Essays on Heterogeneous Agent Models

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Abstract

The thesis consists of three chapters. The first chapter examines the behaviour of the New Keynesian (NK) model, which has been extended to include heterogeneous individuals who face idiosyncratic productivity risk. My main focus in this chapter is the heterogeneity innovation of the model, therefore, I consider the relatively simple NK setting where the idiosyncratic income risk follows a two-state Poisson process and in which households can save in productive capital. This model proved to be a good starting point for establishing the methodology and understanding the key properties of the HA class of models.

The second chapter presents a HANK model with government debt which is both nominal and of long maturity. Other key features of the considered model include sticky prices, monetary policy, endogenous labour supply and distortionary taxation. This framework allows us to reconsider the distributional implications of interactions between monetary and fiscal policy.

It has been found that heterogeneous agent economies generate significant differences in the monetary and fiscal policy transmission mechanisms which are likely to have a quantitative impact on the optimal design of the policy.

The third chapter introduces Ramsey type of optimal policy to the HANK model with government bonds. To address the challenge of solving for optimal policy in HA models, I employ a variational approach inspired by Nuno and Thomas [2020], which incorporates the concept of Geautaux derivatives of infinite-dimensional spaces. As a result, I obtain an analytical characterization of optimal monetary policy, where the factors of aggregate wealth and income dispersion, as well as marginal consumption properties, define optimal inflation value. Specifically, the optimality condition for inflation reveals how the central bank trades off the disutility costs of inflation with its benefits.

Notably, my findings align with those of Nuno and Thomas [2020]: the heterogeneous-agent model generates an inflationary bias with redistributive effects, distinct from the inflationary bias observed in classical New Keynesian literature. The extent of this unconventional inflationary bias depends on the stringency of the government budget constraint and income dispersion.

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Declaration

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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Signature:

Analysis of the HANK Model with Capital Accumulation

1.1 Introduction

In this chapter, I analyze the heterogeneous-agent New Keynesian model (HANK) with capital, featuring the continuous-time framework. This type of model became central in discussion of policy implications (some but not all papers include: Kaplan et al. [2018] incorporate different types of assets in the NK setting, Nuno and Thomas [2020] and Nuño and Moll [2018] analyse social optima, Shaker Akhtekhane [2017] and González et al. [2022] extend the approach to heterogeneous firms) because they enable the analysis of welfare implications of certain shocks and policies, where the distributional consideration are crucial for the correct welfare evaluation Deaton [2016] summarizes these points well in his article. Moreover, the heterogeneous-agent models are often used as a connection bridge between microdata findings about individual behaviour and its impact on the aggregate outcome of the economy.

Another reason why heterogeneous-agent models have garnered a considerable amount of attention recently is the newly developed solution by Achdou et al. [2017] for the continuous-time model specification. Before, most of the studies used purely numerical algorithms to characterize distributions generated by models, but such computational methods were often time-consuming and difficult even when applied to relatively simple models. By contrast, Achdou et al. [2017] in their paper recast the Aiyagari-Bewley-Huggett model (Aiyagari [1994], Bewley [1986], Huggett [1993]) in continuous time and obtain a number of theoretical results, including analytical characterization of consumption and savings behaviour, and develop very efficient yet portable algorithm for numerical solution for the wide class of heterogeneous agent model. This algorithm I will apply to analyse the model in this chapter.

Continuous time setting has several computational advantages compared to the discrete-time one which made it possible to develop this algorithm. First of all, it is handling the borrowing constraint: in a continuous time setting, the borrowing constraint never binds in the interior of the state space. Secondly, the solution to the household's problem can be presented as a system of two partial differential equations: the Hamiltonian-Jacobi-Bellman (HJB) equation that characterises the optimal choice of individuals, and the Kolmogorov Forward (KF) equation that characterizes the evolution of households' wealth distribution. The interesting property of this p.d.f. equations are that they are coupled, meaning the characteristic matrix of one equation is the adjoint matrix of another equation, reducing significantly the number of computational steps required to obtain the numerical solution. Moreover. These two equations are characterised by a very sparse matrix that can be handled very efficiently with the use of special sparse matrix methods, this is another reason the developed algorithm is efficient.

In this chapter, I analyse the behaviour of the New Keynesian model, which has been extended to include heterogeneous individuals who face idiosyncratic productivity risk. The New Keynesian framework allows the examination of the explicit as well as implicit transmission mechanisms of monetary policy via interactions between agents in different markets and is a good starting point for later analysis of the different types of fiscal policy. However, to keep the analysis fairly simple, with a main focus on the heterogeneity innovation of the model I consider the NK setting where households' idiosyncratic income risk follows a two-state Poisson process and in which households can save in productive capital. I investigate the characteristics of stationary distributions produced by this model under different scenarios of productivity shock. By modifying the parameters governing productivity shock magnitude and transition probability, I explore how the outcome of the economy changes at the aggregate level and at the level of an individual. In particular, I obtain the distributional results, where consumption and savings functions exhibit linear behaviour when wealth gets substantially high, this outcome is coherent with other literature findings (Achdou et al. [2017], Benhabib et al. [2016]). Another interesting discovery emerges from this analysis: as the variability of productivity shocks decreases, aggregate welfare diminishes; this outcome can be attributed to the unique shape of the utility function and the simple form of labour income, implying that careful calibration of these economic components is essential for accurate welfare evaluation.

Finally, I present the results of model dynamics in response to the cost-push shock: the patterns of the impulse responses are similar to the one observed in the representative-agent New Keynesian literature. However, the heterogeneous agent model generates different welfare outcomes which is sensitive to the characteristics of the idiosyncratic shock. Therefore, I aim to perform a deeper analysis of the distributional properties of the heterogeneous-agent model with an accurately calibrated process of the idiosyncratic productivity shock in the next chapter.

To sum up, this chapter provides a solid foundation for studying the HACT models with complete markets and endogenous labour supply. Therefore, it serves as an initial step in the exploration of more complex HA models and efficient policy settings. The latter is especially beneficial to study within the context of heterogeneous households, as the explicit modelling of the idiosyncratic shock enables me to track not only the aggregate wealth and consumption position but also the overall distribution of assets. Consequently, this approach allows for a more precise evaluation of social welfare and inequality properties. This, in turn, empowers policymakers to pursue more efficient strategies for addressing household inequality and achieving improved social welfare outcomes.

1.2 The HANK Model

I consider a continuous-time Heterogeneous Agent New Keynesian (HANK) model with productive capital, where the source of heterogeneity is an idiosyncratic productivity shock that follows the two-state Poisson process. The model inherited the standard structure from the New Keynesian literature and includes four types of agents:

1. Households (are assumed to be heterogeneous)
2. Intermediate good producers (are subject to nominal rigidity)
3. Final good producer (gains no profit)
4. Monetary Authority (sets monetary policy according to the Taylor rule)

The interactions between these agents occur within the labour, bonds and goods markets.

1.2.1 Households

The economy is populated by a continuum of households indexed by their holdings of liquid assets a and idiosyncratic labour productivity shock z. Individuals have standard preferences over utility from consumption c_t and disutility from supplying labour l_t and face a time discount factor $\rho \geq 0$. So, their objective is to maximise the following lifetime utility

$$E_0 \int_0^\infty e^{-\rho t} u(c_t, l_t)\, dt \tag{1.1}$$

where the expectation is taken over realizations of idiosyncratic productivity shocks.

The function u is strictly increasing and concave in consumption, and strictly decreasing and strictly convex in labour.

Each individual receives labour income income $w_t l_t z_t$, a fraction of the firms' profit $\Pi(z_i) = \frac{z_i}{z_{ave}} \Pi_t$ proportional to their productivity; and interest payments on asset holdings $r_t a_t$; this income stream can be spent on consumption c_t or purchasing more assets $\dot{a}_t$. Therefore, the household's budget constraint (HBC) is given by

$$\dot{a}_t = w_t z_t l_t + r_t a_t + \Pi(z_t) - c_t \tag{1.2}$$

Moreover, the households are subject to the borrowing limit

$$a_t \geqslant \underline{a} \tag{1.3}$$

where $-\infty < \underline{a} < 0$.

Properties of the productivity shock

1) Productivity shock follows the two-state Poisson process: $z_t \in \{z_1, z_2\}$, $z_1 < z_2$;
2) The probability of becoming more productive (i.e. jump from state z_1 to z_2) is λ_1 and the probability of becoming less productive (i.e. jump from state z_2 to z_1) is λ_2.

Thus, individuals maximize (1.1) given wealth a subject to the budget constraint (1.2), borrowing limit (1.3) and the process for z_t to find the optimal level of consumption and labour.

Description of the solution

Households take as given equilibrium paths for the real wage $\{w_t\}_{t\geq 0}$, the real return to liquid assets $\{r_t\}_{t\geq 0}$, dividends $\{\Pi(z_t)\}_{t\geq 0}$ ($\{r_t\}_{t\geq 0}$ and $\{w_t\}_{t\geq 0}$ will be determined by market clearing conditions for capital and labor and $\{\Pi(z_t)\}_{t\geq 0}$ will be obtained as a result of the profit maximization problem of IGPs).

The household's problem can be rewritten recursively with a Hamilton-Jacobi-Bellman and Kolmogorov-Forward equation. In *steady state*, the recursive solution to this problem consists of decision rules for consumption $c(a,z;\Gamma)$ and labour supply $l(a,z;\Gamma)$, with $\Gamma := (r, w, \Pi(z))$. These decision rules imply optimal drifts for liquid assets a and, together with a stochastic process for z, they induce a stationary joint distribution of liquid assets and labour income $g(a,z;\Gamma)$. Outside of the steady state, each of these objects is time-varying and depends on the time path of prices and policies $\{\Gamma_t\}_{t\geq 0} := \{r_t, w_t, \Pi(z_t)\}_{t\geq 0}$.

Solution to the household's problem in transitionary case

The solution to (1.1)-(1.3) the problem can be described with the HJB equation, first-order conditions on labour and capital, state constraint boundary condition and KF equation, which are supplemented with the terminal condition on the HJB equation and initial condition on KF equation.

$$\begin{aligned}
\rho v_j(a,t) = \max_{c,l} u(c,l) &+ \partial_a v_j(a,t)\left(w_t z_j l + r_t a + \Pi_t\left(z_j\right) - c\right) \\
&+ \lambda_j\left(v_{-j}(a,t) - v_j(a,t)\right) + \partial_t v_j(a,t) \qquad (1.4) \\
c_j(a,t) \text{ is s.t. } &\partial_c u(c,l) = \partial_a v_j(a,t) \qquad (1.5) \\
l_j(a,t) \text{ is s.t. } &\partial_l u(c,l) = -w_t z_j \partial_a v_j(a,t) \qquad (1.6) \\
\partial_a v_j(\underline{a},t) &\geq \partial_c u\left(c_j(\underline{a},t), l_j(\underline{a},t)\right) \qquad (1.7)
\end{aligned}$$

here $c_j(\underline{a},t)$ and $l_j(\underline{a},t)$ are determined by

$$\begin{aligned}
c_j(\underline{a},t) &= w_t z_j l_t + r_t \underline{a}_t + \Pi(z_t) \\
l_t(\underline{a},t) &= \arg\max_l u\left(w_t z_j l + r_t \underline{a}_t + \Pi(z_t), l\right)
\end{aligned}$$

Note that the borrowing limit (1.3) transforms into (1.7) due to

$$c_j(\underline{a},t) \leq w_t z_j l_t(\underline{a},t) + r_t \underline{a}_t + \Pi(z_t)$$

for which the FOC condition for consumption (1.5) and the property of utility function being concave in consumption, implies

$$\begin{aligned}\partial_a v_j(\underline{a},t) \geq \partial_c u\left(c_j(\underline{a},t), l_j(\underline{a},t)\right) = \\ = \partial_c u\left(w_t z_j l_t(\underline{a},t) + r_t \underline{a}_t + \Pi(z_t), l_j(\underline{a},t)\right)\end{aligned}$$

The Kolmogorov-Forward equations and saving function are

$$\partial_t g_j(a,t) = -\partial_a\left[s_j(a,t)\, g_j(a,t)\right] - \lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t) \tag{1.8}$$

$$s_j(a,t) = w_t z_j l_j(a,t) + r_t a_t + \Pi(z_t) - c_j(a,t) \tag{1.9}$$

where $c_j(a,t)$ and $l_j(a,t)$ are found from (1.5) and (1.6).

Both HJB and KF equations are time-varying equations, the first one is backward-looking while the second one is forward-looking, therefore, the terminal and initial conditions should be imposed correspondingly

$$\begin{aligned}v_j(a,T) &= v_{j,\infty}(a) \\ g_j(a,0) &= g_{j,0}(a)\end{aligned}$$

The solution to the household's problem in stationary case

A solution to the problem (1.1)-(1.3) in the stationary case can be described with the HJB equation, the first-order conditions on labour and capital, state constraint boundary condition and KF equation

$$\rho v_j(a) = \max_{c,l} u(c,l) + v_j'(a)\left(w z_j l + r a + \Pi(z) - c\right) + \lambda_j\left(v_{-j}(a) - v_j(a)\right) \tag{1.10}$$

with the state-constraint boundary condition on the value function

$$c_j(a) \text{ is s.t. } \partial_c u(c,l) = v_j'(a) \tag{1.11}$$

$$l_j(a) \text{ is s.t. } \partial_l u(c,l) = -w_t z_j v_j'(a) \tag{1.12}$$

$$v_j'(\underline{a}) \geq \partial_c u\left(c_j(\underline{a}), l_j(\underline{a})\right) \tag{1.13}$$

here $c_j(\underline{a})$ and $l_j(\underline{a})$ are determined by

$$\begin{aligned}c_j(\underline{a}) &= w z_j l + r\underline{a} + \Pi(z) \\ l_j(\underline{a}) &= \arg\max_l u\left(w z_j l + r\underline{a} + \Pi(z_j), l\right)\end{aligned}$$

The Kolmogorov-Forward equation is

$$0 = -\frac{d}{da}\left[s_j(a)g_j(a)\right] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a) \tag{1.14}$$

$$s_j(a) = wz_j l_j(a) + ra + \Pi(z) - c_j(a) \tag{1.15}$$

where $c_j(a)$ and $l_j(a)$ are found from (1.11) and (1.12).

1.2.2 Firms

Final Good Producers

A competitive firm solves a profit maximization problem

$$\max_{y_{j,t}} P_t Y_t - \int_0^1 p_{j,t} y_{j,t} dj,$$

subject to the aggregating technology

$$Y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}},$$

here Y_t denotes final goods, $y_{j,t}$ denotes the j'th intermediate input, and $\varepsilon > 1$ governs the elasticity of substitution between any two intermediate inputs.

Profit maximization implies that demand for intermediate good j is

$$y_{j,t} = \left(\frac{p_{j,t}}{P_t}\right)^{-\varepsilon} Y_t, \text{ where } P_t = \left(\int_0^1 p_{j,t}^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}. \tag{1.16}$$

Intermediate Good Producers

Each intermediate good j is produced by a monopolistically competitive producer using labor $n_{j,t}$ under technology A_t according to the linear production function $y_{j,t} = An_{j,t}$. The firm hires labour at a wage w_t in a competitive labour market. As a result, the cost-minimization problem of the firm implies that the marginal cost is common across all producers and given by

$$mc_t = \frac{w_t}{A_t}, \tag{1.17}$$

and the operational real profits of intermediate goods producers are

$$\tilde{\Pi}_{j,t} = \frac{p_{j,t}}{P_t} y_{j,t} - w_t n_{j,t} = \left(\frac{p_{j,t}}{P_t} - mc_t\right) y_{j,t} = \left(\frac{p_{j,t}}{P_t} - mc_t\right)\left(\frac{p_{j,t}}{P_t}\right)^{-\varepsilon} Y_t.$$

Firms are subject to a price adjustment cost following Rotemberg [1982]. These adjustment costs are quadratic in the rate of price change $\frac{\dot{p}_t}{p_t}$ and are expressed as a fraction of aggregate output:

$$\Theta\left(\frac{\dot{p}_{j,t}}{p_{j,t}}\right) = \frac{\theta}{2}\left(\frac{\dot{p}_{j,t}}{p_{j,t}}\right)^2 Y_t.$$

Each intermediate producer chooses its price to maximize profits subject to price adjustment costs,

$$\max_{p_{j,t}} \int_0^\infty e^{-\rho t}\left\{\left(p_{j,t} - P_t mc_t\right)\left(\frac{p_{j,t}}{P_t}\right)^{-\varepsilon} Y_t - \frac{\theta}{2}\left(\frac{\dot{p}_{j,t}}{p_{j,t}}\right)^2 Y_t P_t\right\} dt.$$

Phillips curve equation characterizes the solution to this profit-maximizing problem (see Appendix A.2 for details)

$$\left(\rho - \frac{\dot{Y}_t}{Y_t}\right)\pi_t = \frac{\varepsilon}{\theta}\left(mc_t - \frac{\varepsilon - 1}{\varepsilon}\right) + \dot{\pi}_t, \tag{1.18}$$

here I define inflation as $\pi_t = \frac{\dot{p}_t}{p_t}$.

In a symmetrical equilibrium ($p_{j,t} = p_t = P_t$), the demand for intermediate goods can be written as

$$y_{j,t} = \left(\frac{P_t}{P_t}\right)^{-\varepsilon} Y_t = Y_t,$$

labour demand is

$$n_{j,t} = n_t = N_t = \frac{Y_t}{A_t}, \tag{1.19}$$

and firm profits are given by

$$\Pi_{j,t} = \Pi_t = Y_t - w_t N_t - \frac{\theta}{2}\pi_t^2 Y_t. \tag{1.20}$$

1.2.3 Monetary Authority

The monetary authority sets nominal interest rate i_t using the Taylor rule

$$\frac{1+i_t}{1+i_{ss}} = \left(\frac{1+\pi_t}{1+\pi^T}\right)^\phi, \tag{1.21}$$

here π^T denotes the target level of inflation and a value of $\phi > 1$ implies that the rule satisfies the Taylor Principle whereby real interest rates rise when inflation is above its target, this is because given inflation and the nominal interest rate, the real return on the asset is determined by the Fisher equation $r_t = i_t - \pi_t$.

1.2.4 Markets Equilibrium

Aggregate consumption is obtained by aggregating all the individual consumptions $c_j(a,t)$ using the distribution density function $g_j(a,t)$,

$$C_t = \sum_{j=1}^{2} \int_{\underline{a}}^{\infty} c_j(a,t) g_j(a,t) da. \tag{1.22}$$

The aggregate labor supply, N_t^S, is equal to the labor demand,

$$N_t = N_t^S = \sum_{j=1}^{2} \int_{\underline{a}}^{\infty} z_j l_j(a,t) g_j(a,t) da. \tag{1.23}$$

Similarly, the assets market clearing condition is,

$$K_t = K_t^H = \sum_{j=1}^{2} \int_{\underline{a}}^{\infty} a g_j(a,t) da. \tag{1.24}$$

Finally, aggregation of individuals' budget constraints and taking into account firms' profits and price-adjustment costs yields the resource constraint,

$$Y_t \left(1 - \frac{\theta}{2}\pi_t^2\right) = C_t. \tag{1.25}$$

The system of equations (1.4)-(1.9), (1.17)-(1.25) determines the private sector equilibrium, given boundary condition (1.7) and the stochastic process for the idiosyncratic shock z_t.

1.3 Numerical Analysis

1.3.1 Calibration

I aim to explore the properties of the presented heterogeneous agent model and to study how it behaves when the parameters of the idiosyncratic shock change. Therefore, in this chapter, I consider the values of parameters typical for the standard New Keynesian literature, while the parameters of the idiosyncratic productivity shocks are not backed up with data and are chosen to be rather simple.

The calibration of most parameters is standard and follows Kaplan et al. [2018]. Specifically, the model is calibrated to a quarterly frequency with discount factor $\rho = 0.01$. I work with log-utility, setting $\sigma = 1$, while the inverse of Frisch elasticity of substitution is set at $\psi = 1$. The weight on disutility of labour, $\varphi = 2.2$. This ensures that the average time worked is about 0.5 of the available time in the steady state. I set the elasticity of substitution between goods at $\varepsilon = 11$, implying a mark-up of 10%, as estimated in Krause and Lubik [2007], Chari et al. [2000] and is consistent with Basu and Fernald [1997]. The price adjustment cost parameter is set to $\theta = 100$, consistent with a Calvo parameter implying prices are sticky for, on average four quarters. Intermediate goods producers weight capital by a coefficient $\alpha = 1/3$. I set the Taylor rule coefficient ϕ to 1.5 which is commonly used for New Keynesian models and the steady-state value of inflation is assumed to be zero.

The baseline calibration of the productivity shock assumes the low productivity state ($z_1 = 0.5$) implies 50% of average productivity ($z_{ave} = 1.0$) while the state of the high productivity ($z_2 = 1.5$) is 50% higher than the average. The probability of becoming less productive is $1/3$ and the same is the probability of switching from a low to a high productivity state. The households operate with the asset of capital, so they can only invest it, therefore, the borrowing limit is set to 0. The other alternative scenarios consider a lower variation of the productivity shock ($z_1 = 0.75, z_2 = 1.25$) and higher probability of becoming less productive ($\lambda_2 = 2/3$).

1.3.2 Stochastic Steady State

In this section, I present the analysis of the stochastic steady state under different assumptions of the idiosyncratic productivity shock. In particular, I consider how the smaller variance of the productivity shock and increased probability of becoming a less productive worker affects an aggregate outcome of the economy. Moreover, to understand the transmission mechanism of the changed parameters, I also study the households' distributions and dynamics of a representative individual profile.

Specifically, the baseline model specification assumes $z_1 = 0.5$ and $z_2 = 1.5$, while the alternative scenario considers a shock of a twice smaller magnitude: $z_1 = 0.75$ and $z_2 = 1.25$, moreover, the third scenario uses the following transition probabilities: $\lambda_1 = 1/3$ and $\lambda_2 = 2/3$ instead of the baseline – $\lambda_1 = 1/3$ and $\lambda_2 = 1/3$. Table 1.1 summarises the results.

In Table 1.1, panel (1), the baseline characteristics show some interesting results. First of all, the difference in consumption between low and high states is not large, while the difference in labour supply is huge, this is because consumers are motivated to keep their consumption at a steady level to maximize their welfare, while decreased returns from their labour supply discourage them to work. In the absence of a steady labour income, individuals give up a significant fraction of their capital holdings when they get to the low state. Note, that when the magnitude of the shock is lower (panel (2)), the individuals sacrifice almost two times less capital to maintain the desired level of consumption. Moreover, increased demand for capital pushes the interest rate down. So when the degree of heterogeneity is decreased (panel (2)), the interest rate almost reaches the rate of the household's time discount (1%).

The precautionary savings motive is the key property of the model with idiosyncratic shock, on one hand, the households seek to ensure against idiosyncratic risk - they prefer to save part of their income for the future to smooth out their consumption in the case of a low-productivity state, on the other hand, the borrowing constraint matters, as it limits the households' ability to

Table 1.1: Characteristics of Stochastic Steady State

Productivity shock	z_1, z_2	0.5, 1.5	0.75, 1.25	0.5, 1.5
Jump probability	λ_1, λ_2	$1/3, 1/3$	$1/3, 1/3$	$1/3, 2/3$
		(1)	(2)	(3)
Consumption	C	1.5343	1.4115	1.3140
- low	C_1	1.4719	1.3899	1.2834
- high	C_2	1.5968	1.4331	1.3752
Output	Y	1.9818	1.8059	1.6972
Assets	K	17.9010	15.7759	15.3289
- low	K_1	16.4340	14.9845	14.5631
- high	K_2	19.3679	16.5672	16.8604
Labour	N	0.6594	0.6110	0.5647
- low	N_1	0.1446	0.3316	0.1650
- high	N_2	1.1743	0.8904	1.3643
Interest rate, %	r	0.85	0.97	0.86
Mass on BL	m	0.0082	0.0077	0.0123
Speed to reach BL	v	0.0950	0.0633	0.0778
Time to reach BL	τ	13.7258	15.7912	14.0376
Utility* (d.p.v.)	$\mathscr{U}$	3.2296	-4.5933 $[-7.52\%C]$	-12.3403 $[-14.42\%C]$
- low	$\mathscr{U}_1$	-21.4001	-20.1418 $[1.27\%C]$	-60.1647 $[-32.14\%C]$
- high	$\mathscr{U}_2$	27.8593	10.9552 $[-15.55\%C]$	11.5718 $[-15.03\%C]$

*Note: the numbers in square brackets denote consumption equivalents of the welfare loss compared to the steady state of the baseline model.

access the financial market (in the considered calibration the borrowing constraint is set at zero level, which, despite being far from the average capital holdings, still implies that around 1% of individual hit it in the stationary distribution). According to Achdou et al. [2017]), an individual reaches the borrowing limit in finite time once experiences an unfortunate sequence of low states; in the considered models the average time to reach the borrowing constraint is about 13 to 16 time periods, so the effect of the borrowing constraint is not significant and precautionary savings motive dominates here. Thus, uncertainty about the future is the main explanation for the increased demand for capital and reduced interest rate. Interestingly, however, that interest rate does not decrease but actually increases when $\lambda_2 = 2/3$, this is a consequence of the model general equilibrium - individuals stay in the high state for shorter periods of time, and, as a result, the economy features smaller number of households who are willing to lend their assets at a low rate.

Another finding is related to the welfare effects of the idiosyncratic shock. It is expected to see a decrease in social welfare when the probability of becoming a less productive worker has increased and an increase in social welfare when the variance of the productivity shock is decreased. However, according to Table 1.1, the latter does not hold here. In particular, the welfare in scenario (2) $\mathscr{U}$ has decreased by 7.52% in consumption equivalent, this is because the welfare of low-state individuals is indeed higher by 1.27% than in the baseline model, but the welfare of high-state individuals is substantially lower (by -15.55%). This finding has not been reported in other relevant literature and is observed here due to the property of labour supply being endogenous in this model. It is attributed to the specific form of the utility function: due to quadratic labour disutility, when high-productivity individuals are experiencing lower rewards for their work, they choose to give up working hours and, as a result, get lower income, so their consumption decreases. As a result, their utility drops substantially leading to a decrease in their welfare which outweighs the benefits received by low-productivity workers and, therefore, the total welfare diminishes.

Thus, I can summarise some of the key observations. Firstly, uncertainty about the future leads to higher precautionary savings, which boosts the demand for capital and suppresses the interest rate. Secondly, considering the distribution of households instead of a representative household allows me to identify asymmetrical effects that lead to changes in the aggregate outcome. Thirdly, the heterogeneous-agent environment enables me to compute the aggregate utility more accurately and discover the hidden welfare effects which are attributed to previously unreported wealth transmission mechanisms. As a result, these findings motivate me to use a heterogeneous-agent framework to improve the calibration of the model and to conduct a more comprehensive welfare analysis.

Next, I consider the distributional properties of the model. According to Figure 1.1, consumption and saving functions exhibit linear behaviour when wealth increases. This is consistent with the analytical results obtained by Achdou et al. [2017] for the endowment economy. The distributions of wealth and consumption are characterised by lower variance (their picks become narrower) as the magnitude of the idiosyncratic shock decreases. Moreover, panel (c) of Figure 1.1 demonstrates an interesting result: here, due to increased λ_2 the fraction of low productive people not only

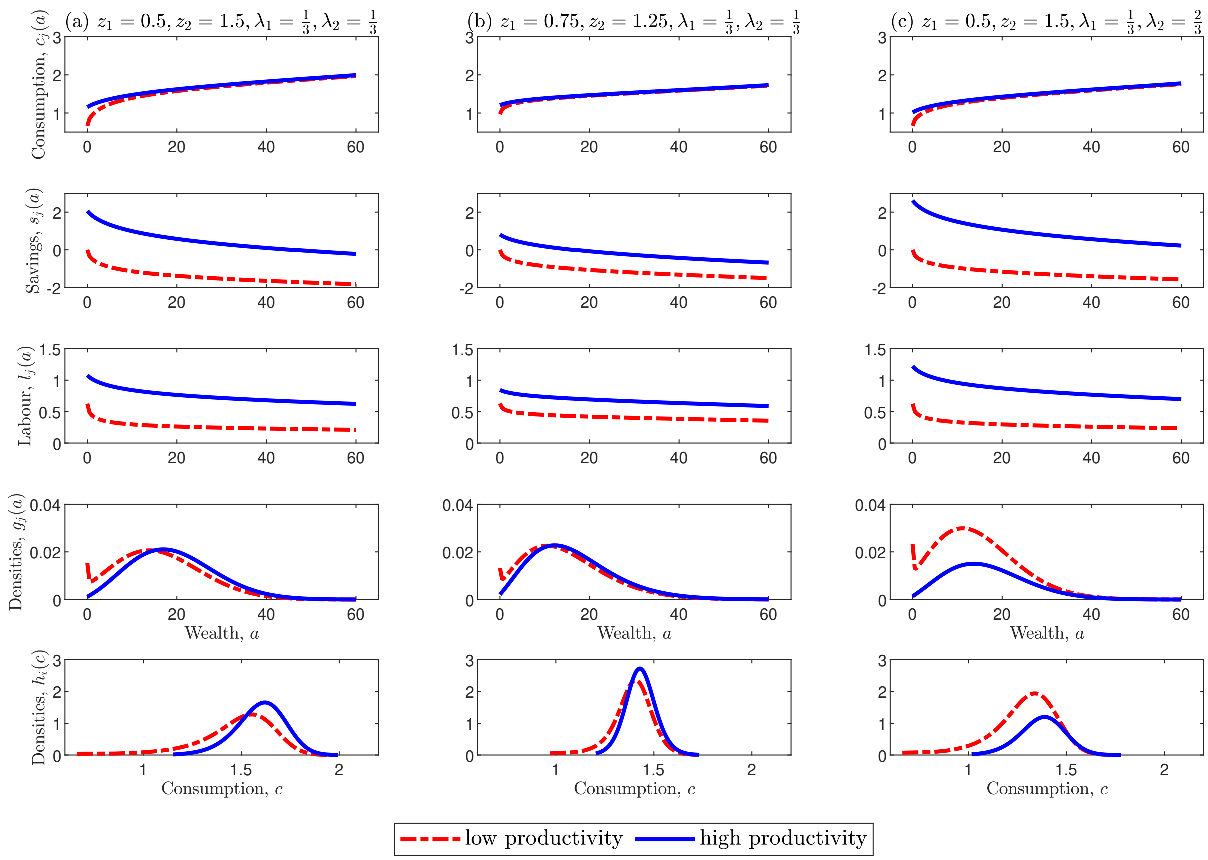

Figure 1.1: Distributional properties of the stochastic steady state

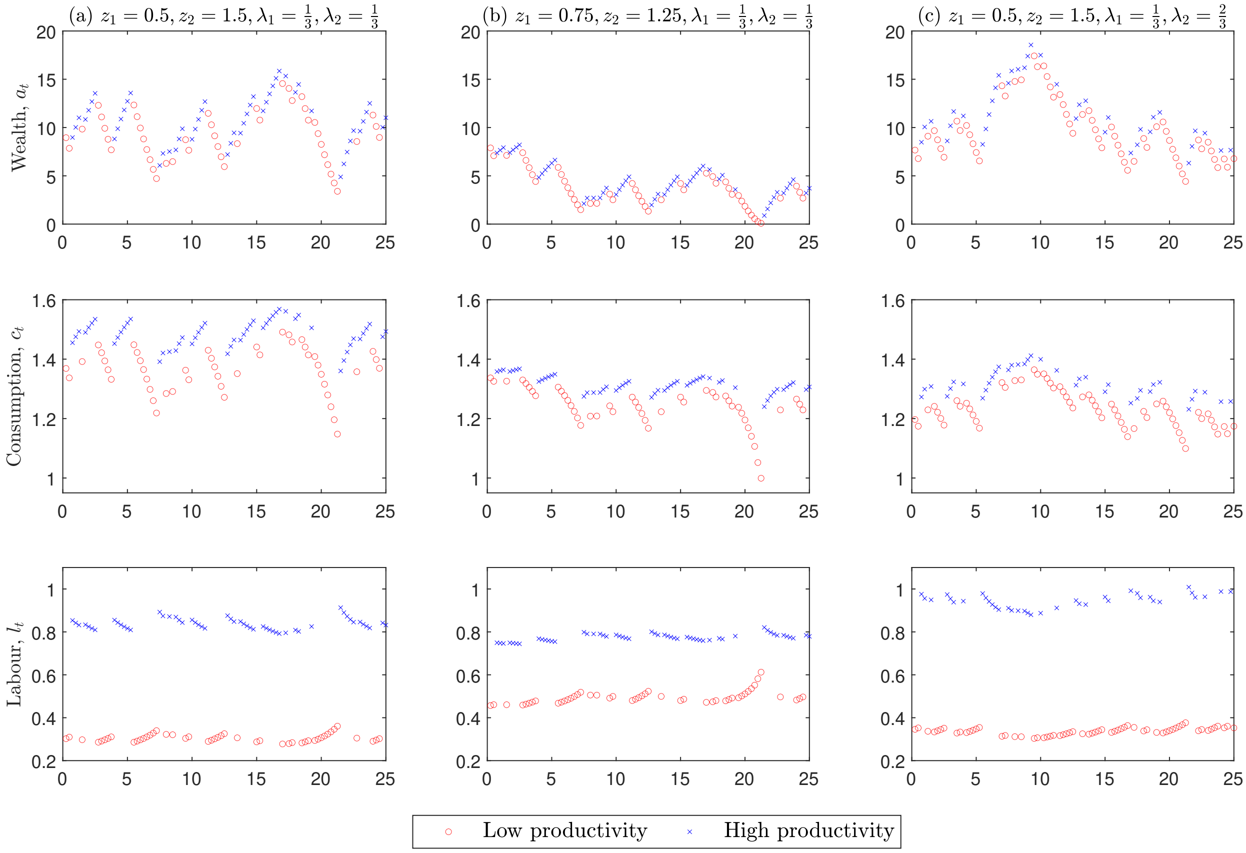

Figure 1.2: Individual profile simulations over 25 years

increased but also their average assets holdings decreased (the wealth distribution line shifted to the left), this observation is supported by the values from Table 1.1, where B_1 is lower and the mass on constraint m is higher by 50%.

