Proof Theoretic Criteria for Logical Constancy

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Submitted in fulfilment of the requirements of the Degree of Doctor of Philosophy

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October 2023

Abstract

Logic concerns inference, and some inferences can be distinguished from others by their holding as a matter of logic itself, rather than say empirical factors. These inferences are known as logical consequences and have a special status due to the strong level of confidence they inspire. Given this importance, this dissertation investigates a method of separating the logical from the non-logical. The method used is based on proof theory, and builds on the work of Prawitz, Dummett and Read. Requirements for logicality are developed based on a literature review of common philosophical use of the term, with the key factors being formality, and the absolute generality / topic neutrality of interpretations of logical constants. These requirements are used to generate natural deduction criteria for logical constancy, resulting in the classification of certain predicates, truth functional propositional operators, first order quantifiers, second order quantifiers in sound and complete formal systems using Henkin semantics, and modal operators from the systems K and S5 as logical constants. Semantic tableaux proof systems are also investigated, resulting in the production of semantic tableaux-based criteria for logicality.

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I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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1. Introduction

Logic concerns inference, or the conclusions which can be drawn from a given set of premises (‘what follows from what’). Some inferences can be distinguished from others by their holding as a matter of logic itself, rather than due to any empirical or other factors. These inferences are known as logical consequences, and have a special status due to the strong, perhaps even absolute, level of confidence they inspire.

This dissertation investigates a method of separating the logical from the non-logical. The method used is based on a proof theoretic approach to logic, which puts the notion of provability of inferences at the forefront. It argues that proof theoretic approaches can provide reliable criteria to identify logical constants. Furthermore, due to the property of formality being central to logic, identifying these logical constants allows the assessment of logicality of examples of inference and formal systems. That the criteria developed in this dissertation capture the nature of logic is assured by undertaking a literature review of common philosophical uses of the term, with the key factors being the absolute generality and topic neutrality of interpretations of logical constants, and the formality (that is, holding due to structure rather than specific content) of examples of logical consequence. The criteria are also precise, in that they provide a clear separation of the logical from the non-logical.

In adopting a proof theoretic approach to logical constancy, this dissertation builds on the work of Prior, Prawitz, Dummett and Read. These authors established many of the methods used in this dissertation, including the critical notion of harmony. This dissertation takes these methods and distils them into a small number of precise and clear criteria to identify logical constants based on their natural deduction operational rules. It also applies similar reasoning and provides criteria for logical constancy based on the semantic tableaux proof system, a topic which has hitherto received little attention. The proof theoretic criteria developed in this dissertation return certain predicates, the common operators of propositional and first order logic and the modal operators □K and □S5 (the former of which is required for any modal logic to be classified as 'normal' (regular and classical) and the latter of which can be interpreted as a strong form of necessity, such as metaphysical or logical necessity) as logical constants. The semantically defined full or unrestricted second order quantifier is deemed non-logical, though semantic incompleteness prevents a proof theoretic conception of it being available for assessment. Other conceptions of the second order quantifier are deemed logical, but do not represent an advance in terms of expressivity compared to systems including only (propositional logic and) the first order quantifier.

The overall aim of this dissertation is to find proof theoretic criteria for logicality. Logicality is linked to formality – an example of inference is logical if it holds simply on the basis of its form or structure. This means that if what dictates the form or structure of an inference consists only of logical constants, then the inference itself can be called logical. Thus the project reduces to the problem of identifying the set of logical constants. In terms of the scope of the dissertation, it addresses only proof theoretic approaches to logical constancy. Proof theoretic approaches are a conception of logic which makes provability the fundamental notion, rather than truth or satisfaction, as is in the case of model theoretic approaches[1]. The proof theoretic approach began with the work of Gentzen, but the specific problem addressed in this dissertation was brought to the fore in Prior (1960). Here, Prior attacked the inferentialist position that logical operators could be defined simply on the basis of their natural deduction operational rules. Responses to Prior’s challenge were developed in Prawitz (1965), Dummett (1991) and Read (1999, 2000, 2008 and 2010). In these works, the notion of harmony, or a kind of balance between the introduction and elimination rules of a logical operator, was introduced and developed in order to respond to the challenge laid down by Prior. This dissertation critically evaluates the approaches taken by these authors and uses them to produce natural deduction criteria for logicality among potential candidates from propositional logic, first order logic, second order logic and modal logic.

While the main focus of this dissertation is natural deduction proof systems, it also investigates criteria for logicality based on semantic tableaux proof systems. In doing so, it produces an analogous conception of natural deduction harmony for

[1] Though note that mathematical approaches are possible which use for example 1 and 0 in place of true/false respectively are also possible, and thus retain the model theoretic approach while avoiding mentions of truth and falsity.

semantic tableaux systems. This is a novel approach, with there being a lack of discussion around logicality and semantic tableaux systems in the literature on the subject.

In terms of wider philosophical issues associated with the precise demarcation of logic, logical, metaphysical, and epistemological ramifications are all involved. While the principal business of this dissertation is the development of relevant and precise criteria for logicality, such wider discussions are important not only for their general philosophical interest, but also to mitigate the risk of developing these criteria being seen as merely an exercise in nomenclature without practical (at least in a philosophical sense) results (the issue of whether the classification into the logical/non logical is ultimately one of a verbal difference between these uses of the word ‘logic’ is taken up further in Section 7).

The methodology adopted by this dissertation begins by investigating the desiderata for criteria for logicality. For proof theoretic methods to successfully provide criteria for logicality, they should be both relevant and precise. Relevance concerns each criterion’s applicability to the intuitive notion of what logic is, or the prevailing current usage of the term in its technical and philosophical sense. Precision concerns each criterion’s ability to clearly distinguish the logical from the non-logical. This dissertation addresses the relevance issue by seeking what will be referred to as the requirements which underpin logicality. These are sought via a literature review of what some prominent authors have claimed to be the hallmarks of logic. This results in the nomination of contentless formality, which is closely linked to absolutely generality and topic neutrality, as the key requirement for logicality[2].

With these requirements for logicality in place, the core business of developing criteria for logical constancy can proceed. Doing so is facilitated by having access to a number of operators which are either known to be logical constants or known not to be logical constants. The criteria can then be developed and adapted to include operators which are logical constants and exclude operators which are not logical constants. This is where the requirements of absolute generality and topic neutrality are used. In many cases, operators in formal systems have interpretations in natural language – for example, the interpretation of the operator $\wedge$ is the natural language term ‘and’. If ‘and’ is a concept which is absolutely general and topic neutral, then $\wedge$ (which is referred to in this dissertation the formalisation of ‘and’) can confidently be called a logical constant. Criteria can then be developed which admit it as such. This is the general methodology adopted in this dissertation: Develop criteria which include operators whose interpretations are clearly absolutely general and topic neutral as logical constants, and exclude those which are not. These criteria can then be applied to more contentious cases, such as that of second order quantification and the modal operators.

[2] As a side note, associating logic with formality aligns with Wittgenstein’s view that logic is contentless and in a sense empty.

The following natural deduction criteria for logical constancy are the results of applying this methodology:

- Natural Deduction Criterion for Logical Constancy 1: Operational rules must not allow the introduced element to appear as the main connective in the antecedent of the I rule and must not allow the eliminated element to appear as the main connective in the consequent of the E rule.
- Natural Deduction Criterion for Logical Constancy 2: The introduction and elimination rules for logical constants must be in (general elimination) harmony.
- Natural Deduction Criterion for Logical Constancy 3: The rules must contain no reference to any non-logical elements external to the operator which the rule defines.

This dissertation argues that each of these three criteria are reasonable expectations for logical constancy. The result of their application leads to certain predicates, all the truth functional operators of propositional logic, the first order quantifier, the second order quantifier (interpreted according to Henkin rather than the full or unrestricted semantics), and the modal operators □K and □S5 (the former of which being an axiom of any logic which is normal, that is to say regular and classical; the latter being interpreted as logical and sometimes metaphysical necessity) being admitted to the set of logical constants. Happily, the interpretations of each of these operators also fulfil the requirements of absolute generality and logical constancy. Just as importantly, these criteria also exclude problematic operators such as Prior's tonk.

It is notable that this dissertation takes what could be described as an analytic approach to providing criteria for logicality. For this reason, the potential pragmatic uses for logic as a basis for seeking criteria are passed over in favour of analysing the inherent nature of logic, as defined by typical usage of the term in the logical and philosophical community. Naturally, given that the logical systems existing today have to a certain extent been developed for pragmatic reasons, it is not possible to completely separate the two. However, the key point is that in this dissertation, the criteria for logical constancy will not be based on the potential utility of candidates.

Even a defensibly successful result for the project is unlikely to satisfy the entire logical community. This is due to the wide variety of natures which logic is purported to have, the uses it is thought to have, and the role it plays in the overall structure of reasoning and thought in general. These latter role-based notions tend to result in pragmatic requirements for logicality (among these is the connection which is commonly thought to exist between logic and mathematics, most notably in the logicist project). However, seeking a foundation for the present project in these informal requirements is intended to provide a strong link between the criteria produced by the project with the usage-based understanding of what logic is, and avoid a major clash with intuition.

The structure of this dissertation is as follows:

- Section 2 sets out background and preliminary information. This includes putting forward the desiderata of precision and relevance for criteria and using a literature review of prominent authors on the subject to suggest that the key requirements for logic are absolute generality, topic neutrality and formality. Further discussion covers potential bearers of logicality (such as formal systems and examples of inference) clarifying some terminology and examining the nature of a potential border between the logical and the non-logical.
- Section 3 discusses natural deduction structural rules. These rules do not concern any particular logical constant (and thus form a rule set apart from operational rules), but rather endow certain properties on the logical consequence relation independent of any constant.
- Section 4 is the main part of the dissertation and develops proof theoretic criteria for logical constancy based on natural deduction systems. After introducing important terminology and discussing natural deduction systems in general, this dissertation first examines propositional logic, taking individual constants, predicates, and connectives in turn. Harmony is a key consideration here, with Read's general elimination harmony (taking the name from Francez and Dyckhoff (2012)) approach being favoured as the correct account of it, based on its ability to find harmony in operational rules for classical negation (¬), along with the other truth functional operators of

propositional logic. First order quantification is considered next, with the criteria developed admitting the universal quantifier as a logical constant. An examination of second order quantification follows this. This case is complicated by the lack of a semantic completeness for the full or unrestricted semantic interpretations of the second order quantifier. However, other conceptions of the second order quantifier, which achieve soundness and completeness with respect to Henkin semantics, adhere to criteria for logical constancy. Finally, modal operators are considered, with this dissertation arguing that only □K and □S5 should be seen as logical constants. Other modal operators such as □KT and □S4 should be excluded based on their operational rules including external reference to a structural rule for the R relation, which is a binary relation which holds between the possible worlds used in modal logics. Comparison with the absolute generality and topic neutrality of the interpretations of various modal operators supports this conclusion.

- Section 5 examines semantic tableaux systems and their potential to provide criteria for logicality[3]. This leads to the production of similar criteria as those given for natural deduction systems and provides similar results in terms of the operators classified as logical constants. Beyond this, the results of the examination of semantic tableaux systems are of interest due to the potentially simpler criteria which they produce, particularly in the case

[3] Note that, in keeping with the proof theoretic approach adopted in this dissertation, and despite the name, semantic tableaux systems can be seen in an entirely proof theoretic manner, with no inherent link to semantic approaches.

of the semantic tableaux analogue of harmony in natural deduction operational rules.

- Section 6 includes a summary of the results of the dissertation, including the natural deduction and semantic tableaux criteria for logical constancy developed in this dissertation, and the set of logical constants that results from their application.
- Section 7 contains a relatively brief discussion of what is at stake when addressing the notion of logicality, and thus examines wider philosophical issues associated with logicality. Areas considered include the logicist and neo-logicist projects, epistemological considerations, and metaphysics, with the latter being of particular importance should the logical realist position be adopted.
- Finally, Section 8 provides some concluding remarks to summarise the results obtained in the dissertation and provides a concise presentation of the criteria developed in this dissertation and the results that they provide in terms of the set of logical constants.

2. Preliminary Information

This part of the dissertation provides some background information and puts in place the groundwork necessary for the detailed investigation of proof theoretic criteria for logical constancy which makes up the main part of the work. This groundwork is required to identify the informal requirements for logicality which are intended to ensure that the criteria developed are relevant to philosophical usage of the term ‘logic’, to introduce the importance of formality to logicality, and from there to introduce the importance of logical constants to logicality in general. The following paragraphs provide an outline of the introductory information included in this part of the dissertation.

Section 2.1 states the desiderata for criteria for logicality, namely relevance and precision. Pursuing each of these desiderata will ensure that the criteria produced both reflect (are relevant to) the nature of logic; and also allow the exact (precise) evaluation of each potential candidate for logicality. The chosen means of addressing the relevance desideratum in the dissertation is through grounding the criteria in what will be termed the informal requirements for logicality (these requirements are discussed in detail in Section 2.2). After all, criteria for logicality cannot be produced entirely *ex nihilo*, and some basis for them must be provided – specifically, the criteria must return results which reflect typical philosophical usage of the term ‘logic’. The approach taken is to undertake a survey of pertinent literature to identify proposed requirements, then to synthesise them into one or more fundamental requirement(s). If criteria can be found which respect these synthesised requirements, they can stake a legitimate claim to being criteria for (relevant to) logicality. The criteria should also provide precise adjudication of each potential case of logicality. This is to avoid any vagueness by providing a clear border between the logical and the non-logical, and thus return a set of logical constants whose status as such is indisputable to the greatest extent possible.

Section 2.2 discusses the informal requirements for logicality. This section introduces an important distinction which will be relevant to the discussion throughout – that is, the distinction between the elements of formal systems; and the concepts discussed in natural language which are interpretations of them. It is the former which can be described as logical, while the latter will be described as informally logical. For criteria which fulfil the relevance desideratum discussed above:

- If the criteria classify an element of a formal system as logical, a correct interpretation of it should be informally logical.
- If a concept discussed in natural language is informally logical, the criteria should classify its formalisation as an element of a formal system as logical.

After discussion of various candidates for these requirements (identified via a survey of pertinent literature on the subject), this dissertation will settle on formality, absolute generality and topic neutrality as the key requirements for logicality. This means that for an operator contained in a formal system to be logical, its interpretation should be absolutely general and topic neutral. This is a key anchor point for this dissertation, with the absolute generality and topic neutrality of interpretations of operators allowing the development of criteria for logicality which reflect the nature of logic. This is the general methodology adopted in this dissertation: Develop criteria which include operators whose interpretations are clearly absolutely general and topic neutral as logical constants and exclude those which are not. These criteria can then be applied to more contentious cases, such as that of second order quantification and the modal operators.

Section 2.3 of the dissertation addresses the question of the bearers of logicality. The adjective 'logical' can be applied to a variety of nouns, including examples of inference and truths. However, the conclusion of Section 2.2 that formality is the basic requirement for logicality means that inferences which hold as a matter of logic (logical inferences) do so due to their form rather than their content. The form of an inference is given by certain purely logical elements which it contains, which are called logical constants. Hence, identifying the examples of inference which are logical requires identifying the logical constants they contain, noting that logical constants here are purely formal objects, free of semantic identification or interpretation; see Section 2.5 for discussion of the logicality of formal objects and the 'informal logicality' of their interpretations. Since the approach adopted in this dissertation focuses on proof theory, a potential objection is that in doing so it reduces the symbols and proofs involved to some kind of meaningless game. While this is true when considered *purely* proof theoretically, sufficient discussion of the interpretations of these symbols is included in this dissertation to provide meaning, and thus philosophical importance.

Since evaluating the logicality of examples of inference and of truths (which themselves are premiss-less inferences) reduces to identifying the structure of the inferences, and hence which elements of them are logical constants, the focus of the present dissertation can be summed up as using proof theoretic methods to produce criteria for logical constancy. Given the formality-based argument presented above, producing these criteria for logical constancy will go a long way towards producing criteria for logicality in general.

Section 2.4 of the dissertation focuses on discussion of the nature of the border between the logical and the non-logical. This involves questions such as: Does a sharp border exist between the logical and the non-logical? And if so, can proof theoretic means provide the proper criteria for separating cases on one side of the divide from the other? Or is the border between the logical and non-logical vague rather than sharp? And in this case, can proof theoretic criteria still be provided which can classify the majority of cases (or perhaps all interesting cases)? Or is the purported border between the two categories in some way incoherent? If this is the case, then no criteria (proof theoretic or otherwise) would be suitable to delimit it. Investigation of the nature of the border between the logical and the non-logical will lead to the proposal that to provide a border between the two which is sharp, the criteria in question should not allow variation in the level of logicality within the logical or the non-logical. This section is thus intended to sharpen discussion of the desiderata of precision discussed in Section 2.1, and in doing so make the goals of the dissertation in terms of the production of precise criteria clearer.

Section 2.5 of the dissertation then provides some further terminological notes to aid the clarity of the exposition in the main part of the dissertation, before Section

2.6 discusses logical pluralism, which is approximately the view that multiple formal systems can be accepted as logics, and Section 2.7 discusses a model theoretic criterion for logicality, permutation invariance. Finally, Section 2.8 provides some concluding remarks before the core business of investigating proof theoretic criteria for logicality is addressed in further sections of the dissertation.

Some of the questions considered in this dissertation are both broad and deep, and involve issues relating to the nature of philosophical investigation itself. Naturally, then, their treatment in this chapter will be less than complete; but will be presented in sufficient detail to provide a groundwork and orientation for later chapters. Also, alternatives exist to some methodological decisions taken regarding how criteria for logicality should be pursued. The path followed below thus represents one of many, but, as the remainder of this section will hopefully show, a worthwhile one.

2.1. Desiderata for Criteria

Two desiderata will be used in this dissertation to guide the selection of criteria for logicality. The first of these is relevance, with relevant criteria being understood to mean those which are as representative as possible of what is understood to be logic by the logical community. Satisfying this desideratum is impeded by the wide variety of uses of the term logic by different members and groups across philosophy and logic. Exemplifying this variety is the following list[4] of interpretations of logic from a range of philosophers and logicians:

> Pure general logic ... is a canon of understanding and of reason, but only in respect to what is formal in their employment. (Kant (1781), First Critique, A53).
>
> On the basis of linguistic images which accompany basic mathematical truths in actual mathematical structures, it is sometimes possible to build up linguistic structures, sequences of sentences, proceeding according to logical laws. (Brouwer (1907), Collected Works, page. 75).
>
> To discover truths is the task of all sciences; it falls to logic to discern the laws of truth. (Frege (1918), Collected Papers, page. 351).
>
> Logic deals with every possibility and all possibilities are its facts. (Wittgenstein (1922), 2.0121).

Many other defensible suggestions are possible. This variety suggests that universal satisfaction across all potential stakeholders will be difficult to achieve. An early decision is therefore necessary to fix a certain approach, to allow later focus on the principal business of the dissertation, the identification of proof theoretic criteria for logicality. This means that the conclusions reached in this dissertation will be relative to the approach to relevance taken, reinforcing the impossibility of universal satisfaction.

[4] Selected from the listing provided by Wang (1994), Page 261, individual citations also after this resource.

The chosen approach will begin with a literature survey to identify the informal requirements for logicality. This will result in a listing of what various members of the logical and philosophical community take to be the attributes of logicality. These will then be compared, and if possible synthesised into a core group of requirements. Again, this approach may not satisfy all readers – in particular, those of Platonist tendencies, who may consider that logic has an objective nature which is independent of the linguistic use of the term, may be dissatisfied. Thus, for such readers, the literature survey-based methodology described above may compromise the goal of understanding the 'true' nature of logic rather than that which is mediated by discussion of logic in natural language.

By way of an apology to these objectors, this dissertation offers the following quotation from Tarski (1986), who holds that such Platonist approaches:

> …seem to aim at something very different (…); people speak of catching the proper, true meaning of a notion, something independent of actual usage, and independent of any normative proposals, something like the platonic idea behind the notion. This last approach is so foreign and strange to me that I shall simply ignore it for I cannot say anything intelligent on such matters[5].

The author of this dissertation shares Tarski's intellectual limitations. Should this mean that this dissertation's conclusions are best considered hypothetical rather than categorical (that is, 'these are the proof theoretic criteria for logicality which

[5] Tarski (1986). Page 145.

result if their relevance is developed from a literature survey of its informal requirements'), then so be it.

Turning to precision, this is less likely to generate controversy compared to establishing relevance. The guiding idea here is that the criteria produced should pronounce clearly on each case, leave no potential candidate for logicality beyond the scope of its assessment, and thus result in the categorisation of every potential candidate as logical or non-logical (note that the nature of the cases on which the criteria pronounce is discussed in Section 2.3). This strictly binary result leaves no room for vagueness, for example any candidates which lie halfway between logic and non-logic, candidates which are both logic and non-logic, or candidates which are neither logic nor non-logic.

The motivation for employing the two chosen desiderata is simply that they both seem necessary to provide respectable criteria for any form of classification. Precision permits clear adjudication, and relevance ensures that this adjudication accurately reflects the concept of logicality. However, it is notable that the two desiderata may be in tension. For example, if the informal requirements (which are used to discern the informally logical from the informally non-logical) are imprecise, moving to more precise criteria (which discern the formally logical from the formally non-logical) based on them may compromise relevance to some extent.

However, even if relevance does suffer to a certain extent, providing precise criteria for something at least resembling logic would represent a significant contribution. This is because achieving the clear delimitation of a certain area of investigation would stand in opposition to the thesis put forward in for example Shapiro (1991), that no sharp borders exist between the different branches of knowledge (or, in the terminology adopted in this dissertation, that precise criteria for logicality do not exist). Scepticism regarding this point is also expressed in Barwise (1985)[6]:

> ... in basic logic courses... we attempt to draw a line between 'logical concepts', as embodied in the so-called 'logical constants', and all the rest of the concepts of mathematics. [W]e do not so much question the placement of this line, as question whether there is such a line.

To be able to provide criteria which clearly cleave logic from, for example, mathematics (even if that area of knowledge so cleaved may not correspond precisely to a natural conception of what logic is) would constitute a rebuttal to these notions of a continuous, so called, web of belief.

In this, the chosen approach has similarities with the method of Carnapian explication. This approach, described in for example Carnap (1947), avoids the problems associated with attempting to provide precise criteria for a vague, informal concept by replacing the latter (known as the explicandum), with a new

[6] Page 5.

concept which is more precisely defined (the explicatum). In the present study, the explicandum corresponds to the informal requirements identified in the literature review, whereas the explicatum corresponds to the project's goal of precise criteria. That there can be a level of tension between the explicandum and explicatum has been brought up as a criticism of the Carnapian approach. Strawson[7] puts the point as follows:

> …it follows that typical philosophical problems about the concepts used in non-scientific discourse cannot be solved by laying down the rules of use of exact and fruitful concepts in science. To do this last is not to solve the typical philosophical problems, but to change the subject.

A means of responding to this critique can be found in Carnap (1947)[8], which states:

> Generally speaking, it is not required that an explicatum have, as nearly as possible, the same meaning as the explicandum; it should, however, correspond to the explicandum in such a way that it can be used instead of the latter.

While the first part of the above quote acknowledges that differences in meaning between explicatum and explicandum are to be expected, the second part shows that the intention of the method is to retain a claim to relevance to the original, vague, informal concept. Thus all claims to relevance would not be abandoned by

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[7] Strawson (1963). Page 506.

[8] Carnap (1947). Page 8.

following this project's chosen methodology. What is needed, however, is a careful synthesis of the informal requirements sourced from the literature review, and an equally careful development of criteria based on that synthesis. These notions will thus guide the present study.

2.2. Requirements for Logicality

This section of the dissertation addresses the requirements for logicality. Drawing on the distinction made in the introductory remarks to Section 2, these requirements concern the concepts discussed in natural language which are interpretations of formal elements of logical systems. It is therefore a discussion of what will be termed in this dissertation *informal logicality*. The purpose of identifying these requirements is therefore to determine which concepts are informally logical. This can then guide the identification of elements which are formalisations of these concepts and are therefore classified as logical by the criteria for logicality the development of which is the principal business of this dissertation.

The requirements for informal logicality have been discussed by a wide range of thinkers. This means that the present survey cannot pretend to be exhaustive, but rather to be representative of some common views on the subject. Each view will simply be stated to begin with, with comparison deferred until each is put forward.

In Beale and Restall's (2006) study of logical pluralism, the authors frame the discussion in terms of the validity of arguments, or the notion of logical consequence. They put forward the following three core features of logicality:

- Necessity: "The truth of the premises of a valid argument *necessitates* the truth of the conclusion of that argument."[9]
- Normativity: "In an important sense, if an argument is valid, then you somehow go wrong if you accept the premises but reject the conclusion."[10]
- Formality: Here, Beale and Restall's discussion follows Łukasiewicz (1956), and comes to the conclusion that "Łukasiewicz speaks for many in the 20th century: it has been a commonplace to characterise the formality of logic in terms of its being schematic. Logic does not speak at first of individual concrete arguments. Instead, it categorises forms."[11] Further discussion is offered regarding the notion of formality within the context of logic being interpreted as "the science of the forms of thought"[12], but given Frege's widely accepted rebuttal of the psychologist position, this line of thought is not pursued here.

Linnebo's (2017) article on plural quantification also discusses three informal requirements for logicality. These are:

[9] Beale and Restall (2006). Page 14.
[10] Beale and Restall (2006). Page 16.
[11] Beale and Restall (2006). Page 19.
[12] Beale and Restall (2006). Page 19.

- Absolute Generality: “A logical principle is valid in any kind of discourse, no matter what kind of objects this discourse is concerned with. For instance, modus ponens is valid not only in physics and mathematics but in religion and in the analysis of works of fiction.”[13] The similarities between this requirement and that of topic neutrality, discussed below in the requirements of Haack (1978), are notable.
- Formality: “the truth of a principle of logic is guaranteed by the *form* of thought and/or language and does not in any way depend upon its *matter*.”[14] However, this is offered with the qualification that “What this feature amounts to will obviously depend on how the distinction between form and matter is understood”.[15] This informal requirement results in two potential formal criteria for logicality:
    - Ontological innocence: A principle of logic should not introduce new ontological commitments. For example, its potential lack of such innocence is given as a key point against the logicality of second order quantification. Hume’s principle provides another example, with those in the neo-logicist school being seen as requiring an argument that it is analytic, in order to avoid charges of ontological introduction.
    - Permutation invariance: “The basic notions of logic must not discriminate between different objects but must treat them all alike. This … idea is often spelled out as the requirement that logical

[13] Linnebo (2017). Section 3.
[14] Linnebo (2017). Section 3.
[15] Linnebo (2017). Section 3.

notions must be invariant under permutations of the domain of objects"[16].

- Cognitive Primacy: "Primitive logical notions must be completely understood, and our understanding of them must be direct in the sense that it doesn't depend on or involve an understanding of notions that must be classified as extra-logical. Assume, for instance, that certain set theoretic principles must be regarded as extra-logical. Then our understanding of the primitive logical notions cannot depend on or involve any of these principles."[17]

  This requirement is perhaps more controversial than the aforementioned two. Though the use of the terms 'understood' and 'understanding' is not entirely clear, Linnebo's claim seems to be connected to the claim that logic is a priori, but this would imply its unrevisability, which is disputed in the influential work of Quine (who held that all knowledge, including the logical, is in principle revisable). While it is not clear that logic can claim to have cognitive primacy as defined above, it does appear to have a comparatively stronger claim to it than other fields of knowledge such as the sciences. For example, physics requires an understanding of notions that must be classified as extra-physical, such as mathematics.

The well-known introduction to the philosophy of logic, Haack (1978), puts forward the following informal requirements of logic. Her account differs from the above

[16] Linnebo (2017). Section 3, with the idea credited by Linnebo to Tarski (1986).

[17] Linnebo (2017). Section 3.

two in that it discusses the logicality of formal systems, rather than single instances of consequence or the validity of arguments:

- Topic Neutrality: “The traditional idea that logic is concerned with the validity of arguments as such, irrespective, that is, of their subject matter – that logic is, as Ryle neatly puts it, ‘topic-neutral’ – could be thought to offer a principle on which to delimit the scope of logic. On this account, those systems which are *applicable to reasoning irrespective of its subject-matter* would count as logics.”[18] As mentioned above, this requirement is very close to that of absolute generality suggested in Linnebo (2017).
- Formality: “Logic applies to reasoning irrespective of its subject-matter because it is concerned with the *form* of arguments rather than their *content*.”[19]

Returning to a consideration of consequence, another well-known introduction to the philosophy of logic, Read (1995), states “We now have a clear conception of the account of logical consequence supplied by classical logic. Logical consequence is a matter of form: one proposition is a logical consequence of others if all propositions of the same form are consequences of others of the same form”[20]. Thus formality represents a key requirement for Read, though it should be underlined that this is only with reference to the *classical* conception of logic, and he argues that formality both over- and under-generates in different cases.

[18] Haack (1978). Page 5.
[19] Haack (1978). Page 5.
[20] Read (1995). Page 49.

Quine's wide-ranging influence in the philosophy of logic means that no survey of this type could claim to be representative without including his view. Quine (1986) contains the following (which deals with logical implication, though in general Quine discusses the issue in terms of logical truth):

> It is not for logic to settle what sequences satisfy the simple sentences, but rather, contingently on such information, to settle what compound sentences will be true or what sequences will satisfy them. Equally logic explores these connections in reverse: given that a compound sentence is true, or given what satisfies it, to settle what alternatives are left open for the simple sentences. Indirectly also, through these dependences upward and downward, there are transverse interdependences to explore between one compound sentence and another.
>
> A familiar connection of the kind is logical implication. One closed sentence logically implies another when, on the assumption that the one is true, the structures of the two sentences assure that the other is true. The crucial restriction here is that no supporting supplementary assumption or information be invoked as to the truth of additional sentences. Logical implication rests wholly on how the truth functions, quantifiers, and variables stack up. It rests wholly on what we may call, in a word, the logical structure of the two sentences.[21]

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[21] Quine (1986). Page 48.

Quine also presents a number of alternative means of distinguishing logical truths, such as in terms of substitution, proof, and grammar. However, they identify the same truths as logical; and furthermore, they have the same fundamental basis. Quine writes:

> We have seen several ways of defining logical truth. They are extensionally equivalent: they all declare the same sentences logically true (supposing the object language reasonably rich in predicates). They differ markedly in their apparatus, but they all hinge upon sameness of structure in respect of three grammatical constructions that are local to the object language: negation, conjunction, quantification.[22,23]

Thus for Quine it is structure which is the key factor which endows a truth with logicality. Comparison of the text cited above with those discussed previously shows that structure is virtually synonymous with form for Quine.

In addition, Quine states that "I have not said which particular predicates are to be present in the language - whether 'walks', 'is red', 'is heavier than', 'is divisible by', etc.; for the point is indifferent to the logical structure of the language. "[24] This

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[22] Quine (1986). Page 58.

[23] The issue of which grammatical constructions are local to object language (which Quine nominates here without explanation to be negation, conjunction and quantification) will be discussed in more detail in later chapters.

[24] Quine (1986). Page 29.

disregard of the importance of particular predicates implies the view that topic neutrality is also a defining feature of logic.

This concludes the survey of five different accounts of the requirements for informal logicality. With this in place, the process of synthesis can proceed. The intention of this is to compare each requirement put forward and attempt to find any links between them. It is hoped that this will allow a more compact set of requirements to be synthesised, rather than the quite expansive list which results from the aggregation of all requirements mentioned above. In purely numerical terms, formality is explicitly cited in all five of the accounts, with each of necessity, normativity, and cognitive primacy receiving one direct 'vote' each and, absolute generality / topic neutrality (grouped due to their similarity) receiving two direct 'votes'. The results of the survey can be summarised in the following table:

Beall and Restall (2006)	Linnebo (2017)	Haack (1978)	Read (1995)	Quine (1986)
Necessity	Absolute Generality	Topic Neutrality	Formality	Formality
Normativity	Formality	Formality		Topic Neutrality
Formality	Cognitive Primacy			

Various means of synthesising these requirements present themselves, based on simple (in that they disregard any qualities associated with each requirement) set theoretic approaches. These, and the results that they give, are:

- Taking the disjunctive union of each author's list of requirements: Here, for a potential candidate to be logical it must be necessary *or* normative *or* formal *or* absolutely general / topic neutral *or* cognitively primal.
- Taking the conjunctive union of each author's list of requirements: Here, for a potential candidate to be logical it must be necessary *and* normative *and* formal *and* absolutely general / topic neutral *and* cognitively primal.
- Taking the conjunctive intersection of each author's list of requirements: Here, for a potential candidate to be logical it must be formal, given that this the only requirement shared by all authors.
- Taking the disjunctive intersection of each author's list of requirements: Here, for a potential candidate to be logical it must be formal, again due to formality's presence on all lists.

This list is ordered by descending liberality – the disjunctive union is the least restrictive simple approach to synthesis, in that it would tend to increase the cardinality of the set of the logical, whereas the conjunctive and disjunctive intersections are the most restrictive, and thus would tend to decrease the cardinality of the set of the logical.

Due to the importance of the synthesis of requirements to be performed (particularly in light of Strawson's objections to the Carnapian approach's move from explicandum to explicatum, see Section 2.1), it is worthwhile attempting to bring some more sophistication to the analysis. To do this, the links between each of the requirements suggested by the range of authors canvassed will be examined, to determine if any stand out as being more important, more fundamental, or more suited to acting as a final requirement than any of the simple approaches given above; or if any links of extensional agreement or synonymy can be found between them. The discussion will be undertaken in terms of examples of consequences as the objects which are potential candidates for logicality.

As will become clear during the discussion, this dissertation holds that the key requirement for logicality is formality. In terms of a definition for the purposes of this dissertation, formality in logic is the view that the validity of an argument depends only on its structure or form (as defined by grammatical particles called logical constants), rather than its content (which can be seen as defining its subject matter). Because of this, each of the other requirements are discussed in terms of their relationship to formality. The first three requirements considered in this way are cognitive primacy, normativity, and necessity. In each of these cases, it will be argued that if an example of consequence is formal (holds in virtue of its form or structure rather than its content), it will also have cognitive primacy, normativity, and necessity. This means that formality can retain the focus of the dissertation as the more fundamental requirement. It will further be argued that cognitive primacy, normativity, and necessity are also features of other, non-logical objects. Thus, although they may pick out certain interesting features of logicality, they do not represent the key requirements for logicality upon which criteria can be built.

2.2.1. Cognitive Primacy

The cognitive primacy requirement differs somewhat from the others in that its specification by Linnebo references logicality itself (since it uses the term *extra-logical*). This means that it operates 'downstream' of the others, in that any other requirements for logicality must be previously identified in order to assess whether it also holds. Cognitive primacy then further requires that the class of logical expressions can be understood without reference to anything outside the class, and thus that the class is cognitively self-contained.

This dissertation asserts that if an example of consequence is formal then it will also have cognitive primacy. The argument for this assertion is as follows: Assume that all examples of a certain type of inference are purely formal; that is, they hold in virtue of their form without any need for 'input' from the content of the inference. Acknowledging the point above regarding the recursive nature of this requirement, it will also be assumed that elements of an example of inference which correspond to its form or structure rather than its content do not have to be classified as extra-logical. Thus the lack of a need for input from the content of the inference (which represents the remainder of the elements of it once its form or structure has been identified) implies that an understanding that "doesn't depend on or involve an understanding of notions that must be classified as extra-logical"[25] is possible. Thus an inference which is a potential candidate for logicality's (that is, being a logical inference) being formal is sufficient for its cognitive primacy.

In addition to the reasoning above, it also seems to be the case that cognitive primacy is a characteristic of other things, not just examples of logical consequence. Logic's fundamental status with respect to the understanding complicates the task of providing an example of a field which can be understood in a way that does not depend on exterior notions – namely, logical notions themselves. Notwithstanding this point, it may be suggested that the truths of pure mathematics can be understood without a dependence on extra-mathematical notions. Assuming that logic and mathematics are two different fields (an issue which will be discussed elsewhere in this dissertation), this means that while cognitive primacy may be a characteristic of logicality, it is not exclusive to logicality.

According to the above discussion, an example of consequence's being formal is sufficient for its having also cognitive primacy. Also, some examples of consequence have cognitive primacy without being logical. Thus, while this dissertation agrees that cognitive primacy has some importance from the perspective of logicality, the notion of 'direct access' it involves is better

[25] Linnebo (2017). Section 3.

considered from the point of view of formality rather than cognitive primacy. This is because the stipulation that an example of consequence holds irrespective of its non-structural content provides a more fundamental insight into logic's directly accessible nature than cognitive primacy does.

The above discussion shows that consideration of cognitive primacy as a requirement for logicality can be dispensed with in this dissertation, with the discussion focussing instead on formality.

2.2.2. Necessity

This dissertation asserts that if an example of consequence is formal then it will also hold by necessity. A similar argument to that used in Section 2.2.1 will be used to show this, by showing that any example of consequence which holds for purely formal reasons will also hold of necessity. Making use of the concept of possible worlds, an example of consequence holding by necessity means that it holds in all possible worlds. Now, the evaluation of an example of consequence's content could potentially interact with states of affairs that constitute the variations across possible worlds. The evaluation of its form, on the other hand, would have no such interaction. Thus if an example of consequence held solely on the basis of its form, it would hold across all possible worlds; and therefore it would also hold by necessity. This implies that formality implies necessity.

The above reasoning could be objected to on the basis of Putnam (1968), where it is argued that developments in quantum physics cast doubt on principles of logic that have been accepted as fundamental. These notions are therefore revisable due to evidence from scientific observation, and thus it may be argued that logic is in fact empirical, since it would depend on what are taken as the best (in terms of their ability to describe the world) laws of nature.

However, the reasoning above is not intended to demonstrate that the fundamental notions of logic hold by necessity, but rather that any principle that holds for formal reasons also holds by necessity. If Putnam's point that these principles are actually empirical is correct, then they would not be formal either. Thus Putnam's argument is not a point against the claim that formality implies necessity, but rather against the claim that any examples of consequence are formal (and thus, according to the reasoning contained in this dissertation), not logical.

A different objection could be based on a suggested class of sentences which are in some sense universally true but not necessarily true which can be found in Kaplan (1979). Kaplan's discussion centres around sentences which involve demonstrative terms, such as 'I am here now'. Kaplan contends that such sentences are "deeply, and in some sense universally, true. One need only understand the meaning of ['I am here now'] to know that it cannot be uttered falsely"[26]. However, such a sentence also "rarely or never expresses a necessary proposition"[27]. The talk of meaning in the first quotation above shows that Kaplan

[26] Kaplan (1979). Page 82.
[27] Kaplan (1979). Page 84.

feels that the correct analysis of ‘I am here now’ is that it is analytically true but not necessarily true.

Does Kaplan’s reasoning present a counter example to this dissertation’s assertion that formality is fundamental to necessity? That is, does ‘I am here now’ represent a logical truth but not a necessary truth? The first thing to note is that this dissertation’s position is that truths are logical if their truth is guaranteed by their form. This is debateable in the case of ‘I am here now’. Such a sentence has the form Lxyz, or ‘x is at location y at time y’. On the face of this simple formalisation, this sentence is not true in virtue of its form. Of course, it is the indexicality of ‘I’, ‘here’, and ‘now’ that makes this a special case which is in some sense always true.

Zalta (1988) provides three reasons to doubt that ‘I am here now’ is a logical but not necessary truth. These are:

1. ‘I am here now’ requires a powerful logical apparatus, which includes indexicals among other elements. Such an apparatus is more powerful than that considered in this dissertation, and also more powerful that what is required to construct the definitions of logical and necessary truths.
2. ‘I am here now’ being a logical but not necessary truth requires accepting a certain metaphysical thesis, namely that there are no non-existent objects, or at least that these non-existent objects cannot make for real utterances. This is because if this thesis is accepted and ‘I’ denotes a non-existent

object, there will be interpretations and contexts where ‘I am here now’ will be false. While there being non-existent objects is controversial, the point remains that Kaplan’s examples require a certain metaphysical standpoint.

3. There are reasons to doubt even the analyticity of ‘I am here now’, since context must be appealed to in order to consider its truth. Thus ‘I am here now’ is no longer true simply in virtue of the meanings of the words it includes, but requires the context of their utterance. This is an expanded definition of analyticity compared to the ‘traditional’ definition.

For these reasons, a case can be made that objections based on ‘I am here now’ to the idea that if an example of consequence is formal then it will also hold by necessity can be rejected.

In addition to the reasoning above, it also seems to be the case that necessity is a characteristic of other things, not just examples of logical consequence. A potential counter example to sufficiency of necessity for logicality is ‘Water is H_2O’, which is discussed extensively in Kripke (1980). Here, this sentence is held to be necessarily true, but is knowable only via empirical investigation. This precludes its logicality.

According to the above discussion, an example of consequence’s being formal is sufficient for its being necessary. Also, some examples of consequence have necessity without being logical. Thus, while this dissertation agrees that necessity has some importance from the perspective of logicality, the notion of holding under all conditions it involves is better considered from the point of view of formality rather than necessity.

The above discussion shows that consideration of necessity as a requirement for logicality can be dispensed with in this dissertation, with the discussion focussing instead on formality.

2.2.3. Normativity

This dissertation asserts that if an example of consequence is formal then it will also be normative. A similar argument to that used in Sections 2.2.1 and 2.2.2 will be used to show this, by showing that any example of consequence which holds for purely formal reasons will also be normative.

The link between formality and normativity is perhaps harder to establish than those discussed above. To do so, it is necessary to provide some maxim which allows an inference to be drawn regarding the normative nature of logic (how agents *ought to* reason) from the factual nature of the logical (how logic *is*). To facilitate this, some of the reasoning contained in MacFarlane (2004), which draws a link between the requirements of formality and normativity via the notion of transparency, will be utilised. MacFarlane writes:

> ...why is it important that logical validity be transparent in this way? I would like to suggest that it is important because of the normative implications of logical validity ... we require logical validity to be formal because we require

> it to be transparent, and we require it to be transparent because of the reasons and responsibilities to which it gives rise.[28]

The use of the term 'transparent' in the above quotation is unclear. It may be possible to equate it with 'obviousness', and thus for an example of consequence to be logical it must obviously hold. This may seem too strong a requirement; examples involving multiple, complex premises may hold, but may not be obvious. It could be argued that they may however be obvious when considered in a 'step by step' manner, where each step in for example a proof is 'obvious'. However, this still retains a certain subjectivity it may be wise to avoid. Other candidates include transparent examples of consequence which are knowable a priori or result proof theoretically from axiomatizable formal systems. Both of these options will be considered in the following reasoning.

While MacFarlane hesitates to endorse this line of reasoning, it can form the basis for an argument for the fundamentality of formality with respect to normativity. The reasoning here is as follows: Assume that a given example of consequence is logical, and thus that it holds due to its form and independent of content. Given this reliance on form, that the consequence holds should, as is pointed out by MacFarlane, be transparent. That is, if transparency is interpreted as a priori knowability, then the lack of any reliance on content should prevent the fact that the consequence holds from being obscured by empirical considerations which

[28] MacFarlane (2004). Page 20.

may not be clear through inspection of the example of consequence itself (note also in this context the relationship between formality and cognitive primacy discussed in Section 2.2.1). If transparency is interpreted as axiomatisability, then the connection is admittedly less clear. In this case, formality, and thus logicality, would be restricted only to axiomatizable systems.

Consider now the following proposed maxim: If a consequence holds transparently (in the way discussed in the above paragraph), then belief in it ought to be assented to (or, in the words of Beall and Restall, anyone who rejects it somehow goes wrong). This maxim seems reasonable – if it is clear that an example of consequence holds, there would not seem to be any reason to deny it. Furthermore, if this maxim is accepted, then formality can be seen to be fundamental to normativity: If an example of consequence holds due to its form rather than its content, then it holds transparently; and if it holds transparently, it ought to be assented to. A further way of seeing the issue (which includes the semantic notion of truth, though not illegitimately since such semantic notions can be called upon when dealing with normative issues) is that if a consequent holds due to its form, then if the premises are true, the conclusion is true; therefore if one accepts the premises, one ought to accept the conclusion.

In addition to the reasoning above, it also seems to be the case that normativity is a characteristic of other things, not just examples of logical consequence. Any example of a thesis whose normativity is owed to a certain field of endeavour or subject matter rather than the most general principles of reasoning provides a counter example to the sufficiency of normativity to logicality. Also, if logicality is restricted to the deductive case, then the principles of abductive reason can be considered normative, but not logically true.

