# On right-veering diffeomorphisms and binding sums

by

**Miguel Orbegozo Rodriguez**

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# Abstract

This thesis consists of two parts, more or less independent of each other but both devoted to the study of open book decompositions of 3-manifolds and their relationship to contact structures.

Right-veering diffeomorphisms are related to tightness of contact structures due to a result of Honda, Kazez, and Matić but it is in general difficult to determine whether a given diffeomorphism is right-veering. In the first part we prove that the right-veering property can be detected in a combinatorial way.

In the second part, we explore properties of an operation of contact manifolds (with a prescribed open book) called the binding sum. We show that even if the summands are Stein fillable the result of the sum need not be, and indeed can be overtwisted. We also provide an explicit computation of vanishing of the Heegaard Floer contact class for an infinite family of such sums where the summands are Stein fillable.

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# Author's declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

# Introduction

Although the study of contact geometry started in the late 1800s, recently it has become a more popular area of study. It can be used to study symplectic manifolds, as a symplectic structure on a manifold with boundary induces a contact structure on its boundary, but also is an interesting object of study in its own right. Moreover, what started as a geometric field began developing topological methods. One of the most important, due to Giroux [19] and central to this thesis, establishes, in dimension 3, a close relationship between contact manifolds and mapping class groups of surfaces. This implies that 3-dimensional contact structures can be studied with purely topological tools. It is therefore contact 3-manifolds that we will be studying, and the tools that we will use are diffeomorphisms of compact surfaces with boundary, up to isotopy. More precisely, we will study *open books*, which are a pair $(\Sigma, \varphi)$ where $\Sigma$ is a compact surface with boundary and $\varphi$ is a mapping class of $\Sigma$. In particular, we will focus on two topics; the right-veering property, and binding sums.

## § 1.1 | Right-veering diffeomorphisms

There is a fundamental dichotomy in contact structures between those that admit an embedding of a disc $D^2$ whose boundary is tangent to the contact structure, called *overtwisted*, and those that do not, called *tight*. Overtwisted contact structures, up to isotopy, are known by work of Eliashberg [6] to correspond to homotopy classes of plane fields, which are well understood. On the other hand, there is much we do not yet know about tight contact structures. Thus being able to distinguish between tight and overtwisted contact structures becomes an important problem.

A result of Honda, Kazez, and Matić [21] states that a contact structure is tight if and only if all its supporting open books are *right-veering*. However, this property is often difficult to verify, and arguments tend to be case specific. Our methods provide a general combinatorial way to check the right-veering property. More precisely, we show that a left-veering arc exists if an only if there exists a collection of regions, which we call *extended towers*, with specific properties. This implies the existence of an algorithm given by simply running a search along all possibilities. If the search terminates without finding the desired collection, then the open book is right-veering.

We note that this algorithm is very inefficient. A recent result of Baldwin, Ni, and Sivek [2] characterised right-veering diffeomorphisms of surfaces with connected boundary using an invariant coming from Knot Floer homology. However, this invariant can be difficult to compute, and their proof relies on the relationship between Heegaard Floer homology and Symplectic Floer homology proved by Lee-Taubes [29] and Kutluhan-Lee-Taubes [27], and on computations of Symplectic Floer homology by Cotton-Clay [4]. Our methods are more elementary, and we hope they can prove Baldwin, Ni, and Sivek's result without using Symplectic Floer homology; this is the subject of work in progress. We also note that our result does not restrict to the connected boundary case.

## § 1.2 | Binding sums

An operation of contact manifolds similar to the connected sum is the *binding sum*. While the effect of the connected sum on the properties of contact structures is well understood, the same cannot be said of the binding sum. Using a result of Klukas [26] that allows us to obtain an open book for the sum in terms of the open books being summed, we provide some examples that illustrate that properties such as tightness and symplectic fillability are not necessarily preserved, and give an explicit computation that shows vanishing of the contact class for an infinite family of sums where the summands are Stein fillable and thus have non-vanishing contact class in Heegaard Floer homology.

## Thesis structure

Chapter 2 contains preliminary notions for the rest of the thesis. First, we introduce contact structures on 3-manifolds and give some of its properties, the main ones being tightness and various notions of fillability. Then we turn our attention to open books, and outline their relationship with contact structures, culminating with the statement of Giroux correspondence. Finally, we introduce Heegaard Floer homology and the contact class, which we will use for computations in Chapter 4.

Chapter 3 is dedicated to proving that the right-veering property can be detected using *extended towers*. We first show that in simple cases regions can detect left-veering arcs. We then introduce extended towers as a way to generalise this situation, and we prove inductively that they detect left-veering arcs.

Finally, Chapter 4 is devoted to defining the binding sum, exploring its properties, and computing the contact class for an infinite family of binding sums. This computation also corrects an error in [24].

Chapter 2

# Preliminaries

## § 2.1 | Contact structures

We start by providing an overview of contact structures in 3-manifolds, following Etnyre's notes [10, 11], and Geiges' book [15].

Let $M$ be a smooth, oriented 3-manifold. A *contact* structure $\xi$ on $M$ is a maximally non-integrable plane field. Equivalently, we may express $\xi$ as $\xi = \ker \alpha$ for some 1-form $\alpha$ such that $\alpha \wedge d\alpha > 0$. Such a form $\alpha$ is called a *contact form*. Note that contact forms for a given contact structure are not unique. Indeed, if $\alpha$ is a contact form for $\xi$, and $f$ is a non-vanishing smooth function, then $\alpha' = f\alpha$ is also a contact form for $\xi$.

*Remark.* These are sometimes referred to as *positive* contact structures, as opposed to *negative* contact structures where $\alpha \wedge d\alpha < 0$. However we will only consider positive ones and so we will refer to them simply as contact structures.

A classical result of Martinet states that every oriented 3-manifold admits a contact structure [35]. We will now see two basic examples for $M = \mathbb{R}^3$.

**Example 2.1.1.**

1. Let $M = \mathbb{R}^3$ and $\xi = \ker(dz - ydx)$, where $(x, y, z)$ are the standard Cartesian coordinates on $\mathbb{R}^3$. Then $\alpha = dz - ydx$ is a contact form because $\alpha \wedge d\alpha = dx \wedge dy \wedge dz$ is a volume form for $\mathbb{R}^3$. Therefore $\xi$ is a contact structure, called the *standard* contact structure on $\mathbb{R}^3$ and denoted $\xi_{\mathrm{std}}$.

2. Let $M = \mathbb{R}^3$ and $\xi = \ker(\cos r dz + r \sin r d\theta)$, where $(r, \theta, z)$ are cylindrical coordinates. Then $\alpha = \cos r dz + r \sin r d\theta$ is also a contact form since $\alpha \wedge d\alpha = (r + \sin r \cos r) dr \wedge d\theta \wedge dz = (1 + \frac{\sin r \cos r}{r}) d\mathrm{vol}$ and $1 + \frac{\sin r \cos r}{r}$ is always positive. We denote this contact structure by $\xi_{\mathrm{ot}}$.

We say that two 3-manifolds $(M_1, \xi_1)$ and $(M_2, \xi_2)$ are *contactomorphic* if there exists a diffeomorphism $f : M_1 \to M_2$ such that $f_*(\xi_1) = \xi_2$. We call such an $f$ a *contactomorphism*. Locally all contact 3-manifolds are contactomorphic (and the local model is given by $(\mathbb{R}^3, \xi_{\mathrm{std}})$ from Example 2.1.1), this is known as Darboux's Theorem. A proof of this result can be found in [15]. Note that this implies that we can perform the connect sum of two contact manifolds and the contact structures on the manifolds will glue together to a contact structure on the summed manifold. In Chapter 4 we will explore a similar operation to the connect sum, called the *binding sum*.

Given a smooth manifold $M$, two contact structures $\xi_0$ and $\xi_1$ are said to be *isotopic* if there exists a homotopy from $\xi_0$ to $\xi_1$ through contact structures, i.e a one-parameter family $\{\xi_t \mid t \in [0,1]\}$ of plane fields with $\xi_t$ a contact structure for $t \in [0,1]$. If $M$ is closed, and $\xi_0$ and $\xi_1$ are isotopic contact structures on $M$, *Gray's theorem* states that there exists a one parameter family of diffeomorphisms $\{f_t : M \to M \mid t \in [0,1]\}$ such that $f_0 = \mathrm{Id}_M$ and $(f_t)_*(\xi_0) = \xi_t$ for all $t \in [0,1]$. In particular, there exists a contactomorphism $f_1 : (M, \xi_0) \to (M, \xi_1)$. Gray's theorem was originally proved by Gray in [20], although a simpler proof using *Moser's trick* can be found in [15]. The theorem is not necessarily true if $M$ is not closed, for example, Eliashberg in [8] showed that it does not hold $S^1 \times \mathbb{R}^2$ (although he showed that it does hold in $\mathbb{R}^3$).

However, we will always work with closed 3-manifolds, and we will be interested in contact structures up to isotopy. The two contact manifolds from Example 2.1.1, although clearly diffeomorphic as smooth manifolds, are not isotopic or contactomorphic, as proved by Bennequin [3]. Indeed, they are examples of a fundamental dichotomy of contact structures, that of tight versus overtwisted.

**Definition 2.1.2.** Let $(M, \xi)$ be a contact manifold. An *overtwisted disc* in $M$ is an embedded $D^2$ such that its boundary is tangent to the contact planes. If such a disc exists, the contact manifold (or the contact structure) is said to be *overtwisted*, otherwise it is said to be *tight*.

The contact structure $\xi_{\mathrm{std}}$ from Example 2.1.1 is tight, while $\xi_{\mathrm{ot}}$ is overtwisted. Overtwisted contact structures were completely classified by Eliashberg, who in [6] proved that given a closed smooth 3-manifold $M$, the natural inclusion of overtwisted contact structures into plane fields is a homotopy equivalence. This means that the study of isotopy classes of overtwisted contact structures reduces to the study of homotopy classes of plane fields, which are well understood through the Thom-Pontryagin construction (see [36]). Tight contact structures, however, are not as well understood, and their classification remains an interesting problem.

This is not the only distinction we can make between contact structures, as we can fit tight contact structures into a hierarchy which gives some measure of how rigid they are.

A *symplectic structure* on a smooth 4-manifold $X$ is a 2-form $\omega$ that is closed and non-degenerate, that is, $d\omega = 0$ and $\omega \wedge \omega > 0$. We then call the pair $(X, \omega)$ a *symplectic manifold*. Then, a contact manifold $(M, \xi)$ is *weakly symplectically fillable* if there exists a symplectic 4-manifold $(X, \omega)$ such that $\partial X = M$ and $\omega_{|\xi} > 0$. If $(X, \omega)$ is compact and such that $\partial X = M$ and there exists a transverse vector field $v$ pointing out of $X$ along $M$ such that the flow of $v$ dilates $\omega$, then $(M, \xi)$ is said to be *strongly symplectically filled* by $(X, \omega)$.

Finally, a *Stein domain* is a complex 4 manifold $W$ together with a Morse function $\psi : W \to \mathbb{R}$ that is bounded below, and such that $\omega_\psi(v, w) = -d(d\psi \circ J)(v, w)$ –where $J$ is the complex structure– is non-degenerate. Suppose that a contact manifold $(M, \xi)$ is a noncritical level set, that is $M = \psi^{-1}(c)$ for $c$ a regular value, and $-(d\psi \circ J)$ is a contact form for $\xi$. Then we say $(M, \xi)$ is *Stein fillable*. Sometimes Stein fillable contact manifolds are also called *holomorphically fillable*.

**Example 2.1.3.** The contact manifold $(S^3, \xi_{\text{std}})$ is Stein fillable, as it is the boundary of the standard 4-ball in $\mathbb{C}^2$.

Then on one end of the hierarchy we have Stein fillable contact structures. These are contained in the set of strongly fillable contact structures, which in turn are contained in the set of weakly fillable contact structures, which are tight. However, none of these inclusions are strict. Eliashberg showed in [9] the first example of a weakly fillable contact structure on $T^3$ that is not strongly fillable, and Ding and Geiges [5] built on this to provide infinitely many such examples on torus bundles over $S^1$. Strongly fillable contact structures that are not Stein fillable were discovered by Ghiggini [16]. There are also tight contact structures that are not fillable in any sense, these were first discovered by Etnyre and Honda [12]. We can see this schematically below. Overtwisted contact structures are not tight and thus also not fillable in any sense.

$$\text{Stein fillable} \subsetneq \text{Strongly fillable} \subsetneq \text{Weakly fillable} \subsetneq \text{Tight} \mid \text{OT}$$

Another notion that we will use in Chapter 4 is the following. A contact manifold $(M, \xi)$ has *Giroux torsion* if it admits a (contact) embedding of a *Giroux torsion domain* $(T^2 \times [0, 1], \ker(\cos(2\pi t)dx - \sin(2\pi t)dy))$ into $(M, \xi)$. By a result of Gay [14] we know that manifolds with Giroux torsion cannot be strongly fillable.

## § 2.2 | Open book decompositions

We now turn our attention to open book decompositions, also somewhat following Etnyre's notes [11]. They will be our main tool for studying contact manifolds. We give some preliminary definitions and examples, and highlight their relationship to contact structures.

Let $M$ be a closed, oriented 3-manifold. An open book decomposition for $M$ is a pair $(B, \pi)$, where $B$, called the *binding*, is an oriented link in $M$, and $\pi : M \setminus B \to S^1$ is a fibration such that for every $x \in S^1$, $\pi^{-1}(x)$ is the interior of a compact surface with boundary $\Sigma_x$ (called the *page*), such that $\partial\Sigma_x = B$.

Alexander showed that every smooth, closed and oriented 3-manifold admits an open book decomposition [1].

**Example 2.2.1.** Let $M = S^3 = \mathbb{R}^3 \cup \{\infty\}$. Then the $z$ axis union the point at infinity is a knot $K$ (in fact it is the unknot). Moreover, using again cylindrical coordinates on $\mathbb{R}^3 \setminus \{z \text{ axis}\} = S^3 \setminus K$, the map $\pi : S^3 \setminus K \to S^1$ given by $\pi((r, \theta, z)) = \theta$ is a fibration whose fibres are half planes, which in $S^3$ are the interior of discs with common boundary $K$. Thus $(K, \pi)$ is an open book decomposition for $S^3$.

This construction is determined up to diffeomorphism by the diffeomorphism type of the page and the return map of the fibration, and so we can specify an open book by a pair $(\Sigma, \varphi)$, where $\Sigma$ is a compact surface with boundary and $\varphi$ is a self-diffeomorphism of $\Sigma$ fixing the boundary pointwise. Indeed, from this data we can form the mapping torus $\Sigma_\varphi = \frac{\Sigma\times[0,1]}{(x,1)\sim(\varphi(x),0)}$. This gives us a 3-manifold whose boundary is a disjoint union of tori. Filling in these tori by attaching discs along $\{\mathrm{pt}\}\times S^1$ we get a closed 3-manifold $M_\varphi$ that is diffeomorphic to the original manifold. The pair $(\Sigma, \varphi)$ is called an *abstract open book*, where we still refer to $\Sigma$ as the *page*, and we will call $\varphi$ the *monodromy*. There is a subtle difference between open book decompositions and abstract open books; in the former we can consider the binding and pages up to isotopy, while in latter we can only do so up to diffeomorphism. However for our purposes we will not need to distinguish between them and thus we will often refer to either as open books.

*Remark.* If $\varphi, \varphi'$ are isotopic diffeomorphisms of $\Sigma$ then $(\Sigma, \varphi)$ and $(\Sigma, \varphi')$ give diffeomorphic 3-manifolds, so we can allow isotopies when discussing abstract open books. This means that we can consider the monodromy of an open book to be a *mapping class*.

**Definition 2.2.2.** Let $\Sigma$ be a compact surface with nonempty boundary. The *mapping class group of* $\Sigma$, denoted $MCG(\Sigma)$, is the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma$ that fix $\partial\Sigma$ pointwise. Its elements are called *mapping classes*.

**Example 2.2.3.** The open book decomposition for $S^3$ from Example 2.2.1 has an abstract open book description $(D^2, \mathrm{Id})$. Indeed, the pages are discs, and $\mathrm{MCG}(D^2)$ is trivial (see for example [13] for a proof, called the *Alexander trick*).

**Definition 2.2.4.** Let $\Sigma = S^1 \times [0, 1]$ be an annulus, and define a diffeomorphism $\tau : \Sigma \to \Sigma$ by $\tau(\theta, t) = (\theta + 2\pi t, t)$. We call $\tau$ a *positive Dehn twist*. Its inverse $\tau^{-1}$ is a *negative Dehn twist*.

More generally, we can define Dehn twists on any surface. Let $a$ be a simple closed curve in a surface $\Sigma$. Then a tubular neighbourhood of $a$ can be identified with the annulus $a \times [0, 1]$. We can define the *positive Dehn twist along* $a$, denoted $\tau_a$, by setting $\tau_a$ to be a positive Dehn twist in the tubular neighbourhood of $a$ and the identity away from it. Its inverse is again called the *negative Dehn twist along* $a$ and denoted $\tau_a^{-1}$. The mapping class of these diffeomorphisms depends only on the isotopy class of the curve $a$, and we will in general not distinguish between mapping classes and their representatives. In particular, the mapping class of a Dehn twist will also be called a Dehn twist. By a result of Lickorish in [31], the mapping class group of a compact surface with boundary $\Sigma$ is generated by Dehn twists along non-separating or boundary parallel curves.

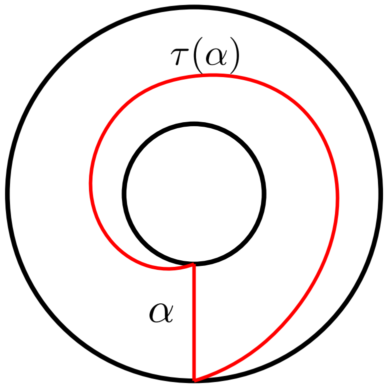


Figure 2.1: The effect of a positive Dehn twist $\tau$ on the arc $\alpha$.

We will now see that there is a relationship between open books and contact structures. We begin with the following definition.

**Definition 2.2.5.** A contact manifold $(M, \xi)$ is *supported* by an open book decomposition $(B, \pi)$ of $M$ if $\xi$ can be isotoped through contact structures so that there is a contact form $\alpha$ such that

- $d\alpha$ is a positive area form for each page of the open book.
- $\alpha > 0$ on $B$.

It is a classical result of Thurston and Winkelnkemper in [42] that every open book $(\Sigma, \varphi)$ supports a contact structure on the manifold $M_\varphi$ obtained from the mapping torus construction as above. However, the relationship between open books and contact structures is even closer. To explore it we first need to introduce the following operation of open books.

**Definition 2.2.6.** Let $(\Sigma, \varphi)$ be an open book decomposition, and let $\alpha$ be a properly embedded arc in $\Sigma$, i.e an embedding of the unit interval in $\Sigma$ such that the boundary of the interval gets mapped to the boundary of $\Sigma$. Then the *(positive) stabilisation* of $(\Sigma, \varphi)$ is the open book $(\Sigma', \tau_a \circ \varphi)$ where

- The page $\Sigma'$ is the surface obtained by attaching a 1-handle to $\Sigma$ along $\partial\alpha$.
- The monodromy $\tau_a \circ \varphi$ is the original monodromy (extended over the handle by the identity) composed with a positive Dehn twist along a closed curve $a$, which is the union of $\alpha$ with the core of the 1-handle.

We can see a stabilisation in Figure 2.2.

**Example 2.2.7.** Take the open book $(D^2, \mathrm{Id})$ from Example 2.2.3. Let $\alpha$ be a properly embedded arc in $D^2$, and perform a positive stabilisation along $\alpha$. The result of attaching a 1-handle to $D^2$ is an annulus, and the monodromy is now a positive Dehn twist along the core of the annulus. Thus the stabilised open book is $(S^1 \times [0, 1], \tau)$.

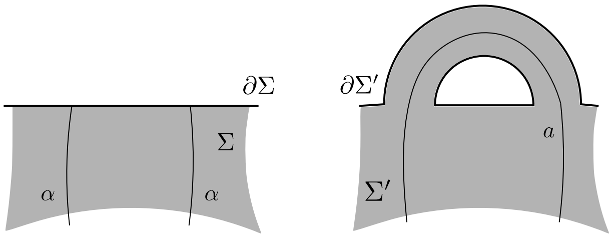


Figure 2.2: The effect of a stabilisation on an open book. Here both endpoints of the arc $\alpha$ belong to the same boundary component, but they can also belong to different boundary components.

There is an analogous definition for *negative* stabilisations replacing the positive Dehn twist with a negative Dehn twist, but we will not need it. The importance of stabilisations comes from the fact that they do not change the underlying 3-manifold, as they correspond to a connected sum with $S^3$ (thus Example 2.2.7 gives another open book for $S^3$). Moreover, as we shall see now, they also do not change the contact structure supported by the open book. Negative stabilisations, on the other hand, do change the contact structure (indeed they make it overtwisted).

We are finally ready to state the fundamental Giroux correspondence theorem.

**Theorem 2.2.8.** *[19] There exists a one-to-one correspondence between contact structures up to isotopy and open books up to positive stabilisation.*

**Example 2.2.9.**
- The open book $(D^2, \mathrm{Id})$, which is an open book for $S^3$, supports the contact manifold $(S^3, \xi_{\mathrm{std}})$. By Giroux's theorem this means that the stabilised open book $(S^1 \times [0,1], \tau)$ also supports $(S^3, \xi_{\mathrm{std}})$.
- The open book $(S^1 \times [0,1], \mathrm{Id})$ supports the standard Stein fillable contact structure on $S^1 \times S^2$.

The Giroux correspondence theorem means that contact 3-manifolds can be studied by the means of surface diffeomorphisms. For example, we have the following result, which can be found in [19] and [33] and that we will use in Chapter 4.

**Theorem 2.2.10.** *A contact* 3*-manifold* $(M, \xi)$ *is Stein fillable if and only if it admits a supporting open book decomposition* $(\Sigma, \varphi)$*, where* $\varphi$ *is a product of positive Dehn twists.*

Another result due to Honda, Kazez, and Matić [21] deals with detection of tight contact structures via a property of mapping classes called *right-veering*. This property will be defined in Chapter 3, and indeed is the focus of that Chapter.

**Theorem 2.2.11.** *A contact manifold* $(M, \xi)$ *is tight if and only if all of its supporting open books are right-veering.*

## § 2.3 | The Heegaard Floer contact invariant

We now introduce Heegaard Floer homology with the aim of defining an invariant of contact structures, usually referred to as the *contact class.* This invariant can provide some information on tightness and fillability of contact manifolds, and we will compute it for an infinite family of manifolds in Chapter 4.

Heegaard Floer homology is a package of 3-manifold invariants defined by Ozsváth and Szabó in [37] and [38]. We will focus on the simplest version, called the "hat" version and denoted $\widehat{HF}$, because the invariant of contact structures that we will use lies in it. We will further simplify the discussion by not mentioning $\text{Spin}^c$ structures or gradings, since we do not need them. Similarly, we will not worry about admissibility of the Heegaard diagrams (which is a necessary condition for Heegaard Floer homology to be well defined), because the diagrams we will use will come from open book decompositions and will, by [23], be admissible. Finally, to avoid sign issues, we will work with $\mathbb{F}_2$ coefficients.

Let $M$ be a closed smooth 3-manifold. The construction of $\widehat{HF}(M)$ starts with a Heegaard diagram for $M$, for which we need to first define Heegaard decompositions.

**Definition 2.3.1.** A *genus $g$ handlebody $U$* is a 3-manifold diffeomorphic to a neighbourhood of a wedge of $g$ circles in $\mathbb{R}^3$. Its boundary is a closed oriented surface of genus $g$. A *Heegaard decomposition* of a closed 3-manifold $M$ is then a decomposition of $M$ into two genus $g$ handlebodies, $M = U_1 \cup_{\Sigma_g} U_2$, where $\Sigma_g$ is the common boundary of $U_1$ and $U_2$ (with the orientation induced by $U_1$).

A Heegaard decomposition for a 3-manifold $M$ can be encoded in the surface $\Sigma_g$ (called the *Heegaard surface*) by means of a *Heegaard diagram.*

**Definition 2.3.2.** A *Heegaard diagram* of a Heegaard decomposition $M = U_1 \cup_{\Sigma_g} U_2$ is a triple $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta})$, where

- $\Sigma_g$ is the Heegaard surface of genus $g$.
- $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\} \subset \Sigma$ is a $g$-tuple of pairwise disjoint simple closed curves, bounding pairwise disjoint embedded discs in $U_1$ and such that $\Sigma \setminus \boldsymbol{\alpha}$ is connected. The curves are called the *$\alpha$ curves.*
- $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\} \subset \Sigma$ is a $g$-tuple of pairwise disjoint simple closed curves, bounding pairwise disjoint embedded discs in $U_2$ and such that $\Sigma \setminus \boldsymbol{\beta}$ is connected. The curves are called the *$\beta$ curves.*

We can recover the Heegaard decomposition (and thus the original manifold) by forming the product $\Sigma_g \times [-1, 1]$, attaching discs to $\Sigma_g \times \{-1\}$ along the $\alpha$ curves and to $\Sigma_g \times \{1\}$ along the $\beta$ curves, and finally attaching 3-balls along the resulting $S^2$ boundaries.

**Example 2.3.3.** Since $S^3$ is the union of two 3-balls along their common boundary $S^2$, we have a trivial Heegaard diagram for $S^3$ where the Heegaard surface is a sphere and there are no $\alpha$ or $\beta$ curves.

**Example 2.3.4.** Let $\Sigma_1 = S^1 \times S^1$, $\alpha_1 = \{\text{pt}\} \times S^1$ and $\beta_1 = S^1 \times \{\text{pt}\}$. Then $(\Sigma_1, \alpha_1, \beta_1)$ is a Heegaard diagram for $S^3$.

By a result of Singer [41], any closed 3-manifold admits a Heegaard decomposition (and thus a Heegaard diagram). Moreover, any two Heegaard diagrams for a 3-manifold $M$ are related by a sequence of moves called *isotopies*, *handleslides*, and *stabilisations*. An isotopy corresponds to an isotopy of the $\alpha$ and $\beta$ curves. A handleslide of a curve $\alpha_i$ over a curve $\alpha_j$ is replacing $\alpha_i$ with the curve $\alpha_k$ such that $\alpha_i, \alpha_j, \alpha_k$ bound a pair of pants in in the Heegaard surface disjoint from the other $\alpha$ curves (analogously we can do a handleslide of $\beta$ curves). Finally, a stabilisation corresponds to a connect sum with the Heegaard diagram from Example 2.3.4 (where the connect sum is performed away from the $\alpha$ and $\beta$ curves of both diagrams).