1.3.3 Dynamics in Stochastic Steady State

To better understand the individual's adjustment process who faces the productivity idiosyncratic shocks, I performed the simulations for an individual profile with 50% of average assets holding. These simulations have been applied for the same three model specifications as I analysed in the previous section. Figure 1.2 presents the evolution of wealth, consumption and labour over the period of 25 years[1]. As we can see, the periods of high productivity are very frequently changed with the periods of low productivity, this is due to the high probability of transition between states. On one hand, such frequent transitions between the productivity states do not allow an individual to accumulate the desired level of assets; on the other hand, the low-productivity state does not last long, so the individual reaches the borrowing limit rather rarely.

[1]here, the sequence of low and high productivity states has been generated in a consistent manner with the use of parameters λ_1 and λ_2, with a low productivity state as a starting point

Overall, all the panels in Figure 1.2 demonstrate the expected results: as heterogeneity decreases in panel (b), the movements of all the variables become less steep (the households experience smaller idiosyncratic shock and do not need to adjust their labour, consumption and consequently wealth accumulation to maintain the desired level of utility), panel (c) demonstrates a distinct evolution of wealth - in this scenario, the individuals are in a high productivity state for a shorter period of time and thus have to accumulate wealth faster and, at the same time, in the low productivity state, they are not rushing to spend their wealth as quickly as in scenario (a). These observations are supported by the values from Table 1.1, specifically the time and speed to reach the borrowing limit are consistent with the individual profile simulations. The baseline model has the shortest time to reach the borrowing limit, while in the other two scenarios, an individual reaches the borrowing limit more slowly. In the second scenario (Figure1.2, panel (b)), a smaller magnitude of the productivity shock makes individuals reluctant to spend a substantial fraction of their accumulated capital to smooth their consumption, while in the third scenario (Figure1.2, panel (c)), households cannot rely on the prolonged streams of labour income, thus they have to spend their capital accumulations cautiously.

Considering the graphs describing the consumption choice of an individual in Figure 1.2, we see that consumption demonstrates big deviations when the productivity state is low and small deviations when the productivity state is high. This is attributed to the diminishing marginal utility property and precautionary savings motive (at a high productivity state, individuals are more willing to save instead of consuming so that in a low productivity state they can use their savings). The graphs of labour dynamics in Figure 1.2, however, exhibit opposite behaviour: labour deviations are higher when individuals are more productive and vice versa. This can be explained by the quadratic form of the disutility function that implies increasing marginal disutility.

Overall, in all the considered scenarios, the probability of staying in a low-productivity state for prolonged periods of time is rather small, as a result, the household's wealth, consumption and labour do not deviate much from their average values. However, this may not be the case when other types of low-productivity shock are applied. For example, in the next chapter, I consider an unemployment shock: it implies the prolonged periods of low-income states and, as a result, leads to substantial changes in savings, consumption and labour of an individual over time.

1.3.4 Transition Dynamics

In this section, I explore the HANK model behaviour in the transition, more specifically - I analyse the dynamic response to a surprise reduction in the elasticity of substitution between differentiated goods, which can be interpreted as a positive cost-push shock.

Figure 1.3 shows the response of the key variables to a cost-push shock during the first 100 periods. More specifically, I assume a one-time, unanticipated reduction of 3% in ε_t[2], after which

[2]this is the same type of shock as the one used in the Himmels and Kirsanova [2013] paper, however, its magnitude

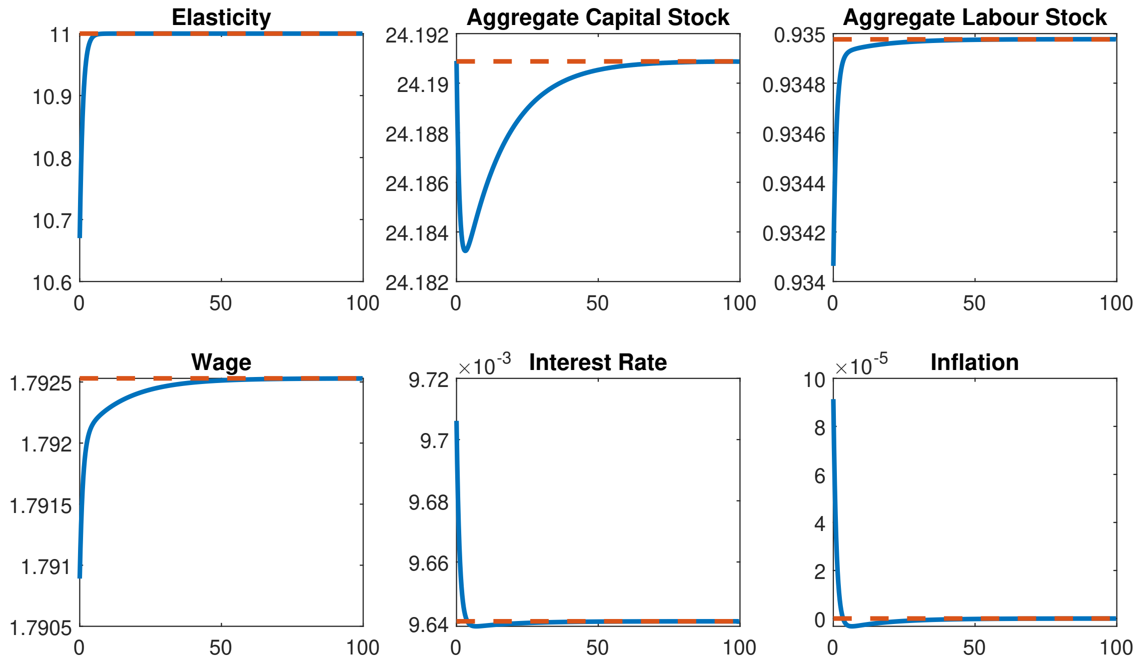

Figure 1.3: Impulse responses to a cost-push shock*

Notes, * a blue solid line represents the variable's deviation from its steady-state level which is represented by a red dashed line.

it returns gradually to its steady-state value $\varepsilon_{ss} = 11$ according to

$$d\varepsilon_t = \eta_\varepsilon(\varepsilon_{ss} - \varepsilon_t)dt, \tag{1.26}$$

with $\eta_\varepsilon = 0.8$.

Focusing first on the goods market, it can be seen that inflation rises following the cost-push shock so that the policy responds by raising interest rate which defers consumption. As a result, a decrease in consumption causes output to lower, so wages go down, labour income reduces, and demand for capital goes down too (note, that the decision about the current capital holdings has been made one period before, therefore capital adjustments occur a period later). In the subsequent periods, although the interest rate is returning to its steady-state value to stimulate the economy, the capital stock takes a longer time to return to its initial level.

Such a shock brings significant welfare losses for the households, usually, inflation can help to mitigate its negative impact and in the heterogeneous model can be even more efficient in doing so due to an extra wealth redistributive channel. However, this is possible when individuals operate with nominal claims (for example, government bonds), not capital, so, in this case, the role of inflation is rather limited.

Overall, the presented impulse responses are consistent with the standard New Keynesian literature results. However, it is worth looking into more details about the distributional properties

has been increased from 1% to 3% to generate a more substantial deviation of inflation and other key variables generated by the aggregate shocks and consequent effects of the implemented policies – this question will be explored in more detail in the next chapter where I consider the model with nominal government bonds and where the interplay of tax and inflation occurs.

In addition, I have generated the impulse responses to a total factor productivity (TFP) shock of 3% for the baseline HA model, the results are presented in the Appendix A.3. They are also consistent with the common New Keynesian literature findings but, again, more research on the distributive properties of this model is needed to fully understand the implication of heterogeneity for the appropriate policy setting.

1.4 Conclusion

In this chapter, I consider the New Keynesian model with capital accumulation, where households face idiosyncratic labour productivity shock. In particular, the developed model features complete markets and an endogenous labour supply. I have explored the properties of the stationary distributions generated by this model under different scenarios of productivity shock and performed individual-level simulations. Also, I have examined the model dynamics in response to the aggregate shocks.

My key findings are that the properties of households' wealth and consumption distribution are coherent with the results of the relevant HACT papers and the behaviour of the model in the transition is consistent with the standard New Keynesian literature. Moreover, it is particularly interesting to study the welfare implications in this framework, as the HACT model enables me to track not only the aggregate wealth, consumption and labour position but also their overall distributions and account for a borrowing limit. Consequently, it allows for a more precise evaluation of social welfare and inequality, making this model particularly promising for studying different policy settings and their welfare implications. I have found that changing the parameters of the idiosyncratic shock (while all other characteristics of the model are kept the same) affects significantly the consumption and labour of high- and low-productive workers, so that the individual as well as the aggregate welfare changes dramatically. In particular, one interesting finding is the decreased value of the aggregate utility when the magnitude of the productivity shock is diminished - this is attributed to the unique form of the utility function and has not been thoroughly studied in continuous-time and discrete-time heterogeneous-agent literature. This result signifies the importance of an accurate calibration of the utility function and income process.

Thus, my findings demonstrate that the HANK framework has great potential for exploring previously undiscovered welfare puzzles and their implications under different policy and calibration settings. Moreover, the model set-up that enables a negative borrowing limit, financial markets and taxation can capture hidden transmission mechanisms, open extra wealth redistribution channels and signify the policy impact. Therefore, the next chapter considers an extended model with government debt and fiscal policy where households are subject to an unemployment shock.

Monetary and Fiscal Policy Interactions in the HANK Model with Government Bonds

2.1 Introduction

Since Leeper [1991] it has been shown that two permutations of monetary and fiscal policy rules are capable of ensuring a determinate equilibrium in a representative agent economy. The conventional policy mix assumes that the fiscal authorities adopt a passive fiscal policy which adjusts fiscal instruments to stabilize government debt. In doing so this gives the monetary authorities the freedom to pursue an active monetary policy which raises nominal interest rates in response to inflation being above target by enough to ensure that real interest rates rise. However, there is another mechanism through which determinacy can be achieved. When the fiscal authorities do not act to use their available fiscal policy instruments to stabilize government debt, this can still support a determinate equilibrium provided the monetary authorities abandon their active targeting of inflation to facilitate debt stabilization. This is achieved through a combination of inflation surprises (as emphasized by the Fiscal Theory of the Price Level - FTPL) and, when prices are sticky, suppression of real interest rates, and as a result, debt service costs.[1] Analysis of the welfare properties of these two stable regimes tends to suggest that the conventional policy mix is welfare superior to the less conventional FTPL regime, although this ranking may depend upon the maturity of government debt. With single-period debt, typically the inflation fluctuations implied by the passive monetary-active fiscal regime are costly in the context of sticky-price New Keynesian models. However, this result has been shown in the context of representative agent models only.

In this chapter I explore the properties of these regimes but in the context of a heterogenous agent New Keynesian model. This matters since the uninsurable idiosyncratic employment risk faced by households implies that there is a distribution of wealth in the economy, including some indebted poorer households who have entered a prolonged spell of unemployment. In such an economy, the inflation surprises needed to stabilize aggregate government debt when fiscal policy is not doing so will also have distributional consequences. Specifically, it will transfer resources from

[1] Other permutations are possible. An active-active policy mix leads to instability and a passive-passive mix implies indeterminacy.

asset-rich households to indebted households whose real debt will be reduced in much the same way as government debt. Can such redistribution effects reverse the welfare ranking of Leeper's policy regimes?

My benchmark model is in the spirit of Huggett [1993] and Aiyagari [1994] and, more specifically, follows Kaplan et al. [2018] who develops a heterogeneous agent model in continuous time with sticky prices and real bonds, using which I will analyze the distributional consequences of monetary policy[2]. I extend this model to include long-maturity nominal government bonds (which is consistent with the majority of observed government debt throughout the world and allows the mechanisms of the FTPL to operate through bond prices and not just current-period inflation surprises as in the single-period debt case) and a fully articulated fiscal policy including transfers, government consumption and a distortionary labour income tax. Individual households are then subject to idiosyncratic shocks which can move them from a state of employment to unemployment. In doing so I have developed an environment where monetary and fiscal interactions have distributional consequences. My contribution lies in analyzing the distributional impacts of shocks under different policy regimes and assessing their welfare implications.

I begin the chapter by outlining the continuous-time heterogeneous-agent economy. Then, I consider the micro and macro-economic consequences of idiosyncratic employment risk, before turning to the implications of alternative policy regimes for the responses to shocks.

2.2 The Model

The economy is populated by heterogenous households, firms and two policy-makers, monetary and fiscal, respectively. Individuals can be either employed or unemployed and randomly move between the two states. There is no insurance against unemployment and households do not pool resources through the kinds of 'large family' assumptions often found in the job-search literature (see, for example, Walsh [2005]). When employed, they supply labour elastically and pay distortionary labour income taxes. In contrast to much of the literature, it is important to formulate private sector decision problems in nominal terms to enable the FTPL mechanisms to operate within the economy.

2.2.1 Households

Income and Assets Consider a continuous-time economy with incomplete markets and stochastic shocks to incomes following a two-state Markov process, as in Huggett [1993]. There is a continuum of infinitely-lived individuals with different quantities of nominal bonds, b^n and the idiosyncratic shock realization z. The two-state Poisson process z jumps between states $z_t \in \{0,1\}$, intensity to jump from state 0 to 1 is q_1, and intensity to jump from state 1 to 0 is q_2.

[2] See also Nuno and Thomas [2020] for analysis of the distributional consequences of monetary policy.

In state $z_t = 1$ the individual is employed, they supply labor l_t, receive post-tax labor income of $(1-\tau_t)\, w_t l_t$, where the real wage rate is given by w_t and firms' profits redistributed to employed households is Π_t. labour income is taxed at rate τ_t. In state $z_t = 0$ the individual is unemployed, and receives real unemployment benefit payment κ_t. The household budget constraint can be written as

$$P_t^M \dot{b}_t^n = P_t\left(\left[(1-\tau_t)\, w_t l_t + \Pi_t\right] z_t + \kappa_t\,(1-z_t) + T_t - c_t\right) + \left(1-\delta P_t^M\right) b_t^n, \tag{2.1}$$

where c_t is real consumption, T_t are lump sum transfers and P_t is the price level. The income process is uninsurable, and individuals can only lend or borrow in the form of non-contingent long-term bonds b at price P_t^M which is determined in equilibrium. Individuals face borrowing constraint,

$$b_t^n \geq P_t \underline{b}, \tag{2.2}$$

where the exogenous nominal borrowing limit $\underline{b}$ is tighter than the 'natural' borrowing limit, $-P_t\,(\kappa_t + T_t)\,/\left(1-\delta P_t^M\right) < \underline{b} < 0$ i.e. the maximum debt the household could sustain if they were permanently in a state of unemployment.

The households' bond portfolio consists entirely of maturing bonds b_t^n, for which I assume that the maturity structure is declining at a constant rate δ. In this case, the average maturity of the portfolio is $-1/\log\left((1-\rho)\,(1-\delta)\right)$. By using such a simple maturity structure only a single bond needs to be priced, since any existing bond issued s periods ago is worth $e^{-\delta s}$ of new bonds. Thus, the households can purchase $\dot{b}_t^n$ of new bonds for the price P_t^M up to an exogenous limit $P_t\underline{b}$ and receive a return of $\left(1-\delta P_t^M\right)$ on bond holdings.[3]

The joint probability distribution of income z_j and wealth x is denoted $G_j\,(x,t)$, and the corresponding density function is $g_j\,(x,t),\ j = 1,2$. I assume that the total nominal bond supply, B^n, is determined by the government and

$$\int_\Omega x g_1(x)\, dx + \int_\Omega x g_2(x)\, dx = B^n, \tag{2.3}$$

where Ω denotes the asset distribution domain.

Preferences and Decision problem Individuals have standard preferences over utility from consumption c_t, disutility from supplying labor l_t and they face a time discount factor $\rho \geq 0$. So, their objective is to maximize the following lifetime utility

$$\mathbb{E}_{0,z} \int_0^\infty e^{-\rho t} U(c_t, l_t)\, dt, \tag{2.4}$$

[3] More details about the general bonds maturity structure and derivations of the final form of the budget constraint can be found in the Appendix.

where the expectation is taken over by realizations of idiosyncratic unemployment shocks and period utility is given by,

$$U(c,l)=\frac{c^{1-1/\sigma}}{1-1/\sigma}-\varphi\frac{l^{1+\psi}}{1+\psi}.$$

Individual optimal consumption and saving decision is described by the Hamilton-Jacobi-Bellman (HJB) equation,

$$\begin{aligned}\rho W_{jt}(b^n)=\max_{c,l}u(c,l)+\frac{\partial}{\partial b^n}\left(\frac{W_{jt}(b^n)}{P_t^M}P_t\left([(1-\tau_t)w_tl_t+\Pi_t]z_t+\kappa_t(1-z_t)+T_t-c_t\right)\right)\\+\frac{\partial}{\partial b^n}\left(\frac{W_{jt}(b^n)}{P_t^M}\left(1-\delta P_t^M\right)b^n\right)+\lambda_j\left(W_{-j}(b^n,t)-W_{jt}(b^n)\right)+\dot{W}_{jt}(b^n),\end{aligned}\tag{2.5}$$

and the Kolmogorov Forward (KF) equation

$$\frac{\partial}{\partial t}g_j(b^n,t)=-\frac{\partial}{\partial b^n}\left[s_j^n(b^n,t)g_j(b^n,t)\right]-\lambda_jg_j(b^n,t)+\lambda_{-j}g_{-j}(b^n,t)\tag{2.6}$$

where $W_{jt}(b^n)$ is the value function, $j=\{1,2\}$ denotes the employment state, and index $-j$ means 'other than j'.

The HJB equation is a continuous-time analogue of the Bellman equation on the value function, and it incorporates additional terms that arise due to the continuous-time nature of the optimization problem. These additional terms capture the continuous evolution of the system over time and reflect the dynamics of the variables involved. As income is subject to a continuous flow of shocks, so is wealth. Partial derivatives with respect to states capture the continuous dynamics more accurately.

The KF equation describes the evolution of the probability density function (p.d.f.) of wealth, $g_j(b^n,t)$, over time. It allows me to understand how this distribution evolves due to various factors such as saving and income processes. By solving the KF equation, I can obtain the time-dependent p.d.f. of the state, which will provide insight into statistical behaviour and distributional properties of wealth and, hence, all other variables.

The saving function is,

$$s_j^n(b^n,t)=\frac{1}{P_t^M}\left[P_t\left([(1-\tau_t)w_tl_t+\Pi_t]z_t+\kappa_t(1-z_t)-c_t+T_t\right)+\left(1-\delta P_t^M-P_t^M\hat{\pi}_t\right)b_t^n\right].\tag{2.7}$$

Finally, there are boundary conditions. Both the HJB and KF equations are time-varying equations, the first one is backward-looking while the second one is forward-looking and, therefore, they require terminal and initial conditions, respectively,

$$W_j(b^n,T)=W_{j,\infty}(b^n),\tag{2.8}$$

$$g_j(b^n,0)=g_{j,0}(b^n),\tag{2.9}$$

where $W_{j,\infty}(b^n)$ and $g_{j,0}(b^n)$ are given. Also, the following state-constraint boundary condition is applied

$$\frac{\partial}{\partial b^n} W_j(\underline{b},t) \geq \frac{\partial}{\partial c_j} U\left(c_j(\underline{b},t), l_j(\underline{b},t)\right) P_t^M, \tag{2.10}$$

which implies $s_j(b,t) \geq 0$ at the boundary and ensures that the borrowing constraint is never violated.

Maximization of (2.5) yields,

$$\frac{\partial}{\partial c_j} U\left(c_j, l_j\right) P_t^M = P_t \frac{\partial W_{jt}(b^n)}{\partial b^n}, \tag{2.11}$$

$$\frac{\partial}{\partial l_2} U\left(c_2, l_2\right) P_t^M = -\left(1-\tau_t\right) w_t z_2 P_t \frac{\partial W_{2t}(b^n)}{\partial b^n}. \tag{2.12}$$

Therefore, household consumption rises as government bond prices increase and decline with the steepness of the value function. In simple terms, when bond prices are higher (or yields are lower), households are motivated to save less and consume more. Conversely, a steeper value function makes saving more appealing in order to accumulate greater net bond holdings. The household choice of labour has exactly opposite properties where, also, lower taxes and higher wages encourage individuals to work more.

Household Problem in Real Terms Although the above relationships are derived in nominal terms to highlight the revaluation effects crucial to the operation of the FTPL (see Appendix), I present their transformed versions, written in real terms, where I define a measure of the real debt quantity, $b_t = \frac{b_t^{nom}}{P_t}$. This term measures the quantity of nominal bonds per unit of currency, while the real value of debt is defined by $b_t P_t^m$.[4] Thus, the system of equations that describes the optimal decisions of individuals can be written as follows. The HJB equation is determined as, [5]

$$\begin{aligned} \rho V_j(b,t) = {} & U(c,l) + \partial_b V_j(b,t) \frac{1}{P_t^M} \left[\left[\left(1-\tau_t\right) v_t l_t + \Pi_t\right] z_t + \kappa_t \left(1-z_t\right)\right] \\ & + \partial_b V_j(b,t) \frac{1}{P_t^M} \left[\left(1 - \delta P_t^M - P_t^M \hat{\pi}_t\right) b_t + T_t - c_t\right] \\ & + \lambda_j \left(V_{-j}(b,t) - V_j(b,t)\right) + \partial_t V_j(b,t), \end{aligned} \tag{2.13}$$

where consumption and labour satisfy

$$\partial_c U(c,l) P_t^M = \partial_b V_j(b,t), \tag{2.14}$$

$$\partial_l U(c,l) P_t^M = -\left(1-\tau_t\right) w_t z_j \partial_b V_j(b,t). \tag{2.15}$$

[4]While all FOCs are derived in nominal terms, I numerically solve the system after it has been transformed into real terms.

[5]To shorten notation I use $\partial_x()$ to denote partial derivative $\frac{\partial}{\partial x}()$.

The KF equation is

$$\partial_t g_j(b,t) = -\partial_b\left[s_j(b,t)\,g_j(b,t)\right] - \lambda_j g_j(b,t) + \lambda_{-j} g_{-j}(b,t), \tag{2.16}$$

where the real-valued saving function is

$$s_j(b,t) = \frac{1}{P_t^M}\left[\left[(1-\tau_t)\,w_t l_t + \Pi_t\right] z_t + \kappa_t(1-z_t) - c + T_t + \left(1-\delta P_t^M - P_t^M\hat{\pi}_t\right) b_t\right], \tag{2.17}$$

and the household's budget constraint in real terms is given by

$$\dot{b}_t = \frac{1}{P_t^M}\left[\left(\left[(1-\tau_t)\,w_t l_t + \Pi_t\right] z_t + \kappa_t(1-z_t) + T_t - c_t\right) + \left(1-\delta P_t^M - P_t^M\pi_t\right) b_t\right]. \tag{2.18}$$

The budget constraint highlights that with positive inflation households have to raise the quantity of nominal bonds to keep the same real income from their bond holdings.

When I have transformed the decision problem into real terms, two distinct types of bond interest payment have emerged, the *ex-ante* interest payment – $\left(1-\delta P_t^M - P_t^M\hat{\pi}_t\right)$, with inflation featuring the right derivative of the price level (i.e. the expected rate of price change from the current time onward) and *ex-post* interest payment – $\left(1-\delta P_t^M - P_t^M\pi_t\right)$, with inflation featuring the left derivative of the price level (that follows the standard definition of inflation). One way of thinking about the operation of the FTPL is that inflation surprises create differences between *ex-ante* and *ex-post* real interest rates and through this mechanism government debt can be stabilized without fiscal backing. This is true in the considered model, although unexpected movements in bond prices can enhance these effects (see Leeper and Leith [2016] for a discussion of the impact of debt maturity on the mechanisms underpinning the FTPL) as bondholders experience unexpected gains/losses on their portfolios. In my sticky price model, there will also be wedges between ex-ante and ex-post real interest rates when monetary policy is active and fiscal policy passive, although these are likely to be of a different order of magnitude.[6] It is important to stress that, in contrast to representative-agent models, these effects will have different impacts across the wealth distribution in the heterogeneous agent economy. As a result, I need to consider how different households are affected by the different policy regimes in order to assess the desirability of either regime.

2.2.2 Firms

There are monopolistically competitive intermediate goods firms that produce differentiated goods subject to price adjustment cost *a la* Rotemberg [1982]. Competitive final good firms aggregate a continuum of differentiated goods in a single final good priced P_t. The firm's problem is standard and a similar continuous-time exposition can be found in Kaplan et al. [2018].

[6] See Sims [1999], Sims [2011] for the effects of disturbance created by the difference between ex-ante and ex-post interest rates.

Final good producers A competitive firm solves a profit maximization problem

$$\max_{y_{j,t}} P_t Y_t - \int_0^1 p_{j,t} y_{j,t} dj,$$

subject to the aggregating technology

$$Y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

here Y_t denotes final goods, $y_{j,t}$ denotes the j'th intermediate input, and $\varepsilon > 1$ governs the elasticity of substitution between any two intermediate inputs.

Profit maximization implies that demand for intermediate good j is

$$y_{j,t} = \left(\frac{p_{j,t}}{P_t} \right)^{-\varepsilon} Y_t, \text{ where } P_t = \left(\int_0^1 p_{j,t}^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}. \tag{2.19}$$

Intermediate good producers Each intermediate good j is produced by a monopolistically competitive producer using labour $n_{j,t}$ under technology A_t according to the linear production function $y_{j,t} = An_{j,t}$. The firm hires labour at a wage w_t in a competitive labour market. As a result, the cost-minimization problem of the firm implies that the marginal cost is common across all producers and given by

$$mc_t = \frac{w_t}{A_t}, \tag{2.20}$$

and the operational real profits of intermediate goods producers are

$$\tilde{\Pi}_{j,t} = \frac{p_{j,t}}{P_t} y_{j,t} - w_t n_{j,t} = \left(\frac{p_{j,t}}{P_t} - mc_t \right) y_{j,t} = \left(\frac{p_{j,t}}{P_t} - mc_t \right) \left(\frac{p_{j,t}}{P_t} \right)^{-\varepsilon} Y_t.$$

Firms are subject to a price adjustment cost which is quadratic in the rate of price change $\frac{\dot{p}_t}{p_t}$ and is expressed as a fraction of aggregate output Y_t,

$$\Theta \left(\frac{\dot{p}_{j,t}}{p_{j,t}} \right) = \frac{\theta}{2} \left(\frac{\dot{p}_{j,t}}{p_{j,t}} \right)^2 Y_t.$$

Each intermediate producer chooses its price to maximize profits subject to price adjustment costs,

$$\max_{p_{j,t}} \int_0^\infty e^{-\rho t} \left\{ \left(p_{j,t} - P_t mc_t \right) \left(\frac{p_{j,t}}{P_t} \right)^{-\varepsilon} Y_t - \frac{\theta}{2} \left(\frac{\dot{p}_{j,t}}{p_{j,t}} \right)^2 Y_t P_t \right\} dt.$$

The solution to this profit-maximizing problem is characterized by the Phillips curve equation[7],

$$\left(\rho - \frac{\dot{Y}_t}{Y_t}\right)\pi_t = \frac{\varepsilon}{\theta}\left(mc_t - \frac{\varepsilon - 1}{\varepsilon}\right) + \dot{\pi}_t. \tag{2.21}$$

In a symmetrical equilibrium ($p_{j,t} = p_t = P_t$), the demand for intermediate goods is,

$$y_{j,t} = \left(\frac{P_t}{P_t}\right)^{-\varepsilon} Y_t = Y_t,$$

labour demand is

$$n_{j,t} = n_t = N_t = \frac{Y_t}{A_t}, \tag{2.22}$$

and firm profits are given by,

$$\Pi_{j,t} = \Pi_t = Y_t - w_t N_t - \frac{\theta}{2}\pi_t^2 Y_t. \tag{2.23}$$

2.2.3 Government

The government budget constraint in real terms [8] is given by

$$\dot{B}_t = \frac{1}{-P_t^M}\left(\tau_t w_t N_t - G_t - T_t^T - \left(1 - \delta P_t^M - P_t^M \pi_t\right) B_t\right), \tag{2.24}$$

where τ_t is a rate of income tax coefficient, G_t is government spending which I set equal to a constant fraction of the output $G_t = GY_t$. The total transfers are the sum of unemployment benefit transfers, received only by the unemployed, and regular transfers received by all households,

$$T_t^T = T_t + \int_\Omega \kappa_t g_2(b,t)\, db = T_t + \kappa_t u_t.$$

Also, the government sets the income tax value τ_t according to the fiscal policy rule which I describe in Section 2.2.5.

2.2.4 Markets Clearing and Private Sector Equilibrium

Aggregate consumption is obtained by aggregating all the individual consumptions $c_j(b,t)$ using the distribution density function $g_j(b,t)$,

$$C_t = \sum_{j=1}^{2} \int_\Omega c_j(b,t) g_j(b,t) db. \tag{2.25}$$

[7] see Appendix A.2 for details.

[8] the budget constraint in nominal terms and its transformation into real terms are presented in the Appendix B.

while the aggregate labor supply, N_t^S, is equal to labor demand, so that labour market clears,

$$N_t = N_t^S = \sum_{j=1}^{2} \int_{\Omega} z_j l_j(b,t) g_j(b,t) db. \tag{2.26}$$

The bonds market clearing condition is,

$$B_t = B_t^H = \sum_{j=1}^{2} \int_{\Omega} b g_j(b,t) db. \tag{2.27}$$

Finally, aggregation of individual and government budget constraints, taking into account firms' profits and government transfers yields the resource constraint,

$$Y_t \left(1 - \frac{\theta}{2}\pi_t^2\right) - G_t = C_t. \tag{2.28}$$

The private sector equilibrium is determined by the system of equations, (2.13)-(2.17), (2.20)-(2.28), given boundary conditions (2.8)-(2.10), the choice of policy instruments $\{i_t, \tau_t\}$ and the stochastic process for the idiosyncratic shock z_t.

2.2.5 Policy Rules

The monetary authority sets nominal interest rate i_t on one-period bonds using the Taylor rule,

$$\frac{1+i_t}{1+i_{ss}} = \left(\frac{1+\pi_t}{1+\pi^T}\right)^{\phi}. \tag{2.29}$$

where π^T denotes a target inflation level and i_{ss} denotes steady-state interest rate.

A value of $\phi > 1$ implies that the rule satisfies the Taylor Principle whereby nominal interest rates are increased by more than excess inflation, such that real interest rates rise when inflation is above its target. This is the definition of an active monetary policy. The short-term interest rate is linked to the bond price P_t^M via the no-arbitrage condition

$$\dot{P}_t^m = P_t^m (\delta + i_t) - 1, \tag{2.30}$$

or, after integration,

$$P_t^M = \int_0^{\infty} e^{-\int_t^{t+s} i_{\xi} d\xi} e^{-\delta s} ds, \tag{2.31}$$

meaning that bond prices depend on the time path of the inverse of the nominal interest rate over the entire duration of debt and shows that the lower value of δ (which implies longer average maturity) the greater the impact of future nominal interest rates on the bond prices. Therefore, a prolonged increase in nominal interest rates will depress bond prices more if the maturity of bonds is higher.

The government sets the labour income tax rate according to the fiscal policy rule as a feedback to the debt value $P_t^M B_t$,

$$\tau_t = \left(\frac{P_t^M B_t}{P_{ss}^M B_{ss}}\right)^{\xi} \tau^T, \tag{2.32}$$

here τ^T denotes the value of tax imposed at the steady state, and the subscript ss denotes target levels, consistent with the stochastic steady state.

In the monetary dominance regime, to ensure the policy stabilizes debt, the parameter ξ must be sufficiently large to ensure that the primary balance rises by more than debt service costs whenever debt rises above its target. This defines a passive fiscal policy. A failure to achieve this threshold means that the fiscal policy is active and monetary policy must become passive to ensure a determinate equilibrium so that the fiscal dominance regime is established.