The above discussion shows that consideration of normativity as a requirement for logicality can be dispensed with in this dissertation, with the discussion focussing instead on formality.

2.2.4. Topic Neutrality

This dissertation asserts that if an example of consequence is formal then it will also be topic neutral. A similar argument to that used in Sections 2.2.1 to 2.2.3 will be used to show this, by showing that any example of consequence which holds for purely formal reasons will also be topic neutral.

The argument for this assertion is as follows. Assume again that all examples of a certain type of consequence hold purely due to their form. Assume also that form and matter can be entirely distinguished from each other, to which Linnebo (2017) in fact does assert (though noting his comments regarding how this distinction is understood). If so, it seems reasonable to hold that those elements of an example of consequence which are formal contribute nothing to the subject matter of the consequence; and this subject matter can be reasonably maintained to be rather contributed (or in fact to be made up of) by its content rather than its form or structure. Thus any example of consequence which holds due to its formality would also hold irrespective of subject-matter, and so a potential candidate for logicality's being formal is sufficient for its topic neutrality[29].

However, in contrast to the cases of cognitive primacy, necessity and normativity, this dissertation asserts that if an example of consequence is topic neutral, it is also formal. This means that topic neutrality and formality are co-extensive across examples of consequence. The argument for this is as follows. Assume that an example of consequence is topic neutral. If so, it must not be the case that it holds due to any specific content referring to elements other than those required to define the structure of the consequence. This is because any such non-structural content would commit the example of consequence to a certain topic and compromise its topic neutrality. The example of consequence holding due only to its structural elements means that it holds due to its form, and hence it holds because it is formal.

The above discussion shows that consideration of topic neutrality as a requirement for logicality could be dispensed with in this dissertation, with the discussion focussing instead on formality. However, this dissertation will at times also refer to topic neutrality, since its co-extensivity with formality means that in certain cases, it can provide useful alternative insight into logicality.

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[29] To the potential objection that the topic of logic is not null, but rather has the context expressed by 'not', 'if-then', etc. it can be replied that these are *interpretations* of the symbols of a logical system.

2.2.5. Absolute Generality

This dissertation asserts that if an example of consequence is formal then it will also be absolutely general. A similar argument to that used in Sections 2.2.1 to 2.2.4 will be used to show this, by showing that any example of consequence which holds for purely formal reasons will also be absolutely general.

There is significant similarity between Linnebo's requirement of absolute generality and Haack's requirement of topic neutrality. Thus similar reasoning to that applied above can therefore be used to argue that formality is also fundamental to absolute generality: Assume an example of consequence holds for purely formal reasons, and the non-structural content of the elements in the example of consequence do not contribute to its holding. Given that any specificity introduced into an example of consequence must come from its content rather than its form or structure, the lack of content means that there is no means by which the consequence's scope of application can be narrowed, and thus nothing to compromise its lack of general application. Thus, purely formal examples of consequence are "valid in any kind of discourse, no matter what kind of objects this discourse is concerned with"[30], and are thus absolutely general.

However, in contrast to the cases of cognitive primacy, necessity and normativity, and similar to the case of topic neutrality, this dissertation asserts that if an example of consequence is absolutely general, it is also formal. This means that

[30] Linnebo (2017). Section 3.

absolute generality and formality (and topic neutrality) are co-extensive across examples of consequence. The argument for this is as follows. Assume that an example of consequence has an absolutely general field of application. If so, it must lack content other than that required to define the structure of the example of consequence, since any specific non-structural content introduced into it would reduce its field of application from being absolutely general to being specific in some way. The example of consequence holding due only to its structural elements means that it holds due to its form, and hence it holds because it is formal.

The above discussion shows that consideration of absolute generality as a requirement for logicality could be dispensed with in this dissertation, with the discussion focussing instead on formality. However, this dissertation will at times also refer to absolute generality, since its co-extensivity with formality means that in certain cases, it can provide useful alternative insight into logicality. This is particularly the case when considering the interpretations of elements of formal systems. For example, the propositional connective $\wedge$ contributes only the form or structure of an inference (satisfying the formality requirement) and so is a logical constant; its interpretation, ‘and’, is absolutely general and topic neutral, and so is informally logical.

Before moving on to some concluding remarks regarding the requirements for logicality, it is worth pointing out that the discussion in Sections 2.2.4 to 2.2.5 imply that, due to the co-extensive relationship with formality of both topic neutrality and absolute generality, these two notions are also themselves co-extensive. This appears to be entirely reasonable, since both topic neutrality and absolute generality are characterised by the absence of any specific subject matter.

Finally, a potential objection to the notion of absolute generality being included as a requirement for logicality is that the concept of absolute generality is itself incoherent. Extensive debate of this point is beyond the scope of this dissertation. It is, however, sufficient to note that quantification over absolutely everything is supported by certain philosophers, including in Williamson (2003). This dissertation thus proceeds under the assumption that absolute generality is indeed a coherent position.

2.2.6. Concluding Remarks

The conclusion based on the above analysis is that the fundamental and key requirement for logicality for the five authors surveyed is formality. Not only is it cited by all five authors, but it appears to be either fundamental or equivalent to the other requirements cited – if an example of consequence satisfies the formality requirement, it satisfies all other requirements also.

Formality also represents a convenient synthesised requirement upon which to base precise criteria for logicality, since it reduces the vagueness associated with the diverse conceptions of the requirements of logic across different authors (including those not surveyed above) by focussing them on a single central requirement. The full extent to which moving to formality permits precise determination of the logical / non-logical divide through the production of criteria corresponding to it is one of the principal concerns of this dissertation. Given the above reasoning, this dissertation will proceed on the basis that formality will act as the key requirement for logicality, for which proof theoretic criteria will be sought.

That formality does in fact facilitate this assessment becomes clear when deciding upon a methodology for assessing logicality. Using formality to develop a strategy to approach the problem of providing criteria for logic permits the following reasoning: If an example of consequence holds after its elements which are *non-structural* – that is, the elements which contribute to the example of consequence's form rather than its content – have been replaced by variables or individual constants, then the example of consequence in question is logical. If this process removes information which is necessary for the consequence to hold, then it is the content of the consequence rather than its form which drives it; and thus it is a non-logical consequence[31]. Adopting this strategy to assessing logicality also represents a means of adding precision to the requirement of formality, in response to Beall and Restall's point regarding formality's disparate nature.

What is needed then is a methodology to distinguish the elements in each example of consequence which correspond to its structure (form) and those which

[31] Such a strategy can be found in Quine (1986), page 59: "a logical truth is a truth that cannot be turned false by substituting for lexicon".

correspond to its content. Given that their meaning does not change across schematised examples of consequence, elements which are retained after schematisation are referred to in the literature as *logical constants*. Thus, if formality is the key to logicality, the task of providing criteria for logicality reduces to that of finding a means of identifying logical constants. This approach has been adopted by various thinkers attempting to address the question, and it is the one which will also be followed in this dissertation[32]. The link between the logicality of examples of consequence and logical constants will be revisited in Section 2.3.

This section of the dissertation also argued that absolute generality and topic neutrality are co-extensive with formality. This is useful because if an element is formal (contributes only to the structure of inferences in which it appears), then its interpretations can be expected to be absolutely general and topic neutral. The utility here is that assessments of logical constancy for elements of formal systems can be compared to absolute generality and topic neutrality of the interpretations of these elements. Agreement of the logical constancy of a formal element and the absolutely generality / topic neutrality of its interpretation thus gives confidence that the criteria used to assess logical constancy are correct.

[32] There has also been pessimism expressed regarding possibility of a satisfactory resolution to this problem, for example in Gomez-Torrente (2002), which states (page 1) that “most conceptions of the problem of logical constants involve requirements of a philosophically demanding nature which are probably not satisfiable by any minimally adequate theory”.

Before moving on, it is worth pausing to respond to a potential objection. The above reasoning has identified formality, absolute generality and topic neutrality as the key requirements for logicality, meaning the purpose of pursuing proof theoretic criteria based on formality may be queried – in other words, why expend the intellectual labour on producing criteria if a fundamental requirement has already been identified? The response to this can be based on the desiderata for criteria for logicality developed in Section 2.1. That is, while attempting to sort the logical from the non-logical based only on formality is clearly advantageous in terms of relevance, it is of insufficient precision.

That this is the case is demonstrated by the existence of various examples whose logicality is controversial, some of which are the subject of long running disputes. There are two notable examples which will be discussed in some detail in this dissertation:

- Second order quantification. While the status of first order quantification (quantification over objects) as logical is not controversial, that of second order quantification (quantification into predicate position) has been the subject of a long running dispute, with key contributions made in for example Quine (1986) and Boolos (1975).
- Modal operators: The alethic notions of necessity and possibility also appear to have a certain claim to absolute generality / topic neutrality, and thus the modal operators which formalise them are worth investigating for logicality. However, how their logicality can be maintained while other

modal notions (formalised in epistemic, deontic, and other (at least so-called)) logics are more doubtful requires explanation.

Beyond justifying the importance of demanding criteria over and above requirements for informal logicality, these two examples will be investigated in more detail once proposed criteria for logical constancy have been developed. In fact, the opportunity to adjudicate on their logicality using the criteria developed represents a key methodological point of difference between the present project and the way such questions are typically addressed. Taking second order quantification as an example, the debate around its logicality typically centres on (for the ‘opponent’ of second order quantification) the purported ontological commitment involved in second order quantification (put very simply, that formal systems including second order quantification reify properties) and its links with the purportedly ontologically costly set theory; put against (for the supporters of the logicality of second order quantification) its structural similarity with first order quantification and its seeming necessity to improve logic’s mathematical utility.

In contrast, the approach taken here will be to develop criteria for logical constancy based on examples whose logicality is much less controversial (such as those involving only propositional and first order logic), thus allowing their ‘calibration’ according to known results. These calibrated criteria can then be applied to the controversial cases given above, with confidence that they are known to provide correct results for other, less controversial candidates. This avoids having to delve into complex debates regarding ontological commitment and the like as detailed above, and instead will (if calibrated criteria can be successfully produced) represent an advancement on the debate by working from more solid foundations.

2.3. Bearers of Logicality

The discussion in Section 2.2 shows that logicality can be a property of a variety of classes of objects. This section of the dissertation discusses a range of these. It will be useful here to recall the distinction between informal logicality and logicality *tout court*, made in Section 2.1: Informal logicality is a property of concepts in natural language (for example, necessity) and according to the arguments of the previous section, informal logicality equates to absolute generality and topic neutrality. On the other hand, logicality is a property of formal systems (for example, the modal logic system S5) or elements of logical systems (for example, $\Box$). Concepts in natural language can be interpretations of elements of formal systems, elements of formal systems can be formalisations of concepts in natural language.

First, logicality can be a property of certain formal systems. Following Shapiro (1991), it will be assumed that for a formal system to be considered a candidate for logicality, it must consist of a symbolic language (including an alphabet of symbols and grammatical rules for constructing well-formed formulas from those symbols), and either or both of a proof system and a set of semantic definitions.

Due to the importance of the requirement of formality discussed in Section 2.2 and in further detail below, formal systems which are logical would then be those which contain only elements which are logical. The discussion of formal systems in terms of logicality is a popular approach and is used in relatively well-known works such as Tharp (1975) and Shapiro (1991).

Second, logicality can be proposed as a property of those sentences which are true under any interpretation of the formal system. Holding that a certain truth is logical then amounts to holding that its truth is assured as a matter of logic alone. The notion of logicality with respect to truths is typically tied to other aspects regarding the metaphysical or epistemological status of truths, such as necessity or universality – that is, those requirements discussed in Section 2.2. Logical truths are contrasted with examples whose truth is assured by non-logical means, such as metaphysical truths (which may be necessary but not logical), physical truths (those assured by the laws of physics), and empirical truths assured by the accidental features of the world. Truths expressed in natural language which are the interpretation of logical truths expressed in formal language are themselves informally logical.

Third, logicality is a characteristic of certain examples of the consequence relation. Consequence for current purposes is a relation which holds between premises (a (possibly empty) set of propositions) and the conclusion (a single proposition). The consequence relation implies that the conclusion in some way holds given, or because of, the premises; or that the conclusion follows in some way from the premises. In the case of logical consequence this relation holds due only to logic itself. Consequences expressed in natural language which are the interpretation of logical consequences expressed in formal language are themselves informally logical.

Fourth, logicality is a characteristic of certain elements of formal languages. Building on the discussion in the previous section, language elements which are ascribed the property of logicality are those which contribute to the form rather than the content of sentences in which they appear. These elements are commonly referred to as logical constants. As will become evident through subsequent discussion in this dissertation, this will be the most useful means of analysing logicality via proof theoretic methods and is the principal subject of investigation in this study.

There are various connections between these four uses of the term ‘logic’ as an adjective. Logical truth is a special case of logical consequence; that for which the premises are the empty set, meaning that the conclusion holds in all cases. The converse, however, does not in general hold, since for non-compact systems (those which include examples of consequence for which an infinite set of premises are required for the consequence to hold), examples of consequence cannot be converted to examples of truth. This means that regardless of any specific importance they are seen to have among other examples of logical consequence, the study of logicality for truths can be subsumed under the study of logicality for examples of consequence; consequence is the more fundamental notion.

There is also a close connection between the logicality of examples of consequence and that of formal systems. One way of casting the relationship between them is to hold that formal systems which purport to be logics are intended to provide proofs of (in the case of systems consisting of a language and a proof system) only those examples of consequence which hold for logical reasons. This analysis treats consequence as fundamental, with formal systems being constructed with the intention of validating or proving those consequences which are in maximal extensional agreement with the consequences which hold for logical reasons. Alternatively, fundamentality could be bestowed on formal systems by holding that those examples of consequence which are logical are so because they form part of the set of provable consequences generated by formal systems which are bona fide logics. Put simply, if all the operators (that is, the elements excluding meta logical symbols such as $\vdash$ and $\vDash$ and names for constants and variables) which make up the formal system are logical constants, then the system itself is a logical system. Then, the logicality of any new symbol which is added to the formal system can be assessed (using the criteria developed in this dissertation) by examining its natural deduction introduction and elimination rules (which the symbol requires to have meaning, according to the terms of inferentialism, discussed later in this dissertation). This allows evaluation of whether the logicality of the formal system is maintained through the addition of further symbols for new operators.

Investigating the logicality of the consequence relation rather than the logicality of formal systems seems attractive because while the precise nature of the consequence relation is far from immediately clear, that it does have an objective nature is more plausible than holding that constructed formal systems do. However, the latter conception does have important methodological advantages, due to the metalogical properties of formal systems being relatively clear and precise, which could in turn facilitate the production of precise criteria for logicality. By contrast, the nature of the logical consequence relationship itself is perhaps harder to analyse using existing logical tools.

The logicality of elements of formal systems, and that of formal systems themselves, is also closely linked, since if all the operators (that is, the elements excluding meta logical symbols such as $\vdash$ and $\vDash$ and names for constants and variables) which make up the formal system are logical constants, then the system itself is a logical system. The converse also applies – any system classified as logical should be expected to only contain logical constants among its elements.

Mention should also be made of the nature of the elements which make up the premises and conclusion of examples of consequence. Three options for this are considered:

- Sentences: The premises and conclusion of examples of consequence are linguistic objects, namely the specifically worded sentences which appear on the page when the example of consequence is written; or the specific words spoken when it is uttered.
- Propositions: The premises and conclusion of examples of consequence are propositions, one potential interpretation of which is that they are roughly the meanings of the sentences used when expressing the example of consequence.
- Schemata: The premises and conclusion of examples of consequence can be seen as ‘placeholders’ for, or which can be replaced by, either specific instances of propositions or sentences.

The advantages and disadvantages of employing propositions as an additional ontological entity over and above sentences has been debated extensively (see McGrath and Frank (2018)). However, in the present case, the elements which make up examples of consequences will be assumed to simply be sentences. There seems to be little advantage for the analysis of consequence and logicality undertaken in this dissertation in postulating the existence of propositions. This means that the labour associated with defending propositions will be strategically avoided by taking the elements of examples of consequence at face value, as sentences.

There are also other potential concerns which can be side-stepped by avoiding propositions. For example, the structure of propositions is potentially unclear, while that of sentences is evident, since it is defined by the grammar of the language in question. This structural clarity is particularly beneficial for the present study given the structural schematisation-based methodology suggested in the previous section. Also, given that investigations of logicality take place within the sphere of formal languages rather than natural languages, there is no need to invoke propositions as a means of explaining what links two sentences which have the same meaning (express the same proposition). Here, formal semantics are used to give an account of the 'meaning' of a sentence in a formal system.

Similar comments regarding ontological concerns can also be directed at schemata. The ontological status of schemata is complicated by their complex nature – each consists of the schema template itself (the syntactic string consisting of the schema's symbols); but also a side condition which specifies for example what objects can replace the schema's placeholders[33]. While the schema template is simply a string of characters, it has the same ontological status as sentences. However, the side condition is an intensional entity which is comparable to a proposition. As such, in order to follow the strategy of the dissertation's avoiding any controversy regarding propositions, sentences are again preferred to schemata.

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[33] Corcoran (2016).

Summarising the above discussion, the following points are key:

- Formal systems which are logical are those which consist of only elements which are logical.
- Consequences which are logical are those which, after abstraction of all contentful (that is, non-formal or non-structural) elements, contain only elements which are logical.

The above reduction of the problem of identifying the logicality of both formal systems and examples of consequence to that of identifying the logicality of the elements which formal system contains means that the focus of the dissertation should be on these elements. This has the advantage of limiting the number of candidates for logicality which must be assessed – the set of examples of consequence in a formal system can be countably infinite, but the set of fundamental elements such as connectives, quantifiers, etc. it contains is finite. This approach is supported by a review of literature on the subject, since the identification of the logicality of elements of formal systems is a well-known topic, known as the problem of logical constants. The key objective of this dissertation is to solve this problem by finding proof theoretic criteria for logical constancy.

Seeking criteria for logical constancy also aligns with identification of formality as the key requirement of formality for logicality in general. This is because logical constants are the elements of an example of consequence which dictate its form or structure, and hence determine whether an example of consequence holds formally / structurally; or whether it requires some non-formal elements. Thus, even if consequence is held to be the fundamental *bearer* of logicality, if formality is accepted as the fundamental *requirement* for logicality, then logical constancy remains key.

2.3.1. A Note on Parsimony

In addition to the technical requirements discussed above, a further potentially desirable characteristic applicable to the overall set of logical constants is parsimony. Seeking parsimony among the set of logical constants is equivalent to minimising the total number of constants admitted as logical, perhaps by showing that certain constants are definable using others, rather than being primitive. In contrast to the other requirements investigated in this section, parsimony would operate as a 'meta-requirement' on the set of constants as a whole, rather than on the basis of each's claim to logicality.

A diverse range of philosophers from Aristotle to Kant[34] have championed simplicity, and thus parsimony, as an advantage in theorising. However, in the case of this dissertation's criteria-based approach to logical constancy, parsimony does not have a role to play. This is because a lack of parsimony among the set of logical constants does not undermine the claims that any single logical constant has to formality and absolute generality / topic neutrality, and thus to logicality. For this reason, no attempt will be made in this dissertation to exclude any candidate

[34] See Baker (2016), Section 1.

for logical constancy on the basis of parsimony, for example due to its being expressible in terms of other constants, or its lack of deductive utility.

2.4. The Border between the Logical and the Non-Logical

Underlying the search for requirements and criteria for logicality is the assumption that the set of examples of the logical can be clearly demarcated from the set of examples that are non-logical (noting that effectively demarcating the set of the logical will also demarcate the set of the non-logical, since the latter consists of simply those examples which are not logical). Again with the aim of properly orientating the present project, this section of the dissertation takes up questions regarding the nature of each of these sets and the border between them. It involves the following questions:

> Q1: Do the criteria developed allow for variation in the degree of logicality between instances of the logical? Are some instances deemed to be, for example, entirely logical while others only just achieve logicality?
>
> Q2: Do the criteria developed provide a sharp border between the logical and the non-logical, without the existence of any borderline cases which resist classification?
>
> Q3: Do the criteria developed allow for variation in the degree of logicality between instances of the non-logical? Are some instances deemed to be, for example, entirely non-logical and others only just fail to achieve logicality?

The analysis of these questions and the arguments developed around them are intended to show that the answers to them have a bearing on the nature of potential criteria for logicality. Specifically, the aim of this section will be to develop and defend the following hypothesis:

> SB: For the criteria developed to provide a sharp border between the logical and the non-logical, they should not allow for any variation in the degree of logicality between instances of the logical; and they should not allow for any variation in the degree of logicality between instances of the non-logical.

That is, criteria capable of providing an affirmative response to Q2 should provide a negative response to Q1 and to Q3. That an affirmative response to Q2 is desirable is suggested by the discussion in Section 2.1, where, along with relevance, precision (the ability to clearly determine the logicality of each potential candidate) was identified as a key desideratum for criteria for logicality. In the absence of precise criteria, the resulting borderline cases which resist classification suggest that the criteria developed do not represent a great advancement compared to knowledge simply of Section 2.2's informal requirements for logicality, rather than the criteria sought in this project.

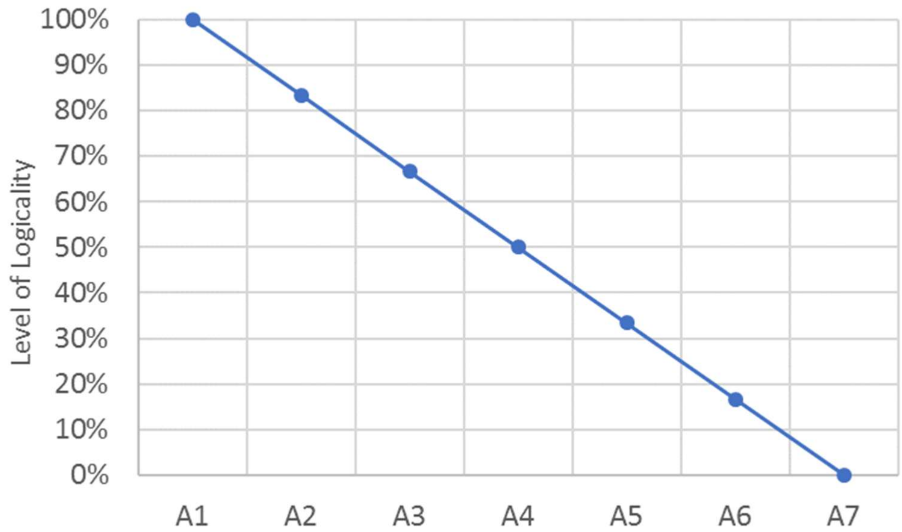
However, it is not universally held that a sharp border amenable to the production of precise criteria exists for logicality. The following quote is taken from Shapiro (1991):

> In particular, Quine holds that there is no sharp border between logic and mathematics on the one hand and, say, physics on the other. He points out that one cannot understand physics without mathematics, and in accepting physics one is accepting the ontology of much mathematics. In this book, Quinean holism is extended to logic itself. There is no sharp border between mathematics and logic, especially the logic of mathematics. One cannot expect to do logic without incorporating some mathematics and accepting at least some of its ontology.[35]

In order to assess SB, three different situations will be considered. These situations describe hypothetical criteria for logicality which provide different response combinations to Q1, Q2, and Q3. Their purpose is to show, in line with SB, that it seems unreasonable to hold that criteria which purport to describe a sharp border between the logical and the non-logical can provide an affirmative/negative/affirmative set of responses to Q1/Q2/Q3. To do this, an artificial, though informative, method of graphical analysis of the logicality of examples of inference denoted A1 to A7 will be used. The statement of these examples of inference is unimportant, the reader is simply requested to accept that A1 is definitely logical, A7 is definitely not logical, and the 'level of logicality' descends from A1, through A2, A3 and so on until A7. The nature of this descent in the level of logicality is the subject matter of the remainder of this section.

[35] Shapiro (1991), page vi.

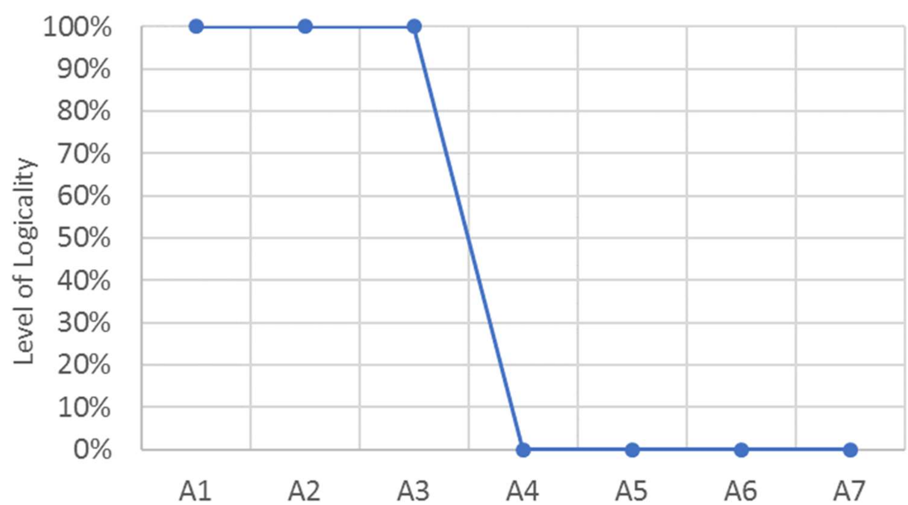
Consider first the following graph:



This represents the set of responses affirmative/negative/affirmative to Q1/Q2/Q3. Thus, the hypothetical criteria assessed here permit variation in the level of logicality within both the logical (those points at the left-hand side of the graph) and the non-logical (those points at the right-hand side of the graph); and also do not provide a sharp border between the logical and non-logical. The artificiality of the graph is in the 'logicality percentage' given in the y-axis, since typically, criteria (such as the proof theoretic criteria discussed later in this dissertation) do not provide a means of precisely assessing an example of consequence's score in this way, and it is unrealistic to suggest that they could. However, for the purposes of this discussion, this represents a fruitful analysis tool.

That a proposed set of criteria should produce an analysis of logicality represented in this graph is a reasonable outcome. Logicality progressively declines through both the logical and non-logical, and also in the potential borderline cases represented by the points around the middle of the graph. This progressive decline over borderline cases implies that the proposed criteria do not provide a sharp border between the logical and the non-logical – it would appear arbitrary to claim that such a border exists between, for example A3 and A4 or A4 and A5, due to the relatively small decline in the level of logicality between them.

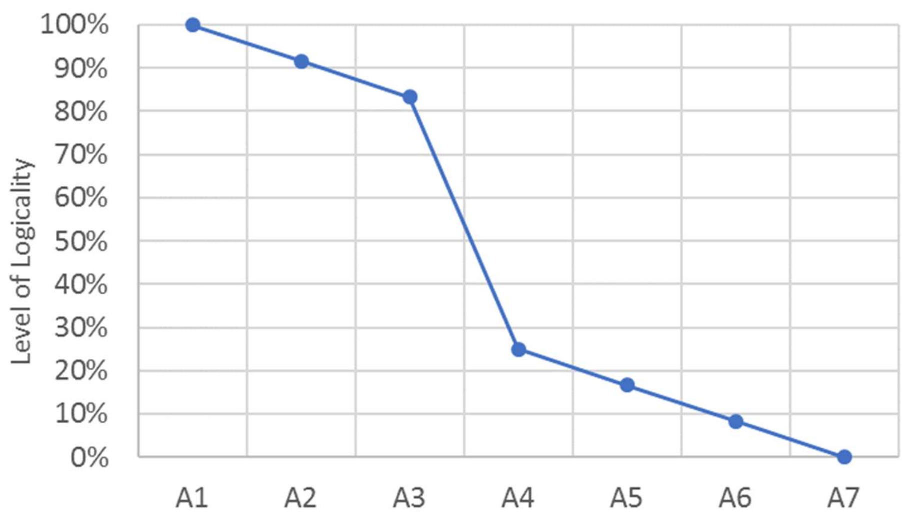
Consider next the following graph:



This represents the set of responses negative/affirmative/negative to Q1/Q2/Q3. Thus the hypothetical criteria here do not permit variation in the level of logicality within either the logical (those points at the left side of the graph) and the non-logical (those points at the right-hand side of the graph); but they do provide a sharp border between the logical and non-logical.

Like the previous case, that a proposed set of criteria should produce an analysis of logicality represented in this graph is a reasonable outcome. Logicality is clearly possessed by examples of consequence A1 to A3, but not by examples A4 to A7. Thus the existence of a sharp border between the logical and the non-logical is clear, and is not compromised by any borderline cases.

However, consider the following graph:



This represents the set of responses affirmative/affirmative/affirmative to Q1/Q2/Q3. Thus the hypothetical criteria here permit variation in the level of logicality within the logical or the non-logical, but also purport to provide a sharp border between the logical and the non-logical.

In contrast to the previous two cases, this does not represent an outcome which could be reasonably expected from produced criteria. While the above graph suggests that a sharp border exists between A3 and A4, that there should be a gradual decline in the level of logicality among cases of the logical, followed by a conveniently placed steep decline, followed by a further gradual decline among cases of non-logicality does not seem plausible. If variation can exist in the level of logicality within the logical and the non-logical, it seems more reasonable that this incremental decline would continue across the gap between the two cases and across a series of borderline cases – that is, as per the first graph considered. For example, there could be a case A3.5 which modified case A3 to somewhat weaken its claims to logicality (or modified case A4 to somewhat strengthen its claims to logicality), allowing it to be placed in between the two in terms of its level of logicality. This would then represent a borderline case, compromising the proposed criteria's claims to providing a sharp border. This situation can be contrasted with the second graph considered. The proposed criteria's 'all or nothing' assessment of logicality does not suggest that modifications could be made to the examples under consideration which would put them on the borderline between logic and non-logic. Here, in this second graph, it seems reasonable to hold that the existence of such a 'case A3.5' could be denied.

The three cases examined above support hypothesis SB by showing that variations predicted by proposed criteria in the level of logicality in the logical and non-logical would seem to bleed across into the logical/non-logical divide. Naturally, this does not represent a deductive argument for SB. That a situation like that given in the third graph could exist is possible; the conclusion here is that is less plausible than the first or second graph. To put it in the terms of this dissertation ‘x is a criterion of logicality which permits variation in the level of logicality within the logical and non-logical; therefore, x does not provide a sharp border between them’ is not a logical consequence. It is, however, supported by the graphical analysis above. Thus the maxim that if criteria are desired which provide a sharp border *between* the logical and the non-logical, then any variation in their assessment of the level of logicality *within* the logical and the non-logical should be avoided.

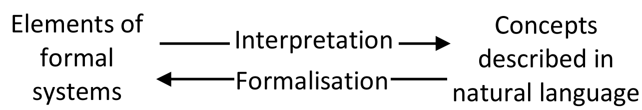
2.5. Terminological Notes

It is important to reiterate an important terminological distinction that will appear throughout the subsequent sections of this dissertation. The preceding sections of this dissertation have established that the property of logicality can be attributed to examples of inference, specifically inferences which hold on the basis of form rather than content. Given the importance of formality to logicality, identifying those examples of inference which are logical reduces to a large extent to identifying a set of elements which determine its logical structure or form, with this set known as logical constants.

Logical constants are elements of formal languages. Well known candidates for logical constancy include $\vee$ and $\forall$, the claims for logicality of both of which will be investigated later sections of this dissertation. In many cases, elements of formal languages have relatively standard and well-known translations into elements of natural language – if asked, most who are familiar with first order logic could be reasonably expected to reply that they can be translated into ‘or’ and ‘all’ (or ‘every’) respectively.

The translation between formal element and element of natural language is not entirely straightforward (examples include: ∨ translates to ‘inclusive or’ rather than simply ‘or’, the latter term encompassing either inclusive or exclusive meanings, depending on context; and → fails to capture the nuances of natural language if-then statements, since it only models the material conditional, meaning that statements with false antecedents are always true). However, it will in any case be convenient to label some elements of natural language as logical also, namely those for which the translation into a formal element satisfies the criteria for logicality. Where confusion may arise, such elements of natural language will be called *informally logical*, and as per the discussion in Section 2.2, these correspond to those describing concepts which are absolutely general and topic neutral. Thus, elements of formal systems are either logical or non-logical; concepts described in natural language are informally logical or informally non-logical.

The term ‘translation’ is used in the previous paragraph. Essentially, this is a two-place relation between an element of formal language and an element of natural language whose meaning is the same. However, it will be useful at various points in this dissertation to distinguish the two directions of translation. Specifically, it will be said that an element of formal language is a *formalisation* of an element of natural language, and an element of natural language is an *interpretation* of an element of formal language:



The notions of formalisation and interpretation are important when considering what is at stake when criteria for logicality are established. It is an important task of both formal and natural languages (at least when interpreted) to describe the world. Formal languages, due to their less ambiguous and more regimented nature, permit clearer analysis in terms of issues such as logicality, while due to their greater familiarity through day-to-day use, natural languages present a clearer connection to the concepts in the world for which descriptions are sought.

This suggests two potentially fruitful methodologies:

1. Investigate natural language to find terms describing absolutely general and topic neutral concepts. That these concepts possess these characteristics suggests that they can be formalised as logical constants. Determine the natural deduction rules for these potential constants and develop criteria (or apply criteria developed in this dissertation as it progresses) to assess them. Similar comments apply for natural language terms which do not describe absolutely general and topic neutral concepts. The formalisations

of these would not be expected to be logical constants, so criteria for logical constancy can be developed which exclude them.

2. Analyse the elements of formal language and use the criteria developed in this dissertation to evaluate their logical constancy. Then investigate the interpretations of these potential logical constants. If these interpretations are concepts which are absolutely general and topic neutral, modify the criteria developed to include them as logical constants; if they are not absolutely general and topic neutral, modify the criteria to exclude them.

These methodological approaches can be used in relatively clear-cut cases, and then the criteria developed applied to more contentious cases of logical constancy.

Turning to formal languages themselves, some notes regarding nomenclature are provided below.

- Lower case letters from the start of the roman alphabet (a, b, c, …) are individual constants.
- Lower case letters close to the end of the roman alphabet (starting at t, u, v, …) are first order variables.
- Upper case letters from the start of the roman alphabet (A, B, C, …) are predicates.
- Upper case letters close to the end of the roman alphabet (starting at T, U, V, …) are second order variables.

- Lower case letters from the Greek alphabet (φ, ψ, χ, …) are metalinguistic variables ranging over object-language formulas, with a subscript to denote a longer series of formulas where appropriate (for example: φ_1, φ_2, φ_3, …).
- Upper case letters from the Greek alphabet (Γ, Δ, …) are sets of formulas.

2.6. Logical Pluralism

Put crudely, logical pluralism is the view that there is more than one type of logical consequence relation, or more than one type of inference which can be classified as logical; and thus more than one type of formal system which can correctly be called logic. To provide a more precise definition of logical pluralism for discussion in this section, consider the following statements from Beale and Restall (2006):

> A valid argument is one whose conclusion is true in every case in which all its premises are true. We hold that deductive validity is a matter of the preservation of truth in all cases. An argument is valid when there is no counterexample to it: that is, there is no case in which the premises are true and in which the conclusion is not true [36]

> Generalised Tarski Thesis (gtt): An argument is valid$_x$ if and only if, in every case$_x$ in which the premises are true, so is the conclusion.[37]

[36] Beale and Restall (2006). Page 23.

[37] Beale and Restall (2006). Page 29.

> Logical pluralism is the claim that at least two different instances of gtt provide admissible precisifications of logical consequence. Unlike the restricted Tarski Thesis, which admits only one instance of $case_x$ (Tarski's models), the pluralist endorses at least two instances. This gives rise to two different accounts of deductive logical consequence (for the same language), two different senses of 'follows from'.[38]

In contrast to the proof theoretic approach taken in this dissertation, the above definitions are given model theoretically, in terms of truth preservation. As such, the remainder of the discussion in this section will be discussed from a model theoretic point of view, but in the context of the set of logical constants.

Logical pluralism can manifest itself in different ways. One is via *extensions* to logic. Here, a core set of logical constants are accepted, and then a further set or sets of logical constants have perhaps claims to logicality in different contexts, or under different interpretations of the logical consequence relation. An example here would be acceptance of propositional logical constants and the logical constants included in first order quantification, with contextual acceptance of the logical constants of second order quantification. Another is via *alternative* logics. Here, there may be a choice between two candidates for logical constancy, both of which claim to represent formalisations of a certain concept. Examples here include choosing between intuitionistic or paraconsistent logics and classical

[38] Beale and Restall (2006). Page 29.

logic[39]. The pluralist may hold that in certain contexts the classical formalisation is appropriate, whereas in other contexts the intuitionistic formalisation should be used – but that their use in both cases represents a legitimate treatment of logical consequence.

In terms of extensions to logic, the approach taken in this dissertation does not align with the description of logical pluralism given above. The operators involved in each extension to for example propositional logic can be assessed using the criteria developed and accepted or rejected on that basis. There is no room for contextual interpretation; given the precision desiderata discussed in Section 2.1, potential logical constants are clearly assessed using the produced criteria.

However, in the case of alternative logics, the approach taken in this dissertation does align with the description of logical pluralism given above. For example, the natural deduction rules for both intuitionistic propositional logic and classical propositional logic can be assessed using the produced criteria, and this assessment can (and in fact, does) return both as consisting of logical constants. In such cases, this dissertation is silent with respect to which is the 'correct' account of negation. Because of this, logical pluralism (in the sense of alternative logics) is a potential outcome of the approach taken in this dissertation.

[39] Russell, G. (2021). Section 1.

2.7. Permutation Invariance

While the focus of this dissertation is on proof theoretic means of assessing logicality, passing mention will be made of an analogous model theoretic approach. Just as proof theoretic methods such as the examination of natural deduction rules can be used to evaluate the potential logicality of the elements of formal systems, alternative methods have been proposed to evaluate logical constancy using a model theoretic approach. One example of this approach is based on the notion of *permutation invariance*, with the following expression of it found in Tarski (1986)[40]: "I suggest that … we call a notion 'logical' if it is invariant under all possible one-one transformations of the world onto itself". This a maximal generalisation of the concept of a transformation. Less general examples of transformations can be found in the field of geometry: Euclidean transformations preserve proportion, affine transformations preserve collinearity and betweenness, and topological transformations preserve connectedness and closedness. Logical notions for Tarski are then those notions which are preserved for all transformations of objects, where these transformations are understood to be (not just in a specific field such as geometry) but in the most general sense possible.

This succinct description of permutation invariance of course omits much detail but will be sufficient for present purposes. While notions such as harmony are applied to natural deduction rules, permutation invariance is applied to the semantic definitions for logical constants, since from the model theoretic point of view these

[40] Page 149.

definitions also provide an exhaustive definition of the operator in question. As in the case of the proof theoretic approach, permutation invariance has received criticism and subsequent refinement, has been calibrated to improve its agreement with intuition regarding the elements whose logicality is relatively secure, and has subsequently been applied to more problematic or controversial cases.

2.8. Concluding Remarks

The intention of this part of the dissertation was to strategically orientate the subsequent investigation of criteria for logicality which makes up most of the project. In doing so, it has led to a number of recommendations regarding how a study of this type should best proceed, which are worth summarising here and bearing in mind as the investigation of criteria for logicality progresses.

First (Section 2.1), it was highlighted that the criteria developed should balance the competing desiderata of relevance to logic's informal requirements with the necessary precision to allow each potential candidate for logicality to be clearly categorised.

Second (Section 2.2), a methodological choice regarding the development of criteria for logicality was taken. This was to first identify informal requirements for logicality, via a survey of the thoughts of prominent authors in the field, and to use that to provide a basis from which to develop criteria. It was then argued that a synthesis of these informal requirements suggests that formality is a fundamental requirement for logicality. This is based on a survey of five prominent views on the subject, all of which reference formality; and formality's capacity to underpin (be fundamental to) the other requirements suggested. That an example of consequence being logical if it holds in virtue of its form or structure is therefore used as a working synthesis of the informal requirements for logicality found via a literature survey. Given this choice, a proposed methodology for distinguishing the logical from the non-logical is to schematise (abstract all elements from it except for those which set out its form or structure) the example of consequence of interest, and if it still holds, claim that it must do so for formal reasons alone. This in turn requires a methodology for distinguishing which elements of a sentence expressing a consequence belong to its form or structure; and which elements belong to its content, with the former being referred to as logical constants.

The links between formality and the further requirements of absolute generality and topic neutrality were also highlighted. It was proposed that, due to their seeming extensional equivalence, an attractive way of looking at these three requirements is that absolute generality and topic neutrality represent the more intuitively appealing requirement (tapping perhaps into the notion of the fundamentality of logic), whereas formality permits easier assessment of potential candidates for logicality, since it provides the link between logical consequence and logical constants. This suggests that elements of formal systems which should be considered logical constants are those whose interpretations are of concepts which are absolutely general and topic neutral.

Third (Section 2.3), it was pointed out that there are a number of classes of nouns to which the adjective ‘logical’ can be applied, including truths, logical systems, examples of the consequence relation, and elements of formal systems. Elements of formal systems which are ascribed the property of logicality are those which contribute the form or structure of the examples of consequence in which they appear. These elements are commonly referred to as logical constants and identifying them within formal systems is the most useful means of analysing logicality via proof theoretic methods, and thus the principal subject of investigation in this study is the development of proof theoretic criteria for logical constancy.

Fourth (Section 2.4), it was argued that for the criteria developed to provide a sharp border between the logical and the non-logical, they should not allow for any variation in the degree of logicality between instances of the logical; and they should not allow for any variation in the degree of logicality between instances of the non-logical.

In Section 2.5 and Section 2.6, some terminological notes were made for the purpose of clarity in the remainder of the dissertation, and logical pluralism was discussed. Section 2.7 then briefly discussed the topic of permutation invariance, a model theoretic criterion for logicality.

This dissertation will now turn to a search for proof theoretic criteria for logical constancy. While much of the preceding discussion was informal and based on the informal requirements for logicality discussed in natural language, the following sections will proceed based on discussion of the formal elements of logics, mostly as defined in natural deduction systems.

3. Natural Deduction System Structural Rules

Before discussing logical constancy based on natural deduction operational rules, mention will be made of what are known as the *structural rules* which govern logical consequence. These rules do not concern any particular logical constant (and thus form a rule set apart from operational rules), but rather endow certain properties on the logical consequence relation independent of any constant.

The most basic of these properties include weakening (basically, that the effectiveness of a proof is not compromised by the addition of premises not required for it) and contraction (basically, that the effectiveness of a proof is not compromised by 'reusing' premises in it), with other, derivable, properties including the transitivity of consequence.

The most explicit statement of these rules comes not from natural deduction systems, but from an alternative proof system, sequent calculus. This, like natural deduction systems, eschews Hilbert-style axioms in favour of deductive rules. In contrast to natural deduction systems, each line of a sequent calculus proof includes the entire sequent which is established at each point in it. Thus it is not a top to bottom list of formulas as per natural deduction, but a developing list of sequents, each of which includes all premises required for the conclusion of the sequent.

Sequent calculi include the equivalent of natural deduction's operational rules (typically known as 'left' and 'right' rules) for proof steps intended to introduce or eliminate certain constants, but also the aforementioned structural rules. In sequent calculus notation, the three principle structural rules to be discussed here are (where Γ and Δ denote finite sequences of sets of formulas, and φ and ψ denote single formulas):

- Weakening $\dfrac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta}$

- Contraction $\dfrac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}$

- Exchange $\dfrac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \psi, \varphi \vdash \Delta}$

Note that in the above table, only sequent calculus rules concerning the manipulation of premises (on the left of the turnstile, '⊢') are included, since those concerning manipulation of conclusions (on the right of the turnstile) do not have natural deduction analogues. This is because another key difference between sequent calculi and natural deduction systems is that the former permit multiple conclusions, thus producing ‘right’ rules (at least in the classical case, since for example Gentzen’s sequent calculus LK for classical logic permits multiple conclusions but his calculus LJ for intuitionist logic does not). When these rules are removed or supressed in some way, substructural logics, weaker than logics which include all these structural rules, result.

