Essays on cost and surplus sharing games and contest theory

Thesis by Ngoc Anh Pham

Submitted in fulfilment of the requirements for the
Degree of Doctor of Philosophy

Adam Smith Business school
University of Glasgow
September 2022

Abstract

This thesis contains three essays on cost and surplus-sharing games. In the first chapter, *Lorenz comparison between increasing serial and Shapley value cost-sharing rule*, I consider the cost-sharing problem using a cooperative approach and compare the Shapley value and the Moulin-Shenker's (Increasing) serial rule in the Lorenz sense. The result allows me to provide the complete ordering in inequality among the four popular sharing rules: The average share, the Shapley value, the Increasing serial, and the Decreasing serial sharing rule.

In the later two chapters, I study the Tullock contest, which is a different interpretation of the non-cooperative surplus sharing game with the average rule. My second chapter, *Underperforming and Outperforming contestant: Who to support?*, consider a repeated Tullock contest where the designer has the option to favour either the early winner or the loser in the subsequent round, aiming to maximise the total effort. In the later section of the chapter, the contest designer is granted the additional capability to determine the prize distribution between the two stages. My work focuses on the optimal biasing decision and allocation of prizes to achieve the highest level of effort.

In the third chapter, *Benefits of Intermediate Competition with Non-Monetary Incentive*, the underlying assumption is that the designer places importance not only on the total effort invested in the contest but also on the contest's selection accuracy. I focus on investigating the impact of introducing an intermediate stage with a non-monetary incentive on the contest's performance in both dimensions. By analysing the contest from this perspective, my work provides insights into the potential advantages and enhancements that can be achieved through the inclusion of intermediate competition and non-monetary incentives.

Contents

List of Figures

Acknowlegement

First and foremost, I need to express deep gratitude to my supervisors, Professor Anna Bogomolnaia and Professor Hervé Moulin, for their invaluable guidance and support at every step of my PhD journey. Hervé's undergraduate Game theory and Mechanism design courses were where I developed an interest and found inspiration to learn Game theory and pursue my PhD in Microeconomics theory. I have greatly benefited from his insightful comments and sharp advice. Undoubtedly, Hervé is the most inspiring teacher I have ever had during my time as a student, and I feel fortunate and honored to be one of his students. I am also immensely grateful to Anna for being such a motivating and understanding supervisor. She has given in me confidence, encouragement, and much more. Her positive energy, open-mindedness, and kindness have taught me invaluable lessons about mentoring that I will surely carry with me throughout life.

During my time of study at the University of Glasgow, I have been fortunate to interact with brilliant academic researchers and receive valuable comments and suggestions in lectures, seminars, and workshops. In particular, I would like to express my gratitude to Prof. Takashi Hayashi, Dr Francesca Flamini, Dr John Levy, Dr Constantine Sorokin, and Prof. Sayantan Goshal. Additionally, I would like to extend my appreciation to Professor Subhasish M. Chowdhury for his valuable comments and suggestions, which have significantly improved my thesis.

On a different note, I am grateful to all the friends and colleagues I have met during this challenging PhD journey, who have made it enjoyable and less burdensome. This includes Maria, Anna, Ashraf, Rohan, Mary, Helena, Faii, a.Trên, Elizaveta, Jerome, Damiano, Fivos, Nikolas, Jinglin, Mumin, Toon, and all others whom I may have inadvertently omitted. I would also like to express my thanks to the supportive and helpful administrative staff, Sophie Watson and Christine Arthone. Additionally, I would like to acknowledge my Undergraduate tutor, Dr Josué Ortega, for introducing me to the MRes programme, where I acquired the skills and knowledge necessary to successfully complete my PhD. Josué has been a great mentor,

providing invaluable support during my final year of the PhD, and I am sincerely grateful for his guidance.

During my MRes and PhD studies, I had the opportunity to serve as a GTA (Graduate Teaching Assistant), which provided me with valuable insights and experiences. I would like to extend my gratitude to Dr Geethanjalt Selvaretnam and Dr Eva Pocher for their helpful tips and advice during my initial years as a GTA. Their guidance played a crucial role in shaping the beginning of my teaching career.

A special thanks from my heart to Professor Robert Downie and Arsenii Paramonov, Robin and Senya to me, for being with me throughout the journey. Robin is the most wonderful friend, teacher, and grandfather figure, and I am deeply grateful for his kindness. He and his family have warmly embraced me as part of their own, providing a sense of home while I am away. I am also incredibly thankful to Senya for being a sincere friend and a kind-hearted companion. Number 17 Ruskin Lane holds a special place in my heart, as it is filled with cherished memories with them. These memories will be forever treasured. I am certain that Robin would be delighted to know that I am doing well and have developed an interest in philosophy. I will always fondly remember the moments we shared, particularly our piano lessons at Half past four.

Last but definitely not least, I would like to give my gratitude to my family. I could never reach this far without their support. Con cám ơn gia đình luôn yêu thương và quan tâm con. Con biết ơn, và yêu bố mẹ nhiều.

Authors declaration

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Ngoc Anh Pham
September 2022

Introduction

Numerous decision-making scenarios entail the distribution of shared expenses or resources in situations involving both cooperative and non-cooperative behaviors. This can range from allocating water, data resources, or electricity costs among users to assigning bonuses within a firm or sharing market revenue among companies.The allocation problem revolves around agents sharing a joint cost (surplus) subjected to their different demands (inputs). This problem has been extensively examined and finds application across various domains within the economics literature. In my thesis, I focus on a setting with a one-input-one-output technology, where agents contribute homogeneous inputs and consume a homogeneous output commodity.

Since the seminal work by Shapley, Roth, et al. (1988), there has been a significant body of research dedicated to the axiomatic analysis of sharing mechanisms using a cooperative approach (Moulin, 2002). A major focus of these studies has been evaluating the properties of sharing rules in settings involving production with variable returns. The most widely used rule is the average rule, where agents receive a share proportional to their inputs or demands, resulting in an equal average cost or surplus allocation. However, despite its popularity, the average rule has raised various concerns (Moulin and Shenker, 1994).

To address the limitations observed with the average rule, several alternative rules have been introduced. Among these, the most prominent are the serial rules proposed by Moulin and Shenker (1992) and the decreasing serial rule developed by de Frutos (1998). These rules aim to mitigate the issues arising from the average rule. In addition to the axiomatic characterization of these popular rules, recent literature has examined the inequality resulting from their respective sharing allocations (Chen et al., 2020; Hougaard and Østerdal, 2009).

In the first chapter, I examine the cost-sharing problem through a cooperative approach and compare the allocation inequality in the the Lorenz sense between the Shapley value and Moulin-Shenker's (Increasing) serial rule. The concept of Lorenz dominance is utilized to determine which sharing allocation is more egalitarian. When one sharing allocation Lorenz dominates another, it indicates that the corresponding sharing rule is considered more egalitarian. I find that when the cost function is concave (convex), representing a positive externality of production, the increasing serial rule is more egalitarian than the Shapley value. Conversely, when the cost function is convex, indicating a negative externality of production, the increasing serial rule is less egalitarian than the Shapley value. Additionally, I provide a direct comparison between de Frutos's (Decreasing) serial rule and the Shapley value using a similar proof technique. These findings enable me to establish a complete ordering of inequality among the four popular sharing rules: the average share, the Shapley value, the Increasing serial rule, and the Decreasing serial rule. This ordering allows for the recommendation of the most appropriate rule in situations where equality is considered a social value.

In the second and third chapters of my thesis, I am focusing on the lottery Tullock contest within a framework of perfect information. The Tullock rent-seeking model, which was first introduced in 1980, has gained significant interest as a contest model. In my models, agents exert effort to compete for a fixed prize while incurring linear costs of effort. This setup is analytically identical to the non-cooperative surplus sharing game with a fixed surplus and a proportional sharing rule. The contest models presented in my later two chapters capture many real-life situations, including competitions for job promotions, educational contests, and races in research and development (R&D). With the two-stage contest framework, I aim to provide insights and analyses that can be applied to a wide range of practical situations. The framework belongs to the broader discussion in dynamic contests (see K. Konrad, 2009 for a survey), particularly where there is a biasing element in the later stage. It is also relates to the ongoing research interest in heterogeneity and handicapping policy in contests, as discussed in Chowdhury et al. (2023).

In the second chapter, I study a repeated Tullock contest with two heterogeneous agents. Specifically, I explore the scenario where the contest designer has the option to favor either the early winner or loser by reducing their marginal cost of effort in the subsequent round. The objective is to determine the optimal decision that maximizes total effort. Through my analysis, I discover that the optimal decision depends on the level of heterogeneity between the players. At each cost reduction level, there exists a threshold for player heterogeneity, beyond which the optimal decision switches from favoring the loser to favoring the winner. In general, if the players do not exhibit significant differences, it is more advantageous to favor the winner. However, if there is substantial heterogeneity, it is often optimal to favor the loser. Additionally, when the contest designer has the ability to determine the prize distribution across the two stages, I find that it is preferable to allocate all the prize to the second stage when the players are similar in terms of their characteristics. Conversely, when the players display significant differences, it is optimal to have a smaller prize in the first stage, regardless of which player is favoured.

By exploring these dynamics, I provide insights into the optimal strategies for contest designers in determining how to allocate resources and influence effort provision based on player heterogeneity in a repeated Tullock contest. My research makes a contribution to the ongoing discourse on optimal favoritism in contests (Franke, 2012; Franke et al., 2018; Nti, 2004 . Many existing papers on this topic assume that the designer has full information about the players' types (Beviá and Corchón, 2013; J. Thomas and Wang, 2017). However, such information may not be readily available in real-life situations. In my model, the designer makes the decision of whom to favor solely based on the observed outcomes of the previous stage, without having direct knowledge of the players' types. This approach aligns with the assumption made by Ridlon and Shin (2013) in studying optimal handicapping policies, examining whether the optimal policy is in favor the early winner or the early loser. Their paper; therefore, also assume that the contest designer has full control over the impact of the handicapping, and decides the handicapping level to maximise total effort given the heterogeneity of the contestants. Meanwhile, in my work, the contest designer takes the handicapping factor as given, and only chooses the recipient. Therefore, his decision depends on both the level of heterogeneity and the magnitude of the bias/handicap factor. However, the fact that it is often effort maximising to favour the winner regardless of the bias factor, and only beneficial to favor the loser when the players exhibit extreme differences suggests that the designer may not require detailed information about the participants to make the optimal decision.

In the third chapter, I investigate further into the effects of the winning advantage on contest performance and introduce two additional properties to the set of objectives for the contest designer. I demonstrate that the inclusion of an intermediate stage and a winning advantage can lead to a (weak) improvement in total effort, selection accuracy (probability of choosing the most able candidate), and participation rate compared to a static contest. By allowing players to compete in the intermediate stage and subsequently reducing their costs in the following stage, there exists a range of cost reduction factors that enables the contest designer to achieve a higher effort level than that of the original static contest, without compromising the selection accuracy.

This finding emphasises the potential advantages of introducing an intermediate stage and providing winning advantages, which take the form of non-monetary rewards, in contests. It contributes to the ongoing discussion on dynamic contests, where early results have an impact on subsequent performance (Iluz and Sela, 2018; Klein and Schmutzler, 2017), particularly regarding the benefits of winning advantages (Clark and Nilssen; Clark et al.; Möller (2018, 2020, 2012)). It allows the contest designer to enhance overall contest performance by increasing effort levels, maintaining or even improving selection accuracy. These insights can inform the design and implementation of contests in various domains where these objectives are of importance.

It is worth noting that not many papers within the contest literature have delved into the trade-off between different contest properties (Bimpikis et al., 2019; Stracke et al., 2015; Tsoulouhas et al., 2007). Moreover, this work stands out as one of the few papers that explicitly highlight the possibility of improving both the incentive and selection properties when transitioning from one contest structure to another. Additionally, this paper proposes the intriguing topic of participation grouping. These discussions offer further avenues for exploration and potential extensions of the research, allowing for a deeper understanding of how contest structures and participant groupings can impact contest outcomes. Furthermore, this study offers theoretical hypotheses for future experimental research. The potential applications of this research extend to educational and workplace settings, where the findings can be utilised to enhance contest design, particularly in participant's performance, motivation, and selection processes.

Chapter 1

Lorenz comparison between increasing serial and Shapley value cost-sharing rule

Abstract

[1] We consider the cost (surplus) sharing problem when a coalition of agents operates under a joint production technology and share the total cost (resp. output) subjected to their individual demands (resp. input). We consider the four most popular rules: the average (proportional) sharing rule, the Shapley value, the Moulin-Shenker's (Increasing) serial rule and the decreasing serial rule and compare the allocation inequality among the resulting sharing allocations in the Lorenz sense. We provide the complete Lorenz ordering in inequality among the four sharing allocations by showing that Shapley value dominates (is dominated by) the increasing serial shares when the marginal is decreasing (increasing).

[1] A shorter version of this work was published in Pham (2019)

1.1 Introduction

We consider the problem where agents share the common technology and discuss the fair distribution of total cost (output). In the cost sharing problem, each agent i has demand for output x_i, and pay the share of the total cost $C(x_N)$ for $x_N = \sum_{i=1}^{n} x_i$. Example includes dividing the telephone bills (Billera et al., 1978), sharing the cost of water supply or electricity lines among users (Shubik, 1962). Cost sharing problem could also be seen in the computer network, telephone system or chain stores (Moulin and Shenker, 1992). Meanwhile, in the surplus sharing problem, agents input resources or labours x_i and share the total production (surplus) of $F(x_N)$. A sharing rule assigns to agent i the cost (surplus) y_i given the demand (input) profiles so that $\sum_{i=1}^{n} y_i = C(x_N)$ (resp. $F(x_N)$). The examples could be the producer cooperative, where the total production is distributed according to labour (Sen, 1966).

When the cost (production) function is linear, there is no externality. The most natural solution to the sharing problem is to assign every agent his own cost (surplus). When the marginal cost (surplus) varies, suggesting the externality of production, the problem becomes less trivial. It is, therefore, not straightforward to determine whether one sharing rule is fairer than the other. Overall, there are several tests for a rule to be considered plausible, for example, the core, the stand-alone test, and the unanimity bound. We discuss the allocation inequality test, which is not normatively one-sided and often depends on the context. We will consider two benchmark cases: when the marginal is increasing and when the marginal is decreasing, corresponding to positive externality (increasing marginal production and decreasing marginal cost) and negative externality (the reverse).

For example, when the cost function is marginally increasing, production has negative externalities. A less egalitarian rule (than average share) is more sensible in this case. Consider the cost function $C(x) = x^2$ and two agents whose demands are $x_1 = 1, x_2 = 9$. The average rule results in the share allocation (10,90). The allocation is considered unfair for the small agent, given that he only needs to pay $C(1) = 1$ if he produces by himself. The small agent, in this case, is overpaying because of the high average cost caused by the big user. On the other hand, when the cost function is marginally decreasing, which suggests positive externalities of production, a more egalitarian rule should be more plausible. In the situation where two agents with demands $x_1 = 1, x_2 = 99$ use the common technology costing $C(x) = \sqrt{x}$, the average share gives (0.1,0.99). Given the cost $C(1) = 1, C(99) = 9.94$ when the two agents produce alone, the average share is unfair for the agent 2 when he is more accountable for the cost reduction but benefits less from it, compared to the agent 1. The Shapley value rules resulting in allocation (0.52,9.27) still lets the agent 2 benefits more from the cost reduction. The increasing serial share rule suggests (0.7,9.3) and appears to be the most equitable. The most egalitarian allocation (2.96,7.03) given by the decreasing serial rule can be considered too hard for the agent 1, and violate the Stand alone test. The decreasing serial rule should only be used when the Stand alone test is satisfied, or is not relevant. For example, when the production requires a huge sunk cost, or when the agents do not have the option of producing themselves (examples include sharing cost of water system, or setting the electric lines).

In the context of surplus sharing, the normative arguments depends greatly on the interpretation of input. Consider the production function $F(x) = x^2$, suggesting positive externality. For the two agents with demand $x_1 = 1$, $x_2 = 9$, the average rule results in the allocation (10, 90). The additional 18 units gained from the increasing marginal productivity is divided equally. If the input the level of effort, one might suggest that the hard working agent with higher effort level should benefit more from the gain. A less egalitarian allocation than average share should be used. Meanwhile, with negative externality illustrated by the production function $F(x) = \sqrt{x}$ with $x_1 = 1$, $x_2 = 99$, the average allocation (0.1, 9.9) is the least favourable for agent 1. If the input is again effort, the allocation can be considered unfair for the small agent. Nonetheless, the allocation can be meaningful if the inputs are endowments or human capacity which the agents are not responsible for.

We compare the four sharing rules: Average share, Shapley value, and Increasing and Decreasing serial share in the egalitarian point of view, using the concept of Lorenz dominance. The concept of Lorenz comparison is the most unambiguous and standard instrument to evaluate distribution inequality. We say that one sharing rule is more egalitarian than the other if the resulting sharing allocation is Lorenz dominating. It should be noted that the Lorenz domination is not complete. We will give several examples to illustrate that the sharing rules are not always pairwise comparable. Recent literature has found that the comparison among Average-Increasing serial-Decreasing serial depends on the convexity of the cost (production) function (Hougaard and Thorlund-Petersen, 2001), and the comparison between Average-Shapley depends on the convexity of the marginal cost (Chen et al., 2020). We complete the comparison of the four rules with the pair Increasing serial-Shapley, and find that if the cost (production) is concave (convex), the increasing serial share rule is more (less) egalitarian than the Shapley value.

Marginals	Convex	Concave
Increasing $Average \succeq_L I.serial \succeq_L D.serial$, $Shapley \succeq_L I.serial$	$Shapley \succeq_L Average \succeq_L I.serial \succeq_L D.serial$	$Average \succeq_L Shapley \succeq_L I.serial \succeq_L D.serial$
Decreasing $D.serial \succeq_L I.serial \succeq_L Average$, $I.serial \succeq_L Shapley$	$D.serial \succeq_L I.serial \succeq_L Shapley \succeq_L Average$	$D.serial \succeq_L I.serial \succeq_L Average \succeq_L Shapley$

The requirement of convexity of the marginal cost for the comparison between Shapley-Average limits the number of cases where we can have the complete ordering of the four rules. However, the property is not critical since the Shapley value is often in the justifiable end. The comparison between Shapley value and Decreasing serial sharing rule is the direct result of the comparisons between Shapley value and Increasing serial and between Increasing and Decreasing serial rule, using the transitivity of the Lorenz order. It could also be directly proved using the similar arguments as the pair Shapley-Increasing serial. We will present the proof in the Appendix, together the alternative elementary proof for the comparison between the average and serial sharing rules.

Example 1.1. *Consider the cost function* $C(x) = x^{\alpha}$ *and two agents with demand* $x_1 = \lambda \leq 0.5, x_2 = 1-\lambda$.
The average rule gives $\phi_1^a = \lambda, \quad \phi_2^a = 1-\lambda$

The increasing serial rule gives $\phi_1^r = \frac{(2\lambda)^{\alpha}}{2}, \quad \phi_2^r = 1 - \frac{(2\lambda)^{\alpha}}{2}.$

The decreasing serial rule gives $\phi_1^d = 1 - \frac{(2-2\lambda)^{\alpha}}{2}, \quad \phi_2^d = \frac{(2-2\lambda)^{\alpha}}{2}.$

The Shapley value is $\phi_1^{Sh} = \frac{1}{2} + \frac{1}{2}\lambda^{\alpha} - \frac{1}{2}(1-\lambda)^{\alpha}, \quad \phi_2^{sh} = \frac{1}{2} - \frac{1}{2}(1-\lambda)^{\alpha} + \frac{1}{2}\lambda^{\alpha}.$

When $\alpha \in (0,1]$*, i.e, C is concave and C' is convex,*

$$\phi_1^d = 1 - \frac{(2-2\lambda)^{\alpha}}{2} \geq \phi_1^r = \frac{(2\lambda)^{\alpha}}{2} \geq \phi_1^{Sh} = \frac{1}{2} + \frac{1}{2}\lambda^{\alpha} - \frac{1}{2}(1-\lambda)^{\alpha} \geq \phi_1^a = \lambda$$

When $\alpha \in (1,2]$*, i.e., C is convex and C' is concave*

$$\phi_1^a = \lambda \geq \phi_1^{Sh} = \frac{1}{2} + \frac{1}{2}\lambda^\alpha - \frac{1}{2}(1-\lambda)^\alpha \geq \phi_1^r = \frac{(2\lambda)^\alpha}{2} \geq \phi_1^d = 1 - \frac{(2-2\lambda)^\alpha}{2}$$

When $\alpha > 2$. i,e., both C and C' is convex

$$\phi_1^{Sh} = \frac{1}{2} + \frac{1}{2}\lambda^\alpha - \frac{1}{2}(1-\lambda)^\alpha \geq \phi_1^a = \lambda \geq \phi_1^r = \frac{(2\lambda)^\alpha}{2} \geq \phi_1^d = 1 - \frac{(2-2\lambda)^\alpha}{2}$$

Example 1.2. *Consider the cost function $C(x) = max(x-10,0)$ and two input vectors $x = (2,3,5,10)$ and $x' = (4,4,6,6)$.*
The corresponding average allocation are $\phi^a = (1,1.5,2.5,5)$ and $\phi'^a = (2,2,3,3)$.
The corresponding Shapley values are $\phi^{Sh} = (4/3,1.5,3.25,47/12)$ and $\phi'^{Sh} = (4/3,4/3,11/3,11/3)$.
We can see that $\phi^{Sh} \succeq_L \phi^a$, but $\phi'^a \succeq_L \phi'^{Sh}$

1.2 The four sharing rules

The equitable division of a joint cost among agents with different demands for output is a central theme of fair division, as well as cooperative game theory. Several cost-sharing rules have been discussed in the literature, among which the average cost-sharing rule is one of the most popular. The rule distributes the cost proportionally to the demands and is the direct application of the proportional sharing rule. It is often viewed as "the single compelling cost-sharing method in the case of single homogeneous good" (Moulin and Shenker, 1994). The rule gains its popularity from the accounting simplicity and other desirable features. For example, average cost sharing is the only cost sharing mechanism that can avoid the threat of reallocation problem, where a group of agents can benefit from misreporting a different distribution of their total demands. There is no advantages gained from either merging or splitting. Average cost sharing is also the only cost sharing rule that satisfies both monotonicity (agent with higher demand should pay higher cost) and separable cost (if the cost is separable, it should be so allocated) properties (Moulin, 2002). However, there are certain concerns about the fairness of the rule. Moulin and Shenker (1994) argue that when the cost of production varies with the quantity produced, then two agents with different demands should not be equally responsible for the average return. When the cost function is decomposable, the average rule might result in an agent paying the cost higher than the cost of producing himself (Hougaard, 2009).

Moulin and Shenker (1992) introduced the serial cost sharing mechanism, which gives a more sensible allocation when the cost varies with quantity produced, and also a better one when the agents are strategic. When the return to scale is not constant, each agent with positive demand imposes externalities on others, for example, the case of sharing the waiting time in the computer network or telephone system. The serial rule divides the cost of such externalities so that each agent pays the equal share of the incremental cost, and is not responsible for the demands higher than of his own. It satisfies many desirable properties, both axiomatic and strategic ones. For example, the increasing serial share is the only rule allowing zero cost to the small agents whose demand is so small that the cost is still zero if all other agents consume as much as he does. It also ensures a unique non-cooperative equilibrium under convex cost and convex preferences (Moulin, 1996; Moulin and Shenker, 1992; Moulin and Shenker, 1994).

Following the arguments about the production externalities, de Frutos (1998) focused particularly on the situation of cost sharing under economies of scale and introduce the decreasing serial cost sharing rule which allocates the highest surplus to the one who has the highest demand, since she creates the largest production externalities. Unlike the Moulin-Shenker's serial rule, the Frutos's decreasing cost sharing rule allocates cost in the decreasing order of demand and has a normative drawback of violating the Stand Alone test. On the other hand, it is proved to satisfy attractive efficiency, strategic and welfare properties (de Frutos, 1998; Hougaard and Østerdal, 2009).

While the average and serial cost-sharing rules are particularly used for the fair division problem, the Shapley value is applied in many other different contexts in cooperative game theory. The Shapley value cost-sharing rule assigns each agent to his Shapley value of the induced cost sharing game. There is an extensive literature of the Shapley value and its use as a cost sharing rule (Aumann, 1994; Roth and Verrecchia, 1979; Shapley, Roth, et al., 1988), which discuss its properties, axiomatic characterisation and comparisons with other mechanisms (Moulin, 1996; Moulin, 2002; Moulin and Shenker, 1994).

1.3 Allocation inequality

Suppose there are n users $S_1, S_2, ...S_n$, which have demands $x_1, x_2, ...x_n$ for a certain commodity. Assume $x_1 \leq x_2 \leq x_3 \leq ..... \leq x_n$. Let x = ($x_1, x_2, ..., x_n$) and $x_N = \sum_{i=1}^{n} x_i$. The cost function $C : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function with C(0) = 0. We call $< C, x >$ a cost sharing problem.

Definition 1.1. *A cost sharing rule ϕ is a map from the domain of all cost sharing problems to $\mathbb{R}^n$, where $\phi(< C, x >) = (\phi_i)_{i=1,...,n}$, ϕ_i is the cost that agent i has to pay so that* $\sum_{i=1}^{n} \phi_i = C(x_N)$

We only sharing rules that satisfy the fundamental properties: (1) Dummy: $x_i = 0 \Rightarrow \phi_i = 0$, (2) Symmetry: $x_i = x_j \Rightarrow \phi_i = \phi_j$, (3) Order preserving $x_i \leq x_j \Rightarrow \phi_i \leq \phi_j$ for all i,j = 1,..,n.

The average cost sharing rule is given by

$$\phi_i^a = \frac{C(x_N)}{x_N} x_i \qquad for \;\; i = 1,...,n$$

where $\frac{C(x_N)}{x_N}$ can be interpreted as average cost.

The Moulin-Shenker's (increasing) serial cost sharing rule is defined as

$$\phi_i^r = \sum_{j=1}^{i} \frac{C(q_j) - C(q_{j-1})}{n+1-j} \qquad for \;\; i = 1,...,n$$

where $q_1 = nx_1, \;\; q_2 = x_1 + (n-1)x_2, \;\; ...., \;\; q_i = x_1 + x_2 + ... + x_{i-1} + (n+1-i)x_i, \;\; ..., \;\; q_n = x_1 + x_2 + ... + x_n = \sum_{i=1}^{n} x_i$.

The decreasing cost sharing rule is given by

$$\phi_{n-i+1}^d = \sum_{j=1}^{i} \frac{C(s_j) - C(s_{j-1})}{n+1-j} \qquad for \;\; i = 1,...,n$$

where $s_n = nx_n, \;\; s_(n-1) = (n-1)x_{n-1} + x_n, \;\; ...., \;\; s_i = ix_i + x_{i+1} + x_{i+2} ... + x_n, \;\; ..., \;\; s_1 = x_1 + x_2 + ... + x_n = \sum_{i=1}^{n} x_i$.

The Shapley value is given by

$$\phi_i^{sh} = \sum_{S \subset N \setminus \{i\}} \frac{|S|! \;\; (n-|S|-1)!}{n!} \;\; [C(x_{S \cup i}) - C(x_S)]$$

Definition 1.2. *Consider the cost sharing problem $< C, x >$ and ϕ and θ two cost sharing rules. The cost allocation $\phi(< C, x >)$ Lorenz dominates $\theta(< C, x >)$ if:*

$$\sum_{i=1}^{j} \phi_i \geq \sum_{i=1}^{j} \theta_i \;\; \forall j = 1,...,n$$

Chen et al. (2020) and Hougaard and Thorlund-Petersen (2001) has recently provided the comparisons between the Average rule and the other three and between Increasing and

Decreasing serial. We will provide the comparison between the Increasing serial share with the Shapley value. Our comparison complete the ordering among the four rules [2]. Arguably, our comparison has the simplest proof, compared to the proofs for the others. We can also use the approach to directly compare the Decreasing serial rule the Shapley value [3].