2.3 Important Benchmark Case, RANK

A representative agent version of our model (RANK) offers a convenient benchmark against which I can evaluate the effects of uninsurable unemployment risk. The key and only difference between the RANK and HANK models is the household's problem is that in the RANK version of our model, I allow households to pool resources and, thereby, effectively insure themselves against the consumption consequences of unemployment. Therefore, instead of a continuum of individuals with different wealth holdings, there is only one representative *household*, members of which are almost fully insured against unemployment shock. Here, unlike the HANK model, the members of a household share labor income $w_t l_t$ (it is subject to the same tax rate τ_t), profits Π_t and bond holdings b_t, and choose to consume the same amount. However, the disutility of providing labour is asymmetrical, and is faced only by employed individuals within the household, therefore, the disutility of the labour supply is normalized by the proportion of household members that are employed, $(1-u)$, where u is the unemployment rate.

Thus, a representative household chooses consumption c_t, labor l_t and wealth b_t to maximize the household's lifetime utility

$$\int_0^{\infty} e^{-\rho t} \left(\frac{c^{1-1/\sigma}}{1-1/\sigma} - \varphi (1-u) \frac{l^{1+\psi}}{1+\psi}\right) dt, \tag{2.33}$$

subject to the *nominal* budget constraint

$$\dot{b}_t^n = \frac{1}{P_t^M} \left[P_t \left[(1-u)(1-\tau_t) w_t l_t + T_t^T + \Pi_t - c_t\right] + b_t^n \left(1 - \delta P_t^M\right)\right]. \tag{2.34}$$

To solve this problem, I define a Hamiltonian,

$$
\begin{aligned}
L = \max_{c,b,l} \int_0^\infty e^{-\rho t} & \left(\frac{c^{1-1/\sigma}}{1-1/\sigma} - \varphi\left(1-u\right)\frac{l^{1+\psi}}{1+\psi} \right) \\
& + \lambda_t \left[\frac{1}{P_t^M} \left[P_t \left[(1-u)(1-\tau_t) w_t l_t + T_t^T + \Pi_t - c_t \right] + b_t^n \left(1 - \delta P_t^M\right) \right] - \dot{b}_t^n \right] dt.
\end{aligned} \tag{2.35}
$$

Optimizing with respect to b_t^n, c_t, l_t yields the following first-order conditions

$$
\left(\frac{1}{P_t^M} - \delta - \rho \right) + \frac{\hat{\lambda}_t}{\lambda_t} = 0, \tag{2.36}
$$

$$
c_t^{-1/\sigma} - \frac{\lambda_t P_t}{P_t^M} = 0, \tag{2.37}
$$

$$
-\varphi l_t^{\psi} + \frac{\lambda_t P_t}{P_t^M} (1-\tau_t) w_t = 0. \tag{2.38}
$$

Together with the household's budget constraint (2.34), these FOCs solve for optimal values of λ_t, c_t, l_t and b_t^n. Here, the choice of optimal consumption is inversely proportional to the value of the multiplier λ_t, while the choice of labour is proportional to the multiplier λ_t. This implies that the marginal rate of substitution between consumption and leisure is proportional to the after-tax wage rate.

I transform these equations into real terms and denote $\mu_t = \lambda_t P_t$ to obtain the household's optimal consumption/savings decision and labour supply condition,

$$
\left(\frac{1}{P_t^M} \left(1 - \delta P_t^M - \hat{\pi}_t P_t^M\right) - \rho \right) + \frac{\hat{\mu}_t}{\mu_t} = 0, \tag{2.39}
$$

$$
c_t^{-1/\sigma} - \frac{\mu_t}{P_t^M} = 0, \tag{2.40}
$$

$$
-\varphi l_t^{\psi} + \frac{\mu_t}{P_t^M} (1-\tau_t) w_t = 0. \tag{2.41}
$$

While the household budget constraint in real terms can be written as,

$$
\dot{b}_t = \frac{1}{P_t^M} \left[(1-u)(1-\tau_t) w_t l_t + T_t^T + \Pi_t + \left(1 - \delta P_t^M - P_t^M \pi_t\right) b_t - c_t \right]. \tag{2.42}
$$

Note that here, μ_t is a forward-looking variable and features a *right* derivative and inflation π_t is the right derivative of the price level. Moreover, equations (2.39) and (2.42) are dynamic and require boundary conditions. Equation (2.39) requires terminal condition $\lim_{t\to\infty} \mu_t = \mu_{ss}$, while equation (2.42) requires initial condition $b_0 = b_{ss}$.

Similarly to the HANK model, the budget constraint equation features ex-post interest payments $\left(1 - \delta P_t^M - P_t^M \pi_t\right)$, while the optimality conditions include $\left(1 - \delta P_t^M - P_t^M \hat{\pi}_t\right)$ – ex-ante interest payments. This implies that surprises surrounding households' anticipation of future inflation and interest rates (via the price of long-maturity bonds) will create a wedge between the ex-ante and ex-post returns that the household receives.

In the RANK model, the level of consumption is the same across all individuals, implying aggregate consumption to be the same as individual,

$$C_t = c_t,$$

labour is supplied only by employed members of the household, therefore the labour market clearing condition is,

$$N_t = (1-u)\, l_t.$$

and the bond market clearing condition remains,

$$B_t = b_t.$$

2.4 Calibration

The model is calibrated to a quarterly frequency with a discount factor $\rho = 0.01$. The calibration of most parameters is standard and follows Kaplan et al. [2018]. Specifically, I work with log-utility, setting $\sigma = 1$, while the inverse of Frisch elasticity of substitution is set at $\psi = 1$. The weight on disutility of labour, $\varphi = 2.2$. I set the elasticity of substitution between goods at $\varepsilon = 11$, implying a mark-up of 10% and the price adjustment cost parameter $\theta = 100$, consistent with a Calvo parameter implying prices are sticky for, on average four quarters. Parameter $\rho^B = 1 - \delta$ regulates debt maturity, it is set to 0.95 which generates an average maturity of about 5 years, consistent with the US data.[9]

The calibration of the labour market is essential for evaluating the effects of the idiosyncratic unemployment shock. Here, I interpret 'unemployment' as measuring the degree of economic inactivity. In the US this implies a rate for u of 26%.

Table 2.1 reports some important ratios, based on the US data from 2000-2018, which I use to calibrate the model[10].

There is no data on the average value of benefits that economically inactive individuals receive, however, using the value of transfers to GDP ratio I have calibrated κ_t to target the replacement ratio $\lambda_r = \kappa_t / (w_t l_t)$; I assume that economically inactive households on average receive 33.25% of their income during employment, this helps me match the other aggregate ratios contained in Table 2.1. There is also no data about the frequency of becoming economically inactive, therefore, I adopt the assumption that half of the individuals who are unemployed for a year become economically inactive, this way the implied value of a separation rate at quarterly frequency is ζ. These values

[9] A more detailed explanation of the average maturity calibration can be found in Section B.8.

[10] The data is taken from the OECD database (https://data.oecd.org).

Table 2.1: Calibrated Parameters

Labor market parameters		
Fraction of economically inactive people	u	26%
Job finding rate at quarterly frequency	λ_f	47%
Replacement ratio for economically inactive	λ_r	33.25%
Separation rate for economically inactive	ζ	2.74%
Policy parameters		
Government spending to GDP ratio	G/Y	5.6%
Government debt to GDP ratio	$P^M B/Y$	45.0%
Total tax revenues to GDP ratio	tax/Y	17.5%
Total transfers to GDP ratio	$transfers/Y$	12.5%
Policy feedback coefficients		
Active monetary policy feedback on inflation	ϕ_a	1.5
Passive monetary policy feedback on inflation	ϕ_p	0.5
Active fiscal policy feedback on debt	ξ_p	0.0
Passive fiscal policy feedback on debt	ξ_a	0.2

are used to calibrate the Poisson parameters q_1 and q_2.

The remaining parameters are fiscal policy parameters. Their values are computed as averages of the US fiscal policy parameters during 2000-2018.[11] I consider a range of borrowing constraints, ranging from a very tight constraint, which excludes borrowing, to a very loose one with the borrowing limit close to its natural level.

2.5 Numerical Results

2.5.1 Stochastic Steady State

Table 2.2 presents the main characteristics of the stochastic steady state. Within this steady state, there are no aggregate shocks, but a perpetual movement of individuals between the states of economically active and inactive. Within this steady state, the configuration of monetary and fiscal policy rules does not matter. Nevertheless, it is helpful to understand the features of the underlying stochastic steady-state before I turn to explore the implications of alternative policy regimes for the micro- and macro-economic response to aggregate shocks. Figure 2.1 illustrates the distributions of assets and consumption, allowing me to highlight the link between macroeconomic aggregate and individual asset distributions. Additionally, Figure 2.2 provides a visualization of the stochastic dynamics. In each case, I vary the tightness of the borrowing constraint.

Consider columns (1) and (2) in Table 2.2 which contrast the stochastic steady-state within the RANK model to that obtained under the HANK model with a tight borrowing limit of zero. With the ability to pool risk in the HANK model, there is a significant degree of precautionary savings behaviour which significantly impacts aggregate variables. Within the RANK model the

[11] The data is taken from the FRED database (https://fred.stlouisfed.org).

equilibrium real interest rate is consistent with the household's rate of time preference implying an annualized real interest rate of 4%. With heterogeneous agents and a tight borrowing constraint, this falls to 1.36%, without any change in individuals' time preferences. Employed workers in this economy wish to accumulate assets to protect themselves should they enter a state of inactivity. With such individuals chasing a finite stock of government debt, this raises the price of that debt, pushing down interest rates and discouraging the accumulation of assets which individuals would otherwise seek to achieve in the absence of a fall in returns. Most inactive individuals run out of assets fairly quickly and there is a mass on the borrowing constraint. These individuals use their entire income to finance consumption. To put it another way, with a tight borrowing limit individuals cannot protect themselves against a fall in consumption when inactive, and equilibrium interest rates reflect this fact. The lower interest rate means that the government can service a far higher stock of debt at the same tax rate.

As we move along the Table, the borrowing limit gets more relaxed. This enables individuals to smooth consumption far more effectively across the two states of activity, bringing the equilibrium interest rate closer to the household's rate of time preference. As a result average consumption, conditioned on the state of activity/inactivity is increasingly similar and that and other macroeconomic aggregates approach the values attained under the RANK model. Reducing the borrowing constraint improves welfare, although even with a relatively slack borrowing constraint welfare there is still a significant mass of people hitting the constraint and welfare is still well below that achieved with full risk sharing under the RANK economy.

The first three rows of Figure 2.1 show how household-level variables vary with the wealth of the household, while the final two rows display the densities of wealth and consumption. In all cases, I condition on activity/employment status with the solid-blue line denoting unemployed individuals and the dashed-red line, the employed. As we move across the columns the tightness of the borrowing constraint is relaxed. Consider the first row and column which shows how consumption varies with household wealth and employment status when borrowing is excluded. If the household has been lucky enough to accumulate a large enough stock of assets then they would be able to smooth consumption across the two states of employment. However, when asset levels fall, their ability to do so drops. As they approach the borrowing limit, there is a sizeable difference in consumption levels between the employed and unemployed. If we then look at the associated distribution of wealth in the fourth row, then we see that the distribution of wealth is concentrated very close to the borrowing limit. Individuals are discouraged from accumulating assets by the low returns on government bonds, and, as a result, when they enter a period of unemployment they very quickly use up their resources with the mass of unemployed individuals holding no wealth. This figure also shows that the unemployed are period-dissavers, while the employed are period-savers unless they achieve very high wealth levels. As a result, they move into dis-saving when their wealth levels are particularly high, implying that very few households reach this point, consistent

Table 2.2: RANK and HANK Characteristics of Stochastic Steady State

		RANK	HANK with borrowing limit			
Borrowing limit	$\underline{b}$	–	0.0	-0.1	-0.2	-1.0
		(1)	(2)	(3)	(4)	(5)
Consumption	C	0.4829	0.4485	0.4524	0.4564	0.4688
- unemployed	C_1	–	0.2798	0.3059	0.3325	0.4089
- employed	C_2	–	0.5078	0.5038	0.4999	0.4898
Output, labor	$Y = N_2$	0.5114	0.4873	0.4915	0.4959	0.5093
Assets	B	0.0562	0.1223	0.0824	0.0677	0.0554
- unemployed	B_1	–	0.0620	0.0047	-0.0280	-0.0922
- employed	B_2	–	0.1435	0.1096	0.1013	0.1073
Assets Value	$P^M B$	0.9198	2.2428	1.4585	1.1675	0.9163
- unemployed	$P^M B_1$	–	1.1370	0.0832	-0.4829	-1.5249
- employed	$P^M B_2$	–	2.6316	1.9400	1.7470	1.7746
Debt to GDP, %	$\frac{P^M B}{Y}$	44.99	115.08	74.14	58.86	45.00
Bond price	P^M	16.3672	18.3388	17.7007	17.2455	16.5391
Interest rate, %	i	1.00	0.34	0.54	0.69	0.94
Mass on BL	m	–	0.0666	0.0448	0.0276	0.0005
Ave. time on BL	τ_b	–	12.9152	12.8390	12.8364	12.7589
Ave. time to BL	τ_c	–	0	1	1	7.5364
Utility*	$\mathscr{U}$	-101.5883	-120.1217 [16.9%C]	-119.1119 [16.1%C]	-118.2114 [15.3%C]	-116.4899 [13.8%C]

Notes, * the number in brackets is the per cent of steady state consumption that a representative agent would be willing to sacrifice to eliminate the effect of idiosyncratic shocks.

with the patterns found for assets and consumption.[12]

As we move along the columns of Figure 2.1 the borrowing constraint gets more relaxed. This allows individuals to continue to smooth consumption across states of employment at lower levels of wealth, although eventually the constraint bites and differences in consumption levels across employment status widen. Since the equilibrium interest rate is far closer to the households' rate of time preference as the borrowing constraint is relaxed, more employed individuals are able to accumulate wealth and we start to see some employed households consuming significantly higher amounts than others. Overall, when the borrowing limit is relaxed, the average labour supply and level of consumption increase, driven primarily by the unemployed, while the average consumption of the employed decreases. Therefore, relaxing the borrowing constraint reduces consumption inequality but increases income inequality, with benefit income remaining the same but labour supply and thus earnings rising.

While I could infer much of the dynamic response to shocks to employment status from Figure 2.1, it is helpful to confirm that intuition by formally representing the typical dynamics of wealth, consumption, and income for an individual starting with average wealth and encountering a series

[12]The density functions are zero below the borrowing limit and above the point where the saving of the employed becomes zero, $s_2 = 0$. See Achdou et al. [2017] for discussion.

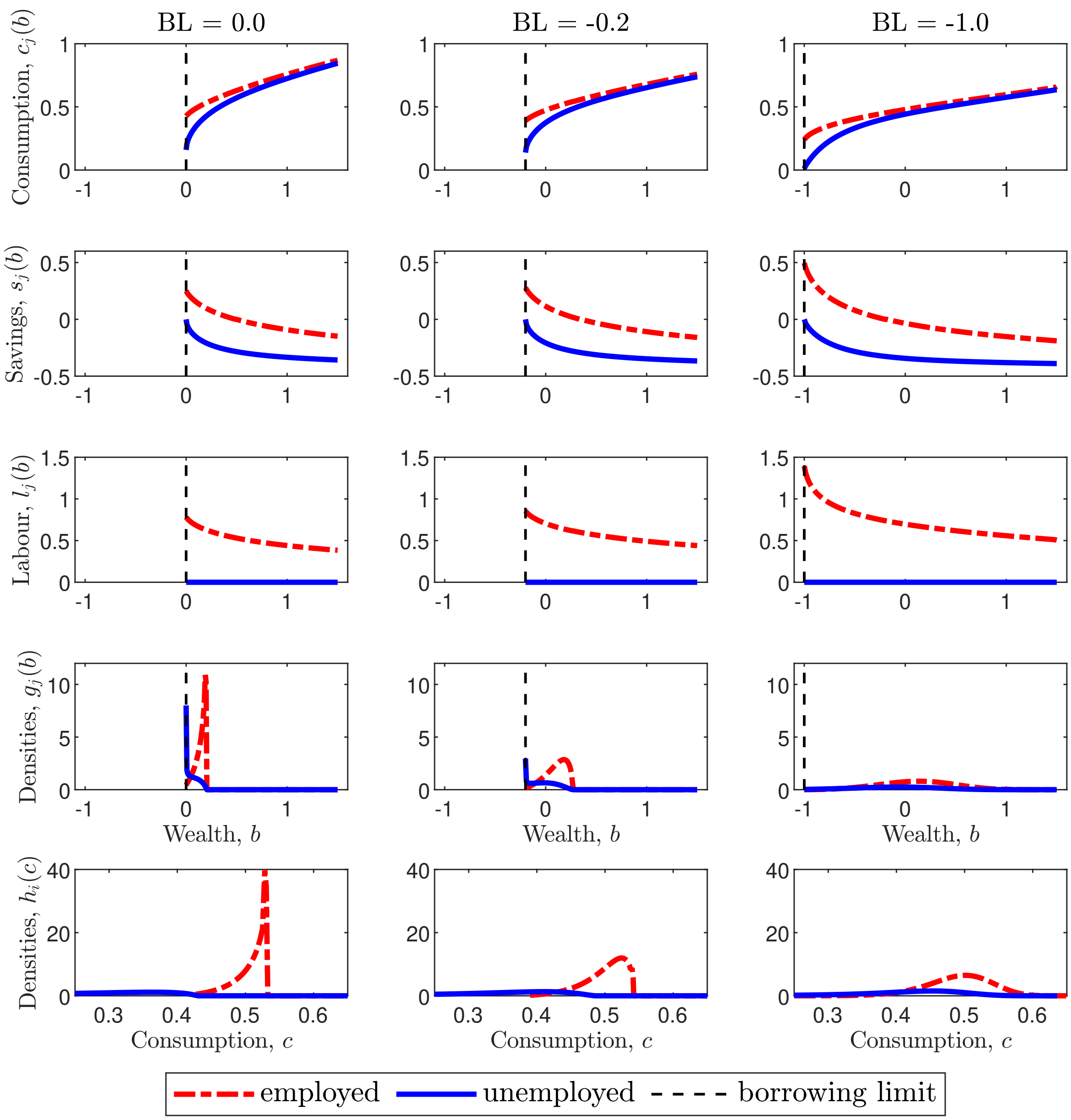

Figure 2.1: Distributions in stochastic steady state

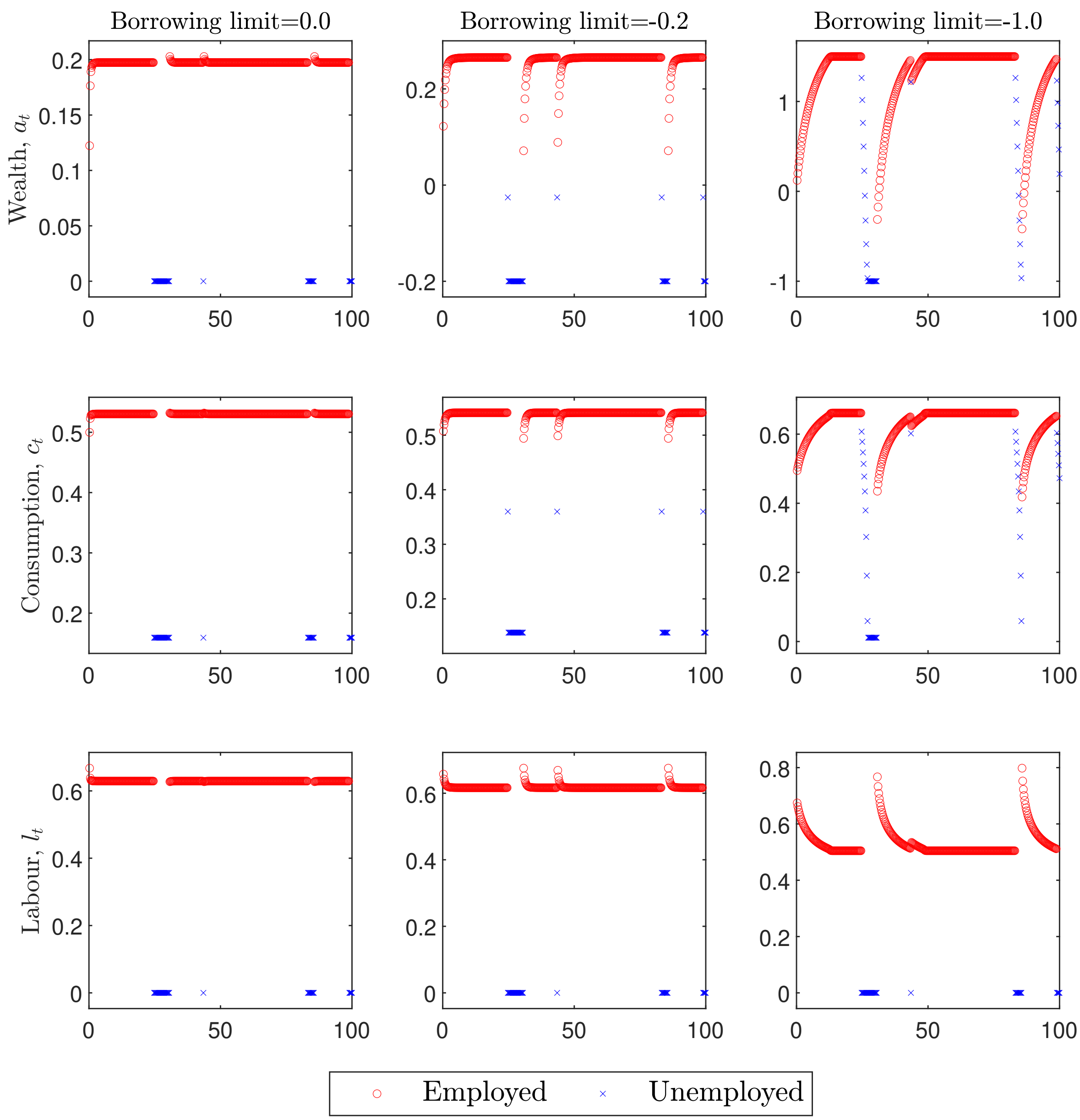

Figure 2.2: Dynamics in stochastic steady state

of idiosyncratic shocks. This is what I do in Figure 2.2. As before, I progressively relax the borrowing constraint as we move across the columns.

Starting with the first column, I am considering a very tight borrowing limit which prohibits borrowing. As shown in Table 2.2, in this case, the time to reach the borrowing limit upon entering the state of unemployment is zero. In other words, upon being made unemployed any resources you have are almost instantaneously consumed and the unemployed individual hits the borrowing constraint. As a result individuals essentially jump between wealth, consumption and labour supply levels implied by each state with little heterogeneity within the population of each group.

As I relax the borrowing constraint, resource depletion in the state of unemployment remains rapid, although is not quite as aggressive as it was before. Similarly, upon exiting unemployment there is a more gradual accumulation of wealth, and individuals take longer to reach their steady-state stock of assets than they do to run them down when faced with a negative shock. This is further enhanced when I relax the borrowing constraint more, and the time taken to hit the borrowing limit is significantly longer, and the time to reach the desired level of assets when in employment is still longer. This desire to gradually accumulate assets upon exiting the unemployed state leads to a period of subdued consumption and a greater willingness to supply labour to accumulate savings. Not all households will remain in a state of employment for long enough to reach this desired level of wealth. As a result, the distribution of consumption becomes more dispersed but with a higher mean, as illustrated in Figure 2.1.

Finally, I compute a measure of social welfare using the following formula,

$$\mathscr{U}_t = \sum_{j=1}^{2} \int_{\Omega} \int_{t}^{\infty} e^{-\rho s} U_j(c(b,s), l(b,s)) g_j(b,s)\, ds db, \tag{2.43}$$

which captures the distribution and fluctuations over time in individual levels of consumption and labour supply. I present levels and consumption equivalents of the difference between the stochastic steady state in the HANK and RANK models. The quantitative difference is substantial, see Table 2.2. It can be attributed to both, heterogeneity and the presence of the borrowing constraint. The rough computations show that these losses are of a different order, the presence of heterogeneity reduces utility by about 10% in consumption units while tightening a borrowing constraint by about 0.1 generates a loss in the order of 0.5% in consumption equivalent.

2.5.2 Monetary and Fiscal Policy Interactions

Having discussed the properties of the stochastic state of the model as a means of highlighting the underlying distributional consequences of uninsured movements between employed and unemployed, I turn to assess how distributional factors affect the operation of alternative monetary and fiscal policy regimes, as they respond to aggregate shocks. As noted above I am interested in two regimes. Firstly, I wish to consider the conventional policy assignment where monetary policy is actively targeting inflation by raising interest rates in response to excess inflation sufficiently aggressively to raise real interest rates and dampen aggregate demand. The implications for debt service costs of such a policy would lead to unstable debt dynamics unless the fiscal authority supports monetary policy by adopting a passive fiscal policy which adjusts fiscal instruments (in this case taxation) in order to ensure fiscal solvency. This is the Active MP-Passive FP regime. The alternative policy regime, stressed by Leeper [1991] and various authors in expositions of the FTPL literature, instead assume that the government fails to adjust fiscal instruments in order to stabilize debt – the fiscal policy is 'active'. This would destabilize the economy unless the monetary authority drops its active pursuit of inflation and moderates its policy so that a combination of inflation surprises and reduced debt service costs can ensure fiscal solvency in the face of a negative shock to the government's finances. This is the Passive MP-Active FP regime.

It has been decided to apply a cost-push shock to analyse the off steady-state dynamics and welfare implications of the model. This is because the literature that studies the time dynamics of the New Keynesian models or explores the properties of the FTPL regime very often employs a monetary policy or a cost-push shock (e.g., see Cochrane [2011], Kirsanova and Wren-Lewis [2012], Leeper and Leith [2016], Kaplan et al. [2018], Nuno and Thomas [2020], González et al. [2022]). This is considered to be a very convenient setting as these types of shocks have direct effects on the inflation level and on the interest rate, therefore, it is easy to interpret the results, such as a jump in inflation level, change in interest rate and subsequent change in the demand for assets. As a result, I can predict and analyze the convergence of the outstanding debt, its implication for the goods market and private agents' choices in a transparent way. The inclusion of the Phillips curve to the model and frequently observed changes in the elasticity of substitution between differentiated goods has motivated me to choose a cost-push shock among these two types of shocks.

Figure 2.3 plots dynamic responses to a one-off surprise mean-reverting reduction in the elasticity of substitution between differentiated goods, which can be interpreted as a positive cost-push shock. More specifically, I assume a one-time, unanticipated reduction of 30%[13] in ε_t, after which it returns gradually to its steady-state value $\varepsilon_{ss} = 11$ with persistence rate $\eta_\varepsilon = 0.8$ according to

$$d\varepsilon_t = \eta_\varepsilon(\varepsilon_{ss} - \varepsilon_t)dt. \tag{2.44}$$

I consider two representative cases, AM-PF (Active Monetary Policy, Passive Fiscal Policy) and PM-AF (Passive Monetary Policy, Active Fiscal Policy). The policy coefficients for each case are provided in Table 2.1. I consider the case with a relatively tight borrowing limit $\underline{b} = -0.2$. The responses under the RANK model are given by the solid blue line and those under the HANK model are the dashed-red line. Under the conventional policy mix, adopted in the LHS column, the cost-push shock raises inflation. The monetary authority responds by raising interest rates by more

[13]The size of the shock is calibrated to generate inflation of about half per cent.

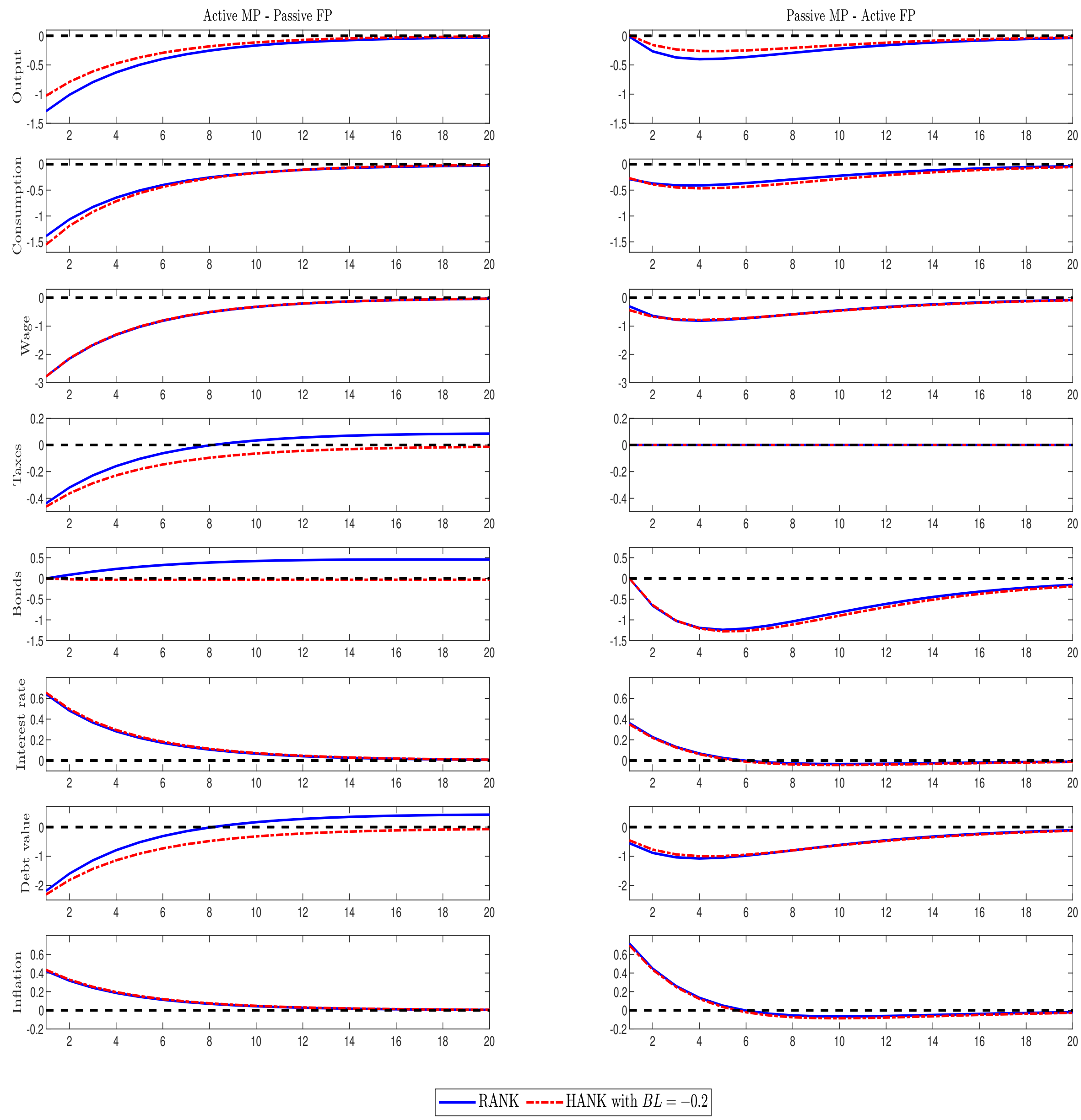

Figure 2.3: Impulse responses to a cost-push shock

Table 2.3: RANK and HANK Social Welfare Results

	steady state	active MP-passive FP	passive MP-active FP
RANK	-101.5883	-101.9396	-101.9305
		$[-0.3506\ \%C]$	$[-0.3416\ \%C]$
HANK, $BL = -0.2$	-116.4899	-116.8698	-116.8660
		$[-0.3791\ \%C]$	$[-0.3753\ \%C]$

Note: the numbers in square brackets denote consumption equivalents of the welfare loss compared to the steady state in the HANK and RANK models.

than the rise in inflation so that real interest rates rise. Normally, in the presence of single-period debt, I would expect taxes to rise to offset the debt service costs this would imply. However, with a plausible debt maturity structure, the sustained rise in nominal interest rates leads to a sharp fall in bond prices. This surprise capital loss amounts to a redistribution of wealth from bondholders to the government to such an extent that the real value of the government debt falls despite the ongoing recession reducing the size of the tax base. The fiscal authority then cuts taxes which in combination with higher debt service costs and a smaller tax base return debt to its steady-state value.

When I consider the same policy combination but in the heterogeneous-agent economy, there are significant differences. The stock of debt is higher in the heterogeneous agent economy, as lower interest rates allow the same primary surplus to sustain a higher stock of debt and the unemployed are typically borrowers while the employed have accumulated positive wealth. Therefore, the same shock to interest rates and bond prices has a greater impact on government debt and household wealth in the HANK economy. Specifically, there is a redistribution of wealth from the wealthy employed to both the government and debtors. In combination with the higher interest rate, it discourages employed households' consumption as they increase their precautionary savings and their labour supply (which is further encouraged by a reduction in distortionary labour income tax rates), explaining the patterns in output and consumption relative to the RANK case.

When I turn to the passive-monetary and active-fiscal policy combination under the RANK model, the inflation response to the same cost-push shock is greater, but the passive monetary policy implies a more modest monetary policy response such that the fall in bond prices is muted. As a result, we do not see such a decline in the value of government debt and the decrease in consumption and output are, correspondingly, more modest. Turning to the heterogeneous agent economy, the fact that the revaluation of bonds is less than in the conventional policy regime means that the distributional consequences are smaller. Therefore, the differences between paths of output and consumption across the RANK and HANK economies have the same sign as before but are smaller in magnitude.