Natural deduction systems do not include an explicit statement of structural rules. However, each of the above structural rules can be proved in natural deduction systems. Taking contraction as an example, to see how it is incorporated into the natural deduction proof calculi, consider the following proof of the corresponding structural rule:

Assume that $\Gamma, \varphi, \varphi \vdash \Delta$

Assume that $\Gamma^{\wedge} \wedge (\varphi \wedge \varphi) \vdash \Delta$ (where $\Gamma^{\wedge}$ is the conjunction of the elements in Γ)

Proof: Let $\Gamma_0 \ldots \Gamma_n$ be the elements of $\Gamma^{\wedge}$. Now consider the following natural deduction proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_0 \qquad \Gamma_1}{\Gamma_0 \wedge \Gamma_1}(\wedge I) \quad \ldots \quad \Gamma_n}{(\Gamma_0 \wedge \Gamma_1) \wedge \ldots \wedge \Gamma_n}(\wedge I)}{\Gamma^{\wedge}} \qquad \dfrac{\varphi \text{ (assumption)} \qquad \varphi \text{ (assumption)}}{(\varphi \wedge \varphi)}(\wedge I)}{\Gamma^{\wedge} \wedge (\varphi \wedge \varphi)}(\wedge I)$$

Thus $\Gamma, \varphi, \varphi \vdash \Gamma^{\wedge} \wedge \varphi \wedge \varphi$, and by transitivity of consequence: $\Gamma^{\wedge} \wedge \varphi \wedge \varphi \vdash \Delta$

Now consider the following natural deduction proof, using Γ^ and φ as premises:

$$\dfrac{\Gamma^{\wedge} \qquad \dfrac{\varphi \text{ (assumption)} \qquad \varphi \text{ (assumption)}}{(\varphi \wedge \varphi)}\,(\wedge I)}{\Gamma^{\wedge} \wedge (\varphi \wedge \varphi)}\,(\wedge I)$$

And given that Γ, φ, φ ⊢ Δ, the above is sufficient to establish that Γ, φ ⊢ Δ

It is notable that this proof succeeds only because the premise φ can be used multiple times in the deduction. If a methodological caveat was added to the natural deduction system prohibiting multiple uses of the same premise in a deduction, this would have the same effect, and thus produce the same substructural logic, as deleting the structural rule of contraction in a sequent calculus system.

The reasoning above gives an indication of how structural rule manipulation can be carried out in natural deduction systems. Similar comments apply to the structural rules of weakening and exchange. The following table shows each structural rule and how this manifests itself in natural deduction systems:

Property	Sequent Calculus Structural Rule	Natural Deduction Manifestation
Weakening	$\dfrac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta}$	Vacuous discharge of assumption is permitted
Contraction	$\dfrac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}$	Discharge of multiple occurrences of an assumption may be carried out in a single application of a rule
Exchange	$\dfrac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \psi, \varphi \vdash \Delta}$	Use of premises in any order is permitted in proofs[41]

Thus the nature of the logical consequence relation (and the set of examples of consequences which hold as a matter of logic) is determined partly by the action of these structural rules. The influence of these structural rules governing deduction is a potential cause for concern in the context of the current project. This is because the methodology adopted here includes the view that in order to determine logicality, it was sufficient to identify the set of elements of inferences which are logical constants. However, the existence of structural rules suggests that there is more to the question of logical consequence than simply the question of logical constancy.

A choice also needs to be made regarding which structural rules are admitted into a proof system, not simply which of the operational rules which define logical constants are admitted. For example, Belnap (1962) states that:

[41] While the order of premises in tree-style natural deduction proofs is less clear than in linear-style natural deduction proofs, a stipulated ordering system could be adopted, taking e.g. the left most branch of the tree as 'first', then etc. to cover the entire tree.

> …we are not defining our connectives ab initio, but rather in terms of an antecedently given context of deducibility, concerning which we have some definite notions[42]

This context of deducibility may provide a means of choosing which structural rules are admitted. However, the point here is that the above analysis shows that these structural issues are not managed in natural deduction systems at the level of logical constants and operational rules. Rather, they are managed by putting in place various discharge policies for the premises (or assumptions) used in the deduction. These policies operate at a level 'above' the operational rules (in that they govern their overall use), and thus escape the methodology focussing on logical constants used in this project, since the tools developed do not apply to them.

The analysis of the logicality of the structural rules with respect to formality and absolute generality / topic neutrality will be restricted to the following comments. Each of the structural rules discussed above seem to fare well when assessed according to these developed requirements for logicality. Whether a premise is used, how many times it is used, and the order in which premises are used would seem to be unimportant to whether an inference holds when inference and argumentation is taken at its most general level. Substructural logics which remove or in some way supress the rule of contraction are used for example in

[42] Belnap (1962). Page 131.

more specific applications such as information handling where certain information resources can only be used a certain number of times due to the limits on the capability of physically implemented computer systems.

This reasoning does not exclude substructural logics from logicality, which are more restricted in terms of theorem generation than non-substructural logics. However, it does suggest that there is no reason to deny logicality to any of the structural rules. While the conclusion here is that the scope of logic should not be restricted to any substructural logic, the possibility that such a restriction could be applied should not be ruled out by any approach involving criteria used to determine logical constancy. In the case of the natural deduction systems which are the main focus of investigation in the present work, any restriction on premise use must be included in operational rule definitions (since the operational rules of a natural deduction system essentially exhaustively define the system). However, in the case of sequent calculi, analysis for logicality would seemingly require analysis also of structural rules, and thus presentation alone of criteria for logical constancy would be insufficient.

4. Natural Deduction Criteria for Logical Constancy

In the previous chapter, formality was identified as the key requirement for logicality. This led to the idea that the problem of providing criteria for logicality reduces to a large extent to that of determining which elements of formal languages are logical constants. One means of doing so is by simply providing a list of logical constants. In the absence of providing explicitly stated criteria to evaluate the logical constancy (or absence thereof) of an element of a formal system, this is the default method for doing so. Given that there are currently no criteria for logical constancy which enjoy widespread acceptance (though the permutation invariance criterion based on the model theoretic approach to logic discussed in Section 2.7 is popular), the existing core set of what are typically seen as being logical constants are simply listed in this way, though naturally not on an entirely arbitrary basis, with each purported constant's position on this list being presumably due to various intuitive, pragmatic, historical, etc. reasons. The drawback of this method corresponds to a key motivation for studies such as the present, namely that (though it is effective in for example the case of truth functions, where, as will be seen in 4.4.2.3, a connective is truth functional if and only if it picks out some subset of truth-conditions based only the truth conditions of the arguments) it does not provide a means of assessing newly considered, borderline, or controversial cases of potential logical constancy.

Producing criteria for logical constancy should, on the other hand, provide a means for assessing these problematic cases. Furthermore, assuming the criteria can themselves be linked back to the requirements for logicality, the latter will provide justification for the presence of each element in the set of logical constants. For this to hold, it is necessary that the criteria produced are motivated by the requirements for logicality discussed in Section 2.2. This part of the dissertation concerns providing these criteria for natural deduction systems, which link back to the key requirement of formality, and its equivalent requirements of absolute generality and topic neutrality.

4.1. Introduction to Proof Theoretic Criteria

Proof theory is the study of proofs as mathematical objects. It provides a means of studying logical consequence via purely syntactic means, in contrast to semantic approaches which are adopted in for example model theoretic analysis of logic. Thus, the key notion which is used to analyse logical consequence in the former approaches is that of proof – a logical consequence holds if a proof exists to demonstrate that fact.

In order to properly study the notion of proof, concrete manifestations of it known as proof calculi must be produced. There are a number of these proof calculi, each of which can potentially be investigated to determine which elements in them are logical constants. Two of these proof calculi have been strategically selected for analysis in this project. The first is natural deduction, selected due to the already significant body of literature examining it from a structural point of view, particularly (for the purposes of this project) the characteristics of its so-called operational rules for introducing and eliminating logical constants in proofs. The second is the method of semantic tableaux, which represents a novel approach, in that much less literature is available regarding its examination from the point of view of logical constancy. This is carried out in Section 5 of this dissertation.

4.2. Natural Deduction

Systems of natural deduction, as the name suggests, endeavour to present proofs which mimic to a certain extent the natural way in which human agents undertake deduction. In this, they contrast with the Hilbert-style or axiomatic approaches developed before them, which, while generally being able to present the overall characteristics of logical systems more succinctly, produce proofs which are less easy to follow and deviate further from the actual practice of reasoning. Natural deduction systems grew out of a dissatisfaction with the ‘artificial’ nature of Hilbert-style systems; in which the proof of logical truths and examples of consequence bear very little resemblance to the intuitive reasoning processes used to arrive at them. Natural deduction systems consist of no axioms, but a series of rules of inference which are used to manipulate the premises of an argument in an attempt to arrive at its conclusion.

The relationship between the advantages of Hilbert-style and natural deduction systems are analogous to an often-observed trade-off between simplicity of vocabulary or lexicon (referred to as ‘alphabet’ here) and simplicity of expression. Hilbert-style systems have the benefit of a relatively simple alphabet, in that the axioms and rules which define the logical system in question are very compact. However, their unwieldy proofs mean they suffer from a lack of expressional simplicity. In contrast, while the rules which make up a natural deduction system for a given logic have less compact alphabets, the proofs which can be developed from them can be expressively simpler.

Foreshadowing later discussions, a perhaps counter-intuitive impression which may be had is that the move from Hilbert-style systems to natural deduction systems increases the number of logical constants. Consider the following Hilbert-style axiomatisation of propositional logic[43]:

Axiom 1: $\varphi \rightarrow (\psi \rightarrow \varphi)$

Axiom 2: $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$

Axiom 3: $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$

Rule: Modus ponens: From $\varphi \rightarrow \psi$ and φ, deduce ψ

Two connectives appear in this axiomatisation, → and ¬, which seem apt for characterisation as logical constants. However, as will be seen subsequently, natural deduction treatments of propositional logic typically include many more candidates for logical constancy, notably including ∧ and ∨. This variation is not, however, restricted to the move from Hilbert-style to natural deduction systems, since an equivalent axiomatisation of propositional logic to that presented above can be produced using just a single connective, the Sheffer stroke (also known as nand), symbolically represented as ↑, as follows:

$$(\varphi \uparrow (\psi \uparrow \omega)) \uparrow ((\varphi \uparrow (\omega \uparrow \varphi)) \uparrow ((\chi \uparrow \psi) \uparrow ((\varphi \uparrow \chi) \uparrow (\varphi \uparrow \chi))))$$

The key reason for this is that certain connectives can be defined in terms of others; for example, $\varphi \vee \psi$ can be seen as an abbreviation of $\neg(\neg\varphi \wedge \neg\psi)$. Natural

43 Imai et al (1966). Page 437.

deduction systems include rules for all common truth functional operators to support their desired intention of being deductively clear and in line with natural reasoning.

Starting in the 1960's, systems of natural deduction have received significant attention with respect to the question of logical constancy. Key contributions have been made in Prawitz (1965), Dummett (1991), and Read (1999, 2000, 2004, 2008 and 2010). The majority of these contributions centre on the notion of harmony, which is inarguably a key tool for the assessment of the logical constancy of potential candidates. However, it is the contention of this dissertation that interesting and informative results concerning natural deduction-based criteria for logical constancy can be developed at a more fundamental level than harmony. For this reason, the methodological approach taken at the outset in the following sections will be one of an intentional naivety with respect to the contributions made by the sources cited above. The intention of this is, before diving directly into considerations of harmony, to ask what the very basic characteristics on an element of a formal language must be for it to qualify as a logical constant.

4.3. Systems of Natural Deduction

Full presentations of natural deduction systems typically include a set of formation rules, usually given as inductive rules defining properly constructed statements of the language. Readers are referred to Tennant (1978)[44] for an example of this. Operational rules are also included. Introduction rules (I rules) are used to combine well-formed formulas into longer well-formed formulas, and elimination rules (E rules) are used to break down well-formed formulas into their constituent parts. The application of each rule allows the construction of tree-shaped proofs. Square brackets, '[' and ']' are used to denote assumptions which are discharged as part of the application of the rule. The rules for a simple system, including only →, ¬, ⊥, ∀ and ∀ are as follows:

$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \to \psi}\ (\to I) \qquad\qquad \dfrac{\varphi \qquad \varphi \to \psi}{\psi}\ (\to E)$$

$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \bot\end{array}}{\neg \varphi}\ (\neg I) \qquad\qquad \dfrac{\varphi \qquad \neg\varphi}{\bot}\ (\neg E)$$

$$\dfrac{\bot}{\varphi}\ (\bot E)$$

[44] Page 19.

$$\frac{\varphi(a)}{\forall x\varphi(x)}\ (\forall I)$$

where every occurrence of a in φ(a) is replaced by x, and a must not occur in any assumption on which φ(a) depends. The variable x must not be bound by any quantifier in φ(a) that has a within its scope[45]

$$\frac{\forall x\varphi(x)}{\varphi(a)}\ (\forall E)$$

In applying this rule one replaces every free occurrence of x in φ(x) by a [46]

$$\frac{\varphi(a)}{\exists x\varphi(x)}\ \exists I$$

where in φ(a) no occurrence of a which is to be replaced by x occurs within the scope of any quantifier binding x. Note also that in applying this rule one need not replace every occurrence of the term a in the sentence φ(a) with an occurrence of the variable x[47]

$$\frac{\exists x\varphi(x) \qquad \begin{matrix}\varphi(a)\\ \vdots \\ \psi\end{matrix}}{\psi}\ \exists E$$

where a does not occur in ∃xφ(x), a does not occur in ψ, and a does not occur in any assumptions, other than φ(a) on which the upper occurrence of ψ depends[48]

The above system is a basic presentation of a natural deduction system, particularly in that it only contains a very few truth functional operators. It is intended to give the reader a view of the fundamental ‘ingredients’ of such systems. The focus of this

[45] Condition taken from Tennant (1978). Page 42.

[46] Condition taken from Tennant (1978). Page 41.

[47] Condition taken from Tennant (1978). Page 41.

[48] Condition taken from Tennant (1978). Page 46.

dissertation is on the operational rules for candidates for logical constancy, and these will be explored in more detail in the coming sections of this dissertation.

The above is one of a variety of means of presenting natural deduction systems. Bostock (1997) offers six points "giving a general characterization of what is nowadays called 'natural deduction'"[49]. As a means of introducing the technical characteristics of natural deduction systems, and to facilitate their later analysis with respect to the provision of criteria for logical constancy, these points are paraphrased below; and will be followed by comments orienting these six points within the context of the current project.

1. The basic notion is that of a proof from assumptions.[50]
2. There will accordingly be no axioms (as traditionally understood) but a number of rules of inference for use in such proofs.
3. We shall expect to find, for each truth functor or quantifier in the language being considered, rules that specifically concern it, and no other truth-functor or quantifier…for each truth-functor or quantifier concerned, there will be one or two rules that are counted as its introduction rules, and one or two that are counted as its elimination rules, and no other rules.[51]

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[49] Bostock (1997). Page 240 to 242.

[50] "Proof from assumptions" being a device used to simplify Hilbert style axiomatic proofs.

[51] Noting, however, that Bostock follows this up by stating that this "is more a requirement of elegance than a condition on what can be counted as natural deduction, and certainly systems have been proposed which one would wish to call systems of natural deduction even though they do not entirely conform to it… Again, there are well-known systems which do not entirely conform to this, but it is what one expects nowadays"

4. (a) The introduction and elimination rules for any one sign be complete for that sign, in the sense that all correct sequents involving only that sign be provable from those rules alone.

   (b) Combining the introduction and elimination rules for any two or more signs yields a system complete for those signs together, again in the sense that all correct sequents containing only those signs be provable from those rules alone.
5. The rules for each sign be 'natural', in the sense that inferences drawn in accordance with them strike us as 'natural' ways of arguing and inferring.
6. So long as the sequent that we are trying to prove is 'not too complicated', there should be a proof of it which is 'reasonably short' and uses only the rules initially adopted.

The key points among the above with respect to the current project seem to be 3 and, to a lesser extent, 4 (both of which, interestingly, Bostock claims are more “requirements of elegance” than anything else). Point 3 states that truth functors (a term which translates, in the current context, to potential candidates for logical constancy) should have rules which concern them and them alone; and that these rules are also expected to entirely fulfil the requirements for the manipulation of potential logical constants in proofs. The rules thus entirely and only define the potential constants. Because of this, they take the form of rules to introduce and eliminate formulas containing the potential constants into proofs and are known as operational rules.

Point 4a's 'completeness' requirement in effect states that the operational rules for a single constant should be such that there is no need to call upon the rules for additional constants in order to prove an example of consequence containing that constant alone. Point 4b extends this to requiring the lack of deviations through additional constants for a given group of constants. A notable example which appears to violate this requirement in the case of classical propositional logic is Peirce's Law, since $\vdash ((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi)$ is a provable result in it, but one which cannot be demonstrated without using the rules for ¬, which do not figure in it (this result is also evident in the fact that Peirce's Law is not provable in intuitionistic propositional logic, which varies from classical logic only in its rules for ¬).

As Bostock states, the characteristics he lists are reports of the nature of modern natural deduction systems. A key result for the present dissertation is to demonstrate that the systems which possess these characteristics can justifiably be said to be useful in generating criteria for logicality. Given the above discussion of point 3 of Bostock's characterisation, this would seem to reduce to the question of why the operational rules for each potential constant may also be said to be a potential target for criteria-based evaluation for logical constancy. Note that this is not yet the question of which elements natural deduction systems imply are logical constants (this will come later and will involve careful analysis of the operational rules themselves, according to principles such as the aforementioned harmony);

rather it is a demonstration that natural deduction systems are in general amenable to such analysis.

One potential reason that these operational rules are in such a position is given in Bostock's third point: It is simply because they exhaustively define the potential constant in question. They therefore necessarily provide all that could be required for an evaluation of logical constancy. However, the position that the operational rules are all that is required for a full understanding of the candidate for logical constancy in question itself requires justification. This position is known in the literature on the subject as logical inferentialism; and the above point can thus be stated as follows: Evaluating logical constancy on the basis of operational rules can be justified by demonstrating the veracity of logical inferentialism.

Logical inferentialism can be traced back to the following oft-quoted passage from Gentzen (1969)[1935][52]:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions.

This quote suggests a particularly strong version of logical inferentialism, since it holds that potential constants are defined by their *introduction* rules only; with elimination rules being defined in terms of these. Other versions of logical

[52] Page 80.

inferentialism (such as in Dummett (1991)[53] and Milne (1994)[54]) take it that in some cases the introduction rule should be given precedence; while in others it is rather the elimination rule which better defines the use of the constant in question. The following general definition of logical inferentialism (from Rossberg and Cohnitz (2009)[55]) will be sufficient for the present purposes:

> Inferentialism insists that the meaning of the logical constants is determined by their introduction and elimination-rules, and that these rules (so far as they are the correct ones) are self-justifying. No further appeal to model-theoretic semantics, truth-tables or the like is needed in order to argue for the validity of the rules.

This definition makes it clear that the logical inferentialist position does maintain that the operational rules for each constant exhaustively define them. Thus, any debate regarding the logicality or otherwise of the constants must be resolvable on the basis of the rules alone, with no recourse to other information such as from semantic and model theoretic considerations being required. If proof theoretic criteria for logical constancy are to exist, and operational rules define constants from a proof theoretic point of view, such criteria must be based on these operational rules[56]. It is also clear from the definition that it concerns logicality

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[53] Page 280.

[54] Page 56.

[55] Page 153.

[56] It is conceivable that an alternative justification could be based on the notion that even if the operational rules do not fully define the constant, they do provide sufficient information to permit evaluation in terms of their logical constancy. This possibility is not further pursued here.

rather than informal logicality – that is, the definition claims that introduction and elimination rules entirely define elements of formal systems, not that these rules make any claim about the informal logicality of concepts expressed in natural language.

In terms of a justification of logical inferentialism, it forms part of a wider inferentialist theory of meaning which holds that meaning is obtained not through the truth conditions, but rather through the inferential connections between sentences. In the case of logical constants, this point is worth bringing out in more detail, since it will provide a useful means of attempting to develop operational rules for candidates for logical constancy. The following points provide this detail:

- Introduction rules provide the set of sufficient conditions (that is, the grounds) for asserting formulas including the constant as its main operator.
- Elimination rules provide the set of necessary consequences of (that is, what follows from) the assertion of the formula with the constant as its main operator[57])

It is important that inferentialism holds in the case of the candidates for logical constancy considered in this dissertation. This is because stipulating that natural deduction rules provide definitions of the operators in question means that logical constancy assessments of these operators can take place based only on these rules, with the inferentialist position guaranteeing that there is no information out

[57] Brandom (2001). Page 62.

with these rules which could escape consideration in the logical constancy assessment. Investigating logical inferentialism is bound up in attempts to provide coherent criteria for logical constants on the basis of their operational rules. This will take place in the following section of this dissertation.

4.4. Natural Deduction Criteria for Logical Constancy

This section of the dissertation will get down to the business of identifying natural deduction-based criteria for logical constancy. The objective is to produce a succinct set of criteria which classify those elements of formal systems which are formal (contribute only to the structure of the sentences in which they appear), and whose interpretations are absolutely general and topic neutral as logical constants; and dismiss those which are not from such a classification.

In terms of the types of candidates for logical constancy evaluated, the order followed will be:

1. Section 4.4.2: Those included in first order logic, namely individual constants, predicates, connectives, and the first order quantifier. In general, the logical constancy or otherwise of these is less controversial than the types of candidates considered in 2 and 3 below. Addressing these less controversial cases first will allow criteria to be developed based on a more solid foundation, before they are applied to the more controversial cases in 2 and 3 below.

2. Section 0: Those which are introduced in second order logic, namely cardinality quantifiers and different versions of the second order quantifier. Here, the criteria developed in the evaluation of first order candidates for logical constancy will be applied to the more controversial second order case.
3. Section 4.4.4: Those which are introduced in the various systems of modal logic. Again, the insights regarding criteria obtained from 1 and 2 above will apply to these more controversial cases.

4.4.1. A Note Regarding Strategy

In terms of the strategy used to develop the criteria, no decisive list of natural deduction criteria for logical constancy will be suggested at the outset. Rather, the criteria will be suggested and refined as each category of candidates for logical constancy discussed in the previous section are evaluated. To do this, some insight regarding the logical constancy of the candidates will be required. This insight will be provided by:

- The requirements for logicality identified in Section 2.2. Specifically, that the potential logical constant is formal (contributes only to the structure of the inference), and that its interpretation is absolutely general and topic neutral.
- The nature of natural deduction systems, as given by Bostock's characterisation of natural deduction systems provided in Section 4.3.

The strategy adopted in this dissertation will therefore be to investigate natural deduction systems for first order logic, second order logic, and modal logic, assessing the potential logical constancy of the elements of them. Each element will be assessed according to the requirements for logicality and Bostock's characterisation of natural deduction systems, and criteria for logical constancy suggested and refined (if necessary) so that they admit elements which are formal and have absolutely general and topic neutral interpretations and exclude elements which do not.

The assessment will take place in the order of first order logic, second order logic, and modal logic, since first order logic forms a basis for second order and modal logic (second order logic is an extension of first order logic obtained by adding second order quantifiers to it, and the modal logics considered in this dissertation are extensions of the propositional part of first order logic).

4.4.2. First Order Logic

In the case of first order logic, there are four general categories from which candidates for logical constancy will be discussed in this dissertation. These are:

- Individual constants.
- Predicates.
- Connectives.
- Quantifiers.

Of these, the connectives are the most familiar source of logical constants. Formal conjunction (∧) and disjunction (∨) are perhaps those which spring to mind most naturally. The material conditional (→) would typically qualify also, though its status may be slightly more questionable, due to some of the purported paradoxes (for example involving relevance) associated with it. Negation (¬) is another strong candidate, though one whose logical constancy may be compromised by the controversy between intuitionistic and classical interpretations of it. However, the discussion will proceed in the order above, because this allows criteria to be developed before the more complex case of the connectives and quantifiers are considered.

4.4.2.1. Individual Constants

At first glance, the prospects for finding a logical constant among individual constants seem poor. Consider the following statement from Steinberger (2009)[58], writing on inferentialism in general:

> By contrast, inferentialism becomes more problematic especially in its stronger variants when applied to expressions that are more intimately hooked up with the world because of their content or indeed because of their grammatical category. (Proper names are a case in point.)

While proper names are particularly problematic, the issues with respect to inferentialism that individual constants suffer can be extended to the general case. As Steinberger points out, this is due to their ‘intimate’ relation with the world; presumably this refers to the seemingly clear referential role that they play in language. In terms of logical constancy and the requirements for logicality, it is

[58] Page 25.

unclear how any element of formal language could be formal rather than contentful if its role is to refer to specific objects in the domain of quantification. Further support for these poor general prospects can be found in the fact that there are no individual constants which are generally accepted to be logical constants in natural deduction systems.

It seems clear, then, that individual constants should be excluded from logical constancy. The methodology described in Section 4.4 dictates that it is thus necessary to legislate against this through proof theoretic criteria imposed on the potential operational rules for them. Before this, however, some comment is required regarding the nature of operational rules themselves. Dummett (1991)[59] contains the following regarding the specification of introduction and elimination rules in the most general sense:

> The terms 'introduction rule' and 'elimination rule' themselves may be explained in a very general way. A rule of inference may be called an introduction rule for a logical constant c if its conclusion is required to have c as principal operator; it may be called an elimination rule for c if one of its premises is required to have c as principal operator, relative to which that will be the 'major premise'.

The intuitive attraction of this definition is clear – any introduction rule for a logical constant worthy of the name should include the constant in its conclusion (that is,

[59] Page 256.

it should introduce it); and its elimination rule should include the constant in its premise (so it can eliminate it). However, that they should appear only as the principal operator is perhaps less clear, and as will be seen presently, seems too restrictive in that it would eliminate candidates which appear to have a good claim to logicality. It also does not regulate against an introduction rule of the following form for the unary connective ↀ, which (foreshadowing criterion 1 below) would not intuitively be thought of as introducing ↀ due to its presence of ↀ in the antecedent:

$$\frac{ↀↀↀ\varphi}{ↀ\varphi}\ ↀI$$

Furthermore, consider the following suggested operational rules for a logical constant c:

$$\frac{Pc}{Pc}\ cI \qquad\qquad \frac{Pc}{Pc}\ cE$$

Where P is an arbitrary predicate. While neither premise nor conclusion in the above contain any operators (unless predication itself could be considered an operation), it does not conform to the letter of Dummett's requirement. However, it would be question begging to rule it out on this basis alone (since that would immediately restrict logical constancy to operators), and it could be argued that cI and cE do conform to Dummett's requirement in that that which is introduced and eliminated does appear in the conclusion of the introduction rule and the premise of the elimination rule respectively.

It could be argued that they should be excluded from any legitimate list of such rules due to their lack of utility: that Pc appears in the premise of a rule purporting to introduce it and in the conclusion of a rule purporting to eliminate it renders them useless for deductive purposes. This is true in a general sense for rules which are vertically symmetrical, since they do not contribute anything in terms of advancing the progress of a deduction. However, a lack of deductive utility should not preclude an element from being a logical constant, since according to the terms of this dissertation, logicality should be based on formality and absolute generality / topic neutrality, not utility.

However, while their lack of deductive utility means that they are relatively benign, there remains an important reason that operational rules such as the above should be legislated against. Should they be permitted, it would mean that any formal element at all could be seen as a logical constant, by giving vertically symmetrical rules such as these for it. This would be an undesirable situation, since it would trivialise the task of categorising the logical constants.

Given the above, a criterion to rule out operational rules taking this form is required. The following possibilities come to mind:

- Introduction rules must differ from elimination rules in any operational rule pair.
- Operational rules must provide some level of utility in deductions.

But these attempts seem to be too directed towards the specific form used for cI and cE above. Even at this early stage, it would be useful to try to exclude them on the basis of something more general. Thus the following will be used:

- Natural Deduction Criterion for Logical Constancy 1: Operational rules must not allow the introduced element to appear as the main connective in the antecedent of the I rule and must not allow the eliminated element to appear as the main connective in the consequent of the E rule.

This is essentially the complement of Dummett's most general requirement provided above; combined they require that the element in question does appear in the conclusion but does not appear in the premise of an introduction rule; and does appear in the premise but does not appear in the conclusion of an elimination rule.

With the above criterion in place, providing operational rules for an individual constant, while maintaining the required formality, absolute generality and topic neutrality becomes more difficult. Perhaps the only potential means of meeting this challenge would be to attempt to develop operational rules which define the introduction and elimination of an individual constant which is absolutely general in its reference. That is, it refers to an object, but one which is entirely arbitrary. In this way, its topic neutrality could be maintained within the constraints of individual constancy.

The natural language equivalent of this entirely arbitrary individual constant is difficult to identify. Terms such as ‘this’, ‘it’, or ‘thing’ come to mind, but each seems unsatisfactory. In terms of the potential referent of the arbitrary individual constant, a candidate would seem to be the arbitrary objects discussed in Fine and Tennant (1983), which includes the following:

> With each arbitrary object is associated an appropriate range of individual objects, its values...An arbitrary object has properties common to the individual objects in its range. So an arbitrary number is odd or even, an arbitrary man is mortal, since each individual number is odd or even, each individual man is mortal[60].

Fine himself acknowledges that this view has “fallen into complete disrepute”[61]. Furthermore, while the notion of the arbitrary man does introduce some generality, it is not generality in an absolute sense. For this, an entirely arbitrary object, which is a further level of generality beyond the arbitrary X, where X is a class of objects, would be required. This means that their existence could be seen as being even more precarious than the arbitrary man.

Notwithstanding the above, consider the following attempt at providing introduction and elimination rules for an arbitrary individual constant c:

$$\frac{\quad}{Pc}\ cI \qquad\qquad \frac{Pc}{\quad}\ cE$$

[60] Page 55.

[61] Fine and Tennant (1983), page 55.

These rules permit the introduction or elimination of the constant c at any point in any deduction, noting that they should include the stipulation that P (a schematic for any predicate letter in the formal language) is a non-complex predicate, to avoid the introduction of a contradiction. Thus they treat the sentence Pc as a logical truth, to be used at will in deductions. It is the introduction rule which is problematic here: What sentence, which simply predicates a property (denoted by P) of an object (denoted by c), could possibly represent a logical truth? Given that the predicate P would be left undefined here, c can be seen as an individual constant to which any predicate at all can be applied. However, the arbitrary objects suggested by Fine do not seem to fit this requirement, since their generality means rather that no predicate can be conclusively applied to them rather than any predicate at all; but also no predicate can be conclusively said to not apply to them (thus meaning ¬Pc cannot be treated as a logical truth either). Later in this dissertation, this issue will be revisited when absolutely general predicates rather than objects are discussed.

Consider next the following proposed operational rules for c, this time involving the two-place predicate = (where φ(t) results from replacing some, not necessarily all, occurrences of c in φ(c) with t):

$$\frac{}{c = c} =\mathrm{I} \qquad\qquad \frac{(c = t) \wedge \varphi(c)}{\varphi(t)} =\mathrm{E}$$

These rules are recognisable as those commonly given for = rather than for an individual constant. The introduction rule =I essentially states that any object is identical to itself; with the elimination rule =E stating that should two constants refer to the same object (the first conjunct in the premise), then their names can be substituted in any formula in which one of them appears. This is the basis of an objection to cI and cE providing the sought-after operational rules for an arbitrary individual constant – they actually provide operational rules for =, not for c. However, it is difficult to see the basis (beyond familiarity with their defining = rather than c) upon which this claim would be justified. The essence of the claim would seem to be that the rules are about identity rather than individual constancy, but this must be made more precise if it is to be effective. The clearest way to do this might be to claim that = plays the role of principal operator in the rules in a way analogous to that of the truth functional connectives in operational rules for conjunction, implication, etc. The problem with this claim is that, given the statement of the rules provided, it is not true, since in the case of cE, ∧ is the main connective. However, the rule can be modified to the following (where φ(t) results from replacing some, not necessarily all, occurrences of c in φ(c) with t):

$$\frac{}{c = c}\ {=}I \qquad \frac{(c = t) \qquad \varphi(c)}{\varphi(t)}\ {=}E$$

This converts =E to a multi-premise rule, with = appearing as the main connective. This means that the rules can be considered as rules governing = rather than the individual constant, and they conform to Natural Deduction Criterion for Logical Constancy 1. Further discussion of them is deferred until Section 4.4.2.2.

4.4.2.2. Predicates

In terms of their general prospects for logical constancy, predicates appear to suffer from similar objections as those levied against individual constants in the previous section. Given that they are used to formalise the combination of certain particular objects with certain particular properties, the meaning of each predicate is strongly tied to the property in question. Predicates therefore appear to violate the requirements for logical constancy of formality, absolute generality and topic neutrality.

This is particularly the case due to the extensional nature of quantified formal systems. In extensional terms, one place predicates are equivalent to subsets of the domain of quantification of the system in question, two place predicates are sets of ordered pairs, and so on for predicates of increasing arity. This suggests that the prospects for logical constancy of predicates are as meagre as those of individual constants, since while the latter denote single objects in the domain, and are thus contentful rather than formal, predicates denote sets of objects in the domain (or sets of ordered pairs, triples, etc.), meaning they could be expected to be contentful as well.

Despite the above comments, three predicates are commonly accepted to represent strong candidates for logical constancy, each of which are considered in turn in the following paragraphs. These are:

- The identity operator (=), a two-place predicate.
- Verum (⊤), a zero-place (meaning that it acts as a sentence letter) predicate.
- Falsum (⊥), a zero-place (meaning that it acts as a sentence letter) predicate.

That fact that the identity operator is the only non-zero-place predicate which is included as a logical constant in standard treatments of first order logic is curious. To resist allegations that its inclusion is ad hoc, it is necessary to investigate whether identity has any special properties which particularly recommend it for inclusion as a logical constant, and which are lacking in other predicates. If no such properties are identified, its inclusion may be at least partly due to for example historical or pragmatic factors, rather than legitimate logicality-based reasons. Alternatively, if such properties are identified, it is worthwhile considering whether other non-zero place predicates also have these properties, and thus also have potential for logical constancy[62].

One potential historical factor which may be involved here is the link between logic and mathematics. Given the prominence of the logicist project in the early development of modern formal logic, the role of logic as providing a foundation for mathematics has long been seen as a key to the very nature of logic (Shapiro

[62] It is notable that according to the criterion of permutation invariance (see Section 2.7), the formalisations of identity and its negation (distinctness) and also the universal property (existence) and its negation (non-existence) are logical constants.

(1991) provides a relatively recent take on this approach). Since the notion of identity (typically referred to by mathematicians as 'equality') is key to mathematics, it is not unreasonable to imagine that even if it did not entirely meet what is expected of a logical constant, the additional mathematical utility it provides would motivate its inclusion in the realm of logic. This approach deviates from that taken in this dissertation, which associates logicality with absolute generality and topic neutrality rather than mathematical utility.

Quine (1986)[63] also provides an argument for including the identity predicate as a logical constant. Quine states his point as follows:

> One respect in which identity theory seems a nearer neighbour to logic than to mathematics is its completeness. Complete proof procedures are available not only for quantification theory but for quantification theory and identity theory together.

The 'quantification theory' referred to in the above is first order logic. Thus Quine argues that since the addition of identity to systems containing only what are uncontroversially logical constants (which is his view of standard systems of first order logic) does not significantly alter the metalogical property of semantic completeness, identity should also be accorded logical constancy. This contrasts with other operators (such as second order quantifiers), whose addition to formal

[63] Page 61 to 64.

systems causes the loss of semantic completeness, which Quine therefore considers to be outside the realm of logic.

As it stands, this is not a particularly strong argument. The simple fact that extending first order logic via the addition of identity means that it retains one of the many metalogical properties attributed to it is not very convincing. Quine's case would be stronger if it included a demonstration that completeness is in some way characteristic of logic; but Quine (at least in the work from which the citation above originates) does not provide this. Without such a demonstration, questions could be raised regarding why such stock should be put in completeness instead of another metalogical property, which may be lost if identity is added to first order logic. Boolos (1975)[64] uses such a strategy to argue against the denial of the logicality of the second order quantifier on the basis that it's addition to a system including first order quantification results in a loss of semantic completeness, asking:

> We have seen, first, that monadic logic differs from full first order logic on the score of decidability… how, then can the *semi*-effectiveness of the set of first-order logical truths be thought to provide much of a reason for distinguishing it from mathematics? Why *completeness* rather than *decidability* or *interpretation*?

[64] Page 51.

It may be that Quine considered that the inclusion or exclusion of ‘=’ in logic is simply a convenience since completeness is retained in systems including it, and thus that the argument simply turns on nomenclature. However, consideration of Quine’s point highlights an important point regarding the approach to logicality this dissertation takes. Instead of attempting to sort the logical from the non-logical at the level of metalogical properties, this dissertation uses accepted (at least according to the literature survey contained in Section 2.2) requirements for logicality to produce proof theoretic criteria and assesses candidates for logicality according to them. This keeps the core notions of logicality – formality, absolute generality and topic neutrality – at the centre of the debate, and thus keeps the argument rooted in fundamentally key notions, rather than metalogical properties.

Moving on, when discussing identity, it is important to draw the distinction between numerical and qualitative identity. The former can the characterised as the only relation which every object holds in relation to itself and only to itself, while the latter holds whenever two objects share all properties. In common language, the difference between numerical rather than qualitative identity is sometimes clarified by stating that two objects are not just the *same*, but the *very same*.

Consideration of the nature of numerical identity with respect to the requirements of formality, absolute generality and topic neutrality suggests that there is a strong case to include it as a logical constant. Quine (1986) also includes the following:

> Another respect in which identity theory seems more like logic than mathematics is universality; it treats of all objects impartially. Any theory can indeed likewise be formulated with general variables, ranging over everything, but still the only values of the variables that matter to number theory, for instance, or set theory, are the numbers and the sets; whereas identity theory knows no preferences[65].

In contrast to the previously considered semantic completeness-based argument provided by Quine, this passage aligns well with the present project. In numerical identity's typical characterisation as a relationship which holds between every object and itself (and only itself), the use of the word 'every' implies generality. Recall the following natural deduction operational rules (where φ(t) results from replacing some, not necessarily all, occurrences of c in φ(c) with t):

$$\frac{}{c = c}\ {=}\mathrm{I} \qquad\qquad \frac{(c = t) \qquad\qquad \varphi(c)}{\varphi(t)}\ {=}\mathrm{E}$$

The aforementioned aspect of identity is embodied in the =I rule, since the empty premise of this rule shows that the conclusion (stating that every object is identical with itself) can be introduced regardless of the nature of the constant in question; thus implying that the rule can be applied in an absolutely general way.

The above reasoning suggests that the admission of numerical identity as a logical constant is justified. Before evaluating its operational rules with respect to Natural

[65] Page 62.

Deduction Criterion for Logical Constancy 1, the possibility that identity can be defined using other constants should be considered. The following two statements provide connections between numerical and qualitative identity:

- The indiscernibility of identicals: $\forall x \forall y[x = y \rightarrow \forall V(Vx \leftrightarrow Vy)]$
- The identity of indiscernibles: $\forall x \forall y[\forall V(Vx \leftrightarrow Vy) \rightarrow x = y)]$

Should both the indiscernibility of identicals and the identity of indiscernibles hold, then numerical identity could be introduced using the following definition:

$$(c = d) =_{def} \forall X(Xc \leftrightarrow Xd)$$

This reduction of = presents two problems. First, the truth of the identity of indiscernibles is questionable. Black (1952) invokes a theoretical universe which includes only two spheres, each of which is a perfect likeness of the other. Black then argues that in such a universe, the two spheres would be non-identical but entirely indiscernible. While objections to Black's argument have been put forward (such as in Hacking (1975)), concerning consideration of the situation as a single sphere in non-Euclidean space (Forrest (2020)), detailed discussion of this point is beyond the scope of this dissertation.

Second, the definition above involves second order quantification. As discussed in Section 0 of this dissertation, the universal second order quantifier is by no means non-controversial in terms of its logical constancy. It should therefore be avoided when attempting to define a further operator which is considered a candidate for logical constancy.

Given this apparent failure of introducing numerical identity by definition, focus will turn to its introduction as a primitive operator and its evaluation according to the criteria established in this dissertation. These rules conform to Natural Deduction Criterion for Logical Constancy 1. Given that it conforms to both the requirements for logicality and the criterion so far established, this dissertation concludes that identity should be accepted as a logical constant. Since, as previously noted, it appears odd that only one (non-zero place) predicate should be a logical constant, other such predicates should also be investigated. Working from the example set by identity, neighbouring concepts which may have potential for logicality are the comparative predicates greater than and less than. However, these can be quickly dismissed on the basis that the variables used cannot range over every different kind of object, since for an object to be coherently included in a greater than or less than relation, it must have a magnitude – something which objects in the most general sense do not possess. This in turn presents an opportunity to test and refine the criteria for logicality, since the fact that it is not a logical constant should mean that these criteria exclude it based on its operational rules in natural deduction systems. The following operational rules give three well known properties of the greater than relation[66]:

[66] *Less than* will not be investigated here; however, similar comments to those made regarding *greater than* apply to it.

- Asymmetry $\dfrac{c > t}{\neg(t > c)}$

- Transitivity $\dfrac{(c > t) \land (t > d)}{c > d}$

- Irreflexivity $\dfrac{}{\neg(c > c)}$

The proposed rules for asymmetry and transitivity, cast as either introduction or elimination rules, can be excluded on the already-established Natural Deduction Criterion for Logical Constancy 1, that operational rules must not allow the introduced element to appear as the main connective in the antecedent of the I rule and must not allow the eliminated element to appear as the main connective in the consequent of the E rule.

This leaves the potential operational rule based on irreflexivity, whose empty premise means it can only reasonably be cast as an introduction rule. One means of excluding this rule from logicality would be to follow Dummett, and establish a criterion based on the greater than predicate's falling within the scope of the negation operator (¬), and thus not being the main operator (that into the scope of which the other elements of the formula fall) of the sentence in the introduction rule. However, this would also rule out the =E rule, and thus it appears to be too restrictive. Modifying this criterion so that the main operator requirement applies only to introduction and not to elimination rules lacks a logical basis and seems somewhat ad hoc.

An alternative criterion to rule out the irreflexivity rule might therefore be sought. The fact that the other two rules giving the defining characteristics of the greater than relation have been excluded on the basis of Natural Deduction Criterion for Logical Constancy 1 means that it has been left without a proposed elimination rule. This suggests a criterion along the lines of: Operational rules must include at least one introduction rule and at least one elimination rule. Such a criterion has some claims to legitimacy, since the inferentialist position implies that both introduction and elimination rules are necessary to fully define the meaning of a logical constant. Dummett (1991)[67] provides a general analysis of this idea, using ∗ as an arbitrary 2-place connective:

> The canonical grounds for the truth of A ∗ B will be given by the introduction rules governing it, and its canonical consequences will be drawn by means of the elimination rules governing it.

However, this would also have the unwanted consequence of ruling out the logical constancy of the existence predicate E, which has a reasonable claim to logicality. In any case, the irreflexivity rule can be used as an introduction rule for the concept of distinctness, and can paired with a suitable elimination rule to fully define that concept. This is discussed in further detail below.

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[67] Page 247.

With greater than and less than thus discounted, are there any predicates apart from identity which are reasonable candidates for logical constancy? The requirements for logicality put forward in this dissertation mean that the investigation should be directed towards those which apply universally across all objects. Given this, a verb whose formalisation may have potential for logical constancy is *exists*, with its formalisation being some kind of existence predicate. The reasoning here is that it is a presupposition of standard model theoretic semantics for intuitionist and classical first-order logic that denotations for individual constants must exist in all interpretations of the language. Of course, existence is typically treated as a quantifier rather than a predicate, so quantification represents perhaps an orthodox treatment of the concept. However, the predicate interpretation is worth exploring in the context of the present investigation, and the quantifier interpretation is in any case explored in later sections of this dissertation.

Given this potential, and using E to denote an existence predicate and c an arbitrary individual constant, the following points state the nature of introduction and elimination rules for logical constants, and develop operational rules for E based on them.

- Introduction rules provide the sets of sufficient conditions (that is, the grounds) for asserting formulas including the constant as its main operator.

In this case, its universal application means that existence can always be predicated of any given object, regardless of the nature of the object in question. This suggests the following introduction rule:

$$\frac{}{Ec}\ \text{EI}$$

The empty premise in this rule is reminiscent of the =I rule; this is reasonable because the self-identity of any object is universal and thus can be asserted in the absence of any sufficient conditions.

- Elimination rules provide the set of necessary consequences of (that is, what follows from) the assertion of the formula with the constant as its main operator)

  In this case, the fact that an object can be said to exist has no consequences, at least from a logical (formal) point of view – again a product of existence's universality. This suggests that the E predicate should have no elimination rule (as is the case for ⊤, discussed later in this dissertation).

