# Flat coordinates of algebraic Frobenius manifolds in small dimensions

by

**Johan Wright**

(supervised by Prof. M. V. Feigin and Dr. D. Valeri)

Submitted in fulfilment of the requirements for the Degree of Doctor of Philosophy

School of Mathematics and Statistics
College of Science and Engineering
University of Glasgow

April 2024

# Abstract

The orbit space of the reflection representation of a finite irreducible Coxeter group $W$ gives a polynomial Frobenius manifold. The intersection form $g$ on these Frobenius manifolds is given by the $W$-invariant inner product on the reflection representation, while the metric $\eta$ is a Lie derivative of $g$. Both metrics $g$ and $\eta$ are flat and flat coordinates of the metric $\eta$ expressed via the flat coordinates of $g$ are Saito polynomials, which are a distinguished set of basic invariants of the Coxeter group.

Algebraic Frobenius manifolds are typically related to quasi-Coxeter conjugacy classes in finite irreducible Coxeter groups. This class of Frobenius manifolds comes naturally after the polynomial case but it is understood less well. In this thesis we are interested in the relations between the flat coordinates of the flat metrics $\eta$ and $g$ on the algebraic Frobenius manifolds.

We find explicit relations between flat coordinates of the metric $\eta$ and flat coordinates of the intersection form $g$ for most known examples of algebraic Frobenius manifolds up to dimension 4. In all the cases, flat coordinates of the metric $\eta$ appear to be algebraic functions on the orbit space of a Coxeter group.

The dual prepotentials for the polynomial Frobenius manifolds are easy to write down explicitly in terms of root systems of Coxeter groups. We find dual prepotentials for a particular family of two-dimensional algebraic Frobenius manifolds. Special functions are needed to give the answer already in this case.

# Contents

# Acknowledgements

Firstly, I would like to thank my supervisors Misha and Daniele for the support and guidance they have given to me over the past 4 years, I am grateful for the time they have spent helping me throughout this PhD. I would also like to thank Christian for the conversations we have had and the encouragement he has given me.

Secondly, I would like to thank EPSRC for their financial support for the first 3.5 years of my PhD, without which it would have been difficult for me to complete this doctoral project. I am also grateful to the School of Mathematics and Statistics at the University of Glasgow for their assistance in helping me to attend workshops and conferences related to my research area.

# Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

# Chapter 1

# Introduction

## 1.1 Origins of Frobenius manifolds

Frobenius manifolds as a concept came from a variety of sources within the mathematical and theoretical physics literature. One of the origins of Frobenius manifolds can be traced back to the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations, which were first discussed in 1990 by Witten [53] and Dijkgraaf, Verlinde and Verlinde [6]. They are a system of nonlinear third order differential equations that arise from the study of two-dimensional topological quantum field theory, which is a metric-independent approach to studying a "toy model" of quantum field theory.

A topological quantum field theory is defined by a set of (physical) operators on a space-time manifold $M$ and a set of correlation functions which are assumed to be independent of the metric on $M$ and can be factorised in a precise sense. One can give a category-theoretic interpretation of these ideas, where a topological quantum field theory is a symmetric monoidal functor from the category of cobordisms to the category of vector spaces. Here, physically, the cobordism represents space-time, while its boundary represents space alone, and the vector spaces are possible configuration spaces for a quantum system. It was shown (see for example [37]) that the category of two-dimensional cobordisms is categorically equivalent to the category of commutative Frobenius algebras. Frobenius algebras are associative, unital algebras equipped with a symmetric, non-degenerate inner product $\langle \cdot, \cdot \rangle$ that satisfies the following invariance property:

$$\langle x \cdot y,\, z\rangle = \langle x,\, y \cdot z\rangle,$$

for all elements $x$, $y$, $z$ in the Frobenius algebra.

The WDVV equations arise by considering the two-point and three-point correlation functions in a topological quantum field theory. Let $\phi_i$, $i = 1, \ldots, n$, be the primary fields of such a theory. Then, the two-point and three-point correlation functions for a genus zero surface can be denoted as $\eta_{ij}$ and $c_{ijk}$, respectively. These can then be used to define an operator algebra of the primary fields by defining the product

$$\phi_i \times \phi_j := \sum_{k,l} \eta^{kl} c_{ijk} \phi_l,$$

where $\eta^{ij}$ is the inverse of the matrix $\eta_{ij}$. One can then show that this operator algebra is commutative and associative, which gives the following system of relations

$$\sum_{\lambda,\mu} c_{ij\lambda} \eta^{\lambda\mu} c_{\mu kl} = \sum_{\lambda,\mu} c_{kj\lambda} \eta^{\lambda\mu} c_{\mu il}, \tag{1.1.1}$$

together with symmetry of $\eta^{\lambda\mu}$, $\eta_{ij}$ and $c_{ijk}$ in their respective indices. In the two-dimensional case we can investigate perturbations of these topological quantum field theories by introducing coupling constants $t^1, \ldots, t^n$ for each of the primary fields and defining new perturbed correlation functions so that, in particular, the three-point correlation functions $c_{ijk}$ depend on $t^1, \ldots, t^n$, where $t = 0$ is the unperturbed case. Investigating the properties of these functions, one then finds that there must exist a function $F = F(t^1, \ldots, t^n)$ such that

$$c_{ijk}(t) = \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k}.$$

The tensor field $\eta_{ij}$ can also be expressed via the third order derivatives of $F$. The set of equations from formula (1.1.1) then becomes the following system of nonlinear third order differential equations, which are now known as the WDVV equations

$$\sum_{\lambda,\mu} \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^k \partial t^l} = \sum_{\lambda,\mu} \frac{\partial^3 F}{\partial t^k \partial t^j \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^i \partial t^l}. \tag{1.1.2}$$

Some key structures of Frobenius manifolds in an important special case appeared earlier in the work of Saito (see [46]) in relation to singularity theory. Consider the case of the $A_N$ singularity $f(x,\, y,\, z) = x^{N+1} + y^2 + z^2$. Here, one deals with its semiuniversal deformation, which is essentially the function

$$\lambda(x,\, a) = x^{N+1} + \sum_{i=1}^{N} a_i x^{N-i},$$

where $a = (a_1, \ldots, a_N) \in \mathbb{C}^N$. The parameters $a_i$ may be thought of as elementary symmetric polynomials on the orbit space $M = \mathbb{C}^{N+1}/S_{N+1}$, or rather on a hypersurface $\pi$ in $M$ given by the vanishing of the first elementary symmetric polynomial. Then the Saito metric on $\pi$ is given by the residue formula

$$\eta(\partial_i,\, \partial_j) = \operatorname*{res}_{x=\infty} \frac{\partial_i \lambda \, \partial_j \lambda}{\lambda'(x)}\, dx.$$

In the case of a more general isolated singularity the superpotential $\lambda$ may be a multi-variable function, and the differential $dx$ may be replaced by a suitable "primitive form" studied by Saito in [44]. Saito showed in [45] that for isolated singularities the residue pairings were flat and proved the existence of primitive forms. In [46] he proved flatness of a metric on the Coxeter orbit spaces which coincides with the residue pairing in the ADE cases. Saito, Yano and Sekiguchi [47] later found flat coordinates of these metrics to be distinguished sets of basic invariants of the corresponding Coxeter groups for all the irreducible cases except for $E_7$ and $E_8$. These latter cases were completed by Abriani [1] and Talamini [51] (see also earlier work by Kato [36]).

Another place from which examples of Frobenius manifolds arise is the area of quantum cohomology. Quantum cohomology can be interpreted as a deformation of the De Rham cohomology of a complex manifold $M$. We start with the cohomology algebra $H^*M := \oplus_{i=0}^{n} H^{2i}(M;\, \mathbb{C})$, we then replace the "cup product" $\bullet$ on the cohomology algebra with a family of "quantum products" $\bullet_x$ where $x \in H^*M$ that reduces to the cup product $\bullet$ when $x = 0$. These quantum cohomologies are directly related to the third-order derivatives of the generating function $F$ of Gromov-Witten invariants. One can then show that such an $F$ is a solution of the WDVV equations (see for example [39]).

In an earlier development in 1983, Dubrovin and Novikov [19] constructed local Poisson brackets of hydrodynamic type and showed that they were equivalent to flat metrics. They started with an $n$-dimensional manifold $M$ on which they consider the loop space $\mathcal{L}(M) = C^\infty(S^1,\, M)$. One can then define the space of local functionals to be the set of functions $I : \mathcal{L}(M) \to \mathbb{R}$ of the form

$$I[\phi] = \int_{S^1} P(\phi(s))ds,$$

where $\phi \in \mathcal{L}(M)$ and $P = P(x; x^i_s, x^i_{ss}, \dots)$ is a polynomial of the derivatives of some coordinates $x^i$, $i = 1, \dots, n$, on the manifold $M$ with coefficients of the derivatives allowed to be smooth functions on the manifold. A local Poisson bracket is then a Lie bracket on the space of local functionals of the form

$$\{I_1,\, I_2\} = \int_{S^1} \frac{\delta I_1}{\delta x^i(s)} A^{ij} \frac{\delta I_2}{\delta x^j(s)} ds, \tag{1.1.3}$$

where $A^{ij} = a^{ij}_k \partial^k_s$ is a differential operator with coefficients $a^{ij}_k$ being polynomials of the derivatives of $x^i$, like $P$ explained above, and $\frac{\delta I_1}{\delta x^i(s)}$, $\frac{\delta I_2}{\delta x^j(s)}$ stand for variational derivatives. Here and below we assume the Einstein summation convention, meaning that summation is assumed over any repeated indices. Dubrovin and Novikov were particularly interested in the local Poisson brackets of hydrodynamic type, that is,

$$A^{ij} = g^{ij}\frac{\partial}{\partial s} + b^{ij}_k x^k_s, \tag{1.1.4}$$

where $g^{ij}$ and $b^{ij}_k$ are some smooth functions on $M$. The condition of a Lie bracket for a local Poisson bracket given by formulas (1.1.3) and (1.1.4) is, by definition, that one has bilinearity, skew-symmetry and the Jacobi identity (but not the Leibniz rule). This constrains how $g^{ij}$ and $b^{ij}_k$ are related to each other, and in fact Dubrovin and Novikov showed that if $\det(g^{ij}) \neq 0$, then $g^{ij}$ defines a flat metric on $M$ with contravariant Christoffel symbols $b^{ij}_k(x) = -g^{il}(x)^g\Gamma^j_{kl}(x)$, where ${}^g\Gamma^j_{kl}$ are the usual Christoffel symbols for the metric $g$. They then showed how these local Poisson brackets are related to well-known integrable systems like the KdV equation and the nonlinear Klein-Gordon equation.

Integrable systems often admit bihamiltonian structure (see [38]). For local Poisson brackets the concept of a bihamiltonian structure can be explained as follows. Consider a pair of local Poisson brackets of hydrodynamic type $\{\cdot,\cdot\}_1$, $\{\cdot,\cdot\}_2$ that are non-degenerate and compatible, meaning that

$$\{\cdot,\cdot\}^\lambda = \{\cdot,\cdot\}_1 - \lambda\{\cdot,\cdot\}_2$$

is a local Poisson bracket for all $\lambda \in \mathbb{C}$. The flat metrics $\eta$ and $g$ for the brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$, respectively, form a flat pencil of metrics. That is,

$$g^{\alpha\beta}_\lambda = g^{\alpha\beta} - \lambda\eta^{\alpha\beta}$$

is a flat metric for all $\lambda \in \mathbb{C}$ and $g_\lambda$ has the contravariant Christoffel symbols

$${}^{g_\lambda}\Gamma^{ij}_k = {}^g\Gamma^{ij}_k - \lambda{}^\eta\Gamma^{ij}_k,$$

where ${}^g\Gamma^{ij}_k$ and ${}^\eta\Gamma^{ij}_k$ are the contravariant Christoffel symbols of the metrics $g$ and $\eta$, respectively. Dubrovin showed in [15] that such pencils are closely related to Frobenius manifolds which he first introduced in 1992 [13].

The concept of Frobenius manifolds is a geometric, coordinate-free interpretation of the WDVV equations. Despite this geometric construction, much of the interest in Frobenius manifolds concerns its local structure.

A Frobenius manifold $M$ is a complex manifold equipped with a flat metric $\eta$, which defines a non-degenerate symmetric bilinear form on the tangent spaces $T_xM$, $x \in M$. Moreover, the tangent spaces $T_xM$ are a family of commutative, associative algebras with a product which we denote by $\bullet$, and there exists a special vector field $E$ on $M$ which is called the Euler vector field. These ingredients of a Frobenius manifold satisfy various properties. One of the most important properties is the Frobenius algebra property which states that for any vector fields $X$, $Y$, $Z$ on $M$ we have that

$$\eta(X \bullet Y,\, Z) = \eta(X,\, Y \bullet Z).$$

We can use this property to define a symmetric (0, 3) tensor field

$$c(X,\, Y,\, Z) := \eta(X \bullet Y,\, Z).$$

For a Frobenius manifold we assume that the (0, 4) tensor field

$$({}^{\eta}\nabla_W c)(X,\, Y,\, Z)$$

is also symmetric, where ${}^{\eta}\nabla_W$ is a covariant derivative along the vector field $W$ by the Levi-Civita connection associated with metric $\eta$. This property along with the fact that $\eta$ is flat allows us to locally define a function $F$ on $M$ such that in a flat coordinate system $t^1, \ldots, t^n$ we have

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} = c(\partial_{t^\alpha},\, \partial_{t^\beta},\, \partial_{t^\gamma}),$$

for all $\alpha,\, \beta,\, \gamma = 1, \ldots, n$. This function $F$ is called a prepotential of the Frobenius manifold and it is a solution of the WDVV equations (1.1.2). Moreover, $F$ is quasihomogeneous, meaning that

$$\mathcal{L}_E F(t) = (3-d)F(t) + Q(t),$$

where $Q(t)$ is a quadratic function and $d \in \mathbb{C}$ is a constant. The constant $d$ is called the charge of the Frobenius manifold. Typically, the Euler vector field $E$ will have the form

$$E(t) = \sum_{i=1}^{n} d_i t^i \partial_{t^i},$$

for some constants $d_i \in \mathbb{C}$ which we call the degrees of the Frobenius manifold.

There are many structures that one can define on a Frobenius manifold. For instance, the intersection form $g$ which is

$$g(X,\, Y) = \eta(X,\, Y \bullet E^{-1}),$$

where $E^{-1}$ is the inverse of the Euler vector field with respect to the multiplication $\bullet$. The set of points $M_*$ where $E$ is invertible is open and dense in $M$ and the intersection form can be shown to be a flat metric on this subset $M_*$. One can then show that $\eta$ and $g$ form a flat pencil of metrics on $M$.

Dubrovin also defined a new multiplication $\star$ on $T_xM$ as follows:

$$X \star Y = X \bullet Y \bullet E^{-1},$$

which is considered for $x \in M_*$, $X$, $Y \in T_xM_*$. The product $\star$, along with the intersection form $g$, gives us a new structure on $M_*$ known as an almost Frobenius manifold [17]. An almost Frobenius manifold has many of the properties of a Frobenius manifold including a flat metric, a commutative, associative multiplication, the Frobenius algebra property, the Euler vector field $E$ and a special additional vector field $e$, which we call the Liouville vector field. The Liouville vector feild $e$ is the identity element for the multiplication $\bullet$ on the Frobenius manifold $M$, while the Euler vector field $E$ is the identity element for the multiplication $\star$ on the almost Frobenius manifold $M_*$. Frobenius manifolds and almost Frobenius manifolds are dual to one another, in the sense that for any Frobenius manifold one can construct an almost Frobenius manifold as explained above, and vice versa. Moreover, in an analogous way to a prepotential $F$ of a Frobenius manifold $M$, one can associate a dual prepotential $F_*$ to an almost Frobenius manifold $M_*$. The dual prepotential is also a solution of the WDVV equations and in some situations it has a simpler form than the prepotential for the Frobenius manifold $M$.

## 1.2 Polynomial Frobenius manifolds

Let us consider a finite, irreducible Coxeter group $W$ and the orbit space of a complexified reflection representation $V$ of $W$. It was shown in [46], see also [47], that given the flat metric $g_{ij}$ which is simply the $W$-invariant inner product inherited from $V$, the metric $\eta$ defined as

$$\eta^{\alpha\beta} = (\mathcal{L}_e g)^{\alpha\beta}$$

appears to be flat for an appropriate choice of vector field $e$. Here, the notation $\mathcal{L}_e$ means we are taking the Lie derivative along the vector field $e$. This vector field, which nowadays is sometimes referred to as the Saito primitive vector field, is a constant vector field on $V/W$ given by $e = \frac{\partial}{\partial t^1}$, where $t^1$ is a basic invariant of $W$ with highest degree. Moreover, an explicit choice of a distinguished set of basic invariants $t^1, \ldots, t^n$, for which $\eta^{\alpha\beta}(t) = \delta^{\alpha+\beta,n+1}$, was found in [47] in all the irreducible cases except $E_7$ and $E_8$. The latter cases were later dealt with by Abriani [1] and Talamini [51].

These distinguished polynomials are known as Saito polynomials and are important for the representation theory of rational Cherednik algebras. These algebras $\mathcal{H}_c(W)$ are defined by a $W$-invariant function $c : R \to \mathbb{C}$ on a root system $R$ of the Coxeter group $W$ [24]. They have a faithful representation $\rho : H_c(W) \to \mathrm{End}(\mathbb{C}[x])$, where $\mathbb{C}[x] = \mathbb{C}[x^1, \ldots, x^n]$. The representation $\rho$ is irreducible for generic $c$. Saito polynomials allow us to describe all singular vectors in the representation $\mathbb{C}[x]$ which belong to reflection representations of $W$ when the representation $\mathbb{C}[x]$ becomes reducible [25]. The corresponding parameters $c$ can be given explicitly in terms of the degrees of the Coxeter group $W$.

The flat metrics $\eta$ and $g$ form a flat pencil of metrics on the Coxeter orbit space. Moreover, Dubrovin showed that the Coxeter orbit space has the structure of a Frobenius manifold [14]. Here, the Saito primitive vector field $e$ is the identity element for the multiplication on $M$ and the Euler vector field has the form

$$E(t) = \frac{1}{h}\sum_{i=1}^{n} d_i t^i \partial_{t^i},$$

where $d_i$ are the degrees of the basic invariants $t^i$, and $h$ is the Coxeter number of $W$.

Another interesting feature of this class of Frobenius manifolds is that they have polynomial prepotentials. Dubrovin conjectured that the Coxeter orbit spaces were the only semisimple Fronbenius manifolds with polynomial prepotentials (with positive degrees). This was proved by Hertling in 2002 [31]. For example, the Coxeter orbit spaces for rank 3, $A_3$, $B_3$ and $H_3$, have the following prepotentials [13], [14]

$$\begin{aligned} F_{A_3}(t) &= \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^2 t_3^2}{4} + \frac{t_3^5}{60}, \\ F_{B_3}(t) &= \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^3 t_3}{6} + \frac{t_2^2 t_3^3}{6} + \frac{t_3^7}{210}, \\ F_{H_3}(t) &= \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^3 t_3^2}{6} + \frac{t_2^2 t_3^5}{20} + \frac{t_3^{11}}{3960}. \end{aligned}$$

Polynomial prepotentials are not intuitively related to their associated Coxeter group and their complexity increases as the rank increases. Whereas, the dual prepotentials of the Coxeter orbit spaces have the following very simple form [17]:

$$F_*(x) = \sum_{\alpha \in R_+} \frac{(\alpha,\, x)^2}{(\alpha,\, \alpha)} \log(\alpha,\, x), \tag{1.2.1}$$

where $R_+$ is a positive root system of the Coxeter group $W$, $x = (x^1, \ldots, x^n)$ and $x^1, \ldots, x^n$ are flat coordinates of the intersection form $g$.

Solutions of the WDVV equations of the form (1.2.1), where $R_+$ is a special collection of vectors which may not be a positive root system, exist [52]. Some of them are related to Dubrovin's duality on discriminant strata [28].

## 1.3 Algebraic Frobenius manifolds

The next case to be considered that extends the class of Frobenius manifolds with polynomial prepotentials would be the class of Frobenius manifolds whose corresponding prepotentials $F(t^1, \ldots, t^n)$ are algebraic functions of the flat coordinates $t^1, \ldots, t^n$. The first non-rational algebraic Frobenius manifolds were found by Dubrovin and Mazzocco in 2000, which they derived from the Coxeter group $H_3$ in relation to Painlevé VI equation [18]. Explicit prepotentials of these Frobenius manifolds, denoted as $(H_3)'$ and $(H_3)''$, were given more recently by Kato, Mano and Sekiguchi [35] (see also Remark 6.1 in that paper).

The local monodromy group of a semisimple Frobenius manifold is generated by finitely many reflections [16]. It comes together with a particular set of generating reflections $R_1, \ldots, R_n$, where $n$ is the dimension of the Frobenius manifold. In the case of an algebraic Frobenius manifold there is a finite orbit of the braid group $\mathfrak{B}_n$ acting on $n$-tuples of reflections in the monodromy group, this action is known as the Hurwitz action [18]. The local monodromy group is then necessarily a finite group [41], and the product of reflections $R_i$ gives a quasi-Coxeter element $w$ in this group. An equivalent property of $w$ is that it does not belong to any proper reflection subgroup of the Coxeter group (see [12]).

It is expected that irreducible semisimple algebraic Frobenius manifolds are closely related to the quasi-Coxeter conjugacy classes of finite irreducible Coxeter groups, where polynomial Frobenius manifolds correspond to the conjugacy class of a Coxeter element. The following conjecture was originally stated by Dinar in [7] and is attributed to Dubrovin (see also discussion in [16]).

**Conjecture 1.3.1.** [7] *The irreducible, semisimple algebraic Frobenius manifolds with positive degrees correspond to quasi-Coxeter conjugacy classes of finite, irreducible Coxeter groups.*

We recall some findings of algebraic Frobenius manifolds below together with their links to quasi-Coxeter elements. It seems not clear though whether these constructions give the same quasi-Coxeter conjugacy class as described above following [12].

Pavlyk constructed bihamiltonian structures of hydrodynamic type by considering the dispersionless limit of generalised Drinfeld–Sokolov hierarchies associated to a regular element of a Heisenberg subalgebra $\mathcal{H}_w$ of an affine Lie algebra $\widehat{\mathfrak{g}}$ [43]. To make the dispersionless limit finite one has to restrict analysis to a suitable submanifold of the phase space. In this construction the Heisenberg subalgebra $\mathcal{H}_w$ is associated with a regular quasi-Coxeter element $w$ of the Weyl group of the finite-dimensional Lie algebra $\mathfrak{g}$ (in general, non-equivalent Heisenberg subalgebras are in one-to-one correspondence with conjugacy classes of the Weyl group [34]). As we mentioned above, Dubrovin had previously shown that bihamiltonian structures of hydrodynamic type have a correspondence with Frobenius manifolds [15]. Pavlyk claimed that his construction produces algebraic Frobenius manifolds and he gave an explicit expression for the prepotential in the case of the conjugacy class $D_4(a_1)$ [43] (in the notation for conjugacy classes of Weyl groups from Carter [3]).

Dinar also gave a construction of algebraic Frobenius manifolds [10]. Starting with a regular quasi-Coxeter element $w$ in a Weyl group there is a distinguished nilpotent element $e$ in the associated simple Lie algebra $\mathfrak{g}$ [5], [49]. Dinar constructed a bi-Hamiltonian structure of hydrodynamic type on a subvariety of the Slodowy slice $\mathcal{S}_e \subseteq \mathfrak{g}^*$ using Dirac reduction and gave an explicit expression for the prepotential in the case of nilpotent orbit $F_4(a_2)$ [7] (in the notation for nilpotent orbits from [4]). He also derived prepotentials for $D_4(a_1)$ [9] and $E_8(a_1)$ [8], the latter of which was simplified in a joint work with Sekiguchi [11]. The eigenvalues of the quasi-Coxeter element $w$ have the form $e^{\frac{2\pi i}{|w|}\eta_j}$, where $|w|$ denotes the order of $w$ and $0 \leq \eta_j \leq |w|-1$. The degrees $d_j$ of the corresponding Frobenius manifold are $d_j = \frac{\eta_j+1}{|w|}$. In the case of a subregular, nilpotent element $e$, Dinar elaborated his construction in [9], where he showed the existence of an algebraic prepotential in the cases $E_6(a_1)$ and $E_7(a_1)$, in addition to $D_4(a_1)$ and $E_8(a_1)$.

Two algebraic prepotentials related to Weyl groups $E_6$ and $E_7$, and seven algebraic prepotentials related to the Coxeter group $H_4$ were found by Sekiguchi [48], who used degrees of the latter Frobenius manifolds conjectured by Douvropoulos (see further details in [12]). The former prepotentials related to $E_6$ and $E_7$ are expected to be the prepotentials for the algebraic Frobenius manifolds found by Dinar in the case of subregular nilpotent elements [9]. The latter prepotentials related to $H_4$ are denoted by $H_4(k)$, where $k = 1, 2, 3, 4, 6, 7, 9$.

## 1.4 Main results I: Flat coordinates of algebraic Frobenius manifolds

We are interested in the relations between the flat coordinates of the metric $\eta$ and those of the intersection form $g$. As we mentioned above, for polynomial Frobenius manifolds expressing flat coordinates of $\eta$ via that of $g$ gives a distinguished set of basic invariants of a Coxeter group, known as Saito polynomials [47]. It is a complicated problem in general to express one flat coordinate system in terms of the other. The main results of this thesis relate flat coordinates of the metric $\eta$ with flat coordinates of the intersection form $g$ for most of the known algebraic Frobenius manifolds in dimensions 3 and 4. These results are published in [26].

### 1.4.1 Two-dimensional case

Let us review the case of two-dimensional algebraic Frobenius manifolds. Prepotentials for two-dimensional (semisimple) algebraic Frobenius manifolds have the following form [13]:

$$F(t) = \frac{1}{2}t_1^2 t_2 + c t_2^{k+1}, \tag{1.4.1}$$

where $k \in \mathbb{Q} \setminus \{-1, 0, 1\}$ and $c \in \mathbb{C}$ is a nonzero constant. The Coxeter orbit spaces for the dihedral groups $I_2(k)$ have polynomial prepotentials of the form (1.4.1). For the algebraic case we have that $k = \frac{m}{l}$ where $m$, $l \in \mathbb{Z} \setminus \{0\}$ with $m$ and $l$ coprime and $m > 0$. Using a method analogous to Douvropoulos' work on $H_4$ (see [12]), one can associate each of these algebraic prepotentials to quasi-Coxeter element in $I_2(m)$ when $1 \leq l \leq \frac{m}{2}$, with the $l = 1$ case giving the Coxeter element of $I_2(m)$. We then have the following proposition which relates the flat coordinates of the two flat metrics $\eta$ and $g$ via the use of basic invariants of $I_2(m)$.

**Proposition 1.4.1.** *Let $x_1$, $x_2$ be flat coordinates of the intersection form $g$ for the prepotential (1.4.1) and define $y_1$, $y_2$ to be the following basic invariants of $I_2(m)$ :*

$$y_1 = (x_1 + i x_2)^m + (x_1 - i x_2)^m, \qquad y_2 = \frac{x_1^2 + x_2^2}{2m}. \tag{1.4.2}$$

*Then we can express the basic invariants (1.4.2) in terms of the flat coordinates of the metric $\eta$ in the following form:*

$$y_1 = \left( \frac{t_1 + \sqrt{t_1^2 - \frac{4c(m^2 - l^2)}{ml} t_2^{\frac{m}{l}}}}{2a} \right)^l + \left( \frac{t_1 - \sqrt{t_1^2 - \frac{4c(m^2 - l^2)}{ml} t_2^{\frac{m}{l}}}}{2a} \right)^l, \qquad y_2 = \frac{t_2}{l},$$

*where $a^2 = \frac{2c(k^2-1)}{(2k)^{k+1}}$.*

A proof of this proposition is presented in Section 4.1 where we use the well known relations, in the two-dimensional case, between the flat coordinates $t_1$, $t_2$ of the metric $\eta$ and the flat coordinates $x_1$, $x_2$ of the intersection form $g$.

### 1.4.2 3-dimensional and 4-dimensional cases

We deal with 9 out of 11 known algebraic Frobenius manifolds in dimensions 3 and 4. We start with their prepotentials, which can be expressed as polynomials of the $t$ coordinates and an additional variable $Z$, which is algebraic in the $t$ coordinates. Each Frobenius manifold is associated to a conjugacy class of a Coxeter group $W$. Motivated by this, we define coordinates $y^1, \dots, y^n$ to be some basic invariants of the group $W$ which are $W$-invariant polynomials of the flat coordinates $x^1, \dots, x^n$ of the intersection form $g$. Then, it appears to be possible to relate $t^1, \ldots, t^n, Z$ and $y^1, \ldots, y^n$.

In the polynomial case, the $t$ coordinates are themselves particular basic invariants of the $x$ coordinates known as Saito polynomials, thus the $y$ coordinates would be polynomials of the $t$ coordinates. For the algebraic case we assume that the $y$ coordinates may be expressed as polynomials of the $t$ coordinates and $Z$, which appears to be true for the algebraic Frobenius manifolds that we deal with. As a result, the $t$ coordinates turn out to be algebraic functions on the orbit space $\mathbb{C}^n/W$.

Let us explain our results in an example.

**Example 1.4.2.** Let us consider the Frobenius manifold $(H_3)''$. This has the prepotential [35]

$$F(t) = \frac{1}{2}\left(t_2^2 t_1 + t_3 t_1^2\right) + \frac{4063}{1701} t_3^7 + \frac{19}{135} t_3^5 Z^2 - \frac{73}{27} t_3^3 Z^4 + \frac{11}{9} t_3 Z^6 - \frac{16}{35} Z^7,$$

where

$$Z^2 + t_2 - t_3^2 = 0.$$

The Euler vector field has the form

$$E(t) = t^1 \partial_{t^1} + \frac{2}{3} t^2 \partial_{t^2} + \frac{1}{3} t^3 \partial_{t^3},$$

and the charge is $d = \frac{2}{3}$. Let us choose the following basic invariants of $H_3$ :

$$\begin{aligned} y_1 &= 95\epsilon_2\epsilon_3 - 32\epsilon_1^2\epsilon_3 - 5\epsilon_1\epsilon_2^2 + 2\epsilon_1^3\epsilon_2 + 3\sqrt{5}\delta\epsilon_2, \\ y_2 &= \sqrt{5}\delta + \epsilon_1\epsilon_2 - 11\epsilon_3, \\ y_3 &= \epsilon_1, \end{aligned}$$

where

$$\begin{aligned} \epsilon_1 &= x_1^2 + x_2^2 + x_3^2, \\ \epsilon_2 &= x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2, \\ \epsilon_3 &= x_1^2 x_2^2 x_3^2, \\ \delta &= (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2). \end{aligned}$$

We find that

$$\begin{aligned} y_1 = \frac{288}{25}&\left(135 t_1 t_2 + 405 t_1 t_3^2 - 90 t_2^2 t_3 - 108 t_2^2 Z + 1070 t_2 t_3^3 \right. \\ &\left. + 216 t_2 t_3^2 Z + 2292 t_3^5 - 108 t_3^4 Z\right), \end{aligned} \tag{1.4.3}$$

$$y_2 = \frac{8}{5}\left(27 t_1 + 90 t_2 t_3 + 160 t_3^3\right), \tag{1.4.4}$$

$$y_3 = 12 t_3. \tag{1.4.5}$$

Inverting these relations to find the $t$ coordinates in terms of the $y$ coordinates then gives us that

$$t_1 = \frac{5}{23328}\left(108y_2 - 25y_3^3 + 1296y_3Z^2\right), \tag{1.4.6}$$

$$t_2 = \frac{1}{144}y_3^2 - Z^2, \tag{1.4.7}$$

$$t_3 = \frac{1}{12}y_3, \tag{1.4.8}$$

where $Z$ satisfies the equation

$$31104Z^5 + 12960Z^4y_3 + (900y_2 - 360y_3^3)Z^2 + (25y_1 - 25y_2y_3^2 + 2y_3^5) = 0.$$

Let us comment briefly on how we find these relations. The main idea is to calculate the intersection form $g^{\alpha\beta}$ in two ways. On the one hand we can assume that $g^{ij}(x) = \delta^{ij}$, which can be used to calculate $g^{ij}(y)$, which only depends on Coxeter group information. In the settings of Example 1.4.2 we have

$$g^{ij}(y) = \begin{pmatrix} 30y_2^3 + 36y_2^2y_3^3 + 8y_1y_3^4 & 28y_2^2y_3 + 8y_2y_3^4 & 20y_1 \\ 28y_2^2y_3 + 8y_2y_3^4 & 8y_1 + 8y_2y_3^2 & 12y_2 \\ 20y_1 & 12y_2 & 4y_3 \end{pmatrix}.$$

On the other hand, we can calculate the intersection form $g^{\alpha\beta}(t)$ in the $t$ coordinates using only the prepotential and Euler vector field. Precisely, we use the relation

$$g^{\alpha\beta}(t) = E^{\gamma}(t)c_{\gamma}^{\alpha\beta}(t),$$

where $E$ is the Euler vector field and $c_{\gamma}^{\alpha\beta}$ are the cotangent structure constants of the Frobenius manifold. If we express the $y$ coordinates as arbitrary polynomials of the $t$ coordinates and $Z$ of specified degrees, then $g^{\alpha\beta}(t)$ can be tensorially transformed into $g^{\alpha\beta}(y)$ expressed in the $t$ coordinates. Comparing the two expressions for $g^{\alpha\beta}(y(t))$ with one another we can find what values the coefficients in the polynomials must take and thus find the expressions of the $y$ coordinates in terms of the $t$ coordinates.

To make this calculation tractable and to fix the unknown coefficients, we use the Frobenius structure to find the result of applying the Laplace operator $\Delta = \sum_{i=1}^{n} \partial_{x^i}^2$ to the coordinates $t^i$.

We find relations of type (1.4.3)–(1.4.5) for the algebraic Frobenius manifolds $(H_3)'$, $(H_3)''$, $D_4(a_1)$, $F_4(a_2)$ and $H_4(k)$ for $k = 1$, 2, 3, 4 and 7. We also find the inverse relations of type (1.4.6)–(1.4.8) for all these cases except $H_4(4)$ and $H_4(7)$, where the calculations become too involved. Calculations are performed by Mathematica, and a notebook of these calculations is available online [27]. The only cases of algebraic prepotentials in dimension 4 that are known that we do not consider are $H_4(6)$ and $H_4(9)$, for which our methods appear not to work.

## 1.5 Main results II: Almost duality

We also investigate the dual prepotentials for some of the algebraic Frobenius manifolds. It is known in the polynomial case that the dual prepotentials have the form (1.2.1), while for the algebraic case, no explicit formulas for $F_*$ are known.

We find the dual prepotentials for the two-dimensional algebraic Frobenius manifolds (1.4.1) in the cases when $k = \pm 1/l$ and $l \geq 2$ is an integer. Thus, we prove the following theorems

**Theorem 1.5.1.** [26] *Let $M$ be a two-dimensional Frobenius manifold with prepotential (1.4.1) with $k = l^{-1}$, where $l \in \mathbb{Z}_{\geq 2}$. Then the dual prepotential of $M$ has the form*

$$F_*(x) = \frac{x_2^2}{l}\log x_2 + \frac{\overline{z}^2}{4l}\log \overline{z} + \frac{z^2}{4l}\log z + \sum_{j=1}^{l-1}\frac{\overline{z}^{\frac{j}{l}}}{4j}\left(\frac{lx_1 + (l-2j)ix_2}{(j-l)z^{\frac{j}{l}-1}} + (2ix_2)^{2-\frac{j}{l}}\,{}_2F_1\left(\frac{j}{l}, \frac{j}{l}; \frac{j}{l}+1; \frac{ix_1+x_2}{2x_2}\right)\right), \quad (1.5.1)$$

*where ${}_2F_1(a, b; c; w)$ is the hypergeometric function and $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$.*

**Theorem 1.5.2.** [26] *Let $\widetilde{M}$ be a two-dimensional Frobenius manifold with prepotential (1.4.1) with $k = -l^{-1}$, where $l \in \mathbb{Z}_{\geq 2}$. Then the dual prepotential of $\widetilde{M}$ has the form*

$$\widetilde{F_*}(x) = F_*(x) - \frac{x_1^2 + x_2^2}{2l}\log(x_1^2 + x_2^2),$$

*where $F_*(x)$ is the function given by formula (1.5.1).*

We also make a few observations concerning the dual prepotentials $F_*$ in the algebraic cases $(H_3)''$ and $D_4(a_1)$. Notice that, in the polynomial case the dual prepotential has the following third order derivatives

$$\frac{\partial^3 F_*}{\partial x^i \partial x^j \partial x^k} = \sum_{\alpha \in R_+} \frac{2\alpha_i\alpha_j\alpha_k}{(\alpha,\, \alpha)}\frac{1}{(\alpha,\, x)},$$

where $\alpha_i = (\alpha, e_i)$ and $e_i$ are orthonormal basis vectors of the complexified reflection representation $V$ of the Coxeter group $W$ with the root system $R$. The sets of points $\Pi_\alpha = \{x \in V \mid (\alpha, x) = 0\}$ are known as mirrors of the group $W$. So, the third order derivatives of $F_*$ have simple poles on the mirrors and their residues on the mirrors are constant, in other words

$$\left((\alpha,\, x)\frac{\partial^3 F_*}{\partial x_i \partial x_j \partial x_k}\right)\bigg|_{(\alpha,\, x)=0} = \frac{2\alpha_i\alpha_j\alpha_k}{(\alpha,\, \alpha)}, \tag{1.5.2}$$

for all $i,\, j,\, k$.

For the algebraic cases of $(H_3)''$, $D_4(a_1)$ and the two-dimensional algebraic cases, we investigate the residues of the third order derivatives of their dual prepotentials on the mirrors of the associated Coxeter groups $H_3$, $D_4$ and $I_2(m)$, respectively.

For the cases of $(H_3)''$ and $D_4(a_1)$, we find that the algebraic variable $Z$ is algebraic in the $x$ coordinates, meaning it satisfies a polynomial relation $P(x,\, Z) = 0$. We find that on the mirrors of their respective groups $H_3$ and $D_4$, this polynomial factors into a linear branch squared and a nonlinear branch. On the linear branch, we find that

$$\left((\alpha,\, x)\frac{\partial^3 F_*}{\partial x_i \partial x_j \partial x_k}\right)\bigg|_{(\alpha,\, x)=0} = 0, \tag{1.5.3}$$

for all $\alpha \in R$ and all $i,\, j,\, k$. On the nonlinear branch, we find the same relation as in the polynomial case, namely, formula (1.5.2) holds. Calculations of these mirror residues are done by Mathematica and can be found in the notebook [27], see also [26]. In the two-dimensional cases, we also have relations (1.5.2) and (1.5.3) depending on which branch we take on the mirrors for an algebraic function.

## 1.6 Thesis structure

In Chapter 2 we give introductory material on Frobenius manifolds, Coxeter groups and Lie algebras that is necessary for understanding later chapters.

More specifically, in Section 2.1 we go over some conventional notation, particularly in relation to tensor fields on complex manifolds.

In Section 2.2 we go over the basic definitions and theorems that relate to Frobenius manifolds. This section is divided into subsections which roughly correspond to differing structures that one can relate to Frobenius manifolds.

In Subsection 2.2.1 we go over the definition of a local diffeomorphism and a flat metric on a complex manifold. In Subsection 2.2.2 we give introductory notes on Frobenius manifolds, prepotentials and their correspondence with one another. In Subsection 2.2.3 we go over the symmetries of Frobenius manifolds which are the Legendre transformations and the inversion symmetry. In Subsection 2.2.4 we consider the dual structures on a Frobenius manifold and review a relation between Frobenius manifolds and their almost duals. In Subsection 2.2.5 we present another approach to Frobenius manifolds, based on flat pencils of metrics. In Subsection 2.2.6 we present yet another interpretation of a

Frobenius manifold, this time going over the construction via bihamiltonian structures, which involves constructing compatible local Poisson brackets of hydrodynamic type. In Subsection 2.2.7 we present and discuss the concept of a semisimple Frobenius manifold and superpotentials of Frobenius manifolds which is a useful technical tool to define the structures of a Frobenius manifold. In Subsection 2.2.8 we present reformulations of semisimple Frobenius manifolds, via Darboux–Egoroff systems and Egoroff potentials. In Subsection 2.2.9 we go over the concept of a monodromy group of a semisimple Frobenius manifold and explore some of its properties.

Material in Subsections 2.2.1–2.2.7 is largely well-known and was mostly developed by Dubrovin in [13]. While Darboux–Egoroff systems have been discussed in detail in the literature before, Egoroff potentials have not been given the same treatment despite their simpler features and properties. In particular we find, using Egoroff potentials, that the inversion of a reducible semisimple Frobenius manifold, when it can be done, is always irreducible. We also show that any nonzero flat, homogeneous vector field on an irreducible Frobenius manifold is invertible which allows us to show that any irreducible, semisimple Frobenius manifold with real degrees and charge can be Legendre transformed into a Frobenius manifold with strictly positive degrees. These statements presented in Subsection 2.2.8 are new, as far as we can tell.

In Section 2.3 we go over some introductory notes on finite Coxeter groups, in particular quasi-Coxeter elements and basic invariants.

In Section 2.4 we present some features of Lie algebras that will become important later in the thesis, including the construction of local Poisson brackets on Slodowy slices.

In Chapter 3 we go over some particular classes and constructions of Frobenius manifolds. Thus, in Section 3.1 we go over the construction of Coxeter orbit spaces, which are polynomial Frobenius manifolds. In Section 3.2 we then consider the case of algebraic Frobenius manifolds, where we discuss constructions given by Dubrovin and Mazzocco [18], Dinar [7] and Sekiguchi [48].

In Subsection 3.2.1 we present the context for Dubrovin and Mazzocco's construction for algebraic Frobenius manifolds in 3 dimensions which they related to the Coxeter group $H_3$. These were the first known examples of irreducible non-rational algebraic Frobenius manifolds in dimension greater than two, and is credited as the origin of Dubrovin's conjecture for algebraic Frobenius manifolds. In Subsection 3.2.2 we go over the construction of algebraic Frobenius manifolds given by Pavlyk and Dinar. We follow only Dinar's construction in this section as it is simpler and Pavlyk's construction gives the same Frobenius manifolds. These Frobenius manifolds are constructed using distinguished nilpotent elements of semisimple type in simple Lie algebras. In Subsection 3.2.3 we go over the construction of algebraic Frobenius manifolds found by Sekiguchi using some combinatorial information, given by Douvropoulos, related to regular quasi-Coxeter elements of

Coxeter groups. These constructions give explicit formulas for the prepotentials which are related to $E_6$, $E_7$ and $H_4$. In Subsection 3.2.4 we go over the known information in the two-dimensional cases and relate that information to quasi-Coxeter elements in the rank 2 Coxeter groups. This motivates us to modify the Dubrovin conjecture for algebraic Frobenius manifolds.

In Chapter 4 we present the first part of our main results, which concern the Frobenius manifolds $(H_3)'$, $(H_3)''$, $D_4(a_1)$, $F_4(a_2)$ and $H_4(k)$ for $k = 1$, 2, 3, 4 and 7. In particular, we present relations between the flat coordinates of the intersection form and the flat coordinates of the metric for each of these Frobenius manifolds. These results are published in [26]. Some of the relations are too involved and are not included in the paper [26], we present these relations in the Appendix rather than in Chapter 4.

In Chapter 5 we present the second part of our results, namely we present some features of the dual prepotentials of some algebraic Frobenius manifolds. In particular, we find the dual prepotentials for a family of two-dimensional algebraic Frobenius manifolds and we find some properties the third order derivatives of the dual prepotentials satisfy for the $(H_3)''$ and $D_4(a_1)$ cases, as well as the two-dimensional cases. Most of these results are published in [26].

# Chapter 2

# Background material

## 2.1 Notation

Throughout this thesis I use a variety of notations that are used by others but may not be understood by all readers. For clarity's sake, I will briefly explain a few of these notations.

### 2.1.1 Einstein summation convention

When two terms are written next to one another and each has the same index variable, one in the upper indices of one of the terms and one in the lower indices of the other, and that index variable is nowhere else defined, we use the Einstein summation convention. This tells us to identify what is written with the summation of the product of those terms over the values of the index variable. For example if we see the term $a_i b^i$ and we know that $i$ is nowhere defined, but can vary in the range $1, \dots, n$, then this product is equal to

$$a_i b^i = \sum_{i=1}^{n} a_i b^i = a_1 b^1 + a_2 b^2 + \cdots + a_n b^n.$$

### 2.1.2 Smooth functions, vector fields and tensor fields

Let $M$ be a complex manifold. We denote $C^\infty(M)$ to be the set of smooth (equivalently, holomorphic or analytic) functions on $M$. We denote $\mathfrak{X}(M)$ to be the vector fields on $M$ and we denote $\Omega^1(M)$ to be the one-forms on $M$.

Let $U \subseteq M$ be an open chart of $M$ with coordinate system $u = (u^1, \dots, u^n)$, and let $T$ be a rank $(r, s)$ tensor field on $M$. When we write $T$ in the $u$ coordinates we sometimes wish to take derivatives of their components. We denote this as follows:

$$T^{a_1 \dots a_r}_{b_1 \dots b_s; i}(u) := \frac{\partial}{\partial u^i} T^{a_1 \dots a_r}_{b_1 \dots b_s}(u),$$

for all $i = 1, \dots, n$. If we wish to consider the tensor field $T$ only at a point $p \in M$, then we write $T_p$ to denote this.

### 2.1.3 Direct sums of tensor fields

Let $M$ and $N$ be complex manifolds and let $T$ and $S$ be tensor fields on $M$ and $N$, respectively, both of rank $(r,\, s)$. Then, for each $m \in M$ and $n \in N$, we denote $\mathcal{T}_s^r(T_m M)$ and $\mathcal{T}_s^r(T_n N)$ to be the spaces of rank $(r,\, s)$ tensors of the tangent spaces $T_m M$ and $T_n N$, respectively. Then we can define the direct sum $T \oplus S$ to be a tensor field of rank $(r,\, s)$ on $M \times N$ as follows:

$$(T \oplus S)_{(m,\, n)} := \psi(T_m,\, S_n),$$

where $\psi : \mathcal{T}_s^r(T_m M) \times \mathcal{T}_s^r(T_n N) \to \mathcal{T}_s^r(T_{(m,\, n)}(M \times N))$ is the natural isomorphism.

### 2.1.4 Lie derivatives

Let $M$ be a smooth manifold and let $X$ be a vector field on $M$ defined in some open chart $U \subseteq M$ and let $T$ be a tensor field on $M$ of rank $(r,\, s)$. If we let $u = (u^1, \dots, u^n)$ be local coordinates of $U$ with

$$X = X^c(u)\partial_{u^c},$$

then the Lie derivative $\mathcal{L}_X T$ of $T$ along $X$ is a tensor field on $M$ of rank $(r,\, s)$, defined in local coordinates $u^1, \dots, u^n$ as

$$\begin{aligned}(\mathcal{L}_X T)^{a_1 \dots a_r}_{b_1 \dots b_s}(u) := X^c(u) T^{a_1 \dots a_r}_{b_1 \dots b_s ; c}(u) - \sum_{i=1}^{r} X^{a_i}_{;c}(u) T^{a_1 \dots a_{i-1}\, c\, a_{i+1} \dots a_r}_{b_1 \dots b_s}(u) \\ + \sum_{j=1}^{s} X^c_{;b_j}(u) T^{a_1 \dots a_r}_{b_1 \dots b_{j-1}\, c\, b_{j+1} \dots b_s}(u). \qquad (2.1.1)\end{aligned}$$

### 2.1.5 Christoffel symbols and the Levi-Civita connection

Let $M$ be a complex manifold with a metric $g$, that is a non-degenerate, symmetric rank (0, 2) or rank (2, 0) tensor field. Let $U \subseteq M$ be an open chart of $M$ with coordinate system $u = (u^1, \dots, u^n)$, then the Christoffel symbols ${}^g\Gamma^k_{ij}(u)$ for the metric $g$ are defined in the local coordinates to be

$${}^g\Gamma^k_{ij}(u) = \frac{1}{2} g^{kl}(u)(g_{li;j}(u) + g_{lj;i}(u) - g_{ij;l}(u)),$$

for all $i, j, k = 1, \dots, n$. These are not tensor fields, and thus they don't transform as tensor fields do. Let $v = (v^1, \dots, v^n)$ be another coordinate system on $U$. Then the Christoffel symbols transform between the $u$ and $v$ coordinates by the following transformation law:

$$ {}^g\Gamma^k_{ij}(v) = {}^g\Gamma^\gamma_{\alpha\beta}(u)\frac{\partial v^k}{\partial u^\gamma}\frac{\partial u^\alpha}{\partial v^i}\frac{\partial u^\beta}{\partial v^j} + \frac{\partial^2 u^\lambda}{\partial v^i \partial v^j}\frac{\partial v^k}{\partial u^\lambda}, \tag{2.1.2} $$

for all $i, j, k = 1, \dots, n$.

Given a metric $g$ on a complex manifold $M$ and local coordinates $u = (u^1, \dots, u^n)$, we can define the Levi-Civita connection ${}^g\nabla_X T$ of a vector field $X$ with a tensor field $T$ of rank $(r,\, s)$ to be a tensor field of rank $(r,\, s)$ on $M$, which we define in the local coordinates $u^1, \dots, u^n$ as

$$ \begin{aligned} ({}^g\nabla_X T)^{a_1 \dots a_r}_{b_1 \dots b_s}(u) := X^c(u) T^{a_1 \dots a_r}_{b_1 \dots b_s; c}(u) &+ \sum_{i=1}^r X^c(u) {}^g\Gamma^{a_i}_{dc}(u) T^{a_1 \dots a_{i-1}\, d\, a_{i+1} \dots a_r}_{b_1 \dots b_s}(u) \\ &- \sum_{j=1}^s X^c(u) {}^g\Gamma^d_{b_j c}(u) T^{a_1 \dots a_r}_{b_1 \dots b_{j-1}\, d\, b_{j+1} \dots b_s}(u). \end{aligned} $$

### 2.1.6 Laplacian and gradient

Let $U \subseteq \mathbb{C}^n$ be a non-empty and open set and consider a smooth function $f : U \to \mathbb{C}$, then the Laplacian and gradient of $f$ are respectively notated as

$$ \Delta(f) := \sum_{i=1}^n \frac{\partial^2 f}{(\partial x^i)^2}, \qquad \nabla(f) := \left(\frac{\partial f}{\partial x^i}\right)_{i=1,\dots,n}. $$

If $f$ is a smooth function such that $\Delta(f) = 0$, then $f$ is known as a harmonic function.

## 2.2 Frobenius manifolds

### 2.2.1 Local diffeomorphisms and flat metrics

Let us recall the definition of a local diffeomorphism between two complex manifolds.

**Definition 2.2.1.** Let $M$ and $N$ be two complex manifolds and let $\phi : M \to N$ be a function. We say that $\phi$ is a *local diffeomorphism* if for every $x \in M$ there exists an open neighbourhood $U$ of $x$ such that $\phi(U)$ is open and $\phi|_U : U \to \phi(U)$ is a diffeomorphism.

Note that any local diffeomorphism is a smooth map, since it is locally smooth. A smooth map $\phi : M \to N$ induces differential maps $\phi_* : T_t M \to T_{\phi(t)} N$ which are linear maps between the tangent spaces, and if $\phi$ is a local diffeomorphism then $\phi_*$ is also invertible for all $t \in M$.

**Definition 2.2.2.** Let $M$ be a complex manifold and let $g$ be a rank $(0,\, 2)$ tensor field. We say that $g$ is *non-degenerate* if for all $X \in \mathfrak{X}(M)$ such that $g(X,\, Y) = 0$ for all $Y \in \mathfrak{X}(M)$

we have $X = 0$. We say that $g$ is a *metric* if it is symmetric and non-degenerate. A metric is called *flat* if in every open chart of $M$, there exists local coordinates $x = (x^1, \dots, x^n)$ such that $g_{\alpha\beta}(x)$ is constant for all $\alpha, \beta = 1, \dots, n$.

If $g_{\alpha\beta}$ is a non-degenerate rank (0, 2) tensor field, then there exists a unique rank (2, 0) tensor field $g^{\beta\gamma}$ such that $g_{\alpha\beta}g^{\beta\gamma} = \delta_\alpha^\gamma$. These tensor fields are known as inverses of one another, and if $g^{\alpha\beta}$ is a rank (2, 0) tensor field whose inverse $g_{\beta\gamma}$ is a metric, then we call $g^{\alpha\beta}$ a contravariant metric. Since metrics and contravariant metrics always co-exist and are unique, when we refer to a contravariant metric we may simply refer to it as a metric.

**Definition 2.2.3.** Let $M$ be a complex manifold. We say that a function $\varphi : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a *pointwise* $C^\infty(M)$ *linear map* if it is $C^\infty(M)$ linear and $X_p = Y_p$ implies that $\varphi(X)_p = \varphi(Y)_p$ for all $p \in M$ and all $X$, $Y \in \mathfrak{X}(M)$.

Pointwise $C^\infty(M)$ multilinear maps are defined to be functions $\phi : \mathfrak{X}(M)^n \to \mathfrak{X}(M)$ which are pointwise $C^\infty(M)$ linear in each of their components. A pointwise $C^\infty(M)$ multilinear map is obviously $C^\infty(M)$ multilinear, however the converse is not true, not every $C^\infty(M)$ multilinear map is pointwise $C^\infty(M)$ multilinear. Pointwise $C^\infty(M)$ multilinear maps allow us to define rank (1, $n$) tensor fields $T$ on $M$, and vice versa, as

$$T_p(\omega_p, X_p^1, \dots, X_p^n) = \omega_p(\varphi(X^1, \dots, X^n)_p),$$

where $X^i$ are chosen so that $(X^i)_p = X_p^i$ for all $i = 1, \dots, n$. Note also that any pointwise $C^\infty(M)$ multilinear map can be restricted to any non-empty open subset $U \subseteq M$ to give a pointwise $C^\infty(U)$ multilinear map in a unique way.

### 2.2.2 Frobenius manifolds and prepotentials

In this section the notion of a Frobenius manifold and its prepotential are presented, which are the central objects that I investigate throughout the rest of this thesis. We begin with the definition of a Frobenius algebra.

**Definition 2.2.4.** [13, Lecture 1] A *Frobenius algebra* is a finite-dimensional associative algebra $A$ over $\mathbb{C}$ with unit $e$ and nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ such that

$$\langle uv, w \rangle = \langle u, vw \rangle, \tag{2.2.1}$$

for all $u, v, w \in A$.

Frobenius algebras may not necessarily be commutative, but the category of commutative Frobenius algebras is equivalent to the category of two-dimensional topological quantum field theories [37]. Frobenius manifolds are moduli spaces of commutative Frobenius algebras, and so are of interest to anyone studying 2-dimensional quantum gravity.

**Definition 2.2.5.** [13, Lecture 1] A *Frobenius manifold* $M$ is a complex manifold together with a flat metric $\eta$, a pointwise $C^\infty(M)$ bilinear, commutative multiplication $\circ$ on $\mathfrak{X}(M)$ and a vector field $E \in \mathfrak{X}(M)$, which we call the Euler vector field, such that

**1)** There exists an identity element $e$ for the multiplication $\circ$ that is covariantly constant with respect to the Levi-Civita connection for the metric $\eta$, meaning that

$$^{\eta}\nabla_X e = 0, \tag{2.2.2}$$

for all $X \in \mathfrak{X}(M)$. We will refer to $e$ as the unity vector field.

**2)** The tangent spaces of $M$ with multiplication and bilinear form defined by the restriction of $\circ$ and $\eta$, respectively, are Frobenius algebras, meaning that the multiplication $\circ$ is associative and

$$\eta(X \circ Y,\, Z) = \eta(X,\, Y \circ Z),$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

**3)** We define the following tensor field:

$$c(X, Y, Z) := \eta(X \circ Y,\, Z) \tag{2.2.3}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. We require that

$$\left(^{\eta}\nabla_W c\right)(X, Y, Z) = \left(^{\eta}\nabla_X c\right)(W, Y, Z) \tag{2.2.4}$$

for all $W, X, Y, Z \in \mathfrak{X}(M)$.

**4)** For all $X,\, Y \in \mathfrak{X}(M)$ with $^{\eta}\nabla_Z Y = 0$ for all $Z \in \mathfrak{X}(M)$, we have that

$$^{\eta}\nabla_X \left(^{\eta}\nabla_Y E\right) = 0. \tag{2.2.5}$$

**5)** There exists a constant $d \in \mathbb{C}$, which we call the charge, such that

$$\mathcal{L}_E c = (3 - d)c,$$

and

$$\mathcal{L}_E e = -e.$$

**Example 2.2.6.** Let $A$ be a Frobenius algebra with commutative multiplication $\bullet$ and a symmetric bilinear form $\langle \cdot,\, \cdot \rangle$. Let $e_1, \ldots, e_n$ be a basis of $A$ with identity element $e = e_1$ and let $M = \mathbb{C}^n$ with coordinates $t^1, \ldots, t^n$. Define $\eta : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ and $\circ : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $C^\infty(M)$-bilinearly extending the following:

$$\eta(\partial_{t^i}, \partial_{t^j}) := \langle e_i,\, e_j \rangle, \qquad \partial_{t^i} \circ \partial_{t^j} := c_{ij}^k \partial_{t^k},$$

for all $i, j = 1, \ldots, n$, where $e_i \bullet e_j = c_{ij}^k e_k$. Then, this gives the structure of a Frobenius manifold on $M$ with unity vector field $e = \partial_{t^1}$, Euler vector field $E(t) = t^\alpha \partial_{t^\alpha}$ and charge $d = 0$.

Frobenius manifolds have various structures that we can define on them. Below we give definitions for some commonly used tensor fields on Frobenius manifolds.

**Definition 2.2.7.** Let $M$ be a Frobenius manifold with metric $\eta$. Then, we define the following tensor fields:

$$c_{\alpha\beta}^{\gamma} := \eta^{\gamma\delta} c_{\alpha\beta\delta}, \qquad c_{\gamma}^{\alpha\beta} := \eta^{\alpha\delta} c_{\delta\gamma}^{\beta}, \tag{2.2.6}$$

where $c_{\alpha\beta\gamma}$ is the rank (0, 3) tensor field defined in formula (2.2.3) and $\eta^{\alpha\beta}$ is the inverse of the metric $\eta_{\alpha\beta}$. We call $c_{\alpha\beta}^{\gamma}$ the *structure constants* of $M$, and $c_{\gamma}^{\alpha\beta}$ are the *cotangent structure constants* of $M$.

Note that, by the commutativity of the multiplication and the symmetry of the metric, the tensor fields $c_{\alpha\beta\gamma}$ and $c_{\alpha\beta}^{\gamma}$ are symmetric in the lower indices and $c_{\gamma}^{\alpha\beta}$ is symmetric in the upper indices.

**Proposition 2.2.8.** *Let $M$ be a Frobenius manifold with metric $\eta$, Euler vector field $E$, structure constants $c_{\alpha\beta}^{\gamma}$ and cotangent structure constants $c_{\gamma}^{\alpha\beta}$. Then, the following relations hold:*

$$c_{\alpha\beta\epsilon} c_{\gamma\delta}^{\epsilon} = c_{\gamma\beta\epsilon} c_{\alpha\delta}^{\epsilon}, \quad c_{\alpha\beta\epsilon} c_{\gamma}^{\epsilon\delta} = c_{\gamma\beta\epsilon} c_{\alpha}^{\epsilon\delta}, \quad c_{\alpha\epsilon}^{\beta} c_{\gamma\delta}^{\epsilon} = c_{\gamma\epsilon}^{\beta} c_{\alpha\delta}^{\epsilon}, \quad c_{\alpha\epsilon}^{\beta} c_{\gamma}^{\epsilon\delta} = c_{\gamma\epsilon}^{\beta} c_{\alpha}^{\epsilon\delta}, \tag{2.2.7}$$

$$(\mathcal{L}_E \eta)_{\alpha\beta} = (2-d)\eta_{\alpha\beta}, \qquad (\mathcal{L}_E \eta)^{\alpha\beta} = (d-2)\eta^{\alpha\beta}, \tag{2.2.8}$$

$$(\mathcal{L}_E c)_{\alpha\beta}^{\gamma} = c_{\alpha\beta}^{\gamma}, \qquad (\mathcal{L}_E c)_{\gamma}^{\alpha\beta} = (d-1) c_{\gamma}^{\alpha\beta}, \tag{2.2.9}$$

*for all $\alpha, \beta, \gamma, \delta = 1, \ldots, n$, where $c_{\alpha\beta\gamma}$ is the rank (0, 3) tensor field defined in (2.2.6).*

Relations (2.2.7) can be deduced from the associativity of the multiplication on the vector fields of a Frobenius manifold. Relations (2.2.8) and (2.2.9) are derived from condition **4)** of Definition 2.2.5 of a Frobenius manifold.

**Definition 2.2.9.** Let $M$ be a Frobenius manifold with metric $\eta$ and unity vector field $e$. If $\eta(e, e) = 0$, then we call $M$ a Frobenius manifold of *first type*, and if $\eta(e, e) \neq 0$, then we call $M$ a Frobenius manifold of *second type*.

The majority of the Frobenius manifolds that we will consider, and that are studied in the wider literature, are of first type. Next, we present the definition of equivalence, and local equivalence, between two Frobenius manifolds.

**Definition 2.2.10.** [13, Lecture 1] Let $M$ and $N$ be Frobenius manifolds with flat metrics $\eta$ and $\zeta$, and Euler vector fields $E_M$ and $E_N$, respectively. Let $\phi : M \to N$ be a smooth map such that the differential $\phi_* : T_tM \to T_{\phi(t)}N$ is an algebra isomorphism for all $t \in M$ with $\phi_*(E_M) = E_N$, and

$$\zeta_{\phi(t)}(\phi_*(X),\, \phi_*(Y)) = c^2\eta_t(X,\, Y),$$

for all $X, Y \in \mathfrak{X}(M)$ and all $t \in M$, where $c \in \mathbb{C}$ is some nonzero constant. If $\phi$ is a diffeomorphism, we say that $M$ and $N$ are *equivalent* and if $\phi$ is a local diffeomorphism, we say that $M$ *is locally equivalent to* $N$.

This definition differs slightly from the one given in Lecture 1 of [13], where the condition $\phi_*(E_M) = E_N$ is not stated. We have chosen to include this condition here because the way in which the Euler vector field is referred to in most of the literature, it is an intrinsic part of the Frobenius manifold and two Frobenius manifolds which differ only by their Euler vector fields are not typically treated as being equivalent to one another. If one removes the condition $\phi_*(E_M) = E_N$ from the definition of a (local) equivalence it is possible to show that $\phi_*(E_M)$ satisfies all of the conditions that the Euler vector field on $N$ also satisfies. However, even if (local) equivalence maps exist between $M$ and $N$, it may be the case that for all (local) equivalence maps $\phi$ we have $\phi_*(E_M) \neq E_N$.

Note that, if $M$ is equivalent to $N$ then $M$ is also locally equivalent to $N$. Also, any open submanifold of a Frobenius manifold $M$ is locally equivalent to $M$.

**Definition 2.2.11.** Let $M$ and $N$ be Frobenius manifolds with metrics $\eta$, $\zeta$, multiplications $\circ$, $\bullet$, unity vector fields $e$, $\widetilde{e}$ and Euler vector fields $E$, $\widetilde{E}$, respectively. If $M$ and $N$ have the same charge $d$, then we define the *product of* $M$ *and* $N$ to be the Frobenius manifold $M \times N$ with metric $\eta \oplus \zeta$, multiplication $\circ \oplus \bullet$, unity vector field $e \oplus \widetilde{e}$, Euler vector field $E \oplus \widetilde{E}$ and charge $d$. Here the multiplications are interpreted as rank (1, 2) tensor fields, so that their direct sum can be defined on $M \times N$. Moreover, if $M$ is a Frobenius manifold that is locally equivalent to a product of Frobenius manifolds, then we say that $M$ is *reducible*, and we say that $M$ is *irreducible* if it is not reducible.

**Remark 2.2.12.** Note that if $M$ is locally equivalent to $\widetilde{M}$ and $N$ is locally equivalent to $\widetilde{N}$, it is not necessarily the case that $M \times N$ is locally equivalent to $\widetilde{M} \times \widetilde{N}$, one can see this by scaling one of the metrics by an arbitrary constant. However, reducibility (and irreducibility) is preserved under local equivalence.

We will now explain what the WDVV equations are, so that we may define a prepotential. Solutions of the WDVV equations are a topic of study in their own right, but in this thesis we will not be investigating any solutions which are not also prepotentials.

**Definition 2.2.13.** Let $F = F(t^1, \ldots, t^n)$ be a thrice-differentiable function on a complex manifold $U$ with a coordinate system $t = (t^1, \ldots, t^n)$ which is defined on all of $U$. Let $F_\alpha$

be the $n \times n$ matrices with entries

$$(F_\alpha)_{\gamma\delta} := \frac{\partial^3 F}{\partial t^\alpha \partial t^\gamma \partial t^\delta}. \tag{2.2.10}$$

Then, the *WDVV* (Witten-Dijkgraaf-Verlinde-Verlinde) *equations* are the following system of equations:

$$F_\alpha A^{-1} F_\beta = F_\beta A^{-1} F_\alpha, \quad (\alpha, \beta = 1, \ldots, n) \tag{2.2.11}$$

where $A$ is an invertible $n \times n$ matrix of the form

$$A = \sum_{i=1}^{n} a_i F_i, \tag{2.2.12}$$

with $a_i \in C^\infty(U)$. If $F$ solves the system of equations (2.2.11), for some matrix $A$ of the form (2.2.12), we say that $F$ is a *solution of the WDVV equations.*

From the definition of a solution of the WDVV equations, it appears that a solution would depend on the choice of the matrix $A$. However, the following proposition shows that this is not the case.

**Proposition 2.2.14.** *(cf.* [29], [40]*) Let $F : U \to \mathbb{C}$ be a solution of the WDVV equations and $B = \sum_{i=1}^{n} b_i F_i$, with $b_i \in C^\infty(U)$. If $B$ is invertible, then*

$$F_\alpha B^{-1} F_\beta = F_\beta B^{-1} F_\alpha,$$

*for all $\alpha, \beta = 1, \ldots, n$.*

*Proof.* Let $A = \sum_{i=1}^{n} a_i F_i$, with $a_i \in C^\infty(U)$ such that $A$ is invertible and

$$F_\alpha A^{-1} F_\beta = F_\beta A^{-1} F_\alpha,$$

for all $\alpha, \beta = 1, \ldots, n$. Then, by linearity, we see that

$$F_\alpha A^{-1} B = \sum_{i=1}^{n} b_i F_\alpha A^{-1} F_i = \sum_{i=1}^{n} b_i F_i A^{-1} F_\alpha = B A^{-1} F_\alpha,$$

for all $\alpha = 1, \ldots, n$, and thus

$$\begin{aligned} F_\alpha B^{-1} F_\beta &= F_\alpha B^{-1} F_\beta A^{-1} B B^{-1} A = F_\alpha B^{-1} B A^{-1} F_\beta B^{-1} A = F_\alpha A^{-1} F_\beta B^{-1} A \\ &= F_\beta A^{-1} F_\alpha B^{-1} A = F_\beta B^{-1} A A^{-1} B A^{-1} F_\alpha B^{-1} A \\ &= F_\beta B^{-1} A A^{-1} F_\alpha A^{-1} B B^{-1} A = F_\beta B^{-1} F_\alpha, \end{aligned}$$

for all $\alpha, \beta = 1, \ldots, n$. □

The above proposition is slightly more general than the ones given in [29] and [40], where it is additionally assumed that each of the matrices $F_\alpha$ are invertible. We now define another central object of this thesis, the prepotential, which is a solution of the WDVV equations with some additional properties.

**Definition 2.2.15.** [13, Lecture 1] A *prepotential* $F : U \to \mathbb{C}$ is a solution of the WDVV equations, together with a vector field $E \in \mathfrak{X}(U)$, which we call the Euler vector field, such that

**1)** The matrix $F_1$ is nondegenerate and has constant entries.

**2)** $E(t)$ has the form

$$E(t) = \left(q^\alpha_\beta t^\beta + r^\alpha\right) \partial_{t^\alpha},$$

where $q^\beta_\alpha, r^\alpha \in \mathbb{C}$ are constants for all $\alpha, \beta = 1, \dots, n$, such that

$$\mathcal{L}_E F(t) = (3 - d)F(t) + Q(t) \tag{2.2.13}$$

where $Q(t) = Q(t^1, \dots, t^n)$ is some quadratic function in the $t$ coordinates and $d \in \mathbb{C}$ is a constant, which we call the charge. We call the eigenvalues of the matrix $q^\alpha_\beta$ the degrees of $E$ and are denoted $d_1, \dots, d_n$.

**3)** We have

$$\mathcal{L}_E \partial_{t^1} = -\partial_{t^1}.$$

In the literature, prepotentials are often simply refered to as solutions of the WDVV equations. I have opted to distinguish between solutions of the WDVV equations and prepotentials, so that the extra properties of prepotentials are never forgotten in the discussion.

**Example 2.2.16.** Let $n = 1$. If $F$ is a one-dimensional prepotential, then it must be of the form

$$F(t^1) = \frac{C}{6}(t^1)^3 + P(t^1),$$

where $C \in \mathbb{C}$ is a nonzero constant and $P$ is a quadratic function. The Euler vector field has the form

$$E(t) = (t^1 + r^1)\partial_{t^1},$$

where $r^1 \in \mathbb{C}$ is a constant and the charge is $d = 0$.

**Example 2.2.17.** (cf. [13, Lecture 1]) Let $n = 2$. One can check that $F$ is of one of the following forms:

$$\begin{aligned} F(t^1, t^2) &= A((1-d)t^2 + a)^{\frac{3-d}{1-d}} + \frac{t^2}{b}(bt^1 + ct^2)^2 + P(t^1, t^2), \\ F(t^1, t^2) &= A(t^1)^3 + B(t^1)^2 t^2 + Ct^1(t^2)^2 + D(t^2)^3 + P(t^1, t^2), \end{aligned}$$

$$
\begin{aligned}
F(t^1,t^2) &= Ae^{\alpha t^2} + \frac{t^2}{b}(bt^1+ct^2)^2 + P(t^1,t^2),\\
F(t^1,t^2) &= A(2t^2+a)^2\ln(2t^2+a) + \frac{t^2}{b}(bt^1+ct^2)^2 + P(t^1,t^2),\\
F(t^1,t^2) &= A\ln(-2t^2+a) + \frac{t^2}{b}(bt^1+ct^2)^2 + P(t^1,t^2),
\end{aligned}
$$

where $A, B, C, a, b, c, d, \alpha \in \mathbb{C}$ are constants with $b \neq 0$, $\alpha \neq 0$, $3AC - B^2 \neq 0$, $d \neq 1$ and $P$ is a quadratic function. The Euler vector fields have the following respective forms:

$$
\begin{aligned}
E(t) &= \left(t^1 + \frac{dc}{b}t^2 + r^1\right)\partial_{t^1} + ((1-d)t^2 + a)\partial_{t^2},\\
E(t) &= (t^1 + r^1)\partial_{t^1} + (t^2 + r^2)\partial_{t^2},\\
E(t) &= \left(t^1 + \frac{c}{b}t^2 + r^1\right)\partial_{t^1} + \frac{2}{\alpha}\partial_{t^2},\\
E(t) &= \left(t^1 - \frac{c}{b}t^2 + r^1\right)\partial_{t^1} + (2t^2 + a)\partial_{t^2},\\
E(t) &= \left(t^1 + \frac{3c}{b}t^2 + r^1\right)\partial_{t^1} + (-2t^2 + a)\partial_{t^2},
\end{aligned}
$$

where $r^1, r^2 \in \mathbb{C}$ are constants and the charges are $d$, 0, 1, $-1$ and 3, respectively.

Frobenius manifolds and prepotentials are different expressions for the same mathematical object. The following theorem specifies exactly how Frobenius manifolds and prepotentials are related to one another.

**Theorem 2.2.18.** [13, Lecture 1] *Let $F : U \to \mathbb{C}$ be a prepotential with Euler vector field $E$ and charge $d$. Define $\eta : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U)$ and $\circ : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ by $C^\infty(U)$-bilinearly extending the following:*

$$
\eta(\partial_{t^\alpha}, \partial_{t^\beta}) = (F_1)_{\alpha\beta}, \qquad \partial_{t^\alpha} \circ \partial_{t^\beta} = c^\gamma_{\alpha\beta}(t)\partial_{t^\gamma},
$$

*for all $\alpha, \beta = 1, \ldots, n$. Then, $U$ with metric $\eta$ and multiplication $\circ$ is a Frobenius manifold with unity vector field $e = \partial_{t^1}$, Euler vector field $E$ and charge $d$. We refer to this as the Frobenius manifold of the prepotential $F$.*

*Conversely, let $M$ be a Frobenius manifold with metric $\eta$, multiplication $\circ$, unity vector field $e$, Euler vector field $E$ and charge $d$. Let $U \subseteq M$ be an open chart with flat coordinate system $t = (t^1, \ldots, t^n)$, where $e|_U = \partial_{t^1}$. If we define the tensor field*

$$
c(X, Y, Z) := \eta(X \circ Y,\, Z),
$$

*for all $X, Y, Z \in \mathfrak{X}(U)$, then there exists a function $F : U \to \mathbb{C}$ such that $F$ is a prepotential* *with Euler vector field* $E|_U$*, charge* $d$ *and*

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} = c(\partial_{t^\alpha}, \partial_{t^\beta}, \partial_{t^\gamma}), \tag{2.2.14}$$

*for all* $\alpha, \beta, \gamma = 1, \ldots, n$*. We call* $F$ *a prepotential of the Frobenius manifold* $M$*.*

*Furthermore, the Frobenius manifold* $U$ *of a prepotential* $F$ *of the Frobenius manifold* $M$ *is locally equivalent to* $M$*, and a prepotential* $\widetilde{F}$ *of a Frobenius manifold* $U$ *of the prepotential* $F$ *is equal to a restiction of* $F + Q$ *to a non-empty, open subset of* $U$*, where* $Q$ *is a quadratic function.*

A proof can be found in [13] within Lecture 1. This correspondence between Frobenius manifolds and prepotentials allows us to transfer concepts for Frobenius manifolds into concepts for prepotentials, and vice versa.

**Definition 2.2.19.** Let $F$ and $\widetilde{F}$ be prepotentials and let $M$ and $\widetilde{M}$ be the Frobenius manifolds of $F$ and $\widetilde{F}$, respectively. Then, we say $F$ is *(locally) equivalent* to $\widetilde{F}$ if $M$ is (locally) equivalent to $\widetilde{M}$. Moreover, a prepotential is called *reducible* (respectively, *irreducible*) if their Frobenius manifold is reducible (respectively, irreducible).

Let $M$ and $N$ have prepotentials $F$ and $\widetilde{F}$, respectively. Then the function $\widehat{F}(m, n) = F(m) + \widetilde{F}(n)$ is a prepotential of the product $M \times N$.

**Definition 2.2.20.** Let $F$ be a prepotential of the Frobenius manifold $M$. If $F$ is polynomial, rational, algebraic, trigonometric, etc., then we say that $M$ is a *polynomial, rational, algebraic, trigonometric, etc. Frobenius manifold*, respectively.

If $M$ is a Frobenius manifold of first type, then we often consider a prepotential $F$ of $M$ in so-called normalised coordinates $t$, where $\eta_{ij}(t) = \delta_{i+j,n+1}$.

### 2.2.3 Symmetries of prepotentials

To better understand and quantify prepotentials, it is preferable to introduce symmetries beyond the already discussed notion of equivalence. The symmetries we will discuss are presented in [13] and, in particular, they preserve the multiplication of the Frobenius manifold that they transform.

**Definition 2.2.21.** Let $M$ be a Frobenius manifold with metric $\eta$, multiplication $\circ$ and Euler vector field $E$. A vector field $L \in \mathfrak{X}(M)$ is called a *Legendre vector field* if it is invertible under the multiplication $\circ$ and

$${}^{\eta}\nabla_X L = 0,$$

for all $X \in \mathfrak{X}(M)$ and

$$\mathcal{L}_E L = \mu L,$$

for some constant $\mu \in \mathbb{C}$. We call $\mu$ the degree of $L$.

Legendre vector fields help us to define new Frobenius manifolds from known Frobenius manifolds.

**Proposition 2.2.22.** *Let $M$ be a Frobenius manifold with metric $\eta$, multiplication $\circ$, unity vector field $e$, Euler vector field $E$ and charge $d$, and let $L$ be a Legendre vector field of $M$ of degree $\mu$. We can define a new metric $\widetilde{\eta}$ on $M$ to be*

$$\widetilde{\eta}(X, Y) := \eta(X \circ L, Y \circ L), \tag{2.2.15}$$

*for all $X, Y \in \mathfrak{X}(M)$. Then, $\widetilde{M} = M$ together with metric $\widetilde{\eta}$ and multiplication $\circ$ is a Frobenius manifold, with unity vector field $e$, Euler vector field $E$ and charge $\widetilde{d} = d - 2\mu - 2$. We call the Frobenius manifold $\widetilde{M}$ a Legendre transformation of the Frobenius manifold $M$.*

*Proof.* By definition of $\widetilde{\eta}$ we have that

$$\widetilde{\eta}(X \circ Y, Z) = \eta(X \circ Y \circ L, Z \circ L) = \eta(X \circ L, Y \circ Z \circ L) = \widetilde{\eta}(X, Y \circ Z),$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Let $t^1, \dots, t^n$ be flat coordinates of $M$. Note that $\partial_{t^\delta} c^\alpha_{\beta\gamma}(t)$ is symmetric in the indices $\beta, \gamma, \delta$ and $L^\gamma(t)$ is constant, so we can define functions $\widetilde{t}^\alpha$ such that

$$\frac{\partial \widetilde{t}^\alpha}{\partial t^\beta} = L^\gamma(t) c^\alpha_{\beta\gamma}(t), \tag{2.2.16}$$

for all $\alpha, \beta = 1, \dots, n$. By the invertibility of $L$ these functions define a set of coordinates on $M$. Also,

$$\begin{aligned}
\widetilde{\eta}(\partial_{\widetilde{t}^\alpha}, \partial_{\widetilde{t}^\beta}) &= \widetilde{\eta}\left(\frac{\partial t^\gamma}{\partial \widetilde{t}^\alpha}\partial_{t^\gamma}, \frac{\partial t^\delta}{\partial \widetilde{t}^\beta}\partial_{t^\delta}\right) = \frac{\partial t^\gamma}{\partial \widetilde{t}^\alpha}\frac{\partial t^\delta}{\partial \widetilde{t}^\beta}\widetilde{\eta}\left(\partial_{t^\gamma}, \partial_{t^\delta}\right) \\
&= (L^{-1})^i(t) c^\gamma_{\alpha i}(t) (L^{-1})^j(t) c^\delta_{\beta j}(t) \eta(\partial_{t^\gamma} \circ L, \partial_{t^\delta} \circ L) \\
&= (L^{-1})^i(t) c^\gamma_{\alpha i}(t) (L^{-1})^j(t) c^\delta_{\beta j}(t) L^k(t) c^\mu_{\gamma k}(t) L^l(t) c^\nu_{\delta l}(t) \eta(\partial_{t^\mu}, \partial_{t^\nu}) \\
&= \eta(\partial_{t^\alpha}, \partial_{t^\beta}),
\end{aligned}$$

since $(L^{-1})^i c^\gamma_{\alpha i} L^k c^\mu_{\gamma k} = (L^{-1})^i c^\gamma_{ki} L^k c^\mu_{\gamma\alpha} = e^\gamma c^\mu_{\gamma\alpha} = \delta^\mu_\alpha$. Thus, $\widetilde{\eta}$ is a flat metric on $M$ and $\widetilde{t}^1, \dots, \widetilde{t}^n$ are flat coordinates for $\widetilde{\eta}$. The unity vector field for $\widetilde{M}$ is $e$, since the multiplication is unchanged, also

$$e^i(\widetilde{t}) = e^\alpha(t) \frac{\partial \widetilde{t}^i}{\partial t^\alpha} = e^\alpha(t) L^\gamma(t) c^i_{\gamma\alpha}(t) = L^i(t),$$

so ${}^{\widetilde{\eta}}\nabla_X e = 0$ for all $X \in \mathfrak{X}(M)$. We see that

$$\begin{aligned}\frac{\partial}{\partial \widetilde{t}^j}(E^i(\widetilde{t})) &= \frac{\partial t^\alpha}{\partial \widetilde{t}^j}\frac{\partial}{\partial t^\alpha}\left(E^\beta(t)\frac{\partial \widetilde{t}^i}{\partial t^\beta}\right) = (L^{-1})^\gamma(t) c^\alpha_{j\gamma}(t)\frac{\partial}{\partial t^\alpha}(E^\beta(t)L^\delta(t)c^i_{\beta\delta}(t))\\ &= (L^{-1})^\gamma c^\alpha_{j\gamma}\left(E^\beta_{;\alpha}L^\delta c^i_{\beta\delta} + E^\beta L^\delta_{;\alpha} + E^\beta L^\delta c^i_{\beta\delta;\alpha}\right)\\ &= (L^{-1})^\gamma c^\alpha_{j\gamma}\left((\mathcal{L}_E L)^\delta c^i_{\delta\alpha} + L^\delta(\mathcal{L}_E c)^i_{\delta\alpha} + E^i_{;\beta}L^\delta c^\beta_{\delta\alpha}\right)\\ &= (\mu+1)(L^{-1})^\gamma c^\alpha_{j\gamma}L^\delta c^i_{\delta\alpha} + (L^{-1})^\gamma c^\alpha_{j\gamma}E^i_{;\beta}L^\delta c^\beta_{\delta\alpha}\\ &= (\mu+1)\delta^i_j + E^i_{;j},\end{aligned}$$

where all tensor fields and their derivatives are expressed in the $t$ coordinates, the notation of which we have opted to remove for clarity of the equations. Thus ${}^{\widetilde{\eta}}\nabla_X({}^{\widetilde{\eta}}\nabla_Y E) = 0$, for all $X, Y \in \mathfrak{X}(M)$ such that ${}^{\widetilde{\eta}}\nabla_Z Y = 0$ for all $Z \in \mathfrak{X}(M)$, as required. Now, define $\widetilde{c}(X,Y,Z) := \widetilde{\eta}(X \circ Y, Z)$. Note that this is a symmetric tensor, and so

$$\widetilde{\eta}^{\alpha\delta}c^\epsilon_{\delta\gamma}\widetilde{\eta}_{\epsilon\beta} = \widetilde{\eta}^{\alpha\delta}\widetilde{c}_{\delta\gamma\beta} = \widetilde{\eta}^{\alpha\delta}\widetilde{c}_{\beta\gamma\delta} = \widetilde{\eta}^{\alpha\delta}\widetilde{\eta}_{\epsilon\delta}c^\epsilon_{\beta\gamma} = c^\alpha_{\beta\gamma}. \tag{2.2.17}$$

We see that

$$\begin{aligned}&\widetilde{\eta}_{\gamma\epsilon}(t)\left({}^{\widetilde{\eta}}\Gamma^\epsilon_{\delta\sigma}(t)c^\sigma_{\alpha\beta}(t) - {}^{\widetilde{\eta}}\Gamma^\sigma_{\delta\beta}(t)c^\epsilon_{\alpha\sigma}(t)\right)\\ &\quad= \widetilde{\eta}_{\gamma\epsilon}(t)\left(\frac{1}{2}\widetilde{\eta}^{\epsilon\lambda}(t)\left(\widetilde{\eta}_{\lambda\sigma;\delta}(t) + \widetilde{\eta}_{\delta\lambda;\sigma}(t) - \widetilde{\eta}_{\delta\sigma;\lambda}(t)\right)c^\sigma_{\alpha\beta}(t)\right.\\ &\qquad\qquad\left.-\frac{1}{2}\widetilde{\eta}^{\sigma\lambda}(t)\left(\widetilde{\eta}_{\lambda\beta;\delta}(t) + \widetilde{\eta}_{\delta\lambda;\beta}(t) - \widetilde{\eta}_{\delta\beta;\lambda}(t)\right)c^\epsilon_{\alpha\sigma}(t)\right)\\ &\quad= \frac{1}{2}\left(\left(\widetilde{\eta}_{\gamma\sigma;\delta}(t) + \widetilde{\eta}_{\delta\gamma;\sigma}(t) - \widetilde{\eta}_{\delta\sigma;\gamma}(t)\right)c^\sigma_{\alpha\beta}(t) - \left(\widetilde{\eta}_{\lambda\beta;\delta}(t) + \widetilde{\eta}_{\delta\lambda;\beta}(t) - \widetilde{\eta}_{\delta\beta;\lambda}(t)\right)c^\lambda_{\alpha\gamma}(t)\right),\end{aligned}$$

by formula (2.2.17). We know that $\widetilde{\eta}_{\alpha\beta} = \eta_{ij}c^i_{\alpha k}L^k c^j_{\beta l}L^l$ by formula (2.2.15), and so the above expression is equal to

$$\begin{aligned}&\frac{1}{2}\eta_{ij}L^kL^l\left(c^i_{\gamma k;\delta}(t)c^j_{\sigma l}(t)c^\sigma_{\alpha\beta}(t) + c^i_{\gamma k}(t)c^j_{\sigma l;\delta}(t)c^\sigma_{\alpha\beta}(t) + c^i_{\delta k;\sigma}(t)c^j_{\gamma l}(t)c^\sigma_{\alpha\beta}(t)\right.\\ &\qquad + c^i_{\delta k}(t)c^j_{\gamma l;\sigma}(t)c^\sigma_{\alpha\beta}(t) - c^i_{\delta k;\gamma}(t)c^j_{\sigma l}(t)c^\sigma_{\alpha\beta}(t) - c^i_{\delta k}(t)c^j_{\sigma l;\gamma}(t)c^\sigma_{\alpha\beta}(t)\\ &\qquad - c^i_{\sigma k;\delta}(t)c^j_{\beta l}(t)c^\sigma_{\alpha\gamma}(t) - c^i_{\sigma k}(t)c^j_{\beta l;\delta}(t)c^\sigma_{\alpha\gamma}(t) - c^i_{\delta k;\beta}(t)c^j_{\sigma l}(t)c^\sigma_{\alpha\gamma}(t)\\ &\qquad \left.-c^i_{\delta k}(t)c^j_{\sigma l;\beta}(t)c^\sigma_{\alpha\gamma}(t) + c^i_{\delta k;\sigma}(t)c^j_{\beta l}(t)c^\sigma_{\alpha\gamma}(t) + c^i_{\delta k}(t)c^j_{\beta l;\sigma}(t)c^\sigma_{\alpha\gamma}(t)\right)\\ &= L^kL^l\left(c_{i\gamma k}(t)c^i_{\sigma\delta;l}(t)c^\sigma_{\alpha\beta}(t) - c_{i\sigma k}(t)c^i_{\delta k;\beta}(t)c^\sigma_{\alpha\gamma}(t)\right)\\ &= L^kL^l\left((c_{i\gamma k}c^i_{\sigma\delta}c^\sigma_{\alpha\beta})_{;l}(t) - c_{i\gamma k;l}(t)c^i_{\sigma\delta}(t)c^\sigma_{\alpha\beta}(t) - c_{i\gamma k}(t)c^i_{\sigma\delta}(t)c^\sigma_{\alpha\beta;l}(t)\right.\\ &\qquad\left.-c_{i\sigma\gamma}(t)c^i_{\delta k;\beta}(t)c^\sigma_{\alpha l}(t)\right)\\ &= L^kL^l\left((c_{i\gamma k}c^i_{\sigma\delta}c^\sigma_{\alpha\beta})_{;l}(t) - c_{i\gamma k;l}(t)c^i_{\sigma\delta}(t)c^\sigma_{\alpha\beta}(t) - c_{i\sigma\gamma}(t)(c^i_{\delta k}c^\sigma_{\alpha l})_{;\beta}(t)\right).\end{aligned}$$

This final expression is symmetric in $\alpha$ and $\delta$, thus

$$\widetilde{\eta}_{\gamma\epsilon}(t)\left({}^{\widetilde{\eta}}\Gamma^{\epsilon}_{\delta\sigma}(t)c^{\sigma}_{\alpha\beta}(t) - {}^{\widetilde{\eta}}\Gamma^{\sigma}_{\delta\beta}(t)c^{\epsilon}_{\alpha\sigma}(t)\right) = \widetilde{\eta}_{\gamma\epsilon}(t)\left({}^{\widetilde{\eta}}\Gamma^{\epsilon}_{\alpha\sigma}(t)c^{\sigma}_{\delta\beta}(t) - {}^{\widetilde{\eta}}\Gamma^{\sigma}_{\alpha\beta}(t)c^{\epsilon}_{\delta\sigma}(t)\right).$$

Therefore,

$$\begin{aligned}({}^{\widetilde{\eta}}\nabla_{\delta}\,\widetilde{c})_{\alpha\beta\gamma}(t) &= ({}^{\widetilde{\eta}}\nabla_{\delta}\,\widetilde{\eta})_{\gamma\epsilon}(t)c^{\epsilon}_{\alpha\beta}(t) + \widetilde{\eta}_{\gamma\epsilon}(t)({}^{\widetilde{\eta}}\nabla_{\delta}\,c)^{\epsilon}_{\alpha\beta}(t) = \widetilde{\eta}_{\gamma\epsilon}(t)({}^{\widetilde{\eta}}\nabla_{\delta}\,c)^{\epsilon}_{\alpha\beta}(t)\\ &= \widetilde{\eta}_{\gamma\epsilon}(t)c^{\epsilon}_{\alpha\beta;\delta}(t) - \widetilde{\eta}_{\gamma\epsilon}(t){}^{\widetilde{\eta}}\Gamma^{\sigma}_{\delta\alpha}(t)c^{\epsilon}_{\sigma\beta}(t) + \widetilde{\eta}_{\gamma\epsilon}(t)\left({}^{\widetilde{\eta}}\Gamma^{\epsilon}_{\delta\sigma}(t)c^{\sigma}_{\alpha\beta}(t) - {}^{\widetilde{\eta}}\Gamma^{\sigma}_{\delta\beta}(t)c^{\epsilon}_{\alpha\sigma}(t)\right)\\ &= \widetilde{\eta}_{\gamma\epsilon}(t)c^{\epsilon}_{\delta\beta;\alpha}(t) - \widetilde{\eta}_{\gamma\epsilon}(t){}^{\widetilde{\eta}}\Gamma^{\sigma}_{\alpha\delta}(t)c^{\epsilon}_{\sigma\beta}(t) + \widetilde{\eta}_{\gamma\epsilon}(t)\left({}^{\widetilde{\eta}}\Gamma^{\epsilon}_{\alpha\sigma}(t)c^{\sigma}_{\delta\beta}(t) - {}^{\widetilde{\eta}}\Gamma^{\sigma}_{\alpha\beta}(t)c^{\epsilon}_{\delta\sigma}(t)\right)\\ &= \widetilde{\eta}_{\gamma\epsilon}(t)({}^{\widetilde{\eta}}\nabla_{\alpha}\,c)^{\epsilon}_{\delta\beta}(t) = ({}^{\widetilde{\eta}}\nabla_{\alpha}\,\widetilde{\eta})_{\gamma\epsilon}(t)c^{\epsilon}_{\delta\beta}(t) + \widetilde{\eta}_{\gamma\epsilon}(t)({}^{\widetilde{\eta}}\nabla_{\alpha}\,c)^{\epsilon}_{\delta\beta}(t)\\ &= ({}^{\widetilde{\eta}}\nabla_{\alpha}\,\widetilde{c})_{\delta\beta\gamma}(t),\end{aligned}$$

for all $\alpha, \beta, \gamma, \delta = 1, \ldots, n$. Now,

$$\begin{aligned}(\mathcal{L}_E\widetilde{\eta})_{\alpha\beta} &= (\mathcal{L}_E\eta)_{ij}c^{i}_{\alpha k}L^k c^{j}_{\beta l}L^l + \eta_{ij}(\mathcal{L}_E c)^{i}_{\alpha k}L^k c^{j}_{\alpha l}L^l + \eta_{ij}c^{i}_{\alpha k}(\mathcal{L}_E L)^k c^{j}_{\beta l}L^l\\ &\quad + \eta_{ij}c^{i}_{\alpha k}L^k(\mathcal{L}_E c)^{j}_{\beta l}L^l + \eta_{ij}c^{i}_{\alpha k}L^k c^{j}_{\beta l}(\mathcal{L}_E L)^l\\ &= (2 - d + 1 + \mu + 1 + \mu)\eta_{ij}c^{i}_{\alpha k}L^k c^{j}_{\beta l}L^l = (4 + 2\mu - d)\widetilde{\eta}_{\alpha\beta}.\end{aligned}$$

Thus,

$$(\mathcal{L}_E\widetilde{c})_{\alpha\beta\gamma} = (\mathcal{L}_E\widetilde{\eta})_{\delta\gamma}c^{\delta}_{\alpha\beta} + \widetilde{\eta}_{\delta\gamma}(\mathcal{L}_E c)^{\delta}_{\alpha\beta} = (4 + 2\mu - d + 1)\widetilde{\eta}_{\delta\gamma}c^{\delta}_{\alpha\beta} = (3 - (d - 2\mu - 2))\widetilde{c}_{\alpha\beta\gamma}.$$

□

The fact that Legendre vector fields are covariantly constant with respect to the metric is necessary to show that the new metric defines a new Frobenius manifold. This, in turn, gives us a new prepotential which is a particular solution of the WDVV equations, thus transforming one solution of the WDVV equations into another. In fact, one can show that any solution of the WDVV equations can be transformed into another solution of the WDVV equations using a Legendre vector field [50].