Moving on to the welfare implications of the obtained results, I compute utility along the transition path using the formula (2.43) and report the obtained results in Table 2.3. From this table we can see that an unanticipated 30% reduction in the elasticity of substitution between differentiated goods results in a loss of approximately 0.3-0.4% in consumption equivalent relative to the corresponding steady state. However, both the RANK and HANK models have a superior social welfare outcome for the monetary dominance policy regime compared to the fiscal dominance regime. This result is contradictory to the standard RANK model outcome, for which it is well known that the welfare loss of macroeconomic volatility in the PM-AF policy regime exceeds that of the AM-PF regime (see, e.g. Schmitt-Grohé and Uribe [2007], Cochrane [2011], Kirsanova and Wren-Lewis [2012]).

We observe this welfare ranking of the policy regimes due to a few factors that increase inflation's importance as a debt stabilizer and they apply to both, RANK and HANK models. These factors include the absence of aggregate uncertainty, the maturity structure of the debt, the costliness of inflation variability and the level of outstanding government debt (Leeper and Leith [2016] discuss the effects of those factors in their work). In particular, the calibration of my model features an average maturity of about 4.5 years which is much higher than that of single-period or even one-year period bonds. Moreover, the nominal debt, being above 45% also contributes to the "fiscal cushion"[14]. Furthermore, in the case of a stochastic shock, an active monetary policy supported by passive fiscal policy is often considered to be beneficial; this is because a negative shock combined with uncertainty causes private agents to suppress their spending, so a more prolonged period of higher inflation is expected. This produces a sizeable debt stabilization bias that drives policymakers to reduce debt levels rapidly, at a larger cost in terms of social welfare, to avoid the high equilibrium rates of inflation. In my model, the shock is deterministic and therefore this effect is reversed. As a result, all these factors together with costly inflation lead to the regime of fiscal dominance being superior to the regime of monetary dominance in terms of social welfare.

Interestingly, both – the AM-PF and PM-AF policy regimes – deliver similar levels of welfare, with PM-AF being slightly better: in the RANK model, the fiscal dominance regime generates better welfare by 9.0e-3 per cent in consumption equivalent, while in the HANK model, we observe the difference between the two regimes of 3.8e-3 per cent. This difference in welfare values is attributed to inflation response, and its ability to stabilize debt and redistribute wealth. Higher inflation transfers wealth from lenders and government to debtors, reducing wealth inequality, however, an increase in the utility of debtors does not fully cover the decreased welfare of lenders, as a result, the overall utility and thus social welfare gets smaller, making the welfare losses of the two policy regimes very similar. This observation is coherent with the finding from the previous chapter where the reduced variance of the productivity shock decreases the income of high-productive households by more than the extra benefit of low-productive households, in other words, the implied reduced inequality does not necessarily lead to higher social welfare.

In addition to the comparison of the HANK and RANK model dynamics in transition, I have conducted the analysis of the HANK model with a relaxed borrowing limit (the impulse responses Figure B.1 and the welfare reporting Table B.1 can be found in Appendix B.7). I have applied the five times more relaxed borrowing limit: $\underline{b} = -1.0$ instead of $\underline{b} = -0.2$ and have found that the

[14] the term is introduced in Sims [2013] paper.

HANK model with a relaxed borrowing constraint generally demonstrates the same patterns as the HANK model with a tighter borrowing constraint. However, the behaviour of interest rate, debt value, bonds quantity and, as a result, taxes are more reminiscent of those in the RANK model. This is not a surprising result, as a relaxed borrowing limit makes it much easier to access assets, so the financial markets are capable of operating almost at the same capacity level as they would under the RANK model. Regarding the social welfare outcome, the lowered borrowing limit implies the same ranking of the AM-PF and PM-AF policy regimes, however, the difference between these two regimes is even smaller now. This is because more open access to financial markets reduces the households' need for inflation wealth redistribution, so the regime of fiscal dominance brings slightly less welfare gains and thus it becomes closer to the monetary dominance regime.

2.6 Conclusion

I explored the individual- and aggregate-level behaviour contained within a heterogeneous agent economy subject to uninsurable idiosyncratic employment risk. Aside from the heterogeneous agent elements, the model also includes sticky prices, monetary policy, endogenous labour supply, distortionary taxation and government debt which is both nominal and of long-maturity in line with observed practice.

This framework enabled me to analyze the properties of stochastic steady state and the effect of a relaxed borrowing limit. Moreover, postulating and solving the household's problem in nominal terms allowed me to obtain the second stable equilibrium related to the FTPL policy regime. This policy regime has not been explored in the heterogeneous-agent literature, as in addition to the computational complexity of solving the heterogeneous-agent models, the researcher would have to deal with the stability sensitivity of this equilibrium. Thus, my goal was to develop a stable numerical algorithm that would enable me to analyze the dynamics and welfare properties of the FTPL policy regime in the heterogeneous-agent environment.

Comparing the commonly used policy regime of monetary dominance with the policy regime of fiscal dominance, I have found that the two policy regimes generate qualitatively very similar results, which is attributed to several factors of my model, such as the absence of aggregate uncertainty, debt maturity and a high level of outstanding debt. However, the borrowing limit affects the private agents' decision making and inflation has an additional wealth redistribution channel within the heterogeneous-agent framework, which resulted in welfare difference between the two policy regimes of the HANK model being much smaller compared to the RANK model. A similar result has been obtained for the HANK model with a relaxed borrowing limit, however, open access to financial markets made the debt value and interest rate responses to a cost-push shock more similar to those of the RANK model. These findings are interesting and suggest that the optimal policy design is likely to have properties of both policy regimes.

Optimal Monetary Policy in HANK Model with Government Bonds

3.1 Introduction

There has been increasing attention to the distributional effects of monetary policy in both policy discussions and academic research. With the aid of new computational techniques, researchers have analyzed the transmission of monetary policy in macroeconomic models that incorporate diverse household characteristics. The focus of this literature has primarily been on positive inquiries, such as examining the various channels through which monetary policy affects redistribution and its overall impact (e.g. Auclert [2019], Kaplan et al. [2018]).

Recently, computational advancements have brought models with heterogeneous agents to the forefront of academic research and policy discussions. These models feature realistic representations of household heterogeneity and robust micro-foundations, making them suitable for policy analysis. Notably, a central topic of current discussion pertains to the distributive nature of policies. For instance, studies by Auclert [2019], Kaplan et al. [2018], Nuno and Thomas [2020] have explored the different redistributive channels of monetary policy and their implications for aggregate outcomes. However, because solving for optimal policy involves the state variable of higher complexity - endogenously-evolving wealth distribution, little work has been done on the analysis of the optimal policy setting in the HA framework: Nuno and Thomas [2020] and Bhandari et al. [2021] consider the incomplete markets environment to simplify the dimensionality of the problem while Le Grand et al. [2020] consider the truncation of idiosyncratic histories. Thus, in this chapter, I demonstrate the approach of solving for optimal commitment policy in the relatively standard New Keynesian framework with complete markets and analyse the numerical properties of the steady state.

My benchmark model is in the spirit of Huggett [1993] and Aiyagari [1994] and, more specifically, it follows Kaplan et al. [2018] who develops a heterogeneous agent model in continuous time with real bonds. I extend this model to include long-maturity nominal government bonds and distortionary taxation. In this framework, I analyze optimal monetary policy setting under commitment assumption. To address the challenge of solving for optimal policy in such models, I employ a variational approach inspired by Nuno and Thomas [2020], which incorporates the concept of Geautaux derivatives of infinite-dimensional spaces. This novel approach enables me to derive analytical first-order conditions for the optimal policy under commitment. Such an approach provides an analytical characterization of optimal monetary policy, where the factors of aggregate wealth and wealth dispersion as well as marginal consumption properties define optimal inflation value. Specifically, the optimality condition for inflation reveals how the disutility costs of inflation are traded off against its benefits by the policy-setting central bank.

Notably, my findings align with those of Nuno and Thomas [2020]: the heterogeneous-agent model generates an inflationary bias with redistributive effects, distinct from the inflationary bias observed in classical New Keynesian literature. Given that debtors exhibit lower consumption and a higher marginal utility of consumption compared to creditors, the central bank has an incentive to inflate and decrease the initial price of long-term nominal bonds, effectively redistributing resources from creditors to debtors. The extent of this unconventional inflationary bias depends on the stringency of the government budget constraint and income dispersion. Even when considering a net nominal asset position, the optimal inflation will still be positive in the HANK model as the wealth redistribution would be able to increase the social welfare (when consumption is redistributed from the lenders to debtors, the latter achieve higher welfare gains than losses of lenders). This result is distinct from the RANK model where the assumptions of the model lead to a zero-level optimal inflation at the steady state.

However, the use of a more complex New Keynesian model environment, which features complete markets, endogenous labour supply and sticky prices has led to a significant discrepancy in the inflation optimality condition compared with the one in Nuno and Thomas [2020] paper. First of all, in my model, inflation affects private agents' decisions via implied price adjustment cost, not via inflation disutility - this is directly reflected in the inflation optimality condition. Secondly, a complete financial market implies that the tightness of budget constraint has a direct effect on optimal inflation as the Lagrange multiplier associated with the government budget constraint enters the inflation optimality condition. Thirdly, the market clearing conditions imply deviation of the Lagrange multiplier associated with the KF equation from the value function of the HJB equation.

All these innovations of my model have also led to the values of other Lagrange multipliers to be substantially distinct from the ones of the Nuno and Thomas model, for instance, they obtain that the HJB equation is slack, while in my model this is true only for the initial time period. Thus, the analysed optimal policy in this chapter has inherited most of the properties from the Nuno and Thomas [2020] paper but, also, due to a more complex model setting, the optimal policy is now influenced by more factors, such as stringency of the government budget constraint, inflation price adjustment cost, labour market tightness and other indirect effects that these factors cause.

3.2 The HANK Model

I consider a continuous-time Heterogeneous Agent New Keynesian (HANK) model with long-term government bonds, where the source of heterogeneity is an unemployment shock that follows the two-state Poisson process. The model inherited the standard structure from the New Keynesian literature and includes five types of agents:

1. Households
2. Intermediate good producers (are subject to nominal rigidity)
3. Final good producer (gains no profit)
4. Government
5. Monetary Authority

The interactions between these agents occur within the labour, bonds and goods markets.

3.2.1 Households

The economy is populated by a continuum of households indexed by their holdings of government bonds b^{nom} and idiosyncratic unemployment shock z. Individuals have standard preferences over utility from consumption c_t and disutility from supplying labour l_t and face a time discount factor $\rho \geq 0$. So, their objective is to maximise the following lifetime utility

$$E_0 \int_0^\infty e^{-\rho t} u(c_t, l_t)\, dt \tag{3.1}$$

where the expectation is taken over realizations of idiosyncratic unemployment shocks. The function u is strictly increasing and concave in consumption, and strictly decreasing and strictly convex in labour.

The household bond portfolio consists entirely of maturing bonds b_t^{nom}, for which I assume that the maturity structure is declining at a *constant* rate δ. In this case, the average maturity of the portfolio is $-1/\log(\beta(1-\delta))$. By using such a simple maturity structure only a single bond should be priced since any existing bond issued s periods ago is worth $e^{-\delta s}$ of new bonds. Thus, the households can purchase $\dot{b}_t^{nom}$ of new bonds for the price P_t^M up to an exogenous limit $P_t \underline{b}$ and receive $\left(1-\delta P_t^M\right)$ return on bonds holdings.[1]

Each employed individual receives labour income $w_t l_t$ taxed at rate τ_t and a fraction of the firms' profit Π_t; each unemployed individual receives benefit payments ben_t; on top of that each

[1] More details about general bonds maturity structure and derivations of the final form of the budget constraint can be found in Appendix B.

individual receives government transfers Tr_t and interest payments on bonds holdings; this income stream can be spent on consumption or purchasing more nominal bonds. Therefore, household's *nominal* budget constraint (HBC) is given by

$$P_t^M \dot{b}_t^{nom} = P_t \left(\left[(1-\tau_t) w_t l_t + \Pi_t \right] z_t + \left[ben_t \right] (1-z_t) + Tr_t - c_t \right) + \left(1 - \delta P_t^M \right) b_t^{nom} \tag{3.2}$$

Properties of the unemployment shock

1) Unemployment shock follows the two-state Poisson process: $z_t \in \{0,1\}$;

2) The probability of becoming unemployed (i.e. jump from state 0 to 1) is λ_1 and the probability of finding a job (i.e. jump from state 1 to 0) is λ_2.

Dividing both sides of HBC equation by P_t^M, I derive the following form of the household's budget constraint

$$\dot{b}_t^{nom} = \frac{1}{P_t^M} \left[P_t \left(\left[(1-\tau_t) w_t l_t + \Pi_t \right] z_t + \left[ben_t \right] (1-z_t) + Tr_t - c_t \right) + \left(1 - \delta P_t^M \right) b_t^{nom} \right] \tag{3.3}$$

Thus, individuals maximize (3.1) given wealth b^{nom} subject to the budget constraint (3.3), borrowing limit and the process for z_t to find the optimal level of consumption and labour.

Solution to the household's problem in transitionary case in nominal terms

To define the transitionary equilibrium, I introduce two functions: $h_j(b^{nom},t),\ j = 1,2$ – the joint distribution of employment status z_j and wealth b^{nom}; and $w_j(b^{nom},t)$ – the HJB value function that also uses employment status z_j and wealth b^{nom} as state variables. The evolution of the joint distribution and value function is determined from the two differential equations: a Hamilton-Jacobi-Bellman (HJB) equation and a Kolmogorov Forward (or Fokker-Planck) equation.

Thus the solution to (3.1)-(3.3) can be expressed as a system of equations: HJB equation, first-order conditions on labour and consumption and KF equation supported by the state constraint boundary condition. The HJB equation is presented below

$$\begin{aligned}
\rho v_j (b^{nom},t) = & \max_{c,l} u(c,l) \\
& + \frac{\partial}{\partial b_{nom}} w_j (b^{nom},t) \frac{1}{P_t^M} \left[P_t \left(\left[(1-\tau_t) w_t l_t + \Pi_t \right] z_t + \left[ben_t \right] (1-z_t) + Tr_t - c_t \right) \right] \\
& + \frac{\partial}{\partial b_{nom}} w_j (b^{nom},t) \frac{1}{P_t^M} \left[\left(1 - \delta P_t^M \right) b_t^{nom} \right] \\
& + \lambda_j \left(w_{-j} (b^{nom},t) - w_j (b^{nom},t) \right) + \partial_t w_j (b^{nom},t)
\end{aligned} \tag{3.4}$$

The FOCs for c and l are given by;

$$c_j\left(b^{nom},t\right) \text{ is s.t. } \partial_c u\left(c,l\right) P_t^M = P_t \frac{\partial}{\partial b_{nom}} w_j\left(b^{nom},t\right) \tag{3.5}$$

$$l_j\left(b^{nom},t\right) \text{ is s.t. } \partial_l u\left(c,l\right) P_t^M = -\left(1-\tau_t\right) w_t z_j P_t \frac{\partial}{\partial b_{nom}} w_j\left(b^{nom},t\right) \tag{3.6}$$

These equations imply that household consumption rises as government bond prices increase, and declines with the steepness of the value function. In simple terms, when bond prices are higher (or yields are lower), households are motivated to save less and consume more. Conversely, a steeper value function makes saving more appealing to accumulate greater net bond holdings. The household choice of labour has exactly opposite properties where lower taxes and higher wages encourage individuals to work more.

The Kolmogorov-Forward equation is

$$\partial_t h_j\left(b^{nom},t\right) = -\frac{\partial}{\partial b_{nom}}\left[s_j^{nom}\left(b^{nom},t\right) h_j\left(b^{nom},t\right)\right] - \lambda_j h_j\left(b^{nom},t\right) + \lambda_{-j} h_{-j}\left(b^{nom},t\right) \tag{3.7}$$

here $c_j\left(b^{nom},t\right)$ and $l_j\left(b^{nom},t\right)$ are found from (3.12) and (3.13), and

$$s_j^{nom}\left(b^{nom},t\right) = \frac{1}{P_t^M}\left[P_t\left(\left[\left(1-\tau_t\right) w_t l_t + \Pi_t\right] z_j + ben_t z_{-j} - c + Tr_t\right) + \left(1 - \delta P_t^M - P_t^M \hat{\pi}_t\right) b_t^{nom}\right] \tag{3.8}$$

denotes savings function.

Both HJB and KF equations are time-varying equations, the first one is backward-looking while the second one is forward-looking and, therefore, they require terminal and initial conditions correspondingly:

$$w_j\left(b^{nom},T\right) = w_{j,\infty}\left(b^{nom}\right) \tag{3.9}$$

$$h_j\left(b^{nom},0\right) = h_{j,0}\left(b^{nom}\right) \tag{3.10}$$

Solution to the household's problem in transitionary case in real terms

To simplify the household's solution and reduce the number of the state variables, I introduce a new variable $b_t = \frac{b_t^{nom}}{P_t}$, it denotes the quantity of the nominal bonds per unit of currency, (this new variable allows to combine two state variables b_t^{nom} and P_t in one variable b_t, reducing the dimensionality of the problem).

Making the necessary transformations, the HJB, FOCs and KF equations are now rewritten as

$$\begin{aligned}\rho v_j(b,t) = & \max_{c,l} u(c,l) \\ & + \partial_b v_j(b,t)\frac{1}{P_t^M}\left[\left[(1-\tau_t)v_t l_t + \Pi_t\right]z_t + \left[ben_t\right](1-z_t)\right] \\ & + \partial_b v_j(b,t)\frac{1}{P_t^M}\left[\left(1-\delta P_t^M - P_t^M\hat{\pi}_t\right)b_t + Tr_t - c_t\right] \\ & + \lambda_j\left(v_{-j}(b,t) - v_j(b,t)\right) + \partial_t v_j(b,t) \end{aligned} \tag{3.11}$$

$$c_j(b,t) \text{ is s.t. } \partial_c u(c,l)P_t^M = \partial_b v_j(b,t) \tag{3.12}$$

$$l_j(b,t) \text{ is s.t. } \partial_l u(c,l)P_t^M = -(1-\tau_t)v_t z_j \partial_b v_j(b,t) \tag{3.13}$$

$$\partial_t g_j(b,t) = -\partial_b\left[s_j(b,t)g_j(b,t)\right] - \lambda_j g_j(b,t) + \lambda_{-j}g_{-j}(b,t) \tag{3.14}$$

$$s_j(b,t) = \frac{1}{P_t^M}\left[\left[(1-\tau_t)w_t l_t + \Pi_t\right]z_t + \left[ben_t\right](1-z_t) - c + Tr_t + \left(1-\delta P_t^M - P_t^M\hat{\pi}_t\right)b\right] \tag{3.15}$$

with the household's budget constraint in *real* terms given by

$$\dot{b}_t = \frac{1}{P_t^M}\left[\left(\left[(1-\tau_t)w_t l_t + \Pi_t\right]z_t + \left[ben_t\right](1-z_t) + Tr_t - c_t\right) + \left(1-\delta P_t^M - P_t^M\pi_t\right)b\right] \tag{3.16}$$

Note that here I have introduced the new value function $v_j(b,t)$ and probability distribution $g_j(b,t)$ which now depend on b instead of b^{nom}.[2] According to this new household's budget constraint in real terms, when inflation becomes positive, households have to raise the quantity of nominal bonds to keep the same real income from their bond holdings.

When I have rewritten the solution in nominal terms to the solution in real terms, the two types of bond interest payment have emerged: *ex-ante* interest payment $\left(1-\delta P_t^M - P_t^M\hat{\pi}_t\right)$ with inflation featuring the right derivative of the price level (i.e. the expected rate of price change from the current time onward) and *ex-post* interest payment $\left(1-\delta P_t^M - P_t^M\pi_t\right)$ with inflation featuring the left derivative of the price level (the standard definition of inflation). As a result, the individual's budget constraint adjusts to the current value of inflation while the individual's optimal choice of labour and consumption relies on expectations about future inflation.

This difference in the interest payments is not impactful for the monetary dominance regime but is essential for the fiscal theory of the price level (FTPL). The later regime requires inflation to respond in a debt-stabilizing manner. So, due to HBC featuring backward-looking inflation, the current price level can respond immediately to adjust the market value of debt after the shock occurs, at the same time, this surprise inflation at the initial period does not directly impact the

[2]More details on the definition of these functions and derivation of these equations can be found in Appendix B.

household's optimal decision allowing the real economy to be more stable. Although nominal rigidities generate a more sustained rise in inflation, they also allow fiscal policy to have more real effects on the economy,

Another debt stabilizing channel emerges due to the debt maturity structure, even if the overall size of the debt is the same, its maturity composition affects private wealth and is crucial for the fiscal dominance regime. Bond prices reflect the entire expected inflation path and depend on the bond's average maturity with higher bond maturity having more impact on the bond price level and, as a result, fiscal disturbances have a smaller impact on the debt. Such composition of the bond prices allows monetary policy (via future inflation path) to have more impact on the current value of debt and be more capable of stabilizing it. Overall, adding maturity to the model reduces the initial jump in inflation (as now both inflation and bond prices can affect the current value of the debt stock) but makes inflation dynamics more persistent.

The borrowing constraint

Instead of the borrowing constraint $b_t \geq \underline{b}$, the state-constraint boundary condition is applied

$$\partial_b w_j(\underline{b},t) \geq \partial_c u\left(c_j(\underline{b},t), l_j(\underline{b},t)\right) P_t^M \tag{3.17}$$

This is the appropriate boundary condition because the first-order condition still holds at $b = \underline{b}$ (in the continuous-time framework, the borrowing constraint never binds in the interior of the state space, i.e. for $b > \underline{b}$ and, as a result, a first-order condition holds everywhere including $b = \underline{b}$). The boundary condition (3.17) therefore implies $s_j(b,t) \geq 0$, i.e. it ensures that the borrowing constraint is never violated.

The solution to the household's problem in the stationary case in real terms

In the stationary equilibrium the solution to (3.1)-(3.3) problem is the time-independent analogue of the system (HJB)-(SCBC)

$$\begin{aligned}\rho v_j(b) = \max_{c,l} u(c,l) \qquad (3.18)\\ + v_j'(b)\left(\frac{1}{P^M}\left[\left[(1-\tau)wl+\Pi\right]z_j + [ben]\left(1-z_j\right) - c + Tr + \left(1-\delta P^M - P^M\pi\right)b\right]\right)\\ + \lambda_j\left(v_{-j}(b) - v_j(b)\right)\end{aligned}$$

$$c_j(b) \text{ is s.t. } \partial_c u(c,l) P^M = v_j'(b) \tag{3.19}$$

$$l_j(b) \text{ is s.t. } \partial_l u(c,l) P^M = -(1-\tau) w z_j v_j'(b) \tag{3.20}$$

$$v_j'(\underline{b}) \geq \partial_c u\left(c_j(\underline{b}), l_j(\underline{b})\right) P^M$$

$$0 = -\partial_b\left[s_j(b) g_j(b)\right] - \lambda_j g_j(b) + \lambda_{-j} g_{-j}(b) \tag{3.21}$$

where $c_j(b)$ and $l_j(b)$ are found from the stationary FOCs (3.19) and (3.20) and $s_j(b)$ is the savings function for the stationary distribution:

$$s_j(b) = \frac{1}{P^M}\left[\left[(1-\tau)wl+\Pi\right]z_j+[ben]\left(1-z_j\right)-c+Tr+\left(1-\delta P^M-P^M\pi\right)b\right] \tag{3.22}$$

3.2.2 Firms

Final Good Producers

A competitive firm solves a profit maximization problem

$$\max_{y_{j,t}} P_t Y_t - \int_0^1 p_{j,t} y_{j,t} dj,$$

subject to the aggregating technology

$$Y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}},$$

here Y_t denotes final goods, $y_{j,t}$ denotes the j'th intermediate input, and $\varepsilon > 1$ governs the elasticity of substitution between any two intermediate inputs.

Profit maximization implies that demand for intermediate good j is

$$y_{j,t} = \left(\frac{p_{j,t}}{P_t}\right)^{-\varepsilon} Y_t, \text{ where } P_t = \left(\int_0^1 p_{j,t}^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}. \tag{3.23}$$

Intermediate Good Producers

Each intermediate good j is produced by a monopolistically competitive producer using labor $n_{j,t}$ under technology A_t according to the linear production function $y_{j,t} = An_{j,t}$. The firm hires labour at a wage w_t in a competitive labour market. As a result, the cost- minimization problem of the firm implies that the marginal cost is common across all producers and given by

$$mc_t = \frac{w_t}{A_t}, \tag{3.24}$$

and the operational real profits of intermediate goods producers are

$$\tilde{\Pi}_{j,t} = \frac{p_{j,t}}{P_t} y_{j,t} - w_t n_{j,t} = \left(\frac{p_{j,t}}{P_t} - mc_t\right) y_{j,t} = \left(\frac{p_{j,t}}{P_t} - mc_t\right)\left(\frac{p_{j,t}}{P_t}\right)^{-\varepsilon} Y_t.$$

Firms are subject to a price adjustment cost following Rotemberg [1982]. The adjustment cost is quadratic in the rate of price change $\frac{\dot{p}_t}{p_t}$ and is expressed as a fraction of aggregate output Y_t,

$$\Theta\left(\frac{\dot{p}_{j,t}}{p_{j,t}}\right) = \frac{\theta}{2}\left(\frac{\dot{p}_{j,t}}{p_{j,t}}\right)^2 Y_t.$$

Each intermediate producer chooses its price to maximize profits subject to price adjustment costs,

$$\max_{p_{j,t}} \int_0^\infty e^{-\rho t}\left\{\left(p_{j,t} - P_t mc_t\right)\left(\frac{p_{j,t}}{P_t}\right)^{-\varepsilon} Y_t - \frac{\theta}{2}\left(\frac{\dot{p}_{j,t}}{p_{j,t}}\right)^2 Y_t P_t\right\} dt.$$

The solution to this profit-maximizing problem is characterized by Phillips curve equation (see Appendix A.2 for details),

$$\left(\rho - \frac{\dot{Y}_t}{Y_t}\right)\pi_t = \frac{\varepsilon}{\theta}\left(mc_t - \frac{\varepsilon - 1}{\varepsilon}\right) + \dot{\pi}_t. \tag{3.25}$$

In a symmetrical equilibrium ($p_{j,t} = p_t = P_t$), the demand for intermediate goods can be written as,

$$y_{j,t} = \left(\frac{P_t}{P_t}\right)^{-\varepsilon} Y_t = Y_t,$$

labour demand is

$$n_{j,t} = n_t = N_t = \frac{Y_t}{A_t}, \tag{3.26}$$

and firm profits are given by,

$$\Pi_{j,t} = \Pi_t = Y_t - w_t N_t - \frac{\theta}{2}\pi_t^2 Y_t. \tag{3.27}$$

3.2.3 Government

The government budget constraint in real terms[3] is given by

$$\dot{B}_t = \frac{1}{-P_t^M}\left(\tau_t w_t N_t - G_t - Tr_t^T - \left(1 - \delta P_t^M - P_t^M \pi_t\right) B_t\right), \tag{3.28}$$

where τ_t is a rate of income tax coefficient, G_t is government spending which I set equal to a constant fraction of the output $G_t = GY_t$. The total transfers are the sum of unemployment benefit transfers, received only by the unemployed, and regular transfers received by all households,

$$Tr_t^T = Tr_t + \int_\Omega ben_t g_2(b,t)\, db = Tr_t + ben_t urate.$$

[3] The budget constraint in nominal terms and its transformation into real terms are presented in the Appendix.

3.2.4 Markets Clearing and Private Sector Equilibrium

Aggregate consumption is obtained by aggregating all the individual consumptions $c_j(b,t)$ using the distribution density function $g_j(b,t)$,

$$C_t = \sum_{j=1}^{2} \int_{\Omega} c_j(b,t) g_j(b,t) db. \tag{3.29}$$

while, aggregate labor supply, N_t^S, is equal to labor demand for the labour market to clear,

$$N_t = N_t^S = \sum_{j=1}^{2} \int_{\Omega} z_j l_j(b,t) g_j(b,t) db. \tag{3.30}$$

The bonds market clearing condition is,

$$B_t = B_t^H = \sum_{j=1}^{2} \int_{\Omega} b g_j(b,t) db. \tag{3.31}$$

Finally, aggregation of individual and government budget constraints, taking into account firms' profits and government transfers yields the resource constraint,

$$Y_t \left(1 - \frac{\theta}{2} \pi_t^2\right) - G_t = C_t. \tag{3.32}$$

The private sector equilibrium is determined by the system of equations, (3.11)-(3.15), (3.26), (3.24), (3.25), (3.28), (3.29)-(3.32), given boundary conditions (3.9)-(3.17), the choice of policy instruments $\{i_t, \tau_t\}$ and the stochastic process for the idiosyncratic shock z_t.

3.2.5 Policy Setting

Unlike the previous chapter, here, I assume that the monetary authority sets policy optimally, it defines welfare maximizing inflation and sets the interest rate so as to achieve it. The short-term interest rate is linked to the bond price P_t^M via the no-arbitrage condition

$$\dot{P}_t^m = P_t^m (\delta + i_t) - 1, \tag{3.33}$$

or, after integration,

$$P_t^M = \int_0^\infty e^{-\int_t^{t+s} i_\xi d\xi} e^{-\delta s} ds,$$

meaning that bond prices depend on the time path of the inverse of the nominal interest rate over the entire duration of debt and shows that a lower value of δ (a longer average maturity) implies a greater impact of future nominal interest rates on bond prices. Therefore, a prolonged increase in nominal interest rates will depress bond prices more when the maturity of the bonds is higher.

The government sets the labour income tax rate according to the fiscal policy rule, where tax feeds back on the debt value $P_t^M B_t$,

$$\tau_t = \left(\frac{P_t^M B_t}{P_{ss}^M B_{ss}}\right)^{\xi} \tau^T,$$

here τ^T denotes the value of tax imposed at the steady state and the subscript ss denotes target levels, consistent with stochastic steady state. To ensure the policy stabilizes debt in the regime of monetary dominance, the parameter ξ must be sufficiently large to ensure the primary balance rises by more than the debt service costs whenever debt rises above its target.

3.3 The RANK Model

A representative agent version of our model (RANK) offers a convenient benchmark against which I can evaluate the effects of uninsurable unemployment risk. The key and only difference between the RANK and HANK models is the household's problem is that in the RANK version of the model, I allow households to pool resources and, thereby, effectively insure themselves against the consumption consequences of unemployment. Therefore, instead of a continuum of individuals with different wealth holdings, there is only one representative *household*, members of which are almost fully insured against unemployment shock. Here, unlike the HANK model, the members of a household share labour income $w_t l_t$ (it is subject to the same tax rate τ_t), profits Π_t and bond holdings b_t, and choose to consume the same amount. However, the disutility of providing labour is asymmetrical and is faced only by employed individuals within the household, therefore, the disutility of the labour supply is normalized by the proportion of household members that are employed, $(1 - urate)$, where $urate$ is the unemployment rate.

Thus, a representative household chooses consumption c_t, labor l_t and wealth b_t to maximize the household's lifetime utility

$$\int_0^{\infty} e^{-\rho t} \left(\frac{c^{1-1/\sigma}}{1-1/\sigma} - \varphi\,(1 - urate)\,\frac{l^{1+\psi}}{1+\psi}\right) dt, \tag{3.34}$$

subject to the *nominal* budget constraint

$$\dot{b}_t^n = \frac{1}{P_t^M}\left[P_t\left[(1 - urate)(1-\tau_t)\, w_t l_t + T_t^T + \Pi_t - c_t\right] + b_t^n\left(1 - \delta P_t^M\right)\right]. \tag{3.35}$$

To solve this problem, I define a Hamiltonian,

$$\mathscr{H} = \max_{c,b,l} \int_0^\infty e^{-\rho t} \left(\frac{c^{1-1/\sigma}}{1-1/\sigma} - \varphi \left(1 - urate\right) \frac{l^{1+\psi}}{1+\psi} \right) \tag{3.36}$$
$$+ \lambda_t \left[\frac{1}{P_t^M} \left[P_t \left[\left(1 - urate\right) \left(1 - \tau_t\right) w_t l_t + T_t^T + \Pi_t - c_t \right] + b_t^n \left(1 - \delta P_t^M\right) \right] - \dot{b}_t^n \right] dt.$$

Optimizing with respect to b_t^n, c_t, l_t yields the following first-order conditions

$$\left(\frac{1}{P_t^M} - \delta - \rho \right) + \frac{\hat{\lambda}_t}{\lambda_t} = 0, \tag{3.37}$$

$$c_t^{-1/\sigma} - \frac{\lambda_t P_t}{P_t^M} = 0, \tag{3.38}$$

$$-\varphi l_t^\psi + \frac{\lambda_t P_t}{P_t^M} \left(1 - \tau_t\right) w_t = 0. \tag{3.39}$$

Together with the household's budget constraint (3.35), these FOCs solve for optimal values of λ_t, c_t, l_t and b_t^n. Here, the choice of optimal consumption is inversely proportional to the value of the Lagrange multiplier λ_t, while the choice of labour is proportional to the Lagrange multiplier λ_t. This implies that the marginal rate of substitution between consumption and leisure is proportional to the after-tax wage rate.

