \in \mathcal{P}$, for all $C \in \mathsf{T}$ implies $A \in \mathcal{P}$ or $B \in \mathcal{P}$ for all $A, B \in \mathsf{T}$.*

2. *A proper thick ideal $\mathcal{P}$ of T is semiprime if and only if $A \otimes C \otimes A \in \mathcal{P}$, for all $C \in \mathsf{T}$ implies $A \in \mathcal{P}$ for all $A \in \mathsf{T}$.*

3. *The noncommutative Balmer spectrum* $\operatorname{Spc}(\mathsf{T})$ *is always nonempty.*

The formulation of primeness in terms of ideals is due to [BKS07] while the formulation in terms of objects is due to [NVY19].

Just like in the symmetric setting, there is a universal support datum which is the noncommutative analogue to Balmer's notion of support.

Definition 4.1.5. For an essentially small mt-category T, the *small noncommutative support* of an object t is given by

$$\operatorname{supp}(t) = \{\mathcal{P} \in \operatorname{Spc}(\mathsf{T}) \mid t \notin \mathcal{P}\}.$$

Lemma 4.1.6. *The small noncommutative support satisfies the following properties:*

1. $\operatorname{supp}(0) = \varnothing$ *and* $\operatorname{supp}(\mathbf{1}) = \operatorname{Spc}(\mathsf{T})$.
2. $\operatorname{supp}(t \oplus s) = \operatorname{supp}(t) \cup \operatorname{supp}(B)$, *for all* $t, s \in \mathsf{T}$.
3. $\operatorname{supp}(\Sigma t) = \operatorname{supp}(t)$.
4. *If* $t \longrightarrow s \longrightarrow r \longrightarrow \Sigma t$ *is a distinguished triangle, then* $\operatorname{supp}(t) \subseteq \operatorname{supp}(s) \cup \operatorname{supp}(r)$.
5. $\bigcup_{r \in \mathsf{T}} \operatorname{supp}(t \otimes r \otimes s) = \operatorname{supp}(t) \cap \operatorname{supp}(s)$ *for all* $t, s \in \mathsf{T}$.
6. *For all* $t \in \mathsf{T}$ *the subset* $\operatorname{supp}(t)$ *is closed.*

We can now detail some properties of prime ideals which lift to the noncommutative setting.

Theorem 4.1.7. [NVY19, 3.2.3] *Suppose $\mathcal{M}$ is a multiplicative subset of T and suppose $\mathcal{I}$ is a proper thick two-sided tensor ideal of T such that $\mathcal{I} \cap \mathcal{M} = \varnothing$. The set*

$$X(\mathcal{M}, \mathcal{I}) = \{\mathcal{J} \textit{ a thick two-sided tensor ideal of } \mathcal{T} \mid \mathcal{I} \subseteq \mathcal{J}, \mathcal{J} \cap \mathcal{M} = \varnothing\}$$

has a maximal element $\mathcal{P}$ and moreover $\mathcal{P}$ is prime.

Proposition 4.1.8. *Let T be a non-zero mt-category.*

1. *Let $\mathcal{S}$ be a $\otimes$-multiplicative collection of objects which does not contain zero. Then there exists a prime ideal $\mathcal{P}$ such that $\mathcal{P} \cap \mathcal{S} = \varnothing$.*
2. *Let $\mathcal{J} \subsetneq \mathsf{T}$ be a proper thick tensor ideal. Then there exists a maximal proper thick tensor ideal $\mathcal{I} \subsetneq \mathsf{T}$ which contains $\mathcal{J}$.*
3. *Maximal proper thick tensor ideals are prime.*

Corollary 4.1.9. *If an object $t \in \mathsf{T}$ belongs to all primes, then t is tensor nilpotent.*

Proof. The proof is as in [Bal05, 2.4]. We prove the contrapositive. Let t be a non-nilpotent element. Then the set $\mathcal{S} = \{t^{\otimes n} \mid n \geq 0\}$ is a multiplicative subset of T which does not contain zero. By Theorem 4.1.7 and Proposition 4.1.8 there exists a prime ideal $\mathcal{P}$ such that $\mathcal{P} \cap \mathcal{S} = \varnothing$. Therefore $t \notin \mathcal{P}$ and so does not belong to all primes. □

Proposition 4.1.10. *If T is a non-zero mt-category, then there exist minimal primes in T. More precisely, for any prime ideal $\mathcal{P} \subset \mathsf{T}$ there exists a minimal prime $\mathcal{P}' \subset \mathcal{P}$.*