Now let $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram, and pick a point $z \in \Sigma_g \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, which we will refer to as the *basepoint*. Then the tuple $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is called a *pointed Heegaard diagram*. There is an analogous notion of *pointed* isotopies, handleslides, and stabilisations relating any two pointed Heegaard diagrams representing the same (pointed) 3-manifold.

Now consider the space $\text{Sym}^g(\Sigma_g) = \Sigma_g^{\times g}/S_g$, where $S_g$ is the symmetric group on $g$ elements (i.e $\text{Sym}^g(\Sigma_g)$ is the space of unordered $g$ tuples of points in $\Sigma_g$). The action of the symmetric group is not free, but $\text{Sym}^g(\Sigma_g)$ is nevertheless a smooth manifold, and it inherits a complex structure from a complex structure in $\Sigma_g$ (see [34] for a proof of both of these facts). Moreover, since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are collections of pairwise disjoint simple closed curves, they induce a pair of $g$-dimensional tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ inside $\text{Sym}^g(\Sigma_g)$.

The *Heegaard Floer* chain complex associated to the pointed Heegaard diagram $\mathcal{H}$ is the $\mathbb{F}_2$-vector space generated by intersection points $x = (x_1, \dots, x_g) \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, with differentials counting holomorphic discs between these intersection points, and is denoted $\widehat{CF}(\mathcal{H})$. We now describe these differentials, for which we need to define Whitney discs.

**Definition 2.3.5.** Let $\mathcal{H}$ be a pointed Heegaard diagram of a 3-manifold $M$, and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \text{Sym}^g(\Sigma_g)$. A *Whitney disc* from $\boldsymbol{x}$ to $\boldsymbol{y}$ is a continuous map $u : D^2 \to \text{Sym}^g(\Sigma_g)$, where $D^2$ is the unit disc in $\mathbb{C}$, such that

- $u(-i) = \boldsymbol{x}$ and $u(i) = \boldsymbol{y}$.
- $u(\{z \in \partial D^2 \mid Re(z) > 0\}) \subset \mathbb{T}_\alpha$ and $u(\{z \in \partial D^2 \mid Re(z) < 0\}) \subset \mathbb{T}_\beta$.

We denote by $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ the set of homotopy classes of Whitney discs from $\boldsymbol{x}$ to $\boldsymbol{y}$.

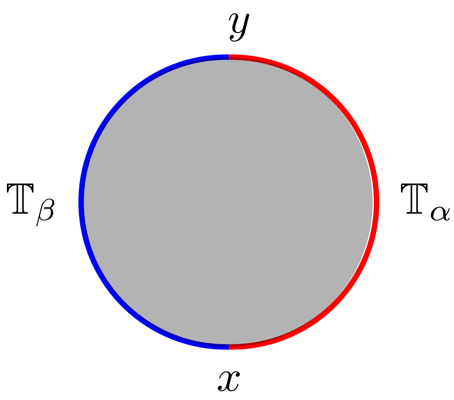


Figure 2.3: A Whitney disc.

Recall that $\mathrm{Sym}^g(\Sigma_g)$ inherits a complex structure from a complex structure on $\Sigma$. Now for $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, let $\mathcal{M}(\phi)$ be the moduli space of holomorphic representatives of $\phi$. This space has an expected dimension called the *Maslov index* and denoted $\mu(\mathcal{M}(\phi))$ or $\mu(\phi)$. Moreover it admits an $\mathbb{R}$ action by considering the unit disc as the infinite strip $[0,1] \times i\mathbb{R} \subset \mathbb{C}$ and then using vertical translations. Thus if $\mu(\mathcal{M}(\phi)) = 1$, the quotient $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$ is a collection of points. For the differential we will use, rather than $\mathcal{M}(\phi)$, the moduli space $\mathcal{M}_0(\phi)$ of holomorphic representatives that miss $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$, where $z$ is the basepoint in $\mathcal{H}$.

**Definition 2.3.6.** Let $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed Heegaard diagram of a 3-manifold $M$. For an intersection point $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we define the differential by

$$\partial \boldsymbol{x} = \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}), \mu(\phi)=1} (\#\widehat{\mathcal{M}}_0(\phi))\boldsymbol{y}$$

Ozsváth and Szabó showed in [37] that the sums in this differential are finite and $\partial^2 = 0$, so $(\widehat{CF}(\mathcal{H}), \partial)$ is a chain complex. We write $\widehat{HF}(\mathcal{H})$ for the homology of this chain complex. Now start with a 3-manifold $M$, and let $\mathcal{H}$ be a Heegaard diagram for $M$. Then one of the main results of [37] is that the isomorphism type of $\widehat{HF}(\mathcal{H})$ does not depend on the various choices made in the definition, for example, the choice of pointed Heegaard diagram for $M$ or the choice of complex structure on the Heegaard surface. Moreover Juhász, Thurston and Zemke [25] show naturality of Heegaard Floer homology, so we can assign to $M$ a concrete group, and not just an isomorphism class. Thus we can refer to this group as the *Heegaard Floer homology* of $M$, denoted $\widehat{HF}(M)$.

This invariant is in general hard to compute, due to the difficulty of identifying holomorphic discs in $\mathrm{Sym}^g(\Sigma_g)$. A useful tool for computations, which we will use in Chapter 4, is the *nicefying* algorithm, defined by Sarkar and Wang in [40]. They show that, using isotopies and handleslides, it is possible to modify a Heegaard diagram into a *nice* diagram, which is one where every connected component of $\Sigma_g \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ (except the one containing the basepoint) is either a rectangle of a bigon. Then, every holomorphic disc in $\mathrm{Sym}^g(\Sigma_g)$ corresponds to a rectangle or a bigon in $\Sigma_g$, and thus

$\widehat{HF}(M)$ can be computed combinatorially.

We will now see that we can compute Heegaard Floer homology from an open book decomposition, and we will use this to define the contact class $c(\xi)$, following the construction in [23]. The original definition of this invariant is due to Ozsváth and Szabó in [39], but the equivalent definition of Honda, Kazez and Matić from [23] is the one that we will use for computations in Chapter 4.

Recall that from an open book $(\Sigma, \varphi)$ for $M$ we can recover the 3-manifold by constructing the mapping torus $\Sigma_\varphi = \frac{\Sigma\times[0,1]}{(x,1)\sim(\varphi(x),0)}$ and filling in the boundary tori, which is topologically the same as further quotienting by the relation $(x,t) \sim (x,t')$ for every $x \in \Sigma$, $t, t' \in [0,1]$. Then we can see a Heegaard decomposition of $M$, where the handlebodies are $H_1 = \Sigma \times [0, \frac{1}{2}]$ and $H_2 = \Sigma \times [\frac{1}{2}, 1]$, and the Heegaard surface is $S_g = (\Sigma \times \{\frac{1}{2}\}) \cup (-\Sigma \times \{0\})$. Now take a basis $\{a_i\}$ for $\Sigma$. Then $a_i \times [0, \frac{1}{2}]$ are embedded discs in $H_1$ whose boundary is in $S_g$, and $a_i \times [\frac{1}{2}, 1]$ are embedded discs in $H_2$ whose boundary is also in $S_g$. Thus the $\alpha$ curves are $\alpha_i = \partial(a_i \times [0, \frac{1}{2}])$ and the $\beta$ curves are $\beta_i = \partial(a_i \times [\frac{1}{2}, 1])$.

We now make some modifications to this so we can compute $\widehat{HF}(M)$ directly on the page of the open book and define the contact invariant. For every $a_i$, let $b_i$ be an arc isotopic to $a_i$ such that

- Its endpoints are obtained from the endpoints of $a_i$ by performing a small isotopy along the boundary of $\Sigma$, in the direction specified by the boundary orientation.
- The arcs $a_i$ and $b_i$ intersect transversely in one point $x_i$ the interior of $\Sigma$, which has positive intersection if we orient $a_i$ and give $b_i$ the induced orientation by the isotopy.

Then if we define $\alpha_i' = \partial(a_i \times [0, \frac{1}{2}])$ and $\beta_i' = \partial(b_i \times [\frac{1}{2}, 1])$, and the Heegaard surface $S_g = (\Sigma \times \{\frac{1}{2}\}) \cup (-\Sigma \times \{0\})$ as before, $(S_g, \alpha_i', \beta_i')$ is still a Heegaard diagram for $M$. We place the basepoint $z$ in the interior of $\Sigma \times \{\frac{1}{2}\}$, outside of the thin strips of isotopy between $a_i$ and $b_i$. Since the discs that we consider for the Heegaard Floer differential must miss $\{z\} \times \mathrm{Sym}^{g-1}(S_g)$, this effectively means that the intersection of any holomorphic disc with $\Sigma \times \{\frac{1}{2}\}$ will be contained in the thin strips of isotopy. Thus we can see all the holomorphic discs in the Heegaard Floer complex by simply looking at the page of the open book, and moreover the points $x_i$ can be seen as the boundary points of the properly embedded arcs of the basis (where we need to consider both endpoints as the same point).

We are now ready to define the contact invariant. Instead of using $(S_g, \alpha_i', \beta_i', z)$, we will use the pointed Heegaard diagram $(S_g, \beta_i', \alpha_i', z)$ (which is a diagram for the manifold with reverse orientation) because the invariant for $M$ lies on $\widehat{HF}(-M)$. Let $\boldsymbol{x} = (x_1, \ldots, x_g)$. Then Honda, Kazez, and Matić show that $\boldsymbol{x}$ is a cycle in $\widehat{CF}(-M)$, and define $EH(\Sigma, \varphi, \{a_1, \ldots, a_g\})$ to be the homology class of $\boldsymbol{x}$. Moreover, they prove that $EH(\Sigma, \varphi, \{a_1, \ldots, a_g\})$ agrees with the contact invariant $c(\xi)$ defined by Ozsváth and Szabó.

It is shown in [39] that this invariant vanishes for overtwisted contact structures and is nonzero for Stein fillable ones. It is also nonzero for strongly fillable contact structures (see [17]). However, it does not completely distinguish between overtwisted and tight contact structures, since there are tight contact structures whose contact invariant vanishes, for example, those with Giroux torsion, as shown in [18] (note that this together with the result in [17] provides another proof of Gay's result that contact manifolds with Giroux torsion cannot be strongly fillable). We will provide an explicit computation of this in Chapter 4.

Chapter 3

# Right-veering diffeomorphisms

## § 3.1 | Introduction

In light of Theorem 2.2.11, we can study tightness of contact structures by looking at the right-veering property of open books. An open book is *right-veering* if its monodromy sends every properly embedded arc to the right (a precise definition will be given in Section 3.2). Therefore existence of a single arc that is not sent to to the right (a *left-veering arc*) is enough to guarantee overtwistedness of the contact structure. However, it is often difficult to determine if such an arc exists. Indeed, if we take a *basis* of the surface (a collection of arcs which cut the surface into a disc), the images of these arcs under a diffeomorphism determine it up to isotopy; therefore it would be reasonable to expect that the right-veering property would be encoded in any such collection. However, it is possible to find monodromies that are not right-veering and yet send each arc of some basis to the right. Looking at more arcs does not necessarily solve the problem, as similar counterexamples can be found, for example, when the collection of arcs we look at is a *complete set*, i.e a maximal collection of pairwise disjoint and non-isotopic arcs.

Thus the usual approach to showing whether an open book is right-veering is to either exhibit an arc that goes to the left, which is found in a non-systematic way, or divide all arcs into different classes and then show that each class can only contain arcs sent to the right. The main issue with this is that in both cases the argument is case dependent.

We show that, given an open book $(\Sigma, \varphi)$ and a basis $\mathcal{B}$ of $\Sigma$, a left-veering arc induces a collection of objects, called *extended towers*, that are constructed using $\mathcal{B}$ and its image under $\varphi$; with some specific properties. Conversely, the existence of such a collection implies the existence of a left-veering arc, which moreover can be constructed from the extended towers. This is a modification of the notion of consistency introduced in [43] and [44] with the aim of detecting tightness.

### § 3.1.1 | Strategy of the algorithm

First we show, in Propositions 3.2.12 and 3.2.16, that using *regions* we can detect a left-veering arc and a fixable arc segment in a simple case, that is, when they are contained in a 6-gon cut out by 3 arcs, two of which belong to our chosen arc collection.

In Subsection 3.4.1 we show that existence of a left-veering arc implies the existence of a *minimal left-veering arc*, which can be divided into arc segments that are either fixable or left-veering and are disjoint from our chosen arc collection except in endpoints that are not on the boundary of the surface.

Then in Subsection 3.4.2 we prove that the regions in Propositions 3.2.12 and 3.2.16 form extended towers. This is our base case.

Our inductive step is then to show that an extended tower supported in an arc collection with $n$ arcs induces an extended tower supported in an arc collection with $n + 1$ arcs, and conversely. Moreover, we show that the properties of the extended tower are preserved. The proof of this inductive step is cumbersome and is broken down into several Lemmas in Subsection 3.4.3 to account for the different cases.

Using this we show that if we have an arc collection that, together with a left-veering arc, cuts out a disc, then the left-veering arc is detected by an incomplete extended tower supported in the arc collection. Similarly, a fixable arc segment is detected by a completed extended tower. This is the content of Theorems 3.4.18 and 3.4.19.

Finally, we show that in the case of the minimal left-veering arc not being disjoint from our chosen basis, we can detect it with a collection of completed extended towers and one incomplete extended tower. This constitutes our main result, Theorem 3.4.22. This implies the existence of an algorithm to check the right-veering property because one a basis is fixed, there exists a finite amount of regions to check.

We can see the strategy in Figure 3.1. Once we have a basis with arcs duplicated, they cut $\Sigma$ into a disc. Moreover, the arcs from the basis divide the minimal left-veering arc $\gamma$ into segments which are either fixable or left-veering, and disjoint from the basis except at endpoints. First, Theorem 3.4.19 shows that $\gamma_1$ being fixable is detected by a completed extended tower supported in the arc collection that, together with $\gamma_1$, cuts out the disc $P_1$ (which exists because the arcs in this arc collection belong to our chosen basis, which cuts the entire surface into a disc), and similarly for $\gamma_2$. Then, Theorem 3.4.18 shows that $\gamma_3$ being left-veering is detected by an incomplete extended tower supported in the arc collection that together with $\gamma_3$ cuts out the disc $P_3$.

### § 3.1.2 | Dictionary

Since a large amount of the terminology used in this Chapter is non-standard, we start by gathering some of the new concepts that we will introduce into a "dictionary", for quick reference. The explanations given here are not meant to be precise definitions, as these are presented throughout the Chapter, but rather provide some intuition as to why they are needed.

The main objects we will use are *extended towers*, collections of regions that aim to

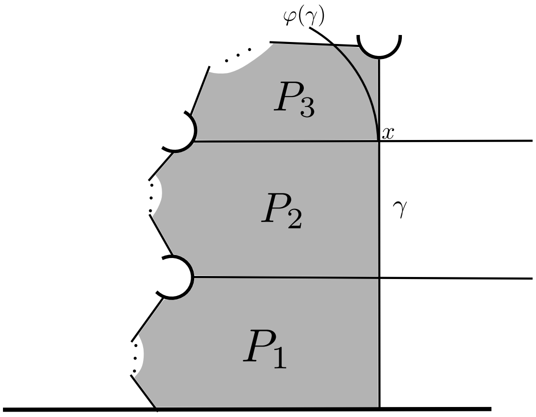


Figure 3.1: A left-veering arc $\gamma$, which we want to detect using extended towers. The arc is minimal with respect to the basis, which means that its image is itself up until the point $x$ and after that it goes to the left.

detect fixable arc segments and left veering arcs. *Regions* are immersed $n$-gons whose boundary edges lie alternatively on arcs from a given arc collection an their images. Vertices of these regions alternate between $\bullet$*-points* (positive intersection points) and $\circ$*-points* (negative intersections points). In terms of Heegaard Floer homology, a region gives a differential from the generator given by the $\circ$-points to the generator given by the $\bullet$-points.

*Remark.* It is important to remark that we will *not* always end up with a Heegaard Floer object. However, we do in some simple cases, which is why an intuition based on Heegaard Floer is provided for several of the definitions here.

Regions are *positive* if the boundary induced by the arcs and arc images is the standard one, and *negative* otherwise.

These regions will be bounded by collections of arcs and their images, with the particular property that the collection of arcs (together with an extra arc not belonging to the collection) cuts out a disc from the surface. Moreover, to obtain a complete characterisation of left-veering arcs we need to impose some further conditions on these extended towers.

1. Repleteness (Definition 3.3.3): If it is possible to add a negative region to the extended tower, then it must be added. A relation to Heegaard Floer homology is the following. If a region $R$ contributes to the differential of a generator $x$, other regions contributing to the differential of $x$ must also be considered.

2. Niceness (Definition 3.3.4): Positive regions are disjoint from the supporting arc collection in their interior, and negative regions are disjoint from the image of the supporting arc collection in their interior. This ensures that the definition of the regions induced by arc-slides, central to the inductive step, is straightforward.

3. Nestedness (Definition 3.3.5): The extended towers are constructed level by level, that is, we start with positive regions whose $\bullet$-points are on the boundary. Then

we get negative regions from the $\circ$-points of these positive regions, then more positive regions from the $\bullet$-points of these positive regions and so on. An analogue coming from Heegaard Floer would be a chain of generators which kills the contact class: First we start with generators whose differential includes the contact class (positive regions), but potentially also more generators (negative regions). Then we consider points whose differential includes these extra generators (positive regions) and so on.

4. Completeness (Definition 3.3.7): Every interior vertex belongs to both a positive and a negative region. This ensures that there exists a fixable arc segment (Definition 3.2.15) because the fact that the regions cover the entire arc collection means the arc segment cannot go to either right or left.

5. Incompleteness (Definition 3.3.8): In every negative region there is a vertex which also belongs to a positive region. In Heegaard Floer language this would mean that the chain given by the collection of regions kills the contact class.

## § 3.2 | Preliminaries

We start with some definitions regarding open books, and we introduce the concept of a region.

**Definition 3.2.1.** Let $\Sigma$ be a compact surface with nonempty boundary. A *properly embedded arc* is the image of an embedding $\alpha : [0,1] \hookrightarrow \Sigma$ such that $\alpha(0), \alpha(1) \in \partial\Sigma$. An *arc segment* is the image of an embedding of the unit interval that is not necessarily proper, i.e we do not require that $\alpha(0), \alpha(1) \in \partial\Sigma$.

**Definition 3.2.2.** Let $(\Sigma, \varphi)$ be an open book. An *arc collection* in $\Sigma$ is a set of pairwise disjoint properly embedded arcs. An arc collection $\mathcal{B}$ such that $\Sigma \setminus \mathcal{B}$ is a disc is called a *basis*.

The significance of bases comes from the fact that a mapping class is uniquely determined by its action on any basis.

**Definition 3.2.3.** Let $(\Sigma, \varphi)$ be an open book, and $\Gamma$ an arc collection in $\Sigma$. If $\Sigma \setminus \Gamma$ contains an $n$-gon component with exactly one edge on each element of $\Gamma$, we say $\Gamma$ *cuts out an $n$-gon.*

**Definition 3.2.4.** Let $\alpha_1, \alpha_2$ be disjoint properly embedded oriented arcs in a compact surface with boundary $\Sigma$, such that there is a boundary arc going from $\alpha_1(1)$ to $\alpha_2(0)$. The *arc-slide* of $\alpha_1$ and $\alpha_2$ is (the isotopy class of) the arc $\beta$ that starts at $\alpha_1(0)$ and ends at $\alpha_2(1)$, such that $\alpha_1, \alpha_2$, and $\beta$ cut out a 6-gon from $\Sigma$ whose standard boundary orientation coincides with the orientation from $\alpha_1$, $\alpha_2$, and $-\beta$.

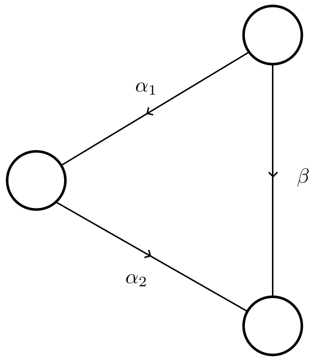


Figure 3.2: The arc-slide $\beta$ of two arcs $\alpha_1$ and $\alpha_2$. Observe we need to reverse the orientation of $\beta$ to obtain an orientation of the boundary of the 6-gon.

Any two bases for $\Sigma$ can be related by a sequence of arc-slides (see [23] for a proof of this fact). We can also extend this definition to larger families of arcs; if a collection of pairwise disjoint arcs $\{\alpha_i\}_{i=1}^n$ is such that there is a boundary arc going from $\alpha_i(1)$ to $\alpha_{i+1}(0)$, then the arc $\beta$ that starts at $\alpha_1(0)$ and ends at $\alpha_n(1)$ and cuts out a disc from $\Sigma$ (whose standard boundary orientation coincides with the orientation from $\alpha_i$ and $-\beta$) is called the *arc-sum* of $\{\alpha_i\}_{i=1}^n$.

We now give the notion of right-veering arcs as introduced in [21]. Our definition is phrased in a slightly different way, in order to be consistent with the orientation convention that we will use, but is equivalent to the one in [21].

**Definition 3.2.5.** Let $(\Sigma, \varphi)$ be an open book decomposition, and let $\alpha$ be an oriented properly embedded arc with starting point $x$. We will adopt the convention that its image $\varphi(\alpha)$ is given the opposite orientation to $\alpha$. We then say that $\alpha$ is *right-veering* (with respect to $\varphi$) if $\varphi(\alpha)$ is isotopic to $\alpha$ or, after isotoping $\alpha$ and $\varphi(\alpha)$ so that they intersect transversely with the fewest possible number of intersections, $(\alpha'(0), \varphi(\alpha)'(1))$ define the orientation of $\Sigma$ at $x$. In this latter case we will say that $\alpha$ is *strictly right-veering*. If $\alpha$ is not right-veering we say it is *left-veering*.

In Figure 3.3 we can see that intuitively a right-veering arc $\alpha$ is such that $\varphi(\alpha)$ is to the right of $\alpha$ near the starting point once we have isotoped them so that they have the fewest possible number of intersections.

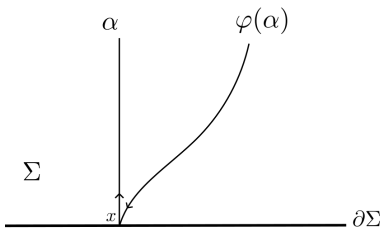


Figure 3.3: A (strictly) right-veering arc $\alpha$.

Although this definition only refers to the starting point of the oriented arc, we will usually say that an arc is right-veering to mean that both itself and the arc with opposite orientation are right-veering (thus referring to both endpoints). However, when we say than an arc is left-veering we only refer to its starting point. This is because one left-veering oriented arc is enough for an open book to support an overtwisted contact structure, by [21], so we only need to detect one.

**Definition 3.2.6.** Let $(\Sigma, \varphi)$ be an open book decomposition. We say that $\varphi$ is *right-veering* if every oriented properly embedded arc in $\Sigma$ is right-veering.

Sometimes we will say that the open book itself is right-veering when the monodromy is right-veering.

Figure 3.4 shows that, to determine if an open book is right-veering, it is not enough to check that every arc of a basis is right-veering. The page is a planar surface with 4 boundary components, and the monodromy is determined by the images of the arcs from the basis.

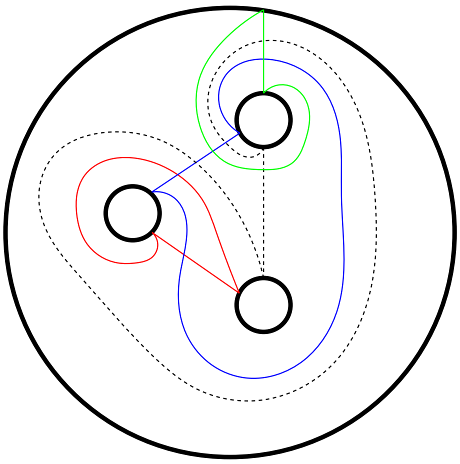

Figure 3.4: A basis of right-veering arcs, and a (dotted) left-veering arc.

**Definition 3.2.7.** Let $(\Sigma, \varphi)$ be an open book. We say that an arc $\gamma$ is *bigon free* with respect to an arc $\alpha$ if $\gamma$ does not form any bigons with $\alpha$. Similarly, we say that an arc collection $\Gamma$ is *bigon free* if for every $\alpha, \beta \in \Gamma$, $\alpha$ does not form any bigons with $\varphi(\beta)$.

Given an arc collection $\Gamma$, we can always isotope $\varphi(\Gamma)$ so that $\Gamma$ is bigon free, and so we will always assume this is the case.

We now recall some definitions and notation from [44].

**Definition 3.2.8.** Let $(\Sigma, \varphi)$ be an open book, $\Gamma = \{\alpha_i\}_{i=1}^n$ an arc collection, and $\varphi(\Gamma) = \{\varphi(\alpha_i)\}_{i=1}^n$ its image under the mapping class $\varphi$ (we orient $\varphi(\alpha_i)$ with the opposite orientation to the one induced by $\alpha_i$). A *region* $R$ *in* $(\Sigma, \varphi, \Gamma)$ (or *supported in* $(\Sigma, \varphi, \Gamma)$) is the image of an immersed $2k$-gon such that:

- The edges are mapped to $\Gamma$ and $\varphi(\Gamma)$ alternatively.

- The orientations of the arcs and their images orient $\partial R$.

- Every corner is acute, i.e for every vertex $x = \alpha_i \cap \varphi(\alpha_j)$, in a neighbourhood of $x$, $R$ only intersects one of the 4 quadrants defined by $\alpha_i$ and $\varphi(\alpha_j)$ at $x$.

- The immersion restricted to the vertices of the $2k$-gon is injective.

A point $\alpha_i \cap \varphi(\alpha_j)$ is *positive* if the tangent vectors of $\alpha_i$ and $\varphi(\alpha_j)$ (in that order) determine the orientation of $\Sigma$ at the intersection point, and *negative* otherwise. Moreover, we will say a region is *positive* if the boundary orientation given by the orientation of the arcs from $\Gamma$ and $\varphi(\Gamma)$ coincides with the usual counterclockwise orientation, and *negative* otherwise.

We will denote positive intersection points by $\bullet$-points and negative intersection points by $\circ$-points. See Figure 3.5 for an example of a region with its positive and negative points labelled. We will also denote the set of $\bullet$-points (respectively $\circ$-points) of a region $A$ by $\mathrm{Dot}(A)$ (respectively $\mathrm{Circ}(A)$), and the set of vertices $\mathrm{Dot}(A) \cup \mathrm{Circ}(A)$ as $\mathrm{V}(A)$.