According to EI, the existence predicate can be applied to any individual constant in the language. This seems reasonable, since it corresponds to simply asserting that the object referred to by the individual constant exists, and thus can be applied with no conditions (which, in terms of operational rules, manifests itself as an empty premise). Also, should existence be predicated of any constant, no further conclusion regarding the constant (or anything else) can be deduced as a matter of logic. Again, this seems reasonable, since mere existence provides no further information regarding an object (which in terms of operational rules, manifests itself as an empty conclusion).

Given this reasoning and the fact that its operational rules conform to Criterion 1, it is difficult to deny logicality to E. The most obvious criticism of EI is that it lacks utility. However, utility was not included as a requirement for logicality in this dissertation, and as such (at least technically) this should not preclude E from logical constancy, since it clearly possesses the requisite formality, absolute generality, and topic neutrality. Thus this dissertation concludes that E, as defined by the operational rule EI, is a logical constant.

Since E has achieved logicality, it is worth investigating other universally applying but deductively useless predicates. Self-identity, the property of an object being identical with itself, is one such candidate. Due to its universal application, it would take nothing to assert its predication of any object, and from it nothing could be deduced. Its rules would therefore be identical with those of E stated above, and for similar arguments as those presented for E, this suggests that it too is a logical constant. However, since inferentialism holds that the meaning of a logical constant is exhausted by its operational rules, meaning that E and the self-identity predicate (and any other predicate applying universally in this way) would have the same meaning, and thus in fact be the very same predicate. According to this reasoning, and given the extensional nature of first order logic, existence and self-identity are simply alternative names for the same predicate, and predicates which are deductively useless in this way can provide only one logical constant.

A similar argument cannot be made regarding the operational rules for identity, =I and =E, and for an arbitrary individual constant, cI and cE, because each of these purports to define a logical constant from a different category. Recall that these identical introduction rules state that, using no antecedent sufficient conditions, it can be asserted that c = c. The syntactical simplicity of =I and cI mean that it is of little use in terms of determining whether the rule is best thought of as an introduction rule for = or for c. However, in the case of the elimination rules =E and cE, the individual constant c appears in the conclusion of the rule, which represents a violation of Natural Deduction Criterion for Logical Constancy 1. Furthermore, in terms of its meaning, it is more or less a statement of a first-order schematic version of the indiscernibility of identicals, a fact which non-controversially concerns identity. Further details on the notions discussed above can be found in Section 4.4.2.6 of this dissertation.

The conclusion of this section thus far is therefore that the natural language category of adjectives provides two logical constants: identity and existence (with the latter being chosen as the representative of universally applying predicates, given the previous remarks regarding the equivalence of such predicates due to their rules being the same). This conclusion is provisional, however, and will be reassessed in later sections of this dissertation as further criteria for logical constancy are added to Natural Deduction Criterion for Logical Constancy 1.

In terms of further predicate-based logical constants, the negations of these two, which will be called distinctness (≠) and non-existence (N) in this dissertation could also be seen as logical constants. The logicality of the former is justifiable on the basis of absolute generality and topic neutrality because it is a concept which does not apply to any object whatsoever. The logicality of the latter is justifiable on the basis that since it is an empty predicate, applying to no objects at all, and thus is absolutely general and topic neutral.

It seems reasonable that there is a principle of compositionality at play here – that is, that elements of formal systems which are definable in terms of other logical constants are themselves logical constants. This is because, due to it being entirely definable by elements which are themselves logical constants, the defined element would also possess the requisite properties of formality (that is, the constituent elements contributing only to the structure of the inference) and absolute generality / topic neutrality.

On the basis of the above, though ≠ and N are derivatively definable using = and E respectively, all four operators have strong claims to logical constancy. Given the comments in Section 2.3.1, their omission from the set of accepted logical constants is not justified on the basis of parsimony-based arguments advisable on the grounds of parsimony. Thus all four can be retained as logical constants.

Out of interest, operational rules for each are presented here. In the case of distinctness, proposed rules are as follows:

$$\frac{\varphi(c) \qquad \neg\varphi(t)}{c \neq t} \neq I1 \qquad\qquad \frac{\neg\varphi(c) \qquad \varphi(t)}{c \neq t} \neq I2$$

$$\frac{c \neq c}{\varphi} \neq E$$

The operational rules for non-existence (N) are:

$$\frac{Nc}{\varphi} NE$$

These sets of rules suggest the inclusion of ≠ and N as logical constants.

4.4.2.3. Connectives

Connectives have an immediate appeal as logical constants due to the fact that they can be attached to any kind of sentence, and are thus general in their application. Deviating somewhat from the proof theoretic focus of this dissertation, the most commonly cited examples of potential logical connectives will be introduced using the apparatus of truth tables (though even such tabular approaches can be regarded as syntactic, with 1 and 0 used rather than truth and falsity). Consider the following truth table style presentation, which includes each variation of zero-, one-, and two-place truth functional connectives, for the truth values of φ and ψ included in its first two rows.

Name	Symbol	Values			
Sentence	φ	T	T	F	F
Sentence	ψ	T	F	T	F
Verum	⊤	T	T	T	T
Falsum	⊥	F	F	F	F
Neutral	-φ	T	T	F	F
Negation	¬φ	F	F	T	T
Conjunction	φ ∧ ψ	T	F	F	F
Nand	φ ↑ ψ	F	T	T	T
Disjunction	φ ∨ ψ	T	T	T	F
Nor	φ ↓ ψ	F	F	F	T
Conditional	φ → ψ	T	F	T	T
Nif	φ ↛ ψ	F	T	F	F
Converse conditional	φ ← ψ	T	T	F	T
Converse Nif	φ ↚ ψ	F	F	T	F
Biconditional	φ ↔ ψ	T	F	F	T
Exclusive Disjunction	φ ↮ ψ	F	T	T	F

The above establishes the classical semantics for truth functional connectives. Natural deduction rules will now be presented for each of these connectives, which together represent a natural deduction system for classical propositional logic. The first 8 rules are of a simpler nature, while the remaining 6 rules are ‘derived’, and based on classical logic equivalences, as noted where necessary with each rule. There is some superfluity among the rules, since if minimal functional completeness is considered, not all rules are necessary for a classical propositional logic natural deduction system. For example, only ↑ is necessary for functional completeness. Again all 16 rules are retained so that there is some correspondence between the truth table semantics given above and the rules provided.

1. Verum

$$\frac{}{\top}\ \top\mathrm{I}$$

2. Falsum

$$\frac{\bot}{\varphi}\ \bot\mathrm{E}$$

3. Neutral

$$\frac{\varphi}{-\varphi}\ -\mathrm{I}$$

$$\frac{-\varphi}{\varphi}\ -\mathrm{E}$$

4. Negation

$$\begin{array}{c}[\varphi] \\ \vdots \\ \dfrac{\bot}{\neg\varphi}\ \neg I\end{array}$$

$$\frac{\varphi \qquad \neg\varphi}{\bot}\ \neg E1$$

$$\frac{\neg\neg\varphi}{\varphi}\ \neg E2$$

5. Conjunction

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\ \wedge I$$

$$\frac{\varphi \wedge \psi}{\varphi}\ \wedge E1$$

$$\frac{\varphi \wedge \psi}{\psi}\ \wedge E2$$

6. Conditional

$$\begin{array}{c}[\varphi] \\ \vdots \\ \dfrac{\psi}{\varphi \rightarrow \psi}\ \rightarrow I\end{array}$$

$$\frac{\varphi \rightarrow \psi \qquad \varphi}{\psi}\ \rightarrow E$$

7. Converse Conditional

$$\dfrac{\begin{matrix}[\psi] \\ \vdots \\ \varphi\end{matrix}}{\varphi \leftarrow \psi}\ \leftarrow\text{I}$$

$$\dfrac{\varphi \leftarrow \psi \qquad \psi}{\varphi}\ \leftarrow\text{E}$$

8. Disjunction

$$\dfrac{\varphi}{\varphi \vee \psi}\ \vee\text{I1}$$

$$\dfrac{\psi}{\varphi \vee \psi}\ \vee\text{I2}$$

$$\dfrac{\varphi \vee \psi \qquad \begin{matrix}[\varphi] \\ \vdots \\ \gamma\end{matrix} \qquad \begin{matrix}[\psi] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \vee\text{E}$$

9. Nif (equivalent to $\varphi \wedge \neg\psi$)

$$\dfrac{\varphi \qquad \begin{matrix}[\psi] \\ \vdots \\ \bot\end{matrix}}{\varphi \nrightarrow \psi}\ \nrightarrow\text{I}$$

$$\dfrac{\varphi \nrightarrow \psi}{\varphi}\ \nrightarrow\text{E1}$$

$$\dfrac{\varphi \nrightarrow \psi \qquad \psi}{\bot}\ \nrightarrow\text{E2}$$

10. Converse Nif

$$\dfrac{\psi \qquad \begin{matrix}[\varphi] \\ \vdots \\ \bot\end{matrix}}{\varphi \nleftarrow \psi}\ {\nleftarrow}\mathrm{I}$$

$$\dfrac{\varphi \nleftarrow \psi}{\psi}\ {\nleftarrow}\mathrm{E1}$$

$$\dfrac{\varphi \nleftarrow \psi \qquad \varphi}{\bot}\ {\nleftarrow}\mathrm{E2}$$

11. Biconditional (equivalent to $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$)

$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix} \qquad \begin{matrix}[\psi] \\ \vdots \\ \varphi\end{matrix}}{\varphi \leftrightarrow \psi}\ {\leftrightarrow}\mathrm{I}$$

$$\dfrac{\varphi \leftrightarrow \psi \qquad \varphi}{\psi}\ {\leftrightarrow}\mathrm{E1}$$

$$\dfrac{\varphi \leftrightarrow \psi \qquad \psi}{\varphi}\ {\leftrightarrow}\mathrm{E2}$$

12. Exclusive Disjunction (equivalent to (φ ∧ ¬ψ) ∨ (¬φ ∧ ψ))

$$\dfrac{\varphi \qquad \begin{array}{c}[\psi]\\ \vdots \\ \bot\end{array}}{\varphi \nleftrightarrow \psi}\ {\nleftrightarrow}\mathrm{I1}$$

$$\dfrac{\psi \qquad \begin{array}{c}[\varphi]\\ \vdots \\ \bot\end{array}}{\psi}\ {\nleftrightarrow}\mathrm{I2}$$

$$\dfrac{\varphi \nleftrightarrow \psi \qquad \varphi \qquad \psi}{\bot}\ {\nleftrightarrow}\mathrm{E1}$$

$$\dfrac{\varphi \nleftrightarrow \psi \qquad \begin{array}{c}[\varphi]\\ \vdots \\ \bot\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots \\ \bot\end{array}}{\bot}\ {\nleftrightarrow}\mathrm{E2}$$

13. Nand (equivalent to ¬(φ ∧ ψ))

$$\dfrac{\begin{array}{c}[\varphi]\\ \vdots \\ \bot\end{array}}{\varphi \uparrow \psi}\ {\uparrow}\mathrm{I1}$$

$$\dfrac{\begin{array}{c}[\psi]\\ \vdots \\ \bot\end{array}}{\varphi \uparrow \psi}\ {\uparrow}\mathrm{I2}$$

$$\dfrac{\varphi \uparrow \psi \qquad \varphi \qquad \psi}{\bot}\ {\uparrow}\mathrm{E}$$

14. Nor (equivalent to $\neg(\varphi \vee \psi)$)

$$\dfrac{\begin{array}{cc} [\varphi] & [\psi] \\ \vdots & \vdots \\ \bot & \bot \end{array}}{\varphi \downarrow \psi}\ {\downarrow}\mathrm{I}$$

$$\dfrac{\varphi \downarrow \psi \qquad \varphi}{\bot}\ {\downarrow}\mathrm{E1}$$

$$\dfrac{\varphi \downarrow \psi \qquad \psi}{\bot}\ {\downarrow}\mathrm{E2}$$

Dummett (1991)[68] refers to rules in which only one (potential) logical constant figures as *pure*. In these terms, it is reasonable to ask whether a criterion for logicality based on purity should be put in place. The intuitive attraction of pure rules is their simplicity; the concern regarding impure rules is the potential that they introduce circularity or regress into the definition they provide for the constant in question. However, such concerns are unwarranted in this case, since falsum is the only other constant referred to in this set of six negated connectives. Reference to the operational rules put forward for falsum give no reason to doubt its logicality, and in any case falsum can be defined as $\neg\varphi \wedge \varphi$. This avoids fears of regress[69], and therefore while falsum's presence in these six rules may reduce their aesthetic appeal, it does not call into question their logicality. This dissertation

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[68] Page 257.

[69] Also, the analysis contained in Section 2.2, in which the requirement of cognitive primacy for logicality specified that the understanding of logical notions must not depend on or involve an understanding of notions that must be classified as extra-logical. Since falsum is not extra-logical, its presence in the operational rules for the negated connectives is not a cause for concern.

will return to the issue of external reference in natural deduction rules in the discussion of modality in Section 4.4.4.

Referring back to the list of truth functional connectives, one of these which typically avoids discussion for obvious reasons is neutral, given the symbol -. In semantic terms, this one-place connective simply returns the truth value possessed by the sentence to which it is applied. Thus, in proof theoretic terms, it can be introduced to or eliminated from sentences in deductions with no restrictions. This is reflected in its operational rules. Like the existence predicate discussed previously, this connective could be criticised on the basis of its lack of utility. However, this does not seem a valid reason for it to be excluded from logical constancy, since it fulfils the criteria thus far put forward.

An example truth table and rules for a 3-place connective are provided below.

φ	ψ	χ	$\wedge^3\varphi\psi\chi$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

$$\frac{\varphi \qquad \psi \qquad \chi}{\wedge^3\varphi\psi\chi}\ \wedge^3\text{I}$$

$$\frac{\wedge^3\varphi\psi\chi}{\varphi}\ \wedge^3\text{E1} \qquad \frac{\wedge^3\varphi\psi\chi}{\psi}\ \wedge^3\text{E2} \qquad \frac{\wedge^3\varphi\psi\chi}{X}\ \wedge^3\text{E3}$$

The connective concerned is named $\wedge^3$, due to its being the 3-place equivalent of $\wedge$. Based on the operational rules themselves, there seems to be no reason to deny logicality to this connective. Including 4-place connectives, 5-place connectives etc. leads to a potential countable infinity of logical constants. However, in practical terms, all 3-place connectives can be reduced to 2-place connectives. In some cases this reduction is clear, as for the above $\wedge^3$, since $\wedge^3\varphi\psi\chi$ is equivalent to both $(\varphi \wedge \psi) \wedge \chi$ and $\varphi \wedge (\psi \wedge \chi)$, and thus its rules can quite obviously be replaced by using repeated applications of the ∧I or ∧E rules. In less clear cases, any 3-place connective can be replaced by 2-place connectives via conversion into disjunctive normal form.

The inclusion of $\perp$ in the set of logical constants is also arguably redundant, since it can be defined as for example $(\varphi \wedge \neg\varphi)$. However, it is included in this dissertation for reasons of elegance of expression. After all, as discussed below in the discussion of ↑, most connectives can be regarded as redundant due to their definability using other connectives, but are likewise retained to streamline expressiveness, and retain natural deduction's objective of closely modelling 'natural' reasoning.

Note also the point previously made regarding a 'principle of compositionality' for logical constancy. This principle means that the three place connectives discussed here qualifies as logical constants, due to their being definable in terms of other logical constants.

It is well known that truth functional connectives can be replaced by a single connective, known as the Sheffer Stroke, referred to in this dissertation as ‘nand’ (to highlight its status as the negation of and), and represented by ↑ (‘nor’, represented in this dissertation by ↓ also has this property, which is known as ‘functional completeness’). This suggests that the most parsimonious presentation of logical constants drawn from connectives would only include nand. Inspection of the operational rules of nand shows that it includes falsum. However, falsum itself can be replaced by nand, given that it is equivalent to the following:

$$(\varphi \uparrow (\varphi \uparrow \varphi)) \uparrow (\varphi \uparrow (\varphi \uparrow \varphi))$$

This means that the following rules can be proposed for nand:

$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ (\varphi \uparrow (\varphi \uparrow \varphi)) \uparrow (\varphi \uparrow (\varphi \uparrow \varphi))\end{matrix}}{\varphi \uparrow \psi} \uparrow I1a \qquad \dfrac{\begin{matrix}[\psi] \\ \vdots \\ (\varphi \uparrow (\varphi \uparrow \varphi)) \uparrow (\varphi \uparrow (\varphi \uparrow \varphi))\end{matrix}}{\varphi \uparrow \psi} \uparrow I2a$$

$$\dfrac{\varphi \uparrow \psi \qquad \varphi \qquad \psi}{(\varphi \uparrow (\varphi \uparrow \varphi)) \uparrow (\varphi \uparrow (\varphi \uparrow \varphi))} \uparrow Ea$$

A similar approach can also be applied to other truth functional connectives which include falsum. The following list shows the equivalences between falsum and some of these connectives.

$\bot \equiv \varphi \nrightarrow \varphi$

$\bot \equiv \varphi \nleftarrow \varphi$

$\bot \equiv \varphi \nleftrightarrow \varphi$

Using these equivalences, falsum can be replaced in each of the operational rules for ↛, ↚, and ↮, to give similar falsum-free operational rules for them. However, these rules violate this dissertation's Natural Deduction Criterion for Logical Constancy 1, which states operational rules must not allow the introduced element to appear as the main connective in the premise of the I rule and must not allow the eliminated element to appear as the main connective in the conclusion of the E rule. While Criterion 1 could be revised at this point so that it did not exclude rules such as ↑I1a, ↑I2a and ↑Ea, the previous discussion regarding falsum's use in ↑I1, ↑I2 and ↑E suggests that this is not necessary, since the reference to falsum is not circular and thus non problematic.

The analysis of connectives conducted thus far has sought potential logical constants using an approach using truth tables, in which those connectives which are truth functional were analysed to evaluate their potential to be logical constants. Now, an approach based on natural deduction rules will be employed. Here, proof rules will be proposed directly and examined to determine whether they define a logical constant.

The most notable example of this approach is the proposed logical connective tonk. tonk was introduced in the influential Prior (1960), and is defined by the following operational rules:

$$\frac{\varphi}{\varphi \text{ tonk } \psi}\text{ tonkI} \qquad\qquad \frac{\varphi \text{ tonk } \psi}{\psi}\text{ tonkE}$$

Prior (1960) is written in a humorous style, which somewhat obscures the author's view on the problem which this connective presents. However, consider the following simple tonk-based reasoning:

$$\cfrac{\cfrac{\varphi}{\varphi \text{ tonk } \psi}\text{ tonkI}}{\psi}\text{ tonkE}$$

This reasoning shows that tonk allows any proposition to be proved on the basis of any other proposition. This is undesirable according to any reasonable interpretation of $\vdash$, since allowing the deduction of any arbitrary $\varphi \vdash \psi$ effectively allows a 'proof theoretic free for all'.

One interpretation of Prior (1960) is that Prior's target is inferentialism, the thesis, important in the context of this dissertation, that the meaning of and justification for the logical constants is entirely determined by their introduction and elimination rules. Since the operational rules given for tonk resemble in general the nature and structure those given for accepted logical constants such as $\wedge$ and $\vee$, the inferentialist seems committed to accepting it as defining a new logical constant, even though the undesirability of the noted explosion in provability would mean it would compromise a proof system to which it was added.

In an early response to Prior (1960), Belnap (1962) agrees with this interpretation, stating:

> A possible moral to be drawn is that connectives cannot be defined in terms of deducibility at all; that, for instance, it is illegitimate to define *and* as that connective such that (1) A-*and*-B ⊢ A, (2) A-*and*-B ⊢ B, and (3) A, B ⊢ A-*and*-B. We must first, so the moral goes, have a notion of what *and* means, independently of the role it plays as premise and as conclusion. Truth-tables are one way of specifying this antecedent meaning[70]

In attempting to diagnose the difficulty with tonk, Belnap points out that the extension to the logical system to which it is added is non-conservative, in that tonk allows sentences which do not contain tonk to be proved and which were not provable before its introduction. Hence while the problem with tonk may not be discernible on the basis of its operational rules themselves, its problems become apparent when considered within the context of the other connectives included in the proof system to which it is added. Thus, Belnap holds onto the inferentialist line of the meaning of a logical constant being fully defined by its operational rules, but rejects connectives such as tonk on the basis that they fail with respect to the totality of other logical constants included in the system.

[70] Belnap (1962). Page 130.

Belnap therefore holds that tonk is problematic in that it provokes a non-conservative extension when added to standard natural deduction systems. However, what it is about the operational rules of tonk, and thus what mechanism allows these operational rules to extend non-conservatively remains unclear in Belnap's analysis. Understanding this mechanism is required to determine how to legitimately exclude tonk using only the tools of proof theory. In the terms used in this dissertation, what is sought is a criterion for operational rules to rule out cases such as tonk (but not rule out any legitimate logical constants), and thus prevent the occurrence of such non-conservative extensions.

Dummett (1991) provides the basis for such a criterion by introducing the notion of harmony. Harmony is based on first distinguishing two different general categories of principles which are embodied in linguistic practice. In Dummett's words, "The first category consists of those that have to do with the circumstances that warrant an assertion, the basis on which we may recognise a statement as having been established"[71]. This is a position which Dummett associated with a verificationist approach to the theory of meaning. However, Dummett goes on to acknowledge that:

> Clearly, however, our use of the language cannot be exhaustively described in terms of our application of principles of verification. If that were all, we

[71] Dummett (1991): Page 211.

> should be skilled at making assertions but incapable of responding to the assertions of others[72].

This leads him to the second category:

> The pragmatists should be understood as making the converse proposal that the content of a statement should be regarded as determined by its consequences for one who accepts it as true: my understanding of the statement consists in my grasp of the difference it would make to me if I were to believe it. A related notion belonging to the same broad category is that of what a speaker commits himself to by making a given assertion[73].

So that linguistic practice functions effectively, the understanding of the content of a statement should balance the verificationist focus on what warrants an assertion and the pragmatist focus on its consequences. This is particularly so in the case of logical constants, since their introduction and elimination rules respectively permit such clear access to what warrants an assertion and what its consequences are. Dummett further argues that the notion of harmony can itself be made precise via Belnap's insight regarding conservative extensions. Dummett explains why harmony between what warrants an assertion and the assertion's consequences ensures that non-conservative extensions are avoided:

> Consider, now, not a formal theory but a natural language; and suppose it contains an expression E such that the conventional consequences of

[72] Dummett (1991): Page 211.

[73] Dummett (1991): Page 211.

> applying E are in disharmony with the conventional warrant for doing so. By means of E, we may be able to say things we should have no way of saying if the language did not contain that expression; but the disharmony means that we are accustomed to draw conclusions from statements made by means of E that what we treat as justifying the assertion of those statements does not entitle us to draw. Now those conclusions, if expressed verbally at all, cannot consist of statements containing E; for the drawing of such conclusions must count as part of our conventions governing the justification of assertions involving E. If there is disharmony, it must manifest itself in consequences not themselves involving the expression E but taken by us to follow from the acceptance of a statement S containing E.[74]

It remains to show how the harmony criterion can be practically implemented in terms of a test for the operational rules defining candidates for logical constancy. Dummett does so in the following passage, in which he provides the analogue of the above analysis in terms of formal logic:

> For an arbitrary logical constant c, [the analogue] is that it should not be possible, by first applying one of the introduction rules for c and then immediately drawing a consequence from the conclusion of that introduction rule by means of an elimination rule of which it is the major premiss, to derive from the premisses of the introduction rule a

[74] Dummett (1991): Page 218.

> consequence that we could not otherwise have drawn. Let us call any part of a deductive inference where, for some logical constant c, a c-introduction rule is followed immediately by a c-elimination rule a 'local peak for c'. Then it is a requirement, for harmony to obtain between the introduction rules and elimination rules for c, that any local peak for c be capable of being levelled, that is, that there be a deductive path from the premisses of the introduction rule to the conclusion of the elimination rule without invoking the rules governing the constant c.[75]

The process of eliminating these local peaks can be demonstrated in the case of ∧ as follows, as defined using the introduction and elimination rules given above. Consider a deduction which includes the following as a part of it (where £ and ¥ represent segments of deductions):

$$\dfrac{\dfrac{\begin{matrix}\text{£} & & \text{¥} \\ \varphi & & \psi\end{matrix}}{\varphi \wedge \psi}\ \wedge I}{\varphi}\ \wedge E$$

In Dummett's words, "it is obvious that this detour through φ ∧ ψ was superfluous"[76]. That is, the conclusion of the above deduction segment could have been obtained in the overall deduction of which it forms part using the following ∧-less proof:

[75] Dummett (1991): Page 247.

[76] Dummett (1991): Page 249.

$$\begin{array}{c} \pounds \\ \varphi \end{array}$$

Consider now a similar analysis involving a local peak created by tonk (where € represents a segment of a deduction):

$$\dfrac{\dfrac{\begin{array}{c}\text{€} \\ \varphi\end{array}}{\varphi \text{ tonk } \psi}\ \text{tonk I}}{\psi}\ \text{tonk E}$$

A similar move to that used to eliminate the local peak created by tonk cannot be used here, since the ultimate conclusion of the proof, ψ, cannot be deduced using only the resources of €.

Hence, for Dummett there are three important considerations regarding tonk:

- The local peak created by tonk cannot always be eliminated.
- The introduction and elimination rules for tonk do not match in terms of the warrant for asserting a tonk expression and the consequences of assertion a tonk expression.
- tonk provokes a non-conservative extension in a logical system.

A combination of these facts indicates what is objectionable about the addition of tonk to a logical system. However, Dummett does not state that each of the three are extensionally equivalent across operators defined in natural deduction systems. In fact, he distinguishes total harmony, which corresponds to a connective's not leading to a non-conservative extension in a system, from partial harmony, which corresponds to the elimination of local peaks.

In various papers such as Read (2010), Read further develops the notion of harmony. His approach to the issue is to develop a concept which he calls general elimination harmony. This presents an essentially algorithmic means of developing a harmonious elimination rule from any given introduction rule (or rules, in cases where multiple introduction rules exist, such as in the operational rules for v). Read (2010) also holds that the existence of the general elimination harmony algorithm also "better matches what Gentzen meant by saying that the introduction rules serve to define the meaning of the logical expressions and that the elimination rules are no more than a consequence of the meaning so conferred"[77].

General elimination harmony operates in the following manner[78].

> Suppose that the grounds for assertion of $\delta\vec{\alpha}$ (some formula with main connective δ) are given schematically as Π_i, where $\Pi_i : 1 \leq i \leq m$ is a collection of subproofs or derivations. We can represent those proofs Π_i as derivations
>
> $$\frac{\pi_{i1} \ \ldots \ \pi_{in_i}}{\delta\vec{\alpha}}\ \delta I$$

[77] Read (2010). Page 575, with the name 'general elimination harmony' taken from Francez and Dyckhoff (2012), page 614.

[78] Read (2010). Page 563.

Which I will write for short as $\pi_{i1}, \ldots, \pi_{in_i} \Rightarrow \delta\vec{\alpha}$, giving the grounds Π_i for the assertion of $\delta\vec{\alpha}$. Then the harmonious form of the elimination-rule is:

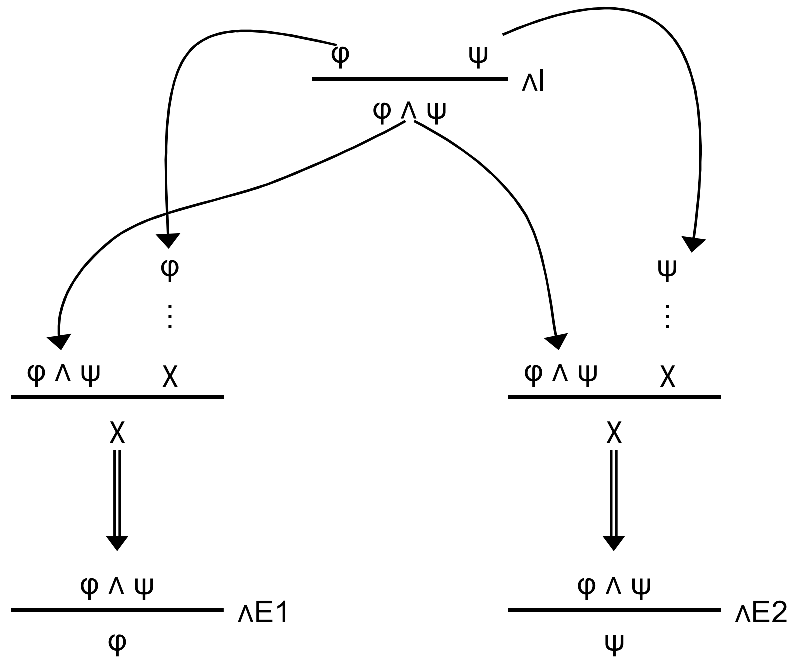
$$\dfrac{\delta\vec{\alpha} \quad \begin{matrix}(\pi_{1j_1}) \\ \vdots \\ \gamma\end{matrix} \quad \ldots \quad \begin{matrix}(\pi_{mj_m}) \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \delta E$$

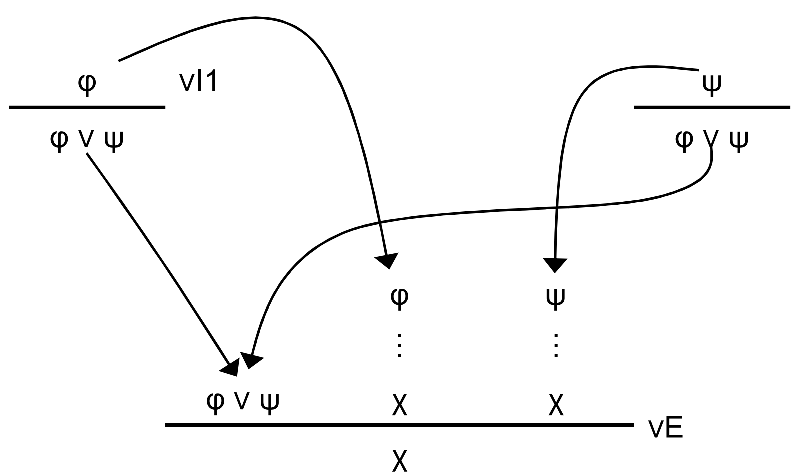
Discharging assumptions π_{ij_i}. That is, given an assertion of $\delta\vec{\alpha}$, and derivation(s) of [some formula] γ from the various grounds for asserting $\delta\vec{\alpha}$, we may infer γ and discharge the assumption of those grounds. Those grounds may be multiple, for there may be several cases of the introduction rule, as in ∨I. The inversion principle[79] requires that, in any application of the E-rule, there be *m* minor premises, each deriving γ from some π_{ij}, that is, for each *i* there needs to be a derivation of γ from π_{ij} for some *j*.

The use of this rule is perhaps best understood through application to the connectives ∧ and ∨, as shown in the following conversions with single line arrows showing how each element of the introduction rule is used in the elimination rule, and double line arrows showing subsequent simplifications of rules (or in the case of ∧, intermediate steps which are simplified to the final forms of the ∧E rules).

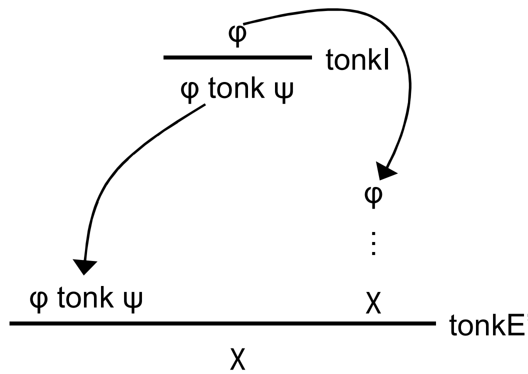
---

[79] Which states that "whatever follows from the direct grounds for asserting a proposition must follow from the proposition." (Negri and von Plato (2001), page 6, and is quoted in (Read (2010), page 562).





Returning to the analysis of tonk, applying general elimination harmony to it gives the following results (again with arrows added to demonstrate the use of the elements in tonk introduction in the elimination rule):



No simplification of this rule is possible. The resulting tonk elimination rule tonkE’ is different from Prior’s tonk elimination rule, which, according to Read, demonstrates that Prior’s rules for tonk are not in harmony. Thus Read agrees with Dummett that tonk as defined by tonkI and tonkE is not a harmonious operator, though Dummett reaches his conclusion due to local peaks containing tonk not being eliminable in the same way that local peaks for ∧ and ∨ can be eliminated. That the local peak formed by using Read’s tonk elimination rule with Prior’s tonk introduction rule is shown below, with the local peak eliminated simply by replacing each instance of γ with an instance of φ:

$$\dfrac{\dfrac{\varphi}{\varphi \text{ tonk } \psi}\,\text{tonkI} \qquad \begin{matrix}\varphi\\ \vdots \\ \varphi\end{matrix}}{\varphi}\,\text{tonkE'}$$

Local peak elimination is possible here by reducing the above to the following:

$$\begin{matrix}\varphi\\ \vdots \\ \varphi\end{matrix}$$

tonk defined by tonkI and tonkE’ is an operator with no deductive utility, since:

- Using φ to deduce φ itself (thus replacing each instance of γ with an instance of φ in the above) does not offer any utility to the deduction, since φ tonk ψ can only be deduced from φ in any case.
- Using φ to deduce φ tonk ψ (by using the tonk introduction rule and thus replacing each instance of γ with an instance of φ tonk ψ in the above) does not offer any utility, since it simply eliminates φ tonk ψ in favour of φ tonk ψ.
- Using φ to deduce any other γ does not offer any utility, since γ could have been arrived at directly from φ without the need to first deduce φ tonk ψ.

The truth functional connectives discussed in this section are in general elimination harmony. However, special comment is required regarding harmony and ¬. Applying Read's general elimination harmony approach to the introduction rule given above for negation leads to the following harmonious pairing:

$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \bot\end{matrix}}{\neg\varphi}\ \neg I \qquad\qquad \dfrac{\varphi \qquad \neg\varphi}{\bot}\ \neg E1$$

The inclusion of ¬ defined by the above pair of rules leads to minimal logic. These rules can then be augmented by to inclusion in the system of ex falso quodlibet, which has previously been discussed in this dissertation as a rule for falsum elimination:

$$\dfrac{\bot}{\varphi}\ \bot E$$

¬I, ¬E1, and ⊥E give intuitionistic logic. For a natural deduction system for classical propositional logic, the following additional rule is required:

$$\dfrac{\neg\neg\varphi}{\varphi}\ \neg E2$$

However, since harmony was achieved for the system containing only ¬ as defined by ¬I, ¬E1 and in the presence of ⊥E, the further addition of ¬E2 means that harmony is seemingly lost. This suggests that logical constancy should only be granted to the truth functional operators defined intuitionistically. This is the route Dummett takes, writing:[80]:

> This more detailed look at classical negation confirms what we had already concluded, that it is not amenable to any proof-theoretic justification procedure based on laws that may reasonably be regarded as self-justifying. That is not, of course, to say that the classical negation-operator cannot be intelligibly explained; it is only to say that it cannot be explained by simply enunciating the laws of classical logic.

However, Dummett continues, "Intuitionistic logic, however, has come out of our enquiry very well"[81], due to the fact that harmony can be found in a straightforward manner between ¬I and ¬E1, the intuitionistic negation elimination rule.

Thus Dummett suggests that the classical version of ¬ is problematic for accounts of harmony. Various strategies present themselves for dealing with this. One is of course to retain the analysis of harmony for ¬, and accept the ramifications implied above regarding the logical constancy of the intuitionistic and classical versions of ¬. This aligns with Dummett's thinking.

Read (2000) takes a different approach, and tries to retain harmony and also the classical version of ¬. The basis of his objection to Dummett's claim is that it

[80] Dummett (1991), page 299.

[81] Dummett (1991), page 299.

“depend[s] crucially on the presentation of logic which is considered”[82]. He points out that the standard sequent calculus system **LK** can be shown to capture classical logic in a harmonious manner, by allowing multiple conclusions. He then acknowledges that[83]:

> Dummett and Prawitz are, of course, not unaware of the existence of LK. They exclude multiple conclusions from consideration because they allow the assertion of disjunctions neither of whose disjuncts is assertible.

Read (1999) contains a further attempt to achieve harmony for the classical version of ¬, while avoiding the problems associated with multiple conclusion systems. His approach here centres on ↑ instead of ¬. However, demonstrating that the operational rules of ↑ are in harmony is sufficient to resolve the problems discussed above, due to the functional completeness of ↑ and ⊥.

Taking his lead from aforementioned success of multiple conclusion logics, Read notes that “the effect of multiple-succedent can be achieved in natural deduction without such a radical departure from the normal single-conclusion format, where what is proved at each juncture is a (single) well-formed formula on certain assumptions”[84]. This is done by introducing ↑ not as the main operator in the formula introduced by the I rule, but as a disjunct in a formula whose main

[82] Read (2000), 143.

[83] Read (2000), 145.

[84] Read (1999). Page 9.

operator is ∨. Read refers to this as introducing ↑ into a 'disjunctive context'. Read proposes the following operational rules for ↑:

$$\frac{\begin{matrix}[\varphi]\\ \vdots \\ \chi\end{matrix}}{(\varphi \uparrow \psi) \vee \chi}\ \uparrow I1(C) \qquad \frac{\begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{(\varphi \uparrow \psi) \vee \chi}\ \uparrow I2(C)$$

$$\frac{\varphi \uparrow \psi \qquad \varphi \qquad \psi}{\chi}\ \uparrow E(C)$$

Using these rules, Read proves the rule of double negation elimination; that is, that φ can be proved from ¬¬φ[85]. This is then sufficient for a proof of completeness for classical logic[86]. This means that if Read can claim to have rehabilitated harmony in the truth functional operators of classical logic, since due to functional completeness, harmony in ↑ is sufficient for all truth functional operators.

Is Read successful in the above endeavour to achieve harmony between the combination of ↑I1(C) and ↑I2(C), and ↑E(C)? A striking thing about ↑I1(C), ↑I2(C), and ↑E(C) is that, in contrast to all other operational rules for connectives, what is introduced is not the same as what is eliminated. The resulting operational rules for classical ↑ do not permit the elimination of local peaks caused by its successive introduction and then elimination, a fact which Read acknowledges. However, Read does not equate harmony with local peak elimination, holding

[85] Read (1999). Page 9.
[86] Read (1999). Page 9 to 11.

instead that “harmony consists rather in the justificatory relation between the I and E rules”[87]. Thus Read holds that the elimination rule can be constructed from the introduction rules by ensuring that it reflects the grounds for asserting φ ↑ ψ. In this case, these are not grounds for assertion for φ ↑ ψ but rather for (φ ↑ ψ) ∨ χ, since it is this which appears in the conclusion of the introduction rule ↑I(C). However, given that (φ ↑ ψ) ∨ χ can be obtained from φ ↑ ψ using a simple application of the ∨I rule, this objection is not problematic and can be dismissed.

The above establishes the basics of harmony between introduction and elimination rules, and Read’s method of general elimination harmony. In the context of the current dissertation the following questions are of interest:

1. Can harmony form the basis of a precise criterion for logical constancy? That is, how clear is the notion of harmony, and can it unambiguously distinguish cases which adhere to the criterion and those which do not?
2. What logical justification can be provided for putting forward a criterion for logicality based on harmony?
3. How should connectives such as tonk whose operational rules are non-harmonious be treated in terms of logicality?

Taking the first question first, one of the attractions of Read’s approach is that it does offer this clarity. This is because it provides a formula for generating a unique elimination rule (or set of elimination rules) from any given introduction rule (or set

[87] Read (1999), Page 12.

of introduction rules). Thus, if Read's general elimination harmony approach is adopted, there is less room for debate regarding whether a pair of introduction and elimination rules are in harmony or not. This is more amenable to precision regarding the border between the logical and the non-logical than Dummett's approach, which includes two different conceptions of harmony: Total harmony, which is the requirement that new rules must not lead to non-conservative extensions; and Intrinsic harmony, which is the requirement that rules must be such that local peaks can be eliminated. However, even in the case of Dummett's approach, that extensions generated are non-conservative and that local peaks can be eliminated are parameters which can be verified. This indicates that harmony does have the requisite precision to form a basis for a criterion for logical constancy. The case for a criterion based on harmony is further strengthened by its link to the key requirement for logicality of formality. As has been shown, non-harmonious operators such as tonk lead to non-conservative extensions of logical systems, and thus can add content to a set of formulas. However, operators that are purely formal could not do this. This means that a harmony-based criterion, in legislating against non-conservative extensions, is fundamentally based on formality. The response to the first question posed above is therefore affirmative.

Turning to the second question, given that absolute generality and topic neutrality were identified as the key requirements for logicality, it does not seem reasonable to object to non-harmonious constants such as tonk simply on that basis that they provoke non-conservative extensions of a logical system. In terms of absolute generality and topic neutrality, tonk seems to score just as well as any of the truth functional connectives discussed thus far.

This reasoning suggests that tonk is best seen as a logical constant, and due to the unwanted results it leads to, its exclusion from logical systems must be based on a different reason. However, Dummett's reasoning regarding the balance between the warrant for asserting an expression with a given connective and the consequences of its assertion which harmony delivers permits a different conclusion. The imbalance between warrant and consequence concerns not the potential for the logicality of a connective, but rather its coherent meaningfulness. Thus harmony actually operates at a more fundamental level than logicality, as long as the reasonable assertion that coherent meaningfulness is prerequisite for logicality is granted. If it is accepted that an element of a formal system must both be meaningful and absolutely general / topic neutral to qualify for logicality, harmony can legitimately be considered a prerequisite for logicality also, even though it does not involve an assessment of the key requirement for logicality itself. Alternatively, it may be claimed that coherence itself implies logicality. This point addresses the third query posed at the outset of this discussion.

However, before confirming harmony as a criterion, it is worth considering the discussion of Dummett on harmony which is found in Read (2010). Read begins by pointing out, in discussing Brandom (2001) and following thought found in Dummett (1991), that requiring harmony between introduction and elimination rules also requires that *both* rules considered in isolation from the other should provide the entire meaning – that is, the sufficient conditions for its assertion and the necessary consequences of its assertion. Specifically, Read writes:

> What was wrong with the analytic validity views which Prior was attacking was the suggestion that the meaning of an expression was given by the totality of rules governing its use. As we saw, Brandom equates the I-rule with the set of sufficient conditions for assertion of a statement containing the expression, and the E-rule with the set of necessary consequences of that assertion. Prior took assertion of φ to be sufficient for inferring φ tonk ψ and assertion of ψ as necessary for it. Hence the necessary and sufficient conditions come apart. If we are to avoid that situation, we need to capture all the meaning of a term in both types of rule. Rather than the one constituting sufficient conditions, the other, necessary conditions, as Brandom claims, each in their totality constitutes both necessary and sufficient conditions.

The fact that both the introduction and eliminate rules encapsulate the meaning of an operator is what permits harmony to obtain between them. It also means that each can be derived from the other by using Read's General Elimination Harmony approach. That elimination rules can be derived from introduction rules is something which has been discussed as far back as in the work of Gentzen[88], but the requirement of harmony means that the converse is also true. In terms of

[88] Gentzen (1969). Page 80.

which is given precedence as that which in a more fundamental sense gives the true (in whatever way 'true' is here interpreted) meaning of an operator could be debated but is perhaps of limited importance. Hodes (2004)[89] holds that introduction rules are more apt as meaning specifications for negation, disjunction and first-order existence; elimination rules are more apt for the conditional and first-order universality; and in the case of conjunction, neither is overtly constitutive of meaning.

In terms of a response to the third question posed above, regarding how non-harmonious connectives should be treated in terms of logicality, it seems justified to hold that logicality should be denied to pairs of introduction and elimination rules which are not in harmony. This is essentially due to the link between harmony and formality, which shows that a harmony-based criterion is based on the fundamental requirement for logicality of formality. The alternative would be to place non-harmonious connectives in a separate category which could be referred to as 'logical but non-admissible'. Such a category could have all the hallmarks which could be reasonably expected from the logical, but for some reason are deemed inadmissible and should thus be excluded from systems of deduction. Examples of this include a lack of harmony, but also perhaps superfluity (such as in the case of the three-place connectives) or a lack of utility (such as in the case of cI and cE considered in Section 0). However, it appears more reasonable to exclude them from logicality all together.

[89] Page 143.