Proof. To apply Zorn's lemma, we will show that for any non-empty chain of prime ideals $C \subset \operatorname{Spc}(\mathsf{T})$, the thick $\otimes$-ideal $\mathcal{Q}' = \bigcap_{\mathcal{Q} \in C} \mathcal{Q}$ is prime. Let $a_1, a_2 \notin \mathcal{Q}'$. There exist $\mathcal{Q}_1, \mathcal{Q}_2 \in C$ such that $a_1 \notin \mathcal{Q}_1$ and $a_2 \notin \mathcal{Q}_2$. So there exists $\mathcal{Q}_0 \in C$ such that $a_1, a_2 \notin \mathcal{Q}_0$. As $\mathcal{Q}_0$ is prime, there exists an object $t \in \mathsf{T}$ such that $a_1 \otimes t \otimes a_2 \notin \mathsf{T}$. Therefore $a_1 \otimes t \otimes a_2 \notin \mathcal{Q}'$. We have shown that if $a_1, a_2 \notin \mathcal{Q}'$ then there exists $t \in \mathsf{T}$ such that $a_1 \otimes t \otimes a_2 \notin \mathcal{Q}'$. This is equivalent to the contrapositive: if for all $t \in \mathsf{T}$ we have $a_1 \otimes t \otimes a_2 \in \mathcal{Q}'$ then $a_1, a_2 \in \mathcal{Q}'$. That is, we have shown that $\mathcal{Q}'$ is prime. □

§ 4.2 | Monoidal actions

Given a tt-category T and a triangulated category K, one can define a left action of T on K [Ste13]. With obvious adjustments one can define a right action. The definitions do not use the symmetric property of the tensor product and so lifts immediately to the monoidal case.

Definition 4.2.1. Let T, R be mt-categories and let K be a triangulated category. We say that K is a *left* T*-module* if there is a left action of T on K. Similarly we say that K is a *right* R*-module* if there is a right action of R on K. We say that K is a T-R*-bimodule* if for each $t \in \mathsf{T}, k \in \mathsf{K}, r \in \mathsf{R}$ there exists an isomorphism

$$\alpha_{t,k,r} : (t *_{\mathsf{T}} k) *_{\mathsf{R}} r \xrightarrow{\sim} t *_{\mathsf{T}} (k *_{\mathsf{R}} r)$$

compatible with both the actions of T and R.

Definition 4.2.2. Let K be a T-R-bimodule. We say $\mathsf{L} \subseteq \mathsf{K}$ is a *subbimodule* if both of the actions $\mathsf{T} \times \mathsf{L} \xrightarrow{*_{\mathsf{T}}} \mathsf{K}$ and $\mathsf{L} \times \mathsf{R} \xrightarrow{*_{\mathsf{R}}} \mathsf{K}$ factor via L.

Definition 4.2.3. Let K be a T-R-bimodule and let $\mathcal{K} \subseteq \mathsf{K}$ be a collection of objects in K. We denote the smallest localising left T-submodule containing $\mathcal{K}$ by $\mathrm{Lloc}^{*_{\mathsf{T}}}(\mathcal{K})$. Similarly denote the smallest localising right R-submodule containing $\mathcal{K}$ by $\mathrm{Rloc}^{*_{\mathsf{R}}}(\mathcal{K})$. We denote the smallest localising T-R-bimodule by $\mathrm{Biloc}^{\mathsf{T}\text{-}\mathsf{R}}(\mathcal{K})$.

Definition 4.2.4. Given collections of objects $\mathcal{T} \subseteq \mathsf{T}$, $\mathcal{K} \subseteq \mathsf{K}$ and $\mathcal{R} \subseteq \mathsf{R}$ we define

$$\mathcal{T} *_{\mathsf{T}} \mathcal{K} = \mathrm{Lloc}^{*_{\mathsf{T}}}(t *_{\mathsf{T}} k \mid t \in \mathsf{T}, k \in \mathsf{K})$$
$$\mathcal{K} *_{\mathsf{R}} \mathcal{R} = \mathrm{Rloc}^{*_{\mathsf{R}}}(k *_{\mathsf{R}} r \mid k \in \mathsf{K}, r \in \mathsf{R}).$$

We have the following collection of lemmas adapted from [Ste13]:

Lemma 4.2.5. *Suppose that $\mathcal{K}$ is a collection of objects of K such that $\mathcal{K}$ is stable under the action of T. Then $\mathrm{loc}(\mathcal{K})$ is a localising left T-submodule. Similarly, if $\mathcal{T}$ is a collection of objects of T and $\mathcal{K}$ is a localising subcategory of K closed under the action of objects in T, then $\mathcal{K}$ is closed under the action of $\mathrm{loc}(\mathsf{T})$. Analogous results hold for right actions.*

Lemma 4.2.6. *Suppose $\mathcal{I} \subseteq \mathsf{T}$ is a localising two-sided $\otimes_{\mathsf{T}}$-ideal and $\mathcal{K}$ is a collection of objects of K. Then there is an equality of localising left T-submodules of K*

$$\mathcal{I} *_{\mathsf{T}} \mathcal{K} = \mathrm{loc}(t *_{\mathsf{T}} k \mid t \in \mathcal{I}, k \in \mathcal{K}).$$