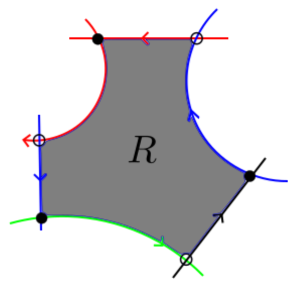


Figure 3.5: A region $R$, which is positive because the arcs orient $\partial R$ counterclockwise. We will use the convention that straight lines represent arcs and curved lines represent their images under $\varphi$.

**Definition 3.2.9.** Let $R$ be a region in $(\Sigma, \varphi, \Gamma)$. We say that $R$ is *completed* if there exists another region $R'$, such that $\mathrm{Circ}(R') \subset \mathrm{Circ}(R)$, but the induced orientation of $\partial R'$ is the opposite orientation to the one in $R$. We will call this region the *completion* of $R$. If no such region exists we say $R$ is *not completed*.

*Remark.* A completion of a region need not use all of the arcs used in the region.

Now our setup differs slightly from the standard version of consistency as defined in [43] and [44]. The reason for this is that, while Honda, Kazez, and Matić's result establishes a relationship between the right-veering property and tightness, the two concepts are not equivalent, since for any overtwisted contact manifold we can find a supporting open book decomposition that is right-veering. The technology in [43] and [44] aims to detect tightness, while we aim to detect the right-veering property, and so it is natural that there will be similarities as well as differences.

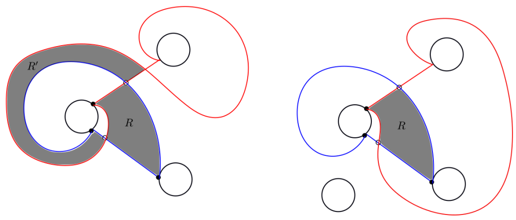


Figure 3.6: On the left, a region $R$ and its completion $R'$. On the right, a region $R$ which is not completed.

**Definition 3.2.10.** Let $(\Sigma, \varphi)$ be an open book, $\Gamma$ an arc collection in $\Sigma$, and $\alpha_1, \alpha_2 \in \Gamma$. Assume there exists a boundary component $B$ of $\Sigma$ which contains an endpoint of $\alpha_1$ and an endpoint of $\alpha_2$. Then assume that $\varphi(\alpha_1)$ is boundary parallel near $B$ until it intersects $\alpha_2$, and does not intersect any other $\alpha \in \Gamma$ before doing so. This creates a triangle (with sides an arc segment of $\varphi(\alpha_1)$, an arc segment of $\alpha_2$ and an arc segment of $B$). We can see such a triangle in Figure 3.7. We will refer to this triangle as a *basepoint triangle*. For an arc collection $\Gamma$, we denote the set of $\circ$-points that are vertices of basepoint triangles by $\mathrm{Circ}_{\partial}(\Gamma)$.

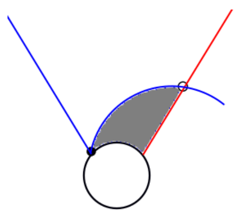

Figure 3.7: A basepoint triangle (shaded).

*Remark.* If there is an arc image $\varphi(\alpha)$ that intersects a basepoint triangle disjoint from $\alpha$, it must do so forming a bigon. Note that since we assume that our collections are bigon free then this cannot happen. We can generalise this situation for the case where, instead of a basepoint triangle, we have a disc component of $\Sigma \setminus (\Gamma \cup \{\alpha\})$ with a unique edge on an arc $\alpha \in \Gamma$ (a basepoint triangle is the simplest case of such a disc, with an edge on an arc, and the other two on the boundary and an arc image respectively).

**Definition 3.2.11.** Let $\Gamma$ be an arc collection in an open book $(\Sigma, \varphi)$, $\{\varphi(\beta_i)\}_{i=1}^{n}$ a subcollection of $\varphi(\Gamma)$, and $a$ an arc segment of an arc $\alpha \in \Gamma$ such that $\{\varphi(\beta_i)\}_{i=1}^{n} \cup \{a\}$ cut out a disc $D$ from $\Sigma$. Then we say $a$ is *restricted in* $\Gamma$.

Similarly as in the previous remark, for a restricted edge $a$ and associated disc $D$, images of arcs disjoint from $D$ cannot intersect the restricted edge because they would then have to form a bigon. Clearly the edge of a basepoint triangle that lies on an arc is restricted, with the disc $D$ being the basepoint triangle itself.

Now we want to show that, in the case where the arc-slide of a pair of arcs –with the opposite orientation– is left-veering, regions with an edge on a basepoint triangle can detect this left-veering arc. We will see later that this is a simple example of an extended tower, and it will be the base case of our induction.

This means that we want to understand what possibilities there are for regions when we look at three arcs $\{\alpha_0, \alpha_1, \alpha_2\}$ that cut out a 6-gon from the surface. Call this 6-gon $P$. Then $\varphi(P)$ must also be a 6-gon. Now consider an arc collection $\mathcal{C}$ (which may also include some of the arcs cutting out $P$). When segments of two of the arc images $\varphi(\alpha_i), \varphi(\alpha_j)$ form opposite sides of a rectangle which is a connected component of $\varphi(P) \setminus \mathcal{C}$, we will say that they are *parallel (with respect to $\mathcal{C}$)* along those segments.

**Proposition 3.2.12.** *Let $\alpha_0, \alpha_1, \alpha_2$ be properly embedded arcs that cut out a 6-gon $P$ from $\Sigma$, oriented counterclockwise, and assume $\alpha_1$ and $\alpha_2$ are right-veering. Then $\alpha_0$ is left-veering if and only if there exists a positive region $R$ in $\{\alpha_1, \alpha_2\}$ contained in $P$, with $\bullet$-points on $\partial\Sigma$ and where one of the edges is the edge of a basepoint triangle, that has no completion.*

*Proof.* First assume that $\alpha_0$ is left-veering, which means (since $\alpha_2$ is right-veering), that it leaves $P$ by intersecting $\alpha_1$ in a point $z$. Since $\alpha_2$ is right-veering, $\varphi(\alpha_2)$ must leave $P$ by intersecting $\alpha_1$ in a point $y$. This in turn means that $\varphi(\alpha_1)$ must leave $P$ by intersecting $\alpha_2$ (as it cannot intersect $\varphi(\alpha_2)$), creating the region $R$, with an edge being an edge of the basepoint triangle formed by $\alpha_2$ and $\varphi(\alpha_1)$. For a contradiction, suppose that this region can be completed with a region $R'$, which must necessarily be a rectangle (it cannot be a bigon because we are assuming our collections are bigon free, and it cannot have more than 4 vertices because $R$ only has 2 $\circ$-points). Moreover, the edge of this rectangle on $\alpha_2$ is restricted, because it is an edge of a basepoint triangle. This in turn means that the edge of the rectangle on $\alpha_1$ is restricted. Then the $\bullet$-point of $R'$ on $\alpha_1$ cannot be between $z$ and $y$, because then $\varphi(\alpha_1)$ would have to form a bigon. This means that it would have to be between $z$ and the other endpoint of $\alpha_1$ (that is, $\alpha_1(0)$). But this means that $\varphi(\alpha_0)$ intersects the restricted edge –a contradiction.

Conversely, assume that there exists a region $R$ satisfying the above conditions, in particular, it has no completion. Suppose for a contradiction that $\alpha_0$ is right-veering. But then, since the image of $P$ must be a disc, there must be an arc segment on $\alpha_1$ cutting out a disc with $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$, so it must be a restricted edge with respect to $\{\alpha_1, \alpha_2\}$. However this in turn means that $R$ has a completion –a contradiction. So $\alpha_0$ must be left-veering, see Figure 3.8.

□

For completeness, we include the case where all three arcs cutting out a 6-gon are right-veering.

**Proposition 3.2.13.** *Let $\alpha_0, \alpha_1, \alpha_2$ be properly embedded arcs cutting out a 6-gon $P$, oriented counterclockwise, and assume they are right-veering. Then we can find a non-empty collection of regions in $\{\alpha_0, \alpha_1, \alpha_2\}$ such that every positive region is completed*

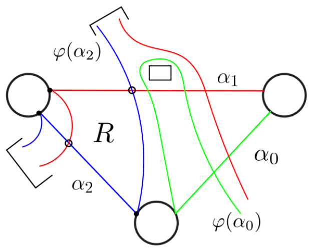


Figure 3.8: The incomplete region $R$ when $\alpha_0$ is left-veering.

*by a negative region, and every interior $\bullet$-point is a vertex of two regions (one positive and one negative).*

*Proof.* Suppose first that the image of each arc leaves $P$ by forming a basepoint triangle. Then we have an initial positive region $R$ (a 6-gon) where the $\bullet$-points are endpoints of the arcs and the $\circ$-points are the intersection points where the arc images leave $P$. To see that this is completed, observe that the image of $P$ is again a 6-gon which is the union of a completing region and the three basepoint triangles, and so we must have a negative region with the same $\circ$-points as $R$ and whose $\bullet$-points are the other endpoints of the arcs.

Now suppose that one arc image (we can assume it is $\varphi(\alpha_2)$) leaves $P$ without forming a basepoint triangle (so in this case, by intersecting $\alpha_1$). This forces the image of $\alpha_1$ to leave $P$ by intersecting $\alpha_2$ and creating a basepoint triangle. This immediately gives a positive region $R_1$. This region has a completion $R_1'$, because if it did not, $\alpha_0$ would have to be left-veering by Proposition 3.2.12. One of the $\bullet$-points is a vertex of the basepoint triangle formed by $\varphi(\alpha_1)$ and the other one is either an endpoint of $\alpha_1$ or an interior point $\alpha_1 \cap \varphi(\alpha_1)$, let us denote it by $x$. In the first case we have that $\alpha_0$ is isotopic to $\varphi(\alpha_0)$, as it cannot be left-veering because it would intersect a restricted edge on $\alpha_1$ but it also cannot be strictly right-veering because $\alpha_0$ with the opposite orientation would also have to be strictly right-veering and it would have to intersect the restricted edge on $\alpha_1$.

For the second case we have that, after $x$, $\varphi(\alpha_1)$ intersects $\alpha_0$ and then it must be parallel to $\varphi(\alpha_0)$ until their other endpoint (because the image of $P$ must be a disc). This gives another two regions, a positive one $R_2$ where the $\bullet$-points are $x$ and an endpoint of $\alpha_0$, and its completion $R_2'$, the part of $\varphi(P)$ where $\varphi(\alpha_0)$ and $\varphi(\alpha_1)$ are parallel. □

We have seen what regions arise in a 6-gon when all arcs are right-veering, and when one of the arcs is left-veering. Eventually we want to detect a left-veering arc by dividing it into segments that can be fixed by the monodromy and a segment that is left-veering. Thus, now we turn our attention to arc segments that can be fixed by the monodromy.

**Definition 3.2.14.** Let $(\Sigma, \varphi)$ be an open book. If two arcs $\alpha_1, \alpha_2$ that cut out a 6-gon $P$ with a third arc $\alpha_0$ (oriented counterclockwise) support a pair of regions $\{R_1, R_1'\}$ as in the second case of Proposition 3.2.13 (i.e there is a positive region $R_1$ in $P$ where one of the sides is a side of the basepoint triangle, and it is completed by a region $R_1'$ with a $\bullet$-point in the interior of $\alpha_1$) we say that $\alpha_2$ is *$\varphi$-contained* in $\alpha_1$ and we call $\{R_1, R_1'\}$ a *positive splitting pair.* We also call $\{R_2, R_2'\}$ a *negative splitting pair*, and we also say that $\alpha_0$ is *$\varphi$-contained* in $\alpha_1$.

**Definition 3.2.15.** Let $\Gamma$ be an arc collection in an open book $(\Sigma, \varphi)$, and $x, y \in \Gamma \cap \varphi(\Gamma)$ be two points (which could be on the boundary or interior points) that are fixed by some representative of $\varphi$ . Let $\gamma$ be an arc segment starting in $x$ and ending in $y$. We say $\gamma$ is *fixable by $\varphi$* if there is a representative of $\varphi$ that fixes it relative to $\partial\gamma$.

This means that $\gamma$ and $\varphi(\gamma)$ bound a collection of bigons that intersect only on points $\gamma \cap \varphi(\gamma)$, see Figure 3.9 for an example.

*Remark.* If both endpoints of an arc segment are fixed by $\varphi$, properties like being right- or left-veering can be defined as for properly embedded arcs.

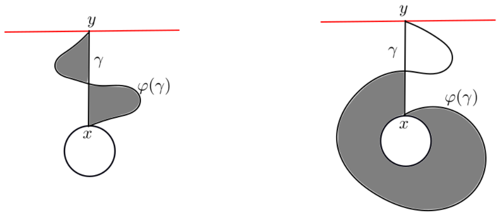


Figure 3.9: On the left, $\gamma$ is fixable because we can isotope $\varphi(\gamma)$ relative to its endpoints to coincide with $\gamma$. On the right, $\gamma$ is not fixable.

In practice, we want our fixed points to come from intersections $\alpha \cap \varphi(\alpha)$ where $\alpha$ is an arc of a chosen basis.

Similarly to Proposition 3.2.12, if we have 3 arcs cutting out a 6-gon, we can use regions to detect a fixable arc inside this 6-gon.

**Proposition 3.2.16.** *Let $(\Sigma, \varphi)$ be an open book, and let $\alpha_0, \alpha_1, \alpha_2$ be properly embedded arcs cutting out a 6-gon $P$ from $\Sigma$, oriented counterclockwise, and assume they are strictly right-veering. Let $\gamma$ be an arc segment contained in $P$ starting on $\partial\Sigma$ between $\alpha_2$ and $\alpha_0$ and ending in an intersection point $x = \alpha_1 \cap \varphi(\alpha_1)$ in the interior of $\alpha_1$. Then $\gamma$ is fixable by $\varphi$ if and only if $\{\alpha_1, \alpha_2\}$ support a positive splitting pair $\{R, R'\}$ such that the $\bullet$-point of $R'$ in the interior of $\Sigma$ is $x$.*

*Proof.* First assume that $\gamma$ is fixable. Then the image of $\alpha_2$ must leave $P$ by intersecting $\alpha_1$, which means that the image of $\alpha_1$ leaves $P$ by intersecting $\alpha_2$ (and forming a basepoint triangle), giving the positive region $R$. Since we are assuming that $\alpha_0$ is strictly right-veering, this region must be completed by a region $R'$, giving the positive splitting pair.

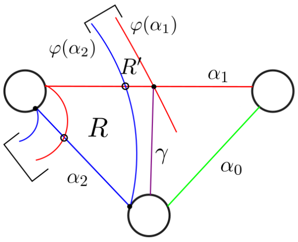


Figure 3.10: The regions $R$ and $R'$ when $\gamma$ is fixable.

Conversely, suppose that we have a positive splitting pair $\{R, R'\}$. Then $\gamma$ cannot be left-veering, because it would have to intersect the edge of $R'$ on $\alpha_1$, which is restricted. However, by Proposition 3.2.13, after the splitting pair (that is, after the interior $\bullet$-point of $R'$) $\varphi(\alpha_1)$ and $\varphi(\alpha_0)$ must be parallel up to the boundary, which means that they form a rectangle with $\alpha_0$ and an edge of a basepoint triangle on $\alpha_1$. Thus the edge of this rectangle on $\alpha_0$ is also restricted. If $\gamma$ were strictly right-veering in its starting point, it would have to intersect this restricted edge. Thus $\gamma$ must be fixable by $\varphi$ (strictly speaking, we might not have that $\varphi(x) = x$, however, in this case, by the same argument, $\gamma, \varphi(\gamma)$, and $\varphi(\alpha_1)$ must bound a disc, and thus we may isotope $\varphi(x)$ by sliding it along $\varphi(\alpha_1)$ to coincide with $x$ and then $\gamma$ is fixable).

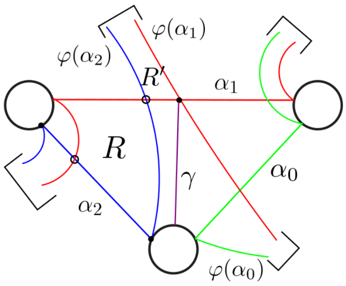


Figure 3.11: Once we have $R$ and $R'$, the image of $\gamma$ cannot go to either side because the edges on $\alpha_0$ and $\alpha_1$ are restricted.

□

## § 3.3 | Extended towers

We now introduce extended towers, which will be our main tool for detecting left-veering arcs. They are inspired by the notion of towers introduced in [44]. However, there are some differences. Towers were defined to detect tightness, and we aim to detect left-veering arcs. Both concepts are related but not equivalent, and the new features of extended towers reflect this. From now on, let $\Gamma$ be an arc collection in a surface with boundary $\Sigma$, and $\alpha_0$ a properly embedded arc, disjoint from $\Gamma$, such that $\Gamma \cup \{\alpha_0\}$ cuts out a disc $P$. Moreover, orient the arcs in the standard counterclockwise orientation of $\partial(P)$.

**Definition 3.3.1.** For a given arc collection $\Gamma$ in an open book $(\Sigma, \varphi)$, we denote the set of regions supported in $\Gamma$ by $\mathcal{R}(\Sigma, \varphi, \Gamma)$. Moreover, for any collection of regions $\mathcal{A}$, the set of positive regions in $\mathcal{A}$ is denoted by $\mathcal{A}^+$, and, similarly, the set of negative regions of $\mathcal{A}$ is denoted by $\mathcal{A}^-$.

**Definition 3.3.2.** An *extended tower* in $(\Sigma, \varphi, \Gamma)$ is a (nonempty) collection $\mathcal{T} \subset \mathcal{R}(\Sigma, \varphi, \Gamma)$ where $\mathrm{Dot}(\mathcal{T}) \subset (\mathrm{Dot}(\mathcal{T}^-) \cup \partial\Sigma)$, $\mathrm{Circ}(\mathcal{T}) \subset (\mathrm{Circ}(\mathcal{T}^+) \cup \mathrm{Circ}_\partial(\Gamma))$, and for all pairs $A, B \in \mathcal{T}$, no corner of $A$ is contained in the interior of $B$. We say that $\Gamma$ *supports* $\mathcal{T}$.

We can see an example of two extended towers in Figure 3.12. These will be our primordial examples, and while we will provide other, simpler examples for some of the properties later on, we will often come back to these ones to illustrate why all the properties are needed. We will see later that the existence of the extended tower on the left implies that the arc segment $\gamma$ is fixable, while the existence of the extended tower on the right implies that the arc $\alpha_5$ is left-veering.

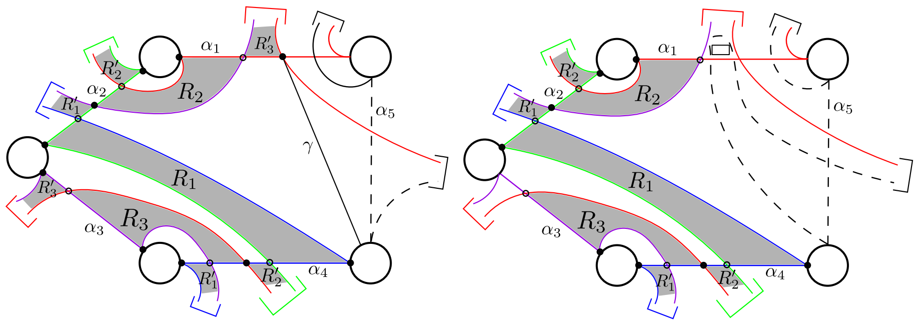


Figure 3.12: Two extended towers which will be our primordial examples to showcase the different properties that we will require.

**Definition 3.3.3.** An extended tower $\mathcal{T}$ in $(\Sigma, \varphi, \Gamma)$ is *replete* if whenever there is a region $A \in \mathcal{R}^-(\Sigma, \varphi, \Gamma)$ which satisfies $\mathrm{Circ}(A) \subset \mathrm{Circ}(\mathcal{T}) \cup \mathrm{Circ}_\partial(\Gamma)$, and $\mathcal{T} \cup A$ is again an extended tower, then $A \in \mathcal{T}$. All extended towers will be assumed replete unless otherwise stated.

The main difference with towers from [44] is that here we allow negative regions with $\circ$-points in $\mathrm{Circ}_\partial(\Gamma)$ as well as $\mathrm{Circ}(\mathcal{T}^+)$. This is because we want to detect left-veering arcs rather than tightness, and we can see an example of this difference in Figure 3.17.

The importance of this definition can now be seen in the extended towers from Figure 3.12. We wanted the extended tower $\{R_1, R_1', R_2, R_2', R_3\}$ on the right side of the figure to correspond to the arc $\alpha_5$ being left-veering. However, if we do not impose the repleteness condition, we can also consider the extended tower $\{R_1, R_1', R_2, R_2', R_3\}$ on the left side of the figure, and there the arc $\alpha_5$ is not left-veering (if it were, it would have to intersect the edges on $R_3'$ or $R_1'$ which are both restricted). Therefore we need to impose the condition that we have to add the region $R_3'$ to this extended tower if possible.

We now impose some further restrictions on our extended towers so that they completely characterise fixable and left-veering arcs.

**Definition 3.3.4.** An extended tower $\mathcal{T}$ in $(\Sigma, \varphi, \Gamma)$, with $\Gamma$ an arc collection as above, is *nice* if for every region $A \in \mathcal{T}^+$ we have $A \subset P$ and for every region $B \in \mathcal{T}^-$ we have that $int(B)$ is disjoint from $\varphi(\Gamma)$. We will assume all extended towers are nice.

*Remark.* We want to consider only nice extended towers because extended towers that are not nice come from collections where there are no left-veering arcs and no fixable arcs. See for example the extended tower in Figure 3.13, it is not nice but conforms to the definition of completed extended tower that we will see next (which is the one we want to identify with fixable arcs, and indeed there is no fixable arc in Figure 3.13). The extended towers in Figure 3.12 on the other hand, are indeed nice, and they do correspond to a fixable arc segment and a left-veering arc.

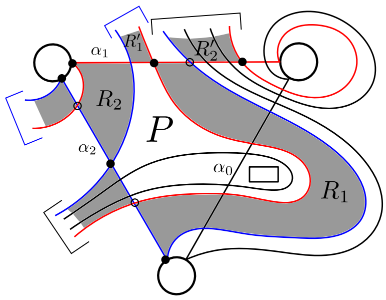


Figure 3.13: The extended tower $\mathcal{T} = \{R_1, R_2, R_1', R_2'\}$ supported in $\alpha_1, \alpha_2$ is not nice because $R_1$ is a positive region that is not contained in $P$.

**Definition 3.3.5.** Let $\Gamma$ be an arc collection in an open book $(\Sigma, \varphi)$ and $\mathcal{T}$ an extended tower in $\Gamma$. We say $\mathcal{T}$ is *nested* if there exist nested subcollections of regions as follows.

- $\mathcal{T}_0^+ = \{R \in \mathcal{T}^+ \mid \text{Dot}(R) \subset \partial\Sigma\}$.
- $\mathcal{T}_0^- = \{R' \in \mathcal{T}^- \mid \text{Circ}(R') \subset \mathcal{T}_0^+ \cup \text{Circ}_\partial(\Gamma)\}$.
- $\mathcal{T}_i^+ = \{R \in \mathcal{T}^+ \mid \text{Dot}(R) \subset \mathcal{T}_{i-1}^- \cup \partial\Sigma\}$.
- $\mathcal{T}_i^- = \{R' \in \mathcal{T}^- \mid \text{Circ}(R) \subset \mathcal{T}_i^+ \cup \text{Circ}_\partial(\Gamma)\}$.
- $\mathcal{T}^+ = \bigcup_i \mathcal{T}_i^+$ and $\mathcal{T}^- = \bigcup_i \mathcal{T}_i^-$

We will assume all extended towers are nested, and we will refer to regions in $\mathcal{T}_i^\pm$ as *being in level i*.

*Remark.* For the arc collections $\Gamma$ as above, a necessary condition for an extended tower to be nested is the existence of a level 0 positive region. Indeed, without a level 0 positive region the only possibility for a level 0 negative region would be one where all $\circ$-points are on basepoint triangles. However, this implies that the arcs supporting this negative region cut out a disc, contradicting the conditions we required for $\Gamma$.

Therefore, we can see that the extended tower from Figure 3.13 is not nested as there is no level 0 positive region. Observe that in terms of Heegaard Floer homology, a level 0 positive region corresponds to a differential to the contact class. We can also see an example of a nice extended tower that is not nested in Figure 3.14.

However, the extended towers in Figure 3.12 are indeed nested. There is a unique level 0 positive region which is $R_1$, and a unique level 0 negative region $R_1'$, because its $\circ$-points are on $R_1$ and on a basepoint triangle. Then there is a unique level 1 positive region $R_2$ because the $\bullet$-point that is not on the boundary belongs to $R_1'$, and a unique level 1 negative region $R_2'$ because its $\circ$-points belong to $R_1$ and $R_2$. Finally, there is a unique level 2 positive region $R_3$ because the $\bullet$-point that is not on the boundary belongs to $R_2'$. On the right hand side there are no more regions, but on the left hand side there is a level 2 negative region $R_3'$ because its $\circ$-points belong to $R_2$ and $R_3$.

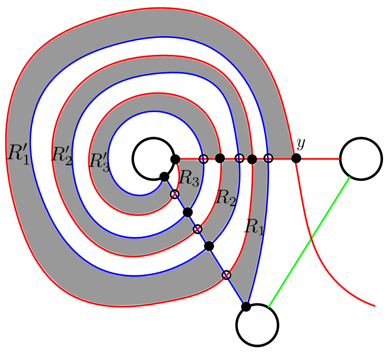


Figure 3.14: An extended tower that is not nested, because there is no level 0 positive region.

**Definition 3.3.6.** Let $\mathcal{T}$ be an extended tower in $(\Sigma, \varphi, \Gamma)$, and let $x$ be an interior point of some $\alpha \in \Gamma$ that is a vertex of a region in $\mathcal{T}$. We say that $x$ is *two-sided* if it is a vertex of exactly two regions of $\mathcal{T}$ (one positive and one negative).