I transform these equations into real terms and denote $\mu_t = \lambda_t P_t$ to obtain the household's optimal consumption/savings decision and labour supply condition,

$$\left(\frac{1}{P_t^M} \left(1 - \delta P_t^M - \hat{\pi}_t P_t^M\right) - \rho \right) + \frac{\hat{\mu}_t}{\mu_t} = 0, \tag{3.40}$$

$$c_t^{-1/\sigma} - \frac{\mu_t}{P_t^M} = 0, \tag{3.41}$$

$$-\varphi l_t^\psi + \frac{\mu_t}{P_t^M} \left(1 - \tau_t\right) w_t = 0. \tag{3.42}$$

While the household budget constraint in real terms can be written as,

$$\dot{b}_t = \frac{1}{P_t^M} \left[\left(1 - urate\right) \left(1 - \tau_t\right) w_t l_t + T_t^T + \Pi_t + \left(1 - \delta P_t^M - P_t^M \pi_t\right) b_t - c_t \right]. \tag{3.43}$$

Note that, μ_t is a forward-looking variable and features a *right* derivative and inflation π_t is the right derivative of the price level. Moreover, equations (3.40) and (3.43) are dynamic and require boundary conditions. Equation (3.40) requires terminal condition $\lim_{t\to\infty} \mu_t = \mu_{ss}$, while equation (3.43) requires initial condition $b_0 = b_{ss}$.

Similarly to the HANK model, the budget constraint equation features $\left(1 - \delta P_t^M - P_t^M \pi_t\right)$ – ex-post interest payments, while the optimality conditions feature $\left(1 - \delta P_t^M - P_t^M \hat{\pi}_t\right)$ – ex-ante interest payments. This implies that surprises surrounding households' anticipation of future inflation and interest rates (via the price of long-maturity bonds) will create a wedge between the ex-ante and ex-post returns the household receives.

In the RANK model, the level of consumption is the same across all individuals,

$$C_t = c_t, \tag{3.44}$$

labour is supplied only by employed members of the household, therefore, the labour market clearing condition is

$$N_t = (1 - urate)\, l_t, \tag{3.45}$$

and the bond market clearing condition remains,

$$B_t = b_t. \tag{3.46}$$

3.4 Optimal Policy under Commitment. HANK

3.4.1 Central Bank Problem

The central bank maximizes social welfare taking into account not just aggregate wealth variable but the whole wealth distribution,

$$\mathscr{W}[g(0,\cdot)] = \max_{\{\pi_s\}_{s=0}^{\infty}} \int_0^{\infty} e^{-\rho s} \sum_{j=1}^{2} \int_{\Phi} U\left(c_j(s,a), l_j(s,a)\right) g_j(s,a)\, dads \tag{3.47}$$

subject to HJB equation (3.4), optimal consumption choice (3.12), optimal labour choice (3.13), KF equation (3.7), bond pricing (3.25), wage setting (3.24), labour market clearing condition (3.30), bonds market clearing condition (3.31) and production function (3.26). The presented Ramsey problem is an optimal control problem, therefore, I apply a variational approach[4] and define a Lagrangian the following way:

[4]more details on variational approach can be found in Appendix C

$$
\begin{aligned}
\mathscr{L}_0 = & \int_0^\infty e^{-\rho t} \left\langle U\left(c_{it}\left(a\right), l_{it}\left(a\right)\right), g\right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t} \left\langle \zeta_{it}\left(a\right), -\frac{\partial}{\partial a}\left[s_{it}\left(a\right) g_{it}\left(a\right)\right] - \lambda_i g_{it}\left(a\right) + \lambda_j g_{jt}\left(a\right) - \frac{\partial g_{it}}{\partial t} \right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t} \mu_t \left(\frac{\varepsilon}{\theta}\left(mc_t - \frac{\varepsilon - 1}{\varepsilon}\right) + \dot{\pi}_t - \left(\rho - \frac{\dot{Y}_t}{Y_t}\right)\pi_t\right) dt \\
& + \int_0^\infty e^{-\rho t} \left\langle \theta_{it}\left(a\right), \frac{\partial v_{it}\left(a\right)}{\partial t} + U\left(\cdot\right) + s_{it}\left(a\right)\frac{\partial v_{it}\left(a\right)}{\partial a} + \lambda_i\left[v_{jt}\left(a\right) - v_{it}\left(a\right)\right] - \rho v_{it}\left(a\right)\right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t} \left\langle \eta_{it}\left(a\right), P_t^M \frac{\partial U\left(c_{it}\left(a\right), l_{it}\left(a\right)\right)}{\partial c} - \frac{\partial v\left(a\right)}{\partial a}\right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t} \left\langle \varphi_{it}\left(a\right), P_t^M \frac{\partial U\left(c_{it}\left(a\right), l_{it}\left(a\right)\right)}{\partial l} + w_t z_i\left(1 - \tau_t\right)\frac{\partial v_{it}\left(a\right)}{\partial a}\right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t} \xi_t \left(\dot{B}_t + \frac{1}{P_t^M}\left(\tau_t w_t N_t - GY_t - Tr_t^T - \left(1 - \delta P_t^M - P_t^M \pi_t\right)B_t\right)\right) dt \\
& + \int_0^\infty e^{-\rho t} \chi_t^w \left(w_t - mc_t A_t\right) dt \\
& + \int_0^\infty e^{-\rho t} \chi_t^N \left(N_t - \sum_{i=1}^2 \int z_i l_{it} g da\right) dt \\
& + \int_0^\infty e^{-\rho t} \chi_t^B \left(B_t - \sum_{i=1}^2 \int b g da\right) dt \\
& + \int_0^\infty e^{-\rho t} \chi_t^Y \left(Y_t - A_t N_t\right) dt
\end{aligned}
$$

where $\langle v, g\rangle_\Phi = \sum_{j=1}^2 \int_\Phi v_j g_j da = \int_\Phi v^T g da$ denotes the inner product of Lebegue-integrable functions from the space $L^2\left(\Phi\right)$, where $\Phi = \{1,2\} \times [\underline{b}, \infty)$.

I substitute Y_t and mc_t to reduce the number of equations,

$$
\begin{aligned}
\mathscr{L}_0 = & \int_0^\infty e^{-\rho t} \langle U(c_{it}(a), l_{it}(a)), g \rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t} \left\langle \zeta_{it}(a), -\frac{\partial}{\partial a}[\mathbf{s}_{it}(a) g_{it}(a)] - \lambda_i g_{it}(a) + \lambda_j g_{jt}(a) - \frac{\partial g_{it}}{\partial t} \right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t} \mu_t \left(\frac{\varepsilon}{\theta} \left(\frac{w_t}{A_t} - \frac{\varepsilon - 1}{\varepsilon} \right) + \dot{\pi}_t - \left(\rho - \frac{\dot{A}}{A} - \frac{\dot{N}_t}{N_t} \right) \pi_t \right) dt \\
& + \int_0^\infty e^{-\rho t} \left\langle \theta_{it}(a), \frac{\partial v_{it}(a)}{\partial t} + U(\cdot) + \mathbf{s}_{it}(a) \frac{\partial v_{it}(a)}{\partial a} + \lambda_i \left[v_{jt}(a) - v_{it}(a) \right] - \rho v_{it}(a) \right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t} \left\langle \eta_{it}(a), \frac{\partial U(c_{it}(a), l_{it}(a))}{\partial c} - \frac{1}{P_t^M} \frac{\partial v(a)}{\partial a} \right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t} \left\langle \varphi_{it}(a), \frac{\partial U(c_{it}(a), l_{it}(a))}{\partial l} + w_t z_i (1 - \tau_t) \frac{1}{P_t^M} \frac{\partial v_{it}(a)}{\partial a} \right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t} \xi_t \left(\dot{B}_t + \frac{1}{P_t^M} \left(\tau_t w_t N_t - G A_t N_t - Tr_t^T - \left(1 - \delta P_t^M - P_t^M \pi_t \right) B_t \right) \right) dt \\
& + \int_0^\infty e^{-\rho t} \chi_t^N \left(N_t - \sum_{i=1}^2 \int z_i l_{it} g da \right) dt \\
& + \int_0^\infty e^{-\rho t} \chi_t^B \left(B_t - \sum_{i=1}^2 \int b g da \right) dt
\end{aligned}
$$

So, after grouping by some of the terms and integrating by parts the KF equation term I get:

$$
\begin{aligned}
\mathscr{L}_0 = \int_0^\infty e^{-\rho t} (\quad & \langle U(c_{it}, l_{it}), g_{it} \rangle_\Phi + \left\langle \frac{\partial \zeta_{it}}{\partial t} + \mathscr{A} \zeta_{it} - \rho \zeta_{it}, g_{it} \right\rangle_\Phi \\
& + \left\langle \theta_{it}, U(c_{it}, l_{it}) + \mathscr{A} v_{it} + \frac{\partial v_{it}}{\partial t} - \rho v_{it} \right\rangle_\Phi \\
& + \mu_t \left(\frac{\varepsilon}{\theta} \left(\frac{w_t}{A_t} - \frac{\varepsilon - 1}{\varepsilon} \right) + \dot{\pi}_t - \left(\rho - \frac{\dot{A}_t}{A_t} - \frac{\dot{N}_t}{N_t} \right) \pi_t \right) \\
& + \left\langle \eta_{it}, \frac{\partial U(c_{it}, l_{it})}{\partial c} - \frac{1}{P_t^M} \frac{\partial v_{it}}{\partial a} \right\rangle_\Phi + \left\langle \varphi_{it}, \frac{\partial U(c_{it}, l_{it})}{\partial l} + w_t z_i (1 - \tau_t) \frac{1}{P_t^M} \frac{\partial v_{it}}{\partial a} \right\rangle_\Phi \\
& + \xi_t \left(\dot{B}_t + \frac{1}{P_t^M} \left(\tau_t w_t N_t - G A_t N_t - Tr_t^T - \left(1 - \delta P_t^M - P_t^M \pi_t \right) B_t \right) \right) \\
& + \chi_t^N (N_t - \langle z_i l_{it}, g_{it} \rangle_\Phi) + \chi_t^B (B_t - \langle b, g_{it} \rangle_\Phi) \quad) dt \\
& + \langle \zeta(0, \cdot), g(0, \cdot) \rangle_\Phi - \lim_{T \to \infty} \left\langle e^{-\rho T} \zeta(T, \cdot) . g(T, \cdot) \right\rangle
\end{aligned} \tag{3.48}
$$

where for shorter notation I omit certain indexes, keeping in mind that Lagrange multipliers $\zeta_{it}(a)$, $\theta_{it}(a), \eta_{it}(a), \varphi_{it}(a) \in L^2([0, \infty) \times \Phi)$, and $\xi_t, \mu_t, \chi_t^N, \chi_t^B \in L^2[0, \infty)$, while the variables $g_{it}(a)$, $v_{it}(a), c_{it}(a), l_{it}(a) \in L^2([0, \infty) \times \Phi)$ and $P_t^M, w_t, N_t, \pi_t \in L^2[0, \infty)$. In addition, I introduce operator

$\mathscr{A}$ and adjoint to it operator $\mathscr{A}^*$,

$$\mathscr{A}\zeta \equiv \begin{pmatrix} \mathbf{s}_{1t}(a)\frac{\partial\zeta_{1t}(a)}{\partial a} + \lambda_1\left[\zeta_{2t}(a) - \zeta_{1t}(a)\right] \\ \mathbf{s}_{2t}(a)\frac{\partial\zeta_{2t}(a)}{\partial a} + \lambda_2\left[\zeta_{1t}(a) - \zeta_{2t}(a)\right] \end{pmatrix}$$

$$\mathscr{A}^*\theta \equiv \begin{pmatrix} -\frac{\partial}{\partial a}\left[\mathbf{s}_{1t}(a)\,\theta_{1t}(a)\right] - \lambda_1\theta_{1t}(a) + \lambda_2\theta_{2t}(a) \\ -\frac{\partial}{\partial a}\left[\mathbf{s}_{2t}(a)\,\theta_{2t}(a)\right] - \lambda_2\theta_{2t}(a) + \lambda_1\theta_{1t}(a) \end{pmatrix}$$

Next, I take the Gateaux derivatives with respect to functionals $g_{it}, v_{it}, c_{it}, l_{it}, \pi_t, w_t, B_t, N_t, P_t^M$ and obtain the optimality conditions by equating these derivatives to zero.

3.4.2 Optimal Inflation Conditions

Here is the system of equations which determines Lagrange multipliers of the Lagrangian function (3.48) and optimal inflation level π_t,

$$\begin{aligned}
0 &= U(c_{it}, l_{it}) + \frac{\partial\zeta_{it}}{\partial t} + \mathscr{A}\zeta_{it} - \rho\zeta_{it} - \chi_t^N z_i l_{it} - \chi_t^B b \\
0 &= \mathscr{A}^*\theta_{it} - \frac{\partial\theta_{it}}{\partial t} + \frac{1}{P_t^M}\left(\frac{\partial\eta_{it}}{\partial b} - \frac{\partial\varphi_{it}}{\partial b}(1-\tau_t)\,w_t z_i\right) \\
0 &= \langle U_c(c_{it}, l_{it}), g_{it}\rangle_\Phi + \langle\eta_{it}, U_{cc}(c_{it}, l_{it})\rangle_\Phi - \left\langle\frac{1}{P_t^M}, \frac{\partial\zeta_{it}}{\partial a} g_{it}\right\rangle_\Phi \\
0 &= \langle U_l(c_{it}, l_{it}), g_{it}\rangle_\Phi + \langle\varphi_{it}, U_{ll}(c_{it}, l_{it})\rangle_\Phi - \left\langle\frac{(1-\tau_t)\,w_t z_{ii}}{P_t^M}, \frac{\partial\zeta_{it}}{\partial a} g_{it}\right\rangle_\Phi + \chi^N\langle z_i, g_{it}\rangle_\Phi \\
0 &= \dot{\mu}_t + \mu_t\left(-\frac{\dot{A}_t}{A_t} - \frac{\dot{N}}{N}\right) + \xi_t B_t - \left\langle\left(\frac{\hat{\theta}\pi A_t N_t}{P_t^M} z_i + b_t\right), \frac{\partial\zeta_{it}}{\partial a} g_{it} + \frac{\partial v_{it}}{\partial a}\theta_{it}\right\rangle_\Phi \\
0 &= \frac{1}{P_t^M}\left\langle(1-\tau_t)\,z_i l - N_t z_i, \theta\frac{\partial v}{\partial a} + g\frac{\partial\zeta}{\partial a}\right\rangle_\Phi + \frac{1}{P_t^M}\left\langle(1-\tau_t)\,z_i\frac{\partial v_{it}}{\partial a}, \varphi_{it}\right\rangle_\Phi + \xi_t\frac{\tau_t N_t}{P_t^M} - \frac{\varepsilon}{\hat{\theta}}\frac{\mu_t}{A_t} \\
0 &= \left\langle\left((1-\tau_t)\,w_t l_{it} z_i + \Pi_t z_i + ben\,(1-z_i) - c_{it} + Tr_t + b_t\right), \frac{\partial\zeta_{it}}{\partial a} g_{it} + \frac{\partial v_{it}}{\partial a}\theta_{it}\right\rangle_\Phi \\
&\quad - \left\langle\eta_{it}, \frac{\partial v_{it}}{\partial a}\right\rangle_\Phi + \left\langle(1-\tau_t)\,w_t z_i\frac{\partial v_{it}}{\partial a}, \varphi_{it}\right\rangle_\Phi + \xi_t\left(\left[\tau_t w_t N_t - GA_t N_t - Tr_t - B_t\right]\right) \\
0 &= \left\langle\frac{1}{P_t^M}\left(\left(1 - \frac{\hat{\theta}}{2}\pi_t\right)A_t - w_t\right) z_i, \frac{\partial\zeta_{it}}{\partial a} g_{it} + \frac{\partial v_{it}}{\partial a}\theta_{it}\right\rangle_\Phi - \frac{\xi_t}{P_t^M}\left(\left[\tau_t w_t - GA_t\right]\right) - \chi_t^N \\
&\quad + \frac{1}{N_t}\left(\dot{\mu}_t\pi_t + \mu_t\dot{\pi}_t - \rho\mu_t\pi_t\right) \\
0 &= \xi_t\left(\rho - \left(\frac{1}{P_t^M} - \delta - \pi_t\right)\right) - \dot{\xi}_t - \chi_t^B
\end{aligned}$$

where the equations defining Lagrange multipliers $\zeta_{it}(a)$, $\theta_{it}(a)$, ξ_t, and μ_t are time-varying and thus are supported with boundary conditions,

$$\begin{aligned}\lim_{t\to\infty}\zeta_{i,t}(a) &= \zeta_{i,ss}(a)\\ \theta_{i,0}(a) &= 0\\ \mu_0 &= 0\\ \lim_{t\to\infty}\xi_t &= \xi_{ss}\end{aligned}$$

3.5 Optimal Policy under Commitment. RANK

3.5.1 Central Bank Problem

The central bank has a standard representative-agent welfare maximizing problem. The lifetime welfare of a representative individual is given by

$$\mathscr{W} = \max_{\{\pi_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} U(c_t, l_t)\, dt \tag{3.49}$$

subject to optimal bonds choice (3.40), optimal consumption choice (3.41), optimal labour choice (3.42), household budget constraint (3.43), bond pricing (3.25), wage setting (3.24), and production function (3.26). The presented Ramsey problem is an optimal control problem, therefore, I define a Lagrangian function to solve it,

$$
\begin{aligned}
\mathscr{L}_0 = & \int_0^\infty e^{-\rho t} U\left(C_t, N_t\right) dt \\
& + \int_0^\infty e^{-\rho t} \zeta_t \left(\frac{1}{P_t^M} \left[\left(1-\tau_t\right) w_t N_t + Tr_t^T + \Pi_t + \left(1 - \delta P_t^M - P_t^M \pi_t\right) B_t - C_t \right] - \dot{B}_t \right) dt \\
& + \int_0^\infty e^{-\rho t} \mu_t \left(\frac{\varepsilon}{\theta} \left(mc_t - \frac{\varepsilon - 1}{\varepsilon} \right) + \dot{\pi}_t - \left(\rho - \frac{\dot{Y}_t}{Y_t} \right) \pi_t \right) dt \\
& + \int_0^\infty e^{-\rho t} \theta_t \left(\frac{1}{P_t^M} - \delta - \hat{\pi}_t - \rho + \frac{\dot{\lambda}_t}{\lambda_t} \right) dt \\
& + \int_0^\infty e^{-\rho t} \eta_t \left(U_C\left(C_t, N_t\right) - \frac{\lambda_t}{P_t^M} \right) dt \\
& + \int_0^\infty e^{-\rho t} \varphi_t \left(U_N\left(C_t, N_t\right) + \frac{w_t \left(1 - \tau_t\right) \lambda_t}{P_t^M} \right) dt \\
& + \int_0^\infty e^{-\rho t} \xi_t \left(\dot{B}_t + \frac{1}{P_t^M} \left(\tau_t w_t N_t - G Y_t - Tr_t^T - \left(1 - \delta P_t^M - P_t^M \pi_t\right) B_t \right) \right) dt \\
& + \int_0^\infty e^{-\rho t} \chi_t^w \left(w_t - mc_t A_t \right) dt \\
& + \int_0^\infty e^{-\rho t} \chi_t^Y \left(Y_t - A_t N_t \right) dt
\end{aligned}
$$

Note that here I already applied bonds and labour market clearing conditions (3.46), (3.45) as well as aggregation of consumption (3.44). Next, I substitute Y_t and mc_t to reduce the number of equations

$$
\begin{aligned}
\mathscr{L}_0 = & \int_0^\infty e^{-\rho t} U\left(C_t, N_t\right) dt \\
& + \int_0^\infty e^{-\rho t} \zeta_t \left(\frac{1}{P_t^M} \left[\left(1-\tau_t\right) w_t N_t + Tr_t^T + \Pi_t + \left(1 - \delta P_t^M - P_t^M \pi_t\right) B_t - C_t \right] - \dot{B}_t \right) dt \\
& + \int_0^\infty e^{-\rho t} \mu_t \left(\frac{\varepsilon}{\theta} \left(\frac{w_t}{A_t} - \frac{\varepsilon - 1}{\varepsilon} \right) + \dot{\pi}_t - \left(\rho - \frac{\dot{A}_t}{A_t} - \frac{\dot{N}_t}{N_t} \right) \pi_t \right) dt \\
& + \int_0^\infty e^{-\rho t} \theta_t \left(\frac{1}{P_t^M} - \delta - \hat{\pi}_t - \rho + \frac{\dot{\lambda}_t}{\lambda_t} \right) dt \\
& + \int_0^\infty e^{-\rho t} \eta_t \left(U_C\left(C_t, N_t\right) - \frac{\lambda_t}{P_t^M} \right) dt \\
& + \int_0^\infty e^{-\rho t} \varphi_t \left(U_N\left(C_t, N_t\right) + \frac{w_t \left(1 - \tau_t\right) \lambda_t}{P_t^M} \right) dt \\
& + \int_0^\infty e^{-\rho t} \xi_t \left(\dot{B}_t + \frac{1}{P_t^M} \left(\tau_t w_t N_t - G A_t N_t - Tr_t^T - \left(1 - \delta P_t^M - P_t^M \pi_t\right) B_t \right) \right) dt
\end{aligned}
$$

Using this Lagrangian, I take derivatives with respect to $\lambda_t, B, C_t, N, \pi_t, w_t, P_t^M$ and obtain the optimality conditions by equating those derivatives to zero.

3.5.2 Optimal Inflation Conditions

Here is the system of equations which determines Lagrange multipliers $\zeta_t, \mu_t, \theta_t, \eta_t, \varphi_t, \xi_t$ and optimal inflation level π_t,

$$
\begin{aligned}
0 &= \varphi_t \frac{(1-\tau_t)\, w_t}{P_t^M} - \frac{\eta_t}{P_t^M} + \rho \frac{\theta_t}{\lambda_t} - \frac{\dot{\theta}_t}{\lambda_t} \\
0 &= U_C\left(C_t, N_t\right) - \frac{\zeta_t}{P_t^M} + \eta_t U_{CC}\left(C_t, N_t\right) \\
0 &= \dot{\mu}_t + \mu_t \left(-\frac{\dot{A}_t}{A_t} - \frac{\dot{N}}{N} \right) + \xi_t B_t - \zeta_t \left(\frac{\hat{\theta} \pi A_t N_t}{P_t^M} + B_t \right) - \theta_t \\
0 &= \frac{1}{P_t^M} \left((1-\tau_t)\, N_t - N_t, \zeta_t \right) + \frac{(1-\tau_t)}{P_t^M} \lambda_t \varphi_t + \xi_t \frac{\tau_t N_t}{P_t^M} - \frac{\varepsilon}{\hat{\theta}} \frac{\mu_t}{A_t} \\
0 &= \left[(1-\tau_t)\, w_t N_t + Tr_t + \Pi_t + B_t - C_t \right] \zeta_t + \theta_t \\
&\quad - \eta_t \lambda_t + (1-\tau_t)\, w_t \lambda_t \varphi_t + \xi_t \left(\left[\tau_t w_t N_t - G A_t N_t - Tr_t^T - B_t \right] \right) \\
0 &= U_N\left(C_t, N_t\right) + \varphi_t U_{NN}\left(C_t, N_t\right) + \frac{1}{P_t^M} \left((1-\tau_t)\, w_t + \left(1 - \frac{\hat{\theta}}{2} \pi_t \right) A_t - w_t \right) \zeta_t \\
&\quad + \frac{\xi_t}{P_t^M} \left(\left[\tau_t w_t - G A_t \right] \right) + \frac{1}{N_t} \left(\dot{\mu}_t \pi_t + \mu_t \dot{\pi}_t - \rho \mu_t \pi_t \right) \\
0 &= \zeta_t \left(\frac{1}{P_t^M} - \delta - \pi_t - \rho \right) + \dot{\zeta}_t + \xi_t \left(\rho - \left(\frac{1}{P_t^M} - \delta - \pi_t \right) \right) - \dot{\xi}_t
\end{aligned}
$$

where the equations defining Lagrange multipliers $\theta_t, \zeta_t, \mu_t, \xi_t$ are time-varying and thus are supported with boundary conditions,

$$\theta(0) = 0 \tag{3.50}$$

$$\lim_{t \to \infty} \zeta_t = \zeta_{ss} \tag{3.51}$$

$$\mu(0) = 0 \tag{3.52}$$

$$\lim_{t \to \infty} \xi_t = \xi_{ss} \tag{3.53}$$

3.5.3 Optimal Inflation at the Steady State

Consider the following set of the optimality conditions defining optimal inflation at the steady state

$$0 = \varphi_t \frac{(1-\tau_t)\, w_t}{P_t^M} - \frac{\eta_t}{P_t^M} + \rho \frac{\theta_t}{\lambda_t}, \tag{3.54}$$

$$\begin{aligned} 0 = &\left[(1-\tau_t)\, w_t N_t + Tr_t + \Pi_t + B_t - C_t\right] \zeta_t + \theta_t \\ &- \eta_t \lambda_t + (1-\tau_t)\, w_t \lambda_t \varphi_t + \xi_t \left(\left[\tau_t w_t N_t - G A_t N_t - Tr_t^T - B_t\right]\right), \end{aligned} \tag{3.55}$$

$$0 = \xi_t B_t - \zeta_t \left(\frac{\hat{\theta} \pi A_t N_t}{P_t^M} + B_t \right) - \theta_t. \tag{3.56}$$

From equation (3.54), I get $-\eta_t \lambda_t + (1-\tau_t)\, w_t \lambda_t \varphi_t = -\rho P_t^M \theta_t$, moreover, I can also apply the definitions of the household's (3.16) and government (3.43) budget constraints, so that the equation (3.55) will be rewritten as

$$0 = \left[P_t^M (\delta + \pi_t) B_t\right] \zeta_t + \xi_t \left(\left[-P_t^M (\delta + \pi_t) B_t\right]\right) + \theta_t - \rho P_t^M \theta_t,$$

or

$$(\xi_t - \zeta_t) \left[P_t^M (\delta + \pi_t) B_t\right] = \left(1 - \rho P_t^M\right) \theta_t,$$

note that at the steady state $\frac{1}{P_t^M} - \delta - \pi_t - \rho = 0$ implies $P_t^M (\delta + \pi_t) = P_t^M \left(\frac{1}{P_t^M} - \rho\right) = \left(1 - \rho P_t^M\right)$, therefore,

$$(\xi_t - \zeta_t) B_t = \theta_t,$$

then, coming back to the inflation optimality condition (3.56) I get

$$\begin{aligned} 0 &= \xi_t B_t - \zeta_t \left(\frac{\hat{\theta} \pi A_t N_t}{P_t^M} + B_t \right) - (\xi_t - \zeta_t) B_t, \\ 0 &= \frac{\hat{\theta} \pi A_t N_t}{P_t^M}. \end{aligned}$$

Consequently, inflation has to be zero. This result has been obtained due to the two key factors. Firstly, the short-term interest rate q_t is defined as $\left(\frac{1}{P_t^M} - \delta - \pi_t\right)$ and is exactly equal to ρ - the households' time preference discount factor; this condition does not hold in the HANK model, where the intertemporal substitution of consumption accounts for idiosyncratic risk that increases demand for the assets and thus decreases interest rate. Secondly, firms are assumed to discount future profits exactly at the same rate as households, otherwise, the extra term, featuring Lagrange multiplier μ_t associated with the Phillips curve, enters the inflation optimality equation and leads to a non-zero optimal inflation value.

3.6 Analytical Characterization of Optimal Inflation in HANK

There are a few factors determining optimal inflation, some of them are related to the completeness of the market, and other factors arise from the heterogeneity assumption - they are the focus of my analysis. As has been mentioned before, one of the main differences between the RANK and HANK models is the equilibrium values of the interest rates: at the steady state in the RANK model, the short-term interest rate is the same as the household's discount factor implying optimal steady-state inflation is equal to zero. This result is consistent with the New Keynesian literature (for example, Benigno and Woodford [2005] have found optimal inflation to be equal to zero and claim that a policymaker should aim to achieve it in the long run), where the desirability of maintaining zero inflation is derived from the balance between the benefits of capitalizing on the temporary trade-off between output and inflation and the drawbacks of exacerbating this trade-off permanently through higher inflation expectations.

The inflation optimality condition can be rewritten the following way,

$$\pi_t = \frac{\xi_t B_t - \left\langle b_t, \frac{\partial \zeta_{it}}{\partial a} g_{it} + \frac{\partial v_{it}}{\partial a}\theta_{it} \right\rangle_{\Phi} + \dot{\mu}_t - \mu_t \frac{\dot{Y}_t}{Y_t}}{\frac{\hat{\theta} Y_t}{P_t^M}\left(\left\langle \frac{\partial \zeta_{it}}{\partial a}, g_{it} \right\rangle_{\Phi} + \left\langle \frac{\partial v_{it}}{\partial a}, \theta_{it} \right\rangle_{\Phi} \right)} \tag{3.57}$$

where the Lagrange multipliers μ_t and θ_{it} are set to be zero at the initial time period according to boundary conditions (3.52) and (3.50).

This optimality condition has a similar structure and some common properties compared with the Nuno and Thomas [2020] result: μ_t and θ_t feature the same initial conditions; moreover, wealth dispersion, marginal consumption and nominal asset position affect optimal inflation the same way and generate redistributive inflationary bias.

However, the condition of optimal inflation also features lots of crucial differences compared to the result of Nuno and Thomas [2020]: the optimal inflation is normalised subject to its adjustment cost, not the disutility parameter; Lagrange multiplier θ_t is not identical to 0 and thus enters the final form of the optimality condition; also, the equation features term $\xi_t B_t$ which captures the effect of the government budget constraint tightness on the optimal inflation level; and, lastly, the Lagrange multiplier $\zeta_t(\cdot) \neq v_t(\cdot)$, making the prediction of optimal inflation level more difficult. Nevertheless, I can make certain conclusions by investigating the form of this optimality condition.

The first component of equation (3.57), $\xi_t B_t$, captures the effect of the government budget constraint tightness on social welfare (the tighter is the budget constraint - the higher the optimal response to inflation proportional to the debt level B_t to achieve the same level of welfare), this component of the optimal inflation equation will be absent in the case of incomplete markets (interestingly, in the case of incomplete markets, the Lagrange multiplier ζ_{it} is equal to the value function v_{it} and coefficient θ_{it} is proved to be zero, as a result, a positive net asset nominal position, $\langle b_t, g_{it} \rangle_{\Phi} > 0$, would imply optimal inflation to be negative, more detailed proof of this can be found in Nuno and Thomas [2020]). The term featuring μ arises due to the presence of the Phillips curve constraint and does not affect the optimal inflation level at the steady state.

An interesting component of the equation (3.57) is $\left\langle b_t, \frac{\partial \zeta_{it}}{\partial a} g_{it} + \frac{\partial v_{it}}{\partial a} \theta_{it} \right\rangle_\Phi$. At this part of the optimality condition, not only the aggregate state of the economy but household's distributional properties start to matter. However, to understand the underlying intuition of this term, I need to apply some simplifying assumptions. The Lagrange multiplier ζ_{it} is determined by the equation, which is very similar to the HJB equation, numerical exercise has shown that substituting ζ_{it} with the value function v_{it} changes inflation by less than 0.001%. Moreover, from the time-zero perspective, coefficient θ_i is equal to 0. Thus, this component of the inflation optimality condition can be simplified to $\left\langle b_t, \frac{\partial v_{it}}{\partial a} g_{it} \right\rangle_\Phi$ or, with the use of the optimal consumption choice equation, $\left\langle P_t^M b, \frac{1}{c_{it}} g_{it} \right\rangle_\Phi$. Consequently, this term represents how the households market value liabilities $P_t^M b$ that are aggregated with the density function g_{it} and weighted by individual i marginal utility of consumption $\frac{1}{c_{it}}$ affect the optimal value of inflation.

More specifically, the considered component of the inflation optimality condition shows how inflation affects social welfare by changing the market value of debt. When households are in debt ($b < 0$), inflation has a positive impact since it diminishes the market value of their debt burden. Conversely, for households that lend money ($b > 0$), the effect is negative. Importantly, this term highlights the central bank's incentive to engage in inflationary measures for the purpose of redistribution. In my model calibration, the debt to GDP target is around 45%, much higher than zero, implying this component has a negative impact on inflation, however, the positive value of the Lagrange multiplier ξ_t associated with the government budget constraint cancels this effect.

On the other hand, the concavity of preferences ($U''_{cc} < 0$) suggests that households in debt experience a greater increase in marginal utility from consumption compared to lending households. Consequently, even when the economy has a balanced net supply of assets, as long as there exists variation in net wealth among households, the central bank has a rationale for redistributing resources from debtors to lenders. Meaning, that heterogeneity among households leads to higher optimal inflation and the stronger idiosyncratic shock faced by the households leads to higher inflation. Later, this point will be proved with the numerical results.