Theorem 1.1. *If the cost function C is concave (convex), the increasing serial cost shares Lorenz dominates (is Lorenz dominated by) the Shapley value.*

Proof. Suppose that the cost function C is concave (The convex case is symmetric).
We wish to show that

$$\sum_{i=1}^{k} \phi_i^r - \sum_{i=1}^{k} \phi_i^{sh} \geq 0$$

for all k=1,...,n □

As it is very difficult to work with Shapley formula directly, we introduce the value $\phi_i^{sh}(k)$ to establish a connection between the Shapley and the Serial cost allocation as following:

Definition 1.3. *For k=1,...,n, for i=1,...,k, let*
$\phi_i^{sh}(k) = \phi_i^{sh}(< C, n, x^k >)$ *where* x^k = $(x_1, x_2, ..., x_{k-1}, x_k, ..., x_k)$
i.e, $\phi_i^{sh}(k)$ *is the Shapley value of the cost sharing problem with cost function C, n users whose demands are* $(x_1, x_2, ..., x_{k-1}, x_k, ..., x_k)$*.*

Then, the Shapley value of the original cost sharing problem ϕ_i^{sh} is actually $\phi_i^{sh}(n)$. Besides, for the demand vector $(x_1, x_2, ..., x_{k-1}, x_k, ..., x_k)$, the total demand is $x_1 + x_2 + ... + (n-k+1)x_k = q_k$. Therefore, for all k=1,...,n

$$C(q_k) = \phi_1^{sh}(k) + \phi_2^{sh}(k) + ... + (n-k+1)\phi_k^{sh}(k) \tag{1.3.1}$$

Lemma 1.1. *When the cost function is concave, if demands for bigger users* $x_{j,j>i}$ *increase then the Shapley value of firm i in will decrease.*

Proof.

$$\phi_1^{sh} = \sum_{S \subset N \setminus \{i\}} \frac{|S|! \ \ (|N| - |S| - 1)!}{|N|!} \ \ [\pi(S \cup \{i\}) - \pi(S)]$$

for $\pi(S) = C(\sum_{j \in S} x_j)$.
Then, $[\pi(S \cup \{i\}) - \pi(S)] = C(\sum_{j \in S} x_j + x_i) - C(\sum_{j \in S} x_j)$

[2]It should be noted that Lorenz dominance is a partial ordering; the relation is transitive but not complete.
[3]See appendix

By concavity of the cost function C, $[\pi(S \cup \{i\}) - \pi(S)]$ is decreasing in $\sum_{j \in S} x_j$. If demands for bigger firms $x_{j,j>i}$ increase to y'_j for $j > i$ and demands for smaller firms $x_{k,k<i}$ remain the same, then $\sum_{j \in S} x'_j \geq \sum_{j \in S} x_j$. Thus, $C(\sum_{j \in S} x'_j + x_i) - C(\sum_{j \in S} x'_j) \leq C(\sum_{j \in S} x_j + x_i) - C(\sum_{j \in S} x_j)$

Therefore, when the cost function is concave, Shapley value of firm i will decrease as demands for bigger firms $x_{j,j>i}$ increase. □

Corollary 1.1.1. *If C is concave, then $\phi_i^{sh}(k) \geq \phi_i^{sh}(j)$ for all $k \leq j \leq n$ and for all i=1,...,k*

We will use the value $\phi_i^{sh}(k)$ to establish a connection between the Shapley value and the Serial cost allocation. For k=1,...,n, we have:

$$\sum_{i=1}^{k} \phi_i^r - \sum_{i=1}^{k} \phi_i^{sh} = \left(\sum_{i=1}^{k} \phi_i^{sh}(i) - \sum_{i=1}^{k} \phi_i^{sh} \right) - \left(\sum_{i=1}^{k} \phi_i^{sh}(i) - \sum_{i=1}^{k} \phi_i^r \right)$$

Consider

$$\sum_{i=1}^{k} \phi_i^r = \sum_{i=1}^{k} \sum_{j=1}^{i} \frac{C(q_j) - C(q_{j-1})}{n+1-j} \quad \text{by definition } \phi_i^r = \sum_{j=1}^{i} \frac{C(q_j) - C(q_{j-1})}{n+1-j}$$

$$= \left(\frac{k}{N} - \frac{k-1}{N-1} \right) C(q_1) + \left(\frac{k-1}{N-1} - \frac{k-2}{N-2} \right) C(q_2) + \ldots$$

$$+ \left(\frac{2}{N-k+2} - \frac{1}{N-k+1} \right) C(q_{k-1}) + \frac{1}{N-k+1} C(q_k)$$

$$= \sum_{i=1}^{k} \left(\frac{k-i+1}{n-i+1} - \frac{k-i}{n-i} \right) C(q_i)$$

$$= \sum_{i=1}^{k} \left(\frac{k-i+1}{n-i+1} - \frac{k-i}{n-i} \right) [\phi_1^{sh}(i) + \phi_2^{sh}(i) + \ldots + (n-i+1)\phi_i^{sh}(i)]$$

$$= \sum_{i=1}^{k} \left(\frac{k-i+1}{n-i+1} - \frac{k-i}{n-i} \right) (n-i+1)\phi_i^{sh}(i) + \sum_{i=1}^{k} \sum_{j=i+1}^{k} \left(\frac{k-j+1}{n-j+1} - \frac{k-j}{n-j} \right) \phi_i^{sh}(j)$$

$$= \sum_{i=1}^{k} \left(\frac{n-k}{n-i} \right) \phi_i^{sh}(i) + \sum_{i=1}^{k} \sum_{j=i+1}^{k} \left(\frac{k-j+1}{n-j+1} - \frac{k-j}{n-j} \right) \phi_i^{sh}(j)$$

Then,

$$
\begin{aligned}
&\sum_{i=1}^{k} \phi_i^{sh}(i) - \sum_{i=1}^{k} \phi_i^{r} \\
&= \sum_{i=1}^{k} \phi_i^{sh}(i) - \left[\sum_{i=1}^{k} \left(\frac{n-k}{n-i} \right) \phi_i^{sh}(i) + \sum_{i=1}^{k} \sum_{j=i+1}^{k} \left(\frac{k-j+1}{n-j+1} - \frac{k-j}{n-j} \right) \phi_i^{sh}(j) \right] \\
&= \sum_{i=1}^{k} \frac{k-i}{n-i} \phi_i^{sh}(i) - \sum_{i=1}^{k} \sum_{j=i+1}^{k} \left(\frac{k-j+1}{n-j+1} - \frac{k-j}{n-j} \right) \phi_i^{sh}(j) \\
&\leq \sum_{i=1}^{k} \frac{k-i}{n-i} \phi_i^{sh}(i) - \sum_{i=1}^{k} \sum_{j=i+1}^{k} \left(\frac{k-j+1}{n-j+1} - \frac{k-j}{n-j} \right) \phi_i^{sh}(n) \\
&= \sum_{i=1}^{k} \frac{k-i}{n-i} \phi_i^{sh}(i) - \sum_{i=1}^{k} \frac{k-i}{n-i} \phi_i^{sh}(n) \\
&= \sum_{i=1}^{k} \frac{k-i}{n-i} \left[\phi_i^{sh}(i) - \phi_i^{sh}(n) \right] \\
&\leq \sum_{i=1}^{k} \left[\phi_i^{sh}(i) - \phi_i^{sh}(n) \right] \\
&= \sum_{i=1}^{k} \phi_i^{sh}(i) - \sum_{i=1}^{k} \phi_i^{sh}
\end{aligned}
$$

Therefore, increasing serial cost sharing allocation Lorenz dominates the Shapley value allocation.

1.4 Conclusion

This paper compares the allocation inequality between the Shapley value and the Increasing serial rule. When the cost is concave (convex), the Increasing serial rule is more (less) egalitarian than the Shapley value. With the comparison, we can provide the ordering in allocation inequality for the four most common sharing rules: the Average rule, the Increasing serial rule, the Decreasing serial rule, and the Shapley value. The comparison between the serial rules and other rules requires the convexity of the cost function, while the comparison between the pair Average-Shapley value requires the convexity of the marginal cost.

When the cost function is convex (concave), but the marginal cost is neither convex nor concave, we can not have the complete ordering among the four rules. However, we can conclude that the decreasing serial rule is the least (most) egalitarian, followed by the increasing serial rule. The increasing serial cost-sharing rule is based on protecting small firms from the negative externalities occurred mainly by large firms. We confirm that it always gives a more favourable distribution to the small firms compared to the Shapley value and Average rule. When there are positive externalities of production and agents benefit from the economies of scale, the decreasing serial rule always gives the most favourable allocation to large firms.

The result can be used in the surplus sharing problem and adds to the discussion about allocation rules in the dimension of inequality. It allows us to recommend the appropriate cost allocation when equality is held as a social value.

Appendix

Here we provide the direct comparison between the Shapley value and the decreasing serial share, and an alternative proof for the pair Serial-Average share

Comparison between Shapley value and decreasing serial share

Theorem 1.2. *If the cost function C is concave (convex), the decreasing serial share Lorenz dominated (is Lorenz dominated by) the Shapley value.*

Assume cost function C is concave. We wish to show that

$$\sum_{i=1}^{k} \phi_i^d \geq \sum_{i=1}^{k} \phi_i^{sh}$$

for all k=1,...,n

We will show

$$1-\sum_{i=1}^{k} \phi_i^d = \sum_{i=k}^{n} \phi_i^d \leq \sum_{i=k}^{n} \phi_i^{sh} = 1-\sum_{i=1}^{k} \phi_i^{sh}$$

for all k=1,...,n

We define $\phi_i'^{sh}(k) = \phi_i^{sh}(< C, n, x^k >)$ where $x^k = (x_k, x_k, ..., x_k, x_{k+1}, ..., x_n)$. Hence, $C(s_k) = k\phi_k'^{sh}(k) + \phi_{k+1}'^{sh}(k) + \phi_{k+2}'^{sh}(k) + ... + \phi_n'^{sh}(k)$. The original Shapley value is then $\phi_i^{sh} = \phi_i'^{sh}(1)$.

We have:

$$\sum_{i=k}^{n}\phi_i^d = \sum_{i=k}^{n}\frac{C(s_i)}{i} - \sum_{i=k}^{n}\sum_{j=i+1}^{n}\frac{C(s_j)}{j(j-1)}$$
$$= \sum_{i=k}^{n}\left(\frac{1}{i} - \frac{i-k}{i(i-1)}\right)C(s_i)$$
$$= \sum_{i=k}^{n}\left(\frac{1}{i} - \frac{i-k}{i(i-1)}\right)\left[i\phi_i'^{sh}(i) + \phi_{i+1}'^{sh}(i) + \phi_{i+2}'^{sh}(i) + \ldots + \phi_n'^{sh}(i)\right]$$

Lemma 1.2. *When the cost function is concave, if demands for smaller users $x_{j,j<i}$ increase then the Shapley value of firm i will decrease.*

Hence we have:

Corollary 1.2.1. *If C is concave, then $\phi_i'^{sh}(j) \leq \phi_i'^{sh}(k)$ for all $j \geq k$ and for all i=k,...,n*

Therefore,

$$\sum_{i=k}^{n}\phi_i^d = \sum_{i=k}^{n}\left(\frac{1}{i} - \frac{i-k}{i(i-1)}\right)\left[i\phi_i'^{sh}(i) + \phi_{i+1}'^{sh}(i) + \phi_{i+2}'^{sh}(i) + \ldots + \phi_n'^{sh}(i)\right]$$
$$\leq \sum_{i=k}^{n}\left(\frac{1}{i} - \frac{i-k}{i(i-1)}\right)\left[i\phi_i'^{sh}(k) + \phi_{i+1}'^{sh}(k) + \phi_{i+2}'^{sh}(i) + \ldots + \phi_n'^{sh}(k)\right]$$
$$= \sum_{i=k}^{n}\left[\sum_{j=k}^{i-1}\left(\frac{1}{j} - \frac{j-k}{j(j-1)}\right) + \left(1 - \frac{i-k}{i-1}\right)\right]\phi_i'^{sh}(k)$$
$$= \sum_{i=k}^{n}\phi_i'^{sh}(k)$$
$$\leq \sum_{i=k}^{n}\phi_i^{sh}$$

Therefore, decreasing serial cost sharing allocation Lorenz dominates the Shapley value allocation.

Alternative proof for the pair Increasing serial-Average share

Theorem 1.3. *If the cost function C is convex (concave), the average share dominates (is dominated by)) the increasing serial share.*

Assuming the cost function C(x) is concave (the proof for convex cost is very similar). Consider

$$
\begin{aligned}
\sum_{i=1}^{k}\phi_i^r-\sum_{i=1}^{k}\phi_i^a &= \frac{1}{N-k+1}C(q_k)+\sum_{i=1}^{k-1}\left(\frac{N-k}{(N-i+1)(N-i)}\right)C(q_i)-\sum_{i=1}^{k}\phi_i^a \\
&\geq \frac{1}{N-k+1}C(q_k)+\sum_{i=1}^{k-1}\left(\frac{N-k}{(N-i+1)(N-i)}\right)C(q_i)-\frac{x_1+x_2+\ldots+x_k}{q_k}C(q_k) \\
&= \sum_{i=1}^{k-1}\frac{N-k}{(N-i+1)(N-i)}C(q_i)-\frac{C(q_k)}{q_k}\frac{(N-k)(x_1+x_2+\ldots+x_{k-1})}{N-k+1} \\
&= (N-k)\left[\sum_{i=1}^{k-1}\frac{1}{(N-i+1)(N-i)}C(q_i)-C(q_k)\left(\frac{1}{N-k+1}-\frac{x_k}{q_k}\right)\right]
\end{aligned}
$$

For k=2:

$$
\begin{aligned}
&(N-k)\left[\sum_{i=1}^{k-1}\frac{1}{(N-i+1)(N-i)}C(q_i)-C(q_k)\left(\frac{1}{N-k+1}-\frac{x_k}{q_k}\right)\right] \\
=&\frac{1}{N(N-1)}C(q_1)-C(q_2)\left(\frac{1}{N-1}-\frac{x_2}{q_2}\right) \\
=&\frac{1}{N(N-1)}c(q_1)-C(q_2)\frac{x_1}{(N-1)q_2} \\
\geq&0
\end{aligned}
$$

Suppose that

$$
\sum_{i=1}^{j-1}\frac{1}{(N-i+1)(N-i)}C(q_i)-C(q_j)\left(\frac{1}{N-j+1}-\frac{x_j}{q_j}\right)\geq 0
$$

Then,

$$
\begin{aligned}
&\sum_{i=1}^{(j+1)-1}\frac{1}{(N-i+1)(N-i)}C(q_i)-C(q_{j+1})\left(\frac{1}{N-(j+1)+1}-\frac{x_{j+1}}{q_{j+1}}\right) \\
=&\sum_{i=1}^{j}\frac{1}{(N-i+1)(N-i)}C(q_i)-\frac{C(q_{j+1})}{q_{j+1}}\left(\frac{x_1+x_2+\ldots+x_j}{N-j}\right) \\
\geq&\sum_{i=1}^{j}\frac{1}{(N-i+1)(N-i)}C(q_i)-\frac{C(q_j)}{q_j}\left(\frac{x_1+x_2+\ldots+x_j}{N-j}\right) \\
\geq&\sum_{i=1}^{j-1}\frac{1}{(N-i+1)(N-i)}C(q_i)-C(q_j)\left(\frac{1}{N-j+1}-\frac{x_j}{q_j}\right) \\
\geq&0
\end{aligned}
$$

Hence,

$$
(N-k)\left[\sum_{i=1}^{k-1}\frac{1}{(N-i+1)(N-i)}C(q_i)-C(q_k)\left(\frac{1}{N-k+1}-\frac{x_k}{q_k}\right)\right]\geq 0
$$

for all k=2,..,N by induction. Therefore, $\sum_{i=1}^{k}\phi_i^r > \sum_{i=1}^{k}\phi_i^a$.

Chapter 2

Underperforming and Outperforming contestant: Whom to support?

Abstract

In this paper, we consider a repeated contest with the aim of determining the optimal strategy for favouring either the early winner or loser in the subsequent round to maximize total effort. Through our analysis, we find that the optimal decision is dependent upon the heterogeneity of players. In most cases, it is effort maximising to favour the winner unless the players are extremely different. When the contest designer has control over the prize distribution, we recommend allocating all of the prize in the second stage when players are similar, and having a smaller prize in the first stage when players differ significantly. The paper contributes to the discussion of optimal favoritism, particularly when the contest designer only has control over the recipient of the favorable advantages and not their impact.

2.1 Introduction

Competitions are often used as a tool to encourage effort. It is extremely common for the contestant to have different abilities. Different students have different strengths in certain subjects; employees in an organisations can be significantly different in productivity. These differences in ability are apparent in various competitions, including sports, politics, and R&D. In many situations, the designer has limited resource and needs to decide whether to support the underdogs or to reward the excellent performance.

It is common practice to observe the allocation of resources and favourable advantages to both under-performing and outperforming contestants. This phenomenon is not limited to a single arena, as illustrated by examples in various contexts such as in sports, research and development, education, job recruitment, and organisational settings. For instance, top teams in sports competition may avoid other top-performing teams. Winning a small prize or acquiring patents can enhance the credibility of research and development efforts, thereby increasing the chance to receive funding. Similarly, high-performing students in prior academic achievements are more likely to be accepted into prestigious universities and internship schemes, and individuals with a stronger educational background tend to receive favourable bias from recruiters in the job market. Successful sales agents within an organisational setting often have access to a broader network of connections, back-office resources, and training, which ultimately enhance their productivity. Subsequently, such employees receive more recognition, further solidifying their position for promotion.

The provision of extra resources and leniency to the laggards is also a widespread phenomenon. It is typical for under-performers to receive assistance to level the playing field in various settings, including sports and education or occupational environments. For instance, highly ranked athletes in golf and horse racing often face handicaps to equalize the competition. Similarly, educators frequently dedicate additional attention to students who struggle academically. In workplaces, subpar employees may receive increased feedback, training, and support, while top performers may go unnoticed by managers.

The existing literature on Tullock type contests suggests that both favoring the strong player and supporting the weak player can enhance effort provision in a competition. Favoring top performers creates an incentive for individuals to work harder, aiming to prevail in the competition (Clark and Nilssen, 2018). On the other hand, supporting the underdog increases their chances of winning, leading to higher motivation and effort. It also mitigates the discouragement effect caused by high levels of heterogeneity, which have been shown to negatively impact total effort (Cornes and Hartley, 2005; Nitzan, 1994). Additionally, improving the skill levels of the less proficient candidates reduces the ability gap, fostering stronger competition and encouraging the more capable candidate to strive harder (Baik, 1994).

As a result, it remains uncertain whether the contest designer should allocate a greater share of resources to top performers or underperforming individuals to maximise total effort, a critical question in many cases. For instance, teachers' priority is often to maximise students' effort; however, with limited time, finding the balance between encouraging good students and supporting struggling ones becomes a dilemma. Similarly, the primary aim of a contest designer in an organisational setting is often to maximise the level of effort exerted. Due to budget constraints, the designer may need to choose whether to provide supplementary resources and training for either the best worker or the underperforming one.

Our paper addresses the question of how contest designers should allocate limited resources based on previous performance and provides recommendations for optimal allocation using a two stage Tullock lottery contest. The resources in question consist of training sessions, supervising time, feedback, back-office support, and opportunity, which collectively enhance the productivity of resource recipients and are thus referred to as favorable advantages. Specifically, we focus on situations where the contest designer can only determine the recipient of the advantage, not its magnitude. When the advantages are in the form of recognition and bias, the designer can determine how much more effective the efforts of a favored contestant become compared to the other contestants. Ridlon and Shin (2013) have previously studied optimal handicapping under such conditions, finding that handicapping could favor either the early winner or early loser depending on the level of heterogeneity. However, in many cases, particularly when the favorable advantage in question is a resource as described above, the designer has no control over its impact and must treat it as a given. Thus, we introduce the impact of the advantage as a parameter and analyse the total effort associated with allocating it to either the winner or the loser of the previous contest. Our findings provide recommendations for the most efficient allocation of these resources when the objective is to maximise aggregate effort.

The study of optimal favouritism in contests has received significant attention, with seminal work dating back to Baye et al. (1993) with an all pay auction setting. This topic remains a subject of debate, particularly surrounding the topic of levelling the playing field. The literature reveals that the decision to increase or remove heterogeneity depends on the contest structure and the mechanism employed. There are various mechanisms for favoring a candidate, each results in different recommendations for optimal bias depending on the contest format (Chowdhury et al., 2023). Within all-pay auctions with complete information, literature often supports leveling the playing field (Che and Gale, 1998; Lu, 2021). Fu (2006) studies a two-player asymmetric all-pay auction with multiplicative bias in performance, namely, an effort impact function, and demonstrates that optimal bias favours the disadvantaged player. When the advantage is granted as an additive head start, biasing the weaker player can generate higher effort level. Moreover, combining both head start and optimal bias could further improve aggregated effort level and is the optimal instrument (Franke et al., 2018). A recent paper by Zhu (2021) considers a two-player all-pay auction wherein the contest designer can bias a player through either a head start or a handicap, and concludes that it is effort maximising to level the playing field, irrespective of the instrument utilised. Conversely, when players possess private information regarding their valuation, leveling the playing field may not always be optimal. Kirkegaard (2012) shows that, in a two-player all-pay auction, providing the weaker player with a head start is always revenue maximising, while handicapping the stronger player with a multiplicative bias is not.

In the symmetric setting, introducing asymmetry in tournaments and Tullock contests by biasing the early winner could generate higher effort level. Meyer (1992) examines symmetric two-player two-stage tournament and demonstrates that the contest designer can maximise total effort by favouring the winner with appropriate bias factor. The "ratcheting" effect causing lower effort in the first stage if favouring the early loser can dominate the moral hazard problem leading to lower effort in the second stage if favouring the winner. Clark et al. (2020) extend this result to n symmetric players using a Tullock lottery contest and provide a similar finding, further supporting the notion that introducing bias as an winning advantage can positively impact effort levels. Studying a class of biased contests including Tullock contest and Lazear and Rosen (1981) type tournament, Drugov and Ryvkin (2017) argue that arbitrary favouritism may lead to higher effort levels, challenging the assumption that ex-post symmetric contestants are always optimal.

On the other hand, affirmative action in favour of the the weak player, or the early loser, could be optimal in an asymmetric Tullock contest. Franke et al. (2013) demonstrate that when employing a multiplicative bias, partially equalising the playing field can maximise effort. In an optimally biased contest, it may be appropriate to include weaker contestants, resulting in a larger number of participants compared to an unbiased contest. However, the optimal favouritism often depends on various factors, including the instrument utilised, the heterogeneity level of the players, and the number of participants. With regard to the number of players, Franke (2012) illustrates that implementing an affirmative action through a multiplicative bias, which equalises the chance of winning for the same disutility of effort, can lead to varying effects on the total effort level depending on the number of participants. While affirmative action is optimal in a two-player contest, it may reduce the overall effort in a multi-player contest with highly variable costs. Various bias instruments can also yield different recommendations. For instance, levelling the playing field using optimal bias generates more effort and is often aligned with effort maximisation, while a head-start in favour of the weaker player might increase or decrease the aggregate effort (Franke et al., 2018). In a two-stage contest using a subsidy (cost reimbursement) scheme, J. Thomas and Wang (2017) highlight the impact of the noises in the contest. Favouring the loser is only optimal when the contest is accurate, hence it is optimal to subsidise the weakest player in the Tullock lottery contest with multi players. The optimal favouritism also depends on the heterogeneity level of the players. The paper by Ridlon and Shin (2013) investigates the optimal handicapping policy in a two-stage lottery contest with two asymmetric players, and finds that it is effort maximising to handicap against the winner when the abilities are different. Meanwhile, the optimal handicap will be in favour of the early winner if the players are sufficiently similar.

In this paper, we examine a repeated Tullock lottery contest involving two risk-neutral players that exhibit asymmetry in their marginal costs of effort. These players compete to win the prizes and pay costs that are linear to the effort they exert. The primary objective of the contest designer is to maximise the total effort expended by the players. To achieve this objective, the designer faces the decision of whether to provide an advantage to the early winner or the early loser. This advantage comes in the form of a multiplicative cost reduction. The contest is analytically equivalent to the contest where players exert effort and ability is presented through effort impact function (Kirkegaard, 2012). Overall, it is easier to implement a change in cost than a change in effort impact function. Recent paper by J. Thomas and Wang (2017) has also considered the setting with cost function when investigating a similar question, specifically the optimal subsidy policy to maximise total effort. Moreover, this model is relevant to our motivating example of distributing limited resources such as training, back-office support and opportunity, which can enhance an individual's productivity and enable them to attain comparable performance levels with less work and time than before. As a result, with the same level of observable effort, which determines their chances of receiving bonuses and promotion, the player's cost is marginally decreased.

We compare aggregated effort levels and examine the optimal decision between favouring the early winner or early loser in the repeated contest with no discounting factor [1]. Our findings reveal that the optimal decision depends on the heterogeneity of participants. In particular, for each cost reduction level, there exists a threshold for the marginal cost ratio which determines the decision switch. If the marginal cost ratio falls below this threshold, it is optimal to favour the loser, otherwise favouring the winner is recommended. When the participants are relatively similar, favouring the winner is always effort maximising, regardless of the cost reduction level. However, when the players are substantially different, it is often optimal to favour the loser, except when the cost reduction is very small. Overall, it is often effort maximising to favouring the winner in the early stage unless when the ex-post heterogeneity level is remarkably high. The results offer recommendations for the optimal allocation of indivisible resources in situations where the provider has to base their decision on previous performance and lacks control over the impact of those resources.

We then analyse the effort maximising prize allocation when the advantage is rewarded to either the winner or the loser. The optimal distribution of prizes also depends on the bias tool used in the contest. Clark and Nilssen (2021) examine the optimal prize distribution in an all-pay auction where the underdog initially receives a head start in the first stage, and the early winner has a head start in the second stage. The findings reveal that the optimal prize distribution is contingent upon the level of heterogeneity and the assigned head start in each stage. In a similar setting, but with multiplicative bias rather than additive head start, the prize distribution also depends on the ex-ante heterogeneity level and the bias factor, as well as the relationship between the bias gained from winning the first stage and the corresponding prize (Clark and Nilssen, 2020). In both cases, it is found to be optimal to allocate the entire budget to the first stage when the heterogeneity level is high. Meanwhile, Klein and Schmutzler (2017) reach a different conclusion by using a dynamic tournament and allowing first stage effort to impact second stage performance. Their paper suggests that putting all the prize in the second stage is optimal when the chance of winning in the second stage is positively influenced by the effort in the first stage. In our model of Tullock contest with multiplicative cost reduction, the optimal prize allocation depends on the heterogeneity of players and the cost reduction level in both cases. When the players are symmetric, it is optimal to allocate all the prize to the last stage (Clark et al., 2020). We will show that the recommendation is different in an asymmetric setting: a small prize in the early round can be ideal when the difference in costs of effort is significant.

[1] assuming no time value of money and the players have certainty regarding the number of stages and when the game will end

2.2 Model

We adopt the standard Tullock contest probability function: If player i exerts effort x_i and pay the cost $c_i x_i$, his probability of winning is:

$$P(i) = \begin{cases} \dfrac{x_i}{x_i + x_{-i}} & \text{if } x_i + x_{-i} \neq 0 \\ 0.5 & \text{if } x_i = x_{-i} = 0 \end{cases}$$

Suppose that there are two players H and L, whose marginal cost is $c_H < c_L$. Let the relative cost $\sigma = \frac{c_H}{c_L} \in (0,1)$. The player i who is favoured will have his marginal cost reduced to γc_i for $\gamma \in (0,1)$.

We first consider a repeated contest where the two players compete for the same prize V, normalised to 1 at each stage.

2.2.1 When the advantage is given to the winner

Stage 1: At stage 1, two players compete for the prize of 1. The pay-off for stage 1 will then be:

$$\pi_1(H) = \frac{x_H}{x_H + x_L} - c_H x_H \tag{2.2.1}$$

$$\pi_1(L) = \frac{x_L}{x_H + x_L} - c_L x_L \tag{2.2.2}$$

Stage 2: At stage 2, they again compete for the prize of 1. The winner receive advantage. Then, If i is the winner of the first contest then the pay off of i in stage 2 is:

$$\pi_2(i) = \frac{y_i}{y_i + y_j} - \gamma c_i y_i \tag{2.2.3}$$

The pay off of the loser j is:

$$\pi_2(j) = \frac{y_j}{y_i + y_j} - c_j y_j \tag{2.2.4}$$

where $\gamma < 1$ is an improvement in effort efficiency.

2.2.2 When the advantage is given to the loser

Stage 1: At stage 1, two players compete for the prize of 1. The pay-off for stage 1 will then be:

$$\pi_1(H) = \frac{x_H}{x_H + x_L} - c_H x_H$$
$$\pi_1(L) = \frac{x_L}{x_H + x_L} - c_L x_L$$

Stage 2: At stage 2, they again compete for the prize of 1. The loser (failer) receives advantage. Then,

If i is the winner of the first contest then the pay off of i in stage 2 is:

$$\pi_2(i) = \frac{y_i}{y_i + y_j} - c_i y_i$$

The pay off of the loser j is:

$$\pi_2(j) = \frac{y_j}{y_i + y_j} - \gamma c_j y_j$$

where $\gamma < 1$ is an improvement in effort efficiency.