**Definition 3.3.7.** Let $\mathcal{T}$ be an extended tower in $(\Sigma, \varphi, \Gamma = \{\alpha_i\}_{i=1}^n)$, where $\Gamma \cup \{\alpha_0\}$ cuts out a disc $P$ for some properly embedded arc $\alpha_0$ disjoint from $\Gamma$, and the arcs are oriented labelled counterclockwise. We say that $\mathcal{T}$ is *completed* if every interior vertex of $\mathcal{T}$ is two-sided, with the exception of a single $\bullet$-point $y_0 \in \alpha_1 \cap \varphi(\alpha_1)$, which we call a *connecting vertex.*

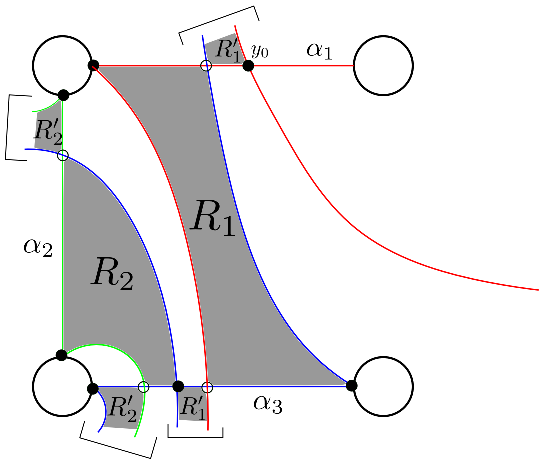


Figure 3.15: A simple example of a completed extended tower, which is also nice and replete, where the only interior vertex that is not two-sided is the •-point $y_0 \in \alpha_1 \cap \varphi(\alpha_1)$.

This is a much more restricted notion than that of completed tower in [44]. The reason for this is we want completed extended towers to correspond exactly to fixable arc segments, and to determine them uniquely. It is also clear that completed extended towers are those where every point of every arc $\alpha_2, \ldots, \alpha_n$ (as long as they are all strictly right-veering) belongs to a region, and every point on $\alpha_1$ from $y_0$ to $\alpha_1(1)$ also belongs to a region.

Once again, if we return to our primordial example in Figure 3.12, we can see that the extended tower on the left is completed. Indeed, every interior vertex is two-sided with the exception of a vertex $\alpha_1 \cap \varphi(\alpha_1)$, and the arc segment $\gamma$ is fixable. However, the extended tower on the right is not completed, as there are interior ∘-points on $R_2$ and $R_3$ that are not two-sided. In this case there is no fixable arc segment (and indeed the arc $\alpha_5$ is left-veering).

**Definition 3.3.8.** Let $\mathcal{T}$ be an extended tower in $\Gamma$. We say that $\mathcal{T}$ is *incomplete* if for every negative region $A \in \mathcal{T}^-$, there exists a vertex $x \in \mathrm{Dot}(A)$ that is two-sided, i.e. there exists a positive region $B \in \mathcal{T}^+$ such that $x \in \mathrm{Dot}(B)$.

Once again returning to our primordial example in Figure 3.12, the extended tower on the right is incomplete, because every negative region has a •-point which is two-sided. The one on the left, however, is not incomplete since the region $R_3'$ is a negative region without any two-sided •-points.

This mirrors the definition of incomplete tower from [44], because the property it aims to detect, a left-veering arc, implies overtwistedness. The additional conditions imposed to extended towers are what distinguishes this from the definition of incomplete tower. We can see this in Figure 3.17, where $\{R\}$ forms an incomplete tower. Indeed, the open book is known to support an overtwisted contact structure (see [32] and [30]). However, as an extended tower, it is not replete, since we can add the region

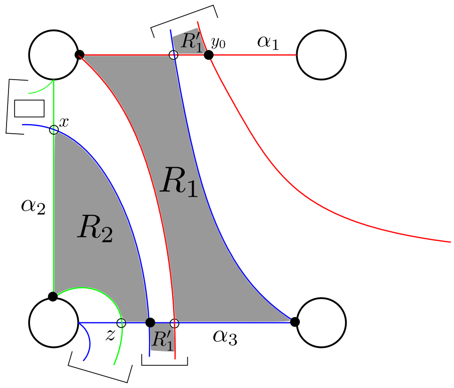


Figure 3.16: A simple example of an incomplete extended tower, because there is only one negative region $R_1'$, which shares a •-point with $R_2$, and the ∘-points $x$ and $z$ are not two-sided.

$R'$ because one of its ∘-points is a vertex of $R$ and the other one is a vertex of a base-point triangle, and the extended tower $\{R, R'\}$ is not incomplete, because $R'$ does not share any •-points with a positive region. Thus this extended tower does not imply the existence of a left-veering arc, and indeed this open book is right-veering by [30].

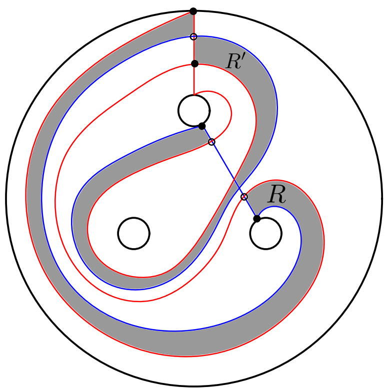


Figure 3.17: The region $R$ forms an incomplete tower but is not a replete extended tower because we can add the region $R'$.

*Remark.* As evidenced by Figure 3.17, we can have extended towers that are neither completed nor incomplete, for instance, if not every interior vertex is two-sided but there exists a negative region with no •-points in common with any positive region. An extended tower where every interior •-point is two-sided is also neither completed nor incomplete. It will follow from our discussion later that in this case the arc $\alpha_0$ is fixable, but we want to detect fixable arc segments rather than arcs and so we exclude this case from our definition of completed extended tower.

In Figure 3.18 we can see two extended towers that are neither completed nor incomplete. On the left, $\mathcal{T}_1 = \{R, R'\}$ is not incomplete because the negative region $R'$ does not have any •-point in common with the unique positive region $R$, but is also not completed because there is a •-point that is not two-sided and is not of the form $\alpha \cap \varphi(\alpha)$. On the right, $\mathcal{T}_2 = \{R_1, R_1', R_2, R_2'\}$ is not completed nor incomplete because every interior vertex is two-sided, and $R_2'$ does not share a $\bullet$-point with a positive region.

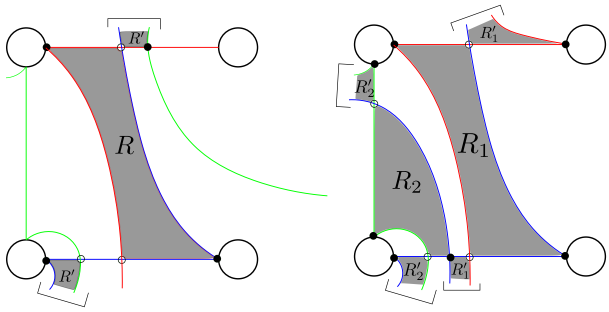


Figure 3.18: Two examples of extended towers which are neither completed nor incomplete.

# § 3.4 | Results

In this section we prove that we can detect a left-veering arc using extended towers. First we show existence of a special type of a left-veering arc that will be easier to detect.

## § 3.4.1 | Minimal left-veering arcs

**Definition 3.4.1.** Let $(\Sigma, \varphi)$ be an open book, and let $\mathcal{B}$ be a basis for $\Sigma$. Let $\gamma$ be a properly embedded arc, which we may assume intersects the basis. This divides $\gamma$ into a collection of arc segments $\gamma_1, \ldots, \gamma_n$, labelled and oriented following the orientation of $\gamma$, which intersect $\mathcal{B}$ only on their endpoints. We say $\gamma$ is *shortened with respect to* $\mathcal{B}$ if $\gamma_1, \ldots, \gamma_{n-1}$ are fixable. If $\gamma$ is left-veering, we call it a *shortened left-veering arc with respect to* $\mathcal{B}$.

Note that a shortened arc $\gamma$ as in Definition 3.4.1 is left-veering if and only if $\gamma_n$ is left-veering.

**Lemma 3.4.2.** *Let $(\Sigma, \varphi)$ be an open book, and let $\mathcal{B}$ be a basis for $\Sigma$. Suppose there exists a left-veering arc $\gamma$ in $(\Sigma, \varphi)$. Then there exists a shortened left-veering arc $\gamma'$ with respect to $\mathcal{B}$.*

*Proof.* We may assume $\varphi(\gamma)$ is bigon free with respect to $\mathcal{B} \cup \gamma$. We define the arc $\gamma'$ as follows. Let $x \in \gamma \cap \mathcal{B}$ be the first intersection point with the basis such that, after $x$, $\gamma$ and $\varphi(\gamma)$ exit the disc cut out by the basis by intersecting different arcs $\alpha_1$ and $\alpha_2$ respectively. Then take the arc $\gamma'$ that is the same as $\gamma$ up to $x$ and ends in the starting point of $\alpha_2$ without having any more intersections with $\mathcal{B}$ or the arc segment of $\gamma$ up to $y$. Clearly $\gamma'$ is shortened with respect to $\mathcal{B}$. To show that it is left-veering, take the arc $\gamma'$ and isotope it slightly so that it lies to the left of $\gamma$. Then its image is fixable and to the left of the image of $\gamma$ up to $x$. Suppose for a contradiction that $\gamma'$ is right-veering. Then $\varphi(\gamma')$ must intersect $\varphi(\gamma)$ so that $\varphi(\gamma)$, $\varphi(\gamma')$, and $\partial\Sigma$ bound a disc. Moreover, this would have to be the image of a disc bounded by $\gamma$, $\gamma'$, and $\partial\Sigma$. However, this gives a contradiction because $\gamma$ and $\gamma'$ are (by construction) disjoint before $\gamma$ intersects $\alpha_1$ so they cannot bound a disc, as we can see in Figure 3.19.

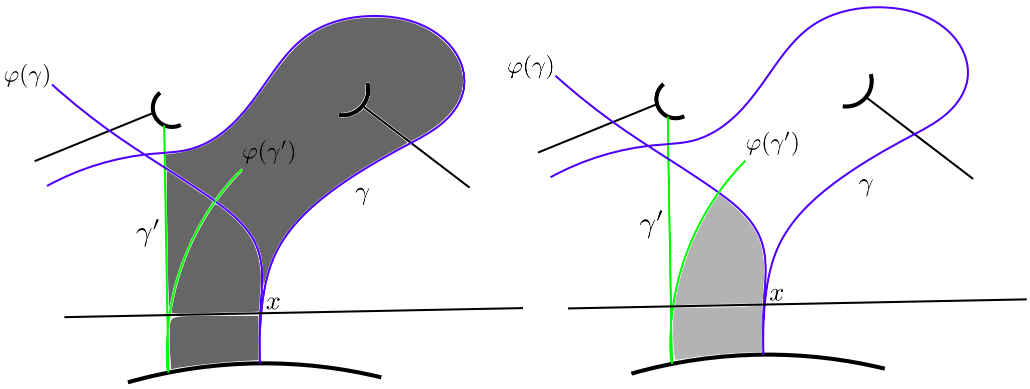


Figure 3.19: If $\gamma'$ were right-veering, the image of the darkly shaded subsurface would have to be the lightly shaded one, a contradiction. Note that in this case $\gamma$ is an arc and not an arc image even though it is not represented with a straight line.

□

**Definition 3.4.3.** Define a length of an arc $\gamma$ with respect to a basis $\mathcal{B}$ by the (unsigned) number of intersections of $\gamma$ with all the arcs of $\mathcal{B}$. Out of all shortened left-veering arcs, we call one which minimises this length a *minimal left-veering arc* with respect to $\mathcal{B}$.

Note that a minimal left-veering arc minimises this distance for all left-veering arcs and not just shortened ones.

We will be interested primarily in minimal left-veering arcs. Moreover, when we have a $2n$-gon $P$ cut out by $\{\alpha_i\}_{i=0}^n$ with $\alpha_0$ left-veering, we can assume there are no left-veering arcs contained in $P$, because otherwise we can find a subset of the arcs cutting out a $2m$-gon (for $m < n$) with a left-veering arc, and we can work with this subset instead. In particular, we can assume that the image of $\alpha_0$ leaves the $2n$-gon by intersecting $\alpha_1$. This is because if it leaves $P$ by intersecting some other arc, say $\alpha_k$, then the arc $\beta_0$ that is obtained by consecutive arcslides of $\alpha_0$ over $\alpha_1, \ldots, \alpha_{k-1}$ must be left-veering, by the same argument we made in Lemma 3.4.2, and then we can focus on the $2(n-k)$-gon cut out by $\alpha_k, \alpha_{k+1}, \ldots, \alpha_n$, and $\beta_0$, which will have the property we want.

### § 3.4.2 | Base Case

In this subsection we show using methods from Section 3.2 that extended towers detect left-veering arcs and fixable arc segments in 6-gons. This will be the base case of our induction.

**Proposition 3.4.4.** *Let $(\Sigma, \varphi)$ be an open book. Let $\alpha_0, \alpha_1, \alpha_2$ be properly embedded arcs cutting out a 6-gon $P$, oriented counterclockwise, and assume $\alpha_1$ and $\alpha_2$ are right-veering. Then $\alpha_0$ is left-veering if and only if $\{\alpha_1, \alpha_2\}$ support an incomplete extended tower that is nested, nice, and replete.*

*Proof.* First suppose that $\alpha_0$ is left-veering. Then Proposition 3.2.12 gives a (positive) region $R$ whose interior is disjoint from $\{\alpha_1, \alpha_2\}$. Since again by Proposition 3.2.12 there are no negative regions, $\mathcal{T} = \{R\}$ forms an incomplete extended tower that moreover is replete and nice, and $R$ is on level zero since the $\bullet$-points of $R$ are on the boundary, so $\mathcal{T}$ is nested.

Now suppose that there exists an incomplete extended tower $\mathcal{T}$ supported in $\{\alpha_1, \alpha_2\}$. We want to show that $\mathcal{T} = \{R\}$, with $R$ the region from Proposition 3.2.12, which shows that $\alpha_0$ is left-veering. There must be a region in $\mathcal{T}_0^+$, and for $\mathcal{T}$ to be nice it must be the region $R$ from Proposition 3.2.12. Now, if there exists a negative region $R'$ in $\mathcal{T}_0^-$, then again by Proposition 3.2.12 every point on $\alpha_2$ belongs to a region, so there can be no more positive regions. But now $R'$ does not have any $\bullet$-points in common with a positive region, so $\mathcal{T}$ is not incomplete –a contradiction. So there does not exist such a negative region, and thus $\mathcal{T} = \{R\}$, and then by Proposition 3.2.12 $\alpha_0$ is left-veering.

□

**Proposition 3.4.5.** *Let $(\Sigma, \varphi)$ be an open book. Let $\alpha_1, \alpha_2, \alpha_0$ be properly embedded strictly right-veering arcs cutting out a 6-gon $P$, oriented counterclockwise. Let $\gamma$ be an arc segment contained in $P$ starting between $\alpha_2$ and $\alpha_0$ and ending in the interior of $\alpha_1$. Then $\gamma$ is fixable by $\varphi$ if and only if $\{\alpha_1, \alpha_2\}$ support a completed extended tower that is nested, nice and replete, and whose connecting vertex coincides with $\gamma \cap \alpha_1$.*

*Proof.* First suppose that $\gamma$ is fixable. Then we have the regions $R$ and $R'$ from Proposition 3.2.16 forming the splitting pair, and we can see that they form a completed extended tower which is nested, replete and nice, and the unique connecting vertex is $\gamma \cap \alpha_1$.

Conversely, suppose that there exists a completed extended tower $\mathcal{T}$ which is nice and replete. By the same reasoning as in Proposition 3.4.4, the positive region $R$ from Proposition 3.2.16 must be in $\mathcal{T}$. Since $\mathcal{T}$ is completed, there must be a negative region $R'$ with the same $\circ$-points as $R$. As one of the $\circ$-points in this negative region is on the basepoint triangle on $\alpha_2$, one of the $\bullet$-points of $R'$ is on the boundary (because there can be no other intersection points between the $\circ$-point on the basepoint triangle and the boundary as our arc collections are bigon free). Now every point of $\alpha_2$ belongs to a region, so there can be no more regions in $\mathcal{T}$. This means that for $\mathcal{T}$ to be completed the other $\bullet$-point of $R'$ must be an interior point $\alpha_1 \cap \varphi(\alpha_1)$, which means that the regions in $\mathcal{T}$ are the regions from Proposition 3.2.16 (i.e the splitting pair), so $\gamma$ is fixable.

□

### § 3.4.3 | Inductive Step

We now have that a left-veering arc is detected by an incomplete extended tower if it cuts out a 6-gon (with the correct orientation) with two arcs from the basis. We want to extend this by induction to the case where the left-veering arc cuts out an $n$-gon with arcs from the basis. Similarly, we have that completed towers detect fixable arc segments when the arc segment is contained in a 6-gon cut out by three arcs, two of which are from our basis, and we want to extend to the case where the arc segment is contained in a $n$-gon, with $n-1$ arcs in our basis. To show this, let us first introduce some notation to be used throughout this subsection.

Let $\alpha_0$, $\alpha_1$, and $\alpha_2$ be properly embedded arcs that cut out a 6-gon $P$, where the arcs are labelled counterclockwise. Also let $\Gamma$ be an arc collection such that there exists an arc $\beta$ disjoint from $\Gamma$ with $\mathcal{C} = \Gamma \cup \{\beta\}$ cutting out a disc $P'$ with disjoint interior with $P$, and $\alpha_0 \in \Gamma$. Again, we will orient this arc collection with the counterclockwise orientation. Moreover, assume that $\alpha_0$ is not the first arc in $\Gamma$ (i.e the next one to $\beta$ as we go counterclockwise through the boundary of the disc cut out by $\mathcal{C}$). This is because we want to detect fixable arcs with an endpoint on the first arc of the collection $\Gamma$, so we do not want this point to change. Let $\Gamma' = (\Gamma \setminus \{\alpha_0\}) \cup \{\alpha_1, \alpha_2\}$. Then $\mathcal{C}' = \Gamma' \cup \{\beta\}$ cuts out a disc $P \cup P'$. Orient the arcs in this collection again counterclockwise (this agrees with the previous orientation).

The idea is that, given an extended tower $\mathcal{T}$ in $\Gamma$, we can slide its regions over $\alpha_0$ to $\alpha_1$ and $\alpha_2$ to obtain an extended tower $\mathcal{T}'$ supported in $\Gamma'$, that will have the same properties as $\mathcal{T}$. We can see a simple example with completed extended towers in Figure 3.20.

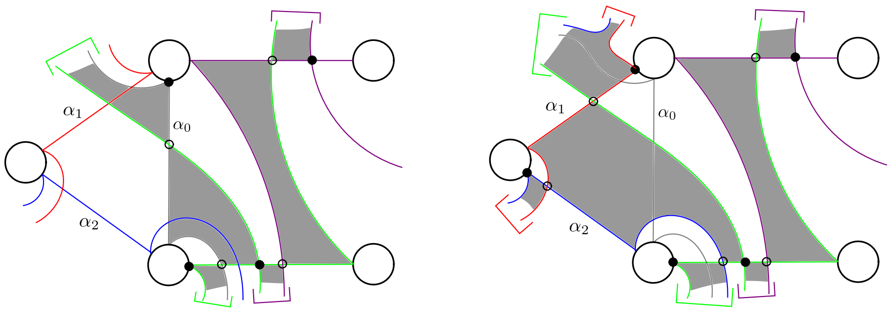


Figure 3.20: The regions for the new extended tower are obtained by sliding the regions from the old extended tower

To make this operation more precise we now define two maps, which we will call *slide maps* and denote by $s^{\pm}$. Given an extended tower $\mathcal{T}$ supported in $\Gamma$, where $\Gamma$ is an arc collection as above, these maps will send the set of vertices of the positive (respectively negative) regions of $\mathcal{T}$ (denoted by $V(\mathcal{T}^{\pm})$) to intersection points $\Gamma' \cap \varphi(\Gamma')$. These points will determine regions that form the extended tower $\mathcal{T}'$ supported in $\Gamma'$ that will have the same properties as $\mathcal{T}$. Note that most points will be two-sided and thus will be both in $\mathcal{T}^+$ and $\mathcal{T}^-$, which means that they will have an image under $s^+$ and an image under $s^-$. Most of the time these will agree. In fact, the only time they will not agree will be when a component of $P \cap \varphi(P)$ is a 6-gon.

**Definition 3.4.6.** Let $x \in \mathrm{V}(\mathcal{T}^+)$. Then $s^+(x)$ is defined as follows.

1. If $x$ does not lie on $\alpha_0$ or $\varphi(\alpha_0)$, $s^+(x) = x$.

2. If $x$ lies on the intersection of $\varphi(\alpha_0)$ with some other arc $\beta$ from $\Gamma$, then $s^+(x)$ is the intersection point $y$ of $\beta$ with $\varphi(\alpha_1)$ or $\varphi(\alpha_2)$ such that the segment between $x$ and $y$ is contained in $\beta \cap \varphi(P)$.

3. If $x$ lies on the intersection of $\alpha_0$ with the image of some other arc $\beta$ from $\Gamma$, then $s^+(x)$ is the intersection point $y$ of $\varphi(\beta)$ with $\alpha_1$ or $\alpha_2$ such that the segment between $x$ and $y$ is contained in $\varphi(\beta) \cap P$.

4. If $x$ lies on the intersection of $\alpha_0$ with its image, then $s^+(x)$ is the intersection point $y$ of $\alpha_m$ with $\varphi(\alpha_l)$ (where $m$ and $l$ can be 1 or 2 and not necessarily equal), obtained by first going along $\alpha_0$ to $\varphi(\alpha_l)$, and then along $\varphi(\alpha_l)$ to $y$, such that this path is contained in $P \cap \varphi(P)$.

We illustrate the different cases in Figure 3.21, where we can see that, while the definition may seem arbitrary, for an intersection point $x \in \Gamma \cap \varphi(\Gamma)$ we are essentially choosing "the closest point" to $x$ that belongs to $\Gamma' \cap \varphi(\Gamma')$.

**Proposition 3.4.7.** *The map $s^+$ is well defined.*

*Proof.* Case 1 is immediate. Case 2 is well defined because if $\varphi(\alpha_0)$ intersects an arc then either $\varphi(\alpha_1)$ or $\varphi(\alpha_2)$ must also intersect that arc because $\varphi(P)$ is a disc. Moreover, there is a unique segment from $x$ to $y$ contained in $\beta \cap \varphi(P)$. Case 3 is the same as Case 2 but with the roles of the arcs and arc images reversed. Finally, in Case 4, the same argument as for Case 2 shows that either $\varphi(\alpha_1)$ or $\alpha_2$ intersect $\alpha_0$, and there is a unique segment between $x$ and $\varphi(\alpha_m)$ contained in $P \cap \varphi(P)$. Then, $\varphi(\alpha_m)$ must exit $P$ by intersecting either $\alpha_1$ or $\alpha_2$, and going along $\varphi(\alpha_m)$ gives $y$. □

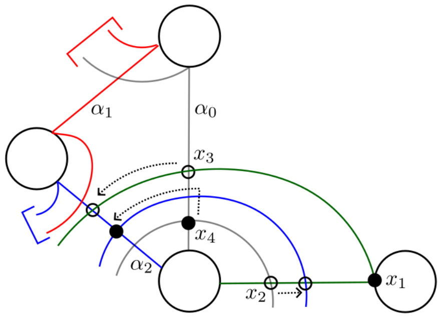


Figure 3.21: The different cases for $s^+$, where the point $x_i$ is of case $i$ in Definition 3.4.6. The dashed arrows indicate the action of $s^+$.

Observe that for the first two cases, both $x$ and $s^+(x)$ belong to an arc that is not in $P$, and for the last two cases $x$ belongs to $\alpha_0$ and $s^+(x)$ belongs to either $\alpha_1$ or $\alpha_2$. We now define $s^-$ similarly.

**Definition 3.4.8.** Let $x \in \mathrm{V}(\mathcal{T}^-)$. Then $s^-(x)$ is defined as:

1. If $x$ does not lie on $\alpha_0$ or $\varphi(\alpha_0)$, $s^-(x) = x$.
2. If $x$ lies on the intersection of $\alpha_0$ with the image of some other arc $\beta$ from $\Gamma$, then $s^-(x)$ is the intersection point $z$ of $\varphi(\beta)$ with $\alpha_1$ or $\alpha_2$ such that the segment between $x$ and $z$ is contained in $P$.
3. If $x$ lies on the intersection of $\varphi(\alpha_0)$ with some other arc $\beta$ from $\Gamma$, then $s^-(x)$ is the intersection point $z$ of $\beta$ with $\varphi(\alpha_1)$ or $\varphi(\alpha_2)$ such that the segment between $x$ and $z$ is contained in $\beta \cap \varphi(P)$.
4. If $x$ lies on the intersection of $\alpha_0$ with its image, then $s^-(x)$ is the intersection point $z$ of $\alpha_m$ with $\varphi(\alpha_l)$ (where $m$ and $l$ can be 1 or 2 and not necessarily equal), obtained by first going along $\varphi(\alpha_0)$ to $\alpha_m$, and then along $\alpha_m$ to $z$, such that this path is contained in $P \cap \varphi(P)$.

Observe that if we reverse the roles of the arcs and arc images, that is, we take our arc collections to be $\varphi(\Gamma)$ and $\varphi(\Gamma')$, and their images to be $\varphi^{-1}(\varphi(\Gamma))$ and $\varphi^{-1}(\varphi(\Gamma'))$, and we also reverse their orientation (so that the negative regions become positive regions), then the definition of $s^-$ is the same as the definition of $s^+$ using the original arc collections. This also means, by Proposition 3.4.7, that $s^-$ is well defined.

Also note that, away from $P \cup \varphi(P)$, the slide map does not change the intersection point. Moreover, if $x$ is a positive (respectively negative) intersection point then $s^+(x)$ and $s^-(x)$ will also be positive (respectively negative).

Finally, the slide maps are injective, so they give a bijection onto their image, and then we can refer to the inverse of these maps. We will use this to show that extended towers in $\Gamma'$ also induce extended towers in $\Gamma$.

Now we want to show that using the maps $s^+$ and $s^-$ we can construct an extended tower $\mathcal{T}'$ supported in $\Gamma'$ that has the same properties as $\mathcal{T}$. The properties that will be preserved will be being nested, being completed/incomplete (or neither), being replete, and being nice. We will focus on the local effect of the slide maps on a neighbourhood of $P$, and a neighbourhood of $\varphi(P)$ (because outside these neighbourhoods the slide maps do not change anything). In particular, this means that if a positive (respectively negative) region $R$ is supported in $\Gamma$ and the image of each of its vertices under $s^+$ (resp. $s^-$) is itself, then $R$ is also supported in $\Gamma'$. If $R$ is not supported in $\Gamma'$, the local effect of the slide map on the vertices will induce one (or more) regions supported in $\Gamma'$.

We will need to check several things. First, that the induced regions form an extended tower, and then, that the properties of being nested, replete, nice, and completed or incomplete (or neither) are preserved.

First we will separate two cases, when $\alpha_2$ is $\varphi$-contained in $\alpha_1$ and when $\alpha_0$ is $\varphi$-contained in $\alpha_2$. The reason for this is that in these cases the image under the slide maps of a boundary point is an interior point (a boundary point is never two sided so we need to consider this separately). We will not consider the case where $\alpha_1$ is $\varphi$-contained in $\alpha_0$, because then the slide maps send an interior point to two boundary points (which are never two-sided). However, we do not need this case.