Similarly, if $\mathcal{J} \subseteq \mathsf{R}$ is a localising two-sided $\otimes_{\mathsf{R}}$-ideal and $\mathcal{K}$ is a collection of objects of K then there is an equality of localising right R-submodules of K

$$\mathcal{K} *_{\mathsf{R}} \mathcal{J} = \mathrm{loc}(k *_{\mathsf{R}} r \mid k \in \mathcal{K}, r \in \mathcal{J}).$$

Lemma 4.2.7. *Given a set of objects $\mathcal{T}$ of T and a set of objects $\mathcal{K}$ of K we have*

$$\mathrm{loc}(t *_{\mathsf{T}} k \mid t \in \mathrm{loc}(\mathcal{T}), k \in \mathrm{loc}(\mathcal{K})) = \mathrm{loc}(t *_{\mathsf{T}} k \mid t \in \mathcal{T}, k \in \mathcal{K}).$$

Similarly given a set of objects $\mathcal{R}$ of R and a set of objects $\mathcal{K}$ of K we have

$$\mathrm{loc}(k *_{\mathsf{R}} r \mid k \in \mathrm{loc}(\mathcal{K}), r \in \mathrm{loc}(\mathcal{R})) = \mathrm{loc}(k *_{\mathsf{R}} r \mid k \in \mathcal{K}, r \in \mathcal{R}).$$

Lemma 4.2.8. *Given a collection of objects of $\mathcal{T}$ of T and a collection of objects $\mathcal{K}$ of K we have*

$$\begin{aligned}\mathrm{loc}^{\otimes}_{\mathsf{T}}(\mathcal{T}) *_{\mathsf{T}} \mathrm{loc}(\mathcal{K}) &= \mathrm{loc}(\mathcal{T}) *_{\mathsf{T}} \mathrm{loc}(\mathcal{K}) \\ &= \mathcal{T} *_{\mathsf{T}} \mathcal{K} \\ &= \mathrm{loc}(s *_{\mathsf{T}} (t *_{\mathsf{T}} k) \mid s \in \mathsf{T}, t \in \mathcal{T}, k \in \mathcal{K}).\end{aligned}$$

Similarly given a collection of objects $\mathcal{R}$ of R and a collections of objects $\mathcal{K}$ of K we have

$$\begin{aligned}\mathrm{loc}(\mathcal{K}) *_{\mathsf{R}} \mathrm{loc}^{\otimes_{\mathsf{R}}}(\mathcal{R}) &= \mathrm{loc}(\mathcal{K}) *_{\mathsf{R}} \mathrm{loc}(\mathcal{R}) \\ &= \mathcal{K} *_{\mathsf{R}} \mathcal{R} \\ &= \mathrm{loc}((k *_{\mathsf{R}} r) *_{\mathsf{R}} s \mid k \in \mathcal{K}, r \in \mathcal{R}, s \in \mathsf{R}).\end{aligned}$$

Now we can make some obvious statements with obvious proofs.

Lemma 4.2.9. *Let $\mathcal{T}, \mathcal{K}$ and $\mathcal{R}$ be collections of objects in T, K and R respectively. Then*

$$(\mathcal{T} *_{\mathsf{T}} \mathcal{K}) *_{\mathsf{R}} \mathcal{R} = \mathcal{T} *_{\mathsf{T}} (\mathcal{K} *_{\mathsf{R}} \mathcal{R}).$$

From now on we can refer to this bimodule as $\mathcal{T} *_{\mathsf{T}} \mathcal{K} *_{\mathsf{R}} \mathcal{R}$ with no confusion.