**Remark 2.2.23.** Slightly relaxing the covariantly constant condition for Legendre vector fields, one can define a generalised Legendre vector field [50], which also transforms solutions of the WDVV equations into solutions of the WDVV equations, but it does not necessarily transform prepotentials into prepotentials.

The next proposition is stated in Appendix B of [13], but no proof is given. We provide a proof here.

**Proposition 2.2.24.** [13, Appendix B] *Let $M$ be a Frobenius manifold with metric $\eta$ and let $\widetilde{M}$ be a Legendre transformation of $M$ by the Legendre vector field $L$. Let $U \subseteq M$ be an* *open chart of $M$ with flat coordinates $t^1, \dots, t^n$ such that $e = \partial_{t^1}$ is the identity element of $M$ and suppose $F : U \to \mathbb{C}$ is a prepotential of $M$. Then, there exists flat coordinates $\widetilde{t}^1, \dots, \widetilde{t}^n$ of $\widetilde{M}$ and a prepotential $\widetilde{F} : U \to \mathbb{C}$ of $\widetilde{M}$ such that*

$$\widetilde{t}^\alpha = L^\gamma(t)\eta^{\alpha\beta}\frac{\partial^2 F}{\partial t^\beta \partial t^\gamma}, \qquad \frac{\partial^2 \widetilde{F}}{\partial \widetilde{t}^\alpha \partial \widetilde{t}^\beta} = \frac{\partial^2 F}{\partial t^\alpha \partial t^\beta}, \tag{2.2.18}$$

*for all $\alpha, \beta = 1, \dots, n$.*

*Proof.* We showed in the proof of Proposition 2.2.22 that $\widetilde{t}^\alpha$ are indeed flat coordinates of $\widetilde{M}$. To check that $\widetilde{F}$ exists and is a prepotential of $\widetilde{M}$ we only need to check that the third order derivatives of $\widetilde{F}(\widetilde{t})$ are equal to $\widetilde{c}_{\alpha\beta\gamma}(\widetilde{t})$. First, note that $\partial_{\widetilde{t}^\alpha} = L^{-1} \circ \partial_{t^\alpha}$ which follows from formula (2.2.16). Thus,

$$\begin{aligned}\widetilde{c}(\partial_{\widetilde{t}^\alpha}, \partial_{\widetilde{t}^\beta}, \partial_{\widetilde{t}^\gamma}) &= \widetilde{\eta}(\partial_{\widetilde{t}^\alpha} \circ \partial_{\widetilde{t}^\beta},\, \partial_{\widetilde{t}^\gamma}) = \eta(\partial_{\widetilde{t}^\alpha} \circ \partial_{\widetilde{t}^\beta} \circ L,\, \partial_{\widetilde{t}^\gamma} \circ L) \\ &= \eta(\partial_{t^\alpha} \circ L^{-1} \circ \partial_{t^\beta},\, \partial_{t^\gamma}) = c(\partial_{t^\alpha} \circ L^{-1}, \partial_{t^\beta}, \partial_{t^\gamma}).\end{aligned}$$

Hence, we see that

$$\frac{\partial^3 \widetilde{F}}{\partial \widetilde{t}^\alpha \partial \widetilde{t}^\beta \partial \widetilde{t}^\gamma} = \frac{\partial t^\delta}{\partial \widetilde{t}^\gamma}\frac{\partial}{\partial t^\delta}\left(\frac{\partial^2 F}{\partial t^\alpha \partial t^\beta}\right) = (L^{-1})^\epsilon(t) c^\delta_{\alpha\epsilon}(t) c_{\delta\beta\gamma}(t) = \widetilde{c}_{\alpha\beta\gamma}(\widetilde{t}),$$

for all $\alpha, \beta, \gamma = 1, \dots, n$, which completes the proof. □

Notice that for every Frobenius manifold, the unity vector field $e$ is a Legendre vector field and the Legendre transformation by this vector field just gives us the same Frobenius manifold. Moreover, a Legendre transformation from $M$ to $\widetilde{M}$ using the Legendre vector field $L$ implies that the vector field $L^{-1}$ is a Legendre vector field on $\widetilde{M}$ and the Legendre transformation by $L^{-1}$ gives us $M$.

**Proposition 2.2.25.** *Legendre transformations form an equivalence relation on Frobenius manifolds. Moreover, if there exists a Legendre transformation from $M$ to $N$ by the Legendre vector field $L$ and there exists a Legendre transformation from $N$ to $O$ by Legendre vector field $K$, then there exists a Legendre transformation from $M$ to $O$ by the Legendre vector field $L \circ K$.*

*Proof.* Reflexivity and symmetry have been shown above, all that remains to show is transitivity, and specifically that $L \circ K$ is a Legendre vector field on $M$. If this is a Legendre vector field, then the Legendre transformation would give $O$ because of the associativity of the multiplication. Since multiplication doesn't change between Legendre transformations and $L^{-1}$, $K^{-1}$ both exist and thus $(L \circ K)^{-1} = K^{-1} \circ L^{-1}$ clearly exists. Moreover, suppose $L$ has degree $\mu$ and $K$ has degree $\nu$, then

$$(\mathcal{L}_E(L \circ K))^\alpha = (\mathcal{L}_E L)^\gamma\, c^\alpha_{\gamma\delta} K^\delta + L^\gamma\, (\mathcal{L}_E c)^\alpha_{\gamma\delta}\, K^\delta + L^\gamma c^\alpha_{\gamma\delta}\, (\mathcal{L}_E K)^\delta = (\mu + 1 + \nu)(L \circ K)^\alpha,$$

so $L \circ K$ has degree $\mu+\nu+1$. Finally, we know that $L$ is covariantly constant with respect to the metric $\eta$ on $M$ and $K$ is covariantly constant with respect to the metric $\widetilde{\eta}$ on $N$, where

$$\widetilde{\eta}(X, Y) = \eta(X \circ L, Y \circ L).$$

Let $t^1, \ldots, t^n$ be flat coordinates for the metric $\eta$, then, recalling that $\widetilde{\eta}_{\alpha\beta} = \eta_{ij} c^i_{\alpha k} L^k c^j_{\beta l} L^l$ and $\widetilde{\eta}^{\alpha\beta} = \eta^{ij} c^\alpha_{ik} (L^{-1})^k c^\beta_{jl} (L^{-1})^l$ we get that

$$\begin{aligned}
{}^{\widetilde{\eta}}\Gamma^\delta_{\alpha\epsilon}(t) c^\beta_{\gamma\delta}(t) &= \frac{1}{2}\widetilde{\eta}^{\delta\sigma}(t)\left(\widetilde{\eta}_{\sigma\epsilon;\alpha}(t) + \widetilde{\eta}_{\alpha\sigma;\epsilon}(t) - \widetilde{\eta}_{\alpha\epsilon;\sigma}(t)\right) c^\beta_{\gamma\delta}(t) \\
&= \frac{1}{2}\widetilde{\eta}^{\delta\sigma}\eta_{ij} L^k L^l \left(c^i_{\sigma k;\alpha} c^j_{\epsilon l} + c^i_{\sigma k} c^j_{\epsilon l;\alpha} + c^i_{\alpha k;\epsilon} c^j_{\sigma l} + c^i_{\alpha k} c^j_{\sigma l;\epsilon} - c^i_{\alpha k;\sigma} c^j_{\epsilon l} - c^i_{\alpha k} c^j_{\epsilon l;\sigma}\right) c^\beta_{\gamma\delta} \\
&= \eta^{ab} c^\delta_{ac} (L^{-1})^c c^\sigma_{bd} (L^{-1})^d \eta_{ij} L^k L^l c^i_{\epsilon k;\alpha} c^j_{\sigma l} c^\beta_{\gamma\delta} = c^\delta_{ic} (L^{-1})^c c^i_{\epsilon k;\alpha} L^k c^\beta_{\gamma\delta}.
\end{aligned}$$

Using the fact that $K$ is covariantly constant with respect to the metric $\widetilde{\eta}$, we find that

$$\begin{aligned}
\left({}^\eta\nabla_\alpha (L \circ K)\right)^\beta (t) &= L^\gamma c^\beta_{\gamma\delta;\alpha} K^\delta + L^\gamma c^\beta_{\gamma\delta} K^\delta_{;\alpha} = L^\gamma \left(c^\beta_{\gamma\epsilon;\alpha}(t) - {}^{\widetilde{\eta}}\Gamma^\delta_{\alpha\epsilon}(t) c^\beta_{\gamma\delta}(t)\right) K^\epsilon \\
&= L^\gamma \left(c^\beta_{\gamma\epsilon;\alpha} - c^\delta_{ic}(L^{-1})^c c^i_{\epsilon k;\alpha} L^k c^\beta_{\gamma\delta}\right) K^\epsilon = L^\gamma (c^\beta_{\gamma\epsilon;\alpha} - c^\beta_{\epsilon\gamma;\alpha}) K^\epsilon = 0.
\end{aligned}$$

□

More recently, some analytic properties of Legendre transformations have been studied by Yang [54]. The Legendre transformations are not the only symmetry of the prepotentials which preserve the multiplication of the Frobenius manifold. There is also the inversion symmetry [13], which is the only non-trivial symmetry which preserves the multiplication and transforms the metric conformally.

**Definition 2.2.26.** [13, Appendix B] Let $F : U \to \mathbb{C}$ be a prepotential of first type written in normalised coordinates $t^1, \ldots, t^n$. We define the *inversion* of $F$ to be the function $\widetilde{F} : V \to \mathbb{C}$, where $V = U \setminus \{t \in U \mid t^n = 0\}$ such that

$$\widetilde{F}(t) := (t^n)^{-2}\left(F(t) - \frac{1}{2} t^1 \eta_{\nu\mu} t^\nu t^\mu\right).$$

**Proposition 2.2.27.** [13, Appendix B] *Let $F$ be a prepotential of first type in normalised coordinates $t^1, \ldots, t^n$ with Euler vector field $E$ and charge $d$, and let $\widetilde{F}$ be an inversion of $F$. If $d \neq 1$ or $\eta(E, \partial_{t^1}) = 0$, then there exists coordinates $\widetilde{t}^1, \ldots, \widetilde{t}^n$ defined as*

$$\widetilde{t}^1 := \frac{\eta_{\nu\mu} t^\nu t^\mu}{2t^n}, \qquad \widetilde{t}^\alpha := \frac{t^\alpha}{t^n}, \qquad \widetilde{t}^n := -\frac{1}{t^n},$$

*for all $\alpha \neq 1, n$, such that $\widetilde{F}(\widetilde{t})$ is a prepotential with Euler vector field $E$ and charge $\widetilde{d} = 2 - d$.*

In appendix B of [13], Dubrovin presents the third order derivatives of an inversion of a prepotential and leaves the checking of the necessary conditions as an exercise to the reader. The inversion symmetry, like the Legendre transformations, can also be applied to solutions of the WDVV equations in general, in order to generate new solutions of the WDVV equations. In this case, the conditions $d \neq 1$ or $\eta(E, \partial_{t^1}) = 0$ are not necessary to impose, however the coordinates must still be normalised for the solution of the WDVV equations that the inversion symmetry is applied to.

### 2.2.4 Almost duality

To each Frobenius manifold we may associate "dual" structures such as a metric, multiplication and a solution of the WDVV equations, which altogether satisfy most, but not all, of the conditions for a Frobenius manifold [17]. We begin with the concept of the intersection form, which is another flat metric for a Frobenius manifold, following lecture 3 in [13].

**Definition 2.2.28.** [13, Lecture 3] Let $M$ be a Frobenius manifold with Euler vector field $E$ and cotangent structure constants $c^{\alpha\beta}_{\gamma}$. The *intersection form* $g$ of $M$ is defined as the rank (2, 0) tensor field

$$g^{\alpha\beta} := E^{\gamma} c^{\alpha\beta}_{\gamma}. \tag{2.2.19}$$

It is easy to see that $g$ is symmetric, and thus it is metric at the points where it is non-degenerate.

**Definition 2.2.29.** Let $M$ be a Frobenius manifold with intersection form $g^{\alpha\beta}$. Then the *discriminant locus* $\Sigma$ of $M$ is the set

$$\Sigma := \{p \in M \mid g^{\alpha\beta}_p \text{ is degenerate}\}.$$

The points where the intersection form is non-degenerate are also points where the Euler vector field $E$ is invertible under the multiplication $\circ$, the converse is not necessarily true.

**Lemma 2.2.30.** *Let $M$ be a Frobenius manifold with discriminant locus $\Sigma$. Then, $M \setminus \Sigma$ is an open, dense subset of $M$.*

*Proof.* Let $E$ be the Euler vector field of $M$ and let $c^{\gamma}_{\alpha\beta}$ be the structure constants of $M$. Consider the tensor field

$$(E\,\circ)^{\alpha}_{\beta} := E^{\gamma} c^{\alpha}_{\beta\gamma}.$$

Note that $(E\,\circ)^{\alpha}_{\beta} = g^{\alpha\delta}\eta_{\delta\beta}$, where $\eta$ is the metric of $M$. Thus, $g^{\alpha\beta}$ is non-degenerate at the point $p \in M$ if and only if $(E\,\circ)$ is non-degenerate at $p \in M$. So, to prove the lemma, we must show that the points where the determinant of $(E\circ)^{\alpha}_{\beta}(t)$ is non-zero form an open, dense subset in some open charts which cover $M$, where $t$ are the coordinates of the open charts.

Let $U \subseteq M$ be an open chart of $M$ with flat coordinates $t^1, \ldots, t^n$ such that $e = \partial_{t^1}$. Then,

$$\begin{aligned}\frac{\partial^n}{\partial(t^1)^n}(\det((E\circ)^{\alpha}_{\beta}(t)) &= \frac{\partial^n}{\partial(t^1)^n}\left(\sum_{\sigma\in S_n}(-1)^{\sigma}\prod_{j=1}^{n}(E\circ)^{\sigma(j)}_{j}(t)\right)\\ &= \sum_{\sigma\in S_n}(-1)^{\sigma}\frac{\partial^n}{(\partial t^1)^n}\left(\prod_{j=1}^{n}E^{\gamma}(t)c^{\sigma(j)}_{j\gamma}(t)\right)\\ &= \sum_{\sigma\in S_n}(-1)^{\sigma}\prod_{j=1}^{n}c^{\sigma(j)}_{j1}(t) = \sum_{\sigma\in S_n}(-1)^{\sigma}\prod_{j=1}^{n}\delta^{\sigma(j)}_{j} = 1,\end{aligned}$$

where we use the fact that $c^{\alpha}_{\beta\gamma}(t)$ does not depend on $t^1$, and $E^{\gamma}_{;1}(t) = \delta^{\gamma}_{1}$. Thus, the determinant of $(E\circ)^{\alpha}_{\beta}(t)$ is a polynomial in $t^1$ of degree $n$, with coefficents being analytic functions in $t^2, \ldots, t^n$. Therefore, the complement of its zero set is open and dense in $U$. Gluing these open, dense subsets of open charts together gives us an open dense subset of $M$. □

Thus we have shown that the points where the intersection form is non-degenerate form an open, dense subset of $M$. Lemma 2.2.30 also allows us to unambiguously define $E^{-1}$ at these points, since we can define $(E^{-1})^{\alpha} := (E\circ)^{\alpha}_{\beta}\, e^{\beta}$, where $e$ is the unity vector field.

**Proposition 2.2.31.** [13, Lecture 3] *Let $M$ be a Frobenius manifold with metric $\eta$, multiplication $\circ$ and Euler vector field $E$. The intersection form $g$ can be written in terms of the metric $\eta$ in the following way:*

$$g_t(X, Y) = \eta_t(X, Y \circ E^{-1}), \tag{2.2.20}$$

*for all $X, Y \in \mathfrak{X}(M)$ and all $t \in M\setminus\Sigma$, where $E^{-1}$ is the inverse of $E$ under multiplication $\circ$ and $\Sigma$ is the discriminant locus of $M$.*

*Proof.* Let $U \subseteq M$ be an open chart with flat coordinates $t^1, \ldots, t^n$ such that $e = \partial_{t^1}$ is the identity element of the multiplication $\circ$. To prove the claim we will show that $\eta(\partial_{t^\alpha}, \partial_{t^\beta} \circ E^{-1}(t))$ is the matrix inverse of $g^{\alpha\beta}(t)$. Indeed,

$$\begin{aligned}\eta(\partial_{t^\alpha}, \partial_{t^\gamma}\circ E^{-1}(t))g^{\gamma\beta}(t) &= c(\partial_{t^\alpha}, \partial_{t^\gamma}, E^{-1})E^{\delta}(t)c^{\gamma\beta}_{\delta}(t)\\ &= c^{\gamma}_{\alpha\epsilon}(t)(E^{-1})^{\epsilon}(t)E^{\delta}(t)c^{\beta}_{\gamma\delta}(t)\\ &= c^{\gamma}_{\delta\epsilon}(t)E^{\delta}(t)(E^{-1})^{\epsilon}(t)c^{\beta}_{\alpha\gamma}(t) = c^{\beta}_{\alpha 1}(t) = \delta^{\beta}_{\alpha},\end{aligned}$$

for all $\alpha, \beta = 1, \dots, n$. □

**Definition 2.2.32.** (cf. [17]) An *almost Frobenius manifold* $M_*$ is a complex manifold together with a flat metric $g$, a pointwise $C^\infty(M_*)$ bilinear, commutative multiplication $\star$ on $\mathfrak{X}(M_*)$ and an invertible vector field $e \in \mathfrak{X}(M_*)$, which we call the Liouville vector field, such that

**1)** There exists an identity element $E$ for the multiplication $\star$ and a constant $d \in \mathbb{C}$ such that

$$ {}^g\nabla_X E = \frac{1-d}{2} X, \tag{2.2.21} $$

for all $X \in \mathfrak{X}(M_*)$. We call $E$ the Euler vector field of $M_*$ and we call $d$ the charge of $M_*$.

**2)** The tangent spaces of $M_*$ with multiplication and bilinear form defined by the restriction of $\star$ and $g$, respectively, are Frobenius algebras, that is the multiplication $\star$ is associative and

$$ g(X \star Y,\, Z) = g(X,\, Y \star Z), $$

for all $X, Y, Z \in \mathfrak{X}(M_*)$.

**3)** We define the following tensor field

$$ \overset{*}{c}(X, Y, Z) := g(X \star Y,\, Z) \tag{2.2.22} $$

for all $X, Y, Z \in \mathfrak{X}(M_*)$. We require that

$$ ({}^g\nabla_W \overset{*}{c})(X, Y, Z) = ({}^g\nabla_X \overset{*}{c})(W, Y, Z) \tag{2.2.23} $$

for all $W, X, Y, Z \in \mathfrak{X}(M_*)$.

**4)** Define the following tensor field:

$$ \overset{*}{c}{}^{\gamma}_{\alpha\beta} := g^{\gamma\delta} \overset{*}{c}_{\alpha\beta\delta}, $$

for all $\alpha, \beta, \gamma = 1, \dots, n$. Let $U \subseteq M_*$ be an open chart with flat coordinates $x^1, \dots, x^n$ of the metric $g$. Define the following sets of functions:

$$ P_\nu := \left\{ p \in C^\infty(U) \,\middle|\, \frac{\partial^2 p}{\partial x^i \partial x^j} = \nu \, \overset{*}{c}{}^k_{ij}(x) \frac{\partial p}{\partial x^k}, \text{ for all } i, j = 1, \dots, n \right\}, $$

for any $\nu \in \mathbb{C}$. Then, $e|_U(P_\nu) \subseteq P_{\nu-1}$, for all $\nu \in \mathbb{C}$.

**5)** Define the following tensor field:

$$ c^{\alpha\beta}_{\gamma} := g^{\alpha\delta} \, \overset{*}{c}{}^{\beta}_{\gamma\delta}, $$

for all $\alpha, \beta, \gamma = 1, \ldots, n$. Then, we require that

$$(\mathcal{L}_e g)^{\alpha\beta} = e^\gamma c_\gamma^{\alpha\beta}.$$

Compare this with the definition of an almost Frobenius manifold given in [17]. Condition **5)** has been added here as it seems necessary to get a Frobenius manifold from an almost Frobenius manifold. Conditions **4)** and **5)** in the above definition of an almost Frobenius manifold can be replaced with coordinate-free descriptions of $e$. In particular, they are equivalent to the conditions given in the next statement.

**Proposition 2.2.33.** *Suppose we have a manifold $M_*$ with flat metric $g$, a pointwise $C^\infty(M_*)$ bilinear, commutative multiplication $\star$ on $\mathfrak{X}(M_*)$ and an invertible vector field $e \in \mathfrak{X}(M_*)$ such that conditions **1)-3)** of Definition 2.2.32 hold. Then $M_*$ is an almost Frobenius manifold if and only if the following conditions hold:*

***a)*** *For all $X, Y \in \mathfrak{X}(M_*)$ where ${}^g\nabla_Z Y = 0$ for all $Z \in \mathfrak{X}(M_*)$, we have that*

$$ {}^g\nabla_X \left({}^g\nabla_Y e\right) + {}^g\nabla_{X\star Y} e = 0. \tag{2.2.24}$$

***b)*** *Define the tensor field*

$$c_\gamma^{\alpha\beta} := g^{\alpha\delta} g^{\beta\epsilon} \overset{*}{c}_{\gamma\delta\epsilon}.$$

*Then*

$$(\mathcal{L}_e c)_\gamma^{\alpha\beta} = 0.$$

***c)*** *We have*

$$\mathcal{L}_E e = -e.$$

*Proof.* Suppose condition **4)** of Definition 2.2.32 holds, and let $U \subseteq M_*$ be an open chart with flat coordinates $x^1, \ldots, x^n$ of the metric $g$. Let $e$ be the Liouville vector field of $M_*$. Note that $x^l \in P_0$ for all $l = 1, \ldots, n$. Thus,

$$\frac{\partial^2 e^l(x)}{\partial x^i \partial x^j} = -\overset{*}{c}{}_{ij}^k(x) \frac{\partial e^l(x)}{\partial x^k},$$

and thus condition **a)** is satisfied. Let $p \in P_\nu$ and $\nu \in \mathbb{C}$ be arbitrary, then

$$\frac{\partial^2}{\partial x^i \partial x^j}\left(e^l(x)\frac{\partial p}{\partial x^l}\right) = (\nu - 1)\, \overset{*}{c}{}_{ij}^k(x) \frac{\partial}{\partial x^k}\left(e^l(x)\frac{\partial p}{\partial x^l}\right). \tag{2.2.25}$$

Let us expand both sides of the equality (2.2.25). We have

$$\begin{aligned}
\frac{\partial^2}{\partial x^i \partial x^j}\left(e^l(x)\frac{\partial p}{\partial x^l}\right) &= \frac{\partial}{\partial x^i}\left(\frac{\partial e^l(x)}{\partial x^j}\frac{\partial p}{\partial x^l} + e^l(x)\frac{\partial^2 p}{\partial x^j \partial x^l}\right) \\
&= \frac{\partial}{\partial x^i}\left(\frac{\partial e^l(x)}{\partial x^j}\frac{\partial p}{\partial x^l} + e^l(x)\nu\, \overset{*}{c}{}_{jl}^k(x)\frac{\partial p}{\partial x^k}\right)
\end{aligned}$$

$$= \frac{\partial^2 e^l(x)}{\partial x^i \partial x^j}\frac{\partial p}{\partial x^l} + \frac{\partial e^l(x)}{\partial x^j}\frac{\partial^2 p}{\partial x^i \partial x^l} + \frac{\partial e^l(x)}{\partial x^i}\nu \overset{*}{c}{}^k_{jl}(x)\frac{\partial p}{\partial x^k} + e^l(x)\nu \overset{*}{c}{}^k_{jl;i}(x)\frac{\partial p}{\partial x^k}$$
$$+ e^l(x)\nu \overset{*}{c}{}^k_{jl}(x)\frac{\partial^2 p}{\partial x^i \partial x^k}$$
$$= -\overset{*}{c}{}^k_{ij}(x)\frac{\partial e^l(x)}{\partial x^k}\frac{\partial p}{\partial x^l} + \nu\frac{\partial e^l(x)}{\partial x^j}\overset{*}{c}{}^k_{il}(x)\frac{\partial p}{\partial x^k} + \nu\frac{\partial e^l(x)}{\partial x^i}\overset{*}{c}{}^k_{jl}(x)\frac{\partial p}{\partial x^k}$$
$$+ \nu e^l(x)\overset{*}{c}{}^k_{ij;l}(x)\frac{\partial p}{\partial x^k} + \nu^2 e^l(x)\overset{*}{c}{}^k_{jl}(x)\overset{*}{c}{}^m_{ik}(x)\frac{\partial p}{\partial x^m}. \tag{2.2.26}$$

Also,

$$(\nu-1)\overset{*}{c}{}^k_{ij}(x)\frac{\partial}{\partial x^k}\left(e^l(x)\frac{\partial p}{\partial x^l}\right) = (\nu-1)\overset{*}{c}{}^k_{ij}(x)\frac{\partial e^l(x)}{\partial x^k}\frac{\partial p}{\partial x^l} + (\nu-1)\overset{*}{c}{}^k_{ij}(x)e^l(x)\frac{\partial^2 p}{\partial x^k \partial x^l}$$
$$= (\nu-1)\overset{*}{c}{}^k_{ij}(x)\frac{\partial e^l(x)}{\partial x^k}\frac{\partial p}{\partial x^l} + \nu(\nu-1)\overset{*}{c}{}^k_{ij}(x)e^l(x)\overset{*}{c}{}^m_{kl}\frac{\partial p}{\partial x^m}. \tag{2.2.27}$$

Substituting (2.2.26) and (2.2.27) into relation (2.2.25) gives

$$\nu(\mathcal{L}_e \overset{*}{c})^k_{ij}(x)\frac{\partial p}{\partial x^k} = \nu\left(-e^l(x)\overset{*}{c}{}^m_{ij}(x)\overset{*}{c}{}^k_{lm}(x)\right)\frac{\partial p}{\partial x^k}.$$

Since this holds for all $p \in P_\nu$ and all $\nu \in \mathbb{C}$, we see that

$$(\mathcal{L}_e \overset{*}{c})^k_{ij} = -e^l\,\overset{*}{c}{}^m_{ij}\,\overset{*}{c}{}^k_{lm}. \tag{2.2.28}$$

Thus by making use of relation (2.2.28), condition **4)** gives us that

$$\begin{aligned}(\mathcal{L}_E e)^i &= -(\mathcal{L}_e E)^i = -\mathcal{L}_e\,\overset{*}{c}{}^i_{jk}E^l E^k \\ &= -(\mathcal{L}_e\overset{*}{c})^i_{jk}E^j E^k - \overset{*}{c}{}^i_{jk}(\mathcal{L}_e E)^j E^k - \overset{*}{c}{}^i_{jk}E^j(\mathcal{L}_e E)^k \\ &= e^l\,\overset{*}{c}{}^m_{jk}\,\overset{*}{c}{}^i_{lm}E^j E^k - 2(\mathcal{L}_e E)^i = e^l\,\overset{*}{c}{}^m_{jl}\,\overset{*}{c}{}^i_{km}E^j E^k + 2(\mathcal{L}_E e)^i \\ &= e^i + 2(\mathcal{L}_E e)^i,\end{aligned}$$

which implies condition **c)**. Now, conditions **4)** and **5)** together also give us that

$$\begin{aligned}(\mathcal{L}_e c)^{\alpha\beta}_\gamma &= \mathcal{L}_e g^{\alpha\delta}\,\overset{*}{c}{}^\beta_{\gamma\delta} = (\mathcal{L}_e g)^{\alpha\delta}\,\overset{*}{c}{}^\beta_{\gamma\delta} + g^{\alpha\delta}(\mathcal{L}_e\overset{*}{c})^\beta_{\gamma\delta} \\ &= e^l c^{\alpha\delta}_l\,\overset{*}{c}{}^\beta_{\gamma\delta} - g^{\alpha\delta}e^l\,\overset{*}{c}{}^k_{\gamma\delta}\,\overset{*}{c}{}^\beta_{kl} = 0,\end{aligned}$$

which is condition **b)**.

Conversely, suppose conditions **b)** and **c)** hold. By formula (2.2.22) we get $g^{\alpha\beta} = E^\gamma c^{\alpha\beta}_\gamma$. Hence

$$(\mathcal{L}_e g)^{\alpha\beta} = (\mathcal{L}_e E)^\gamma c^{\alpha\beta}_\gamma + E^\gamma(\mathcal{L}_e c)^{\alpha\beta}_\gamma = -(\mathcal{L}_E e)^\gamma c^{\alpha\beta}_\gamma = e^\gamma c^{\alpha\beta}_\gamma, \tag{2.2.29}$$

which gives condition **5)**. Now, observe that

$$0 = (\mathcal{L}_e \delta)^\alpha_\beta = (\mathcal{L}_e g)^{\alpha\gamma} g_{\gamma\beta} + g^{\alpha\gamma} (\mathcal{L}_e g)_{\gamma\beta},$$

and so we can deduce that

$$(\mathcal{L}_e g)_{\alpha\beta} = -g_{\alpha\gamma} e^\epsilon c^{\gamma\delta}_\epsilon g_{\delta\beta} = -e^\epsilon \overset{*}{c}_{\alpha\beta\epsilon}, \tag{2.2.30}$$

by use of relation (2.2.29). Then by formula (2.2.30) and condition **b)**

$$(\mathcal{L}_e \overset{*}{c})^\alpha_{\beta\gamma} = (\mathcal{L}_e g)_{\gamma\delta} c^{\delta\alpha}_\beta + g_{\gamma\delta} (\mathcal{L}_e c)^{\delta\alpha}_\beta = -e^\epsilon \overset{*}{c}_{\gamma\delta\epsilon} c^{\delta\alpha}_\beta = -e^\epsilon \overset{*}{c}{}^\delta_{\beta\gamma} \overset{*}{c}{}^\alpha_{\delta\epsilon}. \tag{2.2.31}$$

Now, suppose condition **a)** holds as well as conditions **b)** and **c)** and let $p \in P_\nu$ for some $\nu \in \mathbb{C}$. Then by making use of formula (2.2.31) we get

$$\begin{aligned} \frac{\partial^2}{\partial x^i \partial x^j} \left( e^l(x) \frac{\partial p}{\partial x^l} \right) &= -\overset{*}{c}{}^k_{ij}(x) \frac{\partial e^l(x)}{\partial x^k} \frac{\partial p}{\partial x^l} + \nu \frac{\partial e^l(x)}{\partial x^j} \overset{*}{c}{}^k_{il}(x) \frac{\partial p}{\partial x^k} + \nu \frac{\partial e^l(x)}{\partial x^i} \overset{*}{c}{}^k_{jl}(x) \frac{\partial p}{\partial x^k} \\ &\quad + \nu e^l(x) \overset{*}{c}{}^k_{ij;l}(x) \frac{\partial p}{\partial x^k} + \nu^2 e^l(x) \overset{*}{c}{}^k_{jl}(x) \overset{*}{c}{}^m_{ik}(x) \frac{\partial p}{\partial x^m} \\ &= -\overset{*}{c}{}^k_{ij}(x) \frac{\partial e^l(x)}{\partial x^k} \frac{\partial p}{\partial x^l} + \nu \left( (\mathcal{L}_e \overset{*}{c})^k_{ij}(x) + \overset{*}{c}{}^m_{ij} \frac{\partial e^k(x)}{\partial x^m} \right) \frac{\partial p}{\partial x^k} \\ &\quad + \nu^2 e^l(x) \overset{*}{c}{}^k_{jl}(x) \overset{*}{c}{}^m_{ik}(x) \frac{\partial p}{\partial x^m} \\ &= (\nu - 1) \overset{*}{c}{}^k_{ij}(x) \frac{\partial e^l(x)}{\partial x^k} \frac{\partial p}{\partial x^l} + \nu(\nu - 1) \overset{*}{c}{}^k_{ij}(x) e^l(x) \overset{*}{c}{}^m_{kl} \frac{\partial p}{\partial x^m} \\ &= (\nu - 1) \overset{*}{c}{}^k_{ij}(x) \frac{\partial}{\partial x^k} \left( e^l(x) \frac{\partial p}{\partial x^l} \right), \end{aligned}$$

for all $i, j = 1, \dots, n$, and so $e(P_\nu) \subseteq P_{\nu-1}$, thus condition **4)** is satisfied. □

**Example 2.2.34.** Let $M_* = \mathbb{C}^n \setminus \left\{ x \in \mathbb{C}^n \,\middle|\, \prod_{i=1}^n x^i = 0 \right\}$. Define $g : \mathfrak{X}(M_*) \times \mathfrak{X}(M_*) \to C^\infty(M_*)$ and $\star : \mathfrak{X}(M_*) \times \mathfrak{X}(M_*) \to \mathfrak{X}(M_*)$ to be the $C^\infty(M_*)$-bilinear extensions of the following:

$$g(\partial_{x^i}, \partial_{x^j}) := \delta_{ij}, \qquad \partial_{x^i} \star \partial_{x^j} := \frac{2\delta_{ij}}{x^i} \partial_{x^i},$$

for all $i, j = 1, \dots, n$. Then, $M_*$ with metric $g$ and multiplication $\star$ is an almost Frobenius manifold with Euler vector field $E(x) = \frac{1}{2} x^i \partial_{x^i}$, Liouville vector field $e(x) = \frac{1}{x^i} \partial_{x^i}$ and charge $d = 0$.

Almost Frobenius manifolds have various structures that we can define on them. Below we give definitions for some commonly used tensor fields on almost Frobenius manifolds.

**Definition 2.2.35.** Let $M_*$ be an almost Frobenius manifold with metric $g$. Then, we define the following tensor fields:

$$\overset{*}{c}{}^{\gamma}_{\alpha\beta} := g^{\gamma\delta}\overset{*}{c}_{\alpha\beta\delta}, \qquad c^{\alpha\beta}_{\gamma} := g^{\alpha\delta}\overset{*}{c}{}^{\beta}_{\delta\gamma}, \tag{2.2.32}$$

where $\overset{*}{c}_{\alpha\beta\gamma}$ is the rank (0, 3) tensor field defined in formula (2.2.22) and $g^{\alpha\beta}$ is the inverse of the metric $g_{\alpha\beta}$. We call $\overset{*}{c}{}^{\gamma}_{\alpha\beta}$ the *dual structure constants* of $M_*$, and $c^{\alpha\beta}_{\gamma}$ are the *cotangent structure constants* of $M_*$.

Note that, by the commutativity of the multiplication and the symmetry of the metric, the tensor fields $\overset{*}{c}_{\alpha\beta\gamma}$ and $\overset{*}{c}{}^{\gamma}_{\alpha\beta}$ are symmetric in the lower indices and $c^{\alpha\beta}_{\gamma}$ is symmetric in the upper indices.

**Proposition 2.2.36.** *Let $M_*$ be an almost Frobenius manifold with dual structure constants $\overset{*}{c}{}^{\gamma}_{\alpha\beta}$ and cotangent structure constants $c^{\alpha\beta}_{\gamma}$. Then, the following relations hold:*

$$\overset{*}{c}_{\alpha\beta\epsilon}\overset{*}{c}{}^{\epsilon}_{\gamma\delta} = \overset{*}{c}_{\gamma\beta\epsilon}\overset{*}{c}{}^{\epsilon}_{\alpha\delta}, \quad \overset{*}{c}_{\alpha\beta\epsilon}c^{\epsilon\delta}_{\gamma} = \overset{*}{c}_{\gamma\beta\epsilon}c^{\epsilon\delta}_{\alpha}, \quad \overset{*}{c}{}^{\beta}_{\alpha\epsilon}\overset{*}{c}{}^{\epsilon}_{\gamma\delta} = \overset{*}{c}{}^{\beta}_{\gamma\epsilon}\overset{*}{c}{}^{\epsilon}_{\alpha\delta}, \quad \overset{*}{c}{}^{\beta}_{\alpha\epsilon}c^{\epsilon\delta}_{\gamma} = \overset{*}{c}{}^{\beta}_{\gamma\epsilon}c^{\epsilon\delta}_{\alpha}, \tag{2.2.33}$$

*for all $\alpha, \beta, \gamma, \delta = 1, \ldots, n$, where $\overset{*}{c}_{\alpha\beta\gamma}$ is the rank (0, 3) tensor field defined in (2.2.32).*

Relations (2.2.33) can be deduced from the associativity of the multiplication on the vector fields of an almost Frobenius manifold.

**Theorem 2.2.37.** [17] *Let $M$ be a Frobenius manifold with metric $\eta$, multiplication $\circ$, unity vector field $e$, Euler vector field $E$ and charge $d$. Let $M_* = M \setminus \Sigma$, where $\Sigma$ is the discriminant locus of $M$, and let $E^{-1} \in \mathfrak{X}(M_*)$ be the inverse of $E|_{M_*}$ under the multiplication $\circ$ restricted to $M_*$. Define the rank (0, 2) tensor field $g$ on $M_*$ and the map $\star : \mathfrak{X}(M_*) \times \mathfrak{X}(M_*) \to \mathfrak{X}(M_*)$ as*

$$g_t(X, Y) = \eta_t(X, Y \circ E^{-1}), \qquad (X \star Y)_t = (X \circ Y \circ E^{-1})_t, \tag{2.2.34}$$

*for all $X, Y \in \mathfrak{X}(M_*)$ and $t \in M_*$ where $\circ$ in the relations (2.2.34) is the restriction of the multiplication to $M_*$. Then, $M_*$ with metric $g$ and multiplication $\star$ is an almost Frobenius manifold with Euler vector field $E|_{M_*}$, Liouville vector field $e|_{M_*}$ and charge $d$. We refer to this as the almost dual of the Frobenius manifold $M$.*

*Conversely, let $M_*$ be an almost Frobenius manifold with metric $g$, multiplication $\star$, Euler vector field $E$, Liouville vector field $e$ and charge $d$. Define a rank (0, 2) tensor field $\eta$ on $M_*$ and a map $\circ : \mathfrak{X}(M_*) \times \mathfrak{X}(M_*) \to \mathfrak{X}(M_*)$ as*

$$\eta_t(X, Y) = g_t(X, Y \star e^{-1}), \qquad X \circ Y = X \star Y \star e^{-1},$$

*for all $X, Y \in \mathfrak{X}(M_*)$, where $e^{-1}$ is the multiplicative inverse of $e$ under the multiplication $\star$. Then, $M = M_*$ with metric $\eta$ and multiplication $\circ$ is a Frobenius manifold with unity* *vector field $e$, Euler vector field $E$ and charge $d$. We refer to this as a Frobenius dual of the almost Frobenius manifold $M_*$.*

*Furthermore, the Frobenius dual of the almost dual of a Frobenius manifold $M$ is locally equivalent to $M$, and every almost Frobenius manifold is the almost dual of its Frobenius dual.*

A proof is given in [17]. Note that Frobenius manifolds and almost Frobenius are similarly defined besides a few conditions for each object. Thus, concepts related to Frobenius manifold have their analogues for almost Frobenius manifolds.

**Definition 2.2.38.** Let $M_*$ and $N_*$ be almost Frobenius manifolds with flat metrics $g$ and $\widetilde{g}$ and Liouville vector fields $e_M$ and $e_N$, respectively. Let $\phi : M_* \to N_*$ be a smooth map such that the differential $\phi_* : T_t M_* \to T_{\phi(t)} N_*$ is an algebra isomorphism for all $t \in M_*$ with $\phi_*(e_M) = e_N$, and

$$\widetilde{g}_{\phi(t)}(\phi_*(X),\, \phi_*(Y)) = c^2 g_t(X,\, Y),$$

for all $X, Y \in \mathfrak{X}(M_*)$ and all $t \in M_*$, where $c \in C$ is some nonzero constant. If $\phi$ is a diffeomorphism, we say that $M_*$ and $N_*$ are *equivalent* and if $\phi$ is a local diffeomorphism, we say that $M_*$ *is locally equivalent to* $N_*$.

It would be useful if the concepts of equivalence and local equivalence were compatible between Frobenius manifolds and almost Frobenius manifolds. The following proposition makes this relationship precise.

**Proposition 2.2.39.** *Let $M$ and $N$ be Frobenius manifolds. If $M$ is (locally) equivalent to $N$, then the almost dual of $M$ is (locally) equivalent to the almost dual of $N$.*

*Let $M_*$ and $N_*$ be almost Frobenius manifolds. Then, $M_*$ is (locally) equivalent to $N_*$ if and only if the Frobenius dual of $M_*$ is (locally) equivalent to the Frobenius dual of $N_*$.*

*Proof.* Let $M$ have metric $\eta$, Euler vector field $E_M$, structure constants $c^{\gamma}_{\alpha\beta}$, intersection form $g^{\alpha\beta}$ and discriminant locus $\Sigma_M$. Let $N$ have metric $\zeta$, Euler vector field $E_N$, structure constants $\widetilde{c}^{\gamma}_{\alpha\beta}$, intersection form $\widetilde{g}^{\alpha\beta}$ and discriminant locus $\Sigma_N$.

Let $\phi : M \to N$ be a (local) equivalence map and consider a point $p \in M$. Let $U \subseteq M$ be an open chart with coordinates $t^1, \dots, t^n$ such that $p \in U$, $\phi(U)$ is open and $\phi|_U : U \to \phi(U)$ is a diffeomorphism. We know that this is possible because $\phi$ is a local diffeomorphism. The functions $\widetilde{t^i} := t^i \circ \phi$ define a set of coordinates for the open chart $\phi(U)$ of $N$. Thus,

$$\widetilde{g}^{\alpha\beta}(\widetilde{t})_{\phi(p)} = E^{\gamma}_N(\widetilde{t})_{\phi(p)} \zeta^{\alpha\delta}(\widetilde{t})_{\phi(p)} \widetilde{c}^{\beta}_{\gamma\delta}(\widetilde{t})_{\phi(p)} = c^2 E^{\gamma}_M(t)_p \eta^{\alpha\delta}(t)_p c^{\beta}_{\gamma\delta}(t)_p = c^2 g^{\alpha\beta}(t)_p, \quad (2.2.35)$$

where $c \in \mathbb{C}$ is a nonzero constant. Therefore, $\widetilde{g}^{\alpha\beta}_{\phi(p)}$ is non-degenerate if and only if $g^{\alpha\beta}_p$ is non-degenerate. Thus $\phi(M \setminus \Sigma_M) \subseteq N \setminus \Sigma_N$ and $\phi(\Sigma_M) \subseteq \Sigma_N$, so we can define the map

$\psi = \phi|_{M\setminus\Sigma_M} : M \setminus \Sigma_M \to N \setminus \Sigma_N$. If $\phi$ is a (local) diffeomorphism, then $\psi$ is a (local) diffeomorphism, thus the differential maps are the same at each point $p \in M \setminus \Sigma_M$, and are thus invertible linear maps over $\mathbb{C}$, and $\psi_*(e_M) = e_N$, where $e_M$ is the identity vector field for the multiplication on $M$ and $e_N$ is the identity vector field for the multiplication on $N$. Now, let $\circ$ be the multiplication on $M$ and let $\triangle$ be the multiplication on its almost dual, also let $\bullet$ be the multiplication on $N$ and let $\star$ be the multiplication on its almost dual. Then,

$$\begin{aligned}\psi_*(X \triangle Y) &= \psi_*(X \circ Y \circ E_M^{-1}) = \phi_*(X \circ Y \circ E_M^{-1}) = \phi_*(X) \bullet \phi_*(Y) \bullet \phi_*(E_M^{-1}) \\ &= \phi_*(X) \bullet \phi_*(Y) \bullet E_N^{-1} = \phi_*(X) \star \phi_*(Y) = \psi_*(X) \star \psi_*(Y),\end{aligned}$$

for all $X, Y \in \mathfrak{X}(M\setminus\Sigma_M)$. Therefore $\psi_*$ is an algebra isomorphism at all points $p \in M\setminus\Sigma_M$, and thus because of relation (2.2.35), $\psi$ is a (local) equivalence map from $M\setminus\Sigma_M$ to $N\setminus\Sigma_N$.

Now, to show that $M_*$ is (locally) equivalent to $N_*$ if and only if the Frobenius dual of $M_*$ is (locally) equivalent to the Frobenius dual of $N_*$, we simply take one of the (local) equivalence maps and show that it is a (local) equivalence map in the other sense following the above procedure. □

The concept of taking products and the reducibility of a Frobenius manifold also have their analogous definitions for almost Frobenius manifolds.

**Definition 2.2.40.** Let $M_*$ and $N_*$ be almost Frobenius manifolds with metrics $g$, $\widetilde{g}$, multiplications $\star$, $\bullet$, Euler vector fields $E$, $\widetilde{E}$ and Liouville vector fields $e$, $\widetilde{e}$, respectively. If $M_*$ and $N_*$ have the same charge $d$, then we define the *product of $M_*$ and $N_*$* to be the almost Frobenius manifold $M_* \times N_*$ with metric $g \oplus \widetilde{g}$, multiplication $\star \oplus \bullet$, Euler vector field $E \oplus \widetilde{E}$, Liouville vector field $e \oplus \widetilde{e}$ and charge $d$. Here the multiplications are interpreted as rank (1, 2) tensor fields, so that their direct sum can be defined on $M_* \times N_*$. Moreover, if $M_*$ is an almost Frobenius manifold that is locally equivalent to a product of almost Frobenius manifolds, then we say that $M_*$ is *reducible*, and we say that $M_*$ is *irreducible* if it is not reducible.

These concepts are not just similar to those for Frobenius manifolds, but are linked in a way that is made precise in the following proposition.

**Proposition 2.2.41.** *Let $M$ and $N$ be Frobenius manifolds, then the product of the almost dual of $M$ and the almost dual of $N$ is the almost dual of $M \times N$. Also, if $M$ is reducible then its almost dual is reducible.*

*Let $M_*$ and $N_*$ be almost Frobenius manifolds. Then, the Frobenius dual of $M_* \times N_*$ is the product of the Frobenius dual of $M_*$ and the Frobenius dual of $N_*$. Moreover, $M_*$ is reducible (respectively, irreducible) if and only if its Frobenius dual is reducible (respectively, irreducible).*

*Proof.* To show that the almost dual of $M \times N$ is the product of the almost dual of $M$ and the almost dual of $N$, we must show that the discriminant locus $\Sigma_{M\times N}$ of $M \times N$ is the union of $\Sigma_M \times N$ and $M \times \Sigma_N$, where $\Sigma_M$ is the discriminant locus of $M$ and $\Sigma_N$ is the discriminant locus of $N$. Let $g^{\alpha\beta}$ be the intersection form of $M$ and let $\widetilde{g}$ be the intersection form of $N$. Then,

$$\begin{aligned}\Sigma_{M\times N} &= \{(m,\, n) \in M \times N \mid (g \oplus \widetilde{g})^{\alpha\beta}_{(m,\, n)} \text{ is degenerate}\}\\ &= \{(m,\, n) \in M \times N \mid \psi(g^{\alpha\beta}_m,\, \widetilde{g}^{\alpha\beta}_n) \text{ is degenerate}\}\\ &= \{(m,\, n) \in M \times N \mid g^{\alpha\beta}_m \text{ is degenerate}\} \cup \{(m,\, n) \in M \times N \mid \widetilde{g}^{\alpha\beta}_n \text{ is degenerate}\}\\ &= (\Sigma_M \times N) \cup (M \times \Sigma_N),\end{aligned}$$

where $\psi : \mathcal{T}^2_0(T_m M) \times \mathcal{T}^2_0(T_n N) \to \mathcal{T}^2_0(T_{(m,\, n)}(M \times N))$ is the natural isomorphism. The statement then follows by the definitions of the almost dual of a Frobenius manifold and the direct sum of tensor fields.

Suppose $M$ is reducible, then there exists Frobenius manifolds $M'$ and $M''$ such that $M$ is locally equivalent to $M' \times M''$. The almost dual of $M' \times M''$ is $M'_* \times M''_*$, where $M'_*$ is the almost dual of $M'$ and $M''_*$ is the almost dual of $M''$. By Proposition 2.2.39, the almost dual of $M$ is locally equivalent to $M'_* \times M''_*$ and thus is reducible.

Now, to show that the Frobenius dual of $M_* \times N_*$ is the product of the Frobenius dual of $M_*$ and the Frobenius dual of $N_*$ we simply follow the definition of a Frobenius dual and the definition for the direct sum of tensor fields for the metrics, multiplications and Liouville vector fields.

To show that $M_*$ is reducible if and only if its Frobenius dual is reducible, we do as above for the almost dual and note that an almost Frobenius manifold is the almost dual of its Frobenius dual, so we can construct local equivalence maps. □

Just as the Frobenius manifold has a local concept of a prepotential, the almost Frobenius manifold has the local concept of a dual prepotential, which like the prepotential is a solution of the WDVV equations with some additional conditions.

**Definition 2.2.42.** A *dual prepotential* $F_* : U \to \mathbb{C}$ is a solution of the WDVV equations, where $x^1, \ldots, x^n$ are the coordinates on $U$, together with a vector field $e \in \mathfrak{X}(U)$, which we call the Liouville vector field, such that

**1)** There exists a vector field $E \in \mathfrak{X}(U)$ of the form

$$E(x) = \left(\frac{1-d}{2}x^i + c^i\right)\partial_{x^i},$$

where $c^i, d \in \mathbb{C}$ are constants, such that

$$g_{\alpha\beta} = E^\gamma(x) \frac{\partial^3 F_*}{\partial x^\alpha \partial x^\beta \partial x^\gamma},$$

for all $\alpha, \beta = 1, \ldots, n$, defines a constant, nondegenerate matrix. We refer to $E$ as the Euler vector field and $d$ as the charge of $F_*$.

**2)** The matrix

$$(e\,\star)^\alpha_\beta(x) := e^\gamma(x) g^{\alpha\delta} \frac{\partial^3 F_*}{\partial x^\beta \partial x^\gamma \partial x^\delta} \tag{2.2.36}$$

is invertible at all points $x \in U$, and

$$\frac{\partial^2 e^\alpha}{\partial x^\beta \partial x^\gamma} = -g^{\delta\epsilon} \frac{\partial^3 F_*}{\partial x^\beta \partial x^\gamma \partial x^\delta} \frac{\partial e^\alpha}{\partial x^\epsilon}, \tag{2.2.37}$$

for all $\alpha, \beta, \gamma = 1, \ldots, n$.

**3)** For each pair of $\alpha, \beta = 1, \ldots, n$, there exists a constant $k_{\alpha\beta} \in \mathbb{C}$ such that

$$e^\gamma(x) \frac{\partial^3 F_*}{\partial x^\alpha \partial x^\beta \partial x^\gamma} + g_{\alpha\gamma} \frac{\partial e^\gamma}{\partial x^\beta} + g_{\beta\gamma} \frac{\partial e^\gamma}{\partial x^\alpha} = k_{\alpha\beta}. \tag{2.2.38}$$

**4)** We have

$$\mathcal{L}_E e = -e. \tag{2.2.39}$$

Notice that condition **3)** is equivalent to the condition $(\mathcal{L}_e c)^{\alpha\beta}_\gamma = 0$, where $c^{\alpha\beta}_\gamma$ is defined in the $x$ coordinates as

$$c^{\alpha\beta}_\gamma(x) := g^{\alpha i} g^{\beta j} \frac{\partial^3 F_*}{\partial x^i \partial x^j \partial x^\gamma}.$$

One can see this correspondence by dividing the expression (2.2.38) by $x^\delta$, substituting in the expression for the second order derivative of $e^\alpha(x)$ from relation (2.2.37) and raising indices twice with the metric $g$.

**Example 2.2.43.** Let $n = 1$. If $F_*$ is a one-dimensional dual prepotential, then it is of the form

$$F_*(x^1) = C(x^1 + 2c^1)^2 \ln(x^1 + 2c^1) + P(x^1),$$

where $C, c^1 \in \mathbb{C}$ are constants with $C \neq 0$ and $P$ is a quadratic function. The Euler vector field has the form

$$E(x) = \left(\frac{x^1}{2} + c^1\right) \partial_{x^1},$$

the Liouville vector field has the form

$$e(x) = \frac{A}{x^1 + 2c^1} \partial_{x^1},$$

where $A \in \mathbb{C}$ is a nonzero constant, and the charge is $d = 0$.

**Theorem 2.2.44.** [17] *Let $F_* : U \to \mathbb{C}$ be a dual prepotential with Euler vector field $E$, charge $d$ and Liouville vector field $e$. Define $g : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U)$ and $\star : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ by $C^\infty(U)$-bilinearly extending the following:*

$$g(\partial_{x^\alpha}, \partial_{x^\beta}) = E^\gamma(x) \frac{\partial^3 F_*}{\partial x_\alpha \partial x_\beta \partial x_\gamma}, \qquad \partial_{x^\alpha} \star \partial_{x^\beta} = \overset{*}{c}{}^\gamma_{\alpha\beta}(x) \partial_{x^\gamma},$$

*for all $\alpha, \beta = 1, \ldots, n$. Then, $U$ with metric $g$ and multiplication $\star$ is an almost Frobenius manifold with Euler vector field $E$, charge $d$ and Liouville vector field $e$. We refer to this as the almost Frobenius manifold of the dual prepotential $F_*$.*

*Conversely, let $M_*$ be an almost Frobenius manifold with metric $g$, multiplication $\star$, Euler vector field $E$, charge $d$ and Liouville vector field $e$. Let $U \subseteq M_*$ be an open chart with flat coordinates $x^1, \ldots, x^n$. If we define the tensor field*

$$\overset{*}{c}(X, Y, Z) := g(X \star Y, Z),$$

*for all $X, Y, Z \in \mathfrak{X}(U)$, then there exists a function $F_* : U \to \mathbb{C}$ such that $F_*$ is a dual prepotential with Euler vector field $E|_U$, charge $d$, Liouville vector field $e|_U$ and*

$$\frac{\partial^3 F_*}{\partial x^\alpha \partial x^\beta \partial x^\gamma} = \overset{*}{c}(\partial_{x^\alpha}, \partial_{x^\beta}, \partial_{x^\gamma}),$$

*for all $\alpha, \beta, \gamma = 1, \ldots, n$. We call $F_*$ a dual prepotential of the almost Frobenius manifold $M_*$.*

*Furthermore, the almost Frobenius manifold $U$ of a dual prepotential $F_*$ of the almost Frobenius manifold $M_*$ is locally equivalent to $M_*$, and a dual prepotential $\widetilde{F_*}$ of an almost Frobenius manifold $U$ of the dual prepotential $F_*$ is equal to a restiction of $F_* + Q$ to a non-empty, open subset of $U$, where $Q$ is a quadratic function.*

A proof can be derived fairly easily by comparing conditions for the almost Frobenius manifolds and dual prepotentials and following a parallel procedure to the proof of correspondence between Frobenius manifolds and prepotentials given in [13].

**Definition 2.2.45.** Let $F_*$ and $\widetilde{F_*}$ be dual prepotentials and let $M_*$ and $N_*$ be the almost Frobenius manifolds of $F_*$ and $\widetilde{F_*}$, respectively. Then, we say $F_*$ is *(locally) equivalent* to $\widetilde{F_*}$ if $M_*$ is (locally) equivalent to $N_*$. Moreover, a dual prepotential is called *reducible* (respectively, *irreducible*) if their almost Frobenius manifold is reducible (respectively, irreducible).

Let $M_*$ and $N_*$ have dual prepotentials $F_*$ and $\widetilde{F}_*$, respectively. Then the function $\widehat{F}_*(m, n) = F_*(m) + \widetilde{F}_*(n)$ is a dual prepotential of the product $M_* \times N_*$.

### 2.2.5 Flat pencil of metrics

We introduce another description of Frobenius manifolds, one which will later be used in Section 3.1 to construct an important class of polynomial Frobenius manifolds from Coxeter orbit spaces. Before we do so, we need to introduce a notion related to metrics on complex manifolds.

**Definition 2.2.46.** [15] Let $g$ be a metric on a complex manifold $M$ and let $x^1, \ldots, x^n$ be any local coordinate system on $M$. We define the *contravariant Christoffel symbols* ${}^g\Gamma^{ij}_k(x)$ in the $x$ coordinates to be

$$ {}^g\Gamma^{ij}_k(x) = -g^{il}(x) {}^g\Gamma^j_{lk}(x), $$

where ${}^g\Gamma^j_{lk}(x)$ are the Christoffel symbols for $g$ in the $x$ coordinates.

We now present the definition of a flat pencil of metrics on a complex manifold, making use of the notion of contravariant Christoffel symbols, which was given in [15].

**Definition 2.2.47.** [15] Let $M$ be a complex manifold with a flat metric $\eta^{\alpha\beta}$ and a symmetric (2, 0) tensor field $g^{\alpha\beta}$. We say that $\eta^{\alpha\beta}$ and $g^{\alpha\beta}$ form a *flat pencil of metrics* on $M$ if

**1)** For any $\lambda \in \mathbb{C}$, the tensor field

$$ g^{\alpha\beta}_\lambda := g^{\alpha\beta} - \lambda\eta^{\alpha\beta} $$

is a flat metric on an open dense subset of $M$.

**2)** For any $\lambda \in \mathbb{C}$, let $x^1, \ldots, x^n$ be a set of coordinates on an open chart $U \subseteq M$ where $g^{\alpha\beta}$ and $g^{\alpha\beta}_\lambda$ are always non-degenerate. Then, we have

$$ {}^{g_\lambda}\Gamma^{ij}_k(x) = {}^g\Gamma^{ij}_k(x) - \lambda {}^\eta\Gamma^{ij}_k(x), \tag{2.2.40} $$

where these are the contravariant Christoffel symbols for their respective metrics.

Note that $g^{\alpha\beta}_\lambda$ may be degenerate on some points in $M$, and that includes $g^{\alpha\beta}$, however these points may differ depending on the choice of $\lambda \in \mathbb{C}$. The metric and intersection form of a Frobenius manifold form a flat pencil of metrics, and a flat pencil of metrics is close to being enough to define a Frobenius manifold, but we require some additional properties that the pencil of metrics must satisfy, which we describe below.

**Definition 2.2.48.** [15] Let $\eta^{\alpha\beta}$ and $g^{\alpha\beta}$ form a flat pencil of metrics on a complex manifold $M$ with flat coordinates $t_1, \ldots, t_n$ and $x^1, \ldots, x^n$ respectively. We say that the flat pencil of metrics is *quasihomogeneous* of degree $d \in \mathbb{C}$, if there exists a smooth function $\tau \in C^\infty(M)$, which we call the Egoroff potential, such that the vector fields

$$ E(x) := g^{is}(x)\frac{\partial \tau}{\partial x^s}\partial_{x^i}, \qquad e(t) := \eta^{is}(t)\frac{\partial \tau}{\partial t^s}\partial_{t^i}, $$

satisfy the following conditions:

$$\mathcal{L}_E e = -e, \qquad (\mathcal{L}_E g)^{\alpha\beta} = (d-1)g^{\alpha\beta}, \qquad (\mathcal{L}_e g)^{\alpha\beta} = \eta^{\alpha\beta}, \qquad \mathcal{L}_e \eta = 0.$$

We call $E$ the Euler vector field and we call $e$ the unity vector field of the quasihomogeneous flat pencil of metrics.

The following example is a specific case of a flat pencil of metrics which gives a polynomial Frobenius manifold. We consider these types of examples more generally in Section 3.1.

**Example 2.2.49.** Let $k \in \mathbb{Z}_{\geq 2}$ and define

$$M := \left\{ (r_1 e^{i\theta_1},\, r_2 e^{i\theta_2}) \in \mathbb{C}^2 \,\middle|\, r_1, r_2 > 0 \text{ and } \theta_1, \theta_2 \in \left(0, \frac{\pi}{k}\right) \right\}.$$

Let $\langle \cdot, \cdot \rangle$ be the natural bilinear form on $\mathbb{C}^2$ with orthonormal basis $e_1, e_2$, and define the coordinates $x^i(p) := \langle p, e_i \rangle$ on $M$. We define the rank (2, 0) tensor field $g^{\alpha\beta}(x) := \delta^{\alpha\beta}$ on $M$ and let $t^1, t^2$ be another set of coordinates on $M$ which we express in the $x$ coordinates as

$$t^1(x) := (x^1 + ix^2)^k + (x^1 - ix^2)^k, \qquad\qquad t^2(x) := \frac{(x^1)^2 + (x^2)^2}{2k}.$$

We define the rank (2, 0) tensor field $\eta^{\alpha\beta}$ as

$$\eta^{\alpha\beta} := (\mathcal{L}_{\partial_{t^1}} g)^{\alpha\beta}.$$

This is a flat metric on $M$, where $t^1, t^2$ are flat coordinates for $\eta$. Moreover, $\eta^{\alpha\beta}$ and $g^{\alpha\beta}$ form a quasihomogeneous flat pencil of metrics on $M$ with Egoroff potential $\tau = t^2$, Euler vector field $E(t) = t^1\partial_{t^1} + \frac{2}{k}t^2\partial_{t^2}$ and unity vector field $e(t) = \partial_{t^1}$.

From the notation presented, one can take an educated guess that a Frobenius manifold gives a quasihomogeneous flat pencil of metrics, where $E$ is the Euler vector field, $e$ is the unity vector field and $d$ is the charge. The function $\tau$, known as the Egoroff potential, has some interesting properties of its own when the multiplication on the Frobenius manifold is semisimple, which we will discuss further in Subsection 2.2.8.

Let $x^1_\lambda, \ldots, x^n_\lambda$ be flat coordinates of the metric $g^{\alpha\beta}_\lambda$. By the relation $(\mathcal{L}_{\partial_{t^1}} g)^{\alpha\beta} = \eta^{\alpha\beta}$, we see that

$$g^{\alpha\beta}_\lambda(t) = g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta} = g^{\alpha\beta}(t^1 - \lambda, t^2, \ldots, t^n).$$

Thus, the flat coordinates $x^1_\lambda, \ldots, x^n_\lambda$ of $g^{\alpha\beta}_\lambda$ may be written in terms of the flat coordinates $x^1, \ldots, x^n$ of $g^{\alpha\beta}$ as follows:

$$x^i_\lambda(t^1, t^2, \ldots, t^n) = x^i(t^1 - \lambda, t^2, \ldots, t^n).$$

Now, by the transformation law for Christoffel symbols given by formula (2.1.2) and the relation (2.2.40) between the contravariant Christoffel symbols for flat pencils of metrics, we see that

$$\begin{aligned}\frac{\partial^2 x^i_\lambda}{\partial t^j \partial t^k} &= \frac{\partial^2 x^m_\lambda}{\partial t^j \partial t^k}\frac{\partial t^l}{\partial x^m_\lambda}\frac{\partial x^i_\lambda}{\partial t^l} = {}^{g_\lambda}\Gamma^l_{jk}(t)\frac{\partial x^i_\lambda}{\partial t^l} = -(g_\lambda)_{jm}(t)\,{}^{g_\lambda}\Gamma^{ml}_k(t)\frac{\partial x^i_\lambda}{\partial t^l} \\ &= -\,(g_\lambda)_{jm}(t)\,{}^{g}\Gamma^{ml}_k(t)\frac{\partial x^i_\lambda}{\partial t^l}. \end{aligned} \tag{2.2.41}$$

Now, the contravariant Christoffel symbols ${}^g\Gamma^{ml}_k(t)$ for the intersection form $g$ of a Frobenius manifold have the following form

$${}^g\Gamma^{ml}_k(t) = \frac{d-1}{2}c^{ml}_k(t) + E^l_{;\delta}(t)c^{\delta m}_k(t). \tag{2.2.42}$$

A proof of this claim can be found in Lecture 3 of [13], however the claim is restricted to the cases when the Euler vector field is diagonal in the $t$ coordinates. One can prove the general statement of (2.2.42) using the same method as in [13].

Substituting the relation (2.2.42) into the expression (2.2.41) and raising the indices on both sides by $g^{\alpha j}_\lambda$, we get the relation

$$g^{\alpha j}_\lambda(t)\frac{\partial^2 x^i_\lambda}{\partial t^j \partial t^k} = -\left(\frac{d-1}{2}c^{\alpha l}_k(t) + E^l_{;\delta}(t)c^{\delta\alpha}_k(t)\right)\frac{\partial x^i_\lambda}{\partial t^l}. \tag{2.2.43}$$

Note that since $\frac{\partial x^i_\lambda}{\partial \lambda} = -\frac{\partial x^i_\lambda}{\partial t^1}$ and $c^{\alpha\beta}_1(t) = \eta^{\alpha\beta}$, formula (2.2.43) gives us that

$$g^{\alpha j}_\lambda(t)\frac{\partial^2 x^i_\lambda}{\partial t^j \partial \lambda} = -g^{\alpha j}_\lambda(t)\frac{\partial^2 x^i_\lambda}{\partial t^j \partial t^1} = \left(\frac{d-1}{2}\eta^{\alpha l} + E^l_{;\delta}(t)\eta^{\delta\alpha}\right)\frac{\partial x^i_\lambda}{\partial t^l}. \tag{2.2.44}$$

This equation is known as the *generalised hypergeometric equation associated to the Frobenius manifold* [13].

Not all quasihomogeneous flat pencils of metrics are derivable from Frobenius manifolds. The following definition gives a sufficient condition for a quasihomogenous flat pencil of metrics to have a corresponding Frobenius manifold.

**Definition 2.2.50.** [15] Let $\eta^{\alpha\beta}$ and $g^{\alpha\beta}$ form a quasihomogeneous flat pencil of metrics of degree $d$ on a complex manifold $M$ with Euler vector field $E$. We say that the flat pencil is *regular* if the tensor field

$$R^j_i := \frac{d-1}{2}\delta^j_i + {}^\eta\nabla_i E^j$$

is non-degenerate everywhere on $M$, where ${}^\eta\nabla$ is the Levi-Civita connection for $\eta$.

We now have everything we need to state the correspondence between Frobenius manifolds and flat pencils of metrics.

**Theorem 2.2.51.** [15] *Let $M$ be a Frobenius manifold with metric $\eta$, Euler vector field $E$, unity vector field $e$, intersection form $g^{\alpha\beta}$ and charge $d$. Then, $\eta^{\alpha\beta}$ and $g^{\alpha\beta}$ form a quasihomogeneous flat pencil of metrics of degree $d$, with Euler vector field $E$, unity vector field $e$ and Egoroff potential*

$$\tau(t) = \eta_{1a}t^a,$$

*where $t = (t^1, \dots, t^n)$ are flat coordinates for $\eta$ such that $e = \partial_{t^1}$. We refer to this as the flat pencil of metrics of the Frobenius manifold.*

*Conversely, suppose $\eta^{\alpha\beta}$ and $g^{\alpha\beta}$ form a regular, quasihomogeneous flat pencil of metrics of degree $d$ on a manifold $M$ with Euler vector field $E$ and unity vector field $e$, and let $M_0$ be the open dense subset of $M$ where $g^{\alpha\beta}$ is non-degenerate. We can give $M_0$ a Frobenius structure with metric $\eta$, intersection form $g$, Euler vector field $E$, unity vector field $e$ and charge $d$. We refer to this as the Frobenius manifold of the flat pencil of metrics.*

*Futhermore, a regular, quasihomogeneous flat pencil of metrics is the flat pencil of metrics of its Frobenius manifold.*

A proof can be found in [15]. In Section 3.1 we will give polynomial Frobenius manifolds from flat pencils of metrics on orbit spaces of the finite irreducible Coxeter groups.

### 2.2.6 Bihamiltonian structures

We present yet another interpretation of Frobenius manifolds, that is bihamiltonian structures. These are pairs of local Poisson brackets on a manifold with some additional properties, and they require some added machinery to define. They will be utilised in Subsection 3.2.2 to construct algebraic Frobenius manifolds.

**Definition 2.2.52.** [15] Let $M$ be a complex manifold. We define $\mathcal{L}(M)$ be its *loop space*, in other words

$$\mathcal{L}(M) := C^\infty(S^1, M),$$

where $S^1$ stands for the circle.

Let $s$ be a coordinate on $S^1$ such that $s(S^1 \setminus \{A\}) = (0, 2\pi)$, where $A \in S^1$ is a fixed point. For any smooth function $f \in C^\infty(M)$ and any non-negative integer $j \in \mathbb{Z}_{\geq 0}$, we define the map $f^{(j)} : \mathcal{L}(M) \to \mathcal{L}(\mathbb{C})$ as

$$f^{(j)}(\phi) := \frac{d^j}{ds^j}(f \circ \phi),$$

where $\phi \in \mathcal{L}(M)$. Let $x^1, \dots, x^n$ be a set of coordinates defined on all of $M$. Then, we define $\mathcal{C}(M)$ to be the following ring of maps from $\mathcal{L}(M)$ to $\mathcal{L}(\mathbb{C})$ :

$$\mathcal{C}(M) := C^\infty(M)\left[x^{i(j)}\right]_{i=1,\dots,n}^{j>0},$$

where addition and multiplication of elements in the polynomial ring $\mathcal{C}(M)$ is defined pointwise on elements of $\mathcal{L}(M)$:

$$(P+Q)(\phi)(x) := P(\phi)(x) + Q(\phi)(x), \qquad (P\cdot Q)(\phi)(x) := P(\phi)(x)\, Q(\phi)(x),$$

for all $P$, $Q \in \mathcal{C}(M)$, $\phi \in \mathcal{L}(M)$ and $x \in S^1$. Notice that for any $f \in C^\infty(M)$ we have $f = f^{(0)}$ in $\mathcal{C}(M)$. Both notations will be useful for the Propositions that follow, and so we will switch between them where it is appropriate to do so.

**Proposition 2.2.53.** *For any smooth function $f \in C^\infty(M)$ and any $j \in \mathbb{Z}_{>0}$, the map $f^{(j)}$ is an element of $\mathcal{C}(M)$.*

*Proof.* Note that

$$f^{(1)}(\phi) = \frac{d}{ds}(f\circ\phi) = \left(\frac{\partial f}{\partial x^i}\circ\phi\right)\frac{d}{ds}(x^i\circ\phi) = \left(\frac{\partial f}{\partial x^i}\, x^{i(1)}\right)(\phi),$$

for any $\phi \in \mathcal{L}(M)$, so $f^{(1)} \in \mathcal{C}(M)$. Suppose that $f^{(k)} \in \mathcal{C}(M)$ for all $f \in C^\infty(M)$ for all non-negative integers $k \leq j$ for some $j \in \mathbb{Z}_{>0}$, then

$$\begin{aligned} f^{(j+1)}(\phi) &= \frac{d^{j+1}}{ds^{j+1}}(f\circ\phi) = \frac{d^j}{ds^j}\left(\left(\frac{\partial f}{\partial x^i}\circ\phi\right)\frac{d}{ds}(x^i\circ\phi)\right) \\ &= \sum_{k=0}^{j}\binom{j}{k}\frac{d^k}{ds^k}\left(\frac{\partial f}{\partial x^i}\circ\phi\right)\frac{d^{j+1-k}}{ds^{j+1-k}}(x^i\circ\phi) \\ &= \sum_{k=0}^{j}\binom{j}{k}\left(\left(\frac{\partial f}{\partial x^i}\right)^{(k)} x^{i(j+1-k)}\right)(\phi), \end{aligned}$$

for any $\phi \in \mathcal{L}(M)$, and so by induction, the proposition is proved. □

By proving Proposition 2.2.53, we have shown, in particular, that the derivatives of any coordinate system are elements of $\mathcal{C}(M)$ and thus the set of elements in $\mathcal{C}(M)$ does not depend on our choice of $x$ coordinates. It also allows us to define $\frac{d}{ds}$ to be the derivation (over $\mathbb{C}$) on $\mathcal{C}(M)$ given by extending the following, using the product rule for derivations:

$$\frac{d}{ds}\left(f^{(j)}\right) := f^{(j+1)},$$

for any $f \in C^\infty(M)$ and any $j \in \mathbb{Z}_{\geq 0}$. Alternatively, we can write the derivation $\frac{d}{ds} : \mathcal{C}(M) \to \mathcal{C}(M)$ in the following form:

$$\frac{d}{ds} = \sum_{k=0}^{\infty} x^{i(k+1)}\frac{\partial}{\partial x^{i(k)}},$$

where we define $\frac{\partial}{\partial x^{i(0)}} := \frac{\partial}{\partial x^i}$ to only act non-trivially on the elements of $C^\infty(M)$ in $\mathcal{C}(M)$.

This gives us the following commutation relations:

$$\left[\frac{\partial}{\partial x^{i(j+1)}}, \frac{d}{ds}\right] = \frac{\partial}{\partial x^{i(j)}}, \qquad \left[\frac{\partial}{\partial x^{i(0)}}, \frac{d}{ds}\right] = 0, \tag{2.2.45}$$

for all $j \in \mathbb{Z}_{\geq 0}$. Since $\frac{d}{ds}$ and $\frac{\partial}{\partial x^i}$ commute with each other we have the relation

$$\left(\frac{\partial f}{\partial x^i}\right)^{(k)} = \frac{\partial f^{(k)}}{\partial x^i},$$

for any $k \in \mathbb{Z}_{\geq 0}$ and any $f \in C^\infty(M)$.

**Definition 2.2.54.** Let $P \in \mathcal{C}(M)$. We say that $P$ is *homogeneous of degree $D$* if

$$\sum_{i=1}^{n} \sum_{k=1}^{\infty} k\, x^{i(k)} \cdot \frac{\partial P}{\partial x^{i(k)}} = DP. \tag{2.2.46}$$

Since elements of $\mathcal{C}(M)$ are polynomials in the maps $x^{i(k)}$, only a finite number of these maps occur in the expression of any $P \in \mathcal{C}(M)$. So, the infinite sum in the left hand side of equation (2.2.46) actually becomes a finite sum and is thus well-defined.

**Proposition 2.2.55.** *Let $f \in C^\infty(M)$ and $j \in \mathbb{Z}_{\geq 0}$, then $f^{(j)} \in \mathcal{C}(M)$ is homogeneous of degree $j$.*

*Proof.* The claim is obviously true for $j = 0$, and we see that

$$f^{(1)} = \frac{\partial f}{\partial x^i}\, x^{i(1)}, \tag{2.2.47}$$

so $f^{(1)}$ is homogeneous of degree 1 (in $x$ coordinates). Now, suppose $f^{(k)}$ is homogeneous of degree $k$ for all $f \in C^\infty(M)$ for all non-negative $k \leq j$ for some $j \in \mathbb{Z}_{\geq 0}$. Then since

$$f^{(j+1)} = \sum_{k=0}^{j} \binom{j}{k} \frac{\partial f^{(k)}}{\partial x^i}\, x^{i(j+1-k)}, \tag{2.2.48}$$

and because the degree of a product of two homogeneous terms is the sum of their degrees, we see that $f^{(j+1)}$ must be homogeneous of degree $j+1$. □

A consequence of Proposition 2.2.55 is that an element $P \in \mathcal{C}(M)$ being homogeneous of degree $D$ is independent of the choice of coordinates that we define on $M$. Moreover, for any function $f \in C^\infty(M)$ and any $j$, $l \in \mathbb{Z}_{\geq 0}$, formulas (2.2.47) and (2.2.48) tell us that

$$\frac{\partial f^{(j)}}{\partial x^{i(j+l)}} = \delta^0_l \frac{\partial f}{\partial x^i}. \tag{2.2.49}$$

The following definition of local functionals can be found in [15] and is meant to be an analogue of smooth functions, but defined on the loop space $\mathcal{L}(M)$ rather than $M$ itself.

**Definition 2.2.56.** The set of *local functionals* on $M$ is defined to be the set $\mathcal{F}(M)$ containing all maps $I : \mathcal{L}(M) \to \mathbb{C}$ of the form

$$I[\phi] = \frac{1}{2\pi} \int_0^{2\pi} P(\phi)(s)\, ds, \tag{2.2.50}$$

where $P \in \mathcal{C}(M)$ and $\phi \in \mathcal{L}(M)$.

The set of local functionals $\mathcal{F}(M)$ that we get does not depend on our choice of local coordinates, since by Proposition 2.2.53, any choice of coordinates gives the same polynomial ring $\mathcal{C}(M)$. Any local functional $I \in \mathcal{F}(M)$ can be associated to an element $P \in \mathcal{C}(M)$ such that relation (2.2.50) holds, such a $P$ can be chosen up to the addition of $Q \in \mathcal{C}(M)$ where

$$\int_0^{2\pi} Q(\phi)(s)\, ds = 0,$$

for all $\phi \in \mathcal{L}(M)$.

**Lemma 2.2.57.** *Let $P \in \mathcal{C}(M)$ such that*

$$\int_0^{2\pi} P(\phi)(s)\, ds = 0,$$

*for all $\phi \in \mathcal{L}(M)$. Then,*

$$\sum_{j=0}^{\infty} (-1)^j \frac{d^j}{ds^j} \frac{\partial P}{\partial x^{i(j)}} = 0,$$

*for all $i = 1, \ldots, n$.*

*Proof.* Let $\phi \in \mathcal{L}(M)$ and $\gamma \in \mathcal{L}(\mathbb{C}^n)$. Then, for all $\epsilon \in \mathbb{C}$ small enough, $x^{-1}(x(\phi) + \epsilon\gamma)$ exists in the loop space $\mathcal{L}(M)$. Thus, we can expand $P(x^{-1}(x(\phi) + \epsilon\gamma))$ in powers of $\epsilon$ to get

$$P(x^{-1}(x(\phi) + \epsilon\gamma)) = P(\phi) + \epsilon \sum_{j=0}^{\infty} \frac{\partial P}{\partial x^{i(j)}}(\phi) \frac{d^j}{ds^j}\gamma^i + O(|\epsilon|^2). \tag{2.2.51}$$

Integrating over the range $s \in (0, 2\pi)$ we see that the left hand side and the first term of the right hand side of relation (2.2.51) equal zero, and so we get that

$$\epsilon \int_0^{2\pi} \sum_{j=0}^{\infty} \frac{\partial P}{\partial x^{i(j)}}(\phi)(s) \frac{d^j}{ds^j}\gamma^i(s)\, ds = \int_0^{2\pi} O(|\epsilon|^2)\, ds \tag{2.2.52}$$

Dividing both sides by $\epsilon$ and then taking the limit $\epsilon \to 0$, the right hand side of relation

(2.2.52) equals zero and so we see that

$$\int_0^{2\pi} \sum_{j=0}^{\infty} \frac{\partial P}{\partial x^{i(j)}}(\phi)(s) \frac{d^j}{ds^j}\gamma^i(s)\, ds = \int_0^{2\pi} \sum_{j=0}^{\infty} (-1)^j \frac{d^j}{ds^j} \frac{\partial P}{\partial x^{i(j)}}(\phi)(s)\gamma^i(s)\, ds = 0. \quad (2.2.53)$$

Since $\sum_{j=0}^{\infty}(-1)^j \frac{d^j}{ds^j}\frac{\partial P}{\partial x^{k(j)}}(\phi) \in \mathcal{L}(\mathbb{C})$ it is a periodic function with period $2\pi$, we may write

$$\sum_{j=0}^{\infty}(-1)^j \frac{d^j}{ds^j}\frac{\partial P}{\partial x^{k(j)}}(\phi)(s) = \sum_{m\in\mathbb{Z}} C_m e^{ims},$$

where the constants $C_m \in \mathbb{C}$ are found by the formula

$$C_m = \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=0}^{\infty}(-1)^j \frac{d^j}{ds^j}\frac{\partial P}{\partial x^{k(j)}}(\phi)(s) e^{-ims}\, ds. \quad (2.2.54)$$

If we set $\gamma^i(s) = \delta^{ik}e^{-ims}$ and substitute this into the left hand side of formula (2.2.53) we get that the right hand side of formula (2.2.54) equals zero, and thus $C_m = 0$ for all $m \in \mathbb{Z}$. Therefore,

$$\sum_{j=0}^{\infty}(-1)^j \frac{d^j}{ds^j}\frac{\partial P}{\partial x^{k(j)}}(\phi) = 0.$$

□

Lemma 2.2.57 allows us to make the following definition without having to choose which $P \in \mathcal{C}(M)$ we want to represent our local functional $I \in \mathcal{F}(M)$.

**Definition 2.2.58.** [15] Let $I$ be a local functional of the form (2.2.50). Then the *variational derivative* of $I$ with respect to $x^i$ is defined to be

$$\frac{\delta I}{\delta x^i(s)} = \sum_{j=0}^{\infty}(-1)^j \frac{d^j}{ds^j}\frac{\partial P}{\partial x^{i(j)}}. \quad (2.2.55)$$

Since $P$ is polynomial in $x^{i(j)}$, the infinite sum in equation (2.2.55) becomes a finite sum and so the right hand side is a well-defined element of $\mathcal{C}(M)$. By Lemma 2.2.57, our choice of element $P \in \mathcal{C}(M)$ makes no difference to what the variational derivative of $I$ is, and so it is a well-defined concept. Moreover, by formula (2.2.51) we see that the variational derivative of a local functional appears as a "first order correction" as one changes the loop being evaluated by the local functional.

**Proposition 2.2.59.** *The variational derivative of a local functional $I \in \mathcal{F}(M)$ trans-*

*forms under a change of coordinates on $M$ by the following transformation law:*

$$\frac{\delta I}{\delta y^i(s)} = \frac{\partial x^k}{\partial y^i}\frac{\delta I}{\delta x^k(s)}.$$

*Proof.* Note that if $P \in \mathcal{C}(M)$, then

$$\frac{\partial P}{\partial y^{i(j)}} = \sum_{l=0}^{\infty} \frac{\partial x^{k(l)}}{\partial y^{i(j)}}\frac{\partial P}{\partial x^{k(l)}}.$$

Now, we will prove the following identity:

$$\sum_{m=0}^{\infty}(-1)^m \binom{j+m}{m}\frac{d^m}{ds^m}\left(\frac{\partial x^{k(l)}}{\partial y^{i(j+m)}}\right) = \delta^l_j \frac{\partial x^k}{\partial y^i}, \tag{2.2.56}$$

for any $i$, $k = 1, \ldots, n$, and any $j$, $l \in \mathbb{Z}_{\geq 0}$. Indeed, if $j \geq l$, then this is true by use of relation (2.2.49). Now, suppose $j = 0$ and $l > 0$, then using the commutation relations from formula (2.2.45), we see that

$$\begin{aligned}
\sum_{m=0}^{\infty}(-1)^m \binom{j+m}{m}\frac{d^m}{ds^m}\left(\frac{\partial x^{k(l)}}{\partial y^{i(j+m)}}\right) &= \sum_{m=0}^{\infty}(-1)^m \frac{d^m}{ds^m}\left(\frac{\partial}{\partial y^{i(m)}}\frac{d}{ds}x^{k(l-1)}\right)\\
= \sum_{m=0}^{\infty}(-1)^m \frac{d^{m+1}}{ds^{m+1}}\left(\frac{\partial x^{k(l-1)}}{\partial y^{i(m)}}\right) &+ \sum_{m=1}^{\infty}(-1)^m \frac{d^m}{ds^m}\left(\frac{\partial x^{k(l-1)}}{\partial y^{i(m-1)}}\right)\\
= \sum_{m=0}^{\infty}(-1)^m \frac{d^{m+1}}{ds^{m+1}}\left(\frac{\partial x^{k(l-1)}}{\partial y^{i(m)}}\right) &+ \sum_{m=0}^{\infty}(-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}\left(\frac{\partial x^{k(l-1)}}{\partial y^{i(m)}}\right) = 0.
\end{aligned}$$

Now, suppose for some $j \in \mathbb{Z}_{\geq 0}$ we have

$$\sum_{m=0}^{\infty}(-1)^m \binom{j+m}{m}\frac{d^m}{ds^m}\left(\frac{\partial x^{k(l)}}{\partial y^{i(j+m)}}\right) = 0,$$

for all $l > j$. Then, for $j+1$ and $l > j+1$ we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}(-1)^m \binom{j+1+m}{m}\frac{d^m}{ds^m}\left(\frac{\partial x^{k(l)}}{\partial y^{i(j+1+m)}}\right)\\
&= \sum_{m=0}^{\infty}(-1)^m \binom{j+1+m}{m}\frac{d^m}{ds^m}\left(\frac{\partial}{\partial y^{i(j+1+m)}}\frac{d}{ds}x^{k(l-1)}\right)\\
&= \sum_{m=0}^{\infty}(-1)^m \binom{j+1+m}{m}\frac{d^{m+1}}{ds^{m+1}}\left(\frac{\partial x^{k(l-1)}}{\partial y^{i(j+1+m)}}\right)\\
&\quad + \sum_{m=1}^{\infty}(-1)^m \binom{j+1+m}{m}\frac{d^m}{ds^m}\left(\frac{\partial x^{k(l-1)}}{\partial y^{i(j+m)}}\right) + \frac{\partial x^{k(l-1)}}{\partial y^{i(j)}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} (-1)^{m-1} \binom{j+m}{m-1} \frac{d^m}{ds^m} \left( \frac{\partial x^{k(l-1)}}{\partial y^{i(j+m)}} \right) \\
&\quad + \sum_{m=1}^{\infty} (-1)^{m} \binom{j+1+m}{m} \frac{d^m}{ds^m} \left( \frac{\partial x^{k(l-1)}}{\partial y^{i(j+m)}} \right) + \frac{\partial x^{k(l-1)}}{\partial y^{i(j)}} \\
&= \sum_{m=1}^{\infty} (-1)^{m} \binom{j+m}{m} \frac{d^m}{ds^m} \left( \frac{\partial x^{k(l-1)}}{\partial y^{i(j+m)}} \right) + \frac{\partial x^{k(l-1)}}{\partial y^{i(j)}} \\
&= \sum_{m=1}^{\infty} (-1)^{m} \binom{j+m}{m} \frac{d^m}{ds^m} \left( \frac{\partial x^{k(l-1)}}{\partial y^{i(j+m)}} \right) = 0,
\end{aligned}
$$

since $l-1 > j$. So, we have proved relation (2.2.56). Now, we see that

$$
\begin{aligned}
\frac{\delta I}{\delta y^i(s)} &= \sum_{j=0}^{\infty} (-1)^j \frac{d^j}{ds^j} \frac{\partial P}{\partial y^{i(j)}} = \sum_{j=0}^{\infty} (-1)^j \frac{d^j}{ds^j} \sum_{l=0}^{\infty} \frac{\partial x^{k(l)}}{\partial y^{i(j)}} \frac{\partial P}{\partial x^{k(l)}} \\
&= \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \sum_{m=0}^{j} \binom{j}{m} \frac{d^m}{ds^m} \left( \frac{\partial x^{k(l)}}{\partial y^{i(j)}} \right) \frac{d^{j-m}}{ds^{j-m}} \left( \frac{\partial P}{\partial x^{k(l)}} \right) \\
&= \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{j+m} \binom{j+m}{m} \frac{d^m}{ds^m} \left( \frac{\partial x^{k(l)}}{\partial y^{i(j+m)}} \right) \frac{d^j}{ds^j} \left( \frac{\partial P}{\partial x^{k(l)}} \right) \\
&= \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \delta^l_j \frac{\partial x^k}{\partial y^i} \frac{d^j}{ds^j} \frac{\partial P}{\partial x^{k(l)}} = \frac{\partial x^k}{\partial y^i} \sum_{j=0}^{\infty} (-1)^j \frac{d^j}{ds^j} \frac{\partial P}{\partial x^{k(l)}} = \frac{\partial x^k}{\partial y^i} \frac{\delta I}{\delta x^k(s)}.
\end{aligned}
$$

□

Proposition 2.2.59 is implicitly assumed in [15] when discussing coordinate transformations of local Poisson brackets, which we will soon define. The definition of a local Poisson bracket takes inspiration from the general form of a Poisson bracket on a manifold, where

$$
\{f,\, g\} = \frac{\partial f}{\partial x^i} \{x^i,\, x^j\} \frac{\partial g}{\partial x^j},
$$

for any $f,\, g \in C^\infty(M)$. Such a form tells us that the product rule for the Poisson bracket is automatically satisfied and we just need to check that our choices for the functions $\{x^i,\, x^j\}$ are such that they make the Poisson bracket a Lie bracket on $C^\infty(M)$.