**Lemma 3.4.9.** *Suppose that $\alpha_2$ is $\varphi$-contained in $\alpha_1$. Then if $\mathcal{T}$ is an extended tower in $\Gamma$, there is an extended tower $\mathcal{T}'$ supported in $\Gamma'$ with the same properties as $\mathcal{T}$. Conversely, if $\mathcal{T}'$ is an extended tower supported in $\Gamma'$, there is an extended tower $\mathcal{T}$ supported in $\Gamma$ with the same properties as $\mathcal{T}$.*

*Proof.* First let $\mathcal{T}$ be an extended tower in $\Gamma$. We will now see how each region in $\mathcal{T}$ induces a region supported in $\Gamma'$

First, away from $P$ and $\varphi(P)$ any region $R \in \mathcal{T}$ is unchanged since it is already supported in $\Gamma'$, and so the region induced by the slide maps is $R$ itself. For any region with an edge on the interior of $\alpha_0$, observe that an arc image intersecting $\alpha_0$ must leave $P$ by intersecting $\alpha_1$, and so the images of any vertex on $\alpha_0$ under the slide maps coincide, and is on $\alpha_1$. Then, we obtain the region $R'$ by simply adding or removing rectangles. Similarly, for a region with an edge on $\varphi(\alpha_0)$, we can see reversing the role of arcs and arc images that any arc intersecting $\varphi(\alpha_0)$ must also intersect $\varphi(\alpha_1)$, and so the image under the slide maps of a vertex on $\varphi(\alpha_0)$ lies on $\varphi(\alpha_1)$, and again the induced region is obtained by simply adding or removing rectangles. We can see this in Figure 3.22. Observe that each region $R_i \in \mathcal{T}$ corresponds to a unique region $R_i'$ in $\Gamma'$ (sometimes $R_i = R_i'$), so we define $\mathcal{T}' = \{R_i' \mid R_i \in \mathcal{T}\} \cup \{R_1, R_2\}$, where $\{R_1, R_2\}$ is the splitting pair of $\{\alpha_1, \alpha_2\}$.

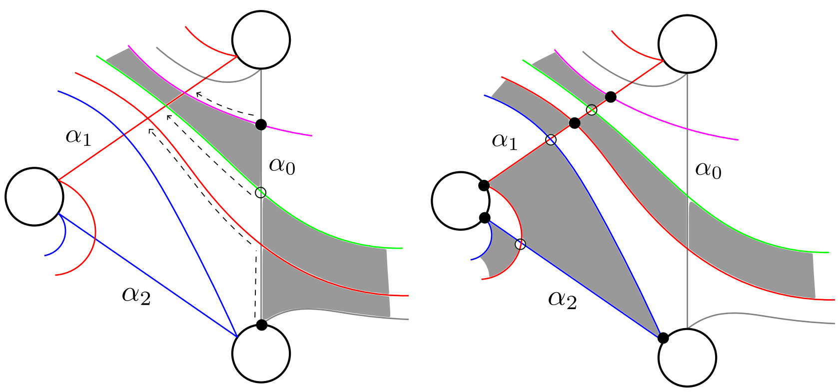


Figure 3.22: The regions obtained by adding or removing rectangles, when $\alpha_2$ is $\varphi$-contained in $\alpha_1$, and the splitting pair. The dashed arrows indicate the action of the slide maps.

Let us now check the properties of $\mathcal{T}'$. The positive regions are by construction contained in $P$ and thus disjoint from $\Gamma'$ in their interior, and we can see by switching the role of arcs and arc images that the interior of the negative regions is disjoint from $\varphi(\Gamma')$, so the extended tower is nice. It is also replete, because we cannot add any negative regions with $\circ$-points in $\mathrm{Circ}(\mathcal{T}') \cup \mathrm{Circ}_\partial(\Gamma)$, since $\mathcal{T}$ is replete and we have already used the $\circ$-point in the basepoint triangle of $\alpha_1$ and $\alpha_2$ for the splitting pair of regions.

The positive region from the splitting pair is a level 0 region, and so is the negative region from the splitting pair. This will imply that some of the regions may go up one level (for example, if there was a level 0 region with a $\bullet$-point on $\alpha_0(1)$, the induced region is now a level 1 region because it as an interior $\bullet$-point that is also a vertex of a level 0 negative region), but if $\mathcal{T}$ was nested then so is $\mathcal{T}'$, as we can construct it level by level from the levels of $\mathcal{T}$.

Finally, being completed, incomplete, or neither comes from which vertices are two-sided. For interior points that are the image of an interior point $x$, they will be two-sided if and only if $x$ is two-sided. So we only need to check the boundary point $y = \alpha_0(1)$ (recall $\alpha_0$ is given its orientation from being an arc of $\Gamma$) because its image $z = s^+(y)$ is an interior point (the image of the other endpoint $\alpha_0(0)$ is not an interior point so we do not need to check if it is two-sided). But $z$ is two-sided because we have included the positive splitting pair of regions in $\{\alpha_1, \alpha_2\}$ in $\mathcal{T}'$. Thus if $\mathcal{T}$ is completed then $\mathcal{T}'$ is completed, and if $\mathcal{T}$ is incomplete then so is $\mathcal{T}'$.

For the converse, notice that the only points in an extended tower that do not have an inverse image are in the splitting pair of $\{\alpha_1, \alpha_2\}$. Now, if we assume that $\mathcal{T}'$ is not supported in just $\{\alpha_1, \alpha_2\}$, then $z = s^+(\alpha_0(1))$ is an interior $\bullet$-point that must be two-sided, so there must be a region $R'$ that is not supported in $\{\alpha_1, \alpha_2\}$. But then we get a region $R$ supported in $\Gamma$. Moreover, note that every point in $\alpha_2$ belongs to a region, and so every other region in $\mathcal{T}'$ has vertices that have an inverse image on $\Gamma$. Moreover, every vertex is two-sided if and only if the preimage is. Notice that we do get an extended tower by taking the regions induced by these points because the preimage of $z$ is on the boundary, so even though the negative region which has $z$ as a vertex does not induce a negative region in $\mathcal{T}$, it does not have to, precisely because $z$ is on the boundary. □

**Lemma 3.4.10.** *Suppose that $\alpha_0$ is $\varphi$-contained in $\alpha_2$. Then if $\mathcal{T}$ is an extended tower in $\Gamma$, there is an extended tower $\mathcal{T}'$ supported in $\Gamma'$ with the same properties as $\mathcal{T}$. Conversely, if $\mathcal{T}'$ is an extended tower supported in $\Gamma'$, there is an extended tower $\mathcal{T}$ supported in $\Gamma$ with the same properties as $\mathcal{T}$.*

*Proof.* The argument is the same as for Lemma 3.4.9, except now the arc images intersecting $\alpha_0$ all intersect $\alpha_2$, and the arcs intersecting $\varphi(\alpha_0)$ all intersect $\varphi(\alpha_2)$. Now we need to check the boundary point $y = \alpha_0(0)$, which is the one that is sent to an interior point. But now we the regions that we include in $\mathcal{T}'$ are the negative splitting pair of $\{\alpha_1, \alpha_2\}$ (if the $\bullet$-point $\alpha_0(0)$ belongs to a region) so its image is two-sided, and it does not affect the properties of the extended tower. Moreover, the regions from the splitting pair now belong to higher levels (specifically, if the positive region containing $\alpha_0(0)$ is on level $i$, then the regions from the splitting pair are on level $i+1$. Indeed, the positive region has a $\bullet$-point on the boundary and another one on a level $i$ region so it is on level $i+1$, and the negative region has a $\circ$-point on $\mathrm{Circ}_\partial(\Gamma')$ and another one on a level $i+1$ positive region). The rest of the regions are on the same levels as before, so $\mathcal{T}'$ is also nested if $\mathcal{T}$ is nested.

For the converse again the argument is the same using the negative splitting pair instead of the positive one.

□

Now suppose none of the arcs $\{\alpha_0, \alpha_1, \alpha_2\}$ is $\varphi$-contained in any of the others, that is, the image of $\alpha_0$ leaves $P$ by intersecting $\alpha_1$ and the image of $\alpha_2$ leaves $P$ by intersecting $\alpha_0$. Again the slide maps are the identity away from a neighbourhood of $P$, and a neighbourhood of $\varphi(P)$. Therefore, we will distinguish three cases, depending on how $\varphi(P)$ intersects $P$.

As before, $\mathcal{T}'$ will be a collection of regions induced by the regions in $\mathcal{T}$. However, in the previous cases every region in $\mathcal{T}$ corresponded to a unique region supported in $\Gamma'$ that was obtained by adding and removing rectangles. Now, the slide maps might "break up" regions into several other regions as vertices connected by an edge in $\alpha_0$, or $\varphi(\alpha_0)$, could be mapped to different arcs, or arc images. In this case the following definition will be useful.

**Definition 3.4.11.** Let $\mathcal{T}$ be an extended tower in $(\Sigma, \varphi, \Gamma)$. Two regions $R_1, R_2 \in \mathcal{T}^\pm$ are said to be *connected by a region* $R \in \mathcal{T}^\mp$ if there exist points $x \in \mathrm{Dot}(R) \cap \mathrm{Dot}(R_1)$ and $y \in \mathrm{Circ}(R) \cap \mathrm{Circ}(R_2)$. We will then refer to the region $R$ as a *connecting region.*

Connecting regions supported in $\{\alpha_1, \alpha_2\}$ might not have all of their vertices be images of a vertex in $\mathcal{T}$ (so they are not induced by regions in $\mathcal{T}$ the way the regions in the previous Lemmas were), but we will include them in some cases to preserve properties of $\mathcal{T}$. Similarly, we might need to add negative regions (whose $\bullet$-points are not images under the slide maps of vertices in $\mathcal{T}$) to ensure the resulting extended tower is replete.

**Lemma 3.4.12.** *Suppose that $P \cap \varphi(P)$ is just the basepoint triangles. Using the maps $s^+$ and $s^-$ we can construct regions giving an extended tower $\mathcal{T}'$ supported in $\Gamma'$ that has the same properties as $\mathcal{T}$.*

*Proof.* Unlike before, given a region $R \in \mathcal{T}$, the image of its vertices under the slide maps might not induce a unique region in $\Gamma'$. We construct the extended tower $\mathcal{T}'$ as follows. For any region $R \in \mathcal{T}^\pm$, if the set of vertices $s^\pm(V(R))$ induces a unique region $R'$ supported in $\Gamma'$, set $R' \in \mathcal{T}'$. So suppose that it does not, that is, points from

$s^{\pm}(V(R))$ lie on two different regions $R_1'$ and $R_2'$ connected by a third region $R_3'$. First assume that $R$ is a negative region. Then if there are two-sided $\bullet$-points $x, y \in V(R)$ such that $s^-(x) \in R_1'$ and $s^-(y) \in R_2'$, we set $R_1', R_2', R_3' \in \mathcal{T}'$. If every two sided point in $R$ has its image in $R_1'$, then set $R_1' \in \mathcal{T}'$ (but not $R_2'$ or $R_3'$). Now assume that $R$ is a positive region. We do the same as before but with $\circ$-points. If there are two-sided $\circ$-points $x, y \in V(R)$ such that $s^+(x) \in R_1'$ and $s^+(y) \in R_2'$, we set $R_1', R_2', R_3' \in \mathcal{T}'$. If every two sided $\circ$-point in $R$ has its image in $R_1'$, then set $R_1' \in \mathcal{T}'$, but not $R_2'$ or $R_3'$.

Finally, if after having done this there is a negative region $R_1'$ supported in $\Gamma'$ such that all its $\circ$-points are vertices of positive regions in $\mathcal{T}'$ or $\circ$-points on a basepoint triangle, set $R_1' \in \mathcal{T}'$. This ensures that $\mathcal{T}'$ is replete. Further, if there exists a positive region $R_3'$ with its $\bullet$-points being on $\alpha_1(1)$ or $\bullet$-points of $R_1'$, set $R_3' \in \mathcal{T}'$. This last case only happens if there is a $\circ$-point $x \in \alpha_0$ in a positive region $R \in \mathcal{T}$ that is not two-sided, but its image $s^+(x)$, which is a vertex of a positive region $R'$, is two sided after applying this rule. We will see later (Figure 3.25) that this ensures that if $\mathcal{T}$ is incomplete then so is $\mathcal{T}'$.

We now show what the induced regions are.

If $P \cap \varphi(P)$ is just the basepoint triangles, for every interior intersection point $x \in \mathcal{T}$ on $\alpha_0$ (which will also be on an arc image $\varphi(\beta)$ with $\beta \neq \alpha_0$)), we have that $s^+(x) = s^-(x)$ and moreover the image under the slide map is obtained by going along $\varphi(\beta)$ until it leaves $P$. We can then see that, for a positive region $R$ with vertices $x, y$ on $\alpha_0$, there are two options.

The first option is that $s^+(x)$ and $s^+(y)$ lie on the same arc, which means that the local effect of the slide map on $R$ is just extending it by a rectangle.

The second option is that $s^+(x)$ lies on $\alpha_1$ and $s^+(y)$ lies on $\alpha_2$. Then the local effect of the slide map on $R$ is extending it by a 6-gon, where the extra sides are given by $\alpha_1$ from $s^+(x)$ to its endpoint, an edge of the basepoint triangle, and $\alpha_2$ from the $\circ$-point on the basepoint triangle to $s^+(y)$, see Figure 3.23. Moreover, observe that this will only happen once.

For a negative region $R$ with vertices $x, y$ on $\alpha_0$, there are again two options.

The first option is that $s^-(x)$ and $s^-(y)$ lie on the same arc, which means that the local effect of the slide map on $R$ is just removing a rectangle. We can see this, together with the positive regions, in Figure 3.23. Observe that in all these first cases the levels are preserved.

The second option is that $s^-(x)$ lies on $\alpha_1$ and $s^-(y)$ lies on $\alpha_2$. In this case, $R$ must have boundary along another arc image intersecting $\alpha_1$ and $\alpha_2$ so that $R$ is a disc. Moreover, because $s^-(x)$ lies on $\alpha_1$ and $s^-(y)$ lies on $\alpha_2$, we do not have a unique induced region now, but two regions $R_1'$ and $R_2'$, one with an edge on $\alpha_2$ between $s^-(y)$ and a $\bullet$-point $b$ and another one with an edge on $\alpha_1$ between a $\circ$-point $a$ and $s^-(x)$, connected by a positive rectangle $R_3'$ with vertices $a, b$, the $\circ$-point on the basepoint triangle, and the boundary point on $\alpha_1$. We can see this in Figure 3.24. Notice that

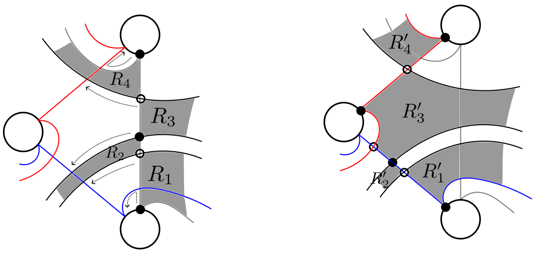


Figure 3.23: Obtaining the new regions by adding or removing rectangles (or a 6-gon in the case of $R_3$). The dashed arrows indicate the action of $s^{\pm}$. Here we do not a priori know if the ∘-point on the basepoint triangle is two-sided, we deal with this case later.

$a$, $b$, and the point on the basepoint triangle are not images of any point in $\mathcal{T}$ under the slide maps but these, together with the analogous case in $\varphi(P)$ instead of $P$, will be the only cases of this.

Because the rectangle $R_3'$ connects $R_1'$ and $R_2'$, both $a$ and $b$ are two-sided. Again, note that there can only be one region $R \in \mathcal{T}$ of this form, otherwise we would have vertices in the interior an edge of of another region, contradicting the definition of extended tower. We now consider the levels of the regions. Assume the region $R$ is on level $i$. Then, since not all ∘-points of $R_1'$ come from vertices of $R$, $R_1'$ will be on level $j$ with $j \leq i$. Then, $R_3'$ will be on level $j+1$, and $R_2'$ will be on level $\max\{j+1, i\}$.

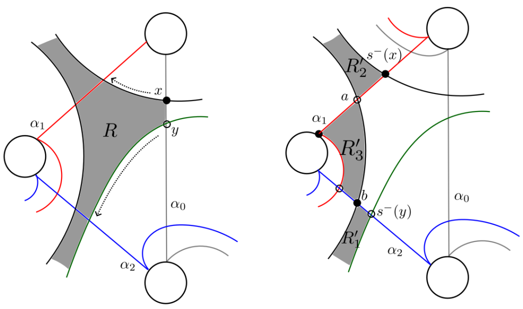


Figure 3.24: Dividing a negative region into two negative regions $R_1'$ and $R_2'$ connected by a positive rectangle $R_3'$.

As for the neighbourhood of $\varphi(P)$, we obtain regions in the same way by reversing the role of the arcs and arc images, and changing the orientation of the arcs. Note that this switches the roles of $\alpha_1$ and $\alpha_2$ (in particular, when we are extending a region by a 6-gon the point on the boundary that we use is on $\alpha_2$ and not $\alpha_1$, but note that neither of these points comes as the image under the slide maps).

Finally, if there is a vertex $x \in \alpha_0$ in a positive region $R \in \mathcal{T}$ that is not two sided, but $s^+(x)$ is two sided because otherwise $\mathcal{T}'$ would not be replete, then $s^+(x)$ is a vertex of a negative region $R_1'$ supported in $\Gamma'$. Then we must have that $s^+(x)$ lies on $\alpha_2$, and $P$ divides what would be a region (but is not) making $x$ two-sided, and $R_1'$ is a region that makes $s^+(x)$ two-sided, see Figure 3.25. Then we also have that the region $R_3'$ has as its •-points the point $\alpha_1(1)$, which is on the boundary, and the point $b$, which is a

•-point on a negative region of $\mathcal{T}'$, so $R_3' \in \mathcal{T}'$. Notice that here the region $R_1'$ is on the same level as $R'$, and the region $R_3'$ is on the level above $R_1'$.

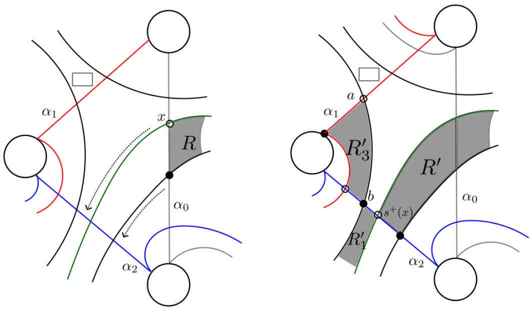


Figure 3.25: The point $x$ is not two-sided but $s^+(x)$ is, because $R_1'$ is a region. However, this induces a positive region $R_3'$ and now $b$ is a two-sided •-point in $R_1'$.

We will now see how the properties of $\mathcal{T}$ are preserved.

As before, because of the way the induced regions are constructed, the result is an extended tower, which moreover is nice. It is also replete by construction.

Also, our discussion on the levels of the regions implies that if $\mathcal{T}$ is nested then so is $\mathcal{T}'$. Being completed (or incomplete or neither) will depend on which vertices are two-sided.

Let us now see how the vertices being two-sided (or not) determines whether their images are two-sided. Again we will focus on a neighbourhood of $P$ and the analogous result for a neighbourhood of $\varphi(P)$ will follow from reversing the role of arcs and arc images.

First suppose that $\mathcal{T}$ is completed. This means that every interior vertex is two-sided, except for the connecting vertex $y_0$. But by construction this means that the images of these interior vertices in $\Gamma'$ are two-sided, except from $y_0$ (we imposed that $\alpha_0$ is not the first arc in $\Gamma$ so $s^-(y_0) = y_0$). Moreover the vertices $a$ and $b$ from the connecting region, if there is one, are also two-sided.

There only remains to show that the ∘-point on the basepoint triangle formed by $\varphi(\alpha_1)$ and $\alpha_2$ is two-sided. Since $\mathcal{T}$ is completed, $\alpha_0(1)$ is the vertex of a positive region in $\mathcal{T}$, which means that $\alpha_2(1) = s^+(\alpha_0(1))$ is the vertex of a positive region in $\mathcal{T}'$. Note that not every arc image intersecting $\alpha_0$ leaves $P$ by intersecting $\alpha_2$, since initially $\varphi(\alpha_0)$ intersects $\alpha_1$ by hypothesis. Since every point in $\alpha_0$ belongs to a region, because $\mathcal{T}$ is completed, there must be a region where the arc images forming the edge on $\alpha_0$ must leave $P$ by intersecting different arcs. If this region is negative, it must split into two negative regions in $\mathcal{T}'$ connected by a (positive) rectangle which has the ∘-point on the basepoint triangle as one of its vertices (see Figure 3.24). If the region is positive, then it is extended by a 6-gon as in Figure 3.23, and one of the vertices is immediately the ∘-point on the basepoint triangle. Therefore the ∘-point on the basepoint triangle is a vertex of a positive region. To see that it is also a vertex of a negative region, reverse the roles of arcs and arc images. Since $\mathcal{T}$ is completed, every point in $\varphi(\alpha_0)$ belongs to a region. But then there must be a region where the arcs forming the edge on $\varphi(\alpha_0)$ leave $\varphi(P)$ by intersecting different arc images. If this region is positive, it must split into two positive regions connected by a (negative) rectangle which has the $\circ$-point on the basepoint triangle as one of its vertices, see Figure 3.26.

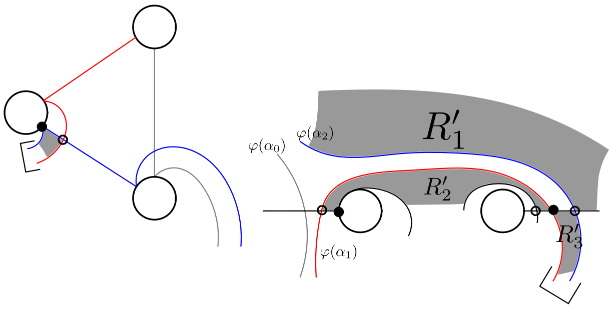


Figure 3.26: When there is a positive region such that the arcs forming the edge on $\alpha_0$ leave $\varphi(P)$ by intersecting different arc images, there are three induced regions $R'_1, R'_2, R'_3$ and the connecting region $R'_3$ uses the basepoint triangle.

If the region is negative, then it is extended by a 6-gon, and one of the vertices is immediately the $\circ$-point on the basepoint triangle. Therefore the $\circ$-point on the basepoint triangle is a vertex of a negative region.

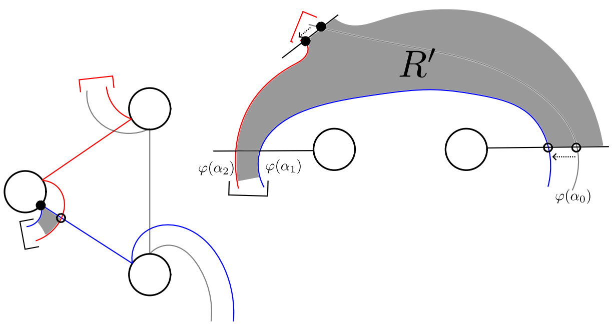


Figure 3.27: When there is a negative region such that the arcs forming the edge on $\alpha_0$ leave $\varphi(P)$ by intersecting different arc images, the induced region $R'$ uses the basepoint triangle.

Now assume that $\mathcal{T}$ is incomplete. Let $R \in \mathcal{T}^-$. Then there exists a $\bullet$-point $x$ in $R$ that is two-sided. If the vertices of $R$ induce a unique region in $\mathcal{T}'$ then $s^-(x)$ is a two-sided $\bullet$-point. If they induce two regions $R'_1, R'_2$ connected by a positive region $R'_2$, as in Figure 3.24, there are three cases to consider.

First, if all the $\bullet$-points in $R$ that are two-sided have their images in $R'_1$, by construction we do not add $R'_2$ or $R'_3$ to $\mathcal{T}'$ and so the property that negative regions have a $\bullet$-point that is two-sided is preserved. The result is still a replete extended tower because the vertex $a$ is not a vertex of a positive region anymore, so we are not forced to add $R'_2$ to make $\mathcal{T}'$ replete, and there are no interior $\bullet$-points in $R'_2$ that are only vertices of positive regions, by hypothesis.

Second, if some •-points in $R$ have their images in $R_1'$ and some in $R_2'$, the property that negative regions have a •-point that is two-sided is immediately preserved.

Third, if all the •-points in $R$ that are two-sided have their images in $R_2'$, note that the vertex $b$ which is a •-point in $R_1'$ is now two-sided, and so the property that negative regions have a •-point that is two-sided is preserved.

Finally, assume that a ∘-point in $\alpha_0$ is not two-sided, but its image is, as in Figure 3.25. Then recall that $R_1', R_3' \in \mathcal{T}'$. Therefore, again the point $b$ is a •-point in the negative region that is two-sided, so the property that negative regions have a •-point that is two-sided is preserved.

Therefore if $\mathcal{T}$ is incomplete $\mathcal{T}'$ is incomplete, and we are done.

□

**Lemma 3.4.13.** *Suppose that the intersection of $P \cap \varphi(P)$ is the basepoint triangles and a collection of rectangles. Using the maps $s^+$ and $s^-$ we can construct regions giving an extended tower $\mathcal{T}'$ supported in $\Gamma'$ that has the same properties as $\mathcal{T}$.*

*Proof.* This case is done in exactly the same way as Lemma 3.4.12. Notice that now in the case where $x \in \varphi(\alpha_0)$, $s^{\pm}(x)$ are not obtained by following an arc image to its intersection with $\alpha_1$ or $\alpha_2$ but by following two sides of a rectangle (which is part of the intersection $P \cap \varphi(P)$). However, locally this only amounts to adding or removing this rectangle from the regions. Moreover, note that in this case we still have that $s^+(x) = s^-(x)$ for all points where both maps are defined. □

**Lemma 3.4.14.** *Suppose that the intersection $P \cap \varphi(P)$ contains a 6-gon. Using the maps $s^+$ and $s^-$ we can construct regions giving an extended tower $\mathcal{T}'$ supported in $\Gamma'$ that has the same properties as $\mathcal{T}$.*

*Proof.* The construction of $\mathcal{T}'$ is done in the same way as Lemma 3.4.12. Let $A$ be the 6-gon contained in $P \cap \varphi(P)$. We only need to focus on the regions given by the intersection points in $A$, because all the others have been covered by Lemmas 3.4.12 and 3.4.13. There are two cases; either $\varphi(\alpha_0)$ intersects $\alpha_0$ as an edge of $A$, or it does not.