Proposition 2.1. *The dynamic contest in which the early winner is favoured results in a higher effort level in both stages (hence higher total effort) compared to the standard repeated contest.*

The dynamic contest in which the early loser is favoured results in a lower effort level in the first stage, higher effort in the second stage and higher total effort compared to the standard repeated contest.[2]

When the cost of effort decreases, players tend to exert more effort. As a result, when a cost reduction is introduced in the second stage of the contest, the level of effort exerted always surpasses that of the original unbiased contest, regardless of who benefits from it. This finding differs from situations where handicaps are employed. Handicapping a player by reducing their effort productivity actually leads to a decrease in total effort (Stein, 2002). Therefore, implementing handicaps in a contest will result in a lower level of effort in the second stage, in comparison to the original unbiased contest.

When a cost reduction serves as a reward for the early winner, players are motivated to compete not only for the prize in the first stage but also for a better chance of winning in the subsequent

[2]The proof of the proposition is provided in Appendix 2.4

stage. Conversely, the advantage gained by the early loser can lead to decreased motivation and, consequently, lower effort levels compared to unbiased static contests with the same prize.

Proposition 2.2. *Favouring the winner results in higher effort level in the first stage. Favouring the loser results in the higher effort level in the second stage.*

There is a trade-off: Favouring the loser helps the Low type contestant (who is more likely to lose) "catch up" in the next stage so that both participants will exert more effort in the second stage. Meanwhile, favouring the winner motivates the players to put effort in the first stage. If the incentives from the winning advantage outweighs the "catch up" effect, favouring the winner will result in higher overall effort level.

Example 2.1. Consider the cost reduction $\gamma = 0.8$ (the person who receives the advantage has his cost saved by 20%).

- When the relative cost of effort is $c_H/c_L = 0.9$: When favouring the winner, effort level is $E_1^W = 0.5845/c_L$ in the first stage and $E_2^W = 0.5846/c_L$ in the second stage. When favouring the loser, the effort levels in first and second stage are $E_1^F = 0.4681/c_L$ and $E_2^F = 0.5850/c_L$. In the standard repeated contest, effort level is $E^N = 0.5263/c_L$ at each stage.
- When the relative cost of effort $c_H/c_L = 0.5$: When favouring the winner, effort level is $E_1^W = 0.7248/c_L$ in the first stage and $E_2^W = 0.7319/c_L$ in the second stage. When favouring the loser, the effort levels in first and second stage are $E_1^F = 0.6073/c_L$ and $E_2^F = 0.7500/c_L$. In the standard repeated contest, effort level is $E^N = 0.6667/c_L$ at each stage.
- When the relative cost of effort $c_H/c_L = 0.1$: When favouring the winner, effort level is $E_1^W = 0.9201/c_L$ in the first stage and $E_2^W = 0.9419/c_L$ in the second stage. When favouring the loser, the effort levels in first and second stage are $E_1^F = 0.8976/c_L$ and $E_2^F = 1.0933/c_L$. In the standard repeated contest, effort level is $E^N = 0.9091/c_L$ at each stage.

	First stage effort	Second stage effort	Total effort
$\sigma = 0.9$	$E_1^W > E^N > E_1^F$	$E_2^F > E_2^W > E^N$	$E^W > E^F > 2E^N$
$\sigma = 0.5$	$E_1^W > E^N > E_1^F$	$E_2^F > E_2^W > E^N$	$E^W > E^F > 2E^N$
$\sigma = 0.1$	$E_1^W > E^N > E_1^F$	$E_2^F > E_2^W > E^N$	$E^F > E^W > 2E^N$

Table 2.1: Effort comparison for $\gamma = 0.8$

Theorem. *For every cost reduction factor γ, there is a threshold $\sigma^*(\gamma)$ so that when $\sigma > \sigma^*$ it is effort maximising to favour the winner, and when $\sigma < \sigma^*$ it is effort maximising to favour the loser. The threshold σ^* is given in figure 2.1.*
When the relative cost of effort $\sigma = \frac{c_H}{c_L}$ is sufficiently high ($\sigma \geq \max \sigma^* = 0.27$), it is always effort maximising to favour the winner. On the other hand, when the relative cost of effort is small, favouring the loser often results in a higher effort level unless the cost reduction is also very small.

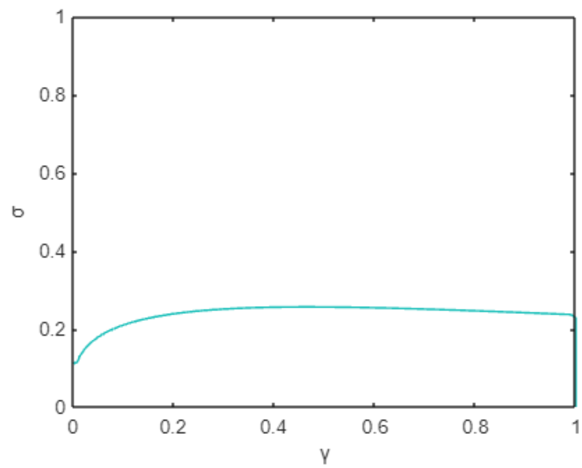

Figure 2.1: Threshold σ^* as a function of γ

2.3 Optimal prize distribution

In some setting, the contest designer can choose the prize distribution and does not need to give an equal prize at each stage. The optimal prize allocation is different when favouring the winner and the loser. Assume that the contest designer has budget normalised to 1 for the whole contest, and can divide it into (V, 1-V) as rewards for the first and the second stage.

2.3.1 Optimal prize distribution when favouring the winner

Lemma 2.1. (*Clark et al., 2020*) *When the players are symmetric and the winner receives favourable advantage, total effort is maximised at V=0*

When the players are initially identical, winning advantage has a definite effect: it creates asymmetry. Therefore, the first round is competitive, and both players will work hard for the advantage because it will make a difference in their position in the next round. Moreover, one prize unit in the second stage will yield a higher effort level than the first stage because the players have become more efficient. It also does not matter who would win because the players are identical. Therefore, the designer should put all prizes in the final stage.

However, it is only true when the players are similar, and therefore the winning advantage has a notable impact on their position. When two players are very different, putting all the prize in the last stage might not be ideal. When the High type worker is remarkably more efficient, and the cost reduction is not too significant, winning advantage is not attractive to him. The improvement in the second stage only has a minor impact on the (already very high) chance of winning, so he will not put much effort into the first stage. A small prize will motivate him to exert effort. Therefore, the total effort will initially increase in the first prize V.

Example 2.2. Consider the situation when the cost reduction $\gamma = 0.8$ is given to the winner, and the designer can choose the distribution (V,1-V).

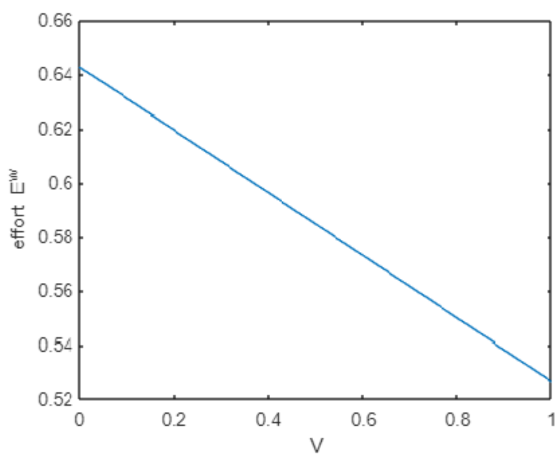

Figure 2.2: E^W when $c_H/c_L = 0.9$

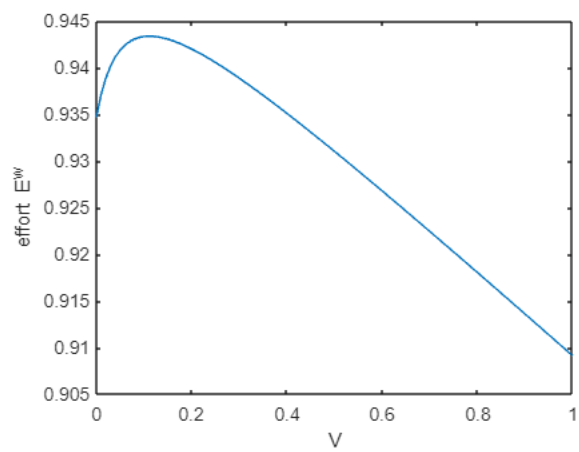

Figure 2.3: E^W when $c_H/c_L = 0.1$

If relative cost is $c_H/c_L = 0.9$, effort level decreases in first stage prize V. It is therefore effort maximising to put all the prize in the second stage. Meanwhile, if the relative cost is $c_H/c_L = 0.1$, putting a small prize in the first round to motivate the two players to exert effort can yield higher effort level than leaving all the prize to the last stage.

Lemma 2.2. *When the cost reduction factor is higher than the cost ratio, i.e, the Low type player with the winning advantage is still less productive than the High type player, total effort is a decreasing function of first stage prize V. It is therefore effort maximising to put all the prize in the second stage*

Proof. See Appendix □

2.3.2 Optimal prize distribution when favouring the loser

When the first stage loser receives advantage for the next stage, both players will have incentive to stay inactive to become loser unless the first stage prize is sufficiently high to induce effort. Any first stage prize that is smaller than the minimum level required for positive first round effort is wasteful. Moreover, when transferring a small portion from the second stage prize to the first stage prize, the loss in effort in the second stage outweighs the gain in the first stage. It is shown by the discontinuity in the graph of E^F in the following example.

Example 2.3. Consider the situation when the cost reduction $\gamma = 0.8$ is given to the loser, and the designer can choose the prize distribution (V,1-V)

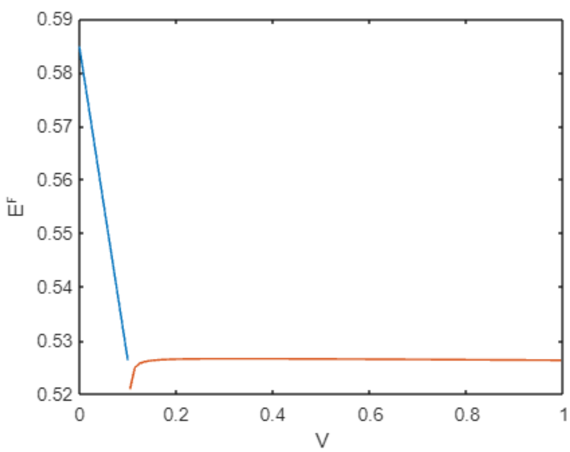

Figure 2.4: E^F when $c_H/c_L = 0.9$

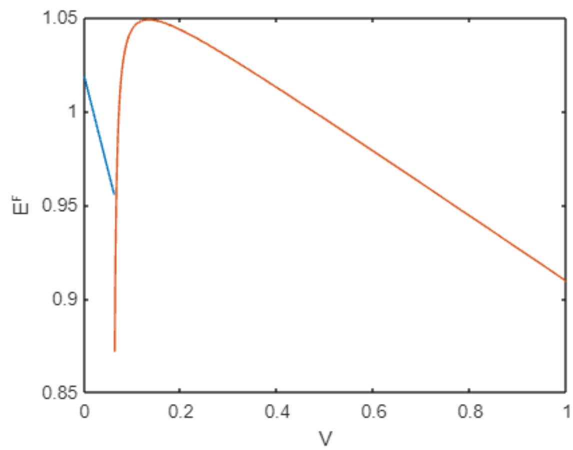

Figure 2.5: E^F when $c_H/c_L = 0.1$

As V increases from 0 to $\frac{\sigma(1-\gamma^2)(2\gamma^2\sigma+2\gamma+\sigma)}{(1+\sigma\gamma)^2+(\sigma+\gamma)^2+\sigma(1-\gamma^2)(2\gamma^2\sigma+2\gamma+\sigma)}$ [3], the effort will be 0 (or $\varepsilon \approx 0$) in the first stage. Total effort is therefore just the second stage effort, which increases in the second stage prize 1-V (thus decreases in the first stage prize V).

As V increases from $\frac{\sigma(1-\gamma^2)(2\gamma^2\sigma+2\gamma+\sigma)}{(1+\sigma\gamma)^2+(\sigma+\gamma)^2+\sigma(1-\gamma^2)(2\gamma^2\sigma+2\gamma+\sigma)}$: The players become active in the first round, resulting in an initial increase in effort. However, as V becomes larger, the decrease in the second stage effort is more significant than the gain from the first stage. Therefore, effort level will gradually decrease.

From the graphs, we can see that a positive first stage prize may, or may not give a higher effort level, compared to the contest with $V = 0$, i.e., single contest with random assignment of the advantage. It is because when the players are similar or the cost reduction is not significant,

[3] The derivation of the threshold for first stage prize is shown in the Appendix.

the second stage effort is notably higher when the High type player is favoured. Therefore, it is optimal to put all the prize in the second stage and assign the cost reduction with probability 0.5. Meanwhile, when the players are particularly asymmetric, there will be a remarkable difference in second stage effort when the High type or the Low type player receives the advantage. We will prefer the Low type player to lose to maximise the effort. A slightly lower second stage prize, but with the more accurate assignment of the advantage, will yield a higher effort level than the whole prize $V = 1$ with random assignment.

Lemma 2.3. *Any prize $V > 0.5$ will result in lower effort level than $V = 0$.*

2.4 Conclusion

This paper provides recommendations for contest designers who are deciding whether to favour underperforming or outperforming participants. This scenario has applications in the workplace, where managers have to decide who to invest in, such as choosing whether to assign training slots to the highest-performing employees or those who are falling behind. There is a trade-off involved since favouring the losing participant decreases initial effort, but can improve competitive balance hence increase effort in the later stage significantly, whereas favouring the winner will motivate participants to exert more effort from the early stage. The result shows that when the relative marginal cost of effort is sufficiently high ($c_H/c_L \geq 0.27$), it is always better to award the cost reduction advantage to the winner. Only when the costs of effort are extremely different it is always better to give the advantage to the loser. There is a threshold of the relative cost of effort for each cost reduction level, where the optimal decision switches.

The paper adds to the discussion of optimal favouritism in dynamic contest. In particular, it allows the contest designer to decide the optimal recipient of the advantage when having no control over its impact. The result provides recommendations for the optimal assignment of indivisible resources such as training slot, opportunity, limited time and back-office support when the provider need to base his decision on previous performance. One possible extension for future research could be to let the resources have different impacts on different participants. In the model, it means that the cost reduction factor will be different for High and Low player. It will be interesting to investigate how such difference will have impact on the optimal decision.

The optimal prize distribution depends on the relative costs of effort and the cost reduction level. The paper shows that when favouring the winner, effort maximisation is achieved by allocating the entire budget for the prize in the final stage for similar players. For significantly more efficient High type player, it is better to allocate a small prize in the first stage. The recommendation is the same when favouring the loser. When players are largely different, the optimal distribution involves allocating a small positive prize in the first stage.

A similar model with more than two players can be considered in future research. In that case, the notion of winner and loser needs to be clearly defined. One can compare total effort when the advantage is given to the players of the different ranks in the early stage, using success functions for rankings in the literature (Drugov and Ryvkin, 2020; Vesperoni, 2016).

Appendix

2.A Comparison between effort levels in the repeated contest

2.A.1 Total effort when the advantage is given to the winner

We calculate the subgame perfect equilibrium.

Stage 2:

If player H wins the first contest, then:

Player H solves

$$\max_{w_H} \frac{w_H}{w_H + f_L} - \gamma c_H \cdot w_H$$

Player L solves

$$\max_{f_L} \frac{f_L}{w_H + f_L} - c_L \cdot f_L$$

Equilibrium effort will be $w_H = \dfrac{c_L}{(\gamma c_H + c_L)^2}$ and $f_L = \dfrac{\gamma c_H}{(\gamma c_H + c_L)^2}$

Payoff will then be $\pi_{H/W} = \dfrac{c_L^2}{(\gamma c_H + c_L)^2}$ and $\pi_{L/F} = \dfrac{\gamma^2 c_H^2}{(\gamma c_H + c_L)^2}$

If L wins the first contest, then:

H solves

$$\max_{f_H} \frac{f_H}{f_H + w_L} - c_H \cdot f_H$$

L solves

$$\max_{w_L} \frac{w_L}{f_H + w_L} - \gamma c_L \cdot w_L$$

Equilibrium effort will then be $f_H = \dfrac{\gamma c_L}{(\gamma c_L + c_H)^2}$ and $w_L = \dfrac{c_H}{(\gamma c_L + c_H)^2}$

Payoff will then be $\pi_{H/F} = \dfrac{\gamma^2 c_L^2}{(\gamma c_L + c_H)^2}$ and $\pi_{L/W} = \dfrac{c_H^2}{(\gamma c_L + c_H)^2}$

Stage 1:

Player H will solve:

$$\max_{x_H} \frac{x_H}{x_H + x_L} + \frac{x_H}{x_H + x_L}\pi_{H/W} + \frac{x_L}{x_H + x_L}\pi_{H/F} - c_H x_H$$

Player L will solve:

$$\max_{x_L} \frac{x_L}{x_H + x_L} + \frac{x_L}{x_H + x_L}\pi_{L/W} + \frac{x_H}{x_H + x_L}\pi_{L/F} - c_L x_L$$

FOCs:

$$\begin{aligned} \frac{x_L}{(x_H + x_L)^2}(1 + \pi_{H/W} - \pi_{H/F}) &= c_H \\ \frac{x_H}{(x_H + x_L)^2}(1 + \pi_{L/W} - \pi_{L/F}) &= c_L \end{aligned} \tag{2.A.1}$$

i.e,

$$\frac{x_L}{(x_H + x_L)^2}\left[1 + \frac{c_L^2}{(\gamma c_H + c_L)^2} - \frac{\gamma^2 c_L^2}{(\gamma c_L + c_H)^2}\right] = c_H \tag{2.A.2}$$

$$\frac{x_H}{(x_H + x_L)^2}\left[1 + \frac{c_H^2}{(\gamma c_L + c_H)^2} - \frac{\gamma^2 c_H^2}{(\gamma c_H + c_L)^2}\right] = c_L \tag{2.A.3}$$

Let $\sigma = \dfrac{c_H}{c_L} < 1$. Then,

$$\begin{aligned} s &= \frac{c_H^2}{(c_H + \gamma c_L)^2} - \frac{\gamma^2 c_H^2}{(c_L + \gamma c_H)^2} = \frac{\sigma^2}{(\sigma + \gamma)^2} - \frac{\gamma^2 \sigma^2}{(1 + \gamma\sigma)^2} \\ &= \frac{\sigma^2(1 - \gamma^2)(\gamma^2 + 2\gamma\sigma + 1)}{(1 + \gamma\sigma)^2(\sigma + \gamma)^2} \end{aligned}$$

$$
\begin{aligned}
t &= \frac{c_L^2}{(c_L+\gamma c_H)^2} - \frac{\gamma^2 c_L^2}{(c_H+\gamma c_L)^2} = \frac{1}{(1+\sigma\gamma)^2} - \frac{\gamma^2}{(\sigma+\gamma)^2} \\
&= \frac{\sigma(1-\gamma^2)(\gamma^2\sigma+2\gamma+\sigma)}{(1+\gamma\sigma)^2(\gamma+\sigma)^2}
\end{aligned}
$$

From the FOCs in 2.A.1, we have,

$$
\frac{x_L}{(x_H+x_L)^2}(1+t) = c_H
$$

$$
\frac{x_H}{(x_H+x_L)^2}(1+s) = c_L
$$

Therefore, stage 1 effort when favouring the winner is:

$$
E_1^W = \frac{(1+t)(1+s)}{(1+t)c_L+(1+s)c_H}
$$

Expected effort level in stage 2 is:

$$
E_2^W = \frac{c_L(1+t)}{c_L(1+t)+c_H(1+s)} \cdot \frac{1}{c_L+\gamma c_H} + \frac{c_H(1+s)}{c_L(1+t)+c_H(1+s)} \cdot \frac{1}{c_H+\gamma c_L}
$$

Total effort if the advantage is given to the winner is:

$$
\begin{aligned}
E^W &= E_2^W + E_1^W \\
&= \frac{c_L(1+t)}{c_L(1+t)+c_H(1+s)} \cdot \frac{1}{c_L+\gamma c_H} + \frac{c_H(1+s)}{c_L(1+t)+c_H(1+s)} \cdot \frac{1}{c_H+\gamma c_L} \\
&+ \frac{(1+t)(1+s)}{(1+t)c_L+(1+s)c_H} \\
&= \frac{1}{c_L}\left[\frac{(1+t)(1+s)}{(1+t)+\sigma(1+s)} + \frac{1+t}{(1+t)+\sigma(1+s)} \cdot \frac{1}{1+\gamma\sigma} + \frac{\sigma(1+s)}{(1+t)+\sigma(1+s)} \cdot \frac{1}{\gamma+\sigma}\right]
\end{aligned}
$$

Lemma 2.4. *Dynamic contest with winning advantage results in higher effort level than the standard repeated contest in both stages.*

Proof. In the repeated contest without any advantage, the equilibrium effort is $\frac{1}{c_H+c_L}$ at each stage.

We can easily show that $E_1^W > \frac{1}{c_H+c_L}$ and $E_2^W > \frac{1}{c_H+c_L}$. Therefore, dynamic contest with winning advantage has higher effort level in both stages. □

2.A.2 Total effort when the advantage is given to the loser

We calculate the subgame perfect equilibrium.

Stage 2:

If player H wins the first contest, then:

Player H solves

$$\max_{w_H} \frac{w_H}{w_H + f_L} - c_H \cdot w_H$$

Player L solves

$$\max_{f_L} \frac{f_L}{w_H + f_L} - \gamma c_L \cdot f_L$$

Equilibrium effort will be $w_H = \dfrac{\gamma c_L}{(c_H + \gamma c_L)^2}$ and $f_L = \dfrac{c_H}{(c_H + \gamma c_L)^2}$

Payoff will then be $\pi_{H/W} = \dfrac{\gamma^2 c_L^2}{(\gamma c_L + c_H)^2}$ and $\pi_{L/F} = \dfrac{c_H^2}{(\gamma c_L + c_H)^2}$

If L wins the first contest, then:

H solves

$$\max_{f_H} \frac{f_H}{f_H + w_L} - \gamma c_H \cdot f_H$$

L solves

$$\max_{w_L} \frac{w_L}{f_H + w_L} - c_L \cdot w_L$$

Equilibrium effort will then be $f_H = \dfrac{c_L}{(\gamma c_H + c_L)^2}$ and $w_L = \dfrac{\gamma c_H}{(\gamma c_H + c_L)^2}$

Payoff will then be $\pi_{H/F} = \dfrac{c_L^2}{(\gamma c_H + c_L)^2}$ and $\pi_{L/W} = \dfrac{\gamma c_H^2}{(\gamma c_H + c_L)^2}$

Stage 1:

Player H will solve:

$$\max_{x_H} \frac{x_H}{x_H+x_L} + \frac{x_H}{x_H+x_L}\pi_{H/W} + \frac{x_L}{x_H+x_L}\pi_{H/F} - c_H x_H$$

Player L will solve:

$$\max_{x_L} \frac{x_L}{x_H+x_L} + \frac{x_L}{x_H+x_L}\pi_{L/W} + \frac{x_H}{x_H+x_L}\pi_{L/F} - c_L x_L$$

FOCs:

$$\frac{x_L}{(x_H+x_L)^2}(1+\pi_{H/W}-\pi_{H/F}) = c_H$$
$$\frac{x_H}{(x_H+x_L)^2}(1+\pi_{L/W}-\pi_{L/F}) = c_L$$

i.e,

$$\frac{x_L}{(x_H+x_L)^2}\left[1-\frac{c_L^2}{(\gamma c_H+c_L)^2}+\frac{\gamma^2 c_L^2}{(\gamma c_L+c_H)^2}\right] = c_H$$
$$\frac{x_H}{(x_H+x_L)^2}\left[1-\frac{c_H^2}{(\gamma c_L+c_H)^2}+\frac{\gamma^2 c_H^2}{(\gamma c_H+c_L)^2}\right] = c_L$$

then,

$$\frac{x_L}{(x_H+x_L)^2}(1-t) = c_H$$
$$\frac{x_H}{(x_H+x_L)^2}(1-s) = c_L$$

From the FOCs, we have [4]

$$\frac{1}{x_H+x_L} = \frac{c_H}{1-t} + \frac{c_L}{1-s} = \frac{(1-s)c_H+(1-t)c_L}{(1-t)(1-s)} \tag{2.A.4}$$

Hence, stage 1 effort level when favouring the loser is:

$$E_1^F = x_H + x_L = \frac{(1-t)(1-s)}{c_L(1-t)+c_H(1-s)}$$

Expected effort level in stage 2 is

$$E_2^F = \frac{c_L(1-t)}{c_L(1-t)+c_H(1-s)} \cdot \frac{1}{c_H+\gamma c_L} + \frac{c_H(1-s)}{c_L(1-t)+c_H(1-s)} \cdot \frac{1}{c_L+\gamma c_H}$$

[4] Here the equilibrium always exists because the expected payoff $1-s$ and $1-t$ are always positive. Hence, by Nti (1999) and notion of perfect subgame, there is a unique equilibrium.

