Quantum Equivariant Kostka Numbers

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Contents

1 Introduction

This thesis will define quantum equivariant Kostka numbers and then go on to look at some of the different methods that exist to calculate them.

Ordinary Kostka numbers, $K_{\lambda,\mu}$ where λ and μ are partitions, count the number of semi-standard Young tableaux of shape λ and weight μ. Here λ can be a skew partition. See the text books [Ful97] and [Mac95] for details. This means that the tableaux have λ_i boxes in the i^{th} row and μ_j of its entries are j. Ordinary Kostka numbers also give the coefficients in the following sum

$$e_{\mu_1}(x)\cdots e_{\mu_l}(x)s_\nu(x) = \sum_\lambda K_{\lambda/\nu,\mu} s_{\lambda'}(x), \tag{1.1}$$

where λ' is the conjugate partition of λ, the e_r are the elementary symmetric polynomials, and the s_λ are the Schur polynomials in the ring of symmetric polynomials [Mac95], [Ful97]. The Grassmannian is the variety of k-dimensional subspaces in an n-dimensional vector space, V, which we will denote as $Gr(k, n)$ or $Gr_k(V)$. The cohomology ring of the Grassmannian has a presentation as a quotient of the ring of symmetric polynomials, and in this presentation Schur polynomials can be identified with Schubert classes in the cohomology ring of the Grassmannian, [Ful97].

The ring of symmetric polynomials can be generalised to the ring of factorial symmetric polynomials, and Schur polynomials can be generalised to factorial Schur polynomials [Mac92]. The equivariant cohomology ring of the Grassmannian has a presentation as a quotient of the ring of factorial symmetric polynomials due to Mihalcea [Mih08]. The factorial Schur polynomials can be identified with Schubert classes in the T-equivariant cohomology ring of the Grassmannian, where T is a maximal torus of $Gl(n)$ for the Grassmannian $Gr_k(\mathbb{C}^n)$. There is, therefore, an analogous definition for equivariant Kostka numbers which involve these factorial Schur polynomials.

The quantum equivariant cohomology ring of the Grassmannian also has a presentation as a quotient of the ring of factorial symmetric polynomials. This quotient involves an extra parameter q, which we may set to 0 to recover the T-equivariant cohomology ring of the Grassmannian. Once again, the factorial Schur polynomials can be identified with the Schubert classes in the quantum equivariant cohomology ring of the Grassmannian, and we can look at the analogous formula for Kostka numbers. This is how we define the quantum equivariant Kostka numbers.

Several ways already exist in the literature to calculate the quantum equivariant Kostka numbers. For example, Bertiger et al have a method in [BEMT22] involving Young diagrams which can be used to calculate the quantum equivariant Kostka numbers. Gorbounov and Korff have a method in [GK14] which utilises lattice models to calculate the quantum equivariant Kostka numbers. Knutson and Tao defined puzzles in [KT03] which can be used to calculate the ordinary Kostka numbers and the equivariant Kostka numbers. Buch has a method involving different puzzles in [Buc15] which can be used to calculate the quantum equivariant Kostka numbers.

We will explore some of these methods before coming up with a new method of our own. This method is combinatorially based on the method of Bertiger et al [BEMT22] but we have translated it onto lattices and the Yang-Baxter algebra of Gorbounov and Korff [GK14]. Our method uses the row-to-row transfer matrix for lattices (defined in Definition 4.1) and its expansion into factorial powers. This transfer matrix will be denoted by $E(x)$ and the coefficient of $(x|y)^{k-r}$ in its expansion into factorial powers will be denoted by E_r. The main result of this masters thesis is the following formula for calculating E_r acting on a vector $|\mu\rangle$.

Proposition 1.1.

$$E_r|\mu\rangle = \sum_{\lambda} wt_r(\mathcal{D}^{\mu}_{\lambda})|\lambda\rangle, \tag{1.2}$$

where $d = 0, 1$ and the sum is over all partitions λ such that $\lambda/d/\mu$ is a vertical t-strip (defined in Definition 2.41) with $0 \leq t \leq r$.

The weights $wt_r(\mathcal{D}^{\mu}_{\lambda})$ are calculated using statistics which we define on single row lattice diagrams and are easy to compute by examination. This is Proposition 5.23 and can be found in Section 5.3.2. We can use this result to calculate quantum equivariant Kostka numbers by using the formula by Gorbounov and Korff.

Corollary 1.2. *[GK14, Corollary 6.27] Let $E_\alpha = E_{\alpha_1} \cdots E_{\alpha_{n-k}}$. Then we have*

$$\langle\lambda|E_\alpha|\mu\rangle = \sum_{d\geq 0} q^d K_{\lambda'/d/\mu',\alpha}(y). \tag{1.3}$$

We hope that this method simplifies the calculation of the quantum equivariant Kostka numbers.

We now give a brief overview of the structure of this masters thesis.

In Section 2 we give the definition of a factorial Schur polynomial and some relevant results involving them due to Macdonald [Mac92] and Molev and Sagan [MS99]. We then give an isomorphism between a quotient of a ring involving factorial symmetric polynomials and the quantum equivariant cohomology ring of the Grassmannian, $QH^*_T(Gr(k,n))$, which is due to Mihalcea [Mih08]. Finally we examine the method that can be used for calculating quantum equivariant Kostka numbers by Bertiger et al [BEMT22].

In Section 3 we give a solution to the Yang-Baxter equation which we wish to work with, followed by an introduction to vertex models. We also give the vertex model of Gorbounov and Korff [GK14] associated with the solution to the Yang-Baxter equation which we will use in the main result of this masters thesis. We then introduce the Yang-Baxter algebra and some of its representations.

In Section 4 we compute the algebraic Bethe ansatz as Gorbounov and Korff do [GK14]. Then we set up the multiplication operators on a subspace of a YB −module, and give the method for calculating quantum equivariant Kostka numbers by Gorbounov and Korff [GK14], which we will need for our result.

In Section 5 we introduce Knutson-Tao puzzles [KT03] and look at variations of these puzzles which can be used to calculate quantum equivariant Kostka numbers. We then give the main result. Finally we describe a bijection between our lattice diagrams and certain sets of puzzles.

2 Symmetric Polynomials

We assume the reader is familiar with some of the basic combinatorics needed in the context of symmetric polynomials and functions as can be found in, for example, the book by Macdonald [Mac95, Chapter I].

2.1 Factorial Schur Polynomials

In this section we define factorial Schur polynomials as found in the 6^{th} variation in the article by Macdonald [Mac92] and introduce some of their basic properties. We then review some known results utilising these polynomials such as Molev and Sagan's formula for Littlewood-Richardson type coefficients [MS99]. We also describe the specialisation of these polynomials to ordinary Schur polynomials found in, for example, Macdonald's book [Mac95, Chapter I].

Before we can define factorial Schur polynomials we must introduce the following:

Let R be a commutative ring, and let $y = (y_n)_{n\in\mathbb{Z}}$ be any (doubly infinite) sequence of elements of R. For every $r \geq 0$, define $(x|y)^r = (x - y_1)(x - y_2)\cdots(x - y_r)$, called the (generalised) falling factorial power.

For each $r \in \mathbb{Z}$, define $\tau^r y$ to be the shifted sequence $(\tau^r y)_n = y_{n+r}$ (so that the n^{th} term of this sequence is y_{n+r}).

Definition 2.1. If λ is a partition of length $\leq n$ and $\delta_n = (n-1, \ldots, 1, 0)$, then the factorial Schur polynomial, $s_\lambda(x|y)$, is

$$s_\lambda(x|y) = \frac{\det\left((x_i|y)^{(\lambda+\delta_n)_j}\right)_{1\leq i,j\leq n}}{\det\left((x_i|y)^{(\delta_n)_j}\right)_{1\leq i,j\leq n}}. \tag{2.1}$$

It is interesting to note that denominator in Definition 2.1 is $\prod_{i<j}(x_i - x_j)$, the Vandermonde determinant, and is therefore independent of the sequence y. It is shown in the article by Macdonald [Mac92] that these factorial Schur polynomials are symmetric in $x_1, \ldots, x_n$, but not homogeneous.

Remark 2.2. We can recover the definition of Schur polynomials as found in, for example, the book by Macdonald [Mac95, Chapter I] from Definition 2.1 by setting each element of the sequence y to 0. Due to this, there is an isomorphism between $s_\lambda(x)$ in the ring of symmetric polynomials and $s_\lambda(x|y)$ in $R[x]^{S_n}$, which makes the $s_\lambda(x|y)$ a basis for $R[x]^{S_n}$.

It will be useful to introduce additional notation for the following special cases for λ:

$$s_{(1^r)}(x|y) = e_r(x|y) \qquad (0 \leq r \leq n), \tag{2.2}$$

and

$$s_{(r)}(x|y) = h_r(x|y) \qquad (r \geq 0). \tag{2.3}$$

Remark 2.3. We once again note that by setting each element of the sequence y to 0 we recover the elementary and complete symmetric polynomials as found in [Mac95, Chapter I].

Hence we will refer to the polynomials defined in (2.2) as factorial elementary symmetric polynomials, and the polynomials defined in (2.3) as factorial complete symmetric polynomials.

We now want to give analogues of the Jacobi-Trudi and Nägelsbach-Kostka determinant formulae for factorial Schur polynomials. In order to do this we will need the following lemmata.

Lemma 2.4. *Let t be another indeterminate and let $f(t) = \prod_{i=1}^{n}(t - x_i)$, then*

$$f(t) = \sum_{r=0}^{n}(-1)^r e_r(x|y)(t|y)^{n-r}. \tag{2.4}$$

Proof. For this proof only, let $x_0 = t$. We have

$$f(t) = \frac{\prod_{i=1}^{n}(t - x_i)\prod_{1\le i<j\le n}(x_i - x_j)}{\prod_{1\le i<j\le n}(x_i - x_j)} = \prod_{i=1}^{n}(t - x_i) = \frac{\det\left((x_i|y)^{(\delta_{n+1})_{j+1}}\right)_{0\le i,j\le n}}{\det\left((x_i|y)^{(\delta_n)_j}\right)_{1\le i,j\le n}}. \tag{2.5}$$

Then using the Laplace expansion to expand $\det\left((x_i|y)^{(\delta_{n+1})_{j+1}}\right)_{0\le i,j\le n}$ along the top row we find

$$f(t) = \sum_{r=0}^{n}(-1)^r (t|y)^{n-r}\frac{\det\left((x_i|y)^{((1)^r+\delta_n)_j}\right)_{1\le i,j\le n}}{\det\left((x_i|y)^{(\delta_n)_j}\right)_{1\le i,j\le n}} = \sum_{r=0}^{n}(-1)^r e_r(x|y)(t|y)^{n-r}, \tag{2.6}$$

as required. □

Lemma 2.5. *Define:*

$$\mathbf{H}(x|y) = \left(h_{j-i}\left(x|\tau^{i+1}y\right)\right)_{i,j\in\mathbb{Z}}, \tag{2.7}$$

$$\mathbf{E}(x|y) = \left((-1)^{j-i}e_{j-i}\left(x|\tau^{j}y\right)\right)_{i,j\in\mathbb{Z}}. \tag{2.8}$$

Then $\mathbf{E}(x|y) = \mathbf{H}(x|y)^{-1}$.

Proof. First note that $h_0 = e_0 = 1$ and $h_{j-i} = e_{j-i} = 0$ when $i > j$, and therefore both matrices are upper unitriangular.

To show that $\mathbf{EH} = I$, we need to show that

$$\sum_j (-1)^{k-j}e_{k-j}(x|\tau^k y)h_{j-i}(x|\tau^{i+1}y) = \delta_{ik} \quad \forall i,k. \tag{2.9}$$

Now, $e_{k-j} = 0$ when $j > k$ and $h_{j-i} = 0$ when $i > j$, so (2.9) $= 0$ when $i > k$.

When $i = k$ we have only one non-zero term in the sum: $(-1)^0 e_0 h_0 = 1$.

Assume $i < k$. Recall that $f(t) = \prod_{i=1}^{n}(t - x_i)$, so $f(x_i) = 0$. Then by using the second formula we obtained for $f(t)$ we find $\sum_{r=0}^{n}(-1)^r e_r(x|y)(x_i|y)^{n-r} = 0$.

Since y is an arbitrary sequence, in the previous sum we may replace y with $\tau^{s-1}y$ for any s to obtain:

$$\sum_{r=0}^{n}(-1)^r e_r(x|\tau^{s-1}y)(x_i|\tau^{s-1}y)^{n-r} = 0. \tag{2.10}$$

Multiplying by $(x_i|y)^{s-1}$ gives:

$$\sum_{r=0}^{n}(-1)^r e_r(x|\tau^{s-1}y)(x_i|y)^{n-r+s-1} = 0. \tag{2.11}$$

This holds for all $s > 0$ and $1 \le i \le n$.

By expanding the determinant $\det\left((x_i|y)^{((m)+\delta)_j}\right)$ down the first column we see that

$$h_m(x|y) = \sum_{i=1}^{n}(x_i|y)^{m+n-1}u_i(x), \tag{2.12}$$

where $u_i(x)$ is a rational function in $x_1, \dots, x_n$.

Now (2.11) $= 0$ for all i, therefore the sum over i of (2.11) multiplied by $u_i(x)$ is 0. Hence

$$\sum_{r=0}^{n}(-1)^r e_r(x|\tau^{s-1}y)h_{s-r}(x|y) = 0. \tag{2.13}$$

Since we assumed $i < k$, we have $k - i > 0$, so let $s = k - i$. We also replace y with $\tau^{i+1}y$ and get

$$\sum_{r=0}^{n}(-1)^r e_r(x|\tau^{k}y)h_{k-i-r}(x|\tau^{i+1}y) = 0. \tag{2.14}$$

With a suitable substitution this becomes

$$\sum_{i\le j\le k}(-1)^{k-j} e_{k-j}(x|\tau^{k}y)h_{j-i}(x|\tau^{i+1}y) = 0. \tag{2.15}$$

Hence we have shown that (2.9) is true. □

We can now give analogues of the Jacobi-Trudi and Nägelsbach-Kostka determinant formulae for factorial Schur polynomials, taken from the 6[th] variation in [Mac92].

Lemma 2.6. *If λ is a partition of length $\le n$, then*

$$\begin{aligned} s_\lambda(x|y) &= \det\left(h_{\lambda_i-i+j}(x|\tau^{1-j}y)\right) \\ &= \det\left(e_{\lambda'_i-i+j}(x|\tau^{j-1}y)\right). \end{aligned} \tag{2.16}$$

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}$. From the previous result we have

$$h_{\alpha_i-n+j}(x|\tau^{1-j}y) = \sum_{k=1}^{n}(x_k|\tau^{1-j}y)^{\alpha_i+j-1}u_k(x) = \sum_{k=1}^{n}(x_k|y)^{\alpha_i}(x_k|\tau^{1-j}y)^{j-1}u_k(x). \tag{2.17}$$

This shows that $H_\alpha := (h_{\alpha_i-n+j}(x|\tau^{1-j}y))_{i,j}$ is the product of the matrices $((x_k|y)^{\alpha_i})_{i,k}$ and $B := \left((x_k|\tau^{1-j}y)^{j-1}u_k(x)\right)_{k,j}$. Then if we take the determinant we have:

$$\det(H_\alpha) = \det((x_k|y)^{\alpha_i})\det(B). \tag{2.18}$$

By taking $\alpha = \delta$ we have $H_\delta = (h_{j-i}(x|\tau^{1-j}y))_{i,j}$. This matrix has 1s on the main diagonal and 0s everywhere below the main diagonal, hence $\det(H_\delta) = 1$. Thus $\det((x_k|y)^{\delta_i})\det(B) = 1$ and so $\det(B) = \frac{1}{\det((x_k|y)^{\delta_i})}$ which implies that

$$\det(H_\alpha) = \det((x_k|y)^{\alpha_i})\det(B) = \frac{\det((x_k|y)^{\alpha_i})}{\det((x_k|y)^{\delta_i})}. \tag{2.19}$$

Taking $\alpha = \lambda + \delta$ we find that $\det(H_{\lambda+\delta}) = s_\lambda(x|y)$ and hence we have the first formula. Lemma 2.5 implies the second formula. □

Remark 2.7. By once again setting each element of the sequence y to 0 we obtain the Jacobi-Trudi and Nägelsbach-Kostka determinant formulae for ordinary Schur polynomials, as given in [Mac95, Chapter I].

There is also a combinatorial definition of the factorial Schur polynomials in terms of Young tableaux, which we will now give. This definition is contained in the article by Molev and Sagan, [MS99, Equation 4].

Proposition 2.8. *The following is an analogous definition for factorial Schur polynomials.*

$$s_\lambda(x|y) = \sum_T \prod_{\alpha\in\lambda} \left(x_{T(\alpha)} - y_{T(\alpha)+c(\alpha)}\right), \tag{2.20}$$

where the sum is over all semistandard tableaux, T, of shape λ with entries in $\{1,\ldots,n\}$, $T(\alpha)$ is the entry of T in cell $\alpha \in \lambda$, and $c(\alpha) = j - i$ is the content of the cell $\alpha = (i,j)$.

Remark 2.9. By setting all elements of the sequence y to 0 we obtain the tableaux definition of Schur polynomials from [Ful97]. This definition can also be extended to include skew factorial Schur polynomials, which is how the definition is given in variation 6 in [Mac92].

The article by Molev and Sagan [MS99] contains two results which we use later on in this thesis. We now state those results.

Theorem 2.10. *[MS99, Theorem 2.1] Let $y_\rho = (y_{\rho_1+n},\ldots,y_{\rho_n+1})$. Given partitions λ,ρ with $l(\lambda), l(\rho) \le n$*

$$s_\lambda(y_\rho|y) = \begin{cases} 0 & \text{if } \lambda \nsubseteq \rho, \\ \prod_{(i,j)\in\lambda}(y_{\lambda_i+n-i+1} - y_{n-\lambda'_j+j}) & \text{if } \lambda = \rho. \end{cases} \tag{2.21}$$

Proof. Assuming that $\lambda \nsubseteq \rho$ and examining the entries of $\left(((y_\rho)_j|y)^{(\lambda+\delta)_i}\right)_{1\le i,j\le n}$ shows that

$$\det\left(((y_\rho)_j|y)^{(\lambda+\delta)_i}\right)_{1\le i,j\le n} = 0, \tag{2.22}$$

as there is a rectangle of 0's in the lower left corner of the matrix with a corner of the rectangle on the diagonal. This proves the first part. To prove the second part, take $\lambda = \rho$. Then the ij^{th} entry of $\left(((y_\lambda)_j|y)^{(\lambda+\delta)_i}\right)_{1\le i,j\le n}$ is $(y_{\lambda_j+n-j+1} - y_1)\cdots(y_{\lambda_j+n-j+1} - y_{\lambda_i+n-i})$, which is 0 for $i < j$ and nonzero for $i = j$. Therefore $\left(((y_\lambda)_j|y)^{(\lambda+\delta)_i}\right)_{1\le i,j\le n}$ is lower triangular with a nonzero diagonal. Calculating $s_\lambda(y_\lambda|y)$ then gives the result. $\square$

Definition 2.11. Define Littlewood-Richardson type coefficients, $c^\lambda_{\mu,\nu}(y)$, by the formula

$$s_\mu(x|y)s_\nu(x|y) = \sum_\lambda c^\lambda_{\mu,\nu}(y)s_\lambda(x|y), \tag{2.23}$$

where μ, ν are partitions.

In later chapters we will devise a combinatorial method that can be used to calculate special cases of these coefficients, but we should first examine the existing formula given in [MS99]. For this we will need the following sequence of partitions $\rho^{(i-1)} \subset \rho^{(i)}$:

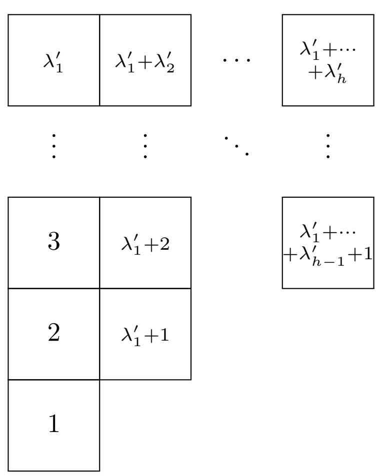
$$P : \nu = \rho^{(0)} \to \rho^{(1)} \to \cdots \to \rho^{(l-1)} \to \rho^{(l)} = \lambda, \tag{2.24}$$

where $|\rho^{(i)}/\rho^{(i-1)}| = 1$. Let r_i be the row number of $\rho^{(i)}/\rho^{(i-1)}$.

Remark 2.12. P is a standard Young tableau. It will later be more convenient for us to use this definition, however.

We will also need the set $\mathcal{T}(\mu, P)$ of semistandard tableaux, T, of shape μ with entries from $\{1, \ldots, n\}$ such that T contains cells $\alpha_1, \ldots, \alpha_l$ with $\alpha_1 < \cdots < \alpha_l$ and $T(\alpha_i) = r_i$ for $1 \le i \le l$, where $<$ is column order (explained below). We mark the entries in cells $\alpha_1, \ldots, \alpha_l$ by barring them. Let $\mathcal{T}(\mu, \lambda/\nu) = \uplus_P \mathcal{T}(\mu, P)$.

Remark 2.13. Here we explain what is meant by column order. Let λ be a partition with $\lambda_1 = h$ and let β_i be the cell in the Young diagram of λ which is labelled by i in the diagram below. The largest cell in the column order is the one labelled by 1, β_1, and the smallest is the one labelled by $\lambda'_1 + \cdots + \lambda'_h$, $\beta_{\lambda'_1+\cdots+\lambda'_h}$. The cells decrease as we go up the column, and the lowermost cell in each column is smaller than the topmost cell in the column immediately to the left.



where the ordering of the cells is $\beta_1 > \cdots > \beta_{\lambda'_1} > \beta_{\lambda'_1+1} > \cdots > \beta_{\lambda'_1+\lambda'_2} > \cdots > \beta_{\lambda'_1+\cdots+\lambda'_h}$.

Example 2.14. Let $n = 2$, $\lambda = (3,3)$, $\nu = (2,2)$, and $\mu = (2,1)$. Then

$$P : \nu = (2,2) \to (3,2) \to (3,3) = \lambda. \tag{2.25}$$

The first box is added to the first row, so $r_1 = 1$, and the second box is added to the second row, so $r_2 = 2$. Therefore $r_1 r_2 = 12$. The ways to bar entries 1 and 2 in the tableaux of μ such that the cell containing 1 is greater than the cell containing 2 in the ordering described by the above remark are given below.

$$\mathcal{T}(\mu, P) = \left\{ \begin{array}{|c|c|}\hline \bar{1} & 1 \\ \hline \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \bar{1} & 2 \\ \hline \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \right\}.$$

For each cell α such that $\alpha_i < \alpha < \alpha_{i+1}$ for $1 \leq i \leq l-1$ we set $\rho(\alpha) = \rho^{(i)}$, for $\alpha < \alpha_1$ we set $\rho(\alpha) = \rho^{(0)}$, and for $\alpha_l < \alpha$ we set $\rho(\alpha) = \rho^{(l)}$.

Example 2.15. We use $\mathcal{T}(\mu, P)$ from Example 2.14 to demonstrate how to assign $\rho(\alpha)$ to a cell α in a Young tableau. For this example α_1 contains the entry $\bar{1}$ and α_2 contains the entry $\bar{2}$. For the first tableau we have $\alpha < \alpha_1$, for the second tableau we have $\alpha_1 < \alpha < \alpha_2$, and for the third tableau we have $\alpha < \alpha_1$ again. Therefore, labelling the cells in the tableaux from Example 2.14 we find that

$$\begin{array}{ccc} \begin{array}{|c|c|}\hline \alpha_1 & \alpha \\ \hline \alpha_2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline \alpha & \alpha_1 \\ \hline \alpha_2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline \alpha_1 & \alpha \\ \hline \alpha_2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \\ \rho(\alpha) = \rho^{(0)}, & \rho(\alpha) = \rho^{(1)}, & \rho(\alpha) = \rho^{(0)}. \end{array}$$

In this example we have $y_{\rho^{(0)}} = y_{(2,2)} = (y_4, y_3)$ and $y_{\rho^{(1)}} = y_{(3,2)} = (y_5, y_3)$.

Theorem 2.16. *[MS99, Theorem 3.1] The coefficient $c^\lambda_{\mu,\nu}(y)$ is 0 unless $\nu \subseteq \lambda$. If $\nu \subseteq \lambda$ then*

$$c^\lambda_{\mu,\nu}(y) = \sum_{T \in \mathcal{T}(\mu, \lambda/\nu)} \prod_{\substack{\alpha \in \nu \\ T(\alpha) \text{ unbarred}}} \left((y_{\rho(\alpha)})_{T(\alpha)} - y_{T(\alpha)+c(\alpha)} \right). \tag{2.26}$$

Sketch of Proof. That the coefficient $c^\lambda_{\mu,\nu}(y)$ is 0 unless $\nu \subseteq \lambda$ is proved using Theorem 2.10. The remainder is proven by Molev and Sagan by a chain of propositions and can be found in [MS99]. They first prove a formula for $\frac{s_\nu(y_\lambda|y)}{s_\lambda(y_\lambda|y)}$ involving a sum of fractions. They then prove a formula for $c^\lambda_{\mu\nu}(y)$ involving similar sums. Next they prove a recurrence relation for the $c^\lambda_{\mu\nu}(y)$. They then give definitions for sums of products similar to the formula in this theorem and give another formula for $c^\lambda_{\mu\nu}(y)$ involving them. To prove this formula they give two more propositions involving various sums of these new sums of products that they just defined. $\square$

Example 2.17. We calculate $c^{\lambda}_{\mu,\nu}(y)$ using Examples 2.14 and 2.15. There is only one term in the product from each tableau as there is only one unbarred cell in each. Therefore we have

$$\begin{aligned} c^{(3,3)}_{(2,1),(2,2)}(y) &= \left(\left(y_{(2,2)}\right)_1 - y_{1+1}\right) + \left(\left(y_{(3,2)}\right)_1 - y_{1+0}\right) + \left(\left(y_{(2,2)}\right)_2 - y_{2+1}\right) \\ &= (y_4 - y_2) + (y_5 - y_1) + (y_3 - y_3). \end{aligned} \tag{2.27}$$

The special case of these coefficients which interests us is an analogue to the Pieri rules for the non-factorial Schur polynomials. We now state this special case as a corollary of Theorem 2.16.