First assume it does, and call this intersection point $x$. Then $s^+(x) \neq s^-(x)$ (if they are both defined), but for a region $R$ (either positive or negative) we will see that the induced region $R'$ is obtained by adding and removing rectangles, see Figure 3.28 for an example.

Now, this case is further divided into two cases. First, if $\varphi(\alpha_1)$ also intersects $\alpha_0$, then the orientation of the arc images forces $x$ to be a •-point, as we can see in Figure 3.29. This means that $x$ is a vertex of a negative region $R_2$, and we get an induced negative region $R_2'$ supported in $\Gamma'$.

Now if $x$ is two-sided, it is also a vertex of a positive region $R_1'$. We also have a positive rectangle $R_3'$ using $\alpha_1$, $\alpha_2$, a side of the basepoint triangle, and either $\varphi(\alpha_1)$ or

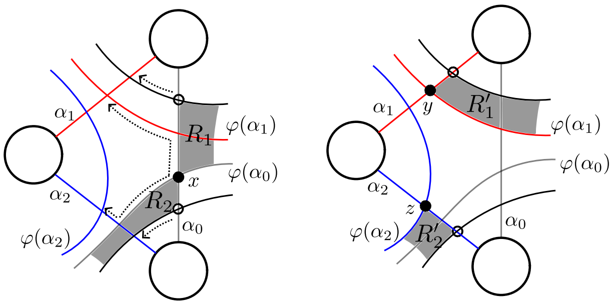


Figure 3.28: The case where $s^+(x) \neq s^-(x)$. Here $y = s^+(x)$ and $z = s^-(x)$. The dashed arrows indicate the action of $s^{\pm}$. The regions $R_1', R_2'$ are obtained from $R_1, R_2$ by adding and removing rectangles.

$\varphi(\alpha_2)$ (depending on which one does not intersect $\alpha_0$). Moreover, one of the vertices of this region is $s^-(x)$. This region can be completed with a rectangle $R_4'$ using the part where $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ are parallel, and one of the vertices of this region is $s^+(x)$. This means that both $s^+(x)$ and $s^-(x)$, together with the new points we have introduced (including the ∘-point on the basepoint triangle) are two-sided. In particular if $\mathcal{T}$ was completed, so is $\mathcal{T}'$, because all new points introduced are two-sided. Similarly, if $\mathcal{T}$ was incomplete, so is $\mathcal{T}'$, because both the negative regions that we have introduced have •-points in common with a positive region. We can see these regions in Figure 3.29.

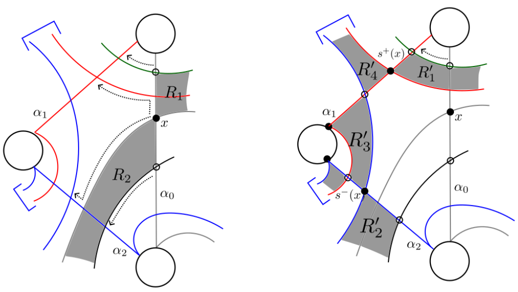


Figure 3.29: The regions induced by a positive region $R_1$ and a negative region $R_2$ with a vertex on $x$. The dashed arrows indicate the action of $s^{\pm}$.

If $x$ is not two-sided, then $\mathcal{T}$ cannot have been completed. So assume it was incomplete. Then $R_2$ has some •-point $y$ in common with a positive region, which means that $R_2'$ has some •-point in common with a positive region, and now by construction we do not include $R_3'$ and $R_4'$ in $\mathcal{T}'$, because $R_4'$ does not have any •-points in common with a positive region in $\mathcal{T}'$. However the result is still an extended tower, which is nice and replete, using the same argument as in Lemma 3.4.12. Moreover, the property that negative regions have a •-point that is two-sided is preserved.

To see that $\mathcal{T}'$ is nested, suppose that $R_2$ is on level $i$. Then $R_1$ is on level $j$, with $j > i$ (because one of its •-points is a vertex of a level $i$ region, but it might have other vertices belonging to regions on higher levels). Then, $R_2'$ is also on level $i$, $R_3'$, is on level $i+1$, and $R_4'$ is also on level $i+1$. Therefore, $R_1'$ is on level $\max\{j, i+2\}$.

Second, if $\varphi(\alpha_2)$ intersects $\alpha_0$, then the orientation of the arc images forces $x$ to be a $\circ$-point. This means that it is a vertex of a positive region $R_1$, which induces a positive region $R_1'$ supported in $\Gamma'$. Now, $s^+(x)$ is two-sided because we can use the negative region $R_4'$ where $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ are parallel, and this in turn gives a positive rectangle $R_3'$ as before, where one of the vertices is $s^-(x)$. This means that the interior $\bullet$-point that we introduced in $R_4'$ is two-sided, so the property that negative regions have a $\bullet$-point that is two-sided is preserved, so if $\mathcal{T}$ is incomplete then so is $\mathcal{T}'$. Moreover, if $\mathcal{T}$ is completed, every interior vertex, in particular $x$, is two-sided, which means that it is a vertex of a negative region $R_2$, which induces a negative region $R_2'$, which means that $s^-(x)$ is two-sided, and $\mathcal{T}'$ is completed. We can see this case in Figure 3.30. Similarly as before we can find the levels of these new regions, and thus $\mathcal{T}'$ is nested.

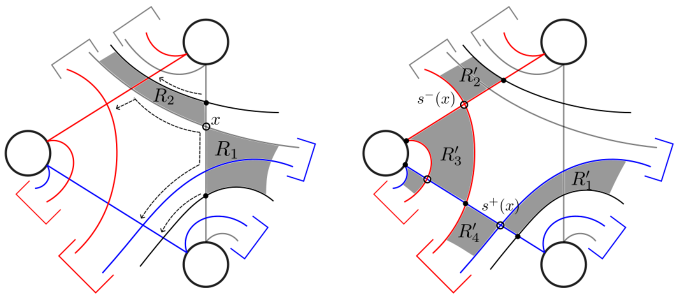


Figure 3.30: The regions induced by a positive region $R_1$ and a negative region $R_2$ with a vertex on $x$. The dashed arrows indicate the action of $s^{\pm}$.

So now assume that $\varphi(\alpha_0)$ does not intersect $\alpha_0$ as an edge of $A$. This means that it intersects $\alpha_1$ and $\alpha_2$. Moreover, the segment of $\varphi(\alpha_0)$ that is an edge of $A$ can only be a part of a negative region $R$. But since this region cannot intersect $\varphi(P)$, the intersection with $P$ must be a rectangle. But since there is no intersection points with $\alpha_0$ there is no action of the slide maps, and the induced region is simply given by extending using a 6-gon. We can see this region in Figure 3.31. There only remains to show that, if $\mathcal{T}$ is completed, the $\circ$-point on the basepoint triangle is two-sided, that is, it is the vertex of a positive region. But if $\mathcal{T}$ is completed, every point on $\alpha_0$ is part of a region, and thus there must be a region in $\mathcal{T}$ with a vertex between $\alpha_0(0)$ and the intersection between $\varphi(\alpha_2)$ and $\alpha_0$, and a vertex between the intersection point of $\varphi(\alpha_1)$ and $\alpha_0$ and $\alpha_0(1)$. Now notice that, because the region $R$ intersects $P$, such a region must necessarily be positive, otherwise we would have a corner of a region in the interior of a region, contradicting the definition of extended tower. But then the induced region supported in $\Gamma'$ uses the $\circ$-point on the basepoint triangle, and we are done.

Again as before we can assign a level to these regions, so $\mathcal{T}'$ is nested.

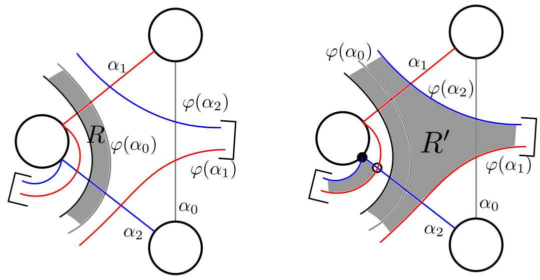


Figure 3.31: If $\varphi(\alpha_0)$ does not intersect $\alpha_0$ we can extend this region using a 6-gon (like we did for positive regions in $P$, see Figure 3.23).

□

We are only left to show that if we have an extended tower $\mathcal{T}'$ supported in $\Gamma'$, there is an extended tower $\mathcal{T}$ supported in $\Gamma$ with the same properties as $\mathcal{T}'$.

Recall that the slide maps do not provide a bijection between vertices in $\mathcal{T}$ and vertices in $\mathcal{T}'$, because there are new vertices in $\mathcal{T}'$ that we have added. However, both $s^+$ and $s^-$ are injective, and so form a bijection with their image. Moreover, the vertices without an inverse are those that lie on endpoints of a segment disjoint from $\alpha_0$ contained in an arc image, or a segment disjoint from $\varphi(\alpha_0)$ contained in an arc. Also, in the definition of the slide maps the extended tower $\mathcal{T}$ is only used to specify the domain (we only consider vertices of regions in $\mathcal{T}$).

Thus, given an extended tower $\mathcal{T}'$ supported in $\Gamma'$, we can define a set of intersection points of $\Gamma \cap \varphi(\Gamma)$ as follows.

Let $R' \in \mathcal{T}'^+$ and $x$ a vertex of $R'$. If the inverse of the slide map makes sense, that is, there exists a vertex $y \in \Gamma \cap \varphi(\Gamma)$ such that $s^+(y) = x$, then define $y = (s^+)^{-1}(x)$ the *preimage* of $x$. We proceed analogously with negative regions and $s^-$. Now consider the set of all such preimages, i.e $\mathcal{V}(\mathcal{T}') = \{y \in \Gamma \cap \varphi(\Gamma) \mid s^+(y) \text{ or } s^-(y) \in V(\mathcal{T}\}$.

**Lemma 3.4.15.** *Let $\mathcal{T}'$ be a completed (respectively incomplete) extended tower in $\Gamma'$, which is not just supported in $\{\alpha_1, \alpha_2\}$. Then the set $\mathcal{V}(\mathcal{T}')$ induces an extended tower $\mathcal{T}$ supported in $\Gamma$ that is completed (respectively incomplete).*

*Proof.* For regions whose entire set of vertices has a preimage we are done in the same way as the previous lemmas. So we only need to focus on regions that have one or more vertices without a preimage under $s^\pm$. We distinguish several different cases.

The first case is when the region is supported in $\{\alpha_1, \alpha_2\}$. There are two options for this. The first one is when the region connects two regions which are not just supported in $\{\alpha_1, \alpha_2\}$, we will deal with this more generally in the second case. If it does not, then the same reasoning as in Lemma 3.4.5 implies that it must be part of a splitting pair (otherwise $\mathcal{T}$ would not be nested), but Lemma 3.4.5 already shows how to get the extended tower $\mathcal{T}$ in this case.

The second case is a vertex belonging to an arc image (different from $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$) that intersects $\alpha_1$ and $\alpha_2$. In this case we see that the arc image provides an edge for a positive region as in the previous case. Moreover, this means that the positive region must be connecting two negative regions, and the points of these regions that do have preimages will give a negative region in $\Gamma$. Essentially we are merging the two negative regions, which is the opposite operation to dividing a region into two connected by a third region as we had done in the previous Lemmas. The level of the new region will be the highest of the levels of the two original regions (this might also increase the level of positive regions with $\bullet$-points in the merged region) so $\mathcal{T}'$ is still nested.

If there are not two regions but just one, then we have a vertex that is not two-sided, but then the preimage of a vertex from the positive region is not two-sided as in Figure 3.25 (in particular neither of the extended towers can be completed, so assume that $\mathcal{T}'$ is incomplete). Moreover this means that we do not add a negative region to $\mathcal{T}$, so this would not affect whether every negative region in $\mathcal{T}$ has a two-sided $\bullet$-point. This could now result in a positive region with $\bullet$-points that do not belong to a negative region. In this case, we also do not add this region to $\mathcal{T}$ to make sure $\mathcal{T}$ is an extended tower. Now this could result in a negative region with $\circ$-points that do not belong to a positive region. We can carry on this procedure, but it must (at the latest) terminate when we reach level 0, where we would have an extended tower $\mathcal{T} = \{R\}$, with $R$ a positive region, so $\mathcal{T}$ is nested, replete, nice, and incomplete.

The third case is analogous to the second one, and happens when both $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ intersect an arc $\beta_j$. Observe that, as before, we can relate this case to the second one by interchanging the roles of $\Gamma'$ and $\varphi(\Gamma')$.

Because all the vertices are inverse images under the slide maps, the resulting collection of regions $\mathcal{T}$ is an extended tower which moreover must be nice. To see whether it is replete, let $A \in \mathcal{R}(\Sigma, \varphi, \Gamma)$ such that $\mathrm{Circ}(A) \subset \mathrm{Circ}(\mathcal{T}) \cup \mathrm{Circ}_\partial(\Gamma)$, and $\mathcal{T} \cup A$ is again an extended tower. Take $s^-(V(A))$. These vertices must also give negative region(s) in $\Gamma'$, so $V(A)$ is the inverse image of vertices of $\mathcal{T}'$ and thus is in $\mathcal{T}$, so $\mathcal{T}$ is replete.

Now suppose that $\mathcal{T}'$ is completed. Then all its interior vertices are two-sided, but this implies that all interior vertices of $\mathcal{T}$ are two-sided, so $\mathcal{T}$ is completed.

So suppose that $\mathcal{T}'$ is incomplete, and let $R$ be a negative region in $\mathcal{T}$. We want to show that it has a two-sided $\bullet$-point. Take $s^-(V(R))$. These points are by construction vertices of regions in $\mathcal{T}'$. The only case where a $\bullet$-point of a region in $\Gamma'$ being two-sided does not imply that its preimage is two-sided is when $P \cap \varphi(P)$ contains a 6-gon. So suppose we have a negative region $R$ supported in $\Gamma$ with a $\bullet$-point $x \in \alpha_0 \cap \varphi(\alpha_0)$ that is not two-sided, and an induced region $R'$ supported in $\Gamma'$ such that $y = s^-(x)$ that is two-sided. This means that $y$ is the vertex of a connecting region $R'_1$ that connects $R'$ to a negative region $R'_2$, which is a rectangle where $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ are parallel.

But now, because $x$ is not two-sided, $R_2'$ has no two-sided $\bullet$-points (it only has two, a boundary point, and the point $z$ that would be $s^+(x)$ if $x$ were two-sided, and none of them can be two-sided). This means that either $\mathcal{T}$ was not incomplete, or $R_2'$ is not in $\mathcal{T}'$. But if $R_2'$ is not in $\mathcal{T}'$, since $\mathcal{T}'$ is replete $R_1'$ also cannot be in $\mathcal{T}'$. But this means that $s^-(x)$ is not two-sided –a contradiction. We can see this in Figure 3.32.

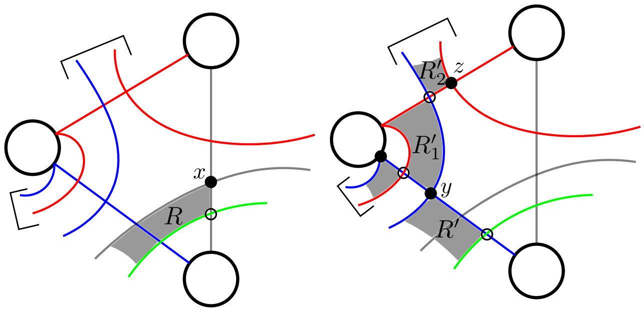


Figure 3.32: If $x$ is not two-sided but $y = s^-(x)$ is, then $\mathcal{T}$ cannot have been incomplete as $R_2'$ does not have any $\bullet$-points in common with any positive region.

There only remains to show that this procedure does not yield an empty extended tower. For a contradiction, assume that it does. Then, there is a positive region $R$ in $\mathcal{T}'$ that does not induce a positive region in $\Gamma$. At least one vertex of $R$ must not have an inverse image under $s^+$. Moreover, $R$ cannot be a connecting region contained in $P$, in this case $R$ does not come from a positive region in $\Gamma$ (it connects two regions induced by a negative region in $\Gamma$) but one of those is in a lower level than $R$ so we can find a positive region in a lower level that also cannot induce a positive region in $\Gamma$ (if we assume that the induced extended tower is empty). Thus the only way this can happen is if we have $R$ and a negative region $R'$ that is not a connecting region, that is, there is not another positive region connected to $R$ by $R'$ (notice that we have encountered this case before with the roles of positive and negative regions reversed). For this to happen, we must have that a vertex $x$ of this region lies either on $\varphi(\alpha_1)$ or $\varphi(\alpha_2)$ and the intersection of the discs cut out by $\Gamma \cup \{\beta\}$ and $\varphi(P)$ contains a 6-gon, see Figure 3.34 for an example.

Suppose $x$ is a $\circ$-point. Then necessarily it lies on $\varphi(\alpha_2) \cap \beta_i$, for some $\beta_i \in \Gamma$. But now the negative region $R'$ does not have any common $\bullet$-points with any positive region, as it has its other $\bullet$-point on the boundary. Indeed, if it does not, we can show $\mathcal{T}'$ is not nested. Suppose that the other $\bullet$-point is not on the boundary. Then either it is not two-sided, in which case we are done because $\mathcal{T}$ is not incomplete, or it is the vertex of a positive region $R_1'$. Notice that the other $\circ$-point of $R'$ must also be a vertex of a positive region $R_2'$. Then, there must be another negative region $R_3'$ where $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ are parallel, until the boundary (if it is not until the boundary we can repeat this argument). Now let us look at the levels of the regions. Let $i$ be the level of $R'$. Then $R_1'$ is on level $j$ with $j > i$, as they share a $\bullet$-point. Similarly, $R_2'$ is on level $k$ with $k \leq i$, as they share a $\circ$-point. Then the same argument shows that $R_3'$ is on level $l$ with $l > k$, and $j \leq l$. But then we have $k \leq i < j \leq l < k$, a contradiction. Thus, the other •-point of $R'$ is on the boundary. We can see this situation in Figure 3.33.

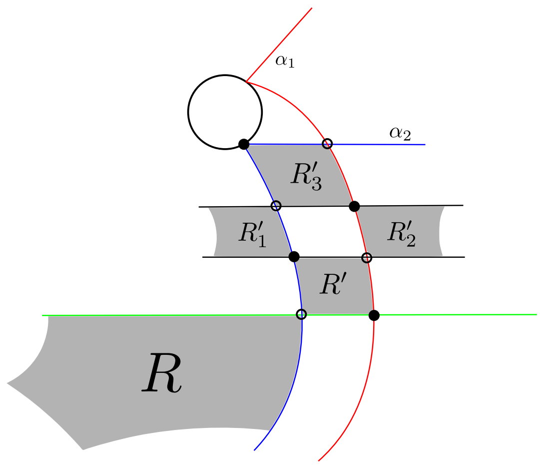


Figure 3.33: We cannot have multiple regions where $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ are parallel because then the extended tower would not be nested. Going from $R'$ to $R'_1$ and then $R'_3$ would increase the level, but going to $R'_2$ and then $R'_3$ would decrease it..

This means that $R'$ has no common •-point with any positive region so $\mathcal{T}'$ is not incomplete. Moreover, if $\mathcal{T}$ is completed, then it must be supported in $\{\alpha_1, \alpha_2\}$ because there is a point in $\varphi(\alpha_1)$ that is not two-sided –a contradiction. We can see this in Figure 3.34.

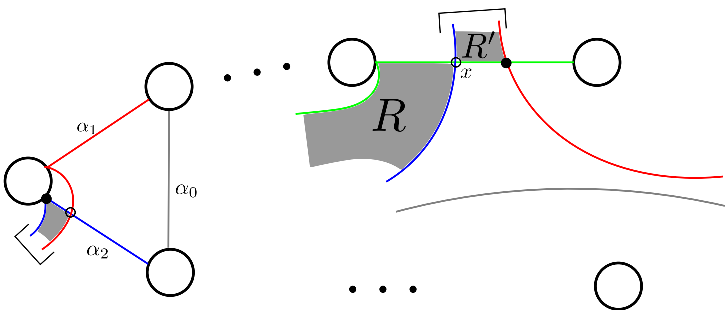


Figure 3.34: If $\mathcal{T}'$ induces an empty extended tower, then it cannot have been incomplete because $R'$ has no common •-point with any positive region but also cannot have been completed unless it is supported in $\{\alpha_1, \alpha_2\}$ because there is a point in $\varphi(\alpha_1)$ that is not two-sided.

Now suppose that $x$ is a •-point. Then the negative region $R'$ has a ∘-point $y$ that is not two-sided (and not on a basepoint triangle), contradicting the definition of extended tower, see Figure 3.35.

□

Recall that our setup is an arc collection $\Gamma$ such that there exists an arc $\beta$ with $\Gamma \cup \beta$ cutting out a disc from the basis. Now let $\alpha_0$ be an arc in $\Gamma$ that is not the next one to $\beta$ as we go along the boundary of the disc cut out by $\Gamma \cup \{\beta\}$, and $\alpha_1, \alpha_2$ arcs

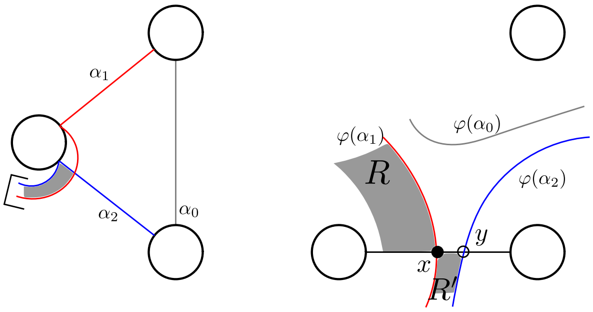


Figure 3.35: The point $y$ is a $\circ$-point of a negative region but not a positive region.

disjoint from this disc such that together with $\alpha_0$ they cut out a 6-gon $P$. Combining the previous Lemmas, we have proved the following Propositions.

**Proposition 3.4.16.** *Let $\mathcal{T}$ be an extended tower in $\Gamma$. If $\alpha_1$ is not $\varphi$-contained in $\alpha_0$, then there is a (nice and replete) extended tower $\mathcal{T}'$ in $\Gamma' = (\Gamma \setminus \{\alpha_0\}) \cup \{\alpha_1, \alpha_2\}$ which is completed (respectively incomplete) if $\mathcal{T}$ is.*

**Proposition 3.4.17.** *Let $\mathcal{T}'$ be an extended tower in $\Gamma'$. If $\alpha_1$ is not $\varphi$-contained in $\alpha_0$, then there is a (nice and replete) extended tower $\mathcal{T}$ in $\Gamma$ which is completed (respectively incomplete) if $\mathcal{T}'$ is.*

### § 3.4.4 | Main Results

Using the base cases and the inductive step we can now show that a collection of arcs $\Gamma$ detects a left-veering arc $\beta$ if $\Gamma \cup \{\beta\}$ cuts out a disc (with the correct orientation), by which we mean that $\Gamma$ supports an incomplete extended tower if and only if $\beta$ is left-veering. Similarly, $\Gamma$ also detects fixable arc segments.

**Theorem 3.4.18.** *Let $\{\alpha_i\}_{i=0}^n$ be a collection of properly embedded arcs cutting out a $(2n+2)$-gon $P$, oriented counterclockwise, and assume $\{\alpha_i\}_{i=1}^n$ are right-veering. Moreover, suppose no arc contained in $P$ is left-veering. Then $\alpha_0$ is left-veering if and only if $\Gamma = \{\alpha_i\}_{i=1}^n$ supports a replete and incomplete extended tower $\mathcal{T}$.*

*Proof.* We argue by induction on the number of arcs in our collection. The case $n = 3$ is given by Proposition 3.4.4. Now assume the result is true for $k$ arcs, with $k < n$. In the $2n$-gon at least one arc image $\varphi(\alpha_{i_0})$ will leave $P$ by intersecting $\alpha_{i_0+1}$ (because each arc image cuts a smaller subsurface inside $P$ so we can apply an innermost disc argument), creating a basepoint triangle. If $\alpha_{i_0} \neq \alpha_1$ then we can apply Propositions 3.4.16 and 3.4.17, and there exists an incomplete extended tower $\mathcal{T}$ supported by $\Gamma$ if and only if there exists an incomplete extended tower $\mathcal{T}$ supported by $\{\alpha_1, \ldots, \alpha_n\} \setminus (\{\alpha_{i_0}, \alpha_{i_0+1}\}) \cup \{\beta\}$, that is nice and replete, where $\beta$ is the arc-sum of $\alpha_{i_0}$ and $\alpha_{i_0+1}$. But by induction this happens if and only if $\alpha_0$ is left-veering.

So now suppose that $\alpha_{i_0} = \alpha_1$, and there are no other cases where the arc images create a basepoint triangle. Then every arc image $\varphi(\alpha_i)$ must leave $P$ by intersecting $\alpha_1$ (an arc image intersecting another arc would cut a smaller disc that does not contain $\alpha_1$

and we could apply our innermost disc argument there). Then $\Gamma$ supports an extended tower $\mathcal{T}$ whose regions are all rectangles as follows. The level0 positive region $R_1$ and its completion $R_1'$ come from the fact that $\alpha_2$ is $\varphi$-contained in $\alpha_1$ (because otherwise the arc-slide of $\alpha_1$ and $\alpha_2$ would be left-veering by Proposition 3.4.4. Then $\varphi(\alpha_1)$ enters $P$ again and must exit by intersecting $\alpha_3$, because $\varphi(\alpha_3)$ leaves $P$ by intersecting $\alpha_1$. This forms another rectangle $R_2$, which must be completed by a rectangle $R_2'$. To see this, suppose for a contradiction that $R_2$ is not completed. Then, $\{R_1, R_1', R_2\}$ would form an incomplete extended tower supported in $\{\alpha_1, \alpha_2, \alpha_3\}$, and then by induction their arc-sum (strictly speaking, the arc-sum with opposite orientation) would be left-veering, which contradicts the assumption that no arc contained in $P$ is left-veering. The rest of the rectangles are obtained in the same fashion. This extended tower is clearly nice and replete. To see that it is nested, observe that $R_1$ is on level 0 (and so is $R_1'$), and then the level increases by 1 with each positive region.

However, if the extended tower is incomplete, the last (positive) rectangle cannot have a completion (otherwise the tower would be completed), and the same argument as in Proposition 3.2.12 shows that $\alpha_0$ is left-veering. Conversely, if $\alpha_0$ is left-veering, suppose for a contradiction that the extended tower is completed, that is, the last positive rectangle does have a completion. But then the edge on $\alpha_1$ must be restricted. However the fact that $\alpha_0$ is left-veering means that its image must intersect this edge –a contradiction. We can see this in Figure 3.36.