**Definition 2.2.60.** [15] A *local Poisson bracket* on $M$ is a Lie bracket $\{\,\cdot, \cdot\,\}$ on the space of local functionals $\mathcal{F}(M)$ of the form

$$
\{I_1,\, I_2\}[\phi] = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\delta I_1}{\delta x^i(s)} A^{ij} \left( \frac{\delta I_2}{\delta x^j(s)} \right) \right) (\phi)(s)\, ds,
$$

where $A^{ij} = \sum\limits_{k=0}^{N} a^{ij}_k \frac{d^k}{ds^k}$ with $a^{ij}_k \in \mathcal{C}(M)$.

The local Poisson bracket is thus defined by the choice of coordinates and the choice of elements $a_k^{ij} \in \mathcal{C}(M)$. Although we use the term local *Poisson bracket*, this is strictly a conventional term as there is no multiplication defined on the space of local functionals and thus a local Poisson bracket is not truly a Poisson bracket, but merely a Lie bracket. There is an alternative notation for local Poisson brackets that are widely used, where instead of using the operator $A^{ij}$ we write

$$\{I_1, I_2\}[\phi] = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\delta I_1}{\delta x^i(s_1)}(\phi)(s_1)\{x^i(s_1), x^j(s_2)\}(\phi) \frac{\delta I_2}{\delta x^j(s_2)}(\phi)(s_2)\, ds_1\, ds_2,$$

where

$$\{x^i(s_1), x^j(s_2)\}(\phi) = \sum_{k=0}^{N} a_k^{ij}(\phi)(s_1)\delta^{(k)}(s_1 - s_2),$$

where $a_k^{ij} \in \mathcal{C}(M)$. The equivalence of these expressions is clear with the understanding that

$$\frac{1}{2\pi} \int_0^{2\pi} f(s_2)\delta^{(k)}(s_1 - s_2)\, ds_2 = f^{(k)}(s_1).$$

The expression $\{x^i(s_1), x^j(s_2)\}$ is often simply referred to as the local Poisson bracket, as it is the essential building block, and when we write it explicitly it is common practice to drop the $\phi$ and simply write

$$\{x^i(s_1), x^j(s_2)\} = \sum_{k=0}^{N} a_k^{ij}(s_1)\delta^{(k)}(s_1 - s_2),$$

for some $a_k^{ij} \in \mathcal{C}(M)$.

**Proposition 2.2.61.** *Let $\{\,\cdot,\cdot\,\}$ be a local Poisson bracket on $M$, and impose that for any two coordinate systems $x^1, \dots, x^n$ and $y^1, \dots, y^n$ defined on all of $M$, we have that*

$$\{x^i(s_1), x^j(s_2)\} = \sum_{k=0}^{N} a_k^{ij}(s_1)\delta^{(k)}(s_1 - s_2), \tag{2.2.57}$$

$$\{y^i(s_1), y^j(s_2)\} = \sum_{k=0}^{N} b_k^{ij}(s_1)\delta^{(k)}(s_1 - s_2), \tag{2.2.58}$$

*such that*

$$b_k^{ij} = \frac{\partial y^i}{\partial x^\alpha} \sum_{l=0}^{N-k} a_{k+l}^{\alpha\beta} \binom{k+l}{l} \frac{\partial y^{j(l)}}{\partial x^\beta}. \tag{2.2.59}$$

*Then, local Poisson bracket $\{\,\cdot,\cdot\,\}$ is independent of the choice of coordinates.*

*Proof.* Using the transformation law for variational derivatives, we see that

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\delta I_1}{\delta y^i(s_1)}(\phi)(s_1)\{y^i(s_1),\, y^j(s_2)\}(\phi)\frac{\delta I_2}{\delta y^j(s_2)}(\phi)(s_2)\,ds_1\,ds_2\\
&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\delta I_1}{\delta y^i(s_1)}(\phi)(s_1)\sum_{k=0}^{N} b_k^{ij}(\phi)(s_1)\delta^{(k)}(s_1-s_2)\frac{\delta I_2}{\delta y^j(s_2)}(\phi)(s_2)\,ds_1\,ds_2\\
&=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\delta I_1}{\delta y^i(s)}\sum_{k=0}^{N}\frac{\partial y^i}{\partial x^\alpha}\sum_{l=0}^{N-k}a_{k+l}^{\alpha\beta}\binom{k+l}{l}\frac{\partial y^{j(l)}}{\partial x^\beta}\frac{d^k}{ds^k}\left(\frac{\delta I_2}{\delta y^j(s)}\right)\right)(\phi)(s)\,ds\\
&=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial y^i}{\partial x^\alpha}\frac{\delta I_1}{\delta y^i(s)}\sum_{k=0}^{N}\sum_{l=0}^{k}a_k^{\alpha\beta}\binom{k}{l}\frac{\partial y^{j(l)}}{\partial x^\beta}\frac{d^{k-l}}{ds^{k-l}}\left(\frac{\delta I_2}{\delta y^j(s)}\right)\right)(\phi)(s)\,ds\\
&=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\delta I_1}{\delta x^\alpha(s)}\sum_{k=0}^{N}a_k^{\alpha\beta}\frac{d^k}{ds^k}\left(\frac{\partial y^j}{\partial x^\beta}\frac{\delta I_2}{\delta y^j(s)}\right)\right)(\phi)(s)\,ds\\
&=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\delta I_1}{\delta x^\alpha(s)}\sum_{k=0}^{N}a_k^{\alpha\beta}\frac{d^k}{ds^k}\left(\frac{\delta I_2}{\delta x^\beta(s)}\right)\right)(\phi)(s)\,ds\\
&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\delta I_1}{\delta x^\alpha(s_1)}(\phi)(s_1)\sum_{k=0}^{N}a_k^{\alpha\beta}(\phi)(s_1)\delta^{(k)}(s_1-s_2)\frac{\delta I_2}{\delta x^\beta(s_2)}(\phi)(s_2)\,ds_1\,ds_2\\
&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\delta I_1}{\delta x^\alpha(s_1)}(\phi)(s_1)\{x^\alpha(s_1),\, x^\beta(s_2)\}(\phi)\frac{\delta I_2}{\delta x^\beta(s_2)}(\phi)(s_2)\,ds_1\,ds_2.
\end{aligned}$$

□

When we change the coordinate system on $M$ to describe a local Poisson bracket, this changes the coefficients $a_k^{ij}$ by the relation (2.2.59). In [15], Dubrovin describes this transformation law using the following notation:

$$\{y^i(s_1),\, y^j(s_2)\} = \frac{\partial y^i}{\partial x^\alpha}(s_1)\frac{\partial y^j}{\partial x^\beta}(s_2)\{x^\alpha(s_1),\, x^\beta(s_2)\}, \tag{2.2.60}$$

where we impose the following definition:

$$\frac{\partial y^j}{\partial x^\beta}(s_2)\delta^{(k)}(s_1-s_2) := \sum_{l=0}^{k}\binom{k}{l}\frac{\partial y^{j(l)}}{\partial x^\beta}(s_1)\delta^{(k-l)}(s_1-s_2).$$

Proposition 2.2.61 then results from this.

**Definition 2.2.62.** Let $\{\cdot,\cdot\}$ be a local Poisson bracket on a complex manifold $M$ and let $x^1,\ldots,x^n$ be coordinates defined on all of $M$ such that

$$\{x^i(s_1),\, x^j(s_2)\} = \sum_{k=0}^{N} a_k^{ij}(s_1)\delta^{(k)}(s_1-s_2).$$

We say this bracket is *graded homogeneous of degree* $D$ if the coefficients $a_k^{ij} \in \mathcal{C}(M)$ are homogeneous of degree $D-k$, for each $k = 0, \ldots, N$.

Graded homogeneous local Poisson brackets are going to be central to constructing Frobenius manifolds, and in particular to contructing algebraic Frobenius manifolds, which we will do in Subsection 3.2.2.

**Lemma 2.2.63.** *Let* $\{x^i(s_1),\, x^j(s_2)\}$ *be a graded homogeneous local Poisson bracket of degree* $D$*, then this local Poisson bracket is graded homogeneous of degree* $D$ *for any choice of coordinates on* $M$*.*

*Proof.* We have that

$$\{x^i(s_1),\, x^j(s_2)\} = \sum_{k=0}^{N} a_k^{ij}(s_1)\delta^{(k)}(s_1 - s_2),$$

for some homogeneous $a_k^{ij} \in \mathcal{C}(M)$ of degrees $D-k$. Also, if $y^1, \ldots, y^n$ is another set of coordinates defined on all of $M$, then

$$\{y^i(s_1),\, y^j(s_2)\} = \sum_{k=0}^{N} b_k^{ij}(s_1)\delta^{(k)}(s_1 - s_2),$$

for some $b_k^{ij} \in \mathcal{C}(M)$. These coefficients transform as

$$b_k^{ij} = \frac{\partial y^i}{\partial x^\alpha} \sum_{l=0}^{N-k} a_{k+l}^{\alpha\beta} \binom{k+l}{l} \frac{\partial y^{j(l)}}{\partial x^\beta}.$$

By Proposition (2.2.55), $a_{k+l}^{\alpha\beta}$ is homogeneous of degree $D-(k+l)$ and $f^{(l)}$ is homogeneous of degree $l$ for any $f \in C^\infty(M)$ in any coordinate system, so $b_k^{ij}$ is homogeneous of degree $D-k$ in any coordinate system, and thus $\{y^i(s_1),\, y^j(s_2)\}$ is graded homogeneous of degree $D$. □

Let $\{x^i(s_1),\, x^j(s_2)\}$ be a local Poisson bracket. Then, we can write

$$\{x^i(s_1),\, x^j(s_2)\} = \sum_{k=-1}^{N} \{x^i(s_1),\, x^j(s_2)\}^{[k]}, \tag{2.2.61}$$

where $\{x^i(s_1),\, x^j(s_2)\}^{[k]}$ is the graded homogeneous component of the local Poisson bracket of degree $k+1$. When we write a local Poisson bracket as a sum of the form (2.2.61), we will call this the *graded decomposition* of the local Poisson bracket.

**Lemma 2.2.64.** [22] *Let* $\{x^i(s_1),\, x^j(s_2)\}$ *be a nonzero local Poisson bracket on a complex manifold* $M$*, and let* $l \in \mathbb{Z}$ *be the smallest integer such that* $\{x^i(s_1),\, x^j(s_2)\}^{[l]} \neq 0$*. Then* $\{x^i(s_1),\, x^j(s_2)\}^{[l]}$ *is a local Poisson bracket on* $M$*.*

The above Lemma allows us to construct new local Poisson brackets by taking the lowest degree graded homogeneous component of an already existing local Poisson bracket.

**Definition 2.2.65.** We say that a local Poisson bracket is *dispersionless* on $M$ if its graded decomposition gives

$$\{x^i(s_1),\, x^j(s_2)\}^{[-1]} = 0, \qquad \{x^i(s_1),\, x^j(s_2)\}^{[0]} \neq 0.$$

Moreover, if a local Poisson bracket is dispersionless, then we define its *dispersionless limit* to be $\{x^i(s_1),\, x^j(s_2)\}^{[0]}$.

By Lemma 2.2.64, the dispersionless limit of a dispersionless local Poisson bracket is itself a local Poisson bracket.

**Example 2.2.66.** Let $\{x^i(s_1),\, x^j(s_2)\}$ be a graded homogeneous local Poisson bracket of degree 0, so it has the form

$$\{x^i(s_1),\, x^j(s_2)\} = h^{ij}(s_1)\delta(s_1 - s_2).$$

The elements $h^{ij} \in \mathcal{C}(M)$ are homogeneous of degree 0, and are thus elements of the set $C^\infty(M)$. These functions then define a Poisson bracket on the complex manifold $M$ where

$$\{x^i,\, x^j\} = h^{ij}.$$

That this is a Poisson bracket results from the fact that we have a local Poisson bracket defined using the same functions.

**Definition 2.2.67.** Let $\{x^i(s_1),\, x^j(s_2)\}$ be a graded homogeneous local Poisson bracket of degree 1, we call such a local Poisson bracket, a *local Poisson bracket of hydrodynamic type.*

A local Poisson bracket of hydrodynamic type has the form

$$\{x^i(s_1),\, x^j(s_2)\} = \eta^{ij}(s_1)\delta^{(1)}(s_1 - s_2) + {}^{\eta}\Gamma^{ij}_{k}(s_1)x^{k(1)}(s_1)\delta(s_1 - s_2),$$

where $\eta^{ij}$, ${}^{\eta}\Gamma^{ij}_{k} \in C^\infty(M)$. In the literature, local Poisson brackets of hydrodynamic type are often referred to as Poisson brackets of hydrodynamic type. We have chosen to include the word "local" when referring to these objects to clarify that this is still a local Poisson bracket, and thus not a Poisson bracket in the usual sense. If $\eta^{ij}$ is non-degenerate as a matrix on an open, dense subset of $M$, then we say that the local Poisson bracket is *non-degenerate.*

**Theorem 2.2.68.** [19] *Let $\{x^i(s_1),\, x^j(s_2)\}$ be a non-degenerate local Poisson bracket of hydrodynamic type on a complex manifold $M$. Suppose this local Poisson bracket has the form*

$$\{x^i(s_1),\, x^j(s_2)\} = \eta^{ij}(s_1)\delta^{(1)}(s_1 - s_2) + {}^{\eta}\Gamma^{ij}_k(s_1)x^{k(1)}(s_1)\delta(s_1 - s_2).$$

*Let $\widetilde{M}$ be the set of points in $M$ where $\eta^{ij}$ is non-degenerate, then $\eta^{ij}$ is a flat metric written in $x$ coordinates on $\widetilde{M}$ and has contravariant Christoffel symbols ${}^{\eta}\Gamma^{ij}_k$ in the $x$ coordinates.*

*Conversely, suppose $\eta$ is a flat metric on a complex manifold $M$ with contravariant Christoffel symbols ${}^{\eta}\Gamma^{ij}_k(x)$ in some coordinate system $x$ defined on all of $M$. Then, the local Poisson bracket defined on $M$ by*

$$\{x^i(s_1),\, x^j(s_2)\} = \eta^{ij}(x)(s_1)\delta^{(1)}(s_1 - s_2) + {}^{\eta}\Gamma^{ij}_k(x)(s_1)\widehat{x}^{k(1)}\delta(s_1 - s_2)$$

*is a non-degenerate local Poisson bracket of hydrodynamic type.*

Since there is a correspondence between non-degenerate local Poisson brackets of hydrodynamic type and flat metrics on $M$ we can simply refer to the corresponding flat metric as the flat metric of the local Poisson bracket, and flat coordinates of the flat metric as flat coordinates of the local Poisson bracket.

**Proposition 2.2.69.** [15] *Let $\{x^i(s_1),\, x^j(s_2)\}$ be a non-degenerate local Poisson bracket of hydrodynamic type, and let $t^1, \ldots, t^n$ be flat coordinates of the Poisson bracket. Then the local functionals $C^1, \ldots, C^n \in \mathcal{F}(M)$ defined as*

$$C^\alpha[\phi] := \frac{1}{2\pi}\int_0^{2\pi} t^\alpha(\phi(s))\, ds$$

*are Casimirs of the local Poisson bracket (meaning that $\{I,\, C^\alpha\} = 0$ for all $I \in \mathcal{F}(M)$).*

*Proof.* In the flat coordinates $t^1, \ldots, t^n$, the local Poisson bracket has the form

$$\{t^i(s_1),\, t^j(s_2)\} = \eta^{ij}(s_1)\delta^{(1)}(s_1 - s_2),$$

where $\eta^{ij}$ is constant for all $i, j = 1, \ldots, n$. The variational derivative of $C^\alpha$ with respect to $t^j$ is then

$$\frac{\delta C^\alpha}{\delta t^j(s)} = \sum_{k=0}^{\infty}(-1)^k \frac{d^k}{ds^k}\frac{\partial t^\alpha}{\partial t^{j(k)}} = \delta^\alpha_j.$$

Let $I \in \mathcal{F}(M)$ be arbitrary, then

$$\begin{aligned}\{I,\, C^\alpha\}[\phi] &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\delta I}{\delta t^i(s)}\eta^{ij}\frac{d}{ds}\left(\frac{\delta C^\alpha}{\delta t^j(s)}\right)\right)(\phi)(s)\, ds \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\delta I}{\delta t^i(s)}\eta^{ij}\frac{d}{ds}\left(\delta^\alpha_j\right)\right)(\phi)(s)\, ds = 0,\end{aligned}$$

since $c^{(1)} = 0$ for any locally constant function $c \in C^\infty(M)$. □

Casimirs are particularly useful for constructing infinite sets of Hamiltonians for compatible local Poisson brackets, which we will define shortly. Being able to easily construct Casimirs gives us a lot of control over computing these Hamiltonians.

**Definition 2.2.70.** [15] Let $\{\cdot,\cdot\}_1$, $\{\cdot,\cdot\}_2$ be a pair of linearly independent local Poisson brackets on a complex manifold $M$. We say that this pair is *compatible* if the bilinear operations

$$\{\cdot,\cdot\}^\lambda := \{\cdot,\cdot\}_1 - \lambda\{\cdot,\cdot\}_2$$

are local Poisson brackets on $M$ for any $\lambda \in \mathbb{C}$.

Let $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ be a pair of compatible local Poisson brackets and let $C$ be a Casimir of the second local Poisson bracket, then under certain non-degeneracy conditions, we can inductively construct an infinite set of commuting Hamiltonians $H^n$, by the relations:

$$H^0 := C, \qquad \{I, H^{n+1}\}_2 = \{I, H^n\}_1,$$

for all $I \in A$ and all $n \in \mathbb{Z}_{\geq 0}$. Note that these Hamiltonians commute for both the first and second local Poisson brackets. This result is known as Magri's theorem, and a proof can be found in [22] (see also [38] for the original statement for ordinary Poisson brackets).

**Definition 2.2.71.** [15] Let $\{\cdot,\cdot\}_1$, and $\{\cdot,\cdot\}_2$ be a pair of compatible non-degenerate local Poisson brackets of hydrodynamic type, we call such a pair a *bihamiltonian structure.*

The name bihamiltonian structure is also used to refer to the infinite set of Hamiltonians that one can construct from the Casimirs of the second local Poisson bracket that one gets using Proposition 2.2.69. Another alternative use of the name bihamiltonian structure is for the integrable hierarchy that one constructs out of this infinite set of commuting Hamiltonians:

$$\frac{\partial X^i}{\partial \tau^{\alpha,p}} = \{X^i,\, H^{\alpha,p}\}_2 = \{X^i,\, H^{\alpha,p-1}\}_1,$$

where $i, \alpha = 1, \ldots, n$, $p = 1, 2, 3, \ldots$ and $X^i[\phi] := \frac{1}{2\pi}\int_0^{2\pi} x^i(\phi(s))\, ds.$

**Theorem 2.2.72.** [15] *Let $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ form a bihamiltonian structure, then the corresponding flat metrics form a flat pencil of metrics.*

*Let $\eta$ and $g$ form a flat pencil of metrics, then their corresponding non-degenerate local Poisson brackets of hydrodynamic type form a bihamiltonian structure.*

To then make a correspondence with Frobenius manifolds, one then has to interpret the quasihomogeneity condition and regularity condition from the world of flat pencils of metrics into the world of bihamiltonian structures. There is no obvious way of doing so, and so when using bihamiltonian structures to construct Frobenius manifolds (as will be done in Subsection 3.2.2), one will need do extra work using the corresponding flat pencil of metrics to prove the construction valid.

### 2.2.7 Semisimplicity and superpotentials

In this subsection we introduce an important class of Frobenius manifolds which have semisimple multiplication on their tangent spaces.

**Definition 2.2.73.** Let $A$ be finite-dimensional commutative associative algebra over a field. We call $A$ *semisimple* if the zero element is the only nilpotent element of $A$.

Semisimplicity tells us that there exists a basis $e_1, \ldots, e_n$ of $A$ such that

$$e_i \cdot e_j = \delta_{ij} e_i,$$

for all $i$, $j = 1, \ldots, n$.

**Lemma 2.2.74.** *Let $M$ be a Frobenius manifold with multiplication $\circ$ on $\mathfrak{X}(M)$. Let $p \in M$ be such that the tangent space $T_pM$ (with the restricted multiplication) is a semisimple $\mathbb{C}$-algebra. Then, there exists an open neighbourhood $U$ of $p$ such that every tangent space $T_xU$ of $U$ is semisimple.*

*Proof.* Let $V \subseteq M$ be an open chart containing $p$ with coordinate system $v = (v^1, \ldots, v^n)$. This allows us to define an open chart $TV \subseteq TM$ with coordinate system $\widetilde{v} : TV \to \mathbb{C}^{2n}$ defined as $\widetilde{v}(X_x) := (v^i(x), X^j(v)_x)$. Define the following sets

$$S := \left\{ \sum_{i=1}^{n} a^i \partial_{v^i} \in \mathfrak{X}(V) \,\middle|\, a^i \in \mathbb{C} \text{ and } \sum_{i=1}^{n} |a^i|^2 = 1 \right\}, \qquad S_x := \{X_x \in T_xV \mid X \in S\},$$

where $x \in V$. Clearly $S_p$ is a compact subset of $TV$. Also, for any $Y_x \in T_xV$, there exists $\lambda \in \mathbb{C}$ and $X_x \in S_x$ such that $Y_x = \lambda X_x$.

Now, a nilpotent element $x$ of an $n$-dimensional associative algebra must satisfy the condition that $x^{n+1} = 0$. Consider the continuous map $f : TV \to TV$ defined as $f(X) = X^{n+1}$, where $n$ is the dimension of $M$. The set of zero vectors of $TV$ forms a closed subset $Z \subseteq TV$ and thus the pre-image $f^{-1}(Z)$ is a closed set in $TV$.

Define the continuous function $g : S_p \to \mathbb{R}$ as

$$g(X_p) := \inf_{Y \in f^{-1}(Z)} \{ |\widetilde{v}(X_p) - \widetilde{v}(Y)| \} .$$

By semisimplicity, $S_p \cap f^{-1}(Z) = \emptyset$ and so by the continuity of $f$, for every point $X_p \in S_p$, there exists an open neighbourhood $U_{X_p} \subseteq TV$ of $X$ such that $U_{X_p} \cap f^{-1}(Z) = \emptyset$ and so $g$

must be nonzero everywhere on $S_p$. The function $g$ is continuous and $S_p$ is compact, so $g$ attains a nonzero minimum. Let $\epsilon := \min\limits_{X_p \in S_p} \{g(X_p)\}$ and define the set $U := v^{-1}(B_\epsilon(v(p)))$. This is an open subset of $V$ containing $p$. Moreover, if $x \in U$, then $v(x) \in B_\epsilon(v(p))$ and so $|v(x) - v(p)| < \epsilon$. Therefore, for any $X \in S$, we have

$$|\widetilde{v}(X_x) - \widetilde{v}(X_p)| = |(v^i(x), X^j(v)_x) - (v^i(p), X^j(v)_p)| = |v(x) - v(p)| < \epsilon.$$

Thus, $f(X_x) \neq 0$ for all $X_x \in S_x$ for all $x \in U$. Now, if $Y_x \in T_xU$ and $Y_x \neq 0$, then there exists $\lambda \in \mathbb{C}^\times$ and $X_x \in S_x$ such that $Y_x = \lambda X_x$ and therefore

$$f(Y_x) = f(\lambda X_x) = \lambda^{n+1} f(X_x) \neq 0.$$

Thus, by definition, $T_xU$ is semisimple for all $x \in U$. □

This means that semisimplicity is an open property of a Frobenius manifold. This property is not usually proven in the literature, but stated as an "obvious" fact. We have chosen to present a proof for completeness.

**Definition 2.2.75.** [13, Lecture 3] Let $M$ be a Frobenius manifold. We define the *nilpotent locus* $\Sigma_{nil}$ of $M$ to be the set

$$\Sigma_{nil} := \{p \in M \mid T_pM \text{ is not semisimple}\}.$$

We say that $M$ is *semisimple* if $\Sigma_{nil}$ is nowhere dense in $M$.

In the literature, when talking about Frobenius manifolds, semisimple Frobenius manifolds are sometimes called *massive* Frobenius manifolds. This is related to the connection that Frobenius manifolds have with two-dimensional quantum gravity via topological quantum field theories. As we do not explore this side of Frobenius manifolds, we will use the adjective semisimple to avoid the use of disparate terminology.

**Lemma 2.2.76.** [13, Lecture 3] *Let $M$ be a semisimple Frobenius manifold with multiplication $\circ$ on $\mathfrak{X}(M)$. Then, on any open chart $U \subseteq M \setminus \Sigma_{nil}$ there exists a set of local coordinates $u = (u^1, \ldots, u^n)$ such that*

$$\partial_{u^i} \circ \partial_{u^j} = \delta_{ij} \partial_{u^i},$$

*for all $i$, $j = 1, \ldots, n$.*

The coordinates $u^1, \ldots, u^n$ are called *canonical coordinates* and are important in the study of semisimple Frobenius manifolds. The multiplication at semisimple points of a manifold can easily be seen to have an identity element of the form $e = \sum\limits_{i=1}^{n} \partial_{u^i}$.

**Proposition 2.2.77.** [13, Lecture 3] *Let $M$ be a semisimple Frobenius manifold with Euler vector field $E$. Then there exists an open chart $U \subseteq M \setminus \Sigma_{nil}$ and canonical coordinates $u^1, \ldots, u^n$ such that the Euler vector field restricted to $U$ has the form*

$$E = \sum_{i=1}^{n} u^i \partial_i. \tag{2.2.62}$$

From now on, when we refer to the canonical coordinates of a semisimple Frobenius manifold, we will assume that these coordinates are such that the Euler vector field has the form expressed in formula (2.2.62). The canonical coordinates are often difficult to find on an arbitrary semisimple Frobenius manifold, the following theorem is useful for finding the canonical coordinates in terms of the flat coordinates of the metric of a Frobenius manifold.

**Theorem 2.2.78.** [13, Lecture 3] *Let $M$ be a semisimple Frobenius manifold with metric $\eta$ and intersection form $g^{\alpha\beta}$. Then there exists canonical coordinates $u^1, \ldots, u^n$ which are the simple roots of the characteristic equation*

$$\det(g^{\alpha\beta}(t) - u\eta^{\alpha\beta}(t)) = 0,$$

*where $t = (t^1, \ldots, t^n)$ are flat coordinates of the metric $\eta$.*

*Conversely, if the roots of this characteristic equation are simple for some point $p \in M$ then $p$ is a semisimple point and the roots $u^1(t), \ldots, u^n(t)$ are canonical coordinates in a neighbourhood of $p$.*

**Example 2.2.79.** Consider a Frobenius manifold in two dimensions with a prepotential of the form

$$F(t^1, t^2) = \frac{1}{2}(t^1)^2 t^2 + c(t^2)^{k+1},$$

where $k \in \mathbb{C} \setminus \{-1, 0, 1\}$ and $c \in \mathbb{C} \setminus \{0\}$ are constants. The metric $\eta$ of this Frobenius manifold has the form $\eta^{\alpha\beta}(t) = \delta^{\alpha+\beta,3}$, while the intersection form $g^{\alpha\beta}$ has the form

$$g^{\alpha\beta}(t) = \begin{pmatrix} 2c(k^2-1)(t^2)^{k-1} & t^1 \\ t^1 & \frac{2}{k}t^2 \end{pmatrix}.$$

The canonical coordinates $u = (u^1, u^2)$ written in terms of the flat coordinates $t^1$, $t^2$ are roots of the characteristic equation

$$\det(g^{\alpha\beta}(t) - u\eta^{\alpha\beta}) = \frac{4c}{k}(k^2-1)(t^2)^k - (t^1 - u)^2 = 0.$$

So the canonical coordinates have the forms

$$u^1(t) = t^1 + \sqrt{\frac{4c}{k}(k^2-1)(t^2)^k}, \qquad u^2(t) = t^1 - \sqrt{\frac{4c}{k}(k^2-1)(t^2)^k}.$$

Let $M$ be a Frobenius manifold with metric $\eta$ and intersection form $g$. Let $U \subseteq M \setminus \Sigma$ be some coordinate neighbourhood of a point $p \in M$ with flat coordinates $t^1, \ldots, t^n$ for the metric $\eta$ and flat coordinates $x^1, \ldots, x^n$ for the intersection form $g^{\alpha\beta}$ such that the unity vector field $e = \partial_{t^1}$. Since $x^1, \ldots, x^n$ are local coordinates, they must be invertible, and so there must exist some $a \in \{1, \ldots, n\}$ such that

$$\frac{\partial x^a}{\partial t^1}(p) \neq 0. \tag{2.2.63}$$

Now, consider the flat coordinates $x_z^1, \ldots, x_z^n$ for the metric $g_z^{\alpha\beta} = g^{\alpha\beta} - z\eta^{\alpha\beta}$, defined on an open dense subset $U_z \subseteq U$. Since, $U_0 = U$, there exists an open neighbourhood $V \subseteq \mathbb{C} \times U$ of the point $(0, p)$ such that the functions $x_z^1, \ldots, x_z^n$ are well-defined on all of $V$. We can now give the definition of the superpotential of a Frobenius manifold.

**Definition 2.2.80.** [13, Appendix I] Let $M$ be a Frobenius manifold and let $p \in M \setminus \Sigma$. Let $V$ be an open neighbourhood of the point $(0, p) \in \mathbb{C} \times M$ such that the functions $x_z^1, \ldots, x_z^n$ are well-defined on all of $V$ and let $a \in \{1, \ldots, n\}$ be such that $\frac{\partial x_0^a}{\partial t^1}(p) \neq 0$. A *superpotential* of $M$ is a function $\lambda : W \to \mathbb{C}$ such that

$$\lambda(x_z^a(u), u) = x^a_{\lambda(z, u)}(u) = z,$$

for all $(z, u) \in V$, where $W \subseteq \mathbb{C} \times M$ is an open subset containing $V$. We call $x_z^a$ the *inverse* of the superpotential.

Superpotentials are a useful technical tool as they obey certain properties which allow us to find what the main structures of a Frobenius manifold look like.

**Lemma 2.2.81.** [13, Appendix I] *If $M$ is a semisimple Frobenius manifold, then there exists a superpotential of $M$.*

A proof of this Lemma can be found in Appendix I of [13], making use of a Lemma in Appendix G that requires $M$ to be semisimple. Proposition 2.2.82 and Theorem 2.2.83, presented below, give us the most important properties of a superpotential for a Frobenius manifold. Proposition 2.2.82 shows us that critical values of a superpotential of a Frobenius manifold $M$ form canonical coordinates on $M$ and Theorem 2.2.83 shows how one can use the superpotential of a Frobenius manifold to calculate its metric, intersection form and structure constants.

**Proposition 2.2.82.** [13, Appendix I] *Let $M$ be a semisimple Frobenius manifold with superpotential $\lambda : U \to M$ and inverse $x^a_z$. Define $q^i : M \to \mathbb{C}$ as*

$$q^i(t) := x^a_{u^i(t)}(t),$$

*where $u^1, \dots, u^n$ are canonical coordinates on $M$. Then $u^i(t) = \lambda(q^i(t),\, t)$ and*

$$\left.\frac{\partial \lambda}{\partial z}\right|_{(z,\,u)=(q^i(t),\,t)} = 0.$$

**Theorem 2.2.83.** [13, Appendix I] *A superpotential $\lambda$ of a Frobenius manifold can be used to find the metric $\eta$, intersection form $g$ and multiplications (via the symmetric 3-tensors $c$ and $\overset{*}{c}$) using the following formulas:*

$$\eta(\partial_i, \partial_j) = \sum_{z_s : \lambda'(z_s)=0} \operatorname*{res}_{z=z_s} \frac{\partial_i(\lambda)\partial_j(\lambda)}{\lambda'(z)}\, dz \tag{2.2.64}$$

$$g(\partial_i, \partial_j) = \sum_{z_s : \lambda'(z_s)=0} \operatorname*{res}_{z=z_s} \frac{\partial_i(\log\lambda)\partial_j(\log\lambda)}{(\log\lambda)'(z)}\, dz \tag{2.2.65}$$

$$c(\partial_i, \partial_j, \partial_k) = \sum_{z_s : \lambda'(z_s)=0} \operatorname*{res}_{z=z_s} \frac{\partial_i(\lambda)\partial_j(\lambda)\partial_k(\lambda)}{\lambda'(z)}\, dz \tag{2.2.66}$$

$$\overset{*}{c}(\partial_i, \partial_j, \partial_k) = \sum_{z_s : \lambda'(z_s)=0} \operatorname*{res}_{z=z_s} \frac{\partial_i(\log\lambda)\partial_j(\log\lambda)\partial_k(\log\lambda)}{(\log\lambda)'(z)}\, dz \tag{2.2.67}$$

As an example we will derive the dual prepotential for the extended affine Weyl group $\widehat{G_2}$ orbit space Frobenius manifold [21]. The corresponding superpotential was recently found in [2].

**Example 2.2.84.** Let $w = (w_0,\, w_1,\, w_2)$ and define $\lambda$ to be the following superpotential

$$\lambda(\mu;\, w) = \frac{w_0}{\mu^2(\mu+1)^2}\left(\mu^6 + 1 + (1-w_1)(\mu^5+\mu) + (1+w_2)(\mu^4+\mu^2) + (1-w_1^2+2w_2)\mu^3\right),$$

and $dz = \frac{d\mu}{\mu}$. Let us define a new set of functions $x_0,\, z_1,\, z_2,\, z_3$ such that $w_0 = e^{x_0}$ and

$$\mu^6 + 1 + (1-w_1)(\mu^5+\mu) + (1+w_2)(\mu^4+\mu^2) + (1-w_1^2+2w_2)\mu^3 = \prod_{i=1}^{3}(\mu - e^{z_i})(\mu - e^{-z_i}).$$

We then have the following relations

$$w_1 = 1 + \sum_{i=1}^{3}(e^{z_i} + e^{-z_i}),$$

$$w_2 = 2 + \sum_{1\le i<j\le 3} (e^{z_i+z_j} + e^{-z_i+z_j} + e^{z_i-z_j} + e^{-z_i-z_j}),$$
$$w_1^2 - 2w_2 = 1 + 2\sum_{i=1}^{3}(e^{z_i} + e^{-z_i}) + \sum_{a,b,c=0}^{1} e^{(-1)^a z_1+(-1)^b z_2+(-1)^c z_3}.$$

Combining these relations we see that $z_3 = \pm z_1 \pm z_2$. Without loss of generality, let us suppose that $z_3 = z_1 + z_2$ and define $x_1 = z_1$, $x_2 = z_2$. We see that $x_0$, $x_1$, $x_2$ are coordinates in $\mathbb{C}^3$ and we will compute the intersection form $g$ and $(0, 3)$ tensor field $\overset{*}{c}_{ijk}$ in these coordinates.

Since $\mu = Ae^z$ for some nonzero constant $A \in \mathbb{C}$, we see that $\frac{\partial\lambda}{\partial z} = \mu\frac{\partial\lambda}{\partial\mu}$. Thus, using the fact that the sum of the residues of a meromorphic one-form, including the residue at infinity, must equal zero, we get the following calculations for the intersection form:

$$\begin{aligned} g(\partial_i, \partial_j) &= \sum_{z_s:\lambda'(z_s)=0} \operatorname*{res}_{z=z_s} \frac{\partial_i(\log\lambda)\partial_j(\log\lambda)}{(\log\lambda)'(z)}\, dz = \sum_{\mu_s:\lambda'(\mu_s)=0} \operatorname*{res}_{\mu=\mu_s} \frac{\partial_i(\lambda)\partial_j(\lambda)}{\lambda\frac{\partial\lambda}{\partial\mu}\mu^2}\, d\mu \\ &= -\operatorname*{res}_{\mu=\infty} \frac{\partial_i(\lambda)\partial_j(\lambda)}{\lambda\frac{\partial\lambda}{\partial\mu}\mu^2}\, d\mu - \operatorname*{res}_{\mu=0} \frac{\partial_i(\lambda)\partial_j(\lambda)}{\lambda\frac{\partial\lambda}{\partial\mu}\mu^2}\, d\mu - \operatorname*{res}_{\mu=-1} \frac{\partial_i(\lambda)\partial_j(\lambda)}{\lambda\frac{\partial\lambda}{\partial\mu}\mu^2}\, d\mu \\ &\quad - \sum_{\mu_s:\lambda(\mu_s)=0} \operatorname*{res}_{\mu=\mu_s} \frac{\partial_i(\lambda)\partial_j(\lambda)}{\lambda\frac{\partial\lambda}{\partial\mu}\mu^2}\, d\mu. \end{aligned}$$

Similarly, for the $\overset{*}{c}_{ijk}$ tensor field, we have

$$\begin{aligned} \overset{*}{c}(\partial_i, \partial_j, \partial_k) &= \sum_{z_s:\lambda'(z_s)=0} \operatorname*{res}_{z=z_s} \frac{\partial_i(\log\lambda)\partial_j(\log\lambda)\partial_k(\log\lambda)}{(\log\lambda)'(z)}\, dz \\ &= \sum_{\mu_s:\lambda'(\mu_s)=0} \operatorname*{res}_{\mu=\mu_s} \frac{\partial_i(\lambda)\partial_j(\lambda)\partial_k(\lambda)}{\lambda^2\frac{\partial\lambda}{\partial\mu}\mu^2}\, d\mu \\ &= -\operatorname*{res}_{\mu=\infty} \frac{\partial_i(\lambda)\partial_j(\lambda)\partial_k(\lambda)}{\lambda^2\frac{\partial\lambda}{\partial\mu}\mu^2}\, d\mu - \operatorname*{res}_{\mu=0} \frac{\partial_i(\lambda)\partial_j(\lambda)\partial_k(\lambda)}{\lambda^2\frac{\partial\lambda}{\partial\mu}\mu^2}\, d\mu \\ &\quad - \operatorname*{res}_{\mu=-1} \frac{\partial_i(\lambda)\partial_j(\lambda)\partial_k(\lambda)}{\lambda^2\frac{\partial\lambda}{\partial\mu}\mu^2}\, d\mu - \sum_{\mu_s:\lambda(\mu_s)=0} \operatorname*{res}_{\mu=\mu_s} \frac{\partial_i(\lambda)\partial_j(\lambda)\partial_k(\lambda)}{\lambda^2\frac{\partial\lambda}{\partial\mu}\mu^2}\, d\mu \end{aligned}$$

Now, the residues at $\mu = -1$ are equal to zero, and the roots of $\lambda$ are known to be $e^{\pm z_i}$. Notice also that $\lambda\left(\frac{1}{\mu};\, w\right) = \lambda(\mu;\, w)$ and thus

$$\left.\frac{\partial\lambda}{\partial\mu}\right|_{\mu\mapsto\frac{1}{\mu}} = -\mu^2\frac{\partial\lambda}{\partial\mu}.$$

Thus, computing the residues at infinity, we find that they are equal to the residues at zero. Computing the residues at zero and the residues at the roots of the superpotential, one can directly calculate the intersection form in the $x$ coordinates. Indeed, we see that

$$g_{ij}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & -2 \\ 0 & -2 & -4 \end{pmatrix}.$$

Thus, the $x$ coordinates are flat coordinates for the intersection form and therefore $\overset{*}{c}_{ijk}(x)$ are the third order derivatives of the dual prepotential. We find that the third order derivatives are

$$\begin{aligned}
\overset{*}{c}_{0ij}(x) =& \, g_{ij}(x), \\
\overset{*}{c}_{111}(x) =& -2(-7e^{x_1} + 4e^{2x_1} + 8e^{x_2} - 5e^{x_1+x_2} + 8e^{2(x_1+x_2)} - 6e^{3(x_1+x_2)} + 4e^{4(x_1+x_2)} \\
& - 7e^{5(x_1+x_2)} + 4e^{2x_1+x_2} - 2e^{3x_1+x_2} + 4e^{4x_1+x_2} - 5e^{x_1+2x_2} - 6e^{3x_1+2x_2} \\
& + 4e^{4x_1+2x_2} - 7e^{5x_1+2x_2} - 7e^{x_1+3x_2} + 4e^{2x_1+3x_2} + 8e^{4x_1+3x_2} - 5e^{5x_1+3x_2} \\
& + 4e^{2x_1+4x_2} - 2e^{3x_1+4x_2} - 5e^{5x_1+4x_2} + 8e^{6x_1+4x_2} + 4e^{4x_1+5x_2}) \\
& /(e^{x_1} - 1)(e^{x_1} - e^{x_2})(e^{x_1+x_2} - 1)(e^{2x_1+x_2} - 1)(e^{x_1+2x_2} - 1), \\
\overset{*}{c}_{112}(x) =& -2(5e^{x_1} - 4e^{x_2} - 3e^{2(x_1+x_2)} + 3e^{3(x_1+x_2)} - 2e^{2x_1+x_2} - e^{3x_1+x_2} + e^{x_1+2x_2} \\
& - 2e^{4x_1+2x_2} + 2e^{x_1+3x_2} - e^{4x_1+3x_2} + e^{2x_1+4x_2} + 2e^{3x_1+4x_2} + 4e^{5x_1+4x_2} \\
& - 5e^{4x_1+5x_2})/(e^{x_1} - e^{x_2})(e^{x_1+x_2} - 1)(e^{2x_1+x_2} - 1)(e^{x_1+2x_2} - 1), \\
\overset{*}{c}_{122}(x) =& -2(4e^{x_1} - 5e^{x_2} + 3e^{2(x_1+x_2)} - 3e^{3(x_1+x_2)} - e^{2x_1+x_2} - 2e^{3x_1+x_2} + 2e^{x_1+2x_2} \\
& - e^{4x_1+2x_2} + e^{x_1+3x_2} - 2e^{4x_1+3x_2} + 2e^{2x_1+4x_2} + e^{3x_1+4x_2} + 5e^{5x_1+4x_2} \\
& - 4e^{4x_1+5x_2})/(e^{x_1} - e^{x_2})(e^{x_1+x_2} - 1)(e^{2x_1+x_2} - 1)(e^{x_1+2x_2} - 1), \\
\overset{*}{c}_{222}(x) =& -2(-8e^{x_1} + 7e^{x_2} - 4e^{2x_2} + 5e^{x_1+x_2} - 8e^{2(x_1+x_2)} + 6e^{3(x_1+x_2)} - 4e^{4(x_1+x_2)} \\
& + 7e^{5(x_1+x_2)} + 5e^{2x_1+x_2} + 7e^{3x_1+x_2} - 4e^{x_1+2x_2} - 4e^{3x_1+2x_2} - 4e^{4x_1+2x_2} \\
& + 2e^{x_1+3x_2} + 6e^{2x_1+3x_2} + 2e^{4x_1+3x_2} - 4e^{x_1+4x_2} - 4e^{2x_1+4x_2} - 8e^{3x_1+4x_2} \\
& - 4e^{5x_1+4x_2} + 7e^{2x_1+5x_2} + 5e^{3x_1+5x_2} + 5e^{4x_1+5x_2} - 8e^{4x_1+6x_2}) \\
& /(e^{x_1} - e^{x_2})(e^{x_2} - 1)(e^{x_1+x_2} - 1)(e^{2x_1+x_2} - 1)(e^{x_1+2x_2} - 1).
\end{aligned}$$

Integrating these expressions, we find the dual prepotential to be

$$\begin{aligned}
F_*(x) =& \, 6\,\mathrm{Li}_3(e^{x_1}) + 6\,\mathrm{Li}_3(e^{x_2}) + 6\,\mathrm{Li}_3(e^{x_1+x_2}) + 2\,\mathrm{Li}_3(e^{x_2-x_1}) + 2\,\mathrm{Li}_3(e^{2x_1+x_2}) \\
& + 2\,\mathrm{Li}_3(e^{x_1+2x_2}) + \frac{x_0^3}{6} - 2x_0x_1^2 + \frac{7x_1^3}{3} - 2x_0x_1x_2 + 5x_1^2x_2 - 2x_0x_2^2 + 4x_1x_2^2 + \frac{8x_2^3}{3},
\end{aligned}$$

where $\mathrm{Li}_3(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^3}$ is the tri-logarithm function.

### 2.2.8 Darboux–Egoroff systems and Egoroff potentials

Semisimple Frobenius manifolds can be equivalently formulated as Darboux–Egoroff systems and as Egoroff potentials. To begin our explanation for these correspondences, let us first consider the metric $\eta$ of a semisimple Frobenius manifold in terms of the canonical coordinates.

**Proposition 2.2.85.** *Let $M$ be a semisimple Frobenius manifold with metric $\eta$ and unity vector field $e$. Let $t^1, \ldots, t^n$ be flat coordinates of the metric such that $e = \partial_{t^1}$ and define the function $\tau := \eta_{1\alpha}(t)t^\alpha$. If $u^1, \ldots, u^n$ are canonical coordinates on $M$, then*

$$\eta_{ij}(u) = \delta_{ij}\frac{\partial \tau}{\partial u^i}.$$

*Proof.* Let $\circ$ be the multiplication on $M$ and see that

$$\begin{aligned}\eta_{ij}(u) &= \eta(\partial_{u^i}, \partial_{u^j}) = \eta(e \circ \partial_{u^i}, \partial_{u^j}) = \eta(e, \partial_{u^i} \circ \partial_{u^j}) = \delta_{ij}\eta(e, \partial_{u^i}) \\ &= \delta_{ij}\eta\left(e, \frac{\partial t^\alpha}{\partial u^i}\partial_{t^\alpha}\right) = \delta_{ij}\frac{\partial t^\alpha}{\partial u^i}\eta_{1\alpha}(t) = \delta_{ij}\frac{\partial \tau}{\partial u^i}.\end{aligned}$$

□

Semisimple Frobenius manifolds have an equivalent formulation as systems of equations in terms of the canonical coordinates, one such system is the Darboux–Egoroff system, which we will define below.

**Definition 2.2.86.** [13, Lecture 3] A *Darboux–Egoroff system* is a system of equations $\gamma_{ij} : U \to \mathbb{C}, (i, j = 1, \ldots, n)$ defined on a complex manifold $U$ with a coordinate system $u = (u^1, \ldots, u^n)$ defined on all of $U$ such that

**1)** Each $\gamma_{ij}$ has zero divergence, meaning that

$$\sum_{k=1}^{n} \frac{\partial \gamma_{ij}}{\partial u^k} = 0.$$

**2)** Each $\gamma_{ij}$ is homogeneous of weight -1, meaning that

$$\sum_{k=1}^{n} u^k \frac{\partial \gamma_{ij}}{\partial u^k} = -\gamma_{ij}.$$

**3)** For all distinct $i, j, k$, we have

$$\frac{\partial \gamma_{ij}}{\partial u^k} = \gamma_{ik}\gamma_{jk}.$$

Semisimple Frobenius manifolds can be used to construct Darboux–Egoroff systems, as will be shown in Theorem 2.2.88. However, we cannot simply go in the opposite direction, in order to construct a Frobenius manifold from a Darboux–Egoroff system we need some genericity conditions.

**Definition 2.2.87.** Let $\gamma_{ij}$ be a Darboux–Egoroff system defined in the coordinates $u^1, \dots, u^n$, and let us define the following matrices:

$$\Gamma(u) := (\gamma_{ij}(u))_{i,j=1,\dots,n}\,, \qquad U := \operatorname{diag}(u^1, \dots, u^n).$$

We say that the Darboux–Egoroff system $\gamma_{ij}$ is *generic* if the matrix

$$V(u) := [\Gamma(u), U]$$

is diagonalisable.

We are now ready to present a theorem of correspondence between Frobenius manifolds and Darboux–Egoroff systems.

**Theorem 2.2.88.** [13, Lecture 3] *Let $\gamma_{ij} : U \to \mathbb{C}$ be a generic Darboux–Egoroff system. Then, $U$ can be given a semisimple Frobenius manifold structure with metric $\eta$ flat coordinates $t^1, \dots, t^n$, canonical coordinates $u^1, \dots, u^n$ and unity vector field $e = \partial_{t^1}$ such that*

$$\tau(u) = \eta_{1\alpha} t^\alpha(u), \qquad \gamma_{ij}(u) = \frac{1}{2} \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}}{\sqrt{\frac{\partial \tau}{\partial u^i} \frac{\partial \tau}{\partial u^j}}},$$

*for all $i, j = 1, \dots, n$. We refer to this as the Frobenius manifold of the Darboux–Egoroff system $\gamma_{ij}$.*

*Conversely, let $M$ be a semisimple Frobenius manifold with metric $\eta$, unity vector field $e$. Let $U \subseteq M$ be an open chart with canonical coordinate system $u = (u^1, \dots, u^n)$ and flat coordinate system $t = (t^1, \dots, t^n)$ such that $e = \partial_{t^1}$. Consider the function $\tau : U \to \mathbb{C}$ defined as*

$$\tau(u) := \eta_{1\alpha} t^\alpha(u),$$

*and define the functions $\gamma_{ij} : U \to \mathbb{C}$ as*

$$\gamma_{ij}(u) := \frac{1}{2} \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}}{\sqrt{\frac{\partial \tau}{\partial u^i} \frac{\partial \tau}{\partial u^j}}},$$

*for $i, j = 1, \dots, n$. Then, $\gamma_{ij}$ is a Darboux–Egoroff system. We call $\gamma_{ij}$ a Darboux–Egoroff system of the Frobenius manifold $M$.*

*Furthermore, the Frobenius manifold $U$ of a Darboux–Egoroff system $\gamma_{ij}$ of the Frobenius manifold $M$ is locally equivalent to $M$, and a Darboux–Egoroff system $\widetilde{\gamma}_{ij}$ of a Frobe-*

*nius manifold $U$ of the Darboux–Egoroff system $\gamma_{ij}$ is equal to a restiction of $\gamma_{ij}$ with possibly permuted coordinates to a non-empty, open subset of $U$.*

This alternative description for semisimple Frobenius manifolds makes use of the function $\tau = \eta_{1\alpha}(t)t^\alpha$. Rather than dressing up this function into a system of equations, one may wish to study this function directly.

**Definition 2.2.89.** An *Egoroff potential* is a function $\tau : U \to \mathbb{C}$ defined on a complex manifold $U$ with a coordinate system $u = (u^1, \ldots, u^n)$ defined on all of $U$ such that

**1)** For all $i = 1, \ldots, n$ and all $u \in U$,

$$\frac{\partial \tau}{\partial u^i}(u) \neq 0.$$

**2)** There exists a constant $C \in \mathbb{C}$ such that

$$\sum_{i=1}^{n} \frac{\partial \tau}{\partial u^i} = C.$$

**3)** There exist constants $d$, $A \in \mathbb{C}$ such that

$$\sum_{i=1}^{n} u^i \frac{\partial \tau}{\partial u^i} = (1-d)\tau + A,$$

we call $d$ the charge of the Egoroff potential.

**4)** For all distinct $i$, $j$, $k$, we have

$$\frac{\partial^3 \tau}{\partial u^i \partial u^j \partial u^k} = \frac{1}{2}\left( \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial^2 \tau}{\partial u^i \partial u^k}}{\frac{\partial \tau}{\partial u^i}} + \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\frac{\partial \tau}{\partial u^j}} + \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^k}\frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\frac{\partial \tau}{\partial u^k}} \right)$$

Although the Egoroff potential has been defined for a Frobenius manifold, it has not been given a definition in the abstract in the literature before, as far as we know.

**Example 2.2.90.** Let $n = 1$. Then any Egoroff potential has the form

$$\tau(u) = au + c,$$

where $a$, $c \in \mathbb{C}$ are constants with $a \neq 0$. These Egoroff potentials all have charge $d = 0$.

Next, consider the case $n = 2$. One can show that any Egoroff potential has one of the following forms:

$$\begin{aligned} \tau(u^1, u^2) &= a(u^1 - u^2)^{1-d} + c, \\ \tau(u^1, u^2) &= au^1 + bu^2 + c, \\ \tau(u^1, u^2) &= a\ln(u^1 - u^2) + c, \end{aligned}$$

where $a$, $b$, $c$, $d \in \mathbb{C}$ are constants such that $a$, $b \neq 0$ and $d \neq 1$. These Egoroff potentials have charges $d$, 0 and 1, respectively.

**Proposition 2.2.91.** *Let $\tau$ be an Egoroff potential with charge $d$ and $C = \sum_{i=1}^{n} \frac{\partial \tau}{\partial u^i}$. Then, $dC = 0$.*

*Proof.* By differentiating the formulas given in conditions **2)** and **3)** from Definition 2.2.89, we get

$$dC = d\sum_{i=1}^{n} \frac{\partial \tau}{\partial u^i} = -\sum_{i=1}^{n}\sum_{j=1}^{n} u^j \frac{\partial^2 \tau}{\partial u^i \partial u^j} = -\sum_{j=1}^{n} u^j \sum_{i=1}^{n} \frac{\partial^2 \tau}{\partial u^i \partial u^j} = 0.$$

□

**Theorem 2.2.92.** *Let $\tau : U \to \mathbb{C}$ be an Egoroff potential with charge $d$. Define $\gamma_{ij} : U \to \mathbb{C}$ as*

$$\gamma_{ij}(u) := \frac{1}{2} \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}}{\sqrt{\frac{\partial \tau}{\partial u^i} \frac{\partial \tau}{\partial u^j}}},$$

*for $i, j = 1, \ldots, n$. Then this is a Darboux–Egoroff system. We refer to this as the Darboux–Egoroff system of the Egoroff potential $\tau$.*

*Conversely, let $\gamma_{ij} : U \to \mathbb{C}$ be a generic Darboux–Egoroff system. Let $\eta$ be the metric of its Frobenius manifold with unity vector field $e$, charge $d$, canonical coordinates $u^1, \ldots, u^n$ and flat coordinates $t^1, \ldots, t^n$ of the metric $\eta$ with $e = \partial_{t^1}$ and consider the function $\tau : U \to \mathbb{C}$ defined as*

$$\tau(u) = \eta_{1\alpha}(t) t^{\alpha}(u).$$

*Then, $\tau$ is an Egoroff potential with charge $d$ and*

$$\gamma_{ij}(u) := \frac{1}{2} \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}}{\sqrt{\frac{\partial \tau}{\partial u^i} \frac{\partial \tau}{\partial u^j}}},$$

*for all $i, j = 1, \ldots, n$. We call $\tau$ an Egoroff potential of the Darboux–Egoroff system $\gamma_{ij}$.*

*Furthermore, the Darboux–Egoroff system $\widetilde{\gamma}_{ij}$ of an Egoroff potential $\tau$ of the Darboux–Egoroff system $\gamma_{ij}$ is the same as $\gamma_{ij}$ with possibly permuted canonical coordinates, and an Egoroff potential $\widetilde{\tau}$ of a Darboux–Egoroff system $\gamma_{ij}$ of the Egoroff potential $\tau$ is equal to $\tau + c$ with possibly permuted canonical coordinates, where $c \in \mathbb{C}$ is a constant.*

*Proof.* Let $\tau$ be an Egoroff potential, then

$$\sum_{k=1}^{n} \frac{\partial \gamma_{ij}}{\partial u^k} = \frac{1}{2} \sum_{k=1}^{n} \frac{\frac{\partial^3 \tau}{\partial u^i \partial u^j \partial u^k}}{\sqrt{\frac{\partial \tau}{\partial u^i} \frac{\partial \tau}{\partial u^j}}} - \frac{1}{2} \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j} \frac{\partial^2 \tau}{\partial u^i \partial u^k} \frac{\partial \tau}{\partial u^j}}{\left(\frac{\partial \tau}{\partial u^i} \frac{\partial \tau}{\partial u^j}\right)^{\frac{3}{2}}} - \frac{1}{2} \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j} \frac{\partial \tau}{\partial u^i} \frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\left(\frac{\partial \tau}{\partial u^i} \frac{\partial \tau}{\partial u^j}\right)^{\frac{3}{2}}} = 0.$$

Also,

$$\begin{aligned}\sum_{k=1}^n u^k \frac{\partial \gamma_{ij}}{\partial u^k} &= \frac{1}{2}\sum_{k=1}^n u^k \frac{\frac{\partial^3 \tau}{\partial u^i \partial u^j \partial u^k}}{\sqrt{\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}}} - \frac{u^k}{2}\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial^2 \tau}{\partial u^i \partial u^k}\frac{\partial \tau}{\partial u^j}}{\left(\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}\right)^{\frac{3}{2}}} - \frac{u^k}{2}\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial \tau}{\partial u^i}\frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\left(\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}\right)^{\frac{3}{2}}} \\ &= \frac{1}{2}\left((-d-1)\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}}{\sqrt{\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}}} + \frac{d}{2}\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}}{\left(\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}\right)^{\frac{3}{2}}} + \frac{d}{2}\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}}{\left(\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}\right)^{\frac{3}{2}}}\right) \\ &= -\frac{1}{2}\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}}{\sqrt{\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}}} = -\gamma_{ij}.\end{aligned}$$

Finally, if $i$, $j$, $k$ are distinct, then

$$\begin{aligned}\frac{\partial \gamma_{ij}}{\partial u^k} &= \frac{1}{2}\left(\frac{\frac{\partial^3 \tau}{\partial u^i \partial u^j \partial u^k}}{\sqrt{\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}}} - \frac{1}{2}\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial^2 \tau}{\partial u^i \partial u^k}\frac{\partial \tau}{\partial u^j}}{\left(\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}\right)^{\frac{3}{2}}} - \frac{1}{2}\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial \tau}{\partial u^i}\frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\left(\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}\right)^{\frac{3}{2}}}\right) \\ &= \frac{1}{4\sqrt{\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}}}\left(\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial^2 \tau}{\partial u^i \partial u^k}}{\frac{\partial \tau}{\partial u^i}} + \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\frac{\partial \tau}{\partial u^j}} + \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^k}\frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\frac{\partial \tau}{\partial u^k}}\right. \\ &\qquad\left. - \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial^2 \tau}{\partial u^i \partial u^k}}{\frac{\partial \tau}{\partial u^i}} - \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\frac{\partial \tau}{\partial u^j}}\right) \\ &= \frac{1}{4\sqrt{\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}}}\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^k}\frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\frac{\partial \tau}{\partial u^k}} = \left(\frac{1}{2}\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^k}}{\sqrt{\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^k}}}\right)\left(\frac{1}{2}\frac{\frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\sqrt{\frac{\partial \tau}{\partial u^j}\frac{\partial \tau}{\partial u^k}}}\right) = \gamma_{ik}\gamma_{jk}.\end{aligned}$$

Thus, $\gamma_{ij}$ is a Darboux–Egoroff system.

Conversely, suppose $\gamma_{ij}$ is a Darboux–Egoroff system and let $M$ be its Frobenius manifold. Define $\tau : M \to \mathbb{C}$ to be

$$\tau(u) := \eta_{1\alpha}(t)t^\alpha(u).$$

Then,

$$\eta_{ii}(u) = \eta(\partial_{u^i}, \partial_{u^i}) = \eta(e, \partial_{u^i}) = \eta\left(e, \frac{\partial t^\alpha}{\partial u^i}\partial_{t^\alpha}\right) = \eta(e, \partial_{t^\alpha})\frac{\partial t^\alpha}{\partial u^i} = \eta_{1\alpha}(t)\frac{\partial t^\alpha}{\partial u^i} = \frac{\partial \tau}{\partial u^i}.$$

Since $\mathcal{L}_E\eta_{\alpha\beta} = (2-d)\eta_{\alpha\beta}$, we see that

$$\begin{aligned}(2-d)\frac{\partial \tau}{\partial u^i} &= (2-d)\eta_{ii}(u) = \mathcal{L}_E\eta_{ii}(u) \\ &= E^\alpha(u)\eta_{ii;\alpha}(u) + E^\alpha_{;i}(u)\eta_{\alpha i}(u) + E^\alpha_{;i}(u)\eta_{i\alpha}(u) \\ &= 2\eta_{ii}(u) + \sum_{k=1}^n u^k\frac{\partial \eta_{ii}(u)}{\partial u^k} = 2\frac{\partial \tau}{\partial u^i} + \sum_{k=1}^n u^k\frac{\partial^2 \tau}{\partial u^i \partial u^k}\end{aligned}$$

and so we have

$$\sum_{k=1}^{n} u^k \frac{\partial^2 \tau}{\partial u^i \partial u^k} = -d \frac{\partial \tau}{\partial u^i},$$

for all $i = 1, \ldots, n$. From this we can conclude that

$$\sum_{i=1}^{n} u^i \frac{\partial \tau}{\partial u^i} = (1-d)\tau + A,$$

for some constant $A \in \mathbb{C}$. Since $\eta$ is a metric on $M$ it is non-degenerate and so

$$\frac{\partial \tau}{\partial u^i}(u) = \eta_{ii}(u) \neq 0,$$

for all $u \in M$ and all $i = 1, \ldots, n$. Now, since $M$ is a Frobenius manifold, we have that

$${}^{\eta}\nabla_{\partial_{u^i}} e = 0,$$

for all $i = 1, \ldots, n$. And, so

$$0 = \left({}^{\eta}\nabla_{\partial_{u^i}} e\right)^{\alpha} = e_i^{\alpha}(u) + {}^{\eta}\Gamma^{\alpha}_{di}(u) e^d(u) = \sum_{d=1}^{n} {}^{\eta}\Gamma^{\alpha}_{di}(u).$$

Now,

$${}^{\eta}\Gamma^{\alpha}_{di}(u) = \frac{1}{2}\eta^{\alpha l}(u)(\eta_{dl;i}(u) + \eta_{li;d}(u) - \eta_{di;l}(u)) = \frac{1}{2}\eta^{\alpha\alpha}(u)(\eta_{d\alpha;i}(u) + \eta_{\alpha i;d}(u) - \eta_{di;\alpha}(u)),$$

noting that there is no summation over $\alpha$. Setting $\alpha = i$, we see that

$$\begin{aligned}
0 &= \sum_{d=1}^{n} {}^{\eta}\Gamma^{i}_{di}(u) = {}^{\eta}\Gamma^{i}_{ii}(u) + \sum_{d \neq i} {}^{\eta}\Gamma^{i}_{di}(u) \\
&= \frac{1}{2}\eta^{ii}(u)\left(\eta_{ii;i}(u) + \eta_{ii;i}(u) - \eta_{ii;i}(u) + \sum_{d\neq i} \eta_{di;i}(u) + \eta_{ii;d}(u) - \eta_{di;i}(u)\right) \\
&= \frac{1}{2}\eta^{ii}(u)\left(\eta_{ii;i}(u) + \sum_{d\neq i}\eta_{ii;d}(u)\right) = \frac{1}{2}\eta^{ii}(u)\sum_{d=1}^{n}\eta_{ii;d}(u) = \frac{1}{2}\eta^{ii}(u)\sum_{d=1}^{n}\frac{\partial^2\tau}{\partial u^i \partial u^d},
\end{aligned}$$

and so

$$\sum_{j=1}^{n} \frac{\partial^2 \tau}{\partial u^i \partial u^j} = 0,$$

for all $i = 1, \ldots, n$. And thus,

$$\sum_{i=1}^{n} \frac{\partial \tau}{\partial u^i} = C,$$

for some constant $C \in \mathbb{C}$. Now, since $\eta$ is a flat metric, the Riemann curvature tensor ${}^{\eta}R^i_{jkl}$ must be zero everywhere on $M$. In particular, if we let $i, j, k$ be distinct, then

$$\begin{aligned}
0 = {}^{\eta}R^i_{jik} &= \partial_{u^i}({}^{\eta}\Gamma^i_{jk}(u)) - \partial_{u^k}({}^{\eta}\Gamma^i_{ij}(u)) + {}^{\eta}\Gamma^i_{il}(u){}^{\eta}\Gamma^l_{jk}(u) - {}^{\eta}\Gamma^i_{kl}(u){}^{\eta}\Gamma^l_{ij}(u) \\
&= -\partial_{u^k}\left(\frac{1}{2}\eta^{ii}(u)\eta_{ii;j}(u)\right) + {}^{\eta}\Gamma^i_{ij}(u){}^{\eta}\Gamma^j_{jk}(u) + {}^{\eta}\Gamma^i_{ik}(u){}^{\eta}\Gamma^k_{jk}(u) - {}^{\eta}\Gamma^i_{ik}(u){}^{\eta}\Gamma^i_{ij}(u) \\
&= -\frac{1}{2}\eta^{ii}(u)\eta_{ii;jk}(u) + \frac{1}{2}\eta^{ii}(u)^2\eta_{ii;k}(u)\eta_{ii;j}(u) + \frac{1}{4}\eta^{ii}(u)\eta^{jj}(u)\eta_{ii;j}(u)\eta_{jj;k}(u) \\
&\quad + \frac{1}{4}\eta^{ii}(u)\eta^{kk}(u)\eta_{ii;k}(u)\eta^{kk;j}(u) - \frac{1}{4}\eta^{ii}(u)^2\eta_{ii;k}(u)\eta_{ii;j}(u) \\
&= \frac{1}{2}\eta^{ii}(u)\left(\frac{1}{2}\left(\frac{\eta_{ii;j}(u)\eta_{ii;k}(u)}{\eta_{ii}(u)} + \frac{\eta_{ii;j}(u)\eta_{jj;k}(u)}{\eta_{jj}(u)} + \frac{\eta_{ii;j}(u)\eta_{kk;j}(u)}{\eta_{kk}(u)}\right) - \eta_{ii;jk}(u)\right).
\end{aligned}$$

And so,

$$\frac{\partial^3\tau}{\partial u^i\partial u^j\partial u^k} = \frac{1}{2}\left(\frac{\frac{\partial^2\tau}{\partial u^i\partial u^j}\frac{\partial^2\tau}{\partial u^i\partial u^k}}{\frac{\partial\tau}{\partial u^i}} + \frac{\frac{\partial^2\tau}{\partial u^i\partial u^j}\frac{\partial^2\tau}{\partial u^j\partial u^k}}{\frac{\partial\tau}{\partial u^j}} + \frac{\frac{\partial^2\tau}{\partial u^i\partial u^k}\frac{\partial^2\tau}{\partial u^j\partial u^k}}{\frac{\partial\tau}{\partial u^k}}\right).$$

Thus, $\tau$ is an Egoroff potential of charge $d$.

These constructions are clearly mutually inverse to one another, except for the possibility of adding a constant to the Egoroff potential or permuting the canonical coordinates. □

We thus have a correspondence between semisimple Frobenius manifolds and Egoroff potentials. In fact, this correspondence can be made stronger, as the assumption of genericity for the Darboux–Egoroff systems unnecessarily restricts the correspondence to only those for which the Darboux–Egoroff system is generic. One can give an unconditional correspondence between semisimple Frobenius manifolds and Egoroff potentials which agrees with the above correspondence when the Darboux–Egoroff potential is generic. This allows us to transfer concepts for Frobenius manifolds into concepts for Egoroff potentials.

**Definition 2.2.93.** Let $\tau$ and $\widetilde{\tau}$ be Egoroff potentials and let $M$ and $\widetilde{M}$ be the Frobenius manifolds of $\tau$ and $\widetilde{\tau}$, respectively. Then, we say $\tau$ is *(locally) equivalent* to $\widetilde{\tau}$ if $M$ is (locally) equivalent to $\widetilde{M}$. Moreover, an Egoroff potential is called *reducible* (respectively, *irreducible*) if their Frobenius manifold is reducible (respectively, irreducible).

**Proposition 2.2.94.** *Let $\tau : M \to \mathbb{C}$ and $\widetilde{\tau} : \widetilde{M} \to \mathbb{C}$ be Egoroff potentials defined in coordinates $u^1, \ldots, u^n$ and $v^1, \ldots, v^n$, respectively. Then, $\tau$ is (locally) equivalent to $\widetilde{\tau}$ if and only if there exists constants $c$, $a \in \mathbb{C}$ and a permutation $\sigma \in S_n$ such that $c \neq 0$ and*

$$\tau(u^1, \ldots, u^n) = c^2\widetilde{\tau}(v^{\sigma(1)}, \ldots, v^{\sigma(n)}) + a.$$

*Proof.* Let $\phi : M \to \widetilde{M}$ be a (local) equivalence map. Then, the differential map $\phi_* :$

$T_pM \to T_{\phi(p)}\widetilde{M}$ is an algebra isomorphism, and thus

$$\phi_*(\partial_{u^i}) \circ \phi_*(\partial_{u^j}) = \phi_*(\partial_{u^i} \circ \partial_{u^j}) = \phi_*(\delta_{ij}\partial_{u^i}) = \delta_{ij}\phi_*(\partial_{u^i}),$$

thus the canonical coordinates of $\widetilde{M}$ are permutations of the canonical coordinates of $M$. Moreover,

$$\frac{\partial \tau}{\partial u^i} = \eta_{ii}(u) = c^2\widetilde{\eta}_{\sigma(i)\sigma(i)}(v) = c^2\frac{\partial \widetilde{\tau}}{\partial v^{\sigma(i)}},$$

where $c \in \mathbb{C}$ is a nonzero constant.

Conversely, suppose $\tau = c\widetilde{\tau} + a$, then define the map $\phi : M \to \widetilde{M}$ as

$$\phi(u^i) = v^{\sigma(i)}.$$

This defines a (local) equivalence map from $M$ to $\widetilde{M}$. □

Let $M$ and $N$ have Egoroff potentials $\tau_M$ and $\tau_N$, respectively. Then the function $\widetilde{\tau}(u, v) = \tau_M(u) + \tau_N(v)$ is an Egoroff potential of the product $M \times N$.

**Proposition 2.2.95.** *Let $\tau$ be an $n$-dimensional Egoroff potential. Then the following are equivalent:*

***1)*** *$\tau$ is reducible.*

***2)*** *There exists a permutation $\sigma \in S_n$ and two Egoroff potentials $\tau_1$, $\tau_2$ of dimensions $1 \leq k$, $n-k < n$, respectively such that*

$$\tau(u^1, \ldots, u^n) = \tau_1(u^{\sigma(1)}, \ldots, u^{\sigma(k)}) + \tau_2(u^{\sigma(k+1)}, \ldots, u^{\sigma(n)}).$$

***3)*** *There exists a non-empty, proper subset $I \subseteq \{1, \ldots, n\}$ such that*

$$\frac{\partial^2 \tau}{\partial u^i \partial u^j} = 0$$

*for all $i \in I$ and all $j \notin I$.*

***4)*** *There exists $i$, $j \in \{1, \ldots, n\}$ such that $i \neq j$ and*

$$\frac{\partial^2 \tau}{\partial u^i \partial u^j} = 0.$$

*Proof.* Suppose **1)**, then its Frobenius manifold $M$ may be written as $M = M_1 \times M_2$, where $M_1$, $M_2$ are Frobenius manifolds of smaller dimension than $M$. Since $M$ is semisimple, $M_1$ and $M_2$ are also semisimple, and thus have Egoroff potentials $\tau_1$ and $\tau_2$, the Egoroff potential of their product is $\widetilde{\tau}(u, v) = \tau_1(u) + \tau_2(v)$, which is an Egoroff potential of $M$ and thus equivalent to $\tau$. Choosing $\tau_1$ and $\tau_2$ so that $\widetilde{\tau}$ and $\tau$ only differ by at most a permutation of the coordinates gives us **2)**.

Conversely, suppose **2)** holds, then $\tau$ is locally equivalent to $\tau \circ \sigma^{-1}$ which is equal to $\tau_1(u^1, \ldots, u^k) + \tau_2(u^{k+1}, \ldots, u^n)$. If $\tau_1$ has Frobenius manifold $M$ and $\tau_2$ has Frobenius manifold $\widetilde{M}$, then $\tau \circ \sigma^{-1}$ is an Egoroff potential of $M \times \widetilde{M}$, which is reducible, and thus $\tau$ is reducible.

Suppose **2)**, then let $I = \{\sigma(i) \mid 1 \leq i \leq k\}$. Then,

$$\frac{\partial^2 \tau}{\partial u^i \partial u^j} = \frac{\partial}{\partial u^j}\left(\frac{\partial \tau_1}{\partial u^i}\right) = 0,$$

for all $i \in I$ and $j \notin I$, which gives **3)**.

Conversely, suppose **3)** holds, then integrating $\frac{\partial \tau}{\partial u^i \partial u^j}$ to find $\tau$ we get that $\tau = \tau_1 + \tau_2$, where $\tau_1$ only depends on $u^i$, for $i \in I$ and $\tau_2$ only depends on $u^j$, for $j \notin I$. We must show that $\tau_1$ and $\tau_2$ are Egoroff potentials, but this immediately follows from the Egoroff potential conditions and since $\tau$ is an Egoroff potential.

Suppose **3)**, then let $i \in I$ and $j \notin I$, then **4)** holds.

Conversely, suppose **4)** holds, then let

$$I := \left\{ i \in \{1, \ldots, n\} \,\middle|\, \frac{\partial^2 \tau}{\partial u^i \partial u^j} = 0 \right\} \setminus \{j\}.$$

Clearly, $I$ is a proper, non-empty subset of $\{1, \ldots, n\}$. Let $k \in I$ and $l \notin I$, then either $l = j$ which implies that

$$\frac{\partial^2 \tau}{\partial u^k \partial u^l} = \frac{\partial^2 \tau}{\partial u^k \partial u^j} = 0,$$

or $l \neq j$ and so

$$0 = \frac{\partial^3 \tau}{\partial u^j \partial u^k \partial u^l} = \frac{1}{2}\left( \frac{\frac{\partial^2 \tau}{\partial u^j \partial u^k}\frac{\partial^2 \tau}{\partial u^j \partial u^l}}{\frac{\partial \tau}{\partial u^j}} + \frac{\frac{\partial^2 \tau}{\partial u^j \partial u^k}\frac{\partial^2 \tau}{\partial u^k \partial u^l}}{\frac{\partial \tau}{\partial u^k}} + \frac{\frac{\partial^2 \tau}{\partial u^j \partial u^l}\frac{\partial^2 \tau}{\partial u^k \partial u^l}}{\frac{\partial \tau}{\partial u^l}} \right) = \frac{\frac{\partial^2 \tau}{\partial u^j \partial u^l}\frac{\partial^2 \tau}{\partial u^k \partial u^l}}{2\frac{\partial \tau}{\partial u^l}},$$

but since $\frac{\partial^2 \tau}{\partial u^l \partial u^j} \neq 0$, we must have $\frac{\partial^2 \tau}{\partial u^k \partial u^l} = 0$, so **3)** holds. □

**Corollary 2.2.96.** *Let $\tau$ be an $n$-dimensional irreducible Egoroff potential. If $n \geq 2$, then*

$$\frac{\partial^2 \tau}{(\partial u^i)^2} \neq 0,$$

*for all $i = 1, \ldots, n$.*

*Proof.* We will prove the contrapositive. Let $i \in \{1, \ldots, n\}$ such that

$$\frac{\partial^2 \tau}{(\partial u^i)^2} = 0.$$

If $n = 2$, then
$$\frac{\partial^2 \tau}{\partial u^i \partial u^j} = -\frac{\partial^2 \tau}{(\partial u^i)^2} = 0,$$
for $i \neq j$, and thus $\tau$ is reducible. If $n \geq 3$, then there exists functions $f$, $g$ such that $\tau = f u^i + g$ with
$$\frac{\partial f}{\partial u^i} = \frac{\partial g}{\partial u^i} = 0.$$
Now, let $j, k \in \{1, \dots, n\}$ such that $i$, $j$, $k$ are distinct, then
$$\begin{aligned}\frac{\partial^2 f}{\partial u^j \partial u^k} = \frac{\partial^3 \tau}{\partial u^i \partial u^j \partial u^k} &= \frac{1}{2}\left(\frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial^2 \tau}{\partial u^i \partial u^k}}{\frac{\partial \tau}{\partial u^i}} + \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\frac{\partial \tau}{\partial u^j}} + \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^k}\frac{\partial^2 \tau}{\partial u^j \partial u^k}}{\frac{\partial \tau}{\partial u^k}}\right) \\ &= \frac{1}{2}\left(\frac{\frac{\partial f}{\partial u^j}\frac{\partial f}{\partial u^k}}{f} + \frac{\frac{\partial f}{\partial u^j}\left(\frac{\partial^2 f}{\partial u^j \partial u^k} u^i + \frac{\partial^2 g}{\partial u^j \partial u^k}\right)}{\frac{\partial f}{\partial u^j} u^i + \frac{\partial g}{\partial u^j}} + \frac{\frac{\partial f}{\partial u^k}\left(\frac{\partial^2 f}{\partial u^j \partial u^k} u^i + \frac{\partial^2 g}{\partial u^j \partial u^k}\right)}{\frac{\partial f}{\partial u^k} u^i + \frac{\partial g}{\partial u^k}}\right).\end{aligned}$$

Multiplying both sides by $2f\left(\frac{\partial f}{\partial u^j} u^i + \frac{\partial g}{\partial u^j}\right)\left(\frac{\partial f}{\partial u^k} u^i + \frac{\partial g}{\partial u^k}\right)$ and then differentiating by $u^i$ twice, we get
$$4f \frac{\partial^2 f}{\partial u^j \partial u^k}\frac{\partial f}{\partial u^j}\frac{\partial f}{\partial u^k} = 2\left(\frac{\partial f}{\partial u^j}\right)^2\left(\frac{\partial f}{\partial u^k}\right)^2 + 2f\frac{\partial^2 f}{\partial u^j \partial u^k}\frac{\partial f}{\partial u^j}\frac{\partial f}{\partial u^k} + 2f\frac{\partial^2 f}{\partial u^j \partial u^k}\frac{\partial f}{\partial u^j}\frac{\partial f}{\partial u^k}.$$

Cancelling alike terms from both sides gives us that $\frac{\partial f}{\partial u^j}\frac{\partial f}{\partial u^k} = 0$ and so there exists $l \neq i$ such that $\frac{\partial f}{\partial u^l} = 0$. Therefore,
$$\frac{\partial^2 \tau}{\partial u^i \partial u^l} = \frac{\partial f}{\partial u^l} = 0,$$
and so $\tau$ is reducible. □

We have thus shown that Egoroff potentials have some easy to understand properties when considering equivalences, products and reducibility. The most interesting applications of Egoroff potentials are when we consider the symmetries of Frobenius manifolds, which we discussed in Section 2.2.3.

**Definition 2.2.97.** Let $\tau$ be an Egoroff potential. We call $\tau$ *invertible* if it is homogeneous and $\sum_{i=1}^{n} \frac{\partial \tau}{\partial u^i} = 0$. If $\tau : U \to \mathbb{C}$ is an invertible Egoroff potential, then the function $-\frac{1}{\tau}$ : $U \setminus \{u \mid \tau(u) = 0\} \to \mathbb{C}$ is called the *inverse* of $\tau$.

Notice that every Egoroff potential $\tau$ with charge $d \neq 0, 1$ we see that $\tau$ is equivalent to an invertible Egoroff potential. Also, the inverse of the inverse of an invertible Egoroff potential $\tau$ is locally equivalent to $\tau$.

**Theorem 2.2.98.** *Let $\tau$ be an invertible Egoroff potential and let $M$ be its Frobenius manifold. Then, the inverse of $\tau$ is an Egoroff potential and, moreover, it is the Egoroff potential of the inversion of $M$.*

*Proof.* Checking that the inverse of $\tau$ is an Egoroff potential is an easy calculation exercise. Let $t^1, \ldots, t^n$ be normalised flat coordinates of $M$, then the normalised flat coordinates of the inversion of $M$ are

$$\widetilde{t}^1 := \frac{\eta_{\nu\mu} t^\nu t^\mu}{2t^n}, \qquad \widetilde{t}^\alpha := \frac{t^\alpha}{t^n}, \qquad \widetilde{t}^n := -\frac{1}{t^n},$$

with $\alpha \neq 1, n$. Since, these are normalised coordinates, the Egoroff potential is

$$\widetilde{\tau} = \widetilde{t}^n = -\frac{1}{t^n} = -\frac{1}{\tau}.$$

□

This is a simple way of seeing the symmetry of inversions that we described in Section 2.2.3. This description also allows us to prove that inversions are equivalence-preserving transformations for semisimple Frobenius manifolds with charge $d \neq 1$, which is difficult for one to see when considering the description given for inversions using the prepotential, as shown in Section 2.2.3.

**Theorem 2.2.99.** *Let $\tau$ and $\widetilde{\tau}$ be invertible Egoroff potentials with charges $d$, $\widetilde{d} \neq 1$. Then, $\tau$ is (locally) equivalent to $\widetilde{\tau}$ if and only if the inverse of $\tau$ is (locally) equivalent to $\widetilde{\tau}$.*

*Proof.* Suppose $\tau$ is (locally) equivalent to $\widetilde{\tau}$, then for some nonzero constants $c \in \mathbb{C}$ and a permutation $\sigma \in S_n$, we have that

$$\tau(u^1, \ldots, u^n) = c^2 \widetilde{\tau}(v^{\sigma(1)}, \ldots, v^{\sigma(n)}).$$

Notice that we do not have the addition of a constant, since adding a nonzero constant to an Egoroff potential with charge $d \neq 1$ removes the homogeneous condition that is necessary for invertibility. Thus, taking the inverse of $\tau$ we see that

$$-\frac{1}{\tau} = -\frac{1}{c^2 \widetilde{\tau}(v^{\sigma(1)}, \ldots, v^{\sigma(n)})} = \frac{1}{c^2}\left(-\frac{1}{\widetilde{\tau}(v^{\sigma(1)}, \ldots, v^{\sigma(n)})}\right),$$

and so the inverse of $\tau$ is (locally) equivalent to the inverse of $\widetilde{\tau}$.

Conversely, suppose the inverse of $\tau$ is locally equivalent to the inverse of $\widetilde{\tau}$. These inversions have charges $2 - d$, $2 - \widetilde{d} \neq 1$ and they are both invertible. Since the inverse of the inverse of $\tau$ is locally equivalent to $\tau$ for any invertible Egoroff potential $\tau$, the theorem is proved using the argument in the first part of the proof. □

We now present a general theorem for Egoroff potentials that essentially tells us that the hope of classifying semisimple Frobenius manifolds by only classifying the irreducibles, does not reduce the classification.

**Theorem 2.2.100.** *Let $\tau$ be an invertible Egoroff potential. If $\tau$ is reducible, then its inverse is irreducible.*

*Proof.* Let $I \subseteq \{1, \ldots, n\}$ be a proper, non-empty subset such that

$$\frac{\partial^2 \tau}{\partial u^i \partial u^j} = 0$$

for all $i \in I$ and all $j \notin I$. Let $\widetilde{\tau}$ be the inverse of $\tau$ and see that

$$\frac{\partial^2 \widetilde{\tau}}{\partial u^i \partial u^j} = \frac{\partial^2}{\partial u^i \partial u^j}\left(-\frac{1}{\tau}\right) = \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}\tau - 2\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}}{\tau^3} = -2\frac{\frac{\partial \tau}{\partial u^i}\frac{\partial \tau}{\partial u^j}}{\tau^3},$$

this is, by definition, nonzero. Now, suppose there exists a proper, non-empty subset $J \subseteq \{1, \ldots, n\}$ such that

$$\frac{\partial^2 \widetilde{\tau}}{\partial u^k \partial u^l} = 0,$$

for all $k \in J$ and all $l \notin J$. Without loss of generality suppose $I \cap J \neq \emptyset$, and let $k \in I \cap J$ and $l \notin J$, then $\frac{\partial^2 \widetilde{\tau}}{\partial u^k \partial u^l} = 0$ and so $l \in I$. Let $m \notin I$, then $\frac{\partial^2 \widetilde{\tau}}{\partial u^l \partial u^m} \neq 0$ and so we must have that $m \notin J$. But we also have that $\frac{\partial^2 \widetilde{\tau}}{\partial u^k \partial u^m} \neq 0$ and so $m \in I$. This is a contradiction and so $\widetilde{\tau}$ must be irreducible. □

What Theorem 2.2.100 tells us is that the set of irreducible semisimple Frobenius manifolds is as complicated as the set of reducible semisimple Frobenius manifolds, and so is an inadequate property for simplifying the study of semisimple Frobenius manifolds. We call an invertible Egoroff potential *doubly irreducible* if itself and its inverse are irreducible. Similarly, we call a semisimple Frobenius manifold doubly irreducible if it has an Egoroff potential which is doubly irreducible.

**Corollary 2.2.101.** *Let $M$ be a semisimple Frobenius manifold of charge $d \neq 0, 1, 2$. Then, there exists doubly irreducible semisimple Frobenius manifolds $M_1, \ldots, M_k$ of charges $d$ or $2-d$ such that $M$ is locally equivalent to a Frobenius manifold constructed from $M_1, \ldots, M_k$ by taking products and inversions a finite number of times. The multiset $\{M_1, \ldots, M_k\}$ is unique for $M$ and is preserved under local equivalence.*

The second type of symmetry of Frobenius manifolds are the Legendre transformations.

**Proposition 2.2.102.** *Let $L$ be a vector field on a semisimple Frobenius manifold $M$ with canonical coordinates $u^1, \ldots, u^n$ on an open chart $U$ and Egoroff potential $\tau$. Then, $L$ is a Legendre vector field if and only if the following conditions hold:*

***1)*** *$L^i(u) \neq 0$ at all points $u \in U$ and for all $i = 1, \ldots, n$.*

***2)*** *We have $\sum\limits_{j=1}^{n} \frac{\partial L^i}{\partial u^j} = 0$ for all $i = 1, \ldots, n$.*

*3) There exists a constant $\mu \in \mathbb{C}$ such that*

$$\sum_{j=1}^{n} u^j \frac{\partial L^i}{\partial u^j} = (\mu+1)L^i,$$

*for all $i = 1, \ldots, n$. We see that $\mu$ is the degree of $L$.*

*4) For all $i$, $j = 1, \ldots, n$ with $i \neq j$ we have*

$$\frac{\partial L^i}{\partial u^j} = \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}}{2\frac{\partial \tau}{\partial u^i}}(L^j - L^i).$$

*Moreover, the Legendre transformation of a semisimple Frobenius manifold $M$ of charge $d$ by a Legendre vector field $L$ of degree $\mu$ is a semisimple Frobenius manifold with Egoroff potential $\widetilde{\tau}$ with charge $\widetilde{d} = d - 2\mu - 2$ such that*

$$\frac{\partial \widetilde{\tau}}{\partial u^i} = (L^i(u))^2 \frac{\partial \tau}{\partial u^i}.$$

*Proof.* A Legendre vector field must be invertible which is equivalent to condition **1)**. A Legendre vector field has degree $\mu$, and thus

$$\sum_{j=1}^{n} u^j \frac{\partial L^i}{\partial u^j} = (\mathcal{L}_E L)^i + L^i = (\mu+1)L^i,$$

so a Legendre vector field must satisfy condition **3)**. Now, by Proposition 2.2.85, the Christoffel symbols of the metric $\eta$ in the canonical coordinates have the form

$$^{\eta}\Gamma^i_{jk}(u) = \frac{1}{2\frac{\partial \tau}{\partial u^i}} \left( \delta_{ij} \frac{\partial^2 \tau}{\partial u^j \partial u^k} + \delta_{ik} \frac{\partial^2 \tau}{\partial u^j \partial u^k} - \delta_{jk} \frac{\partial^2 \tau}{\partial u^i \partial u^j} \right), \tag{2.2.68}$$

and so the condition of a Legendre vector field $L$ being covariantly constant with respect to the metric $\eta$ is equivalent to

$$(^{\eta}\nabla_{\partial_{u^j}} L)^i = \frac{\partial L^i}{\partial u^j} + {}^{\eta}\Gamma^i_{jk}(u) L^k = \frac{\partial L^i}{\partial u^j} + \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^j}}{2\frac{\partial \tau}{\partial u^i}} \left( L^i - L^j \right) = 0, \tag{2.2.69}$$

for $i \neq j$, which is equivalent to condition **4)**, and

$$(^{\eta}\nabla_{\partial_{u^i}} L)^i = \frac{\partial L^i}{\partial u^i} + {}^{\eta}\Gamma^i_{ik}(u) L^k = 0. \tag{2.2.70}$$

Combining formulas (2.2.69) and (2.2.70) we see that

$$\sum_{j=1}^{n} (^{\eta}\nabla_{\partial_{u^j}} L)^i = (^{\eta}\nabla_e L)^i = (^{\eta}\nabla_L e)^i + [e,\, L]^i = \sum_{j=1}^{n} \frac{\partial L^i}{\partial u^j} = 0,$$

which is condition **2)**.

Now, let $\widetilde{\eta}$ be the metric of the Frobenius manifold that is Legendre transformed from $M$ by Legendre vector field $L$, then

$$\frac{\partial\widetilde{\tau}}{\partial u^i} = \widetilde{\eta}(\partial_{u^i},\, \partial_{u^i}) = \eta(\partial_{u^i}\circ L,\, \partial_{u^i}\circ L) = (L^i(u))^2\eta(\partial_{u^i},\, \partial_{u^i}) = (L^i(u))^2\frac{\partial\tau}{\partial u^i}.$$

□

**Remark 2.2.103.** We briefly touched on the concept of a generalised Legendre vector field in Remark 2.2.23. In the context of Egoroff potentials a generalised Legendre vector field is a vector field that satisfies conditions **1)**, **3)** and **4)** from Proposition 2.2.102.

From this framework it is easy to see that Legendre transformations of semisimple, irreducible Frobenius manifolds are irreducible. Indeed, if we let $\tau$ be an Egoroff potential and $\tau^L$ be a Legendre transformation of $\tau$ by a Legendre vector field $L$, then by Propositions 2.2.102 and 2.2.95

$$\begin{aligned}\frac{\partial^2\tau^L}{\partial u^i\partial u^j} &= \frac{\partial}{\partial u^j}\left((L^i)^2\frac{\partial\tau}{\partial u^i}\right) = 2L^i\frac{\partial L^i}{\partial u^j}\frac{\partial\tau}{\partial u^i} + (L^i)^2\frac{\partial^2\tau}{\partial u^i\partial u^j}\\ &= 2L^i\frac{\frac{\partial^2\tau}{\partial u^i\partial u^j}}{2\frac{\partial\tau}{\partial u^i}}(L^j-L^i)\frac{\partial\tau}{\partial u^i} + (L^i)^2\frac{\partial^2\tau}{\partial u^i\partial u^j} = L^iL^j\frac{\partial^2\tau}{\partial u^i\partial u^j}\neq 0,\end{aligned}$$

for any $i,\, j = 1,\ldots,n$ with $i\neq j$.

Moreover, one can see that any nonzero flat vector field of a semisimple, irredicible Frobenius manifold $M$ must be invertible on an open, dense subset of $M$, and thus if it is homogeneous with respect to the Euler vector field, then it is a Legendre vector field. To see this note that if $L$ is a non-invertible flat vector field, then there exist $i$ such that $L^i(u) = 0$. Then by condition **4)** in Proposition 2.2.102,

$$L^j\frac{\frac{\partial^2\tau}{\partial u^i\partial u^j}}{2\frac{\partial\tau}{\partial u^i}} = \frac{\frac{\partial^2\tau}{\partial u^i\partial u^j}}{2\frac{\partial\tau}{\partial u^i}}(L^j-L^i) = \frac{\partial L^i}{\partial u^j} = 0,$$

for all $j\neq i$, so $L^j = 0$ for all $j$ and thus $L = 0$. We also have the following result.