Corollary 2.18. *The coefficients* $c^{\lambda}_{(1^t),\nu}(y)$ *and* $c^{\lambda}_{(t),\nu}(y)$ *for* $t \in \mathbb{N}$ *are given by*

$$c^{\lambda}_{(1^t),\nu}(y) = \sum_{T \in \mathcal{T}((1^t),P)} \prod_{\substack{\alpha \in \nu \\ T(\alpha) \text{ unbarred}}} \left((y_{\rho(\alpha)})_{T(\alpha)} - y_{T(\alpha)+c(\alpha)}\right), \tag{2.28}$$

and

$$c^{\lambda}_{(t),\nu}(y) = \sum_{T \in \mathcal{T}((t),P)} \prod_{\substack{\alpha \in \nu \\ T(\alpha) \text{ unbarred}}} \left((y_{\rho(\alpha)})_{T(\alpha)} - y_{T(\alpha)+c(\alpha)}\right), \tag{2.29}$$

where we note that in each case there is only one possible sequence P for which $\mathcal{T}(\mu, P) \neq \emptyset$. For $c^{\lambda}_{(1^t),\nu}(y)$ this sequence has $r_1 < \cdots < r_l$ and for $c^{\lambda}_{(t),\nu}(y)$ this sequence has $r_1 \geq \cdots \geq r_l$.

Lemma 2.19. *We can simplify the definition of $\rho(\alpha)$ slightly in each case. For $c^{\lambda}_{(1^t),\nu}(y)$ we have $\rho(\alpha) = \rho^{(i)}$ for $r_i < T(\alpha) < r_{i+1}$ and for $c^{\lambda}_{(t),\nu}(y)$ we have $\rho(\alpha) = \rho^{(i)}$ for $r_i \geq T(\alpha) \geq r_{i+1}$ where in both cases $0 \leq i \leq l$ ignoring the half of the inequality involving r_0 and r_{l+1}.*

Proof. For $c^{\lambda}_{(1^t),\nu}(y)$ the Young diagram of $\mu = (1^t)$ is a vertical t-strip. Therefore, since we are using semistandard tableaux which have entries which increase strictly down the columns, to have $r_1, \ldots, r_l$ such that $\alpha_l > \cdots > \alpha_1$ we must have $r_1 < \cdots < r_l$, as notes in Corollary 2.18. Hence for any cell α with an unbarred entry in the tableau to be such that $\alpha_i < \alpha < \alpha_{i+1}$ for some i, we must also have $r_i < T(\alpha) < r_{i+1}$, due to the strictly increasing entries down the single column of T. Thus we must have $\rho(\alpha) = \rho^{(i)}$ for $r_i < T(\alpha) < r_{i+1}$.

For $c^{\lambda}_{(t),\nu}(y)$ the Young diagram of $\mu = (t)$ is a horizontal t-strip. Therefore, since we are using semistandard tableaux which have entries which increase weakly from left to right across the rows, to have $r_1, \ldots, r_l$ such that $\alpha_l > \cdots > \alpha_1$ we must have $r_1 \geq \cdots \geq r_l$, as notes in Corollary 2.18. Hence for any cell α with an unbarred entry in the tableau to be such that $\alpha_i < \alpha < \alpha_{i+1}$ for some i, we must also have $r_i \geq T(\alpha) \geq r_{i+1}$, due to the weakly increasing entries across the single row of T. Thus we must have $\rho(\alpha) = \rho^{(i)}$ for $r_i \geq T(\alpha) \geq r_{i+1}$. $\square$

2.2 Quantum Cohomology as a Frobenius Algebra

We now consider a quotient of a ring involving factorial symmetric polynomials and show that it is a Frobenius algebra. We then state an isomorphism between this ring and the quantum equivariant cohomology ring of the Grassmannian due to Mihalcea [Mih08]. We will be working with the

Grassmannian $Gr(k,n)$, the variety of k-dimensional subspaces in an n-dimensional ambient space. We do this because many of the objects we are interested in within this thesis have a natural interpretation in the quantum equivariant cohomology ring of the Grassmannian as a Frobenius algebra. We first define Frobenius algebras.

Definition 2.20. A Frobenius algebra is a finite-dimensional, unital, associative algebra A defined over a field f and equipped with a nondegenerate bilinear form $\langle\cdot|\cdot\rangle : A \otimes A \to f$ such that $\langle ab|c\rangle = \langle a|bc\rangle$.

We now define the algebra in which we are interested in terms of factorial symmetric polynomials.

Definition 2.21. Let $k \leq n$ be positive integers. Let $\Lambda = \mathbb{Z}[y_1, \ldots, y_n]$. Then we define an algebra

$$\mathfrak{J} = \Lambda[q][\tilde{e}_1, \ldots, \tilde{e}_k]/\langle \tilde{h}_{n-k+1}, \ldots, \tilde{h}_{n-1}, \tilde{h}_n + (-1)^k q\rangle, \tag{2.30}$$

where $\tilde{e}_r = e_r(x_1, \ldots, x_k|y)$ and $\tilde{h}_r = \det\left(e_{1+j-i}(x|\tau^{j-1}y)\right)_{1\leq i,j\leq r}$.

Lemma 2.22. *Let $(k, n-k)$ denote the set of partitions whose Young diagrams are contained within a bounding box of height k and width $n-k$. Then the factorial Schur polynomials $s_\lambda(x|y)$ with $\lambda \in (k, n-k)$ are a $\Lambda[q]$-basis for $\mathfrak{J}$.*

Proof. This was proved by Mihalcea during the proof of Theorem 4.2 in [Mih08]. □

Before we can show that $\mathfrak{J}$ is a Frobenius algebra we must first introduce some results by Gorbounov and Korff [GK14] involving specialisations of factorial Schur polynomials.

Let $z = (z_1, \ldots, z_k)$ be a set of indeterminates which satisfy the following equations

$$\prod_{j=1}^{n}(z_i - y_j) + (-1)^{n-k}q = 0, \qquad i = 1, \ldots, n-k. \tag{2.31}$$

Remark 2.23. We will often set $q = 1$ through the remainder of this thesis. We can do this because it is possible to recover the equations involving q via a rescaling of the z and y indeterminates, provided $q^{\pm\frac{1}{n}}$ exist. Setting $q = 1$, the equation (2.31) becomes

$$\prod_{j=1}^{n}(z_i - y_j) + (-1)^{n-k} = 0, \tag{2.32}$$

and if we then take $q^{-1/n}z_i$ instead of z_i and $q^{-1/n}y_i$ instead of y_i we recover equation (2.31).

Remark 2.24. In order to find all solutions to (2.31), we must consider $z \in \mathbb{F}_q = \mathbb{C}[q^{\pm 1/n}]\hat{\otimes}\mathbb{F}$, where $\mathbb{F} := \mathbb{C}\{\{y_1, \ldots, y_n\}\}$ is the algebraically closed field of Puiseux series, which is the field of all formal power series allowing for negative and fractional powers.

Remark 2.25. There are $\binom{n}{k}$ k-tuples of solutions, $z \in \mathbb{F}_q$, to (2.31). This is the same as the number of partitions α contained within the bounding box of height k and width $n-k$. It will be useful to parametrise these k-tuples, z, of solutions to (2.31) in terms of partitions $\alpha \in (k, n-k)$, as

Gorbounov and Korff do in [GK14, Section 5.3]. The solutions to (2.31) when $q = 0$ are explicitly known; z can be any k-tuple of $y_1, \ldots, y_n$. So we fix a numbering of the n solutions to (2.31) by mapping them onto the known solutions when $q = 0$. Let $z_j = z_j(q)$ be the solution which has $z_j(0) = y_j$ for $j = 1, \ldots, n$. Then we denote by z_α the k-tuple whose i^{th} entry is $z_{\alpha_{k+1-i}+i}$. For any $\alpha \in (k, n-k)$ the $\alpha_{k+1-i} + i$ will be unique for $i = 1, \ldots k$, and we will have $1 \leq \alpha_{k+1-i} + i \leq n$. We can therefore take an indexing set $I(\alpha) \subset \{1, \ldots, n\}$ for each z_α, and hence each indexing set has a complement, $I(\alpha^*)$, in $\{1, \ldots, n\}$.

Let $\lambda^\vee = (n - k - \lambda_k, \ldots, n - k - \lambda_1)$ and $\lambda^* = (\lambda^\vee)'$.

Proposition 2.26. *It can be shown that $I(\alpha^*)$ is indeed the complement of $I(\alpha)$ in $\{1, \ldots, n\}$.*

Proof. First note that $\alpha^* \in (n-k, k)$. The number of parts of α^* which are equal to 0 is α_k. For $i = 1, \ldots, k$, the number of parts of α^* which are equal to i is $\alpha_{k-i} - \alpha_{k-i+1}$. Therefore we can express the set $\{1, \ldots, n\}$ as

$$\begin{aligned}\Big\{&\alpha^*_{n-k} + 1, \ldots, \alpha^*_{n-k+1-\alpha_k} + \alpha_k, \alpha_k + 1, \alpha^*_{n-k+1-(\alpha_k+1)} + \alpha_k + 1, \ldots \\ &\qquad \ldots, \alpha^*_{n-k+1-\alpha_{k-1}} + \alpha_{k-1}, \alpha_{k-1} + 2, \ldots \text{etc}\Big\}.\end{aligned} \tag{2.33}$$

$\square$

The first result from [GK14] which we need is the following orthogonality identity for factorial Schur polynomials evaluated at the z_α.

Proposition 2.27. *[GK14, Corollary 5.12] Let $Y_i := y_{n-i+1}$. For all $\lambda, \mu \in (k, n-k)$ we have the identity*

$$\sum_{\alpha \in (k,n-k)} \frac{s_{\lambda^\vee}(z_\alpha|Y) s_\mu(z_\alpha|y)}{\mathfrak{e}(z_\alpha)} = \delta_{\lambda\mu}, \tag{2.34}$$

where $\mathfrak{e}(z_\alpha) = \prod_{\substack{i \in I(\alpha) \\ j \in I(\alpha^*)}} (z_i - z_j)$ with $I(\alpha)$ being the indexing set of z_α and $I(\alpha^*)$ its complement.

Proposition 2.28. *For all $\alpha, \beta \in (k, n-k)$, we have*

$$\sum_{\lambda \in (k,n-k)} \frac{s_\lambda(z_\alpha|Y) s_{\lambda^\vee}(z_\beta|y)}{\mathfrak{e}(z_\alpha)} = \delta_{\alpha\beta}. \tag{2.35}$$

We prove the above propositions in a later chapter of this thesis.

Corollary 2.29. *[GK14, Corollary 6.6] Define coefficients $C^{\lambda,d}_{\mu\nu}(y)$ by*

$$C^{\lambda,d}_{\mu\nu}(y) = \sum_{\alpha \in (k,n-k)} \frac{s_\mu(z_\alpha|y) s_\nu(z_\alpha|y) s_{\lambda^\vee}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}. \tag{2.36}$$

Then

$$s_\mu(z_\alpha|y) s_\nu(z_\alpha|y) = \sum_{\lambda \in (k,n-k)} C^{\lambda,d}_{\mu\nu}(y) s_\lambda(z_\alpha|y). \tag{2.37}$$

Proof. Multiplying both sides of (2.37) by $\frac{s_{\eta^\vee}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}$ and taking the sum over α gives

$$\sum_{\alpha\in(k,n-k)} \frac{s_\mu(z_\alpha|y)s_\nu(z_\alpha|y)s_{\eta^\vee}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)} = \sum_{\lambda\in(k,n-k)} C^{\lambda,d}_{\mu\nu}(y) \sum_{\alpha\in(k,n-k)} \frac{s_\lambda(z_\alpha|y)s_{\eta^\vee}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}. \tag{2.38}$$

Then applying Proposition 2.27 gives the result. □

Remark 2.30. Equation (2.36) is known as the Bertram-Vafa-Intriligator formula [Ber97], [Vaf92], [Int91].

Let $\mathrm{Func}_{k,n}(\mathbb{F}_q)$ be the set of functions $f : (k, n-k) \to \mathbb{F}_q$. $\mathbb{F}_q$ has a natural algebra structure using pointwise addition and multiplication.

Proposition 2.31 (Mihalcea)**.** *The map $\iota : \mathfrak{J}_q \hookrightarrow Func_{k,n}(\mathbb{F}_q)$ defined via $s_\lambda(x|y) \mapsto \tilde{s}_\lambda$ where $\tilde{s}_\lambda : \alpha \mapsto s_\lambda(z_\alpha|y)$ is an injective algebra homomorphism.*

In what follows we will often identify $\mathfrak{J}_q$ with its image under ι.

We can now prove that $\mathfrak{J} \otimes \mathbb{F}_q$ is a Frobenius algebra.

Proposition 2.32. *[GK14, Prop 6.22] $\mathfrak{J}_q := \mathfrak{J} \otimes \mathbb{F}_q$ with bilinear form*

$$\langle s_\lambda|s_\mu\rangle = \sum_{\alpha\in(k,n-k)} \frac{s_\lambda(z_\alpha|Y)s_\mu(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}, \tag{2.39}$$

is a commutative Frobenius algebra.

Proof. The bilinear form is non-degenerate since the s_λ are a basis for $\mathfrak{J}$, and are therefore linearly independent. So we just need to show that $\langle s_\lambda|s_\mu s_\nu\rangle = \langle s_\lambda s_\mu|s_\nu\rangle$. We have

$$\begin{aligned} \langle s_\lambda|s_\mu s_\nu\rangle &= \sum_{\rho\in(k,n-k)} C^{\rho,d}_{\mu\nu}(Y) \sum_{\alpha\in(k,n-k)} \frac{s_\lambda(z_\alpha|Y)s_\rho(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)} \\ &= \sum_{\alpha,\beta,\rho\in(k,n-k)} \frac{s_\mu(z_\beta|Y)s_\nu(z_\beta|Y)s_{\rho^\vee}(z_\beta|y)s_\lambda(z_\alpha|Y)s_\rho(z_\alpha|Y)}{\mathfrak{e}(z_\beta)\mathfrak{e}(z_\alpha)}. \end{aligned} \tag{2.40}$$

By applying Proposition 2.28 to the last line above we eliminate the sum involving ρ and get a factor of $\delta_{\alpha\beta}$. Hence

$$\langle s_\lambda|s_\mu s_\nu\rangle = \sum_{\alpha\in(k,n-k)} \frac{s_\mu(z_\alpha|Y)s_\nu(z_\alpha|Y)s_\lambda(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}. \tag{2.41}$$

We can deduce that $\langle s_\lambda|s_\mu s_\nu\rangle = \langle s_\lambda s_\mu|s_\nu\rangle$ by noticing that the line above is invariant under permutations of λ, μ, and ν. □

We can also show what happens to s_λ under the Frobenius coproduct.

Lemma 2.33. *[GK14, Lemma 6.23] The image of s_λ under the Frobenius coproduct $\Delta : \mathfrak{J}_q \to \mathfrak{J}_q \otimes \mathfrak{J}_q$ is*

$$\Delta s_\lambda = \sum_{\substack{\mu \in (k,n-k) \\ d \geq 0}} q^d s_{\lambda/d/\mu} \otimes s_\mu, \tag{2.42}$$

where $s_{\lambda/d/\mu}$ is a generalised skew factorial symmetric polynomial given by

$$s_{\lambda/d/\mu}(x|y) = \sum_{\nu \in (k,n-k)} C^{\lambda^\vee, d}_{\mu^\vee \nu^\vee}(y) s_\nu(x|y). \tag{2.43}$$

Proof. Let $\Phi : \mathfrak{J}_q \to \mathfrak{J}_q^*$ denote the Frobenius isomorphism given by $s_\lambda \mapsto \langle s_\lambda | \cdot \rangle$ and $m : \mathfrak{J}_q \otimes \mathfrak{J}_q \to \mathfrak{J}_q$ the multiplication map. Since $\mathfrak{J}_q$ is a Frobenius algebra, the following diagram must commute

$$\begin{tikzcd} \mathfrak{J}_q \arrow[r, "\Delta"] \arrow[d, "\Phi"'] & \mathfrak{J}_q \otimes \mathfrak{J}_q \arrow[d, "\Phi\otimes\Phi"] \\ \mathfrak{J}_q^* \arrow[r, "m^*"] & \mathfrak{J}_q^* \otimes \mathfrak{J}_q^* \end{tikzcd},$$

which means that $m^* \circ \Phi(s_\lambda)(s_\mu \otimes s_\nu) = [\Phi \otimes \Phi(\Delta(s_\lambda))](s_\mu \otimes s_\nu)$. This means that

$$\sum_{\alpha \in (k,n-k)} \frac{s_\mu(z_\alpha|Y) s_\nu(z_\alpha|Y) s_\lambda(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)} = \sum_{\substack{\rho \in (k,n-k) \\ d \geq 0}} q^d \langle s_{\lambda/d/\rho} | s_\mu \rangle \langle s_\rho | s_\nu \rangle. \tag{2.44}$$

Plugging in the definition given in the statement of the lemma for $s_{\lambda/d/\rho}$ into the equation above and taking $q = 1$ shows that it holds. $\square$

Now that we have shown that $\mathfrak{J}_q$ is a Frobenius algebra, we state the isomorphism between $\mathfrak{J}_q$ and $QH_T^*(Gr(k,n))$, the quantum equivariant cohomology ring of the Grassmannian. This isomorphism will be important for future chapters, and is the reason we look at $\mathfrak{J}_q$.

Corollary 2.34 (Mihalcea)**.**

$$QH_T^*(Gr(k,n)) \cong \Lambda[q][\tilde{e}_1, \ldots, \tilde{e}_k]/\langle \tilde{h}_{n-k+1}, \ldots, \tilde{h}_{n-1}, \tilde{h}_n + (-1)^k q \rangle, \tag{2.45}$$

where $\tilde{e}_r = \det\left(h_{1+j-i}(x|\tau^{1-j}y)\right)_{1 \leq i,j \leq r}$ *and* $\tilde{h}_r = \det\left(e_{1+j-i}(x|\tau^{j-1}y)\right)_{1 \leq i,j \leq r}$.

Corollary 2.35. *There is an analogous result for factorial Schur polynomials to the automorphism which maps s_λ to $s_{\lambda'}$ in the ring of symmetric polynomials. Using that result we find that*

$$QH_T^*(Gr(k,n)) \cong \Lambda[q][\tilde{h}_1, \ldots, \tilde{h}_{n-k}]/\langle \tilde{e}_{k+1}, \ldots, \tilde{e}_{n-1}, \tilde{e}_n + (-1)^{n-k} q \rangle. \tag{2.46}$$

Remark 2.36. It is shown by Mihalcea [Mih08] that the basis of factorial Schur polynomials for the Jacobi algebra corresponds to the basis of Schubert classes for $QH_T^*(Gr(k,n))$. Gorbounov and Korff [GK14] use this result by Mihalcea to show that $q^d C^{\lambda,d}_{\mu\nu}(y)$ as defined above are the equivariant Gromov-Witten invariants.

Finally, we define the quantum equivariant Kostka numbers. These will be central to this thesis. We first define

Definition 2.37. We define the equivariant Kostka numbers, $K_{\lambda/d/\mu,\alpha}(y)$, to be 0 unless $nd = |\lambda| - |\mu| - |\alpha|$, in which case

$$h_\alpha(x|y)s_\mu(x|y) = \sum_{\lambda \in (k,n-k)} q^d K_{\lambda/d/\mu,\alpha}(y) s_\lambda(x|y), \tag{2.47}$$

or equivalently

$$e_\alpha(x|y)s_\mu(x|y) = \sum_{\lambda \in (k,n-k)} q^d K_{\lambda'/d/\mu',\alpha}(y) s_{\lambda'}(x|y), \tag{2.48}$$

where $h_\alpha = h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_m}$ and $e_\alpha = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_m}$.

Remark 2.38. In the cases where $d = 0$, by setting the sequence y to 0 we recover the definition of the ordinary Kostka numbers.

2.3 Combinatorial Description of Quantum Equivariant Kostka Numbers

There are several ways of calculating quantum equivariant Kostka numbers already in the literature. We look at the one given by Gorbounov and Korff in [GK14] in more detail in subsequent chapters. Here we give an overview of the combinatorial description of the quantum equivariant Pieri rule on cylindric shapes given by Bertiger et al in [BEMT22], which will be useful for later work. All of the following section comes from [BEMT22].

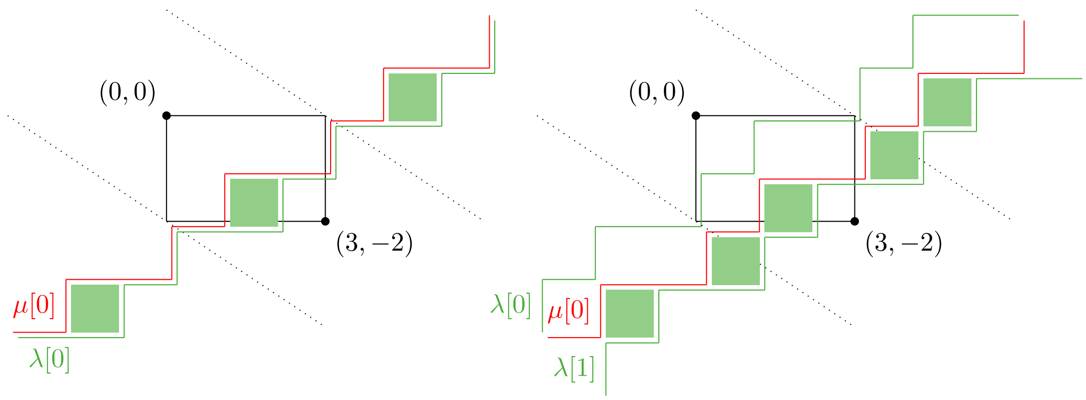
We first describe what is meant by a cylindric shape. These were introduced by Gessel and Krattenthaler [GK97].

Definition 2.39. Let $\mathcal{C}_{kn} := \mathbb{Z}^2/(-k, n-k)\mathbb{Z}$ be the cylinder, $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition whose Young diagram is contained within a bounding box of height k and width $n-k$. In $\mathbb{Z}^2$ we use coordinates (i, j) which follow the convention of matrix coordinates. Define a doubly infinite integer sequence $\lambda[0] = (\dots, l_{-1}, l_0, l_1, \dots)$ by $\lambda[0]_j = \lambda_j$ for $1 \le j \le k$ and $l_{i+k} = l_i - (n-k)$. By plotting (j, l_j) in $\mathbb{Z}^2$ for $j \in \mathbb{Z}$ we obtain a closed loop on the cylinder which outlines the Young diagram of λ inside the k by $n-k$ bounding box with northwest corner at the origin. Finally, define the cylindric loop $\lambda[d]$ by $\lambda[d]_j = l_{j-d} + d$ for $j \in \mathbb{Z}$. Plotting this gives the outline of the Young diagram of λ shifted southeast by d steps.

Definition 2.40. Let $\lambda, \mu \in (k, n-k)$ and $0 \le d \in \mathbb{Z}$ such that $\lambda[d]_i \ge \mu_i$ for $1 \le i \le m$. Then the cylindric diagram $\lambda/d/\mu$ is given by the boxes of $\mathcal{C}_{kn}$ which are located between $\lambda[d]$ and $\mu[0]$.

Definition 2.41. A vertical r-strip is a (skew) Young diagram containing exactly r boxes, no 2 of which are in the same row.

Example 2.42. This example, which is very similar to Figure 2.1 in [BEMT22], demonstrates the definitions above.



On the left we have $\mu = (3, 1)$ and $\lambda = (3, 2)$, with the red line tracing the cylindric loop $\mu[0]$ and the green line tracing $\lambda[0]$, and $\lambda/0/\mu$ indicated by the green boxes. On the right we have $\mu = (2, 1)$ and $\lambda = (1, 0)$, with the red line tracing the cylindric loop $\mu[0]$ and the green lines tracing $\lambda[0]$ and $\lambda[1]$ respectively, and $\lambda/1/\mu$ indicated by the green boxes.

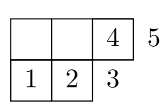
Bertiger et al [BEMT22] use statistics defined on cylindric diagrams in their Pieri rule. We now give their statistics and preliminary definitions. For brevity we omit the case for horizontal strips.

Definition 2.43. [BEMT22, Definition 2.6] Let $\lambda, \mu \in (k, n-k)$ and $0 \leq d \in \mathbb{Z}$. Let $\lambda/d/\mu$ be a vertical r-strip, v^r for some $0 \leq r \leq k$. Fix $r \leq p \leq k$. We say v^r is extendable to a vertical p-strip if we can make a vertical p-strip v^p by adding $p - r$ boxes from μ to $\lambda/d/\mu$, each of which shares a vertical edge with the boundary of $\lambda/d/\mu$. These are called addable boxes, and the extension of v^r to v^p is denoted $v^r \to v^p$.

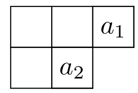
Definition 2.44. [BEMT22, Definition 2.8, Definition 2.9] Given a partition μ within the usual bounding box, number all of the edges in the path tracing the boundary of μ starting in the lower left corner of the k by $n-k$ box. Define $U(\mu)$ to be the up-steps of μ; the set of numbers which index vertical edges. Now consider $\lambda/d/\mu = v^r$, an extension $v^r \to v^p$, and an addable box $a \in v^p/v^r$.

- Define the up-step of a, $u(a)$, to be the index of the vertical edge of the box a from $U(\mu)$.
- Define the row number of a, $r(a)$, to be the row of μ which contains a, where we number the rows starting from the bottom of the k by $n - k$ box.
- Define the number of boxes below a, $b(a)$, to be the number of boxes in v^p which are in rows strictly below a.

Example 2.45. We calculate the above statistics for $\lambda = (3, 2)$, $\mu = (3, 2)$, $d = 0$, $n = 5$, $k = 2$, and $p = 2$. We can see that λ/μ is a vertical 0-strip. Here is the diagram of μ with the edges labelled. The labels are above the horizontal edges and to the right of the vertical edges.



Then we have $U(\mu) = \{3, 5\}$. The addable boxes, a_1 and a_2 are shown below.



The statistics for a_1 and a_2 are as follows.

By examining the above Young diagrams we can see the upstep of each box is

$$u(a_1) = 5, \qquad u(a_2) = 3. \tag{2.49}$$

There are 2 rows in our bounding box. The box a_2 is in the first row from the bottom, and a_1 is on the second, therefore the row number of each box is

$$r(a_1) = 2, \qquad r(a_2) = 1. \tag{2.50}$$

There are 2 boxes in the extension $v^0 \to v^2$, just a_1 and a_2. Therefore there are no boxes below a_2 and there is one box below a_1, namely a_2, and so

$$b(a_1) = 1, \qquad b(a_2) = 0. \tag{2.51}$$

We now use these statistics to define the weight of an addable box.