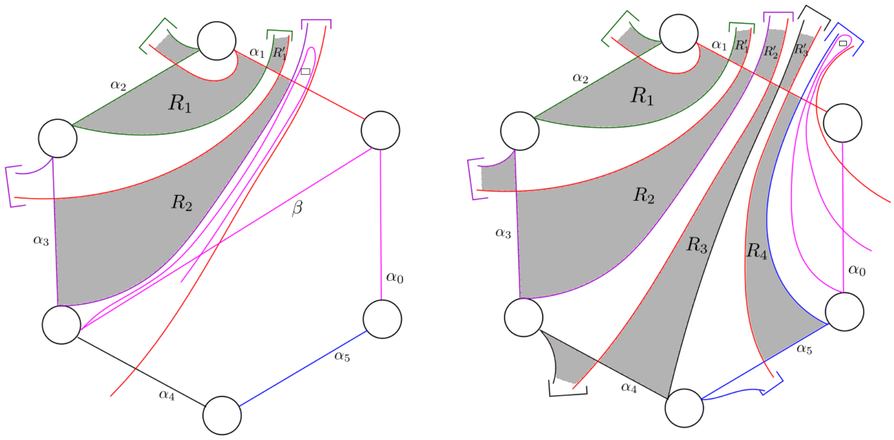


Figure 3.36: On the left, if the incomplete tower is supported in a smaller collection of arcs then their arc-sum $\beta$ (with opposite orientation) is left-veering by induction. On the right, the incomplete extended tower.

□

**Theorem 3.4.19.** *Let $\{\alpha_i\}_{i=0}^n$ be a collection of properly embedded right-veering arcs cutting out a $(2n+2)$-gon $P$, oriented and indexed counterclockwise, and suppose no arc contained in $P$ is left-veering. Let $\gamma$ be an arc segment contained in $P$ starting between $\alpha_n$ and $\alpha_0$ and ending in the interior of $\alpha_1$. Then $\gamma$ is fixable by $\varphi$ if and* *only if* $\Gamma = \{\alpha_i\}_{i=1}^n$ *supports a completed extended tower* $\mathcal{T}$ *whose connecting vertex coincides with* $\gamma \cap \alpha_1$.

*Proof.* We argue by induction on the number of arcs in our collection. The case $n = 3$ is given by Proposition 3.4.5. Now assume the result is true for $k$ arcs, with $k < n$. In the $2n$-gon at least one arc image $\varphi(\alpha_{i_0})$ will leave $P$ by intersecting $\alpha_{i_0+1}$, creating a basepoint triangle (because each arc image cuts a smaller subsurface inside $P$). If $\alpha_{i_0} \neq \alpha_1$ then we can apply Propositions 3.4.16 and 3.4.17, and there exists a completed extended tower $\mathcal{T}$ supported by $\Gamma$ if and only if there exists a completed extended tower $\mathcal{T}$ supported by $\{\alpha_1, \dots \alpha_n\} \setminus (\{\alpha_{i_0}, \alpha_{i_0+1}\}) \cup \{\beta\}$, that is replete, where $\beta$ is the arc-slide of $\alpha_{i_0}$ and $\alpha_{i_0+1}$. But by induction this happens if and only if $\gamma$ is fixable by $\varphi$.

So now suppose that $\alpha_{i_0} = \alpha_1$, and there are no other cases where the arc images create a basepoint triangle. Then every arc image $\varphi(\alpha_i)$ must leave $P$ by intersecting $\alpha_1$. Then the extended tower is a collection of rectangles, obtained as in Theorem 3.4.18, with the difference that now if the extended tower is completed then $\gamma$ must be fixable because it cannot go to either right or left, and conversely if $\gamma$ is fixable then the extended tower must be completed. We can see this last case in Figure 3.37.

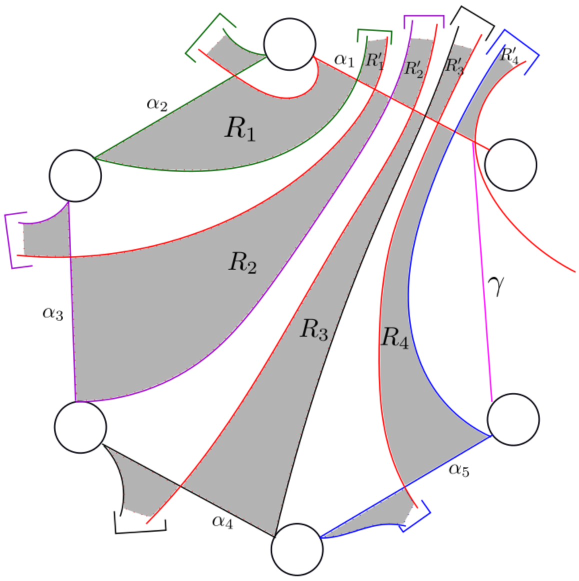


Figure 3.37: The extended tower when the only basepoint triangle is formed by $\alpha_1$ and $\alpha_2$.

□

Our aim is to detect a left-veering arc with a collection of extended towers, each of which will detect a segment of the arc. However, in the setup we have so far, we only detect arcs (or arc segments) with a starting point on the boundary. To get around this, let $\mathcal{C} = \{\alpha_0, \dots, \alpha_n\}$ be an arc collection cutting out a $(2n+2)$-gon $P$, oriented and labelled counterclockwise. Now assume that there is a point $x \in \alpha_n$ that is the endpoint of a fixed arc segment disjoint from $P$, so by Theorem 3.4.19 there exists an extended tower with $x$ as its connecting veretx. Let $\alpha_n'$ be the (oriented) arc segment between $\alpha_n(0)$ and $x$, and let $\alpha_0'$ be the (oriented) arc segment that goes from $x$ to $\alpha_1(0)$. Then $\mathcal{C}' = \{\alpha_0', \alpha_1, \ldots, \alpha_{n-1}, \alpha_n'\}$ is a collection of arc segments that cut out a $(2n+1)$-gon $P'$. Moreover, at $x$, the tangent vector of $\alpha_n'$ followed by the tangent vector of $\varphi(\alpha_n')$ define the orientation of $\Sigma$ (because $x$ is the connecting vertex of a completed extended tower) so in a slight abuse of notation we can say that the arc segment $\alpha_n'$ is right-veering, and we can adapt the terminology and methods of extended towers to $\mathcal{C}'$ (since the only properties we use in the results is that the arcs bound a disc and are disjoint and right-veering). In particular Theorems 3.4.18 and 3.4.19 still hold. We can see this situation in Figure 3.38.

**Definition 3.4.20.** Let $\mathcal{T}$ be an extended tower supported in the collection $\mathcal{C}' \setminus \{\alpha_0'\}$. We say $\mathcal{T}$ is a *partial extended tower*, and $x$ its *starting point*.

We will also say, in a slight abuse of notation, that such a partial extended tower $\mathcal{T}$ is *supported in* $\Gamma = \{\alpha_1, \ldots \alpha_n\}$.

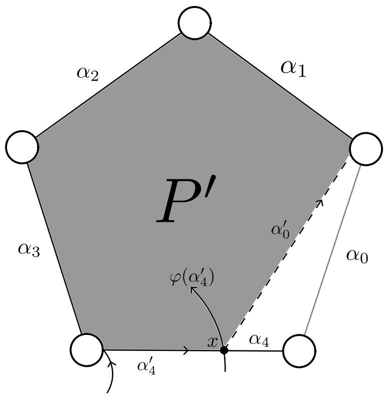


Figure 3.38: The setup for a partial extended tower supported in $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4'\}$. The fact that $x$ is the connecting vertex of an extended tower means that at $x$ the tangent vector of $\alpha_4'$ followed by the tangent vector of $\varphi(\alpha_4')$ define the orientation of $\Sigma$.

We are now almost ready to prove that we can detect the existence of a left-veering arc from a basis of the surface, we just need one more definition. Notice that for our results to work we need the arcs cutting out a disc to be distinct. However, if we simply take a basis as our collection of arcs, this does not necessarily happen. We solve this issue by duplicating every arc from the basis. This has the effect that when a left-veering arc intersects the basis it actually intersects two (isotopic) arcs $\alpha$ and $\beta$. Moreover, the segment of the left-veering arc before this intersection will cut out a disc with a collection containing one of the arcs (say, $\alpha$) and the segment after the intersection will cut out a disc with a collection containing the other arc (say, $\beta$). Then an extended tower in the first collection will have a connecting vertex on $\alpha \cap \varphi(\alpha)$ and an extended tower in the second collection will have a starting point on $\beta \cap \varphi(\beta)$.

Because we want to construct the left-veering arc from the extended towers, we want to relate these two points.

*Remark.* We want to distiguish $\alpha$ and $\beta$ even though they are isotopic because $\alpha$ could also be an arc in the second collection, and we want each arc from the collection supporting an extended tower to be distinct. Note though that we only need to duplicate each arc from a basis because the disc cut by a basis has exactly two copies of each arc, so each extended tower will have at most two isotopic arcs.

**Definition 3.4.21.** Let $\alpha$ and $\beta$ be two isotopic properly embedded arcs. Then $\varphi(\alpha)$ and $\varphi(\beta)$ are always parallel and, for an intersection $x \in \alpha \cap \varphi(\alpha)$, then we have a small rectangle contained in the intersection of the thin strip between $\alpha$ and $\beta$ with its image which has $x$ as a vertex. We call the vertex of this rectangle $y \in \beta \cap \varphi(\beta)$ the *adjacent point to* $x$.

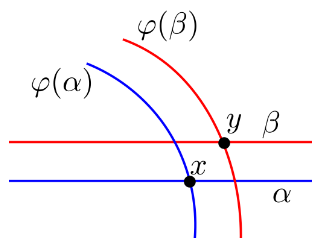


Figure 3.39: The point $y$ that is adjacent to the point $x$.

**Theorem 3.4.22.** *Let $(\Sigma, \varphi)$ be an open book, and $\Gamma$ a basis for $\Sigma$ with all arcs duplicated. Suppose that there exists a left-veering arc $\gamma$, which we can assume to be minimal. Then there exists a collection of extended towers $\{\mathcal{T}_i\}_{i=1}^N$ (where $N$ is the number of intersections between $\gamma$ and the basis) supported in (subcollections of) $\Gamma$ such that:*

- *$\mathcal{T}_1$ is a completed extended tower.*
- *$\mathcal{T}_i$ is a completed partial extended tower, whose starting point is the adjacent point to the connecting vertex of $\mathcal{T}_{i-1}$.*
- *$\mathcal{T}_N$ is a incomplete partial extended tower, whose starting point is the adjacent point to the connecting vertex of $\mathcal{T}_{N-1}$.*

*Conversely, if we have such a collection, then there exists a left-veering arc $\gamma$.*

*Proof.* Cut $\Sigma$ along the arcs $\Gamma$, making a disc, and orient them counterclockwise. Then $\gamma$ is fixable until it intersects one of the arcs (if it is disjoint from the basis then it will cut out a disc with a subcollection of arcs from $\Gamma$ and then Theorem 3.4.18 gives an incomplete extended tower). Then Theorem 3.4.19 gives the first completed extended tower $\mathcal{T}_1$. Then $\gamma$ is again fixable from the adjacent point to this point until the next intersection with $\Gamma$ (and also in the small rectangle between the adjacent points), and now modifying Theorem 3.4.19 for the case where we have an arc segment with fixed endpoints gives the first completed partial extended tower. Repeat until $\varphi(\gamma)$ goes to the left, and then modifying Theorem 3.4.18 gives the incomplete partial extended tower.

For the converse, observe that both Theorem 3.4.19 and Theorem 3.4.18 are if and only if statements, and the left-veering arc is constructed by joining all the fixed arc segments given by the completed extended towers and the left-veering arc segment given by the incomplete one. Observe that the small arc segments between adjacent points necessary to connect all of the arc segments given by Theorems 3.4.19 and 3.4.18 are fixable because their endpoints are fixed points and they lie in the thin strip between isotopic arcs. □

Notice that the number of extended towers $N$ in a collection detecting a left-veering arc in the construction described by Theorem 3.4.22 coincides with the number of intersections of the arc with the basis. Moreover, each of these points corresponds to a point $\alpha \cap \varphi(\alpha)$ for some arc $\alpha$ in the basis. Once we fix a basis, the number of such points is finite and gives an upper bound for $N$. Moreover, each extended tower is also a finite collection of regions. Therefore, Theorem 3.4.22 implies the existence of an algorithm that takes as input a basis of arcs and their images and either produces a collection of extended towers giving a left-veering arc or terminates in a finite number of steps, which means that the monodromy is right-veering.

As a last remark, note that, if we have a basis of right-veering arcs and we find by this algorithm a left-veering arc, we have also computed that the Fractional Dehn Twist Coefficent (FDTC) of the monodromy (with respect to the boundary component where we find the left-veering arc) is 0, since existence of a right-veering arc and a left-veering arc is enough to guarantee that the FDTC is 0 [21]. In cases with higher FDTC, after composing with boundary twists, if we find a left-veering arc with a basis of right-veering arcs, we can also compute that the FDTC is $n$ for any integer $n$. However, notice that we cannot compute it in all cases, since in the case where the FDTC is not an integer we will only be able to provide a bound on it. Moreover, the FDTC can be 0 in a monodromy that is right-veering, and in this case we would not be able to compute it via our methods.

## § 3.5 | Examples

Theorem 3.4.22 does not make any assumptions on the number of extended towers needed to detect a left-veering arc, and a natural question is whether multiple extended towers are always needed, and, if the answer is affirmative, whether there is an upper bound on the number of extended towers that does not depend on the choice of basis. Example 3.5.1 shows that indeed some cases require multiple extended towers and the number of extended towers required can be made arbitrarily large. Recall that the arc segments that form a left-veering arc $\gamma$ are determined by the intersections of $\gamma$ with a basis, and each segment is detected with an extended tower. Therefore for any natural number $n$ we construct an open book which is not right-veering and a basis such that each arc has more than $n$ intersections with every left-veering arc.

**Example 3.5.1.** Let $\Sigma$ be a planar surface with 4 boundary components $\{C_i\}_{i=1}^4$, and $\varphi = \tau_1\tau_2\tau_3\tau_a\tau_b^{-1}$ (where $\tau_i$ represents a positive Dehn twist around the boundary component $C_i$) as shown in Figure 3.40. Then $(\Sigma, \varphi)$ is not a right-veering open book, as the arc $\gamma$ in Figure 3.40 is left-veering. However, any left-veering arc has to start in the boundary component $C_4$, and moreover has to intersect $b$ before it intersects $a$. In particular, this means that it has to intersect the arc $\delta$ going from $C_1$ to $C_3$, and thus the curve $c$ that separates $C_1$ and $C_3$ from $C_2$ and $C_4$. We choose a basis $\mathcal{B}$ of right-veering arcs such that all arcs intersect $c$. For every natural number $n$, let $\mathcal{B}_n = \tau_c^n(\mathcal{B})$. Then $\mathcal{B}_n$ is a basis which is also right-veering and every arc of $\mathcal{B}_n$ intersects every left-veering arc more than $n$ times. But this in turn implies that more than $n$ extended towers are needed for $\mathcal{B}_n$ to detect a left-veering arc.

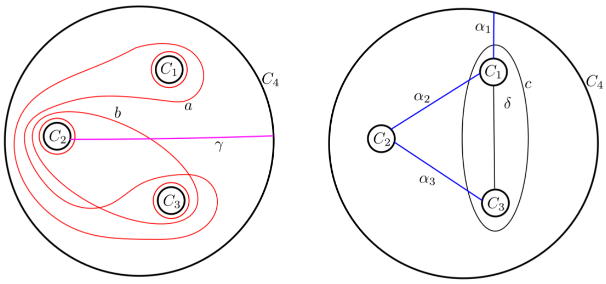


Figure 3.40: On the left, the surface $\Sigma$ with the curves involved in the monodromy, and a left-veering arc $\gamma$. On the right, a basis $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ such that every arc in $\mathcal{B}$ intersects the curve $c$.

We note, however, that this procedure is not very efficient, since it produces an extended tower for each intersection of the left-veering arc with the basis, and it would in some cases be possible to reduce the number of extended towers needed to detect a left-veering arc.

**Example 3.5.2.** Let $(\Sigma, \varphi)$ be the open book from Example 3.5.1, and consider the basis $\{\alpha_1, \alpha_2, \beta_3\}$ on the left-hand side of Figure 3.41. The left-veering arc $\gamma$ has an intersection point $x$ with the basis.

By the procedure we have described, we double the arcs from the basis, and find fixable arc segments and left-veering arc segments disjoint from the arcs in the (duplicated) basis except at their endpoints. Thus, the fact that $\gamma$ is left-veering is detected in two steps. First we detect that $\gamma_1$ is a fixable arc segment with a completed extended tower $\mathcal{T}_1$ supported in $\{\alpha_1, \alpha_2\}$, because they together with the dashed arc cut out a disc $P_1$ from $\Sigma$. This extended tower has $x$ as its connecting vertex. Then, we detect that $\gamma_2$ is a left-veering arc segment with an incomplete extended tower $\mathcal{T}_2$ supported in $\{\beta_1, \beta_2, \beta_3\}$, because they together with $\gamma_2$ cut out a disc $P_2$ from $\Sigma$. This (partial) extended tower has $y$, which is is the adjacent point to $x$, as its starting point. Thus the arc $\gamma$, which is the union of $\gamma_1$ and $\gamma_2$, is left-veering. We can see this on the left hand side of Figure 3.41.

However, we can also detect the fact that $\gamma$ is a left-veering arc directly with a single extended tower. This is because it cuts out a disc $P$ together with $\alpha_1, \beta_1, \beta_3$, so these 3 arcs support an incomplete extended tower. However, the procedure we described will not find it because the interior of $P$ is not disjoint from the basis arcs. We can see this on the right hand side of Figure 3.41.

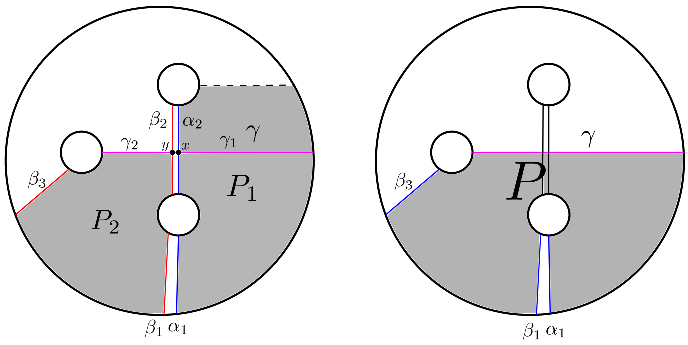


Figure 3.41: On the left, the procedure we described uses two extended towers to detect the fact the $\gamma$ is left-veering. However, the disc $P$ on the right implies that only one extended tower is needed.

# Chapter 4
# Binding sums of contact manifolds

## § 4.1 | Introduction

We now explore properties of an operation of manifolds called the binding sum, which is similar to the connect sum. Contact structures behave well under connect sums because they preserve properties such as tightness and symplectic fillability (see [7] and [45]). This is not the case for binding sums, in particular, properties such as tightness, fillability, or non-vanishing contact invariant are not necessarily preserved under this operation. We exhibit some examples of this, and then provide an explicit chain in Heegaard Floer homology that shows vanishing of the contact invariant for an infinite family of binding sums whose summands have non-vanishing contact class, recovering a result of Wendl [46]. Along the way we correct a computational error in [24].

While the connected sum of two manifolds consists of removing two balls from them and gluing along the $S^2$ boundary, the binding sum consists of removing two solid tori and then gluing along the $T^2$ boundary.

More precisely, for $i = 1, 2$ let $(M_i, \xi_i)$ be two closed contact 3-manifolds, and $(B_i, \pi_i)$ open book decompositions supporting them. Let $K_i$ be a component of the binding $B_i$. Remove a standard tubular neighbourhood of these knots, which we can see as the normal bundle $\nu K_i$, and identify the resulting boundaries via a map that preserves the fibres. Since the contact structure in a neighbourhood of a binding component is standard (more generally, it is standard in a neighbourhood of any transverse knot, see for example [26]), we can glue the contact structures and the resulting manifold inherits a contact structure from the contact structures of the original manifolds.

**Definition 4.1.1.** This operation is called a *binding sum* and will be denoted by $M_1 \boxplus_{K_1,K_2} M_2$.

*Remark.* Observe that we can extend this definition to summing along several binding components and not just one. Moreover, to avoid complicating the notation, we will drop the subscripts $K_i$ when it is clear which binding components are involved in the sum.

We can see from the construction that performing a binding sum interferes with the fibrations of the open book decompositions, so we do not immediately get an open book of the manifold $M_1 \boxplus M_2$. Nevertheless, by a result of Klukas in [26] we can obtain an open book decomposition of the summed manifold from the (abstract) open books of the original manifolds. Indeed, given a binding component $K$, let $K'$ be a transverse knot that intersects each page of the open book exactly once near the boundary. Note that since the monodromy is the identity in a neighbourhood of the boundary, this is indeed a knot. Now change the monodromy of the open book near the boundary as follows. Add a negative Dehn twist along a curve $a$ that goes around the intersection of $K'$ with the page, a boundary parallel positive twist, and a negative twist along a curve $b$ that is boundary parallel but on the other side of the intersection of $K'$ with the page. Call the composition of these Dehn twists $f$. Clearly $f$ is isotopic to the identity, and thus for any open book $(\Sigma, \varphi)$, the open book $(\Sigma, \varphi \circ f)$ is equivalent to it, since $\varphi$ and $\varphi \circ f$ represent the same mapping class. In the language of Klukas, we say that $K$ *admits a navel*, and clearly every binding component of an open book admits a navel. The knot $K'$ is then called the *core of the navel*. We can see this on the left hand side of Figure 4.1.

*Remark.* Klukas' setup is slightly more general than ours, since he considers knots with arbitrary framing, and we will restrict ourselves to knots with 0 framing.

Klukas then shows that a binding component $K$ is transversely isotopic to the core of its navel. Therefore, for open books $(\Sigma_1, \varphi_1)$, $(\Sigma_2, \varphi_2)$ with binding components $K_1, K_2$, instead of removing a neighbourhood of the binding components $K_1, K_2$ and gluing around the resulting boundary torus, we can remove a neighbourhood of the core of their navels $K_1', K_2'$ and glue around the resulting boundary. But now on each page this results in removing a neighbourhood of the intersection point of $K_i'$ with the page, and gluing along the boundary circles. This means that the fibrations of the open book decompositions are preserved, so we do get a new open book. The new page $\Sigma$ is the connect sum of the original pages, and the new monodromy is given by the composition of (the natural inclusions of) $\varphi_1 \circ f_1$ and $\varphi_2 \circ f_2$. Note that $f_1$ and $f_2$ are no longer isotopic to the identity after performing the connect sum, see the right hand side of Figure 4.1.

*Remark.* In the case where we deal with abstract open books (as opposed to ambient ones), we will denote the binding sum as $(\Sigma_1, \varphi_1) \boxplus (\Sigma_2, \varphi_2)$. Again when it is clear which binding components are involved in the sum we will drop them from the notation.

Note that the resulting contact manifold could depend on the choice of open books for the original manifolds, as we can see from the following example.

**Example 4.1.2.** Summing two open books $(D^2, \mathrm{Id})$ (representing the standard Stein fillable $S^3$) gives the manifold determined by the open book $(S^1 \times [0,1], \mathrm{Id})$, which is the standard Stein fillable $S^1 \times S^2$. However, if we stabilize both open books to get

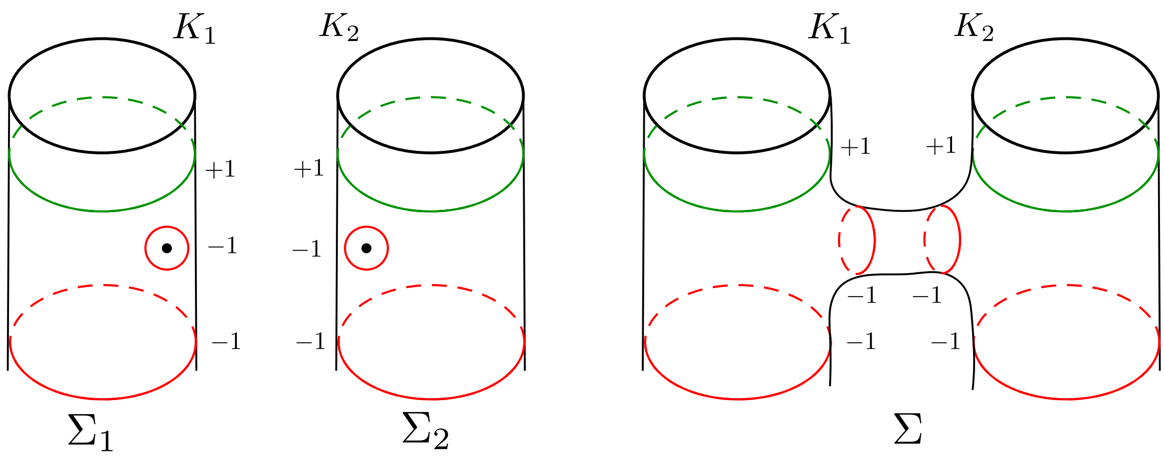


Figure 4.1: On the left, neighbourhoods of the binding components together with their respective navels. The black dots represent the intersections of the knots $K'_1, K'_2$ with the pages. On the right, the new open book near the binding components being summed. Away from this neighbourhood the monodromy is the composition of the original monodromies.

Hopf bands and sum those (along one boundary component), we get an open book that is not right-veering and thus overtwisted by Theorem 2.2.11. We can see this case in Figure 4.2.

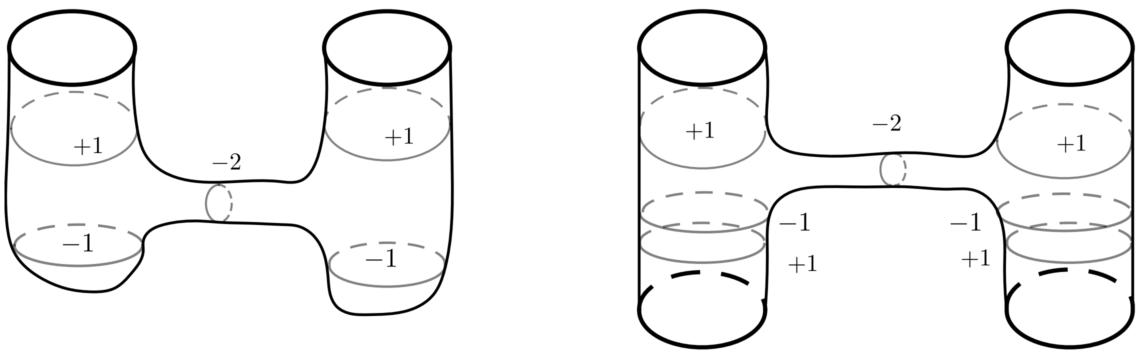


Figure 4.2: On the left, the sum $(D^2, \mathrm{Id}) \boxplus (D^2, \mathrm{Id})$ gives $(S^1 \times [0,1], \mathrm{Id})$. On the right, $(S^1 \times [0,1], \tau) \boxplus (S^1 \times [0,1], \tau)$ gives an open book that is not right-veering (because the twists at the bottom cancel).