**Proposition 2.2.104.** *Let $L$ be a Legendre vector field with degree $\mu$ on a semisimple Frobenius manifold $M$ with charge $d$. Define the matrix $A$ to have the entries*

$$A^i_j = \frac{\frac{\partial^2\tau}{\partial u^i\partial u^j}}{\frac{\partial\tau}{\partial u^i}}(u^i-u^j),$$

*where $\tau$ is an Egoroff potential of $M$ and $u^1,\ldots,u^n$ are canonical coordinates. Then, $L$ is an eigenvector of $A$ with eigenvalue $d-2(\mu+1)$. Moreover, if $M$ is irreducible, then any* *eigenvalue $d-2(\mu+1)$ of $A$ has a flat eigenvector which is a Legendre vector field of $M$ with degree $\mu$.*

*Proof.* If $L$ is a Legendre vector field of degree $\mu$ on a semisimple Frobenius manifold, then it must satisfy conditions **2)**, **3)** and **4)** from Proposition 2.2.102. By also recalling conditions **2)** and **3)** in Definition 2.2.89 of an Egoroff potential, we get

$$\begin{aligned}(\mu+1)L^i &= \sum_{j=1}^{n} u^j \frac{\partial L^i}{\partial u^j} \\ &= \sum_{j=1}^{n}(u^j-u^i)\frac{\frac{\partial^2\tau}{\partial u^i\partial u^j}}{2\frac{\partial\tau}{\partial u^i}}(L^j-L^i) = \sum_{j=1}^{n}\left(-\frac{1}{2}A^i_jL^j\right)+\frac{d}{2}L^i,\end{aligned}$$

which is equivalent to the first claim we want to prove.

To prove the second claim, we need to show that $A$ maps flat vector fields to flat vector fields. Suppose $L$ is a flat vector field on $M$, then it exactly satisfies conditions **2)** and **4)** from Proposition 2.2.102. These conditions can be expressed as the relation

$$\frac{\partial L^i}{\partial u^j} = (B_j)^i_k L^k,$$

for $j=1,\dots,n$, where $B_j$ has the entries

$$(B_j)^i_k = \begin{cases} 0 & \text{if } i\neq j\neq k\neq i, \\ \frac{\frac{\partial^2\tau}{\partial u^i\partial u^j}}{2\frac{\partial\tau}{\partial u^i}} & \text{if } i\neq j=k, \\ -\frac{\frac{\partial^2\tau}{\partial u^j\partial u^k}}{2\frac{\partial\tau}{\partial u^i}} & \text{if } i=j \text{ or } i=k. \end{cases}$$

We can then see that for $i\neq k$, we have

$$\begin{aligned}(d\,\mathrm{Id}-2u^jB_j)^i_k &= -2u^i(B_i)^i_k-2u^k(B_k)^i_k = 2u^i\frac{\frac{\partial^2\tau}{\partial u^i\partial u^k}}{2\frac{\partial\tau}{\partial u^i}}-2u^k\frac{\frac{\partial^2\tau}{\partial u^i\partial u^k}}{2\frac{\partial\tau}{\partial u^i}} \\ &= (u^i-u^k)\frac{\frac{\partial^2\tau}{\partial u^i\partial u^k}}{\frac{\partial\tau}{\partial u^i}} = A^i_k,\end{aligned}$$

and

$$(d\,\mathrm{Id}-2u^jB_j)^i_i = d+2u^j\frac{\frac{\partial^2\tau}{\partial u^i\partial u^j}}{2\frac{\partial\tau}{\partial u^i}} = 0 = A^i_i.$$

Now, consider the Euler vector field $E$. Note that the map $\mathrm{ad}_E$ maps flat vector fields to flat vector fields, and it can be expressed in canonical coordinates as

$$\mathrm{ad}_E(L)^i = u^j\frac{\partial L^i}{\partial u^j}-L^i = u^j(B_j)^i_kL^k-L^i.$$

Thus, $A$ is a linear combination of $\mathrm{ad}_E$ and Id. Hence, $A$ must map flat vector fields to flat vector fields and any eigenvalue of $A$ must have a flat eigenvector with that eigenvalue. Moreover, if $M$ is irreducible, then any nonzero flat eigenvector of $A$ is also invertible. Thus, the only condition left to show that flat eigenvectors of $A$ are Legendre vector fields, is to show homogeneity. Suppose $L$ is a flat eigenvector of $A$ with eigenvalue $d-2(\mu+1)$, then

$$\sum_{j=1}^{n} u^j \frac{\partial L^i}{\partial u^j} = \sum_{j=1}^{n} u^j (B_j)^i_k L^k = \frac{1}{2}(d\,\mathrm{Id} - A)^i_k L^k = (\mu+1)L^i,$$

as required. □

This result helps us to study how the charge changes under Legendre transformations for irreducible semisimple Frobenius manifolds. Since the flat eigenvectors for any eigenvalue of $A$ are Legendre vector fields, we can study the eigenvalues of $A$ to find which charges can be obtained by Legendre transformations. This gives us the following result:

**Proposition 2.2.105.** *Let $M$ be a semisimple, irreducible Frobenius manifold and let $\mathcal{D} = \{D_i\} \subseteq \mathbb{C}$ be the set of charges such that $M$ can be Legendre transformed into a Frobenius manifold with charge $D_i$. Then, $\mathcal{D}$ is equal to the spectrum of $A$. Moreover, if $d \in \mathcal{D}$, then $-d \in \mathcal{D}$, and if $M$ is odd-dimensional, then $0 \in \mathcal{D}$. Furthermore, if $M$ has real degrees and the charge of $M$ is real, then all elements of $\mathcal{D}$ are real and $M$ can be Legendre transformed into a Frobenius manifold with strictly positive degrees.*

*Proof.* Let $M$ have charge $d$ and let $D \in \mathcal{D}$. Then there exists a Legendre vector field $L$ on $M$ with degree $\mu$ such that $D = d - 2(\mu+1)$. By Proposition 2.2.104, $L$ is an eigenvector of $A$ with eigenvalue $d-2(\mu+1)$, thus $D$ belongs to the spectrum of $A$.

Conversely, if $A$ has an eigenvalue $d-2(\mu+1)$, then by Proposition 2.2.104 there exists a Legendre vector field $L$ of $M$ with degree $\mu$. Let $\widetilde{M}$ be the Legendre transformation of $M$ by the Legendre vector field $L$. Then by Proposition 2.2.102, $\widetilde{M}$ must have charge $d-2(\mu+1)$, so the spectrum of $A$ is contained in $\mathcal{D}$.

For any permutation $\sigma \in S_n$ we define the set

$$M(\sigma) := \{i \in \{1,\dots,n\} \mid \sigma(i) \neq i\}.$$

Now, consider the characteristic equation of the matrix $A$ :

$$\begin{aligned} p_A(\lambda) = \det(\lambda\,\mathrm{Id} - A) &= \sum_{k=0}^{n} \lambda^{n-k} \sum_{\substack{\sigma\in S_n \\ |M(\sigma)|=k}} (-1)^\sigma \prod_{i\in M(\sigma)} \frac{\frac{\partial^2 \tau}{\partial u^i \partial u^{\sigma(i)}}}{\frac{\partial \tau}{\partial u^i}} (u^{\sigma(i)} - u^i) \\ &= \sum_{k=0}^{n} \lambda^{n-k} \sum_{\substack{\sigma\in S_n \\ |M(\sigma)|=k}} \frac{(-1)^\sigma}{\prod_{i\in M(\sigma)} \frac{\partial \tau}{\partial u^i}} \prod_{i\in M(\sigma)} \frac{\partial^2 \tau}{\partial u^i \partial u^{\sigma(i)}} (u^{\sigma(i)} - u^i) \end{aligned}$$

Notice that $\sigma$ and $\sigma^{-1}$ must have the same sign and $M(\sigma^{-1}) = M(\sigma)$. Moreover,

$$\prod_{i \in M(\sigma)} \frac{\partial^2 \tau}{\partial u^i \partial u^{\sigma(i)}} (u^{\sigma(i)} - u^i) = (-1)^{|M(\sigma)|} \prod_{i \in M(\sigma^{-1})} \frac{\partial^2 \tau}{\partial u^i \partial u^{\sigma^{-1}(i)}} (u^{\sigma^{-1}(i)} - u^i).$$

Now, if $|M(\sigma)|$ is odd, when we decompose $\sigma$ into disjoint cycles at least one of the cycles must have odd length, and thus $\sigma \neq \sigma^{-1}$. But then, in the characteristic equation, the terms contributed by $\sigma$ will be exactly cancelled by the terms contributed by $\sigma^{-1}$. So, the characteristic equation of $A$ will have the form

$$p_A(\lambda) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{2k} \lambda^{n-2k},$$

where $a_{2k} \in \mathbb{C}$ are constants.

The claims of the proposition now follow naturally. Note that $p_A(\lambda)$ is either an odd polynomial, when $M$ is odd-dimensional, or an even polynomial, when $M$ is even-dimensional. Hence, if $d \in \mathcal{D}$ then $-d \in \mathcal{D}$. Also, if $M$ is odd-dimensional, then $p_A(\lambda)$ is odd and thus $0 \in \mathcal{D}$. Finally, let us consider the case when $M$ has real degrees and charge. Let $M$ have degrees $d_1, \ldots, d_k$, meaning that there exists nonzero flat vector fields $L_1, \ldots, L_k$ such that

$$\mathrm{ad}_E(L_i) = -d_i L_i.$$

Since these are flat eigenvectors of $\mathrm{ad}_E$, they must be flat eigenvectors of $A$ by Proposition 2.2.104 and they have eigenvalues $d - 2(-d_i + 1) = d + 2(d_i - 1)$. Since the eigenvalues of $A$ comprise the set $\mathcal{D}$, all elements of $\mathcal{D}$ must be real. Let us take the smallest element $D_{i_0} \in \mathcal{D}$. We can Legendre transform $M$ to a Frobenius manifold $\widetilde{M}$ with charge $D_{i_0}$. The degrees $\widetilde{d_i}$ of $\widetilde{M}$ are thus found to be

$$\widetilde{d_i} = \frac{1}{2}(D_{\sigma(i)} - D_{i_0} + 2) > 0,$$

where $D_i \in \mathcal{D}$ and $\sigma$ is some permutation. Thus, $\widetilde{M}$ has positive degrees. □

### 2.2.9 The monodromy group

The monodromy group of a Frobenius manifold is an integral part of the motivation for the study of algebraic Frobenius manifolds. To define the monodromy group and analyse some of its properties, we first need to examine monodromy representations of systems of linear differential equations. The following discussion follows Guzzetti's explanation of this topic [30], but we provide a little bit more detail.

Consider the following system of linear differential equations

$$\frac{dY}{dz} = A(z)Y, \tag{2.2.71}$$

where $A$ is an $n \times n$ matrix whose entries are meromorphic functions on $\mathbb{C}$ with a finite number of poles. Let $a_1, \ldots, a_k$ denote the poles of $A(z)$ and let $z_0 \in \mathbb{C} \setminus \{a_1, \ldots, a_k\}$. Let $Y_0(z)$ be a solution of equation (2.2.71) in a small neighbourhood of the point $z_0$. We wish to examine the analytic continuation of $Y_0$ along paths in $\mathbb{C} \setminus \{a_1, \ldots, a_k\}$. What we mean by analytic continuation along a path is expressed in the following definition.

**Definition 2.2.106.** Let $\sigma : [0, 1] \to \mathbb{C}$ be a continuous path. For each $t \in [0, 1]$, let $U_t$ be an open subset of $\mathbb{C}$ such that $\sigma(t) \in U_t$. Let $f_t : U_t \to \mathbb{C}$ be an analytic function for each $t \in [0, 1]$ and suppose that there exists $\delta_t > 0$ such that $|s - t| < \delta_t$ implies that $\sigma(s) \in U_t$ and $f_t(z) = f_s(z)$ for all $z$ in a small neighbourhood of the point $\sigma(s)$. Then, we say that $f_1$ is an *analytic continuation of* $f_0$ *along the path* $\sigma$.

If $f_1$ and $f_1'$ are both analytic continuations of a function $f_0$ along the same path $\sigma$, then by analyticity we must have that $f_1 = f_1'$. In general, it is not always possible to analytically continue a function along any chosen path. However, for solutions of the differential equations (2.2.71) and paths that avoid the poles of $A(z)$, it is possible to define the analytic continuation of the solutions, as the following Lemma makes clear.

**Lemma 2.2.107.** *Let $Y_0$ be a solution of the system of differential equations (2.2.71) defined at some point $z_0 \in \mathbb{C} \setminus \{a_1, \ldots, a_k\}$. Let $\sigma : [0, 1] \to \mathbb{C} \setminus \{a_1, \ldots, a_k\}$ be a continuous path such that $\sigma(0) = z_0$. Then, there exists an analytic continuation $Y_\sigma$ of $Y_0$ along the path $\sigma$. Moreover, if $\sigma' : [0, 1] \to \mathbb{C} \setminus \{a_1, \ldots, a_k\}$ is another continuous path with the same end points as $\sigma$ such that it is homotopic to $\sigma$ in $\mathbb{C} \setminus \{a_1, \ldots, a_k\}$ then $Y_{\sigma'} = Y_\sigma$ in a small neighbourhood of the point $\sigma(1)$.*

The analytic continuation of $Y_0$ along a loop $\gamma$ is related to $Y_0$ by a constant matrix transformation since both give solutions to the same system of linear differential equations. The following lemma gives more precision to this statement.

**Lemma 2.2.108.** *Let $\Phi$ be a fundamental matrix of the system of linear differential equations (2.2.71) defined in a neighbourhood of some point $z_0 \in \mathbb{C} \setminus \{a_1, \ldots, a_k\}$, meaning that $\Phi$ is an $n \times n$ matrix whose columns form a basis for the space of solutions defined in a neighbourhood of the point $z_0$ for (2.2.71). Let $\gamma : [0, 1] \to \mathbb{C} \setminus \{a_1, \ldots, a_k\}$ be a continuous loop starting and ending at the point $z_0$. Then, there exists a constant matrix $M_\gamma$ such that*

$$\Phi_\gamma = \Phi M_\gamma,$$

*where the columns of $\Phi_\gamma$ are the analytic continuations of the columns of $\Phi$ along the loop $\gamma$.*

Since the solutions $Y_\gamma$ do not change if we replace $\gamma$ with a homotopically equivalent loop, Lemma 2.2.108 allows us to define a linear representation of the fundamental group of $\mathbb{C} \setminus \{a_1, \dots, a_k\}$ on the vector space of solutions of the system of equations (2.2.71), namely the map $\gamma \mapsto M_\gamma$. This map is an anti-homomorphism and we refer to it as the *monodromy representation* of the system of equations (2.2.71).

**Definition 2.2.109.** The *monodromy group* of (2.2.71) is defined to be the image of the monodromy representation $\gamma \mapsto M_\gamma$ of the system of linear differential equations (2.2.71) in $GL(n, \mathbb{C})$.

Notice that since any choice of fundamental matrix $\Phi$ of the system of equations (2.2.71) is related to another by the transformation $\Phi \mapsto \Phi C$, where $C$ is a constant non-degenerate matrix, the monodromy representation of the system (2.2.71) is uniquely defined up to conjugation by the constant matrices $C$. Also, note that the fundamental group of $\mathbb{C} \setminus \{a_1, \dots, a_k\}$ is generated by the loops $\gamma_1, \dots, \gamma_k$ that go around the points $a_1, \dots, a_k$. Let us denote $R_1, \dots, R_k$ to be the image of these loops in $GL(n, \mathbb{C})$ under the monodromy representation.

Now, recall from Subsection 2.2.5 that the metric $\eta$ and intersection form $g$ of a Frobenius manifold $M$ form a flat pencil of metrics on $M$, meaning that the tensor fields

$$g_z^{\alpha\beta} = g^{\alpha\beta} - z\eta^{\alpha\beta}$$

are flat metrics on open dense subsets of $M$ for all $z \in \mathbb{C}$. Let $x_z$ be a flat coordinate of $g_z^{\alpha\beta}$ and define $\xi^\alpha : M \times \mathbb{C} \to \mathbb{C}$ as follows:

$$\xi^\alpha(p, z) := \eta^{\alpha\beta} \frac{\partial x_z}{\partial t^\beta}(p). \tag{2.2.72}$$

Then, recalling the generalised hypergeometric equation associated to a Frobenius manifold (2.2.44), we have the following system of equations:

$$g_z^{\beta\gamma}(t)\eta_{\gamma\alpha} \frac{\partial \xi^\alpha}{\partial z} = \left( \frac{d-1}{2} \delta_\alpha^\beta + \eta^{\beta\delta} E^\gamma_{;\delta}(t) \eta_{\gamma\alpha} \right) \xi^\alpha. \tag{2.2.73}$$

Recall from Subsection 2.2.7, that for a semisimple Frobenius manifold there exists canonical coordinates $u^1, \dots, u^n$. In the canonical coordinates we have that $E^\delta(u) = u^\delta$ and $c^\alpha_{\beta\gamma} = 0$ if and only if $\alpha = \beta = \gamma$. Since we know that $g^{\alpha\beta} = E^\gamma c_\gamma^{\alpha\beta}$, we see that

$$g_z^{\beta\gamma}(u)\eta_{\gamma\alpha}(u) = g^{\beta\gamma}(u)\eta_{\gamma\alpha}(u) - z\delta_\alpha^\beta = E^\delta(u) c^\beta_{\delta\alpha}(u) - z\delta_\alpha^\beta = (u^\beta - z)\delta_\alpha^\beta. \tag{2.2.74}$$

Let us define the following matrix:

$$\Psi_\beta^i = \sqrt{\eta_{ii}(u)} \frac{\partial u^i}{\partial t^\beta}.$$

Note that the inverse matrix $\Psi^{-1}$ has entries

$$(\Psi^{-1})^{\alpha}_{j} = \frac{1}{\sqrt{\eta_{jj}(u)}}\frac{\partial t^{\alpha}}{\partial u^{j}}.$$

Using relation (2.2.74) we thus see that

$$\begin{aligned}\Psi^{i}_{\beta}(g^{\beta\gamma}_{z}(t)\eta_{\gamma\alpha})(\Psi^{-1})^{\alpha}_{j} &= \sqrt{\frac{\eta_{ii}(u)}{\eta_{jj}(u)}}\frac{\partial u^{i}}{\partial t^{\beta}}(g^{\beta\gamma}_{z}(t)\eta_{\gamma\alpha}(t))\frac{\partial t^{\alpha}}{\partial u^{j}} = \sqrt{\frac{\eta_{ii}(u)}{\eta_{jj}(u)}}g^{i\gamma}_{z}(u)\eta_{\gamma j}(u)\\ &= \sqrt{\frac{\eta_{ii}(u)}{\eta_{jj}(u)}}(u^{i}-z)\delta^{i}_{j} = (u^{i}-z)\delta^{i}_{j}. \end{aligned} \tag{2.2.75}$$

Similarly, conjugating the expression $\eta^{\beta\delta}E^{\gamma}_{;\delta}(t)\eta_{\gamma\alpha}$ by $\Psi$ and noting that the metric $\eta$ is diagonal in the canonical coordinates, we get

$$\begin{aligned}\Psi^{i}_{\beta}(\eta^{\beta\delta}E^{\gamma}_{;\delta}(t)\eta_{\gamma\alpha})(\Psi^{-1})^{\alpha}_{j} &= \sqrt{\frac{\eta_{ii}(u)}{\eta_{jj}(u)}}\frac{\partial u^{i}}{\partial t^{\beta}}\left(\eta^{\beta\delta}(t)\left({}^{\eta}\nabla_{\delta}E\right)^{\gamma}(t)\eta_{\gamma\alpha}(t)\right)\frac{\partial t^{\alpha}}{\partial u^{j}}\\ = \sqrt{\frac{\eta_{ii}(u)}{\eta_{jj}(u)}}\eta^{i\delta}(u)\left({}^{\eta}\nabla_{\delta}E\right)^{\gamma}(u)\eta_{\gamma j}(u) &= \sqrt{\frac{\eta_{ii}(u)}{\eta_{jj}(u)}}\eta^{ii}(u)\left({}^{\eta}\nabla_{i}E\right)^{j}(u)\eta_{jj}(u)\\ = \sqrt{\frac{\eta_{jj}(u)}{\eta_{ii}(u)}}\left({}^{\eta}\nabla_{i}E\right)^{j}(u). \end{aligned} \tag{2.2.76}$$

Let us define $\phi^{j} := \Psi^{j}_{\beta}\xi^{\beta}$. Thus, multiplying the system of equations (2.2.73) on the left by the matrix $\Psi$ and substituting in the expression $\xi^{\alpha} = (\Psi^{-1})^{\alpha}_{j}\phi^{j}$, we find that, using the relations (2.2.75) and (2.2.76), we have

$$\left(u^{i}-z\right)\delta^{i}_{j}\frac{\partial\phi^{j}}{\partial z} = \left(\frac{d-1}{2}\delta^{i}_{j} + V^{i}_{j}\right)\phi^{j}, \tag{2.2.77}$$

where $V^{i}_{j} = \sqrt{\frac{\eta_{jj}(u)}{\eta_{ii}(u)}}\left({}^{\eta}\nabla_{\partial_{u^{i}}}E\right)^{j}(u)$. Multiplying both sides by the inverse of the matrix $(u^{i}-z)\delta^{i}_{j}$, we can rearrange the system of equations (2.2.77) as

$$\frac{\partial\phi^{i}}{\partial z} = \frac{\frac{d-1}{2}\delta^{i}_{j} + V^{i}_{j}(u)}{u^{i}(p)-z}\phi^{j}. \tag{2.2.78}$$

Note that this system of equations is in the same form as the system (2.2.71), with poles $u^{i}(p)$, thus we can define its monodromy representation. Consider the subset $\widetilde{M} = \{p \in M \mid u^{i}(p) \neq u^{j}(p),\, i \neq j\}$. If $p \in \widetilde{M}$, then the points $u^{i}(p) \in \mathbb{C}$ are distinct poles of the equation (2.2.78). Thus, we define the monodromy representation of a semisimple Frobenius manifold $M$ to be the monodromy representation of the system (2.2.78) at a point $p \in \widetilde{M}$ with poles $u^{1}(p), \ldots, u^{n}(p)$ in $\mathbb{C}$.

**Theorem 2.2.110.** [16] *The monodromy group of the system (2.2.78) is independent of the point* $p \in \widetilde{M}$.

A proof of this theorem can be found in [16] along with an explanation of what we have covered so far in this Subsection. Furthermore, one can see that the monodromy group of the product of two Frobenius manifolds is isomorphic to the product of their respective monodromy groups. We have so far been following Guzzetti's explanation of this topic [30], for the rest of this Subsection, we will be following Dubrovin's explanation [16].

**Definition 2.2.111.** [16] Let $V$ be the space of solutions for the equation (2.2.78). Then, we define the following symmetric bilinear form on $V$ :

$$(\phi_1, \phi_2) = \sum_{i=1}^{n} \phi_1^i \phi_2^i (u^i(p) - z). \tag{2.2.79}$$

We will refer to this bilinear form as the *half-Stokes form*.

Although the definition of this bilinear form can be found in [16], it was not given a name. We have chosen to call it the half-Stokes form as it has a close relationship with the Stokes matrix of the Frobenius manifold. Stokes matrices are part of a more detailed description of the monodromy behaviour of a Frobenius manifold and makes up part of the *monodromy data* of a Frobenius manifold.

**Proposition 2.2.112.** [16] *Let* $M$ *be a semisimple Frobenius manifold with monodromy group* $W(M)$. *Then, the half-Stokes form is independent of the point* $p \in \widetilde{M}$ *and* $z$.

A proof of this proposition can be found in [16]. From now on we will also assume that the half-Stokes form is non-degenerate. The monodromy group, as we have so far described it, is a finitely generated group. This description is too broad, but the following theorem is important as it narrows down what the monodromy group could be.

**Theorem 2.2.113.** [16] *Let* $M$ *be a semisimple Frobenius manifold with non-degenerate half-Stokes form. Let* $R_1, \ldots, R_n$ *be the image of the loops* $\gamma_1, \ldots, \gamma_n$ *around the points* $u^1(p), \ldots, u^n(p)$ *under the monodromy representation. Then, each* $R_i$ *is a reflection with respect to the half-Stokes form (2.2.79).*

Thus, the monodromy group of a semisimple Frobenius manifold is a finitely generated reflection group. We will see in Section 2.3 that such groups are well understood and when they are finite a full classification is known [32].

Notice that the canonical coordinates are well-defined only up to permutation, so different points in $\widetilde{M}$ can give the same set of poles $\{u^1, \ldots, u^n\}$. One may ask what the relationship is between the reflections $R_1, \ldots, R_n$ defined at a point $p$ and the reflections $\widetilde{R}_1, \ldots, \widetilde{R}_n$ defined at the point $\widetilde{p}$, when they have the same set of poles. To analyse this question, we introduce the following action of the fundamental group of $\widetilde{M}/S_n$. Let $\gamma$ be a continuous loop in $\widetilde{M}/S_n$. Then, the loop $\gamma$ can be "lifted" to a continuous path $\widetilde{\gamma}$ in $\widetilde{M}$ as the following lemma makes precise.

**Lemma 2.2.114.** *Let $\widetilde{X}$ be a cover of a connected and locally connected topological space $X$ with covering $\pi : \widetilde{X} \to X$. Fix $x \in X$ and $\widetilde{x} \in \widetilde{X}$ such that $\pi(\widetilde{x}) = x$. Then, for each continuous loop $\gamma : [0,\, 1] \to X$ with $\gamma(0) = x$, there exists a unique continuous path $\widetilde{\gamma} : [0,\, 1] \to \widetilde{X}$ such that $\widetilde{\gamma}(0) = \widetilde{x}$ and $\pi \circ \widetilde{\gamma} = \gamma$. Moreover, if $\gamma' : [0,\, 1] \to X$ is homotopic to $\gamma$, then $\widetilde{\gamma}' : [0,\, 1] \to \widetilde{X}$ is homotopic to $\widetilde{\gamma}$.*

Let $\widetilde{\gamma}$ in $\widetilde{M}$ be the lift of the loop $\gamma$ in $\widetilde{M}/S_n$ and let $p = \widetilde{\gamma}(0)$. The maps $u^i \circ \widetilde{\gamma}$ are continuous paths in $\mathbb{C}$. Let $z_0 \in \mathbb{C}$ such that $z_0 \notin \{u^i(\widetilde{\gamma}(t)) \mid t \in [0,\, 1],\, i = 1, \ldots, n\}$. Consider a loop $\sigma$ in $\mathbb{C}\backslash\{u^1(p), \ldots, u^n(p)\}$ starting and ending at the point $z_0$. Then, there exists a continuous map $\Sigma : [0,\, 1]^2 \to \mathbb{C}$ such that $\Sigma(t,\, 0) = \sigma(t)$, $\Sigma(t,\, s) \neq u^i(\widetilde{\gamma}(s))$ and $\Sigma(0,\, s) = \Sigma(1,\, s) = z_0$ for all $t,\, s \in [0,\, 1]$. Since $\gamma$ is a loop, the set of points $\{u^1(\widetilde{\gamma}(1)), \ldots, u^n(\widetilde{\gamma}(1))\}$ is equal to the set of points $\{u^1(p), \ldots, u^n(p)\}$, thus the map $\Sigma(t,\, 1)$ is a continuous loop in $\mathbb{C} \setminus \{u^1(p), \ldots, u^n(p)\}$ starting and ending at the point $z_0$. Moreover, the continuous loop $\Sigma(t,\, 1)$ is uniquely defined up to homotopy. Thus the fundamental group of $\widetilde{M}/S_n$ can be thought of as acting on the fundamental group of $\mathbb{C}\backslash\{u^1(p), \ldots, u^n(p)\}$ by the map $\sigma(t) \mapsto \Sigma(t,\, 1)$. Thus, by the monodromy representation, the fundamental group of $\widetilde{M}/S_n$ acts on the (local) monodromy group at the point $p \in \widetilde{M}$.

Now, the fundamental group of $\widetilde{M}/S_n$ is isomorphic to the braid group $\mathcal{B}_n$. The braid group $\mathcal{B}_n$ on $n$ strings is defined as the group generated by the elements $\sigma_i$, $i = 1, \ldots, n-1$ such that $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ when $|i - j| = 1$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ otherwise. The braid group has a natural action on $W(M)^n$ as the following definition describes.

**Definition 2.2.115.** Let $G$ be a group and $\mathcal{B}_k$ be the braid group on $k$ strings. Then, there is a natural action of $\mathcal{B}_k$ on $G^k$ where the generator $\sigma_i \in \mathcal{B}_k$ acts as

$$\sigma_i(g_1, \ldots, g_i, g_{i+1}, \ldots, g_k) = \sigma_i(g_1, \ldots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \ldots, g_k),$$

where $i = 1, \ldots, k-1$. This is known as the *Hurwitz action.*

Note that the product $g_1 \cdots g_n \in G$ is preserved by the Hurwitz action. These two actions coincide when we consider the Hurwitz action on the subset of generating reflections, as the following theorem makes clear.

**Theorem 2.2.116.** [16] *Let $W(M)$ be the monodromy group for a semisimple Frobenius manifold $M$ and let $R_1, \ldots, R_n$ be the image, under the monodromy representation, of the loops $\gamma_1, \ldots, \gamma_n$ around the points $u^1(p), \ldots, u^n(p)$, respectively. Then, the action of the fundamental group of $\widetilde{M}/S_n$ on the orbit of $(R_1, \ldots, R_n) \in W(M)^n$ is the Hurwitz action* *restricted onto this orbit, where the generators $\sigma_i$ are given by the loops in $\widetilde{M}/S_n$ whose lifts $\widetilde{\sigma}_i$ into $\widetilde{M}$ swap only $u^i(p)$ and $u^{i+1}(p)$.*

Now, let us consider the monodromy groups of semisimple Frobenius manifolds with algebraic prepotentials. If a Frobenius manifold $M$ has an algebraic prepotential, then its structure constants $c^{\gamma}_{\alpha\beta}(t)$ must also be algebraic. From theorem (2.2.78), we know that the canonical coordinates of $M$ are roots of the characteristic equation

$$\det(g^{\alpha\beta}(t) - u^i(t)\eta^{\alpha\beta}) = 0.$$

Since $g^{\alpha\beta}(t) = E^{\gamma}(t)c^{\alpha\beta}_{\gamma}(t)$ is algebraic in the $t$ coordinates, the canonical coordinates $u^i$ must also be algebraic in the $t$ coordinates. A choice of flat coordinates $x^1_z, \ldots, x^n_z$ of $g^{\alpha\beta}_z$ at a particular point $p \in \widetilde{M}$ gives rise to a fundamental matrix for the system (2.2.73) with columns $\xi^1, \ldots, \xi^n$ given by the equations (2.2.72). We then have the corresponding monodromy matrices $M_1, \ldots, M_n$ from the monodromy representation of the loops around the points $u^1(p), \ldots, u^n(p)$. As we consider a loop $\sigma \in \widetilde{M}/S_n$, the set of flat coordinates $x^1_z, \ldots, x^n_z$ changes to a new branch $\widetilde{x}^1_z, \ldots, \widetilde{x}^n_z$. It has the associated fundamental matrix with columnns $\widetilde{\xi}^1, \ldots, \widetilde{\xi}^n$ and monodromy matrices $\widetilde{M}_1, \ldots, \widetilde{M}_n$. Since in the algebraic case there are only finitely many possible branches for $\widetilde{x}^1_z, \ldots, \widetilde{x}^n_z$ we have finitely many possibilities for the corresponding $n$-tuples of monodromy matrices $(\widetilde{M}_1, \ldots, \widetilde{M}_n)$. Thus, the Hurwitz action on these monodromy matrices must have a finite orbit. It is a result of Michel [41] that if the orbit of a set of reflections by the Hurwitz action is finite, then the group generated by those elements must be finite. The algebraic Frobenius manifolds then must have finite monodromy groups. In 2000, Dubrovin and Mazzocco [18] found all 3-dimensional semisimple algebraic Frobenius manifolds using the fact that the Hurwitz action had finite orbits. They found 5 finite orbits of the Hurwitz action and related each of these to an algebraic Frobenius manifold, 3 of which were polynomial. We will discuss their discoveries in more detail in Subsection 3.2.1.

## 2.3 Finite Coxeter groups

**Definition 2.3.1.** Let $V$ be an $n$-dimensional real vector space with an inner product $(\cdot\,,\cdot)$, and let $R \subseteq V$ be a finite subset of nonzero elements that spans $V$. We call $R$ a *root system* if for every pair of elements $\alpha, \beta \in R$, the vector $s_\alpha(\beta) := \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in R$, and if $\beta = \lambda\alpha$ for some $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$.

Root systems originated in the study of Lie algebras, where they were given the additional constraint that the constants $2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ be integers for all $\alpha, \beta \in R$. If a root system satisfies this condition we call it *crystallographic*, in general we will not restrict ourselves to crystallographic root systems.

**Definition 2.3.2.** Let $R$ be a root system in the $n$-dimensional vector space $V$ and let $S \subseteq R$ be a set of $n$ vectors such that any element of $R$ can be written as a linear combination of elements in $S$ with only non-positive or only non-negative coefficients. We call $S$ a set of *simple roots.*

**Definition 2.3.3.** Let $R$ be a root system for an $n$-dimensional real vector space $V$. For each element $\alpha \in R$ we can define an invertible linear map $s_\alpha : V \to V$, which we call a reflection, where

$$s_\alpha(x) := x - 2\frac{(\alpha,\, x)}{(\alpha,\, \alpha)}\alpha,$$

for all $x \in V$. The *finite Coxeter group* $W$ for the root system $R$ is the subgroup of $GL(V)$ generated by the reflections $\{s_\alpha \mid \alpha \in R\}$. We say that $n$ is the *rank* of the finite Coxeter group $W$. If $R$ is crystallographic, then we call $W$ a *Weyl group.*

Let $S$ be a set of simple roots of a root system $R$ for the vector space $V$. The group generated by the reflections $\{s_\alpha \mid \alpha \in S\}$ is finite Coxeter group $W$ for the root system $R$.

**Definition 2.3.4.** Let $R$ be a root system in a real vector space $V$ and let $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ be the complexification of $V$. For any $\alpha \in R$, the set of points in $V_\mathbb{C}$ that are orthogonal to $\alpha$, namely the set of elements

$$\Pi_\alpha = \{v \in V_\mathbb{C} \mid (\alpha,\, v) = 0\}$$

is called the *mirror* or *hyperplane* associated to the root $\alpha$.

**Example 2.3.5.** Let $V = \mathbb{R}^2$ with the Euclidean inner product $(\,\cdot\,,\cdot\,)$. Let $e_1 = (1,\, 0)$, $e_2 = (0,\, 1)$ be the standard orthonormal basis of $V$. Let $m \in \mathbb{Z}_{\geq 2}$ and define

$$R := \left\{ \sin\left(\frac{k\pi}{m}\right) e_1 + \cos\left(\frac{k\pi}{m}\right) e_2 \,\middle|\, 0 \leq k \leq 2m-1 \right\}. \tag{2.3.1}$$

Then, $R$ is a root system. We can choose simple roots in $R$ to be the vectors with $k = 0$ and $k = 1$. The finite Coxeter group associated to $R$ is the dihedral group of order $2m$. This group acts on a regular $m$-gon by rotations and reflections.

Finite Coxeter groups have another formulation in the combinatorial setting, where Coxeter groups are defined as groups generated by involutions with orders defined for products of these involutions, and finite Coxeter groups are simply Coxeter groups of finite size. Moreover, finite Coxeter groups are groups generated by reflections in finite-dimensional real inner product spaces, and all finite groups that are generated by reflections in finite-dimensional real inner product spaces are finite Coxeter groups. Also, for any finite Coxeter group $W$ there exists a root system $R$ such that $W$ is the finite Coxeter group for $R$.

**Definition 2.3.6.** Let $G$ be a group, we say that $G$ is *reducible* if there exists groups $H$ and $K$ such that $G$ is isomorphic to the product $H \times K$. We say that $G$ is *irreducible*, if it is not reducible.

If $W$ is a finite Coxeter group, then its root system is *reducible* if and only if it can be expressed as the disjoint union of two root subsystems. The finite irreducible Coxeter groups have been classified [32] and belong to one of the following families: $A_n\,(n \geq 1)$, $B_n\,(n \geq 2)$, $D_n\,(n \geq 4)$, $E_6$, $E_7$, $E_8$, $F_4$, $H_3$, $H_4$, $I_2(m)\,(m \geq 5)$, where the subscript denotes the rank of the Coxeter group. This classification is aided by the use of Dynkin diagrams, which are graphs that encode data specific to Coxeter groups and allow for the use of graph theory to deduce the classification.

**Definition 2.3.7.** Let $W$ be a finite Coxeter group for a root system $R$, and let $S$ be a set of simple roots in $R$. An element $c = \prod_{\alpha \in S} s_\alpha$ is called a *Coxeter element* of $W$. The order $h$ of Coxeter elements is always the same, and is called the *Coxeter number* of $W$.

The Coxeter number $h$ can be quickly computed if the size of the root system $R$ is known. Namely, if $n$ is the rank of the finite Coxeter group, then $h = \frac{|R|}{n}$.

**Definition 2.3.8.** Let $W$ be a finite Coxeter group of rank $n$. We say that $w \in W$ is a *quasi-Coxeter element* if there exist reflections $t_1, \ldots, t_n$ such that

$$w = t_1 \cdots t_n$$

and $t_1, \ldots, t_n$ generate the whole Coxeter group $W$.

All Coxeter elements are obviously quasi-Coxeter elements. Quasi-Coxeter elements are also known as *primitive elements* of the Coxeter group and have an alternative definition as elements which do not belong to any subgroups of $W$ which are themselves Coxeter groups.

**Definition 2.3.9.** [49] Let $W$ be a finite Coxeter group with complexified reflection representation $V_\mathbb{C}$ and root system $R$. We say that $w \in W$ is *regular* if it has an eigenvector $v \in V_\mathbb{C}$ that is not contained within any of the mirrors of $R$, meaning that $v \notin \Pi_\alpha$ for all $\alpha \in R$.

Regularity is a preserved property under conjugation, thus one can speak of regular conjugacy classes in Coxeter groups. Quasi-Coxeter elements of Coxeter groups can be regular, indeed the Coxeter elements are always regular and all quasi-Coxeter elements of $H_3$ are regular. However, not all quasi-Coxeter elements are regular, for example $H_4$ has 11 quasi-Coxeter conjugacy classes of which only 10 are regular.

**Definition 2.3.10.** Let $W$ be a finite irreducible Coxeter group of rank $n$ acting on its reflection representation $V$ with inner product $(\cdot, \cdot)$ and orthonormal basis $e_1, \ldots, e_n$. Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $V$ and define functions $x_1, \ldots, x_n$ on $V_{\mathbb{C}}$ as

$$x_j(v \otimes 1 + w \otimes i) := (v, e_j) + i(w, e_j), \tag{2.3.2}$$

for all $v, w \in V$ and all $j = 1, \ldots, n$. We call the functions $x_1, \ldots, x_n$, *flat coordinates* for $W$.

The finite Coxeter group $W$ acts on the set of polynomials $\mathbb{C}[x_1, \ldots, x_n]$ as follows:

$$(w \cdot p)(v) = p(w^{-1} \cdot v)$$

where $w \in W$, $p \in \mathbb{C}[x_1, \ldots, x_n]$ and $v \in V_{\mathbb{C}}$, where the action of $W$ on $V_{\mathbb{C}}$ is a $\mathbb{C}$-linear extension of the action of $W$ on $V$. We denote the algebra of polynomials which are invariant under this action as $\mathbb{C}[x_1, \ldots, x_n]^W$ and a set of homogeneous generators of the algebra $\mathbb{C}[x_1, \ldots, x_n]^W$ are called *basic invariants* of $W$.

**Theorem 2.3.11.** [32] *Let $W$ be a finite irreducible Coxeter group of rank $n$ and let $x_1, \ldots, x_n$ be flat coordinates for $W$, as given in relation (2.3.2). Then, a set of $n$ basic invariants $y_1, \ldots, y_n$ exists and we have an algebra isomorphism*

$$\mathbb{C}[y_1, \ldots, y_n] \cong \mathbb{C}[x_1, \ldots, x_n]^W.$$

*Moreover, the degrees $d_i^W$ of the basic invariants $y_i$ do not depend on the choice of the basic invariants.*

When we consider the degrees of the basic invariants, we typically assume that $y_i$ has degree $d_i^W$ and that $d_1^W \geq d_2^W \geq \cdots \geq d_{n-1}^W > d_n^W = 2$. For the next Lemma, recall the notation and definitions for the Laplacian and gradient of a smooth function from Section 2.1.6.

**Lemma 2.3.12.** *Let $W$ be a finite irreducible Coxeter group of rank $n$ and let $x_1, \ldots, x_n$ be flat coordinates for $W$. If $p, q \in \mathbb{C}[x_1, \ldots, x_n]^W$, then $\Delta(p), \Delta(q) \in \mathbb{C}[x_1, \ldots, x_n]^W$ and $(\nabla(p), \nabla(q)) \in \mathbb{C}[x_1, \ldots, x_n]^W$.*

*Proof.* The first claim follows from the invariance of $\Delta$ under orthogonal transformations. For the second claim, we have

$$(\nabla(p), \nabla(q)) = \frac{1}{2}\Big(\Delta(pq) - \Delta(p)q - p\Delta(q)\Big),$$

which implies the second statement, since $\mathbb{C}[x_1, \ldots, x_n]^W$ is an algebra over $\mathbb{C}$. □

We will use the following statement in Chapter 4 to narrow our search for relations between the two sets of flat coordinates on some Frobenius manifolds.

**Proposition 2.3.13.** *Let $W$ be a finite irreducible Coxeter group of rank $n$ and let $x_1, \dots, x_n$ be flat coordinates for $W$. Then, there exists a set of basic invariants $Y_1, \dots, Y_n \in \mathbb{C}[x_1, \dots, x_n]^W$ such that $\Delta(Y_n) = 1$ and $\Delta(Y_j) = 0$ for $j = 1, \dots, n-1$.*

*Proof.* Let $y_1, \dots, y_n \in \mathbb{C}[x_1, \dots, x_n]^W$ be a set of basic invariants. Define $Y_n$ as

$$Y_n := \frac{1}{2n} \sum_{i=1}^{n} x_i^2,$$

so we have $\Delta(Y_n) = 1$. Now, it is well-known (see e.g. [23]) that

$$\mathbb{C}[x_1, \dots, x_n] = Y_n \, \mathbb{C}[x_1, \dots, x_n] \oplus H, \tag{2.3.3}$$

where $H = \mathrm{Ker}(\Delta)$ is the vector space of harmonic polynomials. Consider the vector spaces $V_k$ of homogeneous $W$-invariant polynomials of degree $k$ and the linear maps

$$\Delta : \mathrm{Span}\{y_j, Y_n \, V_{d_j^W - 2}\} \to V_{d_j^W - 2},$$

for $j = 1, \dots, n-1$. Since the dimension of the domain is larger than the dimension of the range, there must be a nontrivial kernel that is not contained in $Y_n \, V_{d_j^W - 2}$ by the direct sum decomposition (2.3.3). Let $Y_j$ be a nonzero element of this kernel. The polynomials $Y_j$, $1 \le j \le n$, are homogeneous and each basic invariant $y_i$ can be expressed as a polynomial in $Y_j$, thus $Y_j$ generate $\mathbb{C}[x_1, \dots, x_n]^W$ and we have that $\Delta(Y_j) = 0$ for all $j \le n-1$. □

## 2.4 Lie algebras

In this section we go over some concepts involving Lie algebras that we will use in Subsection 3.2.2 to construct algebraic Frobenius manifolds. This includes constructing local Poisson brackets on semisimple Lie algebras and reducing these onto a suitable affine subspace, which we call a Slodowy slice.

**Definition 2.4.1.** Let $\mathfrak{g}$ be a Lie algebra, then we define the *Killing form* of $\mathfrak{g}$ to be the bilinear form $\langle \cdot, \cdot \rangle$ defined as

$$\langle x, \, y \rangle := \mathrm{Trace}(\mathrm{ad}_x \circ \mathrm{ad}_y),$$

where $x, \, y \in \mathfrak{g}$.

The Killing form is a useful tool for Lie algebras and has some nice properties. For example, it is always symmetric and has the so-called *invariance property*:

$$\langle [x,\, y],\, z\rangle = \langle x,\, [y,\, z]\rangle,$$

for all $x,\, y,\, z \in \mathfrak{g}$. Moreover, the Killing form is non-degenerate if and only if $\mathfrak{g}$ is semisimple, and any symmetric bilinear form with the invariance property on a simple Lie algebra is some scalar multiple of the Killing form.

**Definition 2.4.2.** Let $\mathfrak{g}$ be a semisimple Lie algebra, then there exists a root system $R$ which we can associate to $\mathfrak{g}$. This is always a crystallographic root system, and the Coxeter group $W(R)$ is called the *Weyl group* of the Lie algebra $\mathfrak{g}$.

The existence of the root system requires the existence of Cartan subalgebras, which we will not prove here. The decomposition of a semisimple Lie algebra into its simple components also decomposes its Weyl group into a product of irreducible Weyl groups. Thus, the simple Lie algebras can be classified much like the irreducible Weyl groups and belong to one of the following families: $A_n(n \geq 1)$, $B_n(n \geq 2)$, $C_n(n \geq 3)$, $D_n(n \geq 4)$, $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$, where the subscripts denote the rank of their Weyl groups and we note that $C_n$ has the same Weyl group as $B_n$ for all $n \geq 3$. The *rank* of a simple Lie algebra is the same as the rank of its Weyl group.

**Definition 2.4.3.** Let $\mathfrak{g}$ be a Lie algebra, and $x \in \mathfrak{g}$. We say that $x$ is *nilpotent* if there exists $n \in \mathbb{Z}_{>0}$ such that

$$\mathrm{ad}_x^n(y) = 0,$$

for all $y \in \mathfrak{g}$.

The following theorem is known as the Jacobson-Morozov theorem.

**Theorem 2.4.4.** *Let $e \in \mathfrak{g}$ be a nilpotent element of a semisimple Lie algebra $\mathfrak{g}$. Then, there exists elements $f, h \in \mathfrak{g}$ such that*

$$[e,\, f] = h, \quad [h,\, e] = 2e, \quad [h,\, f] = -2f. \tag{2.4.1}$$

We call any set of elements $\{e,\, f,\, h\}$ in a Lie algebra that satisfy the relations (2.4.1), an $\mathfrak{sl}_2$-*triple*. What the Jacobson-Morozov theorem states is that any nilpotent element of a semisimple Lie algebra can be extended to an $\mathfrak{sl}_2$-triple.

**Proposition 2.4.5.** *Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\{e,\, f,\, h\} \subseteq \mathfrak{g}$ be an $\mathfrak{sl}_2$-triple, then we can decompose $\mathfrak{g}$ into integer eigenspaces of* $\mathrm{ad}_h$ :

$$\mathfrak{g} = \bigoplus_{k=-\infty}^{\infty} \mathfrak{g}_k,$$

*where* $\mathfrak{g}_k = \{x \in \mathfrak{g} \mid [h, x] = kx\}$.

Our choice of the elements $f$ and $h$ in an $\mathfrak{sl}_2$-triple containing $e$ are conjugate to one another via some element in the centraliser of $e$. The properties of the eigenspaces of $\mathrm{ad}_h$ that we are interested in are unchanged by conjugation, and thus do not depend on our choice of $f$ and $h$.

**Definition 2.4.6.** Let $\{e, f, h\}$ be an $\mathfrak{sl}_2$-triple inside of a semisimple Lie algebra $\mathfrak{g}$, we say that $e$ is *distinguished* if $\dim(\mathfrak{g}_0) = \dim(\mathfrak{g}_2)$.

**Proposition 2.4.7.** *Let* $\{e, f, h\}$ *be an* $\mathfrak{sl}_2$*-triple inside of a semisimple Lie algebra* $\mathfrak{g}$ *such that* $e$ *is distinguished. Then,* $\mathfrak{g}_k = \{0\}$ *for all odd* $k \in \mathbb{Z}$.

Suppose we have an $\mathfrak{sl}_2$-triple $\{e, f, h\}$ such that $e$ is a distinguished nilpotent element. Let us denote $\eta_r \in \mathbb{Z}$ such that the maximum eigenvalue of $\mathrm{ad}_h$ is $2\eta_r$, so $\mathfrak{g}_{2\eta_r} \neq 0$, but $\mathfrak{g}_k = 0$ for all $k > 2\eta_r$.

**Definition 2.4.8.** Let $\mathfrak{g}$ be a simple Lie algebra of rank $r$, and let $\{e, f, h\}$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. We say that $e$ is *regular* if $\dim(\mathfrak{g}_0) = r$, and we say that $e$ is *subregular* if $\dim(\mathfrak{g}_0) = r + 2$.

Distinguished nilpotent elements of simple Lie algebras will be used in Section 3.2.2 to construct algebraic Frobenius manifolds. The regular nilpotent elements will correspond to polynomial Frobenius manifolds, while the subregular elements will allow for the construction of non-polynomial algebraic Frobenius manifolds. Every simple Lie algebra contains not only a regular nilpotent element but also a subregular nilpotent element [4].

**Definition 2.4.9.** Let $\mathfrak{g}$ be a semisimple Lie algebra and let $x \in \mathfrak{g}$. We say that $x$ is *semisimple* if the linear map $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is diagonalisable. Furthermore, let $\{e, f, h\} \subseteq \mathfrak{g}$ be an $\mathfrak{sl}_2$-triple, we say that $e$ is of *semisimple type* if there exists an element $K_1 \in \mathfrak{g}_{-2\eta_r}$ such that $e + K_1$ is a semisimple element of $\mathfrak{g}$.

A nilpotent, regular element of a simple Lie algebra can be shown to also be a distinguished element of semisimple type. If $\mathfrak{g}$ is a Lie algebra, and $x \in \mathfrak{g}$ is arbitrary, then we can denote the centraliser of $x$ in $\mathfrak{g}$ as $\mathfrak{g}^x$. By definition, we have that $\mathfrak{g}^x = \ker(\mathrm{ad}_x)$.

**Definition 2.4.10.** Let $e$ be a nilpotent element of semisimple type and let $K_1 \in \mathfrak{g}_{-2\eta_r}$ such that $h' = e + K_1$ is a semisimple element of $\mathfrak{g}$. The *opposite Cartan subalgebra* is the subalgebra $\mathfrak{h}' := \mathfrak{g}^{h'}$.

As the name suggests, an opposite Cartan subalgebra is a Cartan subalgebra of the Lie algebra $\mathfrak{g}$, meaning that it is a nilpotent subalgebra such that if $[X, Y] \in \mathfrak{h}'$ for all $X \in \mathfrak{h}'$, then $Y \in \mathfrak{h}'$.

**Definition 2.4.11.** Let $\{e,\, f,\, h\}$ be an $\mathfrak{sl}_2$-triple inside of a semisimple Lie algebra $\mathfrak{g}$. The set

$$S := e + \mathfrak{g}^f = \{x \in \mathfrak{g} \mid [x - e,\, f] = 0\}$$

is called a *Slodowy slice.*

### 2.4.1 Poisson structure on Slodowy slices

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ with Killing form $\langle \cdot, \cdot \rangle$. Then, $\mathfrak{g}$ is a complex manifold and we can define Poisson structures on it. Let $\xi^1, \dots, \xi^n$ be a basis of $\mathfrak{g}$ and let $x^1, \dots, x^n$ be coordinates on $\mathfrak{g}$ defined as:

$$x^i(x) = \langle \xi^i,\, x \rangle, \tag{2.4.2}$$

for each $i = 1, \dots, n$. The non-degeneracy of the Killing form means that these do indeed define coordinates on $\mathfrak{g}$.

**Lemma 2.4.12.** *Let $f \in C^\infty(\mathfrak{g})$, then the map $\nabla f : \mathfrak{g} \to \mathfrak{g}$ defined as*

$$\nabla f(x) := \frac{\partial f}{\partial x^i}(x)\xi^i$$

*satisfies the relation*

$$\left.\frac{d}{dt}\right|_{t=0} f(x + ty) = \langle \nabla f(x),\, y \rangle, \tag{2.4.3}$$

*for all $x,\, y \in \mathfrak{g}$.*

*Proof.* We have that

$$\frac{d}{dt} f(x + ty) = \frac{\partial f}{\partial x^i}(x + ty)\frac{d}{dt}x^i(x + ty) = \frac{\partial f}{\partial x^i}(x + ty)\frac{d}{dt}\langle \xi^i,\, x + ty \rangle = \frac{\partial f}{\partial x^i}(x + ty)\langle \xi^i,\, y \rangle,$$

and so relation (2.4.3) obviously follows. □

We define the following Poisson brackets on $C^\infty(\mathfrak{g})$ :

$$\{f,\, g\}_1^{\mathfrak{g}}(x) = \langle \nabla f(x),\, [\nabla g(x),\, K] \rangle, \tag{2.4.4}$$

$$\{f,\, g\}_2^{\mathfrak{g}}(x) = \langle \nabla f(x),\, [\nabla g(x),\, x] \rangle, \tag{2.4.5}$$

where $K \in \mathfrak{g}$ is a constant element of the Lie algebra. The first Poisson bracket is called the *constant Lie-Poisson bracket* and the second Poisson bracket is called the *standard Lie-Poisson bracket.*

**Lemma 2.4.13.** *Let $I \in \mathcal{F}(\mathfrak{g})$ be a local functional, then the map $\delta I : \mathcal{L}(\mathfrak{g}) \to \mathcal{L}(\mathfrak{g})$*

*defined as*

$$\delta I(\phi) := \frac{\delta I}{\delta x^i(s)}(\phi)\xi^i,$$

*satisfies the following relation*

$$\left.\frac{d}{dt}\right|_{t=0} I(\phi + t\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \langle \delta I(\phi(s)),\, \varphi(s)\rangle\, ds, \tag{2.4.6}$$

*for all* $\phi,\, \varphi \in \mathcal{L}(\mathfrak{g})$.

*Proof.* We have that

$$\begin{aligned}
\frac{d}{dt} I(\phi + t\varphi) &= \frac{d}{dt}\frac{1}{2\pi}\int_0^{2\pi} P(\phi + t\varphi)(s)\, ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=0}^{\infty} \frac{\partial P}{\partial x^{i(j)}}(\phi + t\varphi)(s)\frac{d}{dt}x^{i(j)}(\phi + t\varphi)(s)\, ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=0}^{\infty} \frac{\partial P}{\partial x^{i(j)}}(\phi + t\varphi)(s)\frac{d}{dt}\frac{d^j}{ds^j}(x^i(\phi(s) + t\varphi(s)))\, ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=0}^{\infty} \frac{\partial P}{\partial x^{i(j)}}(\phi + t\varphi)(s)\frac{d^j}{ds^j}\frac{d}{dt}\langle \xi^i,\, \phi(s) + t\varphi(s)\rangle\, ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=0}^{\infty} \frac{\partial P}{\partial x^{i(j)}}(\phi(s) + t\varphi(s))\frac{d^j}{ds^j}\langle \xi^i,\, \varphi(s)\rangle\, ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=0}^{\infty} (-1)^j \frac{d^j}{ds^j}\frac{\partial P}{\partial x^{i(j)}}(\phi(s) + t\varphi(s))\langle \xi^i,\, \varphi(s)\rangle\, ds,
\end{aligned}$$

and so relation (2.4.6) follows. □

We define the following Poisson brackets on $\mathcal{F}(\mathfrak{g})$ :

$${}^{\mathfrak{g}}\{I_1,\, I_2\}_1(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \langle \delta I_1(\phi(s)),\, [\delta I_2(\phi(s)),\, K]\rangle\, ds, \tag{2.4.7}$$

$${}^{\mathfrak{g}}\{I_1,\, I_2\}_2(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \langle \delta I_1(\phi(s)),\, [\delta I_2(\phi(s)),\, \phi(s)] + \frac{d}{ds}\delta I_2(\phi(s))\rangle\, ds, \tag{2.4.8}$$

where $K \in \mathfrak{g}$ is the same constant element that we choose in the definition of the constant Lie-Poisson bracket $\{\,\cdot, \cdot\,\}_1^{\mathfrak{g}}$. One can check that these are actually local Poisson brackets on $\mathfrak{g}$, and so we can decompose them into their graded homogeneous parts.

**Proposition 2.4.14.** *We have that*

$${}^{\mathfrak{g}}\{x^i(s_1),\, x^j(s_2)\}_1^{[-1]} = \{x^i,\, x^j\}_1^{\mathfrak{g}}(s_1)\delta(s_1 - s_2), \tag{2.4.9}$$

$${}^{\mathfrak{g}}\{x^i(s_1),\, x^j(s_2)\}_2^{[-1]} = \{x^i,\, x^j\}_2^{\mathfrak{g}}(s_1)\delta(s_1 - s_2), \tag{2.4.10}$$

*where* $\{\cdot,\cdot\}_1^{\mathfrak{g}}$ *is the constant Lie-Poisson bracket and* $\{\cdot,\cdot\}_2^{\mathfrak{g}}$ *is the standard Lie-Poisson bracket.*

*Proof.* We see that

$$\begin{aligned}
{}^{\mathfrak{g}}\{x^i(s_1),\, x^j(s_2)\}_1 &= \langle \xi^i,\, [\xi^j,\, K]\rangle \delta(s_1 - s_2),\\
{}^{\mathfrak{g}}\{x^i(s_1),\, x^j(s_2)\}_2 &= c_k^{ij} x^k(s_1)\delta(s_1 - s_2) + \langle \xi^i,\, \xi^j\rangle \delta^{(1)}(s_1 - s_2),
\end{aligned}$$

where $[\xi^i,\, \xi^j] = c_k^{ij}\xi^k$. And thus,

$$\begin{aligned}
{}^{\mathfrak{g}}\{x^i(s_1),\, x^j(s_2)\}_1^{[-1]}(\phi) &= \langle \xi^i,\, [\xi^j,\, K]\rangle \delta(s_1 - s_2) = \{x^i,\, x^j\}_1^{\mathfrak{g}}(\phi(s_1))\delta(s_1 - s_2),\\
{}^{\mathfrak{g}}\{x^i(s_1),\, x^j(s_2)\}_2^{[-1]}(\phi) &= c_k^{ij} x^k(\phi(s_1))\delta(s_1 - s_2) = \langle c_k^{ij}\xi^k,\, \phi(s_1)\rangle \delta(s_1 - s_2)\\
&= \langle [\xi^i,\, \xi^j],\, \phi(s_1)\rangle \delta(s_1 - s_2) = \langle \xi^i,\, [\xi^j,\, \phi(s_1)]\rangle \delta(s_1 - s_2)\\
&= \{x^i,\, x^j\}_2^{\mathfrak{g}}(\phi(s_1))\delta(s_1 - s_2),
\end{aligned}$$

for all $\phi \in \mathcal{L}(M)$, which completes the proof. □

Let $N$ be a closed submanifold of $M$. Suppose $\{\cdot,\cdot\}^M$ is a Poisson bracket on $M$ and we want to restrict this to a Poisson bracket $\{\cdot,\cdot\}^N$ on $N$, meaning that

$$\{f|_N,\, g|_N\}^N = \{f,\, g\}^M|_N,$$

for all $f,\, g \in C^\infty(M)$. In general, Poisson brackets cannot be restricted onto submanifolds since we may have that $\{f,\, g\}^M|_N \neq 0$ but $f|_N = 0$. However, it is possible for some Poisson brackets, and so one way of attacking this problem is to alter the Poisson bracket into one which can be restricted without issue. One method of doing so is called *Dirac reduction*. Let $u^1, \ldots, u^n$ be coordinates on $M$ such that $u^1, \ldots, u^r$ are coordinates on $N$ and

$$N = \{m \in M \mid u^{r+1}(m) = \cdots = u^n(m) = 0\}.$$

Suppose the Poisson bracket on $M$ has the form

$$\{u^i,\, u^j\}^M = a^{ij},$$

for some $a^{ij} \in C^\infty(M)$. Then, if the minor matrix $\left(a^{\alpha\beta}\right)_{\alpha,\beta=r+1,\ldots,n}$ is invertible on $N$, we let $s_{\alpha\beta}$ be its inverse on $N$. We can extend $s_{\alpha\beta}$ to all of $M$ by ensuring that $s_{\alpha\beta}$ is independent of the coordinates $u^{r+1}, \ldots, u^n$. Then, we define the Dirac reduction of $\{\cdot,\cdot\}^M$ to be

$$\{u^i,\, u^j\}^{M,N}(m) := a^{ij}(m) - \sum_{\alpha,\beta=r+1}^{n} a^{i\alpha}(m)s_{\alpha\beta}(m)a^{\beta j}(m),$$

for all $m \in M$. We can then restrict this onto $N$ to get the Dirac reduction of $\{\cdot,\cdot\}^M$ onto $N$ which we will denote simply as $\{\cdot,\cdot\}^N$. To see that this is indeed able to be restricted to $N$, note that for $\gamma = r+1,\dots,n$,

$$\begin{aligned}\{u^\gamma,\, u^j\}^{M,N}(x) &= a^{\gamma j}(x) - \sum_{\alpha,\beta=r+1}^{n} a^{\gamma\alpha}(x) s_{\alpha\beta}(x) a^{\beta j}(x)\\ &= a^{\gamma j}(x) - \sum_{\beta=r+1}^{n} \delta^{\gamma}_{\beta} a^{\beta j}(x) = a^{\gamma j}(x) - a^{\gamma j}(x) = 0,\end{aligned}$$

for all $x \in N$.

Suppose now we have a local Poisson bracket ${}^M\{\cdot,\cdot\}$ on $M$. We can do a similar procedure to acquire a local Poisson bracket on a closed submanifold $N$, which we also call a *Dirac reduction*. Suppose the local Poisson bracket on $M$ has the form

$${}^M\{u^i(s_1),\, u^j(s_2)\} = A^{ij}(s_1)\delta(s_1 - s_2),$$

where $A^{ij} = \sum_{k=0}^{N} a^{ij}_k \frac{d^k}{ds^k}$, with $a^{ij}_k \in \mathcal{C}(M)$. If we assume that the minor matrix $\left(A^{\alpha\beta}\right)_{\alpha,\,\beta=r+1,\dots,n}$ is invertible on $N$, we let $S_{\alpha\beta}$ be this inverse matrix on $N$. We can extend $S_{\alpha\beta}$ to all of $M$ by ensuring that $S_{\alpha\beta}$ is independent of the coordinates $u^{r+1}$, $u^n$ and their derivatives. Then we define the Dirac reduction of the local Poisson bracket to be

$${}^{M,N}\{u^i(s_1),\, u^j(s_2)\} := \left(A^{ij}(s_1) - \sum_{\alpha,\,\beta=r+1}^{n} A^{i\alpha}(s_1) S_{\alpha\beta}(s_1) A^{\beta j}(s_1)\right)\delta(s_1 - s_2).$$

This local Poisson bracket can then be restricted to $N$ without issue, and we denote this restriction as ${}^N\{\cdot,\cdot\}$, which we refer to as the Dirac reduction of ${}^M\{\cdot,\cdot\}$ onto $N$.

**Proposition 2.4.15.** *Let ${}^M\{\cdot,\cdot\}$ be a local Poisson bracket on a complex manifold $M$. Suppose in the graded decomposition of the local Poisson bracket, we have*

$${}^M\{u^i(s_1),\, u^j(s_2)\}^{[-1]} = h^{ij}(s_1)\delta(s_1 - s_2), \tag{2.4.11}$$

$${}^M\{u^i(s_1),\, u^j(s_2)\}^{[0]} = \eta^{ij}(s_1)\delta^{(1)}(s_1 - s_2) + {}^{\eta}\Gamma^{ij}_k(s_1)\widehat{x}^{k(1)}(s_1)\delta(s_1 - s_2), \tag{2.4.12}$$

*where $h^{ij}$, $\eta^{ij}$, ${}^{\eta}\Gamma^{ij}_k \in C^\infty(M)$. Let $N$ be a closed submanifold of $M$ defined by the equations $u^{r+1} = \cdots = u^n = 0$. Then the Dirac reduction of the local Poisson bracket onto $N$ has a graded decomposition where*

$$\begin{aligned}{}^N\{u^i(s_1),\, u^j(s_2)\}^{[-1]} &= \widetilde{h}^{ij}(s_1)\delta(s_1 - s_2),\\ {}^N\{u^i(s_1),\, u^j(s_2)\}^{[0]} &= \widetilde{\eta}^{ij}(s_1)\delta^{(1)}(s_1 - s_2) + {}^{\widetilde{\eta}}\Gamma^{ij}_k(s_1)\widehat{x}^{k(1)}(s_1)\delta(s_1 - s_2).\end{aligned}$$

*Let $s_{\alpha\beta}$ be the inverse of the minor matrix $\left(h^{\alpha\beta}\right)_{\alpha\beta=r+1,\dots,n}$ which we assume to be non-degenerate on $N$. We then have that*

$$\begin{aligned}
\widetilde{h}^{ij} &= h^{ij} - h^{i\alpha} s_{\alpha\beta} h^{\beta j}, \\
\widetilde{\eta}^{ij} &= \eta^{ij} - \eta^{i\alpha} s_{\alpha\beta} h^{\beta j} + h^{i\alpha} s_{\alpha\beta} \eta^{\beta\gamma} s_{\gamma\delta} h^{\delta j} - h^{i\alpha} s_{\alpha\beta} \eta^{\beta j}, \\
{}^{\widetilde{\eta}}\Gamma^{ij}_{k} &= {}^{\eta}\Gamma^{ij}_{k} - {}^{\eta}\Gamma^{i\alpha}_{k} s_{\alpha\beta} h^{\beta j} + h^{i\alpha} s_{\alpha\beta} {}^{\eta}\Gamma^{\beta\gamma}_{k} s_{\gamma\delta} h^{\delta j} - h^{i\alpha} s_{\alpha\beta} {}^{\eta}\Gamma^{\beta j}_{k} \\
&\quad - \eta^{i\alpha} s_{\alpha\beta;k} h^{\beta j} - \eta^{i\alpha} s_{\alpha\beta} h^{\beta j}_{;k} + h^{i\alpha} s_{\alpha\beta} \eta^{\beta\gamma} s_{\gamma\delta;k} h^{\delta j} + h^{i\alpha} s_{\alpha\beta} \eta^{\beta\gamma} s_{\gamma\delta} h^{\delta j}_{;k},
\end{aligned}$$

*where summation over the greek letters $\alpha$, $\beta$, $\gamma$, $\delta$ happens only over the range $r+1,\dots,n$.*

We notice immediately that ${}^{N}\{x^i(s_1),\, x^j(s_2)\}^{[-1]} = \{x^i,\, x^j\}^N(s_1)\delta(s_1-s_2)$ and if $h^{i\alpha} = 0$ for all $i = 1,\dots,r$ and $\alpha = r+1,\dots,n$, then

$$\widetilde{h}^{ij} = h^{ij}, \quad \widetilde{\eta}^{ij} = \eta^{ij}, \quad {}^{\widetilde{\eta}}\Gamma^{ij}_{k} = {}^{\eta}\Gamma^{ij}_{k},$$

for all $i,\, j,\, k = 1,\dots,r$.

On the Slodowy slice $S$ we define the Poisson brackets $\{\cdot,\cdot\}^S_1$ and $\{\cdot,\cdot\}^S_2$ by doing Dirac reductions on the constant Lie-Poisson bracket and standard Lie-Poisson bracket, respectively. Also, we do a Dirac reduction on the two local Poisson brackets from $\mathfrak{g}$ onto $S$ to get ${}^S\{\cdot,\cdot\}_1$ and ${}^S\{\cdot,\cdot\}_2$.

**Proposition 2.4.16.** [7] *Let ${}^M\{\cdot,\cdot\}_1$ and ${}^M\{\cdot,\cdot\}_2$ be a pair of compatible local Poisson brackets on a complex manifold $M$ and let $N$ be a closed submanifold of $M$. Then, the Dirac reduction of these local Poisson brackets onto $N$ are compatible.*

Since the local Poisson brackets on $\mathfrak{g}$ are compatible, their Dirac reductions onto $S$ are also compatible, by the above proposition. Also, by Lemma 2.2.64 the Dirac reduction of the Poisson brackets are equal to the $[-1]$ part of the local Poisson brackets and these are also compatible.

# Chapter 3

# Specific classes of Frobenius manifolds

## 3.1 Polynomial Frobenius manifolds

We will let $W$ be a finite irreducible Coxeter group of rank $n$ and let $V$ be its reflection representation, with $V_{\mathbb{C}}$ being the complexification of $V$. Let $x^1, \ldots, x^n$ be flat coordinates for $W$ and let $y^1, \ldots, y^n$ be basic invariants for $W$.

**Definition 3.1.1.** The *space of orbits* or *orbit space* of $W$ is defined to be the space $V_{\mathbb{C}}/W$. The $W$-invariant polynomials in the algebra $\mathbb{C}[x^1, \ldots, x^n]^W$ can be considered as functions on the space of orbits of $W$.

The orbit space for a finite irreducible Coxeter group $W$ is a complex manifold which is diffeomorphic to $\mathbb{C}^n$.

**Definition 3.1.2.** Let $M$ be a space of orbits for $W$, and let $\pi : V_{\mathbb{C}} \to M$ be the natural projection map. We define the *discriminant locus* of $M$ to be the subset $\Sigma \subseteq M$ where

$$\Sigma := \{p \in M \mid |\pi^{-1}(p)| < |W|\}.$$

The inner product on $V$ can be used to define a flat metric on $V_{\mathbb{C}}$ given by

$$(\partial_{x^i}, \partial_{x^j}) := \delta_{ij}.$$

This metric is $W$-invariant and can be pushed forward (using the projection map $\pi$) onto an open dense subset of the space of orbits. The basic invariants of $W$ define a coordinate system on any open chart of $M$.

**Definition 3.1.3.** We define the *intersection form* to be the rank (2, 0) tensor field $g^{\alpha\beta}$ on $M$ given in the $y$ coordinates by

$$g^{\alpha\beta}(y) := (dy^\alpha, dy^\beta) = \sum_{i=1}^{n} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^i}, \tag{3.1.1}$$

for each $\alpha, \beta = 1, \dots, n$.

Note that the right hand side of formula (3.1.1) is a well-defined function on $M$, since by Lemma 2.3.12 it is $W$-invariant and furthermore, it is polynomial in the flat coordinates $x^1, \dots, x^n$ for $W$. Also, it can be shown that the discriminant locus $\Sigma$ can be equivalently defined as the set of points where the intersection form $g^{\alpha\beta}$ is degenerate. On $M \setminus \Sigma$ the intersection form $g^{\alpha\beta}$ is a flat metric.

**Theorem 3.1.4.** [47], [13, Lecture 4] *There exists basic invariants $t^1, \dots, t^n$ defined on $M$ such that the matrix*

$$\eta^{\alpha\beta}(t) := \left(\mathcal{L}_{\partial_{t^1}} g\right)^{\alpha\beta}(t)$$

*is constant and non-degenerate.*

This flat metric $\eta^{\alpha\beta}$ is known as the Saito metric, and the basic invariants $t^1, \dots, t^n$, which are flat coordinates for the Saito metric, are called Saito coordinates.

**Theorem 3.1.5.** [13, Lecture 4] *Let $h$ be the Coxeter number of $W$. Then, the rank $(2, 0)$ tensor fields $\eta^{\alpha\beta}$ and $g^{\alpha\beta}$ form a regular, quasihomogeneous flat pencil of metrics of degree $d = \frac{h-2}{h}$ on $M$ with Euler vector field*

$$E(t) = \frac{1}{h}\sum_{i=1}^{n} d_i^W t^i \partial_{t^i},$$

*unity vector field $e = \partial_{t^1}$ and Egoroff potential $\tau(t) = \eta_{1\alpha} t^\alpha$, where $t = (t^1, \dots, t^n)$ are the Saito coordinates.*

From this theorem, it is clear that $M \setminus \Sigma$ is a Frobenius manifold using Theorem 2.2.51, we call these Frobenius manifolds Coxeter orbit spaces. It can be shown that Coxeter orbit spaces are semisimple Frobenius manifolds and that their prepotentials are polynomial in the $t$ coordinates.

**Example 3.1.6.** [13, Lecture 4] Let $m \in \mathbb{Z}_{\geq 2}$ and recall the root system $R$ in Example 2.3.5 given in formula (2.3.1). The finite Coxeter group $W$ associated to $R$ is a dihedral group of order $2m$ and has Coxeter number $h = m$. Let $x^1$, $x^2$ be the flat coordinates for $W$ and consider basic invariants $t^1$, $t^2$ given by

$$t^1 := (x^1 + ix^2)^m + (x^1 - ix^2)^m, \qquad t^2 := \frac{(x^1)^2 + (x^2)^2}{2m}.$$

These are coordinates on the space of orbits $M = \mathbb{C}^2/W$, and the intersection form in these coordinates is

$$g^{\alpha\beta}(t) = \begin{pmatrix} (2m)^{m+1}(t^2)^{m-1} & t^1 \\ t^1 & \frac{2}{m}t^2 \end{pmatrix}.$$

Moreover one can check that these are Saito coordinates. The Saito metric $\eta^{\alpha\beta}$ and the intersection form $g^{\alpha\beta}$ are thus both flat metrics on $M \setminus \Sigma$, and can be shown to form a regular, quasihomogeneous flat pencil of metrics of degree $d = \frac{m-2}{m}$ on $M$. This quasihomogeneous flat pencil of metrics has Euler vector field

$$E(t) = t^1\partial_{t^1} + \frac{2}{m}t^2\partial_{t^2},$$

unity vector field $e = \partial_{t^1}$ and Egoroff potential $\tau(t) = t^2$. The Frobenius manifold $M \setminus \Sigma$ has charge $d = \frac{m-2}{m}$. The prepotential $F$ of this Frobenius manifold is polynomial in the Saito coordinates and has the form

$$F(t) = \frac{1}{2}(t^1)^2t^2 + \frac{(2m)^{m+1}}{2(m^2-1)}(t^2)^{m+1}.$$

The next theorem gives a partial result in the converse direction to Theorem 3.1.5.

**Theorem 3.1.7.** [31] *Let $M$ be a semisimple Frobenius manifold with metric $\eta$ and Euler vector field $E$. Suppose there exists flat coordinates $t = (t^1, \ldots, t^n)$ for the metric $\eta$ such that*

$$E(t) = \sum_{i=1}^{n} d_i t^i \partial_{t^i},$$

*with $d_i \in \mathbb{R}_{>0}$ and there exists $p \in M$ such that $t(p) = 0$. Then, $M$ uniquely decomposes into a product of Coxeter orbit spaces, each with the same Coxeter number.*

What we see from this theorem is that any semisimple polynomial Frobenius manifold with positive degrees is the product of Coxeter orbit spaces. However, loosening the retriction on the degrees, we are still unclear on what the situation is with semisimple polynomial Frobenius manifolds in general.

## 3.2 Algebraic Frobenius manifolds

The next case to consider beyond polynomial prepotentials could plausibly be the algebraic prepotentials. Indeed, these were some of the first cases to be studied after the polynomial case was considered.

In [18], Dubrovin and Mazzocco found algebraic Frobenius manifolds that could be associated to the Coxeter group $H_3$ and the explicit prepotentials were found for these in [35] by Kato, Mano and Sekiguchi. An explanation of their existence was given by Dubrovin in [18], who explained that their monodromy groups were finite, and one should expect that algebraic Frobenius manifolds would have finite monodromy groups. Moreover, one could associate each of these Frobenius manifolds to a quasi-Coxeter conjugacy class of $H_3$. Recall by Definition 2.3.8 that a quasi-Coxeter element is one that can be written as the product of $n$ reflections, where $n$ is the rank of the Coxeter group such that the reflections also generate the Coxeter group.

It was conjectured by Dinar in [7] that the algebraic Frobenius manifolds (with positive degrees) should be in correspondence with the quasi-Coxeter conjugacy classes of Coxeter groups, which would in turn be the monodromy groups of the algebraic Frobenius manifolds. Dinar attributes this conjecture to Dubrovin, and it is partially discussed in [18].

**Conjecture 3.2.1.** [7] *The irreducible, semisimple Frobenius manifolds with algebraic prepotentials and positive degrees are in one-to-one correspondence with the quasi-Coxeter conjugacy classes of finite irreducible Coxeter groups.*

In [43], Pavlyk presented a general method for constructing algebraic Frobenius manifolds that could be associated to quasi-Coxeter conjugacy classes, and in [7] Dinar gave an alternative method for doing so, which was later shown to be equivalent to Pavlyk's method by Dinar in [7]. These however, were limited to the Weyl groups and only in the case of regular quasi-Coxeter conjugacy classes (see Definition 2.3.9). The authors also gave particular values for the degrees of the corresponding Frobenius manifolds which depended solely on the properties of the quasi-Coxeter elements considered.

Using this as inspiration, Douvropoulos conjectured [12] what the degrees should be for a conjecturally existing Frobenius manifold associated with a regular quasi-Coxeter conjugacy class in a finite irreducible Coxeter group. He also gave a combinatorial interpretation for finding the algebraic degree of the prepotentials [12]. This was then used in [48] by Sekiguchi to construct explicit algebraic prepotentials for $E_6$, $E_7$ and $H_4$. In the case of $H_4$ there are 9 regular quasi-Coxeter (non-Coxeter) conjugacy classes. The corresponding Frobenius manifolds are denoted as $H_4(k)$, where $k = 1, 2, 3, 4, 6, 7, 8, 9$ and 10 (see [48]). Sekiguchi found prepotentials for 7 out of these 9 cases.

We will present below some of these developments of the algebraic case, as well as some information concerning the often omitted two-dimensional case. From the two-dimensional case we will be able to narrow the conjecture, as we believe that the charge of the algebraic Frobenius manifold would need to also be non-negative for it to hold.

### 3.2.1 Dubrovin-Mazzocco's 3-dimensional algebraic Frobenius manifolds

The first non-rational examples of algebraic Frobenius manifolds were found by Dubrovin and Mazzocco in 2000 [18]. They utilised algebraic solutions of the Painlevé VI equation to construct Frobenius manifolds. Prepotentials of these Frobenius manifolds were later found by Kato, Mano and Sekiguchi [35] (see also Remark 6.1 for an earlier work).

The Painlevé equations are nonlinear second-order differential equations of the form

$$\frac{\partial^2 y}{\partial x^2} = R\left(x,\, y,\, \frac{\partial y}{\partial x}\right),$$

where $R$ is some rational function and whose solutions have the Painlevé property, meaning that the movable singularities of $y$ must be poles. Painlevé found that, up to certain transformations, these could be classified into 50 canonical forms (see [33] for a list of these forms). Moreover, he found that solutions of 44 of these forms could be expressed using previously known functions, leaving 6 equations whose solutions required new special functions. These six equations are known as the Painlevé equations and their solutions are known as Painlevé transcendents. These equations have the following forms:

$$\begin{aligned}
\text{Painlevé I}: \frac{\partial^2 y}{\partial x^2} &= 6y^2 + x,\\
\text{Painlevé II}: \frac{\partial^2 y}{\partial x^2} &= 2y^3 + xy + \alpha,\\
\text{Painlevé III}: \frac{\partial^2 y}{\partial x^2} &= \frac{1}{y}\left(\frac{\partial y}{\partial x}\right)^2 - \frac{1}{x}\frac{\partial y}{\partial x} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y},\\
\text{Painlevé IV}: \frac{\partial^2 y}{\partial x^2} &= \frac{1}{2y}\left(\frac{\partial y}{\partial x}\right)^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2-\alpha)y + \frac{\beta}{y},\\
\text{Painlevé V}: \frac{\partial^2 y}{\partial x^2} &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{\partial y}{\partial x}\right)^2 - \frac{1}{x}\frac{\partial y}{\partial x} + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right)\\
&\quad + \gamma\frac{y}{x} + \delta\frac{y(y+1)}{y-1},\\
\text{Painlevé VI}: \frac{\partial^2 y}{\partial x^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{\partial y}{\partial x}\right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)\frac{\partial y}{\partial x}\\
&\quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right),
\end{aligned}$$

where $\alpha,\, \beta,\, \gamma,\, \delta \in \mathbb{C}$ are constants. Although general solutions of the Painlevé equations cannot be expressed using classically known functions, for special values of the constants $\alpha,\, \beta,\, \gamma,\, \delta$, there may be solutions which can be expressed using previously known functions. In [18], Dubrovin and Mazzocco were interested in classifying the algebraic solutions of the Painlevé VI equation with the following constraints:

$$\alpha = \frac{(2\mu-1)^2}{2}, \qquad \beta = \gamma = 0, \qquad \delta = \frac{1}{2}, \tag{3.2.1}$$

where $\mu \in \mathbb{C}$ with $2\mu \notin \mathbb{Z}$. Let $y$ be a solution of the Painlevé VI equation with constraints (3.2.1). Then, following the digression in Lecture 5 of [16], we can construct a 3-dimensional semisimple Frobenius manifold in the following way. Define the following functions

$$q(u^1,u^2,u^3) := u^1 + (u^2-u^1)y\left(\frac{u^3-u^1}{u^2-u^1}\right),$$
$$p(u^1,u^2,u^3) := \frac{1}{2}\left[\frac{P'(u^3)}{P(q)}y'\left(\frac{u^3-u^1}{u^2-u^1}\right) - \frac{1}{q-u^3}\right],$$

where

$$P(\lambda) := (\lambda-u^1)(\lambda-u^2)(\lambda-u^3).$$

One can then show that there exists a function $l(u^1,u^2,u^3)$ with derivatives

$$\frac{\partial l}{\partial u^i} = (2\mu-1)\frac{q-u^i}{P'(u^i)}.$$

Using these functions, we then construct a 3×3 matrix $\Psi$ whose entries satisfy the following conditions

$$\Psi_1^i\Psi_3^i = \frac{u^i-q}{2\mu^2P'(u^i)}Q^i(u),$$
$$(\Psi_1^i)^2 = \frac{u^i-q}{4\mu^4P'(u^i)e^l}Q^i(u)^2,$$
$$(\Psi_3^i)^2 = \frac{e^l(u^i-q)}{P'(u^i)},$$
$$\Psi_2^1 = \epsilon(\Psi_1^2\Psi_3^3 - \Psi_3^2\Psi_1^3),$$
$$\Psi_2^2 = \epsilon(\Psi_3^1\Psi_1^3 - \Psi_1^1\Psi_3^3),$$
$$\Psi_2^3 = \epsilon(\Psi_1^1\Psi_3^2 - \Psi_3^1\Psi_1^2),$$

where $\epsilon = \pm i$ and

$$Q^i(u) = P(q)p^2 + 2\mu\frac{P(q)p}{q-u^i} + \mu^2(q+2u^i-(u^1+u^2+u^3)),$$

for $i = 1, 2, 3$. Now, we will construct a Frobenius manifold $M$ such that the coordinates $u^1,u^2,u^3$ are canonical coordinates of $M$ and the matrix $\Psi$ has entries

$$\Psi_\alpha^i = \sqrt{\eta_{ii}(u)}\frac{\partial u^i}{\partial t^\alpha},$$

where $\eta$ is the flat metric on $M$ and $t^1,t^2,t^3$ are flat coordinates of $\eta$. To do this, note that $\frac{\partial u^i}{\partial t^1} = 1$ for any semisimple Frobenius manifold, thus we must have the following relationship between the flat coordinates of $\eta$ and the canonical coordinates:

$$\frac{\partial u^i}{\partial t^\alpha} = \frac{\Psi_\alpha^i}{\Psi_1^i}.$$

Inverting this matrix to get the Jacobian $\frac{\partial t^\alpha}{\partial u^i}$, one can then find expressions for the $t$ coordinates in terms of the $u$ coordinates. We can then define the third order derivatives of the prepotential of $M$ using the formula

$$c_{\alpha\beta\gamma}(t) = \sum_{i=1}^{3} \frac{\Psi^i_\alpha(u(t))\Psi^i_\beta(u(t))\Psi^i_\gamma(u(t))}{\Psi^i_1(u(t))}.$$

It is then a simple matter of checking the necessary conditions to see that this does define the structure of a Frobenius manifold.

If our solution $y$ of the Painlevé VI equation is algebraic, then all of the functions that we have constructed are also algebraic. This includes any functions that are defined as anti-derivatives, where the condition that $2\mu \notin \mathbb{Z}$ is necessary to show that the anti-derivatives of algebraic functions that we construct are also algebraic. Thus, algebraic solutions of the Painlevé VI equation with constraints (3.2.1) give rise to 3-dimensional algebraic semisimple Frobenius manifolds.

Conversely, if one starts with a 3-dimensional algebraic semisimple Frobenius manifold with charge $d \neq 0$, then it is possible to reverse this construction to get a solution of the Painlevé VI equation with constraints (3.2.1).

In [18], Dubrovin and Mazzocco found five algebraic solutions for the Painlevé VI equation with constraints (3.2.1). Three of the solutions they found correspond to the $A_3$, $B_3$ and $H_3$ polynomial Frobenius manifolds. That left two extra solutions which had relations to $H_3$ and so were labelled as $(H_3)'$ and $(H_3)''$. Their prepotentials are

$$\begin{aligned} F_{(H_3)'}(t) &= \frac{1}{2}\left(t_1t_2^2 + t_1^2t_3\right) - \frac{1}{18}t_3^4 Z_1 - \frac{7}{72}t_3^3 Z_1^4 - \frac{17}{105}t_3^2 Z_1^7 - \frac{2}{9}t_3 Z_1^{10} - \frac{64}{585}Z_1^{13}, \\ F_{(H_3)''}(t) &= \frac{1}{2}\left(t_2^2t_1 + t_3t_1^2\right) + \frac{4063}{1701}t_3^7 + \frac{19}{135}t_3^5 Z_2^2 - \frac{73}{27}t_3^3 Z_2^4 + \frac{11}{9}t_3 Z_2^6 - \frac{16}{35}Z_2^7, \end{aligned}$$

where $Z_1$ and $Z_2$ satisfy the following algebraic equations

$$\begin{aligned} Z_1^4 + t_3 Z_1 + t_2 &= 0, \\ Z_2^2 + t_2 - t_3^2 &= 0. \end{aligned}$$

Recall from Subsection 2.2.9 that for algebraic prepotentials, the Hurwitz action on the generating reflections of the monodromy group has a finite orbit. Michel [41] proved that if a tuple of reflections has a finite orbit under the Hurwitz action, then the group generated by those reflections must itself be finite. Thus, the monodromy group for an algebraic Frobenius manifold must be finite. Now, the product of the tuple of reflections is preserved under the Hurwitz action, and these reflections generate the monodromy group, so their product is a quasi-Coxeter element of the monodromy group. Moreover, this element is conjugate to the product of any cyclic permutations of the generating reflections, but not conjugate generally to the product of any permutation of the reflections. Thus, an algebraic Frobenius manifold can be associated to, possibly multiple, conjugacy classes of quasi-Coxeter elements in a finite Coxeter group. This general argument, along with the particular treatment for the $n = 3$ case that Dubrovin and Mazzocco investigated gives rise to the following conjecture (first presented in [7]).

**Conjecture 3.2.2.** *The irreducible, semisimple Frobenius manifolds with algebraic prepotentials and positive degrees have a correspondence with the quasi-Coxeter conjugacy classes of finite, irreducible Coxeter groups.*

The correspondence is one-to-one, even though multiple conjugacy classes could be associated to the same set of generating reflections. As we will see there have not been any examples of algebraic Frobenius manifolds being associated to more than one quasi-Coxeter conjugacy class. Much of the research on algebraic prepotentials of semisimple Frobenius manifolds has been motivated by this conjecture.

### 3.2.2 Dinar's construction

In this subsection, we will show how to construct a family of algebraic Frobenius manifolds using a method explained by Dinar, first in [7] and then in [10]. A similar method was used earlier by Pavlyk in [43] to construct algebraic Frobenius manifolds. We choose to follow Dinar's construction rather than Pavlyk's, as we feel it is much simpler and both constructions give rise to the same Frobenius manifolds.

Let $e$ be a distinguished nilpotent element of semisimple type in a simple Lie algebra $\mathfrak{g}$ (see Section 2.4). Let $\{e,\, f,\, h\}$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ and let $\eta_r \in \mathbb{Z}$ be such that $2\eta_r$ is the maximum eigenvalue of $\mathrm{ad}_h$. Let $K_1 \in \mathfrak{g}_{-2\eta_r}$ such that $h' = e + K_1$ is a semisimple element of $\mathfrak{g}$. Recall from Section 2.4 that we refer to $\mathfrak{h}' = \mathfrak{g}^{h'}$ as the opposite Cartan subalgebra.

**Definition 3.2.3.** Let $S = e + \mathfrak{g}^f$ be a Slodowy slice. The set of *common equilibrium points* $N$ is defined to be the set

$$N = \{x \in S \mid \mathfrak{g}^x \cap S = \mathfrak{g}^{K_1} \cap S\}.$$

Note from this definition that $N \subseteq \mathfrak{g}^{K_1}$, since $x \in \mathfrak{g}^x$ for any $x \in \mathfrak{g}$.

**Theorem 3.2.4.** [10] *$N$ has dimension equal to the rank of $\mathfrak{g}$.*

In order to prove this, it is necessary to introduce special coordinates on the Slodowy slice and then restrict onto the set of common equilibrium points. We will not show this here, but an explanation can be found in [10]. We can define a pair of compatible local Poisson brackets ${}^S\{\,\cdot,\cdot\,\}_1$, ${}^S\{\,\cdot,\cdot\,\}_2$ as shown in Subsection 2.4.1. We perform a Dirac reduction of these local Poisson brackets onto $N$ and doing so gives us the pair of compatible local Poisson brackets ${}^N\{\,\cdot,\cdot\,\}_1$ and ${}^N\{\,\cdot,\cdot\,\}_2$.

**Proposition 3.2.5.** [7] *The pair* ${}^{N}\{\cdot,\cdot\}_1$ *and* ${}^{N}\{\cdot,\cdot\}_2$ *are well-defined and dispersionless. Furthermore their dispersionless limits are non-degenerate and linearly independent.*

The fact that this pair of local Poisson brackets is well-defined and dispersionless comes from the definition of the special coordinates on $S$.

**Proposition 3.2.6.** [7] *The local Poisson brackets* ${}^{N}\{\cdot,\cdot\}_1^{[0]}$ *and* ${}^{N}\{\cdot,\cdot\}_2^{[0]}$ *form a bihamiltonian structure on* $N$.