Total effort if the advantage is given to the loser is

$$
\begin{aligned}
E^F &= E_2^F + E_1^F \\
&= \frac{c_L(1-t)}{c_L(1-t)+c_H(1-s)} \cdot \frac{1}{c_H+\gamma c_L} + \frac{c_H(1-s)}{c_L(1-t)+c_H(1-s)} \cdot \frac{1}{c_L+\gamma c_H} \\
&+ \frac{(1-t)(1-s)}{(1-t)c_L+(1-s)c_H} \\
&= \frac{1}{c_L}\left[\frac{(1-t)(1-s)}{((1-t)+\sigma(1-s)} + \frac{1-t}{(1-t)+\sigma(1-s)} \cdot \frac{1}{\gamma+\sigma} + \frac{\sigma(1-s)}{(1-t)+\sigma(1-s)} \cdot \frac{1}{1+\gamma\sigma}\right]
\end{aligned}
$$

Lemma 2.5. *The dynamic contest with favourable advantage to early loser results in lower effort level in the first stage, higher effort level in the second stage and higher total effort level compared to the original repeated contest with no advantage.*

Proof. We can show that

$$
\begin{aligned}
E_1^F - \frac{1}{c_H+c_L} &= \frac{(1-t)(1-s)}{c_L(1-t)+c_H(1-s)} - \frac{1}{c_H+c_L} \\
&= \frac{-c_L s(1-t) - c_H t(1-s)}{[c_L(1-t)+c_H(1-s)](c_H+c_L)} < 0
\end{aligned}
$$

and

$$
\begin{aligned}
E_2^F &= \frac{c_L(1-t)}{c_L(1-t)+c_H(1-s)} \cdot \frac{1}{c_H+\gamma c_L} + \frac{c_H(1-s)}{c_L(1-t)+c_H(1-s)} \cdot \frac{1}{c_L+\gamma c_H} \\
&> \frac{1}{c_H+c_L}
\end{aligned}
$$

Consider

$$
\begin{aligned}
&\left(E^F - \frac{2}{c_H+c_L}\right) \cdot c_L \\
&= \frac{(1-t)}{(1-t)+\sigma(1-s)} \cdot \left(\frac{1}{\sigma+\gamma} - \frac{1}{1+\sigma}\right) + \frac{\sigma(1-s)}{(1-t)+\sigma(1-s)} \cdot \left(\frac{1}{1+\sigma\gamma} - \frac{1}{1+\sigma}\right) \\
&- \frac{s(1-t)+\sigma t(1-s)}{[1-t+\sigma(1-s)](1+\sigma)} \\
&= \frac{1-t}{[1-t-\sigma(1-s)](1+\sigma)}\left[\frac{1-\gamma}{\sigma+\gamma} - s\right] - \frac{\sigma(1-s)}{1-t-\sigma(1-s)](1+\sigma)}\left[t - \frac{\sigma(1-\gamma)}{1+\sigma\gamma}\right]
\end{aligned}
$$

Then,

$$\begin{aligned}
&\left(E^F - \frac{2}{c_H + c_L}\right) \cdot c_L \cdot \frac{[1-t-\sigma(1-s)](1+\sigma)(\sigma+\gamma)^2(1+\sigma\gamma)^2}{(1-\sigma)(1-\gamma)} \\
&= (1-t)(\sigma^2\gamma^2 + 2\sigma^2\gamma + 2\sigma\gamma^2 + \sigma\gamma + \sigma + \gamma) - \sigma^2(1-s)(\sigma^2\gamma + 2\sigma\gamma^2 + \sigma\gamma + \sigma + \gamma^2 + \gamma) \\
&= [1-t-\sigma^2(1-s)](\sigma^2\gamma^2 + 2\sigma^2\gamma + 2\sigma\gamma^2 + \sigma\gamma + \sigma + \gamma) - \sigma^2(1-s)(1-\sigma^2)(1+\gamma)\gamma \\
&\geq \frac{(1-\sigma^2)\gamma(\sigma^2 + 2\sigma + \gamma)}{(1+\sigma\gamma)^2}(\sigma^2\gamma^2 + 2\sigma^2\gamma + 2\sigma\gamma^2 + \sigma\gamma + \sigma + \gamma) - \sigma^2(1+\sigma^2)(1-\gamma)\gamma \\
&\geq 0
\end{aligned}$$

Therefore, $E^F \geq \frac{2}{c_H + c_L}$ □

2.A.3 Comparison

$$E^W - E^F = (E_1^W - E_1^F) - (E_2^F - E_2^W)$$

where

$$\begin{aligned}
E_1^W - E_1^F &= \frac{(1+t)(1+s)}{(1+t)c_L + (1+s)c_H} - \frac{(1-t)(1-s)}{(1-t)c_L + (1-s)c_H} \\
&= \frac{1}{c_L} \cdot \left[\frac{(1+t)(1+s)}{(1+t) + (1+s)\sigma} - \frac{(1-t)(1-s)}{(1-t) + (1-s)\sigma}\right] \\
&= \frac{1}{c_L} \cdot \frac{2(1-t^2)s + 2(1-s^2)t\sigma}{[(1-t) + (1-s)\sigma][(1+t) + (1+s)\sigma]}
\end{aligned}$$

and

$$\begin{aligned}
E_2^F - E_2^W &= \frac{c_L(1-t)}{c_L(1-t) + c_H(1-s)} \frac{1}{c_H + \gamma c_L} + \frac{c_H(1-s)}{c_L(1-t) + c_H(1-s)} \frac{1}{c_L + \gamma c_H} \\
&\quad - \frac{c_L(1+t)}{c_L(1+t) + c_H(1+s)} \frac{1}{c_L + \gamma c_H} - \frac{c_H(1+s)}{c_L(1+t) + c_H(1+s)} \frac{1}{c_H + \gamma c_L} \\
&= \frac{1}{c_L(\sigma+\gamma)}\left[\frac{1-t}{(1-t) + \sigma(1-s)} - \frac{(1+s)\sigma}{(1+t) + (1+s)\sigma}\right] \\
&\quad + \frac{1}{c_L(1+\gamma\sigma)}\left[\frac{(1-s)\sigma}{(1-t) + (1-s)\sigma} - \frac{1+t}{(1+t) + (1+s)\sigma}\right] \\
&= \frac{1}{c_L} \cdot \left(\frac{1}{\sigma+\gamma} - \frac{1}{\gamma\sigma+1}\right)\left[\frac{1-t}{(1-t) + (1-s)\sigma} - \frac{(1+s)\sigma}{(1+t) + (1+s)\sigma}\right] \\
&= \frac{1}{c_L} \cdot \left(\frac{1}{\sigma+\gamma} - \frac{1}{\gamma\sigma+1}\right) \frac{(1-t^2) - (1-s^2)\sigma^2}{[(1-t) + (1-s)\sigma][(1+t) + (1+s)\sigma]}
\end{aligned}$$

Proposition 2.3. *Favouring the winner results in higher effort level in the first stage. Favouring the loser results in the higher effort level in the second stage.*

Proof. It is clear that $E_1^W - E_1^F > 0$. For $E_2^F - E_2^W$, we can show that

$$1 - t^2 - \sigma^2(1 - s^2) \geq 0 \tag{2.A.5}$$

Consider

$$1 - t^2 - \sigma^2(1 - s^2) \geq (1 - t^2) - \sigma^2(1 + t)(1 - s) = (1 + t)[1 - t - \sigma^2(1 - s)]$$

then

$$\begin{aligned}
\frac{1 - t^2 - \sigma^2(1 - s^2)}{1 + t} &\geq 1 - t - \sigma^2(1 - s) \\
&= \frac{(1 - \sigma^2)(1 + \sigma\gamma)^2(\sigma + \gamma)^2 - \sigma(1 - \gamma^2)[\sigma^2\gamma + 2\gamma + \sigma - \sigma^3(\gamma^2 + 2\sigma\gamma + 1)]}{(1 + \sigma\gamma)^2(\sigma + \gamma)^2} \\
&= (1 - \sigma^2)\frac{(1 + \sigma\gamma)^2(\sigma + \gamma)^2 - \sigma(1 - \gamma^2)(\sigma\gamma^2 + 2\gamma + \sigma)}{(1 + \sigma\gamma)^2(\sigma + \gamma)^2} \\
&\geq (1 - \sigma^2)\frac{(1 + \sigma\gamma)^2(\sigma + \gamma)^2 - (1 - \gamma^2)(\sigma + \gamma)^2}{(1 + \sigma\gamma)^2(\sigma + \gamma)^2} \\
&\geq 0
\end{aligned}$$

□

Consider

$$\begin{aligned}
E^W - E^L &= (E_1^W - E_1^F) - (E_2^F - E_2^W) \\
&= \frac{1}{c_L} \cdot \frac{1}{[(1 - t) + (1 - s)\sigma][(1 + t) + (1 + s)\sigma]} \cdot A
\end{aligned} \tag{2.A.6}$$

where

$$A = 2(1 - t^2)s + 2(1 - s^2)t\sigma - \left(\frac{1}{\sigma + \gamma} - \frac{1}{\gamma\sigma + 1}\right)[(1 - t^2) - (1 - s^2)\sigma^2]$$

Lemma 2.6. *When the relative marginal cost of effort $\sigma = \frac{c_H}{c_L}$ is sufficiently large ($\sigma > \frac{1+\sqrt{7}}{2} - \frac{\sqrt{2+\sqrt{7}}}{\sqrt{2}} \approx 0.3$), A is always positive.*

Proof. Since $t \geq s$, $1-s^2 \geq 1-t^2$. Therefore,

$$\begin{aligned}
A &= 2(1-t^2)s + 2(1-s^2)\sigma t - \left(\frac{1}{\sigma+\gamma} - \frac{1}{\gamma\sigma+1}\right)[(1-t^2)-(1-s^2)\sigma^2] \\
&\geq 2(1-t^2)s + 2(1-t^2)\sigma t - \left(\frac{1}{\sigma+\gamma} - \frac{1}{\gamma\sigma+1}\right)[(1-t^2)-(1-t^2)\sigma^2] \\
&= (1-t^2)\left[2s + 2\sigma t - \frac{(1-\gamma)(1-\sigma)}{(\sigma+\gamma)(\gamma\sigma+1)}(1-\sigma^2)\right] \\
&= \frac{1-t^2}{(\sigma+\gamma)^2(\gamma\sigma+1)^2}\left[2(1-\gamma^2)\sigma^2(\sigma+1)(\gamma+1)^2 - (1-\gamma)(1-\sigma)^2(1+\sigma)(\sigma+\gamma)(\gamma\sigma+1)\right] \\
&= \frac{(1-t^2)(1-\gamma)(1+\sigma)}{(\sigma+\gamma)^2(\gamma\sigma+1)^2}\left[2(1+\gamma)^3\sigma^2 - (1-\sigma)^2(\sigma+\gamma)(\gamma\sigma+1)\right] \\
&= \frac{(1-t^2)(1-\gamma)(1+\sigma)}{(\sigma+\gamma)^2(\gamma\sigma+1)^2} \cdot m
\end{aligned}$$

where m can be written as a cubic function of γ

$$m = 2\sigma^2\gamma^3 + (-\sigma^3 + 8\sigma^2 - \sigma)\gamma^2 + (-\sigma^4 + 2\sigma^3 + 4\sigma^2 + 2\sigma - 1)\gamma + (-\sigma^3 + 4\sigma^2 - \sigma)$$

If all the coefficients in m is positive, m will be positive for all γ positive. We consider the coefficient of:

- γ^3: $2\sigma^2 > 0$ for all $\sigma \in (0,1]$
- γ^2: $-\sigma^3 + 8\sigma^2 - \sigma > 0$ for all $\sigma \in (4-\sqrt{15}, 1]$
- γ^1: $-\sigma^4 + 2\sigma^3 + 4\sigma^2 + 2\sigma - 1 > 0$ for $\sigma \in \left(\frac{1+\sqrt{7}}{2} - \frac{\sqrt{2+\sqrt{7}}}{\sqrt{2}}, 1\right)$
- γ^0: $-\sigma^3 + 4\sigma^2 - \sigma > 0$ for all $\sigma \in (2-\sqrt{3}, 1]$

Hence, A>0 for $\sigma \in \left(\frac{1+\sqrt{7}}{2} - \frac{\sqrt{2+\sqrt{7}}}{\sqrt{2}}, 1\right)$ □

We can also check that that A is negative for very small σ. For example, when $\sigma = 0$, $A = \dfrac{(1-\sigma)(\gamma-1)}{(\sigma+\gamma)(1+\sigma\gamma)} < 0$.

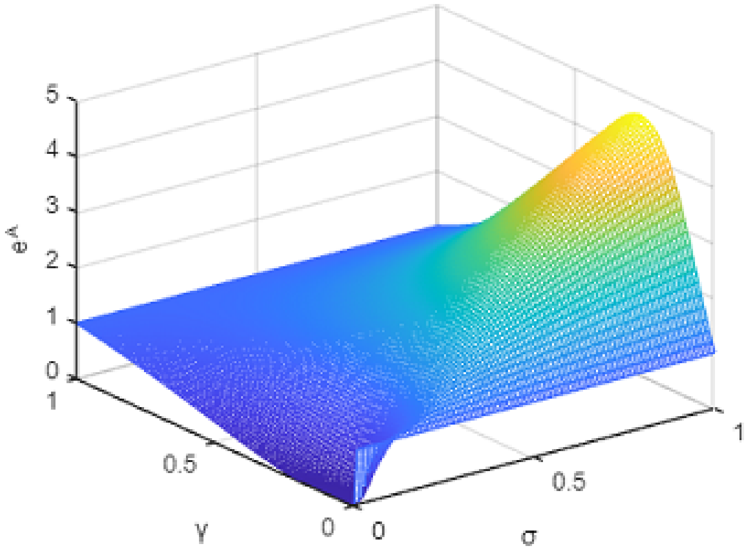

Figure 2.6: Graph of e^A. We can see from the graph that A increases in σ and could be either positive or negative

Proposition 2.4. *A is increasing in* σ *for* $\sigma \in (0,1), \gamma \in (0,1)$.

Consider

$$A = 2(1-t^2)s + 2(1-s^2)\sigma t - \left(\frac{1}{\sigma+\gamma} - \frac{1}{\gamma\sigma+1}\right)[(1-t^2)-(1-s^2)\sigma^2]$$

then

$$\begin{aligned}\frac{\partial A}{\partial \sigma} &= 2(1-t^2)\frac{\partial s}{\partial \sigma} + 2\sigma(1-s^2)\frac{\partial t}{\partial \sigma} + 2t(1-s^2) - 4st\frac{\partial t}{\partial \sigma} - 4\sigma st\frac{\partial s}{\partial \sigma} \\ &\quad + \left(\frac{1}{\sigma+\gamma} - \frac{1}{1+\sigma\gamma}\right)\left[2t\frac{\partial t}{\partial \sigma} + 2\sigma(1-s^2) - 2\sigma^2 s\frac{\partial s}{\partial \sigma}\right] \\ &\quad + \left[\frac{1}{(\sigma+\gamma)^2} - \frac{\gamma}{(1+\gamma\sigma)^2}\right][(1-t^2)-\sigma^2(1-s^2)]\end{aligned} \tag{2.A.7}$$

Recall that

$$s = \frac{\sigma^2}{(\sigma+\gamma)^2} - \frac{\sigma^2\gamma^2}{(1+\sigma\gamma)^2} \Rightarrow \frac{\partial s}{\partial \sigma} = 2\sigma\gamma\left[\frac{1}{(\sigma+\gamma)^3} - \frac{\gamma}{(1+\sigma\gamma)^3}\right] \geq 0$$

and

$$t = \frac{1}{(1+\sigma\gamma)^2} - \frac{\gamma^2}{(\sigma+\gamma)^2} \Rightarrow \frac{\partial t}{\partial \sigma} = 2\gamma\left[\frac{\gamma}{(\sigma+\gamma)^3} - \frac{1}{(1+\sigma\gamma)^3}\right]$$

Hence,

$$\begin{aligned}
s+t &= \frac{\sigma(1-\gamma^2)(\gamma^2\sigma+2\gamma\sigma^2+\sigma+\gamma^2\sigma+2\gamma+\sigma)}{(1+\gamma\sigma)^2(\sigma+\gamma)^2} \\
&= \frac{\sigma(1-\gamma^2)(2\gamma^2\sigma+2\gamma\sigma^2+2\sigma+2\gamma)}{(1+\gamma\sigma)^2(\sigma+\gamma)^2} \\
&= \frac{2\sigma(1-\gamma^2)}{(1+\gamma\sigma)(\sigma+\gamma)}
\end{aligned} \tag{2.A.8}$$

$$\begin{aligned}
\frac{s}{t}-\sigma &= \frac{\gamma^2\sigma+2\gamma\sigma^2+\sigma}{\gamma^2\sigma+2\gamma+\sigma}-\sigma \\
&= \frac{\gamma^2\sigma+2\gamma\sigma^2+\sigma-\gamma^2\sigma^2-2\gamma\sigma-\sigma^2}{\gamma^2\sigma+2\gamma+\sigma} \\
&= \frac{\sigma(1-\sigma)(1-\gamma)^2}{\gamma^2\sigma+2\gamma+\sigma} \\
&\geq 0
\end{aligned} \tag{2.A.9}$$

for all $\gamma \in (0,1)$ and $\sigma \in (0,1]$.

Also,

$$\frac{\partial s}{\partial \sigma}+\frac{\partial t}{\partial \sigma} = 2\gamma\left[\frac{1}{(\sigma+\gamma)^2}-\frac{1}{(1+\sigma\gamma)^2}\right] \geq 0 \tag{2.A.10}$$

$$\frac{\partial s}{\partial \sigma}+\sigma\frac{\partial t}{\partial \sigma} = 2\sigma\gamma(\gamma+1)\left[\frac{1}{(\sigma+\gamma)^3}-\frac{1}{(1+\sigma\gamma)^3}\right] \geq 0 \tag{2.A.11}$$

$$\frac{\partial s}{\partial \sigma}-\sigma\frac{\partial t}{\partial \sigma} = 2\sigma\gamma(1-\gamma)\left[\frac{1}{(\sigma+\gamma)^3}+\frac{1}{(1+\sigma\gamma)^3}\right] \geq 0 \tag{2.A.12}$$

and

$$\begin{aligned}
s+\sigma\frac{\partial t}{\partial \sigma} &= \frac{\sigma(1-\gamma^2)2\sigma^4\gamma^2+\sigma^3\gamma^3+\sigma^3\gamma+\sigma^2\gamma^4-2\sigma^2\gamma^2+\sigma^2+\sigma\gamma^3+\sigma\gamma+2\gamma^2)}{(\sigma+\gamma)^3(\sigma\gamma+1)^3} \\
&\geq 0
\end{aligned} \tag{2.A.13}$$

for all $\sigma \in (0,1]$ and $\gamma \in (0,1)$.

We can show that

$$\begin{aligned}
&\left(2t\frac{\partial t}{\partial \sigma}+\sigma s\frac{\partial s}{\partial \sigma}\right)\cdot\frac{(1+\sigma\gamma)^5(\sigma+\gamma)^5}{2\sigma\gamma(1-\gamma^2)} \\
&= \sigma^3+\sigma\gamma+\sigma^2\gamma-4\sigma^3\gamma+\sigma^4\gamma+\sigma^5\gamma+2\gamma^2+2\sigma\gamma^2-4\sigma^2\gamma^2-2\sigma^3\gamma^2-4\sigma^4\gamma^2+2\sigma^5\gamma^2 \\
&+2\sigma^6\gamma^2+\sigma\gamma^3+\sigma^2\gamma^3-4\sigma^3\gamma^3+\sigma^4\gamma^3+\sigma^5\gamma^3+\sigma^3\gamma^4 \\
&\geq 0
\end{aligned}$$

So that

$$
\begin{aligned}
2t\frac{\partial t}{\partial\sigma}+2\sigma(1-s^2)-2\sigma^2 s\frac{\partial s}{\partial\sigma} &= \left[\sigma(1-s^2)+2t\frac{\partial t}{\partial\sigma}\right]-\sigma\left[2\sigma s\frac{\partial s}{\partial\sigma}-(1-s^2)\right] \\
&\geq \left[\sigma(1-s^2)-2\sigma s\frac{\partial s}{\partial\sigma}\right]-\sigma\left[2\sigma s\frac{\partial s}{\partial\sigma}-(1-s^2)\right] \\
&= (\sigma-1)\left[2\sigma s\frac{\partial s}{\partial\sigma}-(1-s^2)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
2\sigma s\frac{\partial s}{\partial\sigma}-(1-s^2) &= 2\sigma s\frac{\partial s}{\partial\sigma}+s^2-1 \leq \left(s+\sigma\frac{\partial s}{\partial\sigma}\right)^2-1 \\
&= \left(s+\sigma\frac{\partial s}{\partial\sigma}+1\right)\cdot\left(s+\sigma\frac{\partial s}{\partial\sigma}-1\right) \\
&{}^{5}\leq s+\sigma\frac{\partial s}{\partial\sigma}-1 \\
&= \frac{\sigma^2}{(\sigma+\gamma)^2}-\frac{\sigma^2\gamma^2}{(1+\sigma\gamma)^2}+2\sigma^2\gamma\left[\frac{1}{(\sigma+\gamma)^3}-\frac{\gamma}{(1+\sigma\gamma)^3}\right]-1 \\
&= \frac{3\sigma^2}{(\sigma+\gamma)^2}+\frac{3}{(1+\sigma\gamma)^2}-\frac{2\sigma^3}{(\sigma+\gamma)^3}-\frac{2}{(1+\sigma\gamma)^3}-2 \qquad (2.A.14) \\
&= 4-\frac{3(\gamma^2+2\sigma\gamma)}{(\sigma+\gamma)^2}-\frac{3(\sigma^2\gamma^2+2\sigma\gamma)}{(1+\sigma\gamma)^2}-\frac{2\sigma^3}{(\sigma+\gamma)^3}-\frac{2}{(1+\sigma\gamma)^3} \\
&= 4-\frac{2\sigma^3+3(\sigma+\gamma)(\gamma^2+2\sigma\gamma)}{(\sigma+\gamma)^3}-\frac{2+3(1+\sigma\gamma)(\sigma^2\gamma^2+2\sigma\gamma)}{(1+\sigma\gamma)^3} \\
&= 4-\frac{2(\sigma+\gamma)^3+3\sigma\gamma^2+\gamma^3}{(\sigma+\gamma)^3}-\frac{2(1+\sigma\gamma)^3+\sigma^2\gamma^2+3\sigma\gamma}{(1+\sigma\gamma)^3} \\
&\leq 0
\end{aligned}
$$

Hence,

$$2t\frac{\partial t}{\partial\sigma}+2\sigma(1-s^2)-2\sigma^2 s\frac{\partial s}{\partial\sigma}\geq 0$$

[5] since $s\geq 0$ and $\frac{\partial s}{\partial\sigma}\geq 0$

Therefore,

$$
\begin{aligned}
\frac{\partial A}{\partial \sigma} &= 2(1-t^2)\frac{\partial s}{\partial \sigma} + 2\sigma(1-s^2)\frac{\partial t}{\partial \sigma} + 2t(1-s^2) - 4st\frac{\partial t}{\partial \sigma} - 4\sigma st\frac{\partial s}{\partial \sigma} \\
&\quad + \left(\frac{1}{\sigma+\gamma} - \frac{1}{1+\sigma\gamma}\right)\left[2t\frac{\partial t}{\partial \sigma} + 2\sigma(1-s^2) - 2\sigma^2 s\frac{\partial s}{\partial \sigma}\right] \\
&\quad + \left[\frac{1}{(\sigma+\gamma)^2} - \frac{\gamma}{(1+\gamma\sigma)^2}\right][(1-t^2) - \sigma^2(1-s^2)] \\
&\overset{6}{\geq} 2(1-t^2)\frac{\partial s}{\partial \sigma} + 2\sigma(1-s^2)\frac{\partial t}{\partial \sigma} + 2t(1-s^2) - 4st\frac{\partial t}{\partial \sigma} - 4\sigma st\frac{\partial s}{\partial \sigma} \\
&= 2(1-t^2)\frac{\partial s}{\partial t} + 2\left[\sigma(1-s^2) - 2st(1-\sigma)\right]\frac{\partial t}{\partial \sigma} + 2t\left[1 - s^2 - 2s\sigma\left(\frac{\partial s}{\partial \sigma} + \frac{\partial t}{\partial \sigma}\right)\right] \\
&\geq 2(1-t^2)\frac{\partial s}{\partial t} + 2\left[\sigma(1-s^2) - 2st(1-\sigma)\right]\frac{\partial t}{\partial \sigma}
\end{aligned}
$$

since

$$
\begin{aligned}
1 - s^2 - 2s\sigma\left(\frac{\partial s}{\partial \sigma} + \frac{\partial t}{\partial \sigma}\right) &\geq 1 - \left[s + \sigma\left(\frac{\partial s}{\partial \sigma} + \frac{\partial t}{\partial \sigma}\right)\right]^2 \\
&= 1 - \left[\frac{\sigma(1-\gamma^2)(\sigma+2\gamma+2\sigma\gamma^2}{(\sigma+\gamma)^2(1+\sigma\gamma)^2}\right]^2 \\
&\geq 1 - \frac{\sigma(1-\gamma^2)(\sigma+2\gamma+2\sigma\gamma^2}{(\sigma+\gamma)^2(1+\sigma\gamma)^2} \\
&= \frac{\gamma(\sigma^4\gamma + 2\sigma^3\gamma^2 + 2\sigma^3 + 2\sigma^2\gamma^3 + 4\sigma^2\gamma + 4\sigma\gamma^2 + \gamma)}{(\sigma+\gamma)^2(1+\sigma\gamma)^2} \\
&\geq 0
\end{aligned}
\tag{2.A.15}
$$

• Suppose $\frac{\partial t}{\partial \sigma} < 0$. Then,

$$
\begin{aligned}
\frac{\partial A}{\partial \sigma} &\geq 2(1-t^2)\frac{\partial s}{\partial t} + 2\left[\sigma(1-s^2) - 2st(1-\sigma)\right]\frac{\partial t}{\partial \sigma} \\
&= (1-t^2)\left(\frac{\partial s}{\partial \sigma} + \frac{\partial t}{\partial \sigma}\right) + \frac{\partial t}{\partial \sigma}\left[\sigma(1-s^2) - (1-t^2) - 2st(1-\sigma)\right] \\
&\overset{7}{\geq} (1-t^2)\left(\frac{\partial s}{\partial \sigma} + \frac{\partial t}{\partial \sigma}\right) + \frac{\partial t}{\partial \sigma}\left[\sigma(1-\sigma^2t^2) - (1-t^2) - 2\sigma t^2(1-\sigma)\right] \\
&= (1-t^2)\left(\frac{\partial s}{\partial \sigma} + \frac{\partial t}{\partial \sigma}\right) + \frac{\partial t}{\partial \sigma}(1-\sigma)[(\sigma^2-\sigma+1)t^2 - 1] \\
&\geq 0
\end{aligned}
$$

[6] since $1 - t^2 - \sigma^2(1-s^2) \geq 0$ as in 2.A.5

since $\dfrac{\partial s}{\partial \sigma}+\dfrac{\partial t}{\partial \sigma}>0$, and $(\sigma^2-\sigma+1)t^2-1<0$

• Suppose $\dfrac{\partial t}{\partial \sigma}\geq 0$ .Then

$$
\begin{aligned}
\frac{\partial A}{\partial \sigma} &= 2(1-t^2)\frac{\partial s}{\partial \sigma}+2\sigma(1-s^2)\frac{\partial t}{\partial \sigma}+2t(1-s^2)-4st\frac{\partial t}{\partial \sigma}-4\sigma st\frac{\partial s}{\partial \sigma}\\
&\quad +\left(\frac{1}{\sigma+\gamma}-\frac{1}{1+\sigma\gamma}\right)\left[2t\frac{\partial t}{\partial \sigma}+2\sigma(1-s^2)-2\sigma^2 s\frac{\partial s}{\partial \sigma}\right]\\
&\quad +\left[\frac{1}{(\sigma+\gamma)^2}-\frac{\gamma}{(1+\gamma\sigma)^2}\right][(1-t^2)-\sigma^2(1-s^2)]\\
&= 2(1-t^2)\frac{\partial s}{\partial t}+2\left[\sigma(1-s^2)-2st(1-\sigma)\right]\frac{\partial t}{\partial \sigma}+2t\left[1-s^2-2s\sigma\left(\frac{\partial s}{\partial \sigma}+\frac{\partial t}{\partial \sigma}\right)\right]\\
&\quad +\left(\frac{1}{\sigma+\gamma}-\frac{1}{1+\sigma\gamma}\right)\left[2t\frac{\partial t}{\partial \sigma}+\sigma(1-s^2)\right]+\left(\frac{1}{\sigma+\gamma}-\frac{1}{1+\sigma\gamma}\right)\left[\sigma(1-s^2)-2\sigma^2 s\frac{\partial s}{\partial \sigma}\right]\\
&\geq 2(1-t^2)\frac{\partial s}{\partial t}+2\left[\sigma(1-s^2)-2st(1-\sigma)\right]\frac{\partial t}{\partial \sigma}+\left(\frac{1}{\sigma+\gamma}-\frac{1}{1+\sigma\gamma}\right)\left[2t\frac{\partial t}{\partial \sigma}+\sigma(1-s^2)\right]
\end{aligned}
$$

by 2.A.14 and 2.A.15 .
Hence,

$$
\begin{aligned}
\frac{\partial A}{\partial \sigma} &\geq 2(1-t^2)\sigma\frac{\partial t}{\partial t}+2\left[\sigma(1-s^2)-2st(1-\sigma)\right]\frac{\partial t}{\partial \sigma}+\left(\frac{1}{\sigma+\gamma}-\frac{1}{1+\sigma\gamma}\right)\left[2t\frac{\partial t}{\partial \sigma}+\sigma(1-s^2)\right]\\
&\geq 2(1-t^2)\sigma\frac{\partial t}{\partial t}+2\left[\sigma(1-s^2)-2st(1-\sigma)\right]\frac{\partial t}{\partial \sigma}+\left(\frac{1}{\sigma+\gamma}-\frac{1}{1+\sigma\gamma}\right)\left[2t\frac{\partial t}{\partial \sigma}+2\sigma^2 s\frac{\partial t}{\partial \sigma}\right]\\
&\geq 2\frac{\partial t}{\partial \sigma}\left[\sigma(1-s^2)+\sigma(1-t^2)-2st(1-\sigma)+\left(\frac{1}{\sigma+\gamma}-\frac{1}{1+\sigma\gamma}\right)(t+\sigma^2 s)\right]\\
&= 2\frac{\partial t}{\partial \sigma}\left[2\sigma-\sigma s^2-\sigma t^2-2st(1-\sigma)+\left(\frac{1}{\sigma+\gamma}-\frac{1}{1+\sigma\gamma}\right)(t+\sigma^2 s)\right]
\end{aligned}
$$

since

$$
\begin{aligned}
1-s^2-2\sigma s\frac{\partial t}{\partial \sigma} &\geq 1-\left(s+\sigma\frac{\partial t}{\partial \sigma}\right)^2=\left[1-\left(s+\sigma\frac{\partial t}{\partial \sigma}\right)\right]\cdot\left[1+\left(s+\sigma\frac{\partial t}{\partial \sigma}\right)\right]\\
{}^{8} &\geq 1-\left(s+\sigma\frac{\partial t}{\partial \sigma}\right)\\
&= 1-\frac{\sigma^2}{(\sigma+\gamma)^2}+\frac{\sigma^2\gamma^2}{(1+\sigma\gamma)^2}-2\sigma\gamma\left[\frac{\gamma}{(\sigma+\gamma)^3}-\frac{1}{(1+\sigma\gamma)^3}\right]\\
&= 1-\frac{\sigma^3+\sigma^2\gamma+2\sigma\gamma^2}{(\sigma+\gamma)^3}+\frac{\sigma^3\gamma^3+\sigma^2\gamma^2+2\sigma\gamma}{(1+\sigma\gamma)^3}\\
&\geq 0
\end{aligned}
$$