3.7 Numerical Analysis

3.7.1 Calibration

The calibration of most parameters is standard and follows Kaplan et al. [2018]. Specifically, the model is calibrated to a quarterly frequency with discount factor $\rho = 0.01$. I work with log-utility, setting $\sigma = 1$, while the inverse of Frisch elasticity of substitution is set at $\psi = 1$.[5] The weight on disutility of labour, $\varphi = 2.2$. This ensures that the average time worked is about 0.5 of the available

[5]This follows Chetty et al. [2013]. Although many DSGE models use much higher elasticity to match volatilities of aggregate worked hours and of wages (Peterman [2016])), our model accounts for spells of unemployment explicitly, and the lower number is used.

time in the steady state. I set the elasticity of substitution between goods at $\varepsilon = 11$, implying a mark-up of 10%, as estimated in Krause and Lubik [2007], Chari et al. [2000] and is consistent with Basu and Fernald [1997]. The price adjustment cost parameter is set to $\theta = 100$, consistent with a Calvo parameter implying prices are sticky for, on average four quarters (see also estimates in Gust et al. [2017]. Parameter $\rho^B = 1 - \delta$ regulates debt maturity, it is set to 0.95 which generates average maturity of 4.5 years[6].

The calibration of the labour market is essential for evaluating the effects of the idiosyncratic unemployment shock. Panels I-II of Table 3.1 reports some important ratios, based on the US data from 2000-2018, which I use to calibrate the model[7]. In this chapter, I consider two scenarios, the first one uses a standard definition of unemployment (according to the US data its value u is around 6%), while the second one interprets 'unemployment' as the degree of economic inactivity: in the US, this implies a rate for $u^{e.i.}$ of 26%. According to the OECD data, unemployed individuals receive on average benefits of 56% as a share of previous income. Moreover, I used the data about job finding frequency to calibrate the Poisson parameters q_1 and q_2.

Table 3.1: Calibrated Parameters

Panel I: *Labor market parameters*		
Fraction of economically inactive people	$u^{e.i.}$	26%
Replacement ratio for economically inactive	$\lambda_r^{e.i.}$	33.25%
Separation rate for economically inactive	$\zeta_s^{e.i.}$	2.74%
Fraction of unemployed individuals	u	6.1%
Replacement ratio for unemployed	λ_r	56%
Job finding rate at quarterly frequency	ζ_f	47%
Panel II: *Policy parameters*		
Government spending to GDP ratio	G/Y	5.6%
Government debt to GDP ratio	$P^M B/Y$	45.0%
Total tax revenues to GDP ratio	tax/Y	17.5%
Total transfers to GDP ratio	$transfers/Y$	12.5%

There is no data on the average value of benefits that economically inactive individuals receive, however, using the value of transfers to GDP ratio I have calibrated κ_t to target the replacement ratio $\lambda_r = \kappa_t/(w_t l_t)$; I assume that economically inactive households on average receive 33.25% of the income of the economically active which helps me match the other aggregate ratios contained in the Table 3.1. There is also no data about the frequency of becoming economically inactive, therefore, I adopted the assumption that half of the individuals who are unemployed for a year become economically inactive. In particular, I used information about unemployment duration in the Current Population Survey[8]. The probability of becoming economically inactive of 10.5% at

[6]Rhis value is slightly below the average term to maturity of 4.73 years that refers to central government securities for the period 2000-2018 according to OECD and IMF databases.

[7]The data is taken from the OECD database [https://data.oecd.org] and BLS database [https://www.bls.gov].

[8]ca n be found at https://www.bls.gov

annual frequency implies the separation rate of $\zeta = 2.74\%$ at quarterly frequency[9].

In this chapter, I consider scenarios with relaxed borrowing constraints (borrowing constraint is set to -0.1 and -1.0 for the case of $u = 6\%$ and $u = 26\%$ respectively), this way I can focus the analysis about the optimal monetary policy on the wealth redistribution channel with minimal financial frictions.

3.7.2 Stochastic Steady State

Table 3.2 presents the main characteristics of the stochastic steady state under different policy regimes for the different model specifications. Note that for the HANK model, there is a perpetual movement of individuals between the states of economically active and inactive (in the scenario with $u = 26\%$) or between the states of employed and unemployed (in the scenario with $u = 6\%$). In the steady state, the model dynamics and its possible responses to the aggregate shocks cannot be explored, this is because the dynamic case solution has been defined but not solved due to its numerical complexity. Nevertheless, it is insightful to understand the features of the stochastic steady-state and the relevant effects of optimal inflation.

Before I turn to exploring the implications of different monetary policy settings, it is helpful to understand the difference between economies with low ($u = 6\%$), high ($u = 26\%$) and absent unemployment risk (RANK). According to Table 3.2, the difference between the HANK model with $u = 6\%$ and the RANK model is very small and this is not surprising because only a small fraction of the population is unemployed, moreover, the level of the benefits that they receive is high (more than 50% of the average income of employed individuals). Meanwhile, the scenario with $u = 26\%$ generates a much bigger discrepancy with the RANK model, this is due to the reduced labour force and precautionary savings motive: as individuals anticipate a much more prolonged period of being unemployed and receive lower government support, they save substantially more which reduces spending and dampens output.

Figure (3.1) illustrates the distributions of assets and consumption, allowing me to highlight the link between macroeconomic aggregate and individual asset distributions. We can see how the bigger unemployment shock leads to substantially higher savings for the $u = 26\%$ scenario. Moreover, the difference in consumption levels between unemployed states is much bigger, and it is clear that households choose to work more to be able to insure against idiosyncratic risk and be able to smooth out consumption better. Overall, the households' distribution is more uniform across assets and, as a result, the distribution of their consumption is more steady too.

An unemployment shock, that households experience in the $u = 26\%$ scenario, is sufficiently strong to promote negative asset holdings for the unemployed individuals and to create a 37% difference in asset holdings. This variance in asset holdings will be a decisive factor in defining the optimal level of inflation.

[9] here, ζ value is derived from $\sum_{i=1}^{4} \zeta (1-\zeta)^{i-1} = 10.5\%$, where I use an important assumption of the Markov process that the duration of being unemployed does not influence the probability of finding a job.

Table 3.2: RANK and HANK Characteristics of Stochastic Steady State with Optimal and Baseline Policies

	RANK	HANK with $u = 6\%$		HANK with $u = 26\%$	
Inflation,		baseline	optimal	baseline	optimal
annualised	0.0%	0.0%	0.068%	0.0%	0.26%
	(1)	(2)	(3)	(4)	(5)
Consumption	0.544143	0.542174	0.542178	0.468772	0.468734
- unemployed	–	0.527836	0.527820	0.408939	0.408695
- employed	–	0.543090	0.543095	0.489794	0.489829
Output, labor	0.576423	0.577676	0.577681	0.509332	0.509291
Assets	0.0562	0.063431	0.063620	0.055438	0.056022
- unemployed	–	0.040687	0.040816	-0.092163	-0.092869
- employed	–	0.064882	0.065075	0.107298	0.108335
Assets Value	1.019437	1.039427	1.039632	0.916890	0.917520
- unemployed	–	0.666731	0.666987	-1.524282	-1.520999
- employed	–	1.063216	1.063417	1.774600	1.774297
Debt to GDP	44.9923%	44.9831%	44.9916%	0.450045	0.450391
Bond price	16.367171	16.386853	16.341391	16.539054	16.377893
Interest rate	1.00	0.9927	1.00	0.9365	0.9960
Mass on BL	–	0.000787	0.000788	0.000510	0.000568
Utility*	-0.974032	-100.300487	-100.300355 [$0.01\%C$]	-116.489876	-116.488567 [$0.1\%C$]
- employed	–	-102.612893	-102.612542 [$0.35\%C$]	-124.576438	-124.553218 [$2.2\%C$]
- unemployed	–	-64.072797	-64.076097 [$-0.33\%C$]	-93.474277	-93.535330 [$-6.3\%C$]
- debtors	–	-100.300545	-100.300411 [$0.01\%C$]	-132.609311	-132.573565 [$3.6\%C$]
- lenders	–	-91.512343	-91.543041 [$-3.1\%C$]	-103.914482	-103.968086 [$-5.5\%C$]

Notes, * welfare losses are expressed as a % of consumption of the baseline model with zero inflation.

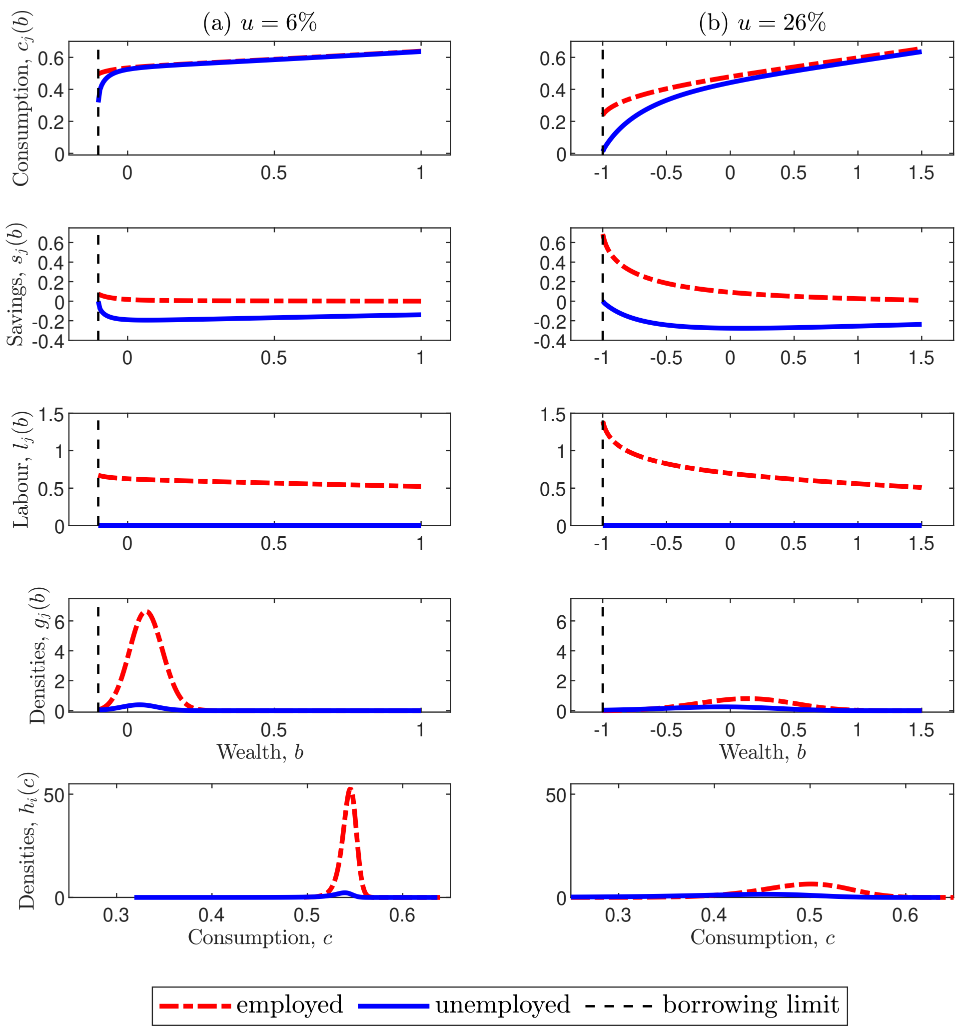

Figure 3.1: Distributions in stochastic steady state

The difference between the baseline zero inflation and optimal inflation is not large, however, it captures some stylized facts about the effect of heterogeneity on the optimal level of inflation. The first observation is that the degree of households' heterogeneity impacts the optimal level of inflation significantly: when I consider the $u = 26\%$ scenario instead of the $u = 6\%$ scenario, the optimal level of inflation is almost 4 times higher. In both cases, an increase in inflation allows unemployed households to borrow more assets and such an increase in demand for bonds leads to an increase in interest rate bringing it very close to 1%. One may think of optimal inflation as the value needed to bring the interest rate close to the household's discount factor and the larger is the interest rate gap generated by implications of the idiosyncratic shock - the larger should be the feedback of the inflation.

Another aspect of the introduction of the optimal policy is its impact on the consumption level of the different categories of individuals. Here, an average employed individual receives a higher level of consumption than the unemployed, which may seem to generate higher inequality in society. However, both employed and unemployed categories of households have debtors and lenders, who experience the direct effect of the change of inflation. Specifically, an increase in inflation leads to a decrease in the income of lenders but reduces the pressure from debtors, so the result of this wealth redistribution channel is the social welfare increase. Interestingly, the welfare gains of this redistributive policy vary a lot depending on the individual wealth state, while the welfare gains of debtors, which are expressed in terms of consumption compensation, is around 3.6%, however, the lenders experience a loss of 5.5%, making such policy not desirable for many households. Thus, even though on the aggregate level the optimal policy does not generate drastically different results, it has a strong redistribution effect that can help to maximize the total welfare and tackle some problems of inequality.

To have a deeper insight into the inflation wealth redistribution channel, consider Figure 3.2. The figure shows the difference in the distribution before and after the optimal policy has been implemented. The blue dashed line represents the average bond holdings of each category of households, for economically inactive households this line is located to the left of the vertical zero line. While employed households have a significant increase in asset lending, unemployed households start to borrow substantially more. The two scenarios exhibit asymmetrical behaviour of the probability density function which is likely to be attributed to the more binding borrowing constraint: while only a small fraction of employed hits the borrowing limit generating a small hump close to the borrowing limit; for the unemployed, this initial significant mass on the constraint is affected by change in inflation, so we observe a substantial increase in the density function at this point. Moreover, the behaviour of the density function in the $u = 6\%$ scenario, is more symmetrical around the average assets line, as in this case inflation operates within a conventional wealth redistributive channel.

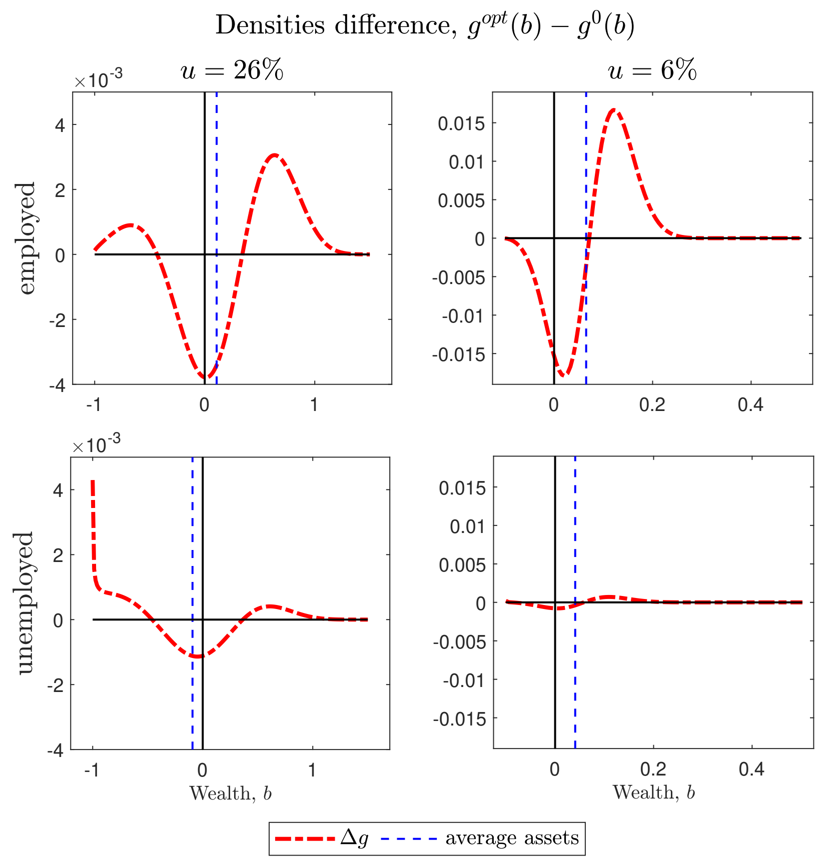

Figure 3.2: Change in probability density function after optimal inflation implementation

3.8 Conclusion

I have explored the optimal monetary policy setting in a continuous-time heterogeneous agent economy subject to uninsurable idiosyncratic employment risk, the key features of the considered model are long-term claims and costly inflation. This framework allows me to reconsider the distributional implications of inflation. Specifically, it has been found that the heterogeneity factor of the model increases the optimal inflation value: the larger the unemployment shock is - the higher the inflation value should be to maximize social welfare. Moreover, the analytical solution for optimal inflation has demonstrated redistributive inflationary bias: given the assumption of concave preferences, individuals in debt exhibit a higher marginal utility compared to lenders, this creates an incentive for the central bank to utilize inflation as a means to redistribute wealth from lenders to debtors.

Another contribution is the analytical solution for the dynamic model with Ramsey-optimal policy and even though it has not been realised numerically yet, it is the first step of the methodology that solves the general equilibrium model with complete markets and, provides a framework for introducing optimal setting of other policy instruments such as taxes and government transfers which have the potential of altering the households' consumption more directly and, therefore, maximize social welfare more efficiently.

To sum up, I have developed the solution for the optimal policy design, which is based on the calculus approach presented in the Nuno and Thomas [2020] paper. However, I have applied it to a more complex, New Keynesian model setting, which is characterised by endogenous labour supply, complete markets and sticky prices. As a result, the characteristics of the inflation optimality condition has inherited many properties from the Nuno and Thomas paper, such as redistributive inflationary bias, application of nominal asset position, wealth dispersion and marginal consumption to define optimal inflation value. And, at the same time, the model modification has led to substantial changes in the inflation optimality condition so now more factors affect optimal trajectory of inflation especially after the initial period. These factors include: price adjustment cost, stringency of the budget constraint, indirect effects of the market clearing conditions and the Phillips curve.

Thus, the model has incorporated the analytical approach of Nuno and Thomas [2020] but obtained the results for a more complex environment, that are likely to be considered by a policymaker as more appropriate for optimal policy design.

General Conclusion

In my thesis, I explore the properties of different heterogeneous agent models, compare their outcome with representative agent models, their welfare properties and implications for policy design. More specifically, I start with a standard New Keynesian model setting that features capital accumulation – this model provides a good foundation for understanding the main principles of the heterogeneous-agent continuous-time New Keynesian framework by establishing the numerical algorithm and solving the model with complete markets under different scenarios of the idiosyncratic shock. Next, I consider the model with nominal government debt and distortionary taxation – this set-up enables me to explore the properties and implications of the conventional and unconventional monetary and fiscal policy mix. I conclude the thesis with the chapter on optimal policy setting in the HANK environment, where I explore an analytical characterization of optimal monetary policy and compare its numerical results with the RANK model.

In Chapter 1, I consider the New Keynesian model with capital accumulation, where households face idiosyncratic labour productivity shock. Here, I explore the properties of the stationary distributions under different scenarios of the productivity shocks and dynamics of the model in response to the aggregate shock. I have found that the properties of households' wealth and consumption distribution are coherent with the results of the relevant HACT papers and the behaviour of the model in transition is consistent with the standard New Keynesian literature. Also, I have found that changing the parameters of the idiosyncratic shock has a significant effect on the consumption and labour choices of high- and low-productive workers, and thus leads to substantial changes in the aggregate welfare. In particular, I have discovered that the decreased magnitude of the productivity shock leads to a diminished aggregate utility, which is attributed to the unique form of the utility function and has not been thoroughly studied in continuous-time and discrete-time heterogeneous-agent literature. This result signifies the importance of an accurate calibration of the utility function and income process.

In Chapter 2, I have explored a continuous-time HANK model, the distinctive features of which include: sticky prices, monetary policy, endogenous labour supply, distortionary taxation and government debt which is both nominal and of long maturity. In this framework, I have analysed the properties of a stochastic steady state and the effect of a relaxed borrowing limit. My main innovation of this chapter is defining the policy regime of fiscal dominance which has not been explored in the heterogeneous-agent literature. To do this, I have developed a stable numerical algorithm, which requires rethinking the standard approach due to the properties of the FTPL policy regime (I had to carefully identify the variables which prevent the debt explosion and incorporate that into the algorithm design). To analyse the dynamics and welfare implications of the monetary and fiscal policy dominance regimes, I consider a cost-push shock. I have found that the two policy regimes of the RANK and HANK models generate qualitatively very similar results, which is attributed to several factors of the considered model: absence of aggregate uncertainty, debt maturity and the high level of outstanding debt. However, the welfare difference between the two regimes in the HANK model is much smaller compared to the RANK model, because inflation has an additional wealth redistribution channel within the heterogeneous-agent framework and because the borrowing limit affects private agents' decision-making. A similar result has been obtained for the HANK model with a relaxed borrowing limit, where the outcomes of some variables get closer to those of the RANK model. These findings are interesting and are likely to have further applications for the optimal policy design.

In Chapter 3, I have explored the optimal monetary policy setting in a continuous-time HANK model with government bonds. In this chapter, I employ a variational approach to define and solve the policymaker problem under commitment assumption. This approach was introduced by Nuno and Thomas [2020], who applied it to a rather simple heterogeneous agent economy; nevertheless, they have obtained interesting analytical results and found that households' wealth dispersion generates a redistributive inflationary bias. My findings align with those presented by Nuno and Thomas, however, a more complex framework has led to numerous discrepancies. In particular, I have found that stringency of the budget constraint and nominal asset position have a direct effect on optimal inflation. Moreover, the introduction of sticky prices has led to the Phillips curve being one of the defining factors of optimal inflation. Also, the inclusion of complete markets of labour and assets has a significant impact on optimal inflation setting. Thus, the developed framework can capture hidden transmission mechanisms of the heterogeneous agent innovation and apply them for optimal policy setting in a more complex, New Keynesian, environment.

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Appendix to Chapter 1

A.1 Derivation of HJB and KF equations

Consider the household's problem in discrete time

$$
\begin{aligned}
E_0 \sum_{i=0}^{\infty} e^{-\rho \Delta i} u\left(c_{\Delta i}, l_{\Delta i}\right) \quad \text{such that} \\
\frac{a_{t+\Delta}-a_t}{\Delta} = w_t z_t l_t + r_t a_t + \Pi\left(z_t\right) - T_t - c_t + a_t \\
a_{t+\Delta} \geq \underline{a}
\end{aligned}
\tag{A.1}
$$

where $\Delta i = t$ is time measured as a number of periods, each of which has length Δ.

The assumption of the Poisson process on z_t implies that the process jumps from state z_1 to state z_2 with intensity λ_1 and vice versa with intensity λ_2, i.e. the households face the same labour productivity z_j with probability $e^{-\lambda_j \Delta}$ and switch to the productivity z_{-j} with probability $1-e^{-\lambda_j \Delta}$. Thus the Bellman equation for the household's problem is

$$
v_j(a,t) = \max_{c,l} u(c,l)\Delta + e^{-\rho\Delta}\left[e^{-\lambda_j\Delta} v_j\left(a_{t+\Delta}, t+\Delta\right) + \left(1-e^{-\lambda_j\Delta}\right) v-_j\left(a_{t+\Delta}, t+\Delta\right)\right]
$$

Immediately taking $\Delta \to 0$, we can rewrite

$$
\begin{aligned}
e^{-\rho\Delta} &\approx 1-\rho\Delta \\
e^{-\lambda_j\Delta} &\approx 1-\lambda_j\Delta
\end{aligned}
$$

So,

$$
v_j(a,t) = \max_{c,l} u(c,l)\Delta + (1-\rho\Delta)\left[\left(1-\lambda_j\Delta\right) v_j\left(a_{t+\Delta}, t+\Delta\right) + \lambda_j \Delta v-_j\left(a_{t+\Delta}, t+\Delta\right)\right]
$$

Then subtract from both sides of the equations $(1-\rho\Delta)v_j(a,t)$ leads to

$$\begin{aligned}
\Delta\rho v_j(a,t) &= \max_{c,l} u(c,l)\Delta + (1-\rho\Delta)\left[v_j(a_{t+\Delta},t+\Delta) - v_j(a,t)\right] \\
&\quad + (1-\rho\Delta)\left[-\lambda_j\Delta v_j(a_{t+\Delta},t+\Delta) + \lambda_j\Delta v-_j(a_{t+\Delta},t+\Delta)\right] \\
\rho v_j(a,t) &= \max_{c,l} u(c,l) + \frac{(1-\rho\Delta)}{\Delta}\left[v_j(a_{t+\Delta},t+\Delta) - v_j(a,t)\right] \\
&\quad + \frac{(1-\rho\Delta)}{\Delta}\left[-\lambda_j\Delta v_j(a_{t+\Delta},t+\Delta) + \lambda_j\Delta v-_j(a_{t+\Delta},t+\Delta)\right]
\end{aligned}$$

1) take the limit when $\Delta \to 0$ of $\left(\frac{1}{\Delta}-\rho\right)\left[\lambda_j\Delta v-_j(a_{t+\Delta},t+\Delta) - \lambda_j\Delta v_j(a_{t+\Delta},t+\Delta)\right]$:

$$\begin{aligned}
&\lambda_j v-_j\left(\underbrace{a_{t+\Delta}}_{=a_t},\underbrace{t+\Delta}_{=t}\right) - \lambda_j v_j\left(\underbrace{a_{t+\Delta}}_{=a_t},\underbrace{t+\Delta}_{=t}\right) - \rho\left[\underbrace{\lambda_j\Delta v-_j(a_{t+\Delta},t+\Delta)}_{=0} - \underbrace{\lambda_j\Delta v_j(a_{t+\Delta},t+\Delta)}_{=0}\right] \\
&\overset{\Delta\to 0}{=} \lambda_j v-_j(a,t) - \lambda_j v_j(a,t)
\end{aligned}$$

2) take the limit when $\Delta \to 0$ of the expression $\left(\frac{1}{\Delta}-\rho\right)\left[v_j(a_{t+\Delta},t+\Delta) - v_j(a,t)\right]$

$$\begin{aligned}
&\lim_{\Delta\to 0}\left\{\frac{1}{\Delta}\left[v_j(a_{t+\Delta},t+\Delta) - v_j(a,t)\right] - \underbrace{\rho\left[v_j\left(\underbrace{a_{t+\Delta}}_{a_t},\underbrace{t+\Delta}_{t}\right) - v_j(a,t)\right]}_{=0}\right\} \\
&= \lim_{\Delta\to 0}\left\{\frac{1}{\Delta}\left[v_j(a_{t+\Delta},t+\Delta) - v_j(a,t)\right]\right\} \\
&= \lim_{\Delta\to 0}\frac{v_j(a_{t+\Delta},t+\Delta) - v_j(a,t)}{\Delta}
\end{aligned}$$

Notice that

$$v_j(a_{t+\Delta},t+\Delta) \approx v_j(a_{t+\Delta},t) + \partial_t v_j(a_{t+\Delta},t)\Delta$$

Therefore,

$$\begin{aligned}
&\lim_{\Delta\to 0}\frac{v_j(a_{t+\Delta},t) - v_j(a,t)}{\Delta} \overset{(A.1)}{=} \\
&= \lim_{\Delta\to 0}\frac{v_j((w_t z_t l_t + r_t a_t + \Pi(z_t) - T_t - c_t)\Delta + a_t, t) - v_j(a,t)}{\Delta} \\
&= \lim_{\Delta\to 0}\frac{v_j(\Delta' + a_t, t) - v_j(a,t)}{\Delta'}(w_t z_t l_t + r_t a_t + \Pi(z_t) - T_t - c_t) \\
&= \partial_a v_j(a,t)\left[w_t z_t l_t + r_t a_t + \Pi(z_t) - T_t - c_t\right]
\end{aligned}$$

Finally, summing up all the results I obtain

$$\frac{(1-\rho\Delta)}{\Delta}\left[v_j(a_{t+\Delta},t+\Delta)-v_j(a,t)-\lambda_j\Delta v_j(a_{t+\Delta},t+\Delta)+\lambda_j\Delta v-_j(a_{t+\Delta},t+\Delta)\right]$$
$$=\lambda_j v-_j(a,t)-\lambda_j v_j(a,t)+\partial_t v_j(a_{t+\Delta},t)+\partial_a v_j(a,t)\left[w_t z_t l_t+r_t a_t+\Pi(z_t)-T_t-c_t\right]$$

Which gives the HJB equation (1.4):

$$\rho v_j(a,t)=\max_{c,l} u(c,l)+\partial_a v_j(a,t)\left(w_t z_j l+r_t a+\Pi_t\left(z_j\right)-T_t-c\right)$$
$$+\lambda_j\left(v_{-j}(a,t)-v_j(a,t)\right)+\partial_t v_j(a,t)$$

In order to obtain the stationary version of this equation, eliminate variable t from the equation and note that $\partial_t v_j(a,t)=0$

$$\rho v_j(a)=\max_{c,l} u(c,l)+v_j'(a)\left(wz_j l+ra+\Pi(z)-T-c\right)+\lambda_j\left(v_{-j}(a)-v_j(a)\right)$$

To derive Kolmogorov-Forward equation (1.8) I will use cumulative density function (rather than just density function) and discrete analogue of savings function equation. Wealth evolves as

$$\dot{a}_t=s_j(a,t)\Rightarrow da_t=s_j(a,t)\,dt \tag{A.2}$$

where the optimal savings s_j is derived from the utility maximization problem:

$$s_j(a,t)=w_t z_j l_j(a,t)+r_t a+\Pi_t\left(z_j\right)-T_t-c_j(a,t)$$

here $c_j(a,t)$ and $l_j(a,t)$ are derived from the value function v using FOCs:

$$\partial_c u(c,l)=\partial_a v_j(a,t)\quad\text{and}\quad\partial_l u(c,l)=-w_t z_j\partial_a v_j(a,t)$$

The discrete time analogue of (A.2) is

$$a_{t+\Delta}=a_t+\Delta s_j(a_t)$$

However, also the analogue is

$$a_{t+\Delta}=a_t+\Delta s_j(a_{t+\Delta})$$
$$a_t=a_{t+\Delta}+\Delta s_j(a_{t+\Delta})$$

because the solution to the differential equation (A.2) implies

$$a_{t+\Delta} = \int_t^{t+\Delta} s_j(a_\tau)\, d\tau \approx \begin{matrix} \Delta s_j(a_t) \\ \Delta s_j(a_{t+\Delta}) \end{matrix}$$

Let's define CDF $G_j(a,t)$

$$G_j(a,t) = \Pr\left(a_t \le a, z_t \le z_j\right)$$

Then the density function $g_j(a,t) = \partial_a G_j(a,t)$.

Assume $s_j(a) \le 0$ and consider the following expression

$$\Pr(a_{t+\Delta} \le a) = \Pr(a_t \le a) + \Pr\left(a \le a_t \le a - \Delta s_j(a)\right) = \Pr\left(a_t \le a - \Delta s_j(a)\right)$$

Note that $a_{t+\Delta} \le a$ => $a_t + \Delta s_j(a_t) \le a$, which under condition $s_j(a) \le 0$ transforms to $a_t \le a$ and $a_t \ge a$ & $a_t + \Delta s_j(a_t) \le a$. However, the tricky part is: why $a_t \le a - \Delta s_j(a_t)$ implies $a_t \le a - \Delta s_j(a)$. Consider $a_t \le a$ it should imply that $s_j(a) \le s_j(a_t)$ (which is an implicit property of s: it is decreasing)

Next, I explore the expression which identifies CDF:

$$\begin{aligned} \Pr\left(a_{t+\Delta} \le a, z_{t+\Delta} = z_j\right) = {} & \left(1 - \Delta\lambda_j\right) \Pr\left[a_t \le a - \Delta s_j(a), z_t = z_j\right] \\ & + \Delta\lambda_j \Pr\left[a_t \le a - \Delta s_{-j}(a), z_t = z_{-j}\right] \end{aligned}$$

it implies that

$$\begin{aligned} G_j(a,t+\Delta) &= \left(1-\Delta\lambda_j\right) G_j\left(a - \Delta s_j(a), t\right) + \Delta\lambda_{-j} G_{-j}\left(a - \Delta s_{-j}(a), t\right) \\ \frac{G_j(a,t+\Delta) - G_j(a,t)}{\Delta} &= \frac{\left(1-\Delta\lambda_j\right) G_j\left(a - \Delta s_j(a), t\right) + \Delta\lambda_{-j} G_{-j}\left(a - \Delta s_{-j}(a), t\right)}{\Delta} - \frac{G_j(a,t)}{\Delta} \\ \frac{G_j(a,t+\Delta) - G_j(a,t)}{\Delta} &= \frac{G_j\left(a - \Delta s_j(a), t\right) - G_j(a,t)}{\Delta} \\ & \quad + \lambda_{-j} G_{-j}\left(a - \Delta s_{-j}(a), t\right) - \lambda_j G_j\left(a - \Delta s_j(a), t\right) \end{aligned}$$

And to derive the final formula take $\Delta \to 0$

$$\begin{aligned} \lim_{\Delta \to 0} \frac{G_j\left(a - \Delta s_j(a), t\right) - G_j(a,t)}{\Delta} &= \lim_{\Delta \to 0} \frac{G_j\left(a - \Delta s_j(a), t\right) - G_j(a,t)}{\Delta\left(-s_j(a)\right)} \left(-s_j(a)\right) \\ &= \lim_{\delta \to 0} \frac{G_j(a+\delta, t) - G_j(a,t)}{\delta} \left(-s_j(a)\right) = \partial_a G_j(a,t)\left[-s_j(a,t)\right] \end{aligned}$$

Therefore,

$$\lim_{\Delta\to 0}\left\{\lambda_{-j}G_{-j}\left(a-\Delta s_{-j}(a),t\right)-\lambda_j G_j\left(a-\Delta s_j(a),t\right)\right\}=\lambda_{-j}G_{-j}(a,t)-\lambda_j G_j(a,t)$$

Thus,

$$\begin{aligned}
\partial_t G_j(a,t) &= -s_j(a,t)\,\partial_a G_j(a,t)+\lambda_{-j}G_{-j}(a,t)-\lambda_j G_j(a,t)\\
\partial_t G_j(a,t) &= -s_j(a,t)\,g_j(a,t)+\lambda_{-j}G_{-j}(a,t)-\lambda_j G_j(a,t)\\
\partial_a\left[\partial_t G_j(a,t)\right] &= \partial_a\left[-s_j(a,t)\,g_j(a,t)+\lambda_{-j}G_{-j}(a,t)-\lambda_j G_j(a,t)\right]\\
\partial_t g_j(a,t) &= -\partial_a\left[s_j(a,t)\,g_j(a,t)\right]+\lambda_{-j}g_{-j}(a,t)-\lambda_j g_j(a,t)
\end{aligned}$$

As a result the KF equation (1.8) is derived.