Proof. Applying the previous lemmas we obtain a string of equalities

$$\begin{aligned}(\mathcal{T} *_{\mathsf{T}} \mathcal{K}) *_{\mathsf{R}} \mathcal{R} &= \mathrm{loc}(s *_{\mathsf{T}} (t *_{\mathsf{T}} k) \mid s \in \mathsf{T}, t \in \mathcal{T}, k \in \mathcal{K}) *_{\mathsf{R}} \mathcal{R} \\ &= \mathrm{loc}(((s *_{\mathsf{T}} (t *_{\mathsf{T}} k)) *_{\mathsf{R}} r) *_{\mathsf{R}} u \mid s \in \mathsf{T}, t \in \mathcal{T}, k \in \mathcal{K}, r \in \mathcal{R}, u \in \mathsf{R}) \\ &= \mathrm{loc}((((s *_{\mathsf{T}} t) *_{\mathsf{T}} k) *_{\mathsf{R}} r) *_{\mathsf{R}} u \mid s \in \mathsf{T}, t \in \mathcal{T}, k \in \mathcal{K}, r \in \mathcal{R}, u \in \mathsf{R}) \\ &= \mathrm{loc}(((s *_{\mathsf{T}} t) *_{\mathsf{T}} (k *_{\mathsf{R}} r)) *_{\mathsf{R}} u \mid s \in \mathsf{T}, t \in \mathcal{T}, k \in \mathcal{K}, r \in \mathcal{R}, u \in \mathsf{R}) \\ &= \mathrm{loc}((s *_{\mathsf{T}} t) *_{\mathsf{T}} ((k *_{\mathsf{R}} r) *_{\mathsf{R}} u) \mid s \in \mathsf{T}, t \in \mathcal{T}, k \in \mathcal{K}, r \in \mathcal{R}, u \in \mathsf{R}) \\ &= \mathrm{loc}(s *_{\mathsf{T}} (t *_{\mathsf{T}} ((k *_{\mathsf{R}} r) *_{\mathsf{R}} u)) \mid s \in \mathsf{T}, t \in \mathcal{T}, k \in \mathcal{K}, r \in \mathcal{R}, u \in \mathsf{R}) \\ &= \mathcal{T} *_{\mathsf{T}} \mathrm{loc}((k *_{\mathsf{R}} r) *_{\mathsf{R}} u \mid k \in \mathcal{K}, r \in \mathcal{R}, u \in \mathsf{R}) \\ &= \mathcal{T} *_{\mathsf{T}} (\mathcal{K} *_{\mathsf{R}} \mathcal{R}).\end{aligned}$$

□

This leads immediately to:

Lemma 4.2.10. *Let $\mathcal{K}$ be a collection of objects in a* T-R*-bimodule* K*. Then*

$$\mathrm{Rloc}^{*\mathsf{R}}(\mathrm{Lloc}^{*\mathsf{T}}(\mathcal{K})) = \mathrm{Lloc}^{*\mathsf{T}}(\mathrm{Rloc}^{*\mathsf{R}}(\mathcal{K})) = \mathrm{Biloc}^{\mathsf{T}\text{-}\mathsf{R}}(\mathcal{K}).$$

Considering [Ste13, 3.13] for left and right actions simultaneously we obtain:

Lemma 4.2.11. *Let* K *be a* T-R*-bimodule. Suppose both* T *and* R *are generated as localising subcategories by their respective tensor units. Then every localising subcategory of* K *is a* T-R*-subbimodule.*

So far we have not mentioned any sort of rigidity conditions or required compact generation. To introduce rigidity we will use the following characterisation of duals.

Definition 4.2.12. Let T be a mt-category. An object $X \in \mathsf{T}$ is *left-dualisable* if there exists an object $X^\vee$ (called the *left dual* of X) together with evaluation and coevaluation maps

$$\mathrm{ev}\colon X^\vee \otimes X \longrightarrow \mathbf{1} \text{ and } \mathrm{coev}: \mathbf{1} \longrightarrow X \otimes X^\vee$$

such that the compositions

$$X \overset{\mathrm{coev}\otimes\mathrm{id}}{\longrightarrow} X \otimes X^\vee \otimes X \overset{\mathrm{id}\otimes\mathrm{ev}}{\longrightarrow} X$$

and

$$X^\vee \overset{\mathrm{id}\otimes\mathrm{coev}}{\longrightarrow} X^\vee \otimes X \otimes X^\vee \overset{\mathrm{ev}\otimes\mathrm{id}}{\longrightarrow} X^\vee$$

are the identity maps on X and $X^\vee$ respectively.

An object $X \in \mathsf{T}$ is *right-dualisable* if there exists an object ${}^\vee X$ together with evaluation and coevalutation maps

$$\mathrm{ev}'\colon X \otimes {}^\vee X \longrightarrow \mathbf{1} \text{ and } \mathrm{coev}'\colon \mathbf{1} \longrightarrow {}^\vee X \otimes X$$

such that the compositions

$$X \overset{\mathrm{id}\otimes\mathrm{coev}'}{\longrightarrow} X \otimes {}^\vee X \otimes X \overset{\mathrm{ev}'\otimes\mathrm{id}}{\longrightarrow} X$$

and

$${}^\vee X \overset{\mathrm{coev}'\otimes\mathrm{id}}{\longrightarrow} {}^\vee X \otimes X \otimes {}^\vee X \overset{\mathrm{id}\otimes\mathrm{ev}'}{\longrightarrow} {}^\vee X$$

are the identity maps on X and ${}^\vee X$ respectively.

We call an object $X \in \mathsf{T}$ *rigid* if it is both left and right dualisable. Note that the left and right duals of an object need not agree in general.

We can now introduce our preferred set of hypotheses on future mt-categories.