Definition 2.46. [BEMT22, Definition 2.10] Let $\lambda, \mu \in (k, n-k)$ and $0 \leq d \in \mathbb{Z}$. Suppose $\lambda/d/\mu = v^r$. The weight of an addable box $a \in v^p/v^r$ is

$$wt(a) := y_{u(a)} - y_{r(a)-b(a)}. \tag{2.52}$$

Example 2.47. We calculate the weight for each addable box in Example 2.45

$$wt(a_1) = y_{u(a_1)} - y_{r(a_1)-b(a_1)} = y_5 - y_{2-1} = y_5 - y_1, \tag{2.53}$$
$$wt(a_2) = y_{u(a_2)} - y_{r(a_2)-b(a_2)} = y_3 - y_{1-0} = y_3 - y_1. \tag{2.54}$$

We now have all of the definitions necessary to state the quantum equivariant Pieri rule by Bertiger et al.

Theorem 2.48. *[BEMT22, Theorem 1.1] For any integer $1 \leq p \leq k$ and any partition μ inside the usual bounding box we have*

$$s_{(1)^p} * s_\mu = \sum_{0 \leq r \leq p} \sum_{\substack{\lambda/d/\mu = v^r \\ v^r \to v^p}} q^d \prod_{a \in v^p/v^r} wt(a) s_\lambda, \tag{2.55}$$

where s_λ is the Schubert class indexed by λ in $QH^*_T(Gr(k,n))$.

Remark 2.49. In the theorem above is it worth noting that $s_{(1)^p}$ is $\tilde{e}_p$ in Definition 2.21.

Example 2.50. Using Examples 2.45 and 2.47 we find the coefficient of $s_{(3,2)}$ in $s_{(1)^2} * s_{(3,2)}$ is $(y_5 - y_1)(y_3 - y_1)$ by applying Theorem 2.48.

Remark 2.51. The equivariant parameters used here have a factor of -1 compared to the ones used later on in this thesis.

Remark 2.52. This quantum equivariant Pieri rule is manifestly positive, by which we mean that the coefficients of s_λ in Theorem 2.48 are polynomials in $(y_i - y_{i-1})$ with coefficients in $\mathbb{Z}_{\geq 0}$.

3 Yang-Baxter Algebras

3.1 Lattice Models

We want to look at Gorbounov's and Korff's [GK14] method of calculating the quantum equivariant Kostka numbers using solutions to the quantum Yang-Baxter equation and lattice models that are associated with them. We first give a description of the solution to the Quantum Yang-Baxter equation that we will be working with and then give an introduction to vertex models. At the end of this section we explain the Yang-Baxter equation in terms of lattice paths.

3.1.1 Solutions of the Quantum Yang-Baxter Equation

Let V be a vector space isomorphic to $\mathbb{C}^2$ and let $V(x)$ be $\mathbb{C}(x) \otimes V$, in which we will denote elements by $f(x)v$ with $f(x) \in \mathbb{C}(x)$ and $v \in V$. Let $R(x, y) : V(x) \otimes V(y) \to V(x) \otimes V(y)$, we can also say that $R(x, y)$ is a rational function in x, y with values in $\mathrm{End}(V \otimes V)$. We will use $\mathbb{1}$ to denote the identity matrix.

If $R = \sum_{i,j} A_i \otimes B_j$ for $A_i, B_j \in \mathrm{End}(V)$, then we introduce the notation:

$$R_{12} = \sum_{i,j} A_i \otimes B_j \otimes \mathbb{1} \qquad R_{13} = \sum_{i,j} A_i \otimes \mathbb{1} \otimes B_j \qquad R_{23} = \sum_{i,j} \mathbb{1} \otimes A_i \otimes B_j.$$

The R-matrix we will be looking at will only depend on the difference $x-y$; so $R(x, y) = R(x-y)$, a rational function in $x - y$. Due to this, we only use a single argument in the R-matrix in the following definition.

Definition 3.1. The Quantum Yang-Baxter Equation is:

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{3.1}$$

Let $\{E_{ij}\}_{i,j}$ be the standard basis for $n \times n$ matrices. It follows that $\{E_{ij} \otimes E_{kl}\}_{0 \le i,j,k,l \le n-1}$ is a basis for the space of $n^2 \times n^2$ matrices. In the proof of the following lemma, and indeed usually throughout the rest of these notes, we set $n = 2$ so that the E_{ij} are 2×2 matrices.

Definition 3.2. Let $R(u) = \sum_{g,h=0,1} E_{gh} \otimes E_{hg} + uE_{00} \otimes E_{11}$.

Lemma 3.3. *$R(u)$ satisfies the QYBE.*

Proof. The proof is a simple calculation.

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = \left(\sum_{k,l=0,1} E_{kl}\otimes E_{lk}\otimes \mathbb{1} + (u-v)E_{00}\otimes E_{11}\otimes \mathbb{1}\right)\cdot$$
$$\cdot\left(\sum_{g,h=0,1} E_{gh}\otimes \mathbb{1}\otimes E_{hg} + uE_{00}\otimes \mathbb{1}\otimes E_{11}\right)\left(\sum_{i,j=0,1} \mathbb{1}\otimes E_{ij}\otimes E_{ji} + v\mathbb{1}\otimes E_{00}\otimes E_{11}\right) =$$
$$= \sum_{g,h,k=0,1} E_{kh}\otimes E_{gg}\otimes E_{hk} + v\sum_{h=0,1} E_{0h}\otimes E_{10}\otimes E_{h1} + u\sum_{i=0,1} E_{i0}\otimes E_{01}\otimes E_{1i} +$$
$$+ uvE_{00}\otimes E_{00}\otimes E_{11} + (u-v)\sum_{h=0,1} E_{0h}\otimes E_{10}\otimes E_{h1} + (u-v)uE_{00}\otimes E_{11}\otimes E_{11}. \quad (3.2)$$

Also

$$R_{23}(v)R_{13}(u)R_{12}(u-v) = \left(\sum_{i,j=0,1} \mathbb{1}\otimes E_{ij}\otimes E_{ji} + v\mathbb{1}\otimes E_{00}\otimes E_{11}\right)\cdot$$
$$\cdot\left(\sum_{g,h=0,1} E_{gh}\otimes \mathbb{1}\otimes E_{hg} + uE_{00}\otimes \mathbb{1}\otimes E_{11}\right)\left(\sum_{k,l=0,1} E_{kl}\otimes E_{lk}\otimes \mathbb{1} + (u-v)E_{00}\otimes E_{11}\otimes \mathbb{1}\right) =$$
$$= \sum_{g,h,l=0,1} E_{gl}\otimes E_{hh}\otimes E_{lg} + (u-v)\sum_{g=0,1} E_{g0}\otimes E_{01}\otimes E_{1g} + u\sum_{l=0,1} E_{0l}\otimes E_{10}\otimes E_{l1} +$$
$$+ (u-v)uE_{00}\otimes E_{11}\otimes E_{11} + v\sum_{g=0,1} E_{g0}\otimes E_{01}\otimes E_{1g} + uvE_{00}\otimes E_{00}\otimes E_{11}. \quad (3.3)$$

We can rearrange (3.2) to get (3.3), hence we have the result. □

If we notice that $V(u)\otimes V(v)\cong \mathbb{C}^4(u,v)$, we can write $R(u)$ as a 4×4 matrix as follows (using the Kronecker product).

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & u & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \quad (3.4)$$

3.1.2 Introduction to Vertex Models

Here we give an introduction to lattice models. These will provide a graphical calculus which will help us to calculate the quantum equivariant Kostka numbers.

The following definitions are taken from the Classical and Quantum Integrable Systems SMSTC course 2020. [FOSK+20].

Let G be an $n \times k$ square lattice. Denote by $Ed(G)$ the set of its edges and denote by $V(G)$ the set of its vertices $v = \langle i,j \rangle$ with i labelling the rows and j labelling the columns. Given a vertex $v = \langle i,j \rangle$ let $N_{\langle i,j \rangle}, E_{\langle i,j \rangle}, S_{\langle i,j \rangle}, W_{\langle i,j \rangle}$ denote its incident north, east, south, and west edges respectively. We are careful to ensure the east vertex always contains the $\langle \rangle$ brackets in the subscript, so as not to be confused with the basis matrices. A lattice configuration is a map $\gamma : Ed(G) \to \{0,1\}$. Let Γ denote the set of all lattice configurations.

We want to assign a weight to each possible lattice configuration.

Let $\{e_0, e_1\}$ be the standard basis for V, then $\{e_g \otimes e_h\}_{g,h=0,1}$ is a basis for $V \otimes V$. We can think of the vertex $\langle i,j \rangle$ in our lattice as corresponding to $R(x_i, y_j)$ acting on the tensor product $V(x_i) \otimes V(y_j)$ as follows.

Definition 3.4. Let $\gamma\left(W_{\langle i,j \rangle}\right) = \sigma$, $\gamma\left(N_{\langle i,j \rangle}\right) = \sigma'$, $\gamma\left(E_{\langle i,j \rangle}\right) = \tau$, and $\gamma\left(S_{\langle i,j \rangle}\right) = \tau'$ where $\sigma, \sigma', \tau, \tau' \in \{0,1\}$. The weight of a vertex is defined via

$$R(x_i, y_j) e_\sigma \otimes e_{\sigma'} = \sum_{\tau, \tau' = 0,1} \mathrm{wt}\left(\begin{array}{ccc} & \sigma' & \\ \sigma & + & \tau \\ & \tau' & \end{array}\right) e_\tau \otimes e_{\tau'}. \tag{3.5}$$

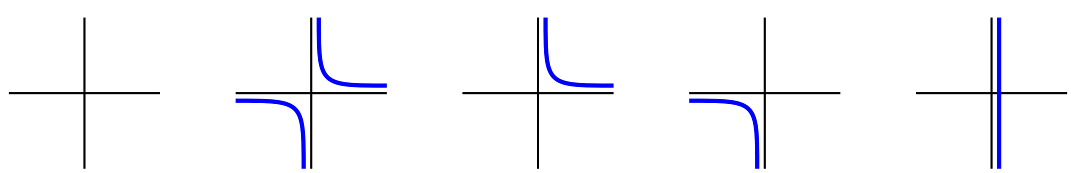
To each lattice configuration we can now assign a weight via a function $\mathrm{wt} : \Gamma \to \mathbb{C}[x_1, \ldots, x_k, y_1, \ldots, y_n]$. This maps γ to $\mathrm{wt}(\gamma) = \prod_{\langle i,j \rangle \in V(G)} \mathrm{wt}\left(\begin{array}{ccc} & N_{\langle i,j \rangle} & \\ W_{\langle i,j \rangle} & + & E_{\langle i,j \rangle} \\ & S_{\langle i,j \rangle} & \end{array}\right)$. The weight of the vertex is an element of $\mathbb{C}[x_i, y_j]$. Since $\mathrm{wt}(\gamma)$ factorises into the weights associated to each of the vertices, we can see why it makes sense to call these vertex models.

Definition 3.5. The partition function is the weighted sum over all lattice configurations:

$$Z(x,y) = \sum_{\gamma \in \Gamma} \mathrm{wt}(\gamma). \tag{3.6}$$

In the following model we use blue lines to indicate where an edge has the value 1, and the absence of a blue line to indicate where an edge has the value 0.

We can deduce that the vertex model for $R(x_i - y_j)$ is as follows (using $u = x_i - y_j$ in Definition 3.2 and $E_{ab} e_c = \delta_{cb} e_a$):



with weights: 1   1   1   1 $x_i - y_j$, All other possible vertex configurations have weight 0. These weights are called Boltzmann weights in statistical mechanics. This lattice model and its R-matrix are equivalent to the ones used by Gorbounov and Korff in [GK14] for their osculating walkers. Their later article, [GK17], uses the same weights and R-matrix as we do here. We choose this version because it requires less rearranging to express the partition function in terms of factorial Schur polynomials.

We will demonstrate in later sections that the partition function for this vertex model can be used to find the quantum equivariant Kostka numbers, as Gorbounov and Korff do in [GK14].

3.2 The Yang-Baxter Algebra

3.2.1 What is the Yang-Baxter Algebra?

We now define the Yang-Baxter Algebra for a general R-matrix.

Definition 3.6. The Yang-Baxter algebra, $\mathrm{YB}(R)$, is the unital associative algebra with generators $\{t_{ij}^{(k)}\}_{i,j=0,1}$, $k \in \mathbb{N}$, and relations given by

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v), \tag{3.7}$$

with the T matrices defined below. This is called the RTT-relation.

Let $R(u,v) : V(u)\otimes V(v) \to V(u)\otimes V(v)$ with $V \cong \mathbb{C}^2$ satisfy the QYBE, and let $T(u) = \sum_{i,j} E_{ij} \otimes t_{ij}(u)$ be a matrix with non-commutative entries $t_{ij}(u) = \sum_{k\geq 0} t_{ij}^{(k)} u^k \in \mathrm{YB}[[u]] = \mathrm{YB} \otimes \mathbb{C}[[u]]$. We need $T_1(u) = \sum_{i,j=0,1} E_{ij} \otimes \mathbb{1} \otimes t_{ij}(u)$ and $T_2(v) = \sum_{k,l=0,1} \mathbb{1} \otimes E_{kl} \otimes t_{kl}(v)$. As the specific R-matrix defined previously which we need to use later on only depends on the difference of the spectral parameters, we will only use one argument in the R-matrix in the general definition of the Yang-Baxter algebra.

Remark 3.7. $\mathrm{YB}(R)$ is well-defined due to the QYBE. To check this we need to check associativity; that $(T_1(u)T_2(v))T_3(w) = T_1(u)(T_2(v)T_3(w))$. This is the same as saying that it doesn't matter which 2 T-matrices we exchange first using (3.7). We will need to use that $R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v)$ since R satisfies (3.1). Then, noting that $R_{ij}T_k = T_kR_{ij}$, where ijk is some permutation of 123, since they do not act on the same tensor components, and repeatedly applying (3.7) we have

$$\begin{aligned} &R_{12}(u-v)R_{13}(u-w)R_{23}(v-w)T_1(u)T_2(v)T_3(w) \\ &= R_{12}(u-v)R_{13}(u-w)T_1(u)T_3(w)T_2(v)R_{23}(v-w) \\ &= R_{12}(u-v)T_3(w)T_1(u)T_2(v)R_{13}(u-w)R_{23}(v-w) \\ &= T_3(w)T_2(v)T_1(u)R_{12}(u-v)R_{13}(u-w)R_{23}(v-w). \end{aligned} \tag{3.8}$$

Similarly we have

$$\begin{aligned} &R_{23}(v-w)R_{13}(u-w)R_{12}(u-v)T_1(u)T_2(v)T_3(w) \\ &= R_{23}(v-w)R_{13}(u-w)T_2(v)T_1(u)T_3(w)R_{12}(u-v) \\ &= R_{23}(v-w)T_2(v)T_3(w)T_1(u)R_{13}(u-w)R_{12}(u-v) \\ &= T_3(w)T_2(v)T_1(u)R_{23}(v-w)R_{13}(u-w)R_{12}(u-v). \end{aligned} \tag{3.9}$$

Then using using $R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v)$ since R satisfies (3.1) we find that (3.8) = (3.9).

Proposition 3.8. *The triple* $(\mathrm{YB}(R), \Delta, \varepsilon)$*, is a well-defined bialgebra with the following algebra homomorphisms:*

$$\textit{coproduct } \Delta : \mathrm{YB}(R) \to \mathrm{YB}(R) \otimes \mathrm{YB}(R), \ \textit{given by } t_{ij}(u) \mapsto \sum_{k=0,1} t_{kj}(u) \otimes t_{ik}(u),$$

$$\textit{counit } \varepsilon : \mathrm{YB}(R) \to \mathbb{C}, \ \textit{given by } t_{ij}(u) \mapsto \delta_{i,j}.$$

Proof. We know that $\mathrm{YB}(R)$ is a unital associative algebra by definition, so we now just need to show that Δ and ε are algebra homomorphisms and that the coalgebra axioms hold for $(\mathrm{YB}(R), \Delta, \varepsilon)$. We denote by id the identity map.

We first show the coalgebra axioms:

$$(\Delta \otimes \mathrm{id})\, \Delta t_{ij}(u) = (\Delta \otimes \mathrm{id}) \sum_{k=0,1} t_{kj}(u) \otimes t_{ik}(u) = \sum_{k,l=0,1} t_{lj}(u) \otimes t_{kl}(u) \otimes t_{ik}(u),$$

$$(\mathrm{id} \otimes \Delta)\, \Delta t_{ij}(u) = (\mathrm{id} \otimes \Delta) \sum_{k=0,1} t_{kj}(u) \otimes t_{ik}(u) = \sum_{k,l=0,1} t_{kj}(u) \otimes t_{lk}(u) \otimes t_{il}(u).$$

Therefore $(\Delta \otimes \mathrm{id})\, \Delta t_{ij}(u) = (\mathrm{id} \otimes \Delta)\, \Delta t_{ij}(u)$ as required.

$$(\mathrm{id} \otimes \varepsilon)\, \Delta t_{ij}(u) = \sum_{k=0,1} t_{kj}(u) \otimes \delta_{i,k} = t_{ij}(u) \otimes 1 \quad \text{and} \quad (\varepsilon \otimes \mathrm{id})\, \Delta t_{ij}(u) = \sum_{k=0,1} \delta_{k,j} \otimes t_{ik}(u) = 1 \otimes t_{ij}(u).$$

Thus the coalgebra axioms hold. More information on coalgebras can be found in [CP95, Chapter 4].

We can see that ε is an algebra homomorphism since $\varepsilon(T(u))$ is the identity matrix, which is the trivial solution of the RTT relation, (3.7).

So we are left to show that Δ is an algebra homomorphism which we will do by showing that $\Delta T(u)$ is a solution of (3.7).

Applying the coproduct, (3.7) becomes

$$R_{12}(u-v)T_{13}(u)T_{14}(u)T_{23}(v)T_{24}(v) = T_{23}(v)T_{24}(v)T_{13}(u)T_{14}(u)R_{12}(u-v), \tag{3.10}$$

where $T_{13}(u) = \sum_{i,j,k=0,1} E_{ij} \otimes \mathbb{1} \otimes t_{kj}(u) \otimes 1$, $T_{14}(u) = \sum_{i,j,k=0,1} E_{ij} \otimes \mathbb{1} \otimes 1 \otimes t_{ik}(u)$, $T_{23}(v) = \sum_{i,j,k=0,1} \mathbb{1} \otimes E_{ji} \otimes t_{kj}(v) \otimes 1$, and $T_{24}(v) = \sum_{i,j,k=0,1} \mathbb{1} \otimes E_{ji} \otimes 1 \otimes t_{ik}(v)$. We need to show that (3.10) does indeed hold.

From these definitions, we can see that $T_{14}(u)T_{23}(v) = T_{23}(v)T_{14}(u)$ and that $T_{13}(u)T_{24}(v) = T_{24}(v)T_{13}(u)$. Hence we have

$$R_{12}(u-v)T_{13}(u)T_{14}(u)T_{23}(v)T_{24}(v) = R_{12}(u-v)T_{13}(u)T_{23}(v)T_{14}(u)T_{24}(v).$$

We can now apply (3.7) to $R_{12}(u-v)T_{13}(u)T_{23}(v)$ to obtain

$$R_{12}(u-v)T_{13}(u)T_{14}(u)T_{23}(v)T_{24}(v) = T_{23}(v)T_{13}(u)R_{12}(u-v)T_{14}(u)T_{24}(v),$$

and then by applying (3.7) to $R_{12}(u-v)T_{14}(u)T_{24}(v)$ we end with

$$R_{12}(u-v)T_{13}(u)T_{14}(u)T_{23}(v)T_{24}(v) = T_{23}(v)T_{13}(u)T_{24}(v)T_{14}(u)R_{12}(u-v).$$

Then using that $T_{13}(u)T_{24}(v) = T_{24}(v)T_{13}(u)$ we obtain (3.10).

Thus Δ is an algebra homomorphism. This completes the proof. $\square$

3.2.2 Representations of the Yang-Baxter Algebra

We shall now give some representations of the Yang-Baxter algebra.

Lemma 3.9. *We express the R-matrix defined in Definition 3.2 as $R(u) = \sum_{i,j=0,1} E_{ij} \otimes r_{ij}(u)$ for ease. The map $\mathrm{YB} \otimes \mathbb{C}[[u]] \to \mathbb{C}[[u]] \otimes End(V(y))$ which sends $t_{ij}(u)$ to $r_{ij}(u-y)$, y a formal variable, is an algebra homomorphism. If we restrict this homomorphism to the coefficients, $t_{ij}^{(k)}$, we get a representation of* YB *from* $\mathrm{YB} \to End(V(y))$.

Proof. We can see that this maps the matrix $T(u)$ to the matrix $R(u-y)$. This means that (3.7) under this map becomes

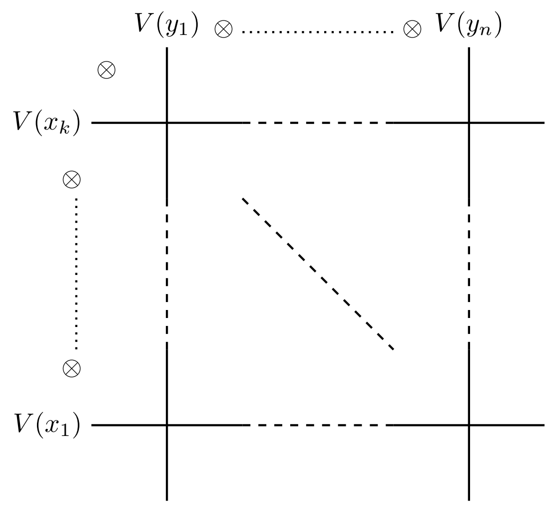
$$R_{12}(u-v)R_{13}(u-y)R_{23}(v-y) = R_{23}(v-y)R_{13}(u-y)R_{12}(u-v). \tag{3.11}$$

This is the QYBE which we know to hold for $R(u)$. Hence this map is an algebra homomorphism. $\square$

Under the representation in Lemma 3.9 it can be seen that all but finitely many of the $t_{ij}^{(k)}$ are mapped to 0, and the rest are mapped to elements of $\mathrm{End}(V(y))$. Therefore we have our first well defined YB $-$module: $V(y)$. We call $V(y)$ the evaluation module.

We can use this representation and the coproduct to find another, which will be more useful with the lattice model that we have introduced earlier.

Some notation first: if $R = \sum_{i,j} A_i \otimes B_j \in \mathrm{End}(V(x_i) \otimes V(y_j))$, then let $R_{g,k+h} = \sum_{i,j} \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes A_i \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes B_j \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \in \mathrm{End}(V(x_1) \otimes \cdots \otimes V(x_k) \otimes V(y_1) \otimes \cdots \otimes V(y_n))$, where A_i is in component g of the tensor and B_j is in component $k+h$. We do this because to each lattice row and each lattice column we wish to assign an evaluation module. The ones for the rows are labelled $V(x_1), \ldots, V(x_k)$ and the ones for the columns are labelled $V(y_1), \ldots, V(y_n)$ as shown below.



We use k rows and n columns because we want to describe the Grassmannian of k-planes in an n-dimensional ambient space.

Now we can give the second representation.

Lemma 3.10. *The map which takes $T(x_i)$ to $R_{i,k+n}(x_i - y_n) \cdots R_{i,k+1}(x_i - y_1) \in End(V(x_i) \otimes V(y_1) \otimes \cdots \otimes V(y_n))$ gives a representation of the Yang-Baxter algebra upon the same restriction as used for the previous representation. This means that $V(y_1) \otimes \cdots \otimes V(y_n)$ is also a* YB*-module.*

Proof. To show that $V(y_1) \otimes \cdots \otimes V(y_n)$ is a YB-module, all we need to do is show that the tensor product of 2 YB-modules is also a YB-module. Then we can apply this result repeatedly to complete the proof.

So, consider the evaluation module $V(y)$ and let M be another YB-module. We consider $V(y) \otimes M$. We know under the representation in Lemma 3.9 that $t_{ij}(u)$ is mapped to $r_{ij}(u-y) \in \mathrm{End}(\mathbb{C}[[u]] \otimes V(y))$. This map takes $T(u)$ to $R(u-y)$. Let $m_{ij}(u) \in \mathrm{End}(\mathbb{C}[[u]] \otimes M)$ be what $t_{ij}(u)$ is mapped to under the representation which gives rise to the YB-module M. Let $m(u) = \sum_{i,j=0,1} E_{ij} \otimes m_{ij}(u)$. This map takes $T(u)$ to $m(u)$.

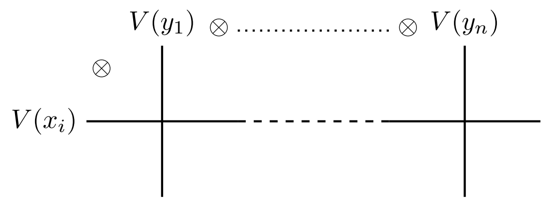
Define a map $\mathrm{YB} \otimes \mathbb{C}[[u]] \to \mathbb{C}[[u]] \otimes \mathrm{End}(V(y) \otimes M)$ by $t_{ij}(u) \mapsto \sum_{a=0,1} r_{aj}(u-y) \otimes m_{ia}(u)$. This map takes $T(u)$ to $R(u-y) \otimes m(u)$. Since we already know that $R(u-y)$ and $m(u)$ satisfy (3.7) because $V(y)$ and M are YB-modules, we can deduce that $R(u-y) \otimes m(u)$ also satisfies (3.7) via a similar calculation to the one used to prove that $\Delta T(u) = T(u) \otimes T(u)$ satisfies (3.7). Therefore $V(y) \otimes M$ is another YB-module.

Noting that the map defined in this proof is the same as the one defined in the statement of the lemma, we can apply this result repeatedly to complete the proof. $\square$

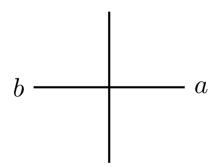
3.3 The RTT Relation on the Square Lattice

So how do these representations relate to the lattice models we introduced previously?

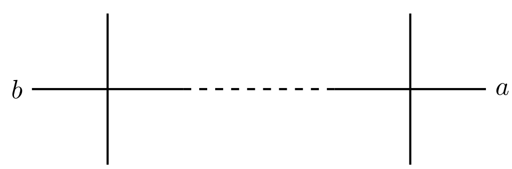
We can think of row i in our $n \times k$ lattice as being $R_{i,k+n}(x_i - y_n) \cdots R_{i,k+1}(x_i - y_1)$ acting on $V(x_1) \otimes \cdots \otimes V(x_k) \otimes V(y_1) \otimes \cdots \otimes V(y_n)$. Here is a diagram of row i of the $n \times k$ lattice to help with visualisation, where the rows which are omitted correspond to the $\mathbb{1}$'s in the tensor product.