By Proposition 2.4.16, the Dirac reduction of compatible local Poisson brackets remain compatible on the submanifold to which we restrict. So, ${}^{N}\{\cdot,\cdot\}_1$ and ${}^{N}\{\cdot,\cdot\}_2$ are compatible local Poisson brackets on $N$. Since, these are compatible on $N$, and by Proposition 3.2.5 they are dispersionless, any nonzero linear combination is also a disperionless local Poisson bracket on $N$. By Lemma 2.2.64, the dispersionless limit of any dispersionless local Poisson bracket is also local Poisson bracket, so any nonzero linear combination of ${}^{N}\{\cdot,\cdot\}_1^{[0]}$ and ${}^{N}\{\cdot,\cdot\}_2^{[0]}$ is a local Poisson bracket on $N$. By Proposition 3.2.5, these are non-degenerate local Poisson brackets of hydrodynamic type and they are also linearly independent, so they are compatible, thus Proposition 3.2.6 obviously follows.

**Theorem 3.2.7.** [10] *The local Poisson brackets* ${}^{N}\{\cdot,\cdot\}_1^{[0]}$ *and* ${}^{N}\{\cdot,\cdot\}_2^{[0]}$ *form a regular, quasihomogeneous bihamiltonian structure of degree* $\frac{\eta_r-1}{\eta_r+1}$ *on the space of common equilibrium points* $N$*. Thus, we have a Frobenius structure on* $N$*.*

Let us define the map

$$w := e^{\frac{\pi i}{\eta_r+1}\mathrm{ad}_h} = \sum_{n=0}^{\infty}\left(\frac{\pi i}{\eta_r+1}\right)^n \frac{\mathrm{ad}_h^n}{n!}.$$

Observe that $w(h') = e^{\frac{2\pi i}{\eta_r+1}}h'$, where $h' = e + K_1$. Indeed,

$$w(e+K_1) = \sum_{n=0}^{\infty}\left(\frac{2\pi i}{\eta_r+1}\right)^n \frac{e}{n!} + \sum_{n=0}^{\infty}\left(\frac{-2\pi i}{\eta_r+1}\eta_r\right)^n \frac{K_1}{n!} = e^{\frac{2\pi i}{\eta_r+1}}(e+K_1).$$

More generally, $w$ can be shown to act on the opposite Cartan subalgebra, meaning that $w(\mathfrak{h}') \subseteq \mathfrak{h}'$. Let $r$ be the rank of $\mathfrak{g}$, so $\mathfrak{h}'$ is $r$-dimensional. Let $\eta_1, \ldots, \eta_{r-1}$ (together with $\eta_r$) be natural numbers such that $e^{\frac{2\pi i}{\eta_r+1}\eta_i}$ are eigenvalues of $w$ acting on $\mathfrak{h}'$. Since $w$ is an action on a Cartan subalgebra of $\mathfrak{g}$, it can be interpreted as an element of the Weyl group $W(\mathfrak{g})$, up to conjugation. We can thus associate each Frobenius manifold constructed in this way with a conjugacy class of the appropriate Weyl group. In particular, these are conjugacy classes of quasi-Coxeter elements and they are unique for each Frobenius manifold.

**Theorem 3.2.8.** [10] *The Frobenius structure on* $N$ *has an algebraic prepotential with charge* $\frac{\eta_r-1}{\eta_r+1}$ *and degrees* $\frac{\eta_i+1}{\eta_r+1}$.

The first prepotential to be found explicitly using this method, was done by Pavlyk in his original paper [43], where he found the prepotential for $D_4(a_1)$. Dinar then used this construction to find the prepotential for $F_4(a_2)$ in [7], and later also for $D_4(a_1)$ in [9]. Dinar also gave an explicit prepotential for $E_8(a_1)$ in [8].

**Theorem 3.2.9.** [20] *Let $e$ be a regular, nilpotent element of a simple Lie algebra $\mathfrak{g}$. Then the resulting Frobenius manifold that one constructs on the space of common equilibrium points $N$ using this method has a polynomial prepotential and is equivalent to the Coxeter orbit space of the Weyl group of $\mathfrak{g}$.*

### 3.2.3 Sekiguchi's prepotentials

In this subsection, we will go over how some 4-dimensional algebraic prepotentials were found by Sekiguchi, using data from quasi-Coxeter elements of $H_4$ provided by Douvropoulos.

In the earlier version of [48], Sekiguchi found two algebraic prepotentials related to Weyl groups $E_6$ and $E_7$, and six algebraic prepotentials related to the Coxeter group $H_4$, a seventh algebraic prepotential related to $H_4$ was found by Sekiguchi in 2023 and can be found in the latest version of [48] online. These were all found using a brute force method, whereby the degrees and algebraic degree of the prepotential were conjectured to have particular values and then prepotentials were found through a computer search.

The conjectured degrees of these Frobenius manifolds were put forward by Douvropoulos in [12] and are determined as follows. For a regular quasi-Coxeter element $w$ with eigenvalues $e^{\frac{2\pi i}{|w|}\eta_j}$, there exists a regular element $w_0 \in W$ such that $w$ is conjugate to $w_0^l$ for some $l \in \mathbb{N}$, $|w| = |w_0|$ and the eigenvalues of $w_0$ have the form $e^{\frac{2\pi i}{|w|}(d_j^W - 1)}$, where $d_j^W$ are the fundamental degrees of $W$, assuming $l$ is the smallest such positive integer. Then, the degrees $d_j$ of the corresponding Frobenius manifolds were conjectured to be

$$d_j = \frac{\widetilde{\eta}_j}{|w|}, \tag{3.2.2}$$

where $\widetilde{\eta}_j$ is the remainder of $\eta_j + l$ modulo $|w|$, which is assumed to be between 1 and $|w|$. The algebraic degree of these Frobenius manifolds also have a combinatorial interpretation given in [12]. For a quasi-Coxeter element $w$, let $F_W^{\text{red}}(w)$ be the number of length $n$ reflection factorisations of $w$. That is,

$$F_W^{\text{red}}(w) = |\{(t_1, \dots, t_n) \in \mathcal{R}^n \mid w = t_1 \dots t_n\}|,$$

where $\mathcal{R}$ is the set of reflections in $W$. Douvropolous then gives the following Proposition.

**Proposition 3.2.10.** [12] *Let $w$ be a regular quasi-Coxeter element in a finite Coxeter* *group $W$ of rank $n$. Then, there exists a positive integer $\delta_w$ such that*

$$F_W^{\mathrm{red}}(w) = \frac{|w|^n n!}{\prod_{i=1}^{n} \widetilde{d_j}} \delta_w,$$

*where $\widetilde{d_j}$ is the remainder of $d_j^W$ modulo $|w|$, which is assumed to be between 1 and $|w|$. Moreover, when $w$ is a Coxeter element, $\delta_w = 1$.*

The positive integer $\delta_w$ is interpreted to be the algebraic degree of the associated prepotential for the quasi-Coxeter element $w$. Although this is only a conjecture, for all algebraic prepotentials that have been found thus far, this appears to hold true.

If we assume that our metric is anti-diagonal, then we can also fix the charge of our prepotential, and thus, using the conjectured degrees and algebraic degree, one can construct a general quasi-homogeneous function. If we impose that the WDVV equations must hold, then by brute force one can find the prepotentials associated to each regular quasi-Coxeter conjugacy class of $H_4$.

In the case of $H_4$ there are 11 quasi-Coxeter conjugacy classes which we denote by $H_4(k)$ with $k = 0, \ldots, 10$, where $H_4(0)$ is the conjugacy class for the Coxeter element of $H_4$. Out of the remaining 10 quasi-Coxeter conjugacy classes, Sekiguchi found prepotentials for 7 of them. The corresponding Frobenius manifolds are denoted $H_4(k)$ with $k = 1, 2, 3, 4, 6, 7$ and 9 [48]. The two regular quasi-Coxeter conjugacy classes for which no prepotential has yet been found are denoted as $H_4(8)$ and $H_4(10)$. For the non-regular quasi-Coxeter conjugacy class $H_4(5)$ there is currently no evidence for the existence of Frobenius manifold which can be related to this quasi-Coxeter conjugacy class.

### 3.2.4 Two-dimensional examples

Recall from Example 2.2.17 that, up to equivalence, prepotentials for two-dimensional (semisimple) algebraic Frobenius manifolds have the following form [13]:

$$F = \frac{1}{2} t_1^2 t_2 + c t_2^{k+1}, \tag{3.2.3}$$

where $k \in \mathbb{Q} \setminus \{-1, 0, 1\}$ and $c \in \mathbb{C}$ is a nonzero constant. The degrees of the Frobenius manifold are $d_1 = 1$ and $d_2 = \frac{2}{k}$, and the charge is $d = \frac{k-2}{k}$.

Now, let $w$ be a quasi-Coxeter element in the dihedral group $I_2(m)$. It must be the product of two reflections that generate $I_2(m)$. Hence $w = C^l$, where $C$ is a Coxeter element of $I_2(m)$ and $(m, l) = 1$. The eigenvalues of $w$ are $e^{\pm \frac{2\pi i}{m} l}$. We can assume $1 \leq l \leq \frac{m}{2}$ as the elements $w = C^l$, $1 \leq l \leq \frac{m}{2}$ give representatives for all the quasi-Coxeter conjugacy classes. Then the smallest positive integer $r$ such that $w$ is conjugate to $C^r$ is $r = l$. Thus, the degrees of the Frobenius manifold, using prescription (3.2.2) and following [12], are

$$d_1 = 1, \qquad d_2 = \frac{2l}{m},$$

since $\eta_1 = m - l$, $\eta_2 = l$ and $|w| = m$. From the general form (3.2.3) of a prepotential of an algebraic two-dimensional Frobenius manifold, we see that $k = \frac{m}{l}$ and thus

$$F = \frac{1}{2}t_1^2 t_2 + c t_2^{\frac{m}{l}+1},$$

and this has charge $d = \frac{m-2l}{m}$. Note that when $l = 1$ we get the polynomial two-dimensional Frobenius manifolds.

Note that in the above analysis we relate an algebraic Frobenius manifold (3.2.3) with a conjugacy class of a quasi-Coxeter element in a dihedral group provided that $k \geq 2$. Two-dimensional Frobenius manifolds with $0 < k < 2$ have positive degrees but a relation to quasi-Coxeter elements seems unclear in this range. Based on the above considerations it seems not possible to relate these cases to a quasi-Coxeter conjugacy class of a suitable dihedral group. Note that the charge $d < 0$ in this case, whereas $d \geq 0$ when $k \geq 2$. The general conjecture on the relation of algebraic Frobenius manifolds with quasi-Coxeter elements in [7] assumes that the degrees are positive. A possible way to exclude the examples (3.2.3) with $0 < k < 2$ is to impose an additional assumption to the conjecture that the charge $d \geq 0$. For $k < 0$ the Frobenius manifolds with prepotential (3.2.3) have $d_2 < 0$, and so are not considered in the conjecture. Also, the data obtained by Douvropoulos for the degrees of algebraic Frobenius manifolds [12] associated to quasi-Coxeter conjugacy classes assumes that the quasi-Coxeter conjugacy class is regular. There are currently no known examples of algebraic Frobenius manifolds that have been associated with non-regular quasi-Coxeter conjugacy classes. Thus, we propose to modify the Dubrovin conjecture 3.2.2 for algebraic Frobenius manifolds to read as follows:

**Conjecture 3.2.11.** *Irreducible, semisimple algebraic Frobenius manifolds with positive degrees and non-negative charge have a one-to-one correspondence with regular quasi-Coxeter conjugacy classes in finite irreducible Coxeter groups.*

# Chapter 4

# Flat coordinates of algebraic Frobenius manifolds

In this chapter, we will be looking specifically at the algebraic Frobenius manifolds of dimensions 2, 3 and 4. Their prepotentials can be written as polynomials in the flat coordinates of the metric $t^1, \ldots, t^n$ and $Z$, which is some algebraic function in the $t$ coordinates. We wish to find relations between the flat coordinates $t^1, \ldots, t^n$ of the metric and the flat coordinates $x^1, \ldots, x^n$ of the intersection form. To do so, we will make use of the basic invariants of the associated Coxeter groups $y^1, \ldots, y^n$, and to help make the search for relations easier we will use the following proposition:

**Proposition 4.0.1.** *Let $M$ be a Frobenius manifold with flat coordinates $t^1, \ldots, t^n$ for the metric and Euler vector field $E$ such that*

$$E(t) = \sum_{\alpha=1}^{n} d_\alpha t_\alpha \partial_{t_\alpha}. \tag{4.0.1}$$

*Let $g$ be the intersection form with flat coordinates $x^1, \ldots, x^n$ such that*

$$g^{ij}(x) = \delta^{ij}. \tag{4.0.2}$$

*For a function $f \in C^\infty(M)$, we have*

$$\Delta(f) = g^{\nu\mu}(t)\frac{\partial^2 f}{\partial t^\nu \partial t^\mu} + \Delta(t^\nu)\frac{\partial f}{\partial t^\nu}, \tag{4.0.3}$$

*where $c_\gamma^{\alpha\beta}$ are the cotangent structure constants and $\Delta(f) := \sum_{i=1}^n \frac{\partial^2 f}{(\partial x^i)^2}$. Furthermore,*

$$\Delta(t^i) = \left(\frac{d-1}{2} + d_i\right) c_\lambda^{i\lambda}(t),$$

*where we sum over the index $\lambda$ and $i = 1, \ldots, n$ is fixed.*

*Proof.* We know that the intersection form in the $t$ coordinates can be calculated as

$$g^{ij}(t) = g^{\alpha\beta}(x)\frac{\partial t^i}{\partial x^\alpha}\frac{\partial t^j}{\partial x^\beta} = \sum_{\alpha=1}^{n}\frac{\partial t^i}{\partial x^\alpha}\frac{\partial t^j}{\partial x^\alpha}. \tag{4.0.4}$$

So, we have

$$\Delta(f) = \sum_{\alpha=1}^{n}\frac{\partial}{\partial x_\alpha}\left(\frac{\partial f}{\partial t_\nu}\frac{\partial t_\nu}{\partial x_\alpha}\right) = \sum_{\alpha=1}^{n}\left(\frac{\partial^2 f}{\partial t_\nu \partial t_\mu}\frac{\partial t_\nu}{\partial x_\alpha}\frac{\partial t_\mu}{\partial x_\alpha} + \frac{\partial f}{\partial t_\nu}\frac{\partial^2 t_\nu}{\partial x_\alpha^2}\right),$$

which gives the equality (4.0.3) by formula (4.0.4). By the transformation law for Christoffel symbols given in relation (2.1.2) and the fact that ${}^g\Gamma^i_{jk}(x) = 0$, we see that

$$\begin{aligned}\Delta(t_i) &= \sum_{\alpha=1}^{n}\frac{\partial^2 t_\lambda}{\partial x_\alpha^2}\delta^i_\lambda = \sum_{\alpha=1}^{n}\frac{\partial^2 t_\lambda}{\partial x_\alpha^2}\frac{\partial x_\mu}{\partial t_\lambda}\frac{\partial t_i}{\partial x_\mu}\\ &= \sum_{\alpha=1}^{n} -{}^g\Gamma^\nu_{\sigma\omega}(t)\frac{\partial t_\sigma}{\partial x_\alpha}\frac{\partial t_\omega}{\partial x_\alpha}\frac{\partial x_\mu}{\partial t_\nu}\frac{\partial t_i}{\partial x_\mu} = -{}^g\Gamma^i_{\sigma\omega}(t)\sum_{\alpha=1}^{n}\frac{\partial t_\sigma}{\partial x_\alpha}\frac{\partial t_\omega}{\partial x_\alpha}.\end{aligned}$$

We also have

$$g^{i\lambda}(z)g_{\lambda j;k}(z) = (g^{i\lambda}g_{\lambda j})_{;k}(z) - g^{i\lambda}_{;k}(z)g_{\lambda j}(z) = \delta^i_{j;k} - g^{i\lambda}_{;k}(z)g_{\lambda j}(z) = -g^{i\lambda}_{;k}(z)g_{\lambda j}(z). \tag{4.0.5}$$

Using equations (4.0.4) and (4.0.5) we get that

$$\begin{aligned}\Delta(t_i) =& -g^{\sigma\omega}(t){}^g\Gamma^i_{\sigma\omega}(t) = -\frac{1}{2}g^{\sigma\omega}(t)g^{i\lambda}(t)\left(g_{\lambda\omega;\sigma}(t) + g_{\sigma\lambda;\omega}(t) - g_{\sigma\omega;\lambda}(t)\right)\\ =&\frac{1}{2}\left(g^{\sigma\omega}(t)g^{i\lambda}_{;\sigma}(t)g_{\lambda\omega}(t) + g^{\sigma\omega}(t)g^{i\lambda}_{;\omega}(t)g_{\sigma\lambda}(t) - g^{\sigma\omega}_{;\lambda}(t)g^{i\lambda}(t)g_{\sigma\omega}(t)\right)\\ =&g^{i\lambda}_{;\lambda}(t) - \frac{1}{2}g^{\sigma\omega}_{;\lambda}(t)g^{i\lambda}(t)g_{\sigma\omega}(t).\end{aligned} \tag{4.0.6}$$

By relation (2.2.19) we can rearrange equation (4.0.6) as

$$\begin{aligned}\Delta(t_i) =& (E^\mu(t)c^{i\lambda}_\mu(t))_{;\lambda} - \frac{1}{2}(E^\mu(t)c^{\sigma\omega}_\mu(t))_{;\lambda}g^{i\lambda}(t)g_{\sigma\omega}(t)\\ =& E^\mu_{;\lambda}(t)c^{i\lambda}_\mu(t) + E^\mu(t)c^{i\lambda}_{\mu;\lambda}(t) - \frac{1}{2}\left(E^\mu_{;\lambda}(t)c^{\sigma\omega}_\mu(t) + E^\mu(t)c^{\sigma\omega}_{\mu;\lambda}(t)\right)g^{i\lambda}(t)g_{\sigma\omega}(t).\end{aligned}$$

From relations (2.2.6) we see that $c^{ij}_{k;l}(t) = c^{ij}_{l;k}(t)$, since $\eta$ is constant in the $t$ coordinates, hence

$$\Delta(t_i) = E^\mu_{;\lambda}(t)c^{i\lambda}_\mu(t) + E^\mu(t)c^{i\lambda}_{\lambda;\mu}(t) - \frac{1}{2}\left(E^\mu_{;\lambda}(t)c^{\sigma\omega}_\mu(t) + E^\mu(t)c^{\sigma\omega}_{\lambda;\mu}(t)\right)g^{i\lambda}(t)g_{\sigma\omega}(t).$$

By relation (2.1.1) the Lie derivative of the tensor field $c_k^{ij}$ has the form

$$(\mathcal{L}_E c)_k^{ij}(t) = E^\lambda(t) c_{k;\lambda}^{ij}(t) - E^i_{;\lambda}(t) c_k^{\lambda j}(t) - E^j_{;\lambda}(t) c_k^{i\lambda}(t) + E^\lambda_{;k}(t) c_\lambda^{ij}(t).$$

Therefore

$$\begin{aligned}\Delta(t_i) = &(\mathcal{L}_E c)_\lambda^{i\lambda}(t) + E^i_{;\mu}(t) c_\lambda^{\mu\lambda}(t) + E^\lambda_{;\mu}(t) c_\lambda^{i\mu}(t) \\ &- \frac{1}{2}\left((\mathcal{L}_E c)_\lambda^{\sigma\omega}(t) + E^\sigma_{;\mu}(t) c_\lambda^{\mu\omega}(t) + E^\omega_{;\mu}(t) c_\lambda^{\sigma\mu}(t)\right) g^{i\lambda}(t) g_{\sigma\omega}(t).\end{aligned}$$

Note that

$$g^{i\lambda} g_{k\mu} c_\lambda^{j\mu} = g^{i\lambda} \overset{*}{c}{}^j_{k\lambda} = g^{i\lambda} \overset{*}{c}{}^j_{\lambda k} = g^{i\lambda} g_{\lambda\mu} c_k^{j\mu} = c_k^{ji}. \tag{4.0.7}$$

By relations (2.2.9) and (4.0.7) we have that

$$\begin{aligned}\Delta(t_i) = &(d-1) c_\lambda^{i\lambda}(t) + E^i_{;\mu}(t) c_\lambda^{\mu\lambda}(t) + E^\lambda_{;\mu}(t) c_\lambda^{i\mu}(t) \\ &- \frac{1}{2}\left((d-1) c_\lambda^{\sigma\omega}(t) + E^\sigma_{;\mu}(t) c_\lambda^{\mu\omega}(t) + E^\omega_{;\mu}(t) c_\lambda^{\sigma\mu}(t)\right) g^{i\lambda}(t) g_{\sigma\omega}(t) \\ = &(d-1) c_\lambda^{i\lambda}(t) + E^i_{;\mu}(t) c_\lambda^{\mu\lambda}(t) + E^\lambda_{;\mu}(t) c_\lambda^{i\mu}(t) \\ &- \frac{d-1}{2} c_\lambda^{i\lambda}(t) - \frac{1}{2} E^\sigma_{;\mu}(t) c_\sigma^{\mu i}(t) - \frac{1}{2} E^\omega_{;\mu}(t) c_\omega^{i\mu}(t) \\ = &\frac{d-1}{2} c_\lambda^{i\lambda}(t) + E^i_{;\mu}(t) c_\lambda^{\mu\lambda}(t).\end{aligned}$$

The statement follows by formula (4.0.1). □

Suppose we can write the prepotential $F$ of a Frobenius manifold $M$ as a polynomial $F(t, Z)$ in the $t$ coordinates and $Z$, where $Z$ satisfies an equation of the form

$$P(t; Z) = \sum_{k=0}^{N} a_k(t) Z^k = 0, \tag{4.0.8}$$

where $a_k \in \mathbb{C}[t_2, \ldots, t_n]$, such Frobenius manifolds are called *algebraic*. We say that an algebraic Frobenius manifold $M$ is *associated to the Coxeter group $W$* if there exist basic invariants $y_1, \ldots, y_n$ in the flat coordinates $x_1, \ldots, x_n$ of the intersection form $g$ which are simultaneously polynomial in $t_1, \ldots, t_n$ and $Z$. All of the examples of Frobenius manifolds which we consider below are associated to Coxeter groups.

We will sometimes need to consider the $t$ coordinates and $Z$ as independent variables (see e.g. Proposition 4.0.2 below). In such cases, for a rational function $f$ of $n+1$ variables, we will write $f^F(t, Z)$ instead of $f(t, Z)$.

Now, for each algebraic Frobenius manifold, their charges $d$ are known and $d \neq 1$ in all cases we consider. It is known [13] that for $d \neq 1$ we have

$$t_n = \frac{1-d}{4}\sum_{i=1}^{n} x_i^2.$$

We also know that the Euler vector field $E$ is diagonal in each case, so we have $\deg t_i(x) = \frac{2d_i}{d_n}$ for all $i$, where $d_i$ are the degrees of the Frobenius manifolds. The general method for finding basic invariants as polynomials $y_i(t, Z)$ below will go through the following steps:

**1)** Set $y_n = \sum_{i=1}^{n} x_i^2 = \frac{4}{1-d}t_n$. Choose $y_1, \ldots, y_{n-1}$ so that $y_1, \ldots, y_n$ form a set of basic invariants for a finite irreducible Coxeter group $W$.

**2)** Let $Y_1, \ldots, Y_n$ be a set of basic invariants such that $\Delta(Y_n) = 1$ and $\Delta(Y_j) = 0$ for $j = 1, \ldots, n-1$, which exist by Proposition 2.3.13. Each $Y_i$ can be expressed as a polynomial in $y_1, \ldots, y_n$. In particular, $Y_n = \frac{1}{2n}y_n$.

**3)** Let $V_j$ be the vector space of polynomials in $t_1, \ldots, t_n$ and $Z$ which are homogenous in the $x$ coordinates of degree $d_j^W$. Find the harmonic elements of $V_j$ using Proposition 4.0.1, for $j = 1, \ldots, n-1$.

**4)** Equate $Y_j = Y_j(y)$ with a general harmonic element of $V_j$. Rearrange these equations to find each $y_j$ as a polynomial in $t_1, \ldots, t_n$ and $Z$ up to some coefficients to be found. This can be done successively for $j = n-1, n-2, \ldots, 1$.

**5)** Find the intersection form $g^{ij}$ in the $y$ coordinates by the formula $g^{ij}(y) = (\nabla(y_i), \nabla(y_j))$, and express the entries as polynomials in the $y$ coordinates, which can be done by Lemma 2.3.12. Substitute the expressions for $y_i(t, Z)$ into these entries, so we have $g^{ij}(y(t))$.

**6)** Calculate the components $g^{ij}(y(t))$ of the intersection form $g$ in the $y$ coordinates by performing a change of coordinates $y = y(t)$ on the intersection form $g^{\lambda\mu}(t)$ given by formula (2.2.19):

$$g^{ij}(y(t)) = g^{\lambda\mu}(t)\frac{\partial y_i}{\partial t_\lambda}\frac{\partial y_j}{\partial t_\mu}.$$

Here, the derivatives $\frac{\partial y_i}{\partial t_\lambda}$ are found via their expressions in the $t$ coordinates and $Z$ which still contain some coefficients to be found.

**7)** We equate the two expressions for $g^{ij}(y(t))$ from steps 5) and 6), and find the values for the remaining coefficients, which is possible in all the examples we consider. Thus we get basic invariants $y_j$ expressed as polynomials in $t_i$ and $Z$. Note that the polynomials we find may not be unique if the Coxeter graph of $W$ has non-trivial symmetries.

One may alternatively try to skip steps 2) and 4), but this increases the difficulty of the calculations needed to equate the two expressions for $g^{ij}(y(t))$.

**Proposition 4.0.2.** *Let $e = e^i(y)\partial_{y_i}$ be the unity vector field of an algebraic Frobenius* *manifold associated to $W$ with prepotential $F(t, Z)$. Then $e^i(y) \in \mathbb{C}[t, Z]$ for each $i = 1, \ldots, n$.*

*Proof.* We know that $e = \partial_{t_1}$. Hence

$$e^i(y) = e^\alpha(t)\frac{\partial y_i}{\partial t_\alpha} = \frac{\partial y_i}{\partial t_1} = \frac{\partial y_i^F}{\partial t_1} + \frac{\partial y_i^F}{\partial Z}\frac{\partial Z}{\partial t_1} \in \mathbb{C}[t, Z]$$

since $\frac{\partial Z}{\partial t_1} = 0$ by relation (4.0.8). □

**Proposition 4.0.3.** *Let $g$ be the intersection form of an algebraic Frobenius manifold associated to $W$ with root system $R_W$. Then*

$$\det(g^{ij}(t)) = \frac{c \prod_{\alpha \in R_W} (\alpha,\, x)}{(\det J)^2},$$

*where $J = \left(\frac{\partial y_i}{\partial t_j}\right)_{i,j=1}^n$ is the Jacobi matrix and $c \in \mathbb{C}$.*

*Proof.* It follows from [32] that

$$\det(g^{\lambda\mu}(y)) = c \prod_{\alpha \in R_W} (\alpha,\, x)$$

for some $c \in \mathbb{C}$. From relation (4.0.4), we see that

$$\det(g^{ij}(t)) = \det\left(g^{\lambda\mu}(y)\frac{\partial t_i}{\partial y_\lambda}\frac{\partial t_j}{\partial y_\mu}\right) = \det(g^{\lambda\mu}(y))\det\left(J^{-1}\right)^2 = \frac{c \prod_{\alpha \in R_W} (\alpha,\, x)}{(\det J)^2}.$$

□

Just as in Section 3.2, we will look at the two-dimensional case first, then we will look at the two $H_3$ cases, then $D_4(a_1)$ and $F_4(a_2)$ and finally we will consider most of the $H_4$ cases, $H_4(k)$, for $k = 1, 2, 3, 4$ and 7.

The only known examples of algebraic Frobenius manifolds in dimensions 3 and 4 which we do not deal with are $H_4(6)$ and $H_4(9)$. Our method does not work for these examples, as we cannot find expressions for the basic invariants of $H_4$ as polynomials of the $t$ coordinates and $Z$. The prepotential for $H_4(9)$ is expressed as a rational function in the $t$ coordinates and the variable $Z$, rather than as a polynomial. As a result, the entries of the intersection form $g^{\alpha\beta}(t)$ for $H_4(9)$ cannot be expressed as polynomials in the $t$ coordinates and $Z$. For $H_4(6)$, although the prepotential is expressable as a polynomial in the $t$ coordinates and the variable $Z$, the entries of the intersection form $g^{\alpha\beta}(t)$ are not polynomial in the $t$ coordinates and $Z$. In all of the other cases that we consider, the entries of the intersection form are polynomials of the $t$ coordinates and the variable

$Z$. This suggests that the main reason why our method does not work for the $H_4(6)$ and $H_4(9)$ cases is because of these entries in their intersection forms which are not polynomial in the $t$ coordinates and the variable $Z$.

## 4.1 Two-dimensional algebraic Frobenius manifolds

Let us explain the relation between the two sets of flat coordinates for two-dimensional algebraic Frobenius manifolds. Prepotentials for two-dimensional (semisimple) algebraic Frobenius manifolds have the following form [13]:

$$F = \frac{1}{2}t_1^2 t_2 + c t_2^{k+1}, \tag{4.1.1}$$

where $k \in \mathbb{Q} \setminus \{-1, 0, 1\}$ and $c \in \mathbb{C}$ is a nonzero constant. The degrees of the Frobenius manifold are $d_1 = 1$ and $d_2 = \frac{2}{k}$, and the charge is $d = \frac{k-2}{k}$. Let $x_1$, $x_2$ be flat coordinates of the intersection form. Then we have the relations

$$t_1 = a\left[(x_1 + ix_2)^k + (x_1 - ix_2)^k\right], \qquad t_2 = \frac{x_1^2 + x_2^2}{2k}, \tag{4.1.2}$$

where $a^2 = \frac{2c(k^2-1)}{(2k)^{k+1}}$. One can check that these relations are correct similarly to the polynomial case $k \in \mathbb{N}$ considered in [13].

Let $k = \frac{m}{l}$ where $m, l \in \mathbb{Z} \setminus \{0\}$ with $m$ and $l$ coprime and $m > 0$. We showed in Subsection 3.2.4 how one can associate such a two-dimensional algebraic prepotential to a quasi-Coxeter element of $I_2(m)$ when $1 \leq l \leq \frac{m}{2}$. The Coxeter group $I_2(m)$ has basic invariants

$$y_1 = (x_1 + ix_2)^m + (x_1 - ix_2)^m, \qquad y_2 = \frac{x_1^2 + x_2^2}{2m}. \tag{4.1.3}$$

The following proposition shows how we can express these basic invariants in terms of the flat coordinates of the metric $\eta$.

**Proposition 4.1.1.** *Let $t_1$, $t_2$ be given by the formulas (4.1.2) and $y_1$, $y_2$ be basic invariants of $I_2(m)$ given by the formulas (4.1.3). Then we can express the basic invariants $y_1$, $y_2$ in terms of $t_1$, $t_2$ in the following form:*

$$y_1 = \left(\frac{t_1 + \sqrt{t_1^2 - \frac{4c(m^2-l^2)}{ml} t_2^{\frac{m}{l}}}}{2a}\right)^l + \left(\frac{t_1 - \sqrt{t_1^2 - \frac{4c(m^2-l^2)}{ml} t_2^{\frac{m}{l}}}}{2a}\right)^l, \qquad y_2 = \frac{t_2}{l}, \tag{4.1.4}$$

*where $a^2 = \frac{2c(k^2-1)}{(2k)^{k+1}}$.*

*Proof.* Using the substitutions $k = \frac{m}{l}$, $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$, formulas (4.1.2) and

(4.1.3) can be expressed as

$$t_1 = a(z^{\frac{m}{l}} + \overline{z}^{\frac{m}{l}}), \quad t_2 = l\frac{z\overline{z}}{2m}, \quad y_1 = z^m + \overline{z}^m, \quad y_2 = \frac{z\overline{z}}{2m}.$$

It is obvious that $y_2 = \frac{t_2}{l}$. Proving the expression for $y_1$ in terms of $t_1$ and $t_2$ can also be seen fairly easily. Indeed,

$$a^2\left(\frac{2m}{l}\right)^{\frac{m}{l}} = \frac{2c(k^2-1)}{(2k)^{k+1}}(2k)^k = \frac{c(k^2-1)}{k} = \frac{c\left(\frac{m^2}{l^2}-1\right)}{\frac{m}{l}} = \frac{c(m^2-l^2)}{ml},$$

and so

$$\begin{aligned} t_1^2 - \frac{4c(m^2-l^2)}{ml}t_2^{\frac{m}{l}} &= a^2\left(z^{\frac{2m}{l}} + 2z^{\frac{m}{l}}\overline{z}^{\frac{m}{l}} + \overline{z}^{\frac{2m}{l}}\right) - 4a^2\left(\frac{2m}{l}\right)^{\frac{m}{l}}\left(\frac{l}{2m}\right)^{\frac{m}{l}} z^{\frac{m}{l}}\overline{z}^{\frac{m}{l}} \\ &= a^2(z^{\frac{2m}{l}} - 2z^{\frac{m}{l}}\overline{z}^{\frac{m}{l}} + \overline{z}^{\frac{2m}{l}}) = a^2(z^{\frac{m}{l}} - \overline{z}^{\frac{m}{l}})^2 = a^2(z^k - \overline{z}^k)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left(\frac{t_1 + \sqrt{t_1^2 - \frac{4c(m^2-l^2)}{ml}t_2^{\frac{m}{l}}}}{2a}\right)^l + \left(\frac{t_1 - \sqrt{t_1^2 - \frac{4c(m^2-l^2)}{ml}t_2^{\frac{m}{l}}}}{2a}\right)^l \\ &= \left(\frac{a(z^k + \overline{z}^k) + a(z^k - \overline{z}^k)}{2a}\right)^l + \left(\frac{a(z^k + \overline{z}^k) - a(z^k - \overline{z}^k)}{2a}\right)^l \\ &= \left(z^k\right)^l + \left(\overline{z}^k\right)^l = \left(z^{\frac{m}{l}}\right)^l + \left(\overline{z}^{\frac{m}{l}}\right)^l = z^m + \overline{z}^m = y_1. \end{aligned}$$

□

Formulas (4.1.4) may be thought of as inverse relations to formulas (4.1.2), where we replace flat coordinates $x_1$, $x_2$ with basic invariants given by (4.1.3). Note also that Proposition 4.1.1 holds for any choice of $m$ including the case when $m = 1$. This is a kind of degenerate case where the Coxeter group $I_2(1)$ is isomorphic to the Coxeter group $A_1$, which is of lower rank. The functions $y_1$ and $y_2$ are still invariant under the $A_1$ transformations $x_2 \mapsto -x_2$, but the $x_1$ coordinate is "free" in this case.

## 4.2 Algebraic Frobenius manifolds related to $H_3$

There are two non-polynomial algebraic Frobenius manifolds which we can be associated to $H_3$, both found by Dubrovin and Mazzocco [18]. Prepotentials of these three dimensional Frobenius manifolds were given explicitly by Kato, Mano and Sekiguchi [35] (see also

Remark 6.1 in [35]). Let $R_{H_3}$ be the following root system for $H_3$:

$$R_{H_3} = \{\pm e_i \mid 1 \le i \le 3\} \cup \left\{ \frac{1}{2}\left(\pm e_{\sigma(1)} \pm \varphi e_{\sigma(2)} \pm \overline{\varphi} e_{\sigma(3)}\right) \middle| \sigma \in \mathfrak{A}_3 \right\},$$

where

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \overline{\varphi} = \frac{1-\sqrt{5}}{2},$$

and $\mathfrak{A}_3$ is the alternating group on 3 elements. Let us introduce the following basic invariants for $H_3$ (cf. [47]):

$$y_1 = 95\epsilon_2\epsilon_3 - 32\epsilon_1^2\epsilon_3 - 5\epsilon_1\epsilon_2^2 + 2\epsilon_1^3\epsilon_2 + 3\sqrt{5}\delta\epsilon_2, \tag{4.2.1}$$

$$y_2 = \sqrt{5}\delta + \epsilon_1\epsilon_2 - 11\epsilon_3, \tag{4.2.2}$$

$$y_3 = \epsilon_1, \tag{4.2.3}$$

where

$$\epsilon_1 = x_1^2 + x_2^2 + x_3^2, \tag{4.2.4}$$

$$\epsilon_2 = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2, \tag{4.2.5}$$

$$\epsilon_3 = x_1^2x_2^2x_3^2, \tag{4.2.6}$$

$$\delta = (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2). \tag{4.2.7}$$

The basic invariants $y_1, y_2, y_3$ have degrees $10, 6, 2$, respectively.

**Lemma 4.2.1.** *(cf. [47]) The intersection form $g^{ij}(y)$ takes the form*

$$g^{ij}(y) = \begin{pmatrix} 30y_2^3 + 36y_2^2y_3^3 + 8y_1y_3^4 & 28y_2^2y_3 + 8y_2y_3^4 & 20y_1 \\ 28y_2^2y_3 + 8y_2y_3^4 & 8y_1 + 8y_2y_3^2 & 12y_2 \\ 20y_1 & 12y_2 & 4y_3 \end{pmatrix}.$$

Consider another set of basic invariants for $H_3$ given by

$$Y_1 = y_1 - \frac{9}{17}y_2y_3^2 - \frac{10}{187}y_3^5, \tag{4.2.8}$$

$$Y_2 = y_2 - \frac{2}{21}y_3^3, \tag{4.2.9}$$

$$Y_3 = \frac{1}{6}y_3. \tag{4.2.10}$$

The following statement can be checked directly.

**Lemma 4.2.2.** *We have $\Delta(Y_3) = 1$ and $\Delta(Y_1) = \Delta(Y_2) = 0$.*

### 4.2.1 $(H_3)'$ example

The prepotential for $(H_3)'$ is

$$F(t) = \frac{1}{2}\left(t_1 t_2^2 + t_1^2 t_3\right) - \frac{1}{18} t_3^4 Z - \frac{7}{72} t_3^3 Z^4 - \frac{17}{105} t_3^2 Z^7 - \frac{2}{9} t_3 Z^{10} - \frac{64}{585} Z^{13},$$

where

$$P(t_2, t_3, Z) := Z^4 + t_3 Z + t_2 = 0. \tag{4.2.11}$$

The Euler vector field is

$$E(t) = t_1 \partial_{t_1} + \frac{4}{5} t_2 \partial_{t_2} + \frac{3}{5} t_3 \partial_{t_3},$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{2}{5}$. The intersection form (2.2.19) is then given by

$$g^{ij}(t) = \begin{pmatrix} \frac{1}{60}(16 t_2 Z^3 + 19 t_2 t_3 - 9 t_3^2 Z) & \frac{1}{5}(2 t_2 Z^2 + t_3 Z^3 + t_3^2) & t_1 \\ \frac{1}{5}(2 t_2 Z^2 + t_3^2 + t_3 Z^3) & t_1 + \frac{Z}{10}(8 t_2 + 3 t_3 Z) & \frac{4}{5} t_2 \\ t_1 & \frac{4}{5} t_2 & \frac{3}{5} t_3 \end{pmatrix}. \tag{4.2.12}$$

We have that $\deg t_1(x) = \frac{10}{3}$, $\deg t_2(x) = \frac{8}{3}$, $\deg t_3(x) = 2$ and $\deg Z(x) = \frac{2}{3}$.

**Proposition 4.2.3.** *Let $V_1 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 10\}$ and let $V_2 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 6\}$. The harmonic elements of $V_1$ are proportional to*

$$2244000 t_1^3 - 628320 t_1 t_2^2 Z^2 - 1168530 t_1 t_2 t_3^2 - 583440 t_1 t_2 t_3 Z^3 + 151470 t_1 t_3^3 Z$$
$$+768944 t_2^3 t_3 + 406912 t_2^3 Z^3 - 311872 t_2^2 t_3^2 Z + 43087 t_2 t_3^3 Z^2 + 32000 t_3^5 + 37103 t_3^4 Z^3,$$

*and the harmonic elements of $V_2$ are proportional to*

$$1260 t_1 t_2 - 224 t_2^2 Z - 154 t_2 t_3 Z^2 - 80 t_3^3 - 35 t_3^2 Z^3.$$

*Proof.* Using Proposition 4.0.1 we can directly calculate

$$\Delta(t_1) = \frac{7}{20} Z^2, \tag{4.2.13}$$

$$\Delta(t_2) = -\frac{1}{2} Z, \tag{4.2.14}$$

$$\Delta(t_3) = \frac{9}{10}. \tag{4.2.15}$$

A general element of $V_1$ is of the form

$$a_1 t_1^3 + a_2 t_1^2 t_2 Z + a_3 t_1^2 t_3 Z^2 + a_4 t_1 t_2^2 Z^2 + a_5 t_1 t_2 t_3^2 + a_6 t_1 t_2 t_3 Z^3 + a_7 t_1 t_3^3 Z$$

$$+a_8t_2^3t_3 + a_9t_2^3Z^3 + a_{10}t_2^2t_3^2Z + a_{11}t_2t_3^3Z^2 + a_{12}t_3^5 + a_{13}t_3^4Z^3, \tag{4.2.16}$$

where $a_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.2.16) using Proposition 4.0.1 and formulas (4.2.13)–(4.2.15) we find that the only harmonic elements of $V_1$ are as claimed. A general element of $V_2$ has the form

$$b_1t_1t_2 + b_2t_1t_3Z + b_3t_2^2Z + b_4t_2t_3Z^2 + b_5t_3^3 + b_6t_3^2Z^3, \tag{4.2.17}$$

where $b_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.2.17) using Propostion 4.0.1 and formulas (4.2.13)–(4.2.15) we find that the only harmonic elements of $V_2$ are as claimed. □

**Theorem 4.2.4.** *We have the following relations*

$$\begin{aligned} y_1 &= \frac{128000}{19683}\left(12000t_1^3 - 3360t_1t_2^2Z^2 - 3390t_1t_2t_3^2 - 3120t_1t_2t_3Z^3 + 810t_1t_3^3Z\right. \\ &\quad \left. +4112t_2^3t_3 + 2176t_2^3Z^3 - 2176t_2^2t_3^2Z - 119t_2t_3^3Z^2 + 200t_3^5 + 119t_3^4Z^3\right), && (4.2.18) \\ y_2 &= \frac{3200}{729}\left(180t_1t_2 - 32t_2^2Z - 22t_2t_3Z^2 - 5t_3^3 - 5t_3^2Z^3\right), && (4.2.19) \\ y_3 &= \frac{20}{3}t_3. && (4.2.20) \end{aligned}$$

*Proof.* Note that $Y_3 = \frac{1}{6}y_3 = \frac{10}{9}t_3$. We now equate $Y_1$ and $Y_2$ given by relations (4.2.8)–(4.2.10) with general harmonic elements of $V_1$ and $V_2$, respectively, given by Proposition 4.2.3. We then rearrange these equations to find $y_i$ in terms of $t_j$ and $Z$. We find

$$\begin{aligned} y_1 &= \frac{25600000}{18711}t_3^5 + \frac{a}{583440}(2244000t_1^3 - 628320t_1t_2^2Z^2 - 1168530t_1t_2t_3^2 \\ &\quad - 583440t_1t_2t_3Z^3 + 151470t_1t_3^3Z + 768944t_2^3t_3 + 406912t_2^3Z^3 \\ &\quad - 311872t_2^2t_3^2Z + 43087t_2t_3^3Z^2 + 32000t_3^5 + 37103t_3^4Z^3) \\ &\quad - \frac{80b}{119}(1260t_1t_2 - 224t_2^2Z - 154t_2t_3Z^2 - 80t_3^3 - 35t_3^2Z^3), && (4.2.21) \\ y_2 &= \frac{16000}{567}t_3^3 - \frac{b}{35}(1260t_1t_2 - 224t_2^2Z - 154t_2t_3Z^2 - 80t_3^3 - 35t_3^2Z^3), && (4.2.22) \\ y_3 &= \frac{20}{3}t_3, && (4.2.23) \end{aligned}$$

where $a, b \in \mathbb{C}$. In order to find $a$ and $b$ we perform steps 5–7 from the introduction of Chapter 4. That is, we transform the intersection form (4.2.12) into $y$ coordinates by applying formulas (4.2.21)–(4.2.23) and compare it with the expression given by Lemma 4.2.1. We find that $a = \frac{133120000}{6561}$ and $b = -\frac{16000}{729}$, which implies the statement. □

**Proposition 4.2.5.** *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, Z]$.

*Proof.* We have $P(t,Z)=0$ by relation (4.2.11). Hence

$$0=\frac{\partial P}{\partial t_j}=\frac{\partial P^F}{\partial t_j}+\frac{\partial P^F}{\partial Z}\frac{\partial Z}{\partial t_j}.$$

Therefore

$$\frac{\partial Z}{\partial t_j}=-\frac{\frac{\partial P^F}{\partial t_j}}{\frac{\partial P^F}{\partial Z}}.$$

We thus have that

$$\frac{\partial y_i}{\partial t_j}=\frac{\partial y_i^F}{\partial t_j}-\frac{\partial y_i^F}{\partial Z}\frac{\frac{\partial P^F}{\partial t_j}}{\frac{\partial P^F}{\partial Z}}. \tag{4.2.24}$$

The first term is polynomial in $t_1,t_2,t_3$ and $Z$. The polynomial $P^F$ is irreducible over $\mathbb{C}[t_1,t_2,t_3]$ and thus $\frac{\partial P^F}{\partial Z}$ is invertible in the field $\mathbb{C}(t_1,t_2,t_3)[Z]/(P^F)$, where $\mathbb{C}(t_1,t_2,t_3)$ is the field of rational functions in $t_1,t_2$ and $t_3$. Hence the second term in equality (4.2.24) can be represented as an element of the ring $\mathbb{C}(t_1,t_2,t_3)[Z]$, when we reduce it modulo $P^F$ as a polynomial in $Z$. It can be checked that it is a polynomial in $t_1,t_2$ and $t_3$. □

**Proposition 4.2.6.** *We have that*

$$\det(g^{ij}(t))=\frac{c\prod\limits_{\alpha\in R_{H_3}}(\alpha,\,x)}{Q(t,Z)^2},$$

*where* $c=-3^{26}\cdot 5$ *and*

$$\begin{aligned}Q(t,Z)=2^3\cdot 5^7\big(&24000t_1^3-10208t_2^3t_3+540t_1t_2t_3^2+(2484t_2^2t_3^2+540t_1t_3^3-9600t_1^2t_2)Z\\&+(3360t_1t_2^2-3600t_1^2t_3-189t_2t_3^3)Z^2+(720t_1t_2t_3-4544t_2^3-81t_3^4)Z^3\big).\end{aligned}$$

By Proposition 4.0.3, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 4.2.4, which leads to Proposition 4.2.6.

In the next statement we express flat coordinates $t_i$ via basic invariants $y_j$ and $Z$, which is an inversion of the formulas from Theorem 4.2.4.

**Theorem 4.2.7.** *We have the following relations:*

$$t_1=-\frac{1}{28800Z(20Z^3+3y_3)}\left(102400Z^9+20160y_3Z^6+1080y_3^2Z^3+729y_2+54y_3^3\right), \tag{4.2.25}$$

$$t_2=-\frac{Z}{20}\left(20Z^3+3y_3\right), \tag{4.2.26}$$

$$t_3=\frac{3}{20}y_3, \tag{4.2.27}$$

*where* $Z$ *satisfies the equation*

$$2^{29}5^{11}Z^{27}+2^{27}3^35^{10}y_3Z^{24}+2^{22}3^45^8151\,y_3^2Z^{21}+2^{18}3^45^7\left(2^27^219\,y_3^3-3^3y_2\right)Z^{18}$$

$$
\begin{aligned}
&+ 2^{13}3^{6}5^{5}\left(60089\, y_3^4 - 2^2 3^3 11\, y_2 y_3\right) Z^{15} + 2^{10}3^{7}5^{3}\left(5^2 2{\cdot}11{\cdot}19{\cdot}41\, y_3^5 - 3^3 263\, y_2 y_3^2 + 2^2 3^7 y_1\right) Z^{12} \\
&+ 2^9 3^7 5^2 \left(2^3 3^6 y_2^2 + 3^9 2\, y_1 y_3 + 3^3 19{\cdot}41\, y_2 y_3^3 + 5^2 2{\cdot}4987\, y_3^6\right) Z^9 + 2^6 3^9 5\left(3^6 7\, y_2^2 y_3 + 2^2 3^8 y_1 y_3^2\right. \\
&\left.+ 2^5 3^3 23\, y_2 y_3^4 + 2^2 5^3 53\, y_3^7\right) Z^6 + 2^3 3^{10}\left(3^6 5\, y_2^2 y_3^2 + 2^3 3^7 y_1 y_3^3 + 2^2 3^3 131\, y_2 y_3^5 - 2^2 5^2 7\, y_3^8\right) Z^3 \\
&+ 3^9\left(3^9 y_2^3 + 3^7 2\, y_2^2 y_3^3 + 2^2 3^4 y_2 y_3^6 + 2^3 y_3^9\right) = 0, \qquad (4.2.28)
\end{aligned}
$$

*and $y_i$ are given by relations (4.2.1)–(4.2.7).*

*Proof.* Formula (4.2.27) follows immediately from Theorem 4.2.4, and formula (4.2.26) follows from relation (4.2.11). Substituting the relations (4.2.26) and (4.2.27) into formula (4.2.19) we get the expression (4.2.25). Finally, substituting relations (4.2.25)–(4.2.27) into formula (4.2.18) we get the formula (4.2.28). □

**Proposition 4.2.8.** *The unity vector field $e = \partial_{t_1}$ in the $y$ coordinates has the form*

$$
e(y) = \frac{64000}{81} t_2 \partial_{y_2} + \frac{1280000}{6561}\left(1200 t_1^2 - 112 t_2^2 Z^2 - 113 t_2 t_3^2 - 104 t_2 t_3 Z^3 + 27 t_3^3\right)\partial_{y_1}.
$$

*Proof.* We have that

$$
e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},
$$

which gives the statement by applying the relations from Theorem 4.2.4. □

### 4.2.2 $(H_3)''$ example

The prepotential for $(H_3)''$ is

$$
F(t) = \frac{1}{2}\left(t_2^2 t_1 + t_3 t_1^2\right) + \frac{4063}{1701} t_3^7 + \frac{19}{135} t_3^5 Z^2 - \frac{73}{27} t_3^3 Z^4 + \frac{11}{9} t_3 Z^6 - \frac{16}{35} Z^7,
$$

where

$$
P(t_2, t_3, Z) := Z^2 + t_2 - t_3^2 = 0. \qquad (4.2.29)
$$

The Euler vector field is

$$
E(t) = t_1 \partial_{t_1} + \frac{2}{3} t_2 \partial_{t_2} + \frac{1}{3} t_3 \partial_{t_3},
$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{2}{3}$. The intersection form (2.2.19) is then given by

$$
g^{ij}(t) = \begin{pmatrix}
\frac{4}{243}(585 t_2^2 t_3 + 3240 t_2 t_3^3 + 4456 t_3^5 - 324 Z(t_2^2 - 7 t_2 t_3^2 + 6 t_3^4)) & -\frac{4}{27}(33 t_2^2 + 4 t_2 t_3(18Z - 13 t_3) - 72 t_3^3(Z + t_3)) & t_1 \\
-\frac{4}{27}(33 t_2^2 + 4 t_2 t_3(18Z - 13 t_3) - 72 t_3^3(Z + t_3) & t_1 - \frac{22}{3} t_2 t_3 + \frac{52}{27} t_3^3 + 4Z(t_2 - t_3^2) & \frac{2}{3} t_2 \\
t_1 & \frac{2}{3} t_2 & \frac{1}{3} t_3
\end{pmatrix}. \qquad (4.2.30)
$$

We have that $\deg t_1(x) = 6$, $\deg t_2(x) = 4$, $\deg t_3(x) = 2$ and $\deg Z(x) = 2$.

**Proposition 4.2.9.** *Let* $V_1 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 10\}$ *and let* $V_2 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 6\}$*. The harmonic elements of* $V_1$ *are proportional to*

$$25245t_1t_2 + 22275t_1t_3^2 - 16830t_2^2t_3 - 20196t_2^2Z + 21890t_2t_3^3 + 40392t_2t_3^2Z - 104196t_3^5 - 20196t_3^4Z,$$

*and the harmonic elements of* $V_2$ *are proportional to*

$$189t_1 + 630t_2t_3 + 400t_3^3.$$

*Proof.* Using Proposition 4.0.1 we can directly calculate

$$\Delta(t_1) = -\frac{5}{27}(33t_2 - 26t_3^2 + 54t_3Z), \tag{4.2.31}$$

$$\Delta(t_2) = \frac{1}{3}(9Z - 11t_3), \tag{4.2.32}$$

$$\Delta(t_3) = \frac{1}{2}. \tag{4.2.33}$$

A general element of $V_1$ is of the form

$$a_1t_1t_2 + a_2t_1t_3^2 + a_3t_1t_3Z + a_4t_2^2t_3 + a_5t_2^2Z + a_6t_2t_3^3 + a_7t_2t_3^2Z + a_8t_3^5 + a_9t_3^4Z, \tag{4.2.34}$$

where $a_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.2.34) using Proposition 4.0.1 and formulas (4.2.31)–(4.2.33) we find that the only harmonic elements of $V_1$ are as claimed. A general element of $V_2$ has the form

$$b_1t_1 + b_2t_2t_3 + b_3t_2Z + b_4t_3^3 + b_5t_2^2Z, \tag{4.2.35}$$

where $b_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.2.35) using Propostion 4.0.1 and formulas (4.2.31)–(4.2.33) we find that the only harmonic elements of $V_2$ are as claimed. ☐

**Theorem 4.2.10.** *We have the following relations*

$$\begin{aligned} y_1 = \frac{288}{25}\big(&135t_1t_2 + 405t_1t_3^2 - 90t_2^2t_3 - 108t_2^2Z + 1070t_2t_3^3 \\ &+216t_2t_3^2Z + 2292t_3^5 - 108t_3^4Z\big), \end{aligned} \tag{4.2.36}$$

$$y_2 = \frac{8}{5}\left(27t_1 + 90t_2t_3 + 160t_3^3\right), \tag{4.2.37}$$

$$y_3 = 12t_3. \tag{4.2.38}$$

*Proof.* Note that $Y_3 = \frac{1}{6}y_3 = 2t_3$. We now equate $Y_1$ and $Y_2$ given by relations (4.2.8)–(4.2.10) with general harmonic elements of $V_1$ and $V_2$, respectively, given by Proposition

4.2.9. We then rearrange these equations to find $y_i$ in terms of $t_j$ and $Z$. We find

$$\begin{aligned} y_1 = &\frac{1990656}{77}t_3^5 + \frac{a}{40392}(25245t_1t_2 + 22275t_1t_3^2 - 16830t_2^2t_3 - 20196t_2^2Z \\ &+ 21890t_2t_3^3 + 40392t_2t_3^2Z - 104196t_3^5 - 20196t_3^4Z) \\ &+ \frac{81b}{425}t_3^2(189t_1 + 630t_2t_3 + 400t_3^3), \end{aligned} \quad (4.2.39)$$

$$y_2 = \frac{1152}{7}t_3^3 + \frac{b}{400}(189t_1 + 630t_2t_3 + 400t_3^3), \quad (4.2.40)$$

$$y_3 = 12t_3, \quad (4.2.41)$$

where $a, b \in \mathbb{C}$. In order to find $a$ and $b$ we perform steps 5–7 from the introduction of Chapter 4. That is, we transform the intersection form (4.2.30) into $y$ coordinates by applying formulas (4.2.39)–(4.2.41) and compare it with the expression given by Lemma 4.2.1. We find that $a = \frac{62208}{25}$ and $b = \frac{640}{7}$, which implies the statement. □

**Proposition 4.2.11.** *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, Z]$.

Proof is similar to the one for Proposition 4.2.5.

**Proposition 4.2.12.** *We have that*

$$\det(g^{ij}(t)) = \frac{c \prod_{\alpha \in R_{H_3}} (\alpha, x)}{Q(t, Z)^2},$$

*where* $c = -2^{14} \cdot 5^5$ *and*

$$Q(t, Z) = 3^6 \left(56t_3^3 + 126t_2t_3 - 27t_1 + 54(t_2 - t_3^2)Z\right).$$

By Proposition 4.0.3, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated Theorem 4.2.10, which leads to Proposition 4.2.12.

In the next statement we express flat coordinates $t_i$ via basic invariants $y_j$ and $Z$, which is an inversion of the formulas from Theorem 4.2.10.

**Theorem 4.2.13.** *We have the following relations:*

$$t_1 = \frac{5}{23328}\left(108y_2 - 25y_3^3 + 1296y_3Z^2\right), \quad (4.2.42)$$

$$t_2 = \frac{1}{144}y_3^2 - Z^2, \quad (4.2.43)$$

$$t_3 = \frac{1}{12}y_3, \quad (4.2.44)$$

*where* $Z$ *satisfies the equation*

$$31104Z^5 + 12960Z^4y_3 + (900y_2 - 360y_3^3)Z^2 + (25y_1 - 25y_2y_3^2 + 2y_3^5) = 0, \quad (4.2.45)$$

*and $y_i$ are given by relations (4.2.1)–(4.2.7).*

*Proof.* Formula (4.2.44) follows immediately from Theorem 4.2.10, and formula (4.2.43) follows from relation (4.2.29). Substituting the relations (4.2.43) and (4.2.44) into formula (4.2.37) we get the expression (4.2.42). Finally, substituting relations (4.2.42)–(4.2.44) into formula (4.2.36) we get the formula (4.2.45). □

**Proposition 4.2.14.** *The unity vector field $e = \partial_{t_1}$ in the $y$ coordinates has the form*

$$e(y) = \frac{216}{5}\partial_{y_2} + \frac{7776}{5}\left(t_2 + 3t_3^2\right)\partial_{y_1}.$$

*Proof.* We have that

$$e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 4.2.10. □

## 4.3 Algebraic Frobenius manifold related to $D_4$

The $D_4(a_1)$ Frobenius manifold has been described by Pavlyk [43] and Dinar [7], with a prepotential given explicitly by Pavlyk. It is a four dimensional Frobenius manifold which can be associated to the Coxeter group $D_4$, it is denoted with the conjugacy class $a_1$ in the Coxeter group $D_4$ [3]. The prepotential for $D_4(a_1)$ is

$$F(t) = \frac{19\,t_4^5}{2^6 3^4 5} + \frac{7\,t_4^3 t_3^2}{2^5 3^3} - \frac{t_4^3 t_2}{2{\cdot}3^3} + \frac{t_4 t_3^4}{2^6 3} + \frac{t_4 t_3^2 t_2}{6} + \frac{t_4 t_2^2}{6} + \frac{t_4 t_1^2}{2} + t_2 t_3 t_1 - \frac{Z^5}{2^3 3^4 5},$$

where

$$P(t_2, t_3, t_4, Z) := Z^2 - (t_4^2 + 3t_3^2 + 24t_2) = 0. \tag{4.3.1}$$

The Euler vector field is

$$E(t) = t_1\partial_{t_1} + t_2\partial_{t_2} + \frac{1}{2}t_3\partial_{t_3} + \frac{1}{2}t_4\partial_{t_4},$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{1}{2}$. We note that slightly different prepotentials and coordinates are used in Pavlyk [43] and Dinar [7]. The intersection form (2.2.19) is then given by

$$g^{11}(t) = \frac{1}{864}\left(t_4(19t_4^2 + 63t_3^2 - 144t_2) - 2Z(4t_4^2 + 3t_3^2 + 24t_2)\right), \tag{4.3.2}$$

$$g^{12}(t) = \frac{1}{96}t_3\left(t_4(7t_4 - 2Z) + 3t_3^2 + 48t_2\right), \tag{4.3.3}$$

$$g^{22}(t) = \frac{1}{288}\left(t_4(7t_4^2 + 27t_3^2 + 144t_2) - 2Z(t_4^2 + 12t_3^2 + 24t_2)\right), \tag{4.3.4}$$

$$g^{13}(t) = \frac{1}{18}\left(6t_2 + 3t_3^2 - t_4(t_4 + Z)\right), \tag{4.3.5}$$

$$g^{23}(t) = t_1 + \frac{1}{6}t_3(2t_4 - Z),\ g^{33}(t) = \frac{1}{6}(t_4 - 2Z), \tag{4.3.6}$$

$$g^{14}(t) = t_1,\ g^{24}(t) = t_2,\ g^{34}(t) = \frac{1}{2}t_3,\ g^{44}(t) = \frac{1}{2}t_4. \tag{4.3.7}$$

Let $R_{D_4}$ be the following root system for $D_4$:

$$R_{D_4} = \{\pm e_i \pm e_j \mid 1 \le i < j \le 4\}.$$

Let us introduce the following basic invariants for $D_4$ (cf. [47]):

$$y_1 = x_1^6 + x_2^6 + x_3^6 + x_4^6, \tag{4.3.8}$$

$$y_2 = x_1 x_2 x_3 x_4, \tag{4.3.9}$$

$$y_3 = x_1^4 + x_2^4 + x_3^4 + x_4^4, \tag{4.3.10}$$

$$y_4 = x_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{4.3.11}$$

The basic invariants $y_1, y_2, y_3, y_4$ have degrees $6, 4, 4, 2$, respectively.

**Lemma 4.3.1.** *(cf. [47]) The intersection form $g^{ij}(y)$ takes the form*

$$g^{ij}(y) = \begin{pmatrix} 30y_1y_3 - 180y_2^2y_4 + 30y_1y_4^2 - 30y_3y_4^3 + 6y_4^5 & 6y_2y_3 & 32y_1y_4 - 96y_2^2 + 12y_3^2 - 24y_3y_4^2 + 4y_4^4 & 12y_1 \\ 6y_2y_3 & \frac{1}{6}\left(2y_1 - 3y_3y_4 + y_4^3\right) & 4y_2y_4 & 8y_2 \\ 32y_1y_4 - 96y_2^2 + 12y_3^2 - 24y_3y_4^2 + 4y_4^4 & 4y_2y_4 & 16y_1 & 8y_3 \\ 12y_1 & 8y_2 & 8y_3 & 4y_4 \end{pmatrix}.$$

Consider another set of basic invariants for $D_4$ given by

$$Y_1 = y_1 - \frac{5}{4}y_3y_4 + \frac{5}{16}y_4^3, \tag{4.3.12}$$

$$Y_2 = y_2, \tag{4.3.13}$$

$$Y_3 = y_3 - \frac{1}{2}y_4^2, \tag{4.3.14}$$

$$Y_4 = \frac{1}{8}y_4. \tag{4.3.15}$$

The following statement can be checked directly.

**Lemma 4.3.2.** *We have $\Delta(Y_4) = 1$ and $\Delta(Y_1) = \Delta(Y_2) = \Delta(Y_3) = 0$.*

We have that $\deg t_1(x) = 4$, $\deg t_2(x) = 4$, $\deg t_3(x) = 2$, $\deg t_4(x) = 2$ and $\deg Z(x) = 2$.

**Proposition 4.3.3.** *Let $V_1 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 6\}$ and let $V_2 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 4\}$. The harmonic elements of $V_1$ are proportional to*

$$216t_1t_3 + 72t_2t_4 + 24t_2Z - 9t_3^2t_4 + 3t_3^2Z + t_4^3 + t_4^2Z,$$

*and the harmonic elements of $V_2$ are of the form*

$$a(4t_1 - t_3t_4) + b(t_4^2 + 3t_3^2 - 8t_2),$$

*where $a$, $b \in \mathbb{C}$ are constants.*

*Proof.* Using Proposition 4.0.1 we can directly calculate

$$\Delta(t_1) = \frac{t_3}{Z}(2Z - t_4), \tag{4.3.16}$$

$$\Delta(t_2) = \frac{1}{4}\left(2t_4 - Z - \frac{3t_3^2}{Z}\right), \tag{4.3.17}$$

$$\Delta(t_3) = -\frac{t_3}{Z}, \tag{4.3.18}$$

$$\Delta(t_4) = 1. \tag{4.3.19}$$

A general element of $V_1$ is of the form

$$\begin{aligned} a_1t_4t_2 + &a_2t_4t_1 + a_3t_4Z + a_4t_3t_2 + a_5t_3t_1 + a_6t_3Z + a_7t_2^3 \\ &+ a_8t_2^2t_1 + a_9t_2^2Z + a_{10}t_2t_1^2 + a_{11}t_2t_1Z + a_{12}t_1^3 + a_{13}t_1^2Z, \end{aligned} \tag{4.3.20}$$

where $a_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.3.20) using Proposition 4.0.1 and formulas (4.3.16)–(4.3.19) we find that the only harmonic elements of $V_1$ are as claimed. A general element of $V_2$ has the form

$$b_1t_4 + b_2t_3 + b_3t_2^2 + b_4t_2t_1 + b_5t_2Z + b_6t_1^2 + b_7t_1Z, \tag{4.3.21}$$

where $b_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.3.21) using Propostion 4.0.1 and formulas (4.3.16)–(4.3.19) we find that the only harmonic elements of $V_2$ are as claimed. □

**Theorem 4.3.4.** *Define*

$$y_1 = -\frac{16}{9}\left(216t_1t_3 - 288t_2t_4 + 24t_2Z + 126t_3^2t_4 + 3t_3^2Z - 44t_4^3 + t_4^2Z\right), \tag{4.3.22}$$

$$y_2 = 4\left(4t_1 - t_3t_4\right), \tag{4.3.23}$$

$$y_3 = 8\left(3t_4^2 - 3t_3^2 + 8t_2\right), \tag{4.3.24}$$

$$y_4 = 8t_4. \tag{4.3.25}$$

*Under the corresponding tensorial transformation the intersection form given by formulas (4.3.2)–(4.3.7) takes the form given in Lemma 4.3.1.*

*Proof.* Note that $Y_4 = \frac{1}{8}y_4 = t_4$. We now equate $Y_1$ with a general harmonic element of $V_1$, and we equate $Y_2$ and $Y_3$ with general harmonic elements of $V_2$, where $Y_1, Y_2$ and $Y_3$

are given by formulas (4.3.12)–(4.3.14) and the harmonic elements of $V_1$ and $V_2$ are given by Proposition 4.3.3. We then rearrange these equations to find $y_i$ in terms of $t_j$ and $Z$. We find

$$\begin{aligned} y_1 =& 160t_4^3 + \frac{a_1}{24}(216t_1t_3 + 72t_2t_4 + 24t_2Z - 9t_3^2t_4 + 3t_3^2Z + t_4^3 + t_4^2Z) \\ &+ 10t_4\left(\frac{a_3}{4}(4t_1 - t_3t_4) + b_3(t_4^2 + 3t_3^2 - 8t_2)\right), \quad (4.3.26) \\ y_2 =& \frac{a_2}{4}(4t_1 - t_3t_4) + b_2(t_4^2 + 3t_3^2 - 8t_2^2), \quad (4.3.27) \\ y_3 =& 32t_4^2 + \frac{a_3}{4}(4t_1 - t_3t_4) + b_3(t_4^2 + 3t_3^2 - 8t_2), \quad (4.3.28) \\ y_4 =& 8t_4, \quad (4.3.29) \end{aligned}$$

where $a_i, b_j \in \mathbb{C}$. In order to find $a_i$ and $b_j$ we perform steps 5–7 from the introduction of Chapter 4. That is, we transform the intersection form (4.3.2)–(4.3.7) into $y$ coordinates by applying formulas (4.3.26)–(4.3.29) and compare it with the expression given by Lemma 4.3.1. A particular solution is given by

$$a_1 = -\frac{128}{3}, \quad a_2 = 16, \quad a_3 = 0, \quad b_2 = 0, \quad b_3 = -8,$$

which implies the statement. □

**Remark 4.3.5.** There are in fact five other ways to choose $y_i$ in Theorem 4.3.4 as polynomials of $t_j$ and $Z$. This non-uniqueness is due to the $S_3$ symmetry of the Coxeter graph of $D_4$.

**Proposition 4.3.6.** *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 4.2.5.

**Proposition 4.3.7.** *We have that*

$$\det(g^{ij}(t)) = \frac{c \prod_{\alpha \in R_{D_4}} (\alpha,\, x)}{Q(t, Z)^2},$$

*where* $c = 9$ *and*

$$Q(t, Z) = 2^{14}\left(12t_1 + 5t_3t_4 + 2t_3Z\right).$$

By Proposition 4.0.3, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 4.3.4, which leads to Proposition 4.3.7.

In the next statement we express flat coordinates $t_i$ via basic invariants $y_j$ and $Z$, which is an inversion of formulas from Theorem 4.3.4.

**Theorem 4.3.8.** *We have the following relations:*

$$t_1 = -\frac{1}{13824y_2}\left(32y_4Z^3 + 24y_4^2Z^2 + 18y_1y_4 - 864y_2^2 - 27y_3y_4^2 + 7y_4^4\right), \tag{4.3.30}$$

$$t_2 = \frac{1}{512}\left(16Z^2 + 2y_3 - y_4^2\right), \tag{4.3.31}$$

$$t_3 = -\frac{1}{432y_2}\left(32Z^3 + 24y_4Z^2 + 18y_1 - 27y_3y_4 + 7y_4^3\right), \tag{4.3.32}$$

$$t_4 = \frac{1}{8}y_4, \tag{4.3.33}$$

*where $Z$ satisfies the equation*

$$\begin{aligned}&2^{10}Z^6 + 2^9 3\, y_4Z^5 + 2^6 3^2 y_4^2 Z^4 + 2^6\left(7\, y_4^3 - 3^3 y_3y_4 + 3^2 2\, y_1\right)Z^3 + 2^4 3\left(7\, y_4^4\right.\\ &\left.- 3^3 y_3y_4^2 + 3^2 2\, y_1y_4 - 2^2 3^4 y_2^2\right)Z^2 + 7^2 y_4^6 - 3^3 2{\cdot}7\, y_3y_4^4 + 2^2 3^2 7\, y_1y_4^3\\ &+ 3^6 y_3^2y_4^2 - 2^3 3^5 y_2^2y_4^2 - 2^2 3^5 y_1y_3y_4 + 2^3 3^6 y_2^2y_3 + 2^2 3^4 y_1^2 = 0,\end{aligned} \tag{4.3.34}$$

*and $y_i$ are given by relations (4.3.8)–(4.3.11).*

*Proof.* Formula (4.3.33) follows immediately from Theorem 4.3.4. Using relations (4.3.1) and (4.3.23) we see that

$$t_1 = \frac{1}{32}\left(2y_2 + t_3y_4\right), \tag{4.3.35}$$

$$t_2 = \frac{1}{24}\left(Z^2 - 3t_3^2 - \frac{1}{64}y_4^2\right). \tag{4.3.36}$$

Substituting the relations (4.3.33) and (4.3.36) into formula (4.3.24) and rearranging, we get

$$t_3^2 = \frac{1}{75}\left(Z^2 + \frac{71}{64}y_4^2 - 3y_3\right). \tag{4.3.37}$$

We can then substitute relations (4.3.33), (4.3.35) and (4.3.36) into formula (4.3.22) and reduce modulo $t_3^2$ using relation (4.3.37) to find a linear equation in $t_3$ which we can rearrange to find relation (4.3.32). Substituting relations (4.3.32) and (4.3.33) into formulas (4.3.35) and (4.3.36) gives us relation (4.3.30) and the following:

$$\begin{aligned}t_2 = -\frac{1}{1492992y_2^2}(&1024Z^6 + 1152y_1Z^3 + 324y_1^2 - 62208y_2^2Z^2 + 1536y_4Z^5 + 864y_1y_4Z^2\\ &- 1728y_3y_4Z^3 - 972y_1y_3y_4 + 576y_4^2Z^4 + 972y_2^2y_4^2 - 1296y_3y_4^2Z^2\\ &+ 729y_3^2y_4^2 + 448y_4^3Z^3 + 252y_1y_4^3 + 336y_4^4Z^2 - 378y_3y_4^4 + 49y_4^6).\end{aligned} \tag{4.3.38}$$

We then substitute relations (4.3.30), (4.3.32), (4.3.33) and (4.3.38) into formula (4.3.22) and we get the formula (4.3.34). Finally, reducing relation (4.3.38) modulo the polynomial (4.3.34) in $Z$ gives us relation (4.3.31). □

**Proposition 4.3.9.** *The unity vector field $e = \partial_{t_1}$ in the $y$ coordinates has the form*

$$e(y) = 16\left(\partial_{y_2} - 24 t_3 \partial_{y_1}\right).$$

*Proof.* We have that

$$e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 4.3.4. □

## 4.4 Algebraic Frobenius manifold related to $F_4$

The $F_4(a_2)$ Frobenius manifold was described by Dinar with a prepotential given explicitly [7] (there seem to be some typos for the prepotential in [7], we include a corrected version below which was communicated to us by Dinar). It is a four dimensional Frobenius manifold which can be associated to the Coxeter group $F_4$, and is denoted by the conjugacy class $a_2$ in the Coxeter group $F_4$ [3]. The prepotential for $F_4(a_2)$ is

$$\begin{aligned}
F(t) = {} & \frac{2^5 3^7 67\cdot 521749}{5^9 7} t_4^7 + \frac{2^4 3^{10} 13\cdot 693097}{5^9 7} t_4^6 t_3 + \frac{2^2 3^8 13^2 23^2 7\cdot 97}{5^9} t_4^5 t_3^2 + \frac{3^7 13^3 18224639}{2^6 5^8 7} t_4^4 t_3^3 \\
& + \frac{3^8 13^4 7243667}{2^{11} 5^8 7} t_4^3 t_3^4 + \frac{3^8 13^5 8754721}{2^{14} 5^9 7} t_4^2 t_3^5 + \frac{3^7 13^6 19\cdot 435503}{2^{18} 5^9 7} t_4 t_3^6 + \frac{3^8 13^7 41\cdot 7129}{2^{22} 5^9 7} t_3^7 \\
& + \frac{2^4 3^4 71\cdot 4259}{5^5 7\cdot 13} t_4^4 t_2 - \frac{2^3 3^4 23\cdot 47}{5^5 7} t_4^3 t_3 t_2 - \frac{3^5 13\cdot 103\cdot 293}{2^3 5^5 7} t_4^2 t_3^2 t_2 - \frac{3^4 13^2 79\cdot 467}{2^6 5^5 7} t_4 t_3^3 t_2 \\
& - \frac{3^4 13^3 157\cdot 383}{2^{12} 5^5 7} t_3^4 t_2 + \frac{2^4}{13^2} t_4 t_2^2 - \frac{7}{13} t_3 t_2^2 + \Bigg( \frac{3^{14} 139}{5^5 2\cdot 7} t_4^5 + \frac{3^{13} 13\cdot 19}{5^4 2\cdot 7} t_4^4 t_3 + \frac{3^{15} 13^3}{2^{11} 5^4} t_4^2 t_3^3 \\
& + \frac{3^{12} 13^3 101}{2^9 5^4 7} t_4^3 t_3^2 + \frac{3^{13} 13^4 31}{2^{16} 5^4 7} t_4 t_3^4 + \frac{3^{12} 13^5 41}{2^{20} 5^5 7} t_3^5 - \frac{3^{11}}{2^2 5\cdot 7\cdot 13} t_4^2 t_2 - \frac{3^{10}}{2^5 5\cdot 7} t_4 t_3 t_2 \\
& - \frac{3^9 13}{2^{10} 5\cdot 7} t_3^2 t_2 \Bigg) Z^2 + \Bigg( \frac{2^2 3^9 89\cdot 11701}{5^7 7} t_4^6 + \frac{3^{11} 13\cdot 68473}{5^7 7} t_4^5 t_3 + \frac{3^{10} 13^3 1949}{2^3 5^6 7} t_4^4 t_3^2 \\
& + \frac{3^9 13^3 131357}{2^8 5^6 7} t_4^3 t_3^3 + \frac{3^{12} 13^4 2729}{2^{12} 5^6 7} t_4^2 t_3^4 + \frac{3^{10} 13^5 15937}{2^{15} 5^7 7} t_4 t_3^5 + \frac{3^9 13^6 29\cdot 43}{2^{20} 5^7} t_3^6 + \frac{3^3 5^2}{13^2 7} t_2^2 \\
& - \frac{2^2 3^6 139}{5^2 7\cdot 13} t_4^3 t_2 - \frac{3^8 23}{5^2 2\cdot 7} t_4^2 t_3 t_2 - \frac{3^7 13\cdot 73}{2^5 5^2 7} t_4 t_3^2 t_2 - \frac{3^6 13^2 41}{2^9 5^2 7} t_3^3 t_2 \Bigg) Z + t_3 t_2 t_1 + \frac{t_4 t_1^2}{2},
\end{aligned}$$

where

$$\begin{aligned}
P(t_2, t_3, t_4, Z) := Z^3 & - \frac{2^3 3^4 13}{5^4}\left(\frac{2^3 3}{13} t_4^2 + t_4 t_3 + \frac{13}{2^5 3} t_3^2\right) Z + \frac{2^2 3\cdot 13}{5^6}\Bigg(\frac{2^6 5^4}{3^3 13^2} t_2 \\
& - \frac{2^7 139}{13} t_4^3 - 2^4 3^2 23\, t_4^2 t_3 - 3\cdot 13\cdot 73\, t_4 t_3^2 - \frac{13^2 41}{2^4} t_3^3\Bigg) = 0. \qquad (4.4.1)
\end{aligned}$$

The Euler vector field is

$$E(t) = t_1 \partial_{t_1} + t_2 \partial_{t_2} + \frac{1}{3} t_3 \partial_{t_3} + \frac{1}{3} t_4 \partial_{t_4},$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{2}{3}$. The intersection form (2.2.19) is then given by

$$
\begin{aligned}
g^{11}(t) = & -\frac{3^4}{2^{13}5^8 13}\big(259200000000Z^2t_2 + 861120000000Zt_2t_3 + 1833744640000t_2t_3^2 \\
& - 94787461372500Z^2t_3^3 - 229118338413900Zt_3^4 - 184708373655429t_3^5 \\
& + 1067520000000Zt_2t_4 + 6818938880000t_2t_3t_4 - 593226777780000Z^2t_3^2t_4 \\
& - 1466753056797600Zt_3^3t_4 - 1556308486273320t_3^4t_4 - 1870069760000t_2t_4^2 \\
& - 996804506880000Z^2t_3t_4^2 - 4153975366694400Zt_3^2t_4^2 - 6641896760778240t_3^3t_4^2 \\
& - 546311900160000Z^2t_4^3 - 6910723746201600Zt_3t_4^3 - 16691391832227840t_3^2t_4^3 \\
& - 5389922879078400Zt_4^4 - 17222218351902720t_3t_4^4 - 6432998677807104t_4^5\big)\,, \qquad (4.4.2)
\end{aligned}
$$

$$
\begin{aligned}
g^{12}(t) = & -\frac{3^4}{2^{17}5^8}\big(86400000000t_2Z^2 + 911040000000t_2t_3Z + 1440474880000t_2t_3^2 \\
& - 44608926082500t_3^3Z^2 - 143790305946300t_3^4Z - 157672431777393t_3^5 \\
& + 1059840000000t_2t_4Z + 4513832960000t_2t_3t_4 - 330765116760000t_3^2t_4Z^2 \\
& - 1081698384499200t_3^3t_4Z - 1043122465279440t_3^4t_4 + 4196270080000t_2t_4^2 \\
& - 700228488960000t_3t_4^2Z^2 - 2682937817164800t_3^2t_4^2Z - 3140123415886080t_3^3t_4^2 \\
& - 382258206720000t_4^3Z^2 - 3411310364467200t_3t_4^3Z - 5919243052769280t_3^2t_4^3 \\
& -2126376537292800t_4^4Z - 7197815592714240t_3t_4^4 - 3176813316538368t_4^5\big)\,, \qquad (4.4.3)
\end{aligned}
$$

$$
\begin{aligned}
g^{22}(t) = & -\frac{3^4 13}{2^{21}5^8}\big(28800000000t_2Z^2 + 511680000000t_2t_3Z + 1958344960000t_2t_3^2 \\
& - 19207343902500t_3^3Z^2 - 78756703307100t_3^4Z - 114276677239881t_3^5 \\
& + 1121280000000t_2t_4Z + 3545784320000t_2t_3t_4 - 158303428920000t_3^2t_4Z^2 \\
& - 707890736966400t_3^3t_4Z - 936110404686480t_3^4t_4 + 2777743360000t_2t_4^2 \\
& - 383440936320000t_3t_4^2Z^2 - 2025454752921600t_3^2t_4^2Z - 2337422808015360t_3^3t_4^2 \\
& - 244681482240000t_4^3Z^2 - 2221827115622400t_3t_4^3Z - 3112967576309760t_3^2t_4^3 \\
& -1046938278297600t_4^4Z - 2618007725998080t_3t_4^4 - 1417602668691456t_4^5\big)\,, \qquad (4.4.4)
\end{aligned}
$$

$$
\begin{aligned}
g^{13}(t) = & \frac{1}{2^6 5^4 13^2}\big(1280000t_2 - 924007500t_3Z^2 - 3897258300t_3^2Z - 2181574863t_3^3 \\
& - 3411720000t_4Z^2 - 9067593600t_3t_4Z - 4518684144t_3^2t_4 - 5620492800t_4^2Z \\
& +9861336576t_3t_4^2 + 50200031232t_4^3\big)\,, \qquad (4.4.5)
\end{aligned}
$$

$$
\begin{aligned}
g^{23}(t) = & \frac{1}{2^{10}5^4 13}\big(8320000t_1 - 8960000t_2 - 308002500t_3Z^2 - 2188871100t_3^2Z \\
& - 3888894321t_3^3 - 1137240000t_4Z^2 - 9593251200t_3t_4Z - 8055045648t_3^2t_4 \\
& -5580057600t_4^2Z - 5561457408t_3t_4^2 + 4045676544t_4^3\big)\,, \qquad (4.4.6)
\end{aligned}
$$

$$
g^{33}(t) = \frac{2}{13^2 3}\left(150Z - 91t_3 + 16t_4\right), \qquad (4.4.7)
$$

$$
g^{14}(t) = t_1, \qquad g^{24}(t) = t_2, \qquad g^{34}(t) = \frac{1}{3}t_3, \qquad g^{44}(t) = \frac{1}{3}t_4. \qquad (4.4.8)
$$

Let $R_{F_4}$ be the following root system for $F_4$:

$$R_{F_4} = \{\pm e_i \mid 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\} \cup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\}.$$

Let us introduce the following basic invariants for $F_4$ (cf. [47]):

$$\begin{aligned} y_1 &= 288\epsilon_2\epsilon_4 - 108\epsilon_1^2\epsilon_4 - 8\epsilon_2^3 + 3\epsilon_1^2\epsilon_2^2, \\ y_2 &= 12\epsilon_4 - 3\epsilon_1\epsilon_3 + \epsilon_2^2, \\ y_3 &= 6\epsilon_3 - \epsilon_1\epsilon_2, \\ y_4 &= \epsilon_1, \end{aligned}$$

where

$$\begin{aligned} \epsilon_1 &= x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ \epsilon_2 &= x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2, \\ \epsilon_3 &= x_1^2x_2^2x_3^2 + x_1^2x_2^2x_4^2 + x_1^2x_3^2x_4^2 + x_2^2x_3^2x_4^2, \\ \epsilon_4 &= x_1^2x_2^2x_3^2x_4^2. \end{aligned}$$

The basic invariants $y_1, y_2, y_3, y_4$ have degrees $12, 8, 6, 2$, respectively.

**Lemma 4.4.1.** *(cf. [47]) The entries of the intersection form $g^{ij}(y)$ are*

$$\begin{aligned} g^{11}(y) &= 1152y_2^2y_3 - 144y_1y_2y_4 + 1152y_2y_3^2y_4 - 144y_1y_3y_4^2 + 288y_3^3y_4^2, \\ g^{12}(y) &= -96y_2^2y_4 - 48y_2y_3y_4^2, \qquad g^{22}(y) = -8y_2y_3 - y_1y_4 + 3y_3^2y_4, \\ g^{13}(y) &= 192y_2^2 + 120y_2y_3y_4 - 12y_1y_4^2 + 12y_3^2y_4^2, \\ g^{23}(y) &= 2y_1 - 6y_3^2 - 8y_2y_4^2, \qquad g^{33}(y) = 20y_2y_4 - 4y_3y_4^2, \\ g^{14}(y) &= 24y_1, \quad g^{24}(y) = 16y_2, \quad g^{34}(y) = 12y_3, \quad g^{44}(y) = 4y_4. \end{aligned}$$

Consider another set of basic invariants for $F_4$ given by

$$Y_1 = y_1 + \frac{1}{3080}y_4\left(2520y_2y_4 + 1708y_3y_4^2 + 61y_4^5\right), \tag{4.4.9}$$

$$Y_2 = y_2 + \frac{1}{160}y_4\left(40y_3 + 3y_4^3\right), \tag{4.4.10}$$

$$Y_3 = y_3 - \frac{1}{8}y_4^3, \tag{4.4.11}$$

$$Y_4 = \frac{1}{8}y_4, \tag{4.4.12}$$

The following statement can be checked directly.

**Lemma 4.4.2.** *We have $\Delta(Y_4) = 1$ and $\Delta(Y_1) = \Delta(Y_2) = \Delta(Y_3) = 0$.*

We have that $\deg t_1(x) = 6$, $\deg t_2(x) = 6$, $\deg t_3(x) = 2$, $\deg t_4(x) = 2$ and $\deg Z(x) = 2$.

**Theorem 4.4.3.** *Define*

$$
\begin{aligned}
y_1 = & \frac{1}{2^{15}3^9 5^6 13^2}\big(2^{17}5^8 13^2 t_1^2 - 2^{22}5^8 13\, t_1 t_2 + 2^{22}5^8 17\, t_2^2 + 2^{14}3^6 5^8 13^2 t_2 t_3 Z^2 \\
& - 2^{12}3^6 5^6 13^3 41\, t_2 t_3^2 Z + 2^{13}3^3 5^4 13^5 7\cdot 29\, t_1 t_3^3 - 2^{12}3^3 5^5 13^4 491\, t_2 t_3^3 - 2^2 3^9 5^4 13^6 41\, t_3^4 Z^2 \\
& + 2^2 3^9 5^2 11^2 13^8 t_3^5 Z + 3^6 13^8 1202837\, t_3^6 + 2^{18}3^7 5^8 13\, t_1 t_3^3 - 2^{17}3^6 5^6 13^2 23\, t_2 t_3 t_4 Z \\
& + 2^{17}3^4 5^4 13^4 59\, t_1 t_3^2 t_4 + 2^{16}3^4 5^5 13^3 67\, t_2 t_3^2 t_4 - 2^8 3^{10} 5^4 13^5 41\, t_3^3 t_4 Z^2 + 2^6 3^9 5^3 13^6 43^2 t_3^4 t_4 Z \\
& + 2^5 3^7 13^8 111347\, t_3^5 t_4 + 2^{20}3^6 5^6 13\cdot 31\, t_2 t_4^2 Z - 2^{19}3^4 5^4 13^3 367\, t_1 t_3 t_4^2 + 2^{20}3^4 5^5 13^4 11\, t_2 t_3 t_4^2 \\
& - 2^{11}3^{10}5^4 13^4 269\, t_3^2 t_4^2 Z^2 + 2^{11}3^9 5^3 13^5 757\, t_3^3 t_4^2 Z + 2^8 11^2 5\cdot 23633\cdot 6892993\, t_3^4 t_4^2 \\
& + 2^{22}3^3 5^4 13^2 43\cdot 61\, t_1 t_4^3 - 2^{25}3^3 5^5 11\cdot 13\cdot 701\, t_2 t_4^3 - 2^{15}3^9 5^4 13^3 1039\, t_3 t_4^3 Z^2 \\
& - 2^{13}3^9 5^3 13^4 9431\, t_3^2 t_4^3 Z + 2^{13}3^6 13^5 5\cdot 1939033\, t_3^3 t_4^3 - 2^{18}3^{10}5^4 13^2 557\, t_4^4 Z^2 \\
& - 2^{22}3^9 5^3 13^3 587\, t_3 t_4^4 Z - 2^{16}3^7 13^4 5\cdot 23\cdot 206351\, t_3^2 t_4^4 - 2^{21}3^9 5^2 13^3 17\cdot 257\, t_4^5 Z \\
& - 2^{22}3^7 13^3 19\cdot 71\cdot 2383\, t_3 t_4^5 + 2^5 3\cdot 43\cdot 103\cdot 149\cdot 2791\cdot 1285517\, t_4^6\big), \qquad (4.4.13)\\
y_2 = & \frac{1}{2^{12}3^5 5^4 13}\big(2^{13}5^4 13^2 t_1 t_3 - 2^{13}5^4 7\cdot 13\, t_2 t_3 + 2^2 3^5 5^4 13^3 t_3^2 Z^2 - 2^9 3^4 13^3 47\cdot 593\, t_3^2 t_4^2 \\
& + 2^2 3^4 5^2 13^4 41\, t_3^3 Z + 3^3 13^5 11\cdot 1171\, t_3^4 + 2^{17}5^4 13\, t_1 t_4 - 2^{17}5^6 t_2 t_4 + 2^7 3^6 5^4 13^2 t_3 t_4 Z^2 \\
& + 2^6 3^5 5^2 13^3 73\, t_3^2 t_4 Z + 2^6 3^3 13^4 17\cdot 79\, t_3^3 t_4 + 2^{10}3^7 5^4 13\, t_4^2 Z^2 + 2^{10}3^6 5^2 13^2 23\, t_3 t_4^2 Z \\
& - 2^{12}5^6 3\, t_2 Z + 2^{13}3^4 5^2 13\cdot 139\, t_4^3 Z - 2^{15}3^3 13^2 23\cdot 1303\, t_3 t_4^3 - 2^{16}3^3 13\cdot 62539\, t_4^4\big), \qquad (4.4.14)\\
y_3 = & \frac{1}{2^4 3^4 5\cdot 13}\big(-2^5 5\cdot 13\, t_1 + 2^9 3\cdot 5\, t_2 + 3^3 13^4 t_3^3 + 2^4 3^5 13^3 t_3^2 t_4 + 2^6 3^4 13^2 11\, t_3 t_4^2 \\
& - 2^9 3^3 13\cdot 79\, t_4^3\big), \qquad (4.4.15)\\
y_4 = & 12 t_4. \qquad (4.4.16)
\end{aligned}
$$

*Under the corresponding tensorial transformation the intersection form given by formulas (4.4.2)–(4.4.8) takes the form given in Lemma 4.4.1.*

A proof of this theorem is given in Appendix A as the formulas involved are too long to present here. The method of the proof is similar to those given for Theorems 4.2.4, 4.2.10 and 4.3.4. No proof was presented in [26], where these results were first published.