More could be said regarding tonk and harmony, both in terms of the philosophical motivations for requiring it and the best means of implementing it. Due to the interest created by tonk, the harmony criterion for logical constancy has received significant attention. Literature about it thus abounds, with useful summaries being contained in Hjortland (2010) and Sternberger (2009). However, from the point of view of this dissertation, the key conclusions are that the inclusion of a harmony-based criterion for logical constancy is required to exclude connectives such as tonk, which can be defined using natural deduction operational rules but which should not be afforded logical constancy; and that Read's general elimination harmony approach provides the requisite features on which to base a precise criterion for logical constancy.

In light of these conclusions, this dissertation offers the following, second natural deduction criterion for logical constancy:

- Natural Deduction Criterion for Logical Constancy 2: The introduction and elimination rules for logical constants must be in (general elimination) harmony.

Note that general elimination harmony guarantees that introduction and elimination rule pairings must be define the same operator, since the elimination rule is derivable using the general elimination approach from the introduction rule, and vice versa. Note also that natural deduction systems for *both* intuitionistic and classical propositional logic (and also minimal logic) are returned as including only logical constants when assessed using this harmony-based criterion.

Before moving on to further candidates for logical constancy, mention (now that both concepts have been introduced) of harmony with respect to identity is required. Recall the operational rules for E and = from Section 4.4.2.2:

$$\frac{}{c = c}\ {=}\mathrm{I} \qquad\qquad \frac{(c = t) \qquad\qquad \varphi(c)}{\varphi(t)}\ {=}\mathrm{E}$$

$$\frac{}{Ec}\ \mathrm{EI}$$

Here, =I is a notational variant of EI. This is problematic for the purported harmony of = because, if the combination of introduction rules and the definition of general elimination harmony define the harmonic elimination rules for an operator, how can it be that =I is paired with =E; but EI has no elimination rule?

Read (2004) discusses the potential logical constancy of = and offers a solution to the problem noted above by modifying the EI rule. His suggestion is as follows (with notation adapted to suit that which is used in this dissertation):

$$\frac{\begin{array}{c}[Fc]\\ \vdots \\ Ft\end{array}}{c = t}\ {=}\mathrm{I}'$$

This rule is put forward with the proviso that F does not occur (as a predicate variable) in any assumption other than Fa. Read also notes that since it is trivial that Fa ⊢ Fa, =I follows immediately (here, F does not occur in any assumption other than Fa because there are none). Applying general elimination harmony to =I' gives the following elimination rule for =:

$$\dfrac{(c = t) \qquad \begin{matrix}[Fc] \\ \vdots \\ Ft\end{matrix}}{Ft}\ =E'$$

The above should suffice to show that, through Read (2004)'s modifications to the operational rules for =, harmony can be achieved for it, and the previously noted problem of = and E having the same introduction but a differing (or, in the case of E, no) elimination rule is resolved. The reader is referred to Read (2004) for some further detail required to also generalise the = rules to an arbitrary context (and Kremer (2007) also contains some simplifications to Read (2004)'s reasoning).

4.4.2.4. Quantifiers

The inclusion of a quantifier as a logical constant in a system represents the move from propositional logic to first order logic. In the case of ∀, it is universal quantification which is concerned. First order universal quantification can be given in a natural deduction system using the following rules:

$$\frac{\varphi(a)}{\forall x\varphi(x)}\ \forall I \qquad\qquad \frac{\forall x\varphi(x)}{\varphi(a)}\ \forall E$$

Elimination of local peaks of the following type formed by the sequential application of ∀I and ∀E is a simple affair:

$$\dfrac{\dfrac{\varphi(a)}{\forall x\varphi(x)}\ \forall I}{\varphi(a)}\ \forall E$$

However, elimination of local peaks of the following type is more complex:

$$\dfrac{\dfrac{\varphi(a)}{\forall x\varphi(x)}\ \forall I}{\varphi(b)}\ \forall E$$

Assuming that correct substitution of all occurrences of variables and individual constants have been done in accordance with the relevant proviso for the ∀ rule (see Section 4.3), this local peak can be eliminated also. That this is true is supported by the following lemma, proved in Tennant (1978)[90]:

> If there is a proof Π involving the formula φ and based on assumptions contained in Δ and u is a *closed term*, then there is a proof Π_u^b (with the superscript and subscript notation being the result of replacing u at all its closed occurrences in Π with b, where b is understood not to occur in Π)

[90] Tennant (1978). Pages 66 to 69.

involving the formula φ^b_u (with the same superscript and subscript notation) and based on assumptions contained in Δ^b_u. A *closed term* here is understood to mean the following:

"An application of the rule:

$$\frac{\begin{array}{c}\Pi \\ \varphi\end{array}}{\forall x \varphi^a_t}$$

is said to close all occurrences of a in Π. Likewise, an application of the rule:

$$\frac{\begin{array}{c}\Pi_1 \\ \exists x \varphi\end{array} \qquad \begin{array}{c}\varphi^x_a \\ \Pi_2 \\ \psi\end{array}}{\psi}$$

is said to close all occurrences of a in Π_2"[91]

Also, ∀E can be arrived at from ∀I using Read's general elimination harmony approach. Furthermore, they do not violate this dissertation's Natural Deduction Criterion for Logical Constancy 1. As such its logicality does not appear to be questionable. This accords with the widely accepted view that first order logic is firmly implanted in the realms of logic.

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[91] Tennant (1978). Page 66.

In terms of ∃, first order existential quantification can be given in a natural deduction system using the following rules:

$$\frac{\varphi(a)}{\exists x\varphi(x)}\ \exists I \qquad\qquad \frac{\exists x\varphi(x) \qquad \begin{array}{c}\varphi(a)\\ \vdots \\ \psi\end{array}}{\psi}\ \exists E$$

Here, a simple example of local peak elimination is as follows:

$$\frac{\dfrac{\varphi(a)}{\exists x\varphi(x)}\ \exists I \qquad \begin{array}{c}\varphi(a)\\ \vdots \\ \psi\end{array}}{\psi}\ \exists E$$

As in the case of the universal quantifier, more complex cases can be suggested, but Tennant's lemma can again be used to dispense with them.

None of the above is particularly controversial. More controversial are other examples of determiners which are not expressible in first order logic. For example, first-order logic has very limited expressive power on finite structures[92]. The investigation of this in this dissertation is deferred until Section 0.

4.4.2.5. Potential Further Logical Constants

The discussion in Section 4.4.2 up to this point used formal systems as a starting point, investigating the potential for logical constants to be found among the individual constants, predicates, connectives, and quantifiers of first order logic.

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92 Kolatis and Vardy (1990)

The following sections of this dissertation address the issue using natural language as a starting point – that is, investigating various parts of natural language, to determine if there are any elements of formal systems which correspond to them which are potential candidates for logical constancy.

Underlining the above point, it is of course not words in natural language which are logical constants, but elements of formal systems. Hence the discussion of natural language is simply used to stimulate the search of potential logical constants. Also, there are various well-known problems associated with the accurate and complete formalisation of natural language words into elements of formal systems. The use of natural language can be argued to include many complications which mean that accurate formalisation is not possible. These complications include for example conversational implicatures and vagueness in predicates (natural language predicates often have extensions without precise boundaries).

However, this is not relevant in the present study, because what is important is the nature of the elements of formal systems which arise from their discussion, not that these elements of formal systems are perfect formalisations of the natural language words in question.

An adverb can be defined as follows[93]:

[93] https://www.lexico.com/en/definition/adverb

> A word or phrase that modifies or qualifies an adjective, verb, or other adverb or a word group, expressing a relation of place, time, circumstance, manner, cause, degree, etc. (e.g., 'gently', 'quite', 'then', 'there').

Given the above definition, the formalisation of an adverb in a logical language would attach to a predicate in the same way that a predicate attaches to an individual constant. Their use in formal languages could however, be avoided by expanding the non-logical vocabulary of the language by enriching it with more predicates. For example, if a formalisation of the verb 'goes' is included as a predicate in a formal language, and 'goes quickly' is also required, a formalisation of 'goes quickly' as a separate adjective could be added to the language.

However, this results in some loss of information in the translation from natural to formal language, since the link between the two predicates (that is, that all individual constants to which the formalisation of 'goes quickly' is attached must also have the formalisation of 'goes' attached to them also) is lost. Cresswell (1974)[94] puts the point as follows

> It may be thought that words with related meanings should be so expressed that their connections are made quick, e.g. 'quickly' and 'quick' or 'runs' and 'runner' should not appear as independent lexical items.

[94] Page 460.

That is to say, a formalisation of 'If x runs quickly then x runs' as 'Fx → Gx' fails to capture it as a logical truth. In terms of existent treatments of the logic of adverbs, Cresswell (1974)[95] also contains the following:

> There are two basic approaches to the analysis of adverbial constructions in formalised representations of English. One is to follow Richard Montague and treat them as sentential operators of the same syntactical category as 'not'. The other is to follow Donald Davidson and represent them in the predicate calculus as with the aid of an extra argument place in the verb to be modified.

Cresswell also supplies the semantics and syntax for these two approaches. Adverbs in natural language can have multiple arguments, and thus, like predicates, their formalisations can have an arity greater than one. Examples of adverbs of higher arities can be drawn from the comparative adverbs, such as 'faster than' or 'as fast as'.

Just as the fact that predicates receive a treatment in logical systems does not mean that they immediately have potential for logical constancy, the available treatments for adverbs does not necessarily imply their potential logical constancy. In searching for potential logical constants among adverbs, the approach taken above for predicates can be followed. A predicate which is logical is one which is absolutely general in its application, in that it can be applied to absolutely any

[95] Page 455.

individual constant. Thus an adverb whose formalisation is logical would also be absolutely general in application, in that it can be applied to absolutely any predicate. Furthermore, an absolutely general adverb would need to be applicable to predicates of all arities, which is a complication not encountered in the case of predicates, since the concept of arity does not apply to individual constants. As will be seen in the discussion which follows, this is a significant barrier to the formalisation of any adverb achieving logical constancy.

A reasonable starting point would be to seek logical constants among the formalisations of the adverbial forms of those adjectives whose corresponding predicates were identified as logical constants above, that is, identity and existence. In the case of the former, intuition suggests that the adverb in question would be 'identically'. However, the identity predicate refers to numerical rather than qualitative identity, and numerical identity is a concept which is applicable to the objects denoted by individual constants in formal systems, but not the adjectives and verbs denoted by predicates. Thus there can be no formal correlate for an adverb which corresponds to the numerical identity predicate.

In the case of qualitative identity, extensional formal systems can define predicate identity (where í is used to denote qualitative identity of predicates), using only the resources of first order quantification. For example, $\forall x(Px \leftrightarrow Qx)$ states that predicates P and Q are extensionally equivalent.

Since qualitative identity can be defined using more fundamental logical constants, its addition as a separate logical constant would be superfluous. Thus this dissertation maintains that no logical constants based on formalisations of the adverb ‘identically’ are required.

Turning to the existence predicate, the reasoning in Section 4.4.2.2 found that it can be attached to absolutely any individual constant, but that it was essentially deductively useless. Thus it achieved logical constancy, albeit in a somewhat trivial way. While there is no natural language adverb which corresponds exactly to this adjective, there may be potential for an analogue in terms of universality. This would take the form of an adverb which simply affirms the attribution of the adjective (or verb) in question to the object in question. Among natural language terms, this may be something like ‘really’, in the sense of employed when someone asks ‘is it really?’ when the question is posed seeking confirmation of an already stated fact. Note that this usage of course varies significantly from the use of ‘really’ which is closer to ‘very’.

Conceived in such a way, this is a universally applying adverb, applicable to any adjective. But, like the existence predicate, it is also trivial, simply because its application is defined as being universal. However, if the formal correlate of this adverb were symbolised by ρ, its operational rules would be as follows (assuming that such adverb correlates were added to the language used in the expected way, i.e. as taking predicates as arguments):

$$\frac{\qquad}{\rho P}\ \rho I$$

Like the existence predicate E, this potential logical constant would have no elimination rule. This rule conforms to the criteria already developed for such rules in this dissertation.

Beyond the above, the author is unable to suggest any adverbs which recommend themselves as providing potential leads for logical constants. Inspection of lists of common English adverbs reveal that many are specific to spatial (‘slowly’, ‘below’) or temporal (‘always’, ‘never’) concerns and thus lack the absolute generality required for logical constancy. In the temporal case, these are typically treated as modal operators, and thus further discussion of them is deferred until Section 4.4.4, as are adverbs of an alethic nature (‘certainly’, ‘possibly’).

The very high frequency of use in a wide variety of linguistic contexts of some prepositions in natural language suggests that they may have the required generality for potential eligibility as logical constants. However, many common prepositions have spatial (‘above’, ‘beside’, etc.) or temporal (‘after’, ‘during’, etc.) aspects which restrict their applicability to spatially extended or temporally existing objects.

One preposition which does appear to have a high level of generality, given that it does not have this spatial or temporal component, is ‘with’. ‘With’ can be used in a variety of ways. Perhaps the use which has the most potential to generate a logical constant is that which is connected to the notion of ‘accompaniment’. While this sense of ‘with’ is perhaps most closely associated with accompaniment by persons, it can be applied to objects in general. However, it is the contention of this dissertation that no new logical constant is justified in this case, since any situation where ‘with’ is used can be formalised using the existing apparatus of first order logic.

This is essentially because ‘with’ as accompaniment can be expressed by using a one place predicate to state that objects share a common property, then using a two-place relation to state a relevant connection between the two objects in accordance with the context of the natural language usage in question. Consider the following natural language example:

John went to university with Sarah

A first attempt at formalisation for this, with the individual constants j and s being interpreted as John and Sarah, and W being interpreted as ‘went’, is as follows:

$jWu \wedge sWu$

However, while this specifies the necessary condition that both parties went to university, it does not (even if the W predicate included the requirement that John and Sarah’s going to university occurred at the same time) capture the essence of accompaniment implied by the use of 'with' in this case. Consider next the following formalisation:

jWu ∧ sWu ∧ jAPs ∧ sAPj

Here, the predicate xAPy means something like 'x acknowledges the presence of y'. This is intended to play the role of the context-according connection between the two objects mentioned above, which demonstrates that they are not only doing the same action at the same time, but that they are doing it together (*with* each other). While it could be argued that merely acknowledging the presence of another is insufficient to entirely describe this togetherness, the point is that correctly formalising 'with' in this case would seem only to depend on the correct choice of predicate, and would not require the inclusion of a formal correlate of 'with' as a separate logical constant.

This dissertation therefore asserts that the specific case of the use of a preposition can be formalised using existing resources of first order logic. Turning to the general case, consider the following definition of a preposition:

> A word governing, and usually preceding, a noun or pronoun and expressing a relation to another word or element in the clause[96]

Thus prepositions govern nouns or pronouns (the individual constants of first order logic) and express relations (the predication of first order logic) to other elements.

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[96] https://www.lexico.com/en/definition/preposition

In the case that these other elements are adjectives or adverbs, this is a simple case of predication in first order logic. In the case that these elements are other nouns, the analysis of the above example suggests that this too can be formalised using the resources of first order logic.

For the above reasoning, there does not seem to be a need to posit any logical constants based on prepositions. The case of 'with' seems a good candidate due to its absolute generality and topic neutrality, but the discussion above shows that it can be formalised using the resources of first order logic, and thus its addition to the set of logical constants is not required. It is furthermore held that this kind of analysis can be extended to other prepositions which may be suggested.

Natural language conjunctions can be categorised into two groups: Coordinating conjunctions, which introduce two parts of a sentence of equal rank (that is, where there is no relation of dependency between the two), and subordinating conjunctions, which introduce a sentence which is a dependent clause and, like adverbs (to which they are similar) they are placed in natural language sentences in front of that clause.

There are seven coordinating conjunctions in English. These are: 'for', 'and', 'nor', 'but', 'or', 'yet', and 'so'. Ignoring those which have been examined in the previous discussion of truth functional connectives, there remain four to consider: 'for', 'but', 'yet' and 'so'.

When it is used as a conjunction in natural language, the word ‘for’ indicates a causal link between the two parts of a sentence that it connects (as opposed to its prepositional use, in which it has various meanings such as ‘in support of’, ‘in favour of’, ‘affecting’, ‘with regard to’, ‘in respect of’, ‘on behalf of’, and ‘to the benefit of’[97]). Take for example the following sentence.

The window shattered, for the window was hit by a flying hammer

This way of using ‘for’ is somewhat archaic, and in more modern language the word ‘because’ is typically used in preference to it while retaining the same meaning:

The window shattered because the window was hit by a flying hammer

In order to better bring out the cause-to-effect ordering of events described in this sentence, the following analysis of it will be undertaken by substituting (again, without any change in meaning) the verb ‘cause’ as a stand-in for the converse of the natural language conjunction ‘because’ in the following manner:

The window was hit by a flying hammer (cause) The window shattered

Investigation of this (substitute) natural language connective (which will be denoted using the symbol $\mathbb{C}$) will begin by seeking a means of expressing it using the truth functional logical constants already discussed. Intuitively, the closest truth

[97] https://www.lexico.com/definition/for

functional approximation for $\mathbb{C}$ is ∧, due to the truth of sentences involving it requiring the truth of both of the constituent elements of the sentence ('the window was hit by a flying hammer' and 'the window shattered') in the example above. However, this falls short of accurately modelling causation, since it does not distinguish cases of true causation in which a true causal link exists, from those of simple correlation.

In fact, no truth functional connective, or combination of them, provides an accurate means of modelling $\mathbb{C}$. This is because the mere transmission of truth from a purported cause to a purported effect does not do the concept of causation justice. What is missing is the *metaphysical* element of causation, and this introduction of metaphysical notions calls into doubt the logicality of any causal operator. Furthermore, attempting to find operational rules for $\mathbb{C}$ is hampered by the general lack of agreement regarding the metaphysical nature of causation, resulting in any attempt to analyse it logically will be controversial.

The natural language connective 'but' is very similar to the natural language connective 'and'. The latter is usually formalised by the ∧ operator of propositional logic, and as such has been considered in this dissertation in Section 4.4.2.3. The key difference between the two natural language conjunctions is that 'but' is "used to introduce a phrase or clause contrasting with what has already been mentioned"[98], whereas in the case of 'and', the clauses involved do not present such a contrast.

In terms of the requirements for logicality of absolute generality and topic neutrality, a connective corresponding to 'but' fares well, since a wide range of sentences can be conjoined in natural language using it. However, the essential inclusion of the notion of contrast mentioned above compromises this. For two elements to contrast is for them to *differ strikingly*[99]. That the difference which exists between contrasting objects is striking brings with it a psychological aspect into the analysis, since it plays on the expectations of the utterer, the expectations of the utterer's audience, or both. Expectations themselves are then based on the notion of coherence with existing beliefs. For this reason, the prospects of a logical constant based on 'but', though differing from ∧, are poor.

The natural language connective 'yet' can be defined as "but at the same time; but nevertheless"[100]. There are therefore close similarities between the use of 'yet' and the use of 'but'. Similar comments apply to 'yet' as were made in reference to 'but' in the previous section thus apply.

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[98] https://www.lexico.com/definition/but

[99] https://www.lexico.com/definition/contrast

[100] https://www.lexico.com/definition/yet

The natural language conjunction ‘so’ can be defined as “and for this reason; therefore”[101]. Therefore, ‘so’ expresses the concept of consequence itself, and thus the element to which it corresponds most closely in formal systems is a metalogical one, namely the semantic or syntactic turnstile, $\vDash$ or $\vdash$. This suggests that no logical constant can be based on ‘so’, even though of course the concept of consequence is central to logic. An example of an attempts to model consequence at the level of object language rather than meta language is the strict conditional:

$$\Box(\varphi \rightarrow \psi)$$

Inspection of the above shows that it includes only $\Box$ and $\rightarrow$, the logical constancy of both of which is considered elsewhere in this dissertation.

In addition to these seven coordinating conjunctions, there are a wide range of subordinating conjunctions. These typically have less healthy prospects in terms of generating logical constants, due to their less general range of application. The following tables presents various subordinating conjunctions grouped by category, and notes for each category how they can be reduced to logical constants already discussed in this dissertation.

[101] https://www.lexico.com/definition/so

Category	Examples	Logical Treatment
Concession	Though, although, even though, while	Can be treated in essentially the same manner as ‘but’
Condition	If, only if, unless, until, provided that, assuming that, even if, in case (that), lest	The formalisation of these corresponds to one of the logical constants →, ←, or ↔
Comparison	Than, rather than, whether, as much as, whereas	The absolute generality / topic neutrality (and thus potential for logical constancy) of these is questionable, since they only apply to sentences which possess the quality of magnitude.
Time	After, as long as, as soon as, before, by the time, now that, once, since, till, until, when, whenever, while	The absolute generality / topic neutrality (and thus potential for logical constancy) of these is questionable, due to their temporal focus. These are typically treated in formal systems using the resources of modal logic (though some temporal relations can be formalised adequately in first order logic).
Reason	Because, since, so that, in order (that), why	Can be treated in essentially the same manner as ‘for’.
Place	Where, wherever	The absolute generality / topic neutrality (and thus potential for logical constancy) of these is questionable, due to their spatial focus.

Manner	How, as though, as if	The absolute generality / topic neutrality (and thus potential for logical constancy) of these is questionable, due to not all sentences have a 'manner' which can be compared to others. For example, 'Adam saw God as if he was present before his eyes' is coherent, but 'The aeroplane crashed as if 1 + 1 = 2' is not.

This dissertation therefore concludes that subordinating conjunctions do not provide potential logical constants, due either to their reducibility to coordinating conjunctions or their lack of absolute generality/topic neutrality.

4.4.2.6. Uniqueness

Belnap (1962) discusses two characteristics which logical operators can possess. The first is existence, which relates to the connective not provoking non-conservative extensions of the logic, and has been discussed in this dissertation extensively in the analysis of tonk and harmony. The second is uniqueness. Here, Belnap states the following (with the notation adapted to that used in this dissertation):

> The mathematical analogy leads us to ask if we ought not also to add uniqueness as a requirement for connectives introduced by definitions in terms of deducibility (although clearly this requirement is not as essential as [harmony], or at least not in the same way). Suppose, for example, that I propose to define a connective plonk by specifying that $\psi \vdash \varphi$ plonk ψ. The

extension is easily shown to be conservative, and we may, therefore, say 'There is a connective having these properties '. But is there only one? It seems rather odd to say we have defined plonk unless we can show that φ plonk ψ is a function of φ and ψ, i.e., given φ and ψ, there is only one proposition φ plonk ψ. But what do we mean by uniqueness when operating from a synthetic, contextualist point of view? Clearly that at most one inferential role is permitted by the characterisation of plonk; i.e., that there cannot be two connectives which share the characterisation given to plonk but which otherwise sometimes play different roles. Formally put, uniqueness means that if exactly the same properties are ascribed to some other connective, say plink, then φ plink ψ will play exactly the same role in inference as φ plonk ψ, both as premiss and as conclusion. To say that plonk (characterised thus and so) describes a unique way of combining φ and ψ is to say that if plink is given a characterisation formally identical to that of plonk, then:

1. φ1, ... , φ2 plonk φ3 ,... , φ4 ⊦ ψ if and only if φ1, ... , φ2 plink φ3 ,... , φ4 ⊦ ψ; and
2. φ1, ... , φ2 plink φ3 ,... , φ4 ⊦ ψ if and only if φ1, ... , φ2 plonk φ3 ,... , φ4 ⊦ ψ[102]

Thus if an operator ϴ does not possess uniqueness, then it can have the same rules as an operator ϴ', but play a different role in inference in a natural deduction

[102] Page 133.

system. Uniqueness can be seen as the converse of harmony. The levelling of local peaks which is central to harmony shows that the meaning of the operator given by the elimination rule does not outstrip the meaning given by its introduction rule. Thus the levelling of local peaks is demonstrated by sequential application of the introduction rule followed by the elimination rule. Uniqueness can be demonstrated using a similar technique of sequentially applying elimination rule followed by the introduction rule.

The case of conjunction provides an illustrative example. Consider the operator ∧', defined by the following operational rules:

$$\frac{\varphi \qquad \psi}{\varphi \wedge' \psi} \wedge' I$$

$$\frac{\varphi \wedge' \psi}{\varphi} \wedge' E1$$

$$\frac{\varphi \wedge' \psi}{\psi} \wedge' E2$$

Sequential application of the elimination and introduction rules of ∧' then ∧ and vice versa gives the following:

$$\dfrac{\dfrac{\varphi \wedge' \psi}{\varphi}\,\wedge' E1 \qquad \dfrac{\varphi \wedge' \psi}{\psi}\,\wedge' E2}{\varphi \wedge \psi}\,\wedge I$$

$$\dfrac{\dfrac{\varphi \wedge \psi}{\varphi}\,\wedge E1 \qquad \dfrac{\varphi \wedge \psi}{\psi}\,\wedge E2}{\varphi \wedge' \psi}\,\wedge' I$$

Thus ∧ and ∧' are interchangeable in any argument. Since ∧' is only notationally different rules compared to ∧, the above shows that ∧ has the property of uniqueness.

For an example where uniqueness fails, consider the following example, provided in discussion with Peter Milne:

> Suppose that we have two modal operators, $\Box$ and $\Box'$, governed by the S5 rules. The question is this: do the rules fix it that, for any φ, $\Box\varphi \vdash \Box'\varphi$ and $\Box'\varphi \vdash \Box\varphi$? It seems very unlikely that they do because we have S5 completeness in possible world semantics when the accessibility relation is any equivalence relation; associating $\Box$ and $\Box'$ with different equivalence relations – e.g. identity for one, the universal relation for the other – should be enough to make it the case that at least one of the required patterns of entailment fails to hold for all φ.

In terms of the objectives of this dissertation, the pertinent question is whether uniqueness should, like harmony, be the basis of a criterion for logical constancy. On the surface, due to the similarity (as noted previously, they are converses of each other) with harmony, it would appear that it should. However, consider the following reasoning from Restall (2010), after a discussion of the potential non-uniqueness of ∀ and ∃:

> So what, then, of Belnap's criteria? Do quantifiers pass the uniqueness test? Does the possibility of having two different universal quantifiers mean that we should banish ∀ and ∃ from the canon of logical constants? Surely such a conclusion is too extreme. However, we must acknowledge that in the face of considerations like this, the choice of quantifiers as logical constants is relative to the syntax of the language under consideration. Once we identify a category of singular terms (of names or variables or whatever), then relative to this choice, the quantifiers (for that category) are logical constants. Existence and uniqueness proofs work, and the meanings of the quantifiers are fixed.[103]

This dissertation concurs with the view described in the paragraph above, and holds that while uniqueness is an important consideration, it is not necessary to add it as a criterion for logical constancy. Note also that this view aligns also with Belnap's own view, where he states that uniqueness is "clearly … not as essential as [harmony], or at least not in the same way".

[103] Page 212

4.4.2.7. Concluding Remarks

The above analysis of first order logic led to the production of two natural deduction criteria for logical constancy. The first established the requirements for operational rules to act as true introduction and elimination rules for logical constants, the second was introduced as a response to the threat posed by tonk. The resulting criteria are as follows:

- Natural Deduction Criterion for Logical Constancy 1: Operational rules must not allow the introduced element to appear as the main connective in the antecedent of the I rule and must not allow the eliminated element to appear as the main connective in the consequent of the E rule.
- Natural Deduction Criterion for Logical Constancy 2: The introduction and elimination rules for logical constants must be in (general elimination) harmony.

These criteria appear coherent with respect to the expectations for logicality. The first stipulates what it means to be defined in natural deduction terms, via introduction and elimination rules. The second stipulates, via the notion of harmony, that the meanings established by the introduction and elimination rules should cohere with each other (in fact, according to Read’s analysis, each should establish the same meaning for the constant). Both criteria furthermore offer the desired precision for sorting the logical from the non-logical. That is, both criteria allow clear and unequivocal evaluation of candidates for logical constancy. In the case of Natural Deduction Criterion for Logical Constancy 1, this is clear. In the case of Natural Deduction Criterion for Logical Constancy 2, the requisite precision is achieved by stipulating that it is Read's general elimination harmony which is the correct account of harmony. The algorithmic nature of Read's approach means that this criterion can precisely separate the logical from the non-logical.

These criteria lead to the categorisation of all the operators of both intuitionistic and classical (and minimal) propositional logic as logical constants when assessed using this harmony-based criterion, along with the universal quantifier, the existential quantifier, and the identity predicate of first order logic. Other candidates which are also classed as logical using these criteria include examples such as the existence and non-existence predicates (see Section 4.4.2.2), which are strictly logical but which lack utility. Also, there is a 'principle of compositionality' for logical constancy at play here. That is, that elements of formal systems which are definable in terms of other logical constants are themselves logical constants. This is because, due to it being entirely definable by elements which are themselves logical constants, the defined element would also possess the requisite properties of formality (that is, the constituent elements contributing only to the structure of the inference) and absolute generality / topic neutrality. This means that the three place connectives discussed in this section qualify as a logical constants, due to their being definable in terms of other logical constants.

4.4.3. Second order quantification.

First order quantification was discussed in Section 4.4.2.4 of this dissertation. The results of this discussion were that first order ∀ and ∃ are unproblematically logical constants, on account of the fact that they satisfy Natural Deduction Criterion for Logical Constancy 1 and 2; and their interpretations are absolutely general and topic neutral. Furthermore, this accords with the general view of first order quantification, namely that it is uncontroversially logical.

Second order quantification is a more complex topic, at least partly because there are a number of ways in which second order quantification can be conceived. Understanding the second order quantifier's potential for logical constancy can therefore be facilitated by preliminary discussion. Unlike in previous sections of this dissertation, this will include discussion of model theoretic approaches to logic, in addition to the proof theoretic approaches thus far considered. Proof theoretic approaches to logic are based on an examination of logical syntax and places the notion of *provability* (derivability and deducibility) at the centre of endeavours to analyse logical consequence. In contrast, the model theoretic approach is an examination of the semantics of logic and puts the notion of *satisfaction* or *truth* (true of) at the centre of analysis. The relative extents of the model theoretic and the proof theoretic approaches to logical consequence can be informatively elucidated using the concepts of semantic completeness and incompleteness, a subject to which this dissertation will now turn.

4.4.3.1. Semantic Completeness

A well-known metalogical result[104] concerning formal systems which include only first order quantification is that proof systems can be produced for them which means that they are semantically complete. Thus, for such systems, for any example of consequence which holds according to the system's semantics, given symbolically as:

$\Gamma \vDash \varphi$ Where: $\vDash$ denotes implication defined according to the system's semantics

A corresponding proof can be found, given symbolically as:

$\Gamma \vdash \varphi$ Where: $\vdash$ denotes implication according to an effective proof calculus

In both cases, Γ is possibly empty. This is known as the completeness theorem[105] for first order logic, given symbolically as:

$$\Gamma \vDash \varphi \rightarrow \Gamma \vdash \varphi$$

While this result was first arrived at in Gödel (1929), significant simplification of the method used to demonstrate first order completeness was achieved in Henkin

[104] Originally proved in Gödel's doctoral dissertation, with a more succinct published version appearing in Gödel, K (1930).

[105] This is actually the *strong* completeness theorem; a weaker version of it is as stated above but where $\Gamma = \varnothing$.

(1949). The heart of Henkin's methodology is proving that if a set of sentences is maximally syntactically consistent (a set of terms and formulas Γ in a countable language L is maximally consistent iff Γ is consistent and for any formula φ in L, if $\varphi \notin \Gamma$ then $\Gamma \cup \{\varphi\}$ is inconsistent, or for no formula φ is φ and its negation provable) and witnessed (A set of formulas Γ is witnessed if for every formula of type $\exists x\varphi(x)$ where $\varphi(x) \in \Gamma$, $\varphi(c) \in \Gamma$ where c is a constant of L) then it has a countable model. To prove this using the Henkin method, the very elements (that is, the witness constants) which appear in the sentences of Γ are used to populate the domain of the model in question[106], then an inductive proof is used to show that the structure thus created really does satisfy Γ.

Some qualifying comments are required here. The importance in the definition of completeness of the requirement that proof systems be *effective* can be brought out by considering the trivial nature of the task of producing weakly complete non-effective proof systems. Such a proof system can be produced for any given semantics simply by defining the axioms of the proof system to be the set of theorems which are true according to those semantics. If the set of true sentences in question is not recursively enumerable, such an axiomatisation lacks practical utility[107]. The notion of effectiveness is thus introduced to eliminate such cases:

[106] Noting that this is true if the language does not contain identity; it is not strictly true when the language contains identity, the domain then being formed of equivalence classes of constants under the relation of provable equality in the maximal consistent extension under consideration.
[107] Strictly, this has to be the set of numerical codes of members of the set of true sentences in question, the codes being given by some "effective" encoding procedure, where effective is used here in an intuitive sense meaning, roughly, performable by some well-defined step-by-step procedure.

Effective deductive systems are those which can be used to recursively enumerate each theorem from a recursive set of axioms.

Completeness considered in isolation from its converse property of soundness is equally trivial. This is because an effective proof system which simply generated all possible sentences of the language used would be classed as complete, since it would thus include all *true* sentences. Soundness in isolation is also trivial, since providing a proof system which could prove no consequences at all would imply the soundness of a formal system. Requiring that a system is both sound and complete ensures that the set of valid semantic consequences is extensionally equivalent to the set of provable consequences: All valid semantic consequences are provable, and all provable instances of consequence are semantically valid. In the remainder of this dissertation, the term completeness will be used with the tacit understanding that soundness is also assumed; 'complete' is thus used as a shorthand for 'sound and complete'.

4.4.3.2. Semantic Incompleteness

As stated above, proof systems which are sound and complete with respect to the first order semantics can be constructed. However, some unfortunate mathematical facts intervene in the case of some conceptions of second order logic. The central problem is the fact that semantic descriptions of some potential logical constants can in a sense 'outstrip' their proof theoretic counterparts. This is essentially due to the influence of the concept of the infinite on models versus its influence on proofs; or more specifically to the stipulation that proofs must be finite objects under a standard interpretation of the notion of proof. The result of this is that, in general, semantic approaches can formalise examples of the consequence relation including second order quantifiers which are out of reach of proof theoretic approaches.

Second order quantification can be defined semantically in a number of ways. In the following discussion, it is the 'full' semantics which are under consideration, which can be defined as follows (at least for the monadic case, and for structures of the type $\langle d, I \rangle$, where d is the non-empty domain of the structure and I is an interpretation function which assigns elements of the domain d to individual constants, assigns subsets of the cartesian product d^n to (first-order) n-place predicates, and functions from d^n into d to n-ary function symbols):

> The sentence $\exists X_i \varphi(X_i)$ is true in structure $\mathfrak{A}$ iff for some P_j which does not occur in $\varphi(X_i)$ and some interpretation I_P (in which P is interpreted, and which differs from I (if at all) only in the element of the (second order) domain (subset of the domain) it associates with P_j, $\varphi(P_j)$ is true in $\mathfrak{A}$.

When second order quantification is semantically defined in this way, the resulting system is neither weakly nor strongly complete – no proof calculus can be developed with which all semantic consequences or logical truths can be proved. This is an implication of two key results. The first is Gödel's incompleteness theorem (Gödel (1931)), which concerns not semantic but negation completeness, which is the property possessed by a formal system if for each sentence φ of the language of the system either φ or ¬φ is a theorem of the system. The second is Dedekind's (1888) proof of the categoricity of arithmetic, which shows that there is a unique second order structure which satisfies the conjunction of the second order Peano axioms of arithmetic.

It is notable here that the Gödel result is very general, and applies widely across formal systems. The Dedekind result, however, is more specific, in that it can be proved for certain formal systems, but not others. The upshot of this is that if a formal system can provide a categorical characterisation of the natural numbers, then no effective proof system can exist for it. However, while second order logic defined using the 'full' semantics given above does allow for a categorical characterisation of natural numbers, there are various conceptions of the second order quantifier which, when added to logical systems instead of the 'full' version of the quantifier, do not permit such characterisation.

This introduces a difficulty with respect to the approach taken in this dissertation. Because of the issues associated with semantic incompleteness raised above, no set of proof rules can be produced which result in a semantically sound and complete logic which includes the semantic definition of full second order quantification. This then precludes the assessment of the natural deduction operational rules for the second order quantifier using the criteria developed thus far in this dissertation. However, it would be hasty to hold that failure of the method used in this dissertation should not immediately preclude the possibility of the logicality of the second order quantifiers, See Section 4.4.3.8 for further discussion.

4.4.3.3. ‘Maximal’ Second Order Proof Calculi

The conclusions of the previous section suggest the following approach: Investigate the extent to which first order logic can be strengthened in terms of expressivity towards second order logic, while retaining the possibility of semantic completeness. Such an investigation would ideally reveal an extension of first order logic which is ‘maximal’ in the sense that any further increase in expressivity results in a loss of semantic completeness, thus providing a maximum of expressivity while retaining the possibility of defining the operators involved using proof rules and investigating them for logical constancy using the proof theoretic criteria developed so far in this dissertation. While the Gödel/Dedekind result shows that completeness fails for formal systems which include full (unrestricted) second order quantification, there is perhaps some kind of semantically complete system between first order and second order logic which is of interest.

There is a lot to unpack in the above paragraph. First, what does it mean to say that a system is ‘between’ first and second order logic? Following now-familiar lines in terms of the present discussion, this claim can be cast both proof theoretically and model theoretically. Shapiro (2001) offers the following regarding each:

> Proof theory: “The logician begins with an ordinary, second-order language of a particular theory, such as arithmetic or analysis, and studies sub-systems of the full second-order deductive system for that theory. A typical focus is on restricted versions of the comprehension scheme, for example limiting it to Δ_1^0-formulas, or to Π_1^1-formulas. Logicians also consider restrictions on the axiom of choice, and restrictions on the schemes used to characterize various structures, such as the induction principle for arithmetic and the completeness principle for analysis.”[108]
>
> Model theory: “a potpourri of different logical operators which can be added to a standard, first-order language. Most of the languages have a model-theoretic semantics over the same class of models as first-order and second-order logic, and each of the logics can make more distinctions among models than can be done in first-order logic. That is, each language has more expressive resources than the corresponding first-order language.” [109]

Thus Shapiro’s conception here is that while the proof theorist begins with second order logic and restricts it in some way to move ‘down’ towards first order logic; the model theorist adds semantically defined quantifiers to first order logic to move ‘up’ towards second order logic. Other conceptions of how to modify the strength of logical systems defined either proof theoretically or model theoretically are possible. For example, in line with Shapiro’s point above (that logical operators

[108] Page 135

[109] Page 135

can be added to a first order language to strengthen it for a model theoretic system), logical operators can be added to proof theoretic systems to strengthen them, such as by defining them using natural deduction operational rules.

The point here is that strengthening or weakening of logical systems defined either proof theoretically or model theoretically is possible. In contrast with previous sections, whose focus has been strictly proof theoretical, the question of semantic completeness and incompleteness means that parts of the discussion in this section of this dissertation will be cast in model theoretic terms. In any case, since the maximal semantically complete system is being sought, model theoretic analysis will of course be indispensable.

Shapiro's analysis brings some clarity to the notion of 'betweenness'. Further precision can be added to it by calling upon some resources from abstract model theory, "the general study of model theoretic properties of extensions of first order logic"[110]. In this context, the relative strength of logical systems can be analysed as follows[111]:

> A logical system $\mathcal{L}$' (which consists of a function L (which associates with every symbol set S a set L(S), the set of S-sentences of $\mathcal{L}$) and a binary relation $\vDash_{\mathcal{L}}$) is at least a strong as a logical system $\mathcal{L}$ (written $\mathcal{L} \leq \mathcal{L}'$) iff for every S and every $\varphi \in$ L(S) there is a $\psi \in$ L'(S) such that:

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110 García-Matos M., Väänänen J. (2005)

111 Definition and subsequent analysis of relative strength taken from Ebbinghaus et. al. (1984). Page 194.

$$\{\mathfrak{A} \mid \mathfrak{A} \text{ is an S-structure and } \mathfrak{A} \models_{\mathcal{L}} \varphi\} = \{\mathfrak{A} \mid \mathfrak{A} \text{ is an S-structure and } \mathfrak{A} \models_{\mathcal{L}'} \psi\}$$

This definition of relative strength is based on the notion that the set of structures $\mathfrak{A}$ for which $\mathfrak{A} \models_{\mathcal{L}} \varphi$ holds gives a precise representation of the meaning of φ. Thus if there exists, for every sentence φ of the system $\mathcal{L}$, a counterpart sentence ψ in $\mathcal{L}'$, then $\mathcal{L}'$ can be said to be at least as strong as $\mathcal{L}$.

With this definition in hand, it appears that the relative strength of progressively stronger complete logical systems can be evaluated, until completeness is lost as expressivity crosses a certain threshold between first order logic and second order logic. However, this expectation must be tempered by the fact that the relative strength definition provided above may not impose a total ordering on the systems between first and second order logic. This means that a categorically maximal single system in terms of expressive strength between first and second order logic may not exist. Instead, consider a system $\mathcal{L}_1$ for which a semantically complete proof calculus can be produced. The expressivity of this system may be augmented in two different ways, to produce $\mathcal{L}_2$ and $\mathcal{L}_3$. Completeness may be possible for both $\mathcal{L}_2$ and $\mathcal{L}_3$, but not for any further extensions to them. However, the definition provided above for expressive strength does not provide any guarantee that the situation in which both $\mathcal{L}_2 \leq \mathcal{L}_3$ and $\mathcal{L}_3 \leq \mathcal{L}_2$ will be avoided, and thus that there will be a single maximally expressive extension of first order logic which is semantically sound and complete.

There is a certain key limitative result which can guide this investigation, one of which provides an upper bound on the expressivity of formal systems with certain metalogical properties. This is Lindström's Theorem, which can be stated as follows[112]:

> There is no logical system with more expressive power than first order logic, which is both compact (that is, for any ψ and φ, $\psi \vDash \varphi$ iff there is a finite $\psi_0 \subset \Phi$ such that $\psi_0 \vDash \varphi$) and the Löwenheim-Skolem property (if a set Γ of sentences is satisfiable by an infinite structure, then it is satisfiable by a structure with any infinite cardinality) holds.

One way of formalising the notion of expressive power is to say that logic L1 has more expressive power than logic L2 if L1 can define all of the classes of structures that L2 can defined, and that there are some classes of structures that L1 can defined that L2 cannot.

Of particular relevance here is the compactness aspect. This is due to the close correspondence between compactness and semantic completeness, which, in all but contrived examples, can be seen as equivalent. This is shown by the following reasoning.

> Completeness → Compactness: Consider a semantic consequence $\Gamma \vDash \varphi$, in which Γ is an infinite set, and assume that it is valid in a certain formal

[112] Adapted from Ebbinghaus et. al. (1984). Page 193.

system. Assume then that there is proof calculus for this system which means that completeness is achieved. It is therefore possible to produce a proof, $\Gamma \vdash \varphi$. Since proofs are by definition finite, this proof cannot require the infinite resources of Γ. Therefore, there must be a finite Γ' such that $\Gamma' \subseteq \Gamma$ and $\Gamma' \vdash \varphi$. Assuming also that soundness is also achieved, , $\Gamma' \vDash \varphi$ must also be valid in it. Thus, if a system is complete, for any $\Gamma \vDash \varphi$ in it, a finite Γ' can be found such that $\Gamma' \subseteq \Gamma$ and $\Gamma' \vDash \varphi$. Since this is compactness (as defined in the statement of Lindström's Theorem above), completeness has been shown to imply compactness.

Compactness → Completeness: Consider again a semantic consequence $\Gamma \vDash \varphi$ in which Γ is an infinite set. Assume also that the system in question is compact, meaning that there is a finite Γ' such that $\Gamma' \subseteq \Gamma$ and $\Gamma' \vDash \varphi$. Is it then possible in all cases to produce a proof of $\Gamma' \vdash \varphi$? Given that Γ' is known to be finite, this has certainly removed one of the key obstacles to doing this, namely the possibility that the validity of the semantic consequence somehow requires the infinite resources of Γ, which would clash with the restriction of proofs to finiteness. But could some other factor prevent the recursive enumerability of each instance of consequence, as is demanded by completeness?

To see how this could occur, consider the following formal system S. S extends standard first order logic by adding a set of new logical constants

> a_x and a one-place predicate F, with x ranging over the natural numbers. These constants are then interpreted with respect to a non-recursively enumerable subset of the natural numbers $A \subseteq \mathbb{N}$, by stipulating that Fa_x is true if $x \in A$ and false if $x \notin A$. The resulting system is compact because each new formula employing F and one of the new constants is equivalent to a sentence of first order logic. However, due to the stipulation that the set of all a_x's is not recursively enumerable, no recursive enumeration of all logical consequences in the system is possible. Thus the system is not complete.