Under the representation in Lemma 3.9 we can think of $t_{ab}(x)$ as being a map from the north edge of a vertex to the south edge such that the east edge has value b and the west edge has value a:



Under the representation in Lemma 3.10 we can think of $t_{ab}(x)$ as being a map from the set of north edges to the set of south edges in a row of our lattice such that the east edge has value b and the west edge has value a:



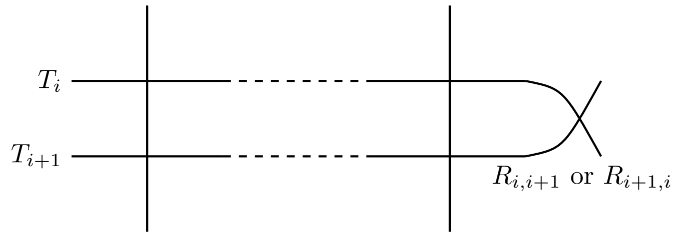
In this case $T(x)$ is known as a row monodromy matrix, and the RTT-relation can be thought of way to braid the lattice rows. The RTT-relation for rows i and $i+1$, for $1 \leq i \leq k-1$, is

$$R_{i,i+1}(x_i - x_{i+1})T_i(x_i)T_{i+1}(x_{i+1}) = T_{i+1}(x_{i+1})T_i(x_i)R_{i,i+1}(x_i - x_{i+1}), \tag{3.12}$$

or

$$R_{i+1,i}(x_{i+1} - x_i)T_{i+1}(x_{i+1})T_i(x_i) = T_i(x_i)T_{i+1}(x_{i+1})R_{i+1,i}(x_{i+1} - x_i). \tag{3.13}$$

$R_{i,i+1}$ and $R_{i+1,i}$ swap rows i and $i+1$, so T_i and T_{i+1} must swap places when $R_{i,i+1}$ or $R_{i+1,i}$ is on the other side of them:



Whether we use (3.12) or (3.13) depends on whether we work up or down the lattice rows. It makes more sense with the graphical calculus to work down, i.e. applying T_i first then T_{i+1} and then swapping the rows. In this case the above diagram depicts the LHS of (3.13). For the RHS, we would put $R_{i+1,i}$ on the left of the rows, and then working down we would have T_{i+1} then T_i (since the rows have swapped places).

We now have a nice graphical calculus for working out matrix elements of $T(x)$ applied to any basis vector of our YB-modules.

Throughout the rest of this thesis we will use the following expression for T:

$$T(x) = E_{00} \otimes A(x) + E_{01} \otimes B(x) + E_{10} \otimes C(x) + E_{11} \otimes D(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}. \tag{3.14}$$

From now on we assume that $T(x) = R_{i,k+n}(x_i - y_n) \cdots R_{i,k+1}(x_i - y_1)$, so that $A(x)$, $B(x)$, $C(x)$, and $D(x)$ are operators acting on $f(y_1, \ldots, y_n) v_1 \otimes \cdots \otimes v_n \in V(y_1) \otimes \cdots \otimes V(y_n)$.

Proposition 3.11. *For $R(u)$ the relation (3.7) is equivalent to the following commutation relations for $A(x)$, $B(x)$, $C(x)$, and $D(x)$.*

$$\begin{aligned}
&A(u)A(v) = A(v)A(u), && B(u)B(v) = B(v)B(u),\\
&C(u)C(v) = C(v)C(u), && D(u)D(v) = D(v)D(u),\\
&A(u)C(v) = A(v)C(u), && B(u)C(v) = B(v)C(u),\\
&D(u)C(v) = D(v)C(u), && B(u)A(v) = B(v)A(u),\\
&B(u)D(v) = B(v)D(u), && D(v)A(u) = (u-v)B(u)C(v) + D(u)A(v),\\
&D(v)B(u) = (u-v)B(u)D(v) + D(u)B(v), && C(v)A(u) = (u-v)A(u)C(v) + C(u)A(v),\\
&A(u)B(v) = (u-v)B(v)A(u) + A(v)B(u), && A(u)D(v) = (u-v)B(v)C(u) + A(v)D(u),\\
&C(u)D(v) = (u-v)D(v)C(u) + C(v)D(u), && C(u)B(v) - C(v)B(u) = (u-v)\left(D(v)A(u) - A(u)D(v)\right).
\end{aligned}$$

The proof is a simple calculation and can be done by expanding both sides of the RTT-relation and comparing terms in the resulting matrix.

4 Bethe Vectors and Quantum Cohomology

In this section we define the row-to-row transfer matrix for lattices with quasi-periodic boundary conditions and then find its eigenvectors via the Bethe ansatz, as Gorbounov and Korff do in [GK14]. Then we look at a subspace, $\mathcal{V}_k$, of the YB-module $V(y_1) \otimes \cdots \otimes V(y_n)$ and describe a ring structure on it in order to see that it is isomorphic to the quantum equivariant cohomology ring of the Grassmannian, $QH^*_T(Gr(k,n))$. This ring structure is also due to Gorbounov and Korff [GK14].

4.1 The Bethe Ansatz

We now define the row-to-row transfer matrix using the operators $A(x)$ and $D(x)$ from (3.14).

Definition 4.1. The row-to-row transfer matrix is defined to be:

$$E(x) = A(x) + qD(x). \tag{4.1}$$

We will need to express $E(x)$ in terms of factorial powers shortly, so we define the operator $E_r \in \mathrm{End}(V(y_1) \otimes \cdots \otimes V(y_n))$ via

$$E(x) = \sum_{r=0}^{k} (x|y)^{k-r} E_r. \tag{4.2}$$

We will need the following Bethe vectors and Bethe ansatz equations. We then show how they were obtained via a method of Faddeev [Fad96], and found in [GK14].

Lemma 4.2. *Let $\alpha \in (k, n-k)$. We use α to index the Bethe vectors similarly to what is described in Remark 2.25. The eigenvectors of $E(x)$ are*

$$\Omega = e_1 \otimes \cdots \otimes e_1 \quad \textit{(with } n \textit{ tensor components)}, \tag{4.3}$$

and

$$|z_\alpha\rangle = C\left(z_{\alpha^*_{n-k}+1}\right) \cdots C\left(z_{\alpha^*_{n-k+1-i}+i}\right) \cdots C\left(z_{\alpha^*_{n-k+1-(n-k)}+n-k}\right)\Omega, \tag{4.4}$$

subject to the Bethe ansatz equations

$$(z_{\alpha^*_{n-k+1-i}+i}|y)^n + (-1)^{n-k} q = 0, \quad \textit{for } i = 1, \ldots, n-k. \tag{4.5}$$

It will be convenient to introduce the notation z_α for $\left(z_{\alpha^*_{n-k}+1}, \ldots, z_{\alpha^*_{n-k+1-(n-k)}+n-k}\right)$, where $(z_\alpha)_i = z_{\alpha^*_{n-k+1-i}+i}$.

We will consider the cases for $q = 0$ and $q \neq 0$ separately. First we give the proof for $q = 0$.

Proof. For $q = 0$. First we recall from the previous section that, for $\Omega \in V(y_1) \otimes \cdots \otimes V(y_n)$, when we write $T(x)\Omega$ we mean $\begin{pmatrix} A(x)\Omega & B(x)\Omega \\ C(x)\Omega & D(x)\Omega \end{pmatrix}$. We will introduce some notation to make clear which component of the tensor product $R_{i,k+j}$ is acting on. Using the R-matrix defined in Definition 3.2, we have $R_{i,k+j}(x_i - y_j) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x_i - y_j \end{pmatrix}_j & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_j & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_j \end{pmatrix}$, where the subscript j indicates that the matrix acts on component j. Note that if we are considering R as a 2×2 matrix with 2×2 matrices as entries, then when we multiply $R_{i,k+g}$ and $R_{i,k+h}$ for $g \neq h$ as matrices, the products of their entries $(\ )_g(\ )_h$ is not matrix multiplication as they do not act on the same tensor position.

We will need the following commutation relations from Proposition 3.11.

$$C(u)C(v) = C(v)C(u), \tag{4.6}$$

$$A(u)C(v) = \frac{1}{u-v}\left(C(v)A(u) - C(u)A(v)\right). \tag{4.7}$$

We will look for $\Omega \in V(y_1) \otimes \cdots \otimes V(y_n)$ such that $B(x)\Omega = 0$. As $T(x) = R_{i,k+n}(x - y_n) \cdots R_{i,k+1}(x - y_1)$, we will find such an Ω by looking for a $v_j \in V(y_j)$ such that $R_{i,k+j}(x - y_j)v_j$ is lower triangular, and then taking the tensor product of v_j with itself n times. We have that

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j \begin{pmatrix} 0 \\ 1 \end{pmatrix}_j = 0, \tag{4.8}$$

therefore we take $v_j = e_1$. Hence $\Omega = e_1 \otimes \cdots \otimes e_1$ with n components in the tensor product. Let $*$ denote operators we do not care about. We have

$$R_{i,k+j}(x - y_j)e_1 = \begin{pmatrix} x - y_j & 0 \\ * & 1 \end{pmatrix} e_1, \qquad \text{and} \qquad T(x)\Omega = \begin{pmatrix} (x|y)^n & 0 \\ * & 1 \end{pmatrix} \Omega. \tag{4.9}$$

Hence Ω is an eigenvector of $A(x)$ with eigenvalue $(x|y)^n$.

We look for other eigenvectors of the form

$$|z_\alpha\rangle = C\left(z_{\alpha^*_{n-k}+1}\right) \cdots C\left(z_{\alpha^*_{n-k+1-i}+i}\right) \cdots C\left(z_{\alpha^*_{n-k+1-(n-k)}+n-k}\right)\Omega, \tag{4.10}$$

and expect to find constraints on the $z_{\alpha^*_{n-k+1-i}+i}$. Using (4.6), (4.7), and (4.16), we find that

$$\begin{aligned} &A(x)C\left(z_{\alpha^*_{n-k}+1}\right) \cdots C\left(z_{\alpha^*_{n-k+1-(n-k)}+n-k}\right)\Omega \\ &\qquad = \prod_{j=1}^{n-k} \frac{1}{x - z_{\alpha^*_{n-k+1-j}+j}} (x|y)^n C\left(z_{\alpha^*_{n-k}+1}\right) \cdots C\left(z_{\alpha^*_{n-k+1-(n-k)}+n-k}\right)\Omega \\ &\qquad + \sum_{j=1}^{n-k} M_j\left(x, z_\alpha\right) C\left(z_{\alpha^*_{n-k}+1}\right) \cdots C\left(z_{\alpha^*_{n-k+1-(j-1)}+j-1}\right) C\left(z_{\alpha^*_{n-k+1-(j+1)}+j+1}\right) \cdots \\ &\qquad\qquad \cdots C\left(z_{\alpha^*_{n-k+1-(n-k)}+n-k}\right) C(x)\Omega. \end{aligned} \tag{4.11}$$

Finding M_j directly involves lots of computation, however finding M_1 is fairly simple. We have

$$M_1\left(x, z_\alpha\right) = -\frac{1}{x - z_{\alpha^*_{n-k}+1}} (z_{\alpha^*_{n-k}+1}|y)^n \prod_{j=2}^{n-k} \frac{1}{z_{\alpha^*_{n-k}+1} - z_{\alpha^*_{n-k+1-j}+j}}. \tag{4.12}$$

Notice that by using (4.6) we can exchange the position of $C(z_{\alpha^*_{n-k}+1})$ and $C(z_{\alpha^*_{n-k+1-h}+h})$ in $|z_\alpha\rangle$. Therefore we can find M_h by exchanging 1 and h in M_1. Thus

$$M_h\,(x, z_\alpha) \quad = \quad -\frac{1}{x - z_{\alpha^*_{n-k+1-h}+h}}(z_{\alpha^*_{n-k+1-h}+h}|y)^n \prod_{\substack{j=1\\j\neq h}}^{n-k} \frac{1}{z_{\alpha^*_{n-k+1-h}+h} - z_{\alpha^*_{n-k+1-j}+j}}. \quad (4.13)$$

For $|z_\alpha\rangle$ to be an eigenvector of $A(x)$ we need all the terms containing $C(x)$ in $A(x)|z_\alpha\rangle$ to cancel. Therefore we need $M_h = 0$. We have

$$M_h\,(x, z_\alpha) = \frac{1}{z_{\alpha^*_{n-k+1-h}+h} - x}\left((z_{\alpha^*_{n-k+1-h}+h}|y)^n \prod_{\substack{j=1\\j\neq h}}^{n-k} \frac{1}{z_{\alpha^*_{n-k+1-h}+h} - z_{\alpha^*_{n-k+1-j}+j}}\right) = 0. \quad (4.14)$$

Thus

$$(z_{\alpha^*_{n-k+1-h}+h}|y)^n = 0, \quad (4.15)$$

for $h = 1, \ldots, n-k$, as required. □

Remark 4.3. For the case where $q = 0$ we can solve the Bethe ansatz equations explicitly. We see that $z_{\alpha^*_{n-k+1-i}+i} = y_j$ for $i = 1, \ldots, n-k$ and $j = 1, \ldots, n$ is a solution to (4.15). Therefore, to obtain a set of solutions $z_{\alpha^*_{n-k}+1}, \ldots, z_{\alpha^*_{n-k+1-(n-k)}+n-k}$, we can take a subset of size $n-k$ from $\{y_1, \ldots, y_n\}$. Note that for (4.13) to be well defined we must have $z_{\alpha^*_{n-k+1-i}+i} \neq z_{\alpha^*_{n-k+1-j}+j}$ for $i \neq j$.

We now prove the case where $q \neq 0$.

Proof. For $q \neq 0$. We will need the additional commutation relation from Proposition 3.11.

$$D(v)C(u) = \frac{1}{u-v}\left(C(u)D(v) - C(v)D(u)\right). \quad (4.16)$$

We can deduce from the proof for the case where $q = 0$ that Ω is also an eigenvector of $A(x)+qD(x)$ with eigenvalue $(x|y)^n + q$.

For the other eigenvectors of the form

$$|z_\alpha\rangle = C\left(z_{\alpha^*_{n-k}+1}\right)\cdots C\left(z_{\alpha^*_{n-k+1-i}+i}\right)\cdots C\left(z_{\alpha^*_{n-k+1-(n-k)}+n-k}\right)\Omega, \quad (4.17)$$

we also find for $D(x)|z_\alpha\rangle$

$$\begin{aligned}
&D(x)C\left(z_{\alpha^*_{n-k}+1}\right)\cdots C\left(z_{\alpha^*_{n-k+1-(n-k)}+n-k}\right)\Omega \\
&\qquad\qquad = \prod_{j=1}^{n-k}\frac{1}{z_{\alpha^*_{n-k+1-j}+j}-x}C\left(z_{\alpha^*_{n-k}+1}\right)\cdots C\left(z_{\alpha^*_{n-k+1-(n-k)}+n-k}\right)\Omega \\
&\qquad\qquad + \sum_{j=1}^{n-k}N_j\left(x, z_{\alpha^*_{n-k}+1},\ldots,z_{\alpha^*_{n-k+1-(n-k)}+n-k}\right)C\left(z_{\alpha^*_{n-k}+1}\right)\cdots \\
&\cdots C\left(z_{\alpha^*_{n-k+1-(j-1)}+j-1}\right)C\left(z_{\alpha^*_{n-k+1-(j+1)}+j+1}\right)\cdots C\left(z_{\alpha^*_{n-k+1-(n-k)}+n-k}\right)C(x)\Omega.
\end{aligned}\tag{4.18}$$

In order to find N_j, we first find N_1 below

$$N_1(x,z_\alpha) = -\frac{1}{z_{\alpha^*_{n-k}+1}-x}\prod_{j=2}^{n-k}\frac{1}{z_{\alpha^*_{n-k+1-j}+j}-z_{\alpha^*_{n-k}+1}}.\tag{4.19}$$

Now, similarly to the way we found M_h in the proof for the case where $q=0$, we find N_h by using (4.6) to exchange the position of $C(z_{\alpha^*_{n-k}+1})$ and $C(z_{\alpha^*_{n-k+1-h}+h})$ in $|z_\alpha\rangle$, therefore exchanging 1 and h in N_1. Thus

$$N_h(x,z_\alpha) = -\frac{1}{z_{\alpha^*_{n-k+1-h}+h}-x}\prod_{\substack{j=1\\ j\neq h}}^{n-k}\frac{1}{z_{\alpha^*_{n-k+1-j}+j}-z_{\alpha^*_{n-k+1-h}+h}}.\tag{4.20}$$

For $|z_\alpha\rangle$ to be an eigenvector of $A(x)+qD(x)$ we need all the terms containing $C(x)$ in $(A(x)+qD(x))|z_\alpha\rangle$ to cancel. Therefore we need $M_h+qN_h=0$. We have

$$\begin{aligned}
M_h(x,z_\alpha)+qN_h(x,z_\alpha) = \frac{1}{x-z_{\alpha^*_{n-k+1-h}+h}}\Bigg(&q\prod_{\substack{j=1\\ j\neq h}}^{n-k}\frac{1}{z_{\alpha^*_{n-k+1-j}+j}-z_{\alpha^*_{n-k+1-h}+h}} \\
&-(z_{\alpha^*_{n-k+1-h}+h}|y)^n\prod_{\substack{j=1\\ j\neq h}}^{n-k}\frac{1}{z_{\alpha^*_{n-k+1-h}+h}-z_{\alpha^*_{n-k+1-j}+j}}\Bigg).
\end{aligned}\tag{4.21}$$

So we need

$$(z_{\alpha^*_{n-k+1-h}+h}|y)^n = q\prod_{\substack{j=1\\ j\neq h}}^{n-k}\frac{z_{\alpha^*_{n-k+1-h}+h}-z_{\alpha^*_{n-k+1-j}+j}}{z_{\alpha^*_{n-k+1-j}+j}-z_{\alpha^*_{n-k+1-h}+h}} = q(-1)^{n-k-1}.\tag{4.22}$$

Thus

$$(z_{\alpha^*_{n-k+1-h}+h}|y)^n+(-1)^{n-k}q=0,\tag{4.23}$$

for $h=1,\ldots,n-k$, as required. □

Remark 4.4. Unlike the case where $q=0$, we cannot easily find explicit solutions to (4.5). In order to find all solutions to (4.5), we must consider $z\in\mathbb{F}_q=\mathbb{C}[q^{\pm1/n}]\hat{\otimes}\mathbb{F}$, where $\mathbb{F}:=\mathbb{C}\{\{y_1,\ldots,y_n\}\}$, the algebraically closed field of Puiseux series, which is the field of all formal power series allowing for negative and fractional powers.

4.2 Bethe Vectors and Lattice Models

The Bethe ansatz in the previous section will work for any choice of k such that $0 \leq k \leq n$. We now want to fix k and look at a particular subspace of $V(y_1) \otimes \cdots \otimes V(y_n)$.

Let $\mathcal{V}_k$ be the subspace of $V(y_1)\otimes\cdots\otimes V(y_n)$ which has a basis $\{e_{i_1}\otimes\cdots\otimes e_{i_n}\}_{i_j=0,1}$ where exactly k of the i_j are equal to 1, with the other $n-k$ equal to 0. For convenience we now use bra-ket notation for vectors $v_\lambda \in \mathcal{V}_k$, i.e. $v_\lambda = e_{i_1} \otimes \cdots \otimes e_{i_n} = |\lambda\rangle$ where $i_j = 1$ for $j = \lambda_{k+1-h} + h$, $h = 1, \ldots, k$ and $i_j = 0$ otherwise. We also introduce the dual basis of the $\{|\lambda\rangle\}_{\lambda\in(k,n-k)}$ vectors denoted by $\{\langle\lambda|\} \subset \mathcal{V}_k^*$.

The following 2 propositions are taken from the article by Gorbounov and Korff [GK14].

Proposition 4.5. *[GK14, Prop 5.1] Recall that $\lambda^* = (\lambda^\vee)'$ from Section 2.2. We can express $|z_\alpha\rangle$ in terms of the $|\lambda\rangle$ as follows*

$$|z_\alpha\rangle = \sum_{\lambda\in(k,n-k)} s_{\lambda^*}(z_\alpha|y)|\lambda\rangle. \tag{4.24}$$

Proof. We can prove this by looking at what the $C(z_i)$ do to Ω on the lattice, as Gorbounov and Korff did in [GK14], and then establishing a bijection between n by $n-k$ lattices with paths according to our vertex model in the previous section and tableaux whose shape is in $(n-k,k)$. We know that

$$\begin{aligned} |z_\alpha\rangle &= C\left(z_{\alpha^*_{n-k}+1}\right)\cdots C\left(z_{\alpha^*_{n-k+1-i}+i}\right) e_1\otimes\cdots\otimes e_1 \\ &= C\left(z_{\alpha^*_{n-k+1-i}+i}\right)\cdots C\left(z_{\alpha^*_{n-k}+1}\right) e_1\otimes\cdots\otimes e_1, \end{aligned} \tag{4.25}$$

using (4.6). Therefore we may apply $C\left(z_{\alpha^*_{n-k}+1}\right)$ first so that the first row of the lattice corresponds to $z_{\alpha^*_{n-k}+1}$. Each $C(z_{\alpha^*_i})$ operator removes an e_1 from the end of the row above thereby increasing the number of e_0 in the row below by one, therefore $C\left(z_{\alpha^*_{n-k+1-i}+i}\right)\cdots C\left(z_{\alpha^*_{n-k}+1}\right) e_1\otimes\cdots\otimes e_1 \in \mathcal{V}_k$. We show how to obtain a tableaux of shape λ^* from a lattice whose first row is Ω and last row is $|\lambda\rangle$, and vice versa, and then that the term contributed by the lattice to the coefficient of $|\lambda\rangle$ in (4.24) is the same as the term contributed by the corresponding tableau to s_{λ^*}.

Consider the path that begins at the top of column j of the lattice, for some $0 \leq j \leq k$. This path will end in column $\lambda_{k+1-j} + j$. Therefore there are λ_{k+1-j} rows of the path which do not contain a straight vertical line, and $n-k-\lambda_{k+1-j}$ which do. There are $n-k-\lambda_{k+1-j}$ boxes in the j^{th} column of the tableaux of λ^*. Therefore we can map between this lattice and tableau. By examining the allowed vertex configurations, we can deduce that the i^{th} straight line in the j^{th} path must be in a row lower than or equal to the i^{th} straight line in the $(j-1)^{\text{st}}$ path. Hence if we put the row numbers of the straight lines in the j^{th} path in the lattice into the j^{th} column of the Young diagram of λ^* for each $0 \leq j \leq k$ we obtain a valid tableau.

Since the positions of the straight lines uniquely determine each lattice configuration, we can reverse this process given a tableau and obtain a unique lattice configuration. Therefore we have a bijection. It only remains to check that the weights contributing to the respective coefficients match.

Each box, (a, b) in the tableau, T, of shape λ^* contributes $((z_\alpha)_{T(a,b)} - y_{T(a,b)+b-a})$ to $s_{\lambda^*}(z_\alpha|y)$, where $(z_\alpha)_i = z_{\alpha^*_{n-k+1-i}+i}$ as defined after Lemma 4.2. Each straight line, in position (i, k), in the lattice contributes $((z_\alpha)_i - y_k)$ in the coefficient of $|\lambda\rangle$. Now, by construction, we have $i = T(a, b)$ for the a^{th} straight line in path b, and, by examination, $k = b + i - a$. Hence these contributions are the same. Since we calculate both $s_{\lambda^*}(z_\alpha|y)$ and the coefficient of $|\lambda\rangle$ by taking the product of all the contributions from the boxes and straight lines respectively, and then taking the sum over all possible tableaux of shape λ^* and lattices whose first row is Ω and last row is $|\lambda\rangle$ respectively, we have completed the proof. □

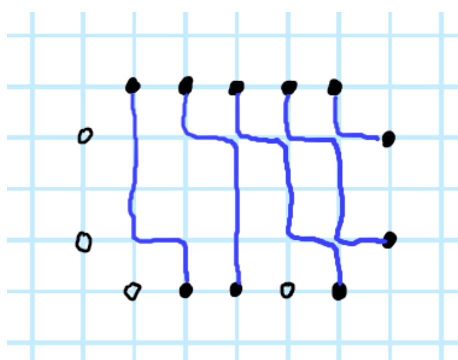
We want to give an example to illustrate the proof above, but we first need another definition.

Definition 4.6. A Maya diagram is a sequence of n black and white go-stones arranged in a horizontal line. We can think of the black go-stones as being "1" and the white go-stones as being "0" to obtain a bijection with binary strings. The weight of a Maya diagram λ is the number of black go-stones, or equivalently $||\lambda|| := \sum_{i=1}^{n} \lambda_i = k$.

Example 4.7. An example of a Maya diagram μ: $\bullet\bullet\circ\bullet\bullet\circ\bullet$. Equivalently $\mu = 1101101$. Here $n = 7$ and $||\mu|| = 5$.

Example 4.8. We can now give an example to illustrate the proof of Proposition 4.5.

Let $n = 5$ and $k = 3$. Take $\lambda = (2, 1, 1)$ and therefore $\lambda^* = (2)$. Below is the lattice configuration which corresponds to the tableau $\boxed{1 \mid 2}$.



The first path starts in column 1 and ends in column $\lambda_{3+1-1} + 1 = \lambda_3 + 1 = 1 + 1 = 2$. There is one row which contains a vertical line: row 1. Therefore we put a 1 in the only box in the first column of λ^*.

The second path starts in column 2 and ends in column $\lambda_{3+1-2} + 2 = \lambda_2 + 2 = 1 + 2 = 3$. There is one row which contains a vertical line: row 2. Therefore we put a 2 in the only box in the second column of λ^*.

The third path starts in column 3 and ends in column $\lambda_{3+1-3} + 3 = \lambda_1 + 3 = 2 + 3 = 5$.

Proposition 4.9. *[GK14, Prop 5.10]*

$$\langle z_\alpha| = \sum_{\lambda\in(k,n-k)} \frac{s_{\lambda'}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}\langle\lambda|, \tag{4.26}$$

where $\mathfrak{e}(z_\alpha) = \prod_{\substack{i\in I(\alpha)\\ j\in I(\alpha^*)}}(z_i - z_j)$ *with* $I(\alpha)$ *being the indexing set of* z_α *and* $I(\alpha^*)$ *its complement as defined in Remark 2.25.*

Proof. This is proved by Gorbounov and Korff in [GK14]. □

We are now able to give proofs of Propositions 2.27 and 2.28.

Proof of Proposition 2.27.