**Remark 4.4.4.** There is in fact one other way to choose $y_i$ in Theorem 4.4.3 as polynomials of $t_j$ and $Z$. This non-uniqueness is due to the $\mathbb{Z}_2$ symmetry of the Coxeter graph of $F_4$.

**Proposition 4.4.5.** *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 4.2.5.

**Proposition 4.4.6.** *We have that*

$$\det(g^{ij}(t)) = \frac{c \prod_{\alpha \in R_{F_4}} (\alpha,\, x)}{Q(t, Z)^2},$$

*where* $c = 2^{36}3^{42}5^{8}13^{4}$ *and*

$$\begin{aligned}
Q(t,Z) = & -2^{16}5^{6}13^{2}t_1^2 + 2^{17}5^{6}7\cdot 13\, t_1t_2 + 2^{21}5^{6}t_2^2 - 2^{8}3^{6}5^{6}13^{3}Z^2t_1t_3 + 2^{12}3^{7}5^{6}13^{2}Z^2t_2t_3 \\
& - 2^{8}3^{5}5^{4}13^{4}41\, Zt_1t_3^2 - 2^{14}3^{5}5^{4}13^{4}Zt_2t_3^2 + 2^{6}3^{3}5^{2}13^{5}17\cdot 797\, t_1t_3^3 - 2^{10}3^{3}5^{2}13^{5}919\, t_2t_3^3 \\
& - 2^{2}3^{9}5^{4}13^{6}47\, Z^2t_3^4 + 2^{2}3^{8}5^{2}13^{7}17\cdot 41\, Zt_3^5 - 3^{6}13^{8}5\cdot 89\cdot 97\, t_3^6 - 2^{12}3^{7}5^{6}13^{2}Z^2t_1t_4 \\
& + 2^{14}3^{6}5^{6}11\cdot 13\, Z^2t_2t_4 - 2^{13}3^{5}5^{4}13^{3}73\, Zt_1t_3t_4 + 2^{13}3^{5}5^{4}13^{2}229\, Zt_2t_3t_4 \\
& + 2^{10}3^{4}5^{2}13^{5}919\, t_1t_3^2t_4 - 2^{12}3^{4}5^{2}13^{3}72889\, t_2t_3^2t_4 - 2^{3}3^{9}5^{4}13^{5}19\cdot 149\, Z^2t_3^3t_4 \\
& + 2^{3}3^{8}5^{2}13^{6}41^{2}17\, Zt_3^4t_4 + 3^{7}13^{7}2\cdot 17\cdot 23\cdot 37\cdot 59\, t_3^5t_4 - 2^{16}3^{6}5^{4}13^{2}23\, Zt_1t_4^2 \\
& + 2^{17}3^{6}5^{4}13\cdot 227\, Zt_2t_4^2 + 2^{14}3^{4}5^{2}13^{3}19\cdot 1039\, t_1t_3t_4^2 - 2^{17}3^{4}5^{2}13^{2}64871\, t_2t_3t_4^2 \\
& - 2^{7}3^{10}5^{6}13^{4}83\, Z^2t_3^2t_4^2 + 2^{9}3^{9}5^{2}13^{5}7\cdot 11\cdot 41\, Zt_3^3t_4^2 + 2^{5}3^{7}13^{6}5\cdot 41\cdot 37649\, t_3^4t_4^2 \\
& + 2^{20}3^{3}5^{2}13^{3}17\cdot 47\, t_1t_4^3 - 2^{20}3^{3}5^{2}13\cdot 188701\, t_2t_4^3 - 2^{11}3^{9}5^{4}13^{3}3571\, Z^2t_3t_4^3 \\
& + 2^{13}3^{8}5^{2}13^{4}11\cdot 97\, Zt_3^2t_4^3 + 2^{17}3^{6}13^{5}5\cdot 52057\, t_3^3t_4^3 - 2^{17}3^{9}5^{4}13^{2}11\, Z^2t_4^4 \\
& - 2^{14}3^{8}5^{2}13^{3}11\cdot 37\cdot 139\, Zt_3t_4^4 + 2^{13}3^{7}13^{4}5\cdot 17\cdot 499\cdot 659\, t_3^2t_4^4 - 2^{18}3^{11}5^{3}13^{2}197\, Zt_4^5 \\
& + 2^{17}3^{7}13^{3}5\cdot 11\cdot 247439\, t_3t_4^5 + 2^{22}3^{6}7^{2}13^{2}19\cdot 41\cdot 61\, t_4^6.
\end{aligned}$$

By Proposition 4.0.3, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 4.4.3, which leads to Proposition 4.4.6.

The relations for $t_i, Z$ in terms of the $y$ coordinates can be found in Appendix A.1, along with the expression for $e$ in terms of the $y$ coordinates. We do not present these here as they are too long and for this reason we chose not to include these in our paper [26].

## 4.5 Algebraic Frobenius manifolds related to $H_4$

There are 6 known non-polynomial algebraic Frobenius manifolds which can be associated to $H_4$, they are each four-dimensional and their prepotentials have been listed by Sekiguchi [48]. Let $R_{H_4}$ be the following root system for $H_4$:

$$R_{H_4} = \{\pm e_i \mid 1 \le i \le 4\} \cup \left\{\frac{1}{2}\left(\pm e_1 \pm e_2 \pm e_3 \pm e_4\right)\right\} \cup \left\{\frac{1}{2}\left(\pm e_{\sigma(2)} \pm \varphi e_{\sigma(3)} \pm \overline{\varphi} e_{\sigma(4)}\right) \,\middle|\, \sigma \in \mathfrak{A}_4\right\},$$

where

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \overline{\varphi} = \frac{1-\sqrt{5}}{2},$$

and $\mathfrak{A}_4$ is the alternating group on 4 elements. Let us introduce the following basic invariants for $H_4$ (cf. [47]):

$$\begin{aligned}
y_1 = & \frac{32}{3}x_1^{24}h_2^3 - 40x_1^{22}\left(2h_2^4 + 3h_2h_6\right) + x_1^{20}\left(360h_{10} + \frac{1344}{5}h_2^5 + 672h_2^2h_6\right) \\
& + x_1^{18}\left(1080h_6^2 - 1608h_2^3h_6 - \frac{1328}{3}h_2^6 - 2880h_{10}h_2\right) + x_1^{16}\left(10024h_{10}h_2^2 + 272h_2^7\right.
\end{aligned}$$

$$+1248h_2^4h_6 - 5628h_2h_6^2\big) + x_1^{14}\left(18588h_2^2h_6^2 + 272h_2^8 - 7620h_{10}h_6 - 16856h_{10}h_2^3\right)$$
$$+ x_1^{12}\left(14216h_{10}h_2^4 + 23508h_{10}h_6h_2 - \frac{1328}{3}h_2^9 - 1248h_6h_2^6 - 27396h_6^2h_2^3 - 5796h_6^3\right)$$
$$+ x_1^{10}\Big(3240h_{10}^2 - 7160h_{10}h_2^5 - 25332h_{10}h_6h_2^2 + \frac{1344}{5}h_2^{10} + 1608h_2^7h_6 + 19968h_2^4h_6^2$$
$$+7350h_2h_6^3\Big) + x_1^8\big(2144h_{10}h_2^6 - 3232h_{10}^2h_2 + 10908h_{10}h_2^3h_6 - 906h_{10}h_6^2 - 80h_2^{11}$$
$$-672h_2^8h_6 - 6924h_2^5h_6^2 - 1956h_2^2h_6^3\big) + x_1^6\Big(1168h_{10}^2h_2^2 - 344h_{10}h_2^7 - 2172h_{10}h_2^4h_6$$
$$-1908h_{10}h_2h_6^2 + \frac{32}{3}h_2^{12} + 120h_2^9h_6 + 1332h_2^6h_6^2 + 288h_2^3h_6^3 + 2394h_6^4\Big)$$
$$+ x_1^4\big(348h_{10}^2h_6 - 152h_{10}^2h_2^3 + 16h_{10}h_2^8 + 60h_{10}h_6h_2^5 + 408h_{10}h_6^2h_2^2 - 84h_2^7h_6^2$$
$$+84h_2^4h_6^3 - 909h_2h_6^4\big) + x_1^2\left(8h_{10}^2h_2^4 - 42h_{10}h_6^2h_2^3 - 87h_{10}h_6^3 - 6h_2^5h_6^3 + 135h_2^2h_6^4\right)$$
$$+ \frac{4}{3}h_{10}^3 - 3h_{10}h_2h_6^3 + \frac{9}{5}h_6^5, \tag{4.5.1}$$

$$y_2 = 4x_1^{16}h_2^2 - 10x_1^{14}\left(2h_2^3 + 3h_6\right) + x_1^{12}\left(44h_2^4 + 138h_2h_6\right) + x_1^{10}\left(180h_{10} - 44h_2^5 - 402h_2^2h_6\right)$$
$$+ x_1^8\left(44h_2^6 - 464h_{10}h_2 + 402h_2^3h_6 + 294h_6^2\right) + x_1^6\left(296h_{10}h_2^2 - 20h_2^7 - 138h_2^4h_6 - 306h_2h_6^2\right)$$
$$+ x_1^4\left(4h_2^8 - 76h_{10}h_2^3 - 114h_{10}h_6 + 30h_2^5h_6 + 168h_2^2h_6^2\right) + x_1^2\left(4h_{10}h_2^4 - 21h_2^3h_6^2 + \frac{57}{2}h_6^3\right)$$
$$+ h_{10}^2 - \frac{3}{2}h_2h_6^3, \tag{4.5.2}$$

$$y_3 = -2x_1^{10}h_2 + 6x_1^8h_2^2 + x_1^6\left(33h_6 - 14h_2^3\right) - x_1^4\left(33h_2h_6 - 6h_2^4\right) + x_1^2\left(11h_{10} - 2h_2^5\right)$$
$$- h_{10}h_2 + \frac{3}{2}h_6^2, \tag{4.5.3}$$

$$y_4 = x_1^2 + h_2, \tag{4.5.4}$$

where

$$h_2 = \epsilon_1, \tag{4.5.5}$$
$$h_6 = \sqrt{5}\delta + \epsilon_1\epsilon_2 - 11\epsilon_3, \tag{4.5.6}$$
$$h_{10} = 95\epsilon_2\epsilon_3 - 32\epsilon_1^2\epsilon_3 - 5\epsilon_1\epsilon_2^2 + 2\epsilon_1^3\epsilon_2 + 3\sqrt{5}\delta\epsilon_2, \tag{4.5.7}$$

and

$$\epsilon_1 = x_2^2 + x_3^2 + x_4^2, \tag{4.5.8}$$
$$\epsilon_2 = x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2, \tag{4.5.9}$$
$$\epsilon_3 = x_2^2x_3^2x_4^2, \tag{4.5.10}$$
$$\delta = (x_2^2 - x_3^2)(x_2^2 - x_4^2)(x_3^2 - x_4^2). \tag{4.5.11}$$

The basic invariants $y_1, y_2, y_3, y_4$ have degrees $30, 20, 12, 2$, respectively.

**Lemma 4.5.1.** *(cf. [47]) The entries of the intersection form $g^{ij}(y)$ are*

$$
\begin{aligned}
g^{11}(y) &= \frac{928}{3}y_2y_3^3y_4 + 240y_1y_3^2y_4^2 + 96y_2^2y_3y_4^3 + 160y_1y_2y_4^4,\\
g^{12}(y) &= -32y_3^4 - 112y_2y_3^2y_4^2 - 120y_1y_3y_4^3 + 48y_2^2y_4^4,\\
g^{22}(y) &= \frac{152}{3}y_3^3y_4 - 56y_2y_3y_4^3 + 20y_1y_4^4,\\
g^{13}(y) &= -80y_2^2 - \frac{16}{3}y_3^3y_4^2 - 16y_2y_3y_4^4 - 40y_1y_4^5,\\
g^{23}(y) &= -30y_1 + 8y_3^2y_4^3 - 24y_2y_4^5, \qquad g^{33}(y) = 44y_2y_4 - 8y_3y_4^5,\\
g^{14}(y) &= 60y_1, \qquad g^{24}(y) = 40y_2, \qquad g^{34}(y) = 24y_3, \qquad g^{44}(y) = 5y_4.
\end{aligned}
$$

Consider another set of basic invariants for $H_4$ given by

$$Y_1 = y_1 - \frac{y_4^3}{30030}\left(4y_4^{12} + 320y_3y_4^6 + 7051y_2y_4^2 - 715y_3^2\right), \tag{4.5.12}$$

$$Y_2 = y_2 + \frac{y_4^4}{748}\left(3y_4^6 + 110y_3\right), \tag{4.5.13}$$

$$Y_3 = y_3 + \frac{y_4^6}{14}, \tag{4.5.14}$$

$$Y_4 = \frac{1}{8}y_4. \tag{4.5.15}$$

The following statement can be checked directly.

**Lemma 4.5.2.** *We have $\Delta(Y_4) = 1$ and $\Delta(Y_1) = \Delta(Y_2) = \Delta(Y_3) = 0$.*

### 4.5.1 $H_4(1)$ example

The prepotential for $H_4(1)$ is

$$
\begin{aligned}
F(t) = {} & t_1t_2t_3 + \frac{1}{2}t_1^2t_4 + \frac{3356}{665}t_4^{21} + \frac{64}{5}t_3t_4^{16} + \frac{472}{11}t_3^2t_4^{11} + \frac{16}{3}t_3^3t_4^6 + 28t_3^4t_4 - \frac{16}{15}t_2t_4^{15}\\
& + 8t_2t_3t_4^{10} + 32t_2t_3^2t_4^5 - \frac{8}{3}t_2t_3^3 + \frac{19}{18}t_2^2t_4^9 - t_2^2t_3t_4^4 + \frac{1}{6}t_2^3t_4^3 + \frac{1}{105}Z^7,
\end{aligned}
$$

where

$$P(t_2, t_3, t_4, Z) := Z^2 - 4t_4(t_4^5 - 3t_3) - t_2 = 0. \tag{4.5.16}$$

The Euler vector field is

$$E(t) = t_1\partial_{t_1} + \frac{3}{5}t_2\partial_{t_2} + \frac{1}{2}t_3\partial_{t_3} + \frac{1}{10}t_4\partial_{t_4},$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{9}{10}$. The intersection form (2.2.19) is then given by

$$\begin{aligned} g^{11}(t) =& \frac{1}{10}\left(228t_2t_3^2Z + 19t_2^3t_4 - 2736t_3^3t_4Z - 228t_2^2t_3t_4^2 + 12160t_2t_3^2t_4^3\right.\\ &+ 76t_2^2t_4^4Z + 3040t_3^3t_4^4 - 2736t_2t_3t_4^5Z + 22800t_3^2t_4^6Z + 1444t_2^2t_4^7\\ &+ 13680t_2t_3t_4^8 + 89680t_3^2t_4^9 + 1520t_2t_4^{10}Z - 21888t_3t_4^{11}Z\\ &\left.-4256t_2t_4^{13} + 58368t_3t_4^{14} + 4864t_4^{16}Z + 40272t_4^{19}\right), \end{aligned} \tag{4.5.17}$$

$$\begin{aligned} g^{12}(t) =& -\frac{3}{5}\left(t_2^2Z - 280t_3^3 - 54t_2t_3t_4Z + 504t_3^2t_4^2Z + 10t_2^2t_4^3 - 800t_2t_3t_4^4\right.\\ &-240t_3^2t_4^5 + 68t_2t_4^6Z - 936t_3t_4^7Z - 200t_2t_4^9 - 2360t_3t_4^{10}\\ &\left.+256t_4^{12}Z - 512t_4^{15}\right), \end{aligned} \tag{4.5.18}$$

$$\begin{aligned} g^{22}(t) =& -\frac{2}{5}\left(44t_2t_3 - 924t_3^2t_4 - 33t_2t_4^2Z + 396t_3t_4^3Z - 176t_2t_4^5 - 88t_3t_4^6\right.\\ &\left.-132t_4^8Z - 236t_4^{11}\right), \end{aligned} \tag{4.5.19}$$

$$\begin{aligned} g^{13}(t) =& \frac{7}{10}\left(-2t_2t_3Z + 24t_3^2t_4Z + 3t_2^2t_4^2 - 16t_2t_3t_4^3 + 320t_3^2t_4^4 + 4t_2t_4^5Z\right.\\ &\left.-56t_3t_4^6Z + 38t_2t_4^8 + 160t_3t_4^9 + 16t_4^{11}Z - 32t_4^{14}\right), \end{aligned} \tag{4.5.20}$$

$$g^{23}(t) = t_1 - 8t_3^2 - t_2t_4Z + 12t_3t_4^2Z - 2t_2t_4^4 + 64t_3t_4^5 - 4t_4^7Z + 8t_4^{10}, \tag{4.5.21}$$

$$g^{33}(t) = \frac{1}{40}\left(3t_2Z - 36t_3t_4Z + 36t_2t_4^3 - 72t_3t_4^4 + 12t_4^6Z + 76t_4^9\right), \tag{4.5.22}$$

$$g^{14}(t) = t_1, \quad g^{24}(t) = \frac{3}{5}t_2, \quad g^{34}(t) = \frac{1}{2}t_3, \quad g^{44}(t) = \frac{1}{10}t_4. \tag{4.5.23}$$

We have that $\deg t_1(x) = 20$, $\deg t_2(x) = 12$, $\deg t_3(x) = 10$, $\deg t_4(x) = 2$ and $\deg Z(x) = 6$.

**Proposition 4.5.3.** *Let* $V_1 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 30\}$, *let* $V_2 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 20\}$ *and let* $V_3 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 12\}$. *The harmonic elements of* $V_1$ *are proportional to*

$$\begin{aligned} &27027t_2^2Z - 1801800t_1t_3 - 6806800t_3^3 - 648648t_2t_3t_4Z + 3891888t_3^2t_4^2Z + 32175t_2^2t_4^3\\ &+ 13556400t_2t_3t_4^4 - 2335080t_1t_4^5 + 90256320t_3^2t_4^5 + 216216t_2t_4^6Z - 25594592t_3t_4^7Z\\ &- 4591440t_2t_4^9 + 35834480t_3t_4^{10} + 432432t_4^{12}Z - 864864t_4^{15}, \end{aligned}$$

*the harmonic elements of* $V_2$ *are proportional to*

$$561t_1 + 7106t_3^2 - 627t_2t_4^4 - 37620t_3t_4^5 - 12350t_4^{10},$$

*and the harmonic elements of* $V_3$ *are proportional to*

$$21t_2 + 308t_3t_4 - 220t_4^6.$$

*Proof.* Using Proposition 4.0.1 we can directly calculate

$$\Delta(t_1) = -\frac{19}{10}\left(t_2 Z - 30t_3t_4Z + 8t_2t_4^3 - 320t_3t_4^4 + 40t_4^6Z - 80t_4^9\right), \tag{4.5.24}$$

$$\Delta(t_2) = -\frac{11}{5}\left(8t_3 - 9t_4^2Z - 32t_4^5\right), \tag{4.5.25}$$

$$\Delta(t_3) = -\frac{9}{20}t_4\left(3Z + 4t_4^3\right), \tag{4.5.26}$$

$$\Delta(t_4) = \frac{1}{5}. \tag{4.5.27}$$

A general element of $V_1$ is of the form

$$a_1t_1t_3 + a_2t_1t_4^5 + a_3t_1t_4^2Z + a_4t_2^2t_4^3 + a_5t_2^2Z + a_6t_2t_3t_4^4 + a_7t_2t_3t_4Z + a_8t_2t_4^9 + a_9t_2t_4^6Z + a_{10}t_3^3 + a_{11}t_3^2t_4^5 + a_{12}t_3^2t_4^2Z + a_{13}t_3t_4^{10} + a_{14}t_3t_4^7Z + a_{15}t_4^{15} + a_{16}t_4^{12}Z, \tag{4.5.28}$$

where $a_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.5.28) using Proposition 4.0.1 and formulas (4.5.24)–(4.5.27) we find that the only harmonic elements of $V_1$ are as claimed. A general element of $V_2$ has the form

$$b_1t_1 + b_2t_2t_4^4 + b_3t_2t_4Z + b_4t_3^2 + b_5t_3t_4^5 + b_6t_3t_4^2Z + b_7t_4^{10} + b_8t_4^7Z, \tag{4.5.29}$$

where $b_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.5.29) using Propostion 4.0.1 and formulas (4.5.24)–(4.5.27) we find that the only harmonic elements of $V_2$ are as claimed. A general element of $V_3$ has the form

$$c_1t_2 + c_2t_3t_4 + c_3t_4^6 + c_4t_4^3Z, \tag{4.5.30}$$

where $c_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.5.30) using Propostion 4.0.1 and formulas (4.5.24)–(4.5.27) we find that the only harmonic elements of $V_3$ are as claimed. □

**Theorem 4.5.4.** *We have the following relations*

$$y_1 = \frac{2^{30}5^9}{3}\left(27t_2^2Z - 1800t_1t_3 - 6800t_3^3 - 648t_2t_3t_4Z + 3888t_3^2t_4^2Z + 12600t_2t_3t_4^4 + 6120t_1t_4^5 + 190320t_3^2t_4^5 + 216t_2t_4^6Z - 2592t_3t_4^7Z - 42840t_2t_4^9 - 953520t_3t_4^{10} + 432t_4^{12}Z + 1309136t_4^{15}\right), \tag{4.5.31}$$

$$y_2 = 2^{20}5^7\left(3t_1 + 38t_3^2 - 21t_2t_4^4 - 460t_3t_4^5 + 950t_4^{10}\right), \tag{4.5.32}$$

$$y_3 = 2^{11}5^4\left(3t_2 + 44t_3t_4 - 260t_4^6\right), \tag{4.5.33}$$

$$y_4 = 40t_4. \tag{4.5.34}$$

*Proof.* Note that $Y_4 = \frac{1}{8}y_4 = 5t_4$. We now equate $Y_1$, $Y_2$ and $Y_3$ given by relations (4.5.12)–

(4.5.14) with general harmonic elements of $V_1$, $V_2$ and $V_3$, respectively, given by Proposition 4.5.3. We then rearrange these equations to find $y_i$ in terms of $t_j$ and $Z$. We find

$$
\begin{aligned}
y_1 = & \frac{2^{42}3^25^{14}13}{7^311}t_4^{15} + \frac{a}{2^33^37\cdot 11\cdot 13}\left(27027t_2^2Z - 1801800t_1t_3 - 6806800t_3^3 - 648648t_2t_3t_4Z\right. \\
& + 3891888t_3^2t_4^2Z + 32175t_2^2t_4^3 + 13556400t_2t_3t_4^4 - 2335080t_1t_4^5 + 90256320t_3^2t_4^5 \\
& + 216216t_2t_4^6Z - 2594592t_3t_4^7Z - 4591440t_2t_4^9 + 35834480t_3t_4^{10} + 432432t_4^{12}Z \\
& \left.-864864t_4^{15}\right) - \frac{131276800b}{67431}t_4^5\left(561t_1 + 7106t_3^2 - 627t_2t_4^4 - 37620t_3t_4^5 - 12350t_4^{10}\right) \\
& + \frac{2^{23}5^82251c}{302379}t_4^9\left(21t_2 + 308t_3t_4 - 220t_4^6\right) - \frac{80c^2}{2541}t_4^3\left(21t_2 + 308t_3t_4 - 220t_4^6\right)^2, \qquad (4.5.35) \\
y_2 = & \frac{2^{29}5^{10}}{77}t_4^{10} + \frac{b}{12350}\left(-561t_1 - 7106t_3^2 + 627t_2t_4^4 + 37620t_3t_4^5 + 12350t_4^{10}\right) \\
& + \frac{320000c}{187}t_4^4\left(21t_2 + 308t_3t_4 - 220t_4^6\right), \qquad (4.5.36) \\
y_3 = & -\frac{2^{17}5^6}{7}t_4^6 + \frac{c}{220}\left(-21t_2 - 308t_3t_4 + 220t_4^6\right), \qquad (4.5.37) \\
y_4 = & 40t_4, \qquad (4.5.38)
\end{aligned}
$$

where $a, b, c \in \mathbb{C}$. In order to find $a, b$ and $c$ we perform steps 5–7 from the introduction of Chapter 4. That is, we transform the intersection form (4.5.17)–(4.5.23) into $y$ coordinates by applying formulas (4.5.35)–(4.5.38) and compare it with the expression given by Lemma 4.5.1. We find that

$$
a = 2^{33}3^25^9, \quad b = -\frac{2^{21}5^913\cdot 19}{11\cdot 17}, \quad c = -\frac{2^{13}5^511}{7},
$$

which implies the statement. ☐

**Proposition 4.5.5.** *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 4.2.5.

**Proposition 4.5.6.** *We have that*

$$
\det(g^{ij}(t)) = \frac{c \prod_{\alpha \in R_{H_4}} (\alpha, x)}{Q(t, Z)^2},
$$

*where* $c = 5$ *and*

$$
Q(t, Z) = 5^{20}\left(5t_1 - 70t_3^2 + 5t_2t_4Z - 60t_3t_4^2Z - 35t_2t_4^4 + 140t_3t_4^5 + 20t_4^7Z + 42t_4^{10}\right).
$$

By Proposition 4.0.3, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 4.5.4, which leads to Proposition 4.5.6.

In the next statement we express flat coordinates $t_i$ via basic invariants $y_j$ and $Z$, which is an inversion of the formulas from Theorem 4.5.4.

**Theorem 4.5.7.** *We have the following relations:*

$$t_1 = \frac{1}{2^{27}5^{12}3y_4^2}\left(-2^{26}3^2 5^{12}19\, Z^4 + 2^{16}5^8 3\cdot 19\, y_3 Z^2 - 2^4 5^4 19\, y_3^2 \right.$$
$$\left. + 2^7 5^5 y_2 y_4^2 + 2^{13}5^6 3\cdot 7\cdot 19\, y_4^6 Z^2 + 2^2 3^4 5^2 7\, y_3 y_4^6 + 3^2 7\cdot 47\, y_4^{12}\right), \tag{4.5.39}$$

$$t_2 = \frac{1}{2^{15}5^7}\left(2^{13}5^6 11\, Z^2 + 2^2 5^2 3\, y_3 + 23 y_4^6\right), \tag{4.5.40}$$

$$t_3 = \frac{1}{2^{15}5^6 y_4}\left(-2^{14}5^6 3\, Z^2 + 2^3 5^2 y_3 + 17 y_4^6\right), \tag{4.5.41}$$

$$t_4 = \frac{1}{40} y_4, \tag{4.5.42}$$

*where $Z$ satisfies the equation*

$$2^{34}3^3 5^{12} Z^6 - 2^{30}3^3 5^9 y_4^3 Z^5 - 2^{23}3^3 5^8 y_3 Z^4 + 2^{12}3^2 5^4 y_3^2 Z^2$$
$$- 2^{12}3^2 5^4 y_2 y_4^2 Z^2 - 2y_3^3 + 6 y_2 y_3 y_4^2 + 3 y_1 y_4^3 = 0, \tag{4.5.43}$$

*and $y_i$ are given by relations (4.5.1)–(4.5.11).*

*Proof.* Formula (4.5.42) follows immediately from Theorem 4.5.4. Using relations (4.5.16) and (4.5.33) we see that

$$t_2 = Z^2 + \frac{3}{10} t_3 y_4 - \frac{y_4^6}{2^{16}5^6},$$
$$t_3 = \frac{1}{2^{15}5^4 11\, y_4}\left(-2^{16}5^5 3\, t_2 + 160 y_3 + 13 y_4^6\right).$$

We can solve this system of equations to find $t_2$ and $t_3$ which gives us formulas (4.5.40) and (4.5.41). Substituting formulas (4.5.40)–(4.5.42) into relation (4.5.32) and solving for $t_1$ we get formula (4.5.39). Finally, substituting relations (4.5.39)–(4.5.42) into formula (4.5.31) we get the formula (4.5.43). □

**Proposition 4.5.8.** *The unity vector field $e = \partial_{t_1}$ in the $y$ coordinates has the form*

$$e(y) = 2^{20}5^7 3\, \partial_{y_2} - 2^{33}5^{10} 3\left(5t_3 - 17 t_4^5\right)\partial_{y_1}.$$

*Proof.* We have that

$$e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 4.5.4. □

### 4.5.2 $H_4(2)$ example

The prepotential for $H_4(2)$ is

$$\begin{aligned}F(t) = &-\frac{66084040}{73920}t_4^{16}+\frac{143564400}{73920}t_3^2t_4^{11}-\frac{40727610}{73920}t_3^4t_4^6-\frac{392931}{73920}t_3^6t_4+t_1t_2t_3+\frac{1}{2}t_1^2t_4\\ &-\frac{3}{4}t_4^4\left(2288t_4^{10}-1620t_3^2t_4^5-27t_3^4\right)Z-760t_4^{12}Z^2+\left(\frac{1744}{48}t_4^{10}-\frac{4860}{48}t_3^2t_4^5-\frac{81}{48}t_3^4\right)Z^3\\ &+140t_4^8Z^4+24t_4^6Z^5-\frac{53}{6}t_4^4Z^6-\frac{10}{7}t_4^2Z^7+\frac{Z^8}{4},\end{aligned}$$

where

$$P(t_2,t_3,t_4,Z) := Z^3-12t_4^4Z-11t_4^6+\frac{27}{4}t_3^2t_4-t_2=0. \qquad (4.5.44)$$

The Euler vector field is

$$E(t)=t_1\partial_{t_1}+\frac{4}{5}t_2\partial_{t_2}+\frac{1}{3}t_3\partial_{t_3}+\frac{2}{15}t_4\partial_{t_4},$$

the unity vector field is $e(t)=\partial_{t_1}$, and the charge is $d=\frac{13}{15}$. The intersection form (2.2.19) is then given by

$$\begin{aligned}g^{11}(t)=&\frac{1}{6}\left(-32Zt_2^2+567Z^2t_3^4+1440Zt_2t_3^2t_4-112t_2^2t_4^2-10530Zt_3^4t_4^2-20160t_2t_3^2t_4^3\right.\\ &-2784Z^2t_2t_4^4+20979t_3^4t_4^4+27864Z^2t_3^2t_4^5+2208Zt_2t_4^6-23976Zt_3^2t_4^7\\ &\left.+40416t_2t_4^8+648000t_3^2t_4^9-7776Z^2t_4^{10}-186624Zt_4^{12}+182736t_4^{14}\right), \qquad (4.5.45)\\ g^{12}(t)=&\frac{3}{4}t_3\left(-20Z^2t_2+81t_3^4+360Z^2t_3^2t_4+300Zt_2t_4^2-2925Zt_3^2t_4^3-1840t_2t_4^4\right.\\ &\left.+1170t_3^2t_4^5+1980Z^2t_4^6-1980Zt_4^8+18720t_4^{10}\right), \qquad (4.5.46)\\ g^{22}(t)=&-\frac{3}{10}\left(99t_2t_3^2+44Z^2t_2t_4-891t_3^4t_4-1287Z^2t_3^2t_4^2-220Zt_2t_4^3+5445Zt_3^2t_4^4\right.\\ &\left.+528t_2t_4^5+5841t_3^2t_4^6-396Z^2t_4^7+396Zt_4^9-3744t_4^{11}\right), \qquad (4.5.47)\\ g^{13}(t)=&-9Z^2t_3^2-8Zt_2t_4+90Zt_3^2t_4^2-8t_2t_4^3-468t_3^2t_4^4-72Z^2t_4^5+72Zt_4^7+792t_4^9, \qquad (4.5.48)\\ g^{23}(t)=&\frac{1}{4}\left(4t_1-27t_3^3-60Z^2t_3t_4+240Zt_3t_4^3-240t_3t_4^5\right), \qquad (4.5.49)\\ g^{33}(t)=&\frac{8}{27}\left(2Z^2-8t_4^2Z-19t_4^4\right), \qquad (4.5.50)\\ g^{14}(t)=&t_1, \quad g^{24}(t)=\frac{4}{5}t_2, \quad g^{34}(t)=\frac{1}{3}t_3, \quad g^{44}(t)=\frac{2}{15}t_4. \qquad (4.5.51)\end{aligned}$$

We have that $\deg t_1(x)=15$, $\deg t_2(x)=12$, $\deg t_3(x)=5$, $\deg t_4(x)=2$ and $\deg Z(x)=4$.

**Proposition 4.5.9.** *Let $V_1=\{p\in\mathbb{C}[t_1,t_2,t_3,t_4,Z]\mid \deg p(x)=30\}$, let $V_2=\{p\in\mathbb{C}[t_1,t_2,t_3,t_4,Z]\mid \deg p(x)=20\}$ and let $V_3=\{p\in\mathbb{C}[t_1,t_2,t_3,t_4,Z]\mid \deg p(x)=12\}$. The harmonic elements of $V_1$ are proportional to*

$$5765760t_1^2-51891840Z^2t_2t_3^2+188107920t_1t_3^3+302837535t_3^6+30750720Zt_2^2t_4$$

$$+ 350269920Z^2t_3^4t_4 - 155675520Zt_2t_3^2t_4^2 + 274743040t_2^2t_4^3 - 350269920Zt_3^4t_4^3 + 3090653280t_2t_3^2t_4^4 + 337438464Z^2t_2t_4^5 + 558122400t_1t_3t_4^5 + 33046299000t_3^4t_4^5 - 1810683072Z^2t_3^2t_4^6 + 96349440Zt_2t_4^7 - 1117385280Zt_3^2t_4^8 + 1662275328t_2t_4^9 + 185379977376t_3^2t_4^{10} + 1391157504Z^2t_4^{11} + 11062884096Zt_4^{13} - 21529753344t_4^{15},$$

*the harmonic elements of $V_2$ are proportional to*

$$47872Z^2t_2 - 269280t_1t_3 - 984555t_3^4 - 323136Z^2t_3^2t_4 - 239360Zt_2t_4^2 + 1615680Zt_3^2t_4^3 - 3512256t_2t_4^4 - 25208172t_3^2t_4^5 - 430848Z^2t_4^6 + 430848Zt_4^8 - 35274672t_4^{10},$$

*and the harmonic elements of $V_3$ are proportional to*

$$112t_2 + 2079t_3^2t_4 + 5940t_4^6.$$

*Proof.* Using Proposition 4.0.1 we can directly calculate

$$\Delta(t_1) = \frac{28t_3}{4t_2 - 27t_3^2t_4 - 20t_4^6}\left(-4Z^2t_2 + 45Z^2t_3^2t_4 + 48Zt_2t_4^2 - 360Zt_3^2t_4^3 - 208t_2t_4^4 + 1260t_3^2t_4^5 + 100Z^2t_4^6 - 400Zt_4^8 + 400t_4^{10}\right), \tag{4.5.52}$$

$$\Delta(t_2) = -\frac{11}{10\left(4t_2 - 27t_3^2t_4 - 20t_4^6\right)}\left(108t_2t_3^2 + 80Z^2t_2t_4 - 729t_3^4t_4 - 1260Z^2t_3^2t_4^2 - 320Zt_2t_4^3 + 3600Zt_3^2t_4^4 + 320t_2t_4^5 + 3060t_3^2t_4^6 - 400Z^2t_4^7 + 1600Zt_4^9 - 1600t_4^{11}\right), \tag{4.5.53}$$

$$\Delta(t_3) = \frac{64t_3t_4\left(2t_4^2 + Z\right)\left(4t_4^2 - Z\right)}{3\left(4t_2 - 27t_3^2t_4 - 20t_4^6\right)}, \tag{4.5.54}$$

$$\Delta(t_4) = \frac{4}{15}. \tag{4.5.55}$$

A general element of $V_1$ is of the form

$$a_1t_1^2 + a_2t_1t_3^3 + a_3t_1t_3t_4^5 + a_4t_1t_3t_4^3Z + a_5t_1t_3t_4Z^2 + a_6t_2^2t_4^3 + a_7t_2^2t_4Z + a_8t_2t_3^2t_4^4 + a_9t_2t_3^2t_4^2Z + a_{10}t_2t_3^2Z^2 + a_{11}t_2t_4^9 + a_{12}t_2t_4^7Z + a_{13}t_2t_4^5Z^2 + a_{14}t_3^6 + a_{15}t_3^4t_4^5 + a_{16}t_3^4t_4^3Z + a_{17}t_3^4t_4Z^2 + a_{18}t_3^2t_4^{10} + a_{19}t_3^2t_4^8Z + a_{20}t_3^2t_4^6Z^2 + a_{21}t_4^{15} + a_{22}t_4^{13}Z + a_{23}t_4^{11}Z^2, \tag{4.5.56}$$

where $a_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.5.56) using Proposition 4.0.1 and formulas (4.5.52)–(4.5.55) we find that the only harmonic elements of $V_1$ are as claimed. A general element of $V_2$ has the form

$$b_1t_1t_3 + b_2t_2t_4^4 + b_3t_2t_4^2Z + b_4t_2Z^2 + b_5t_3^4 + b_6t_3^2t_4^5 + b_7t_3^2t_4^3Z + b_8t_3^2t_4Z^2 + b_9t_4^{10} + b_{10}t_4^8Z + b_{11}t_4^6Z^2, \tag{4.5.57}$$

where $b_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.5.57) using Proposition 4.0.1 and formulas (4.5.52)–(4.5.55) we find that the only harmonic elements of $V_2$ are as claimed. A general element of $V_3$ has the form

$$c_1 t_2 + c_2 t_3^2 t_4 + c_3 t_4^6 + c_4 t_4^4 Z + c_5 t_4^2 Z^2, \tag{4.5.58}$$

where $c_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.5.58) using Proposition 4.0.1 and formulas (4.5.52)–(4.5.55) we find that the only harmonic elements of $V_3$ are as claimed. □

**Theorem 4.5.10.** *We have the following relations*

$$\begin{aligned}
y_1 = & 2^8 3^2 5^9 \big(5760t_1^2 - 51840Z^2t_2t_3^2 + 187920t_1t_3^3 + 302535t_3^6 + 30720Zt_2^2t_4 \\
& + 349920Z^2t_3^4t_4 - 155520Zt_2t_3^2t_4^2 + 266240t_2^2t_4^3 - 349920Zt_3^4t_4^3 + 2782080t_2t_3^2t_4^4 \\
& - 393216Z^2t_2t_4^5 + 4665600t_1t_3t_4^5 + 45198000t_3^4t_4^5 + 3120768Z^2t_3^2t_4^6 \\
& + 3747840Zt_2t_4^7 - 25764480Zt_3^2t_4^8 + 75859968t_2t_4^9 + 952477056t_3^2t_4^{10} \\
& + 7962624Z^2t_4^{11} + 4478976Zt_4^{13} + 4105230336t_4^{15}\big), \qquad (4.5.59) \\
y_2 = & - 2^4 3^2 5^6 \big(256Z^2t_2 - 1440t_1t_3 - 5265t_3^4 - 1728Z^2t_3^2t_4 - 1280Zt_2t_4^2 \\
& + 8640Zt_3^2t_4^3 - 31488t_2t_4^4 - 370656t_3^2t_4^5 - 2304Z^2t_4^6 + 2304Zt_4^8 - 2566656t_4^{10}\big), \qquad (4.5.60) \\
y_3 = & - 2^3 5^4 3 \big(16t_2 + 297t_3^2t_4 + 4320t_4^6\big), \qquad (4.5.61) \\
y_4 = & 30t_4. \qquad (4.5.62)
\end{aligned}$$

*Proof.* Note that $Y_4 = \frac{1}{8}y_4 = \frac{15}{4}t_4$. We now equate $Y_1$, $Y_2$ and $Y_3$ given by relations (4.5.12)–(4.5.14) with general harmonic elements of $V_1$, $V_2$ and $V_3$, respectively, given by Proposition 4.5.9. We then rearrange these equations to find $y_i$ in terms of $t_j$ and $Z$. We find

$$\begin{aligned}
y_1 = & \frac{2^{12}3^{17}5^{14}13}{7^3 11}t_4^{15} + \frac{a}{2^8 3 \cdot 11 \cdot 59 \cdot 677}\big(5765760t_1^2 - 51891840Z^2t_2t_3^2 \\
& + 188107920t_1t_3^3 + 3028375 35t_3^6 + 30750720Zt_2^2t_4 + 350269920Z^2t_3^4t_4 \\
& - 155675520Zt_2t_3^2t_4^2 + 274743040t_2^2t_4^3 - 350269920Zt_3^4t_4^3 + 3090653280t_2t_3^2t_4^4 \\
& + 337438464Z^2t_2t_4^5 + 558122400t_1t_3t_4^5 + 33046299000t_3^4t_4^5 - 1810683072Z^2t_3^2t_4^6 \\
& + 96349440Zt_2t_4^7 - 1117385280Zt_3^2t_4^8 + 1662275328t_2t_4^9 + 185379977376t_3^2t_4^{10} \\
& + 1391157504Z^2t_4^{11} + 11062884096Zt_4^{13} - 21529753344t_4^{15}\big) \\
& - \frac{3^2 5^4 641 b}{7 \cdot 13 \cdot 29 \cdot 8447}t_4^5\big(47872Z^2t_2 - 269280t_1t_3 - 984555t_3^4 - 323136Z^2t_3^2t_4 \\
& - 239360Zt_2t_4^2 + 1615680Zt_3^2t_4^3 - 3512256t_2t_4^4 - 25208172t_3^2t_4^5 - 430848Z^2t_4^6 \\
& + 430848Zt_4^8 - 35274672t_4^{10}\big) - \frac{2^5 3^5 5^8 2251 c}{7^2 11^2 17}t_4^9\big(112t_2 + 2079t_3^2t_4 + 5940t_4^6\big) \\
& - \frac{5c^2}{2^2 3^4 11^2 7}t_4^3\big(112t_2 + 2079t_3^2t_4 + 5940t_4^6\big)^2, \qquad (4.5.63)
\end{aligned}$$

$$y_2 = \frac{2^9 3^{10} 5^{10}}{77} t_4^{10} + \frac{b}{2^4 3^2 29 \cdot 8447} \left(-47872 Z^2 t_2 + 269280 t_1 t_3 + 984555 t_3^4 + 323136 Z^2 t_3^2 t_4 + 239360 Z t_2 t_4^2 - 1615680 Z t_3^2 t_4^3 + 3512256 t_2 t_4^4 + 25208172 t_3^2 t_4^5 + 430848 Z^2 t_4^6 - 430848 Z t_4^8 + 35274672 t_4^{10}\right) - \frac{3750c}{187} t_4^4 \left(112 t_2 + 2079 t_3^2 t_4 + 5940 t_4^6\right), \tag{4.5.64}$$

$$y_3 = -\frac{2^5 3^6 5^6}{7} t_4^6 + \frac{c}{5940}\left(112 t_2 + 2079 t_3^2 t_4 + 5940 t_4^6\right), \tag{4.5.65}$$

$$y_4 = 30 t_4, \tag{4.5.66}$$

where $a, b, c \in \mathbb{C}$. In order to find $a, b$ and $c$ we perform steps 5–7 from the introduction of Chapter 4. That is, we transform the intersection form (4.5.45)–(4.5.51) into $y$ coordinates by applying formulas (4.5.63)–(4.5.66) and compare it with the expression given by Lemma 4.5.1. We find that

$$a = \frac{2^{16} 3^3 5^9 59 \cdot 677}{91}, \quad b = \frac{2^8 3^4 5^6 29 \cdot 8447}{11 \cdot 17}, \quad c = -\frac{2^5 3^4 5^5 11}{7},$$

which implies the statement. □

**Proposition 4.5.11.** *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 4.2.5.

**Proposition 4.5.12.** *We have that*

$$\det(g^{ij}(t)) = \frac{c \prod_{\alpha \in R_{H_4}} (\alpha, x)}{Q(t, Z)^2},$$

*where* $c = 2^7 2^5$ *and*

$$Q(t, Z) = 3^7 5^{20} \left(4 t_1^2 + 36 Z^2 t_2 t_3^2 - 72 t_1 t_3^3 + 324 t_3^6 + 60 Z^2 t_1 t_3 t_4 - 783 Z^2 t_3^4 t_4 + 1080 Z t_2 t_3^2 t_4^2 - 240 Z t_1 t_3 t_4^3 - 5130 Z t_3^4 t_4^3 + 4932 t_2 t_3^2 t_4^4 - 1380 t_1 t_3 t_4^5 - 20871 t_3^4 t_4^5 + 11016 Z^2 t_3^2 t_4^6 - 11016 Z t_3^2 t_4^8 - 194076 t_3^2 t_4^{10}\right).$$

By Proposition 4.0.3, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 4.5.10, which leads to Proposition 4.5.12.

In the next statement we express flat coordinates $t_i$ via basic invariants $y_j$ and $Z$, which is an inversion of the formulas from Theorem 4.5.10.

**Theorem 4.5.13.** *We have the following relations:*

$$t_1 = \frac{1}{2^8 3^{11} 5^9 \sqrt{3} y_4^{\frac{3}{2}} \sqrt{-2^7 3^6 5^6 Z^3 - 3^5 5^2 y_3 + 2^5 3^3 5^2 y_4^4 Z - 518 y_4^6}} \left(-2^{14} 3^{12} 5^{12} 13\, Z^6 - 2^8 3^{11} 5^8 13\, Z^3 y_3 - 3^{10} 5^4 13\, y_3^2 + 2^{12} 3^{14} 5^{11} Z^5 y_4^2 + 3^{12} 5^5\, y_2 y_4^2\right.$$

$$
\begin{aligned}
&- 2^{10}3^9 5^8 571\, Z^4 y_4^4 + 2^6 3^8 5^4 13\, Z y_3 y_4^4 - 2^8 3^6 5^6 11\cdot 2269\, Z^3 y_4^6 + 2^3 3^5 5^2 17\cdot 19\cdot 23\, y_3 y_4^6 \\
&+ 2^9 3^6 5^4 199\, Z^2 y_4^8 + 2^6 3^3 5^2 22259\, Z y_4^{10} + 2^4 7\cdot 180569\, y_4^{12}\big), \qquad (4.5.67)
\end{aligned}
$$

$$
t_2 = \frac{1}{2^6 3^7 5^7}\left(2^6 3^6 5^6 11\, Z^3 - 3^5 5^2 2\, y_3 - 2^4 3^3 5^2 11\, y_4^4 Z - 1201 y_4^6\right), \qquad (4.5.68)
$$

$$
t_3 = \frac{1}{3^4 5^3 2\sqrt{3}}\sqrt{\frac{-2^7 3^6 5^6 Z^3 - 3^5 5^2 y_3 + 2^5 3^3 5^2 y_4^4 Z - 518 y_4^6}{y_4}}, \qquad (4.5.69)
$$

$$
t_4 = \frac{1}{30} y_4, \qquad (4.5.70)
$$

*where $Z$ satisfies the equation*

$$
\begin{aligned}
&2^{28}3^{24}5^{24} Z^{12} + 2^{23}3^{23}5^{20} Z^9 y_3 + 2^{15}3^{23}5^{16} Z^6 y_3^2 + 2^9 3^{21} 5^{12} Z^3 y_3^3 + 3^{20}5^8 y_3^4 \\
&+ 2^{27}3^{25}5^{22} Z^{11} y_4^2 - 2^{15}3^{23}5^{16} Z^6 y_2 y_4^2 + 2^{21}3^{24}5^{18} Z^8 y_3 y_4^2 - 2^9 3^{22} 5^{12} Z^3 y_2 y_3 y_4^2 \\
&+ 2^{13}3^{23}5^{14} Z^5 y_3^2 y_4^2 - 3^{21}5^8 2\, y_2 y_3^2 y_4^2 - 2^8 3^{22} 5^{12} Z^3 y_1 y_4^3 - 3^{21} 5^8 2\, y_1 y_3 y_4^3 \\
&+ 2^{24}3^{21}5^{20} 353\, Z^{10} y_4^4 - 2^{13}3^{23}5^{14} Z^5 y_2 y_4^4 - 3^{21}5^8 y_2^2 y_4^4 + 2^{20}3^{21}5^{16} 13\, Z^7 y_3 y_4^4 \\
&- 2^{11}3^{20}5^{12} 23\, Z^4 y_3^2 y_4^4 - 2^7 3^{18} 5^8 Z y_3^3 y_4^4 + 2^{25}3^{18}5^{19} 223\, Z^9 y_4^6 + 2^{11}3^{20}5^{12} 23\, Z^4 y_2 y_4^6 \\
&+ 2^{18}3^{18}5^{14} 53\, Z^6 y_3 y_4^6 + 2^7 3^{19} 5^8 Z y_2 y_3 y_4^6 - 2^9 3^{17} 5^{10} 17\cdot 41\, Z^3 y_3^2 y_4^6 + 2^3 3^{15} 5^6 7\cdot 37\, y_3^3 y_4^6 \\
&+ 2^6 3^{19} 5^8 Z y_1 y_4^7 + 2^{20}3^{19}5^{17} 7\cdot 19\, Z^8 y_4^8 + 2^9 3^{17} 5^{10} 17\cdot 41\, Z^3 y_2 y_4^8 - 2^{15}3^{19}5^{12} 59\, Z^5 y_3 y_4^8 \\
&- 2^3 3^{16} 5^6 7\cdot 37\, y_2 y_3 y_4^8 + 2^{10}3^{17}5^8 7\, Z^2 y_3^2 y_4^8 - 2^2 3^{16} 5^6 7\cdot 37\, y_1 y_4^9 - 2^{19}3^{16}5^{15} 2029\, Z^7 y_4^{10} \\
&- 2^{10}3^{17}5^8 7\, Z^2 y_2 y_4^{10} - 2^{13}3^{15}5^{10} 7\cdot 191\, Z^4 y_3 y_4^{10} + 2^7 3^{14} 5^6 7^2 13\, Z y_3^2 y_4^{10} \\
&- 2^{16}3^{13}5^{15} 2213\, Z^6 y_4^{12} - 2^7 3^{14} 5^6 7^2 13\, Z y_2 y_4^{12} + 2^{16} 3^{12} 5^8 7^2 11\, Z^3 y_3 y_4^{12} \\
&- 2^8 3^{11} 5^4 7^3 y_3^2 y_4^{12} + 2^{16}3^{13}5^{11} 7\cdot 11\cdot 43\, Z^5 y_4^{14} + 2^8 3^{11} 5^4 7^3 y_2 y_4^{14} + 2^{12}3^{12}5^6 7^3 Z^2 y_3 y_4^{14} \\
&+ 2^{14}3^{10}5^9 7^2 1171\, Z^4 y_4^{16} - 2^{11}3^9 5^4 7^4 Z y_3 y_4^{16} - 2^{16}3^6 5^7 7^3 163\, Z^3 y_4^{18} + 2^9 3^5 5^2 7^5 y_3 y_4^{18} \\
&- 2^{12}3^7 5^4 7^4 47\, Z^2 y_4^{20} + 2^{12}3^3 5^2 7^5 41\, Z y_4^{22} - 2^{10} 7^6 11\, y_4^{24} = 0, \qquad (4.5.71)
\end{aligned}
$$

*and $y_i$ are given by relations (4.5.1)–(4.5.11).*

*Proof.* Formula (4.5.70) follows immediately from Theorem 4.5.10. Using relations (4.5.44) and (4.5.60) we see that

$$
\begin{aligned}
t_1 = &-\frac{1}{2^9 3^{11} 5^{11} t_3}\big(-2^{12}3^9 5^{10} Z^5 + 2^4 3^{13} 5^{11} 13 t_3^4 - 3^7 5^4 y_2 + 2^{10}3^7 5^9 y_4^2 Z^4 + 2^8 3^8 5^7 y_4^4 Z^3 \\
&+ 2^4 3^{10} 5^7 y_4^5 t_3^2 - 2^9 3^3 5^5 y_4^6 Z^2 - 2^6 5^3 3\cdot 7\cdot 11 y_4^8 Z + 2^2 7^2 59 y_4^{10}\big), \qquad (4.5.72)
\end{aligned}
$$

$$
t_2 = \frac{1}{2^6 3^6 5^6}\left(2^6 3^6 5^6 Z^3 + 2^3 3^8 5^5 t_3^2 y_4 - 2^4 3^3 5^2 y_4^4 Z - 11 y_4^6\right). \qquad (4.5.73)
$$

We also have relation (4.5.61), which together with relations (4.5.72) and (4.5.73) gives formulas (4.5.67)–(4.5.69). Finally, by substituting relations (4.5.67)–(4.5.70) into formula (4.5.59) we get the formula (4.5.71). □

**Proposition 4.5.14.** *The unity vector field $e = \partial_{t_1}$ in the $y$ coordinates has the form*

$$
e(y) = 2^9 3^4 5^7 t_3 \partial_{y_2} - 2^{12} 3^4 5^{10}\left(16 t_1 + 261 t_3^3 + 6480 t_3 t_4^5\right)\partial_{y_1}.
$$

*Proof.* We have that

$$e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 4.5.10. □

### 4.5.3 $H_4(3)$ example

The prepotential for $H_4(3)$ is

$$\begin{aligned} F(t) = & \frac{2016569088}{43793750}t_4^{13} + \frac{7929073152}{43793750}t_4^{10}t_3 + \frac{11291664384}{43793750}t_4^7t_3^2 - \frac{6228045824}{43793750}t_4^4t_3^3 \\ & - \frac{1582124544}{43793750}t_4t_3^4 + t_1t_2t_3 + \frac{1}{2}t_1^2t_4 - \frac{256}{9375}t_4^3\left(t_4^3 - t_3\right)\left(779t_4^6 + 20532t_4^3t_3 - 18480t_3^2\right) \\ & + \frac{32256}{3125}t_4^2\left(17t_4^3 - 10t_3\right)\left(t_4^3 - t_3\right)^2 - \frac{7168}{125}t_4\left(t_4^3 - t_3\right)^3 Z^3 + \frac{96}{5}t_4\left(t_4^3 - t_3\right)^2 Z^6 \\ & - \frac{8}{2625}\left(1573t_4^9 - 27588t_4^6t_3 + 25536t_4^3t_3^2 - 2352t_3^3\right)Z^4 + \frac{544}{175}\left(t_4^3 - t_3\right)^2 Z^7 \\ & - \frac{288}{125}t_4^2\left(17t_4^3 - 10t_3\right)\left(t_4^3 - t_3\right)Z^5 + \frac{9}{70}t_4^2\left(17t_4^3 - 10t_3\right)Z^8 + \frac{50}{1911}Z^{13} \\ & - \frac{15}{7}t_4\left(t_4^3 - t_3\right)Z^9 - \frac{10}{21}\left(t_4^3 - t_3\right)Z^{10} + \frac{125}{1568}t_4Z^{12}, \end{aligned}$$

where

$$P(t_2, t_3, t_4, Z) := Z^4 - \frac{224}{25}\left(t_4^3 - t_3\right)Z + \frac{48}{25}t_4^4 + \frac{224}{25}t_4t_3 - t_2 = 0. \tag{4.5.74}$$

The Euler vector field is

$$E(t) = t_1\partial_{t_1} + \frac{2}{3}t_2\partial_{t_2} + \frac{1}{2}t_3\partial_{t_3} + \frac{1}{6}t_4\partial_{t_4},$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{5}{6}$. The intersection form (2.2.19) is then given by

$$\begin{aligned} g^{11}(t) = & -\frac{2}{765625}\left(4812500t_2^2t_3 - 10780000Zt_2t_3^2 + 36220800Z^2t_3^3 - 1203125Z^2t_2^2t_4\right. \\ & + 10010000Z^3t_2t_3t_4 + 36005200t_2t_3^2t_4 + 305220608Zt_3^3t_4 + 1375000Zt_2^2t_4^2 \\ & + 40040000Z^2t_2t_3t_4^2 - 193177600Z^3t_3^2t_4^2 - 1491331072t_3^3t_4^2 - 24234375t_2^2t_4^3 \\ & - 43120000Zt_2t_3t_4^3 - 236297600Z^2t_3^2t_4^3 - 33110000Z^3t_2t_4^4 - 607006400t_2t_3t_4^4 \\ & - 1372388864Zt_3^2t_4^4 + 36190000Z^2t_2t_4^5 + 260198400Z^3t_3t_4^5 + 1541281280t_3^2t_4^5 \\ & - 13200000Zt_2t_4^6 - 395225600Z^2t_3t_4^6 + 119891200t_2t_4^7 + 2865967104Zt_3t_4^7 \\ & + 184307200Z^3t_4^8 - 5516836864t_3t_4^8 - 91660800Z^2t_4^9 - 1231515648Zt_4^{10} \\ & \left. -5094508544t_4^{11}\right), \end{aligned} \tag{4.5.75}$$

$$\begin{aligned} g^{12}(t) = & -\frac{8}{21875}\left(3125Zt_2^2 - 45500Z^2t_2t_3 + 156800Z^3t_3^2 + 592704t_3^3 + 28125t_2^2t_4\right. \\ & - 119000Zt_2t_3t_4 + 619360Z^2t_3^2t_4 + 45000Z^3t_2t_4^2 + 119700t_2t_3t_4^2 \end{aligned}$$

$$+ 2847488Zt_3^2t_4^2 + 75500Z^2t_2t_4^3 - 851200Z^3t_3t_4^3 - 7902720t_3^2t_4^3$$
$$- 66000Zt_2t_4^4 - 619360Z^2t_3t_4^4 - 673200t_2t_4^5 - 3351936Zt_3t_4^5$$
$$+204800Z^3t_4^6 + 4163712t_3t_4^6 - 326400Z^2t_4^7 + 2147328Zt_4^8 - 4627456t_4^9\big), \qquad (4.5.76)$$

$$g^{22}(t) = -\frac{32}{9375}\big(-1250Z^3t_2 + 6300t_2t_3 - 56448Zt_3^2 - 4375Z^2t_2t_4 + 30800Z^3t_3t_4$$
$$+ 91728t_3^2t_4 - 5250Zt_2t_4^2 + 56840Z^2t_3t_4^2 + 3325t_2t_4^3 + 159936Zt_3t_4^3$$
$$-17200Z^3t_4^4 - 369600t_3t_4^4 - 9240Z^2t_4^5 - 46368Zt_4^6 + 80672t_4^7\big), \qquad (4.5.77)$$

$$g^{13}(t) = \frac{1}{9800}\big(3125t_2^2 - 22400Zt_2t_3 + 75264Z^2t_3^2 - 179200t_2t_3t_4 + 200704Zt_3^2t_4$$
$$+ 28000Z^2t_2t_4^2 - 125440Z^3t_3t_4^2 - 551936t_3^2t_4^2 - 19200Zt_2t_4^3 - 261632Z^2t_3t_4^3$$
$$+ 225600t_2t_4^4 + 215040Zt_3t_4^4 + 125440Z^3t_4^5 + 2867200t_3t_4^5 - 118272Z^2t_4^6$$
$$+36864Zt_4^7 - 604160t_4^8\big), \qquad (4.5.78)$$

$$g^{23}(t) = \frac{1}{175}\big(175t_1 - 125Z^2t_2 + 560Z^3t_3 - 300Zt_2t_4 + 2128Z^2t_3t_4 - 1200t_2t_4^2$$
$$+2688Zt_3t_4^2 - 560Z^3t_4^3 - 4928t_3t_4^3 - 768Z^2t_4^4 + 576Zt_4^5 + 6400t_4^6\big), \qquad (4.5.79)$$

$$g^{33}(t) = \frac{1}{1568}\big(250Zt_2 - 840Z^2t_3 + 625t_2t_4 - 2240Zt_3t_4 - 8960t_3t_4^2 + 840Z^2t_4^3$$
$$-480Zt_4^4 + 4512t_4^5\big), \qquad (4.5.80)$$

$$g^{14}(t) = t_1, \quad g^{24}(t) = \frac{2}{3}t_2, \quad g^{34}(t) = \frac{1}{2}t_3, \quad g^{44}(t) = \frac{1}{6}t_4. \qquad (4.5.81)$$

We have that $\deg t_1(x) = 12$, $\deg t_2(x) = 8$, $\deg t_3(x) = 6$, $\deg t_4(x) = 2$ and $\deg Z(x) = 2$.

**Theorem 4.5.15.** *We have the following relations*

$$y_1 = \frac{32768}{45}\big(273437500Zt_1t_2^2 + 166015625Z^3t_2^3 - 7717500000t_1^2t_3 - 1684375000Z^2t_1t_2t_3$$
$$+ 2810937500t_2^3t_3 + 3430000000Z^3t_1t_3^2 + 10143000000Zt_2^2t_3^2 + 44562560000t_1t_3^3$$
$$+ 13088880000Z^2t_2t_3^3 - 98467891200Z^3t_3^4 - 1066905133056t_3^5 - 3691406250t_1t_2^2t_4$$
$$- 615234375Z^2t_2^3t_4 - 4900000000Zt_1t_2t_3t_4 + 656250000Z^3t_2^2t_3t_4$$
$$+ 15092000000Z^2t_1t_3^2t_4 + 17272500000t_2^2t_3^2t_4 - 75075840000Zt_2t_3^3t_4$$
$$- 117276364800Z^2t_3^4t_4 + 1289062500Zt_2^3t_4^2 - 85995000000t_1t_2t_3t_4^2$$
$$- 4633125000Z^2t_2^2t_3t_4^2 + 21952000000Zt_1t_3^2t_4^2 + 74499600000Z^3t_2t_3^2t_4^2$$
$$+ 1219784832000t_2t_3^3t_4^2 + 2152574484480Zt_3^4t_4^2 + 7717500000t_1^2t_4^3$$
$$+ 1684375000Z^2t_1t_2t_4^3 - 7557031250t_2^3t_4^3 - 6860000000Z^3t_1t_3t_4^3$$
$$- 17206000000Zt_2^2t_3t_4^3 - 1196603520000t_1t_3^2t_4^3 - 40481840000Z^2t_2t_3^2t_4^3$$
$$+ 27536588800Z^3t_3^3t_4^3 - 2854462464000t_3^4t_4^3 - 1050000000Zt_1t_2t_4^4$$
$$- 6075000000Z^3t_2^2t_4^4 - 11858000000Z^2t_1t_3t_4^4 - 237331500000t_2^2t_3t_4^4$$
$$- 361141760000Zt_2t_3^2t_4^4 + 1152093644800Z^2t_3^3t_4^4 + 166320000000t_1t_2t_4^5$$
$$+ 24714375000Z^2t_2^2t_4^5 + 9408000000Zt_1t_3t_4^5 - 163279200000Z^3t_2t_3t_4^5$$
$$- 4338329856000t_2t_3^2t_4^5 - 5691700510720Zt_3^3t_4^5 + 3430000000Z^3t_1t_4^6$$

$$
\begin{aligned}
&- 59062000000Zt_2^2t_4^6 + 4552719360000t_1t_3t_4^6 + 290661840000Z^2t_2t_3t_4^6 \\
&- 604041267200Z^3t_3^2t_4^6 - 1144791531520t_3^3t_4^6 - 3234000000Z^2t_1t_4^7 \\
&+ 699646500000t_2^2t_4^7 + 1311466240000Zt_2t_3t_4^7 - 4066807449600Z^2t_3^2t_4^7 \\
&+ 1008000000Zt_1t_4^8 + 147735600000Z^3t_2t_4^8 + 15995339136000t_2t_3t_4^8 \\
&+ 7407908372480Zt_3^2t_4^8 - 3837646400000t_1t_4^9 - 481752880000Z^2t_2t_4^9 \\
&+ 1597027532800Z^3t_3t_4^9 + 130060834283520t_3^2t_4^9 + 105855360000Zt_2t_4^{10} \\
&+ 2986145792000Z^2t_3t_4^{10} - 21627170112000t_2t_4^{11} - 10675438878720Zt_3t_4^{11} \\
&- 1135868723200Z^3t_4^{12} - 398693684838400t_3t_4^{12} + 838213017600Z^2t_4^{13} \\
&+2299551252480Zt_4^{14} + 318244776083456t_4^{15}\Big) , \qquad (4.5.82)
\end{aligned}
$$

$$
\begin{aligned}
y_2 =& \frac{256}{3}\Big(656250t_1t_2 + 109375Z^2t_2^2 - 910000Z^3t_2t_3 - 16503200t_2t_3^2 - 18966528Zt_3^3 \\
&- 500000Zt_2^2t_4 + 22344000t_1t_3t_4 + 1120000Z^2t_2t_3t_4 + 1881600Z^3t_3^2t_4 \\
&+ 66382848t_3^3t_4 + 3093750t_2^2t_4^2 + 8960000Zt_2t_3t_4^2 - 18816000Z^2t_3^2t_4^2 \\
&+ 910000Z^3t_2t_4^3 + 56022400t_2t_3t_4^3 + 16758784Zt_3^2t_4^3 - 29484000t_1t_4^4 \\
&- 3500000Z^2t_2t_4^4 + 6137600Z^3t_3t_4^4 + 298923520t_3^2t_4^4 + 1920000Zt_2t_4^5 \\
&+ 25446400Z^2t_3t_4^5 - 164559200t_2t_4^6 - 74102784Zt_3t_4^6 - 8019200Z^3t_4^7 \\
&-3012660224t_3t_4^7 + 6316800Z^2t_4^8 + 17123328Zt_4^9 + 3641568256t_4^{10}\Big) , \qquad (4.5.83)
\end{aligned}
$$

$$
y_3 = 64\left(875t_1 + 8624t_3^2 + 4125t_2t_4^2 + 66528t_3t_4^3 - 196992t_4^6\right), \qquad (4.5.84)
$$

$$
y_4 = 24t_4. \qquad (4.5.85)
$$

A proof of this theorem is given in Appendix A as the formulas involved are too long to present here. The method of the proof is similar to those given for Theorems 4.2.4, 4.2.10, 4.3.4, 4.5.4, 4.5.10. No proof was presented in [26], where these results were first published.

**Proposition 4.5.16.** *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 4.2.5.

The expression for the discriminant of $H_4(3)$ as well as the relations for $t_i, Z$ in terms of the $y$ coordinates can be found in Appendix A.2, along with the expression for $e$ in terms of the $y$ coordinates. We do not present these here as they are too long and for this reason we chose not to include these in our paper [26].

### 4.5.4 $H_4(4)$ example

The prepotential for $H_4(4)$ is

$$
\begin{aligned}
F(t) = &-\frac{25}{33}Z^{11} + \frac{1}{2}Z^{10}(-4t_3 - 5t_4) - \frac{125}{9}Z^9(t_3 + t_4)(t_3 + 3t_4) - 5Z^8(8t_3^3 + 57t_3^2t_4 \\
&+ 124t_3t_4^2 + 95t_4^3) - \frac{35}{3}Z^6(t_3 + t_4)(t_3 + 3t_4)(12t_3^3 + 123t_3^2t_4 + 336t_3t_4^2 + 305t_4^3)
\end{aligned}
$$

$$
\begin{aligned}
&- 10Z^7(5t_3^4 + 64t_3^3t_4 + 236t_3^2t_4^2 + 344t_3t_4^3 + 175t_4^4) + Z^5(135t_3^6 - 840t_3^5t_4 \\
&- 17130t_3^4t_4^2 - 80920t_3^3t_4^3 - 174765t_3^2t_4^4 - 181952t_3t_4^5 - 74420t_4^6) \\
&+ 5Zt_4(t_3+t_4)^2(t_3+3t_4)^2(180t_3^5 + 2115t_3^4t_4 + 11760t_3^3t_4^2 + 30270t_3^2t_4^3 \\
&+ 32316t_3t_4^4 + 9035t_4^5) + \frac{5}{3}Z^3(t_3+t_4)(t_3+3t_4)(-45t_3^6 + 1260t_3^5t_4 + 13185t_3^4t_4^2 \\
&+ 42360t_3^3t_4^3 + 58125t_3^2t_4^4 + 37788t_3t_4^5 + 15815t_4^6) - 100Z^4(-6t_3^7 - 87t_3^6t_4 \\
&- 450t_3^5t_4^2 - 1015t_3^4t_4^3 - 650t_3^3t_4^4 + 1283t_3^2t_4^5 + 2450t_3t_4^6 + 1195t_4^7) \\
&- \frac{5}{6}Z^2(540t_3^9 + 9315t_3^8t_4 + 73440t_3^7t_4^2 + 356940t_3^6t_4^3 + 1252440t_3^5t_4^4 \\
&+ 3321450t_3^4t_4^5 + 6239424t_3^3t_4^6 + 7356636t_3^2t_4^7 + 4563564t_3t_4^8 + 994315t_4^9) \\
&+ \frac{1}{198}(198t_1t_2t_3 + 99t_1^2t_4 - 4455t_3^{10}t_4 - 178200t_3^9t_4^2 - 2569050t_3^8t_4^3 \\
&- 21740400t_3^7t_4^4 - 120561210t_3^6t_4^5 - 458678880t_3^5t_4^6 - 1191552120t_3^4t_4^7 \\
&- 2004227280t_3^3t_4^8 - 1955070535t_3^2t_4^9 - 858257224t_3t_4^{10} - 45669270t_4^{11}),
\end{aligned}
$$

where

$$
\begin{aligned}
P(t_2,t_3,t_4,Z) :=\, & Z^5 - t_2 + 15t_3^4t_4 + 120t_3^3t_4^2 + 530t_3^2t_4^3 + 1160t_3t_4^4 + 843t_4^5 \\
& + 10Z^3(t_3+t_4)(t_3+3t_4) - 15Z(t_3+t_4)^2(t_3+3t_4)^2 \\
& + 20Z^2t_4(3t_3^2 + 12t_3t_4 + 13t_4^2) = 0.
\end{aligned}
$$

The Euler vector field is

$$
E(t) = t_1\partial_{t_1} + t_2\partial_{t_2} + \frac{1}{5}t_3\partial_{t_3} + \frac{1}{5}t_4\partial_{t_4},
$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{4}{5}$. The intersection form (2.2.19) is then given by

$$
\begin{aligned}
g^{11}(t) =\, & -15\,\big(-2Z^4t_2 - 24Z^3t_2t_3 + 116Z^2t_2t_3^2 + 192Zt_2t_3^3 - 956t_2t_3^4 - 3360Z^4t_3^5 \\
& + 12560Z^3t_3^6 + 2880Z^2t_3^7 - 14826Zt_3^8 + 108t_3^9 - 78Z^3t_2t_4 + 368Z^2t_2t_3t_4 \\
& + 1584Zt_2t_3^2t_4 - 4096t_2t_3^3t_4 - 48360Z^4t_3^4t_4 + 106920Z^3t_3^5t_4 + 104472Z^2t_3^6t_4 \\
& - 194592Zt_3^7t_4 + 17013t_3^8t_4 + 276Z^2t_2t_4^2 + 2976Zt_2t_3t_4^2 + 2824t_2t_3^2t_4^2 \\
& - 238560Z^4t_3^3t_4^2 + 221106Z^3t_3^4t_4^2 + 806976Z^2t_3^5t_4^2 - 962208Zt_3^6t_4^2 + 199488t_3^7t_4^2 \\
& + 1048Zt_2t_4^3 + 27264t_2t_3t_4^3 - 526312Z^4t_3^2t_4^3 - 417856Z^3t_3^3t_4^3 + 2293152Z^2t_3^4t_4^3 \\
& - 1907424Zt_3^5t_4^3 + 1003828t_3^6t_4^3 + 24108t_2t_4^4 - 527680Z^4t_3t_4^4 - 2204820Z^3t_3^2t_4^4 \\
& + 1151040Z^2t_3^3t_4^4 + 655860Zt_3^4t_4^4 + 2033688t_3^5t_4^4 - 193800Z^4t_4^5 - 2871240Z^3t_3t_4^5 \\
& - 6809880Z^2t_3^2t_4^5 + 10362720Zt_3^3t_4^5 - 3485610t_3^4t_4^5 - 1183430Z^3t_4^6 - 13507200Z^2t_3t_4^6 \\
& + 20695840Zt_3^2t_4^6 - 29726240t_3^3t_4^6 - 7758720Z^2t_4^7 + 19063840Zt_3t_4^7 \\
& -68995740t_3^2t_4^7 + 7219830Zt_4^8 - 73570020t_3t_4^8 - 30991507t_4^9\big)\,, \qquad (4.5.86)
\end{aligned}
$$

$$\begin{aligned}
g^{12}(t) = & -5\big(-4Z^4t_2 - 18Z^3t_2t_3 + 184Z^2t_2t_3^2 + 144Zt_2t_3^3 - 1792t_2t_3^4 - 2520Z^4t_3^5 \\
& + 21760Z^3t_3^6 + 2160Z^2t_3^7 - 26880Zt_3^8 + 81t_3^9 - 72Z^3t_2t_4 + 664Z^2t_2t_3t_4 \\
& + 1968Zt_2t_3^2t_4 - 11672t_2t_3^3t_4 - 49920Z^4t_3^4t_4 + 228270Z^3t_3^5t_4 + 148320Z^2t_3^6t_4 \\
& - 398112Zt_3^7t_4 + 29796t_3^8t_4 + 552Z^2t_2t_4^2 + 5352Zt_2t_3t_4^2 - 18832t_2t_3^2t_4^2 \\
& - 288120Z^4t_3^3t_4^2 + 813000Z^3t_3^4t_4^2 + 1444824Z^2t_3^5t_4^2 - 2417184Zt_3^6t_4^2 \\
& + 422196t_3^7t_4^2 + 3536Zt_2t_4^3 + 5448t_2t_3t_4^3 - 708320Z^4t_3^2t_4^3 + 915172Z^3t_3^3t_4^3 \\
& + 5680992Z^2t_3^4t_4^3 - 7655496Zt_3^5t_4^3 + 2785376t_3^6t_4^3 + 20448t_2t_4^4 \\
& - 776504Z^4t_3t_4^4 - 881952Z^3t_3^2t_4^4 + 10163184Z^2t_3^3t_4^4 - 13244448Zt_3^4t_4^4 \\
& + 10650846t_3^5t_4^4 - 307680Z^4t_4^5 - 2577210Z^3t_3t_4^5 + 6008160Z^2t_3^2t_4^5 \\
& - 11434800Zt_3^3t_4^5 + 23025000t_3^4t_4^5 - 1401160Z^3t_4^6 - 4244760Z^2t_3t_4^6 \\
& - 1991200Zt_3^2t_4^6 + 20880740t_3^3t_4^6 - 5101920Z^2t_4^7 + 4450040Zt_3t_4^7 \\
& - 12170880t_3^2t_4^7 + 2911200Zt_4^8 - 39000375t_3t_4^8 - 21754220t_4^9\big)\,, \qquad (4.5.87)
\end{aligned}$$

$$\begin{aligned}
g^{22}(t) = & \, 5\big(2Z^4t_2 - 92Z^2t_2t_3^2 + 896t_2t_3^4 - 10880Z^3t_3^6 + 13440Zt_3^8 + 18Z^3t_2t_4 \\
& - 368Z^2t_2t_3t_4 - 552Zt_2t_3^2t_4 + 7168t_2t_3^3t_4 + 12360Z^4t_3^4t_4 - 130560Z^3t_3^5t_4 \\
& - 59040Z^2t_3^6t_4 + 215040Zt_3^7t_4 - 14169t_3^8t_4 - 348Z^2t_2t_4^2 - 2208Zt_2t_3t_4^2 \\
& + 17408t_2t_3^2t_4^2 + 98880Z^4t_3^3t_4^2 - 570750Z^3t_3^4t_4^2 - 708480Z^2t_3^5t_4^2 \\
& + 1432368Zt_3^6t_4^2 - 226704t_3^7t_4^2 - 1984Zt_2t_4^3 + 12288t_2t_3t_4^3 + 284680Z^4t_3^2t_4^3 \\
& - 1084400Z^3t_3^3t_4^3 - 3305976Z^2t_3^4t_4^3 + 5146176Zt_3^5t_4^3 - 1665604t_3^6t_4^3 \\
& - 1512t_2t_4^4 + 347680Z^4t_3t_4^4 - 691908Z^3t_3^2t_4^4 - 7555008Z^2t_3^3t_4^4 \\
& + 10855464Zt_3^4t_4^4 - 7291824t_3^5t_4^4 + 150456Z^4t_4^5 + 337008Z^3t_3t_4^5 \\
& - 8461536Z^2t_3^2t_4^5 + 13837632Zt_3^3t_4^5 - 19824414t_3^4t_4^5 + 416810Z^3t_4^6 \\
& - 3634560Z^2t_3t_4^6 + 10628480Zt_3^2t_4^6 - 31855600t_3^3t_4^6 + 143640Z^2t_4^7 \\
& + 4582400Zt_3t_4^7 - 26339220t_3^2t_4^7 + 771720Zt_4^8 - 6866640t_3t_4^8 + 1889215t_4^9\big)\,, \qquad (4.5.88)
\end{aligned}$$

$$\begin{aligned}
g^{13}(t) = & -5\big(t_2 - 4Z^4t_3 - 8Z^3t_3^2 + 24Z^2t_3^3 - 6Zt_3^4 - 6Z^4t_4 - 64Z^3t_3t_4 + 132Z^2t_3^2t_4 \\
& - 96Zt_3^3t_4 + 60t_3^4t_4 - 104Z^3t_4^2 + 216Z^2t_3t_4^2 - 636Zt_3^2t_4^2 + 720t_3^3t_4^2 + 108Z^2t_4^3 \\
& - 1856Zt_3t_4^3 + 2240t_3^2t_4^3 - 1686Zt_4^4 + 1600t_3t_4^4 - 1444t_4^5\big)\,, \qquad (4.5.89)
\end{aligned}$$

$$\begin{aligned}
g^{23}(t) = & \, t_1 - 4t_2 + 10Z^4t_3 - 60Z^2t_3^3 + 20Z^4t_4 + 80Z^3t_3t_4 - 360Z^2t_3^2t_4 + 120Zt_3^3t_4 \\
& + 160Z^3t_4^2 - 660Z^2t_3t_4^2 + 720Zt_3^2t_4^2 - 600t_3^3t_4^2 - 360Z^2t_4^3 + 2120Zt_3t_4^3 \\
& - 3600t_3^2t_4^3 + 2320Zt_4^4 - 5600t_3t_4^4 - 1600t_4^5, \qquad (4.5.90)
\end{aligned}$$

$$g^{33}(t) = -\frac{1}{5}\left(2Z + 4t_3 + 5t_4\right), \qquad (4.5.91)$$

$$g^{14}(t) = t_1, \quad g^{24}(t) = t_2, \quad g^{34}(t) = \frac{1}{5}t_3, \quad g^{44}(t) = \frac{1}{5}t_4. \qquad (4.5.92)$$

We have that $\deg t_1(x) = 10$, $\deg t_2(x) = 10$, $\deg t_3(x) = 2$, $\deg t_4(x) = 2$ and $\deg Z(x) = 2$.

**Theorem 4.5.17.** *We have the following relations*

$$
\begin{aligned}
y_1 = -\frac{2^{19}5^9}{3^{13}7}\big(&42t_1^2t_2 - 168t_1t_2^2 + 168t_2^3 - 210Z^4t_1t_2t_3 + 420Z^4t_2^2t_3 + 1050Z^3t_2^2t_3^2 \\
&+ 2100Z^2t_1t_2t_3^3 - 4200Z^2t_2^2t_3^3 + 27300Zt_2^2t_3^4 + 89250t_1t_2t_3^5 - 178500t_2^2t_3^5 \\
&+ 228620Z^4t_2t_3^6 + 114240Z^3t_1t_3^7 - 228480Z^3t_2t_3^7 + 2465680Z^2t_2t_3^8 \\
&- 141120Zt_1t_3^9 + 282240Zt_2t_3^9 - 5667634t_2t_3^{10} - 13482560Z^3t_3^{12} \\
&+ 42611520Zt_3^{14} - 420Z^4t_1t_2t_4 + 1050Z^4t_2^2t_4 - 1890Z^3t_1t_2t_3t_4 + 7980Z^3t_2^2t_3t_4 \\
&+ 12600Z^2t_1t_2t_3^2t_4 - 20790Z^2t_2^2t_3^2t_4 - 104580Zt_1t_2t_3^3t_4 + 427560Zt_2^2t_3^3t_4 \\
&- 164430t_1^2t_3^4t_4 + 1550220t_1t_2t_3^4t_4 - 2334570t_2^2t_3^4t_4 - 46620Z^4t_1t_3^5t_4 \\
&+ 2836680Z^4t_2t_3^5t_4 + 1599360Z^3t_1t_3^6t_4 - 2240700Z^3t_2t_3^6t_4 - 15120Z^2t_1t_3^7t_4 \\
&+ 39481120Z^2t_2t_3^7t_4 - 2540160Zt_1t_3^8t_4 + 38725680Zt_2t_3^8t_4 + 28513800t_1t_3^9t_4 \\
&- 170380280t_2t_3^9t_4 + 16561860Z^4t_3^{10}t_4 - 323581440Z^3t_3^{11}t_4 - 64723680Z^2t_3^{12}t_4 \\
&+ 1193122560Zt_3^{13}t_4 - 29292690t_3^{14}t_4 - 3780Z^3t_1t_2t_4^2 + 8820Z^3t_2^2t_4^2 \\
&+ 23100Z^2t_1t_2t_3t_4^2 - 28560Z^2t_2^2t_3t_4^2 - 627480Zt_1t_2t_3^2t_4^2 + 1892100Zt_2^2t_3^2t_4^2 \\
&- 1315440t_1^2t_3^3t_4^2 + 8585220t_1t_2t_3^3t_4^2 - 11043480t_2^2t_3^3t_4^2 - 466200Z^4t_1t_3^4t_4^2 \\
&+ 15817200Z^4t_2t_3^4t_4^2 + 9705150Z^3t_1t_3^5t_4^2 - 7914060Z^3t_2t_3^5t_4^2 - 211680Z^2t_1t_3^6t_4^2 \\
&+ 289891980Z^2t_2t_3^6t_4^2 - 17575740Zt_1t_3^7t_4^2 + 573477240Zt_2t_3^7t_4^2 + 513248400t_1t_3^8t_4^2 \\
&- 1955803360t_2t_3^8t_4^2 + 331237200Z^4t_3^9t_4^2 - 3429078870Z^3t_3^{10}t_4^2 - 1553368320Z^2t_3^{11}t_4^2 \\
&+ 14721448140Zt_3^{12}t_4^2 - 820195320t_3^{13}t_4^2 + 12600Z^2t_1t_2t_4^3 - 3290Z^2t_2^2t_4^3 \\
&- 1244460Zt_1t_2t_3t_4^3 + 3290280Zt_2^2t_3t_4^3 - 3844260t_1^2t_3^2t_4^3 - 450Z^5t_2t_3^2t_4^3 \\
&+ 21037800t_1t_2t_3^2t_4^3 - 24410930t_2^2t_3^2t_4^3 - 1773240Z^4t_1t_3^3t_4^3 + 49466480Z^4t_2t_3^3t_4^3 \\
&+ 225Z^8t_3^4t_4^3 + 33077100Z^3t_1t_3^4t_4^3 - 11938395Z^3t_2t_3^4t_4^3 - 3845520Z^2t_1t_3^5t_4^3 \\
&+ 1272065200Z^2t_2t_3^5t_4^3 - 2700Z^6t_3^6t_4^3 - 56395080Zt_1t_3^6t_4^3 + 3895595560Zt_2t_3^6t_4^3 \\
&+ 4000050600t_1t_3^7t_4^3 - 11987148880t_2t_3^7t_4^3 + 2939165645Z^4t_3^8t_4^3 \\
&- 21122966200Z^3t_3^9t_4^3 - 17379662880Z^2t_3^{10}t_4^3 + 105145262880Zt_3^{11}t_4^3 \\
&- 6999178480t_3^{12}t_4^3 - 815640Zt_1t_2t_4^4 + 2080680Zt_2^2t_4^4 - 4853520t_1^2t_3t_4^4 \\
&- 1800Z^5t_2t_3t_4^4 + 23828490t_1t_2t_3t_4^4 - 26013980t_2^2t_3t_4^4 - 3180240Z^4t_1t_3^2t_4^4 \\
&+ 91783860Z^4t_2t_3^2t_4^4 + 1800Z^8t_3^3t_4^4 + 69522180Z^3t_1t_3^3t_4^4 - 11884320Z^3t_2t_3^3t_4^4 \\
&+ 5400Z^7t_3^4t_4^4 - 29988000Z^2t_1t_3^4t_4^4 + 3617141020Z^2t_2t_3^4t_4^4 - 32400Z^6t_3^5t_4^4 \\
&- 38118780Zt_1t_3^5t_4^4 + 15323659800Zt_2t_3^5t_4^4 - 25650Z^5t_3^6t_4^4 + 17678161200t_1t_3^6t_4^4 \\
&- 44692143980t_2t_3^6t_4^4 + 15227879120Z^4t_3^7t_4^4 - 83502915015Z^3t_3^8t_4^4 \\
&- 119765904000Z^2t_3^9t_4^4 + 469572887280Zt_3^{10}t_4^4 + 2620343040t_3^{11}t_4^4 - 2164806t_1^2t_4^5 \\
&- 1350Z^5t_2t_4^5 + 10008684t_1t_2t_4^5 - 11210172t_2^2t_4^5 - 2653140Z^4t_1t_3t_4^5 \\
&+ 96693240Z^4t_2t_3t_4^5 + 4950Z^8t_3^2t_4^5 + 92617560Z^3t_1t_3^2t_4^5 - 43361430Z^3t_2t_3^2t_4^5 \\
&+ 43200Z^7t_3^3t_4^5 - 113047200Z^2t_1t_3^3t_4^5 + 6749433120Z^2t_2t_3^3t_4^5 - 121500Z^6t_3^4t_4^5
\end{aligned}
$$

$$
\begin{aligned}
&+ 357293160Zt_1t_3^4t_4^5 + 38092662600Zt_2t_3^4t_4^5 - 307800Z^5t_3^5t_4^5 + 49036806000t_1t_3^5t_4^5 \\
&- 108705602416t_2t_3^5t_4^5 + 50844400500Z^4t_3^6t_4^5 - 219447220080Z^3t_3^7t_4^5 \\
&- 570111869640Z^2t_3^8t_4^5 + 1249790969280Zt_3^9t_4^5 + 479633884512t_3^{10}t_4^5 - 803880Z^4t_1t_4^6 \\
&+ 45585960Z^4t_2t_4^6 + 5400Z^8t_3t_4^6 + 74024790Z^3t_1t_3t_4^6 - 107328900Z^3t_2t_3t_4^6 \\
&+ 126000Z^7t_3^2t_4^6 - 225570240Z^2t_1t_3^2t_4^6 + 7964410020Z^2t_2t_3^2t_4^6 - 108000Z^6t_3^3t_4^6 \\
&+ 1399923420Zt_1t_3^3t_4^6 + 61840178440Zt_2t_3^3t_4^6 - 1415250Z^5t_3^4t_4^6 + 89821989600t_1t_3^4t_4^6 \\
&- 183359111760t_2t_3^4t_4^6 + 113517812720Z^4t_3^5t_4^6 - 379424004920Z^3t_3^6t_4^6 \\
&- 1998548319360Z^2t_3^7t_4^6 + 1114813850860Zt_3^8t_4^6 + 4204075665280t_3^9t_4^6 + 2025Z^8t_4^7 \\
&+ 27351660Z^3t_1t_4^7 - 87125115Z^3t_2t_4^7 + 158400Z^7t_3t_4^7 - 231759360Z^2t_1t_3t_4^7 \\
&+ 5344717200Z^2t_2t_3t_4^7 + 337500Z^6t_3^2t_4^7 + 2400609960Zt_1t_3^2t_4^7 + 65461451880Zt_2t_3^2t_4^7 \\
&- 3114000Z^5t_3^3t_4^7 + 113150026920t_1t_3^3t_4^7 - 234714455760t_2t_3^3t_4^7 + 169129871150Z^4t_3^4t_4^7 \\
&- 396171632160Z^3t_3^5t_4^7 - 5361270379440Z^2t_3^6t_4^7 - 5723934259520Zt_3^7t_4^7 \\
&+ 20067909135750t_3^8t_4^7 + 70200Z^7t_4^8 - 97251840Z^2t_1t_4^8 + 1534267620Z^2t_2t_4^8 \\
&+ 831600Z^6t_3t_4^8 + 2057206620Zt_1t_3t_4^8 + 42875701800Zt_2t_3t_4^8 - 3307950Z^5t_3^2t_4^8 \\
&+ 103042663920t_1t_3^2t_4^8 - 249632019090t_2t_3^2t_4^8 + 160462051440Z^4t_3^3t_4^8 \\
&- 159576392790Z^3t_3^4t_4^8 - 11200275624000Z^2t_3^5t_4^8 - 29740325063280Zt_3^6t_4^8 \\
&+ 62134558132320t_3^7t_4^8 + 535500Z^6t_4^9 + 712487160Zt_1t_4^9 + 13938177080Zt_2t_4^9 \\
&- 1452600Z^5t_3t_4^9 + 68111856120t_1t_3t_4^9 - 203506674360t_2t_3t_4^9 + 84058972000Z^4t_3^2t_4^9 \\
&+ 151452210000Z^3t_3^3t_4^9 - 18131069931360Z^2t_3^4t_4^9 - 75535900424000Zt_3^5t_4^9 \\
&+ 130861824336350t_3^6t_4^9 - 125550Z^5t_4^{10} + 25090865520t_1t_4^{10} - 86556812352t_2t_4^{10} \\
&+ 12789015040Z^4t_3t_4^{10} + 241285828350Z^3t_3^2t_4^{10} - 22060526165760Z^2t_3^3t_4^{10} \\
&- 125256834868940Zt_3^4t_4^{10} + 186992066838888t_3^5t_4^{10} - 5145601875Z^4t_4^{11} \\
&+ 114867728760Z^3t_3t_4^{11} - 18961639423440Z^2t_3^2t_4^{11} - 142926704208480Zt_3^3t_4^{11} \\
&+ 171096028587180t_3^4t_4^{11} + 11533193965Z^3t_4^{12} - 10237991411520Z^2t_3t_4^{12} \\
&- 110581088197120Zt_3^2t_4^{12} + 78681803681760t_3^3t_4^{12} - 2609003447640Z^2t_4^{13} \\
&- 53312566785280Zt_3t_4^{13} - 16250437730220t_3^2t_4^{13} - 12255008872620Zt_4^{14} \\
&-46426601177520t_3t_4^{14} - 22667569904962t_4^{15}\big),
\end{aligned}
$$

$$
\begin{aligned}
y_2 = &-\frac{2^{12}5^6}{3^8}\big(t_1^2 - 4t_1t_2 + 3t_2^2 - 45Z^3t_2t_3^2 - 810Zt_2t_3^4 - 1026t_1t_3^5 + 2052t_2t_3^5 \\
&- 405Z^4t_3^6 - 4050Z^2t_3^8 + 10044t_3^{10} + 10Z^4t_2t_4 - 180Z^3t_2t_3t_4 - 100Z^2t_2t_3^2t_4 \\
&- 6480Zt_2t_3^3t_4 - 10260t_1t_3^4t_4 + 12220t_2t_3^4t_4 - 4860Z^4t_3^5t_4 - 9085Z^3t_3^6t_4 \\
&- 64800Z^2t_3^7t_4 + 33450Zt_3^8t_4 + 200880t_3^9t_4 - 225Z^3t_2t_4^2 - 400Z^2t_2t_3t_4^2 \\
&- 16080Zt_2t_3^2t_4^2 - 30780t_1t_3^3t_4^2 - 4840t_2t_3^3t_4^2 - 22485Z^4t_3^4t_4^2 - 109020Z^3t_3^5t_4^2 \\
&- 511200Z^2t_3^6t_4^2 + 535200Zt_3^7t_4^2 + 1253640t_3^8t_4^2 - 300Z^2t_2t_4^3 - 12480Zt_2t_3t_4^3 \\
&- 20520t_1t_3^2t_4^3 - 163520t_2t_3^2t_4^3 - 50280Z^4t_3^3t_4^3 - 550695Z^3t_3^4t_4^3 - 2505600Z^2t_3^5t_4^3
\end{aligned}
$$

$$
\begin{aligned}
&+3797100Zt_3^6t_4^3+773760t_3^7t_4^3+2410Zt_2t_4^4+46710t_1t_3t_4^4-380460t_2t_3t_4^4\\
&-55975Z^4t_3^2t_4^4-1498360Z^3t_3^3t_4^4-8058900Z^2t_3^4t_4^4+15594000Zt_3^5t_4^4\\
&-27933080t_3^6t_4^4+60588t_1t_4^5-287748t_2t_4^5-29020Z^4t_3t_4^5-2341695Z^3t_3^2t_4^5\\
&-17008800Z^2t_3^3t_4^5+39864360Zt_3^4t_4^5-162541344t_3^5t_4^5-6375Z^4t_4^6\\
&-2031420Z^3t_3t_4^6-22768320Z^2t_3^2t_4^6+62921280Zt_3^3t_4^6-465541320t_3^4t_4^6\\
&-778805Z^3t_4^7-17671680Z^2t_3t_4^7+56624500Zt_3^2t_4^7-804407360t_3^3t_4^7\\
&-6115770Z^2t_4^8+23108560Zt_3t_4^8-886614660t_3^2t_4^8+1355790Zt_4^9\\
&-625055760t_3t_4^9-240893344t_4^{10}\big),
\end{aligned}
$$

$$
\begin{aligned}
y_3=&-\frac{2^7 5^4}{3^5}\big(-Zt_2-6t_1t_3+12t_2t_3+Z^4t_3^2-6Z^2t_3^4-297t_3^6-12t_1t_4+30t_2t_4\\
&+4Z^4t_3t_4+12Z^3t_3^2t_4-48Z^2t_3^3t_4+15Zt_3^4t_4-3564t_3^5t_4+3Z^4t_4^2+48Z^3t_3t_4^2\\
&-132Z^2t_3^2t_4^2+120Zt_3^3t_4^2-22365t_3^4t_4^2+52Z^3t_4^3-144Z^2t_3t_4^3+530Zt_3^2t_4^3\\
&-83880t_3^3t_4^3-54Z^2t_4^4+1160Zt_3t_4^4-166515t_3^2t_4^4+843Zt_4^5-147084t_3t_4^5\\
&-20311t_4^6\big),\\
y_4=&\,20t_4.
\end{aligned}
$$

A proof of this theorem is given in Appendix A as the formulas involved are too long to present here. The method of the proof is similar to those given for Theorems 4.2.4, 4.2.10, 4.3.4, 4.5.4, 4.5.10. No proof was presented in [26], where these results were first published.