[7] since $s\geq\sigma t$ and $\frac{\partial t}{\partial \sigma}<0$

Consider

$$\begin{aligned}
&2\sigma - \sigma s^2 - \sigma t^2 - 2st(1-\sigma) + \left(\frac{1}{\sigma+\gamma} - \frac{1}{1+\sigma\gamma}\right)(t+\sigma^2 s) \\
&\geq 2\sigma - \sigma s^2 - \sigma t^2 - 2st(1-\sigma) + \left(\frac{1}{\sigma+\gamma} - \frac{1}{1+\sigma\gamma}\right)(\sigma t+\sigma s) \\
&= 2\sigma - \sigma(t-s)^2 - 2st + \left(\frac{\sigma}{\sigma+\gamma} - \frac{\sigma}{1+\sigma\gamma}\right)(s+t) \\
&\geq 2\sigma - \sigma(t-s)^2 - \frac{(s+t)^2}{2} + \left(\frac{\sigma}{\sigma+\gamma} - \frac{\sigma}{1+\sigma\gamma}\right)(s+t) \\
&= 2\sigma + (s+t)\left(\frac{\sigma}{\sigma+\gamma} - \frac{\sigma}{1+\sigma\gamma} - \frac{s+t}{2}\right) - \sigma(t-s)^2 \\
&\geq 2\sigma + (s+t)\left(\frac{\sigma}{\sigma+\gamma} - \frac{\sigma}{1+\sigma\gamma} - \frac{s+t}{2}\right) - \sigma(t-s) \\
&= 2\sigma - \frac{2\sigma^2(1-\gamma)^2(1+\gamma)}{(\sigma+\gamma)(1+\sigma\gamma)^2} - \frac{2\sigma^2\gamma(1-\sigma^2)(1-\gamma^2)}{(\sigma+\gamma)^2(1+\sigma\gamma)^2} \\
&= 2\sigma\left[1 - \frac{\sigma(1-\gamma)^2(1+\gamma)}{(\sigma+\gamma)(1+\sigma\gamma)^2} - \frac{\sigma\gamma(1-\sigma^2)(1-\gamma^2)}{(\sigma+\gamma)^2(1+\sigma\gamma)^2}\right] \\
&= \frac{2\sigma\gamma(\sigma^4\gamma+\sigma^3\gamma^2+3\sigma^3+\sigma^2\gamma^3-\sigma^2\gamma^2+5\sigma^2\gamma+\sigma^2-\sigma\gamma^3+4\sigma\gamma^2+\sigma\gamma+\gamma)}{(\sigma+\gamma)^2(1+\sigma\gamma)^2} \\
&\geq 0
\end{aligned}$$

Therefore,

$$\frac{\partial A}{\partial \sigma} \geq 0$$

for all $\sigma \in (0,1]$, $\gamma \in (0,1)$

Proposition 2.5. *For each γ, there exist $\sigma^*(\gamma)$ such that $A(\sigma^*,\gamma) = 0$. If $\sigma > \sigma^*(\gamma)$ then $A > 0$, and if $\sigma < \sigma^*(\gamma)$ then $A < 0$.*

Proof. It can be shown that A is continuous on $\sigma \in [0.1)$ for $\gamma \in (0,1]$. We have showed that $A(\sigma,\gamma) > 0$ for all $\gamma \in (0,1]$, $\sigma \in [0.3,1)$, and that $A(0,\gamma) < 0$ for all $\gamma \in (0,1)$. By proposition 2.1, A is increasing in σ for $\sigma \in (0,1), \gamma \in (0,1)$. Therefore, for any $\gamma \in (0,1)$, there will exist $\sigma^*(\gamma)$ such that $A(\sigma^*,\gamma) = 0$.
Since $\dfrac{\partial A}{\partial \sigma} > 0$, $A(\sigma,\gamma) > 0$ for all $\sigma > \sigma^*(\gamma)$. If $\sigma < \sigma^*(\gamma)$, $A(\sigma,\gamma) < 0$. □

Theorem 2.1. *If the relative cost of effort $\sigma > \max \sigma^*(\gamma) \approx 0.27$, it is always better to favour the winner of the first stage. If the relative cost of effort $\sigma < \min \sigma^*(\gamma) \approx 0$, it is always better to favour the loser of the first stage.*

[8] since $s + \sigma\dfrac{\partial t}{\partial \sigma} \geq 0$ by 2.A.13

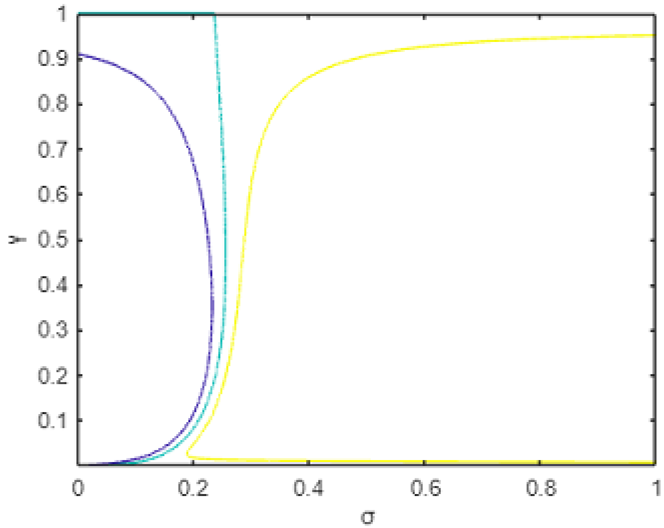

Figure 2.7: Level curves of A at A=-0.1, A=0, A=0.1 respectively

Proof. Let Δ be the difference in total effort when the winner has an advantage and when the favour is given to the loser. Then, by 2.A.6 $sgn\Delta = sgnA$. As $A(\sigma,\gamma) > 0$ for all $\sigma > \max\sigma^*(\gamma)$,it is therefore better to favour the winner. If $\sigma < \min\sigma^*(\gamma)$, $A(\gamma,\sigma) < 0$ for all $\gamma \in (0,1)$. It is then better to favour the loser. □

2.B Optimal prize distribution

2.B.1 When favouring the winner

Solving the equilibrium efforts , we can calculate the total effort level when is favourable advantage is given to the winner:

$$E^W \cdot c_L = \frac{[t+V(1-t)][s+V(1-s)]}{\sigma[s+V(1-s)]+[t+V(1-t)]} + \frac{t+V(1-t)}{\sigma[s+V(1-s)]+[t+V(1-t)]} \cdot \frac{1-V}{\sigma\gamma+1} + \frac{\sigma[s+V(1-s)]}{\sigma[s+V(1-s)]+[t+V(1-t)]} \cdot \frac{1-V}{\sigma+\gamma}$$

Consider

$$\begin{aligned}
&\frac{\partial E^W}{\partial V}\cdot c_L[\sigma s+t+V(1+\sigma-t-\sigma s)]^2(\sigma+\gamma)(1+\sigma\gamma)\\
&=V^2(1+\sigma-t-\sigma s)\big[(1-t)(1-s)(\sigma+\gamma)(1+\sigma\gamma)-(1-t)(\sigma+\gamma)-(1-s)\sigma(1+\sigma+\gamma)\big]\\
&\quad+2V(t+\sigma s)\big[(1-t)(1-s)(\sigma+\gamma)(1+\sigma\gamma)-(1-t)(\sigma+\gamma)-(1-s)\sigma(1+\sigma+\gamma)\big]\\
&\quad+(t^2+\sigma^2s^2-\sigma s^2t-st^2)(\sigma+\gamma)(1+\sigma\gamma)+(t-s-st-\sigma s^2)\sigma(1+\sigma\gamma)\\
&\quad-[t^2+\sigma st+\sigma(t-s)](\sigma+\gamma)\\
&=[V^2(1+\sigma-t-\sigma s)+2V(t+\sigma s)]\cdot k+r
\end{aligned}$$

for

$$\begin{aligned}
k&=(1-t)(1-s)(\sigma+\gamma)(1+\sigma\gamma)-(1-t)(\sigma+\gamma)-(1-s)\sigma(1+\sigma\gamma)\\
&=(1-s)\sigma\big[(1-t)(\sigma+\gamma)\gamma-((1+\sigma\gamma)\big]-s(1-t)(\sigma+\gamma)\\
&\leq(1-s)\sigma\big[(\sigma+\gamma)\gamma-(1+\sigma\gamma)\big]-s(1-t)(\sigma+\gamma)\\
&\leq 0
\end{aligned}$$

and

$$\begin{aligned}
r=&(t^2+\sigma s^2-\sigma s^2t-st^2)(\sigma+\gamma)(1+\sigma\gamma)+(t-s-st-\sigma s^2)\sigma(1+\sigma\gamma)\\
&-[t^2+\sigma st+\sigma(t-s)](\sigma+\gamma)
\end{aligned}$$

We can see that

$$\frac{\partial E^W}{\partial V}|_{V=0}=\frac{r}{c_L(\sigma s+t)(\sigma+\gamma)(1+\sigma\gamma)}$$

When σ is significantly smaller than γ, r is positive. For example, when $\sigma=0.1$, $\gamma=0.8$, $r=0.0013$. Therefore, a small prize V>0 at stage 1 would yields higher effort level than V=0.

On the other hand, when σ is sufficiently large compared to γ, $r<0$. Therefore, $\frac{\partial E^W}{\partial V}<0$.

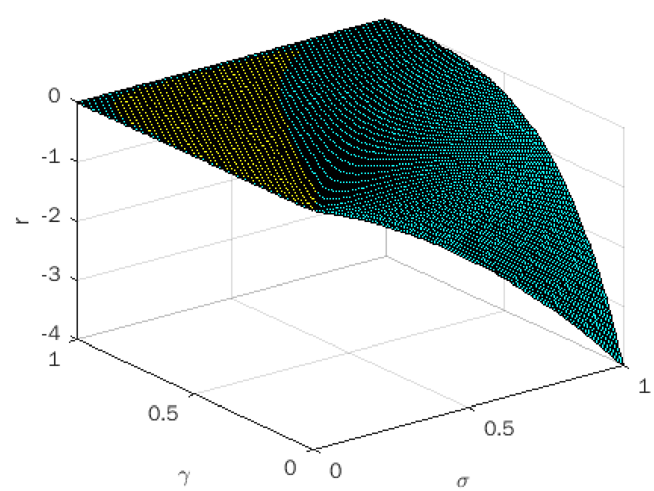

Figure 2.8: Graph of r. The yellow region is where r>0

From the graph, we can see that $r < 0$ when $\sigma > \gamma$. Therefore, $\frac{\partial E^W}{\partial V} < 0$. As total effort level decreases in the first stage prize V, it is effort maximising to put all the prize in the second stage.

2.B.2 When favouring the loser

With the prize distribution (V,1-V), the effort in the second stage will be $\frac{1-V}{c_L+\gamma c_H}$ if Low type player win the first stage, and $\frac{1-V}{c_H+\gamma c_L}$ if High type player win the first stage.
The FOCs for the first stage effort becomes

$$\frac{x_L}{(x_H+x_L)^2}[V-t(1-V)] = c_H$$
$$\frac{x_H}{(x_H+x_L)^2}[V-s(1-V)] = c_L$$

From the FOCs, we can see that positive expected pay-off for both players is only guaranteed when $V-s(1-V) = V(1+s)-s \geq V(1+t)-t = V-t(1-V) > 0$.

If $(1+t)V-t \leq (1+s)V-s < 0$, i.e., the first stage prize V is small, both player will stay inactive hoping to get the advantage to earn big prize in the later stage.

If $(1+t)V-t < 0 < (1+s)V-s$, High type player will stay inactive.The Low type player will put in effort ε to secure the prize V because that will give him higher pay-off in expectation compared to staying inactive and gaining advantage by half a chance.

Therefore, we can see that any prize $V \in \left(0, \frac{t}{1+t}\right]$ is wasteful because it does not induce any first stage effort. It is then better to put V=0.

Consider the prize distribution V such that $V > \frac{t}{1+t}$. Then, the total effort in the two stage contest when favouring the loser is:

$$E^F \cdot c_L = \frac{[V(1+t)-t][V(1+s)-s]}{\sigma[V(1+s)-s]+[V(1+t)-t]} + \frac{[V(1+s)-s]\sigma}{\sigma[V(1+s)-s]+[V(1+t)-t]} \cdot \frac{1-V}{\sigma\gamma+1}$$
$$+ \frac{[V(1+t)-t]}{\sigma[V(1+s-s)]+[V(1+t)-t]} \cdot \frac{1-V}{\sigma+\gamma}$$

We can see that

$$E^F \cdot c_L|_{V=\frac{t}{1+t}+\varepsilon} < \frac{1}{2}\left(1-\frac{t}{1+t}\right)\left(\frac{1}{\sigma+\gamma}+\frac{1}{1+\sigma\gamma}\right)$$

as $e \to 0$. Therefore, when the first stage prize is just enough to motivate the players to put effort, the effort level will be lower than when they are inactive in the first round.

Lemma 2.7. *Any $V > 0.5$ results in lower effort level than $V = 0$.*

Proof. When $V = 0$, both players will be inactive in the first stage. The advantage for the loser will be given randomly to one of the players. They will then compete in the second stage for the prize of 1. Total effort will then be

$$E_0^F \cdot c_L = \frac{1}{2}\frac{1}{\sigma+\gamma} + \frac{1}{2}\frac{1}{1+\sigma\gamma}$$

When $V > 0.5 > \frac{t}{1+t}$, the effort level is

$$\begin{aligned} E_V^F \cdot c_L = & \frac{[V(1+t)-t][V(1+s)-s]}{\sigma[V(1+s)-s]+[V(1+t)-t]} + \frac{[V(1+s)-s]\sigma}{\sigma[V(1+s)-s]+[V(1+t)-t]} \cdot \frac{1-V}{\sigma\gamma+1} \\ & + \frac{[V(1+t)-t]}{\sigma[V(1+s)-s]+[V(1+t)-t]} \cdot \frac{1-V}{\sigma+\gamma} \end{aligned}$$

Consider

$$\begin{aligned} (E_V^F - E_0^F) \cdot c_L = & \frac{[V(1+t)-t][V(1+s)-s]}{\sigma[V(1+s)-s]+[V(1+t)-t]} + \frac{[V(1+s)-s]\sigma}{\sigma[V(1+s)-s]+[V(1+t)-t]} \cdot \frac{1-V}{\sigma\gamma+1} \\ & + \frac{[V(1+t)-t]}{\sigma[V(1+s)-s]+[V(1+t)-t]} \cdot \frac{1-V}{\sigma+\gamma} - \frac{1}{2(\sigma+\gamma)} - \frac{1}{2(1+\sigma\gamma)} \end{aligned}$$

Hence

$$\begin{aligned} & (E_V^F - E_0^F) \cdot c_L \cdot (\sigma+\gamma)(1+\sigma\gamma)[V(\sigma+1+\sigma s+t)-\sigma s-t] \\ = & [V(1+s)-s][V(1+t)-t](\sigma+\gamma)(1+\sigma\gamma) + [V(1+t)-t][(1-2V)(1+\sigma\gamma)-(\sigma+\gamma)] \\ & + [V(1+s)-s]\sigma[(\sigma+\gamma)(1-2V)-(1+\sigma\gamma)] \\ \leq & [V(1+s)-s][V(1+t)-t](\sigma+\gamma)(1+\sigma\gamma) - [V(1+t)-t][(1+\sigma\gamma)+(\sigma+\gamma)] \\ & - [V(1+s)-s]\sigma[(\sigma+\gamma)+(1+\sigma\gamma)] \\ \leq & 0 \end{aligned}$$ [9]

for all $V > 0.5$ □

[9] Since the expression is a quadratic function of V and is negative at $V = 0$ and $V = 1$, hence is negative for all $V \in [0,1]$.

Chapter 3

Benefits of intermediate competition with non monetary incentive

Abstract

Studies on contest design frequently reveal a trade-off between total effort and selection accuracy. This paper demonstrates that contest performance can be enhanced in both dimensions by adding an intermediate competition with non-monetary benefits, ow winning advantage. By allowing losers to progress to the next stage and granting early winners a reduced cost of effort, the contest designer can foster higher effort levels and increase the likelihood of selecting the most capable candidate. Additionally, our findings indicate that the comparison between dynamic and static contests is contingent on how participants are grouped in the first stage. To our knowledge, our work is among a limited number of papers that have identified the potential for improving both selection accuracy and aggregated effort by transitioning from one contest structure to another.

3.1 Introduction

Most studies on contest design primarily focus on maximising effort. However, for promotion tournaments, selecting the best candidate can be a crucial objective for the designer. Promotion contests are often employed to achieve two objectives: to motivate participants to exert effort (incentive objective) and to select the best candidate for the position (selection accuracy objective). However, contest designers frequently face a trade-off between these two objectives. A contest that results in a higher effort level also increases the probability of selecting an unsuitable candidate. Tsoulouhas et al. (2007) examine the competition for a CEO position using a biased Tullock contest, and find a trade-off between incentives and selection when deciding whether to handicap the insider or the outsider. Similarly, Stracke et al. (2015) compare static contests with dynamic up-or-out promotion contests and demonstrate that it is not possible to improve both attributes by making the contest more dynamic. Empirical evidence further supports the existence of this trade-off in competitions. For instance, Boudreau et al. (2011) show that adding a competitor decreases the effort level but increases the likelihood of a participant producing an exceptional solution.

In many competitions, the designer is also interested in participation rates. Examples include crowdsourcing, early-stage school competitions, and "quit and win" contests in healthcare (with the most popular being smoking cessation). Companies that implement crowdsourcing competitions may desire a large number of participants to maximize public attention and the diversity of received opinions and ideas. Depending on the context, the dropout rate (participation) can serve as an indicator of the contest's performance. The designer seeks to maintain the participation rate while enhancing incentives. Therefore, the contest should not be excessively competitive to discourage less capable contestants (Bimpikis et al., 2019; Fershtman and Gneezy, 2011).

This raises the question of whether it is possible to improve all dimensions simultaneously. Assuming the designer aims to enhance both incentive and selection properties while also increasing participation rates without compromising incentive and selection accuracy, we investigate whether this objective can be attained via winning advantage.

The concept that effort provision can only be induced through monetary incentives is a notion that has been challenged by various empirical findings. Intrinsic motivations can encourage engagement in work (Burgess and Ratto, 2003). Studies involving contests have revealed persistent over dissipation with regards to observed effort levels, which tend to be higher than anticipated by theoretical models. For example, Sheremeta (2010b) observes that the two stage elimination contest, in contrary to the theory, generates higher effort than the equivalent one stage contest, suggesting the joy of winning is a component of utility. Erkal et al. (2018) confirms the role of non-monetary incentives in effort provision. The non-monetary incentives in the paper, which include task enjoyment, capture part of real-life motivation for participation. However, it is important to recognise that that individuals are often motivated not only by immediate rewards and enjoyment but also by their desire to achieve benefits that can grant them advantageous positions in future contests. The non-monetary benefits include training, flexible working hours, opportunities for advancement, or simply recognition. These benefits may entail expenses for firms (such as external training courses) or may come at no cost (such as recognition, positive bias, or flexible working hours).

Studies in psychology and organisational behaviour have suggested that those non-monetary benefits cancan significantly enhance organisational performance (Heyman and Ariely, 2004). Research has shown that symbolic awards, along with improvements in confidence and recognition, can increase motivation, effort, and overall performance (Gallus and Frey, 2016). According to W. Thomas (2009) non-monetary rewards act as positive reinforcement, effectively encouraging employee efforts.[1] Therefore, a competition that incorporates symbolic awards and non-monetary benefits, or winning advantages, can be considered a plausible and effective means of encouraging efforts.

The advantages of winning are also widely observed in various contexts. For instance, in the NBA league, the team with a superior win-loss record is granted the home-court advantage. While this advantage does not significantly increase the likelihood of winning the game, it is shown to have a positive impact on team performance (Leota et al., 2022). Similarly, in education, healthy competition can validate students' sense of efficacy. Winning a competition, even a weekly class quiz, leads to recognition, increased motivation, and enhanced self-esteem, which strengthens engagement (Strong et al., 1995). The winner can receive praise and attributional feedback from teachers, which supports their confidence and competence (Schunk, 1983). In sports education, competitions are generally considered a useful means of promoting engagement. Shindler (2009) suggests that if competitions are used effectively and appropriately as tools to challenge children to focus their attention and refine their skills, then they become valuable instruments for personal development.

We employ the Tullock contest model to compare static contest with dynamic contest with

[1]For a comprehensive review of the impact of non-monetary rewards in the workplace, refer Gallus (2017)

winning advantage. The primary aim of the contest designer is to enhance effort provision and increase the accuracy of contestant selection. Additionally, the contest designer might also have an interest in promoting a high participation rate. To accomplish these goals, the designer introduces intermediate stages of competition and implements non-monetary incentives such as recognition, and training that are rewarded to the winners. In the dynamic contest, there is no intermediate monetary reward, only a symbolic prize, and several winners are selected from smaller groups. The early winners become more efficient, resulting in a reduction in their marginal cost of effort. The reduction in cost is analytically equivalent to the bias on the effort impact, a commonly employed tool to introduce favourable bias (Kirkegaard, 2012), particularly within organisational setting (Ridlon and Shin, 2013; Tsoulouhas et al., 2007). Cost reduction is common practice in sport tournament, and easier to implement than changes in effort impact function (Chowdhury et al., 2023). This setting with cost reduction effectively captures informal competitions within firms and organizations. In such scenarios, managers publicly acknowledge exceptional employees within each department and assign them as team leaders for various projects. These employees often receive additional training and favorable treatment in subsequent promotion competitions[2]. Analytically, the Tullock contest is strategically equivalent to tournament, which is also a common framework to study competition in organisational setting (Baye and Hoppe, 2003).

Our comparison suggests that the additional informal competition can enhance the effort level and the likelihood of selecting the best candidate for the position. We discover that there exists a range of cost reduction where no trade-off exists between the rate of participation, incentive, and selection properties. If total effort and selection accuracy are the only two objectives, the contest designer can achieve strictly better outcomes in both dimensions using the dynamic contest, provided that the cost reduction is significant enough to maintain the participation rate. The dynamic contest with moderate winning advantage in our model satisfies the proposed characteristics of a "healthy competition" by educationists [3]. Our equilibrium analysis corresponds with empirical studies that indicate an improvement in effort and attainment levels when small competitions are introduced (Shindler, 2009; Verhoeff, 1997).

Our findings suggest that firms can enhance their competition's performance without altering the prize distribution or budget by introducing intermediate stages for the team leader and monthly awards for the best employee before the promotion or end-of-year bonus sharing. While K. A.

[2]In the context of internal promotion competitions, contestants often have knowledge of the productivity levels of their fellow competitors, who are also their co-workers. Thus, we consider a game with perfect information, where the players possess complete knowledge about each other's types or abilities.

[3]The prize is symbolic or little important, and all students should they that they have a chance to win (Shindler, 2009)

Konrad and Kovenock (2009) explore a similar scenario using an all-pay auction race with component contests, our model differs in that the attainment of the grand prize depends solely on the final effort level, rather than the number of component contests won. In many cases, the likelihood of promotion or the share of a bonus is largely influenced by performance closer to the events or the effort exerted in the final stage. Additionally, the intermediate winners in our model receive favourable treatment and occupy better positions in subsequent contests.

3.2 Related literature

There are two strands of literature on contest design related to our work. The first one is concerned with the incentive and selection properties in the organizational setting, particularly promotion tournaments, starting from the work by Lazear and Rosen (1981). Their paper considers rank-order tournaments as an alternative form of labor contract and discusses the efficiency of production and the potential issue of adverse selection when using the scheme. Following a similar setting, Rosen (1986) examines the incentive property of the prize in the sequential elimination tournament and finds that elevating the later prize is essential to maintain the motivation of the high-rank survivors. Stracke et al. (2015) adopt the model for the up-or-out promotion contest and find that the elimination contest indeed results in better effort provision compared to the one-stage contest. However, the dynamic structure, which can reduce the effective degree of heterogeneity, results in lower selection accuracy.

Tsoulouhas et al. (2007) develop a model for a CEO contest using a similar approach: internal workers compete in an elimination contest to become candidates for the CEO position, with the final round open to external applicants. This paper examines the optimal handicapping mechanism and concludes that favoring insiders serves as an incentive for current employees to exert effort, but it hinders the selection of more capable external candidates for the CEO role. Our study shares similarities with their work, as it also involves a two-stage contest and a favorable bias towards certain candidates. However, our model differs in that the loser of the first round is not eliminated, and there are no external candidates. While our model may not be the most suitable for CEO selection, it is more applicable than the model proposed by Tsoulouhas et al. (2007) for internal promotions. Furthermore, our model has a wide range of applications, including sales force management, education, and politics.

In a similar vein of comparison, Morgan (2003) conducts a comparative analysis of welfare, specifically allocative efficiency and total effort outlays, between sequential and simultaneous rent-seeking contests. The study focuses on the competition between two asymmetric players for an object with either a high or low value, which is realized after receipt. In this particular setting, the paper demonstrates that sequential contests dominate simultaneous contests in terms of ex-ante Pareto optimality. However, should be highlighted that in Morgan's (2003) study, the term "sequential contest" refers to a scenario where players sequentially choose their effort outlays and only play once, unlike our dynamic setting where players compete simultaneously in different stages.

The second stream of literature related to our work is dynamic contests where the early result impacts later performance. The dynamic contest, particularly equilibrium strategies in dynamic contests, has been increasingly studied lately in Tullock contest and all-pay auction setups [4]. The recent work by Iluz and Sela (2018) recently considers a lottery contest where the player with the highest effort in the first round wins the intermediate prize, and the one with the highest total effort wins the grand prize and gives the characterisation of the subgame perfect equilibrium. The impact of early effort on later performance has been explored using different contest structures and factors. For example, Klein and Schmutzler (2017) analyse the situation when the designer can decide the prize distribution and the weight of early effort in the second period and find that it is effort maximizing to have a grand prize in the last stage, with a positive first performance weight.

In our paper, the performance in the later stage does not depend on the early effort, but the early result. Clark and Nilssen (2018) have previously discussed the head-start as a winning advantage with all pay auction settings. They find that when the winner of intermediate rounds benefits from a head start, the losers might be motivated to exert more effort to win the following stage, which can lead to outperforming the winner at some stages, particularly closer to the final round. These results contrast with our findings while using handicapping as the winning advantage. When the advantage is a multiplicative rather than an additive factor, the early winners always exert higher effort. This is because the head start acts as "free effort," enabling the early winner to relax and put in less effort as the series of contests proceeds. This motivates the laggard to exert higher effort and outperform in the later stages. Conversely, the benefit of the multiplicative handicap requires effort provision, and the improvement in effort productivity encourages the recipient to exert more effort to take advantage of it.

In the study by Möller (2012), a two-stage dynamic contest was analysed, where winning the first stage prize can improve the player's ability in the next contest. The winning advantage enlarges the difference in efforts between the winner and loser, resulting in a competitive

[4] see K. Konrad (2009) for a comprehensive survey

imbalance. Although the setting is relatively similar to ours, the paper's focus is on the size of the first prize that determines the improvement in ability. Additionally, the paper aims to maximize the total effort by finding the prize distribution between stages. In comparison, Megidish and Sela (2014) examine the total effort in a repeated Tullock contest, where the marginal values for the prize can either increase or decrease, and winning the earlier stage can act as either an advantage or disadvantage in the later stage. The paper analyses the equilibrium and the role of budget constraints, showing that a non-restrictive budget leads to higher (lower) total effort when the prize's marginal valuation increases (decreases). When the budget is non-restrictive, players distribute their effort equally among the two stages. This part of the study is relatively similar to our work, as winning the first stage increases the expected payoff in the next stage.

The comparison between the dynamic contest where the players are initially divided into small group and the static contest can also be linked to the comparison between smaller and larger contests, particularly in terms of the total effort exerted in a grand contest compared to a set of sub-contests when multiple winners are involved. When there are multiple winners, several mechanisms can be employed to select the winners. Clark and Riis (1996) propose a nested contest with successive rounds of selection. The first prize winner is chosen from the entire pool of participants and subsequently eliminated. The second prize winner is then selected based on the contest success probability function and the effort profile of all participants except the first prize winner, and so on. Using this mechanism, Fu and Lu (2009) discover that a grand contest elicits higher effort levels compared to any set of sub-contests when the players are symmetric. In contrast, Chowdhury and Kim (2017) find the opposite result when employing a sequential loser elimination mechanism, proposed by Chowdhury and Kim (2014). In their model, the players are successively eliminated from the pool of the survivors using the contest *failure* function, and the prizes are rewarded to the final survivors. The mechanism is equivalent to the mechanism proposed by Berry (1993), where the winning probability of a player for one of k prizes is defined as the sum of efforts exerted by any group of k players that includes him, divided by the total effort of all combinations of k players. It is worth noting that the contests examined in the comparisons by Fu and Lu (2009) and Chowdhury and Kim (2017), despite the successive rounds in the choosing procedure, are one-stage unbiased contests, where players only provide efforts once. On the other hand, our comparison involves a dynamic contest where the players are initially grouped in the first stage to compete for a favourable bias, and then exert effort in the subsequent stage. It should also be noted that unlike the aforementioned papers, our model considers asymmetric players and feature several intermediate winners who receive advantages but ultimately compete for a single final prize.