Proof. Since the Bethe vectors $|z_\alpha\rangle$ and $\langle z_\alpha|$ are eigenbases, we can sum over all $\alpha\in(k,n-k)$ to get a resolution of the identity $\mathbb{1} = \sum_{\alpha\in(k,n-k)}|z_\alpha\rangle\langle z_\alpha|$. Hence we have

$$\mathbb{1} = \sum_{\alpha\in(k,n-k)}|z_\alpha\rangle\langle z_\alpha| = \sum_{\alpha\in(k,n-k)}\sum_{\lambda\in(k,n-k)} s_{\lambda^*}(z_\alpha|y)|\lambda\rangle \sum_{\mu\in(k,n-k)}\frac{s_{\mu'}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}\langle\mu|. \tag{4.27}$$

In order for this to be true, we need all terms for which $\mu\neq\lambda$ to be 0, and all terms for which $\mu=\lambda$ to be 1. Hence we have

$$\sum_{\alpha\in(k,n-k)}\frac{s_{\lambda^*}(z_\alpha|y)s_{\mu'}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)} = \delta_{\lambda\mu}. \tag{4.28}$$

This is equivalent to the statement in the proposition, hence it is proved. □

Proof of Proposition 2.28.

Proof. Since by Proposition 4.9 we know that $\langle z_\alpha|z_\beta\rangle = \delta_{\alpha\beta}$ we have

$$\begin{aligned}\delta_{\alpha\beta} = \langle z_\alpha|z_\beta\rangle &= \sum_{\lambda\in(k,n-k)}\frac{s_{\lambda'}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}\langle\lambda|\sum_{\mu\in(k,n-k)}s_{\mu^*}(z_\beta|y)|\mu\rangle \\ &= \sum_{\lambda\in(k,n-k)}\sum_{\mu\in(k,n-k)}\frac{s_{\lambda'}(z_\alpha|Y)s_{\mu^*}(z_\beta|y)}{\mathfrak{e}(z_\alpha)}\langle\lambda|\mu\rangle = \sum_{\lambda\in(k,n-k)}\frac{s_{\mu'}(z_\alpha|Y)s_{\mu^*}(z_\beta|y)}{\mathfrak{e}(z_\alpha)},\end{aligned} \tag{4.29}$$

using that $\langle\lambda|\mu\rangle = \delta_{\lambda\mu}$ since they are dual. This is equivalent to the statement in the proposition, hence the proof is complete. □

We can now express $|\lambda\rangle$ in terms of the Bethe vectors, $|z_\alpha\rangle$, which will be of use to us in the following subsection.

Lemma 4.10. *We have*

$$|\lambda\rangle = \sum_{\alpha\in(k,n-k)} \frac{s_{\lambda'}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}|z_\alpha\rangle. \tag{4.30}$$

Proof. We know that $|z_\alpha\rangle = \sum_{\mu\in(k,n-k)} s_{\mu^*}(z_\alpha|y)|\mu\rangle$. If we use this definition to expand the Bethe vectors in (4.30) we get

$$\sum_{\alpha\in(k,n-k)} \frac{s_{\lambda'}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}|z_\alpha\rangle = \sum_{\mu,\alpha\in(k,n-k)} \frac{s_{\lambda'}(z_\alpha|Y)s_{\mu^*}(z_\alpha|y)}{\mathfrak{e}(z_\alpha)}|\mu\rangle = \sum_{\mu\in(k,n-k)} \delta_{\lambda\mu}|\mu\rangle = |\lambda\rangle, \tag{4.31}$$

where we have used Proposition 2.27, as required. □

4.3 Multiplication Operators Using the Transfer Matrix

We now define operators on $\mathcal{V}_k$ which correspond to multiplication by a Schubert class in $QH_T^*(Gr_k(\mathbb{C}^n))$.

Definition 4.11. Define an operator $S_\lambda : \mathcal{V}_k \to \mathcal{V}_k$ by

$$S_\lambda := \det\left(\tau^{j-1}E_{\lambda'_i-i+j}\right)_{1\le i,j\le n-k}. \tag{4.32}$$

We want to show that the matrix elements $\langle\nu|S_\lambda|\mu\rangle$ are Gromov-Witten invariants. We will need the following lemma from [GK14].

Lemma 4.12. *[GK14, Lemma 6.2] Consider the (unique) extension of S_λ to $\mathcal{V}_k^{\mathbb{F}_q} := \mathcal{V}_k \otimes \mathbb{F}_q$. (i) The Bethe vectors, $|z_\alpha\rangle$, are eigenvectors of S_λ and on each $\mathcal{V}_k^{\mathbb{F}_q}$ we have the eigenvalue equation $S_\lambda|z_\alpha\rangle = s_{\lambda'}(z_\alpha|Y)|z_\alpha\rangle$. (ii) Let $|\emptyset\rangle = v_1\otimes\cdots\otimes v_1\otimes v_0\otimes\cdots\otimes v_0 \in \mathcal{V}_k$ be the unique basis vector which corresponds to the empty partition. Then $S_\lambda|\emptyset\rangle = |\lambda\rangle$.*

Proof. Proof of part (i) can be found within [GK14] using a result from [Mac92]. Part (ii) can be seen using (4.30):

$$S_\lambda|\emptyset\rangle = \sum_{\alpha\in(k,n-k)} \frac{s_{\emptyset'}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}S_\lambda|z_\alpha\rangle = \sum_{\alpha\in(k,n-k)} \frac{s_{\lambda'}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}|z_\alpha\rangle = |\lambda\rangle. \tag{4.33}$$

□

In order to make $\mathcal{V}_k$ a ring, we must define a product.

Theorem 4.13. *[GK14, Theorem 6.5] Define a product on $\mathcal{V}_k$ by setting*

$$|\lambda\rangle \circledast |\mu\rangle = S_\lambda|\mu\rangle. \tag{4.34}$$

Then $(\mathcal{V}_k, \circledast)$ is a commutative ring.

Proof. Consider again the extension of S_λ to $\mathcal{V}_k \otimes \mathbb{F}_q$. Then we have

$$\begin{aligned} S_\lambda|\mu\rangle &= \sum_{\alpha\in(k,n-k)} \frac{s_{\mu'}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)} S_\lambda|z_\alpha\rangle = \sum_{\alpha\in(k,n-k)} \frac{s_{\mu'}(z_\alpha|Y)s_{\lambda'}(z_\alpha|Y)}{\mathfrak{e}(z_\alpha)}|z_\alpha\rangle \\ &= \sum_{\nu,\alpha\in(k,n-k)} \frac{s_{\mu'}(z_\alpha|Y)s_{\lambda'}(z_\alpha|Y)s_{\nu^*}(z_\alpha|y)}{\mathfrak{e}(z_\alpha)}|\nu\rangle. \end{aligned} \tag{4.35}$$

The last line of the equation above is symmetric in λ and μ, hence the product must be commutative. It remains to check associativity. Since (4.1) is commutative by Proposition 3.11, we must have that $S_\lambda S_\mu = S_\mu S_\lambda$. Hence

$$|\lambda\rangle \circledast (|\mu\rangle \circledast |\nu\rangle) = |\lambda\rangle \circledast (|\nu\rangle \circledast |\mu\rangle) = S_\lambda S_\nu|\mu\rangle = S_\nu S_\lambda|\mu\rangle = |\nu\rangle \circledast (|\lambda\rangle \circledast |\mu\rangle) = (|\lambda\rangle \circledast |\mu\rangle) \circledast |\nu\rangle. \tag{4.36}$$

□

We now want to show that there is an isomorphism between $\mathcal{V}_k$ and our Frobenius algebra defined in an earlier section. Gorbounov and Korff prove the following result in [GK14].

Theorem 4.14. *[GK14, Theorem 6.13] The map from $\mathcal{V}_k \to \mathfrak{J}_q$ which maps $|\lambda\rangle \to s_\lambda(x|y)$ for all $\lambda \in (k, n-k)$ is an algebra isomorphism. In particular, the Bethe vectors with a renormalisation factor, $|z_\alpha\rangle \mathfrak{e}(z_\alpha)^{-1}$ are mapped onto the idempotents of $\mathfrak{J}_q$, and the matrix elements $q^d C^{\nu,d}_{\lambda\mu}(y) = \langle\nu|S_\lambda|\mu\rangle$ with $dn = |\lambda| + |\mu| - |\nu|$ are the equivariant Gromov-Witten invariants.*

Remark 4.15. This proves there is an isomorphism between $\mathcal{V}_k \otimes \mathbb{F}_q$ and the quantum cohomology ring of the Grassmannian, $QH^*_T(Gr(k,n))$, as we already know from Corollary 2.34 that $\mathfrak{J}_q$ is isomorphic to $QH^*_T(Gr(k,n))$. Corollary 6.14 in [GK14] defines an isomorphism that takes $S_\lambda \in \mathrm{End}(\mathcal{V}_k)$ to the Schubert class $\sigma_\lambda \in QH^*_T(Gr(k,n))$.

This gives us a way to directly calculate the quantum equivariant Kostka numbers as defined in Definition 2.37 earlier. The following results are again due to Gorbounov and Korff [GK14].

Proposition 4.16. *[GK14, Proposition 6.25] The partition function for lattices with periodic boundary conditions is related to the coproduct of $QH^*_T(Gr(k,n))$ as a Frobenius algebra.*

$$\langle\lambda|E(x_1)\cdots E(x_k)|\mu\rangle = \sum_{d\geq 0} q^d s_{\lambda^*/d/\mu^*}(x|y). \tag{4.37}$$

Remark 4.17. $\langle\lambda|E(x_1)\cdots E(x_k)|\mu\rangle$ is the partition function for lattice configurations which have the 01-word taken from the indices in $|\mu\rangle = v_{i_1} \otimes \cdots v_{i_n}$ along the top and the 01-word taken from the indices in $|\lambda\rangle$ along the bottom.

Corollary 4.18. *[GK14, Corollary 6.27] Let $E_\alpha = E_{\alpha_1} \cdots E_{\alpha_{n-k}}$. Then we have*

$$\langle\lambda|E_\alpha|\mu\rangle = \sum_{d\geq 0} q^d K_{\lambda'/d/\mu',\alpha}(y). \tag{4.38}$$

Sketch of Proof. From (2.43) we know that $s_{\lambda^*/d/\mu^*}(x|y) = \sum_{\nu\in(n-k,k)} C^{\lambda',d}_{\mu'\nu'}(y)s_{\nu^*}(x|y)$. We also know from (2.37) that $C^{\lambda',d}_{\mu'\nu'}(y)$ is the coefficient of $s_{\lambda'}(x|y)$ in the expansion of $s_{\nu'}(x|y)s_{\mu'}(x|y)$.

From Definition 2.37 we know that $q^d K_{\lambda'/d/\mu',\alpha}(y)$ is the coefficient of $s_{\lambda'}(x|y)$ in the expansion of $e_\alpha(x|y)s_\mu(x|y)$. Note that $e_r(x|y) = s_{(1^r)}(x|y)$. Therefore to obtain $K_{\lambda'/d/\mu',\alpha}(y)$ from $C^{\lambda',d}_{\mu'\nu'}(y)$ we can repeatedly apply the result for $C^{\lambda',d}_{\mu'(1^{\alpha_i})}(y)$ to build $K_{\lambda'/d/\mu',\alpha}(y)$ recursively. Noting that $C^{\lambda',d}_{\mu'(1^{\alpha_i})}(y) = \langle\lambda|E_{\alpha_i}|\mu\rangle$, we can deduce the result. $\square$

5 Calculating Quantum Equivariant Kostka Numbers

In this section we detail several ways to calculate quantum equivariant Kostka numbers and give a method of our own. This new method is the main result of this masters thesis. It is unpublished.

5.1 Using Lattices and the Yang-Baxter Algebra to Calculate Quantum Equivariant Kostka Numbers - an Example

We now give an example to explain exactly how to use the graphical calculus method on lattices of Gorbounov and Korff [GK14] to calculate quantum equivariant Kostka numbers. We will give the complete example for E_r in $Gr(2,5)$. First we will give their combinatorial formula to calculate $E(x)|\mu\rangle$.

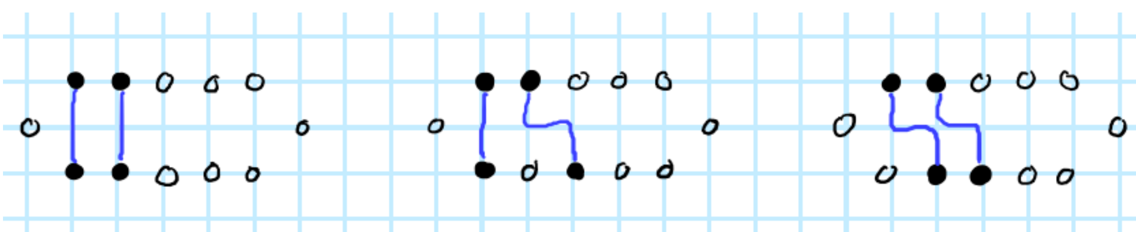
Proposition 5.1. *[GK14, Proposition 4.6] Let $\mu \in (k, n-k)$. We have the following combinatorial action of the transfer matrix.*

$$E(x)|\mu\rangle = \sum_{d=0,1} q^d \sum_{\lambda'/d/\mu' \ hor\ strip} \prod_{j \in J_{\lambda'/d/\mu'}} (x - Y_j)|\lambda\rangle, \tag{5.1}$$

where λ' and μ' are conjugate partitions, $Y_j = y_{n+1-j}$ and $J_{\lambda'/d/\mu'}$ consists of the diagonals $j - i + n - k$ of the bottom square (i, j) in each column of the diagram of μ with $\lambda'/d/\mu'$ added which does not intersect with $\lambda'/d/\mu'$. If a column contains no boxes add $j + n - k$ to $J_{\lambda'/d/\mu'}$.

Example 5.2. Recall that $E(x) = A(x) + qD(x) = \sum_{r=0}^{k} (x|y)^{k-r} E_r$. We will use lattices to calculate $E_r|\lambda\rangle$ for $0 \le r \le 2$ and $\lambda \in (k, n-k) = (2,3)$, from which we can then calculate any quantum equivariant Kostka number for $Gr(2,5)$.

For $E(x)|\emptyset\rangle$ we have:



Using the Boltzmann weights for vertices from Section 3 we can determine the weights of the lattice configurations above.

In the first lattice configuration there are 2 vertical blue lines in positions 1 and 2. The Boltzmann weight of a vertex with a vertical blue line is $x_i - y_j$ where i is the row number and j is the column number. Therefore the first configuration above has weight $(x_1 - y_1)(x_1 - y_2)$.

In the second configuration there is one vertex with a vertical blue line in position 1, so this configuration has weight $(x_1 - y_1)$.

The third configuration above has no vertices with vertical blue lines, therefore this configuration has weight 1.

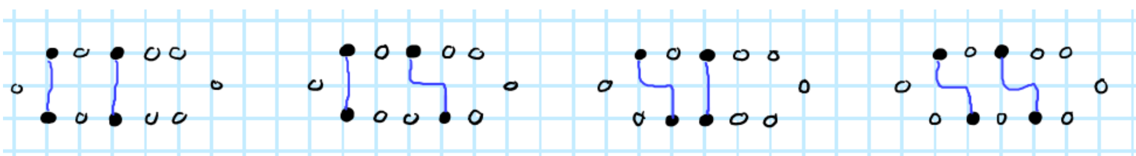
Therefore, from the lattice configurations above we have calculated that

$$E(x)|\emptyset\rangle = (x-y_1)(x-y_2)|\emptyset\rangle + (x-y_1)|(1)\rangle + |(1,1)\rangle. \tag{5.2}$$

To find $E_r|\emptyset\rangle$ for $r=0,1,2$ we must look at the coefficients of the factorial powers $(x|y)^r$ in the expression for $E(x)$ above. Thus we deduce that

$$E_0|\emptyset\rangle = |\emptyset\rangle, \tag{5.3}$$
$$E_1|\emptyset\rangle = |(1)\rangle, \tag{5.4}$$
$$E_2|\emptyset\rangle = |(1,1)\rangle. \tag{5.5}$$

For $E(x)|(1)\rangle$ we have:



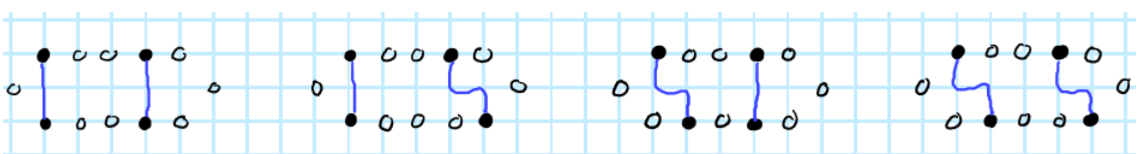
From the lattice configurations above we calculate that

$$\begin{aligned} E(x)|(1)\rangle &= (x-y_1)(x-y_3)|(1)\rangle + (x-y_1)|(2)\rangle + (x-y_3)|(1,1)\rangle + |(2,1)\rangle \\ &= (x|y)^2|(1)\rangle + (x|y)\left((y_2-y_3)|(1)\rangle + |(2)\rangle + |(1,1)\rangle\right) + (y_1-y_3)|(1,1)\rangle + |(2,1)\rangle, \end{aligned} \tag{5.6}$$

from which we deduce that

$$E_0|(1)\rangle = |(1)\rangle, \tag{5.7}$$
$$E_1|(1)\rangle = (y_2-y_3)|(1)\rangle + |(2)\rangle + |(1,1)\rangle, \tag{5.8}$$
$$E_2|(1)\rangle = (y_1-y_3)|(1,1)\rangle + |(2,1)\rangle. \tag{5.9}$$

For $E(x)|(2)\rangle$ we have:



From the lattice configurations above we calculate that

$$\begin{aligned} E(x)|(2)\rangle &= (x-y_1)(x-y_4)|(2)\rangle + (x-y_1)|(3)\rangle + (x-y_4)|(2,1)\rangle + |(3,1)\rangle \\ &= (x|y)^2|(2)\rangle + (x|y)\left((y_2-y_4)|(2)\rangle + |(3)\rangle + |(2,1)\rangle\right) + (y_1-y_4)|(2,1)\rangle + |(3,1)\rangle, \end{aligned} \tag{5.10}$$

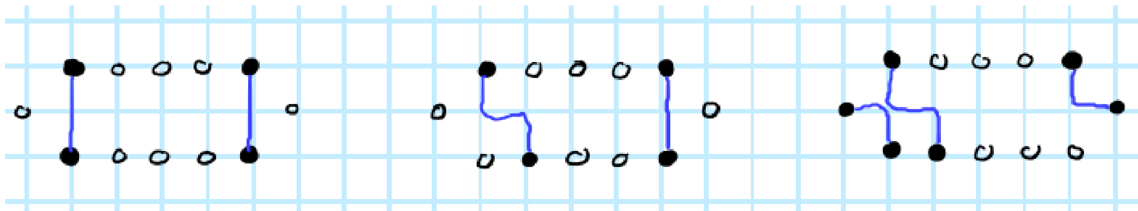
from which we deduce that

$$E_0|(2)\rangle = |(2)\rangle, \tag{5.11}$$

$$E_1|(2)\rangle = (y_2 - y_4)|(2)\rangle + |(3)\rangle + |(2,1)\rangle, \tag{5.12}$$

$$E_2|(2)\rangle = (y_1 - y_4)|(2,1)\rangle + |(3,1)\rangle. \tag{5.13}$$

For $E(x)|(3)\rangle$ we have:



From the lattice configurations above we calculate that

$$\begin{aligned} E(x)|(3)\rangle &= (x-y_1)(x-y_5)|(3)\rangle + (x-y_5)|(3,1)\rangle + q|\emptyset\rangle \\ &= (x|y)^2|(3)\rangle + (x|y)\left((y_2-y_5)|(3)\rangle + |(3,1)\rangle\right) + (y_1-y_5)|(3,1)\rangle + q|\emptyset\rangle, \end{aligned} \tag{5.14}$$

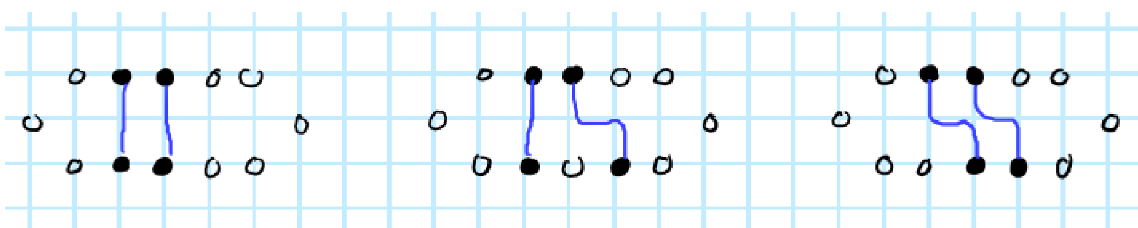
from which we deduce that

$$E_0|(3)\rangle = |(3)\rangle, \tag{5.15}$$

$$E_1|(3)\rangle = (y_2 - y_5)|(3)\rangle + |(3,1)\rangle, \tag{5.16}$$

$$E_2|(3)\rangle = (y_1 - y_5)|(3,1)\rangle + q|\emptyset\rangle. \tag{5.17}$$

For $E(x)|(1,1)\rangle$ we have:



From the lattice configurations above we calculate that

$$\begin{aligned} E(x)|(1,1)\rangle &= (x-y_2)(x-y_3)|(1,1)\rangle + (x-y_2)|(2,1)\rangle + |(2,2)\rangle \\ &= (x|y)^2|(1,1)\rangle + (x|y)\left((y_1-y_3)|(1,1)\rangle + |(2,1)\rangle\right) \\ &\quad + (y_1-y_2)(y_1-y_3)|(1,1)\rangle + (y_1-y_2)|(2,1)\rangle + |(2,2)\rangle, \end{aligned} \tag{5.18}$$

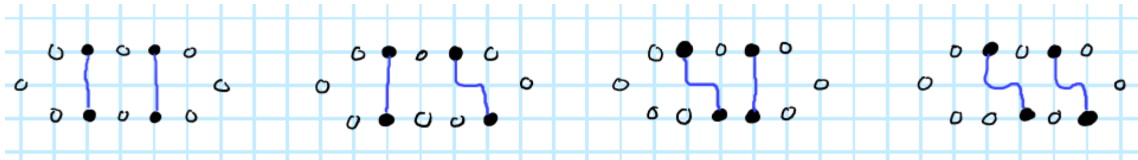
from which we deduce that

$$E_0|(1,1)\rangle = |(1,1)\rangle, \tag{5.19}$$

$$E_1|(1,1)\rangle = (y_1 - y_3)|(1,1)\rangle + |(2,1)\rangle, \tag{5.20}$$

$$E_2|(1,1)\rangle = (y_1 - y_2)(y_1 - y_3)|(1,1)\rangle + (y_1 - y_2)|(2,1)\rangle + |(2,2)\rangle. \tag{5.21}$$

For $E(x)|(2,1)\rangle$ we have:



From the lattice configurations above we calculate that

$$\begin{aligned}E(x)|(2,1)\rangle &= (x-y_2)(x-y_4)|(2,1)\rangle + (x-y_2)|(3,1)\rangle + (x-y_4)|(2,2)\rangle + |(3,2)\rangle \\ &= (x|y)^2|(2,1)\rangle + (x|y)\left((y_1-y_4)|(2,1)\rangle + |(3,1)\rangle + |(2,2)\rangle\right) \\ &\quad + (y_1-y_2)(y_1-y_4)|(2,1)\rangle + (y_1-y_2)|(3,1)\rangle + (y_1-y_4)|(2,2)\rangle + |(3,2)\rangle,\end{aligned} \tag{5.22}$$

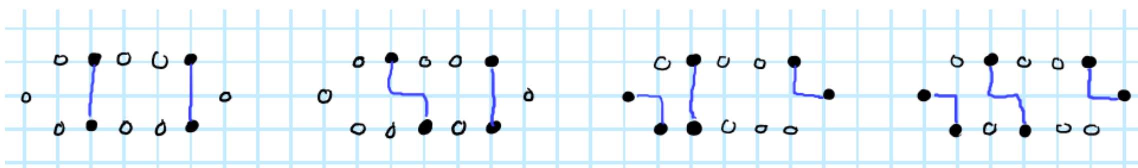
from which we deduce that

$$E_0|(2,1)\rangle = |(2,1)\rangle, \tag{5.23}$$

$$E_1|(2,1)\rangle = (y_1-y_4)|(2,1)\rangle + |(3,1)\rangle + |(2,2)\rangle, \tag{5.24}$$

$$E_2|(2,1)\rangle = (y_1-y_2)(y_1-y_4)|(2,1)\rangle + (y_1-y_2)|(3,1)\rangle + (y_1-y_4)|(2,2)\rangle + |(3,2)\rangle. \tag{5.25}$$

For $E(x)|(3,1)\rangle$ we have:



From the lattice configurations above we calculate that

$$\begin{aligned}E(x)|(3,1)\rangle &= (x-y_2)(x-y_5)|(3,1)\rangle + (x-y_5)|(3,2)\rangle + q(x-y_2)|\emptyset\rangle + q|(1)\rangle \\ &= (x|y)^2|(3,1)\rangle + (x|y)\left((y_1-y_5)|(3,1)\rangle + |(3,2)\rangle + q|\emptyset\rangle\right) \\ &\quad + (y_1-y_2)(y_1-y_5)|(3,1)\rangle + (y_1-y_5)|(3,2)\rangle + q(y_1-y_2)|\emptyset\rangle + q|(1)\rangle,\end{aligned} \tag{5.26}$$

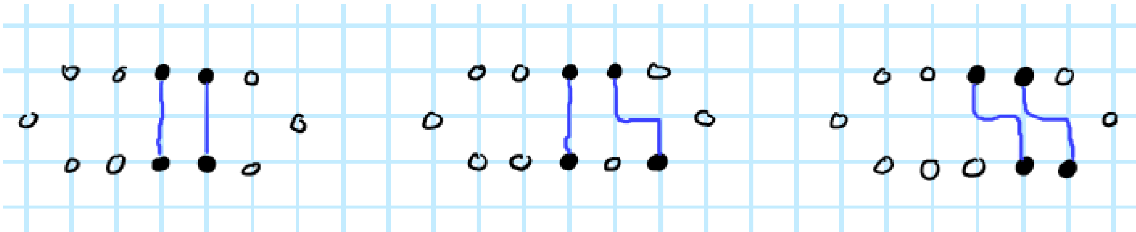
from which we deduce that

$$E_0|(3,1)\rangle = |(3,1)\rangle, \tag{5.27}$$

$$E_1|(3,1)\rangle = (y_1-y_5)|(3,1)\rangle + |(3,2)\rangle + q|\emptyset\rangle, \tag{5.28}$$

$$E_2|(3,1)\rangle = (y_1-y_2)(y_1-y_5)|(3,1)\rangle + (y_1-y_5)|(3,2)\rangle + q(y_1-y_2)|\emptyset\rangle + q|(1)\rangle. \tag{5.29}$$

For $E(x)|(2,2)\rangle$ we have:



From the lattice configurations above we calculate that

$$\begin{aligned}E(x)|(2,2)\rangle &= (x-y_3)(x-y_4)|(2,2)\rangle + (x-y_3)|(3,2)\rangle + |(3,3)\rangle \\ &= (x|y)^2|(2,2)\rangle + (x|y)\left((y_1-y_3+y_2-y_4)|(2,2)\rangle + |(3,2)\rangle\right) \\ &\quad + (y_1-y_3)(y_1-y_4)|(2,2)\rangle + (y_1-y_3)|(3,2)\rangle + |(3,3)\rangle,\end{aligned} \tag{5.30}$$

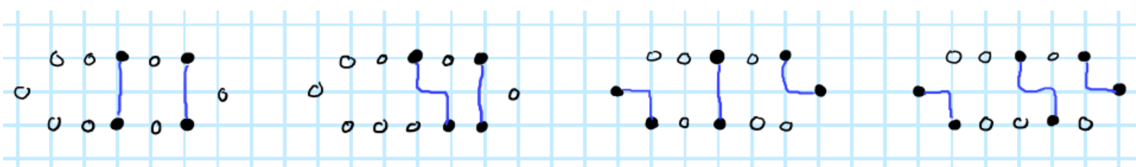
from which we deduce that

$$E_0|(2,2)\rangle = |(2,2)\rangle, \tag{5.31}$$

$$E_1|(2,2)\rangle = (y_1-y_3+y_2-y_4)|(2,2)\rangle + |(3,2)\rangle, \tag{5.32}$$

$$E_2|(2,2)\rangle = (y_1-y_3)(y_1-y_4)|(2,2)\rangle + (y_1-y_3)|(3,2)\rangle + |(3,3)\rangle. \tag{5.33}$$

For $E(x)|(3,2)\rangle$ we have:



From the lattice configurations above we calculate that

$$\begin{aligned}E(x)|(3,2)\rangle &= (x-y_3)(x-y_5)|(3,2)\rangle + (x-y_5)|(3,3)\rangle + q(x-y_3)|(1)\rangle + q|(2)\rangle \\ &= (x|y)^2|(3,2)\rangle + (x|y)\left((y_1-y_3+y_2-y_5)|(3,2)\rangle + |(3,3)\rangle + q|(1)\rangle\right) \\ &\quad + (y_1-y_3)(y_1-y_5)|(3,2)\rangle + (y_1-y_5)|(3,3)\rangle + q(y_1-y_3)|(1)\rangle + q|(2)\rangle,\end{aligned} \tag{5.34}$$

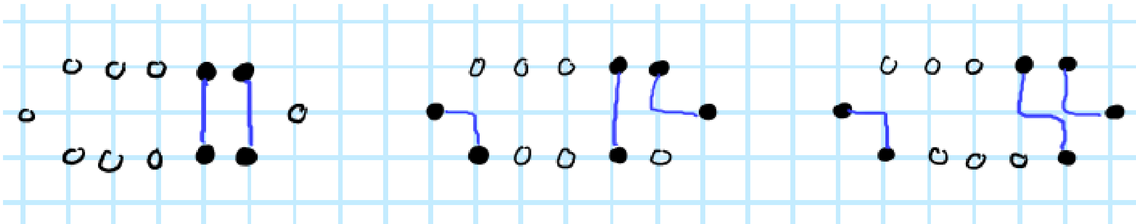
from which we deduce that

$$E_0|(3,2)\rangle = |(3,2)\rangle, \tag{5.35}$$

$$E_1|(3,2)\rangle = (y_1-y_3+y_2-y_5)|(3,2)\rangle + |(3,3)\rangle + q|(1)\rangle, \tag{5.36}$$

$$E_2|(3,2)\rangle = (y_1-y_3)(y_1-y_5)|(3,2)\rangle + (y_1-y_5)|(3,3)\rangle + q(y_1-y_3)|(1)\rangle + q|(2)\rangle. \tag{5.37}$$

For $E(x)|(3,3)\rangle$ we have:



From the lattice configurations above we calculate that

$$
\begin{aligned}
E(x)|(3,3)\rangle &= (x-y_4)(x-y_5)|(3,3)\rangle + q(x-y_4)|(2)\rangle + q|(3)\rangle \\
&= (x|y)^2|(3,3)\rangle + (x|y)\left((y_1-y_4+y_2-y_5)|(3,3)\rangle + q|(2)\rangle\right) \\
&\quad + (y_1-y_4)(y_1-y_5)|(3,3)\rangle + q(y_1-y_4)|(2)\rangle + q|(3)\rangle,
\end{aligned} \tag{5.38}
$$

from which we deduce that

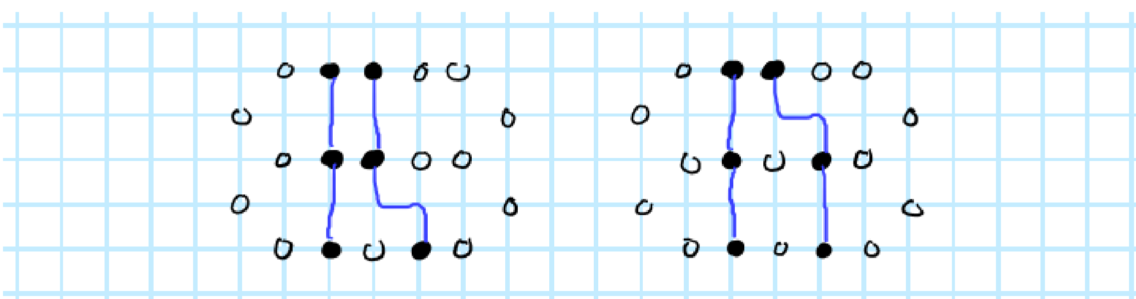
$$E_0|(3,3)\rangle = |(3,2)\rangle, \tag{5.39}$$

$$E_1|(3,3)\rangle = (y_1-y_4+y_2-y_5)|(3,3)\rangle + q|(2)\rangle, \tag{5.40}$$

$$E_2|(3,3)\rangle = (y_1-y_4)(y_1-y_5)|(3,3)\rangle + q(y_1-y_4)|(2)\rangle + q|(3)\rangle. \tag{5.41}$$

We will now give a brief example to demonstrate the calculation of the quantum equivariant Kostka numbers.

Example 5.3. We will calculate $\langle(2,1)|E_2E_1|(1,1)\rangle = K_{(2,1)/0/(2),(2,1)}(y)$. The lattices we need to do this, which have the 01-word for $(1,1)$ labelling the top and the 01-word for $(2,1)$ labelling the bottom are



We need to find E_1 for the first row and E_2 for the second row of both lattices.

The first row of the first lattice is $\langle(1,1)|E_1|(1,1)\rangle$ which we calculated in the previous example to be y_1-y_3. The second row of the first lattice is $\langle(1,1)|E_2|(2,1)\rangle$ which we calculated in the previous example to be y_1-y_2. Hence the weight of this lattice is $(y_1-y_3)(y_1-y_2)$.

The first row of the second lattice is $\langle(1,1)|E_1|(2,1)\rangle$ which we calculated in the previous example to be 1. The second row of the second lattice is $\langle(2,1)|E_2|(2,1)\rangle$ which we calculated in the previous example to be $(y_1-y_2)(y_1-y_4)$. Hence the weight of this lattice is $(y_1-y_2)(y_1-y_4)$.

Thus we have $K_{(2,1)/0/(2),(2,1)}(y) = (y_1-y_3)(y_1-y_2) + (y_1-y_2)(y_1-y_4)$.

Remark 5.4. If we take $-y_i$ instead of y_i, it is not clear that this method would yield a manifestly positive result.

We summarise Example 5.2 in the table below. We use Young diagrams to depict $E_r|\mu\rangle$

$E_r\vert\mu\rangle$	Result
$\emptyset \circledast \emptyset$	$\emptyset$
$\ydiagram{1} \circledast \emptyset$	$\ydiagram{1}$
$\ydiagram{1,1} \circledast \emptyset$	$\ydiagram{1,1}$
$\emptyset \circledast \ydiagram{1}$	$\ydiagram{1}$
$\ydiagram{1} \circledast \ydiagram{1}$	$(y_2 - y_3)\ydiagram{1} + \ydiagram{2} + \ydiagram{1,1}$
$\ydiagram{1,1} \circledast \ydiagram{1}$	$(y_1 - y_3)\ydiagram{1,1} + \ydiagram{2,1}$
$\emptyset \circledast \ydiagram{2}$	$\ydiagram{2}$
$\ydiagram{1} \circledast \ydiagram{2}$	$(y_2 - y_4)\ydiagram{2} + \ydiagram{3} + \ydiagram{2,1}$
$\ydiagram{1,1} \circledast \ydiagram{2}$	$(y_1 - y_4)\ydiagram{2,1} + \ydiagram{3,1}$
$\emptyset \circledast \ydiagram{3}$	$\ydiagram{3}$
$\ydiagram{1} \circledast \ydiagram{3}$	$(y_2 - y_5)\ydiagram{3} + \ydiagram{3,1}$
$\ydiagram{1,1} \circledast \ydiagram{3}$	$(y_1 - y_5)\ydiagram{3,1} + q\emptyset$
$\emptyset \circledast \ydiagram{1,1}$	$\ydiagram{1,1}$
$\ydiagram{1} \circledast \ydiagram{1,1}$	$(y_1 - y_3)\ydiagram{1,1} + \ydiagram{2,1}$

$\ydiagram{1,1} \circledast \ydiagram{1,1}$	$(y_1 - y_2)(y_1 - y_3)\ydiagram{1,1} + (y_1 - y_2)\ydiagram{2,1} + \ydiagram{2,2}$
$\emptyset \circledast \ydiagram{2,1}$	$\ydiagram{2,1}$
$\ydiagram{1} \circledast \ydiagram{2,1}$	$(y_1 - y_4)\ydiagram{2,1} + \ydiagram{3,1} + \ydiagram{2,2}$
$\ydiagram{1,1} \circledast \ydiagram{2,1}$	$(y_1 - y_2)(y_1 - y_4)\ydiagram{2,1} + (y_1 - y_2)\ydiagram{3,1} + (y_1 - y_4)\ydiagram{2,2} + \ydiagram{3,2}$
$\emptyset \circledast \ydiagram{3,1}$	$\ydiagram{3,1}$
$\ydiagram{1} \circledast \ydiagram{3,1}$	$(y_1 - y_5)\ydiagram{3,1} + \ydiagram{3,2} + q\emptyset$
$\ydiagram{1,1} \circledast \ydiagram{3,1}$	$(y_1 - y_2)(y_1 - y_5)\ydiagram{3,1} + (y_1 - y_5)\ydiagram{3,2} + q(y_1 - y_2)\emptyset + q\ydiagram{1}$
$\emptyset \circledast \ydiagram{2,2}$	$\ydiagram{2,2}$
$\ydiagram{1} \circledast \ydiagram{2,2}$	$(y_1 - y_3 + y_2 - y_4)\ydiagram{2,2} + \ydiagram{3,2}$
$\ydiagram{1,1} \circledast \ydiagram{2,2}$	$(y_1 - y_3)(y_1 - y_4)\ydiagram{2,2} + (y_1 - y_3)\ydiagram{3,2} + \ydiagram{3,3}$
$\emptyset \circledast \ydiagram{3,2}$	$\ydiagram{3,2}$
$\ydiagram{1} \circledast \ydiagram{3,2}$	$(y_1 - y_3 + y_2 - y_5)\ydiagram{3,2} + \ydiagram{3,3} + q\ydiagram{1}$
$\ydiagram{1,1} \circledast \ydiagram{3,2}$	$(y_1 - y_3)(y_1 - y_5)\ydiagram{3,2} + (y_1 - y_5)\ydiagram{3,3} + q(y_1 - y_3)\ydiagram{1} + q\ydiagram{2}$
$\emptyset \circledast \ydiagram{3,3}$	$\ydiagram{3,3}$

$\ydiagram{1} \circledast \ydiagram{3,3}$	$(y_1 - y_4 + y_2 - y_5)\ydiagram{3,3} + q\ydiagram{2}$
$\ydiagram{1,1} \circledast \ydiagram{3,3}$	$(y_1 - y_4)(y_1 - y_5)\ydiagram{3,3} + q(y_1 - y_4)\ydiagram{2} + q\ydiagram{3}$

5.2 Knutson-Tao Puzzles

This section will introduce Knutson-Tao puzzles [KT03]. We start with a description of the ordinary puzzles and then go on to a description of the equivariant puzzles.

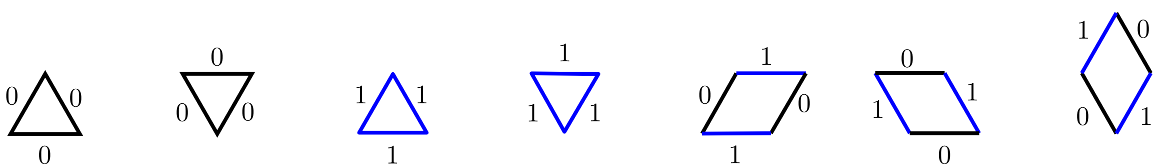
5.2.1 Ordinary Knutson-Tao Puzzles

These puzzles are interesting combinatorial objects as counting these puzzles with fixed edges gives the Littlewood-Richardson coefficients. We should first recall the definition of Littlewood-Richardson coefficients and then give the definition of the puzzles that count them.

Definition 5.5. [Mac95, Chapter I, (5.2)] Let s_λ be the Schur polynomial indexed by the partition λ. The Littlewood-Richardson coefficients are given by

$$s_\mu s_\lambda = \sum_\nu c^\nu_{\mu,\lambda} s_\nu. \tag{5.42}$$

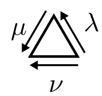
Definition 5.6. [KT03, Section 1.1] A puzzle is an equilateral triangle with 01-words along its outer edges and a filling using the following puzzle pieces:



Before we describe how to use these puzzles to calculate the Littlewood Richardson coefficients, we must first recall how to convert between 01-words and partitions. If λ is the partition which corresponds to a 01-word, then λ_1 is the number of 0's to the left of the final (k^{th}) 1, λ_2 is the number of 0's to the left of the $k-1^{\text{st}}$ 1, and so on.

The following theorem was proved by Knutson and Tao [KT03]

Theorem 5.7. *[KT03, Theorem 1] The Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ is the number of puzzles such that the* 01*-word for μ is along the NW side, the* 01*-word for λ is along the NE side, and the* 01*-word for ν is along the bottom side as follows:*



Remark 5.8. This theorem is significant because it proves that the Littlewood-Richardson coefficients are positive integers.

We can use these puzzles together with an extra puzzle piece, the equivariant piece, to obtain puzzles which calculate equivariant Littlewood-Richardson coefficients. These puzzles are described below.

5.2.2 Equivariant Knutson-Tao Puzzles

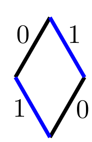
Some of these puzzles will later be seen to be in bijection with our A operator from the Yang-Baxter algebra. To motivate the definition of these puzzles recall the definition of factorial Schur polynomials, $s_\lambda(x|y)$, from the article by MacDonald [Mac92] as described in Section 2.1.

Definition 5.9. [MS99, Section 3, (8)] The factorial Littlewood-Richardson coefficients $c^{\nu}_{\mu,\lambda}(y)$, which are polynomials in $y_1, \dots, y_n$, are given by

$$s_\mu(x|y)s_\lambda(x|y) = \sum_{\nu} c^{\nu}_{\mu,\lambda}(y)s_\nu(x|y). \tag{5.43}$$

The puzzles which need to be counted to find this type of Littlewood-Richardson coefficient involve an additional piece. This piece is called the equivariant piece as the geometric interpretation involves the T-equivariant cohomology of the Grassmannian.

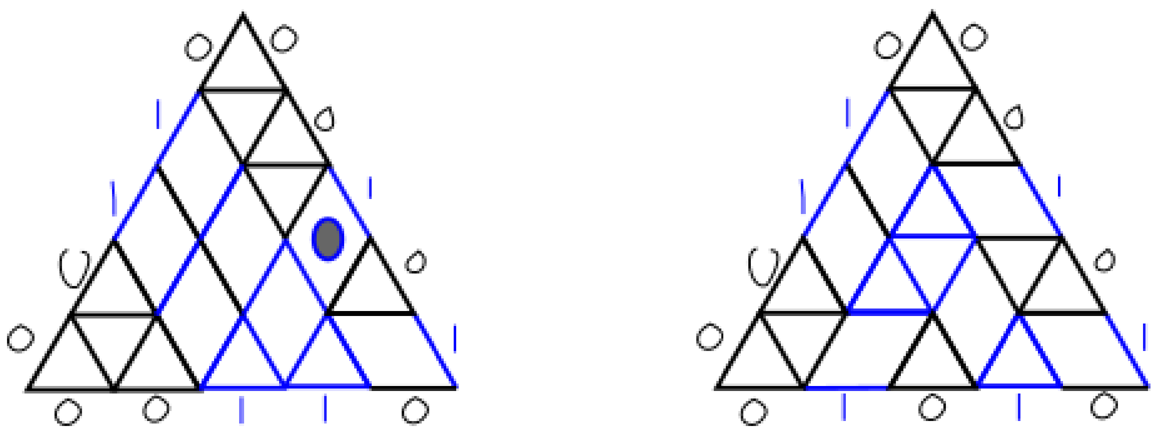
Definition 5.10. [KT03, Section 1.2] An equivariant puzzle is an equilateral triangle with 01-words along its outer edges and a filling using the same puzzle pieces as in Definition 5.6, with the following additional piece also allowed.



Knutson and Tao prove in Theorem 2 of [KT03] that equivariant puzzles as described above can be used to calculate equivariant Littlewood-Richardson coefficients.

Instead of counting the number of puzzles, we now have a weighted sum of puzzles with each equivariant piece in a puzzle giving a factor of the form $y_i - y_j$. To find i, we look at the lower right diagonal edge of the equivariant piece and follow it down the diagonal column in the same direction until we reach the bottom edge of the puzzle. Then i is the position we end up at in the 01-word from right to left. To find j, we look at the lower left diagonal edge of the equivariant piece and follow it down the diagonal column in the same direction until we reach the bottom edge of the puzzle. Then j is the position we end up at in the 01-word from right to left. Then, the total weight of the puzzle is the product of the factors obtained from all its equivariant pieces. We demonstrate this with an example.

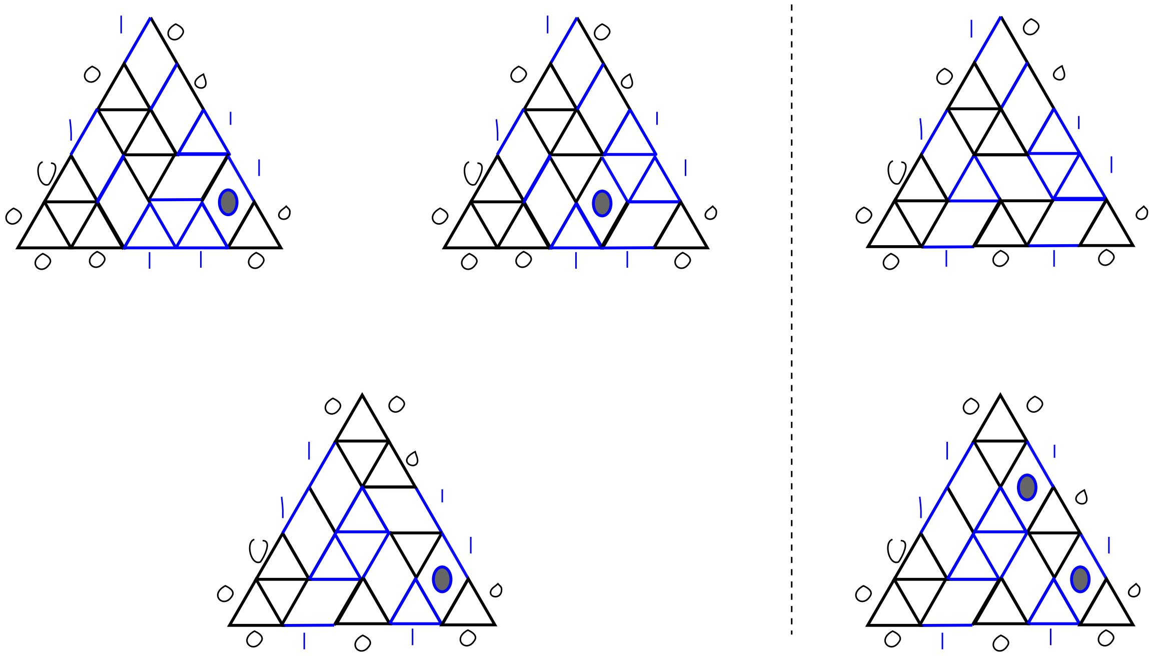
Example 5.11. Here are the Knutson-Tao puzzles to calculate the coefficients for the product $s_{(1,1)}s_{(1)}$. We now know this corresponds to $E_2|(1)\rangle$ in Example 5.2. We mark the equivariant pieces with a circle to identify them more easily within the puzzle.



Looking at the first puzzle above, there is one equivariant piece. If we follow the lower right diagonal side down to the bottom of the puzzle we end up in column 1. If we follow the lower left diagonal side down to the bottom of the puzzle we end up in column 3. Therefore this piece contributes a factor of $y_1 - y_3$ to the weight of the puzzle. Since there are no other equivariant pieces in either puzzle, we conclude that $s_{(1,1)}s_{(1)} = (y_1 - y_3)|(1,1)\rangle + |(2,1)\rangle$. Checking this against Example 5.2, we see that this is as expected.

In order to calculate the quantum equivariant Kostka numbers, we must apply the puzzle rule several times. To do Example 5.3 using puzzles we need one puzzle for each lattice row.

Example 5.12. We calculate $K_{(2,1)/0/(2),(2,1)}(y)$ using puzzles. The puzzles we need are below.



For the top row we put the 01-word for $(1,1)$ on the NE edge and the 01-word for (1) on the NW edge. The puzzles in the second row are the second application of the puzzle rule, multiplying the factorial Schur polynomial corresponding to the 01 from the S edge of the puzzles above by e_2, so we put the 01-word which is on the south side of the puzzles above on the NE edge of a puzzle in the second row with the 01-word for (2) on the NW edge. We are looking for the coefficient of $s_{(2,1)}$, so we put the 01-word for $(2,1)$ on the S side of the puzzles in the bottom row.

To get the result we take the product of the weights of the top and bottom left puzzles, top middle and bottom left puzzles, and top right and bottom right puzzles and then add them together.

The top left puzzle has weight $y_2 - y_1$. The top bottom left puzzle has weight $y_2 - y_1$.

The top middle puzzle has weight $y_3 - y_2$.

The top right puzzle has weight 1. The bottom right puzzle has weight $(y_4 - y_1)(y_2 - y_1)$.

Therefore we find that $K_{(2,1)/0/(2),(2,1)}(y) = (y_2 - y_1)(y_3 - y_2 + y_2 - y_1) + (y_4 - y_1)(y_2 - y_1) = (y_2 - y_1)(y_3 - y_1) + (y_4 - y_1)(y_2 - y_1)$.

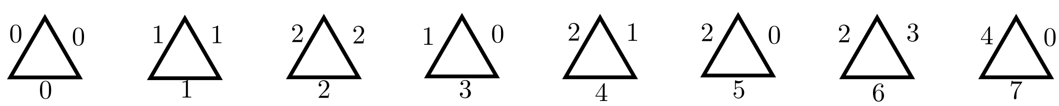
5.2.3 Quantum Equivariant Puzzles

The equivariant puzzles we have just described can only be used to find the Littlewood-Richardson coefficients when $q = 0$. There are more general puzzles defined in the paper by Buch [Buc15] which can be used for the case where $q \neq 0$. We will refer to these puzzles as quantum equivariant puzzles.

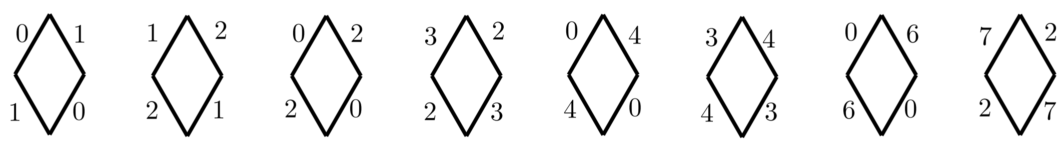
The outer edges of quantum equivariant puzzles are 012-words, so before defining the puzzles we will first explain how to convert 01-words we have previously obtained for partitions into these 012-words. The puzzles we will be considering here will always have a vertical strip on the NW side, so we will give the conversion rule only for this case. It is slightly different for the general case, which can be found in [Buc15]. As we are only interested in modified versions of these puzzles, we omit the general case.

Definition 5.13. [Buc15, Section 2.3] Let w_λ be the 01-word for a partition λ. Define a map $J_{NE,NW}$ from the set of 01-words for partitions in $(k, n-k)$ to the set of 012-words as follows. $J_{NE,NW}(w_\lambda)$ is w_λ with all the 1's replaced by 2's, and then replacing the first 0 with a 1 and the last 2 with a 1. Define a second map J_S on the same sets as follows. $J_S(w_\lambda)$ os w_λ with all the 1's replaced by 2's, and then replacing the last 0 with a 1 and the first 2 with a 1.

Definition 5.14. [Buc15, Section 2.2] A quantum equivariant puzzle is an equilateral triangle with north-east and north-west edges given by $J_{NE,NW}(w_\lambda)$ for certain partitions λ, and south edge give by $J_S(w_\lambda)$ for some partition λ, in the same directions as depicted in Theorem 5.7 and filled using the triangular pieces, for which we allow rotation:



and the equivariant pieces, for which we do not allow rotation:



Buch proves in Corollary 2.4 of [Buc15] that these puzzles can be used to calculate the Gromov-Witten invariants for the quantum equivariant cohomology ring of the Grassmannian. If we multiply the y_i's by a factor of -1, then the way in which the Gromov-Witten invariants are calculated here means that they are manifestly positive.

Remark 5.15. To reduce the puzzles defined above to the equivariant case we allow the pieces whose sides are labelled with 0's. 2's and 5's only, and disregard the remaining pieces. We must also use 02-words for the partitions obtained by converting the 1's to 2's in the first step described in the maps in Definition 5.13, but omitting the second step of swapping one 0 and one 2 on each side with 1's. To then obtain the equivariant puzzles we change the 2's to 1's and omit any line which is labelled by a 5.

5.3 Statistics on Diagrams

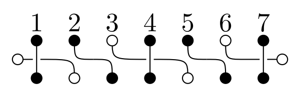
We are now ready to give the main result of this thesis, so we will now define our statistics on lattice diagrams. This result is new.