Definition 4.2.13. A *rigidly-compactly generated monoidal triangulated category* is a triple $(\mathsf{T}, \otimes, \mathbf{1})$ where T is a compactly generated monoidal triangulated category, and $(\otimes, \mathbf{1})$ is a monoidal structure on T such that the tensor product $\otimes$ is a coproduct preserving exact functor in each variable, and the compact objects T^c form a rigid tensor subcategory. In particular we require $\mathbf{1}$ to be compact. We will refer to such a category T as a *big mt-category.*

In the presence of rigidity and the strongly dualisable condition, we can restrict actions to compact objects.

Lemma 4.2.14. *Suppose* $\mathsf{T} \times \mathsf{K} \xrightarrow{*} \mathsf{K}$ *is a left action where* T *is rigidly-compactly generated and* K *is compactly generated. Then the action restricts to an action at the level of compact objects* $\mathsf{T}^c \times \mathsf{K}^c \xrightarrow{*} \mathsf{K}^c$*. Similarly if* $\mathsf{K} \times \mathsf{R} \xrightarrow{*} \mathsf{K}$ *is a right action where* R *is rigidly-compactly generated and* K *is compactly generated then the action restricts to an action at the level of compact objects* $\mathsf{K}^c \times \mathsf{R}^c \xrightarrow{*} \mathsf{K}$*.*

Proof. The proof is identical to [Ste13, 4.6]. The only subtlety is that as we are not in the symmetric setting, in the case of the left action we use the left dual, while in the case of the right action we use the right dual. □

We have the following lemma for localising subcategories generated by compact objects. This is the two-sided, non-symmetric version of [Ste13, 4.9].

Proposition 4.2.15. *Suppose* K *is a* T-R*-bimodule, with* K, T*, and* R *compactly generated, in such a way that we may restrict and consider* K^c *as a* T^c-R^c*-bimodule. Let* $\mathcal{L} \subseteq \mathsf{T}$ *be a left* $\otimes_{\mathsf{T}}$*-ideal generated, as a localising sucategory, by compact objects of* T*. Let* $\mathcal{M} \subseteq \mathsf{K}$ *be a localising subcategory generate by objects of* K^c*, and finally let* $\mathcal{N} \subseteq \mathsf{R}$ *be a right* $\otimes_{\mathsf{R}}$*-ideal generated, as a localising subcategory, by compact objects of* R*. Then the subcategory* $\mathcal{L} *_{\mathsf{T}} \mathcal{M} *_{\mathsf{R}} \mathcal{N}$ *is also generated, as a localising subcategory, by compact objects of* K*.*

Proof. The proof is identical to [Ste13] Proposition 4.9, applied to each side individually. □

Here is a two-sided ideal version of smashing localisation:

Theorem 4.2.16. *Let* T *be a rigidly-compactly generated mt-category,* $\mathcal{S} \subseteq \mathsf{T}^c$ *a set of compact objects, and* $\mathsf{S} = \mathrm{Biloc}(\mathcal{S})$ *i.e* S *is the smallest two-sided ideal containing* $\mathcal{S}$*. Consider the corresponding smashing localisation sequence*

$$\mathsf{S} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathsf{T} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathsf{S}^{\perp} \qquad (i_* \dashv i^!,\ j^* \dashv j_*)$$

Then

1. $\mathsf{S}^{\perp}$ *is a localising two-sided ideal of* T*.*

2. *there are isomorphisms of functors* $i_*i^!\mathbf{1} \otimes (-) \cong i_*i^!$ *and* $j_*j^*\mathbf{1} \otimes (-) \cong j_*j^*$*.*

3. *the objects* $i_*i^!\mathbf{1}$ *and* $j_*j^*\mathbf{1}$ *satisfy*

$$\begin{aligned} i_*i^!\mathbf{1} \otimes i_*i^!\mathbf{1} &\cong i_*i^!\mathbf{1}, \\ j_*j^*\mathbf{1} \otimes j_*j^*\mathbf{1} &\cong j_*j^*\mathbf{1}, \\ i_*i^!\mathbf{1} \otimes j_*j^*\mathbf{1} &\cong 0, \\ j_*j^*\mathbf{1} \otimes i_*i^!\mathbf{1} &\cong 0. \end{aligned}$$

Proof. First note that as $\mathrm{Biloc}(\mathcal{S}) = \mathsf{T} * \mathcal{S} * \mathsf{T}$ then by the previous proposition $\mathrm{Biloc}(\mathcal{S})$ is generated as a localising subcategory by objects of T^c and so the smashing localisation sequence exists. We now proceed with the proofs.