**Proposition 4.5.18.** *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1,t_2,t_3,t_4,Z]$.

Proof is similar to the one for Proposition 4.2.5.

The expression for the discriminant of $H_4(4)$ as well as the expression for $e$ in terms of the $y$ coordinates can be found in Appendix A.3. We do not present these here as they are too long and for this reason we chose not to include these in our paper [26]. We do not give expressions for the $t$ coordinates and $Z$ in terms of the $y$ coordinates, as this was too complicated even for computation.

### 4.5.5 $H_4(7)$ example

The prepotential for $H_4(7)$ is

$$
\begin{aligned}
F(t)=&\,t_1t_2t_3+\frac{1}{2}t_1^2t_4-\frac{4096}{135}t_3t_4\left(315t_3^3+32t_4^5\right)+\frac{32768}{1125}t_4^2\left(75t_3^3+2t_4^5\right)Z^2\\
&-\frac{32768}{225}t_3\left(75t_3^3+2t_4^5\right)Z^3-\frac{16384}{5625}t_4\left(375t_3^3+14t_4^5\right)Z^5+\frac{34816}{225}t_3t_4^4Z^6\\
&-\frac{116736}{175}t_3^2t_4^2Z^7+\frac{256}{75}\left(220t_3^3+3t_4^5\right)Z^8-\frac{118784}{945}t_3t_4^3Z^9+\frac{44544}{175}t_3^2t_4Z^{10}\\
&-\frac{5632}{1575}t_4^4Z^{11}+\frac{832}{225}t_3t_4^2Z^{12}+\frac{17664}{455}t_3^2Z^{13}-\frac{352}{315}t_4^3Z^{14}+\frac{1568}{225}t_3t_4Z^{15}
\end{aligned}
$$

$$+\frac{496}{2975}t_4^2Z^{17}+\frac{496}{945}t_3Z^{18}+\frac{71}{1575}t_4Z^{20}+\frac{16}{7245}Z^{23},$$

where

$$P(t_2,t_3,t_4,Z):=Z^8+\frac{32}{5}t_4Z^5+64t_3Z^3-\frac{64}{5}t_4^2Z^2+128t_3t_4-t_2=0.$$

The Euler vector field is

$$E(t)=t_1\partial_{t_1}+\frac{4}{5}t_2\partial_{t_2}+\frac{1}{2}t_3\partial_{t_3}+\frac{3}{10}t_4\partial_{t_4},$$

the unity vector field is $e(t)=\partial_{t_1}$, and the charge is $d=\frac{7}{10}$. The intersection form (2.2.19) is then given by

$$\begin{aligned}g^{11}(t)=&\frac{128}{703125}\left(-1250Zt_2^2+11250Z^4t_2t_3+1850000Z^7t_3^2+153000000Z^2t_3^3\right.\\&+5375Z^6t_2t_4+758000Zt_2t_3t_4+13878000Z^4t_3^2t_4-32200Z^3t_2t_4^2\\&+745600Z^6t_3t_4^2-117344000Zt_3^2t_4^2-754420t_2t_4^3-1829120Z^3t_3t_4^3\\&\left.-819072Z^5t_4^4-121034240t_3t_4^4+1223424Z^2t_4^5\right),\end{aligned}\tag{4.5.93}$$

$$\begin{aligned}g^{12}(t)=&\frac{128}{9375}\left(-125Z^7t_2-11500Z^2t_2t_3+16000Z^5t_3^2+5400000t_3^3\right.\\&-450Z^4t_2t_4+28400Z^7t_3t_4+3272000Z^2t_3^2t_4+5280Zt_2t_4^2\\&\left.+202480Z^4t_3t_4^2+4448Z^6t_4^3-1155840Zt_3t_4^3+3584Z^3t_4^4-256000t_4^5\right),\end{aligned}\tag{4.5.94}$$

$$\begin{aligned}g^{22}(t)=&\frac{512}{1875}\left(-25Z^5t_2-4250t_2t_3-40000Z^3t_3^2-295Z^2t_2t_4+2480Z^5t_3t_4\right.\\&\left.+622000t_3^2t_4+788Z^7t_4^2+123760Z^2t_3t_4^2+5324Z^4t_4^3-20800Zt_4^4\right),\end{aligned}\tag{4.5.95}$$

$$\begin{aligned}g^{13}(t)=&\frac{1}{1875}\left(25Z^4t_2-4000Z^7t_3-288000Z^2t_3^2-560Zt_2t_4-20160Z^4t_3t_4\right.\\&\left.+544Z^6t_4^2+148480Zt_3t_4^2-2048Z^3t_4^3-51200t_4^4\right),\end{aligned}\tag{4.5.96}$$

$$\begin{aligned}g^{23}(t)=&\frac{1}{75}\left(75t_1+20Z^2t_2-320Z^5t_3-38400t_3^2-128Z^7t_4-12160Z^2t_3t_4\right.\\&\left.-704Z^4t_4^2+2560Zt_4^3\right),\end{aligned}\tag{4.5.97}$$

$$g^{33}(t)=\frac{Z}{600}\left(5Z^6+420Zt_3+35Z^3t_4-112t_4^2\right),\tag{4.5.98}$$

$$g^{14}(t)=t_1,\quad g^{24}(t)=\frac{4}{5}t_2,\quad g^{34}(t)=\frac{1}{2}t_3,\quad g^{44}(t)=\frac{3}{10}t_4.\tag{4.5.99}$$

We have that $\deg t_1(x)=\frac{20}{3}$, $\deg t_2(x)=\frac{16}{3}$, $\deg t_3(x)=\frac{10}{3}$, $\deg t_4(x)=2$ and $\deg Z(x)=\frac{2}{3}$.

**Theorem 4.5.19.** *We have the following relations*

$$\begin{aligned}y_1=&-\frac{5^4}{3^{28}7}\left(2^33^45^{18}7\,Z^7t_1^3t_2-2^33^75^{15}7\,Zt_1^2t_2^3+3^35^{14}7\cdot11\cdot31\,Z^3t_1t_2^4\right.\\&-5^{13}7\cdot1831\,Z^5t_2^5-2^43^65^{20}7\,t_1^4t_3+2^43^45^{18}7^3Z^2t_1^3t_2t_3+3^55^{15}2\cdot7\cdot967\,Z^4t_1^2t_2^2t_3\end{aligned}$$

$$- 3^4 5^{14} 2\cdot 7\cdot 2383\, Z^6 t_1 t_2^3 t_3 + 2^2 5^{13} 7\cdot 19121\, t_2^5 t_3 - 2^8 3^4 5^{18} 7^2 Z^5 t_1^3 t_3^2$$
$$- 2^6 3^6 5^{15} 7\cdot 283\, Z^7 t_1^2 t_2 t_3^2 + 2^5 3^4 5^{16} 7\cdot 11\cdot 17\, Z t_1 t_2^3 t_3^2 - 2^6 5^{13} 3\cdot 7\cdot 31547\, Z^3 t_2^4 t_3^2$$
$$- 2^{12} 3^5 5^{18} 7\cdot 61\, t_1^3 t_3^3 - 2^{12} 3^5 5^{15} 7\cdot 257\, Z^2 t_1^2 t_2 t_3^3 - 2^8 3^3 5^{14} 7\cdot 47\cdot 421\, Z^4 t_1 t_2^2 t_3^3$$
$$- 2^7 5^{13} 7\cdot 131\cdot 2647\, Z^6 t_2^3 t_3^3 - 2^{14} 3^5 5^{15} 7\cdot 1153\, Z^5 t_1^2 t_3^4 - 2^{21} 3^3 5^{14} 7\cdot 239\, Z^7 t_1 t_2 t_3^4$$
$$+ 2^{11} 5^{13} 3\cdot 7\cdot 181\cdot 6269\, Z t_2^3 t_3^4 - 2^{17} 3^6 5^{15} 7\cdot 13\cdot 1291\, t_1^2 t_3^5 + 2^{16} 3^5 5^{14} 7\cdot 16447\, Z^2 t_1 t_2 t_3^5$$
$$- 2^{13} 5^{14} 3\cdot 7\cdot 3183137\, Z^4 t_2^2 t_3^5 - 2^{20} 3^5 5^{15} 7\cdot 17\cdot 643\, Z^5 t_1 t_3^6 + 2^{18} 5^{14} 7\cdot 1747\cdot 1789\, Z^7 t_2 t_3^6$$
$$- 2^{24} 3^5 5^{15} 7\cdot 34649\, t_1 t_3^7 + 2^{23} 5^{14} 7\cdot 2511151\, Z^2 t_2 t_3^7 - 2^{26} 5^{15} 7^2 112687\, Z^5 t_3^8$$
$$- 2^{28} 3^3 5^{15} 7\cdot 530807\, t_3^9 + 3^5 5^{17} 7^3 2\, Z^4 t_1^3 t_2 t_4 + 2^2 3^7 5^{14} 7\cdot 67\, Z^6 t_1^2 t_2^2 t_4$$
$$- 3^4 5^{13} 7\cdot 41\, t_1 t_2^4 t_4 - 5^{13} 2\cdot 7\cdot 4871\, Z^2 t_2^5 t_4 - 2^6 3^4 5^{17} 7^2 13\, Z^7 t_1^3 t_3 t_4$$
$$+ 2^5 3^5 5^{14} 7\cdot 23\cdot 199\, Z t_1^2 t_2^2 t_3 t_4 - 2^4 3^3 5^{14} 7\cdot 1091\, Z^3 t_1 t_2^3 t_3 t_4 - 2^5 5^{13} 7\cdot 11\cdot 3119\, Z^5 t_2^4 t_3 t_4$$
$$- 2^{11} 3^4 5^{18} 7^3 Z^2 t_1^3 t_3^2 t_4 + 2^7 3^6 5^{14} 7\cdot 10169\, Z^4 t_1^2 t_2 t_3^2 t_4 - 2^7 3^4 5^{13} 7\cdot 53\cdot 401\, Z^6 t_1 t_2^2 t_3^2 t_4$$
$$+ 2^6 5^{12} 7\cdot 4272533\, t_2^4 t_3^2 t_4 - 2^{12} 3^5 5^{14} 7\cdot 37\cdot 367\, Z^7 t_1^2 t_3^3 t_4 - 2^{12} 3^3 5^{13} 7\cdot 127\cdot 251\, Z t_1 t_2^2 t_3^3 t_4$$
$$- 2^{10} 5^{12} 7^2 3\cdot 19\cdot 48157\, Z^3 t_2^3 t_3^3 t_4 + 2^{19} 3^5 5^{15} 7\cdot 257\, Z^2 t_1^2 t_3^4 t_4 - 2^{13} 3^3 5^{13} 7\cdot 1883993\, Z^4 t_1 t_2 t_3^4 t_4$$
$$+ 2^{13} 5^{13} 7^2 3\cdot 397\cdot 557\, Z^6 t_2^2 t_3^4 t_4 - 2^{18} 3^3 5^{14} 7\cdot 67\cdot 293\, Z^7 t_1 t_3^5 t_4 - 2^{17} 5^{13} 13^2 3\cdot 7\cdot 22063\, Z t_2^2 t_3^5 t_4$$
$$- 2^{23} 3^5 5^{14} 7\cdot 16447\, Z^2 t_1 t_3^6 t_4 + 2^{19} 5^{13} 73^2 7\cdot 12347\, Z^4 t_2 t_3^6 t_4 - 2^{24} 5^{14} 7^2 3\cdot 31\cdot 6673\, Z^7 t_3^7 t_4$$
$$- 2^{30} 5^{14} 7\cdot 2511151\, Z^2 t_3^8 t_4 - 2^5 3^7 5^{16} 7^2 Z t_1^3 t_2 t_4^2 + 2^5 3^7 5^{13} 7\cdot 367\, Z^3 t_1^2 t_2^2 t_4^2$$
$$+ 2^5 3^4 5^{13} 7\cdot 2099\, Z^5 t_1 t_2^3 t_4^2 - 2^4 5^{12} 7\cdot 11\cdot 3517\, Z^7 t_2^4 t_4^2 - 2^7 3^4 5^{16} 7^2 173\, Z^4 t_1^3 t_3 t_4^2$$
$$- 2^6 3^5 5^{13} 7\cdot 19\cdot 29\cdot 59\, Z^6 t_1^2 t_2 t_3 t_4^2 + 2^5 3^4 5^{12} 7\cdot 407717\, t_1 t_2^3 t_3 t_4^2 - 2^5 3^3 5^{13} 7\cdot 149\cdot 271\, Z^2 t_2^4 t_3 t_4^2$$
$$- 2^{11} 3^5 5^{13} 7\cdot 19\cdot 3119\, Z t_1^2 t_2 t_3^2 t_4^2 + 2^9 3^4 5^{12} 7\cdot 61\cdot 48311\, Z^3 t_1 t_2^2 t_3^2 t_4^2 + 2^9 5^{13} 7\cdot 11\cdot 19\cdot 479\, Z^5 t_2^3 t_3^2 t_4^2$$
$$- 2^{13} 3^5 5^{13} 7\cdot 311\cdot 439\, Z^4 t_1^2 t_3^3 t_4^2 - 2^{14} 3^4 5^{13} 7\cdot 61\cdot 4663\, Z^6 t_1 t_2 t_3^3 t_4^2 + 2^{11} 5^{12} 7\cdot 13\cdot 3501503\, t_2^3 t_3^3 t_4^2$$
$$+ 2^{17} 3^4 5^{13} 7\cdot 14029\, Z t_1 t_2 t_3^4 t_4^2 - 2^{15} 5^{12} 3\cdot 7\cdot 11\cdot 13\cdot 104297\, Z^3 t_2^2 t_3^4 t_4^2 - 2^{19} 3^3 5^{13} 7\cdot 4814471\, Z^4 t_1 t_3^5 t_4^2$$
$$- 2^{18} 3^2 5^{13} 7\cdot 3231829\, Z^6 t_2 t_3^5 t_4^2 + 2^{23} 5^{13} 3\cdot 7\cdot 47\cdot 263303\, Z t_2 t_3^6 t_4^2 - 2^{25} 3^2 5^{13} 7\cdot 1951\cdot 29207\, Z^4 t_3^7 t_4^2$$
$$+ 2^6 3^4 5^{15} 7^2 17\, Z^6 t_1^3 t_4^3 - 2^4 3^7 5^{12} 7\cdot 11\cdot 79\, t_1^2 t_2^2 t_4^3 + 2^4 3^3 5^{12} 7\cdot 89\cdot 6449\, Z^2 t_1 t_2^3 t_4^3$$
$$+ 2^4 5^{11} 3\cdot 7\cdot 211\cdot 6449\, Z^4 t_2^4 t_4^3 + 2^{12} 3^7 5^{16} 7^2 Z t_1^3 t_3 t_4^3 - 2^{10} 3^5 5^{12} 7\cdot 208253\, Z^3 t_1^2 t_2 t_3 t_4^3$$
$$- 2^{15} 3^3 5^{12} 7\cdot 1429\, Z^5 t_1 t_2^2 t_3 t_4^3 - 2^8 5^{11} 7\cdot 31\cdot 181\cdot 3659\, Z^7 t_2^3 t_3 t_4^3 + 2^{12} 3^5 5^{14} 7\cdot 13\cdot 37\cdot 59\, Z^6 t_1^2 t_3^2 t_4^3$$
$$+ 2^{16} 3^2 5^{13} 23^2 Z^{10} t_2^2 t_3^2 t_4^3 + 2^{12} 3^3 5^{12} 7^2 1178167\, t_1 t_2^2 t_3^2 t_4^3 - 2^{10} 5^{11} 2107634939\, Z^2 t_2^3 t_3^2 t_4^3$$
$$+ 2^{18} 3^8 5^{13} 7\cdot 1033\, Z t_1^2 t_3^3 t_4^3 - 2^{17} 3^3 5^{12} 7\cdot 13\cdot 23\cdot 4099\, Z^3 t_1 t_2 t_3^3 t_4^3 + 2^{18} 5^{11} 107\cdot 330509\, Z^5 t_2^2 t_3^3 t_4^3$$
$$- 2^{18} 3^4 5^{13} 7\cdot 377563\, Z^6 t_1 t_3^4 t_4^3 + 2^{24} 3^2 5^{16} 23\, Z^8 t_2 t_3^4 t_4^3 + 2^{16} 5^{13} 491035057\, t_2^2 t_3^4 t_4^3$$
$$- 2^{24} 3^3 5^{13} 7\cdot 373\cdot 523\, Z t_1 t_3^5 t_4^3 - 2^{22} 5^{12} 3\cdot 658847099\, Z^3 t_2 t_3^5 t_4^3 + 2^{24} 5^{12} 7\cdot 25229\cdot 55439\, Z^6 t_3^6 t_4^3$$
$$- 2^{30} 5^{13} 3\cdot 7\cdot 23\cdot 71\cdot 5791\, Z t_3^7 t_4^3 - 2^{12} 3^4 5^{15} 7^2 Z^3 t_1^3 t_4^4 + 2^9 3^{12} 5^{12} 7\, Z^5 t_1^2 t_2 t_4^4$$
$$+ 2^7 3^3 5^{11} 7\cdot 13\cdot 17\cdot 5981\, Z^7 t_1 t_2^2 t_4^4 + 2^{12} 3^2 5^{12} 13\, Z^9 t_2^3 t_4^4 + 2^8 5^{10} 1553\cdot 74317\, Z t_2^4 t_4^4$$
$$- 2^{10} 3^5 5^{12} 7\cdot 23\cdot 97849\, t_1^2 t_2 t_3 t_4^4 - 2^{16} 3^2 5^{12} 23\, Z^{12} t_2^2 t_3 t_4^4 + 2^9 3^4 5^{11} 7^2 644789\, Z^2 t_1 t_2^2 t_3 t_4^4$$
$$- 2^{10} 5^{10} 684923563\, Z^4 t_2^3 t_3 t_4^4 + 2^{16} 3^5 5^{13} 7\cdot 11\cdot 23\cdot 157\, Z^3 t_1^2 t_3^2 t_4^4 + 2^{14} 3^5 5^{11} 7\cdot 239\cdot 1361\, Z^5 t_1 t_2 t_3^2 t_4^4$$
$$- 2^{15} 5^{10} 3\cdot 7\cdot 105745993\, Z^7 t_2^2 t_3^2 t_4^4 + 2^{24} 3^4 5^{14} Z^{10} t_2 t_3^3 t_4^4 + 2^{17} 3^5 5^{13} 7^2 144073\, t_1 t_2 t_3^3 t_4^4$$
$$- 2^{15} 5^{10} 31\cdot 2275515659\, Z^2 t_2^2 t_3^3 t_4^4 - 2^{25} 3^3 5^{12} 7^2 19\cdot 1987\, Z^3 t_1 t_3^4 t_4^4$$

$$
\begin{aligned}
&- 2^{20}5^{10}7{\cdot}17{\cdot}683{\cdot}247601\, Z^5 t_2 t_3^4 t_4^4 + 2^{30}3^2 5^{17} Z^8 t_3^5 t_4^4 - 2^{22}5^{11}56634970591\, t_2 t_3^5 t_4^4 \\
&+ 2^{28}3^2 5^{11}59{\cdot}56802241\, Z^3 t_3^6 t_4^4 - 2^{16}3^4 5^{15}7{\cdot}97\, t_1^3 t_4^5 + 2^{10}3^{12}5^{10}7{\cdot}103\, Z^2 t_1^2 t_2 t_4^5 \\
&+ 2^{14}3^2 5^{11} Z^{14} t_2^2 t_4^5 + 2^8 3^3 5^9 7{\cdot}17{\cdot}6436589\, Z^4 t_1 t_2^2 t_4^5 + 2^8 5^8 3{\cdot}83{\cdot}113{\cdot}283487\, Z^6 t_2^3 t_4^5 \\
&- 2^{15}3^{10}5^{10}7^2 367\, Z^5 t_1^2 t_3 t_4^5 + 2^{12}3^3 5^{10}7{\cdot}47{\cdot}409{\cdot}1801\, Z^7 t_1 t_2 t_3 t_4^5 - 2^{15}3^3 5^{11}89{\cdot}241\, Z^9 t_2^2 t_3 t_4^5 \\
&+ 2^{15}5^8 19^2 379{\cdot}72043\, Z t_2^3 t_3 t_4^5 - 2^{17}3^5 5^{12}7{\cdot}67{\cdot}151{\cdot}443\, t_1^2 t_3^2 t_4^5 + 2^{20}3^2 5^{12}29{\cdot}173\, Z^{12} t_2 t_3^2 t_4^5 \\
&+ 2^{17}3^3 5^9 7{\cdot}242346031\, Z^2 t_1 t_2 t_3^2 t_4^5 - 2^{14}5^8 3{\cdot}17{\cdot}29{\cdot}307{\cdot}2364179\, Z^4 t_2^2 t_3^2 t_4^5 \\
&+ 2^{21}3^3 5^9 7{\cdot}151247651\, Z^5 t_1 t_3^3 t_4^5 + 2^{18}5^9 21157{\cdot}4697569\, Z^7 t_2 t_3^3 t_4^5 + 2^{27}3^2 5^{15}13{\cdot}17\, Z^{10} t_3^4 t_4^5 \\
&+ 2^{24}3^3 5^{13}7{\cdot}2004073\, t_1 t_3^4 t_4^5 - 2^{22}5^8 3{\cdot}883{\cdot}159406111\, Z^2 t_2 t_3^4 t_4^5 - 2^{29}5^8 3343{\cdot}11789651\, Z^5 t_3^5 t_4^5 \\
&- 2^{29}5^{11}85909378939\, t_3^6 t_4^5 + 2^{14}3^{14}5^9 7\, Z^7 t_1^2 t_4^6 + 2^{14}3^2 5^{10}1523\, Z^{11} t_2^2 t_4^6 \\
&- 2^{11}3^3 5^8 7{\cdot}13{\cdot}73{\cdot}627131\, Z t_1 t_2^2 t_4^6 + 2^{11}5^7 6691{\cdot}10945717\, Z^3 t_2^3 t_4^6 - 2^{16}3^{10}5^{10}7{\cdot}739\, Z^2 t_1^2 t_3 t_4^6 \\
&- 2^{19}3^3 5^{11}73\, Z^{14} t_2 t_3 t_4^6 + 2^{13}3^3 5^8 7{\cdot}6560672719\, Z^4 t_1 t_2 t_3 t_4^6 - 2^{13}5^8 3{\cdot}19{\cdot}107{\cdot}271{\cdot}4603\, Z^6 t_2^2 t_3 t_4^6 \\
&- 2^{21}3^4 5^8 7{\cdot}195868609\, Z^7 t_1 t_3^2 t_4^6 + 2^{21}3^2 5^{10}173{\cdot}2039\, Z^9 t_2 t_3^2 t_4^6 - 2^{17}5^7 3{\cdot}175949{\cdot}534167\, Z t_2^2 t_3^2 t_4^6 \\
&+ 2^{26}3^3 5^{13}73\, Z^{12} t_3^3 t_4^6 - 2^{25}3^3 5^9 7{\cdot}307854203\, Z^2 t_1 t_3^3 t_4^6 + 2^{19}5^7 17{\cdot}1171{\cdot}755025827\, Z^4 t_2 t_3^3 t_4^6 \\
&+ 2^{26}5^7 3{\cdot}11{\cdot}29{\cdot}1522702439\, Z^7 t_3^4 t_4^6 + 2^{28}5^8 17{\cdot}23{\cdot}99971{\cdot}187073\, Z^2 t_3^5 t_4^6 + 2^{17}3^{14}5^9 7\, Z^4 t_1^2 t_4^7 \\
&+ 2^{12}3^3 5^7 7{\cdot}17{\cdot}61{\cdot}545533\, Z^6 t_1 t_2 t_4^7 + 2^{12}3^2 5^9 1747{\cdot}2179\, Z^8 t_2^2 t_4^7 + 2^{12}5^6 3{\cdot}7{\cdot}25411781761\, t_2^3 t_4^7 \\
&- 2^{18}3^2 5^9 7{\cdot}31{\cdot}101{\cdot}109\, Z^{11} t_2 t_3 t_4^7 + 2^{18}3^4 5^7 7{\cdot}3491{\cdot}87421\, Z t_1 t_2 t_3 t_4^7 \\
&+ 2^{16}5^6 3{\cdot}11{\cdot}41{\cdot}625672687\, Z^3 t_2^2 t_3 t_4^7 + 2^{22}3^4 5^{11}73^2 Z^{14} t_3^2 t_4^7 \\
&- 2^{20}3^3 5^7 7^2 13^2 607{\cdot}45503\, Z^4 t_1 t_3^2 t_4^7 - 2^{18}3^3 5^6 128237{\cdot}6437531\, Z^6 t_2 t_3^2 t_4^7 \\
&- 2^{28}3^3 5^{11}6197\, Z^9 t_3^3 t_4^7 - 2^{24}5^6 3{\cdot}443{\cdot}8098119613\, Z t_2 t_3^3 t_4^7 \\
&+ 2^{26}5^6 3{\cdot}7{\cdot}715783571707\, Z^4 t_3^4 t_4^7 + 2^{19}3^2 5^8 7541\, Z^{13} t_2 t_4^8 \\
&- 2^{19}3^3 5^6 7^2 17^2 19{\cdot}31{\cdot}199\, Z^3 t_1 t_2 t_4^8 + 2^{15}5^6 3{\cdot}23{\cdot}29{\cdot}219793529\, Z^5 t_2^2 t_4^8 \\
&+ 2^{20}3^3 5^6 7{\cdot}23{\cdot}67{\cdot}1297{\cdot}3301\, Z^6 t_1 t_3 t_4^8 - 2^{20}3^4 5^9 28807\, Z^8 t_2 t_3 t_4^8 \\
&+ 2^{17}5^5 3{\cdot}7{\cdot}2544911109121\, t_2^2 t_3 t_4^8 + 2^{25}3^2 5^9 19{\cdot}43{\cdot}829\, Z^{11} t_3^2 t_4^8 \\
&+ 2^{26}3^3 5^7 7{\cdot}53{\cdot}167{\cdot}116381\, Z t_1 t_3^2 t_4^8 - 2^{26}3^2 5^5 167{\cdot}5651{\cdot}157637\, Z^3 t_2 t_3^2 t_4^8 \\
&+ 2^{26}3^2 5^5 19{\cdot}47{\cdot}359{\cdot}6805069\, Z^6 t_3^3 t_4^8 + 2^{32}5^6 3{\cdot}31{\cdot}1249{\cdot}8516173\, Z t_3^4 t_4^8 \\
&+ 2^{18}3^3 5^7 11{\cdot}151{\cdot}2281\, Z^{10} t_2 t_4^9 - 2^{18}3^3 5^5 7^2 1453{\cdot}9140533\, t_1 t_2 t_4^9 \\
&+ 2^{16}5^5 8861954242873\, Z^2 t_2^2 t_4^9 - 2^{23}3^3 5^8 73{\cdot}7541\, Z^{13} t_3 t_4^9 \\
&- 2^{25}3^3 5^5 7{\cdot}4139{\cdot}129119\, Z^3 t_1 t_3 t_4^9 - 2^{23}5^5 151{\cdot}5680110559\, Z^5 t_2 t_3 t_4^9 \\
&- 2^{26}3^2 5^7 40185941\, Z^8 t_3^2 t_4^9 + 2^{24}5^4 17{\cdot}23621998776373\, t_2 t_3^2 t_4^9 \\
&+ 2^{30}5^4 11{\cdot}764381{\cdot}5742169\, Z^3 t_3^3 t_4^9 - 2^{24}3^{14}5^4 7{\cdot}11{\cdot}863\, Z^5 t_1 t_4^{10} \\
&+ 2^{20}5^3 17{\cdot}233941{\cdot}6976421\, Z^7 t_2 t_4^{10} - 2^{25}3^2 5^6 7{\cdot}1381231\, Z^{10} t_3 t_4^{10} \\
&- 2^{25}3^3 5^5 7{\cdot}7457{\cdot}33020201\, t_1 t_3 t_4^{10} - 2^{25}5^3 58812201936403\, Z^2 t_2 t_3 t_4^{10} \\
&+ 2^{29}5^3 39878573708081\, Z^5 t_3^2 t_4^{10} + 2^{32}5^4 3909211{\cdot}4328957\, t_3^3 t_4^{10} \\
&+ 2^{22}3^2 5^5 7541^2 Z^{12} t_4^{11} + 2^{24}3^{13}5^4 7^2 5399\, Z^2 t_1 t_4^{11} - 2^{28}3^2 5^5 7{\cdot}53{\cdot}7541 Z^9 t_4^{12} \\
&+ 2^{22}5^2 225349{\cdot}1564621501\, Z^4 t_2 t_4^{11} - 2^{27}5^2 3{\cdot}107{\cdot}127{\cdot}4920163967\, Z^7 t_3 t_4^{11}
\end{aligned}
$$

$$
\begin{aligned}
&- 2^{31}5^3 13\cdot 1583\cdot 4421\cdot 158759\, Z^2 t_3^2 t_4^{11} - 2^{26}5\cdot 296367312358063\, Z t_2 t_4^{12} \\
&- 2^{32}5\cdot 1383659\cdot 123837583\, Z^4 t_3 t_4^{12} + 2^{30}7\cdot 249677\cdot 73045429\, Z^6 t_4^{13} \\
&+ 2^{33}5\cdot 296367312358063\, Z t_3 t_4^{13} - 2^{32}11\cdot 43\cdot 104455205831\, Z^3 t_4^{14} \\
&-2^{50}5^{10}7\cdot 349\, t_4^{15}\big),
\end{aligned}
$$

$$
\begin{aligned}
y_2 = \frac{5^3}{2^3 3^{18}}\,&\big(2^2 3^3 5^{13} t_1^3 - 3^2 5^{10} 17\, Z^4 t_1 t_2^2 - 5^8 2\cdot 3\cdot 7\, Z^6 t_2^3 - 2^5 3^3 5^{10} 11\, Z^7 t_1 t_2 t_3 \\
&+ 2^5 3^3 5^8 31\, Z t_2^3 t_3 + 2^8 3^4 5^{12} 19\, t_1^2 t_3^2 - 2^9 3^2 5^{10} 353\, Z^2 t_1 t_2 t_3^2 - 2^6 3^2 5^8 367\, Z^4 t_2^2 t_3^2 \\
&+ 2^{15} 3^2 5^{10} 29\, Z^5 t_1 t_3^3 - 2^{11} 5^8 3\cdot 4327\, Z^7 t_2 t_3^3 + 2^{14} 3^3 5^{10} 1319\, t_1 t_3^4 \\
&- 2^{15} 5^8 3\cdot 109\cdot 137\, Z^2 t_2 t_3^4 - 2^{21} 5^{11} 3\cdot 11\, Z^5 t_3^5 + 2^{20} 3^2 5^{10} 97\, t_3^6 - 2^4 5^7 5023\, Z^3 t_2^3 t_4 \\
&+ 2^4 3^2 5^9 11\cdot 13\, Z t_1 t_2^2 t_4 - 2^7 3^3 5^{10} 7^2 Z^4 t_1 t_2 t_3 t_4 + 2^7 5^7 3\cdot 4817\, Z^6 t_2^2 t_3 t_4 \\
&+ 2^{15} 3^2 5^9 53\, Z^7 t_1 t_3^2 t_4 - 2^{10} 5^7 3\cdot 13\cdot 397\, Z t_2^2 t_3^2 t_4 + 2^{16} 3^2 5^{10} 353\, Z^2 t_1 t_3^3 t_4 \\
&- 2^{13} 5^7 19\cdot 41\cdot 317\, Z^4 t_2 t_3^3 t_4 + 2^{19} 5^8 11\cdot 1723\, Z^7 t_3^4 t_4 + 2^{22} 5^8 3\cdot 109\cdot 137\, Z^2 t_3^5 t_4 \\
&- 2^5 3^2 5^8 547\, Z^6 t_1 t_2 t_4^2 - 2^4 3^2 5^6 7\cdot 13\cdot 167\, t_2^3 t_4^2 - 2^{10} 3^2 5^8 11\cdot 103\, Z t_1 t_2 t_3 t_4^2 \\
&- 2^9 5^6 3\cdot 7\cdot 3907\, Z^3 t_2^2 t_3 t_4^2 + 2^{15} 3^2 5^8 881\, Z^4 t_1 t_3^2 t_4^2 + 2^{11} 3^2 5^7 31883\, Z^6 t_2 t_3^2 t_4^2 \\
&- 2^{16} 5^8 43\cdot 461\, Z t_2 t_3^3 t_4^2 + 2^{22} 5^7 127\cdot 1301\, Z^4 t_3^4 t_4^2 + 2^9 3^2 5^7 823\, Z^3 t_1 t_2 t_4^3 \\
&- 2^8 5^6 3\cdot 7\cdot 37\cdot 53\, Z^5 t_2^2 t_4^3 - 2^{13} 3^2 5^8 7\cdot 53\, Z^6 t_1 t_3 t_4^3 - 2^{12} 5^6 3\cdot 73\cdot 1877\, t_2^2 t_3 t_4^3 \\
&- 2^{17} 3^5 5^8 11\, Z t_1 t_3^2 t_4^3 + 2^{17} 3^2 5^6 13\cdot 137\, Z^3 t_2 t_3^2 t_4^3 - 2^{19} 3^4 5^7 641\, Z^6 t_3^3 t_4^3 \\
&+ 2^{23} 5^7 96601\, Z t_3^4 t_4^3 + 2^8 3^2 5^6 7\cdot 13\cdot 67\cdot 79\, t_1 t_2 t_4^4 - 2^{10} 5^5 7^2 10343\, Z^2 t_2^2 t_4^4 \\
&+ 2^{14} 3^2 5^6 19\cdot 443\, Z^3 t_1 t_3 t_4^4 + 2^{13} 5^5 7^2 29\cdot 409\, Z^5 t_2 t_3 t_4^4 - 2^{14} 5^5 17\cdot 379\cdot 17209\, t_2 t_3^2 t_4^4 \\
&- 2^{20} 5^5 17\cdot 19\cdot 13931\, Z^3 t_3^3 t_4^4 + 2^{13} 3^7 5^5 79\, Z^5 t_1 t_4^5 - 2^{12} 5^3 13\cdot 59\cdot 5623\, Z^7 t_2 t_4^5 \\
&+ 2^{15} 3^2 5^6 13\cdot 103\cdot 883\, t_1 t_3 t_4^5 + 2^{15} 5^3 11\cdot 5478043\, Z^2 t_2 t_3 t_4^5 \\
&- 2^{19} 5^3 11\cdot 83\cdot 32839\, Z^5 t_3^2 t_4^5 + 2^{21} 5^5 5189507\, t_3^3 t_4^5 - 2^{14} 3^7 5^5 59\, Z^2 t_1 t_4^6 \\
&- 2^{12} 5^2 327966773\, Z^4 t_2 t_4^6 + 2^{18} 5^2 3\cdot 83\cdot 593\cdot 613\, Z^7 t_3 t_4^6 \\
&+ 2^{20} 5^3 1249\cdot 3677\, Z^2 t_3^2 t_4^6 + 2^{17} 5\cdot 139571863\, Z t_2 t_4^7 + 2^{19} 5\cdot 1099915517\, Z^4 t_3 t_4^7 \\
&-2^{18} 13\cdot 49685821\, Z^6 t_4^8 - 2^{24} 5\cdot 139571863\, Z t_3 t_4^8 + 2^{23} 19\cdot 1661677\, Z^3 t_4^9 + 2^{35} 5^7 17\, t_4^{10}\big),
\end{aligned}
$$

$$
\begin{aligned}
y_3 = -\frac{5^3}{2^2 3^{11}}\,&\big(3^3 5^6 t_1 t_2 - 2^2 5^4 13\, Z^2 t_2^2 + 2^6 5^4 23\, Z^5 t_2 t_3 + 2^6 5^4 587\, t_2 t_3^2 \\
&+ 2^{13} 5^7 Z^3 t_3^3 - 2^5 5^3 Z^7 t_2 t_4 + 2^7 3^3 5^5 11\, t_1 t_3 t_4 + 2^9 5^3 53\, Z^2 t_2 t_3 t_4 \\
&+ 2^{12} 5^5 Z^5 t_3^2 t_4 + 2^{13} 5^4 23\cdot 131\, t_3^3 t_4 - 2^4 5^2 1523\, Z^4 t_2 t_4^2 + 2^9 5^3 3\cdot 73\, Z^7 t_3 t_4^2 \\
&- 2^{13} 5^3 29\, Z^2 t_3^2 t_4^2 + 2^8 5\cdot 31\cdot 41\, Z t_2 t_4^3 + 2^{11} 5\cdot 7\cdot 491\, Z^4 t_3 t_4^3 - 2^9 7541\, Z^6 t_4^4 \\
&- 2^{15} 5\cdot 31\cdot 41\, Z t_3 t_4^4 + 2^{14} 7\cdot 53\, Z^3 t_4^5 + 2^{21} 5^3 11\, t_4^6\big),
\end{aligned}
$$

$$
y_4 = \frac{40}{3} t_4.
$$

A proof of this theorem is given in Appendix A as the formulas involved are too long to present here. The method of the proof is similar to those given for Theorems 4.2.4, 4.2.10, 4.3.4, 4.5.4, 4.5.10. No proof was presented in [26], where these results were first published.

**Proposition 4.5.20.** *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 4.2.5.

The expression for the discriminant of $H_4(7)$ as well as the expression for $e$ in terms of the $y$ coordinates can be found in Appendix A.4. We do not present these here as they are too long and for this reason we chose not to include these in our paper [26]. We do not give expressions for the $t$ coordinates and $Z$ in terms of the $y$ coordinates, as this was too complicated even for computation.

# Chapter 5

# Almost duality results

Let us define on a Frobenius manifold $M$ the following tensor fields:

$$\overset{*}{c}_{ijk} := g_{i\lambda}\overset{*}{c}{}^{\lambda}_{jk}, \qquad \overset{*}{c}{}^{ijk} := g^{i\lambda}c^{jk}_{\lambda}, \tag{5.0.1}$$

where $\overset{*}{c}{}^{\lambda}_{jk}$ and $c^{jk}_{\lambda}$ are given by formulas (2.2.32) and (2.2.6), respectively. It can be shown that $\overset{*}{c}_{ijk}(x) = \frac{\partial^3 F_*}{\partial x_i \partial x_j \partial x_k}$ for a function $F_*(x)$, which is the dual prepotential of $M$ [17].

For irreducible polynomial Frobenius manifolds, it was shown in [17] that their dual prepotentials (up to rescaling) have the following simple form:

$$F_*(x) = \sum_{\alpha \in R_+} \frac{(\alpha,\, x)^2}{(\alpha,\, \alpha)} \log(\alpha,\, x), \tag{5.0.2}$$

where $R_+$ is a positive root system for the associated Coxeter group $W$. Below we give some partial results about dual prepotentials for some algebraic Frobenius manifolds.

## 5.1 Two-dimensional algebraic examples

A two-dimensional (semisimple) algebraic Frobenius manifold has a prepotential of the form

$$F(t) = \frac{1}{2}t_1^2 t_2 + \frac{k(2k)^k}{k^2-1}t_2^{k+1}, \tag{5.1.1}$$

with $k \in \mathbb{Q}\backslash\{-1, 0, 1\}$ (see [13]). This has degrees $d_1 = 1$ and $d_2 = \frac{2}{k}$, and charge $d = \frac{k-2}{k}$. The choice of the coefficient of $t_2^{k+1}$ in formula (5.1.1) is convenient for having a simple relation between coordinates $t_1$, $t_2$ and the flat coordinates of the intersection form $x_1$, $x_2$. Using formulas (2.2.14), (4.0.1) and (2.2.19), we find the intersection form:

$$g^{ij}(t) = \begin{pmatrix} (2k)^{k+1}t_2^{k-1} & t_1 \\ t_1 & \frac{2}{k}t_2 \end{pmatrix}.$$

Using formulas (2.2.6), (2.2.32) and (5.0.1), we get

$$\overset{*}{c}_{111}(t) = -\,4k^{-1}t_1t_2D, \qquad \overset{*}{c}_{112}(t) = \left(4(2k)^k t_2^k + t_1^2\right)D, \tag{5.1.2}$$

$$\overset{*}{c}_{122}(t) = -\,2(2k)^{k+1}t_1t_2^{k-1}D, \qquad \overset{*}{c}_{222}(t) = (2k)^k k^2 t_2^{k-2}\left(4(2k)^k t_2^k + t_1^2\right)D, \tag{5.1.3}$$

where $D = \det\left(g^{ij}(t)\right)^{-2} = \left(4(2k)^k t_2^k - t_1^2\right)^{-2}$. Similar to the polynomial case $k \in \mathbb{Z}_{\geq 2}$ considered in [13], the flat coordinates of the metric $t_1$, $t_2$ are related to the flat coordinates of the intersection form $x_1$, $x_2$ by the following formulas:

$$t_1 = z^k + \overline{z}^k, \qquad t_2 = \frac{z\overline{z}}{2k}, \tag{5.1.4}$$

where $z := x_1 + ix_2$ and $\overline{z} := x_1 - ix_2$.

Performing a tensorial transformation of (5.1.2) and (5.1.3) with the relations (5.1.4), we get the following third order derivatives of the dual prepotential:

$$\overset{*}{c}_{111}(x) = \frac{k\,x_1(x_1^2+3x_2^2)}{(z\overline{z})^2} + \frac{2ki\,x_2^3}{(z\overline{z})^2}\,\frac{\overline{z}^k+z^k}{\overline{z}^k-z^k}, \quad \overset{*}{c}_{112}(x) = \frac{k\,x_2(x_2^2-x_1^2)}{(z\overline{z})^2} - \frac{2ki\,x_1x_2^2}{(z\overline{z})^2}\,\frac{\overline{z}^k+z^k}{\overline{z}^k-z^k}, \tag{5.1.5}$$

$$\overset{*}{c}_{122}(x) = \frac{k\,x_1(x_1^2-x_2^2)}{(z\overline{z})^2} + \frac{2ki\,x_1^2x_2}{(z\overline{z})^2}\,\frac{\overline{z}^k+z^k}{\overline{z}^k-z^k}, \quad \overset{*}{c}_{222}(x) = \frac{k\,x_2(x_2^2+3x_1^2)}{(z\overline{z})^2} - \frac{2ki\,x_1^3}{(z\overline{z})^2}\,\frac{\overline{z}^k+z^k}{\overline{z}^k-z^k}. \tag{5.1.6}$$

Since $k$ is a nonzero rational number there exists $m,\, l \in \mathbb{Z}\setminus\{0\}$ such that $k = \frac{m}{l}$ and $(m,\, l) = 1$. Here, and in the theorems that follow, we are assuming that when taking powers of $\frac{1}{l}$ we are working in an open set $U \subseteq \mathbb{C}$ which contains the points $z$, $\overline{z}$, $2k$ and $\frac{z\overline{z}}{2k}$. In the open set $U$ we choose a single branch of the function $f(w) = w^{\frac{1}{l}}$ so that we have the relation $f(z)f(\overline{z}) = f(2k)f(\frac{z\overline{z}}{2k})$. For example, we could assume that $U$ does not contain the non-positive imaginary axis which can be achieved for $k > 0$ by taking $|\mathrm{Re}(x_1)|$, $|\mathrm{Im}(x_1)| < 1$ and $\mathrm{Re}(x_2)$, $\mathrm{Im}(x_2) > 1$. Similarly, for $k < 0$ we could assume that $U$ does not contain the non-negative imaginary axis and take the same conditions for $x_1$ and $x_2$.

Recall that for a polynomial Frobenius manifold associated to a Coxeter group $W$ with root system $R = R_W$, the dual prepotential has the form (5.0.2). Let $\alpha \in R$ and define $\alpha_i = (\alpha,\, e_i)$, then we have the following relations (for generic points on $(\alpha,\, x) = 0$):

$$\left((\alpha,\, x)\frac{\partial^3 F_*}{\partial x_i \partial x_j \partial x_k}\right)\Bigg|_{(\alpha,\, x)=0} = \frac{2\alpha_i\alpha_j\alpha_k}{(\alpha,\, \alpha)},$$

for all $i,\, j,\, k = 1, \ldots, n$. Let us now investigate the case of two-dimensional algebraic Frobenius manifolds. The positive half of the root system $R_W$ for $W = I_2(m)$ may be taken as

$$\alpha = \alpha^{(a)} = \sin\left(\frac{\pi a}{m}\right) e_1 + \cos\left(\frac{\pi a}{m}\right) e_2, \tag{5.1.7}$$

for $a = 0, \ldots, m-1$ (see Example 2.3.5). The mirrors for $I_2(m)$ are the hyperplanes $(\alpha,\, x) = 0$ which can be equivalently formulated as the subsets where $\overline{z} = e^{\frac{2\pi i a}{m}} z$. If $e^{\frac{2\pi i a}{m}} z \in U$ for all $a$ then we have

$$f(e^{\frac{2\pi i a}{m}} z) = \xi_a f(z),$$

where $\xi_a$ is an $(ml)$'th root of unity and $f$ is the particular branch of the function $f(w) = w^{\frac{1}{l}}$ that we have chosen for taking powers of $\frac{1}{l}$. If $\xi_a^m = 1$, then we will say that $\alpha^{(a)}$ *is consistent with $f$ at $z$*. For example, when $a = 0$, $\alpha^{(a)}$ is consistent with any branch of $f$ for all $z$. The property of consistency is not affected by changing $z$ to a point in a small neighbourhood of $z$. The notion of consistency is useful for analysing singularities in the formulas (5.1.5)–(5.1.6). Thus, on the complex line $\overline{z} = e^{\frac{2\pi i a}{m}} z$, we have

$$\overline{z}^k - z^k = f(e^{\frac{2\pi i a}{m}} z)^m - f(z)^m = (\xi_a^m - 1) f(z)^m,$$

which is equal to 0 precisely when $\alpha^{(a)}$ is consistent with $f$ at $z$. For the theorem that follows we first need the following technical result.

**Lemma 5.1.1.** *Let $\alpha \in R_W$ be a root of the Coxeter group $W = I_2(m)$ as given by equation (5.1.7) and let $b = 0,\, 1,\, 2$ or 3. Then,*

$$\left(\frac{2mi(-x_1)^b x_2^{3-b}(\alpha,\, x)}{(z\overline{z})^2(\overline{z}^m - z^m)}\right)\Bigg|_{(\alpha,\, x)=0} = \frac{\alpha_1^{3-b}\alpha_2^b}{z^m}\Bigg|_{(\alpha,\, x)=0}.$$

*Proof.* The root system $R_W$ has mirrors given by the equations $\overline{z} = e^{\frac{2\pi i a}{m}} z$, where $a = 0, \ldots, m-1$. We can factor $\overline{z}^m - z^m$ into these mirrors:

$$\overline{z}^m - z^m = \prod_{a=0}^{m-1} \left(\overline{z} - e^{\frac{2\pi i a}{m}} z\right). \tag{5.1.8}$$

Consider the following identity

$$\prod_{\substack{b=0\\ b\neq a}}^{m-1} \left(\overline{z} - e^{\frac{2\pi i b}{m}} z\right) = e^{-\frac{2\pi i a}{m}} \sum_{c=0}^{m-1} \overline{z}^c z^{m-1-c} e^{-\frac{2\pi i a c}{m}}, \tag{5.1.9}$$

which can be checked by evaluating the right hand side at $\overline{z} = e^{\frac{2\pi i b}{m}} z$ for $b \neq a$, and by comparing the coefficients of $\overline{z}^{m-1}$. Evaluating (5.1.9) at $\overline{z} = e^{\frac{2\pi i a}{m}} z$ and factoring out $z^{m-1}$

we get

$$\prod_{\substack{b=0\\b\neq a}}^{m-1}\left(e^{\frac{2\pi ia}{m}}-e^{\frac{2\pi ib}{m}}\right)=me^{-\frac{2\pi ia}{m}}. \tag{5.1.10}$$

To complete the proof we will need the following trigonometric relations:

$$1-e^{\frac{2\pi ia}{m}}=-2i\sin\left(\frac{\pi a}{m}\right)e^{\frac{\pi ia}{m}}, \qquad 1+e^{\frac{2\pi ia}{m}}=2\cos\left(\frac{\pi a}{m}\right)e^{\frac{\pi ia}{m}}.$$

The mirrors of $I_2(m)$ are also given by the equations $(\alpha^{(a)},\, x)=0$ and we can relate this to the expression $\overline{z}=e^{\frac{2\pi ia}{m}}z$ by the ratio

$$\begin{aligned}\frac{(\alpha^{(a)},\, x)}{\overline{z}-e^{\frac{2\pi ia}{m}}z}&=\frac{\sin\left(\frac{\pi a}{m}\right)x_1+\cos\left(\frac{\pi a}{m}\right)x_2}{\left(1-e^{\frac{2\pi ia}{m}}\right)x_1-i\left(1+e^{\frac{2\pi ia}{m}}\right)x_2}\\&=\frac{\sin\left(\frac{\pi a}{m}\right)x_1+\cos\left(\frac{\pi a}{m}\right)x_2}{-2ie^{\frac{\pi ia}{m}}\left(\sin\left(\frac{\pi a}{m}\right)x_1+\cos\left(\frac{\pi a}{m}\right)x_2\right)}=\frac{i}{2}e^{-\frac{\pi ia}{m}}.\end{aligned} \tag{5.1.11}$$

By formulas (5.1.8), (5.1.10) and (5.1.11) we get

$$\begin{aligned}\frac{(\alpha^{(a)},\, x)}{\overline{z}^m-z^m}\Big|_{(\alpha,\,x)=0}&=\frac{(\alpha^{(a)},\, x)}{\overline{z}-e^{\frac{2\pi ia}{m}}z}\prod_{\substack{b=0\\b\neq a}}^{m-1}\frac{1}{\overline{z}-e^{\frac{2\pi ib}{m}}z}\Big|_{(\alpha,\,x)=0}\\&=\frac{i}{2}e^{-\frac{\pi ia}{m}}\frac{1}{me^{-\frac{2\pi ia}{m}}z^{m-1}}\Big|_{(\alpha,\,x)=0}=\frac{ie^{\frac{\pi ia}{m}}}{2mz^{m-1}}\Big|_{(\alpha,\,x)=0}.\end{aligned} \tag{5.1.12}$$

Now, for $a=0$, the equation $(\alpha^{(a)},\, x)=0$ is equivalent to the condition that $x_2=0$ and so we have the following relations for $b=0,\,1,\,2$ and 3:

$$\frac{(-x_1)^b x_2^{3-b}}{(z\overline{z})^2}\Big|_{(\alpha,\,x)=0}=\frac{(-x_1)^b\delta_{3b}}{x_1^4}=-\frac{\delta_{3b}}{x_1}=-\frac{\alpha_1^{3-b}\alpha_2^b}{e^{\frac{\pi ia}{m}}z}\Big|_{(\alpha,\,x)=0}. \tag{5.1.13}$$

Similarly, for $a\neq 0$ the equation $(\alpha^{(a)},\, x)=0$ is equivalent to the condition

$$x_1=-\frac{\cos\left(\frac{\pi a}{m}\right)}{\sin\left(\frac{\pi a}{m}\right)}x_2=-\frac{\alpha_2}{\alpha_1}x_2,$$

and thus we get

$$\begin{aligned}z|_{(\alpha,\,x)=0}=(x_1+ix_2)|_{(\alpha,\,x)=0}&=\left(-\frac{\cos\left(\frac{\pi a}{m}\right)}{\sin\left(\frac{\pi a}{m}\right)}+i\right)x_2\\&=-\left(\frac{\cos\left(\frac{\pi a}{m}\right)-i\sin\left(\frac{\pi a}{m}\right)}{\sin\left(\frac{\pi a}{m}\right)}\right)x_2=-\frac{e^{-\frac{\pi ia}{m}}}{\alpha_1}x_2.\end{aligned}$$

Thus, for $a \neq 0$, we have the following relations:

$$\frac{(-x_1)^b x_2^{3-b}}{(z\overline{z})^2}\Big|_{(\alpha,\,x)=0} = \frac{\alpha_2^b x_2^3}{\alpha_1^b e^{\frac{4\pi i a}{m}} z^4}\Big|_{(\alpha,\,x)=0} = \frac{\alpha_1^{3-b}\alpha_2^b x_2^3}{e^{\frac{4\pi i a}{m}}(-x_2^3)e^{-\frac{3\pi i a}{m}} z}\Big|_{(\alpha,\,x)=0}$$
$$= -\frac{\alpha_1^{3-b}\alpha_2^b}{e^{\frac{\pi i a}{m}} z}\Big|_{(\alpha,\,x)=0}, \tag{5.1.14}$$

for $b = 0$, 1, 2 or 3. Formulas (5.1.12), (5.1.13) and (5.1.14) thus imply the statement. □

This allows us to present the following result for two-dimensional algebraic Frobenius manifolds, which closely resembles the polynomial case.

**Theorem 5.1.2.** *Let $F$ be a two-dimensional prepotential of the form (5.1.1) with $k = \frac{m}{l}$, where $m,\, l \in \mathbb{Z}_{>0}$ and $(m,\, l) = 1$. Let $W = I_2(m)$ and $\alpha \in R_W$. Let $z$ be a generic point in the hyperplane $(\alpha,\, x) = 0$. Then, if $\alpha$ is consistent with $f$ at $z$, the third order derivatives $\overset{*}{c}_{ijk}(x)$ of the dual prepotential $F_*$ satisfy the following:*

$$\Big((\alpha,\, x)\,\overset{*}{c}_{ijk}(x)\Big)\Big|_z = 2\alpha_i\alpha_j\alpha_k.$$

*If $\alpha$ is not consistent with $f$ at $z$, then the third order derivatives satisfy the following:*

$$\Big((\alpha,\, x)\,\overset{*}{c}_{ijk}(x)\Big)\Big|_z = 0.$$

*Proof.* Consider the identity

$$\frac{\overline{z}^{\frac{m}{l}} + z^{\frac{m}{l}}}{\overline{z}^{\frac{m}{l}} - z^{\frac{m}{l}}} = \frac{\overline{z}^m + z^m}{\overline{z}^m - z^m} + \frac{2}{\overline{z}^m - z^m}\sum_{j=1}^{l-1} \overline{z}^{\frac{mj}{l}} z^{\frac{m(l-j)}{l}},$$

which can be checked by multiplying the left hand side by $\sum_{j=0}^{l-1} \overline{z}^{\frac{mj}{l}} z^{\frac{m(l-j-1)}{l}}$ and dividing by the same expression. Hence, by the formula (5.1.5)

$$\overset{*}{c}_{111}(x) = \frac{1}{l}\left(\frac{m\,x_1(x_1^2+3x_2^2)}{(z\overline{z})^2} + \frac{2mi\,x_2^3}{(z\overline{z})^2}\frac{\overline{z}^m + z^m}{\overline{z}^m - z^m}\right) + \frac{2mi\,x_2^3}{l(z\overline{z})^2}\frac{2}{\overline{z}^m - z^m}\sum_{j=1}^{l-1} \overline{z}^{\frac{mj}{l}} z^{\frac{m(l-j)}{l}}. \tag{5.1.15}$$

The first term in formula (5.1.15) is equal to $\frac{1}{l}\overset{*}{c}_{111}(x)$ in formula (5.1.5) when $k = m$. Thus, this term is equal to $\frac{1}{l}\sum_{\alpha\in R_+} \frac{2\alpha_1^3}{(\alpha,\,x)}$ by formula (5.0.2) for the dihedral Coxeter case $W = I_2(m)$. So, we see that

$$\overset{*}{c}_{111}(x) = \frac{1}{l}\sum_{\alpha\in R_+}\frac{2\alpha_1^3}{(\alpha,\,x)} + \frac{4mix_2^3}{l(z\overline{z})^2}\frac{1}{\overline{z}^m - z^m}\sum_{j=1}^{l-1} \overline{z}^{\frac{mj}{l}} z^{\frac{m(l-j)}{l}},$$

Similarly, for the other third order derivatives of the dual prepotential given by formulas (5.1.5)–(5.1.6), we have

$$\overset{*}{c}_{112}(x) = \frac{1}{l}\sum_{\alpha\in R_+}\frac{2\alpha_1^2\alpha_2}{(\alpha,\,x)} - \frac{4mix_1x_2^2}{l(z\overline{z})^2}\frac{1}{\overline{z}^m - z^m}\sum_{j=1}^{l-1}\overline{z}^{\frac{mj}{l}}z^{\frac{m(l-j)}{l}},$$
$$\overset{*}{c}_{122}(x) = \frac{1}{l}\sum_{\alpha\in R_+}\frac{2\alpha_1\alpha_2^2}{(\alpha,\,x)} + \frac{4mix_1^2x_2}{l(z\overline{z})^2}\frac{1}{\overline{z}^m - z^m}\sum_{j=1}^{l-1}\overline{z}^{\frac{mj}{l}}z^{\frac{m(l-j)}{l}},$$
$$\overset{*}{c}_{222}(x) = \frac{1}{l}\sum_{\alpha\in R_+}\frac{2\alpha_2^3}{(\alpha,\,x)} - \frac{4mix_1^3}{l(z\overline{z})^2}\frac{1}{\overline{z}^m - z^m}\sum_{j=1}^{l-1}\overline{z}^{\frac{mj}{l}}z^{\frac{m(l-j)}{l}},$$

where $R_+$ is a positive root system of $I_2(m)$. If $\alpha$ is consistent with $f$, then

$$\overline{z}^{\frac{mj}{l}}z^{\frac{m(l-j)}{l}} = f(e^{\frac{2\pi ia}{m}}z)^{mj}f(z)^{m(l-j)} = \xi_a^{mj}f(z)^{mj}f(z)^{m(l-j)} = f(z)^{ml} = z^m,$$

and so, using Lemma 5.1.1,

$$\begin{aligned}\Big((\alpha,\,x)\,\overset{*}{c}_{ijk}(x)\Big)\Big|_{(\alpha,\,x)=0} &= \frac{2\alpha_i\alpha_j\alpha_k}{l} + \frac{2\alpha_i\alpha_j\alpha_k}{l}\frac{1}{z^m}\sum_{j=1}^{l-1}z^m\Big|_{(\alpha,\,x)=0}\\ &= \frac{2\alpha_i\alpha_j\alpha_k}{l}(1+(l-1)) = 2\alpha_i\alpha_j\alpha_k.\end{aligned}$$

If $\alpha$ is not consistent with $f$, then

$$\overline{z}^{\frac{mj}{l}}z^{\frac{m(l-j)}{l}} = f(e^{\frac{2\pi ia}{m}}z)^{mj}f(z)^{m(l-j)} = \xi_a^{mj}f(z)^{mj}f(z)^{m(l-j)} = \xi_a^{mj}f(z)^{ml} = \xi_a^{mj}z^m,$$

where $\xi_a$ is an $(ml)$'th root of unity with $\xi_a^m \neq 1$. Thus, by use of Lemma 5.1.1,

$$\begin{aligned}\Big((\alpha,\,x)\,\overset{*}{c}_{ijk}(x)\Big)\Big|_{(\alpha,\,x)=0} &= \frac{2\alpha_i\alpha_j\alpha_k}{l} + \frac{2\alpha_i\alpha_j\alpha_k}{l}\frac{1}{z^m}\sum_{j=1}^{l-1}\xi_a^{mj}z^m\Big|_{(\alpha,\,x)=0}\\ &= \frac{2\alpha_i\alpha_j\alpha_k}{l}\left(1+\sum_{j=1}^{l-1}\xi_a^{mj}\right) = 0.\end{aligned}$$

□

For the next two theorems we consider only the case when $k = \pm l^{-1}$ with $l \in \mathbb{Z}_{\geq 2}$. We will also need to assume that $2ix_2 \in U$, as we will make use of the hypergeometric function ${}_2F_1(a,b;c;w)$ which is single-valued for the argument $|w| < 1$. This condition holds for $w = \frac{ix_1+x_2}{2x_2}$ when we use the constraints for $U$ specified above.

**Theorem 5.1.3.** *Let $M$ be a two-dimensional Frobenius manifold with prepotential (5.1.1)* *with* $k = l^{-1}$*, where* $l \in \mathbb{Z}_{\geq 2}$*. Then the dual prepotential of* $M$ *has the form*

$$F_*(x) = \frac{x_2^2}{l}\log x_2 + \frac{\overline{z}^2}{4l}\log\overline{z} + \frac{z^2}{4l}\log z \\ + \sum_{j=1}^{l-1}\frac{\overline{z}^{\frac{j}{l}}}{4j}\left(\frac{lx_1 + (l-2j)ix_2}{(j-l)z^{\frac{j}{l}-1}} + (2ix_2)^{2-\frac{j}{l}}\,{}_2F_1\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right)\right), \quad (5.1.16)$$

*where* ${}_2F_1(a,b;c;w)$ *is the hypergeometric function.*

*Proof.* For $k = l^{-1}$, the third order derivatives of the dual prepotential given by formulas (5.1.5)–(5.1.6) may be simplified as

$$\overset{*}{c}_{111}(x) = \frac{x_1}{lz\overline{z}} - \frac{2x_2^2}{l(z\overline{z})^2}\sum_{j=1}^{l-1}\overline{z}^{\frac{j}{l}}z^{\frac{l-j}{l}}, \quad (5.1.17)$$

$$\overset{*}{c}_{112}(x) = \frac{x_2}{lz\overline{z}} + \frac{2x_1x_2}{l(z\overline{z})^2}\sum_{j=1}^{l-1}\overline{z}^{\frac{j}{l}}z^{\frac{l-j}{l}}, \quad (5.1.18)$$

$$\overset{*}{c}_{122}(x) = -\frac{x_1}{lz\overline{z}} - \frac{2x_1^2}{l(z\overline{z})^2}\sum_{j=1}^{l-1}\overline{z}^{\frac{j}{l}}z^{\frac{l-j}{l}}, \quad (5.1.19)$$

$$\overset{*}{c}_{222}(x) = \frac{1}{l}\left(\frac{2}{x_2} - \frac{x_2}{z\overline{z}}\right) + \frac{2x_1^3}{lx_2(z\overline{z})^2}\sum_{j=1}^{l-1}\overline{z}^{\frac{j}{l}}z^{\frac{l-j}{l}}, \quad (5.1.20)$$

where we use the identity

$$\frac{\overline{z}^{\frac{1}{l}} + z^{\frac{1}{l}}}{\overline{z}^{\frac{1}{l}} - z^{\frac{1}{l}}} = \frac{\overline{z}+z}{\overline{z}-z} + \frac{2}{\overline{z}-z}\sum_{j=1}^{l-1}\overline{z}^{\frac{j}{l}}z^{\frac{l-j}{l}}.$$

Let us define the following functions:

$$A(x) := \frac{x_2^2}{l}\log x_2 + \frac{\overline{z}^2}{4l}\log\overline{z} + \frac{z^2}{4l}\log z,$$

$$B_j(x) := \frac{\overline{z}^{\frac{j}{l}}\left(lx_1 + (l-2j)ix_2\right)}{4j(j-l)z^{\frac{j}{l}-1}},$$

$$C_j(x) := \frac{\overline{z}^{\frac{j}{l}}}{4j}(2ix_2)^{2-\frac{j}{l}}\,{}_2F_1\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right),$$

for $j = 1,\ldots,l-1$. Then, we want to show that

$$\frac{\partial^3}{\partial x_a\partial x_b\partial x_c}\left(A(x) + \sum_{j=1}^{l-1}\left(B_j(x) + C_j(x)\right)\right) = \overset{*}{c}_{abc}(x), \quad (5.1.21)$$

where $\overset{*}{c}_{abc}(x)$ are given by formulas (5.1.17)–(5.1.20). The third order derivatives of $A(x)$

are

$$\frac{\partial^3 A}{\partial x_1^3}=\frac{x_1}{lz\overline{z}},\qquad \frac{\partial^3 A}{\partial x_1^2\partial x_2}=\frac{x_2}{lz\overline{z}},\qquad \frac{\partial^3 A}{\partial x_1\partial x_2^2}=-\frac{x_1}{lz\overline{z}},\qquad \frac{\partial^3 A}{\partial x_2^3}=\frac{1}{l}\left(\frac{2}{x_2}-\frac{x_2}{z\overline{z}}\right). \tag{5.1.22}$$

Next, we calculate the third order derivatives of $B_j(x)$ for $j=1,\dots,l-1$ to be

$$\frac{\partial^3 B_j}{\partial x_1^3}=\frac{4ix_2^3\,\overline{z}^{\frac{j}{l}-3}b_j}{l^3 z^{\frac{j}{l}+2}},\qquad \frac{\partial^3 B_j}{\partial x_1^2\partial x_2}=-\frac{4ix_1x_2^2\,\overline{z}^{\frac{j}{l}-3}b_j}{l^3 z^{\frac{j}{l}+2}}, \tag{5.1.23}$$

$$\frac{\partial^3 B_j}{\partial x_1\partial x_2^2}=\frac{4ix_1^2x_2\,\overline{z}^{\frac{j}{l}-3}b_j}{l^3 z^{\frac{j}{l}+2}},\qquad \frac{\partial^3 B_j}{\partial x_2^3}=-\frac{4ix_1^3\,\overline{z}^{\frac{j}{l}-3}b_j}{l^3 z^{\frac{j}{l}+2}}, \tag{5.1.24}$$

where $b_j=l(l-2j)x_1+i(j^2-jl+l^2)x_2$. Now, let us consider the first order derivatives of $C_j(x)$ for $j=1,\dots,l-1$. We get

$$\begin{aligned}\frac{\partial C_j}{\partial x_1}&=\frac{\overline{z}^{\frac{j}{l}-1}}{4l}(2ix_2)^{2-\frac{j}{l}}\,{}_2F_1\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right)\\&\quad-\frac{\overline{z}^{\frac{j}{l}}}{4j}(2ix_2)^{1-\frac{j}{l}}\,{}_2F_1'\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right),\\ \frac{\partial C_j}{\partial x_2}&=\left(\frac{ix_1+x_2}{j}-\frac{ix_1}{2l}\right)\overline{z}^{\frac{j}{l}-1}(2ix_2)^{1-\frac{j}{l}}\,{}_2F_1\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right)\\&\quad+\frac{ix_1}{2j}\overline{z}^{\frac{j}{l}}(2ix_2)^{-\frac{j}{l}}\,{}_2F_1'\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right).\end{aligned}$$

The hypergeometric function has the properties

$$\begin{aligned}{}_2F_1'(a,b;c;w)&=\frac{c-1}{w}({}_2F_1(a,b;c-1;w)-{}_2F_1(a,b;c;w)),\\ {}_2F_1(a,b;b;w)&=(1-w)^{-a},\end{aligned}$$

for all $a,b,c\in\mathbb{C}$. Here the branch of $(1-w)^{-a}$ is the one which equals 1 at $w=0$. When $a=\frac{j}{l}$ and $w=\frac{ix_1+x_2}{2x_2}$ we have the relation

$${}_2F_1\left(\frac{j}{l},b;b;\frac{ix_1+x_2}{2x_2}\right)=\left(1-\frac{ix_1+x_2}{2x_2}\right)^{-\frac{j}{l}}=\frac{(2ix_2)^{\frac{j}{l}}}{z^{\frac{j}{l}}}, \tag{5.1.25}$$

for any $b\in\mathbb{C}$, where the functions $f(t)=t^{\frac{1}{l}}$ on the right-hand side of (5.1.25) are taken on the same branch, which is possible since the open set $U$ contains both $z$ and $2ix_2$. Hence we have

$${}_2F_1'\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right)=\frac{2jx_2}{l(ix_1+x_2)}\left(\frac{(2ix_2)^{\frac{j}{l}}}{z^{\frac{j}{l}}}-{}_2F_1\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right)\right). \tag{5.1.26}$$

Substitution of relation (5.1.26) into the above formulas for the derivatives $\frac{\partial C_j}{\partial x_i}$ gives

$$\frac{\partial C_j}{\partial x_1} = -\frac{x_2^2\,\overline{z}^{\frac{j}{l}-1}}{l z^{\frac{j}{l}}},$$
$$\frac{\partial C_j}{\partial x_2} = \frac{i}{j}\overline{z}^{\frac{j}{l}}(2ix_2)^{1-\frac{j}{l}}{}_2F_1\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right)+\frac{x_1x_2\,\overline{z}^{\frac{j}{l}-1}}{l z^{\frac{j}{l}}}.$$

Since $\frac{\partial C_j}{\partial x_1}$ contains no hypergeometric functions, its derivatives are more easily attainable, and we see that

$$\frac{\partial^3 C_j}{\partial x_1^3} = -\frac{2x_2^2\,\overline{z}^{\frac{j}{l}-3}c_j}{l^3 z^{\frac{j}{l}+2}}, \qquad \frac{\partial^3 C_j}{\partial x_1^2\partial x_2} = \frac{2x_1x_2\,\overline{z}^{\frac{j}{l}-3}c_j}{l^3 z^{\frac{j}{l}+2}}, \qquad \frac{\partial^3 C_j}{\partial x_1\partial x_2^2} = -\frac{2x_1^2\,\overline{z}^{\frac{j}{l}-3}c_j}{l^3 z^{\frac{j}{l}+2}}, \tag{5.1.27}$$

where $c_j = l^2x_1^2 + 2il(l-2j)x_1x_2 - (l^2-2jl+2j^2)x_2^2$. On the other hand, $\frac{\partial C_j}{\partial x_2}$ still contains hypergeometric functions. Looking at the second order derivative, and using relation (5.1.26), we see that

$$\frac{\partial^2 C_j}{\partial x_2^2} = -\frac{2}{j}\overline{z}^{\frac{j}{l}}(2ix_2)^{-\frac{j}{l}}{}_2F_1\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right)+\frac{x_1\,\overline{z}^{\frac{j}{l}-2}}{l^2 z^{\frac{j}{l}+1}}(3lx_1^2+i(l-2j)x_1x_2+2lx_2^2).$$

Differentiating $C_j(x)$ with respect to $x_2$ for the third time and substituting relation (5.1.26) into this expression, we get

$$\frac{\partial^3 C_j}{\partial x_2^3} = \frac{2x_1^3\,\overline{z}^{\frac{j}{l}-3}}{l^3 x_2 z^{\frac{j}{l}+2}}(l^2x_1^2+2il(2j-l)x_1x_2-(l^2-2jl+2j^2)x_2^2). \tag{5.1.28}$$

From relations (5.1.22)–(5.1.24) and (5.1.27)–(5.1.28), one can check directly that formula (5.1.21) holds. □

**Theorem 5.1.4.** *Let $\widetilde{M}$ be a two-dimensional Frobenius manifold with prepotential (5.1.1) with $k=-l^{-1}$, where $l \in \mathbb{Z}_{\geq 2}$. Then the dual prepotential of $\widetilde{M}$ has the form*

$$\widetilde{F_*}(x) = F_*(x) - \frac{x_1^2+x_2^2}{2l}\log(x_1^2+x_2^2),$$

*where $F_*(x)$ is the function given by formula (5.1.16).*

*Proof.* Given a two-dimensional Frobenius manifold $M$, with charge $d \neq 1$ and $\eta_{11}=0$ one can construct a two-dimensional Frobenius manifold $\widetilde{M}$ with charge $\widetilde{d}=2-d$ using a symmetry of the WDVV equations known as an inversion [13]. The flat coordinates $x$ of the intersection form of $M$ may be expressed in terms of the flat coordinates $\widetilde{x}$ of the intersection form of $\widetilde{M}$ via the following relation:

$$x_i = \frac{2\,\widetilde{x}_i}{\left(1-\widetilde{d}\right)\left(\widetilde{x}_1^2+\widetilde{x}_2^2\right)},$$

for $i = 1, 2$. Moreover, the dual prepotential $\widetilde{F_*}$ of $\widetilde{M}$ may be expressed as

$$\widetilde{F_*}(\widetilde{x}) = \frac{4F_*(x(\widetilde{x}))}{(1-d)^2\left(x_1(\widetilde{x})^2 + x_2(\widetilde{x})^2\right)^2}, \tag{5.1.29}$$

where $F_*$ is the dual prepotential of $M$ [42]. In two dimensions, semisimple Frobenius manifolds with $d \neq 1$ and $\eta_{11} = 0$ are uniquely parametrized, up to isomorphism, by their charge [13]. A Frobenius manifold with prepotential (5.1.1) has charge $d = \frac{k-2}{k}$. Let $M$ be the Frobenius manifold with prepotential (5.1.1) with $k = l^{-1}$, thus $M$ has charge $d = 1 - 2l$. We know from Theorem 5.1.3 that this Frobenius manifold has a dual prepotential of the form (5.1.16). The inversion $\widetilde{M}$ must have charge $\widetilde{d} = 2l + 1$ and therefore its prepotential must be of the form (5.1.1) with $k = -l^{-1}$. The dual prepotential of $\widetilde{M}$ is given by equation (5.1.29) from which the statement follows. □

## 5.2 $(H_3)''$ and $D_4(a_1)$

Below we give related results for the algebraic Frobenius manifolds $(H_3)''$ and $D_4(a_1)$.

**Proposition 5.2.1.** *Let $P^M(x, Z)$ be the polynomial from relation (4.2.45) for $M = (H_3)''$ and let it be the polynomial from relation (4.3.34) for $M = D_4(a_1)$ expressed in the $x$ coordinates. Then for each $\alpha \in R$, where $R = R_{H_3}$ for $M = (H_3)''$ and $R = R_{D_4}$ for $M = D_4(a_1)$, we have that*

$$P^M(x, Z)|_{(\alpha,\, x)=0} = K^M_\alpha (L^M_\alpha)^2,$$

*where $K^M_\alpha, L^M_\alpha \in \mathbb{C}[x; Z]$ and $L^M_\alpha$ is linear in $Z$. $K^M_\alpha$ is cubic in $Z$ for $M = (H_3)''$ and quartic in $Z$ for $M = D_4(a_1)$.*

To check that the polynomial $P^M(x, Z)$ factorises on the hyperplanes $(\alpha,\, x) = 0$ we first substitute the expressions for $y_i(x)$ from relations (4.2.1)–(4.2.7), or (4.3.8)–(4.3.11), into the left-hand side of equation (4.2.45), or equation (4.3.34), respectively. We then restrict to the hyperplane $(\alpha,\, x) = 0$ and see that the polynomial factorises as claimed.

**Proposition 5.2.2.** *Let $\alpha \in R$, where $R = R_{H_3}$ for $M = (H_3)''$ and $R = R_{D_4}$ for $M = D_4(a_1)$. The third order derivatives $\overset{*}{c}_{ijk}(x)$ of the dual prepotential $F_*$ of $M = (H_3)''$ or $M = D_4(a_1)$ satisfy*

$$\left((\alpha,\, x)\,\overset{*}{c}_{ijk}(x)\right)\Big|_{(\alpha,\, x)=0} = 0$$

*if* $L_\alpha^M(x, Z) = 0$*. If* $K_\alpha^M(x, Z) = 0$ *then we have*

$$\Big((\alpha,\, x)\, \overset{*}{c}_{ijk}(x)\Big)\Big|_{(\alpha,\, x)=0} = \frac{2\alpha_i\alpha_j\alpha_k}{(\alpha,\, \alpha)}.$$

*Proof.* By formulas (4.0.2) and (5.0.1) we have

$$\frac{\partial^3 F_*}{\partial x_i \partial x_j \partial x_k} = \overset{*}{c}_{ijk}(x) = g_{i\lambda}(x) g_{j\mu}(x) g_{k\nu}(x) \overset{*}{c}^{\lambda\mu\nu}(x) = \overset{*}{c}^{ijk}(x).$$

Then

$$\frac{\partial^3 F_*}{\partial x_i \partial x_j \partial x_k} = \overset{*}{c}^{\alpha\beta\gamma}(t) \frac{\partial x_i}{\partial t_\alpha} \frac{\partial x_j}{\partial t_\beta} \frac{\partial x_k}{\partial t_\gamma} = g^{\alpha\delta}(t) c_\delta^{\beta\gamma}(t) \frac{\partial x_i}{\partial y_r} \frac{\partial y_r}{\partial t_\alpha} \frac{\partial x_j}{\partial y_s} \frac{\partial y_s}{\partial t_\beta} \frac{\partial x_k}{\partial y_t} \frac{\partial y_t}{\partial t_\gamma}. \tag{5.2.1}$$

Now we express the right-hand side of (5.2.1) in $x$ coordinates and $Z$. For the terms $g^{\alpha\delta}(t)$ and $c_\delta^{\beta\gamma}(t)$ we apply Theorem 4.2.13. The derivatives $\frac{\partial x_i}{\partial y_r}$, $\frac{\partial x_j}{\partial y_s}$ and $\frac{\partial x_k}{\partial y_t}$ can be found by inverting the Jacobi matrix $J = \left(\frac{\partial y_i}{\partial x_j}\right)$. The derivatives $\frac{\partial y_r}{\partial t_\alpha}$, $\frac{\partial y_s}{\partial t_\beta}$ and $\frac{\partial y_t}{\partial t_\gamma}$ can be found by Theorem 4.2.10. We then reduce the resulting expression for $\overset{*}{c}_{ijk}(x)$ as a polynomial in $Z$ modulo the relation (4.2.45) for $M = (H_3)''$, or modulo the relation (4.3.34) for $M = D_4(a_1)$.

Then, for any $\alpha \in R$ we get $(\alpha,\, x)\, \overset{*}{c}_{ijk}(x)$ which can be restricted to $(\alpha,\, x) = 0$. Using Proposition 5.2.1 we can then reduce the restricted expression as a polynomial in $Z$ modulo $K_\alpha^M$ or modulo $L_\alpha^M$ depending on which branch of $Z$ we consider on the hyperplane. This leads to the claim. □

# Chapter 6

# Concluding remarks

In this thesis we studied the relations between flat coordinates of the metric $\eta$ and flat coordinates of the intersection form $g$ for algebraic Frobenius manifolds in dimensions 3 and 4 and we obtained explicit formulas for the relations. The complexity of these formulas changes as the underlying Frobenius manifolds changes and it generally increases with the increase of the algebraic degree of the prepotential. In all cases the flat coordinates of $\eta$ appear to be algebraic functions on the Coxeter orbit space for a suitable Coxeter group $W$.

In the polynomial case, Saito polynomials appear as the flat coordinates of the metric $\eta$ expressed in terms of the flat coordinates of the intersection form $g$. They have applications in relation to the representation theory of rational Cherednik algebras [25]. It would be interesting to understand a representation theoretic meaning of the flat coordinates of algebraic Frobenius manifolds.

The class of algebraic Frobenius manifolds is considerably more complicated than the polynomial one. The classification of algebraic Frobenius manifolds remains a key challenge in the area. Our brief exploration of two-dimensional algebraic Frobenius manifolds in Subsection 3.2.4 suggested that the Dubrovin conjecture on algebraic Frobenius manifolds [7], [10] should be modified to include the assumption that the charge $d$ must be non-negative. Also, the data suggested by Douvropoulos for the degrees of algebraic Frobenius manifolds [12] assumes that the quasi-Coxeter conjugacy class is regular. There seems to be no known examples of algebraic Frobenius manifolds associated with non-regular quasi-Coxeter conjugacy classes. We propose to modify the conjecture so that it reads as follows:

**Conjecture 6.0.1.** *Irreducible, semisimple algebraic Frobenius manifolds with positive degrees and non-negative charge correspond to regular quasi-Coxeter conjugacy classes in finite irreducible Coxeter groups.*

The dual prepotentials for the two-dimensional algebraic Frobenius manifolds that we found are rather involved and require the use of special functions to express them. We were able to deal with a particular class of two-dimensional examples which may be associated to the Coxeter group $W = A_1$. It may be instructive to try to obtain the dual prepotentials when $W$ has rank 2. Further investigation of the two-dimensional cases may also shed some light on the dual prepotentials in higher dimensions.

We observed a certain structure in the residues on the Coxeter mirrors of the third order derivatives of the dual prepotentials for $(H_3)''$ and $D_4(a_1)$ which also agrees with the two-dimensional examples and may be expected to hold in general. Furthermore, we also found the Liouville vector field $e$ in all the cases of algebraic Frobenius manifolds we considered, which is an important ingredient of the almost dual structure.

# Appendix A

# Appendix

## A.1 Extra formulas for $F_4(a_2)$

Recall from Section 4.4 that the degrees of the $t$ coordinates and $Z$ are $\deg t_1(x) = 6$, $\deg t_2(x) = 6$, $\deg t_3(x) = 2$, $\deg t_4(x) = 2$ and $\deg Z(x) = 2$. This allows us to deduce which harmonic polynomials of the $t$ coordinates and $Z$ have the same degrees as the basic invariants of $F_4$, which the following Proposition makes precise.

**Proposition A.1.1.** *Let $V_1 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 12\}$, let $V_2 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 8\}$ and let $V_3 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 6\}$. The harmonic elements of $V_1$ are of the form*

$$\begin{aligned}
&a(3999427165146316800 0t_1^2 - 80904225849081856000t_1t_2 + 91628316367781888000t_2^2 \\
&\quad + 39691987478208 \times 10^8 t_2t_3Z^2 - 21155829325884864 \times 10^6 t_2t_3^2Z \\
&\quad + 5509343267986188533760 0t_1t_3^3 - 3415310196236774184960 0t_2t_3^3 \\
&\quad - 4902119686649079092250 0t_3^4Z^2 + 977984833875970910804100t_3^5Z \\
&\quad + 311092186219696155229523 1t_3^6 + 146555030688768 \times 10^8 t_2t_4Z^2 \\
&\quad - 29213302783961088 \times 10^6 t_2t_3t_4Z + 855552822598840025088 00t_1t_3^2t_4 \\
&\quad - 957748959639116955648 00t_2t_3^2t_4 - 724005369105094758240000t_3^3t_4Z^2 \\
&\quad + 707435652730860272512800 0t_3^4t_4Z + 2114774127022999514941065 6t_3^5t_4 \\
&\quad - 8834398276534272 \times 10^6 t_2t_4^2Z + 743019013393899061248 00t_1t_3t_4^2 \\
&\quad - 621927482894271302860800t_2t_3t_4^2 - 2693340299619017568 \times 10^6 t_3^2t_4^2Z^2 \\
&\quad + 8762848369368650105702400t_3^3t_4^2Z + 76905929434144514727538944t_3^4t_4^2 \\
&\quad + 1238495422217491316736 00t_1t_4^3 - 978296108365586733465600t_2t_4^3 \\
&\quad - 2934732272971723960320000t_3t_4^3Z^2 - 16590944187768816265420800t_3^2t_4^3Z \\
&\quad + 6466070203950441122072985 6t_3^3t_4^3 - 1450851560860623667200000t_4^4Z^2 \\
&\quad - 5450188860933339721236480 0t_3t_4^4Z - 10965117567024849492377 6t_3^2t_4^4 \\
&\quad - 4470116718838046706892800 0t_4^5Z - 9012167797860654843219148 8t_3t_4^5)
\end{aligned}$$

$$
\begin{aligned}
&+b(1665664000t_1^2 - 12300288000t_1t_2 + 22708224000t_2^2 - 1235069035200t_1t_3^3\\
&\quad + 4560254899200t_2t_3^3 + 288947062809465t_3^6 - 13680764697600t_1t_3^2t_4\\
&\quad + 50513592729600t_2t_3^2t_4 + 5072058006855840t_3^5t_4 + 4245696 \times 10^6 t_2t_4^2Z\\
&\quad - 34568645529600t_1t_3t_4^2 + 67292583321600t_2t_3t_4^2 - 29513723557500t_3^2t_4^2Z^2\\
&\quad - 209744195415300t_3^3t_4^2Z + 30958248927422817t_3^4t_4^2 - 16707033907200t_1t_4^3\\
&\quad + 20023079731200t_2t_4^3 - 217947497040000t_3t_4^3Z^2 - 1378881164606400t_3^2t_4^3Z\\
&\quad + 59383246493608128t_3^3t_4^3 - 402364609920000t_4^4Z^2 - 1604093578214400t_3t_4^4Z\\
&\quad + 63810842541368832t_3^2t_4^4 - 662858438860800t_4^5Z + 67385500072869888t_3t_4^5\\
&\quad + 40644595177684992t_4^6),
\end{aligned}
$$

*where $a$, $b \in \mathbb{C}$ are constants, the harmonic elements of $V_2$ are proportional to*

$$
\begin{aligned}
&865280000t_1t_3 - 192 \times 10^6 t_2Z - 465920000t_2t_3 + 1334677500t_3^2Z^2\\
&+ 9485108100t_3^3Z + 129130878591t_3^4 + 465920000t_1t_4 + 163840000t_2t_4\\
&+ 9856080000t_3t_4Z^2 + 62356132800t_3^2t_4Z + 288371962944t_3^3t_4\\
&+ 18195840000t_4^2Z^2 + 72540748800t_3t_4^2Z - 79360280064t_3^2t_4^2\\
&+ 29975961600t_4^3Z - 1705504260096t_3t_4^3 - 2383159394304t_4^4,
\end{aligned}
$$

*and the harmonic elements of $V_3$ are proportional to*

$$
2080t_1 - 7680t_2 - 771147t_3^3 - 8541946t_3^2t_4 - 9637056t_3t_4^2 - 3998592t_4^3.
$$

*Proof.* Using Proposition 4.0.1 we can directly calculate

$$
\begin{aligned}
\Delta(t_1) = &-\frac{81}{8000Q_1(t)Q_2(t)}\big(576 \times 10^{10}t_2^2Z^2 + 485888 \times 10^8 t_2^2t_3Z + 54815488 \times 10^6 t_2^2t_3^2\\
&- 1475975358 \times 10^6 t_2t_3^3Z^2 - 6098353859520000t_2t_3^4Z - 4312468094407200t_2t_3^5\\
&+ 401400694536075 00t_3^6Z^2 + 169301892939882300t_3^7Z + 94770406646119503t_3^8\\
&+ 565248 \times 10^8 t_2^2t_4Z + 159850496 \times 10^6 t_2^2t_3t_4 - 10369910304 \times 10^6 t_2t_3^2t_4Z^2\\
&- 46636996423680000t_2t_3^3t_4Z - 35184390037056000t_2t_3^4t_4\\
&+ 526623988130662500t_3^5t_4Z^2 + 2284832220068899800t_3^6t_4Z\\
&+ 1312847276928293691t_3^7t_4 + 129630208 \times 10^6 t_2^2t_4^2\\
&- 18401520384 \times 10^6 t_2t_3t_4^2Z^2 - 98430133201920000t_2t_3^2t_4^2Z\\
&- 66357828605952000t_2t_3^3t_4^2 + 2305480912207200000t_3^4t_4^2Z^2\\
&+ 10670351180656752000t_3^5t_4^2Z + 5900835212770893408t_3^6t_4^2\\
&- 8953970688 \times 10^6 t_2t_4^3Z^2 - 92208473210880000t_2t_3t_4^3Z\\
&+ 22800849297408000t_2t_3^2t_4^3 + 3539747954527200000t_3^3t_4^3Z^2\\
&+ 19290935595705696000t_3^4t_4^3Z + 6816070148589823680t_3^5t_4^3
\end{aligned}
$$

$$
\begin{aligned}
&- 40173136773120000t_2t_4^4Z + 149486891433984000t_2t_3t_4^4\\
&- 3909678274944000000t_3^2t_4^4Z^2 + 38426674020433920000t_3^3t_4^4Z\\
&- 178671741761428531200t_3^4t_4^4 + 78222734445772800t_2t_4^5\\
&- 6031027762606080000t_3t_4^5Z - 37639199254793011200t_3^2t_4^5Z\\
&- 561477079299099525120t_3^3t_4^5 - 4727890612224 \times 10^6t_4^6Z^2\\
&- 5440084592146513920\,0t_3t_4^6Z - 468322054413136035840t_3^2t_4^6\\
&- 231981832706457600000t_4^7Z + 2162153488801333248t_3t_4^7\\
&+16819243213809254400t_4^8\big)\,, \qquad \text{(A.1.1)}\\
\Delta(t_2) = &- \frac{1053}{128000Q_1(t)Q_2(t)}\big(192 \times 10^{10}t_2^2Z^2 + 272896 \times 10^8t_2^2t_3Z + 77399296 \times 10^6t_2^2t_3^2\\
&- 569106486 \times 10^6t_2t_3^3Z^2 - 3402855789840000t_2t_3^4Z - 7185639666722400t_2t_3^5\\
&+ 13380023151202500t_3^6Z^2 + 95087364527879100t_3^7Z + 168938550977865201t_3^8\\
&+ 598016 \times 10^8t_2^2t_4Z + 134930432 \times 10^6t_2^2t_3t_4 - 4310830368 \times 10^6t_2t_3^2t_4Z^2\\
&- 30022665154560000t_2t_3^3t_4Z - 56468639477952000t_2t_3^4t_4\\
&+ 169823370765262500t_3^5t_4Z^2 + 1415249648779859100t_3^6t_4Z\\
&+ 2377501071078737097t_3^7t_4 + 98369536 \times 10^6t_2^2t_4^2 - 8557705728 \times 10^6t_2t_3t_4^2Z^2\\
&- 78506238320640000t_2t_3^2t_4^2Z - 113377017821184000t_2t_3^3t_4^2\\
&+ 662931324572400000t_3^4t_4^2Z^2 + 7581006758299104000t_3^5t_4^2Z\\
&+ 11703954629799315936t_3^6t_4^2 - 4170756096 \times 10^6t_2t_4^3Z^2\\
&- 75311922216960000t_2t_3t_4^3Z - 70992614744064000t_2t_3^2t_4^3\\
&+ 508647944282400000t_3^3t_4^3Z^2 + 17080376315195232000t_3^4t_4^3Z\\
&+ 23876892382455927360t_3^5t_4^3 - 29090685911040000t_2t_4^4Z\\
&+ 33521273929728000t_2t_3t_4^4 - 1809325549324800000t_3t_4^5Z^2\\
&+ 10778537934987264000t_3^3t_4^4Z + 17101711725365928960t_3^4t_4^4\\
&+ 28526685821337600t_2t_4^5 - 3408486672015360000t_3t_4^5Z^2\\
&- 19873390910187110400t_3^2t_4^5Z - 11424221360726114304t_3^3t_4^5\\
&- 1942325858304 \times 10^6t_4^6Z^2 - 39315036239128166400t_3t_4^6Z\\
&- 25446262590965219328t_3^2t_4^6 - 20349963495014400000t_4^7Z\\
&-15334581962784374784t_3t_4^7 - 3407968112895590400t_4^8\big)\,, \qquad \text{(A.1.2)}\\
\Delta(t_3) = &- \frac{2}{39Q_1(t)Q_2(t)}\big(1792 \times 10^6t_2^2 + 41067 \times 10^6t_2t_3Z^2 + 145924740000t_2t_3^2Z\\
&- 227025676800t_2t_3^3 - 6861550333950t_3^5Z + 4162673869263t_3^6\\
&+ 151632 \times 10^6t_2t_4Z^2 + 639550080000t_2t_3t_4Z - 1717498598400t_2t_3^2t_4\\
&+ 3045066716250t_3^3t_4Z^2 - 72574090070625t_3^4t_4Z + 49128700391082t_3^5t_4\\
&+ 372003840000t_2t_4^2Z - 3397850726400t_2t_3t_4^2 + 33729969780000t_3^2t_4^2Z^2\\
&- 233861123808000t_3^3t_4^2Z + 1444092439514400t_3^4t_4^2 - 3074944204800t_2t_4^3
\end{aligned}
$$

$$+\,95712392880000t_3t_4^3Z^2 - 210375070776000t_3^2t_4^3Z - 214805772979200t_3^3t_4^3$$
$$+\,46836092160000t_4^4Z^2 + 82412598528000t_3t_4^4Z - 1626554086410240t_3^2t_4^4$$
$$+180304762060800t_4^5Z - 2664913466621952t_3t_4^5 - 1267254332817408t_4^6\big)\,, \tag{A.1.3}$$
$$\Delta(t_4) = \frac{2}{3}, \tag{A.1.4}$$

where

$$Q_1(t) = 12800t_2 - 771147t_3^3 - 5694624t_3^2t_4 - 10513152t_3t_4^2 - 8985600t_4^3,$$
$$Q_2(t) = 20000t_2 - 771147t_3^3 - 4093011t_3^2t_4 + 1314144t_3t_4^2 + 7795008t_4^3.$$

A general element of $V_1$ is of the form

$$\sum_{k=0}^{2} \sum_{\substack{0 \le i_1, i_2, i_3, i_4 \\ 2k + \sum_{j=1}^{4} d_j i_j = 12}} a_{(i_1, i_2, i_3, i_4, k)} t_1^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4} Z^k, \tag{A.1.5}$$

where $a_{(i_1, i_2, i_3, i_4, k)} \in \mathbb{C}$. By calculating the Laplacian of this general element (A.1.5) using Proposition 4.0.1 and formulas (A.1.1)–(A.1.4) we find that the only harmonic elements of $V_1$ are as claimed. A general element of $V_2$ has the form

$$\sum_{k=0}^{2} \sum_{\substack{0 \le i_1, i_2, i_3, i_4 \\ 2k + \sum_{j=1}^{4} d_j i_j = 8}} b_{(i_1, i_2, i_3, i_4, k)} t_1^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4} Z^k, \tag{A.1.6}$$

where $b_{(i_1, i_2, i_3, i_4, k)} \in \mathbb{C}$. By calculating the Laplacian of this general element (A.1.6) using Propostion 4.0.1 and formulas (A.1.1)–(A.1.4) we find that the only harmonic elements of $V_2$ are as claimed. A general element of $V_3$ has the form

$$\sum_{k=0}^{2} \sum_{\substack{0 \le i_1, i_2, i_3, i_4 \\ 2k + \sum_{j=1}^{4} d_j i_j = 6}} c_{(i_1, i_2, i_3, i_4, k)} t_1^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4} Z^k, \tag{A.1.7}$$

where $c_{(i_1, i_2, i_3, i_4, k)} \in \mathbb{C}$ are constants. By calculating the Laplacian of this general element (A.1.7) using Propostion 4.0.1 and formulas (A.1.1)-(A.1.4) we find that the only harmonic elements of $V_3$ are as claimed. $\square$

Proposition A.1.1 thus allows us to give a proof of Theorem 4.4.3, in which the basic invariants $y_i$ are expressed as polynomials in terms of the $t$ coordinates and the variable $Z$. We now present this proof.