5.3.1 Definitions

Definition 5.16. A diagram $\mathcal{D}_\lambda^\mu$ consists of a pair λ, μ of Maya diagrams (or equivalently binary strings) of length n and an allowed lattice configuration using the vertices or . μ gives the top values of the vertical edges and λ gives the bottom values. We call a connecting line and we call an avoidance. We allow an avoidance to have any arrangement of black and white go-stones, as long as stones connected by a line are the same colour. Define the position of a go-stone in a diagram to be the column the go-stone occupies, numbering from left to right. When discussing the position of a vertex, we mean the position of the go-stones on the vertical edges. Define $\mathscr{C}(\mathcal{D}_\lambda^\mu) \subset \mathbb{N}$ to be the set of positions of all connecting lines in the diagram $\mathcal{D}_\lambda^\mu$.

Remark 5.17. Our lattice diagrams here do not involve Boltzmann weights, but we can see that they have the same allowed vertex configurations as the vertex model from section 1.

Example 5.18. An example of a diagram $\mathcal{D}_\lambda^\mu$:



where we have labelled the positions. Here $\mathscr{C}(\mathcal{D}_\lambda^\mu) = \{1, 4, 7\}$. We will use this as our guiding example in this section.

Definition 5.19. We call a connecting line in position i admissible if there is an avoidance in position j with $j < i$ in the lattice row. Define the admissible set $\mathscr{A}(\mathcal{D}_\lambda^\mu) \subseteq \mathscr{C}(\mathcal{D}_\lambda^\mu)$ to be the set of positions of admissible connecting lines in a given diagram $\mathcal{D}_\lambda^\mu$. Let $\mathcal{P}(\mathscr{A}(\mathcal{D}_\lambda^\mu))$ denote the power set of $\mathscr{A}(\mathcal{D}_\lambda^\mu)$. Let $S \in \mathcal{P}(\mathscr{A}(\mathcal{D}_\lambda^\mu))$, then for $s \in S$ define $\mathcal{L}(s)$ to be 1+ the number of connecting lines $a \in \mathscr{C}(\mathcal{D}_\lambda^\mu) \setminus S$, such that $a < s$.

Example 5.20. For the diagram in our guiding example, Example 5.18, we have $n = 7$. The connecting lines in positions 4 and 7 are admissible. So $\mathscr{A}(\mathcal{D}_\lambda^\mu) = \{4, 7\}$. Let $S = \{4\}$, then $\mathcal{L}(4) = 1$. If we instead take $S = \{7\}$ we have $\mathcal{L}(7) = 2$.

Definition 5.21. Fix an integer $r \geq 0$ and let $\lambda/d/\mu$ be a vertical t-strip with $0 \leq t \leq r$. This means that $|\lambda/d/\mu| := |\lambda| + dn - |\mu| = t$. The r-weight of $\mathcal{D}_\lambda^\mu$ is defined as follows:

$$wt_r(\mathcal{D}_\lambda^\mu) = q^d \sum_{\substack{S \in \mathcal{P}(\mathscr{A}(\mathcal{D}_\lambda^\mu)) \\ |S| = r-t}} \prod_{s \in S} (y_{\mathcal{L}(s)} - y_s). \tag{5.44}$$

Example 5.22. Using our guiding example, Example 5.18, and taking $r = 3$ and $t = 2$, we find that $wt_3(\mathcal{D}^{\mu}_{\lambda}) = y_2 - y_4 + y_3 - y_7$.

We now have all the definitions we require.

5.3.2 The Main Result

We claim that we have the following.

Proposition 5.23. *Recall the definition of the operator E_r from* (4.2) *and let $d = 0, 1$. Then we have:*

$$E_r|\mu\rangle = \sum_{\lambda} wt_r(\mathcal{D}^{\mu}_{\lambda})|\lambda\rangle, \tag{5.45}$$

where the sum is over all partitions λ such that $\lambda/d/\mu$ is a vertical t-strip with $0 \leq t \leq r$.

Proof. Recall that $E(x)|\mu\rangle = \sum_r (x|y)^{k-r} E_r|\mu\rangle$. If we expand this we get

$$E(x)|\mu\rangle = \sum_{\lambda} \left(\sum_{s=0}^{k-|\lambda/d/\mu|} (x|y)^{k-|\lambda/d/\mu|-s} f_{\lambda/d/\mu}(s) \right) |\lambda\rangle, \tag{5.46}$$

where the sum is over all partitions λ such that $\lambda/d/\mu$ is a vertical t-strip with $0 \leq t \leq r$.

We would like to prove that $wt_r(\mathcal{D}^{\mu}_{\lambda}) = f_{\lambda/d/\mu}(r-t)$ where $f_{\lambda/d/\mu}(r-t)$ is the coefficient of $(x|y)^{k-r}$ in the expansion of the partition function $\langle\lambda|E(x)|\mu\rangle$ of $\mathcal{D}^{\mu}_{\lambda}$ into factorial powers.

For fixed λ and μ, let $\mathscr{A}(\mathcal{D}^{\mu}_{\lambda}) = \{a_1, \ldots, a_l\}$ with $a_1 < \cdots < a_l$ be the set of positions of admissible lines in $\mathcal{D}^{\mu}_{\lambda}$ and let a_0 be the position of the rightmost connecting line which is not admissible, if there are no inadmissible connecting lines, take $a_0 = 0$. Then the partition function for $\mathcal{D}^{\mu}_{\lambda}$ is

$$\langle\lambda|E(x)|\mu\rangle = q^d (x|y)^{a_0} (x - y_{a_1}) \cdots (x - y_{a_l}). \tag{5.47}$$

We expand this into factorial powers as follows. First we do the expansion:

$$\begin{aligned} \langle\lambda|E(x)|\mu\rangle &= q^d (x|y)^{a_0} (x - y_{a_0+1} + y_{a_0+1} - y_{a_1})(x - y_{a_2}) \cdots (x - y_{a_l}) = \\ & q^d (x|y)^{a_0+1} (x - y_{a_2}) \cdots (x - y_{a_l}) + (y_{a_0+1} - y_{a_1}) q^d (x|y)^{a_0} (x - y_{a_2}) \cdots (x - y_{a_l}). \end{aligned} \tag{5.48}$$

We then repeat this expansion process on the brackets $(x - y_{a_i})$ for $i = 2, \ldots, l$ in increasing order for the terms obtained in the previous expansion step. This process leaves only terms which are factorial powers. We know from the definition of the partition function that each bracket $(x - y_{a_i})$ corresponds to an admissible line. We will now show that there is a bijection between the set of coefficients $q^d \prod_i (y_{a_0+j_{a_i}} - y_{a_i})$ of the factorial powers in the final expansion and $\mathcal{P}(\mathscr{A}(\mathcal{D}^{\mu}_{\lambda}))$. Note that $j_{a_i} = 1+$ the number of a_k with $k < i$ such that the term $(y_{a_0+j_{a_k}} - y_{a_k})$ is not contained in the product. This is evident from the way we have constructed the expansion.

Let $S \in \mathcal{P}(\mathscr{A}(\mathcal{D}^{\mu}_{\lambda}))$ be the subset corresponding to an arbitrary term under this bijection. If at the first step of the expansion we choose $(x - y_{a_0+1})$ then $a_1 \notin S$, and if we choose $(y_{a_0+1} - y_{a_1})$ then $a_1 \in S$. At each subsequent step we make the same choice for each a_i. We see that this choice at each step is exactly how $\mathcal{P}(\mathscr{A}(\mathcal{D}^{\mu}_{\lambda}))$ can be constructed, so the set of coefficients $q^d \prod_i (y_{a_0+j_i} - y_{a_i})$ and $\mathcal{P}(\mathscr{A}(\mathcal{D}^{\mu}_{\lambda}))$ are constructed identically, and hence the identification we have just described is indeed a bijection. Let $S = \{s_1, \dots, s_{r-t}\}$. If $|S| = r - t$ then the term which S corresponds to under the bijection is of degree $k - r$. Hence

$$\langle \lambda | E(x) | \mu \rangle = q^d \sum_r (x|y)^{k-r} \sum_{\substack{S \in \mathcal{P}(\mathscr{A}(\mathcal{D}^{\mu}_{\lambda})) \\ |S| = r-t}} (y_{a_0+j_{s_1}} - y_{s_1}) \cdots (y_{a_0+j_{s_{r-t}}} - y_{s_{r-t}}). \tag{5.49}$$

Noting that $a_0 + j_s = 1 + |\{a \in \mathbb{C}(\mathcal{D}^{\mu}_{\lambda}) \setminus S : a < s\}| = \mathcal{L}(s)$ proves the proposition. $\square$

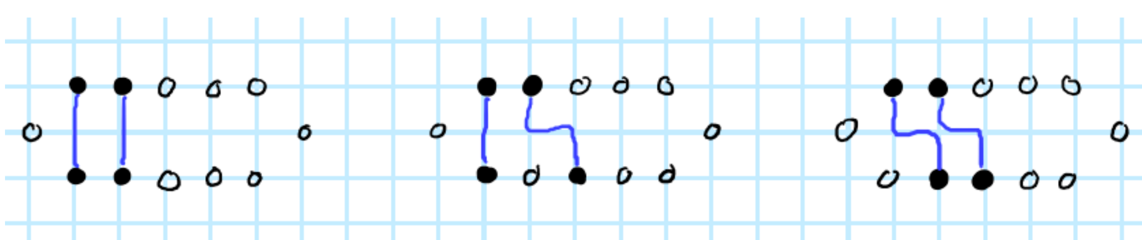
Corollary 5.24. *By using Corollary 4.18 we obtain a way to calculate the quantum equivariant Kostka numbers via our statistics, using diagrams that can have more than one lattice row and then calculating the statistics for each row.*

Remark 5.25. When using this method to calculate the equivariant Gromov-Witten invariants, it appears that we can account for the τ^{j-1} operator by adding $j - 1$ to the $\mathcal{L}$ statistic. In terms of the lattice j will be the number of the row in the lattice that the operator $\tau^{j-1} E_{\lambda' - i + j}$ is acting on. If this is the case, it would make calculating the equivariant Gromov-Witten invariants using this new method simpler than some of the others we have looked at. For the method of Gorbounov and Korff [GK14] we would need to expand the lattice weight for each row separately into shifted factorial powers, which is computation intensive. For the method of Bertiger et al [BEMT22] we would need a series of Young diagrams as their method is only for the Pieri rule, which would need to be applied repeatedly.

We now revisit Example 5.2 to compare our method with the method from [GK14].

Example 5.26. We will once again calculate $E(x)|\mu\rangle$, this time using our statistics. When calculating wt_0 in this example we need a subset of $2 - t$ admissible lines, wt_1 needs $1 - t$, and wt_2 needs $0 - t$. If the necessary size of the subset is negative, that means we did not have the condition $t \leq r$, so the weight is 0. Recall t is the height of the vertical strip $\lambda/d/\mu$. For this example, $q = 0$ when the horizontal boundary of the lattice is labelled with a 0 and $q = 1$ when it is labelled with a 1. We show how to calculate the statistics in detail for $E(x)|(1,1)\rangle$.

For $E(x)|\emptyset\rangle$ we have



$\mathcal{D}^{\emptyset}_{\emptyset}$ $\mathcal{D}^{\emptyset}_{(1)}$ $\mathcal{D}^{\emptyset}_{(1,1)}$

For the first lattice above $t = 0$, for the second $t = 1$, and for the third $t = 2$.

The only non-zero weights in this case are

$$wt_0\left(\mathcal{D}^{\emptyset}_{\emptyset}\right) = 1, \quad wt_1\left(\mathcal{D}^{\emptyset}_{(1)}\right) = 1, \quad wt_2\left(\mathcal{D}^{\emptyset}_{(1,1)}\right) = 1. \tag{5.50}$$

Therefore we find that $E_0|\emptyset\rangle = |\emptyset\rangle, E_1|\emptyset\rangle = |(1)\rangle$, and $E_2|\emptyset\rangle = |(1,1)\rangle$.

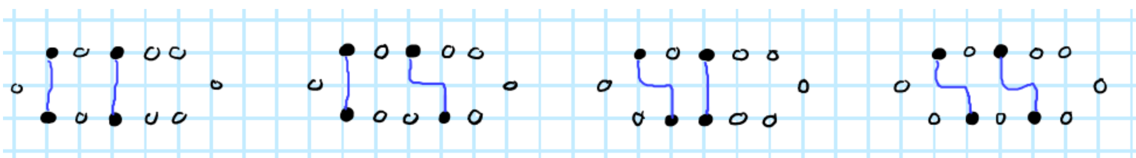
We will also give this result using Young diagrams to depict $E_r|\emptyset\rangle$ as in the table at the end of Section 5.1 in order to make it easier to see what is happening in terms of partitions.

$$\emptyset \circledast \emptyset = \emptyset,$$

$$\ydiagram{1} \circledast \emptyset = \ydiagram{1},$$

$$\ydiagram{1,1} \circledast \emptyset = \ydiagram{1,1}.$$

For $E(x)|(1)$ we have



$\mathcal{D}^{(1)}_{(1)}$ $\qquad$ $\mathcal{D}^{(1)}_{(2)}$ $\qquad$ $\mathcal{D}^{(1)}_{(1,1)}$ $\qquad$ $\mathcal{D}^{(1)}_{(2,1)}$

For the first lattice above $t = 0$, for the second and third $t = 1$, and for the fourth $t = 2$.

The only non-zero weights in this case are

$$\begin{aligned} wt_0\left(\mathcal{D}^{(1)}_{(1)}\right) = 1, \quad wt_1\left(\mathcal{D}^{(1)}_{(1)}\right) = y_2 - y_3, \quad wt_1\left(\mathcal{D}^{(1)}_{(2)}\right) = 1, \\ wt_1\left(\mathcal{D}^{(1)}_{(1,1)}\right) = 1, \quad wt_2\left(\mathcal{D}^{(1)}_{(1,1)}\right) = y_1 - y_3, \quad wt_2\left(\mathcal{D}^{(1)}_{(2,1)}\right) = 1. \end{aligned} \tag{5.51}$$

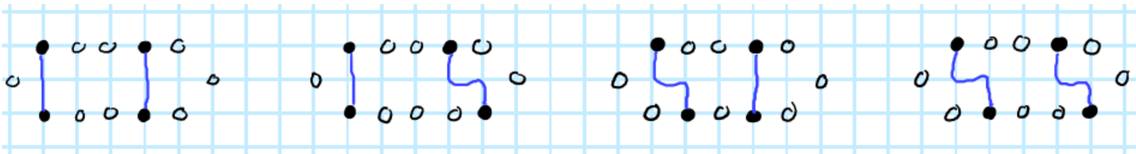
Therefore we find that $E_0|(1)\rangle = |(1)\rangle$, $E_1|(1)\rangle = (y_2 - y_3)|(1)\rangle + |(2)\rangle + |(1,1)\rangle$, and $E_2|(1)\rangle = (y_1 - y_3)|(1,1)\rangle + |(2,1)\rangle$, and also

$$\emptyset \circledast \ydiagram{1} = \ydiagram{1},$$

$$\ydiagram{1} \circledast \ydiagram{1} = (y_2 - y_3)\ydiagram{1} + \ydiagram{2} + \ydiagram{1,1},$$

$$\ydiagram{1,1} \circledast \ydiagram{1} = (y_1 - y_3)\ydiagram{1,1} + \ydiagram{2,1}.$$

For $E(x)|(2)$ we have



$\mathcal{D}^{(2)}_{(2)}$ $\mathcal{D}^{(2)}_{(3)}$ $\mathcal{D}^{(2)}_{(2,1)}$ $\mathcal{D}^{(2)}_{(3,1)}$

For the first lattice above $t = 0$, for the second and third $t = 1$, and for the fourth $t = 2$.

The only non-zero weights in this case are

$$\begin{aligned} wt_0\left(\mathcal{D}^{(2)}_{(2)}\right) = 1, \quad wt_1\left(\mathcal{D}^{(2)}_{(2)}\right) = y_2 - y_4, \quad wt_1\left(\mathcal{D}^{(2)}_{(3)}\right) = 1, \\ wt_1\left(\mathcal{D}^{(2)}_{(2,1)}\right) = 1, \quad wt_2\left(\mathcal{D}^{(2)}_{(2,1)}\right) = y_1 - y_4, \quad wt_2\left(\mathcal{D}^{(1)}_{(3,1)}\right) = 1. \end{aligned} \tag{5.52}$$

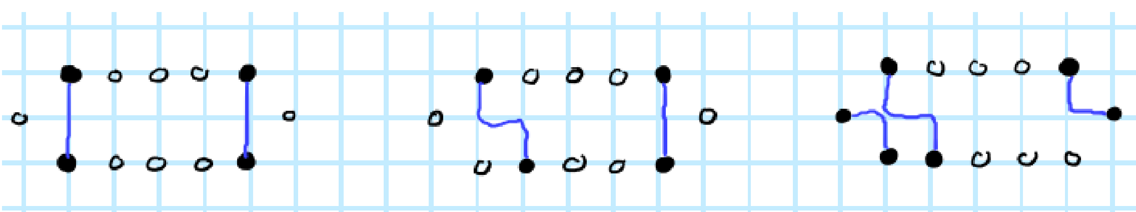
Therefore we find that $E_0|(2)\rangle = |(2)\rangle$, $E_1|(2)\rangle = (y_2 - y_4)|(2)\rangle + |(3)\rangle + |(2,1)\rangle$, and $E_2|(2)\rangle = (y_1 - y_4)|(2,1)\rangle + |(3,1)\rangle$, and also

$$\emptyset \circledast \ydiagram{2} = \ydiagram{2},$$

$$\ydiagram{1} \circledast \ydiagram{2} = (y_2 - y_4)\ydiagram{2} + \ydiagram{3} + \ydiagram{2,1},$$

$$\ydiagram{1,1} \circledast \ydiagram{2} = (y_1 - y_4)\ydiagram{2,1} + \ydiagram{3,1}.$$

For $E(x)|(3)$ we have



$\mathcal{D}^{(3)}_{(3)}$ $\mathcal{D}^{(3)}_{(3,1)}$ $\mathcal{D}^{(3)}_{\emptyset}$

For the first lattice above $t = 0$, for the second $t = 1$, and for the third $t = 2$.

The only non-zero weights in this case are

$$\begin{aligned} wt_0\left(\mathcal{D}^{(3)}_{(3)}\right) = 1, \quad wt_1\left(\mathcal{D}^{(3)}_{(3)}\right) = y_2 - y_5, \quad wt_1\left(\mathcal{D}^{(3)}_{(3,1)}\right) = 1, \\ wt_2\left(\mathcal{D}^{(3)}_{(3,1)}\right) = y_1 - y_5, \quad wt_2\left(\mathcal{D}^{(3)}_{\emptyset}\right) = q. \end{aligned} \tag{5.53}$$

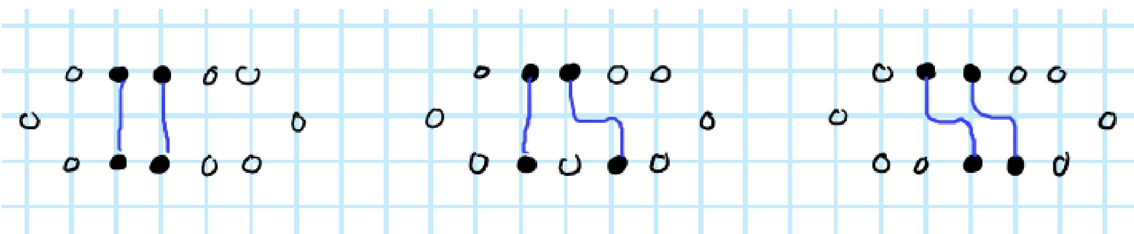
Therefore we find that $E_0|(3)\rangle = |(3)\rangle$, $E_1|(3)\rangle = (y_2 - y_5)|(3)\rangle + |(3,1)\rangle$, and $E_2|(3)\rangle = (y_1 - y_5)|(3,1)\rangle + q|\emptyset\rangle$, and also

$$\emptyset \circledast \ydiagram{3} = \ydiagram{3},$$

$$\ydiagram{1} \circledast \ydiagram{3} = (y_2 - y_5)\ydiagram{3} + \ydiagram{3,1},$$

$$\ydiagram{1,1} \circledast \ydiagram{3} = (y_1 - y_5)\ydiagram{3,1} + q\emptyset.$$

For $E(x)|(1,1)$ we have



$\mathcal{D}^{(1,1)}_{(1,1)}$ $\mathcal{D}^{(1,1)}_{(2,1)}$ $\mathcal{D}^{(1,1)}_{(2,2)}$

For the first lattice above $t = 0$, for the second $t = 1$, and for the third $t = 2$. All of the connecting lines in these diagrams are admissible.

For $\mathcal{D}^{(1,1)}_{(1,1)}$ the set of admissible lines is $\mathscr{A}(\mathcal{D}^{(1,1)}_{(1,1)}) = \{2,3\}$. When $r = 0$ the size of the subset of admissible lines, S, that we need is $r - t = 0$. When $r = 1$ the size of the subset of admissible lines, S, that we need is $r - t = 1$ and so the possibilities for S are $\{2\}$, with $\mathcal{L}(2) = 1$ and $\{3\}$ with $\mathcal{L}(3) = 2$. When $r = 2$ the size of the subset of admissible lines, S, that we need is $r - t = 2$ and so the only possibility for S is $\{2,3\}$ with $\mathcal{L}(2) = 1$ and $\mathcal{L}(3) = 1$.

For $\mathcal{D}^{(1,1)}_{(2,1)}$ the set of admissible lines is $\mathscr{A}(\mathcal{D}^{(1,1)}_{(2,1)}) = \{2\}$. When $r = 0$ we have $r - t = -1 < 0$ therefore the weight is 0. When $r = 1$ the size of the subset of admissible lines, S, that we need is $r - t = 0$. When $r = 2$ the size the size of the subset of admissible lines, S, that we need is $r - t = 1$ and so S must be $\{2\}$ with $\mathcal{L}(2) = 1$.

For $\mathcal{D}^{(1,1)}_{(2,2)}$ there are no connecting lines, so no admissible lines. When $r = 0$ and $r = 1$ we have $r - t < 0$ and so these weights are 0. When $r = 2$ the size of the subset of admissible lines, S, that we need is $r - t = 0$.

Therefore, the only non-zero weights in this case are

$$\begin{aligned} &wt_0\left(\mathcal{D}^{(1,1)}_{(1,1)}\right) = 1, \quad wt_1\left(\mathcal{D}^{(1,1)}_{(1,1)}\right) = y_1 - y_2 + y_2 - y_3, \quad wt_1\left(\mathcal{D}^{(1,1)}_{(2,1)}\right) = 1, \\ &wt_2\left(\mathcal{D}^{(1,1)}_{(1,1)}\right) = (y_1 - y_2)(y_1 - y_3), \quad wt_2\left(\mathcal{D}^{(1,1)}_{(2,1)}\right) = y_1 - y_2, \quad wt_2\left(\mathcal{D}^{(1,1)}_{(2,2)}\right) = 1. \end{aligned} \tag{5.54}$$

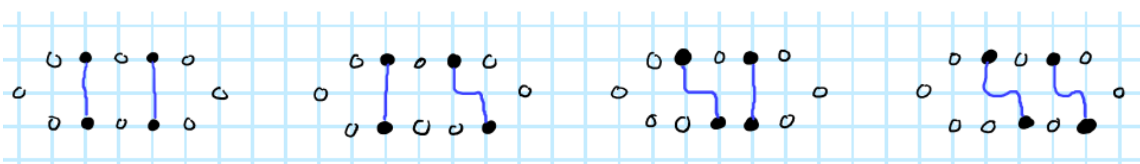
Therefore we find that $E_0|(1,1)\rangle = |(1,1)\rangle$, $E_1|(1,1)\rangle = (y_1 - y_3)|(1,1)\rangle + |(2,1)\rangle$, and $E_2|(1,1)\rangle = (y_1 - y_2)(y_1 - y_3)|(1,1)\rangle + (y_1 - y_2)|(2,1)\rangle + |(2,2)\rangle$, and also

$$\emptyset \circledast \ydiagram{1,1} = \ydiagram{1,1},$$

$$\ydiagram{1} \circledast \ydiagram{1,1} = (y_1 - y_3)\ydiagram{1,1} + \ydiagram{2,1},$$

$$\ydiagram{1,1} \circledast \ydiagram{1,1} = (y_1 - y_2)(y_1 - y_3)\ydiagram{1,1} + (y_1 - y_2)\ydiagram{2,1} + \ydiagram{2,2}.$$

For $E(x)|(2,1)$ we have



$\mathcal{D}^{(2,1)}_{(2,1)}$ $\mathcal{D}^{(2,1)}_{(3,1)}$ $\mathcal{D}^{(2,1)}_{(2,2)}$ $\mathcal{D}^{(2,1)}_{(3,2)}$

For the first lattice above $t = 0$, for the second and third $t = 1$, and for the fourth $t = 2$.

The only non-zero weights in this case are

$$\begin{gathered} wt_0\left(\mathcal{D}^{(2,1)}_{(2,1)}\right) = 1, \quad wt_1\left(\mathcal{D}^{(2,1)}_{(2,1)}\right) = y_1 - y_2 + y_2 - y_4, \quad wt_1\left(\mathcal{D}^{(2,1)}_{(3,1)}\right) = 1, \\ wt_1\left(\mathcal{D}^{(2,1)}_{(2,2)}\right) = 1, \quad wt_2\left(\mathcal{D}^{(2,1)}_{(2,1)}\right) = (y_1 - y_2)(y_1 - y_4), \quad wt_2\left(\mathcal{D}^{(2,1)}_{(3,1)}\right) = y_1 - y_2, \\ wt_2\left(\mathcal{D}^{(2,1)}_{(2,2)}\right) = y_1 - y_4, \quad wt_2\left(\mathcal{D}^{(2,1)}_{(3,2)}\right) = 1. \end{gathered} \tag{5.55}$$

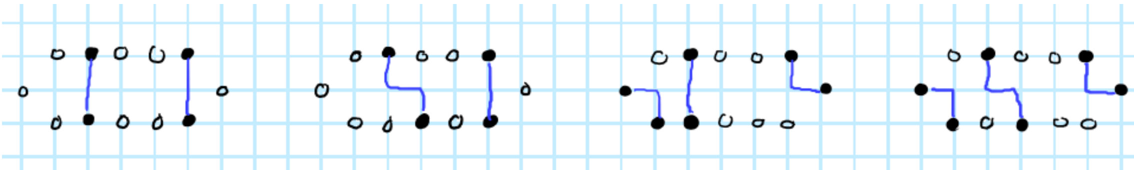
Therefore we find that $E_0|(2,1)\rangle = |(2,1)\rangle$, $E_1|(2,1)\rangle = (y_1 - y_4)|(2,1)\rangle + |(3,1)\rangle + |(2,2)\rangle$, and $E_2|(2,1)\rangle = (y_1 - y_2)(y_1 - y_4)|(2,1)\rangle + (y_1 - y_2)|(3,1)\rangle + (y_1 - y_4)|(2,2)\rangle + |(3,2)\rangle$, and also

$$\emptyset \circledast \ydiagram{2,1} = \ydiagram{2,1},$$

$$\ydiagram{1} \circledast \ydiagram{2,1} = (y_1 - y_4)\ydiagram{2,1} + \ydiagram{3,1} + \ydiagram{2,2},$$

$$\ydiagram{1,1} \circledast \ydiagram{2,1} = (y_1 - y_2)(y_1 - y_4)\ydiagram{2,1} + (y_1 - y_2)\ydiagram{3,1} + (y_1 - y_4)\ydiagram{2,2} + \ydiagram{3,2}.$$

For $E(x)|(3,1)$ we have



$\mathcal{D}^{(3,1)}_{(3,1)}$ $\mathcal{D}^{(3,1)}_{(3,2)}$ $\mathcal{D}^{(3,1)}_{\emptyset}$ $\mathcal{D}^{(3,1)}_{(1)}$

For the first lattice above $t = 0$, for the second and third $t = 1$, and for the fourth $t = 2$.