1. We first show that $\mathsf{S}^\perp$ is a *left* ideal. Consider the full subcategory $LM = \{X \in \mathsf{T} \mid X \otimes \mathsf{S}^\perp \subseteq \mathsf{S}^\perp\}$. As the tensor product is exact and coproduct preserving, Lemma 2.1.18 applies to the family of functors $(-\otimes S)_{S \in \mathsf{S}^\perp}$, and so M is localising. Now consider a compact object $t \in \mathsf{T}^c$ and an object $Y \in \mathsf{S}^\perp$. Then for all $Z \in \mathsf{S}$ we have

$$\begin{aligned} \mathrm{Hom}_\mathsf{T}(Z, t \otimes Y) &\cong \mathrm{Hom}_\mathsf{T}(t^\vee \otimes Z, Y) \text{ as } t \text{ is left dualisable} \\ &\cong 0 \text{ as } \mathsf{S} \text{ is a left ideal and } Y \in \mathsf{S}^\perp. \end{aligned}$$

   Therefore $t \otimes Y \in \mathsf{S}^\perp$ for all $Y \in \mathsf{S}^\perp$ and so $\mathsf{T}^c \subseteq M$. In particular $M = \mathsf{T}$ and so $\mathsf{S}^\perp$ is a *left* ideal. The proof that $\mathsf{S}^\perp$ is a right ideal is almost identical: we lay it out below. Consider the full subcategory $RM = \{X \in \mathsf{T} \mid \mathsf{S}^\perp \otimes X \subseteq \mathsf{S}^\perp\}$. As the tensor product is exact and coproduct preserving, Lemma 2.1.18 applies to the family of functors $(S \otimes -)_{S \in \mathsf{S}^\perp}$, and so M is localising. Now consider a compact object $t \in \mathsf{T}^c$ and an object $Y \in \mathsf{S}^\perp$. Then for all $Z \in \mathsf{S}$ we have

$$\begin{aligned} \mathrm{Hom}_\mathsf{T}(Z, Y \otimes t) &\cong \mathrm{Hom}_\mathsf{T}(Z \otimes {}^\vee t, Y) \text{ as } t \text{ is right dualisable} \\ &\cong 0 \text{ as } \mathsf{S} \text{ is a right ideal and } Y \in \mathsf{S}^\perp. \end{aligned}$$

   Therefore $Y \otimes t \in \mathsf{S}^\perp$ for all $Y \in \mathsf{S}^\perp$ and so $\mathsf{T}^c \subseteq M$. In particular $M = \mathsf{T}$ and so $\mathsf{S}^\perp$ is a *right* ideal. We have shown that $\mathsf{S}^\perp$ is both a left ideal and a right ideal, and so we conclude that $\mathsf{S}^\perp$ is a two-sided ideal.

2. Consider the localising triangle for the tensor unit $\mathbf{1}$:

$$i_*i^!\mathbf{1} \longrightarrow \mathbf{1} \longrightarrow j_*j^*\mathbf{1} \longrightarrow \Sigma i_*i^!\mathbf{1}.$$

   We can tensor this triangle with any object $X \in \mathsf{T}$ *on the right* to obtain a triangle

$$i_*i^!\mathbf{1} \otimes X \longrightarrow X \longrightarrow j_*j^*\mathbf{1} \otimes X \longrightarrow \Sigma i_*i^!\mathbf{1} \otimes X.$$

   Now $i_*i^!\mathbf{1} \otimes X \in \mathsf{S}$ and $j_*j^*\mathbf{1} \otimes X \in \mathsf{S}^\perp$ so by uniqueness of localisation triangles the triangle is isomorphic to the localisation triangle

$$i_*i^!X \longrightarrow X \longrightarrow j_*j^*X \longrightarrow \Sigma i_*i^!X.$$

Therefore $i_*i^!\mathbf{1} \otimes X \cong i_*i^!X$ and $j_*j^*\mathbf{1} \otimes X \cong j_*j^*X$, and these isomorphisms assemble to an isomorphism of the corresponding functors $i_*i^!\mathbf{1} \otimes - \cong i_*i^!$ and $j_*j^*\mathbf{1} \otimes - \cong j_*j^*$. The other direction can be obtained by tensoring *on the left* to obtain a triangle

$$X \otimes i_*i^!\mathbf{1} \longrightarrow X \longrightarrow X \otimes j_*j^*\mathbf{1} \longrightarrow X \otimes \Sigma i_*i^!\mathbf{1}.$$

and by the same argument we conclude that $-\otimes i_*i^!\mathbf{1} \cong i_*i^!$ and $-\otimes j_*j^*\mathbf{1} \cong j_*j^*$.