This example shows that semantically complete logics cannot be found for certain compact systems. To this it may be objected that S is quite clearly a contrived system, since the failure of completeness for it relies on the *stipulated* non-recursive enumerability of A. This effectively imports an arbitrary or random element into the set of valid logical consequences, thus rendering any attempt to develop an effective proof system for it a non-starter. Whether there is a 'natural' system with any form of actual logical utility or philosophical interest which is compact but not complete is highly questionable; in any case the author of this dissertation is not aware of one, and it is difficult to conceive how any natural system which purported to be a logic could include a justification of the inclusion of an arbitrary set such as A.

Given this and the previous implication from completeness to compactness, it could be maintained that the two are extensionally equivalent to each other as far as natural systems are concerned. Lindström's theorem thus also puts a limit on the expressivity of semantically complete formal systems. It is also worth pointing out that alternative statements of Lindström's theorem involve completeness rather than compactness. For example, Shapiro (2001)[113] contains the following, originally due to Lindström himself:

- Lindström's Theorem, Version 1: If a logic has the finite occurrence property, is countably compact, and has the downward Löwenheim-Skolem property, then L[K] is first-order equivalent.
- Lindström's Theorem, Version 2: Let L be an effectively regular logic. Then if L has the downward Löwenheim-Skolem property and the upward Löwenheim-Skolem property, then L is first-order equivalent.
- Lindström's Theorem, Version 3: Let L be an effectively regular logic. If L has the downward Löwenheim-Skolem property and is weakly complete then L is first-order equivalent, and, moreover, there is a recursive function f such that for every sentence φ of L, f(φ) is a sentence of L1 that has exactly the same models as φ.

Comparison of Theorem 1 and Theorem 3 shows that all that is required to move from (countable) compactness to (weak) completeness is the move from a logic having the finite occurrence property (roughly, that if L has the finite occurrence

[113] Page 137.

property, then each formula of L involves only finitely many non-logical items[114]) to its being effectively regular (a logic L is effectively regular if (essentially) the collection of formulas of L is a recursive set of strings). Thus, as in the example produced above concerning the set A, as long as non-recursivity can be avoided, compactness and completeness are essentially extensionally equivalent to each other. Given that there does not seem to be any evident reason to *require* a system to be non-recursive, it would appear that, as maintained above, the two are entirely extensionally equivalent to each other as far as natural systems are concerned.

That a failure of recursion is the only way that a compact system could fail to have a semantically complete proof system is supported by the following reasoning. In general, how could a semantics be such that it is not possible to construct a proof system for it which results in completeness? There seem to be only two avenues which permit this. The first concerns considerations centring on the infinite – if an example of semantic consequence is such that any corresponding proof would be non-finite, since this transgresses the typical requirement of finiteness imposed on proofs. However, this avenue is closed by stipulating that the semantics in question is compact. With finiteness of proofs thus imposed, the only other way that producing a proof system could be rendered impossible would be via a lack of clarity regarding for example whether a given proof could be algorithmically

[114] Definition taken from Shapiro (1991), Page 158.

demonstrated to be correct, and this is remediated by requiring that proof systems have recursive axiomatisations.

Thus, a system which is compact and for which the Löwenheim-Skolem theorem holds has a definite and significant limitation in terms of expressivity. Each of compactness and Löwenheim-Skolem contribute to this expressive limitation in their own way. Compactness means that finiteness in the first order domain of quantification is not definable, since compactness means that any sentences with arbitrarily large finite models will also have infinite models[115] (strictly, compactness means that no sentence in a compact system is true in all and only finite domains). The fact that, as noted, a system's being compact also leads to semantic completeness, clearly demonstrates the trade-off that exists between expressivity and proof theoretic tractability in logical systems. Compactness is also advantageous model theoretically since it can be used to for example provide a means of constructing models based on finite consistency. It is also mathematically fruitful, since it leads to phenomena such as the existence of non-standard models of arithmetic, non-isomorphic structures in which all the sentences in the language of first-order arithmetic true in the standard model are true. In the case of the Löwenheim-Skolem theorem (upwards and downwards), the expressive limitation here is the fact that in systems possessing this property, any theory satisfiable by an infinite structure can also be satisfied by a structure of any infinite cardinality (provided that the language in which the theory is expressed

[115] Note that a theory (a set of sentences) is termed categorical if, up to isomorphism, it has a unique model

is countable). This means that these systems cannot distinguish infinite cardinalities.

A key specific application of the above concerns the categoricity of arithmetic results for second order logic already mentioned above. A notable fact about the overall proof of second order semantic incompleteness is that the Gödel result is independent of any specific logical system, and thus applies across them all. A proof system which results in completeness therefore cannot be found for any logical system whose semantics can provide a categorical description of (a unique model up to isomorphism for) the natural numbers. Such systems can be instructively described as inherently incomplete.

While the link between compactness and completeness means that Lindström's Theorem is instructive in terms of the present search for the expressively maximal complete extension of first order logic, it does not provide a full answer to the question. Lindström's Theorem assumes that any compact and Löwenheim-Skolem property-possessing logic is expressively equivalent to first order logic. However, it does not dictate the extent to which expressivity can be increased while retaining only compactness, while accepting the loss of the Löwenheim-Skolem property. It is notable, however, that the very formulation of Lindström's theorem in terms of both the compactness and Löwenheim-Skolem properties suggests that the expressivity of systems retaining only compactness is greater than that of first order logic (if it did not, why include the Löwenheim-Skolem stipulation?). Examples of such systems are certain instances of the cardinality quantifiers discussed later in this dissertation.

With the above in place, the following sections of this dissertation consider a range of extensions of first order logic and evaluate them for their potential to be the maximally expressive complete system which is sought.

4.4.3.4. Monadic Second Order Logic

A possible restricted conception of second order quantification is to restrict full second order quantification to the monadic case. Here, while first order quantification remains unrestricted, second order variables and quantification over them is restricted to the one place relations. This means that quantification is permitted only over sets, rather than relations of any arity.

Analogous approaches in first order logic which restrict the stock of predicates to one-place (monadic) predicates perhaps provide cause for optimism here. This is because, in the first order case, such restrictions do result in what could be interpreted as 'improved' metalogical properties on the count of decidability (effective determination of membership in the set of logically valid formulas). It could therefore be hoped that a similar restriction in the second order case could lead to the possibility of 'improved' metalogical properties, hopefully in the form of semantic completeness, and thus the possibility of evaluating logical constancy via the proof theoretic means studied in this dissertation.

However, relatively simple reasoning can be used to demonstrate that this is not the case. Recall that the portion of the Dedekind/Gödel argument for the semantic incompleteness of (full) second order logic specific to the formal system under consideration requires only demonstration that there is a unique second order structure which satisfies the conjunction of the second order Peano axioms of arithmetic. The notable fact here is that the only use of second order quantification in the Peano axioms is in the induction axiom, which can be stated as follows:

$$\forall X[(X0 \land \forall x(Xx \rightarrow Xsx)) \rightarrow \forall xXx].$$

This axiom involves only monadic second order quantification. Hence the Dedekind result, and thus the overall semantic incompleteness result, holds in the case of monadic second order logic. Expanding on this somewhat, consider the following from Shapiro (1991): “In short, then, the categoricity of arithmetic theorems fail in the first-order cases because certain subsets of the domains, constructed in the metatheory, may not be first-order definable”[116]. In the case of first order and monadic second order quantification, all subsets of the domain of quantification fall within the range of the second order quantifier, including the set that contains the denotations of the representations of the natural numbers (0, s0, ss0, … where s is the successor function) and nothing else. Thus the categoricity proof succeeds for first order and monadic second order logic, and monadic second order logic is therefore not the semantically complete extension of second

[116] Page 112.

order logic which is being sought. This means that evaluation of proof rules for the monadic second order quantifier are not available for evaluation using the criteria developed in this dissertation. This does not immediately preclude the possibility of the logicality of the monadic second order quantifier, see Section 4.4.3.8 for further discussion.

It is notable that the move from monadic first order logic (in which all relation symbols take only one argument) to polyadic first order logic (in which relation symbols can take one *or more* arguments) engenders a significant change in metalogical properties of the logical system which includes them. Specifically, systems which include first order quantification are decidable (the set of examples of logical consequence can be effectively determined) – in fact, as discussed in the following paragraph, they are expressively equivalent to first order logic. In contrast, there may not be a corresponding significant change in the metalogical properties of the corresponding systems when moving from monadic second order quantification to polyadic second order quantification. The above example shows that the possibility of semantic completeness is already lost as soon as second order quantification (interpreted along full / non-Henkin lines) is introduced, even in its monadic form. In terms of other metalogical properties, decidability is already lost as mentioned, and as discussed previously, compactness is asserted in this dissertation to rise and fall with semantic completeness for natural systems; thus monadic second order logic is non-compact.

There is no immediately clear philosophical conclusion to be drawn from the above, but it does represent an interesting dissimilarity between first and second order quantification. In terms of an explanation, the decidability of systems with only monadic first order quantification is due to its expressive equivalence to propositional logic (which is itself decidable), since any proposition (or sentence, or whatever logical formulas are taken to express) which is expressible in monadic predicate logic can also be expressed in propositional logic without any loss in terms of inferential relations. However, the loss of semantic completeness from systems with first order quantification to systems with (first and) monadic second order quantification shows that a similar reduction of the latter to the former is not possible.

4.4.3.5. Second Order Logic with Henkin Semantics

The full or unrestricted version of second order quantification is typically characterised by the following statement of truth conditions (this being, for the sake of simplicity of the explanation which follows, a slightly 'stripped down' version, in that it restricts predication to the monadic case, and taking ∃ as an example):

> The sentence $\exists X_i \varphi(X_i)$ is true in structure $\mathfrak{A}$ iff for some P_j not in $\varphi(X_i)$ and some interpretation I_P (in which P is interpreted, and which differs from I (if at all) only in the element of the (second order) domain (subset of the domain) it associates with P_j, then $\varphi(P_j)$ is true in $\mathfrak{A}$.

This definition is evaluated over structures of the form ⟨d, I⟩, where d is the structure's domain and I is an interpretation function which assigns elements of the domain d to individual constants; it assigns subsets of the cartesian product d^n to (first-order) n-place predicates, and functions from d^n into d to n-ary function symbols. Thus, individual constants in the language are assigned to individual elements in d, and predicate constants are assigned to sets of elements in d. First order variables then range over the elements of d, and second order variables range over every subset of d, meaning that the range of second order variables is the powerset of d (symbolically, $\wp$(d)). In cases in which the cardinality of d is finite; the cardinality of the second order domain $\wp$(d) will also be finite, albeit larger. In cases in which the cardinality of d is countably infinite (that is, $\aleph 0$, the cardinality of the natural numbers), the cardinality of $\wp$(d) will be uncountably infinite (or $2^{\aleph 0}$, that of the real numbers). This interpretation of second order quantification in which no restrictions are placed on the second order domain is known in the literature as full second order quantification and is that which is discussed up to this point in this dissertation.

It is with respect to formal systems involving full second order quantification that semantic incompleteness arises. Fundamentally, completeness fails in the case of full second order systems due to the requirement that proofs are finite. Proof theoretic analyses of logical consequence are, at least by standard definitions, finite sequences of sentences. Model theoretic analyses of logical consequence are subject to no such constraint regarding finiteness.

One alternative semantic account of second order quantification is that which is provided by what is known as Henkin semantics. The key difference between it and the full second order semantic analysis presented above is in the nature of the structures involved. In the case of Henkin semantics, evaluation of second order quantification occurs over structures of the form ⟨d, D, I⟩. Here, a second order domain, D, is specified, rather than simply assumed to be ℘(d). For each first order domain d, there are therefore multiple[117] Henkin models – one of these with D = ℘(d), and others with any combination of elements of d (though the null set is excluded by stipulation).

Thus, Henkin semantics present what appears to be an alternative interpretation of second order quantification. Furthermore, in contrast to the full second order semantics presented previously, semantic completeness for second order systems involving Henkin semantics can be achieved[118]. Intuitively, part of the reason that the semantic incompleteness of full second order quantification is avoided is that, for a given example of entailment, there are more Henkin models available in which the example of inference can fail to hold, thus reducing the set of logical consequences.

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[117] For a first order domain d of cardinality n, there are $(2^{(2^n-1)} - 1)$ Henkin models – that is, the powerset of the powerset, with subtractions occurring to adjust for the fact that the empty set is excluded by stipulation.

[118] A proof of this is available in Shapiro (1991), starting on page 89. This source also includes information on a caveat, unimportant for the present discussion, involving only 'faithful' models which is required to the ensure soundness in addition to completeness of the proof calculus presented there.

The above analysis appears to suggest that second order quantification with Henkin semantics is a candidate for a system with greater expressivity than first order logic, but for which a completeness can be achieved. Natural deduction operational rules which correspond to (are sound and complete with respect to) second order quantification with Henkin semantics are as follows (taking ∀ as an example):

$$\frac{\varphi(A)}{\forall X \varphi(X)} \ (\forall_2 I)$$

where every occurrence of A in φ(A) is replaced by X, and A must not occur in any assumption on which φ(A) depends. The variable x must not be bound by any quantifier in φ(A) that has A within its scope[119]

$$\frac{\forall X \varphi(X)}{\varphi(A)} \ (\forall_2 E)$$

In applying this rule one replaces every free occurrence of X in φ(X) by a

These rules satisfy both of the natural deduction criteria for logical constancy developed thus far in this dissertation. Thus they should be accepted as logical constants. However, their acceptance as such is a hollow victory. The reason for this is that second order systems with quantification interpreted in the Henkin style also possess the compactness and the Löwenheim-Skolem property[120], and therefore, due to Lindström's Theorem, such systems are no more expressive than first order logic – in fact, they can be reduced to equivalent systems involving multi sorted first order quantification[121]. Thus, contrary to initial appearances, no

[119] Condition taken from Tennant (1978). Page 42.

[120] Shapiro (1991), from Page 92.

[121] Shapiro (1991), Page 76 states that "Henkin semantics and [multi sorted] first order semantics are pretty much the same".

advance in terms of a semantically complete system exceeding the expressivity of first order logic is made here. This means that including only a multi sorted version of $\forall$ (the first order universal quantifier) as a logical constant is required to obtain equivalent expressivity as adding $\forall_{2H}$ would achieve.

4.4.3.6. Second Order Logic with Faithful Henkin Semantics

Shapiro (1991) puts forward an axiomatic proof theory for second order logic which includes the following:

Axiom 1: $\forall_{2HF}X\varphi(X) \rightarrow \varphi(T)$, where T is either an n-place relation variable free for X in φ or a non-logical n-place relation letter

Axiom 2: $\forall_{2HF}f\varphi(f) \rightarrow \varphi(p)$, where p is either an n-place function variable free for X in φ or a non-logical n-place function letter

Comprehension: $\exists X\forall x(Xx \leftrightarrow \varphi(x))$

Choice: $\forall_{2HF}Y(\forall x\exists yYxy \rightarrow \exists f\forall xYxf(x))$

Rule of Inference 1: From $\varphi \rightarrow \psi(X)$ infer $\varphi \rightarrow \forall X_{2HF}\psi(X)$, provided that X does not occur free in φ or in any premise of the deduction.

Rule of Inference 2: From $\varphi \rightarrow \psi(f)$ infer $\varphi \rightarrow \forall X_{2HF}\psi(f)$, provided that X does not occur free in φ or in any premise of the deduction.

These axioms serve to introduce an alternative version of the second order quantifier, $\forall_{2HF}$. In comparison to the first set of four laws (which have structural similarities with the introduction and elimination rules of natural deduction systems), comprehension and choice seem to represent more substantive claims about second order quantification. These are therefore not in line with the formality required for purely logical constants. Furthermore, the inclusion of these axioms means that it is difficult (seemingly impossible) to demonstrate that, taken as a whole, the natural deduction rules which represent this axiomatisation are in harmony, and thus in accordance with Natural Deduction Criterion for Logical Constancy 2. In any case, attempts to justify their inclusion (historically, debate regarding the axiom of choice has been particularly extensive) are somewhat unimportant, since, as will become evident, the resulting semantics are not an extension in terms of expressivity compared to first order logic.

If the axioms of comprehension and choice are added to the proof system, the proof system is strengthened so that more examples of logical consequence can be proved. However, as Shapiro points out, this strengthening goes too far with respect to Henkin semantics, since soundness with respect to Henkin semantics is lost. This is because there are some Henkin models in which the axiom of comprehension does not hold, and other Henkin models in which the axiom of choice does not hold.

The remedy for this is to restrict Henkin semantics to exclude models in which comprehension and choice do not hold. Shapiro calls the resulting semantics *faithful* Henkin semantics. In the case of the axiom of comprehension, the philosophical justification for this move (and thus also the justification for the addition of comprehension to the above axiomatisation) is that it guarantees a certain level of richness in the second order domain. This is because without comprehension, there is no guarantee that the second order domain is not impoverished, or even empty. With comprehension, the second order domain must at least be populated by those subsets of the first order domain which can be specified by formulas.

In terms of assessing $\forall_{2HF}$ for logical constancy, this quantifier falls foul of the same problems noted for $\forall_{2H}$. This is because Shapiro proves that faithful Henkin semantics are compact, and that the Löwenheim-Skolem theorem holds for them. Thus Lindström's Theorem again dictates that they do not represent an extension in terms of expressivity compared to first order logic. Therefore, while they satisfy the criteria for logical constancy produced in this dissertation, they do not extend the expressivity of first order logic.

4.4.3.7. Cardinality Quantifiers: Infinitely / Finitely / Uncountably Many

The above means of restricting the expressivity of (full) second order logic have either failed to improve on the expressivity of first order logic (Henkin semantics) or failed to limit expressivity sufficiently to achieve semantic completeness (monadic second order logic). A slightly different approach is taken below, in that the expressivity of first order logic is augmented through the additional of a cardinality quantifier. Cardinality quantifiers[122] are unary quantificational operators which impose a condition of cardinality on the extension of a formula. A familiar example of this is the quantifier ∃!n, which is interpreted as exactly n objects. However, this particular example is an abbreviation of convenience, since it is definable in terms of identity, negation, the standard existential quantifier and, for convenience (though not strictly necessary) the universal quantifier, and thus does not extend the expressivity of first order logic.

A more pertinent cardinality quantifier example for the current investigation is that of Qx(φ), which has the following semantic definition[123]:

> M, s ⊨ Qx(φ) iff there are infinitely many distinct assignments s' such that s agrees with s' on every variable except possibly x, and M, s' ⊨ Qx(φ).

Thus Qx(φ) can be read as 'for infinitely many x, φ'. Just as Qx(φ) thus expresses the infiniteness of the extension of φ, the finiteness of the extension of φ (that is, the fact that φ can be true in all and only finite domains) can be expressed by ¬Qx(φ). Given the aforementioned restrictions on first order systems due to the Löwenheim-Skolem theorem holding for them, this capability of first order logic augmented by the Q quantifier does represent a proper extension of pure first

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[122] Uzquiano (2018). Section 3.1.1.

[123] Shapiro (2001). Page 153.

order logic[124]. However, this increased expressivity is gained through the downfall of the system in terms of its metalogical properties, since the set of valid sentences of first-order logic augmented by this quantifier is not effectively enumerable, and hence semantic completeness is lost, as stated in Fuhrken (1971)[125]. Discussion of the relationship between incompleteness and logicality is contained in Section 4.4.3.8 of this dissertation.

A more modest cardinality quantifier is thus needed in order to retain semantic completeness. A possibility is offered in Keisler (1970), which provides a semantic completeness proof based on an axiomatisation for a cardinality quantifier expressing the notion "for uncountably many". To do this, Keisler develops a language L(Q), which contains all the familiar elements of first order logic, with the addition of a new quantifier Qx, meaning 'there are uncountably many x'. The propositional and first order existential and universal operators are defined in the usual way, leaving Keisler with the tasks of adding a semantic definition for Qx, and axioms which describe it, to a standard axiomatisation of first order logic. The Qx clause Keisler provides is as follows:

> $(\mathfrak{A}, q) \vDash (Qv_m)\varphi[a_1, \ldots, a_n]$ if and only if $\{b \in A: (\mathfrak{A}, q) \vDash \varphi[a_1, \ldots, a_{m-1}, b, a_{m+1}, \ldots, a_n]\} \in q$
>
> Where φ(v1, …, vn) is a formula of L(Q) and m ≤ n.

[124] As stated in Shapiro (2001). Page 153.

[125] Fuhrken (1971). Page 685.

This is the satisfaction clause for what Keisler calls a weak model (a pair (A, q) such that A is a model for the first-order language L and q is a set of subsets of the universe A of A, i.e. q ⊂ S(A).). Thus q denotes an arbitrary set of subsets of the domain (or universe) A of the model $\mathfrak{A}$.

Keisler then states the following lemmas on the basis of induction on the complexity of φ:

> If all the free variables of $\varphi(v_1, \ldots, v_n)$ are among $v_{i_1}, \ldots, v_{i_m}$, and if $a_{i_1} = b_{i_1}, \ldots, a_{i_m} = b_{i_m}$, then $(\mathfrak{A}, q) \vDash \varphi[a_1, \ldots, a_n]$ if and only if $(\mathfrak{A}, q) \vDash \varphi[b_1, \ldots, b_n]$.
>
> Let $(\mathfrak{A}, q)$ be a weak model, let $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ be a formula of L(Q), and form ψ by replacing each fee occurrence of $y_1, \ldots, y_n$ in φ by constants $c_1, \ldots, c_n$. If $d_1, \ldots, d_n$ are the interpretations of $c_1, \ldots, c_n$ in $\mathfrak{A}$ then for all $a_1, \ldots, a_m \in A$, then$(\mathfrak{A}, q) \vDash \varphi[a_1, \ldots, a_m, d_1, \ldots, d_n]$ if and only if $(\mathfrak{A}, q) \vDash \psi[a_1, \ldots, a_m]$.

This then permits Keisler to state: "Let $\mathfrak{A}$ be a model for L. We shall write $\mathfrak{A} \vDash \varphi[a_1, \ldots, a_n]$ iff $(\mathfrak{A}, q) \vDash \varphi[a_1, \ldots, a_n]$ where q is the set of all uncountable subsets of A. We shall say that A is a standard model of a sentence φ iff $\mathfrak{A} \vDash \varphi$ in the above sense. Thus $\mathfrak{A}$ is a standard model of φ just in case φ holds in $\mathfrak{A}$ with (Qx) interpreted by 'there exist uncountably many x'"[126]. This dissertation will accept without further comment that this definition correctly represents the semantics of uncountably many.

[126] Keisler (1970). Page 5.

Turning to the axiomatisation of L(Q), in addition to a standard axiomatisation for first order logic with identity, Keisler puts forward the following "simple schemes of formulas which are obviously true in all standard models". Also included are Keisler's suggestions for the intuitive content of each axiom:

Axiom 1. $\neg(Qx)(x = y \vee x = z)$

"Every set of power ≤ 2 is uncountable"

Axiom 2. $(\forall x)(\varphi \rightarrow \psi) \rightarrow ((Qx)\varphi \rightarrow (Qx)\psi)$, where φ, ψ are formulas of L(Q).

"Every set which has an uncountable subset is uncountable"

Axiom 3. $(Qx)\varphi(x \ldots) \leftrightarrow (Qy)\varphi(y \ldots)$, where $\varphi(x \ldots)$ is a formula of L(Q) in which y does not occur, and $\varphi(y \ldots)$ is obtained by replacing each free occurrence of x by y.

No description of this axiom's intuitive content is provided by Keisler, presumably because it is clearly a stipulation of equivalence through substitution of variables.

Axiom 4. $(Qy)(\exists x)\varphi \rightarrow (\exists x)(Qy)\varphi \vee (Qx)(\exists y)\varphi$, where φ is a formula of L(Q).

"If $\cup_{x \in X} a_x$ is uncountable then either some a_x is uncountable or X is uncountable". This is equivalent to: "The union of countably many countable sets is countable"[127]

Much of the remainder of Keisler's paper is devoted to proving the soundness and completeness of these axioms with respect to the semantics given for them.

[127] Keisler (1970). Page 6.

Again, the correctness of this proof will be taken on faith in this dissertation. The focus here will instead be on attempting to evaluate the potential that Q as defined by Keisler's axioms is a logical constant.

Keisler's proof theoretic definition of Q is given as a Hilbert-style axiomatisation, rather than as natural deduction rules. This is problematic in terms of assessing Q's potential for logical constancy because (based on the inferentialist view that the meaning of a logical constant is entirely contained in its introduction and elimination rules) the criteria for logicality developed in this dissertation are based on the evaluation of constants defined in natural deduction systems. The options open to assess Q's claims to logical constancy are therefore to either develop a set of criteria for assessment of constancy on the basis of definition by axiomatisation, or to find a means of either converting the axioms of Q into natural deduction rules which permit assessment via the criteria already developed. The second of these options will be pursued below.

von Plato (2014) presents a means of adapting axioms into natural deduction rules. The following axiom to natural deduction rule conversion is provided as an example (though with the symbols used adapted to those used in this dissertation), with the method then applied to Keisler's axioms given above.

Axiom: $\varphi \to (\psi \to \varphi)$

Corresponding Rule of Inference:

$$\frac{\varphi \to (\psi \to \varphi) \qquad \varphi}{(\psi \to \varphi)}$$

Simplified Rule of Inference:

$$\frac{\varphi}{(\psi \to \varphi)}$$

Applying this technique to the Axioms 1, 2, 3 and 4 from Keisler gives the following rules of inference (with only the left to right 'half' of Axiom 3 being analysed):

Axiom 1

$$\frac{(Qx)(x = y \vee x = z)}{\bot}$$

Axiom 2 $(\forall x)(\varphi \to \psi) \to ((Qx)\varphi \to (Qx)\psi)$,

Corresponding rule of inference:

$$\frac{(\forall x)(\varphi \to \psi) \to ((Qx)\varphi \to (Qx)\psi) \qquad (\forall x)(\varphi \to \psi)}{(Qx)\varphi \to (Qx)\psi)}$$

Simplified rule of inference:

$$\frac{(\forall x)(\varphi \to \psi)}{(Qx)\varphi \to (Qx)\psi}$$

Axiom 3 $(Qx)\varphi(x \ldots) \to (Qy)\varphi(y \ldots)$

Corresponding rule of inference:

$$\frac{(Qx)\varphi(x \ldots)}{(Qy)\varphi(y \ldots)}$$

Axiom 4 $(Qy)(\exists x)\varphi \rightarrow (\exists x)(Qy)\varphi \vee (Qx)(\exists y)\varphi$

Corresponding rule of inference:

$$\frac{(Qy)(\exists x)\varphi}{(\exists x)(Qy)\varphi \vee (Qx)(\exists y)\varphi}$$

Simplified rule of inference:

$$\frac{(Qy)(\exists x)\varphi \qquad \neg((\exists x)(Qy)\varphi \vee (Qx)(\exists y)\varphi)}{\bot}$$

Simplified rule of inference:

$$\frac{(Qy)(\exists x)\varphi \qquad \neg(\exists x)(Qy)\varphi \qquad \neg(Qx)(\exists y)\varphi)}{\bot}$$

Summarising the above, this technique arrives at the following four rules of inference:

Q1:

$$\frac{(Qx)(x = y \vee x = z)}{\bot}$$

Q2:

$$\frac{(\forall x)(\varphi \rightarrow \psi)}{(Qx)\varphi \rightarrow (Qx)\psi}$$

$$\text{Q3:}\quad \frac{(Qx)\varphi(x \ldots)}{(Qy)\varphi(y \ldots)}$$

$$\text{Q4:}\quad \frac{(Qy)(\exists x)\varphi \qquad \neg(\exists x)(Qy)\varphi \qquad \neg(Qx)(\exists y)\varphi}{\bot}$$

In terms of their general form, Q2 can be seen as an introduction rule for Q, while Q1 and Q4 can be seen as elimination rules for Q. Rules Q1, Q2, and Q4 also adhere to the following criterion, established previously in this dissertation:

- Natural Deduction Criterion for Logical Constancy 1: Operational rules must not allow the introduced element to appear as the main connective in the antecedent of the I rule and must not allow the eliminated element to appear as the main connective in the consequent of the E rule.

However, Q3 does not adhere to this criterion, since it includes Q in both its antecedent and its consequent. Intuitively, this is perhaps not a particularly important failing, since it represents a stipulation regarding substitution of variables, rather than a substantive fact about Q (like Q1, Q2, and Q4), but the point stands nonetheless.

However, the rules fare poorly in the case of the following criterion, also established previously in this dissertation:

- Natural Deduction Criterion for Logical Constancy 2: The introduction and elimination rules for logical constants must be in (general elimination) harmony.

This is true for what this dissertation has established as the accepted account of harmony (Read's general elimination harmony), or other accounts of it such as local peak elimination. Denying logical constancy to Q seems reasonable, since the need for relatively complex axioms to define Q, which lead to corresponding natural deduction rules which violate the established criteria, means that they should be precluded from logical constancy in any case. This is because operators which fulfil the requirements of absolute generality, topic neutrality and formality could be expected to be simple rather than complex. This aligns with the intuitive notion that an operator for 'uncountably many' does not have the generality required to be considered a legitimate candidate for logical constancy.

4.4.3.8. Concluding Remarks

The investigations above have only resulted in conceptions of second order quantification which are not an advance in terms of expressivity with respect to first order logic (second order logic with Henkin semantics, and second order logic with faithful Henkin semantics) or for which soundness and completeness fail (monadic second order logic and full or unrestricted second order logic). In the case of the first category, given that the natural deduction operational rules involved adhere to the criteria developed in this dissertation, they are acceptable as logical constants. However, since they do not represent an advance in terms of expressivity compared to first order quantification, this is somewhat of a 'hollow victory'.

For those conceptions for which no sound and complete proof rules are available, the situation is more complex. Here, the conceptions of the second order quantifier in question simply escape evaluation in terms of the criteria developed in this dissertation, since these criteria are based on the evaluation of natural deduction operational rules, and these are not available for this evaluation due to incompleteness. This raises the question, should the potential logical constancy of these conceptions of second order quantification be dismissed on this basis? Or does their semantic definition mean that they may be logical constants, but are not assessable using proof theoretic tools?

This is a significant question, and one which means taking a position on wider issues than the choice of natural deduction criteria for logical constancy. The position adopted in this dissertation is that the very lack of an effective proof system for systems including full second order quantification precludes them from logical constancy. Putting the point in a perhaps blunt but intuitive manner, any example of logical consequence for which a proof cannot be offered is not a legitimate example of logical consequence; thus, any example of consequence the holding of which relies on second order quantification is non-logical; and hence the second order quantifier cannot be a logical constant.

The inferentialist position holds that natural deduction operational rules provide the meaning of logical constants. However, no such introduction and elimination rules can be produced for the second order quantifier which result in soundness and completeness relative to the semantic definition of full second order quantification. If the second order universal quantifier (denoted as $\forall_2$ below) is added to the natural deduction system for first order logic discussed in Section 4.2 in the same way in which the first order quantifier was added, the following rules result (taking the universal quantifier as an example):

$$\frac{\varphi(A)}{\forall X\varphi(X)}\ (\forall_2 I)$$

where every occurrence of A in φ(A) is replaced by X, and A must not occur in any assumption on which φ(A) depends. The variable x must not be bound by any quantifier in φ(A) that has A within its scope[128]

$$\frac{\forall X\varphi(X)}{\varphi(A)}\ (\forall_2 E)$$

In applying this rule one replaces every free occurrence of X in φ(X) by a

The similarity between these rules and the rules for the first order universal quantifier, ∀I and ∀E, shows that $\forall_2$ is a logical constant according to Natural Deduction Criterion for Logical Constancy 1 and 2. However, the Gödel/Dedekind result also shows that systems which include them are not sound and complete relative to the full semantics for second order quantification – thus underlining the

[128] Condition taken from Tennant (1978). Page 42.

impossibility of having a discussion based on proof theoretic criteria for logical constancy which addresses the same notion of the concept as put forward by model theorists when they discuss full second order quantification.

Further criticism of the potential logical constancy of the second order quantifier can also be directed at the fact that significant metaphysical (or at least mathematical) theses can be derived through its use. An example of such a thesis is the continuum hypothesis (CH), which states that there is no set whose cardinality lies between $\aleph_0$ (the cardinality of the natural numbers) and $2^{\aleph_0}$ (the cardinality of the real numbers). For seemingly any reasonable axiomatisation of set theory, such as the standard Zermelo-Fraenkel with Choice (ZFC), stated using the resources of only first order logic, neither the continuum hypothesis nor its negation results as a consequence (that is, ZFC $\nvDash$ CH and ZFC $\nvDash$ ¬CH). Thus it can be argued that first order logic (and in fact first order set theory) does not pronounce truth or falsity of CH. Given that CH is seemingly a substantial mathematical thesis, the argument continues, since logic is absolutely general and topic neutral, this lack of commitment either for or against CH is as it should be. However, CH can be formulated in the language of second order logic[129], and the (logical) truth of it or otherwise is simply a (semantic) consequence of the set theoretic metatheory applied to the domains of quantification used for the semantic models (Koellner (2019) contains examples of systems which include either CH or ¬CH is a theorem; given again that CH is a substantial mathematical thesis, any of

[129] See Shapiro (1991), page 105 for this formulation.

these systems should not be considered logics). This commitment of ‘pure’ (that is, independent of any additional axioms) second order logic to CH arguably means it oversteps the boundaries of logic. It is notable that the position described above is in line with generally verificationist views of mathematics and logic, in that they regard definitions involving the full powerset of the natural numbers with scepticism. This agrees with the general thrust of the position adopted in this thesis – as described in Section 4.4.2.3, verificationism was part of the underpinning of Dummett’s view of logical harmony.

Given the arguments made in the preceding paragraphs, this dissertation concludes that while the second order quantifier defined according to for example Henkin semantics is a logical constant, it does not represent an advance in terms of expressivity compared to the first order quantifier. In the case of the full or unrestricted second order quantifier, the very lack of sound and complete operational rules precludes it from logical constancy. However, it is important to note that this conclusion does not imply that the second order quantifier as defined using the full or unrestricted semantic definition of it is entirely meaningless. The very existence and general intelligibility of the semantic definition of full second order quantification shows that it is not without *meaning*. Rather, the lack of a proof theoretic definition of it means that it is not *logical*.

4.4.4. Modality

This section of the dissertation analyses extensions to propositional logic through the addition of modal operators. As their name suggests, modal operators are elements of formal systems which can be interpreted as representing aspects of reality which display modal behaviour; or as providing a formalisation of modals in natural language. Most prominent among the modal concepts expressed in natural language are the alethic modalities of necessity and possibility, though various others exist also and will be discussed in this dissertation.

Modal operators can also be added to systems which include quantification, resulting in quantified modal logic. However, due to the additional complexity introduced by quantified modal logic compared to modal extensions of propositional logic, consideration of only the latter only will be included here. Additionally, discussion here will be limited to modal extensions of classical propositional logic, rather than intuitionistic propositional logic.

Unlike the case of second order quantification discussed in the previous section of this dissertation, if semantic definitions for modal operators are added to those of propositional logic, propositional proof systems can be extended so they are semantically sound and complete with respect to them. The following sections of this dissertation will assess various formal modal operators based on the criteria for logicality established in previous sections and discuss a selection of modal concepts and their relative claims to informal logicality (which the reader will recall is the term used in this dissertation to denote concepts expressed in natural language which exhibit the requirements of absolute generality and topic neutrality, and are thus usually the interpretations of logical constants).

Before launching into a discussion of modal concepts and their claims to informal logicality, mention should be made of the relationship between logical inferentialism and modal operators. Recall the following definition of logical inferentialism (from Rossberg and Cohnitz (2009)[130]) used in this dissertation:

> Inferentialism insists that the meaning of the logical constants is determined by their introduction and elimination-rules, and that these rules (so far as they are the correct ones) are self-justifying. No further appeal to model-theoretic semantics, truth-tables or the like is needed in order to argue for the validity of the rules.

This approach will again be followed in the following sections – the operational rules for logical operators are sufficient to determine their meaning. Thus, the entirety of the meaning of the modal operator □ is given by □I and □E. This is the same as in the case of the propositional connectives (the entirety of the meaning of ∧ is given by ∧I and ∧E and so on).

In the case of modal operators, the inferentialist claim may present more concerns than in the case of the propositional, etc. operators discussed up to this point in this dissertation, connected in the main to the concept of uniqueness discussed in Section 4.4.2.6 of this dissertation. Thus it may be doubted whether the rules for the modal operator fully capture its meaning. However, given the approach applied

[130] Page 153.

thus far in the dissertation, the inferentialist position will be adopted in the case of modality also, to evaluate the results produced (with the above concern borne in mind throughout)[131].

The examination of modality will begin with a presentation of model theoretic and proof theoretic approaches to it. An analysis of the logical constancy of the operational rules for modal operators will follow this. This will be followed by a discussion of possible interpretations of modal operators (alethic, deontic, etc.), and a comparison of the results of the logical constancy analysis with the topic neutrality and absolute generality of the concepts which are interpretations of them.

The main modal operator which will be discussed in this dissertation is □. Given that a variety of conceptions of □ exist, where necessary each of which can be defined using natural deduction operational rules, suffixes will be used to distinguish between them, for example □S5 and □KD for the modal systems S5 and KD respectively.

4.4.4.1. Possible Worlds Semantics

While the focus of this dissertation is on proof theory, mention of the semantic definitions used for the modal operators □ and ◊ is made here. This is because

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[131] The author would like to acknowledge Stephan Leuenberger for his contribution to this discussion.

they facilitate the understanding of modal proof theory, especially according to Read's treatment of it discussed in later sections of this dissertation.

While propositional logic assigns a simple truth value to each propositional variable, the semantics of modal logic use the concept of possible worlds and allow the truth of propositional variables to be evaluated at each of these possible worlds independently. The additional mechanics of possible worlds do not materially affect the evaluation of the truth value of sentences without modal operators. However, for the relatively simple modal logic system S5, sentences involving modal operators are evaluated as follows (where for example v(□φ,w) is read as 'the truth value of □φ at world w ∈ W'):

- v(□φ,w) = T iff for every world w′ in W,v(φ,w′) = T
- v(◊φ,w) = T iff for some world w′ in W,v(φ,w′) = T

Modal logics other than S5 exist, and their semantics are distinguished from it via the introduction of the notion of accessibility between possible worlds, typically denoted using R. R is a binary relation between possible worlds. Using this mechanism, possible worlds are R-accessible by (world v is R-accessible from world w if wRv) certain other worlds and the truth value of □ and ◊ are then evaluated as follows:

- v(□φ,w) = T iff for every world w′ in W such that wRw', v(φ,w′) = T
- v(◊φ,w) = T iff for some world w′ in W such that wRw', v(φ,w′) = T

That is, □φ is true at world w if φ is true at every world in which w is in an R relation (that is, every world which is R-accessible from w). ◊φ is true if φ is true at least one world which is R-accessible from w.

Different modal logics are then specified by putting different conditions on the R relation. The following table displays some common modal logics and their associated R relation conditions.

Modal Logic	Condition on R Relation
K	No condition imposed
KD	R is serial (∀w∃uwRu)
KT	R is reflexive (∀wwRw)
K4	R is transitive ((wRv&vRu) ⇒ wRu)
KB	R is symmetric (wRu ⇒ uRw)
K5	R is Euclidean ((wRv&wRu) ⇒ uRv)
S5	R is an equivalence relation, and thus reflexive, symmetrical, and transitive

The ordering in terms of the relative strengths of these logics (understood in terms of the semantics presently under discussion as logic L1 being stronger than logic L2 if all the theorems of L1 include all the theorems of L2) is quite complex, and multiple conditions can be imposed to form logics such as KT4, etc. K is the weakest logic and S5 is the strongest logic which will be discussed in this dissertation. As will be seen in the discussion of Read (2008), recognition of the nature of the R-relation in the semantic definition can be used to develop natural deduction rules for various modal logics.

4.4.4.2. Hilbert Systems for Modal Logic

Proof theoretic treatments of modal notions are often undertaken using Hilbert systems, due to some strong links between the addition of modal axioms to form new logics and the conditions on the R relation described in the previous section. For this reason, a preliminary presentation of modal logic proof theory is undertaken using Hilbert systems, before later moving on to natural deduction systems which will be used in the evaluation of the operators' potential for logical constancy.

Modal logics of progressively greater strength (understood here proof theoretically as logic L1 being stronger than L2 if every consequence which is provable in L2 is also provable in L1; and there are consequences which are provable in L1 that are not provable in L2 (that is, if $\Gamma \vdash_{L2} \varphi$ then $\Gamma \vdash_{L1} \varphi$, and for some Δ and ψ, $\Delta \vdash_{L1} \psi$ but it is not the case that $\Delta \vdash_{L2} \psi$) are generated by the addition of new modal axioms to a standard axiomatisation of propositional logic. These axioms can be added independently to propositional logic, or multiple axioms can be added to further strengthen the resulting logic. In addition to these axioms, the following inference rule is also added:

- Necessitation Rule: If φ is a theorem of the modal logic in question, then so is $\Box\varphi$.

The following table presents some common modal axioms which are used.

Axiom Name	Axiom
K	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
D	$\Box\varphi \rightarrow \Diamond\varphi$
T	$\Box\varphi \rightarrow \varphi$
4	$\Box\varphi \rightarrow \Box\Box\varphi$
B	$\varphi \rightarrow \Box\varphi$
5	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$

The reader may have noticed the link between the nomenclature used for the names of the axioms and the modal logics referenced in the corresponding table in Section 4.4.4.1. Modal logic K, which includes only Axiom K and the Necessitation Rule (in addition to the axioms and rules of propositional logic), represents a basic version of modal logic in that it is sound and complete with respect to possible worlds semantics with no condition imposed on R. Axiom K is a prerequisite for qualifying as a so-called 'normal' modal logic which conforms to the possible worlds semantics as described in the previous section. The modal logic K can then be extended through the addition of Axiom D (to form logic KD), Axiom T (to form logic KT), and so on. The addition of these axioms results in further strengthening of modal logic K into systems which are sound and complete with possible worlds semantics with various conditions imposed on R, such as reflexivity (via Axiom T), transitivity (via Axiom 4) or symmetry (via Axiom B).

The addition of multiple axioms to K produces logics which are sound and complete with respect to semantics which have multiple conditions imposed on the R relation. Thus for example the logic KD4 is sound and complete with respect to semantics with both seriality and transitivity imposed on the R relation. However, as already noted, ordering the logics by strength is complex, and some axioms are made redundant by the addition of other axioms. For example, if Axiom T and Axiom 5 are both added, the addition of any other axiom appearing in the table above is redundant, since the resulting system S5 is sound and complete with respect to semantics in which the R relation is an equivalence relation.

4.4.4.3. Logical Constancy of Modal Operators

With the above conclusions in place, attention will now turn to discussion of the potential logical constancy of the modal operators discussed above. As per previous sections of this dissertation, this will be based on investigation of natural deduction operational rules, using the criteria for logical constancy already developed in this dissertation.

Prawitz (1965) provides the following natural deduction operational rules for □ in S5:

$$\frac{\varphi}{\Box\varphi}\ \Box S5I \qquad \frac{\Box\varphi}{\varphi}\ \Box S5E$$

Like the ∀I rule, the □S5I rule operates under a proviso: Every open assumption that φ depends on must have □S5 as its principal operator or be the negation of a formula with □S5 as its principal operator.

There are clear similarities between the operational rules for □S5 and those for ∀, including the fact that a proviso is placed on (and only on) the introduction rule for both. Referring back to the semantic definition of □S5, the R relation is an equivalence relation. Because of this, the R relation can be used to define different 'restricted' necessity operators, which represent an analogy to first order logic's restricted quantifiers. This similarity appears to support the notion that □S5 is a logical constant, because given the aforementioned similarity, it would be difficult to develop criteria which exclude □S5 from logical constancy while simultaneously including ∀.