Our representation of winning advantage is similar to previous work by Clark et al. (2020), who examine the optimal prize distribution in a Tullock contest with symmetric players. They find that the introduction of asymmetry can increase total effort, and proposed the optimal prize distribution for such cases. Before Clark et al. (2020), Ridlon and Shin (2013) have previously considered the improvement in effort efficiency as a result of favourable bias better back-office resources and argued that it is commonly observed in the organisational setting. However, on optimising favouritism to maximise total effort, allowing the advantage to be given to the loser. In contrast, our study only considers the impact of favouring the winner on contest performance.

Our work adds to the ongoing discussion about the role of non-monetary incentives in contests and tournaments. Several experimental studies have suggested that non-monetary factors may contribute to the commonly observed overbidding phenomenon (Erkal et al., 2018; Parco et al., 2005; Schmitt et al., 2004; Sheremeta, 2010a, 2010b). These factors may include identification, as proposed by Mago et al. (2016), or intrinsic motivations, as suggested by Sheremeta (2010b). We investigate another non-monetary motivation for players to exert effort in the early stage, which is the benefits of an advanced position in the future contest. Besides, our paper also relates to the discussion of feedback, as the non-monetary benefit in our model can capture performance appraisals and improvement in confidence. However, previous studies on tournaments have mainly focused on the private information setting, where feedback acts as a signal of ability, and the optimal information reveal is the main concern (Ederer, 2010; Kuhnen and Tymula, 2012).

In our model, we consider a contest with four players, categorised into two types: High and Low. These players differ in their marginal costs of effort. Initially, the players are paired up and engage in a first-stage competition, which does not offer any monetary reward. The winners of each pair are granted an advantage that enhances their efficiency in exerting effort. This advantage is represented as a multiplicative cost reduction. Following the first stage, all four players proceed to the second stage, where they compete for the grand prize. Our analysis reveals that the total effort exerted in the two-stage contest surpasses that of the one-stage contest, provided that all four players actively participate in the second stage. As the cost reduction becomes more significant, the participation rate may change due to changes in the heterogeneity level. We demonstrate that, for moderate cost reduction factors that maintain a participation rate similar to that of the one-stage contest, the dynamic contest enhances the likelihood of the ultimate winner belonging to the High type. Consequently, by transitioning from a one-stage contest to a two-stage contest with a moderate winning advantage, the designer can achieve improved effort provision and enhanced accuracy in selecting the winner. Our findings highlight the advantages of implementing a two-stage contest model over a one-stage contest.

3.3 Model

Consider the setting with four players of two types High and Low. Marginal cost of effort is $c_H < c_L$.

• In the static contest, the players compete simultaneously for the prize of 1. If player i exerts effort x_i and pay the cost $c_i x_i$, his probability of winning is:

$$P(i) = \begin{cases} \dfrac{x_i}{x_i + x_{-i}} & \text{if } x_i + x_{-i} \neq 0 \\ 0.5 & \text{if } x_i = x_{-i} = 0 \end{cases}$$

• In the dynamic contest, the players are initially grouped in pair. The winner i of each group has his cost reduced to γc_i for $\gamma \in (0,1)$. All players then participate in the second stage and exert effort y_i to win the prize.

i) When there are two players of each type, i.e., (H,H,L,L) the players can be grouped either homogeneously of heterogeneously.

In the situation of homogeneous grouping, the players are grouped as (H, H) and (L, L).

The probability that the player H_i wins the first stage is

$$P(H_i) = \begin{cases} \dfrac{x_{H_i}}{x_{H_i} + x_{H_j}} & \text{if } x_{H_i} + x_{H_j} \neq 0 \\ 0.5 & \text{if } x_{H_i} = x_{H_j} = 0 \end{cases}$$

The probability that the player L_i wins the first stage is

$$P(L_i) = \begin{cases} \dfrac{x_{L_i}}{x_{L_i} + x_{L_j}} & \text{if } x_{L_i} + x_{L_j} \neq 0 \\ 0.5 & \text{if } x_{L_i} = x_{L_j} = 0 \end{cases}$$

The first stage winners contain 1 High type and 1 Low type player. The four players whose marginal costs now are $c_H, c_L, \gamma c_H, \gamma c_L$ then compete for the prize of 1 in the second stage.

In the situation of heterogeneous grouping, there are two groups 1, 2, each of which contains one High type and one Low type player. Players compete within their group for the winning advantage.

The probability that the player H in group i wins the first stage is

$$P(H_i) = \begin{cases} \dfrac{x_{H_i}}{x_{H_i} + x_{L_i}} & \text{if } x_{H_i} + x_{L_i} \neq 0 \\ 0.5 & \text{if } x_{H_i} = x_{L_i} = 0 \end{cases}$$

The probability that the player L_i wins the first stage is

$$P(L_i) = \begin{cases} \dfrac{x_{L_i}}{x_{H_i} + x_{L_i}} & \text{if } x_{H_i} + x_{L_i} \neq 0 \\ 0.5 & \text{if } x_{L_i} = x_{H_i} = 0 \end{cases}$$

There could be situations where 2 High type players, 2 Low type players, or 1 High type and 1 Low type player win the first stage and have their marginal costs decrease from c_i to γc_i. Four players will compete in the second stage for the prize.

Lemma 3.1. *When there are two High type and two Low type players and $c_L < 2c_H$, all four players are active in the static contest. When $c_L \geq 2c_H$, only the High type players are active in the static contest.*

Depending on the heterogeneity of the players, the cost reduction rewarded to the first stage winner can increase or decrease the participation rate. When $2c_H > c_L$, all four players are active in the static contest. γ being too low will discourage the early loser(s) from participating in the next round. Meanwhile, when $2c_H \leq c_L$, only two High type players are active in the static contest. Very small γ gives the Low type players incentive to compete, knowing that the ability gap will be narrowed significantly once they obtain it, increasing the contest's participation rate.

Lemma 3.2. *When the players are homogeneously grouped as (H,H), (L,L) in the first stage, the cost reduction factor maintains the participation rate is*

$$\begin{cases} \gamma > \dfrac{2c_L - c_H}{c_L + c_H} & \textit{if } 2c_H > c_L \\ \gamma > \dfrac{c_H}{c_L - c_H} & \textit{if } 2c_H \leq c_L \end{cases}$$

When the players are heterogeneously grouped as (H,L), (H,L) in the first stage, the cost *reduction factor that maintain the participation rate is*

$$\begin{cases} \gamma > \dfrac{2c_L - c_H}{c_L + c_H} & \textit{if } 2c_H > c_L \\ \gamma > \dfrac{2c_H}{c_L} & \textit{if } 2c_H \leq c_L \end{cases}$$

ii) When there are three players of one type and one player of the other type, i.e., (H,H,H,L) or (L,L,H,H), there will be one symmetric pair and one asymmetric pair in the first stage.

Consider the situation involving three High type players and one Low type player. The two pairs competing in the first stage then are (H,H) and (H,L). There could be two High type, or one High type and one Low type first stage winner. The marginal costs of the players in the second stage could then be either $(\gamma c_H, \gamma c_H, c_H, c_L)$ or $(\gamma c_H, \gamma c_L, c_H, c_H)$.

Lemma 3.3. *When there are three High type and one Low type players and $c_L < 1.5c_H$, all four players are active in the static contest. When $c_L \geq 1.5c_H$, only the High type players are active in the static contest.*

The cost reduction factor that maintains the participation rate is

$$\begin{cases} \gamma > \dfrac{2c_L - c_H}{2c_H} & \textit{if } 1.5c_H > c_L \\ \gamma > \dfrac{2c_H}{2c_L - c_H} & \textit{if } 2c_H \leq c_L \end{cases}$$

Similarly, in the situation involving three Low type players and one High type players, the two pairs competing in the first stage then are (L,L) and (H,L). There could be two Low type, or one High type and one Low type first stage winner. The marginal costs of effort in the second stage could be either $(\gamma c_L, \gamma c_L, c_L, c_H)$ or $(\gamma c_L, \gamma c_H, c_L, c_L)$.

Lemma 3.4. *When there are one High type and three Low type players, all four players will be active in the static contest regardless the marginal cost ratio. The cost reduction to maintain the full participation rate is*

$$\gamma > \frac{2c_L - c_H}{2c_L}$$

iii) When there are four players of the same type, i.e., (L,L,L,L) or (H,H,H,H), the players compete pairwise in a symmetric contest before moving to the second stage. The marginal cost of the four players in the second stage become either $(\gamma c_L, \gamma c_L, c_L, c_L)$ or $(\gamma c_H, \gamma c_H, c_H, c_H)$.

Lemma 3.5. *When there are four players of the same type, full participation in the second stage* *of the dynamic contest happens if any only if* $\gamma > 0.5$

3.3.1 Performance of dynamic contest with homogeneous grouping when there are two High type and two Low type players

Example 3.1. *Let* $c_L = 1.5c_H$ *so that all four players are active in the static contest and yields total effort* $0.6/c_H$*. Let the players be grouped heterogeneously as (H,L), (H,L) to initially compete for the cost reduction* γ*.*

- *When* $\gamma = 0.75$*, all four players are active in the second stage. The total effort is* $0.9/c_H$*. The selection accuracy of the dynamic contest is 0.8, precisely the same as the static contest.*
- *When* $\gamma = 0.5$*, three players are active in the second stage. The total effort is* $1.01/c_H$ *. The selection accuracy is 0.67, lower than the original static contest.*
- *When* $\gamma = 0.3$*, only two winners are active in the second stage. The selection accuracy is 0.6, lower than the original static contest.*

When players have similar abilities ($2c_H > c_L$), all four will participate in the static contest. When players are homogeneously grouped, one High and one Low type winner will receive a cost reduction. When γ is extremely small, only two early winners, one High and one Low type, participate in the later round, both receiving the same cost reduction. The dynamic contest will have inferior selection properties compared to the static contest. The static contest with two Low and two High type players has a better chance of selecting the High type than a contest with only one of each type with the same relative cost $\frac{c_H}{c_L}$. In the case of three participants, with two High and one early winner Low type player, a small cost reduction significantly improves the Low type player's performance and chance of winning. Therefore, the dynamic contest is less accurate than the static contest.

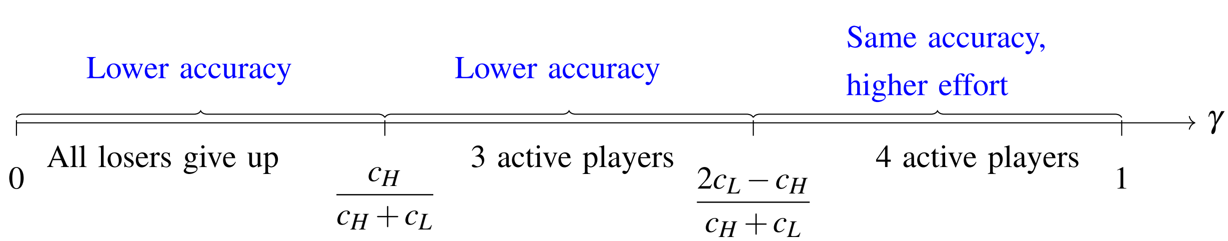

Figure 1a: γ with homogeneous grouping when $c_L < 2c_H$

In contrast, when the players have substantially different abilities ($2c_H < c_L$), only two High type players will compete in the static contest, resulting in a 100% selection accuracy. In the dynamic contest, the winning advantage narrows the ability gap between the Low type early winner and the High type early loser, and a small cost reduction factor γ may encourage the Low type early winner to participate in the second stage, decreasing contest accuracy. Hence, γ should be sufficiently high to prevent a trade-off between participation rate and selection accuracy.

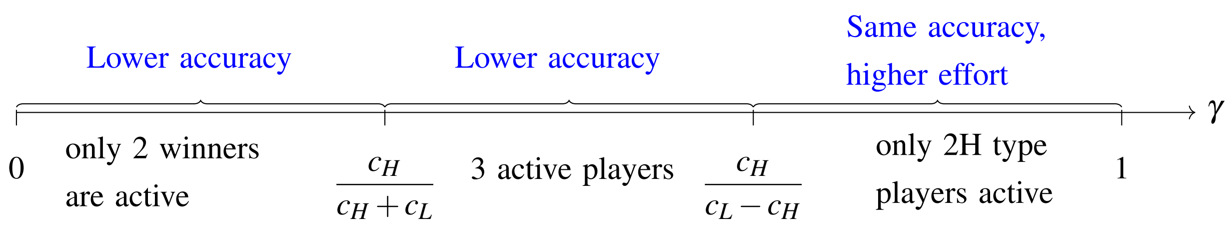

Figure 1b: γ with homogeneous grouping when $c_L > 2c_H$

Proposition 3.1. *Consider the situation involving two High type and two Low type players and homogeneously grouped in pairs.*

For sufficiently high γ which maintain the participation rate similar to static contest (as in lemma 3.2), the dynamic contest results in a higher effort level and the same selection accuracy level compared to the static contest.

When the players are similar, small γ that decreases the participation rate will lead to lower selection accuracy.

When the players are significantly different, small γ that improves the participation rate will result in lower selection accuracy than in the static contest.

The proof of the result is shown in section 3.B of the appendix, part 3.5 and 3.6.

3.3.2 Performance of dynamic contest with heterogeneous grouping when there are two High type and two Low type players

When the players are heterogeneously grouped as (H, L), (H, L), there are three possible possibilities: either two High type players, two Low type players, or one High type and one

Low type player become first stage winner.

Example 3.2. *Suppose that $c_L = 1.5c_H$ so that all four players are active in the static contest. Let the players be grouped heterogeneously as (H,L), (H,L) to initially compete for the cost reduction.*

- *When $\gamma = 0.8$, all four players compete in the second stage. Selection accuracy of the dynamic contest is 0.94, higher than in the static contest (and dynamic contest with homogeneous grouping). The effort level in the second stage of the dynamic contest is $0.7/c_H$, which is already higher than the total effort in the static contest.*

- *When $\gamma = 0.7$, there will be 2,3,4 active players respectively if 2 High type players, 1 High type, 1 Low type player and 2 Low type players win the first stage. The selection accuracy is 0.82, slightly higher than the static contest.*

- *When $\gamma = 0.5$, there will be 2,3,4 active players respectively if 2 High type players, 1 High type, 1 Low type player and 2 Low type players win the first stage. The selection accuracy is 0.76, lower than the static contest.*

- *When $\gamma = 0.4$, the losers of the first stage will quit whenever there is a High type winner. The selection accuracy is 0.79, slightly lower than the static contest.*

- *When $\gamma = 0.3$, only the early winners compete in the second stage. The selection accuracy is 0.84, higher than the static contest, but still lower than the selection accuracy when $\gamma = 0.8$.*

The example highlights the possibility of lower selection accuracy in dynamic contests when γ is extremely small and discourages the early loser from participating in the second stage. The result could be a chance of selection accuracy of 1 if only two High type early winners compete in the second stage, but a significant decrease in selection accuracy whenever a Low type player wins the first stage. Thus, the designer faces a trade-off between incentive and selection. The dynamic contest with a small cost reduction may sometimes be more accurate than the static contest (as in examples 3.2 when $\gamma = 0.7$ and $\gamma = 0.3$). However, the designer will still face a trade-off between participation and selection, and the selection accuracy is not as high as when γ is high enough to keep all four players active in the second stage.

When the players are significantly different ($2c_H < c_L$) so that only two High type winners are active in the dynamic contest, the outcome for heterogeneous grouping is akin to homogeneous grouping. Any γ that encourages the Low type players to participate in the second stage will decrease the selection accuracy.

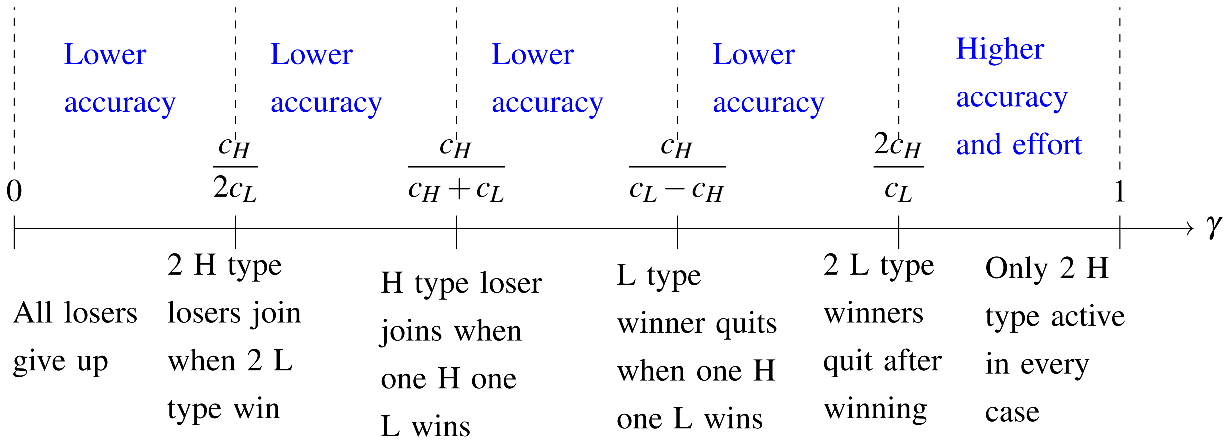

Figure 2b: γ with heterogeneous grouping when $c_L > 2c_H$

Proposition 3.2. *Consider the situation involving two High type and two Low type players and heterogeneously grouped in pairs.*

For sufficiently high γ which maintains the participation rate similar to static contest (as in lemma 3.2), the dynamic contest performs strictly better than the static contest in the incentive and selection dimension when the players are relatively similar.

When the players are significantly different, small γ that improves the participation rate will result in lower selection accuracy than in the static contest.

The proof of the result in shown in section 3.C of the appendix, part 3.7 and 3.8

3.3.3 Performance of dynamic contest when there are three players of the same type

Example 3.3. *Suppose that there are three High type and one Low type players, and $c_L = 1.25c_H$ so that all four players are active in the static stage. The selection accuracy of the static contest is 0.882. The players are grouped as (H,H), (H,L) to compete for the cost reduction γ.*

- *When $\gamma = 0.8$, all four players compete in the second stage, the selection accuracy is 0.9.*
- *When $\gamma = 0.7$, only three High type players participate in the second stage if the early winners are both High type. The probability of the winner being High type is in this case*

is 1. If the Low type player wins the first stage, there will be four active players in the first stage. The probability of the winner being High type is then 0.71. The expected selection accuracy of the dynamic contest is 0.78, lower than the the static contest.

- *When* $\gamma = 0.5$, *only two winners participate in the second stage if the early winners are both High type. The probability of the winner being High type is in this case is 1. If the Low type player wins the first stage, there will be four active players in the first stage. The probability of the winner being High type is then 0.56. The expected selection accuracy of the dynamic contest is 0.84, lower than the the static contest.*
- *When* $\gamma = 0.4$, *only two early winner participate in the second stage. The expected selection accuracy of the dynamic contest is 0.86, lower than the static contest.*

Similar to the previous analysis, the low cost reduction results in a significant decrease in selection accuracy whenever a Low type player wins the first stage. Therefore, the selection accuracy of the whole contest might decrease.

When there are only three active High type players in the static contest, the selection accuracy is 100%. Any cost reduction that encourage the Low type player participate in the second stage of the dynamic contest will decrease the overall selection accuracy.

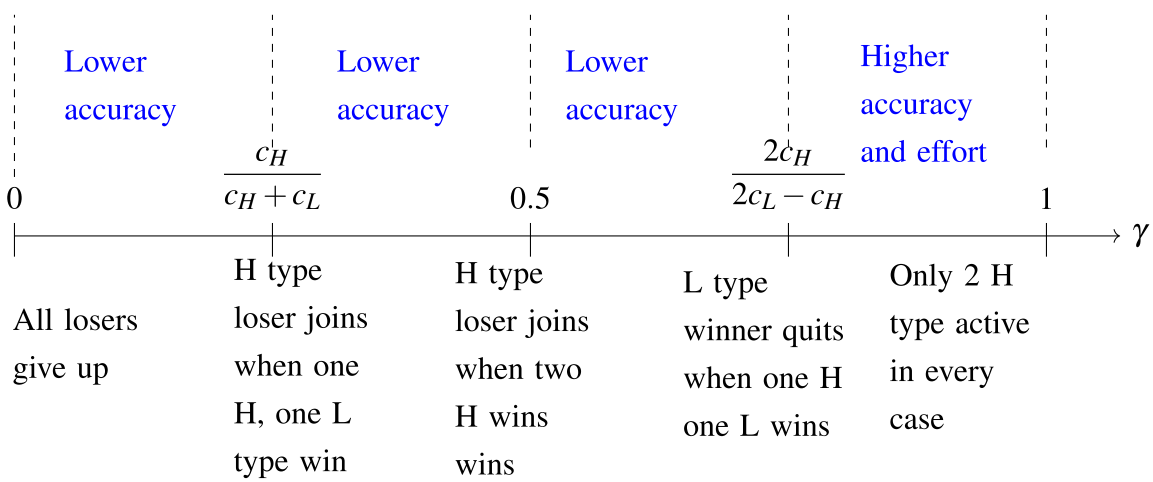

Figure 2b: γ with heterogeneous grouping when $c_L > 1.5c_H$

In comparison, all four players will three of them are Low type. A significant cost reduction can discourage the Low type player to participate in the second stage, leaving the High type to compete with only two more Low type people. However, such cost reduction will also remarkably increase the chance that the Low type early winner gets selected in the final stage. The overall selection accuracy might then decrease.

Proposition 3.3. *Consider the situation involving three players of one type and one player of the other type.*

For sufficiently high γ which maintains the full participation rate as in the static contest (as in lemma 3.3 and 3.4), the dynamic contest results in strictly better effort provision and selection accuracy level.

When there are three High type players and one Low type players who are significantly different, small γ that improves the participation rate will result in lower selection accuracy compared the static contest.

3.3.4 Performance of the dynamic contest when all four players are of the same type

In this situation, the selection accuracy is either 100% or 0%, regardless of the number of active player in the second stage. As the winning advantage reduce the cost of the effort, the effort provision is always higher than the static contest. Therefore, the dynamic contest will result in higher effort level and similar selection accuracy compared to the static contest.

Proposition 3.4. *When the players are symmetric, introducing an intermediate stage with a winning advantage leads to increased effort provision while maintaining the same level of accuracy selection.*

3.3.5 Contest design to improve both incentive and selection

Depending on the designer's objective, the range for γ to achieve improvement will be different. If the designer is interested in the participation rate and has disutility from players dropping out, he should choose γ, which maintains the participation rate. For such γ, there will be no trade-off between incentive and selection. The dynamic contest will have better performance than the static contest.

Theorem 3.1. *The dynamic contest with moderate cost reduction which maintain the same participation rate as the static contest always perform better in terms of both selection accuracy and effort provision. Therefore, there will be no trade-off among effort provision, selection and participation rate when moving from static contest to dynamic contest.*

The dynamic contest will perform strictly better than the static contest in both effort provision and selection accuracy property if there is at least one heterogeneous pair in the first stage.

If the players are paired with another player of the same type in the first stage, the dynamic contest with moderate cost reduction will yield higher effort level and have the same selection accuracy as the static contest.

The lower bound for the moderate cost reduction which maintain the participation rate, thus ensure no trade-off among the properties are as below:

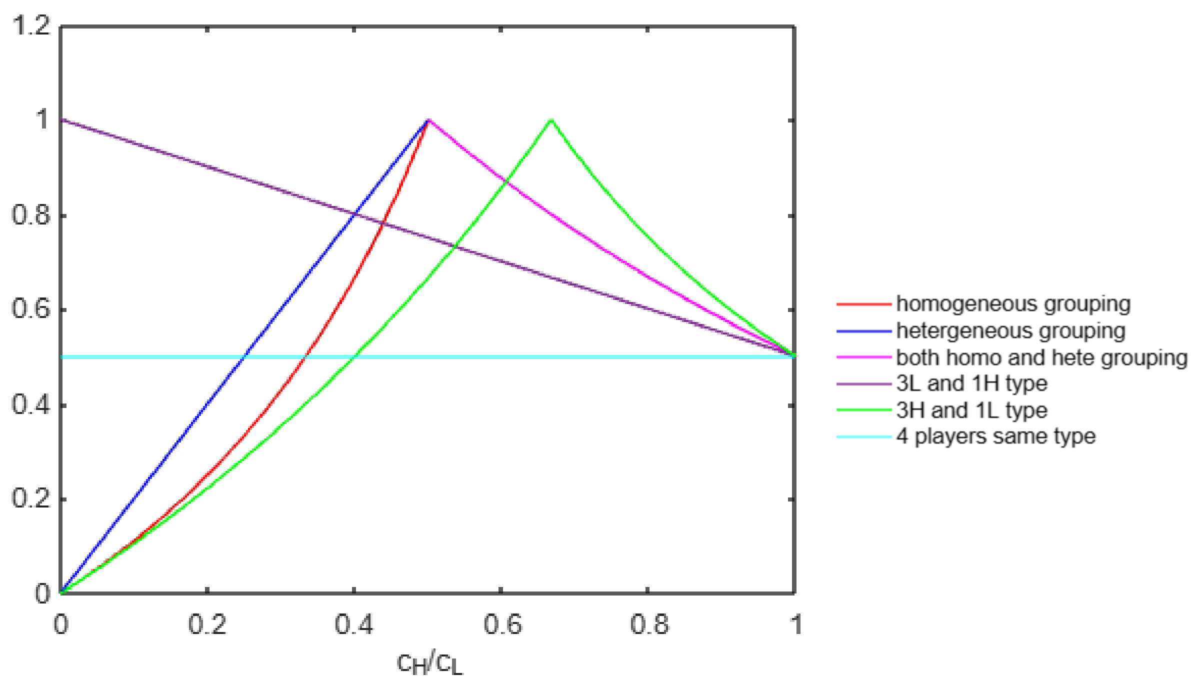

Figure 3.1: lower bound of γ that ensures no trade-off

The graph presents the minimum level of cost reduction necessary for the dynamic contest to maintain a consistent participation rate, a higher effort level, and at least weakly improved selection accuracy compared to the static contest. In fact, the selection accuracy will be strictly better, except in cases of homogeneous grouping. The graph indicates that the contest designer can generally enhance the contest's performance by introducing an intermediate stage with an appropriate winning advantage, except at the "peak" of the graph, where the lower bound is 1. In such situations, any winning advantage will motivate low-type players who initially abstained from participating in the static contest to join the dynamic contest.

The winning advantage can take various forms, such as training, recognition, bias, or resources, but it should not be excessively significant to discourage early losers from participating or alter the cost ranking of the players in a way that prompts initially inactive participants to join the competition after obtaining the advantage. In the context of workplace, it is recommended to provide moderate non-monetary rewards to the early top performers while ensuring that these rewards do not significantly affect the employees' incentive to compete in subsequent competitions. Excessive bias, highly unequal training provisions and the allocation of back office resources based on previous performance could be harmful if the contest designer's goal is to select the best candidate for the final contest.

3.4 Concluding remarks

The paper investigates a dynamic contest where participants compete in groups during the early stage to gain a favorable advantage in the subsequent stage. The findings reveal that the dynamic contest, with a slight reduction in costs awarded to the group winners, outperforms the static contest in terms of both incentives and selection properties. Consequently, it is recommended to introduce "healthy" competitions where winners improve their effort efficiency while losers are not discouraged from participating in future contests.

In an educational context, this implies providing appraisals and positive feedback for exemplary performance without promoting unhealthy discrimination. A competition that enhances confidence, fosters the joy of winning, and incentivises engagement from all students including ones who are falling behind can have a positive impact on students' achievement levels. The paper suggests that ducationists can structure competitions that effectively motivate students, improve their effort provision, and facilitate the identification of high-achieving individuals in different subjects.

Similarly, in a workplace setting, organizing small competitions for mentorship or team leader positions, coupled with additional back-office resources and training opportunities, can bolster both effort provision and the chances of selecting the most capable employees for promotions. However, the contest designer needs to be aware of the possibility of having less able employee winning the early stage, and the fact that excessively unequal treatments can discourage subsequent participation and ultimately lead to a decline in selection accuracy. Therefore, when implementing an intermediate competition, it is crucial to ensure a moderate non-monetary benefit that sustains the participation rate.

It would be interesting to explore how the results vary when contest designers alter the rules for selecting advantaged contestants. In this paper, the intermediate winners are chosen using a similar approach to the shortlisting methods employed in Amegashie (1999) and Gradstein and Konrad (1999). For future research, alternative methods from the contest literature could be considered to select intermediate winners, such as dividing players into groups for competition and rewarding advantages to the winning groups.