Note that all the derivations can be repeated for the case $s_j(a)\geq 0$.

To obtain the stationary version of the KF equation, i.e. equation (1.14), we have to exclude variable t from the equation and put all the time derivatives be equal to 0, in this case the following expression will be obtained:

$$0=-\frac{d}{da}\left[s_j(a)\,g_j(a)\right]-\lambda_j g_j(a)+\lambda_{-j}g_{-j}(a)$$

A.2 Derivation of the Phillips curve

Each intermediate producer chooses its price to maximize profits subject to price adjustment costs

$$\begin{aligned}
&\int_0^\infty e^{-\int_0^t \rho ds}\left(\tilde{\Pi}_{j,t}-\Theta\left(\pi_{j,t}\right)\right)\\
&=\int_0^\infty e^{-\rho t}\left\{\left(\frac{p_{j,t}}{P_t}-mc_t\right)\left(\frac{p_{j,t}}{P_t}\right)^{-\varepsilon}Y_t-\frac{\theta}{2}\left(\frac{\dot{p}_{j,t}}{p_{j,t}}\right)^2 Y_t\right\}
\end{aligned}$$

Let's introduce the function $J(p,t)$, it is the real value of a firm with price p.

$$\begin{aligned}
\rho J(p,t) &= \max_\pi\left(\frac{p}{P(t)}-mc(t)\right)\left(\frac{p}{P(t)}\right)^{-\varepsilon}Y(t)-\frac{\theta}{2}\pi^2 Y(t)+J_p(p,t)\,\dot{p}+J_t(p,t)\\
&= \max_\pi\left(\frac{p}{P(t)}-mc(t)\right)\left(\frac{p}{P(t)}\right)^{-\varepsilon}Y(t)-\frac{\theta}{2}\pi^2 Y(t)+J_p(p,t)\,p\pi+J_t(p,t)
\end{aligned}$$

The first order and envelope conditions for the firm are

$$J_p(p,t) = \theta p Y(t)$$
$$(\rho - \pi(t)) J_p(p,t) = -\left(\frac{p}{P(t)} - mc(t)\right)\left(\frac{p}{P(t)}\right)^{-\varepsilon-1} \times \frac{Y(t)}{p} + \left(\frac{p}{P(t)}\right)^{-\varepsilon} \frac{Y(t)}{p}$$
$$+ J_{pp}(p,t)\, p\pi(t) + J_{tp}(p,t)$$

For simpler notations I omit "(t)" for the following variables: $Y(t), \pi(t), mc(t), P(t)$:

$$pJ_p(p,t) = \theta \pi Y$$
$$(\rho - \pi) J_p(p,t) = -\left(\frac{p}{P} - mc(t)\right)\left(\frac{p}{P}\right)^{-\varepsilon-1} \times \frac{Y}{P} + \left(\frac{p}{P}\right)^{-\varepsilon} \frac{Y}{P} + J_{pp}(p,t)\, p\pi + J_{tp}(p,t)$$

In a symmetric equilibrium we will have $p_j = p = P$, and hence

$$J_p(p,t) = \frac{\theta \pi Y}{p} \tag{A.3}$$
$$(\rho - \pi) J_p(p,t) = -(1 - mc(t)) \times \frac{Y}{P} + \frac{Y}{P} + J_{pp}(p,t)\, p\pi + J_{tp}(p,t) \tag{A.4}$$

Differentiating (A.3) with respect to time gives

$$J_{pp}(p,t)\dot{p} + J_{pt}(p,t) = \frac{\theta \dot{\pi} Y}{p} + \frac{\theta \pi \dot{Y}}{p} - \frac{\theta \pi Y}{p}\frac{\dot{p}}{p}$$

Substituting into the envelope condition (A.4) and dividing by $\theta Y/p$ implies the needed Phillips curve equation:

$$\left(\rho - \frac{\dot{Y}}{Y}\right)\pi = \frac{1}{\theta}(1 - (1 - mc)\varepsilon) + \dot{\pi}$$

A.3 Transition dynamics of the HANK model with capital accumulation. TFP shock

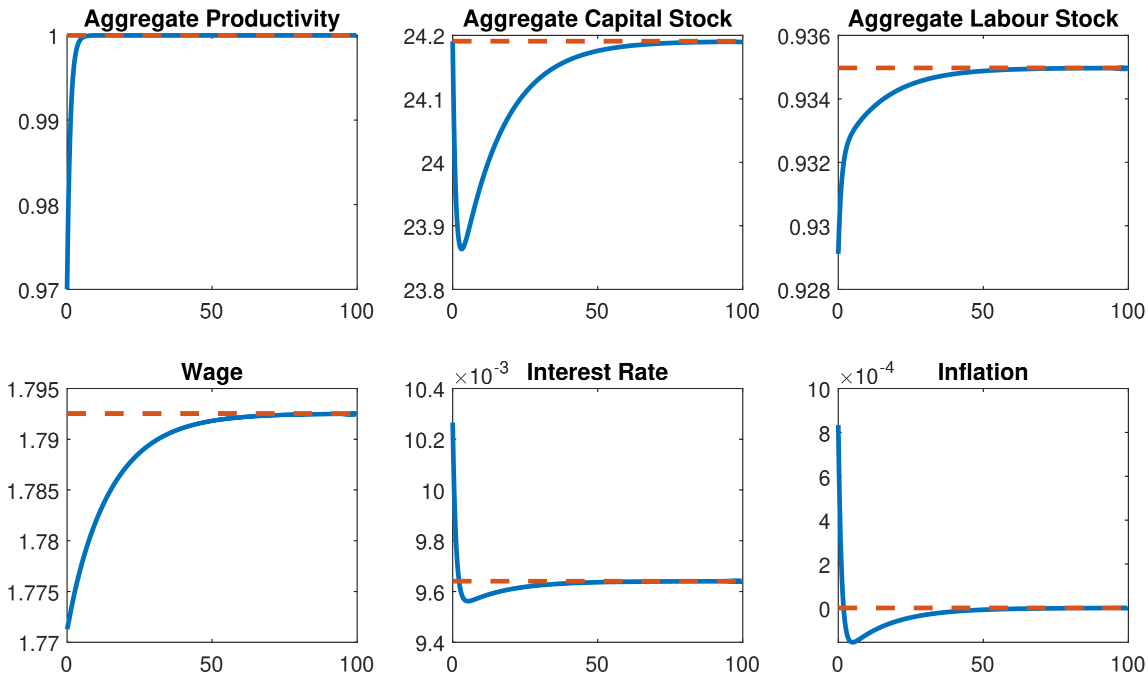


Figure A.1: Impulse responses to a 3% TFP shock*

Notes, * a blue solid line represents the variable's deviation from its steady-state level which is represented by a red dashed line.

Appendix to Chapter 2

B.1 Derivation of continuous-time budget constraint

This section aims to derive the household's budget constraint using long-term bonds and a continuous-time framework.

In this section, I use the following notations:

- b_t^{t+s} is a **face value** of zero-coupon bond outstanding at date t, redeemable at date $t+s$,
- $e^{-\int_t^{t+s} q(\xi)d\xi}$ is the discount factor applied to nominal payoffs obtained in period $t+s$.

At time t the government buys bonds issued at time $t-dt$, b_{t-dt}^{t+s} and pays $\int_0^\infty e^{-\int_t^{t+s} q(\xi)d\xi} b_{t-dt}^{t+s} ds$. It also receives the primary *nominal* surplus $S_t dt$ (the HANK and RANK models have the same government budget constraint that features a surplus $S_t = P_t\left(\tau_t w_t N_t - G_t - Tr_t\right)$) and covers the deficit by new long-term bonds $\int_{dt}^\infty e^{-\int_t^{t+s} q(\xi)d\xi} b_t^{t+s} ds$. So, the budget constraint *in nominal terms* is

$$\int_0^\infty e^{-\int_t^{t+s} q(\xi)d\xi} b_{t-dt}^{t+s} ds = S_t dt + \int_{dt}^\infty e^{-\int_t^{t+s} q(\xi)d\xi} b_t^{t+s} ds \tag{B.1}$$

this can be rewritten as,

$$\begin{aligned}
\int_0^\infty e^{-\int_t^{t+s} q(\xi)d\xi} b_{t-dt}^{t+s} ds - \int_{dt}^\infty e^{-\int_t^{t+s} q(\xi)d\xi} b_t^{t+s} ds &= S_t dt \\
\int_0^{dt} e^{-\int_t^{t+s} q(\xi)d\xi} b_{t-dt}^{t+s} ds + \int_{dt}^\infty e^{-\int_t^{t+s} q(\xi)d\xi} \left(b_{t-dt}^{t+s} - b_t^{t+s}\right) ds &= S_t dt \\
e^{-\int_t^{t+s^*} q(\xi)d\xi} b_{t-dt}^{t+s^*} dt + \int_{dt}^\infty e^{-\int_t^{t+s} q(\xi)d\xi} \left(b_{t-dt}^{t+s} - b_t^{t+s}\right) ds &= S_t dt
\end{aligned}$$

where $s^* \in [0, dt]$, which implies

$$\begin{aligned}
e^{-\int_t^{t+dt} q(\xi)d\xi} b_{t-dt}^{t+dt} dt \ &\geq I \geq \ e^{-\int_t^{t} q(\xi)d\xi} b_{t-dt}^{t} dt \\
e^{-q(\xi^*)dt} b_{t-dt}^{t+dt} dt \ &\geq I \geq \ e^{0} b_{t-dt}^{t} dt
\end{aligned}$$

where $\xi^* \in [t, t+dt]$. As dt is close to zero, $dt^2 \approx 0$

$$e^{-q(\xi^*)dt} \approx 1 - q(\xi^*)dt$$
$$(1 - q(\xi^*)dt)\, b_{t-dt}^{t+dt} dt = b_{t-dt}^{t+dt} dt - q(\xi^*) b_{t-dt}^{t+dt} dt^2 \approx b_{t-dt}^{t+dt} dt$$

therefore,

$$I = b_{t-dt}^{t+s^*} dt, \quad s^* \in [0, dt]$$

As a result, we can rewrite the constraint as

$$b_{t-dt}^{t+s^*} dt + \int_{dt}^{\infty} e^{-\int_t^{t+s} q(\xi) d\xi} \left(b_{t-dt}^{t+s} - b_t^{t+s} \right) ds = S_t dt$$

Let's introduce new variable B_t^M where 'M' is not an index but notation showing that these are bonds with variable maturity. We assume that the maturity structure of bonds is declining at *constant* rate δ and

$$b_{t-dt}^{t+s} = e^{-\delta s} B_{t-dt}^M \tag{B.2}$$

so,

$$b_{t-dt}^{t} = B_{t-dt}^M$$

and

$$b_{t-dt}^{t-dt+s+dt} = e^{-\delta s} B_{t-dt}^M$$
$$p = t - dt$$
$$b_p^{p+s+dt} = e^{-\delta s} B_p^M$$
$$w = s + dt$$
$$b_p^{p+w} = e^{-\delta (w-dt)} B_p^M$$
$$b_t^{t+s} = e^{-\delta (s-dt)} B_t^M$$

substitute this into the budget constraint and get

$$b_{t-dt}^{t+s^*} dt + \int_{dt}^{\infty} e^{-\int_t^{t+s} q(\xi) d\xi} \left(b_{t-dt}^{t+s} - b_t^{t+s} \right) ds = S_t dt$$
$$e^{-\delta s^*} B_{t-dt}^M dt + \int_{dt}^{\infty} e^{-\int_t^{t+s} q(\xi) d\xi} \left(e^{-\delta s} B_{t-dt}^M - e^{-\delta (s-dt)} B_t^M \right) ds = S_t dt$$
$$B_{t-dt}^M \left(e^{-\delta s^*} dt + \int_{dt}^{\infty} e^{-\int_t^{t+s} q(\xi) d\xi} e^{-\delta s} ds \right) - B_t^M \int_{dt}^{\infty} e^{-\int_t^{t+s} q(\xi) d\xi} e^{-\delta (s-dt)} ds = S_t dt \tag{B.3}$$

Now define the variable of bond price P_t^M

$$P_t^M = \int_{dt}^{\infty} e^{-\int_t^{t+s} q(\xi) d\xi} e^{-\delta (s-dt)} ds \tag{B.4}$$

so that the budget constraint becomes

$$S_t dt = B^M_{t-dt}\left(e^{-\delta s^*}dt + \int_{dt}^{\infty} e^{-\int_t^{t+s} q(\xi)d\xi} e^{-\delta(s-dt)} e^{-\delta dt} ds\right)$$
$$- B^M_t \int_{dt}^{\infty} e^{-\int_t^{t+s} q(\xi)d\xi} e^{-\delta(s-dt)} ds$$
$$S_t dt = B^M_{t-dt}\left(e^{-\delta s^*}dt + e^{-\delta dt}P^M_t\right) - B^M_t P^M_t$$

this expression can be simplified

$$e^{-\delta dt} \approx 1 - \delta dt$$

so, the budget constraint is now

$$B^M_{t-dt}\left(e^{-\delta s^*}dt + [1-\delta dt]\, P^M_t\right) - B^M_t P^M_t = S_t dt$$
$$P^M_t \frac{B^M_{t-dt} - B^M_t}{dt} + \frac{B^M_{t-dt}\left(e^{-\delta s^*}dt - \delta dt P^M_t\right)}{dt} = \frac{S_t dt}{dt}$$
$$-\dot{B}^M_{t-dt} P^M_t + B^M_{t-dt}\left(e^{-\delta s^*} - \delta P^M_t\right) = S_t$$

Note that $s^* \in [0, dt]$ and $dt \to 0$, which means $e^{-\delta s^*} = e^0 = 1$, therefore

$$-\dot{B}^M_t P^M_t + B^M_t\left(1 - \delta P^M_t\right) = S_t \qquad \text{(B.5)}$$

or by using the de3finition of the surplus S_t, the budget constraint is

$$-\dot{B}^M_t P^M_t + B^M_t\left(1 - \delta P^M_t\right) = P_t\left(\tau_t w_t N_t - G_t - Tr_t\right) \qquad \text{(B.6)}$$
$$\dot{B}^M_t = -\frac{1}{P^M_t}\left[P_t\left(\tau_t w_t N_t - G_t - Tr_t\right) - B^M_t\left(1 - \delta P^M_t\right)\right] \qquad \text{(B.7)}$$

which is the exact budget constraint we use in the HANK and RANK models.

Similarly, I derive household's budget constraint in nominal terms for the HANK model. At time t the household buys $\int_{dt}^{\infty} e^{-\int_t^{t+s} q(\xi)d\xi} b^{t+s}_t ds$ using the surplus S^H_t and income from the bonds holdings $\int_0^{\infty} e^{-\int_t^{t+s} q(\xi)d\xi} b^{t+s}_{t-dt} ds$. In the case of the HANK model the surplus is defined the following way: $S^H_t = P_t\left(\left[(1-\tau_t)\, w_t l_t + \Pi_t\right] z_t + [ben_t]\,(1 - z_t) + Tr_t - c_t\right)$; in the case of the RANK model it is defined as $S^H_t = P_t\left((1-\tau_t)\, w_t l_t + Tr_t + \Pi_t - c_t\right)$.

Then the household's budget constraint is

$$\int_{dt}^{\infty} e^{-\int_t^{t+s} q(\xi)d\xi} b^{t+s}_t ds = S^H_t dt + \int_0^{\infty} e^{-\int_t^{t+s} q(\xi)d\xi} b^{t+s}_{t-dt} ds \qquad \text{(B.8)}$$

I then define bonds price P^M the same way as before (B.4) and the new variable for the household's holdings of bonds b^M_t: $b^{t+s}_{t-dt} = e^{-\delta s} b^M_{t-dt}$. Performing the same transformations, I obtain the following household's budget constraint:

$$\dot{b}_t^M P_t^M = S_t^H + b_t^M \left(1 - \delta P_t^M\right) \tag{B.9}$$

which in the case of the RANK model is written as

$$\begin{aligned}
\dot{b}_t^M P_t^M &= P_t\left((1-\tau_t) w_t l_t + Tr_t + \Pi_t - c_t\right) + b_t^M \left(1 - \delta P_t^M\right) \\
\dot{b}_t^M &= \frac{1}{P_t^M}\left[P_t\left((1-\tau_t) w_t l_t + Tr_t + \Pi_t - c_t\right) + b_t^M \left(1 - \delta P_t^M\right)\right]
\end{aligned}$$

and in the case of the HANK model has the following form

$$\begin{aligned}
\dot{b}_t^M P_t^M &= P_t\left([(1-\tau_t) w_t l_t + \Pi_t] z_t + [ben_t](1-z_t) + Tr_t - c_t\right) + b_t^M \left(1 - \delta P_t^M\right) \\
\dot{b}_t^M &= \frac{1}{P_t^M}\left[P_t\left([(1-\tau_t) w_t l_t + \Pi_t] z_t + [ben_t](1-z_t) + Tr_t - c_t\right) + b_t^M \left(1 - \delta P_t^M\right)\right]
\end{aligned}$$

B.2 Derivation of continuous-time no-arbitrage condition

To find the connection between bond price P_t^M and short-term interest rate q_t, consider bond price at period t and bond price at period $t+dt$

$$\begin{aligned}
P_t^M &= \int_a^\infty e^{-\int_t^{t+s} q(\xi)d\xi} e^{-\delta(s-a)} ds \\
P_{t+dt}^M &= \int_a^\infty e^{-\int_{t+dt}^{t+dt+s} q(\xi)d\xi} e^{-\delta(s-a)} ds
\end{aligned}$$

then

$$P_t^M = \int_a^\infty e^{-\int_t^{t+s} q(\xi)d\xi} e^{-\delta(s-a)} ds = \int_{a-dt}^\infty e^{-\int_t^{t+k+dt} q(\xi)d\xi} e^{-\delta(k+dt-a)} dk$$

here I substitute variable s with k: $s = k + dt$, $k = s - dt$, $dk = ds$.

$$\begin{aligned}
P_t^M &= \int_{a-dt}^\infty e^{-\int_t^{t+dt} q(\xi)d\xi} e^{-\int_{t+dt}^{t+k+dt} q(\xi)d\xi} e^{-\delta(k+dt-a)} dk \\
&= e^{-\delta dt} e^{-\int_t^{t+dt} q(\xi)d\xi} \int_{a-dt}^\infty e^{-\int_{t+dt}^{t+k+dt} q(\xi)d\xi} e^{-\delta(k-a)} dk \\
&= e^{-\delta dt} e^{-\int_t^{t+dt} q(\xi)d\xi} \left(\int_{a-dt}^a e^{-\int_{t+dt}^{t+k+dt} q(\xi)d\xi} e^{-\delta(k-a)} dk + \int_a^\infty e^{-\int_{t+dt}^{t+k+dt} q(\xi)d\xi} e^{-\delta(k-a)} dk\right) \\
&= e^{-\delta dt} e^{-q(\xi^*)dt} \left(e^{-q(\xi^*)dk^*} e^{-\delta(k^*-a)} dt + P_{t+dt}^M\right)
\end{aligned}$$

where $\xi^* \in [t, t+dt]$ and $k^* \in [a-dt, a] \overset{a=dt}{=>} k^* \in [0, dt]$.

Next, I approximate the expression with a first-order Taylor expansion using that dt is near zero, and so is k^*

$$
\begin{aligned}
P_t^M &= (1-\delta dt)(1-q(\xi^*)dt)\left((1-q(\xi^*)dk^*)(1-\delta(k^*-dt))dt + P_{t+dt}^M\right)\\
P_t^M &= (1-\delta dt)(1-q(\xi^*)dt)(1-q(\xi^*)dk^*)(1-\delta(k^*-dt))dt\\
&\quad + (1-\delta dt)(1-q(\xi^*)dt)P_{t+dt}^M\\
P_t^M &= (1-\delta dt - q(\xi^*)dt - q(\xi^*)dt - \delta(k^*-dt))dt + P_{t+dt}^M(1-\delta dt - q(\xi^*)dt)
\end{aligned}
$$

$$
\begin{aligned}
P_{t+dt}^M(1-\delta dt - q(\xi^*)dt) - P_t^M(1-\delta dt - q(\xi^*)dt) &= P_t^M - P_t^M(1-\delta dt - q(\xi^*)dt)\\
&\quad - (1-\delta dt - q(\xi^*)dt - q(\xi^*)dt - \delta(k^*-dt))dt
\end{aligned}
$$

$$
(1-\delta dt - q(\xi^*)dt)\frac{P_{t+dt}^M - P_t^M}{dt} = P_t^M(\delta + q(\xi^*)) - (1-\delta dt - q(\xi^*)dt - q(\xi^*)dt - \delta(k^*-dt))
$$

after that, I apply the limit $dt \to 0$ and get

$$
\widehat{\dot{P}}_t^M = P_t^M(\delta + q_t) - 1 \tag{B.10}
$$

Thus, the no-arbitrage condition for the P_t^M is obtained. It links bond price P_t^M and short term interest rate q_t. Here $\widehat{\dot{P}}$ denotes the right (forward-looking) derivative of the variable P_t^M ($\widehat{\dot{P}}_t^M = \lim_{t\to 0}\frac{P_{t+dt}^M - P_t^M}{dt}$).

B.3 Derivation of bonds maturity coefficient

The bonds average maturity is defined as:

$$
AM = \frac{\int_{j=dt}^{\infty}\beta^j j\rho^j B_t^m dj}{\int_{j=dt}^{\infty}\beta^j\rho^j B_t^m dj} = \frac{\int_{j=dt}^{\infty} jx^j dj}{\int_{j=dt}^{\infty} x^j dj}
$$

here $x = \beta\rho < 1$. By introducing the notation $S(x)$ for the expression $\int_{j=dt}^{\infty} x^j dj$, the expression above becomes $\frac{S'(x)}{S(x)}$. Let's have a closer look at its numerator

$$
S(x) = \int_{j=dt}^{\infty} x^j dj = \frac{x^j}{\log(x)}|_{j=dt, j=\infty} = 0 - \frac{x^{dt}}{\log(x)}
$$

and now let's have a closer look at its denominator

$$
\begin{aligned}
S'(x) &= \int_{j=dt}^{\infty} jx^{j-1}dj = \frac{\partial}{\partial x}\left(\int_{j=dt}^{\infty} x^j dj\right) = \frac{\partial}{\partial x}\left(-\frac{x^j}{\log(x)}\right)\\
&= \frac{jx^{j-1}\log(x) - x^{j-1}}{\log^2(x)}|_{j=1, j=\infty} = 0 - \frac{dtx^{dt-1}\log(x) - x^{dt-1}}{\log^2(x)}
\end{aligned}
$$

therefore,

$$AM = \frac{S'(x)}{S(x)} = \frac{-\frac{dtx^{dt-1}\log(x)-x^{dt-1}}{\log^2(x)}x}{-\frac{x^{dt}}{\log(x)}} = -\frac{dtx^{dt-1}\log(x)-x^{dt-1}}{\log^2(x)}x \times \frac{\log(x)}{x^{dt}}$$
$$= -\frac{dtx^{-1}\log(x)-x^{-1}}{\log(x)}x \times \frac{1}{1} = -\frac{dt1\log(x)-1}{\log(x)} \overset{dt\to 0}{=} 0 - \frac{1}{\log(x)}$$

As a result, we obtain a concise definition of the average bond's maturity in terms of household's discount factor β and bond maturity coefficient ρ

$$AM = -\frac{1}{\log(\beta\rho)}$$

which allows us to express the bond maturity coefficient ρ from the bonds average maturity

$$\rho = \frac{1}{\beta}\exp\left(\frac{1}{-AM}\right)$$

and consequently bonds depreciation rate δ is

$$\delta = 1-\rho = 1-\frac{1}{\beta}\exp\left(\frac{1}{-AM}\right)$$

B.4 Derivation of GBC and HBC in real terms

Consider the government budget constraint with a variable of the government surplus S_t

$$B_t^M\left(1-\delta P_t^M\right) = S_t + \dot{B}_t^M P_t^M$$

and define the nominal bonds quantity per unit of currency $B_t^{re} = \frac{B_t^M}{P_t}$,so $B_t^M = P_t B_t^{re}$, therefore

$$P_t B_t^{re}\left(1-\delta P_t^M\right) = S_t + P_t^M \frac{d}{dt}\left[P_t B_t^{re}\right]$$
$$P_t B_t^{re}\left(1-\delta P_t^M\right) = S_t + P_t^M\left[\dot{P}_t B_t^{re} + P_t \dot{B}_t^{re}\right]$$
$$B_t^{re}\left(1-\delta P_t^M - \frac{\dot{P}_t}{P_t}P_t^M\right) = \frac{S_t}{P_t} + P_t^M \dot{B}_t^{re}$$

with the notation for inflation $\pi_t = \frac{\dot{P}_t}{P_t}$ and $S_t = P_t\left(\tau_t w_t N_t - G_t - Tr_t\right)$ I get

$$B_t^{re}\left[1-\delta P_t^M - \pi_t P_t^M\right] = \left(\tau_t w_t N_t - G_t - Tr_t\right) + \dot{B}_t^{re} P_t^M$$

or

$$\dot{B}_t^{re} = -\frac{1}{P_t^M}\left[(\tau_t w_t N_t - G_t - Tr_t) - B_t^{re}\left[1 - \delta P_t^M - \pi_t P_t^M\right]\right]$$

In a similar way, I obtain the expression for the household's budget constraint in real terms:

$$\dot{b}_t^{re} = \frac{1}{P_t^M}\left[\frac{S_t^H}{P_t} + b_t^{re}\left[1 - \delta P_t^M - \pi_t P_t^M\right]\right]$$

Note that here inflation uses the left derivative of price with respect to time, however, when I define the household's optimal condition, the inflation will feature the right (forward-looking) derivative of the price level.

B.5 Derivation of household's FOCs for the RANK model

Individuals maximize lifetime utility

$$\int_0^\infty e^{-\rho t}\left(\frac{c^{1-1/\sigma}}{1-1/\sigma} - \varphi\left(1 - urate\right)\frac{l^{1-\psi}}{1-\psi}\right)dt$$

subject to the household's budget constraint

$$\begin{aligned}
\dot{b}_t^{nom} P_t^M &= P_t\left(\left(1 - urate\right)\left(1 - \tau_t\right) w_t l_t + Tr_t + \Pi_t - c_t\right) + b_t^{nom}\left(1 - \delta P_t^M\right) \\
\dot{b}_t^{nom} &= \frac{1}{P_t^M}\left[P_t\left(\left(1 - urate\right)\left(1 - \tau_t\right) w_t l_t + Tr_t + \Pi_t - c_t\right) + b_t^{nom}\left(1 - \delta P_t^M\right)\right]
\end{aligned}$$

To solve this problem, we set up the *Hamiltonian*

$$\begin{aligned}
H = \max_{c,l,b^{nom}} \int_0^\infty e^{-\rho t}\left(\frac{c^{1-1/\sigma}}{1-1/\sigma} - \varphi\left(1 - urate\right)\frac{l^{1-\psi}}{1-\psi}\right) + \\
+ e^{-\rho t}\lambda_t\left(\frac{1}{P_t^M}\left[P_t\left(\left(1 - urate\right)\left(1 - \tau_t\right) w_t l_t + Tr_t + \Pi_t - c_t\right) + b_t^{nom}\left(1 - \delta P_t^M\right)\right] - \dot{b}_t^{nom}\right) dt
\end{aligned}$$

Taking derivatives with respect to c, l and b I obtain the household's optimality conditions

$$c_t^{-1/\sigma} - \frac{\lambda_t P_t}{P_t^M} = 0 \tag{B.11}$$

$$-\varphi l_t^{-\psi} + \frac{\lambda_t P_t}{P_t^M}\left(1 - \tau_t\right) w_t = 0 \tag{B.12}$$

$$\left(\frac{1}{P_t^M} - \delta - \rho\right) + \frac{\hat{\lambda}_t}{\lambda_t} = 0 \tag{B.13}$$

Note, that the last equation was obtained the following way:

$$\frac{\partial}{\partial b^{nom}}\left\{\int_0^\infty e^{-\rho t}\lambda_t\left(\frac{1}{P_t^M}\left[b_t^{nom}\left(1-\delta P_t^M\right)\right]-\dot{b}_t^{nom}\right)dt\right\}=0$$
$$\int_0^\infty e^{-\rho t}\lambda_t\frac{\left(1-\delta P_t^M\right)}{P_t^M}dt-\frac{\partial}{\partial b^{nom}}\left\{\int_0^\infty e^{-\rho t}\lambda_t\dot{b}_t^{nom}dt\right\}=0$$

where

$$\int_0^\infty e^{-\rho t}\lambda_t\dot{b}_t^{nom}dt=-\int_0^\infty e^{-\rho t}\hat{\lambda}_t b_t^{nom}dt+\rho\int_0^\infty e^{-\rho t}\lambda_t b_t^{nom}dt+\lim_{t\to\infty}e^{-\rho t}\lambda_t b_t^{nom}-e^0\lambda_0 b_0^{nom}$$

here b_0^{nom} is fixed, but b_∞^{nom} is variable, so, a transversality condition $\lim_{t\to\infty}e^{-\rho t}\lambda_t B_t^{nom}=0$ is imposed, therefore,

$$\int_0^\infty\left(\lambda_t\left(\frac{1}{P_t^M}-\delta\right)e^{-\rho t}+\hat{\lambda}_t e^{-\rho t}-\rho\lambda_t e^{-\rho t}\right)dt=0$$

and the FOC for b^{nom} is

$$\lambda_t\left(\frac{1}{P_t^M}-\delta-\rho\right)+\hat{\lambda}_t=0$$
$$\left(\frac{1}{P_t^M}-\delta-\rho\right)+\frac{\hat{\lambda}_t}{\lambda_t}=0$$

here, λ_t is forward looking variables and features a *right* derivative ($\hat{\lambda}_t=\lim_{\Delta t\to 0}\frac{\lambda_{t+\Delta t}-\lambda_t}{\Delta t}$).

Now, let's obtain the solution for the household's problem *in real terms*. To do so, I introduce a new variable which replaces the current Lagrangian multiplier $\mu_t=\lambda_t P_t$, so

$$\lambda_t=\frac{\mu_t}{P_t}$$
$$\frac{\hat{\lambda}_t}{\lambda_t}=\frac{P_t}{\mu_t}\frac{\partial}{\partial t}\left(\frac{\mu_t}{P_t}\right)=\frac{P_t}{\mu_t}\left(\frac{\hat{\mu}_t}{P_t}-\frac{\mu_t}{P_t}\frac{\hat{P}_t}{P_t}\right)=\left(\frac{\hat{\mu}_t}{\mu_t}-\frac{\hat{P}_t}{P_t}\right)$$

denoting $\frac{\hat{P}_t}{P_t}$ as $\hat{\pi}_t$, the final system of the FOCs becomes

$$c_t^{-1/\sigma}-\frac{\mu_t}{P_t^M}=0 \tag{B.14}$$
$$-\varphi l_t^{-\psi}+\frac{\mu_t}{P_t^M}\left(1-\tau_t\right)w_t=0 \tag{B.15}$$
$$\left(\frac{1}{P_t^M}-\hat{\pi}_t-\delta-\rho\right)+\frac{\hat{\mu}_t}{\mu_t}=0 \tag{B.16}$$

B.6 Transforming HANK solution in nominal terms to real terms

The HJB equation and FOCs are

$$\rho V_j(B,t) = \max_{c,l} u(c,l) + \partial_B V_j(B,t)\frac{1}{P_t^M}\left[S_t^H + B\left(1-\delta P_t^M\right)\right] \\ + \lambda_j\left(V_{-j}(B,t) - V_j(B,t)\right) + \partial_t V_j(B,t)$$

$$c_j(B,t) \text{ is s.t. } \partial_c u(c,l) = P_t \partial_b V_j(B,t)$$
$$l_j(B,t) \text{ is s.t. } \partial_l u(c,l) = -P_t(1-\tau_t) w_t z_j \partial_b V_j(B,t)$$

where B denotes nominal bonds.