□S5I and □S5E also conform to Natural Deduction Criterion for Logical Constancy 1 developed in previous sections of this dissertation. However, determining whether the operational rules for □S5 are in harmony is more complex. Given that it is the introduction rule which works under a proviso, demonstrating that local introduction / elimination peaks associated with □S5 can be eliminated is an exceedingly simple affair, just as it was in the case of ∀:

$$\frac{\varphi}{\Box S5\varphi}\ \Box S5I \qquad \text{(every open assumption that } \varphi \text{ depends on has the form } \Box \text{ or } \neg\Box)$$

$$\frac{\Box S5\varphi}{\varphi}\ \Box S5E$$

This result suggests that the operational rules for □S5 are in harmony, that □S5 therefore satisfies Criterion 2, and is, according to the criteria thus far developed, a logical constant. However, Read (2008) disagrees. As mentioned in Section 4.4.2.3, Read does not equate harmony with local peak elimination, holding instead that "harmony consists rather in the justificatory relation between the -I and -E rules"[132]. In the case of modal operators, the force of Read's point is apparent via a comparison to other systems of modal logic beyond S5. Specifically, he states the following rules for the logic S4, a logic which is weaker than S5 and is equivalent to KT4:

$$\frac{\varphi}{\Box S4\varphi}\ \Box S4I \qquad\qquad \frac{\Box S4\varphi}{\varphi}\ \Box S4E$$

Where the proviso is that: Every open assumption that φ depends on must have □ as its principal operator (thus notably excluding negations of formulas with □ as its principal operator, which are permitted in the □S5I rule).

Given this, Read poses the following questions[133]:

1. If logics have different I-rules, should we not expect harmony to yield different E-rules?
2. If logics share I-rules, should they not be the same logic unless disharmonious? - that is, if one has the E-rule, the other not, must not at least one of them be disharmonious?

[132] Read (1999), Page 12.

[133] Read (2008). Page 13.

The focus here will be on the first of Read's questions, since it is sufficient to bring out the points applicable to the present analysis. Read's first question is based on the following argument: The meaning of a logical constant is wholly contained in its introduction rule. This is because, if the general elimination harmony approach developed by Read is accepted, any introduction rules and elimination rules for a constant which are in harmony should be inter derivable. Thus the meaning of a logical constant is wholly contained in its elimination rule also. This equivalency then means that if two introduction rules are different, their meaning must be different (and thus, according to the thesis of inferentialism, the operators defined by them must not be the same), and thus their elimination rules (again, assuming harmony) should also be different.

The problem here of course arises when the operational rules for □S5 and □S4 are compared, since (while the introduction rules differ due to the different proviso which is placed on them) □S5E and □S4E are identical. Furthermore, this would not appear to be an isolated case, since operational rules for other logics could be suggested which place other provisos on the introduction rule, but which again leave the elimination rule unchanged. This would mean that Read's concern is not unique to the operational rules for S4 and S5, but rather it appears that elimination rules could be shared across a wide range of modal operators, the meaning of each of which differs for the inferentialist because their introduction rules differ.

Given the similarity between the operational rules for ∀ and those suggested above for □S5, it may be wondered why a similar problem does not arise in the logic of quantification. This is at least partly because defining a ∀I rule with an adequate proviso is sufficient for the definition of a range of other sentences which express quantificational concepts, such as some, no/none, exactly, and at least. Examples regarding other cardinalities such as countability are considered in Section 0. However, in the case of modal logics, proof systems which are sound and complete for models with restricted R relations, such as the restriction of S4 reflexivity and transitivity relative to S5's equivalence R relation are not definable using S5 (compared to concepts such as some, none, and exactly are definable using ∀). Hence the proliferation of structurally similar □ rules for modal logics with different provisos, required to introduce soundness and completeness with respect to the correct R relation restriction.

Returning to Read's first question, the above analysis supports Read's assertion that harmony should not be equated with the ability to remove local peaks in proofs. In the case of the operational rules for □S5, this local peak flattening is clearly possible. However, as argued above, the same elimination rules being paired with different introduction rules permits local peak elimination but implies a difference of meaning between the introduction and elimination rules. Since it is this sameness of meaning which is the essence of harmony (with local peak elimination being *in most cases* a convenient proxy for this), then this implies that in general, the rules for □S5 and □S4 (and those for other variants of modal logic, some of which will be discussed below) are not in harmony.

One means of addressing this issue would be to investigate whether the □S4E rule (or equivalently the □S5E rule) endows □S4 with the same meaning as one of the □I rules. If such an equivalence of meaning was found, □S4E (or □S5E) could then be pair them off with the ‘correct’ introduction rule, and then an alternative elimination rule sought for the other introduction rules. However, Read adopts a different approach. This is to use a labelled deductive system, in which each formula in a proof has a label attached to it. Understood in terms of possible worlds semantics, these labels reflect the truth value of a formula at a world given in the index (thus φ_i can be read as ‘φ is true at world i’). The operational rules for non-modal formulas are unaffected by this new element, but in order to state modal laws a further symbol is required, which is a relation between these indices symbolised using ‘R’, and thus appearing in the form iRj, where i and j are these labels. Note that statements of the type ‘iRj’ are not strictly formulas of the language of propositional modal logic. Read explains their use as follows, with notation adapted to that used in this dissertation[134]:

> We are setting up a formal system, in which labels and ‘R’ is at best a useful metaphor, whose meaning, if any, is conferred by the rules…a similar point is often made about such an expression as:
>
> $$\lim_{n \to \infty} a_n$$

[134] Read (2008). Page 15.

> where '∞' does not refer to a value of n, but indicates rather that there is no greatest value of n. One may ask how [rules involving e.g. iRj] relate to our inferential practice. That is part of a wider question, how any part of the theoretical systematization of logic relates to practice. More needs to be said, but not here.

Using this newly defined symbol and equipped with the explanation of it given above, Read presents the following rules for □K, the modal operator for the logic which corresponds to adding only the axiom K to a Hilbert-style axiomatisation of propositional logic (where the symbol ⇒ is used as a shorthand for 'there being a proof', so $\varphi \Rightarrow \psi$ is read as 'there is a proof of ψ from φ'):

$$\dfrac{\begin{matrix}(iRj)\\ \vdots \\ \varphi_j\end{matrix}}{\Box\varphi_i}\ \Box KI \qquad\qquad \dfrac{\Box\varphi_i \qquad \begin{matrix}(iRj \Rightarrow \varphi_j)\\ \vdots \\ \psi_k\end{matrix}}{\psi_k}\ \Box KE$$

The rule operates with the additional condition that $i \neq j$ and j must not appear in any other assumption on which φ_j depends.

These operational rules are in harmony according to Read's general elimination harmony approach, and they also allow the elimination of local peaks, as demonstrated by the fact that the following proof:

$$\dfrac{\dfrac{\begin{matrix}(iRj)\\ \vdots \\ \varphi_j\end{matrix}}{\Box\varphi_i}\,\Box KI \qquad \begin{matrix}(iRj \Rightarrow \varphi_j)\\ \vdots \\ \psi_k\end{matrix}}{\psi_k}\,\Box KE$$

reduces to the following, local peak-less proof:

$$\begin{matrix}(iRj)\\ \vdots \\ \varphi j\\ \vdots \\ \psi_k\end{matrix}$$

These operational rules are retained in the other normal modal logics, with the difference between them being produced by different rules used for the R relation. The R rules for each modal logic are based on the conditions imposed on the R relation given in the table in Section 4.4.4.1. They are as follows:

- K (where no condition is imposed on the R relation): In this case, no condition is placed on the R relation, since any one instance of the R relation does not imply a further instance of it due to seriality, transitivity, etc.
- KD (where R is serial):

$$\dfrac{\begin{matrix}(\exists j(iRj))\\ \vdots \\ \varphi_i\end{matrix}}{\varphi_l}\,RD$$

- KT (where R is reflexive):

$$\dfrac{\begin{matrix}(iRi)\\ \vdots \\ \varphi_i\end{matrix}}{\varphi_l}\ RT$$

- K4 (where R is transitive):

$$\dfrac{iRj \quad jRk \quad \begin{matrix}(iRk)\\ \vdots \\ \varphi_l\end{matrix}}{\varphi_l}\ R4$$

- KB (where R is symmetric):

$$\dfrac{iRj \quad \begin{matrix}(jRi)\\ \vdots \\ \varphi_l\end{matrix}}{\varphi_l}\ RB$$

- K5 (where R is Euclidean):

$$\dfrac{iRj \quad iRk \quad \begin{matrix}(jRk)\\ \vdots \\ \varphi_m\end{matrix}}{\varphi_m}\ R5$$

- S5 (where R is an equivalence relation): No specific stipulation is required for the R relation, and the following natural deduction rule holds:

$$\dfrac{\varphi}{\Box\varphi}\ \Box S5I \qquad \dfrac{\Box\varphi}{\varphi}\ \Box S5E$$

Where every open assumption that φ depends on must have □S5 as its principal operator or be the negation of a formula with □S5 as its principal operator.

With these structural rules, the link between Read's natural deduction treatment of modal logic using R and the semantic definitions involving the R relation becomes evident. For example, the R4 rule imposes transitivity on R, just as the addition of Axiom 4 (□φ → □□φ) to an axiomatisation results in soundness and completeness with respect to possible worlds semantics with a transitive R relation; and R5 imposes a Euclidean relation on R and its analogue is Axiom 5 (◊φ → □◊φ), whose addition to an axiomatisation results in soundness and completeness with respect to possible worlds semantics with a Euclidean R relation.

Thus, using Read's approach, □ can be defined in a way which both guarantees local peak elimination and disposes of the aforementioned problem of different introduction rules being paired with the same elimination rule. However, it does so (except for K and S5) by defining □ through its reference to the new symbol R. This is reminiscent of the reference to ⊥ which many theorists use in the operational rules for ↑ ('nand'). However, there is a key difference here: ⊥ is itself a logical constant, and thus ↑ can be defined (and defined without circularity since the operational rules for ⊥ do not include ↑), by referring to it while keeping the definition entirely within the scope of the logical constants.

In the case of Read's definition of □, reference to what he terms the structural rule R is required. It is the variation in this structural rule which modifies the characteristics of □, and thus allows different modal logics to be developed. Since reference here is to a structural rule, this conflicts with the inferentialist position that the meanings of the logical constants are contained within the definitions provided by their operational rules. It therefore undermines the notion that distinguishing those examples which do represent logical consequence out of the wider set of inferences in general is possible based on operational rules alone.

Recalling previous discussion in this dissertation, the reasoning supporting this concern is as follows. Logical consequence can be distinguished by the fact that it is purely formal. Thus, an example of logical consequence will still hold after all of its contentful components (for example, its components which refer to particular physical objects) have been abstracted from it by replacing them with variables. Justification that the purported example of logical consequence holds must then be provided by what remains after this abstraction has occurred – that is, the logical constants. The inferentialist view is that the meaning of the constants is given entirely by their operational rules, and these rules further allow a deduction to be constructed which provides, from a proof theoretic perspective, justification that the example of consequence does hold.

This is definitely a defensible position, and one which has a lot to recommend it. In terms of logical constancy, this position results in only □K and □S5 being accepted as logical constants, defined by the following rules:

$$\dfrac{\begin{matrix}(iRj)\\ \vdots \\ \varphi j\end{matrix}}{\Box\varphi i}\ \Box KI \qquad\qquad \dfrac{\Box\varphi_i \qquad \begin{matrix}(iRj \Rightarrow \varphi_j)\\ \vdots \\ \psi_k\end{matrix}}{\psi_k}\ \Box KE$$

$$\dfrac{\varphi}{\Box\varphi}\ \Box S5I \qquad\qquad \dfrac{\Box\varphi}{\varphi}\ \Box S5E$$

This result also gels well with intuitions regarding the modal operators. □S5 and □K stand out from the other modal operators in that they have an all (S5) or nothing (K) approach to the R relation – in the case of S5, the R relation is an equivalence relation, whereas in the case of K, it is not necessary that any worlds are R related at all.

Even though the conclusion above is reasonable, it is worth also investigating further to determine if more nuance can be introduced into the situation. In terms of the meaning of R, Read compares it to the condition which is attached to the ∀I rule, that x may not occur free in any hypothesis on which φ(a) depends. Likewise, RB, RT, R4, RB and R5 state conditions for the use of □BI, □TI, □4I, □BI and □5I respectively. The difference is that the condition is more complex, and its statement is facilitated by introducing R.

It is notable that each of the R rules put forward above conform to Criterion 1's requirement for elimination rules, since R does not appear in the antecedent in any case. This raises the possibility that they could be treated as elimination rules for

R, rather than as structural rules as Read asserts. This would then put the □I and □E rules in a similar position to those rules for propositional logical constants, which have external reference in their rules, but external reference to something which is itself a logical constant (⊥). While it is acknowledged that this appears to be a somewhat strange approach, since R is more part of the 'logical background' rather than itself a candidate for logical constancy, the reader is requested to bear with the approach for the moment.

Producing introduction rules for each case of R which conform to Natural Deduction Criterion for Logical Constancy 1 and are in harmony with the existing 'elimination' rules would then allow each case of R to be considered as a logical constant in its own right rather than a structural rule. To produce these introduction rules, the basic notion that the meaning of introduction and elimination rules must be the same in order to produce a harmonic operational rule can be employed. Taking R4 and R5 as examples, this leads to the following introduction rules, R4I and R5I respectively (the R4 and R5 rules given previously will henceforth be referred to as R4E and R5E):

$$\frac{}{(iRj) \wedge (jRk) \rightarrow (iRk)}\ \text{R4I} \qquad \frac{}{(iRj) \wedge (iRk) \rightarrow (jRk)}\ \text{R5I}$$

However, establishing these introduction rules does not allow local peak elimination of R4 and R5. This is essentially due to the non-standard nature of the elimination rules. While they do conform to Natural Deduction Criterion for Logical

Constancy 1, they do not conform to the general structure of introduction and elimination rules as laid out by Read in his theory of general elimination harmony:

$$\frac{\pi_{i1} \ \ldots \ \pi_{in_i}}{\delta\vec{\alpha}}\ \delta I \qquad \frac{\delta\vec{\alpha} \quad \begin{matrix}(\pi_{1j_1})\\ \vdots \\ \gamma\end{matrix} \quad \ldots \quad \begin{matrix}(\pi_{mj_m})\\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \delta E$$

According to the above, the introduction rule for δ introduces $\delta\vec{\alpha}$ with its grounds for assertion given by $\pi_{i1} \ldots \pi_{in_i}$. However, R4I and R5I both have no grounds for assertion, meaning that $\pi_{i1} \ldots \pi_{ini}$ is empty. However, R4E and R5E use iRk and jRk to fill the places represented by $\pi_{i1} \ldots \pi_{ini}$. Given the conditional structure of R4I and R5I, a potential modification of them to fit into Read's casting of introduction rules would be as follows:

$$\frac{(iRj) \qquad (jRk)}{(iRk)}\ R4I' \qquad \frac{(iRj) \qquad (iRk)}{(jRk)}\ R5I'$$

However, this modification both violates Natural Deduction Criterion for Logical Constancy 1 and does not lead to local peak elimination either. In any case, the appearance of both iRj and jRk in the case of R4E; and iRj and iRk in the case of R5E in the place of $\delta_{\vec{\alpha}}$ again does not fit the structure of Read's general elimination harmony format for elimination rules for a logical constant. This and the aforementioned structural differences between the operational rules suggested for R4 and R5 and Read's generalised approach to stating such rules display serious problems in treating R4 and R5 as potential logical constants.

Thus the attempt to introduce nuance into the various systems of modal logic by evaluating the potential logical constancy of R has failed. In any case, as previously mentioned, R operates more at the level of a structural rule than as a logical operator, meaning that even if successful, demonstrating that a conception of R can be shown to adhere to Natural Deduction Criteria for Logical Constancy 1 and 2 is unconvincing even if successful. Intuitively, logical constancy fails for definitions of □ which require external definitions of the R relation is because such definitions are substantive metaphysical (or other substantive) commitments

In summary, the conclusion of this proof theoretic analysis of modal operators results in only □K and □S5 being accepted as logical constants, as defined by the following natural deduction operational rules:

$$\dfrac{\begin{matrix}(iRj)\\ \vdots \\ \varphi j\end{matrix}}{\Box\varphi i}\ \Box KI \qquad \dfrac{\Box\varphi_i \qquad \begin{matrix}(iRj \Rightarrow \varphi_j)\\ \vdots \\ \psi_k\end{matrix}}{\psi_k}\ \Box KE$$

$$\dfrac{\varphi}{\Box\varphi}\ \Box S5I \qquad \dfrac{\Box\varphi}{\varphi}\ \Box S5E$$

4.4.4.4. Interpretations of Modal Operators

Now that logical constancy evaluations of the modal operators have been completed in the previous section, this dissertation can proceed with comparison of these results with the natural language conceptions which are interpretations of them. If the logical constancy evaluations have been successful, then the interpretations (where they exist) of □K and □S5 should be concepts which are absolutely general and topic neutral, and the interpretations of the other modal operators should not be absolutely general and topic neutral.

A factor which complicates the above approach is the varying levels of agreement regarding the correct natural language interpretations of the modal operators, or equivalently, the correct formalisations of modal concepts in natural language. The arguments leading to the lack of consensus in each case are complex, and this dissertation's focus on proof theoretic analysis of logical constancy means that there is insufficient scope herein to do them justice. Because of this, the correct interpretation/formalisation in each case will be based on what seems in each case to be a defensible position found via a review of prominent literature on the subject.

The alethic modality of necessity can be qualified in a number of ways, resulting in a hierarchy of necessities. For example, biological necessities can be easily imagined, such as the prerequisites for certain life forms (oxygen is a biological necessity for humans, for example). Physical necessities are also abundantly evident (massive bodies necessarily move at sub-luminal velocities, for example). The hierarchical aspect mentioned above means that all biological entities are subject to physical necessities also. What is considered here is however is logical necessity, which operates at the most fundamental level of the necessity hierarchy.

According to the arguments put forward in this dissertation, evaluating logical necessity's potential to be formalised by a logical constant should proceed according to the requirements of absolute generality, topic neutrality and formality, as follows:

- Absolute generality: Being the most fundamental level of athletic modality, the concept of logical necessity applies generally. This can be seen via the apparatus of possible worlds, since for example logical truth is truth in *all* possible worlds, whereas concepts such as obligation concern only possible worlds which are morally permissible.
- Topic neutrality: Rather than applying to only a certain topic (as was the case for physical or biological necessity), logical necessity is entirely topic neutral.
- Formality: The grammatical structure of sentences expressing logical necessity suggest that their truth is a matter of form rather than content.

In terms of the correct formalisation of logical necessity, Girle (2009) states that "The S5 modal logic is often suggested as the system for logical possibility and necessity"[135]. As mentioned previously, whether S5 is the correct formalisation of logical necessity and possibility could be further debated but is not the focus of the

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[135] Page 139.

present dissertation. If it is taken as true that S5 is the system for logical possibility and necessity, the proof theoretic results obtained in this section are in line with expectations. That is, logical necessity is absolutely general and topic neutral, and its formalisation $\Box$S5 is a logical constant.

Compared to the logical necessity and possibility considered above, the deontic notions of obligation and permission appear to have limited prospects in terms of informal logicality. Examining these notions with respect to the requirements for logicality established in this dissertation results in the following:

- Absolute generality: Obligation and permission are applicable only in the sphere of ethics, and thus have a scope which falls short of absolute generality.
- Topic neutrality: Obligation and permission concern the specific topic of ethics, and thus cannot be considered to be topic neutral.
- Formality: Due to their contentful nature, obligation and permission seem to contribute more than simply structure or form of an inference.

These points depend to a certain extent on the way in which obligation and permission are construed. For example, on a divine command theory, 'it ought to be the case that' should be understood as 'it has been commanded by God that'. Another option may be that ‘it ought to be the case that’ should be understood as ‘social norms and institutions require that’. However, under either of these means of construing the concept, the lack of topic neutrality is maintained.

Certain thinkers object to the very possibility of deontic logics. An example of this following Jörgensen (1937) is as follows. Taking the (admittedly controversial) position that the sentences associated with morality are normative rather than factual, this implies that they cannot legitimately have a truth value attached to them. If logical consequence concerns the transmission of truth from premises to consequence, then a logic based on morality seems impossible. However, on the other hand, certain normative sentences do seem to be a consequence of certain other sentences, in the way characteristic of logics in general, which is puzzling for those who take a normative view of moral statements.

On balance, then, it seems that deontic modalities have less claim to informal logicality than the alethic modalities. This is most evident when considered from the point of view of the informal requirement of topic neutrality – the concept of logical necessity is neutral in terms of topic or subject matter, whereas that of obligation is restricted to ethics. This specialisation of subject matter is also evident in terms of the hierarchy of necessity described in the previous section of this dissertation, since obligation can be seen as ethical necessity, meaning it occupies a less fundamental position than the concept of logical necessity considered in the previous section, and thus that it could not be expected to formalised by a logical constant.

In contrast to logical necessity, then, the expectation would then be that however obligation is developed as a formal notion, evaluation of it will suggest that it is not a logical constant. What is known as 'standard' deontic logic is KD[136], the axiomatisation of which includes all the axioms and rules of propositional logic, plus Axiom K, Axiom D, and the rule of necessitation. As mentioned previously, whether KD is the correct formalisation of deontic obligation could be further debated but is not the focus of the present dissertation. If it is taken as true that KD is the system for deontic obligation, the proof theoretic results obtained in this section are in line with expectations. That is, deontic obligation is not absolutely general and topic neutral, and its formalisation $\Box_{KD}$ is not a logical constant. As discussed in the relevant section, this is due to its reliance on a specific definition of the R relation.

Like deontic modality, the epistemic notion of knowledge has limited prospects in terms of informal logicality. Applying again the examination of these notions with respect to the requirements for logicality established in this dissertation results in the following:

- Absolute generality: Knowledge is only applicable in the sphere of epistemology, and thus has a scope which falls short of absolute generality.
- Topic neutrality: Knowledge concerns the specific topic of epistemology, and thus cannot be considered to be topic neutral.

[136] McNamara, P. (2019). Section 2.1.

- Formality: Due to its contentful nature, knowledge contributes more than simply structure or form to an inference.

The logical system which represents the best formalisation of the intuitive concept of knowledge is controversial, with objections possible to many of the logical axioms which can be suggested as providing a characterisation of knowledge. The epistemic Axiom T ($\Box\varphi \rightarrow \varphi$) is, however, broadly accepted[137]. This appears reasonable, since the truth of a sentence is a necessary condition for knowledge of it. Its equivalent form, $\neg\varphi \rightarrow \neg\Box\varphi$, which states that if a proposition is false then agent does not know it, is also reasonable.

In contrast to its alethic and deontic counterparts, however, the epistemic axiom K ($\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$) is disputable. This is because it represents a closure principle for knowledge – if an agent knows that φ implies ψ, then the agent's knowing φ is sufficient for their knowing ψ. This appears to be a highly idealised view, and implies that a sort of perfect rational capacity, in which agents are able to deduce all the ramifications of each proposition they know. Practical experience of errors made by agents who do not realise the ramifications of their existing knowledge bases suggests that this idealised view is dubious at best.

Axiom 4 ($\Box\varphi \rightarrow \Box\Box\varphi$) states that if an agent knows something, they also know that they know it (this is known as positive introspection). While this may be deemed

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[137] Rendsvig et al (2019). Section 2.6.

appropriate, since it seems reasonable that agents can have introspective knowledge in this way, this is however, controversial. Axiom 5 ($\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$) states that if an agent does not know something, then they know that they don't know it (negative introspection). This axiom seems even more dubious than Axiom 4, since it implies that agents cannot be mistaken about gaps in their own knowledge.

What seems certain though, is that neither K nor S5 represent formalisations of the concept of knowledge of an ordinary agent. Given that these were the only modal operators identified as logical constants, it does not seem that the concept of knowledge can be formalised using a logical constant. As in the case of alethic and deontic modality, this is in line with expectations – the concept of knowledge is not absolutely general and topic neutral, so it is to be expected that it cannot be formalised by a logical constant.

4.4.4.5. Modal Contingency

The arguments of the previous three sections support the view that logical necessity is absolutely general and topic neutral, while deontic obligation and the epistemic modality of knowledge are not. This is on the basis of logical necessity being applicable to sentences in the most general sense, while the applicability of deontic or epistemic necessity are more restricted. However, the concept of contingency provides a seeming means of maintaining a universal applicability of each of these modal operators. In alethic terms, a sentence is contingent if necessity is denied to both a sentence and its negation (symbolically, $\nabla\varphi =_{def} \neg\Box\varphi \wedge \neg\Box\neg\varphi$). Examples of alethically, deontically, and epistemically contingent sentences are given below:

- It is not logically necessary that the flag above Buckingham Palace is at half-mast; and it is also not logically necessary that the flag above Buckingham Palace is not at half-mast.
- It is not (deontically) obligatory that massive bodies mutually attract; and it is also not obligatory that massive bodies do not mutually attract.
- John does not know that God exists; and John does not know that God does not exist.

Giving an evaluation of contingency to any sentence which does not involve the relevant subject matter for the modal operator in question thus appears to provide a means of maintaining their equivalent, and universal applicability. This in turn suggests that each modal operator has an equal claim to logical constancy, when judged on the basis of this absolute generality / topic neutrality. This is problematic in the context of this dissertation, since as maintained in the relevant section above, logical necessity does appear to be in some way more widely applicable than deontic obligation and knowledge.

However, recourse to the idea of topic neutrality provides a means of maintaining the arguments in the previous sections that logical necessity is logical, whereas deontic obligation is not. Consider the example of the attraction of massive bodies.

A non-contingent deontic evaluation of this sentence would be difficult to justify, since this statement about the physical world seems entirely out of the scope of deontology. However, this is not so in the case of logical necessity, which seems more universally applicable to such statements. This point establishes the wider generality of non-contingent evaluations of logical necessity compared to deontic obligation, and thus the greater potential for logical constancy of the former compared to the latter will be maintained in this dissertation.

4.4.4.6. Multiple Interpretations

As was alluded to in Section 4.4.4.4, the correct natural language interpretations of modal operators and the correct formalisations of modal concepts in natural language is less clear and more open to debate than in other areas discussed in this dissertation. For example, that $\wedge$ is a formalisation of the natural language term ‘and’ is relatively uncontroversial (though even in this case there may be some controversy, associated with example collective subjects). On the other hand, whether $\Box$KD is the correct formalisation of the notion of obligation in a deontic setting is much more open to debate, with little consensus being achieved in the literature. Similar points can be made regarding other modal operators and modal concepts.

In addition, the possibility of multiple interpretations of the same operator in a formal system seems to be a more reasonable possibility in the modal setting than for the propositional and quantificational operators considered previously in this dissertation. The existence of such multiple interpretations would be problematic for this dissertation. This is because the production of natural deduction criteria for logical constancy relies on the thesis of inferentialism, which states that the meaning of the logical constants is determined by, and only by, their introduction and elimination rules. Allowing multiple interpretations of a logical constant would subvert this unity of meaning required by inferentialism.

The author is not aware of any cases in which multiple interpretations can be said to exist with confidence for any logical operator, modal or otherwise. However, an instructive situation exists in the case of □S5. As has been discussed previously, while □S5 is argued in the literature to be the correct formalisation of logical necessity, there are also views which argue that it is also the correct formalisation of *metaphysical* necessity. This would of course represent a multiple interpretation of □S5, which would be of concern for the views espoused in this dissertation.

It appears that these results support the conclusion that conclusion that □S5 is not the correct formalisation of metaphysical necessity. This is due to the assessment of □S5 as a logical constant, which itself demonstrates that □S5 cannot formalise metaphysical necessity. This is because, given that it is concerns the metaphysical and not just the logical, metaphysical necessity cannot be formalised by a logical constant like □S5 (to clarify this statement proof theoretically, the meaning is that there are facts regarding metaphysical necessity which are not provable theorems of S5 logic; or there are theorems of S5 logic which conflict with facts regarding metaphysical necessity). That □S5 and logical necessity exist in the formalisation / interpretation relationship is to be expected, due to their logical constancy and informal logicality respectively. This would not be the case for □S5 and metaphysical necessity.

This is an interesting case of the application of the results obtained in this dissertation. Note that this involves two claims, both of which this dissertation holds to be true:

1. Logical constants can only formalise informally logical concepts.
2. Informally logical concepts are only formalised by logical constants.

However, there are significant reasons to pause regarding such a conclusion, which would show that a view with significant support in contemporary discussions of modality is wrong.[138]

First, it may be the case that there is no distinction between logical and metaphysical necessity, in that there is nothing that is metaphysically necessary but not logically necessary or logically necessary but not metaphysically necessary. If true, this would mean that the point above regarding S5, logical necessity and metaphysical necessity is not of any significance.

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[138] The author would like to acknowledge Stephan Leuenberger for his contribution to this discussion.

Second, it is true that logical necessity conforms to the logical system S5. However, with the R relation in S5 being an equivalence relation, there are other types of necessity (including perhaps metaphysical necessity) which conform to not the logical system S5 *as a whole*, but rather to logical systems which partition the space of possible worlds into equivalence classes of worlds. It might be that logical necessity is universal and metaphysical necessity corresponds to a partition of the space of worlds (into clusters of worlds that have the same laws of metaphysics, for example). Given this partitioning, the resulting □ operators for partitioned S5 logics may not be logical constants, which would conform to the view that metaphysical necessity is not informally logical. Thus metaphysical necessity may be formalised by a □ operator which is S5-like but only within a certain subclass of worlds, meaning a formal system which includes it can be seen as a theory rather than as a logic itself.

Notwithstanding these comments, the above reasoning involves the notion that demonstrating that □S5 is a logical constant means that it cannot be the correct formalisation of any modal concept which is itself informally logical (an application of 2 above). The same conclusion can also be applied to □K – since it is a logical constant, it cannot be the correct formalisation of any non-informally logical modal concept (though no interpretation of □K at all is known to the author) (an application of 1 above). Objections to these conclusions should bear in mind that the natural deduction criteria for logical constancy developed in this dissertation have been applied to propositional, first order, and second order logic with results which accord with the absolute generality and topic neutrality of the respective interpretations. This implies that confidence should be granted to the results they produce in the case of the modal operators.

4.4.4.7. Concluding Remarks

The logical constancy analysis conducted in this section of the dissertation concluded that □S5 and □K are logical constants. This was on the basis of their conforming to Natural Deduction Criterion for Logical Constancy 1 and 2, developed previously in this dissertation. Other modal operators which involve more complex stipulations on R, including □KB, □KT, □KD, □K4 and □K5, are not logical constants. The three concepts investigated in this dissertation which represent interpretations of modal operators were in accordance with these results:

- Logical necessity is informally logical (is absolutely general and topic neutral), and assuming it is formalised by □S5, which is a logical constant, this accords with the results of the logical constancy analysis.
- Deontic obligation is not informally logical (it is not absolutely general nor is it topic neutral), and assuming it is formalised by □KD, which is not a logical constant, this accords with the results of the logical constancy analysis.
- Knowledge is not informally logical (is not absolutely general nor is it topic neutral), and while the correct formalisation of knowledge is less clear, it seems sure that it is not formalised by □K nor □S5, the only modal logical

constants. Hence this accords with the results of the logical constancy analysis.

Given the discussion included in this chapter, this dissertation proposes to add the following third criterion for logical constancy:

- Natural Deduction Criterion for Logical Constancy 3: The rules must contain no reference to any non-logical[139] elements external to the operator which the rule defines.

This criterion excludes modal operators with externally defined conditions on R, leaving only $\Box K$ and $\Box S5$ as logical constants. The criterion is furthermore justifiable in and of itself, since it aligns with the general inferentialist approach taken when developing proof theoretic criteria for logical constancy.

Overall, then, the proof theoretic approach adopted to assess logical constancy has fared well when used to analyse modal operators. Those whose natural language interpretations conform to absolute generality and topic neutrality (logical necessity) were adjudicated to be logical constants, while those whose natural language interpretation is not absolutely general, nor topic neutral (deontic and epistemic modalities) were adjudicated not to be logical constants.

[139] Recall that reference to $\bot$ is required in some rules for truth functional logical constants, but also that $\bot$ is itself a logical constant according to the criteria developed in this dissertation.

5. Semantic Tableaux Criteria for Logical Constancy

This dissertation has focussed on producing criteria for logical constancy through analysis of the operational rules of natural deduction systems. In doing so, it has in general followed the literature on the subject, with contributions made by Prior, Prawitz and Dummett all examining natural deduction systems, and the harmony being a property of constants which appear in natural deduction systems.

This approach is reasonable, since natural deduction systems embody the inferentialist position that the meaning of a connective is entirely encapsulated in its operational rules. This in turn means that all the information required to evaluate logical constancy should be available in these rules. However, these facts do not preclude attempts to evaluate logical constancy on the basis of other types of proof systems. In terms of which other types should be examined, Hilbert style axiomatic systems stray too far from inferentialist principles to permit profitable investigation; and sequent calculi are perhaps too similar to natural deduction systems to be genuinely interesting.

This leaves the method of semantic tableaux. That the general structure of semantic tableaux rules is similar to that of natural deduction rules means that analysis of these systems has *prima facie* potential for identifying criteria for logical constancy. The objective which is pursued in this section of this dissertation is therefore to seek such criteria for rules appearing in semantic tableaux systems.

No formal presentation of semantic tableaux will be made here. However, their motivation and the importance of the individual tableaux rules for each operator is summarised in the following, taken from Bostock (1997)[140]:

> A tableau proof is a proof by reductio ad absurdum. One begins with an assumption, and one develops the consequences of that assumption, seeking to derive an impossible consequence. If the proof succeeds, and an impossible consequence is discovered, then, of course, we conclude that the original assumption was impossible…The general method is to argue in this way: if such and such a formula is to be true (or false), then also such and such shorter formulae must be true (or false), and that in turn requires the truth (or falsehood) of yet shorter formulae, and so on, until in the end we can express the requirement in terms of the truth or falsehood of the shortest possible formulae, i.e. atomic formulae. So we need a number of particular rules which state how the truth conditions for longer formulae carry implications for the truth or falsehood of their shorter components.

Despite their similarity to natural deduction systems, inferentialist principles do not apply as strongly in the case of semantic tableaux systems. In the case of natural deduction systems, it is reasonable to hold that the entirety of the meaning of an operator is contained in its operational rules. However, this is less reasonable in the case of the semantic tableaux rules. This is particularly important in the case of

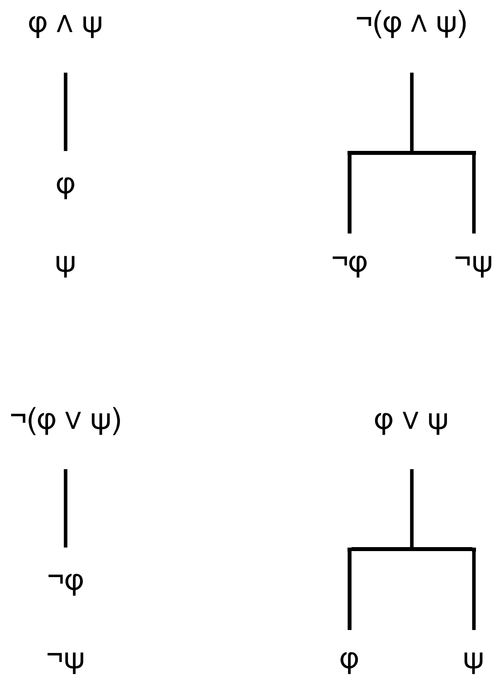
140 Page 141 and 142.

negation, which is presented in (classical) semantic tableaux systems using the following rule:

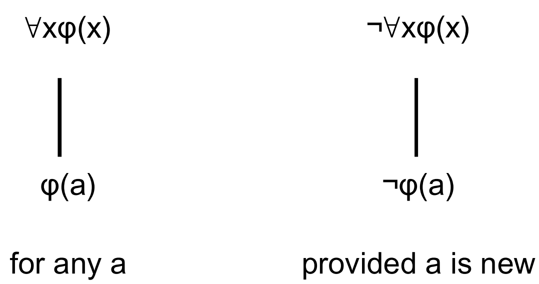
$$\begin{array}{c} \neg\neg\varphi \\ | \\ \varphi \end{array}$$

However, negation also plays an important role in semantic tableaux systems in general, since tableaux branches close when, for some φ, both φ and ¬φ appear on the same branch. This branch closure stipulation provides some of the meaning of ¬ beyond what is contained in the above rule, and thus plays some role in defining it. Thus while it is reasonable to claim that *holistically* the semantic tableaux approach defines negation, the definition of it is not entirely contained in its tree rule. This should be borne in mind in the following discussion of semantic tableaux systems, which is at more of a high-level and less detailed than in the case of previous sections on natural deduction systems.

Despite the name, semantic tableaux systems can be seen in an entirely proof theoretic manner, with no inherent link to semantic approaches. Given the focus of this dissertation, this is the view adopted in this section. The particular semantic tableaux rules which most directly demonstrate the reduction of formulas to their smaller component parts are the rules for ∧ and ∨. These are given below, noting that a negated rule is required in each case to permit reduction of formulas in semantic tableaux in all cases in which formulas can be encountered.



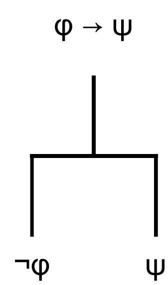
These rules show that they reflect the truth tables for the respective formulas and can be more or less read straight from them. This is possible because ∧ and ∨ are both truth functional. For non-truth functional operators, tableaux rules can still be constructed, with those for universal quantification being as follows:



Similar rules can be defined for the other connectives of first order logic. Before investigating tonk in the context of semantic tableaux systems, an analogue of Natural Deduction Criterion for Logical Constancy 1 should be considered. That is, a criterion should be put in place which ensures that the semantic tableaux rules considered can rightly be considered rules, regardless of considerations of logical constancy – that is, an analogue of criterion 1. Given the nature of semantic tableaux systems, the following is tentatively offered:

- Semantic Tableaux Criterion for Logical Constancy 1 (Tentative): The lower part of the semantic tableaux rule must be of a reduced complexity compared to the upper part of the rule.

However, this depends on how complexity is measured. Consider the following semantic tableaux rule for the conditional:



Given that $\varphi \rightarrow \psi$ and $\neg\varphi$ contain the same number of occurrences of connectives, it can be argued that this rule violates Semantic Tableaux Criterion for Logical Constancy 1 (Tentative). Thus the following is offered to remedy this problem[141]:

- Semantic Tableaux Criterion for Logical Constancy 1: Call the formula that occurs with the operator of interest dominant the *main* formula of the rule; it either occurs as is (in the analogue of an elimination rule) or within the scope of a single occurrence of negation (in the analogue of an introduction

[141] Objection to Semantic Tableaux Criterion for Logical Constancy 1 (Tentative) and the wording of its reformulation in Semantic Tableaux Criterion for Logical Constancy 1 is due to comments provided by Peter Milne.

rule). Then formulas occurring in the lower part of the semantic tableaux rule should be proper subformulas or negations of proper subformulas of the main formula.

This criterion is required to allow the overall semantic tableaux approach, that is the breaking down of formulas into their constituent parts with the goal of seeking a counter example, to properly function. All rules thus far discussed fulfil this criterion.

Since all the connectives discussed thus far are logical constants, at least according to the arguments presented in the main sections of this dissertation, limited information regarding criteria for constancy can be gleaned from their evaluation, since they include no cases in which operators lack logical constancy whose rules can be used to refine the criteria. Thus, as was the case for natural deduction, tonk will be used to stimulate the discussion. Recall that Prior gives the following natural deduction rules for tonk:

$$\frac{\varphi}{\varphi \text{ tonk } \psi}\ \text{tonkI} \qquad\qquad \frac{\varphi \text{ tonk } \psi}{\psi}\ \text{tonkE}$$

Given these rules, this dissertation proposes the following semantic tableaux rules for tonk:

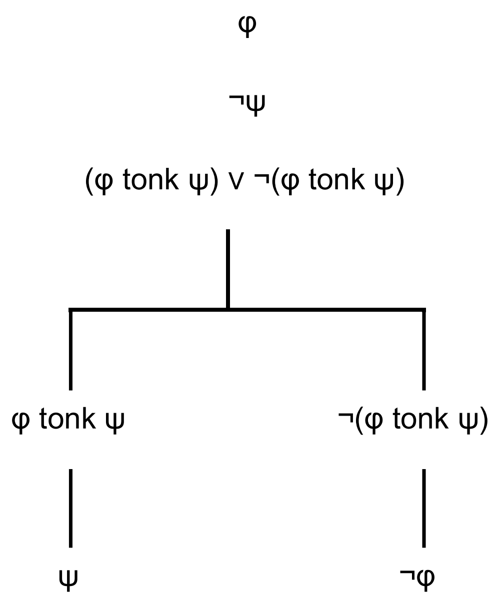
$$\begin{array}{c} \varphi \text{ tonk } \psi \\ | \\ \psi \end{array} \qquad\qquad \begin{array}{c} \neg(\varphi \text{ tonk } \psi) \\ | \\ \neg\varphi \end{array}$$

The left-hand rule states that φ tonk ψ entails ψ, and thus is analogous to the natural deduction rule tonkE, which also quite plainly states that φ tonk ψ entails ψ. The right-hand rule states that ¬(φ tonk ψ) entails ¬φ, and thus is analogous to the natural deduction rule tonkI. Here, the reasoning is that if ¬(φ tonk ψ) holds (that is, φ tonk ψ does not hold), then ¬φ must hold also, since as per tonkI, φ entails φ tonk ψ.

To demonstrate that the semantic tableaux rules described above for tonk are correct, recall that one interpretation of the central problem posed by the natural deduction rules for tonk was that they led to a breakdown of any reasonable notion of entailment, by allowing a proof of the arbitrary entailment φ ⊢ ψ as follows:

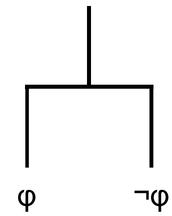
$$\cfrac{\cfrac{\varphi}{\varphi \text{ tonk } \psi}\ \text{tonkI}}{\psi}\ \text{tonkE}$$

The following semantic tableaux proof provides a similar result, using the addition of the premise (φ tonk ψ) ∨ ¬(φ tonk ψ). The other two premises, φ and ¬ψ represent the entailment to be proved, with ψ being negated in keeping with the semantic tableaux methodology of seeking a counter example to the entailment of interest:



Both pathways of the semantic tableaux close, meaning that no counter example to φ ⊢ ψ can be found. Thus the semantic tableau above represents a tonk-based proof of the entailment φ ⊢ ψ, as desired. That the chosen semantic tableaux rules for tonk deliver the same result as the natural deduction rules for tonk increases confidence that the selected tableaux rules for tonk are correct.

Introducing an instance of the excluded middle such as (φ tonk ψ) ∨ ¬(φ tonk ψ) to a semantic tableaux proof is strategically equivalent to invoking what could be called the splitting rule, which states that for any formula φ it is permissible to impose the following branching[142]:



This is obviously sound with respect to classical semantics, and it is provable that it is admissible in semantic tableaux proofs by proving what Bostock (1997)[143] calls the cut principle, which states that with the usual tableaux rules for ¬, ∧, ∨, → ↔, ∀ and ∃, if there is a closed tableau headed by φ and the formulas in Γ and a closed tableau headed by ¬φ and the formulas in Γ then there is a closed tableau headed by just the formulas in Γ.

However, once the rules suggested above for tonk are added to the semantic tableaux system, the permissible addition of instances of the excluded middle (or equivalently, the use of the splitting rule) become essential for the tonk rules to be applicable – it is an indispensable part of the proof system, rather than a provable addition to it. Revisions to this dissertation provided by Peter Milne suggest that because of this:

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[142] The discussion of the splitting rule and the cut principle is based on feedback on this dissertation provided by Peter Milne.

[143] Page 182.

The failure of the Cut principle in the presence of the tableaux rules for tonk is a clear indication that something has gone wrong - but what? Given how tableaux rules relate to introduction and elimination rules in natural deduction, there is a significant analogy between ineliminability of instances of the cut rule with φ tonk ψ and its negation and the failure to level the local peak in the deduction:

$$\dfrac{\dfrac{\varphi}{\varphi \text{ tonk } \psi}\ \text{tonk I}}{\psi}\ \text{tonk E}$$

This analogy is sufficiently strong to make them seem like instances of the same phenomenon. If that is right, the cut principle is the analogue of Prawitz's Normalisation Theorem for natural deduction.