The only non-zero weights in this case are

$$\begin{gathered} wt_0\left(\mathcal{D}^{(3,1)}_{(3,1)}\right) = 1, \quad wt_1\left(\mathcal{D}^{(3,1)}_{(3,1)}\right) = y_1 - y_2 + y_2 - y_5, \quad wt_1\left(\mathcal{D}^{(3,1)}_{(3,2)}\right) = 1, \\ wt_1\left(\mathcal{D}^{(3,1)}_{\emptyset}\right) = q, \quad wt_2\left(\mathcal{D}^{(3,1)}_{(3,1)}\right) = (y_1 - y_2)(y_1 - y_5), \quad wt_2\left(\mathcal{D}^{(3,1)}_{(3,2)}\right) = y_1 - y_5, \\ wt_2\left(\mathcal{D}^{(3,1)}_{\emptyset}\right) = q(y_1 - y_2), \quad wt_2\left(\mathcal{D}^{(3,1)}_{(1)}\right) = q. \end{gathered} \tag{5.56}$$

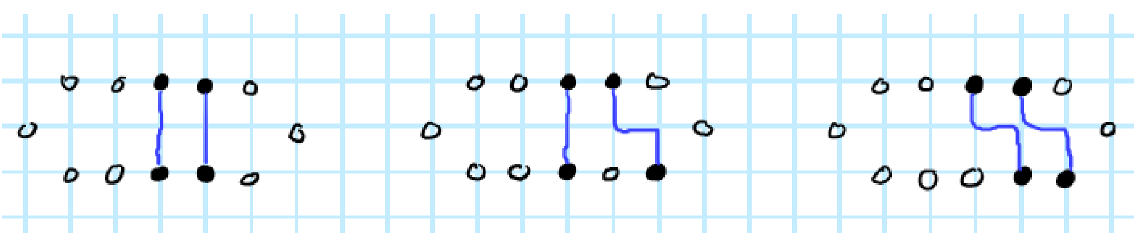
Therefore we find that $E_0|(3,1)\rangle = |(3,1)\rangle$, $E_1|(3,1)\rangle = (y_1 - y_5)|(3,1)\rangle + |(3,2)\rangle + q|\emptyset\rangle$, and $E_2|(3,1)\rangle = (y_1 - y_2)(y_1 - y_5)|(3,1)\rangle + (y_1 - y_5)|(3,2)\rangle + q(y_1 - y_2)|\emptyset\rangle + q|(1)\rangle$, and also

$$\emptyset \circledast \ydiagram{3,1} = \ydiagram{3,1},$$

$$\ydiagram{1} \circledast \ydiagram{3,1} = (y_1 - y_5)\ydiagram{3,1} + \ydiagram{3,2} + q\emptyset,$$

$$\ydiagram{1,1} \circledast \ydiagram{3,1} = (y_1 - y_2)(y_1 - y_5)\ydiagram{3,1} + (y_1 - y_5)\ydiagram{3,2} + q(y_1 - y_2)\emptyset + q\ydiagram{1}.$$

For $E(x)|(2,2)$ we have



$\mathcal{D}^{(2,2)}_{(2,2)}$ $\mathcal{D}^{(2,2)}_{(3,2)}$ $\mathcal{D}^{(2,2)}_{(3,3)}$

For the first lattice above $t = 0$, for the second $t = 1$, and for the third $t = 2$.

The only non-zero weights in this case are

$$wt_0\left(\mathcal{D}^{(2,2)}_{(2,2)}\right) = 1, \quad wt_1\left(\mathcal{D}^{(2,2)}_{(2,2)}\right) = y_1 - y_3 + y_2 - y_4, \quad wt_1\left(\mathcal{D}^{(2,2)}_{(3,2)}\right) = 1,$$
$$wt_2\left(\mathcal{D}^{(2,2)}_{(2,2)}\right) = (y_1 - y_3)(y_1 - y_4), \quad wt_2\left(\mathcal{D}^{(2,2)}_{(3,2)}\right) = y_1 - y_3, \quad wt_2\left(\mathcal{D}^{(2,2)}_{(3,3)}\right) = 1. \tag{5.57}$$

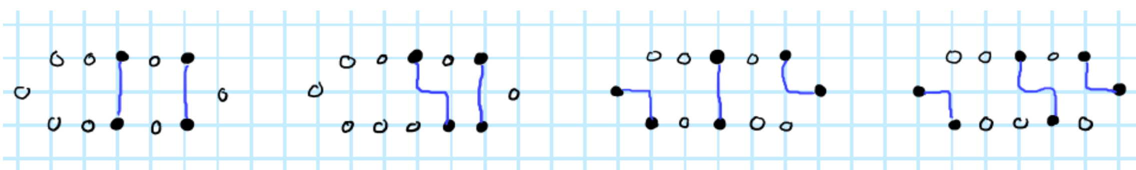
Therefore we find that $E_0|(2,2)\rangle = |(2,2)\rangle$, $E_1|(2,2)\rangle = (y_1 - y_3 + y_2 - y_4)|(2,2)\rangle + |(3,2)\rangle$, and $E_2|(2,2)\rangle = (y_1 - y_3)(y_1 - y_4)|(2,2)\rangle + (y_1 - y_3)|(3,2)\rangle + |(3,3)\rangle$, and also

$$\emptyset \circledast \ydiagram{2,2} = \ydiagram{2,2},$$

$$\ydiagram{1} \circledast \ydiagram{2,2} = (y_1 - y_3 + y_2 - y_4)\ydiagram{2,2} + \ydiagram{3,2},$$

$$\ydiagram{1,1} \circledast \ydiagram{2,2} = (y_1 - y_3)(y_1 - y_4)\ydiagram{2,2} + (y_1 - y_3)\ydiagram{3,2} + \ydiagram{3,3}.$$

For $E(x)|(3,2)$ we have



$\mathcal{D}^{(3,2)}_{(3,2)}$ $\qquad$ $\mathcal{D}^{(3,2)}_{(3,3)}$ $\qquad$ $\mathcal{D}^{(3,2)}_{(1)}$ $\qquad$ $\mathcal{D}^{(3,2)}_{(2)}$

For the first lattice above $t = 0$, for the second and third $t = 1$, and for the fourth $t = 2$.

The only non-zero weights in this case are

$$wt_0\left(\mathcal{D}^{(3,2)}_{(3,2)}\right) = 1, \quad wt_1\left(\mathcal{D}^{(3,2)}_{(3,2)}\right) = y_1 - y_3 + y_2 - y_5, \quad wt_1\left(\mathcal{D}^{(3,2)}_{(3,3)}\right) = 1,$$
$$wt_1\left(\mathcal{D}^{(3,2)}_{(1)}\right) = q, \quad wt_2\left(\mathcal{D}^{(3,2)}_{(3,2)}\right) = (y_1 - y_3)(y_1 - y_5), \quad wt_2\left(\mathcal{D}^{(3,2)}_{(3,3)}\right) = y_1 - y_5, \tag{5.58}$$
$$wt_2\left(\mathcal{D}^{(3,2)}_{(1)}\right) = q(y_1 - y_3), \quad wt_2\left(\mathcal{D}^{(3,2)}_{(2)}\right) = q.$$

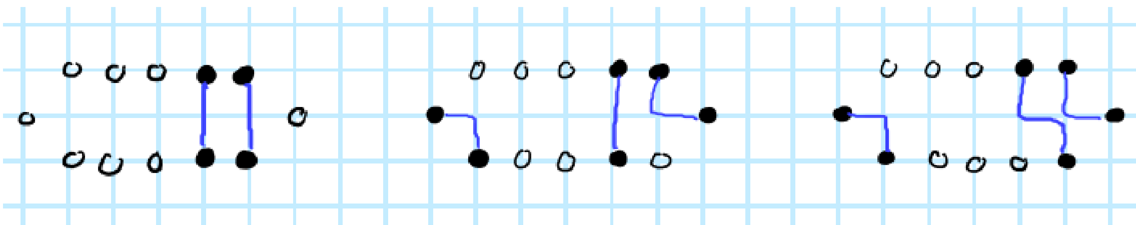
Therefore we find that $E_0|(3,2)\rangle = |(3,2)\rangle$, $E_1|(3,2)\rangle = (y_1 - y_3 + y_2 - y_5)|(3,2)\rangle + |(3,3)\rangle + q|(1)\rangle$, and $E_2|(3,2)\rangle = (y_1 - y_3)(y_1 - y_5)|(3,2)\rangle + (y_1 - y_5)|(3,3)\rangle + q(y_1 - y_3)|(1)\rangle + q|(2)\rangle$, and also

$$\emptyset \circledast \ydiagram{3,2} = \ydiagram{3,2},$$

$$\ydiagram{1} \circledast \ydiagram{3,2} = (y_1 - y_3 + y_2 - y_5)\ydiagram{3,2} + \ydiagram{3,3} + q\ydiagram{1},$$

$$\ydiagram{1,1} \circledast \ydiagram{3,2} = (y_1 - y_3)(y_1 - y_5)\ydiagram{3,2} + (y_1 - y_5)\ydiagram{3,3} + q(y_1 - y_3)\ydiagram{1} + q\ydiagram{2}.$$

For $E(x)|(3,3)$ we have



$\mathcal{D}^{(3,3)}_{(3,3)}$ $\mathcal{D}^{(3,3)}_{(2)}$ $\mathcal{D}^{(3,3)}_{(3)}$

For the first lattice above $t = 0$, for the second $t = 1$, and for the third $t = 2$.

The only non-zero weights in this case are

$$\begin{gathered} wt_0\left(\mathcal{D}^{(3,3)}_{(3,3)}\right) = 1, \quad wt_1\left(\mathcal{D}^{(3,3)}_{(3,3)}\right) = y_1 - y_4 + y_2 - y_5, \quad wt_1\left(\mathcal{D}^{(3,3)}_{(2)}\right) = q, \\ wt_2\left(\mathcal{D}^{(3,3)}_{(3,3)}\right) = (y_1 - y_4)(y_1 - y_5), \quad wt_2\left(\mathcal{D}^{(3,3)}_{(2)}\right) = q(y_1 - y_4), \quad wt_2\left(\mathcal{D}^{(3,3)}_{(3)}\right) = q. \end{gathered} \tag{5.59}$$

Therefore we find that $E_0|(3,3)\rangle = |(3,3)\rangle$, $E_1|(3,3)\rangle = (y_1 - y_4 + y_2 - y_5)|(3,3)\rangle + q|(2)\rangle$, and $E_2|(3,3)\rangle = (y_1 - y_4)(y_1 - y_5)|(3,3)\rangle + q(y_1 - y_4)|(2)\rangle + q|(3)\rangle$, and also

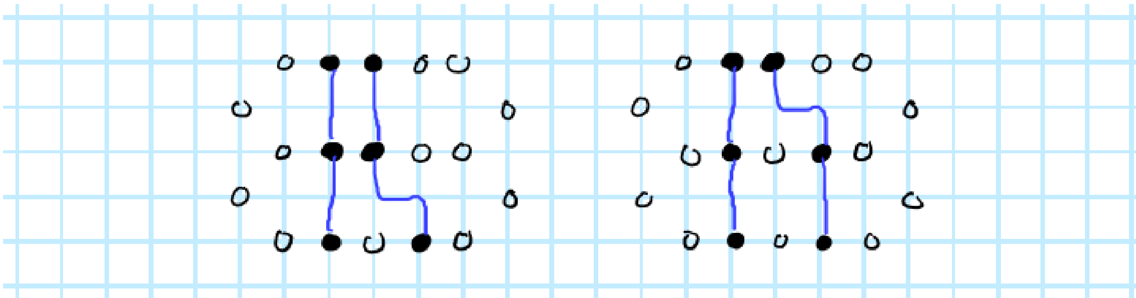
$$\emptyset \circledast \ydiagram{3,3} = \ydiagram{3,3},$$

$$\ydiagram{1} \circledast \ydiagram{3,3} = (y_1 - y_4 + y_2 - y_5)\ydiagram{3,3} + q\ydiagram{2},$$

$$\ydiagram{1,1} \circledast \ydiagram{3,3} = (y_1 - y_4)(y_1 - y_5)\ydiagram{3,3} + q(y_1 - y_4)\ydiagram{2} + q\ydiagram{3}.$$

We will now revisit Example 5.3 with our statistics to calculate $K_{(2,1)/0/(2),(2,1)}(y)$.

Example 5.27. The necessary diagrams are



We must calculate E_1 for the first row and E_2 for the second row of both diagrams.

Using the previous example we find that the first row of the first diagram has weight $y_1 - y_3$, and the second row of the first diagram has weight $y_1 - y_2$. Hence the first diagram has weight $(y_1 - y_3)(y_1 - y_2)$.

Again, using the previous example we find that the first row of the second diagram has weight 1, and the second row of the second diagram has weight $(y_1 - y_2)(y_1 - y_4)$.

Thus we find that $K_{(2,1)/0/(2),(2,1)}(y) = (y_1 - y_3)(y_1 - y_2) + (y_1 - y_2)(y_1 - y_4)$.

Remark 5.28. By examination of the formula for the weight wt_r, we can deduce that this method will give manifestly positive coefficients for $|\lambda\rangle$, if we take $-y_i$ instead of y_i, since we must have $\mathcal{L}(s) \leq s$. However it is not clear that using this method for the quantum equivariant Littlewood-Richardson coefficients would yield a manifestly positive result, due to the shift operator τ^{j-1}.

5.4 Bijection Between Lattice Diagrams and Knutson-Tao Puzzles

For $q = 0$ we can prove directly that there is a bijection between the set of diagrams our method produces and the equivariant Knutson-Tao puzzles.

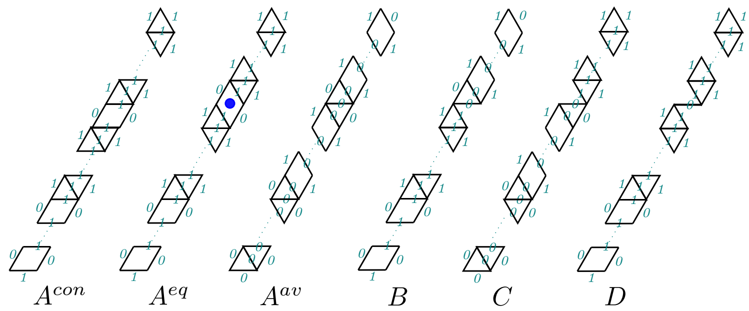
Consider the following sets:

- The set of equivariant Knutson-Tao puzzles with sides of length $n > 0$ whose NW edges correspond to a vertical strip of size $r \geq 0$.
- The set of lattice diagrams $\mathcal{D}_\lambda^\mu$ on n positions which have $r - |\lambda/\mu|$ admissible lines chosen.

Note that each distinct choice of $r - |\lambda/\mu|$ admissible lines in a diagram will give a different element of the above set.

The aim is to prove a bijection between these sets. We will do this using the following lemmata.

Lemma 5.29. *A puzzle can be split into the columns detailed below:*



where we indicate the equivariant piece with a blue dot. The missing edges in the B, C, and D type columns indicates that a diamond shaped piece has been cut in half to make the column. If two columns are touching in a puzzle, the sides that are touching must have the 0's, 1's and missing edges in exactly the same places.

Proof. We find this by examining all possible cases. $\square$

Lemma 5.30. *If the NW edge of the puzzle corresponds to a vertical r-strip then there can be at most one equivariant piece per column.*

Proof. This is immediate from examination of the possible columns determined in Lemma 5.29, but it is an important fact to note. $\square$

Lemma 5.31. *Fix μ, λ, and r. Then in the equivariant puzzle $\underset{\lambda}{1^r\triangle\mu}$ there are exactly $r - |\lambda/\mu|$ equivariant pieces.*

Proof. The NW edge of the puzzle will be $1\cdots101\cdots10\cdots0$ starting at the top of the puzzle and reading downwards, with r 1's in the second group.

Starting at the left side, each column eliminates one entry from the 01-string described above. More specifically:

- A^{con} eliminates a 1 from the first group of 1's.
- A^{eq}, B, and D eliminate a 1 from the second group of 1's.
- A^{av} and C eliminate the last 0.

From this we deduce that there must be a total of r columns of type A^{eq}, B, and D combined.

If we now recall the paths on the puzzle columns, we can see that B and D together count the number of 1's which have moved one place to the left from μ to λ. If we convert μ and λ into Young diagrams we see that this number is $|\lambda/\mu|$. Therefore there must be exactly $r - |\lambda/\mu|$ columns of type A^{eq}, each of which contain exactly one equivariant piece. $\square$

Lemma 5.32. *An equivariant puzzle which has a NW edge corresponding to a vertical r-strip is uniquely determined by its edges and the positions of the equivariant pieces.*

Proof. We can see that each column is determined by its NW edge and end values. By examining the possible columns we see that the only columns which have the same values on the NW edge and at the ends are A^{con} and A^{eq}. So for the puzzle to be uniquely determined by its edges we also need to know the location of the equivariant pieces. $\square$

Lemma 5.33. *There must be at least one occurrence of a pair of B and C type columns or a type A^{av} column to the right of any A^{eq} type column.*

Proof. There cannot be an equivariant piece in the rightmost column as it is a single triangle.

We will prove the general case by contradiction. It is already established that a column containing an equivariant piece cannot occur between type B and C columns. So start with a single A^{con} triangle. There cannot be a D type column immediately to the left of this triangle, so we assume that we have p columns of type A^{con} to the right of an A^{eq} type column in position $p+1$. By inspection, we see that the NW side of the A^{con} type column in position p will only have 1's in its 01-word. Thus we have a contradiction as the A^{eq} type column requires a 0 on its SE side. Hence there must be at least one occurrence of a pair of B and C type columns or a type A^{av} column to the right of any A^{eq} type column, as required. $\square$

Theorem 5.34. *The map which takes the diagram $\mathcal{D}^{\mu}_{\lambda}$ with chosen admissible lines in positions $a_1, \ldots, a_{r-|\lambda/\mu|}$ to the KT-puzzle $\underset{\lambda}{{}^{1^r}\triangle^{\mu}}$ with equivariant pieces in the same positions is a bijection.*

Proof. We must show that this map is well-defined, injective and surjective. An element in the set of diagrams described at the start of this subsection is uniquely determined by μ, λ, r, and the positions of the admissible lines chosen. Lemma 5.32 tells us the same thing for puzzles. Note that not all possible choices will lead to a diagram that exists, this is also true for puzzles. Therefore we must check that such a puzzle exists.

Lemma 5.33 is equivalent to the condition for lines to be admissible. We can see this by examining the entries in μ and λ before the line in the diagram or column in the puzzle.

Lemma 5.30 ensures that we will not need to choose the same admissible line more than once to obtain a puzzle via this map.

Lemma 5.31 ensures that there are the same number of equivariant pieces as there are chosen admissible lines.

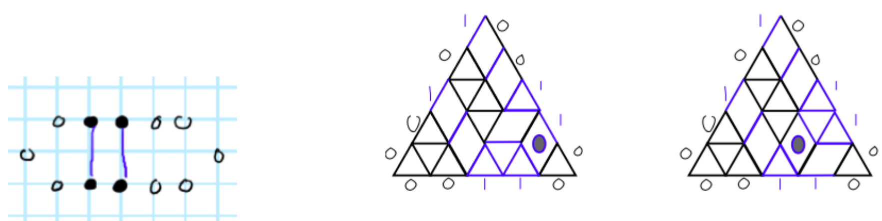
We have seen implicitly by looking at the Young diagrams for μ and λ that a puzzle only exists if the 1's in μ either are in the same position or moved one place to the left in λ. This is equivalent to the conditions imposed upon diagrams by the allowed lattice vertices from which they are constructed.

All of this ensures that a diagram exists for μ, λ, r with chosen admissible lines in positions $a_1, \ldots, a_{r-|\lambda/\mu|}$ if and only if the puzzle for μ, λ, r with equivariant pieces in positions $a_1, \ldots, a_{r-|\lambda/\mu|}$ also exists. The map is therefore well-defined. Injectivity comes from the puzzles and diagrams being uniquely defined by the aforementioned conditions. It is also now clear that the sets described at the beginning of this subsection have the same cardinality, hence we also have surjectivity.

Thus this map is indeed a bijection. □

Example 5.35. Here is an example of this bijection for the diagram $\mathcal{D}^{(1,1)}_{(1,1)}$ with one chosen admissible line.

We have $|(1,1)/(1,1)| = 0$, so with only one admissible line chosen we must have $r = 1$. Here are $\mathcal{D}^{(1,1)}_{(1,1)}$ and the puzzles which have (1) on the NW side, and $(1,1)$ on the NE and S sides.



When the chosen admissible line is the connecting line in position 2 of the lattice configuration, the bijection maps $\mathcal{D}^{(1,1)}_{(1,1)}$ to the puzzle on the left above, and when the chosen admissible line is the connecting line in position 3 of the lattice configuration, the bijection maps $\mathcal{D}^{(1,1)}_{(1,1)}$ to the puzzle on the right above.

6 Conclusion

We have now seen that there are many methods that exist or can be adapted to calculate the quantum equivariant Kostka numbers.

The method of Bertiger et al [BEMT22], which involves statistics on Young diagrams, is used to calculate the Pieri rule for Schubert classes of $QH_T^*(Gr(k,n))$. Thus in order to obtain the quantum equivariant Kostka numbers using this method we have to apply it repeatedly.

The method of Gorbounov and Korff [GK14], which uses Yang-Baxter algebras and lattice models, can calculate the quantum equivariant Kostka numbers directly. However it requires the expansion of the partition function of the lattice into factorial powers. If we then wanted to use it to find the quantum equivariant Littlewood-Richardson coefficients (Gromov-Witten invariants) we would need to expand the partition function for each row separately into shifted factorial powers, which could be calculation intensive.

The puzzle method of Knutson and Tao [KT03] works very efficiently for the case where $q = 0$ and can be generalised, by Buch [Buc15], to include the case where $q \neq 0$. However these generalised puzzles involve far more pieces and the rule for inputting the partitions as 012-strings is slightly more complicated, and we would still need to use this method repeatedly to find the quantum equivariant Kostka numbers since the puzzles are used to compute quantum equivariant Littlewood-Richardson coefficients. In order to get the quantum equivariant Kostka numbers we have to apply the method repeatedly to account for the multiplication by each $e_r(x|y)$.

The method that we introduce allows us to stack lattice rows and read the statistics from the diagram. However we must keep track of which E_r is acting on each row. If it can proved that $\tau^{j-1}E_r$ acting on row j of the lattice affects the statistics by adding $j-1$ to $\mathcal{L}$, then this would seem to be a simpler way to calculate the Gromov-Witten invariants than the other methods which we have explored in this thesis.

References

[BEMT22] Anna Bertiger, Dorian Ehrlich, Elizabeth T Milićević, and Kaisa Taipale. An equivariant quantum Pieri rule for the Grassmannian on cylindric shapes. *Electronic Journal of Combinatorics*, 29(2), 2022.

[Ber97] Aaron Bertram. Quantum Schubert Calculus. *Advances in Mathematics*, 2(128):289–305, 1997.

[Buc15] Anders Skovsted Buch. Mutations of puzzles and equivariant cohomology of two-step flag varieties. *Annals of mathematics*, pages 173–220, 2015.

[CP95] Vyjayanthi Chari and Andrew N Pressley. *A guide to quantum groups*. Cambridge university press, 1995.

[Fad96] LD Faddeev. How Algebraic Bethe Ansatz works for integrable model. *arXiv preprint hep-th/9605187*, 1996.

[FOSK+20] Jose M Figueroa-O'Farrill, Bernd Schroers, Christian Korff, Bart Vlaar, and Robert Weston. Classical and quantum integrable systems, 2020.

[Ful97] William Fulton. *Young tableaux: with applications to representation theory and geometry*, volume 35. Cambridge University Press, 1997.

[GK97] Ira Gessel and Christian Krattenthaler. Cylindric partitions. *Transactions of the American Mathematical Society*, 349(2):429–479, 1997.

[GK14] Vassily Gorbounov and Christian Korff. Equivariant quantum cohomology and Yang-Baxter algebras. *arXiv preprint arXiv:1402.2907*, 2014.

[GK17] Vassily Gorbounov and Christian Korff. Quantum integrability and generalised quantum Schubert calculus. *Advances in Mathematics*, 313:282–356, 2017.

[Int91] Kenneth Intriligator. Fusion residues. *Modern Physics Letters A*, 6(38):3543–3556, 1991.

[KT03] Allen Knutson and Terence Tao. Puzzles and (equivariant) cohomology of Grassmannians. *Duke Mathematical Journal*, 119(2):221–260, 2003.

[Mac92] Ian G Macdonald. Schur functions: theme and variations. *Séminaire Lotharingien de Combinatoire (Saint-Nabor, 1992)*, 498:5–39, 1992.

[Mac95] Ian G Macdonald. *Symmetric functions and Hall polynomials*. Oxford University Press, 1995.

[Mih08] Leonardo Mihalcea. Giambelli formulae for the equivariant quantum cohomology of the Grassmannian. *Transactions of the American Mathematical Society*, 360(5):2285–2301, 2008.

[MS99] Alexander Molev and Bruce Sagan. A Littlewood-Richardson rule for factorial Schur functions. *Transactions of the American Mathematical Society*, 351(11):4429–4443, 1999.

[Vaf92] C Vafa. Topological Mirrors and Quantum Rings. *Essays on Mirror Manifolds*, 1992.