Additionally, the paper puts forth theoretical hypotheses that can serve as a basis for future experimental research. A relationship between join of winning (constant utility derived from winning the early stage) and the improvement in confidence and anticipation of better future position, as well as their impact on the effort provision and selection accuracy would be interesting areas for empirical testing.

Appendix

3.A Equilibrium in static contest

In the static contest, two High type players whose marginal cost is c_H and two Low type players whose marginal costs are $c_L > c_H$ compete for the prize $V = 1$. The pay off function of player i in the static contest is:

$$\pi_i(x_i, X_{-i}) = \frac{x_i}{x_i + X_{-i}} - c_i x_i \tag{3.A.1}$$

Lemma 3.6. *In the lottery Tullock contest with $c_i \le c_j$ for $i < j$, entry will take place until, for some agent (n+1), the marginal cost $c_{n+1} > \frac{\sum\limits_{i=1}^{n} c_i}{n-1}$.*

Proof. The result was proved in Hillman and Riley (1989) and Stein (2002) □

Lemma 3.7. *In a lottery contest with prize V=1 and n asymmetric active players, total effort is*

$$TE = \frac{n-1}{\sum_{i=1}^{n} c_i}$$

The probability that player i will win the contest is

$$Pr(i) = 1 - c_i TE$$

and his expected payoff is

$$\pi_i = (1 - c_i TE)^2$$

Proof. In a contest with n asymmetric active players, the equilibrium effort of player i will satisfy the first order condition:

$$\frac{TE - x_i}{TE^2} = c_i$$

Then,

$$x_i = TE - c_i TE^2$$

i.e,

$$\frac{x_i}{TE} = 1 - c_i TE$$

and

$$\begin{aligned}\pi_i &= \frac{x_i}{TE} - c_i x_i \\ &= 1 - c_i TE - c_i(TE - c_i TE^2) \\ &= 1 - 2c_i TE + c_i^2 TE^2 \\ &= (1 - c_i TE)^2\end{aligned}$$

□

Then, equilibrium effort of the static contest with two High type and two Low type player is:

$$\begin{aligned}TE_1 = 2x_H + 2x_L &= \frac{3}{2(c_H + c_L)} \\ Pr_1(H\ wins) &= \frac{2c_L - c_H}{c_H + c_L}\end{aligned} \tag{3.A.2}$$

when $2c_H > c_L$

and

$$\begin{aligned}TE_2 = 2x_H &= \frac{1}{2c_H} \\ Pr_2(H\ wins) &= 1\end{aligned} \tag{3.A.3}$$

when $2c_H < c_L$.

3.B Dynamic contest with homogeneous grouping

In dynamic contest with homogeneous grouping, the players are grouped as (H,H) and (L,L). They will compete pairwise within the group for the advantage. The winner i of each group has his cost reduced to γc_i for $\gamma \in (0,1)$. All players then compete for the prize in the second stage. The probability that the player H_i wins the first stage is

$$P(H_i) = \frac{x_{H_i}}{x_{H_i} + x_{H_j}}$$

The probability that the player L_i wins the first stage is

$$P(L_i) = \frac{x_{L_i}}{x_{L_i} + x_{L_j}}$$

In the second stage, there will be one High type and one Low type player who has his cost reduced. Pay off functions at the second stage is then as follow:

$$\begin{aligned}
\pi_{H_W} &= \frac{y_{H_W}}{y_{H_W}+y_{H_F}+y_{L_W}+y_{L_F}} - \gamma c_H y_{H_W} \\
\pi_{H_F} &= \frac{y_{H_F}}{y_{H_W}+y_{H_F}+y_{L_W}+Y_{L_F}} - c_H y_{H_F} \\
\pi_{L_W} &= \frac{y_{L_W}}{y_{H_W}+y_{H_F}+y_{L_W}+Y_{L_F}} - \gamma c_L y_{L_W} \\
\pi_{L_F} &= \frac{y_{H_W}}{y_{H_W}+y_{H_F}+y_{L_W}+Y_{L_F}} - c_L y_{L_F}.
\end{aligned} \tag{3.B.1}$$

3.B.1 When all four players are active in static contest

Lemma 3.8. *Suppose* $2c_H > c_L$ *so that all four players are active in the static contest. When* $\gamma \leq \frac{c_H}{c_H+c_L}$, *only two first stage winners are active in the second stage.*

When $\frac{c_H}{c_H+c_L} < \gamma \leq \frac{2c_L-c_H}{c_H+c_L}$, *three players are active in the second stage (all agents except the Low type early loser)*

When $\gamma > \frac{2c_L-c_H}{c_H+c_L}$, *all four players are active in static contest.*

Proof. The result can easily be shown using lemma 3.6. □

Proposition 3.5. *Suppose that the designer moves from the static contest to the dynamic contest with homogeneous grouping and cost reduction* γ *for the group winner. Any* γ *that leads to players quitting in the second stage will decrease the selection accuracy.* γ *that ensures full participation will result in the same selection accuracy and higher total effort than the static contest.*

Proof. • Consider the case where γ is so small that only the first stage winners are active in the second stage. Since the players are grouped homogeneously in the first round, there will be one High type and one Low type player, whose cost are now γc_L and γc_H. The probability that the High type winner wins in the two player contest is

$$P(H) = \frac{c_H}{c_H+c_L} < \frac{2c_L-c_H}{c_H+c_L} = Pr_1(H\ wins)$$

□

• Consider the situation where there are three active players in the second round, i.e, $\frac{c_H}{c_H+c_L} < \gamma < \frac{2c_L-c_H}{c_H+c_L}$. The Low type player who lost the first stage give up, only two High type players and the Low type first stage winner exert efforts which maximises $\pi_{H_W}, \pi_{H_F}, \pi_{L_W}$ as in equation 3.B.1 with $y_{L_F} = 0$. The FOCs give

$$\frac{y_{H_F} + y_{L_W}}{(y_{H_W} + y_{H_F} + y_{L_W})^2} = \gamma c_H$$
$$\frac{y_{H_W} + y_{L_W}}{(y_{H_W} + y_{H_F} + y_{L_W})^2} = c_H$$
$$\frac{y_{H_W} + y_{H_F}}{(y_{H_W} + y_{H_F} + y_{L_W})^2} = \gamma c_L$$

Then,

$$\begin{aligned} P(H) &= \frac{y_{H_W} + y_{H_F}}{y_{H_W} + y_{H_F} + y_{L_W}} \\ &= \frac{2\gamma c_L}{\gamma c_L + (\gamma+1)c_H} \\ &< \frac{2c_L - c_H}{c_H + c_L} = Pr_1(H\ wins) \end{aligned}$$

for $\gamma < \frac{2c_L - c_H}{c_L + c_H}$.

• Consider the situation where all four players are active in the second round. Then, the F.O.C for 3.B.1 becomes

$$\frac{y_{H_F} + y_{L_W} + y_{L_F}}{(y_{H_W} + y_{H_F} + y_{L_W} + y_{L_F})^2} = \gamma c_H$$
$$\frac{y_{H_W} + y_{L_W} + y_{L_F}}{(y_{H_W} + y_{H_F} + y_{L_W} + y_{L_F})^2} = c_H$$
$$\frac{y_{H_W} + y_{H_F} + y_{L_F}}{(y_{H_W} + y_{H_F} + y_{L_W} + y_{L_F})^2} = \gamma c_L$$
$$\frac{y_{H_W} + y_{H_F} + y_{L_W}}{(y_{H_W} + y_{H_F} + y_{L_W} + y_{L_F})^2} = c_L$$

Summing up all the FOCs, we can see that the effort in the second stage will be

$$y_{H_W} + y_{H_F} + y_{L_W} + y_{L_F} = \frac{3}{(\gamma+1)(c_H + c_L)} > TE_1 \tag{3.B.2}$$

The probability that the winner is High type is

$$\frac{y_{H_W} + y_{H_F}}{y_{H_W} + y_{H_F} + y_{L_W} + y_{L_F}} = \frac{2c_L - c_H}{c_H + c_L} = Pr_1(H\ wins)$$

3.B.2 When only two High type players are active in static contest

Lemma 3.9. *When $2c_H < c_L$ so that only High type players are active in the static contest, there can not be an equilibrium when all four players remain active in the second stage of the dynamic contest.*

If $\gamma < \frac{c_H}{c_L+c_H}$, only two winners (one High type one Low type) are active in the second stage.

If $\frac{c_H}{c_H+c_L} < \gamma \leq \frac{c_H}{c_L-c_H}$, there are three active players in the second stage.

If $\gamma \geq \frac{c_H}{c_L-c_H}$, only two High type players are active in the second stage.

Proof. This can be proved easily using lemma 3.6. □

Proposition 3.6. *The dynamic contest with $\gamma > \frac{c_H}{c_L-c_H}$ results in same level of selection accuracy and higher total effort. Any $\gamma \leq \frac{c_H}{c_L-c_H}$ results in lower selection accuracy level.*

Proof. Clearly, when $c_L > 2c_H$ so that only two High type players are active in the static contest, the probability that a High type candidate is chosen is 1. The total effort level in the static contest is

$$TE_2 = \frac{1}{2c_H}$$

In the dynamic contest with $\gamma > \frac{c_H}{c_L-c_H}$, only two High type players are active in the second stage. Selection accuracy is also 100%. Total effort level is

$$y_{H_W} + y_{H_F} = \frac{1}{(\gamma+1)c_H} > TE_2$$

Any $\gamma \leq \frac{c_H}{c_L-c_H}$ means participation of the Low type player, which decreases the selection accuracy. □

3.C Dynamic contest with heterogeneous grouping

In this setting, the players are grouped as (H,L), (H,L). The players compete within group, and the winner i of each group will have his cost reduced to γc_i. All players then compete for the prize in the second stage.

The probability that the player H in group i wins the first stage is

$$P(H_i) = \frac{x_{H_i}}{x_{H_i} + x_{L_i}}$$

The probability that the player L_i wins the first stage is

$$P(L_i) = \frac{x_{L_i}}{x_{H_i} + x_{L_i}}$$

There could be situations where 2 High type players, 2 Low type players, or 1 High type and 1 Low type player win the first stage and receive the advantage in the second stage.

If 2 High type players win, then the second stage effort will be

$$E_2^R = 2y_{H_W} + 2y_{L_F} = \frac{3}{2(\gamma c_H + c_L)}$$

and the probability that H type player wins the second stage is

$$P(H) = \frac{2c_L - \gamma c_H}{\gamma c_H + c_L}$$

when $2\gamma c_H > c_L$.

Similarly,

$$E_2^R = 2y_{H_W} = \frac{1}{2c_H}$$

and

$$P(H) = 1$$

when $2\gamma c_H < c_L$.

If 2 Low type players win, then the second stage equilibrium effort will be:

$$E_2^R = 2y_{H_F} + 2y_{L_W} = \frac{3}{2(c_H + \gamma c_L)}$$

and

$$P(H) = \frac{2\gamma c_L - c_H}{c_H + \gamma c_L}$$

when $2\gamma c_L > c_H > \frac{\gamma}{2} c_L$.

Similarly,

$$E_2^R = 2y_{H_F} = \frac{1}{2c_H}$$

and

$$P(H) = 1$$

when $2c_H < \gamma c_L$.

Finally,

$$E_2^R = 2y_{L_W} = \frac{1}{2c_L}$$

and

$$P(H) = 0$$

when $c_H > 2\gamma c_L$.

If one High type and one Low type winner wins, then the payoff functions is given in equation 3.B.1.

3.C.1 When all 4 players are active in static contest

Let $\pi_{i^W/ik}$ be the second stage pay-off for the early winner of type i when the two early winners are i and k, and $\pi_{i^F/jk}$ be the second stage pay-off for the early loser of type i when the two early winners are j and k. For example, $\pi_{H^W/HL}$ is the second stage pay-off of the High type early winner when one High type, one Low type player win the first stage. $\pi_{H^F/HL}$ is the second stage pay-off of the High type early loser when one High type and one Low type player win the first stage.

Then, the expected pay off of player H_i and L_i in the first stage is given as follow: [5]

$$\begin{aligned}\pi_{H_i} = & \frac{x_{H_i}}{x_{H_i}+x_{L_i}}\left[\frac{x_{H_j}}{x_{H_j}+x_{L_j}}\pi_{H^W/HH}+\frac{x_{L_j}}{x_{H_j}+x_{L_j}}\pi_{H^W/HL}\right] \\ & +\frac{x_{L_i}}{x_{H_i}+x_{L_i}}\left[\frac{x_{H_j}}{x_{H_j}+x_{L_j}}\pi_{H^F/HL}+\frac{x_{L_j}}{x_{H_j}+x_{L_j}}\pi_{H^F/LL}\right]-c_H x_H\end{aligned} \tag{3.C.1}$$

and

$$\begin{aligned}\pi_{L_i} = & \frac{x_{L_i}}{x_{H_i}+x_{L_i}}\left[\frac{x_{H_j}}{x_{H_j}+x_{L_j}}\pi_{L^W/HL}+\frac{x_{L_j}}{x_{H_j}+x_{L_j}}\pi_{L^W/LL}\right] \\ & +\frac{x_{H_i}}{x_{H_i}+x_{L_i}}\left[\frac{x_{H_j}}{x_{H_j}+x_{L_j}}\pi_{L^F/HH}+\frac{x_{L_j}}{x_{H_j}+x_{L_j}}\pi_{L^F/HL}\right]-c_L x_L\end{aligned} \tag{3.C.2}$$

First order conditions is:

$$\begin{aligned}c_H = & \frac{x_{L_i}}{(x_{H_i}+x_{L_i})^2}\left[\frac{x_{H_j}}{x_{H_j}+x_{L_j}}\pi_{H^W/HH}+\frac{x_{L_j}}{x_{H_j}+x_{L_j}}\pi_{H^W/HL}\right. \\ & \left.-\frac{x_{H_j}}{x_{H_j}+x_{L_j}}\pi_{H^F/HL}-\frac{x_{L_j}}{x_{H_j}+x_{L_j}}\pi_{H^F/LL}\right]\end{aligned} \tag{3.C.3}$$

and

$$\begin{aligned}c_L = & \frac{x_{H_i}}{(x_{H_i}+x_{L_i})^2}\left[\frac{x_{H_j}}{x_{H_j}+x_{L_j}}\pi_{L^W/HL}+\frac{x_{L_j}}{x_{H_j}+x_{L_j}}\pi_{L^W/LL}\right. \\ & \left.-\frac{x_{H_j}}{x_{H_j}+x_{L_j}}\pi_{L^F/HH}-\frac{x_{L_j}}{x_{H_j}+x_{L_j}}\pi_{L^F/HL}\right]\end{aligned} \tag{3.C.4}$$

The second order condition is satisfied for any given pair of x_{H_j} and x_{L_j}

Let

$$\begin{aligned}k &= \pi_{L^W/LL}-\pi_{L^F/HL} \\ g &= \pi_{L^W/HL}-\pi_{L^F/HH} \\ m &= \pi_{H^W/HL}-\pi_{H^F/LL} \\ n &= \pi_{H^W/HH}-\pi_{H^F/HL}\end{aligned} \tag{3.C.5}$$

Then, the FOCs can be written as

$$c_L = \frac{x_{L_i}}{(x_{L_i}+x_{H_i})^2}\left[\frac{x_{L_j}}{x_{H_j}+x_{L_j}}\cdot k+\frac{x_{H_j}}{x_{H_j}+x_{L_j}}\cdot g\right]$$

[5] By Cornes and Hartley, 2005; Nti, 1999, contest with two heterogeneous players has a unique, interior solution. There always exist equilibrium effort in each group given the effort of players in the other group. However, we still need to show that there exist an interacting equilibrium, i.e the system of 4 FOCs have solution.

$$c_H = \frac{x_{L_i}}{(x_{L_i}+x_{H_i})^2}\left[\frac{x_{L_j}}{x_{H_j}+x_{L_j}}\cdot m + \frac{x_{H_j}}{x_{H_j}+x_{L_j}}\cdot n\right]$$

From the FOCs, we have:

$$\frac{x_{L_i}}{x_{H_i}} = \frac{c_H}{c_L}\frac{k+\frac{x_{H_j}}{x_{H_j}+x_{L_j}}(g-k)}{m+\frac{x_{H_j}}{x_{H_j}+x_{L_j}}(n-m)} \tag{3.C.6}$$

Using $\frac{x_{H_j}}{x_{H_j}+x_{L_j}} = \frac{1}{1+\frac{x_{L_j}}{x_{H_j}}}$, we can rewrite the equation 3.C.6 as

$$\frac{x_{L_i}}{x_{H_i}} = \frac{c_H}{c_L}\frac{k\frac{x_{L_j}}{x_{H_j}}+g}{m\frac{x_{L_j}}{x_{H_j}}+n} \tag{3.C.7}$$

Similarly, from the optimisation of players in group j, we have

$$\frac{x_{L_j}}{x_{H_j}} = \frac{c_H}{c_L}\frac{k\frac{x_{L_i}}{x_{H_i}}+g}{m\frac{x_{L_j}}{x_{H_j}}+n} \tag{3.C.8}$$

Any pair (x_{L_i}, x_{H_i}) and (x_{L_j}, x_{H_j}) that satisfies 3.C.7 and 3.C.8 will satisfy the original first order conditions. We have just reduced the system of 4 equations with 4 unknown $(x_{L_i}, x_{H_i}), (x_{L_j}, x_{H_j})$ into system of 2 equations with 2 unknowns $\frac{x_{L_i}}{x_{H_i}}, \frac{x_{L_j}}{x_{H_j}}$. The system is of the form

$$\begin{cases} X = \dfrac{kY+g}{mY+n} \\ Y = \dfrac{kX+g}{mX+n} \end{cases}$$

where $X = Y$ is a solution. Therefore, equilibrium for stage 1 effort exist and in equilibrium $\frac{x_{L_i}}{x_{H_i}} = \frac{x_{L_j}}{x_{H_j}}$ [6]

We have just showed that stage 1 equilibrium effort exist and will satisfy $\frac{x_{L_i}}{x_{H_i}} = \frac{x_{L_j}}{x_{H_j}} = \frac{x_L}{x_H}$.

[6] Here we have not proved that the equilibrium is unique. In order to prove that it is unique, we have to show that the system of equations 3.C.7 and 3.C.8 has unique solution on the interval (0,1). However, since we are not interested in finding the equilibrium, but the ratio of efforts in the equilibrium, we care more about the existence of the equilibrium than its uniqueness. It is sufficient for our comparison to have that the equilibrium in stage 1 exists and will satisfy the first order condition, and that the ratio of equilibrium effort is the same in 2 groups.

Let $r = \dfrac{x_L}{x_H}$. Then, equilibrium effort ratio r^* is the solution for

$$r = \frac{c_H}{c_L}\frac{kr+g}{mr+n}$$

i.e.,

$$mc_L r^2 + (nc_L - kc_H)r - gc_H = 0 \tag{3.C.9}$$

Lemma 3.10. *Consider $c_L < 2c_H$ so that all 4 players are active in static contest. When $\frac{c_L}{2c_H} < \gamma < 1$, all 4 players are active in every scenario.*

Proof. This can be proved using lemma 3.6 $\square$

Proposition 3.7. *Assume that $c_L < 2c_H$ so that all 4 players are active in the static contest. When $\gamma > \frac{c_L}{2c_H}$ so that all 4 players are active in every scenario, both selection accuracy and aggregate effort are higher than the static contest.*

Proof. Consider $\gamma > \frac{c_L}{2c_H}$. If the first stage effort are x_H for the H type player and x_L for the Low type players, then by lemma 3.7 the probability of a High type player winning the contest is:

$$\begin{aligned} Pr(H) &= \frac{x_H^2}{(x_H+x_L)^2}\cdot\frac{2c_L-\gamma c_H}{c_L+\gamma c_H} + \frac{2x_H x_L}{(x_H+x_L)^2}\cdot\frac{2c_L-c_H}{c_H+c_L} + \frac{x_L^2}{(x_H+x_L)^2}\cdot\frac{2\gamma c_L - c_H}{c_H+\gamma c_L} \\ &= \frac{2c_L-c_H}{c_H+c_L} + \frac{x_H^2}{(x_H+x_L)^2}\cdot\frac{3c_H c_L(1-\gamma)}{(c_H+c_L)(c_L+\gamma c_H)} - \frac{x_L^2}{(x_H+x_L)^2}\cdot\frac{3c_H c_L(1-\gamma)}{(c_H+c_L)(c_H+\gamma c_L)} \end{aligned}$$

The accuracy of selection in dynamic contest will be higher than in static contest if

$$\frac{x_H^2}{(x_H+x_L)^2}\cdot\frac{3c_H c_L(1-\gamma)}{(c_H+c_L)(c_L+\gamma c_H)} > \frac{x_L^2}{(x_H+x_L)^2}\cdot\frac{3c_H c_L(1-\gamma)}{(c_H+c_L)(c_H+\gamma c_L)}$$

i.e.,

$$\frac{x_L^2}{x_H^2} < \frac{c_H+\gamma c_L}{c_L+\gamma c_H}$$

We will show that

$$\frac{x_L}{x_H} < \frac{c_H}{c_L} < \sqrt{\frac{c_H+\gamma c_L}{c_L+\gamma c_H}} \tag{3.C.10}$$

in equilibrium when $\gamma > \frac{c_L}{2c_H}$. The second inequality is straightforward.

From the first order conditions in 3.C.3 and 3.C.4, we have

$$\frac{x_L}{x_H} = \frac{c_H}{c_L}\frac{Q_1}{Q_2}$$

for

$$Q_1 = \frac{x_L}{x_H + x_L}k + \frac{x_H}{x_H + x_L}g$$
$$Q_2 = \frac{x_L}{x_H + x_L}m + \frac{x_H}{x_H + x_L}n$$

for k,g,m,n defined in 3.C.5. We will show that

$$g - n < 0$$

and

$$k - m < 0$$

for $\gamma > \frac{c_L}{2c_H}$ so that $Q_1 < Q_2$.

By lemma 3.7, the payoff of the second stage when 1 High type player and 1 Low type player win the first stage and all 4 players remain active in the second stage will be:

$$\pi_{H_W/HL} = \left[1 - \frac{3\gamma c_H}{(\gamma+1)(c_H + c_L)}\right]^2 \quad (3.C.11) \quad \pi_{H_F/HL} = \left[1 - \frac{3c_H}{(\gamma+1)(c_H + c_L)}\right]^2 \quad (3.C.12)$$

$$\pi_{L_W/HL} = \left[1 - \frac{3\gamma c_L}{(\gamma+1)(c_H + c_L)}\right]^2 \quad (3.C.13) \quad \pi_{L_F/HL} = \left[1 - \frac{3c_L}{(\gamma+1)(c_H + c_L)}\right]^2 \quad (3.C.14)$$

If two High type players win the first stage and all 4 player remains active in the second stage, the second stage pay-offs are:

$$\pi_{H_W/HH} = \left[1 - \frac{3\gamma c_H}{2(\gamma c_H + c_L)}\right]^2 \quad (3.C.15) \qquad \pi_{L_F/HH} = \left[1 - \frac{3c_L}{2(\gamma c_H + c_L)}\right]^2 \quad (3.C.16)$$

If two Low type players win the first stage and all 4 player remains active in the second stage, the second stage pay-offs are:

$$\pi_{H_F/LL} = \left[1 - \frac{3c_H}{2(c_H + \gamma c_L)}\right]^2 \quad (3.C.17) \qquad \pi_{L_W/LL} = \left[1 - \frac{3\gamma c_L}{2(c_H + \gamma c_L)}\right]^2 \quad (3.C.18)$$

Substitute into 3.C.5,

$$
\begin{aligned}
k &= \left[1-\frac{3\gamma c_L}{2(c_H+\gamma c_L)}\right]^2 - \left[1-\frac{3c_L}{(\gamma+1)(c_H+c_L)}\right]^2 \\
&= \frac{(1-\gamma)(c_H+2c_L)[(3+\gamma)c_H-2c_L]}{4(\gamma+1)^2(c_H+c_L)^2(c_H+\gamma c_L)^2}
\end{aligned}
$$

$$
\begin{aligned}
g &= \left[1-\frac{3\gamma c_L}{(\gamma+1)(c_H+c_L)}\right]^2 - \left[1-\frac{3c_L}{2(\gamma c_H+c_L)}\right]^2 \\
&= \frac{(1-\gamma)(2c_L+c_H)[(3\gamma+1)c_H-2\gamma c_L]}{4(\gamma+1)^2(c_H+c_L)^2(c_L+\gamma c_H)^2}
\end{aligned}
$$

$$
\begin{aligned}
m &= \left[1-\frac{3\gamma c_H}{(\gamma+1)(c_H+c_L)}\right]^2 - \left[1-\frac{3c_H}{2(c_H+\gamma c_L)}\right]^2 \\
&= \frac{(1-\gamma)(2c_H+c_L)[(3\gamma+1)c_L-2\gamma c_H]}{4(\gamma+1)^2(c_H+c_L)^2(c_H+\gamma c_L)^2}
\end{aligned}
$$

$$
\begin{aligned}
n &= \left[1-\frac{3\gamma c_H}{2(\gamma c_H+c_L)}\right]^2 - \left[1-\frac{3c_H}{(\gamma+1)(c_H+c_L)}\right]^2 \\
&= \frac{(1-\gamma)(c_L+2c_H)[(3+\gamma)c_L-2c_H]}{4(\gamma+1)^2(c_H+c_L)^2(c_L+\gamma c_H)^2}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
m-k &= \frac{(1-\gamma)\Big\{(2c_H+c_L)[(3\gamma+1)c_L-2\gamma c_H]-(c_H+2c_L)[(3+\gamma)c_H-2c_L]\Big\}}{4(\gamma+1)^2(c_H+c_L)^2(c_H+\gamma c_L)^2} \\
&= \frac{(1-\gamma)}{4(\gamma+1)^2(c_H+c_L)^2(c_L+\gamma c_H)^2}\times\left[(5+3\gamma)c_L^2-(2-2\gamma)c_Hc_L-(3+5\gamma)c_H^2\right] \\
&> 0
\end{aligned}
$$

and

$$
\begin{aligned}
n-g &= \frac{(1-\gamma)\Big\{(c_L+2c_H)[(3+\gamma)c_L-2c_H]-(2c_L+c_H)[(3\gamma+1)c_H-2\gamma c_L]\Big\}}{4(\gamma+1)^2(c_H+c_L)^2(c_L+\gamma c_H)^2} \\
&= \frac{(1-\gamma)}{4(\gamma+1)^2(c_H+c_L)^2(c_L+\gamma c_H)^2}\times\left[(3+5\gamma)c_L^2+(2-2\gamma)c_Hc_L-(5+3\gamma)c_H^2\right] \\
&> 0
\end{aligned}
$$

for $c_L > c_H$

Therefore, $\dfrac{x_L}{x_H} = \dfrac{c_H}{c_L}\dfrac{Q_1}{Q_2} < \dfrac{c_H}{c_L}$. Inequality 3.C.10 is hence satisfied.

For aggregated effort, we can see that the effort in the second round of the dynamic contest is

$$
\begin{aligned}
E_2^R &= \frac{x_H^2}{(x_H+x_L)^2}\frac{3}{2(\gamma c_H+c_L)} + \frac{2c_Hc_L}{(c_H+c_L)^2}\frac{3}{(\gamma+1)(c_H+c_L)} + \frac{c_L^2}{(c_H+c_L)^2}\frac{3}{2(c_H+\gamma c_L)} \\
&> \frac{3}{2(c_H+c_L)}
\end{aligned}
$$

which is already higher than effort level in the static contest. □

3.C.2 When only two High type players are active in static contest

Lemma 3.11. *Consider the situation where $c_L > 2c_H$ so that only two High type players are active in the static contest.*

If $\gamma < \frac{c_H}{2c_L}$, only two winners are active in the second stage regardless who won the first stage.

If $\frac{c_H}{2c_L} < \gamma \leq \frac{c_H}{c_L+c_H}$, all four players are active when two Low type players won the first stage; otherwise only two winners are active in the second stage.

If $\frac{c_H}{c_H+c_L} < \gamma \leq \frac{c_H}{c_L-c_H}$, there are three active players in the second stage when 1 High type and 1 Low type player win the first stage. All four players are active when 2 Low type players win the first stage. Otherwise only two High type players are active

If $\frac{c_H}{c_L-c_H} < \gamma \leq \frac{2c_H}{c_L}$, only two High type players are active in the second stage unless 2 Low type players won the first stage, in which case all 4 players will be active.

If $\frac{2c_H}{c_L} < \gamma < 1$, only 2 High type players are active in the second stage regardless who won the first stage.