3. Consider the localisation triangle for the unit $\mathbf{1}$:

$$i_*i^!\mathbf{1} \longrightarrow \mathbf{1} \longrightarrow j_*j^*\mathbf{1} \longrightarrow \Sigma i_*i^!\mathbf{1}.$$

Tensoring with $i_*i^!\mathbf{1}$ we obtain

$$i_*i^!\mathbf{1} \otimes i_*i^!\mathbf{1} \longrightarrow i_*i^!\mathbf{1} \longrightarrow i_*i^!\mathbf{1} \otimes j_*j^*\mathbf{1} \longrightarrow i_*i^!\mathbf{1} \otimes \Sigma i_*i^!\mathbf{1}.$$

Now as $i_*i^!\mathbf{1} \in \mathsf{S}$ and $j_*j^*\mathbf{1} \in \mathsf{S}^\perp$, we have $i_*i^!\mathbf{1} \otimes j_*j^*\mathbf{1} \in \mathsf{S} \cap \mathsf{S}^\perp$ and so $i_*i^!\mathbf{1} \otimes j_*j^*\mathbf{1} \cong 0$. The same argument shows that $j_ast \otimes j^*\mathbf{1} \otimes i_*i^!\mathbf{1} \cong 0$. Therefore the localisation triangle reduces to

$$i_*i^!\mathbf{1} \otimes i_*i^!\mathbf{1} \longrightarrow i_*i^!\mathbf{1} \longrightarrow 0 \longrightarrow i_*i^!\mathbf{1} \otimes \Sigma i_*i^!\mathbf{1}.$$

and we conclude that $i_*i^!\mathbf{1} \otimes i_*i^!\mathbf{1} \cong i_*i^!\mathbf{1}$. Finally, to prove that $j_*j^*\mathbf{1} \otimes j_*j^*\mathbf{1} \cong j_*j^*\mathbf{1}$, consider the localisation triangle for the unit $\mathbf{1}$ and tensor with $j_*j^*\mathbf{1}$ to obtain

$$j_*j^*\mathbf{1} \otimes i_*i^!\mathbf{1} \longrightarrow j_*j^*\mathbf{1} \longrightarrow j_*j^*\mathbf{1}j_*j^*\mathbf{1} \longrightarrow j_*j^*\mathbf{1} \otimes \Sigma i_*i^!\mathbf{1}.$$

This reduces to a triangle

$$0 \longrightarrow j_*j^*\mathbf{1} \longrightarrow j_*j^*\mathbf{1}j_*j^*\mathbf{1} \longrightarrow 0$$

and we conclude $j_*j^*\mathbf{1} \otimes j_*j^*\mathbf{1} \cong j_*j^*\mathbf{1}$.

□

§ 4.3 | The non-commutative spectrum of the A_2 enveloping algebra

Consider the following quiver

$$A_2 = 1 \xrightarrow{\alpha} 2.$$

with path algebra $\Gamma = kA_2$. There are right and left projectives $P_i = \Gamma(-,i)$ and $P^i = \Gamma(i,-)$ respectively. In particular we have

$$\begin{aligned} P_1 &= ke_1 \\ P_2 &= k\alpha \oplus ke_2 \\ P^1 &= ke_1 \oplus k\alpha \\ P^2 &= ke_2 \end{aligned}$$

We consider the quiver defining the enveloping algebra, namely $A_2^e = A_2 \otimes A_2^{\mathrm{op}}$. This is given by the following quiver:

$$\begin{array}{ccc} (1,2) & \xrightarrow{\alpha\otimes e_2} & (2,2) \\ {\scriptstyle e_1\otimes\alpha}\downarrow & & \downarrow{\scriptstyle e_2\otimes\alpha} \\ (1,1) & \xrightarrow[\alpha\otimes e_1]{} & (2,1) \end{array}$$

As our particular example is given by A_2^e, we can apply the specific results in [Kel07, 2.9, 4.4], which show that the path algebra kA_2^e is derived equivalent to the path algebra of the Dynkin quiver D_4, shown without orientation below:

$$\begin{array}{ccccc} 1 & — & 2 & — & 3 \\ & & | & & \\ & & 4 & & \end{array}$$

Note that the result used from [Kel07] is specific to A_2^e and D_4. The quiver D_4 has twelve indecomposable modules. As the path algebra of D_4 is finite-dimensional hereditary, every object in $\mathsf{D}(kD_4)$ is quasi-isomorphic to its cohomology complex and so up to suspension $\mathsf{D}(kD_4)$ has twelve indecomposable objects. Therefore by derived equivalence $\mathsf{D}(kA_2^e)$ has twelve indecomposable objects up to suspension. We will describe these indecomposables, as well as their representations and projective resolutions. The maps in the projective resolutions are the obvious ones. Eleven of these are given by the indecomposable modules of the commutative square. There are four indecomposable projective modules, which we denote $P_{(1,2)}, P_{(2,2)}, P_{(1,1)}$ and $P_{(2,1)}$.

1. The module $P_{(1,2)}$ has representation $\begin{array}{ccc} k & \longrightarrow & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \end{array}$

2. The module $P_{(2,2)}$ has representation $\begin{array}{ccc} 0 & \longrightarrow & k \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & k \end{array}$

3. The module $P_{(1,1)}$ has representation $\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \end{array}$

4. The module $P_{(2,1)}$ has representation $\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & k \end{array}$

There are four indecomposable simple modules, one of which is $P_{(2,1)}$. We denote the other three by $S_{(1,2)}, S_{(2,2)}$ and $S_{(1,1)}$.