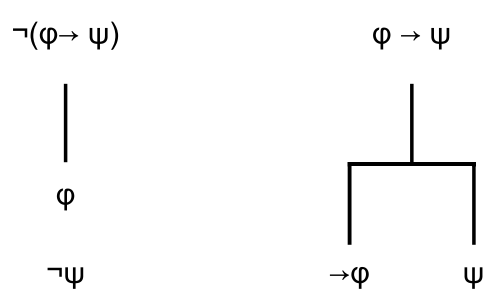
With the *tonk* rules thus established, this dissertation can proceed with investigation of them to seek a semantic tableaux concept similar to that of harmony for natural deduction. As Bostock[144] points out, a difference between the quantifier rules and those for the truth functional operators is that the entailments described by the latter hold bi-directionally. That is, taking the rule for ∧ as an example, φ ∧ ψ entails the truth of both φ and ψ (downwards direction), and if it is the case that both φ is true and ψ is true then this entails the truth of φ ∧ ψ (upwards direction). However, in the case of the rule for ∀, the upwards direction

[144] Page 151.

fails since φ(a) does not entail ∀xφ(x). This is because the lower part of the semantic tableaux rule for ∀, φ(a) (for any a), is only a part of what ∀xφ entails. However, the rule can of course be applied repeatedly in proofs, allowing φ(a) for all individual constants used in the proof to appear.

The fact that the same unidirectional entailment which is present in the rules for tonk cannot therefore be used to preclude tonk's logical constancy – at least under the reasonable view, supported by the reasoning in this dissertation, that the universal quantifier is a logical constant.

However, further inspection reveals that in each of the tableaux rules covering connectives, in which both φ and ψ appear in the upper part of the rule, they both also appear in the lower part of the rule – in a linear fashion for the rules governing both ∧ and the negation of ∨; and in a branching fashion in the rules governing ∨ and the negation of ∧. The same thing is apparent in the semantic tableaux rules for →:



The same applies to rules for the other truth functional connectives not provided in this section but which are available in resources such as Bostock (1997). For the quantifier rules, only φ appears in the upper part, and φ is also present in the lower part of the rule. However, in the case of the semantic tableaux rules for tonk, while both φ and ψ are present in the upper part of the rule, only one of these is present in the lower part – ψ in the base rule and φ in the negated case.

Dummett's view of natural deduction rules, described in the following quote, is pertinent here:

> The canonical grounds for the truth of A ∗ B will be given by the introduction rules governing it, and its canonical consequences will be drawn by means of the elimination rules governing it.

In the case of the semantic tableaux rules for tonk, the rule moving in the downwards direction from φ tonk ψ to ψ corresponds in terms of the information it conveys to the natural deduction elimination rule for tonk, and in light of the quotation above, it gives the canonical consequences of tonk. Put in this way, the problem with tonk cast in terms of semantic tableaux rules is that its canonical consequences are based only on ψ, with no consideration given to φ. Similarly, in the case of the semantic tableaux rule for tonk moving in the downwards direction from ¬(φ tonk ψ) to ¬φ, this corresponds to the natural deduction introduction rule for tonk, and given the above quote, it gives the canonical grounds for the truth of φ tonk ψ. Here, only φ is provided by the semantic tableaux rule, with no consideration given to ψ.

This suggests that the presence of only one of φ and ψ in semantic tableaux rules for tonk is indicative of the same kind of imbalance between the semantic tableaux rules as found in rules which lack harmony in natural deduction systems. This suggests a tentative criterion for logical constancy based on semantic tableaux rules:

- Semantic Tableaux Criterion for Logical Constancy 2 (Tentative): All φ, ψ, … appearing in the upper part of the rule must also appear in the lower part of the rule.

Stipulating this criterion is intended to maintain the balance between the rules required to avoid tonk-like cases, and as stated above, all the rules in propositional and first order logic which are typically accepted to describe logical constants adhere to it. An alternative method, though one not pursued in this dissertation, would be to show that the rules for tonk cannot be sound and complete for any reasonable semantics.

However, consider the following semantic tableaux rules for the operator ∘, which is a binary analogue of the neutral operator introduced in Section 4.4.2.3[145]:

$$\begin{array}{c} \varphi \circ \psi \\ | \\ \varphi \end{array} \qquad\qquad \begin{array}{c} \neg(\varphi \circ \psi) \\ | \\ \neg\varphi \end{array}$$

[145] Objection to Semantic Tableaux Criterion for Logical Constancy 2 (Tentative) and the wording of its reformulation in Semantic Tableaux Criterion for Logical Constancy 1 is due to comments provided by Peter Milne.

Here, the occurrence of ψ contributes nothing, but does not have any harmful deductive consequences, and it is not tonk-like. According to the reasoning put forward in this dissertation, it should therefore be considered a logical constant. However, it violates Semantic Tableaux Criterion for Logical Constancy 2 (Tentative), since ψ makes no appearance in the bottom of the rules.

This is clearly a concern. The lack of deductive utility of ∘ does not provide grounds for is exclusion from logical constancy, given the inclusion of the predicate E and the operator - (neutral) previously discussed in this dissertation. Comparison of the semantic tableaux rules for tonk and for ∘ suggest that the former can be excluded and the latter included in the set of logical constants by amending the criterion as follows:

- Semantic Tableaux Criterion for Logical Constancy 2: All φ, ψ, … appearing in the upper part of the rule must also appear in the lower part of the rule; or if there is a φ that appears in the upper part of one of the rules but not in the lower part of that rule, it must also not appear in the lower part of the other rule.

This criterion serves the purpose of denying logical constancy to tonk but according it to ∘, as desired. However, it may be criticised on the basis that it is ad hoc, and directed specifically at the challenge to Semantic Tableaux Criterion for Logical Constancy 2 (Tentative) mounted by ∘. In its defence, Semantic Tableaux Criterion for Logical Constancy 2 does have some seeming justification, since the ‘proof theoretic free for all’ introduced by tonk requires the appearance of both φ and ψ in the lower half the rules. However, the criterion does admittedly remain to an extent unsatisfactory, since it would be desirable for it to somehow be more explicitly based on a semantic tableaux equivalent of natural deduction’s harmony.

Moving on, an interesting semantic tableaux case is that of negation. Recall that in the case of natural deduction rules, the ¬ operator presented some problems in that which harmony could be relatively easily achieved when the intuitionist conception of its introduction and elimination rules were analysed for harmony, the classical ¬ rules were more problematic. However, consider the following semantic tableaux rule for classical negation:

$$\neg\neg\varphi$$

$$|$$

$$\varphi$$

This rule unproblematically conforms to the proposed criterion for logical constancy based on semantic tableaux rules. The simplicity of the semantic tableaux approach’s validation of classical negation’s logical constancy compared to that of the natural deduction harmony-based approach would be of value to those wishing to defend the logical constancy of the classical negation operator, and thus the logicality of classical logic in general against intuitionist objections. A potential concern is that Semantic Tableaux Criterion for Logical Constancy 2 may be too permissive – that is, while it does endow classical negation with logical constancy, it may also admit other operators which are intuitively non-logical. However, it has already been shown that the criterion is sufficient to deny logical constancy to tonk, and the author of this dissertation is unaware of other cases which would be problematic in this way.

Thus far, then, the evaluation of semantic tableaux in terms of criteria for logical constancy seems to have fared well – criteria were produced which allow evaluation of all propositional and first order operators, with the results that all the accepted operators of classical first order logic are given logical constant status. At the same time, the claims to logical constancy of the problematic tonk operator are rejected. As previously noted in this dissertation, the lack of a semantically sound and complete proof system for formal systems including full second order quantifiers means that proof theoretic evaluation of logical constancy for the full second order quantifier is not possible, regardless of the choice of proof system.

This dissertation will therefore proceed with an analysis of semantic tableaux systems for modal operators. Recall from the previous section that the natural deduction-based evaluation of modal operators proposed logical constancy for □K and □S5 but rejected logical constancy for operators such as □S4. Girle (2009) provides a useful resource for semantic tableaux systems including modal operators, and, with some modifications in terms of nomenclature, the rules presented there will be used in this dissertation, starting with the following rule for □S5:

□S5φ (w)

|

φ (v)

Notable here is the addition of an index on the right-hand side of each formula. These indices refer to the worlds at which the stated formula is true. Due to the equivalence relation between worlds in S5, no stipulation on the relationship between world w and world v is needed – if □S5φ holds at a world w, φ will hold at any world v in the set of worlds considered. Inspection of the rule shows that it adheres to the semantic tableaux criteria thus far developed, since only φ is used, and it appears on both the top and bottom of the rule; and complexity reduction is achieved. As per the discussion regarding modal operators in Section 4.4.4 of this dissertation, this agrees with the natural deduction-based assessment of □S5, and also, again as argued in Section 4.4.4, with intuitive notions regarding modality and logical constancy.

Turning to the rule for □K, Girle presents the following:

□Kφ (w)

wRv

|

φ (v)

Here, a stipulation is required to the effect that world v is accessible via the R relation from world w. This is needed in the system K because there are no prior stipulations regarding the R relations between worlds, so for φ to be true at world v based on □Kφ being true at world w, the R relation must hold between w and v. Again, this rule fares well with respect to the criteria so far developed for logical constancy for semantic tableaux rules, since it conforms to the criteria developed so far. As per the discussion regarding modal operators in Section 4.4.4 of this dissertation, this agrees with the natural deduction-based assessment of □K, and with intuitive notions regarding modality and logical constancy. However, this intuitive agreement is more or less by default, since the author is not aware of any interpretation of □K in terms of modalities in natural language.

In order to produce semantic tableaux proof systems for modal logics other than K and S5, Girle takes (in what he calls the 'orthodox strategy') the following approach. As discussed in Section 4.4.4, each modal logic is associated with a stipulation regarding the R relation which holds between worlds. For each of these stipulations, Girle adds what could be termed an auxiliary (in that it does not concern connectives, but rather the R relation) rule as follows:

(Refl)	(Trans)	(Sym)
	wRv	
	vRt	wRv
|	|	|
wRw	wRt	vRw

These auxiliary rules are then added to a base system including the semantic tableaux rules for propositional logic (referred to below as PL) and the □K rule to produce semantic tableaux rules for different modal logics as follows:

- Semantic tableaux rules for K: PL ∪ {□K}
- Semantic tableaux rules for KT: PL ∪ {□K, Refl}
- Semantic tableaux rules for S4: PL ∪ {□K, Refl, Trans}
- Semantic tableaux rules for S5: PL ∪ {Refl, Trans, Sym}[146]

Recall from Section 4.4.4 that based on the evaluation of natural deduction rules, and in accordance with intuitions based on the common interpretations of the modal logics, only □K and □S5 were accorded logical constancy. This was due to the more complex stipulations regarding the R relation, which moves the operators involved away from absolute generality and topic neutrality. Consistency of results across the sections of this dissertation would therefore demand that similar results are produced here.

The facts stated above suggest the following criteria:

- Semantic Tableaux Criterion for Logical Constancy 1: Call the formula that occurs with the operator of interest dominant the main formula of the rule; it either occurs as is (in the analogue of an elimination rule) or within the scope of a single occurrence of negation (in the analogue of an introduction

[146] Or alternatively, and as discussed previously, the semantic tableaux rules for S5 can be gives PL ∪ □S5}, meaning (importantly in what will follow), S5 can be defined without reference to auxiliary rules.

rule). Then formulas occurring in the lower part of the semantic tableaux rule are proper subformulas or negations of proper subformulas of the main formula.

- Semantic Tableaux Criterion for Logical Constancy 2: All φ, ψ, … appearing in the upper part of the rule must also appear in the lower part of the rule; or if there is a φ that appears in the upper part of one of the rules but not in the lower part of that rule, it must also not appear in the lower part of the other rule.
- Semantic Tableaux Criterion for Logical Constancy 3: The rule for the logical constant must not make any reference to any auxiliary rules.

The first criterion has previously been justified on the basis that it is required for semantic tableaux rules to operate successfully, by breaking down longer formulas into their constituent parts in the search for a counter example. The second criterion has previously been justified on the basis that it is essentially the semantic tableaux analogue of the natural deduction harmony criterion. The third criterion is justifiable on the basis of the inferentialist notion that proof rules should provide the entire meaning of a logical constant without the requirement for external reference, such as the references to stipulations on the R relation Refl, Trans, and Sym.

In conclusion to this section, and updating the natural deduction-based discussion of modal operations in Section 4.4.4, this dissertation accords logical constancy to the modal operators □K and □S5, and to no further modal operator. In the case of semantic tableaux criteria, the following are put forward:

- Semantic Tableaux Criterion for Logical Constancy 1: Call the formula that occurs with the operator of interest dominant the main formula of the rule; it either occurs as is (in the analogue of an elimination rule) or within the scope of a single occurrence of negation (in the analogue of an introduction rule). Then formulas occurring in the lower part of the semantic tableaux rule should be proper subformulas or negations of proper subformulas of the main formula.
- Semantic Tableaux Criterion for Logical Constancy 2: All φ, ψ, … appearing in the upper part of the rule must also appear in the lower part of the rule; or if there is a φ that appears in the upper part of one of the rules but not in the lower part of that rule, it must also not appear in the lower part of the other rule.
- Semantic Tableaux Criterion for Logical Constancy 3: The rule for the logical constant must not make any reference to any auxiliary rules.

It may be objected that these criteria have simply been selected to lead to the desired results from an intuitive standpoint (that is, using a ‘gerrymandering’ process to give what could be intuitively seen as logical constants). This is an important point to consider, given that in a study of this type, it is the building of criteria from first principles (as defined in this dissertation by the requirements for logicality discussed in Section 2.2) which endows the resulting criteria with utility, rather than retrospectively building criteria based on intuitive notions of logicality. Thus in response to this objection, the following justifications are offered for each of the above criteria:

- Semantic Tableaux Criterion for Logical Constancy 1: This criterion is uncontroversial, since it simply defines how semantic tableaux rules must function in order to result in atomic sentences, and thus identify counter examples.
- Semantic Tableaux Criterion for Logical Constancy 2: This criterion can be seen as the semantic tableaux analogue of the natural deduction harmony criterion. Its justification is that (mirroring Dummett's analysis of the nature of natural deduction operational rules) that the lower part of the rule should represent the consequences and/or grounds of the upper part of the rule. Given this, it seems reasonable that if the grounds of the rule include φ, ψ, … then the consequences of the rule should include φ, ψ, … also.
- Semantic Tableaux Criterion for Logical Constancy 3: The mechanism behind Semantic Tableaux Criterion 2 is similar to that of Natural Deduction Criterion 3, that is on the basis of the inferentialist notion that proof rules should provide the entire meaning of a logical constant without the requirement for external reference.

Recall that in the case of natural deduction rules, the following were put forward:

- Natural Deduction Criterion for Logical Constancy 1: Operational rules must not allow the introduced element to appear as the main connective in the antecedent of the I rule and must not allow the eliminated element to appear as the main connective in the consequent of the E rule.
- Natural Deduction Criterion for Logical Constancy 2: The introduction and elimination rules for logical constants must be in (general elimination) harmony.
- Natural Deduction Criterion for Logical Constancy 3: The rules must contain no reference to any non-logical elements external to the operator which the rule defines.

As discussed, both of these sets of criteria return the same results – all truth functional connectives, ∀, □K and □S5 are classified as logical constants, while tonk and modal operators other than □K and □S5 are excluded from logical constancy. As per arguments in previous sections of this dissertation, this accords with intuitive notions regarding the interpretations of these operators.

The semantic tableaux rules are notably simpler than those presented for natural deduction, particularly in terms of the analogue of the harmony criterion – the reader will recall that harmony is a difficult notion to get clarity on, with various interpretations of it such as local peak elimination and Read's general elimination harmony, being offered. This counts as a point in favour of using semantic tableaux-based criteria for logical constancy, and is an interesting result, given that the author is unaware of other published works investigating semantic tableaux-based investigations of logical constancy.

6. Results Summary

This section of the dissertation summarises the results obtained in the previous sections, including:

- The natural deduction criteria for logical constancy.
- The semantic tableaux criteria for logical constancy.
- The resulting set of logical constants.
- The natural deduction operational rules which define these constants.

6.1. Natural Deduction Criteria for Logical Constancy:

- Natural Deduction Criterion for Logical Constancy 1: Operational rules must not allow the introduced element to appear as the main connective in the antecedent of the I rule and must not allow the eliminated element to appear as the main connective in the consequent of the E rule.
- Natural Deduction Criterion for Logical Constancy 2: The introduction and elimination rules for logical constants must be in (general elimination) harmony.
- Natural Deduction Criterion for Logical Constancy 3: The rules must contain no reference to any non-logical elements external to the operator which the rule defines.

6.2. Semantic Tableaux Criteria for Logical Constancy:

- Semantic Tableaux Criterion for Logical Constancy 1: Call the formula that occurs with the operator of interest dominant the main formula of the rule; it either occurs as is (in the analogue of an elimination rule) or within the scope of a single occurrence of negation (in the analogue of an introduction rule). Then formulas occurring in the lower part of the semantic tableaux rule should be proper subformulas or negations of proper subformulas of the main formula.
- Semantic Tableaux Criterion for Logical Constancy 2: All φ, ψ, … appearing in the upper part of the rule must also appear in the lower part of the rule; or if there is a φ that appears in the upper part of one of the rules but not in the lower part of that rule, it must also not appear in the lower part of the other rule.
- Semantic Tableaux Criterion for Logical Constancy 3: The rule for the logical constant must not make any reference to any auxiliary rules.

6.3. Logical Constants Table

Logical Constant	Natural Language Interpretation
E N = ≠	Existence, Non-existence, Identity, Non-identity
⊤ ⊥ - ∧ ∨ → ← ↔ ¬ ↑ ↓ ↛ ↚ ↮	Truth functional connectives of minimal, intuitionistic and classical propositional logic with arities 0, 1, and 2
$\wedge^3$ …	Higher arity truth functional connectives of minimal, intuitionistic and classical propositional logic which can be defined in terms of the above connectives with arities of 0, 1 and 2
∀, ∃	All, Every (first order case)
$\forall_2$, $\exists_2$	All, Every (second order case, sound and complete when paired with Henkin semantics)
□K	There is no accepted interpretation of □K
□S5	Logical necessity

6.4. Logical Constant Rules

Identity

$$\frac{\begin{matrix}\varphi(c)\\ \vdots \\ \varphi(t)\end{matrix}}{c = c}\ {=}I'' \qquad\qquad \frac{(c = t) \qquad \varphi(c)}{\varphi(t)}\ {=}E''$$

Existence

$$\frac{\quad}{Ec}\ EI$$

Distinctness

$$\frac{\varphi(c) \qquad \neg\varphi(t)}{c \neq t}\ {\neq}I1 \qquad\qquad \frac{\neg\varphi(c) \qquad \varphi(t)}{c \neq t}\ {\neq}I2$$

$$\frac{c \neq c}{\varphi}\ {\neq}E$$

Non-Existence

$$\frac{Nc}{\varphi}\ NE$$

Propositional Logic

1. Verum

$$\frac{}{\top}\ \top I$$

2. Falsum

$$\frac{\bot}{\varphi}\ \bot E$$

3. Neutral

$$\frac{\varphi}{-\varphi}\ -I$$

$$\frac{-\varphi}{\varphi}\ -E$$

4. Negation

$$\frac{\begin{array}{c}[\varphi] \\ \vdots \\ \bot\end{array}}{\neg\varphi}\ \neg\text{I}$$

$$\frac{\varphi \qquad \neg\varphi}{\bot}\ \neg\text{E1}$$

$$\frac{\neg\neg\varphi}{\varphi}\ \neg\text{E2}$$

5. Conjunction

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\ \wedge\text{I}$$

$$\frac{\varphi \wedge \psi}{\varphi}\ \wedge\text{E1}$$

$$\frac{\varphi \wedge \psi}{\psi}\ \wedge\text{E2}$$

6. Conditional

$$\frac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \rightarrow \psi}\ \rightarrow\text{I}$$

$$\frac{\varphi \rightarrow \psi \qquad \varphi}{\psi}\ \rightarrow\text{E}$$

7. Converse Conditional

$$\dfrac{\begin{array}{c}[\psi]\\ \vdots\\ \varphi\end{array}}{\varphi \leftarrow \psi}\ {\leftarrow}\mathrm{I}$$

$$\dfrac{\varphi \leftarrow \psi \qquad \psi}{\varphi}\ {\leftarrow}\mathrm{E}$$

8. Disjunction

$$\dfrac{\varphi}{\varphi \vee \psi}\ \vee\mathrm{I1}$$

$$\dfrac{\psi}{\varphi \vee \psi}\ \vee\mathrm{I2}$$

$$\dfrac{\varphi \vee \psi \qquad \begin{array}{c}[\varphi]\\ \vdots\\ \gamma\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots\\ \gamma\end{array}}{\gamma}\ \vee\mathrm{E}$$

9. Nif (equivalent to $\varphi \wedge \neg\psi$)

$$\dfrac{\varphi \qquad \begin{array}{c}[\psi]\\ \vdots\\ \bot\end{array}}{\varphi \nrightarrow \psi}\ {\nrightarrow}\mathrm{I}$$

$$\dfrac{\varphi \nrightarrow \psi}{\varphi}\ {\nrightarrow}\mathrm{E1}$$

$$\dfrac{\varphi \nrightarrow \psi \qquad \psi}{\bot}\ {\nrightarrow}\mathrm{E2}$$

10. Converse Nif

$$\dfrac{\psi \qquad \begin{array}{c}[\varphi] \\ \vdots \\ \bot\end{array}}{\varphi \nleftarrow \psi} \; {\nleftarrow}\mathrm{I}$$

$$\dfrac{\varphi \nleftarrow \psi}{\psi} \; {\nleftarrow}\mathrm{E1}$$

$$\dfrac{\varphi \nleftarrow \psi \qquad \varphi}{\bot} \; {\nleftarrow}\mathrm{E2}$$

11. Biconditional (equivalent to $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$)

$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array} \qquad \begin{array}{c}[\psi] \\ \vdots \\ \varphi\end{array}}{\varphi \leftrightarrow \psi} \; {\leftrightarrow}\mathrm{I}$$

$$\dfrac{\varphi \leftrightarrow \psi \qquad \varphi}{\psi} \; {\leftrightarrow}\mathrm{E1}$$

$$\dfrac{\varphi \leftrightarrow \psi \qquad \psi}{\varphi} \; {\leftrightarrow}\mathrm{E2}$$

12. Exclusive Disjunction (equivalent to $(\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)$)

$$\frac{\varphi \qquad \begin{matrix}[\psi] \\ \vdots \\ \bot\end{matrix}}{\varphi \nleftrightarrow \psi} \nleftrightarrow I1$$

$$\frac{\psi \qquad \begin{matrix}[\varphi] \\ \vdots \\ \bot\end{matrix}}{\varphi \nleftrightarrow \psi} \nleftrightarrow I2$$

$$\frac{\varphi \nleftrightarrow \psi \qquad \varphi \qquad \psi}{\bot} \nleftrightarrow E1$$

$$\frac{\varphi \nleftrightarrow \psi \qquad \begin{matrix}[\varphi] \\ \vdots \\ \bot\end{matrix} \qquad \begin{matrix}[\psi] \\ \vdots \\ \bot\end{matrix}}{\bot} \nleftrightarrow E2$$

13. Nand (equivalent to $\neg(\varphi \wedge \psi)$)

$$\frac{\begin{matrix}[\varphi] \\ \vdots \\ \bot\end{matrix}}{\varphi \uparrow \psi} \uparrow I1$$

$$\frac{\begin{matrix}[\psi] \\ \vdots \\ \bot\end{matrix}}{\varphi \uparrow \psi} \uparrow I2$$

$$\frac{\varphi \uparrow \psi \qquad \varphi \qquad \psi}{\bot} \uparrow E$$

14. Nor (equivalent to ¬(φ ∨ ψ))

$$\frac{\begin{matrix}[\varphi] & & [\psi] \\ \vdots & & \vdots \\ \bot & & \bot\end{matrix}}{\varphi \downarrow \psi}\ {\downarrow}\mathrm{I}$$

$$\frac{\varphi \downarrow \psi \qquad \varphi}{\bot}\ {\downarrow}\mathrm{E1}$$

$$\frac{\varphi \downarrow \psi \qquad \psi}{\bot}\ {\downarrow}\mathrm{E2}$$

Note that rule ¬E2 is omitted for intuitionistic propositional logic.

First Order Quantification

$$\frac{\varphi(a)}{\forall x\varphi(x)}\ (\forall\mathrm{I})$$

where every occurrence of a in φ(a) is replaced by x, and a must not occur in any assumption on which φ(a) depends. The variable x must not be bound by any quantifier in φ(a) that has a within its scope[147]

$$\frac{\forall x\varphi(x)}{\varphi(a)}\ (\forall\mathrm{E})$$

In applying this rule one replaces every free occurrence of x in φ(x) by a [148]

$$\frac{\varphi(a)}{\exists x\varphi(x)}\ \exists\mathrm{I}$$

where in φ(a) no occurrence of a which is to be replaced by x occurs within the scope of any quantifier binding x. Note also that in applying this rule one need not replace every occurrence of the term a in the sentence φ(a) with an occurrence of the variable x

[147] Condition taken from Tennant (1978). Page 42.

[148] Condition taken from Tennant (1978). Page 41.

$$\frac{\exists x\varphi(x) \qquad \begin{matrix}\varphi(a)\\ \vdots \\ \psi\end{matrix}}{\psi}\ \exists E$$

where a does not occur in ∃xφ(x), a does not occur in ψ, and a does not occur in any assumptions, other than φ(a) on which the upper occurrence of ψ depends

Second Order Quantification

$$\frac{\varphi(A)}{\forall X\varphi(X)}\ (\forall_2 I)$$

where every occurrence of A in φ(A) is replaced by X, and A must not occur in any assumption on which φ(A) depends. The variable x must not be bound by any quantifier in φ(A) that has A within its scope[149]

$$\frac{\forall X\varphi(X)}{\varphi(A)}\ (\forall_2 E)$$

In applying this rule one replaces every free occurrence of X in φ(X) by a

$$\frac{\varphi(A)}{\exists X\varphi(X)}\ \exists_2 I$$

where in φ(A) no occurrence of A which is to be replaced by X occurs within the scope of any quantifier binding X. Note also that in applying this rule one need not replace every occurrence of the term A in the sentence φ(A) with an occurrence of the variable x

[149] Condition taken from Tennant (1978). Page 42.

$$\dfrac{\exists X\varphi(X) \qquad \begin{array}{c}\varphi(A)\\ \vdots \\ \psi\end{array}}{\psi}\ \exists_2E$$

where a does not occur in ∃Xφ(X), A does not occur in ψ, and a does not occur in any assumptions, other than φ(A) on which the upper occurrence of ψ depends

Modal Operators

Modal Operator K

$$\dfrac{\begin{array}{c}(iRj)\\ \vdots \\ \varphi j\end{array}}{\Box\varphi i}\ \Box KI \qquad\qquad \dfrac{\Box\varphi_i \qquad \begin{array}{c}(iRj \Rightarrow \varphi_j)\\ \vdots \\ \psi_k\end{array}}{\psi_k}\ \Box KE$$

Modal Operation S5

$$\dfrac{\varphi}{\Box\varphi}\ \Box S5I \qquad\qquad \dfrac{\Box\varphi}{\varphi}\ \Box S5E$$

Every open assumption that φ depends on must have □S5 as its principal operator or be the negation of a formula with □S5 as its principal operator.

7. Wider Philosophical Considerations

It is important to identify what is at stake when delimiting the scope of logic. In the absence of such a discussion, it may be objected that such categorisations are simply exercises in nomenclature, and that no deep importance should be attached to them. If, alternatively, it can be shown that other issues of philosophical (or wider) importance depend on clear definitions of logic and the limits of logicality, then this reinforces the value of undertaking studies such as the present.

The most direct role the project outlined above could play would be to contribute to the resolution of debates regarding the logicality of the potential logical constants considered in this dissertation (second order quantification and modal operators) whose status is controversial. While claiming this result would not avoid the charges of being restricted to nomenclature, it would at the very least liberate philosophers who are concerned with such questions to pursue more fruitful activities.

This dissertation can lay claim to some success in this area. The previous sections developed natural deduction criteria for logical constancy which are relevant and precise, and which delivered reasonable results with respect to the well-known logical constants of first order logic. These criteria were then applied to modal logics, resulting in □K and □S5 being returned as logical constants. In the case of second order quantification, the inherent and unavoidable lack of a semantically sound and complete proof system for systems including full second order quantification meant that its evaluation using the criteria developed in this dissertation was restricted to versions of the second order quantifier which are sound and complete with respect to Henkin semantics. The potential for logical constancy of the full or unrestricted second order quantifier was dismissed based on the impossibility of proof rules being provided for it. One's views of the logical constancy of the second order quantifier are therefore likely to depend on broader issues than logical constancy itself, involving wider ideas on model theory and proof theory, and even more generally on Platonism in mathematics.

A further direct benefit of the project would be (notwithstanding the complexities introduced by translation between natural and formal languages) allow adjudication of whether a given inference holds as a matter of logic or not. By taking the stated premises of arguments as antecedents in a conditionalised form of the argument, the tools developed in this project should allow assessment of the force (logicality or otherwise) of an arbitrary argument. This is useful from a philosophical perspective. A pertinent example given that it heavily involves modality is Gödel's ontological argument[150]. Assessing the logicality of modal operators would assist in evaluating the force of this argument, which of course concerns a very long running debate in the philosophy of religion. On this point, advancement in this dissertation was made in line with the advancement made regarding the problem of logical constants. This is because those inferences which hold as a matter of logic are those which depend only on the structure or form of the inference, as marked out by these logical constants in the form of the inferences in question.

[150] Gödel (1995).

A pertinent example of a specific philosophical project to which clearly defining the boundaries of logic is important is debate concerning the theory of logicism. In its stronger version, logicism holds that all mathematical truths (or at least all truths in certain branches of mathematics) can in some way be reduced to logical truths. Thus, for the strong logicist it is clearly important to be able to demarcate the limits of logical truth compared to any other species of truth (especially mathematical truth), in order to be able to claim to have successfully effected this this reduction. In its weaker version, logicism holds that all provable results in mathematics can again be in some way reduced to logical truths[151]. Again, this means that those who subscribe to logicism in even in its weaker form must properly define the scope of logic. Due to the importance of logic for foundational studies in the philosophy of mathematics, the relationship between logical and mathematical consequence has been extensively examined since Frege. The reason for this is clear, since results in mathematics appear to share many of the informal requirements cited in Section 2.2. Given the close proximity of logic and mathematics, differentiating between them provides a key acid test to the precision of proposed logical criteria.

Until Gödel, it was thought that the weak form of logicism was the same as the strong form of logicism, but whether logicism (that is, whether mathematical truths are reducible to logical truths) had been achieved depended on the view taken of the logical axioms used. For example, in Principia Mathematica, Russell and

[151] Tennant (2014).

Whitehead included axioms related to concepts such as multiplicity, infinity, and reducibility, or accepted that most theorems are implications of these. With the work of Gödel, the focus of logicism changed to attempting to show not just that all provable results in mathematics can again be in some way reduced to logical truths, but simply that all truths of mathematics are provable (in a given system). Due to the success of Gödel's incompleteness theorems, even if mathematics was thought of as logic, one could think of logic as incomplete and incompletable, and thus that the logicist programme should be regarded as a failure.

Thus, logicism could be thought of as more of historical rather than philosophical interest, as a remnant of the work of its early twentieth century proponents. However, while this comment may apply to logicism in its original form, the doctrine has evolved since the setback it suffered via the results of Gödel's work, with a later evolution of it being represented by the neo-logicist school. This position, into which research began in the 1960's and began to gain significant traction during the 1980's, is based on the claim that the logicist programme of reducing mathematics to logic can be resurrected by strategies such as holding that Hume's Principle is an analytic truth[152].

A key example of this approach can be found in Wright (1983), which outlines how the basic axioms of arithmetic could be derived from second order logic with the addition of Hume's Principle (which states that the number of Fs is equal to the number of Gs if and only if there is a one-to-one correspondence between the Fs

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[152] Tennant (2014).

and the Gs) as a replacement of the problematic Basic Law V used by Frege. This represents a particularly pertinent example, due to the controversy surrounding whether formal systems including second order quantification should be included within the scope of logic or not. This is because the addition of these axioms in effect violates the requirements for logicality set down in this dissertation of absolute generality and topic neutrality, and the exclusion of strong forms of the second order quantifier. Thus this clearly demonstrates the importance of the questions considered in this thesis to the neo-logicist programme.

Turning to more general areas of philosophical interest, there are good epistemological grounds for investigating the scope of logic. The basic principles of logic are accorded a special epistemological status, perhaps in terms of their a prioricity or unrevisability (though this has been questioned in works such as Putnam (1968)). The requirements for logicality detailed in Section 2.2 give an indication of the reason for according it this status. For example, logic's purported absolute generality and its resulting independence from any specific objects gives it an independence from specific worldly states of affairs, which reduces potential doubts surrounding its truths. Following the reasoning in Section 2.2, the epistemological security endowed on logic by these requirements can be traced back to its more fundamental requirement of formality. Of course, this perceived security provides the basic motivation of the logicist project mentioned in the previous paragraphs, since founding mathematics on logic provides the epistemological bedrock that the logicists sought.

It is also reasonable to argue that the scope of logic should be of interest to the metaphysician, since logic could also be afforded a special metaphysical status. While metaphysics is notoriously hard to define[153], this dissertation asserts that under most philosophical conceptions of it, it is connected with the fundamental nature of existence. Linking logic to metaphysics is the thesis of logical realism, of which Rush (2014) provides the following definition:

> Logic might chart the rules of the world itself; the rules of rational human thought; or both. The first of these possible roles suggests strong similarities between logic and mathematics: in accordance with this possibility, both logic and mathematics might be understood as applicable to a world (either the physical world or an abstract world) independent of our human thought processes. Such a conception is often associated with mathematical and logical realism.[154]

Thus, in essence, logical realism holds that logic describes principles which exist in the world, and not (only) those of rational human thought. Logic's absolutely general nature would then imply that these principles are those which are of very general importance, and thus of particular metaphysical significance.

Like perhaps all substantial philosophical theses, logical realism is far from non-controversial. For current purposes, all that is be put forward here with respect to

[153] See van Inwagen, P., Sullivan, M. (2018).

[154] Rush (2014). Page 13.

the metaphysical significance of logic is therefore the following conditional claim: Should logical realism hold, delimiting the scope of logic would be of interest to the metaphysician. In terms of the results provided by this dissertation, a particularly interesting example here is that of S5, since its status as a logical constant (according to the criteria put forward in this dissertation) would endow the concept of logical necessity with a certain realism.

A more specific reason to be interested in the scope of logic relates to the Quinean slogan that "to be is to be the value of a variable". In logical terms, this means that the existence of a certain class of entities relates to the legitimacy of placing it within the scope of an existential quantifier. Identifying a certain formal system as logical would then lend credibility to its assessment of the legitimate use of existential quantification. This point relates of course to the common debate regarding the legitimacy of second order quantification (quantification into predicate position), rather than objects (first order quantification). The idea here is that should second order quantification receive a favourable assessment in terms of its logicality, this would lend credence to the claim that sets or properties are legitimately existing entities, given that they are the values of the variables of second order quantifiers. Arguments such as this support the claim that questions of logicality have an important bearing on metaphysics, since they play a determining role in the types of objects which exist.

Finally, it could be hoped that the *process* of investigating criteria for logicality could unearth informative results regarding the general nature of the enterprise of logic. Given that this project will seek criteria based on proof theoretic conceptions of logic, particular learnings could be hoped for in the case of proof theory, and its advantages and limitations. An example of this is the potential limitations of proof theoretic criteria due to the semantic incompleteness of formal systems which include (full) second order quantification.

8. Conclusion

This dissertation investigated proof theoretic means of providing criteria for logicality. It defended the view that proof theoretic systems such as natural deduction and semantic tableaux can be used to generate criteria for logical constancy. These criteria classify the truth functional operators of propositional logic (minimal, intuitionistic and classical), first order quantification, certain types of second order quantification, and the modal operators □K and □S5 as logical constants. This accords with expectations, due to the nature of the interpretations of these elements of formal systems. In addition, the criteria produced exclude the problematic operator *tonk* (and other non-harmonious operators), again in accordance with expectations regarding logical constancy.

Any conceptual analysis, including that of logicality, must start from some kind of basis. In this dissertation, this was obtained via a survey of the thoughts of prominent authors in the field regarding the requirements for logicality. This survey returned the following requirements for logicality:

- Absolute generality, or validity in any kind of discourse regarding any kind of objects.
- Topic neutrality, or validity regardless of subject matter.
- Formality, or that an example of consequence being logical if it holds in virtue of its form or structure and regardless of its content.

These requirements were used throughout the dissertation as a basis for assessing logicality, and thus formed the basis of its methodology. The specific strategy adopted here was to investigate natural deduction systems for first order logic, second order logic, and modal logic, assessing the potential logical constancy of the elements of them. Each element was assessed according to the requirements for logicality, and criteria for logical constancy suggested and refined (if necessary) so that they admit elements which are formal and have absolutely general and topic neutral interpretations and exclude elements which do not.

However, while these requirements are useful in terms of ensuring relevance to logic, they leave much room for argument regarding their application. What was needed, therefore, were precise criteria for logicality, which allowed each potential candidate for logicality to be clearly categorised. Given the acceptance of the importance of formality, this required a methodology for distinguishing which elements of a sentence expressing a consequence belong to its structure or form, with these elements being referred to as *logical constants*. This conclusion brought the focus of the remainder of the dissertation on the search for criteria for these logical constants.

The distinction between requirements for logical constancy and criteria for logical constancy highlights an important distinction apparent throughout the dissertation. While elements of formal systems are the objects which can rightly be described as logical, it is convenient to label some elements of natural language as logical also, namely those which are, as per the requirements identified above, absolutely general and topic neutral. Where confusion may arise, such elements of natural language were called *informally logical*. Thus elements of formal systems are either logical or non-logical; concepts described in natural language are informally logical or informally non-logical.

The issue of parsimony was also discussed as a potentially desirable 'meta-requirement' applicable to the overall set of logical constants. Parsimony as a criterion for logical constancy was however rejected, since it applies to the set of logical constants as a whole, rather than the criteria-based assessment of each individual candidate for logical constancy. That is, a lack of parsimony among the set of logical constants does not undermine the claims any single logical constant has to formality and absolute generality / topic neutrality, and thus to logicality.

With these requirements thus established, the business of establishing criteria for logical constancy began. In terms of the specific implementations of proof theory used, this dissertation first focussed on natural deduction systems to seek criteria for logicality. Systems of natural deduction, as the name suggests, endeavour to present proofs which mimic to a certain extent the natural way in which human agents undertake deduction. In this, they contrast with the Hilbert-style or axiomatic approaches developed previous to them, which, while generally being able to present the overall characteristics of logical systems more succinctly, produce proofs which are less easy to follow and deviate further from the actual practice of reasoning[155]. In addition to their intuitive appeal, natural deduction systems have received significant attention with respect to the question of logical constancy, with key contributions being made in Prawitz (1965), Dummett (1991), and Read (2010). A natural deduction system was presented in this dissertation in Section 4.3.

Central to natural deduction-based proof theoretic approaches to evaluating logical constancy, and through that logicality, is the thesis of inferentialism (understood as per the definition provided by Rossberg and Cohnitz in Section 4.3, rather than in any wide sense intended to serve as a general theory meaning). While various definitions of inferentialism exist, key to it is the notion that operational rules which

[155] Noting, however, that Hilbert himself stated that his formal proofs are "carried out according to certain definite rules, in which *the technique of our thinking* is expressed" (Hilbert, 1928, page 475, emphasis in the original). The author of this dissertation does not agree with Hilbert's contention here and maintains that natural deduction systems are preferable in this area.

are included for each logical operator in a natural deduction system exhaustively define them. The exhaustive definition provided by the operational rules means that the proof theorist can proceed with confidence, since all information required for the evaluation of logical constancy for each operator must be contained in these rules. Should the inferentialist position fail to hold, on the other hand, there may be information regarding the operators which escapes the methods used in this dissertation, which may in turn hinder evaluations of logicality.

This dissertation put forward the following natural deduction operational rule-based criteria for logical constancy:

- Natural Deduction Criterion for Logical Constancy 1: Operational rules must not allow the introduced element to appear as the main connective in the antecedent of the I rule and must not allow the eliminated element to appear as the main connective in the consequent of the E rule.
- Natural Deduction Criterion for Logical Constancy 2: The introduction and elimination rules for logical constants must be in (general elimination) harmony.
- Natural Deduction Criterion for Logical Constancy for Logical Constancy 3: The rules must contain no reference to any non-logical elements external to the operator which the rule defines.

These criteria are coherent with respect to the expectations for logicality. The first stipulates what it means to be defined in natural deduction terms, via introduction and elimination rules. The second stipulates, via the notion of harmony, that the meanings established by the introduction and elimination rules should cohere with each other. They lead to the categorisation of all the operators of first order logic as logical constants, along with the universal quantifier and the identity predicate of first order logic; and also examples such as the existence and non-existence predicates, which are strictly logical, but which lack utility. The third criterion is justifiable since it aligns with the general inferentialist approach taken when developing proof theoretic criteria for logical constancy.

After investigation of propositional and first order candidates for logical constancy, this dissertation examined the potential logical constancy of the second order quantifier. However, the issue here is complicated by the lack of a sound and complete proof system for the full or unrestricted conception of second order quantification. The investigations undertaken resulted in two broad categories of second order quantification:

- Those which are not an advance in terms of expressivity with respect to first order logic (second order logic with Henkin semantics, and second order logic with faithful Henkin semantics) but whose proof rules adhere to this dissertation's criteria for logical constancy.
- Those for which no sound and complete proof rules are available (monadic second order logic and full or unrestricted second order logic). Here, the conceptions of the second order quantifier in question simply escape evaluation in terms of the criteria developed in this dissertation, since these criteria are based on the evaluation of natural deduction operational rules,

and these are not available for this evaluation due to incompleteness. Excluding these as logical constants requires taking the position that the very lack of an effective proof system for systems including full second order quantification precludes them from logical constancy.

Regarding the potential logical constancy of modal operators, the results indicated that both □K and □S5 should be included as logical constants, but not others with more complex structural rules such as □KD or □K4. This is because it seems wise to deny logicality to any operator which requires external reference to an entirely separate structural rule which defines how R, the accessibility relation functions within the logic – that is, the □ operator must be defined in the natural deduction rule without any stipulation regarding the nature of R. Given that □K does not have a well-known interpretation and the interpretation of □S5 is logical necessity, so these results accord with expectations regarding informal logicality.

The potential for criteria for logicality from an alternative (to natural deduction) proof system, that of semantic tableaux, was then considered. This resulted in the following criteria being produced:

- Semantic Tableaux Criterion for Logical Constancy 1: Call the formula that occurs with the operator of interest dominant the main formula of the rule; it either occurs as is (in the analogue of an elimination rule) or within the scope of a single occurrence of negation (in the analogue of an introduction rule). Then formulas occurring in the lower part of the semantic tableaux should be proper subformulas or negations of proper subformulas of the main formula.

- Semantic Tableaux Criterion for Logical Constancy 2: All φ, ψ, … appearing in the upper part of the rule must also appear in the lower part of the rule; or if there is a φ that appears in the upper part of one of the rules but not in the lower part of that rule, it must also not appear in the lower part of the other rule.
- Semantic Tableaux Criterion for Logical Constancy 3: The rule for the logical constant must not make any reference to any auxiliary rules.

The application of the methodology described above resulted in classification of the set of operators in the following table as logical constants.

Logical Constant	Natural Language Interpretation
E N = ≠	Existence, Non-existence, Identity, Non-identity
$\top$ $\bot$ - $\wedge$ $\vee$ $\rightarrow$ $\leftarrow$ $\leftrightarrow$ $\neg$ $\uparrow$ $\downarrow$ $\nrightarrow$ $\nleftarrow$ $\nleftrightarrow$	Truth functional connectives of minimal, intuitionistic and classical propositional logic with arities 0, 1, and 2
$\wedge^3$ …	Higher arity truth functional connectives of minimal, intuitionistic and classical propositional logic which can be defined in terms of the above connectives with arities of 0, 1 and 2
$\forall$, $\exists$	All, Every (first order case)
$\forall_2$, $\exists_2$	All, Every (second order case, sound and complete when paired with Henkin semantics)
□K	There is no accepted interpretation of □K
□S5	Logical necessity

Overall, the results of this dissertation provide an interesting insight into the functioning of proof systems and the nature of logicality. This is particularly the case with respect to the semantic tableaux criteria for logical constancy, which appear to provide a simpler means of assessing it than the natural deduction criteria. Given that the author is unaware of other studies using the semantic tableaux approach, this is an opportunity for further fruitful research into the topic.

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