I introduce the new value function $W(b,t)$ that depends on the nominal bonds quantity per a unit of currency $b = \frac{B_t}{P_t}$ and such that

$$W(b,t) = V(B,t)$$

then

$$\frac{\partial[V(B,t)]}{\partial B} = \frac{\partial[W(b,t)]}{\partial B} = \frac{\partial[W(b,t)]}{\partial b}\frac{\partial b}{\partial B} = \frac{1}{P_t}\partial_b W(b,t)$$

as a result,

$$P_t \partial_b V_j(B,t) = \partial_b W(b,t)$$

Now consider the time derivative of the new value function

$$\begin{aligned}\frac{\partial[V(B,t)]}{\partial t} = \frac{\partial[W(b,t)]}{\partial t} &= \frac{\partial\left[W\left(\frac{B}{P_t},t\right)\right]}{\partial t} \\ &= \frac{\partial[W(b,t)]}{\partial b}\frac{\partial[b]}{\partial P_t}\frac{\partial P_t}{\partial t} + \frac{\partial\left[W\left(\frac{B}{P_t},t\right)\right]}{\partial t} \\ &= -\partial_b W(b,t)\frac{\hat{P}_t}{P_t^2}B_t + \partial_t W(b,t) \\ &= -\partial_b W(b,t)\frac{\hat{P}_t}{P_t}b_t + \partial_t W(b,t)\end{aligned}$$

note, that value function $V(B,t)$ features forward-looking derivative, therefore, the obtained inflation also features the forward-looking derivative.

Thus, by denoting $s_t^H = \frac{S_t^H}{P_t}$ the real surplus, the HJB and FOCs become

$$\begin{aligned}\rho W(b,t) = \max_{c,l} u(c,l) + \partial_B W_j(b,t)\frac{1}{P_t^M}\left[s_t^H + b\left(1-\delta P_t^M - \hat{\pi}_t P_t^M\right)\right] \\ + \lambda_j\left(W_{-j}(b,t) - W_j(b,t)\right) + \partial_t W(b,t)\end{aligned} \tag{B.17}$$

$$c_j(b,t) \text{ is s.t. } \partial_c u(c,l) = \partial_b W(b,t) \tag{B.18}$$

$$l_j(b,t) \text{ is s.t. } \partial_l u(c,l) = -(1-\tau_t)\, w_t z_j \partial_b W(b,t) \tag{B.19}$$

Now consider the Kolmogorov-Forward equations is

$$\partial_t g_j(B,t) = -\partial_B\left[s_j(B,t)\, g_j(B,t)\right] - \lambda_j g_j(B,t) + \lambda_{-j} g_{-j}(B,t)$$

with

$$s_j(B,t) = \frac{1}{P_t^M}\left[S_t^H + B\left(1-\delta P_t^M\right)\right]$$

Let's identify a new function $h(b,t)$ that depends on the nominal bonds quantity per a unit of currency $b = \frac{B_t}{P_t}$ and is such that

$$h(b,t) = g(B,t)$$

then

$$\frac{\partial\left[g(B,t)\right]}{\partial B} = \frac{\partial\left[h(b,t)\right]}{\partial B} = \frac{\partial\left[h(b,t)\right]}{\partial b}\frac{\partial b}{\partial B} = \frac{1}{P_t}\partial_b h(b,t)$$

i.e.

$$P_t \partial_b g_j(B,t) = \partial_b h_j(b,t)$$

And apply the time derivative

$$\begin{aligned}\frac{\partial\left[g(B,t)\right]}{\partial t} = \frac{\partial\left[h(b,t)\right]}{\partial t} &= \frac{\partial\left[h\left(\frac{B}{P_t},t\right)\right]}{\partial t} \\ &= \frac{\partial\left[h(b,t)\right]}{\partial b}\frac{\partial\left[b\right]}{\partial P_t}\frac{\partial P_t}{\partial t} + \frac{\partial\left[h\left(\frac{B}{P_t},t\right)\right]}{\partial t} \\ &= -\partial_b h(b,t)\frac{\hat{P}_t}{P_t^2}B_t + \partial_t h(b,t) \\ &= -\partial_b h(b,t)\frac{\hat{P}_t}{P_t}b_t + \partial_t h(b,t)\end{aligned}$$

Next, consider the savings function

$$s_j(B,t) = \frac{1}{P_t^M}\left[P_t s_t^H + P_t b\left(1-\delta P_t^M\right)\right]$$

so,

$$s_j(b,t) = \frac{1}{P_t^M}\left[P_t s_t^H + P_t b\left(1-\delta P_t^M\right)\right]$$
$$s_j^{re}(b,t) = \frac{s_j(b,t)}{P_t} = \frac{1}{P_t^M}\left[s_t^H + b\left(1-\delta P_t^M\right)\right]$$

and,

$$\frac{\partial\left[s_j(B,t)\right]}{\partial B} = \frac{\partial\left[s_j(b,t)\right]}{\partial B} = \frac{\partial\left[s_j(b,t)\right]}{\partial b}\frac{\partial b}{\partial B} = \frac{1}{P_t}\partial_b s_j(b,t)$$

Thus, the KF equation and savings function can be rewritten as

$$\begin{aligned}
-\partial_b h(b,t)\frac{\hat{P}_t}{P_t}b_t + \partial_t h(b,t) &= -\partial_B\left[\frac{1}{P_t}\partial_b s_j(b,t)\right]h_j(b,t) - \partial_B\left[\frac{1}{P_t}\partial_b h(b,t)\right]s_j(b,t) \\
&\quad - \lambda_j h_j(b,t) + \lambda_{-j}h_{-j}(b,t) \\
&\quad - \partial_B\left[\frac{1}{P_t}\partial_b h(b,t)\right]s_j(b,t) - \lambda_j h_j(b,t) + \lambda_{-j}h_{-j}(b,t) \\
\partial_t h(b,t) &= -\partial_b\left[s_j^{re}(b,t)\right]h_j(b,t) - \partial_b\left[h(b,t)\right]s_j^{re}(b,t) \\
&\quad - \lambda_j h_j(b,t) + \lambda_{-j}h_{-j}(b,t) + \partial_b h(b,t)\frac{\hat{P}_t}{P_t}b_t \\
\partial_t h(b,t) &= -\partial_b\left[\left(s_j^{re}(b,t) - \hat{\pi}_t b_t\right)h(b,t)\right] - \lambda_j h_j(b,t) + \lambda_{-j}h_{-j}(b,t) \qquad \text{(B.20)}
\end{aligned}$$

with

$$s_j^{re}(b,t) - \hat{\pi}_t b_t = \frac{1}{P_t^M}\left[s_t^H + b\left(1 - \hat{\pi}_t P_t^M - \delta P_t^M\right)\right] \qquad \text{(B.21)}$$

B.7 Dynamics figures of RANK and HANK with a relaxed borrowing constraint

Figure B.1: Impulse responses to a cost-push shock featuring HANK with a relaxed borrowing limit

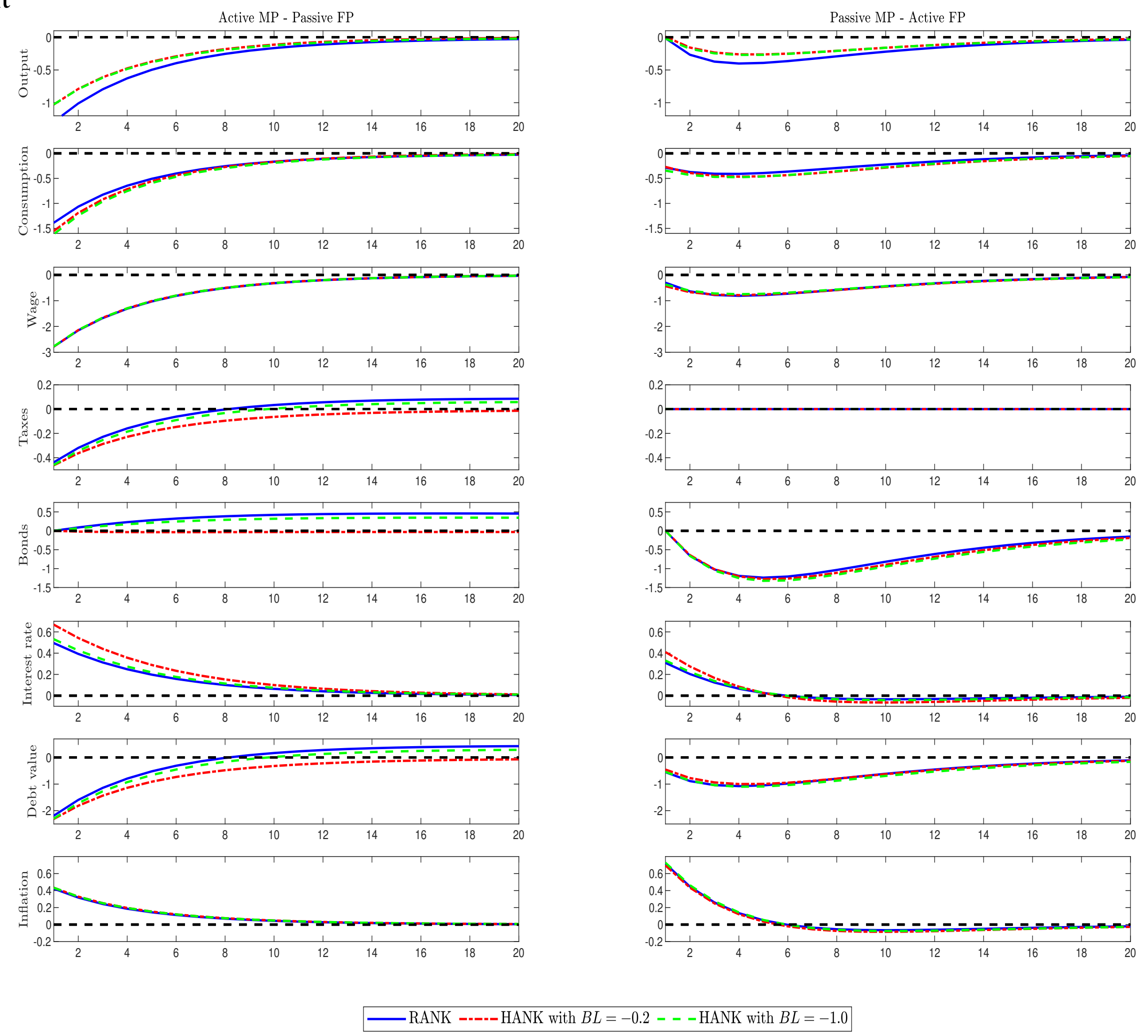
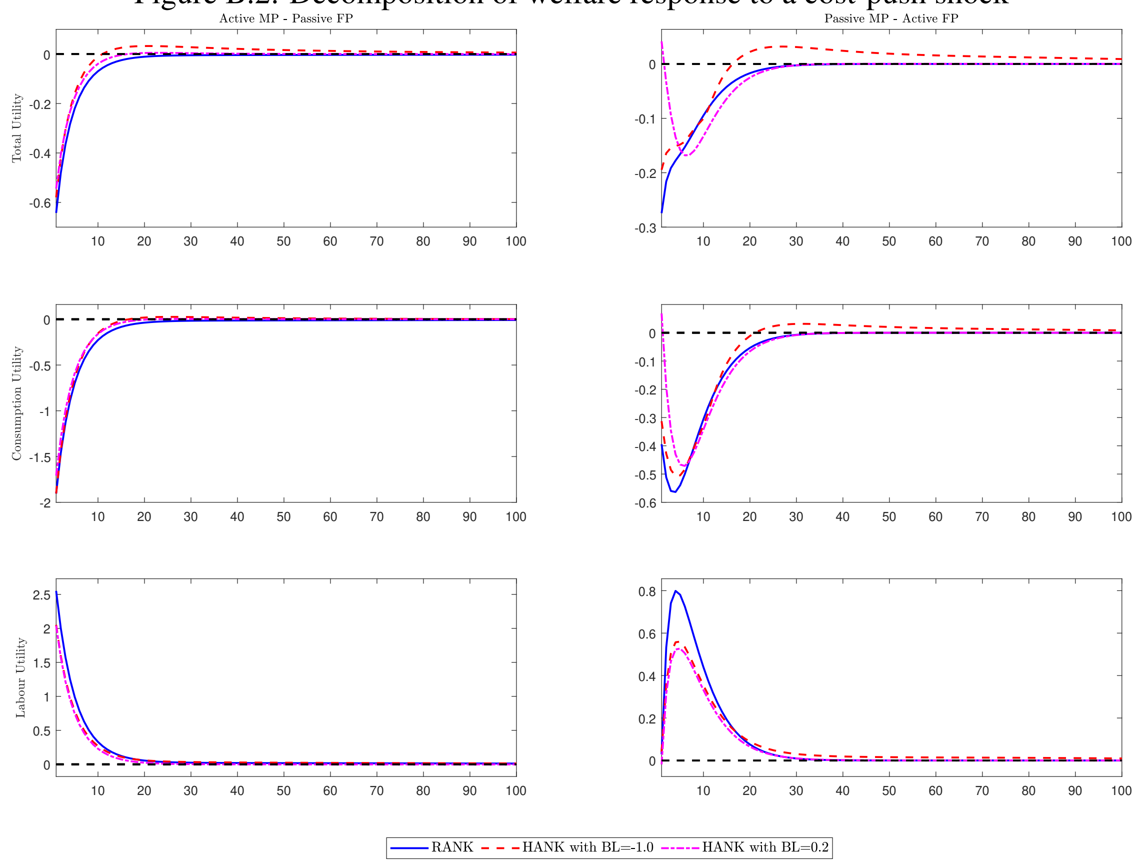


Table B.1: RANK and HANK Social Welfare Results. Tight and Relaxed Borrowing Constraints

	steady state	active MP-passive FP	passive MP-active FP
RANK	-101.5883	-101.9396 $[-0.3506\ \%C]$	-101.9305 $[-0.3416\ \%C]$
HANK, $BL = -0.2$	-116.4899	-116.8698 $[-0.3791\ \%C]$	-116.8660 $[-0.3753\ \%C]$
HANK, $BL = -1.0$	-118.2114	-118.6094 $[-0.3972\ \%C]$	-118.6060 $[-0.3938\ \%C]$

Note: the numbers in square brackets denote consumption equivalents of the welfare loss compared to the steady state in HANK and RANK models

Figure B.2: Decomposition of welfare response to a cost-push shock



B.8 Robustness analysis of the HANK model with government debt. Different maturity

According to Figure B.3, the average term to maturity is high, exceeding the level of 6.1 years recently. However, due to lower values before 2010, the calibrated value that I use in the model is slightly above 4.5 years (4.73 years to be precise). The average term to maturity is characterised by significant volatility across years, making it important to test the effect of different values of the average term to maturity. In this section, I study the effect of a lowered term to maturity on the main variables' outcome.

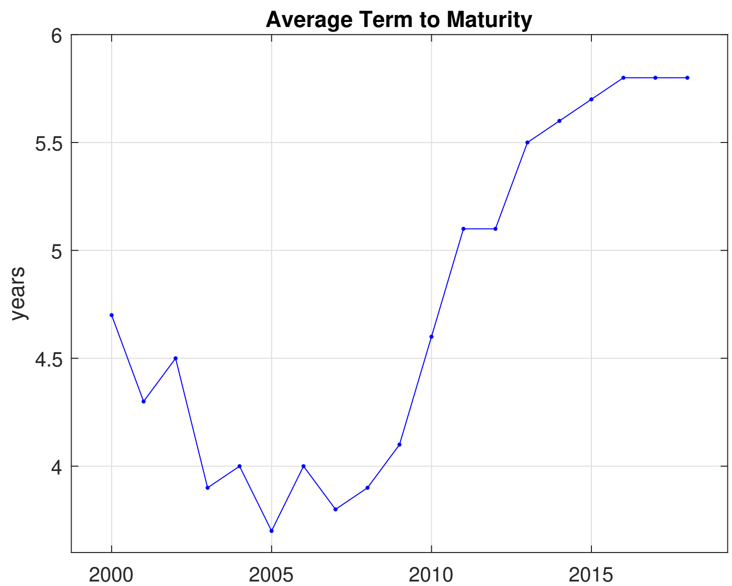


Figure B.3: Average term-to-maturity of the US government bonds in 2000-2018

An important property of the model is the *borrowing limit*. When maturity changes, it is the quantity of bonds rather than the debt value that is affected. Therefore, to generate consistent results, I have to adjust the borrowing limit in accordance with the maturity coefficient. To understand how to adjust the borrowing limit, I consider the budget equation in the steady state

$$P_t^M B_t = \frac{1}{i_t}\left[\tau_t w_t N_t - G_t - T_t^T\right] \tag{B.22}$$

The surplus S_t remains relatively constant but what changes is the bond pricing P_t^M. So, I apply the no-arbitrage condition $P_t^m = \frac{1}{(\delta + i_t)}$ and rewrite the budget equation as

$$B_t = \frac{\delta + i_t}{i_t} S_t \tag{B.23}$$

A smaller bond average term to maturity implies higher bond holdings B_t and lower bond price P_t^m. Therefore, while the interest rate is relatively constant ($i_t \approx 1\%$), the borrowing limit has to be adjusted the following way

$$BL^{new} = \frac{\delta^{new} + i_t}{\delta^{old} + i_t} BL^{old} \tag{B.24}$$

Below, Table B.2 compares the outcome between different bond maturities, specifically, it reports the results of models with 4.73 years and 2 years average term-to-maturities. In this table, the borrowing limit of the baseline HANK model is chosen at -0.1 level to generate a significant mass of households at the borrowing constraint, while the borrowing limit of the lower maturity HANK model is derived using the formula (B.24). According to Table B.2, the HANK model generates the results of the same pattern as the RANK model, the key and only significant difference here is the *asset values* and *bond prices*. Even though the aggregate outcome seems to be almost identical for the different maturities, the distributional effects are visible (see Figure B.4). Also, the HANK model with different maturity generates distinctively different results for the time dynamics of an individual (see Figure B.5 below). Most of this discrepancy is coming from the modified borrowing limit which impacts households' distribution in the HANK model but does not have any effect in the RANK model.

Table B.2: RANK and HANK Characteristics of Stochastic Steady State. Different Maturities

	RANK, m=4.73	RANK, m=2	HANK, m=4.73	HANK, m=2
Borrowing limit	–	–	-0.1	-0.2285
	(1)	(1)	(3)	(4)
Consumption	0.4830	0.4830	0.4575	0.4574
- unemployed	–	–	0.3076	0.3074
- employed	–	–	0.5101	0.5101
Output, labor	0.5112	0.5112	0.4856	0.4855
Assets	0.0516	*0.1179*	0.0600	*0.1461*
- unemployed	–	–	-0.0072	*-0.0154*
- employed	–	–	0.0836	*0.2029*
Assets Market Value	0.9939	0.9939	1.2693	1.2830
- unemployed	–	–	-0.1519	-0.1355
- employed	–	–	1.7687	1.7814
Debt to GDP	48.61	48.61	65.35	66.06
Bond price	19.2676	*8.4325*	21.1628	*8.7812*
Interest rate	1.00	1.00	0.54	0.53
Mass on BL	–	–	0.0447	0.0398
Average Time on BL	–	–	12.9547	12.9123
Average Time to BL	–	–	0	0
Utility*	-101.5094	-101.5094	-118.6270	-117.2428

Lastly, Figure B.6 demonstrates the economy response to a 30% positive cost-push shock under different maturities in the HANK and RANK models. The outcome may look almost identical, however, the difference is substantial for the debt value and quantity of bonds: both the RANK and HANK models show a larger deviation from the steady state once the maturity decreases.

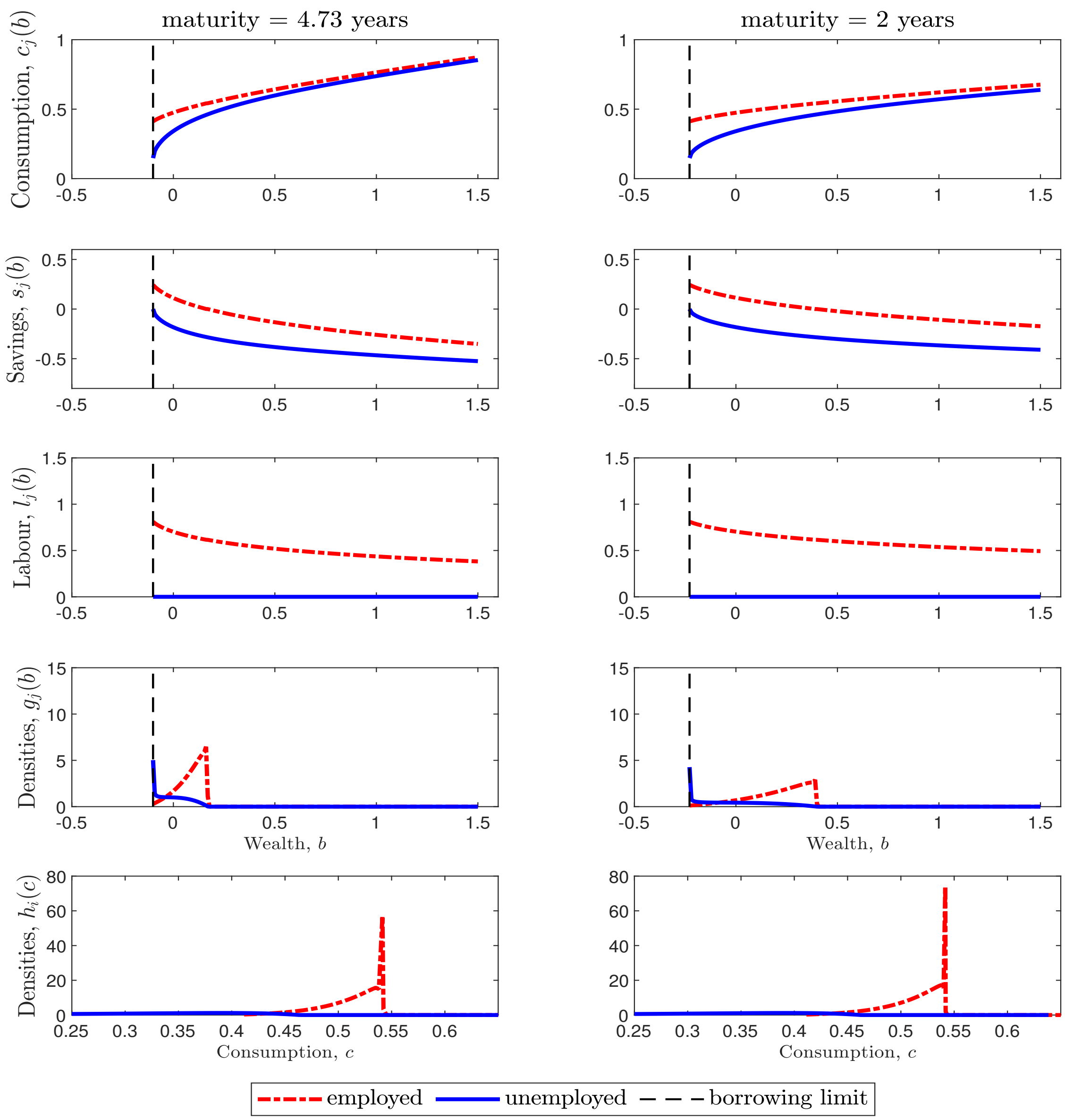


Figure B.4: Distributions in the stochastic steady state with different maturities

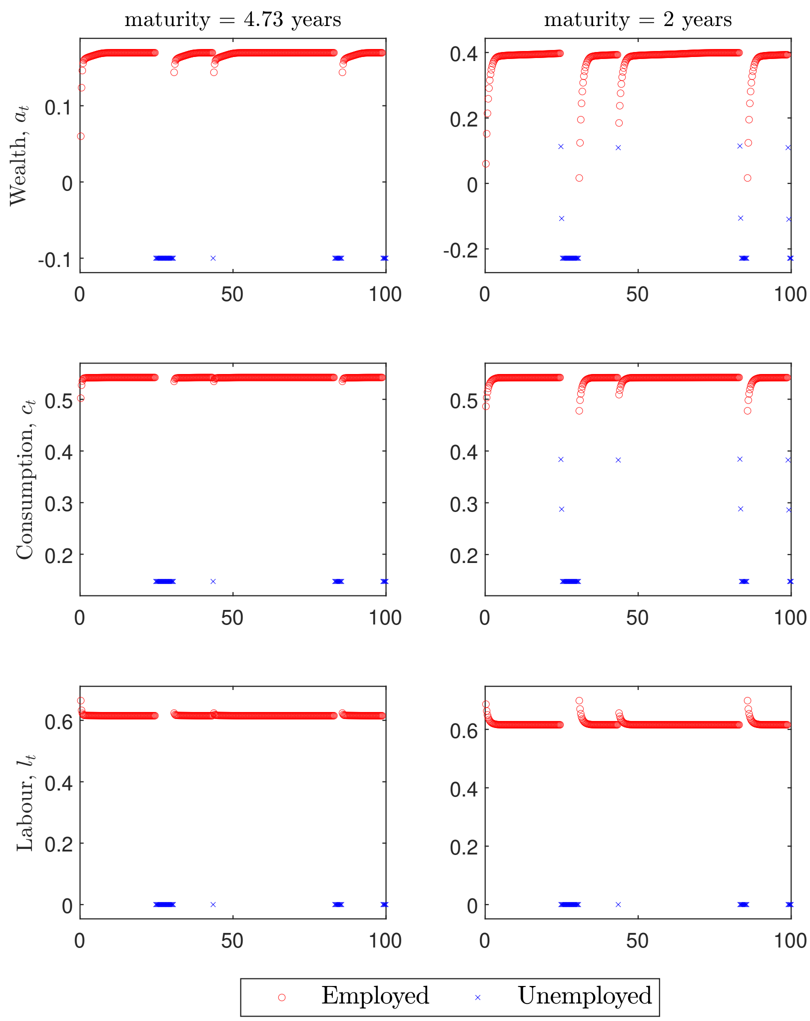


Figure B.5: Dynamics in stochastic steady state with different maturities

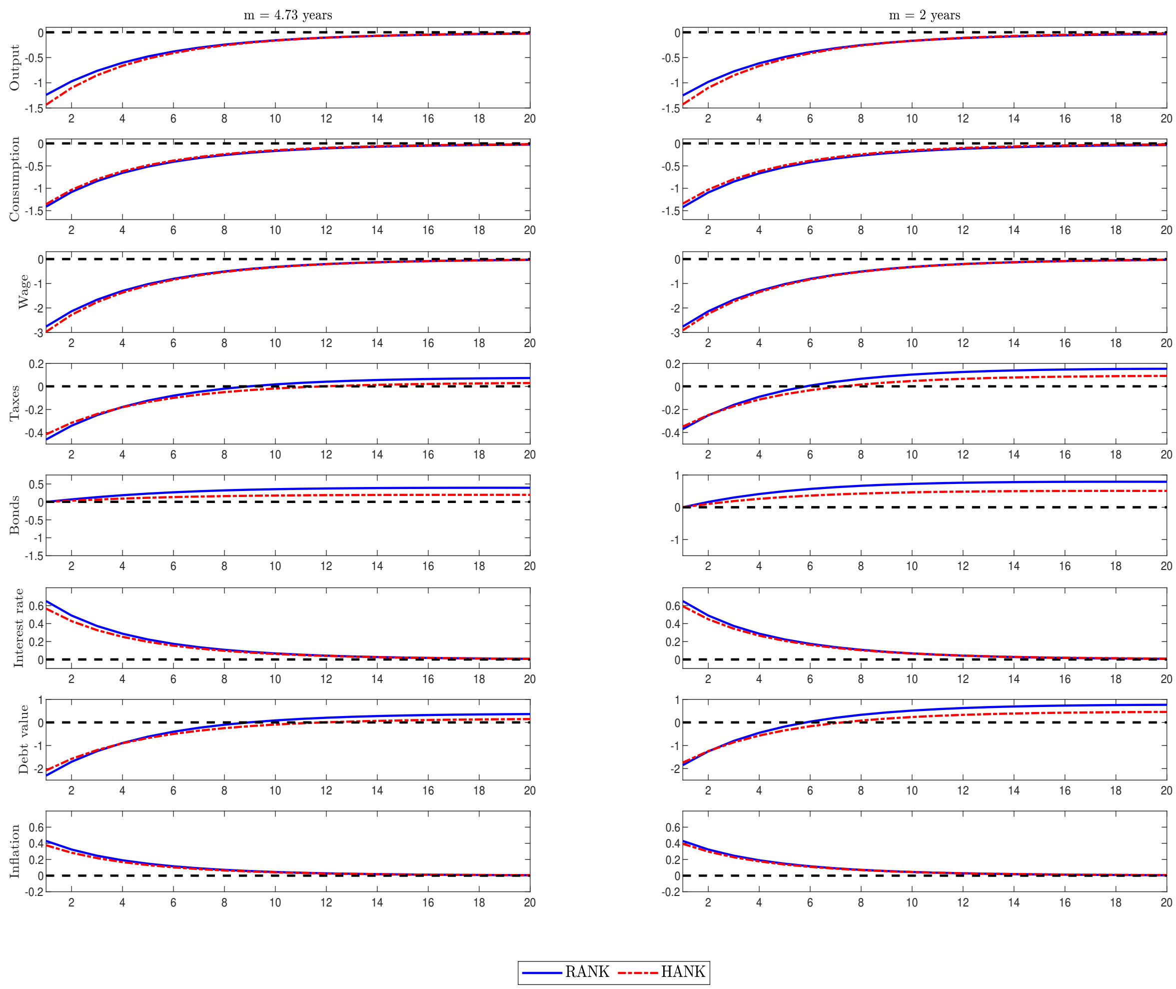


Figure B.6: RANK and HANK impulse responses to a cost-push shock with different maturities

Appendix to Chapter 3

C.1 Review of Calculus about functionals

Let $\Phi := [\underline{a}, \infty)$. The space of Lebegue-integrable functions $L^2(\Phi)$ with the inner product

$$\langle v, g \rangle_\Phi = \sum_{j=1}^{2} \int_\Phi v_j g_j da = \int_\Phi v^T g da, \quad \forall v, g \in L^2(\Phi)$$

is a Hilbert space. Note that we could have alternatively worked with $\Phi = R$ as the density $g(t,a) = 0$ for $a < \underline{a}$.

Next, this chapter introduces the concept of Gateaux and Frechet derivatives in $L^2(\Omega)$, where $R \subset \Omega$ as generalizations of the standard concept of derivative to infinite-dimensional spaces.

Definition 1 (Gateaux derivative) *Let $W[f]$ be a functional and let h be arbitrary in $L^2(\Omega)$. If the limit*

$$\delta W[f;h] = \lim_{\alpha \to 0} \frac{W[f + \alpha h] - W[f]}{\alpha}$$

exists, it is called the Getaus derivative of W at f with increment h.
If the limit (1) exists for each $h \in L^2(\Omega)$, the functional W is said to be Gateaux differentiable at f.

The concept of the Frechet derivative is more restrictive.

Definition 2 (Frechet derivative) *Let h be arbitrary in $L^2(\Omega)$. If for fixed $f \in L^2(\Omega)$ there exists $\delta W[f;h]$ which is linear and continuous with respect to h such that*

$$\lim_{\|h\|_{L^2(\Omega)} \to 0} \frac{|W[f+h] - W[f] - \delta W[f;h]|}{\|h\|_{L^2(\Omega)}} = 0$$

then W is said to be Frechet differentiable at f and $\delta W[f;h]$ is the Frechet derivative of W at f with increment h.

Theorem 1 *If the Frechet derivative of W exists at f, then the Gateaux derivative exists at f and they are equal.*

Theorem 2 *Let W have a Gateaux derivative, a necessary condition for W to have an extremum at f is that $\delta W[f;h] = 0$ for all $h \in L^2(\Omega)$.*

Theorem 3 (Lagrange multipliers) *Let H be a mapping from $L^2(\Omega)$ into R^2. If W has a continuous Frechet derivative, a necessary condition for W to have an extremum at f under the constraint $H[f] = 0$ at the function f is that there exists a function $\eta \in L^2(\Omega)$ such that the Lagrangian functional*

$$L[f] = W[f] + \langle \eta, H[f] \rangle_\Omega$$

is stationary in f, that is, $\delta L[f,h] = 0$.

Theorem 4 (Riesz representation theorem) *Let $\delta W[f;h] : L^2(\Phi) \to R$ be a linear continuous functional. Then there exists a unique function $w[f] = \frac{\delta W}{\delta f}[f]$ such that*

$$\delta W[f;h] = \left\langle \frac{\delta W}{\delta f}, h \right\rangle = \sum_{j=1}^{2} \int_\Phi w_j[f](a)\, h_j(a)\, da$$