## § 4.2 | Examples

If one of the open books has more than one binding component, and we do not perform the binding sum on all boundary components, the result can be an overtwisted structure (with vanishing contact invariant), even if the original manifolds are Stein fillable (with non-vanishing contact invariant), due to the fact that, by Klukas' construction, we may obtain an open book that is not right-veering. We can see this in Example 4.1.2, but it is a more general phenomenon, as we can see in the following example.

**Example 4.2.1.** If we sum two open books $(\Sigma_1, \mathrm{Id})$ and $(\Sigma_2, \varphi)$, where $\Sigma_1$ is any surface which has a boundary component not used in the sum, we get a left-veering arc by connecting the non summed boundary component with a summed one, due to the negative twist at the bottom of Figure 4.1. Note that $(\Sigma_1, \mathrm{Id})$ is Stein fillable (it is the standard Stein fillable contact structure on $\#^k(S^1 \times S^2)$), and the construction does not depend on $\varphi$, so we can choose it to support a Stein fillable contact structure by taking it to be the identity, or a product of positive Dehn twists (by Theorem 2.2.10).

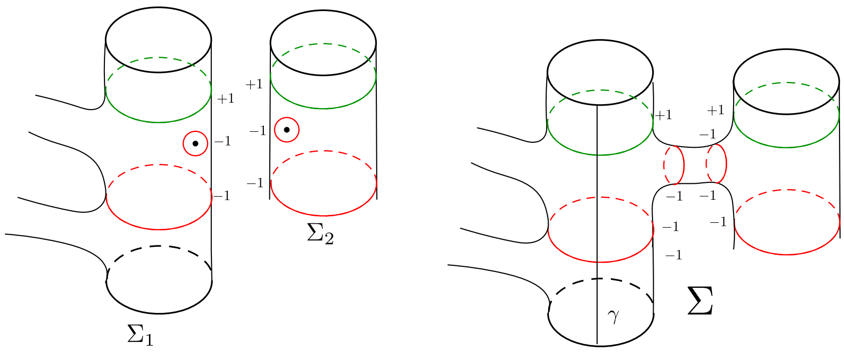


Figure 4.3: On the left the two open books $(\Sigma_1, \mathrm{Id})$ and $(\Sigma_2, \varphi)$ being summed, together with their corresponding navels. On the right, the open book for the binding sum and a left-veering arc $\gamma$.

It would then seem that this happens only when we have an arc that is fixed by the monodromy, which then becomes left-veering after performing the binding sum. However, this is not true.

**Example 4.2.2.** Stabilise the sum from Example 4.2.1 enough to make it right-veering (this can always be done by [21]). Stabilisations do not change the contact manifold, so we still have an overtwisted contact structure. Then, we observe that, if none of the components being summed was a disc (so the left-veering arc happens in a boundary component that was not used in the sum), the stabilisations are performed away from the part of the open book where we have performed the binding sum and, in particular, these two operations commute. This means that we can start with an open book where no arc fixed by the monodromy becomes left-veering (and indeed the open book is right-veering) and obtain an overtwisted contact structure.

Another example is the following.

**Example 4.2.3.** If the open books are $(A, \tau^n)$ and $(A, \tau^2)$, where $A$ is an annulus, the result is an example of a right-veering, non-destabilizable, overtwisted open book shown by Lisca in [32] (note that if we use $(A, \tau)$ instead of $(A, \tau^2)$ we once again get something that is not right-veering). We can see this open book in Figure 4.4.

*Remark.* Observe in this last case, unlike in Example 4.2.2, none of the open books involved are stabilisations (and the monodromies are not the identity), and yet by Theorem 2.2.10 we are summing two Stein fillable open books, but we obtain an overtwisted one.

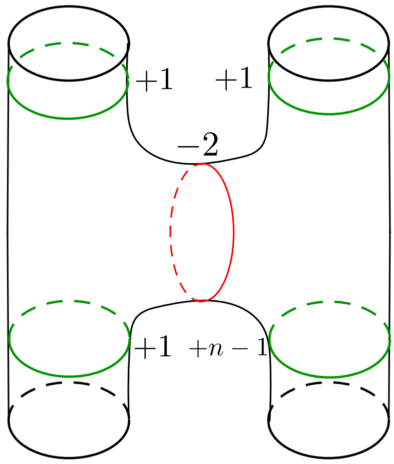


Figure 4.4: Binding sum of $(A, \tau^n)$ and $(A, \tau^2)$.

## § 4.3 | Non-symmetric sums

All of the examples we have studied until now involved having some boundary components of the original open books that are not used in the sum. However, even if we perform the binding sum on all binding components we might end up with a manifold that has vanishing contact invariant. For example, a manifold has Giroux torsion if and only if it can be expressed as the binding sum of three open books, two of which are $(S^1 \times [0,1], \mathrm{Id})$ (see [26]), so if the third is a Stein fillable open book, or indeed one with non-vanishing invariant; we have a sum of manifolds with non-vanishing invariants giving a manifold with vanishing invariant (since manifolds with Giroux torsion have vanishing invariant by [18]).

It is not true, however, that the result of a binding sum is always overtwisted or has vanishing invariant. We know by work of Wendl (see [46] and [26]) that if we sum two copies of the same open book (a *symmetric sum*) with the identity as monodromy along all boundary components, the result is a Stein fillable contact structure; while summing two different surfaces (a *non-symmetric sum*), still with the identity as monodromy, along all boundary components give contact manifolds with vanishing contact invariant.

Wendl's result used $J$-holomorphic curves, and we want to understand this from the open book perspective, using Klukas' result about the neighbourhood of the binding in the binding sum. Many of the examples are too complicated to solve, but we are able to give a chain in an infinite family of sums, namely, summing a genus $g > 0$ surface with two boundary components with an annulus, where both monodromies are the identity.

First consider the sum of two open books $(A, \mathrm{Id})$ where $A$ is an annulus, with the sum being performed on both boundary components. The result is an open book supporting a Stein fillable contact structure on $T^3$ [26], and so it has non-vanishing contact invariant. The open book is given in Figure 4.6, and Figure 4.5 gives the diagram for the surface that we will use, together with a basis of arcs and its image under the monodromy.

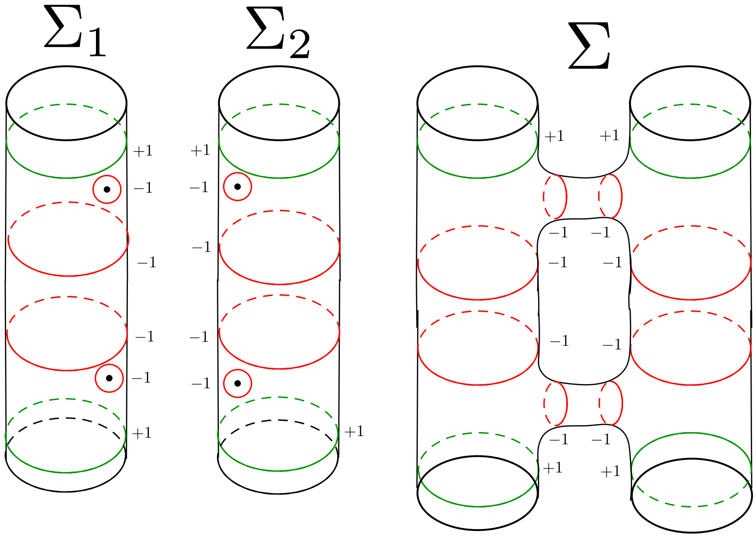


Figure 4.5: On the left, the two annuli with the navels corresponding to all boundary components. On the right, the result of the binding sum.

Now, if we add a boundary component, the resulting partial open book has vanishing contact class because it is the partial open book corresponding to a neighbourhood of a Giroux torsion domain (see [24] and [22]), and we know by [18] that it has vanishing invariant.

We will adopt the following convention for denoting generators of the Heegaard Floer complex (which coincides with the one in [24]). Each generator will be denoted by a tuple where the $i$-th component will be an intersection point lying in $\alpha_i$ (the arcs of the basis), where $\alpha_1$ is the horizontal arc, $\alpha_2$ is the arc immediately below $\alpha_1$ and the rest are ordered from top to bottom. The number in that component will denote the position of the intersection point in $\alpha_i$, with the order going from right to left in $\alpha_1$ and from top to bottom in the rest. In particular the contact class $c(\xi)$ is the point $(1, 1, 1, 1, 1)$.

Now, we can apply the Sarkar-Wang algorithm to obtain a nice diagram (see Figure 4.9), thus turning the problem of finding the differentials into a combinatorial problem, since the only possible domains are rectangles and bigons. Assisted by the computer, the chain we obtain is then the following:

There is a rectangle going from $(1, 2, 2, 1, 1)$ to $(1, 1, 1, 1, 1)$, and the only other domain coming out of this point is a bigon to $(1, 3, 2, 1, 1)$. Next, there is a rectangle from $(2, 4, 2, 1, 1)$ to $(1, 3, 2, 1, 1)$, and the only other domain coming out of it is a bigon to $(3, 4, 2, 1, 1)$. Then, there is a rectangle from $(6, 4, 5, 1, 1)$ to $(3, 4, 2, 1, 1)$, and the only other domain coming out of it is a rectangle to $(9, 1, 5, 1, 1)$. Then, there is a rectangle from $(9, 1, 4, 2, 1)$ to $(9, 1, 5, 1, 1)$, and the only other domain coming out of it is a bigon going to $(9, 1, 3, 2, 1)$. Next, there is a rectangle going from $(9, 15, 2, 2, 1)$ to $(9, 1, 3, 2, 1)$, and the only other domain coming out of it is a bigon going to $(9, 14, 2, 2, 1)$. Then

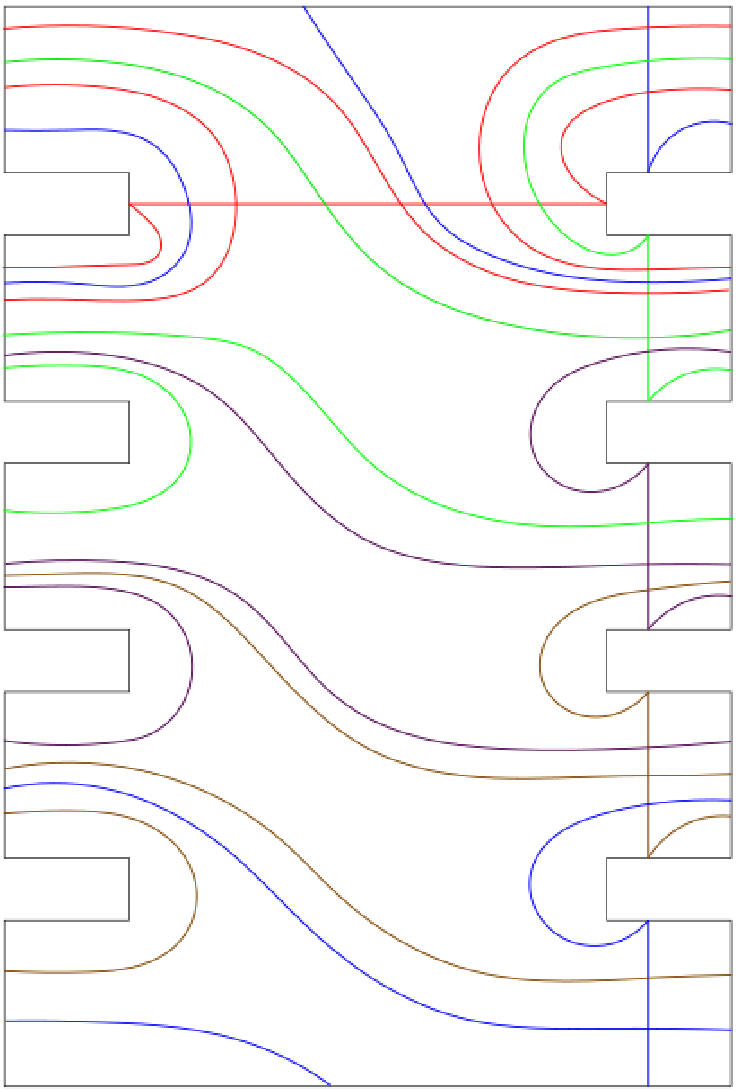

Figure 4.6: The open book for $(A, \mathrm{Id}) \boxplus (A, \mathrm{Id})$, with arcs forming a basis and their images. The top and bottom of the diagram, as well as the left and right, are identified.

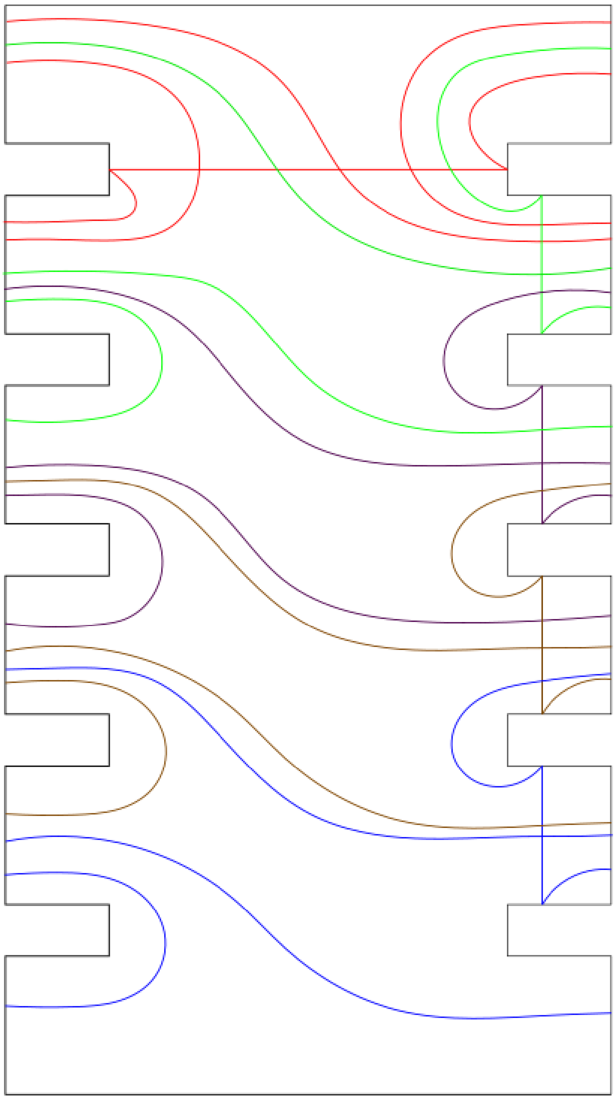

Figure 4.7: The partial open book of a neighbourhood of a Giroux torsion domain as shown in [24]

there is a rectangle from $(9, 11, 2, 5, 1)$ to $(9, 14, 2, 2, 1)$, but now there are two other domains coming out of it, a rectangle going to $(16, 4, 2, 5, 1)$ and a rectangle going to $(3, 5, 2, 5, 1)$. Now there is a bigon from $(2, 5, 2, 5, 1)$ that goes to $(3, 5, 2, 5, 1)$, and the only other domain coming out of it is a rectangle going to $(1, 6, 2, 5, 1)$. We then consider the point $(17, 4, 2, 5, 1)$, which has a bigon to $(16, 4, 2, 5, 1)$ and a rectangle to $(1, 6, 2, 5, 1)$, but also a rectangle to $(18, 3, 2, 5, 1)$. Next, there is a bigon from $(18, 2, 2, 5, 1)$ to $(18, 3, 2, 5, 1)$, and the only other domain coming out of it is a rectangle to $(18, 1, 1, 5, 1)$. Then, there is a rectangle from $(18, 1, 1, 4, 2)$ to $(18, 1, 1, 5, 1)$, and the only other domain coming out of it is a bigon to $(18, 1, 1, 3, 2)$. Next, there is a rectangle from $(18, 1, 10, 2, 2)$ to $(18, 1, 1, 3, 2)$ and the only other domain coming out of it is a bigon to $(18, 1, 9, 2, 2)$. Finally, there is a rectangle from $(18, 1, 5, 6, 2)$ to $(18, 1, 9, 2, 2)$ and there are no other domains coming out of this point. We can see the first few domains in this chain in Figure 4.9. The intersection points that are the sources of a domain are labelled as $\circ$-points and the points that are the targets are labelled as $\bullet$-points.

Note that this is not the chain given in [24]. Indeed, the chain there ends with the point $(9, 11, 2, 3, 2)$, and it is claimed that the only domain coming out of it is a rectangle to $(9, 12, 2, 2, 2)$. However, there also rectangles going to $(16, 4, 2, 3, 2)$ and $(3, 5, 2, 3, 2)$, and so this chain does not work. Juhász and Kang also give an upper bound of 2 for the spectral order of a neighbourhood of a Giroux torsion domain based on this computation. Our computation yields the same upper bound, although it is conjectured that this can be improved to 1 (see [24] and [28]).

In the original diagram, the explicit chain killing the contact class is the following:

There is an embedded annulus from (2,2,3,1,1) to (1,1,1,1,1). The only other domain coming out of (2,2,3,1,1) is a disk going to (3,1,3,1,1). Next, there is an embedded annulus from (3,6,2,3,1). The only other domain coming out of (3,6,2,3,1) is an immersed annulus going to (5,1,1,3,1). Finally, there is an embedded annulus going from (5,1,3,4,2) to (5,1,1,3,1) and no other domains come out of (5,1,3,4,2). This means that

$\partial((2, 2, 3, 1, 1)+(3, 6, 2, 3, 1)+(5, 1, 3, 4, 2)) = (1, 1, 1, 1, 1)+2(3, 1, 3, 1, 1)+2(5, 1, 1, 3, 1) = (1, 1, 1, 1, 1)$ because we are working with $\mathbb{F}_2$ coefficients.

Using this chain, we can show an explicit chain that causes vanishing of the contact class for an infinite family of binding sums. Let $\Sigma_{g,n}$ denote the compact surface of genus $g$ and $n$ boundary components.

**Theorem 4.3.1.** *The chain killing the contact class in the partial open book of Figure 4.7 also appears in the open book for the binding sum $(\Sigma_{g,2}, Id) \boxplus (A, Id)$ for any $g > 0$, where the sum is performed on both boundary components, and also kills the contact class in it.*

*Proof.* Observe that the domains in this partial diagram will also be in the diagram for $T^3$, since we can see the last boundary component as the first one by an identification.

The reason why the contact invariant does not vanish in this open book is that there are more domains coming from the rest of the arcs in the basis once we have gone back to the top. However, these are the only extra domains. This means that, if we do not use domains that require the identification of top and bottom, we can get a chain killing the contact class. But now adding genus on the bottom part of the open book means that there will be no domains that use both top and bottom. Thus, the chain we used before also kills the contact invariant in this case. But now observe that adding genus to the open book in this manner amounts to adding genus to one of the summands in the binding sum, and we can add arbitrarily large genus. Figure 4.8 shows the case where the genus is one, and we can see that all the domains that we computed before are present in this open book, and there are no new ones.

□

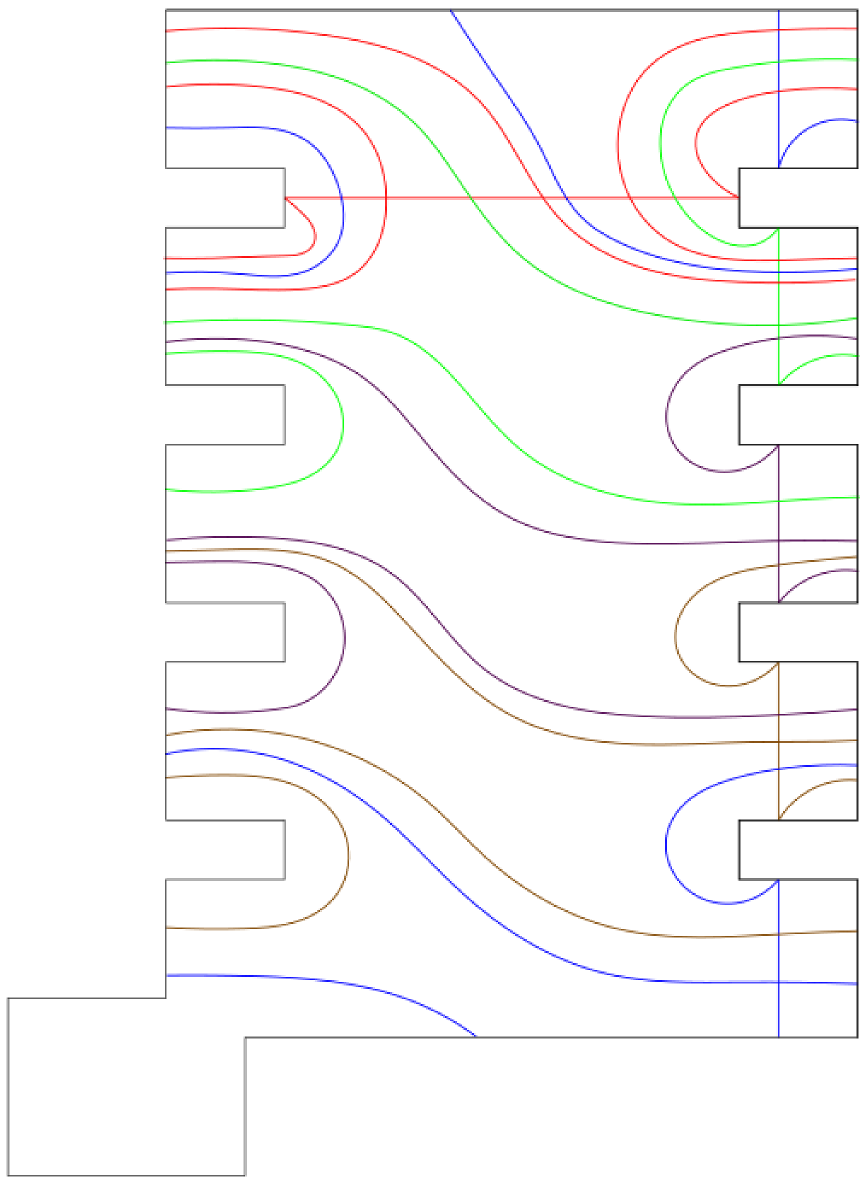

Figure 4.8: An open book for the non-symmetric sum $(\Sigma_{1,2}, \mathrm{Id}) \boxplus (A, \mathrm{Id})$, where the identifications are as before but we also need to identify the boundary of the rectangle at the bottom left of the figure, which we do in a standard way for a torus by identifying top and bottom and left and right. The two extra arcs needed to complete to a basis are omitted for simplicity.

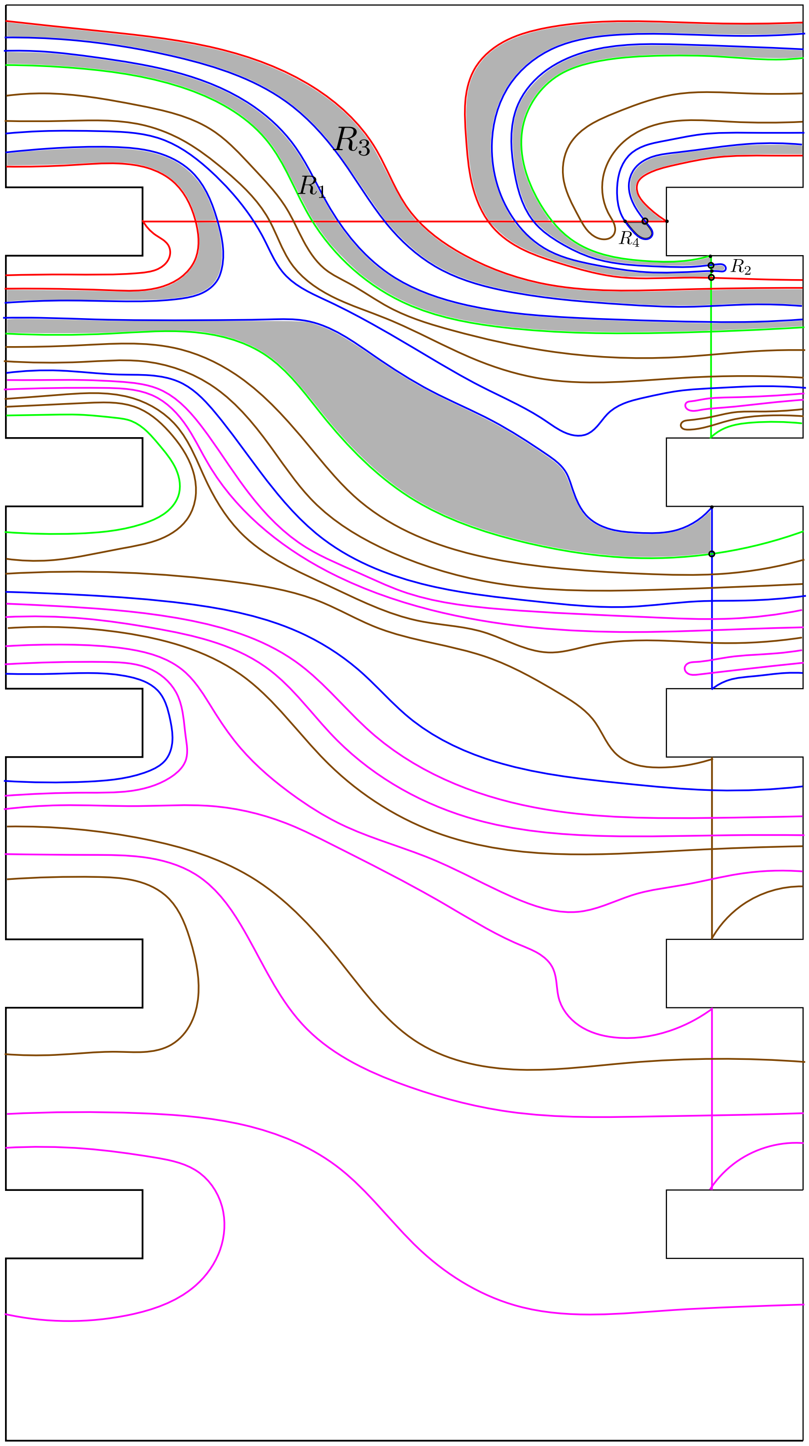


Figure 4.9: Nicefied diagram for the partial open book for the neighbourhood of a Giroux torsion domain, together with the first four domains.

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