1. The module $S_{(1,2)}$ has representation $\begin{array}{ccc} k & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$. The module has projective resolution
$$P_{(2,1)} \longrightarrow P_{(2,2)} \oplus P_{(1,1)} \longrightarrow P_{(1,2)}.$$

2. The module $S_{(2,2)}$ has representation $\begin{array}{ccc} 0 & \longrightarrow & k \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$. The module has projective resolution $P_{(2,1)} \longrightarrow P_{(2,2)}$.

3. The module $S_{(1,1)}$ has representation $\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ k & \longrightarrow & 0 \end{array}$. The module has projective resolution $P_{(2,1)} \longrightarrow P_{(1,1)}$

There are four more indecomposable modules, which we label E, F, I and J.

1. The module E has representation $\begin{array}{ccc} k & \longrightarrow & k \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$. The module has projective resolution $P_{(1,1)} \longrightarrow P_{(1,2)}$.

2. The module F has representation $\begin{array}{ccc} k & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ k & \longrightarrow & 0 \end{array}$. The module has projective resolution $P_{(2,2)} \longrightarrow P_{(1,2)}$.

3. The module I has representation $\begin{array}{ccc} k & \longrightarrow & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & 0 \end{array}$. The module has projective resolution $P_{(2,1)} \longrightarrow P_{(1,2)}$.

4. The module J has representation $\begin{array}{ccc} 0 & \longrightarrow & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \end{array}$. The module has projective resolution $P_{(2,1)} \longrightarrow P_{(2,2)} \oplus P_{(1,1)}$

The final indecomposable object is denoted by K and is given by

$$K = P_{(2,2)} \oplus P_{(1,1)} \longrightarrow P_{(1.2)}.$$

We want to classify the two sided prime ideals of the derived category $\mathsf{D}(A_2^e)$. Such ideals will be entirely determined by which indecomposables they contain. The following will be useful:

Lemma 4.3.1. *Let Q be a (finite) quiver. If S is a two-sided ideal of $\mathsf{D}(kQ^e)$ containing a projective indecomposable $P_{(i,j)}$, then S contains all indecomposable projectives. In particular $\mathsf{S} = \mathsf{D}(kQ^e)$.*

Proof. Consider a projective $P_{(x,y)}$. We calculate the tensor product:

$$\begin{aligned} P_{(x,i)} \otimes P_{(i,j)} \otimes P_{(j,y)} &\cong {P_{(x,j)}}^{\oplus |Q(i,i)|} \otimes P_{(j,y)} \\ &\cong (P_{(x,j)} \otimes P_{(j,y)})^{\oplus |Q(i,i)|} \\ &\cong ({P_{(x,y)}}^{\oplus |Q(j,j)|})^{\oplus |Q(i,i)|} \\ &\cong {P_{(x,y)}}^{\oplus |Q(i,i)||Q(j,j)|}. \end{aligned}$$

where $Q(i,j)$ denotes the set of arrows between vertices i and j in Q. As thick tensor ideals are closed under summands we have $P_{(x,y)} \in \mathsf{S}$. As the choice of projective was arbitrary it follows that S contains all projectives. As $\mathsf{D}(kQ^e)$ is generated by the algebra, which itself is a sum of projectives, it follows that $\mathsf{S} = \mathsf{D}(kQ^e)$. □

We can now compute the spectrum directly in the case

$$A_2 = 1 \xrightarrow{\alpha} 2.$$

Proposition 4.3.2. *The spectrum of prime thick two-sided ideals of $\mathsf{D}(kA_2^e)$ is a point.*

Proof. By the previous lemma, we need only show that for every indecomposable object $X \in \mathsf{D}(kA_2^e)$ the thick two-sided ideal $\mathrm{thick}^{\otimes}(X)$ contains a projective indecomposable. Obviously this holds for the projectives, so we need only check the remaining eight indecomposables.

We calculate directly:

$$\begin{aligned} P_{(1,2)} \otimes S_{(2,2)} \otimes P_{(2,1)} &\cong P_{(1,1)} \\ P_{(1,1)} \otimes S_{(1,1)} &\cong P_{(1,1)} \\ E \otimes P_{(2,1)} &\cong P_{(1,1)} \\ P_{(1,1)} \otimes F &\cong P_{(1,2)} \\ P_{(1,1)} \otimes I &\cong P_{(1,2)} \\ J \otimes P_{(2,1)} &\cong P_{(2,1)} \\ P_{(1,1)} \otimes K \otimes P_{(2,1)} &\cong P_{(1,1)} \\ P_{(1,1)} \otimes S_{(1,2)} \otimes P_{(2,1)} &\cong P_{(1,1)} \end{aligned}$$

from which we conclude that the thick two-sided tensor ideals containing an indecomposable are equal to $\mathsf{D}(kA_2^e)$ and hence only the zero ideal is prime. □

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