Proof. This can be proved easily using lemma 3.6. □

Proposition 3.8. *The dynamic contest with $\gamma > \frac{2c_H}{c_L}$ results in same level of selection accuracy and higher total effort. Any $\gamma \leq \frac{2c_H}{c_L}$ results in lower selection accuracy level.*

Proof. Clearly, when $c_L > 2c_H$ so that only two High type players are active in the static contest, the probability that a High type candidate is chosen is 1. The total effort level in the static contest is

$$TE_2 = \frac{1}{2c_H}$$

In the dynamic contest with $\gamma > \frac{2c_H}{c_L}$, only two High type players are active in the second stage. Selection accuracy is also 100%. Total effort level is

$$y_{H_W} + y_{H_F} = \frac{1}{(\gamma+1)c_H} > TE_2$$

Any $\gamma \leq \frac{2c_H}{c_L}$ means participation of at least one Low type player, which decreases the selection accuracy. □

3.D Dynamic contest with three players of the same type

3.D.1 When all four players are active in the static contest

Consider the situation when there are three High type player and one Low type player (The situation of three Low type and one High type player is symmetric).

By lemma 3.6 and 3.7, all four players will be active in the static contest when $c_L \leq 1.5c_H$. In this case, total effort in the static contest is

$$TE = \frac{3}{3c_H + c_L}$$

The probability that a High type player win is

$$Pr_3(H\ wins) = \frac{c_L}{3c_H + c_L}$$

and the probability that the Low type player win is

$$Pr_3(L\ wins) = \frac{3c_H - 2c_L}{3c_H + c_L}$$

Without loss of generalisation, assume that the players are grouped (H_1, H_2), (H_3, L). There could be situation where 2 High type players, or one High type and one Low type player win the first stage. Using 3.6, when $\gamma > \dfrac{2c_L - c_H}{2c_H}$ as in lemma 3.3, all four players will be active in both situations.

Consider $\gamma > \dfrac{2c_L - c_H}{2c_H}$.

If 2 High type players win, the second stage effort will be

$$2y_{H^W} + y_{H^F} + y_{L^F} = \frac{3}{c_H + c_L + 2\gamma c_H}$$

and the second stage payoff will be

$$\pi_{H^W/HH} = \left(1 - \frac{3\gamma c_H}{c_H + c_L + 2\gamma c_H}\right)^2$$

$$\pi_{H^F/HH} = \left(1 - \frac{3c_H}{c_H + c_L + 2\gamma c_H}\right)^2$$

$$\pi_{L^F/HH} = \left(1 - \frac{3c_L}{c_H + c_L + 2\gamma c_H}\right)^2$$

If 1 High type and 1 Low type player win, the second stage effort will be

$$y_{H^W} + y_{L^F} + 2y_{H^F} = \frac{3}{2c_H + \gamma c_L + \gamma c_H}$$

and the second stage payoff will be

$$\pi_{H^W/HL} = \left(1 - \frac{3\gamma c_H}{2c_H + \gamma c_L + \gamma c_H}\right)^2$$

$$\pi_{H^F/HL} = \left(1 - \frac{3c_H}{2c_H + \gamma c_L + \gamma c_H}\right)^2$$

$$\pi_{L^W/HL} = \left(1 - \frac{3\gamma c_L}{2c_H + \gamma c_L + \gamma c_H}\right)^2$$

The expected pay-offs of the players in the first stage are:

$$
\begin{aligned}
\pi_{H_1} &= \frac{x_{H_1}}{x_{H_1}+x_{H_2}}\left[\frac{x_{H_3}}{x_{H_3}+x_L}\pi_{H^W/HH}+\frac{x_L}{x_{H_3}+x_L}\pi_{H^W/HL}\right] \\
&\quad+\frac{x_{H_2}}{x_{H_1}+x_{H_2}}\left[\frac{x_{H_3}}{x_{H_3}+x_L}\pi_{H^F/HH}+\frac{x_L}{x_{H_3}+x_L}\pi_{H^F/HL}\right]-c_H x_{H_1} \\
\pi_{H_2} &= \frac{x_{H_2}}{x_{H_1}+x_{H_2}}\left[\frac{x_{H_3}}{x_{H_3}+x_L}\pi_{H^W/HH}+\frac{x_L}{x_{H_3}+x_L}\pi_{H^W/HL}\right] \\
&\quad+\frac{x_{H_1}}{x_{H_1}+x_{H_2}}\left[\frac{x_{H_3}}{x_{H_3}+x_L}\pi_{H^F/HH}+\frac{x_L}{x_{H_3}+x_L}\pi_{H^F/HL}\right]-c_H x_{H_2} \\
\pi_{H_3} &= \frac{x_{H_3}}{x_{H_3}+x_L}\pi_{H^W/HH}+\frac{x_L}{x_{H_3}+x_L}\pi_{H^F/HL}-c_H x_{H_3} \\
\pi_L &= \frac{x_{H_3}}{x_{H_3}+x_L}\pi_{LF/HH}+\frac{x_L}{x_{H_3}+x_L}\pi_{L^W/HL}-c_L x_L
\end{aligned}
$$

Taking the FOCS

$$
\begin{aligned}
\frac{x_L}{(x_{H_3}+x_L)^2}\left(\pi_{H^W/HH}-\pi_{H^F/HL}\right) &= c_H \\
\frac{x_{H_3}}{(x_{H_3}+x_L)^2}\left(\pi_{L^W/HL}-\pi_{L^F/HH}\right) &= c_L
\end{aligned}
$$

Therefore,

$$
\frac{x_L}{x_{H_3}} = \frac{c_H(\pi_{L^W/HL}-\pi_{L^F/HH})}{c_L(\pi_{H^W/HH}-\pi_{H^F/HL}}
$$

Consider

$$
\begin{aligned}
&\pi_{L^W/HL}-\pi_{L^F/HH} \\
&= \frac{(2c_H+\gamma c_H-2\gamma c_L)^2(c_H+c_L+2\gamma c_H)^2-(c_H+2\gamma c_H-2c_L)^2(2c_H+\gamma c_H+\gamma c_L)^2}{(2c_H+\gamma c_H+\gamma c_L)^2(c_H+c_L+2\gamma c_H)^2} \\
&< \frac{(2c_H+\gamma c_H-2\gamma c_L)^2-(c_H+2\gamma c_H-2c_L)^2}{(2c_H+\gamma c_H+\gamma c_L)^2(c_H+c_L+2\gamma c_H)^2}\cdot(2\gamma c_H+c_H+c_L)^2
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\pi_{H^W/HH}-\pi_{H^F/HL} \\
&= \frac{(2c_H+\gamma c_H-c_L)^2(c_H+c_L-\gamma c_H)^2-(c_H+2\gamma c_H+c_L)^2(\gamma c_H+\gamma c_L-c_H)^2}{(2c_H+\gamma c_H+\gamma c_L)^2(c_H+c_L+2\gamma c_H)^2} \\
&> \frac{(c_H+c_L-\gamma c_H)^2-(\gamma c_H+\gamma c_L-c_H)^2}{(2c_H+\gamma c_H+\gamma c_L)^2(c_H+c_L+2\gamma c_H)^2}\cdot(2c_H+\gamma c_H+c_L)^2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{x_L}{x_H} &< \frac{c_H}{c_L} \cdot \frac{(2c_H+\gamma c_H-2\gamma c_L)^2-(c_H+2\gamma c_H-2c_L)^2}{(c_H+c_L-\gamma c_H)^2-(\gamma c_H+\gamma c_L-c_H)^2} \cdot \frac{(2\gamma c_H+c_H+c_L)^2}{(2c_H+\gamma c_H+c_L)^2} \\
&= \frac{c_H}{c_L} \cdot \frac{(3c_H-2c_L)(c+H+2c_L)}{c_L(2c_H+c_L)} \cdot \frac{(2\gamma c_H+c_H+c_L)^2}{(2c_H+\gamma c_H+c_L)^2} \\
&< \frac{(2\gamma c_H+c_H+c_L)^2}{(2c_H+\gamma c_H+c_L)^2}
\end{aligned}
$$

The expected probability that the Low type player win the second stage is

$$
\begin{aligned}
P &= \frac{x_L}{x_L+x_{H_3}} \cdot \frac{2c_H+\gamma c_H-2\gamma c_L}{2c_H+\gamma c_H+\gamma c_L} + \frac{x_{H_3}}{x_L+x_{H_3}} \cdot \frac{c_H+2\gamma c_H-2c_L}{c_H+c_L+2\gamma c_H} \\
&= \frac{x_L}{x_L+x_{H_3}} \cdot 3c_L \cdot \left(\frac{1}{3c_H+c_L} - \frac{\gamma}{2c_H+\gamma c_H+\gamma c_L}\right) + \frac{x_{H_3}}{x_L+x_{H_3}} \cdot 3c_L \left(\frac{1}{3c_H+c_L} - \frac{1}{c_H+c_L+2\gamma c_H}\right) \\
&+ \frac{3c_H-2c_L}{3c_H+c_L} \\
&= \frac{1}{x_L+x_{H_3}} \cdot \frac{3c_L}{(3c_H+c_L)(c_H+c_L-2\gamma c_H)} \cdot \left[x_L(2c_H+c_L+\gamma c_H) - x_{H_3}(2\gamma c_H+c_H+c_L)\right] \\
&+ \frac{3c_H-2c_L}{3c_H+c_L} \\
&< \frac{3c_H-2c_L}{3c_H+c_L} = Pr_3(L\,wins)
\end{aligned}
$$

Therefore, the dynamic contest results in lower chance of having the Low type player winning the prize than the static contest, i.e., it performs better than the static contest in terms of selection accuracy.

The expected effort level in the second stage is

$$
\begin{aligned}
&\frac{x_L}{x_L+x_{H_3}} \cdot \frac{3}{2c_H+\gamma c_L+\gamma c_H} + \frac{x_{H_3}}{x_L+x_{H_3}} \cdot \frac{3}{2\gamma c_H+c_H+c_L} \\
&> \frac{x_L}{x_L+x_{H_3}} \cdot \frac{3}{2c_H+c_L+c_H} + \frac{x_{H_3}}{x_L+x_{H_3}} \cdot \frac{3}{2c_H+c_H+c_L} \\
&= \frac{3}{3c_H+c_L} = TE_3
\end{aligned}
$$

3.D.2 When only three High type players are active in the static contest

When there are three High type and one Low type players, there will only be three High type active player when $c_L > 1.5c_H$. The probability of a High type player winning is

$$Pr(H\ wins) = 1$$

The total effort in the static contest is

$$TE_3 = \frac{2}{3c_H}$$

In the dynamic contest when $\gamma > \dfrac{2c_H}{2c_L - c_H}$, there will also be three active High type players in the second stage, regardless of who won the first stage. Hence, the probability of a High type player winning the prize is

$$Pr(H\ wins) = 1$$

Expected second stage effort in dynamic contest is

$$\begin{aligned}
&\frac{x_L}{x_{H_3}} \cdot \frac{2}{(2+\gamma c_H)} + \frac{x_{H_3}}{x_L + x_{H_3}} \cdot \frac{2}{(1+2\gamma c_H)} \\
&> \frac{x_L}{x_{H_3}} \cdot \frac{2}{3c_H} + \frac{x_{H_3}}{x_L + x_{H_3}} \cdot \frac{2}{3c_H} \\
&= \frac{2}{3c_H} = TE_3
\end{aligned}$$

3.E When all four players are of the same type

Consider the situation when there are four High type players (the case with four Low type players is similar).

Clearly, the probability of a High type player winning the prize is always

$$Pr(H\ wins) = 1$$

Total effort in the static contest is

$$TE_4 = \frac{3}{4c_H}$$

Effort in the second stage of the dynamic contest is

$$E = \frac{3}{2(\gamma+1)c_H} > \frac{3}{4c_H}$$

when $1 > \gamma > \frac{1}{2}$, and

$$E = \frac{1}{2\gamma c_H} > \frac{3}{4c_H}$$

when $\gamma < \frac{1}{2}$

Chapter 4

Conclusion

This dissertation examines three cost and surplus-sharing games with different objectives. The first chapter investigates a cooperative cost-sharing game and explores the fairness of various sharing rules. Specifically, it focuses on the allocation inequality of the sharing allocations: the average share, the Shapley value, the increasing serial share, and the decreasing serial share. Recent literature has addressed the Lorenz comparison between the average share and the others, as well as between increasing and decreasing serial shares.

In this study, I analyse the combination of the Shapley value and increasing serial shares and establish the Lorenz order for all four sharing rules. The findings suggest that when cost production is concave (convex), the increasing serial rule is more (less) egalitarian than the Shapley value. Among the four discussed rules, serial shares are the most (least) egalitarian when the cost is concave (convex). In situations where the decreasing serial share is implausible, the increasing serial share emerges as the most appealing choice when production involves negative externalities. It provides the most favourable cost distribution to small firms among the remaining three rules.

The results remain consistent when considering the surplus-sharing game. However, the recommendation for the appropriate rule may vary depending on the fairness arguments. This work contributes to the ongoing discussion on allocation inequality rules and offers guidance for selecting the appropriate rule when equality is regarded as a social value.

In the subsequent two chapters, the focus is on the analysis of the two-stage Tullock lottery contest. Both chapters consider contests with perfect information and involve two types of players.

In the second chapter, the investigation revolves around a contest comprising two players and a designer whose objective is to maximize total effort. The conclusions drawn indicate that when the players do not exhibit significant differences, the designer should consistently prioritize the winner. Conversely, when there is a remarkable level of heterogeneity, it is often more advantageous to favor the loser.

Furthermore, if the designer possesses the authority to determine the prize distribution between the two stages, it is deemed optimal to allocate the entire prize to the second stage when the players are similar. However, in cases where the players display significant differences, allocating a small prize to the first stage is found to be more effective.

The third chapter expands the contest to include four players of two distinct types, while the designer's concerns encompass not only the effort level but also the selection accuracy and participation rate. Previous research has frequently highlighted the trade-off between these objectives: contests with higher effort levels often exhibit lower selection accuracy, and vice versa. A similar relationship has been observed between selection accuracy and the participation rate.

However, my findings demonstrate the potential to enhance contest performance across all dimensions. By introducing an intermediate stage with a winning advantage, the designer can preserve the participation rate while improving at least one of the remaining aspects of the static contest. This advantage, which may take the form of non-monetary incentives such as recognition or favourable bias, can serve as motivation for players to exert greater effort, thereby widening the efficiency gap between players when awarded to those of the "High" type.

It is crucial, however, to maintain a moderate advantage to prevent a decline in participation. The results indicate the benefits of incorporating additional intermediate competition but emphasize the importance of maintaining a "healthy" level where players do not feel discouraged from continued participation after experiencing losses.

This paper contributes to the ongoing discourse on contest design by highlighting the potential for improving contest performance in both selection accuracy and effort provision. It stands out as one of the few papers that address this possibility, offering valuable insights for future research and contest design considerations.

Given the similarity and standard structure of the contest explored in the latter two chapters, a promising direction for future work would involve introducing additional features into the existing model. For instance, one possible extension is to incorporate variations in the prize or marginal costs as effort levels change. By allowing the prize to increase with total effort, the Tullock contest can be transformed into a utility-sharing game, which opens up avenues for investigating a wider range of objectives in both strategic and non-strategic settings. Moreover, it would be valuable to compare the proportional rule employed in the Tullock contest with alternative allocation rules.

Additionally, considering the problem from a normative perspective, akin to the discussion in Chapter one, offers another intriguing avenue for exploration. There may be scenarios where the designer's interests lie in factors such as efficiency, competitive balance, or ensuring a fair chance for all players. For instance, in competitions for funding, where effort is manifested through grant writing, the designer may be concerned with the opportunity cost of exerted effort and the promotion of less "conventional" research. In such cases, maximizing effort is no longer the sole ideal objective. Therefore, conducting a study on optimal contest design with alternative objectives becomes an engaging and valuable future research endeavor. This exploration would involve examining the interplay between different contest objectives and identifying optimal designs that align with the specific goals of the designer. By delving into these aspects, we can gain a deeper understanding of how to structure contests to achieve desired outcomes and promote fairness and efficiency in various contexts. Undertaking such work would contribute to the broader field of contest design and provide valuable insights for policymakers, practitioners, and researchers alike.

Bibliography

Amegashie, J. A. (1999). The design of rent-seeking competitions: Committees, preliminary and final contests. *Public Choice*, *99*(1-2), 63–76.

Aumann, R. J. (1994). The shapley value. In *Game-theoretic methods in general equilibrium analysis* (pp. 61–66). Springer.

Baik, K. H. (1994). Effort levels in contests with two asymmetric players. *Southern Economic Journal*, 367–378.

Baye, M. R., & Hoppe, H. C. (2003). The strategic equivalence of rent-seeking, innovation, and patent-race games. *Games and economic behavior*, *44*(2), 217–226.

Baye, M. R., Kovenock, D., & De Vries, C. G. (1993). Rigging the lobbying process: An application of the all-pay auction. *The American Economic Review*, *83*(1), 289–294.

Berry, S. K. (1993). Rent-seeking with multiple winners. *Public Choice*, *77*(2), 437–443.

Beviá, C., & Corchón, L. C. (2013). Endogenous strength in conflicts. *International Journal of Industrial Organization*, *31*(3), 297–306.

Billera, L. J., Heath, D. C., & Raanan, J. (1978). Internal telephone billing rates—a novel application of non-atomic game theory. *Operations Research*, *26*(6), 956–965.

Bimpikis, K., Ehsani, S., & Mostagir, M. (2019). Designing dynamic contests. *Operations Research*, *67*(2), 339–356.

Boudreau, K. J., Lacetera, N., & Lakhani, K. R. (2011). Incentives and problem uncertainty in innovation contests: An empirical analysis. *Management science*, *57*(5), 843–863.

Burgess, S., & Ratto, M. (2003). The role of incentives in the public sector: Issues and evidence. *Oxford review of economic policy*, *19*(2), 285–300.

Che, Y.-K., & Gale, I. L. (1998). Caps on political lobbying. *The American Economic Review*, *88*(3), 643–651.

Chen, Z., Hu, Z., & Tang, Q. (2020). Allocation inequality in cost sharing problem. *Journal of Mathematical Economics*, *91*, 111–120.

Chowdhury, S. M., Esteve-González, P., & Mukherjee, A. (2023). Heterogeneity, leveling the playing field, and affirmative action in contests. *Southern Economic Journal*, *89*(3), 924–974.

Chowdhury, S. M., & Kim, S.-H. (2017). “small, yet beautiful”: Reconsidering the optimal design of multi-winner contests. *Games and Economic Behavior*, *104*, 486–493.

Chowdhury, S. M., & Kim, S.-H. (2014). A note on multi-winner contest mechanisms. *Economics Letters*, *125*(3), 357–359.

Clark, D. J., & Nilssen, T. (2021). Competitive balance when winning breeds winners. *Social Choice and Welfare*, *56*(2), 363–384.

Clark, D. J., & Nilssen, T. (2020). Creating balance in dynamic competitions. *International Journal of Industrial Organization*, *69*, 102578.

Clark, D. J., & Nilssen, T. (2018). Keep on fighting: The dynamics of head starts in all-pay auctions. *Games and Economic Behavior*, *110*, 258–272.

Clark, D. J., Nilssen, T., & Sand, J. Y. (2020). Gaining advantage by winning contests. *Review of Economic Design*, 1–16.

Clark, D. J., & Riis, C. (1996). A multi-winner nested rent-seeking contest. *Public Choice*, *87*(1-2), 177–184.

Cornes, R., & Hartley, R. (2005). Asymmetric contests with general technologies. *Economic theory*, *26*(4), 923–946.

de Frutos, M. A. (1998). Decreasing serial cost sharing under economies of scale. *Journal of Economic theory*, *79*(2), 245–275.

Drugov, M., & Ryvkin, D. (2017). Biased contests for symmetric players. *Games and Economic Behavior*, *103*, 116–144.

Drugov, M., & Ryvkin, D. (2020). Tournament rewards and heavy tails. *Journal of Economic Theory*, *190*, 105116.

Ederer, F. (2010). Feedback and motivation in dynamic tournaments. *Journal of Economics & Management Strategy*, *19*(3), 733–769.

Erkal, N., Gangadharan, L., & Koh, B. H. (2018). Monetary and non-monetary incentives in real-effort tournaments. *European Economic Review*, *101*, 528–545.

Fershtman, C., & Gneezy, U. (2011). The tradeoff between performance and quitting in high power tournaments. *Journal of the European Economic Association*, *9*(2), 318–336.

Franke, J. (2012). Affirmative action in contest games. *European Journal of Political Economy*, *28*(1), 105–118.

Franke, J., Kanzow, C., Leininger, W., & Schwartz, A. (2013). Effort maximization in asymmetric contest games with heterogeneous contestants. *Economic Theory*, *52*(2), 589–630.

Franke, J., Leininger, W., & Wasser, C. (2018). Optimal favoritism in all-pay auctions and lottery contests. *European Economic Review*, *104*, 22–37.

Fu, Q. (2006). A theory of affirmative action in college admissions. *Economic Inquiry*, *44*(3), 420–428.

Fu, Q., & Lu, J. (2009). The beauty of "bigness": On optimal design of multi-winner contests. *Games and Economic Behavior*, *66*(1), 146–161.

Gallus, J. (2017). Fostering public good contributions with symbolic awards: A large-scale natural field experiment at wikipedia. *Management Science*, *63*(12), 3999–4015.

Gallus, J., & Frey, B. S. (2016). Awards: A strategic management perspective. *Strategic Management Journal*, *37*(8), 1699–1714.

Gradstein, M., & Konrad, K. A. (1999). Orchestrating rent seeking contests. *The Economic Journal*, *109*(458), 536–545.

Heyman, J., & Ariely, D. (2004). Effort for payment: A tale of two markets. *Psychological science*, *15*(11), 787–793.

Hillman, A. L., & Riley, J. G. (1989). Politically contestable rents and transfers. *Economics & Politics*, *1*(1), 17–39.

Hougaard, J. L. (2009). *An introduction to allocation rules*. Springer Science & Business Media.

Hougaard, J. L., & Østerdal, L. P. (2009). Decreasing serial cost sharing: An axiomatic characterization. *International Journal of Game Theory*, *38*(4), 469.

Hougaard, J. L., & Thorlund-Petersen, L. (2001). Mixed serial cost sharing. *Mathematical Social Sciences*, *41*(1), 51–68.

Iluz, A., & Sela, A. (2018). Sequential contests with first and secondary prizes. *Economics Letters*, *171*, 6–9.

Kirkegaard, R. (2012). Favoritism in asymmetric contests: Head starts and handicaps. *Games and Economic Behavior*, *76*(1), 226–248.

Klein, A. H., & Schmutzler, A. (2017). Optimal effort incentives in dynamic tournaments. *Games and Economic Behavior*, *103*, 199–224.

Konrad, K. (2009). Strategy and dynamics in contests. *OUP Catalogue*.

Konrad, K. A., & Kovenock, D. (2009). Multi-battle contests. *Games and Economic Behavior*, *66*(1), 256–274.

Kuhnen, C. M., & Tymula, A. (2012). Feedback, self-esteem, and performance in organizations. *Management Science*, *58*(1), 94–113.

Lazear, E. P., & Rosen, S. (1981). Rank-order tournaments as optimum labor contracts. *Journal of political Economy*, *89*(5), 841–864.

Leota, J., Hoffman, D., Mascaro, L., Czeisler, M. E., Nash, K., Drummond, S. P., Anderson, C., Rajaratnam, S. M., & Facer-Childs, E. R. (2022). Home is where the hustle is: The influence of crowds on effort and home advantage in the national basketball association. *Journal of Sports Sciences*, 1–10.

Lu, D. (2021). On optimal favoritism in asymmetric competitions. *Theoretical Economics Letters*, *11*(1), 74–83.

Mago, S. D., Samak, A. C., & Sheremeta, R. M. (2016). Facing your opponents: Social identification and information feedback in contests. *Journal of Conflict Resolution*, *60*(3), 459–481.

Megidish, R., & Sela, A. (2014). Sequential contests with synergy and budget constraints. *Social Choice and Welfare*, *42*(1), 215–243.

Meyer, M. A. (1992). Biased contests and moral hazard: Implications for career profiles. *Annales d'Economie et de Statistique*, 165–187.

Möller, M. (2012). Incentives versus competitive balance. *Economics Letters*, *117*(2), 505–508.

Morgan, J. (2003). Sequential contests. *Public choice*, *116*(1-2), 1–18.

Moulin, H. (1996). Cost sharing under increasing returns: A comparison of simple mechanisms. *Games and Economic Behavior*, *13*(2), 225–251.

Moulin, H. (2002). Axiomatic cost and surplus sharing. *Handbook of social choice and welfare*, *1*, 289–357.

Moulin, H., & Shenker, S. (1992). Serial cost sharing. *Econometrica: Journal of the Econometric Society*, 1009–1037.

Moulin, H., & Shenker, S. (1994). Average cost pricing versus serial cost sharing: An axiomatic comparison. *Journal of Economic Theory*, *64*(1), 178–201.

Nitzan, S. (1994). Modelling rent-seeking contests. *European Journal of Political Economy*, *10*(1), 41–60.

Nti, K. O. (2004). Maximum efforts in contests with asymmetric valuations. *European Journal of Political Economy*, *20*(4), 1059–1066.

Nti, K. O. (1999). Rent-seeking with asymmetric valuations. *Public Choice*, *98*(3-4), 415–430.

Parco, J. E., Rapoport, A., & Amaldoss, W. (2005). Two-stage contests with budget constraints: An experimental study. *Journal of Mathematical Psychology*, *49*(4), 320–338.

Pham, N. A. (2019). Lorenz comparison between increasing serial and shapley value cost-sharing rules. *Economics Letters*, *179*, 49–52.

Ridlon, R., & Shin, J. (2013). Favoring the winner or loser in repeated contests. *Marketing Science*, *32*(5), 768–785.

Rosen, S. (1986). Prizes and incentives in elimination tournaments. *The American Economic Review*, 701–715.

Roth, A. E., & Verrecchia, R. E. (1979). The shapley value as applied to cost allocation: A reinterpretation. *Journal of Accounting Research*, 295–303.

Schmitt, P., Shupp, R., Swope, K., & Cadigan, J. (2004). Multi-period rent-seeking contests with carryover: Theory and experimental evidence. *Economics of Governance*, *5*, 187–211.

Schunk, D. H. (1983). Ability versus effort attributional feedback: Differential effects on self-efficacy and achievement. *Journal of educational psychology*, *75*(6), 848.

Sen, A. K. (1966). Labour allocation in a cooperative enterprise. *The Review of Economic Studies*, *33*(4), 361–371.

Shapley, L. S., Roth, A. E., et al. (1988). *The shapley value: Essays in honor of lloyd s. shapley*. Cambridge University Press.

Sheremeta, R. M. (2010a). Expenditures and information disclosure in two-stage political contests. *Journal of Conflict Resolution*, *54*(5), 771–798.

Sheremeta, R. M. (2010b). Experimental comparison of multi-stage and one-stage contests. *Games and Economic Behavior*, *68*(2), 731–747.

Shindler, J. (2009). *Transformative classroom management: Positive strategies to engage all students and promote a psychology of success*. John Wiley & Sons.

Shubik, M. (1962). Incentives, decentralized control, the assignment of joint costs and internal pricing. *Management science*, *8*(3), 325–343.

Stein, W. E. (2002). Asymmetric rent-seeking with more than two contestants. *Public Choice*, *113*(3-4), 325–336.

Stracke, R., Höchtl, W., Kerschbamer, R., & Sunde, U. (2015). Incentives and selection in promotion contests: Is it possible to kill two birds with one stone? *Managerial and Decision Economics*, *36*(5), 275–285.

Strong, R., et al. (1995). What do students want?. *Educational leadership*, *53*(1), 8–12.

Thomas, J., & Wang, Z. (2017). Marginal subsidies in tullock contests. *Journal of Public Economic Theory*, *19*(2), 511–526.

Thomas, W. (2009). *Intrinsic motivation at work: What really drives employee engagement*. Berrett-Koehler Publishers.

Tsoulouhas, T., Knoeber, C. R., & Agrawal, A. (2007). Contests to become ceo: Incentives, selection and handicaps. *Economic Theory*, *30*(2), 195–221.

Tullock, G. (1980). Efficient rent seeking. In *Toward a theory of the rent-seeking society*. College Station: Texas A & M University, 97-112.

Verhoeff, T. (1997). The role of competitions in education. *Future world: Educating for the 21st century*, 1–10.

Vesperoni, A. (2016). A contest success function for rankings. *Social Choice and Welfare*, *47*(4), 905–937.

Zhu, F. (2021). On optimal favoritism in all-pay contests. *Journal of Mathematical Economics*, 102472.