*Proof.* Note that $Y_4 = \frac{1}{8}y_4 = \frac{3}{2}t_4$. We now equate $Y_1$, $Y_2$ and $Y_3$ given by relations (4.4.9)–(4.4.11) with general harmonic elements of $V_1$, $V_2$ and $V_3$, respectively, given by Proposition A.1.1. We then rearrange these equations to find $y_i$ in terms of $t_j$ and $Z$. We find

$$
\begin{aligned}
y_1 = {} & \frac{4105728}{35}t_4^6 + \frac{a_1}{39691987478208 \times 10^8}(3999427165146316800 0t_1^2 \\
& - 8090422584908185600 0t_1t_2 + 9162831636778188800 0t_2^2 \\
& + 39691987478208 \times 10^8 t_2t_3Z^2 - 21155829325884864 \times 10^6 t_2t_3^2Z \\
& + 5509343267986188533 7600t_1t_3^3 - 3415310196236774184 9600t_2t_3^3 \\
& - 4902119686649079092 2500t_3^4Z^2 + 9779848338759709108 04100t_3^5Z \\
& + 3110921862196961552 295231t_3^6 + 146555030688768 \times 10^8 t_2t_4Z^2 \\
& - 29213302783961088 \times 10^6 t_2t_3t_4Z + 8555528225988400250 8800t_1t_3^2t_4 \\
& - 9577489596391169556 4800t_2t_3^2t_4 - 7240053691050947582 40000t_3^3t_4Z^2 \\
& + 7074356527308602725 128000t_3^4t_4Z + 2114774127022999514 9410656t_3^5t_4 \\
& - 8834398276534272 \times 10^6 t_2t_4^2Z + 7430190133938990612 4800t_1t_3t_4^2 \\
& - 6219274828942713028 60800t_2t_3t_4^2 - 269334029961901756 8 \times 10^6 t_3^2t_4^2Z^2 \\
& + 8762848369368650105 702400t_3^3t_4^2Z + 7690592943414451472 7538944t_3^4t_4^2 \\
& + 1238495422217491316 73600t_1t_4^3 - 9782961083655867334 65600t_2t_4^3 \\
& - 2934732272971723960 320000t_3t_4^3Z^2 - 1659094418776881626 5420800t_3^2t_4^3 \\
& + 6466070203950441122 0729856t_3^3t_4^3 - 1450851560860623667 200000t_4^4Z^2 \\
& - 5450188860933339721 2364800t_3t_4^4Z - 1096511756702484949 23776t_3^2t_4^4 \\
& - 4470116718838046706 8928000t_4^5Z - 9012167797860654843 2191488t_3t_4^5) \\
& + \frac{b}{40644595177684992}(1665664000t_1^2 - 12300288000t_1t_2 + 22708224000t_2^2 \\
& - 1235069035200t_1t_3^3 + 4560254899200t_2t_3^3 + 288947062809465t_3^6 \\
& - 13680764697600t_1t_3^2t_4 + 50513592729600t_2t_3^2t_4 + 5072058006855840t_3^5t_4 \\
& + 4245696 \times 10^6 t_2t_4^2Z - 34568645529600t_1t_3t_4^2 + 67292583321600t_2t_3t_4^2 \\
& - 29513723557500t_3^2t_4^2Z^2 - 209744195415300t_3^3t_4^2Z + 30958248927422817t_3^4t_4^2 \\
& - 16707033907200t_1t_4^3 + 20023079731200t_2t_4^3 - 217947497040000t_3t_4^3Z^2 \\
& - 1378881164606400t_3^2t_4^3Z + 59383246493608128t_3^3t_4^3 - 402364609920000t_4^4Z^2 \\
& - 1604093578214400t_3t_4^4Z + 63810842541368832t_3^2t_4^4 - 662858438860800t_4^5Z \\
& + 67385500072869888t_3t_4^5 + 40644595177684992t_4^6) \\
& - \frac{4a_2}{45313125}t_4^2(865280000t_1t_3 - 192 \times 10^6 t_2Z - 465920000t_2t_3 + 1334677500t_3^2Z^2 \\
& + 9485108100t_3^3Z + 129130878591t_3^4 + 465920000t_1t_4 + 163840000t_2t_4 \\
& + 9856080000t_3t_4Z^2 + 62356132800t_3^2t_4Z + 288371962944t_3^3t_4 + 18195840000t_4^2Z^2 \\
& + 72540748800t_3t_4^2Z - 79360280064t_3^2t_4^2 + 29975961600t_4^3Z - 1705504260096t_3t_4^3
\end{aligned}
$$

$$- 2383159394304t_4^4) + \frac{7a_3}{46280}t_4^3(2080t_1 - 7680t_2 - 771147t_3^3 - 8541946t_3^2t_4 - 9637056t_3t_4^2 - 3998592t_4^3), \tag{A.1.8}$$

$$y_2 = \frac{1296}{5}t_4^4 + \frac{a_2}{1334677500}(865280000t_1t_3 - 192 \times 10^6 t_2 Z - 465920000t_2t_3 + 1334677500t_3^2Z^2 + 9485108100t_3^3Z + 129130878591t_3^4 + 465920000t_1t_4 + 163840000t_2t_4 + 9856080000t_3t_4Z^2 + 62356132800t_3^2t_4Z + 288371962944t_3^3t_4 + 18195840000t_4^2Z^2 + 72540748800t_3t_4^2Z - 79360280064t_3^2t_4^2 + 29975961600t_4^3Z - 1705504260096t_3t_4^3 - 2383159394304t_4^4) + \frac{a_3}{1332864}t_4(2080t_1 - 7680t_2 - 771147t_3^3 - 8541946t_3^2t_4 - 9637056t_3t_4^2 - 3998592t_4^3), \tag{A.1.9}$$

$$y_3 = -216t_4^3 - \frac{a_3}{3998592}(2080t_1 - 7680t_2 - 771147t_3^3 - 8541946t_3^2t_4 - 9637056t_3t_4^2 - 3998592t_4^3), \tag{A.1.10}$$

$$y_4 = 12t_4, \tag{A.1.11}$$

where $a_i, b \in \mathbb{C}$. In order to find $a_i$ and $b$ we perform steps 5–7 from the introduction of Chapter 4. That is, we transform the intersection form (4.4.2)–(4.4.8) into $y$ coordinates by applying formulas (A.1.8)–(A.1.11) and compare it with the expression given by Lemma 4.4.1. We find that

$$a_1 = \frac{25}{24}, \quad a_2 = \frac{169}{1024}, \quad a_3 = \frac{712}{15}, \quad b = \frac{329462365952}{32484375},$$

which implies the statement. □

In the next statement we express flat coordinates $t_i$ via basic invariants $y_j$ and $Z$, which is an inversion of formulas from Theorem 4.4.3.

**Proposition A.1.2.** *We have the following relations:*

$$t_1 = \frac{1}{40000}\left(-1687500Z^3 + 3080025R^2Z + 22125987R^3 - 1620000y_3 + 1895400Ry_4Z + 17686188R^2y_4 + 291600y_4^2Z + 1674504Ry_4^2 - 144656y_4^3\right), \tag{A.1.12}$$

$$t_2 = -\frac{13}{640000}\left(562500Z^3 - 1026675R^2Z - 2432079R^3 - 631800Ry_4Z - 1332396R^2y_4 - 97200y_4^2Z - 129168Ry_4^2 - 4448y_4^3\right), \tag{A.1.13}$$

$$t_3 = R, \tag{A.1.14}$$

$$t_4 = \frac{1}{12}y_4, \tag{A.1.15}$$

*where*

$$R = R(y, Z) = \frac{2}{39}\left(43188040173339843 75 \times 10^{11}Z^{16} + 41695196778676615429687500000Z^{10}y_1 - 13376868139989070724092218750Z^4y_1^2 + 30351743909545447265625 \times 10^7 Z^{12}y_2\right.$$

$$+ 223453003503291231550331250000Z^6y_1y_2 - 792090821832795226315575984 78y_1^2y_2$$

$$- 207998100058843000575 \times 10^{10}Z^8y_2^2 + 592010641875029717236464 0000Z^2y_1y_2^2$$

$$+ 336058395817981027627 35 \times 10^7Z^4y_2^3 + 506410309822141223670106860 0192y_2^4$$

$$+ 144318722022657421875 \times 10^{10}Z^{13}y_3 + 2578270394116425882009375 00000Z^7y_1y_3$$

$$- 70192912739683257063432 33750Zy_1^2y_3 + 1347373675551383220656 25 \times 10^7Z^9y_2y_3$$

$$+ 18140060848030535238320 5785000Z^3y_1y_2y_3 - 323325921909825190701 99 \times 10^8Z^5y_2^2y_3$$

$$+ 3242363404942230542828 6083200Zy_2^3y_3 + 20896501340052049412109 37500000Z^{10}y_3^2$$

$$+ 26169278652058682377240 8187500Z^4y_1y_3^2 + 16242693415370007120352 68750000Z^6y_2y_3^2$$

$$+ 18988639754443400014584 1924500y_1y_2y_3^2 - 73323242913690044291509 9920000Z^2y_2^2y_3^2$$

$$+ 15002367708441703153246 87500000Z^7y_3^3 + 51466022282163640057848 562500Zy_1y_3^3$$

$$+ 11760844176689179161382 03125000Z^3y_2y_3^3 + 49631103694600216042869 1406250Z^4y_3^4$$

$$+ 26953129596743233400470 0781250y_2y_3^4 + 58567263896436729125097 656250Zy_3^5$$

$$+ 1335792100698242187 5 \times 10^{11}Z^{15}y_4 - 3686883545073852014062 5 \times 10^6Z^9y_1y_4$$

$$- 55170939614090150722767 924375Z^3y_1^2y_4 - 7718873980700475 \times 10^{12}Z^{11}y_2y_4$$

$$- 232908050043718804151874 \times 10^6Z^5y_1y_2y_4 - 211882944575305916220375 \times 10^7Z^7y_2^2y_4$$

$$- 84487382612986982332071 016800Zy_1y_2^2y_4 + 65720695072709224610530 27920000Z^3y_2^3y_4$$

$$+ 453921706314204890625 \times 10^{10}Z^{12}y_3y_4 + 478112102015711370664275 \times 10^6Z^6y_1y_3y_4$$

$$- 69401633326811140866095 860848y_1^2y_3y_4 - 766333931393975131 44 \times 10^{10}Z^8y_2y_3y_4$$

$$+ 78922659546867736308823 585200Z^2y_1y_2y_3y_4 - 412646839445753974819 6173 \times 10^6Z^4y_2^2y_3y_4$$

$$+ 76256702234789396058801 15761664y_2^3y_3y_4 + 634749604797092639104687 5 \times 10^6Z^9y_3^2y_4$$

$$+ 68237526918397639982383 5176250Z^3y_1y_3^2y_4 - 926856258539793292751094 \times 10^6Z^5y_2y_3^2y_4$$

$$- 12128343062489521954429 41151200Zy_2^2y_3^2y_4 + 481016933764706180089867 5 \times 10^6Z^6y_3^3y_4$$

$$+ 33513355341476057847595 3956000y_1y_3^3y_4 - 36159581360788925559068 8170000Z^2y_2y_3^3y_4$$

$$+ 22246755396565358388167 75390625Z^3y_3^4y_4 + 40512814034948437517960 6250000y_3^5y_4$$

$$+ 1843907075350927734375 \times 10^9Z^{14}y_4^2 - 616252913755839985 5 \times 10^{11}Z^8y_1y_4^2$$

$$+ 33135780225705946844214 28950Z^2y_1^2y_4^2 - 77052700743715402031 25 \times 10^8Z^{10}y_2y_4^2$$

$$- 120344562665381362098195 \times 10^6Z^4y_1y_2y_4^2 + 472259027224454120158 5 \times 10^8Z^6y_2^2y_4^2$$

$$- 12151952888838113753711 9629056y_1y_2^2y_4^2 + 38560284506769372458183 0505600Z^2y_2^3y_4^2$$

$$+ 7277397242643146559375 \times 10^9Z^{11}y_3y_4^2 - 784021766003815426802487 \times 10^6Z^5y_1y_3y_4^2$$

$$- 784027284874855640403 75 \times 10^7Z^7y_2y_3y_4^2 - 16750691596545057426856 3171200Zy_1y_2y_3y_4^2$$

$$- 24527917287221253083797 64400000Z^3y_2^2y_3y_4^2 + 952206092401690340562375 \times 10^7Z^8y_3^2y_4^2$$

$$- 10158419242940859446569 1064900Z^2y_1y_3^2y_4^2 - 379766236476658288971519 \times 10^6Z^4y_2y_3^2y_4^2$$

$$- 32629566048373096329349 73313792y_2^2y_3^2y_4^2 + 508793698548534080279985 3 \times 10^6Z^5y_3^3y_4^2$$

$$+ 51992198902302305826247 286400Zy_2y_3^3y_4^2 + 63804664709067624444697 3863750Z^2y_3^4y_4^2$$

$$- 4089472671861328125 \times 10^{11}Z^{13}y_4^3 - 385954076105372423908125 \times 10^6Z^7y_1y_4^3$$

$$
\begin{aligned}
&+ 1628934926139749141858047680 0Zy_1^2y_4^3 - 29937945385114200815625 \times 10^8 Z^9 y_2 y_4^3 \\
&+ 80241006858163315858771596000 0Z^3y_1y_2y_4^3 + 2485124426702571292561995 \times 10^6 Z^5 y_2^2 y_4^3 \\
&- 205515020799801060152221198080 0Zy_2^3y_4^3 + 1440710299200946725 \times 10^{12} Z^{10} y_3 y_4^3 \\
&- 852931789405160573147841 \times 10^6 Z^4 y_1 y_3 y_4^3 - 771711806725426250679 33 \times 10^8 Z^6 y_2 y_3 y_4^3 \\
&+ 76444769887351291650752930304 0y_1y_2y_3y_4^3 + 192386581969529112833423030400 0Z^2y_2^2y_3y_4^3 \\
&+ 463713597068962545102937 5 \times 10^6 Z^7 y_3^2 y_4^3 - 95188847090565169764771685200 00Z^3y_2y_3^2y_4^3 \\
&+ 399343521950795903835451 5 \times 10^6 Z^4 y_3^3 y_4^3 - 46284920003839081371037356349 44y_2y_3^3y_4^3 \\
&+ 1074402738414905625716316566400Zy_3^4y_4^3 - 209755642121173453125 \times 10^{10} Z^{12} y_4^4 \\
&+ 2089324193573502243601875000 00Z^6y_1y_4^4 + 46495633610240475278401992096y_1^2y_4^4 \\
&+ 1581391619744050771616 25 \times 10^7 Z^8 y_2 y_4^4 + 31266414756995074095355200 00Z^2y_1y_2y_4^4 \\
&+ 146327982584274644549938 2 \times 10^6 Z^4 y_2^2 y_4^4 - 50550772334935012200438219924 48y_2^3y_4^4 \\
&- 639142789151707249387 5 \times 10^9 Z^9 y_3 y_4^4 + 18938987542081181889037296000 0Z^3y_1y_3y_4^4 \\
&+ 151301464265658194283124 8 \times 10^6 Z^5 y_2 y_3 y_4^4 - 36275273852664849207564556800Zy_2^2y_3y_4^4 \\
&- 6071013823888793547452362500000Z^6y_3^2y_4^4 - 1341875170940583537760546892 16y_1y_3^2y_4^4 \\
&- 9121045888713156960314459520 0Z^2y_2y_3^2y_4^4 - 21594629106177219112750459200 00Z^3y_3^3y_4^4 \\
&+ 14877842804785315088095756867 2y_3^4y_4^4 - 170671732234553193 75 \times 10^{10} Z^{11} y_4^5 \\
&+ 348874651682777832961761 \times 10^6 Z^5 y_1 y_4^5 + 834322793422889418 21 \times 10^{10} Z^7 y_2 y_4^5 \\
&+ 124029036396931501761701760 0Zy_1y_2y_4^5 + 2306790998477868377873217600 00Z^3y_2^2y_4^5 \\
&- 167071970160906821361 \times 10^{10} Z^8 y_3 y_4^5 + 955669543922381589864372672 00Z^2y_1y_3y_4^5 \\
&+ 107812905154662243014553 6 \times 10^6 Z^4 y_2 y_3 y_4^5 + 14310512331645576885199841157 12y_2^2y_3y_4^5 \\
&- 267487412650020494285611 5 \times 10^6 Z^5 y_3^2 y_4^5 + 9394359539772765003721195968 00Zy_2y_3^2y_4^5 \\
&- 8014242903804444029268343104 00Z^2y_3^3y_4^5 + 108143934311445735937 5 \times 10^9 Z^{10} y_4^6 \\
&+ 116751237125755239283086 \times 10^6 Z^4 y_1 y_4^6 - 705642341596863853120 2 \times 10^8 Z^6 y_2 y_4^6 \\
&- 12423675932947986027734925619 2y_1y_2y_4^6 - 60995326057053869324650003200 0Z^2y_2^2y_4^6 \\
&+ 194090599220237918766 \times 10^{10} Z^7 y_3 y_4^6 + 16226987135066749343416550400Zy_1y_3y_4^6 \\
&- 233616794719929428939 04 \times 10^7 Z^3 y_2 y_3 y_4^6 + 120496139656312722572827 8 \times 10^6 Z^4 y_3^2 y_4^6 \\
&+ 99827342524100423382802470604 8y_2y_3^2y_4^6 + 3056149048260490020107654400 00Zy_3^3y_4^6 \\
&+ 735962709925891050 75 \times 10^{10} Z^9 y_4^7 - 25133261246783816046910992000 0Z^3y_1y_4^7 \\
&- 4888255927933618918192 8 \times 10^7 Z^5 y_2 y_4^7 + 784374148708089670699714252800Zy_2^2y_4^7 \\
&+ 1092312112134919344580 8 \times 10^8 Z^6 y_3 y_4^7 - 84213722079491655183238766592y_1y_3y_4^7 \\
&- 86539459654551984419036160 00Z^2y_2y_3y_4^7 + 100204127910385089394488624000 0Z^3y_3^2y_4^7 \\
&+ 69909599895549171600572375040y_3^3y_4^7 - 617635280526424615 44 \times 10^{10} Z^8 y_4^8 \\
&- 2302600703963837349392893440 0Z^2y_1y_4^8 - 577481245926419433888 \times 10^7 Z^4 y_2 y_4^8 \\
&- 33409520007138422432500506624y_2^2y_4^8 - 309707549577793852204032 \times 10^6 Z^5 y_3 y_4^8 \\
&- 6137253202340425375637913600Zy_2y_3y_4^8 + 5165187056162438242988160000Z^2y_3^2y_4^8
\end{aligned}
$$

$$- 5454074083854102804 \times 10^{11} Z^7 y_4^9 + 39447212783710902645356236800 Z y_1 y_4^9$$
$$+ 30995173623698015971030272000 0 Z^3 y_2 y_4^9 - 524513087880074638889472 \times 10^6 Z^4 y_3 y_4^9$$
$$+ 2395162607081479140305369825 28 y_2 y_3 y_4^9 - 282152758694237921398611456000 Z y_3^2 y_4^9$$
$$+ 3881468268623087677152 \times 10^8 Z^6 y_4^{10} - 1073730093626098371575498 3424 y_1 y_4^{10}$$
$$- 310739360056947343401504768 00 Z^2 y_2 y_4^{10} + 210915414434080879871016960000 Z^3 y_3 y_4^{10}$$
$$+ 91094109062226732586517299 20 y_3^2 y_4^{10} + 10563696264163984145 28 \times 10^8 Z^5 y_4^{11}$$
$$- 192820843272946811570307072 00 Z y_2 y_4^{11} + 66346002965378675008143360000 Z^2 y_3 y_4^{11}$$
$$- 1159117709714762629478 4 \times 10^7 Z^4 y_4^{12} + 14960422004241499110751076352 y_2 y_4^{12}$$
$$- 584194527038286235077967872 00 Z y_3 y_4^{12} + 128466601442307995811 84 \times 10^6 Z^3 y_4^{13}$$
$$+ 225324945704306128264888320 0 y_3 y_4^{13} + 997117846604995055321088000 0 Z^2 y_4^{14}$$
$$- 412786330926174115725312000 0 Z y_4^{15} - 341730799961790275458968326400 Z y_1 y_3^2 y_4^3$$
$$+ 257704505778374571353702400 y_4^{16} \big) \, / \, \big( 689504741064453125 \times 10^{11} Z^{15}$$
$$+ 798419185573154814843750000 0 Z^9 y_1 + 609166919030243699405044125 0 Z^3 y_1^2$$
$$+ 3048703464697120546875 \times 10^7 Z^{11} y_2 + 6060606416185670985644775 0000 Z^5 y_1 y_2$$
$$- 429260244036284126565 \times 10^9 Z^7 y_2^2 + 43991466996451657855402944 00 Z y_1 y_2^2$$
$$- 605719664906364326628784080000 Z^3 y_2^3 + 18834917989394390625 \times 10^{10} Z^{12} y_3$$
$$+ 369987869074609781026875000 00 Z^6 y_1 y_3 + 9305446245780529744207586064 y_1^2 y_3$$
$$+ 309904362929089023333 75 \times 10^7 Z^8 y_2 y_3 + 98416334315830123680275358 00 Z^2 y_1 y_2 y_3$$
$$- 590278112644707839964996 \times 10^6 Z^4 y_2^2 y_3 - 5304277602857905733159124364 80 y_2^3 y_3$$
$$+ 2316598213682937070546875000 00 Z^9 y_3^2 + 63972263997064644352072447500 Z^3 y_1 y_3^2$$
$$+ 240068344337595256473380250000 Z^5 y_2 y_3^2 - 120521924169073341982017984000 Z y_2^2 y_3^2$$
$$+ 167475603869487047676037500000 Z^6 y_3^3 + 18634172756622174816164100000 y_1 y_3^3$$
$$+ 393304422010060723245063750 00 Z^2 y_2 y_3^3 + 643780474972862505404414062 50 Z^3 y_3^4$$
$$+ 932874107142065979515625000 0 y_3^5 + 2762301120380859375 \times 10^{11} Z^{14} y_4$$
$$+ 3854453809356481415625 \times 10^7 Z^8 y_1 y_4 - 6913924346121965243252273 25 Z^2 y_1^2 y_4$$
$$- 2913977860021116421875 \times 10^8 Z^{10} y_2 y_4 + 1194233879356241573308725000 00 Z^4 y_1 y_2 y_4$$
$$- 1141466324532105439091 25 \times 10^7 Z^6 y_2^2 y_4 - 7622743642329595602833432352 y_1 y_2^2 y_4$$
$$- 764175392817100233118178976 00 Z^2 y_2^3 y_4 + 797442479987652956 25 \times 10^{10} Z^{11} y_3 y_4$$
$$+ 2691204285403242259900 2 \times 10^6 Z^5 y_1 y_3 y_4 - 3812783848601766629175 \times 10^8 Z^7 y_2 y_3 y_4$$
$$+ 658555997693697397636248168 00 Z y_1 y_2 y_3 y_4 - 244322075327602735347898716000 0 Z^3 y_2^2 y_3 y_4$$
$$+ 9482471738871692025262 5 \times 10^7 Z^8 y_3^2 y_4 - 865292062648456746840974085 0 Z^2 y_1 y_3^2 y_4$$
$$- 143035461790424020580584500000 Z^4 y_2 y_3^2 y_4 - 124310696278760153520998560272 0 y_2^2 y_3^2 y_4$$
$$+ 425985293579782674682062 \times 10^6 Z^5 y_3^3 y_4 + 216095898368449651410425160 00 Z y_2 y_3^3 y_4$$
$$+ 548511422754031852059465468 75 Z^2 y_3^4 y_4 + 44174742609329882812 5 \times 10^9 Z^{13} y_4^2$$
$$+ 2646504414130152612375 \times 10^7 Z^7 y_1 y_4^2 - 111955395688460084864085030 0 Z y_1^2 y_4^2$$

$$- 6289929404981097935625 \times 10^{8} Z^{9} y_2 y_4^2 + 8436238834745089903833720000 Z^{3} y_1 y_2 y_4^2$$
$$+ 9152965033876812998178 \times 10^{7} Z^{5} y_2^2 y_4^2 - 4972901259776901143246841600 Z y_2^3 y_4^2$$
$$+ 1145848974421217428125 \times 10^{9} Z^{10} y_3 y_4^2 + 2319401871314856709907 4 \times 10^{6} Z^{4} y_1 y_3 y_4^2$$
$$- 1308747345171497309083 05 \times 10^{7} Z^{6} y_2 y_3 y_4^2 + 1311716185590331227107656 12032 y_1 y_2 y_3 y_4^2$$
$$- 2422697325966871488203184000 Z^{2} y_2^2 y_3 y_4^2 + 1396825721127720942 75 \times 10^{10} Z^{7} y_3^2 y_4^2$$
$$+ 2111781931236558007611229800 Z y_1 y_3^2 y_4^2 - 9328309615964921271025003 20000 Z^{3} y_2 y_3^2 y_4^2$$
$$+ 8594765464309756769167 38 \times 10^{6} Z^{4} y_3^3 y_4^2 - 285135549585187434759240975360 y_2 y_3^3 y_4^2$$
$$+ 174138796276580145477533566500 Z y_3^4 y_4^2 + 30138045629786390625 \times 10^{10} Z^{12} y_4^3$$
$$- 6929578312242637788382 5 \times 10^{6} Z^{6} y_1 y_4^3 - 4134728924696760451871189184 y_1^2 y_4^3$$
$$- 6879093454387072557225 \times 10^{8} Z^{8} y_2 y_4^3 - 11491694798860251146136960000 Z^{2} y_1 y_2 y_4^3$$
$$+ 5637774140297914876781 49 \times 10^{6} Z^{4} y_2^2 y_4^3 - 995815428447361487077571 18976 y_2^3 y_4^3$$
$$+ 777862679930665332 75 \times 10^{10} Z^{9} y_3 y_4^3 - 1852124453967891405453325 20000 Z^{3} y_1 y_3 y_4^3$$
$$- 8340163145683538453128 08 \times 10^{6} Z^{5} y_2 y_3 y_4^3 + 438228174910640901137589772800 Z y_2^2 y_3 y_4^3$$
$$+ 8122389460906958195991 75 \times 10^{6} Z^{6} y_3^2 y_4^3 - 1099113615059203136623244808 96 y_1 y_3^2 y_4^3$$
$$- 2151858790212065127329303808 00 Z^{2} y_2 y_3^2 y_4^3 + 6829771117043528679543361200 00 Z^{3} y_3^3 y_4^3$$
$$+ 27856177844129709588543373 3440 y_3^4 y_4^3 - 25073769213912605625 \times 10^{10} Z^{11} y_4^4$$
$$- 2445104040090693162206850 0000 Z^{5} y_1 y_4^4 - 3580925865826675856 0625 \times 10^{7} Z^{7} y_2 y_4^4$$
$$- 1538475379183659741650173 4400 Z y_1 y_2 y_4^4 + 1251065562556236322844453760000 Z^{3} y_2^2 y_4^4$$
$$- 417231020141682779565 \times 10^{9} Z^{8} y_3 y_4^4 + 670928552525286036763922 8800 Z^{2} y_1 y_3 y_4^4$$
$$- 5945591756857810369087 44 \times 10^{6} Z^{4} y_2 y_3 y_4^4 + 1267848939924964025228901814272 y_2^2 y_3 y_4^4$$
$$- 1728790980202944982791225 00000 Z^{5} y_3^2 y_4^4 - 2067166221230380553197827 84000 Z y_2 y_3^2 y_4^4$$
$$- 1822985956877652522215588 1600 Z^{2} y_3^3 y_4^4 - 33696420485940781875 \times 10^{10} Z^{10} y_4^5$$
$$- 1666968072680893928900 4 \times 10^{6} Z^{4} y_1 y_4^5 + 4414336489183298623677 \times 10^{8} Z^{6} y_2 y_4^5$$
$$- 5527031029504079001531957 9648 y_1 y_2 y_4^5 + 4338518266436721989576515 2000 Z^{2} y_2^2 y_4^5$$
$$- 6446940655039246611 \times 10^{11} Z^{7} y_3 y_4^5 + 1483885371397890110748019200 Z y_1 y_3 y_4^5$$
$$+ 4026923435894331035075155 20000 Z^{3} y_2 y_3 y_4^5 - 4229719372928321784323 64 \times 10^{6} Z^{4} y_3^2 y_4^5$$
$$+ 5489645426782940488175003 7504 y_2 y_3^2 y_4^5 - 1007715505076922719477291 90400 Z y_3^3 y_4^5$$
$$- 843198192515952337 5 \times 10^{9} Z^{9} y_4^6 + 3773296246534411396536504 0000 Z^{3} y_1 y_4^6$$
$$+ 1500366005106778246276 8 \times 10^{7} Z^{5} y_2 y_4^6 - 6112545842177572855850703 3600 Z y_2^2 y_4^6$$
$$+ 1165569640250198869656 \times 10^{8} Z^{6} y_3 y_4^6 + 3663855442019549232078439 2192 y_1 y_3 y_4^6$$
$$+ 8924188455773943477177369 6000 Z^{2} y_2 y_3 y_4^6 - 3563116339545578901520296 0000 Z^{3} y_3^2 y_4^6$$
$$- 8538125874445651256562187 4688 y_3^3 y_4^6 + 155864970627225780 24 \times 10^{10} Z^{8} y_4^7$$
$$- 6990197488291098670421376 00 Z^{2} y_1 y_4^7 - 1861424704535040581256 \times 10^{7} Z^{4} y_2 y_4^7$$
$$- 1092316012724951164819732 56192 y_2^2 y_4^7 + 1638565466922927027486 72 \times 10^{6} Z^{5} y_3 y_4^7$$
$$- 9969097308845103988122562 5600 Z y_2 y_3 y_4^7 + 4953728654982805406446368 0000 Z^{2} y_3^2 y_4^7$$

$$+\,8250270962982736464\times10^{10}Z^7y_4^8-109563262080005319210961920 0Zy_1y_4^8$$
$$-\,1008550782572467287316953600 00Z^3y_2y_4^8+17211844506558451106304\times10^6Z^4y_3y_4^8$$
$$-\,518562666246887847935652003 84y_2y_3y_4^8+447770996598062571857372160 0Zy_3^2y_4^8$$
$$-\,801014458799349621792\times10^8Z^6y_4^9+348777509216388550946087731 2y_1y_4^9$$
$$-\,792667921115638195213086720 0Z^2y_2y_4^9-511067495269698429981081600 00Z^3y_3y_4^9$$
$$-\,147046281172825372667213414 4y_3^2y_4^9-228032070586158220704\times10^8Z^5y_4^{10}$$
$$+\,201892269620222029124149248 00Zy_2y_4^{10}-108729088194134532818534400 00Z^2y_3y_4^{10}$$
$$+\,2786488622313686118144\times10^7Z^4y_4^{11}-598515736737178377393419059 2y_2y_4^{11}$$
$$+\,139706286935951853505216512 00Zy_3y_4^{11}-3953810027606047911936\times10^6Z^3y_4^{12}$$
$$-\,409818887516973973163212800y_3y_4^{12}-287200230706866395086848000 0Z^2y_4^{13}$$
$$+1294511155628802272133120000Zy_4^{14}-820997886406790221922304 00y_4^{15}\Big)$$

*and $Z$ satisfies the equation*

$$152587890625\times10^{24}Z^{24}+7602885459228515625\times10^{17}Z^{18}y_1$$
$$-\,19283435970090558016886718 75\times10^8Z^{12}y_1^2+378006848494099420330994356125000 00Z^6y_1^3$$
$$-\,697655443630137459342864388 89645 87y_1^4-1795625244140625\times10^{21}Z^{20}y_2$$
$$+\,296440028792528828203125\times10^{13}Z^{14}y_1y_2-391574108172964418398731375\times10^9Z^8y_1^2y_2$$
$$+\,147071166363697221441453335 6160000Z^2y_1^3y_2-137650537389281835937 5\times10^{16}Z^{16}y_2^2$$
$$+\,29271054326721094395\times10^{17}Z^{10}y_1y_2^2-107787455856647839340053373 16\times10^7Z^4y_1^2y_2^2$$
$$-\,164633068668895850306925\times10^{14}Z^{12}y_2^3-190801191945595818762225763 92\times10^8Z^6y_1y_2^3$$
$$+\,892069191107027891378436546943767936y_1^2y_2^3+12673407786586104652684 2\times10^{14}Z^8y_2^4$$
$$-\,863331723305910907022036979302400 00Z^2y_1y_2^4+613226394064397122777944946944\times10^7Z^4y_2^5$$
$$-\,285164922379733605560061775180258795 52y_2^6+18792333984375\times10^{23}Z^{21}y_3$$
$$+\,30020385852427584375\times10^{17}Z^{15}y_1y_3-460754856638791337232241875\times10^8Z^9y_1^2y_3$$
$$+\,231031959673635225892777889528100 00Z^3y_1^3y_3-8580184512103125\times10^{21}Z^{17}y_2y_3$$
$$+\,950458374248414486264437 5\times10^{12}Z^{11}y_1y_2y_3-38463920155742336380440417 78\times10^8Z^5y_1^2y_2y_3$$
$$-\,5422655695127024688637 5\times10^{15}Z^{13}y_2^2y_3+2656314401118945351826981 32\times10^{10}Z^7y_1y_2^2y_3$$
$$-\,490275349933504819116467103861888 00Zy_1^2y_2^2y_3-548015614391668033160912 52\times10^{12}Z^9y_2^3y_3$$
$$-\,160750848479150352629962372152384 0000Z^3y_1y_2^3y_3+228630846876082162600039046784\times10^8Z^5y_2^4y_3$$
$$+\,336012653690355088171471790089912 3200Zy_2^5y_3+661191544551269531 25\times10^{17}Z^{18}y_3^2$$
$$+\,545947488882489091515820312 5\times10^9Z^{12}y_1y_3^2+479552866587861378497450644612500 000Z^6y_1^2y_3^2$$
$$+\,371821095239677750287010187012601 00y_1^3y_3^2-1397178250325730294609375\times10^{13}Z^{14}y_2y_3^2$$
$$+\,1146017856663319810455046575\times10^{10}Z^8y_1y_2y_3^2-102011813902628256727366282746\times10^6Z^2y_1^2y_2y_3^2$$
$$-\,946937290944988583523108\times10^{14}Z^{10}y_2^2y_3^2+651386955317112727550564642064\times10^6Z^4y_1y_2^2y_3^2$$
$$-\,681548931275960242836185687352\times10^8Z^6y_2^3y_3^2-235130487850459994047687486190505 6000y_1y_2^3y_3^2$$
$$+\,613888459525706326298755772115456 0000Z^2y_2^4y_3^2+116364474343727746875\times10^{17}Z^{15}y_3^3$$
$$+\,578593937356973199254725312 5\times10^9Z^9y_1y_3^3+592726229443074452034415695639750000Z^3y_1^2y_3^3$$

$$
\begin{aligned}
&- 10108239003868506378733125 \times 10^{12} Z^{11} y_2 y_3^3 + 53754453985847270674774271616 \times 10^{8} Z^{5} y_1 y_2 y_3^3 \\
&- 7967703903884351923295097108 \times 10^{10} Z^{7} y_2^2 y_3^3 + 140746604913634720033855853539200000 Z y_1 y_2^2 y_3^3 \\
&- 45348445674656449831825041551184 \times 10^{6} Z^{3} y_2^3 y_3^3 + 120169519389316914213251953125 \times 10^{8} Z^{12} y_3^4 \\
&+ 3688677567260663636153885937187500000 Z^{6} y_1 y_3^4 + 153649976601366611814352638952968750 y_1^2 y_3^4 \\
&- 3087172921103959425142174875 \times 10^{9} Z^{8} y_2 y_3^4 + 7440777771164302229074246419 \times 10^{8} Z^{2} y_1 y_2 y_3^4 \\
&- 31019970056856309743948780013 \times 10^{9} Z^{4} y_2^2 y_3^4 - 1094680582162253252931166499889 \times 10^{7} y_2^3 y_3^4 \\
&+ 76538107963247572305333703125 \times 10^{8} Z^{9} y_3^5 + 1260641253957117784553500642968750000 Z^{3} y_1 y_3^5 \\
&- 174125610121417376166025875 \times 10^{9} Z^{5} y_2 y_3^5 - 4151986297832500533562136805 \times 10^{9} Z y_2^2 y_3^5 \\
&+ 2965936577383044475828760742187500000 Z^{6} y_3^6 + 167847712533271490799643040039062500 y_1 y_3^6 \\
&+ 3239931377420334664496859375 \times 10^{7} Z^{2} y_2 y_3^6 + 640037202014182753762668457031250000 Z^{3} y_3^7 \\
&+ 58356400019624053607544708251953125 y_3^8 - 494384765625 \times 10^{24} Z^{23} y_4 \\
&+ 296164765638369140625 \times 10^{16} Z^{17} y_1 y_4 - 78022414059048532890984375 \times 10^{10} Z^{11} y_1^2 y_4 \\
&+ 3236477112395302046348102625 \times 10^{6} Z^{5} y_1^3 y_4 - 682977413671875 \times 10^{22} Z^{19} y_2 y_4 \\
&+ 7432282606303915860796875 \times 10^{12} Z^{13} y_1 y_2 y_4 - 7386995471520995316432551025 \times 10^{8} Z^{7} y_1^2 y_2 y_4 \\
&- 13039901553230143364415969498686400 Z y_1^3 y_2 y_4 - 3971560600500461296875 \times 10^{16} Z^{15} y_2^2 y_4 \\
&+ 68426954446098218118264 \times 10^{14} Z^{9} y_1 y_2^2 y_4 + 385990077977534387542370025593520000 Z^{3} y_1^2 y_2^2 y_4 \\
&- 22995161299790823185235 \times 10^{15} Z^{11} y_2^3 y_4 + 245962792641836743172130096 \times 10^{8} Z^{5} y_1 y_2^3 y_4 \\
&+ 2666389934328195164609142 \times 10^{13} Z^{7} y_2^4 y_4 + 807019501083025290004031959601049600 Z y_1 y_2^4 y_4 \\
&- 23716599806946390657552345857633280000 Z^{3} y_2^5 y_4 - 24143853515625 \times 10^{23} Z^{20} y_3 y_4 \\
&+ 14697779297106914409375 \times 10^{15} Z^{14} y_1 y_3 y_4 - 185361004856333450331462825 \times 10^{10} Z^{8} y_1^2 y_3 y_4 \\
&+ 18294647618169733528426012005060000 Z^{2} y_1^3 y_3 y_4 - 64546514465911875 \times 10^{21} Z^{16} y_2 y_3 y_4 \\
&+ 29112799843506921522025725 \times 10^{12} Z^{10} y_1 y_2 y_3 y_4 - 809410999391697357036564339258 \times 10^{6} Z^{4} y_1^2 y_2 y_3 y_4 \\
&- 169058296143885796542945 \times 10^{15} Z^{12} y_2^2 y_3 y_4 + 60418194203354320750160015712 \times 10^{8} Z^{6} y_1 y_2^2 y_3 y_4 \\
&+ 568319981336087001024421280691016704 y_1^2 y_2^2 y_3 y_4 - 5212478277415466661964248 \times 10^{13} Z^{8} y_2^3 y_3 y_4 \\
&- 1245105953455827865326014736195840000 Z^{2} y_1 y_2^3 y_3 y_4 + 51720114631568294131769279288448 \times 10^{6} Z^{4} y_2^4 y_3 y_4 \\
&- 36725577222051176038344519967114592256 y_2^5 y_3 y_4 - 413564010040107421875 \times 10^{16} Z^{17} y_3^2 y_4 \\
&+ 259861161753621765514490625 \times 10^{11} Z^{11} y_1 y_3^2 y_4 - 1189554582109416191544245878125 \times 10^{6} Z^{5} y_1^2 y_3^2 y_4 \\
&- 180069744492491984102390625 \times 10^{12} Z^{13} y_2 y_3^2 y_4 + 40067508085150103304421763205 \times 10^{9} Z^{7} y_1 y_2 y_3^2 y_4 \\
&- 134631081779862815466707990372788800 Z y_1^2 y_2 y_3^2 y_4 - 322230194193902746035033828 \times 10^{12} Z^{9} y_2^2 y_3^2 y_4 \\
&- 892258203972546479716155384521760000 Z^{3} y_1 y_2^2 y_3^2 y_4 - 268751106618291757642976225712 \times 10^{8} Z^{5} y_2^3 y_3^2 y_4 \\
&+ 13677293178748991525956666971176448000 Z y_2^4 y_3^2 y_4 - 2149642939612100428125 \times 10^{15} Z^{14} y_3^3 y_4 \\
&+ 217490983809601001410494315 \times 10^{11} Z^{8} y_1 y_3^3 y_4 - 269401038362200597884806384014500000 Z^{2} y_1^2 y_3^3 y_4 \\
&- 242985767684384739965405175 \times 10^{12} Z^{10} y_2 y_3^3 y_4 + 22818067875044783009196629421564 \times 10^{6} Z^{4} y_1 y_2 y_3^3 y_4 \\
&- 3204282882362238028843302140928 \times 10^{8} Z^{6} y_2^2 y_3^3 y_4 - 2478939326379413370028605477821184000 y_1 y_2^2 y_3^3 y_4 \\
&- 1566919544170839113591115512474880000 Z^{2} y_2^3 y_3^3 y_4 + 176901919559807869080140625 \times 10^{10} Z^{11} y_3^4 y_4 \\
&+ 8056716966339780906991739142675 \times 10^{6} Z^{5} y_1 y_3^4 y_4 - 1712194426979098001844992768925 \times 10^{8} Z^{7} y_2 y_3^4 y_4 \\
&+ 4278364589047458199963961135014200000 Z y_1 y_2 y_3^4 y_4 - 166619686942453002322328861617026 \times 10^{6} Z^{3} y_2^2 y_3^4 y_4 \\
&+ 267197130755465668453167975 \times 10^{10} Z^{8} y_3^5 y_4 + 958089765263519239835961694387500000 Z^{2} y_1 y_3^5 y_4 \\
&- 5948282160363815040638625829425 \times 10^{7} Z^{4} y_2 y_3^5 y_4 - 3447039718501132612379244619896 \times 10^{7} y_2^2 y_3^5 y_4
\end{aligned}
$$

$$+ 1125423146845717983972848390625 \times 10^6 Z^5 y_3^6 y_4 - 740846205385950303530438235187 5 \times 10^6 Z y_2 y_3^6 y_4$$
$$+ 13714373003452613663941300781250000 Z^2 y_3^7 y_4 - 13824462890625 \times 10^{22} Z^{22} y_4^2$$
$$+ 3127486417543177734375 \times 10^{15} Z^{16} y_1 y_4^2 - 13161327194746746602002968 75 \times 10^9 Z^{10} y_1^2 y_4^2$$
$$+ 652301519522318620553995188975000 00 Z^4 y_1^3 y_4^2 + 58454060726953125 \times 10^{20} Z^{18} y_2 y_4^2$$
$$+ 602793804778093527102 \times 10^{16} Z^{12} y_1 y_2 y_4^2 - 8981972445397260862820775447 \times 10^8 Z^6 y_1^2 y_2 y_4^2$$
$$- 258844837060442147082528495717037 44 y_1^3 y_2 y_4^2 - 4159493904054581728593 75 \times 10^{14} Z^{14} y_2^2 y_4^2$$
$$+ 8709478902285583220035519 5 \times 10^{11} Z^8 y_1 y_2^2 y_4^2 + 281342538256526808172411927183200 00 Z^2 y_1^2 y_2^2 y_4^2$$
$$+ 2904309303715705322304 \times 10^{15} Z^{10} y_2^3 y_4^2 - 4176519694608985235631415532736 \times 10^6 Z^4 y_1 y_2^3 y_4^2$$
$$+ 361359898368297607619421111 36 \times 10^9 Z^6 y_2^4 y_4^2 + 167798462061921975865852166272643 4816 y_1 y_2^4 y_4^2$$
$$- 1499317350651567888767150853012480000 Z^2 y_2^5 y_4^2 - 82222671796875 \times 10^{23} Z^{19} y_3 y_4^2$$
$$+ 2240685654465490405481 25 \times 10^{14} Z^{13} y_1 y_3 y_4^2 - 34256086785340114084475302575 \times 10^8 Z^7 y_1^2 y_3 y_4^2$$
$$+ 26498014997564578675144263360076800 Z y_1^3 y_3 y_4^2 - 414340975035589725 \times 10^{20} Z^{15} y_2 y_3 y_4^2$$
$$+ 30691332872410102086020508 \times 10^{12} Z^9 y_1 y_2 y_3 y_4^2 - 1956151944559120368060759898333440000 Z^3 y_1^2 y_2 y_3 y_4^2$$
$$- 1389707533031357069613675 \times 10^{14} Z^{11} y_2^2 y_3 y_4^2 + 13001615722090624612367709 2268 \times 10^8 Z^5 y_1 y_2^2 y_3 y_4^2$$
$$- 318221291730851137913114664 \times 10^{11} Z^7 y_2^3 y_3 y_4^2 - 1231483651917923953528153830416793600 Z y_1 y_2^3 y_3 y_4^2$$
$$+ 1015476913445668600979424805270732 80000 Z^3 y_2^4 y_3 y_4^2 - 4836721222814078320312 5 \times 10^{15} Z^{16} y_3^2 y_4^2$$
$$+ 506930346520387677359458312 5 \times 10^{10} Z^{10} y_1 y_3^2 y_4^2 - 3065020281572611276077514093201500000 Z^4 y_1^2 y_3^2 y_4^2$$
$$- 2152797403845706902966 6 \times 10^{16} Z^{12} y_2 y_3^2 y_4^2 + 53027534872984655576566089671 4 \times 10^8 Z^6 y_1 y_2 y_3^2 y_4^2$$
$$- 9399671160794669941582131848239908 48 y_1^2 y_2 y_3^2 y_4^2 - 1836196916339137164366800295 \times 10^{11} Z^8 y_2^2 y_3^2 y_4^2$$
$$+ 28341018988276196735868184243416000 00 Z^2 y_1 y_2^2 y_3^2 y_4^2 - 3868737087872062812568404593491 2 \times 10^6 Z^4 y_2^3 y_3^2 y_4^2$$
$$+ 6431465607053724836520758910778963 9680 y_2^4 y_3^2 y_4^2 - 1098136348572747545364375 \times 10^{14} Z^{13} y_3^3 y_4^2$$
$$+ 5168563018158247415436005158 5 \times 10^9 Z^7 y_1 y_3^3 y_4^2 - 743149060364068744588483451015347200 Z y_1^2 y_3^3 y_4^2$$
$$- 3906485896986089687007485 64 \times 10^{12} Z^9 y_2 y_3^3 y_4^2 + 3969593889940706577876523162259712 0000 Z^3 y_1 y_2 y_3^3 y_4^2$$
$$- 103911023449492192588668727539 6 \times 10^8 Z^5 y_2^2 y_3^3 y_4^2 - 1381811314201354900486052546164531200 0 Z y_2^3 y_3^3 y_4^2$$
$$- 1287288383580857020028476218 75 \times 10^9 Z^{10} y_3^4 y_4^2 + 2478188805872716941945939108932850 0000 Z^4 y_1 y_3^4 y_4^2$$
$$- 3513146540554098031672189271439 \times 10^8 Z^6 y_2 y_3^4 y_4^2 + 1132762228877504374579963410941676 08000 y_1 y_2 y_3^4 y_4^2$$
$$- 1674163961684179861467088006549416 0000 Z^2 y_2^2 y_3^4 y_4^2 - 8200784769266696633631943395 75 \times 10^8 Z^7 y_3^5 y_4^2$$
$$+ 4123754433433384018025842903068480 0000 Z y_1 y_3^5 y_4^2 - 1598682786294193602794013108098 88 \times 10^6 Z^3 y_2 y_3^5 y_4^2$$
$$- 2653133243484136179484176910481250 0000 Z^4 y_3^6 y_4^2 - 3048441354192289298156645867635 5 \times 10^6 y_2 y_3^6 y_4^2$$
$$- 3135526588446157004786857 17 \times 10^{10} Z y_3^7 y_4^2 + 16649560546875 \times 10^{23} Z^{21} y_4^3$$
$$- 1402572611178903515625 \times 10^{14} Z^{15} y_1 y_4^3 - 1270115739470006763557646375 \times 10^9 Z^9 y_1^2 y_4^3$$
$$+ 1904924792966461284617038404446400 00 Z^3 y_1^3 y_4^3 + 189427145116078125 \times 10^{20} Z^{17} y_2 y_4^3$$
$$- 3100246846030642011963 75 \times 10^{13} Z^{11} y_1 y_2 y_4^3 + 306877905973291733285352111 \times 10^8 Z^5 y_1^2 y_2 y_4^3$$
$$- 3169848111098010561225 \times 10^{16} Z^{13} y_2^2 y_4^3 + 474067053222072486891132816 \times 10^{10} Z^7 y_1 y_2^2 y_4^3$$
$$- 2946672309554877025683393910818048 00 Z y_1^2 y_2^2 y_4^3 + 826736211176253038659066 68 \times 10^{12} Z^9 y_2^3 y_4^3$$
$$- 1079073944562042843738798764496384 0000 Z^3 y_1 y_2^3 y_4^3 - 2903894945873922981191715449 28 \times 10^8 Z^5 y_2^4 y_4^3$$
$$+ 1639566078672905754286613113539624 9600 Z y_2^5 y_4^3 + 9981940078125 \times 10^{24} Z^{18} y_3 y_4^3$$
$$+ 887612920407308721107 25 \times 10^{14} Z^{12} y_1 y_3 y_4^3 - 4483604658558787995004954855 2 \times 10^8 Z^6 y_1^2 y_3 y_4^3$$
$$+ 1960534151415347590902607678300769 28 y_1^3 y_3 y_4^3 + 5956050763929668355 \times 10^{19} Z^{14} y_2 y_3 y_4^3$$

$$- 918204626363139867220440 3 \times 10^{12} Z^8 y_1 y_2 y_3 y_4^3 + 2155849156150181405368491845664000 0Z^2 y_1^2 y_2 y_3 y_4^3$$
$$- 833545161518778598230792 \times 10^{14} Z^{10} y_2^2 y_3 y_4^3 + 1377870062022608237947769627664 \times 10^7 Z^4 y_1 y_2^2 y_3 y_4^3$$
$$+ 18213605418868623075106009189 44 \times 10^8 Z^6 y_2^3 y_3 y_4^3 - 1249579181036759533006614201731402 9568 y_1 y_2^3 y_3 y_4^3$$
$$- 18969269952344544647564586988032 \times 10^6 Z^2 y_2^4 y_3 y_4^3 - 40746131382797539453125 \times 10^{14} Z^{15} y_3^2 y_4^3$$
$$+ 3248694898912177432479168525 \times 10^{10} Z^9 y_1 y_3^2 y_4^3 - 549958149563353637896618058680560 0000 Z^3 y_1^2 y_3^2 y_4^3$$
$$+ 740250670371900157684912 5 \times 10^{13} Z^{11} y_2 y_3^2 y_4^3 - 44415637134988127071412526858 \times 10^8 Z^5 y_1 y_2 y_3^2 y_4^3$$
$$- 1097013860979216537868505308 8 \times 10^{10} Z^7 y_2^2 y_3^2 y_4^3 + 6028180454710625912594925711722035200 Z y_1 y_2^2 y_3^2 y_4^3$$
$$+ 17709891118007983276819939227173376 0000 Z^3 y_2^3 y_3^2 y_4^3 - 73137125099118852931297 5 \times 10^{14} Z^{12} y_3^3 y_4^3$$
$$+ 46131841796091059371629764808 \times 10^9 Z^6 y_1 y_3^3 y_4^3 - 2478118946400477355068001264305028608 y_1^2 y_3^3 y_4^3$$
$$+ 40898841091277822746948089 \times 10^{12} Z^8 y_2 y_3^3 y_4^3 + 27436407393832217764214325039936 0000 Z^2 y_1 y_2 y_3^3 y_4^3$$
$$- 7546926577387139148390247551024 \times 10^7 Z^4 y_2^2 y_3^3 y_4^3 + 7501179688285170086942861956084826112 0 y_2^3 y_3^3 y_4^3$$
$$- 13336764768653944094941927837 5 \times 10^9 Z^9 y_3^4 y_4^3 + 305050602151376450135731451828635200 00 Z^3 y_1 y_3^4 y_4^3$$
$$+ 58642164992631116730623408 55 \times 10^8 Z^5 y_2 y_3^4 y_4^3 - 16553552729801924516239227690308736000 Z y_2^2 y_3^4 y_4^3$$
$$- 11032449754697542825779675223 44 \times 10^8 Z^6 y_3^5 y_4^3 + 85894006642675790941922016556759680 00 y_1 y_3^5 y_4^3$$
$$- 4807033297918668246606238364310720000 Z^2 y_2 y_3^5 y_4^3 - 45007399964280553298838586796148 \times 10^6 Z^3 y_3^6 y_4^3$$
$$- 81013615297452183505817248390 8 \times 10^7 y_3^7 y_4^3 - 625361572265625 \times 10^{20} Z^{20} y_4^4$$
$$- 38259096602407396875 \times 10^{17} Z^{14} y_1 y_4^4 + 10405295347673606244705815625 \times 10^8 Z^8 y_1^2 y_4^4$$
$$- 34008998887259913639883410871200 00 Z^2 y_1^3 y_4^4 + 7100741408750859375 \times 10^{17} Z^{16} y_2 y_4^4$$
$$- 554311110482068682957 94 \times 10^{14} Z^{10} y_1 y_2 y_4^4 + 543728149583154660713291631408 \times 10^6 Z^4 y_1^2 y_2 y_4^4$$
$$+ 114137319341452387206 15 \times 10^{15} Z^{12} y_2^2 y_4^4 - 50010668074604241078890520024 \times 10^8 Z^6 y_1 y_2^2 y_4^4$$
$$- 5739577774495297206094667726366745 60 y_1^2 y_2^2 y_4^4 + 40805645097622599503395344 \times 10^{12} Z^8 y_2^3 y_4^4$$
$$+ 732371805528014774446659502963200000 Z^2 y_1 y_2^3 y_4^4 - 469329419853393182837180295154 56 \times 10^6 Z^4 y_2^4 y_4^4$$
$$+ 374237027532684819088141095278319697 92 y_2^5 y_4^4 + 1459005196932421875 \times 10^{19} Z^{17} y_3 y_4^4$$
$$- 158450071227582473482 5 \times 10^{16} Z^{11} y_1 y_3 y_4^4 + 119475813453421786295339482 92 \times 10^8 Z^5 y_1^2 y_3 y_4^4$$
$$+ 186474225534377043375 \times 10^{17} Z^{13} y_2 y_3 y_4^4 - 2023382014795920714325643772 \times 10^{10} Z^7 y_1 y_2 y_3 y_4^4$$
$$+ 3449616505332154775667554051414016 00 Z y_1^2 y_2 y_3 y_4^4 + 985468359226553794754187 48 \times 10^{12} Z^9 y_2^2 y_3 y_4^4$$
$$- 3453467738578669517399176600143360000 Z^3 y_1 y_2^2 y_3 y_4^4 + 507116540446252819366905085824 \times 10^8 Z^5 y_2^3 y_3 y_4^4$$
$$- 2028789587168290639985092357249105920 0 Z y_2^4 y_3 y_4^4 + 646499241484106585625 \times 10^{17} Z^{14} y_3^2 y_4^4$$
$$- 236891499034585552319019833375 \times 10^9 Z^8 y_1 y_3^2 y_4^4 + 1073523169371560490521153456522400 00 Z^2 y_1^2 y_3^2 y_4^4$$
$$+ 718479571555012142527452 \times 10^{14} Z^{10} y_2 y_3^2 y_4^4 - 2205864986264328788214439847 76 \times 10^8 Z^4 y_1 y_2 y_3^2 y_4^4$$
$$+ 1628439430114156486495035816072 \times 10^8 Z^6 y_2^2 y_3^2 y_4^4 + 23931078935285193743380646701047306 24 y_1 y_2^2 y_3^2 y_4^4$$
$$+ 177490919633900168782606417740518400 00 Z^2 y_2^3 y_3^2 y_4^4 + 110099859286126052734 65 \times 10^{16} Z^{11} y_3^3 y_4^4$$
$$- 1327897135746262649846707156 08 \times 10^8 Z^5 y_1 y_3^3 y_4^4 + 1100837063697520711233787851 6 \times 10^{10} Z^7 y_2 y_3^3 y_4^4$$
$$- 6994815358550451399568604192650444800 Z y_1 y_2 y_3^3 y_4^4 + 953020313249764215133872014527411200 00 Z^3 y_2^2 y_3^3 y_4^4$$
$$+ 90940657965025743462876026062 5 \times 10^8 Z^8 y_3^4 y_4^4 - 1093817457326206458215121235264560000 Z^2 y_1 y_3^4 y_4^4$$
$$+ 7504633688793020029757897788996 8 \times 10^6 Z^4 y_2 y_3^4 y_4^4 + 13888157906568898379094146743463953920 y_2^2 y_3^4 y_4^4$$
$$+ 33817964456587749636387047907 6 \times 10^8 Z^5 y_3^5 y_4^4 + 1829337812471054380188720443748249600 0 Z y_2 y_3^5 y_4^4$$
$$+ 256603030689683018789159571727416000 0 Z^2 y_3^6 y_4^4 - 33862497978515625 \times 10^{20} Z^{19} y_4^5$$
$$+ 1178274777148114630875 \times 10^{15} Z^{13} y_1 y_4^5 + 56937262510376613889475661 \times 10^{10} Z^7 y_1^2 y_4^5$$

$$
\begin{aligned}
&- 19610295460709702690314060715750400 Z y_1^3 y_4^5 - 1585286604909261 \times 10^{22} Z^{15} y_2 y_4^5 \\
&- 3112538478561369801672597 \times 10^{12} Z^9 y_1 y_2 y_4^5 + 1038575402697584627432733304483200000 Z^3 y_1^2 y_2 y_4^5 \\
&+ 17176383355075314026895 \times 10^{15} Z^{11} y_2^2 y_4^5 - 61551106601211844689178336608 \times 10^8 Z^5 y_1 y_2^2 y_4^5 \\
&+ 26477486220263229441754668 \times 10^{12} Z^7 y_2^3 y_4^5 + 40034623048061552994734440925 5526400 Z y_1 y_2^3 y_4^5 \\
&- 42534428605043310268517181911961600000 Z^3 y_2^4 y_4^5 - 1675311814140721875 \times 10^{19} Z^{16} y_3 y_4^5 \\
&- 5662256798086115266196 1 \times 10^{14} Z^{10} y_1 y_3 y_4^5 + 1393947686084434874349809933376 \times 10^6 Z^4 y_1^2 y_3 y_4^5 \\
&- 300843287076412277877 6 \times 10^{16} Z^{12} y_2 y_3 y_4^5 - 10407840879767162145941956752 \times 10^9 Z^6 y_1 y_2 y_3 y_4^5 \\
&+ 8113763693061026546904365183782502 40 y_1^2 y_2 y_3 y_4^5 + 530005854119213118191639 76 \times 10^{12} Z^8 y_2^2 y_3 y_4^5 \\
&- 35247775493608602815818315740211200 00 Z^2 y_1 y_2^2 y_3 y_4^5 + 4008200750681702538553273132 6464 \times 10^6 Z^4 y_2^3 y_3 y_4^5 \\
&- 50707422313966906434841493265292984320 y_2^4 y_3 y_4^5 - 1145028834397453855162 5 \times 10^{15} Z^{13} y_3^2 y_4^5 \\
&- 181926375620279945346719825 4 \times 10^{10} Z^7 y_1 y_3^2 y_4^5 + 73608493091774391436770106352755 2000 Z y_1^2 y_3^2 y_4^5 \\
&- 10645224974482657499515 53 \times 10^{12} Z^9 y_2 y_3^2 y_4^5 - 11969999703957779282474579904180480000 Z^3 y_1 y_2 y_3^2 y_4^5 \\
&+ 667201108028099286788712237 6 \times 10^{10} Z^5 y_2^2 y_3^2 y_4^5 + 41899862351595900351485174993842176 00 Z y_2^3 y_3^2 y_4^5 \\
&+ 40202754429075094846737 9 \times 10^{14} Z^{10} y_3^3 y_4^5 - 1730610714216708771524055008755 2 \times 10^6 Z^4 y_1 y_3^3 y_4^5 \\
&+ 29587175292501286214959078934 4 \times 10^8 Z^6 y_2 y_3^3 y_4^5 - 41890549761274901090601630640777297 92 y_1 y_2 y_3^3 y_4^5 \\
&+ 23475363930901911904604608441052160000 Z^2 y_2^2 y_3^3 y_4^5 + 753955351372877605806023752 5 \times 10^{10} Z^7 y_3^4 y_4^5 \\
&- 53579724443174893887617409224328960 00 Z y_1 y_3^4 y_4^5 + 1705229920148189585109087321998784 0000 Z^3 y_2 y_3^4 y_4^5 \\
&+ 4826900141851989425528649617107 2 \times 10^6 Z^4 y_3^5 y_4^5 + 12359073520629677653486750993160355 84 y_2 y_3^5 y_4^5 \\
&+ 11006678956035976790757273582317337600 Z y_3^6 y_4^5 + 733506113056640625 \times 10^{18} Z^{18} y_4^6 \\
&+ 34275899795893487333 55 \times 10^{15} Z^{12} y_1 y_4^6 - 21150250700477748406181275 68 \times 10^8 Z^6 y_1^2 y_4^6 \\
&- 53246773860388988989849831132563456 y_1^3 y_4^6 - 1578432488207080927 5 \times 10^{18} Z^{14} y_2 y_4^6 \\
&+ 33166662826717596244318 92 \times 10^{12} Z^8 y_1 y_2 y_4^6 + 42180291712290372202785481827840000 Z^2 y_1^2 y_2 y_4^6 \\
&+ 122466740221075068555021 \times 10^{14} Z^{10} y_2^2 y_4^6 - 163921359600794355500589167001 6 \times 10^6 Z^4 y_1 y_2^2 y_4^6 \\
&- 27408613168837486569249358848 \times 10^9 Z^6 y_2^3 y_4^6 + 35942883087550754183200694606847836 16 y_1 y_2^3 y_4^6 \\
&+ 56305647053044705517528818097971200 00 Z^2 y_2^4 y_4^6 - 1188264842768004187 5 \times 10^{18} Z^{15} y_3 y_4^6 \\
&+ 89412667126553671718470 56 \times 10^{12} Z^9 y_1 y_3 y_4^6 + 96736634969556511856379217858560000 Z^3 y_1^2 y_3 y_4^6 \\
&- 376295187583592743307 4 \times 10^{16} Z^{11} y_2 y_3 y_4^6 + 254021115496504217029956466 56 \times 10^8 Z^5 y_1 y_2 y_3 y_4^6 \\
&+ 122307068538294549644730700 8 \times 10^{10} Z^7 y_2^2 y_3 y_4^6 + 29863928385440344756029305243566080 0 Z y_1 y_2^2 y_3 y_4^6 \\
&- 42159547542070766265039543618846720000 Z^3 y_2^3 y_3 y_4^6 - 4094855085778328496181 5 \times 10^{15} Z^{12} y_3^2 y_4^6 \\
&+ 76029250203597807519122808288 \times 10^8 Z^6 y_1 y_3^2 y_4^6 + 53828215328911680159511357357249536 0 y_1^2 y_3^2 y_4^6 \\
&- 398368321966588926060785 88 \times 10^{12} Z^8 y_2 y_3^2 y_4^6 - 10962588137533025211897652541952000 0 Z^2 y_1 y_2 y_3^2 y_4^6 \\
&+ 7914018342659387266732539525 12 \times 10^6 Z^4 y_2^2 y_3^2 y_4^6 - 28104255471639359482356527562821713920 y_2^3 y_3^2 y_4^6 \\
&- 516315187197021416910904 8 \times 10^{13} Z^9 y_3^3 y_4^6 + 11299513513389843657805795613491200 00 Z^3 y_1 y_3^3 y_4^6 \\
&- 1713861842241239030707207931 52 \times 10^8 Z^5 y_2 y_3^3 y_4^6 - 48610197897763157426526805675081728 00 Z y_2^2 y_3^3 y_4^6 \\
&- 28643904067570671087638710228 8 \times 10^8 Z^6 y_3^4 y_4^6 - 16195886196007061189600453320097034 24 y_1 y_3^4 y_4^6 \\
&- 18558875617400762792412030589363200 00 Z^2 y_2 y_3^4 y_4^6 - 49589513921778424848884962977177600 00 Z^3 y_3^5 y_4^6 \\
&+ 15414842913145612957379866237830328 32 y_3^6 y_4^6 + 39237072344296875 \times 10^{20} Z^{17} y_4^7 \\
&- 811158007716347301022 5 \times 10^{14} Z^{11} y_1 y_4^7 - 2827469196881458874840018 16 \times 10^9 Z^5 y_1^2 y_4^7 \\
&+ 669629015561065658 4 \times 10^{18} Z^{13} y_2 y_4^7 + 1926521842789719743622946 56 \times 10^{10} Z^7 y_1 y_2 y_4^7
\end{aligned}
$$

$$
\begin{aligned}
&- 55309317587720537151876396211814400Zy_1^2y_2y_4^7 - 765124269164236875995006 4 \times 10^{12}Z^9y_2^2y_4^7\\
&+ 293114167982916514756843108902912000 0Z^3y_1y_2^2y_4^7 - 42451596180967682001342574 08 \times 10^9Z^5y_2^3y_4^7\\
&- 642672875170705490109602281034219520 0Zy_2^4y_4^7 + 130180994585158356 \times 10^{20}Z^{14}y_3y_4^7\\
&+ 108433664600022486986659 2 \times 10^{12}Z^8y_1y_3y_4^7 - 927458096747422972831267037184000 00Z^2y_1^2y_3y_4^7\\
&+ 4888959408708672231741 6 \times 10^{14}Z^{10}y_2y_3y_4^7 + 481539258554972814701510792140 8 \times 10^6Z^4y_1y_2y_3y_4^7\\
&- 142424440632598434296455821107 2 \times 10^8Z^6y_2^2y_3y_4^7 + 193369397368685263274497246470733824y_1y_2^2y_3y_4^7\\
&- 77162697625439367416646477198950400 00Z^2y_2^3y_3y_4^7 + 136944608656834568866275 \times 10^{14}Z^{11}y_3^2y_4^7\\
&+ 388128954294274019165532233 28 \times 10^8Z^5y_1y_3^2y_4^7 - 188572967680255476125137574 4 \times 10^{10}Z^7y_2y_3^2y_4^7\\
&+ 2110643634149656005464610282786816000Zy_1y_2y_3^2y_4^7 - 428528418469981412002480462215168000 0Z^3y_2^2y_3^2y_4^7\\
&- 201212185416230802888835 2 \times 10^{12}Z^8y_3^3y_4^7 + 122918670227903826307829713108992000 0Z^2y_1y_3^3y_4^7\\
&- 275377217224087390002334987760 64 \times 10^6Z^4y_2y_3^3y_4^7 + 761119321809187717466160823295606784 0y_2^2y_3^3y_4^7\\
&- 105970313479578888251768246 64 \times 10^9Z^5y_3^4y_4^7 - 875450792354283786104064452210319360 0Zy_2y_3^4y_4^7\\
&- 315293894392238200167937093287936000 0Z^2y_3^5y_4^7 - 108974105914528125 \times 10^{19}Z^{16}y_4^8\\
&- 139002936267059502358 14 \times 10^{14}Z^{10}y_1y_4^8 - 385929162624171223402316680 32 \times 10^6Z^4y_1^2y_4^8\\
&+ 13782877002554201608 5 \times 10^{17}Z^{12}y_2y_4^8 - 254796712004129134034142489 6 \times 10^9Z^6y_1y_2y_4^8\\
&- 520115302818016900193767890059919 36y_1^2y_2y_4^8 - 184095737496945053938310 4 \times 10^{12}Z^8y_2^2y_4^8\\
&+ 791241153256445593595123874263040000Z^2y_1y_2^2y_4^8 + 35948222064849612166336402882 56 \times 10^6Z^4y_2^3y_4^8\\
&+ 515170737879290763089534613281813299 2y_2^4y_4^8 + 4767187214296752075 \times 10^{18}Z^{13}y_3y_4^8\\
&- 204435344129810629051729152 \times 10^{10}Z^7y_1y_3y_4^8 - 3535771114266682162203744465715200 0Zy_1^2y_3y_4^8\\
&+ 287767401434557612181775 36 \times 10^{12}Z^9y_2y_3y_4^8 - 247274010579413362857564304687104000 0Z^3y_1y_2y_3y_4^8\\
&- 101579955477158375519267552256 \times 10^8Z^5y_2^2y_3y_4^8 + 780501337235248580690733483829493760 0Zy_2^3y_3y_4^8\\
&+ 144969270482036870192466 \times 10^{14}Z^{10}y_3^2y_4^8 - 721151524552676015580724971648 \times 10^6Z^4y_1y_3^2y_4^8\\
&+ 238154461916796997331213541888 \times 10^8Z^6y_2y_3^2y_4^8 - 570519507207812327363270212108812288y_1y_2y_3^2y_4^8\\
&- 558595670009406073960758844704768000 0Z^2y_2^2y_3^2y_4^8 + 123486726233109772059569740 8 \times 10^{10}Z^7y_3^3y_4^8\\
&+ 172049201628357931585963846926336000Zy_1y_3^3y_4^8 + 903914424911570773083517889888256000 0Z^3y_2y_3^3y_4^8\\
&+ 414560618081234206834041154886 4 \times 10^6Z^4y_3^4y_4^8 + 152323926278546186179319767777837056 0y_2y_3^4y_4^8\\
&+ 506987024040627856928934009510297600Zy_3^5y_4^8 - 272864715624383625 \times 10^{19}Z^{15}y_4^9\\
&- 160621606919715696642528 \times 10^{12}Z^9y_1y_4^9 + 163338897098054156731215987548160000Z^3y_1^2y_4^9\\
&- 390372530419113601788 \times 10^{16}Z^{11}y_2y_4^9 - 209092720544976784198921804 8 \times 10^8Z^5y_1y_2y_4^9\\
&+ 523005518852133927722657 28 \times 10^{11}Z^7y_2^2y_4^9 - 966555824633979312691665955867852800Zy_1y_2^2y_4^9\\
&- 129384206430429854154656606404608000 0Z^3y_2^3y_4^9 - 5452215210061695846 \times 10^{18}Z^{12}y_3y_4^9\\
&- 702732057654641319238762598 4 \times 10^8Z^6y_1y_3y_4^9 + 291838068099098865250380132982456 32y_1^2y_3y_4^9\\
&- 412553782162912589838 72 \times 10^{14}Z^8y_2y_3y_4^9 - 331018719986080149220353357250560000Z^2y_1y_2y_3y_4^9\\
&+ 103398234431608393188947947929 6 \times 10^7Z^4y_2^2y_3y_4^9 - 232796838444296012044543816416991641 6y_2^3y_3y_4^9\\
&- 344688243541287274543881 6 \times 10^{12}Z^9y_3^2y_4^9 - 119895196308354986297592673198080000 0Z^3y_1y_3^2y_4^9\\
&+ 111715764122074403816061534 72 \times 10^8Z^5y_2y_3^2y_4^9 + 684393186323631192091446759258193920 0Zy_2^2y_3^2y_4^9\\
&+ 862789206043453447982174515 2 \times 10^8Z^6y_3^3y_4^9 - 775341865161607840365816486800916 48y_1y_3^3y_4^9\\
&+ 181684630956455108076781166985216000 0Z^2y_2y_3^3y_4^9 + 195112046884596234216212215812096000 0Z^3y_3^4y_4^9\\
&- 279454073684653448185793228769853 44y_3^5y_4^9 + 732398532993703125 \times 10^{18}Z^{14}y_4^{10}
\end{aligned}
$$

$$
\begin{aligned}
&+ 4781699167791581675 15328 \times 10^{12} Z^8 y_1 y_4^{10} + 168203486808115542677615077785600000 Z^2 y_1^2 y_4^{10} \\
&- 5379241675962144582 6624 \times 10^{14} Z^{10} y_2 y_4^{10} + 2632015453026468086824799232 \times 10^8 Z^4 y_1 y_2 y_4^{10} \\
&- 1928217287819064819140052 48 \times 10^{10} Z^6 y_2^2 y_4^{10} + 165159626510561421727812571004141568 y_1 y_2^2 y_4^{10} \\
&+ 121456539404632451000778524983296000 0 Z^2 y_2^3 y_4^{10} - 717325244318637812 16 \times 10^{16} Z^{11} y_3 y_4^{10} \\
&+ 1341856767328490376341839872 \times 10^8 Z^5 y_1 y_3 y_4^{10} - 922315519575125257291610112 \times 10^{10} Z^7 y_2 y_3 y_4^{10} \\
&+ 27352274635326220857543161806848 0000 Z y_1 y_2 y_3 y_4^{10} + 644354947482964111117152729169920000 Z^3 y_2^2 y_3 y_4^{10} \\
&- 296703135977130662484 6144 \times 10^{12} Z^8 y_3^2 y_4^{10} - 1133903310156615994480031057510400 00 Z^2 y_1 y_3^2 y_4^{10} \\
&- 56003850270884263820073188 10624 \times 10^6 Z^4 y_2 y_3^2 y_4^{10} - 8522957584879919716963211765297 9712 y_2^2 y_3^2 y_4^{10} \\
&- 928660801251533753434676 4288 \times 10^8 Z^5 y_3^3 y_4^{10} - 18314809041015494192325883242479616 00 Z y_2 y_3^3 y_4^{10} \\
&+ 12374347985183390421428054138880 0000 Z^2 y_3^4 y_4^{10} + 11812521661372809 \times 10^{20} Z^{13} y_4^{11} \\
&+ 22467085824857739575166 72 \times 10^{11} Z^7 y_1 y_4^{11} - 2576410525569073703786826442997 7600 Z y_1^2 y_4^{11} \\
&+ 195328845007896596417 28 \times 10^{14} Z^9 y_2 y_4^{11} + 12983451886384863565944713379840000 Z^3 y_1 y_2 y_4^{11} \\
&- 54332583200299373613648814 08 \times 10^8 Z^5 y_2^2 y_4^{11} - 10197568113238775476354671562457088 00 Z y_2^3 y_4^{11} \\
&+ 144357019248996922176 \times 10^{16} Z^{10} y_3 y_4^{11} + 3216335041967099411592128 1024 \times 10^7 Z^4 y_1 y_3 y_4^{11} \\
&+ 2396078688624490433315082 24 \times 10^{10} Z^6 y_2 y_3 y_4^{11} - 1328717318477719912870911221767864 32 y_1 y_2 y_3 y_4^{11} \\
&- 985664674856765649605868713410560000 Z^2 y_2^2 y_3 y_4^{11} + 2413856607802790516 20224 \times 10^{11} Z^7 y_3^2 y_4^{11} \\
&+ 25390302170592688430203130420920320 0 Z y_1 y_3^2 y_4^{11} + 644353869552964439618867441172480000 Z^3 y_2 y_3^2 y_4^{11} \\
&- 7557676589447735913633182 5152 \times 10^7 Z^4 y_3^3 y_4^{11} + 382069601514117973396057411704324096 y_2 y_3^3 y_4^{11} \\
&- 497508162420339183339129912636211200 Z y_3^4 y_4^{11} - 276864400278689652 \times 10^{18} Z^{12} y_4^{12} \\
&- 2012765807226247916619110 4 \times 10^{10} Z^6 y_1 y_4^{12} + 3520640787319994362435399521927168 y_1^2 y_4^{12} \\
&+ 10728454304090234308101 12 \times 10^{12} Z^8 y_2 y_4^{12} + 644488193755853739496271668838400 00 Z^2 y_1 y_2 y_4^{12} \\
&+ 3778293398144734385264682270 72 \times 10^6 Z^4 y_2^2 y_4^{12} + 21401126449221041608531788882247680 0 y_2^3 y_4^{12} \\
&- 246640948963882318 08 \times 10^{15} Z^9 y_3 y_4^{12} - 6441074911083931405396947763 2 \times 10^6 Z^3 y_1 y_3 y_4^{12} \\
&+ 95759277668775532627517767 68 \times 10^8 Z^5 y_2 y_3 y_4^{12} + 294408291836560147678014148391731200 Z y_2^2 y_3 y_4^{12} \\
&+ 67020706645715381680415232 \times 10^{10} Z^6 y_3^2 y_4^{12} - 1833279802747115673382915099380940 8 y_1 y_3^2 y_4^{12} \\
&+ 649082690037210657923250624921600 00 Z^2 y_2 y_3^2 y_4^{12} + 18169759595838082195104517324 8 \times 10^6 Z^3 y_3^3 y_4^{12} \\
&+ 225342914478287215839999642430341 12 y_3^4 y_4^{12} - 3177105734593008 \times 10^{20} Z^{11} y_4^{13} \\
&- 40936076850770279277310771 2 \times 10^8 Z^5 y_1 y_4^{13} - 508151245902365342592 \times 10^{15} Z^7 y_2 y_4^{13} \\
&- 40465561630120407684323787512217600 Z y_1 y_2 y_4^{13} + 1993846579630752763004269363 2 \times 10^6 Z^3 y_2^2 y_4^{13} \\
&- 229459151342259327744 \times 10^{15} Z^8 y_3 y_4^{13} - 30833990705753252513202862817280000 Z^2 y_1 y_3 y_4^{13} \\
&- 5680331657864450682887208 96 \times 10^9 Z^4 y_2 y_3 y_4^{13} - 6196708067298649231943358480384000 y_2^2 y_3 y_4^{13} \\
&+ 6769153507553955753624207 36 \times 10^8 Z^5 y_3^2 y_4^{13} - 1736206861451313870759729530142720 0 Z y_2 y_3^2 y_4^{13} \\
&+ 825892173585865932201104139878400 00 Z^2 y_3^3 y_4^{13} + 686267035071277996 8 \times 10^{16} Z^{10} y_4^{14} \\
&+ 4638875421128206432568279 04 \times 10^8 Z^4 y_1 y_4^{14} - 1023649181506420003631923 2 \times 10^{10} Z^6 y_2 y_4^{14} \\
&- 2313895332078212176581702451200 00 y_1 y_2 y_4^{14} + 5812466925090890062110060969984000 0 Z^2 y_2^2 y_4^{14} \\
&+ 221467606230843762278 4 \times 10^{13} Z^7 y_3 y_4^{14} + 1800400346913422257454346928128000 0 Z y_1 y_3 y_4^{14} \\
&+ 35116493520374127577075286016 \times 10^6 Z^3 y_2 y_3 y_4^{14} - 13321851265845859223925227 52 \times 10^8 Z^4 y_3^2 y_4^{14} \\
&+ 5280643753377251375022467973120000 y_2 y_3^2 y_4^{14} - 5092269887065147284353917648896000 0 Z y_3^3 y_4^{14} \\
&+ 529481592304488576 \times 10^{17} Z^9 y_4^{15} - 277249308043514672392018329 6 \times 10^6 Z^3 y_1 y_4^{15}
\end{aligned}
$$

$$
\begin{aligned}
&+ 540371002270843265266483 2 \times 10^{10} Z^5 y_2 y_4^{15} - 3617558616942011411228446949376 0000 Z y_2^2 y_4^{15} \\
&+ 19195807702775360716 8 \times 10^{14} Z^6 y_3 y_4^{15} - 837795657785477699476227568435200 y_1 y_3 y_4^{15} \\
&+ 3630871310888559054467314483 2 \times 10^6 Z^2 y_2 y_3 y_4^{15} + 84540416607315409697903738 88 \times 10^6 Z^3 y_3^2 y_4^{15} \\
&+ 238720018900307867720351282626560 0 y_3^3 y_4^{15} - 11765844577464933888 \times 10^{15} Z^8 y_4^{16} \\
&- 3214651664253299127044014 08 \times 10^7 Z^2 y_1 y_4^{16} + 8138442181249946187988992 \times 10^8 Z^4 y_2 y_4^{16} \\
&+ 294750999873552298691854986117120 0 y_2^2 y_4^{16} - 27074851515333750030336 \times 10^{11} Z^5 y_3 y_4^{16} \\
&- 11018629641819244848677388288 \times 10^6 Z y_2 y_3 y_4^{16} + 9169293390649572771590307 84 \times 10^7 Z^2 y_3^2 y_4^{16} \\
&- 52495930190042726 4 \times 10^{16} Z^7 y_4^{17} + 965927196449337650161794416640000 Z y_1 y_4^{17} \\
&- 92107948703098917932236 8 \times 10^9 Z^3 y_2 y_4^{17} - 5875522011293033064038 4 \times 10^{10} Z^4 y_3 y_4^{17} \\
&+ 559667792997568271192508334080000 y_2 y_3 y_4^{17} - 275381893440400285760898465792000 0 Z y_3^2 y_4^{17} \\
&+ 13410830346355644825 6 \times 10^{13} Z^6 y_4^{18} - 557248649917668759849192652800 00 y_1 y_4^{18} \\
&+ 30761918318502394572098764 8 \times 10^6 Z^2 y_2 y_4^{18} + 919771686469295532933 12 \times 10^9 Z^3 y_3 y_4^{18} \\
&+ 159889618195145815831640801280000 y_3^2 y_4^{18} + 26336582128563860275 2 \times 10^{12} Z^5 y_4^{19} \\
&- 13907079216927208511845171 2 \times 10^6 Z y_2 y_4^{19} - 597349600776343529717 76 \times 10^8 Z^2 y_3 y_4^{19} \\
&- 89719365688324050124 8 \times 10^{11} Z^4 y_4^{20} + 916343186856049692993650688000 0 y_2 y_4^{20} \\
&+ 24953734132091028598947 84 \times 10^6 Z y_3 y_4^{20} - 21075072224107010457 6 \times 10^{10} Z^3 y_4^{21} \\
&- 17063227091385363883622 4 \times 10^6 y_3 y_4^{21} + 25690684636889232703488 \times 10^8 Z^2 y_4^{22} \\
&- 24709781201321616998 4 \times 10^9 Z y_4^{23} + 7008592486193040457728 \times 10^6 y_4^{24} = 0. \qquad \text{(A.1.16)}
\end{aligned}
$$

*Proof.* Formula (A.1.15) follows immediately from Theorem 4.4.3. Using relations (4.4.1) and (4.4.15) we see that

$$
\begin{aligned}
t_2 = &-\frac{117}{640000}\left(62500Z^3 - 114075t_3^2 Z - 270231t_3^3 - 842400t_3 t_4 Z\right. \\
&\left. -1776528t_3^2 t_4 - 1555200t_4^2 Z - 2066688t_3 t_4^2 - 854016t_4^3\right), \qquad \text{(A.1.17)} \\
t_1 = &\frac{3}{2080}\left(2560t_2 + 257049t_3^3 + 2847312t_3^2 t_4 + 3212352t_3 t_4^2\right. \\
&\left. -4732416t_4^3 - 28080y_3\right). \qquad \text{(A.1.18)}
\end{aligned}
$$

Substituting these relations and (A.1.15) into formulas (4.4.13) and (4.4.14), and rearranging we get expressions for zero $P_1, P_2$, respectively, in terms of $y_i, Z$ and $t_3$. $P_1$ has algebraic degree 6 in terms of $t_3$, and $P_2$ has algebraic degree 4 in terms of $t_3$. We can then successively reduce these expressions modulo one another, until we get an expression equal to zero that is linear in $t_3$. Solving this expression gives us formula (A.1.14). Substituting relations (A.1.14) and (A.1.15) into formula (A.1.17) gives us formula (A.1.13), and substituting relations (A.1.13)–(A.1.15) into formula (A.1.18) gives us formula (A.1.12). Finally, calculating the resultant of $P_1$ and $P_2$ with respect to $t_3$ gives us formula (A.1.16). □

**Proposition A.1.3.** *The unity vector field $e = \partial_{t_1}$ in the $y$ coordinates has the form*

$$
e(y) = \frac{1}{25587900}\left(260000t_1 - 320000t_2 + 156542841t_3^3 + 167991408t_3^2 t_4\right.
$$

$$-321527232t_3t_4^2+471384576t_4^3\big)\,\partial_{y_1}+\frac{2}{243}\left(13t_3+16t_4\right)\partial_{y_2}-\frac{2}{81}\partial_{y_3}.$$

*Proof.* We have that

$$e=\partial_{t_1}=\frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 4.4.3. □

## A.2 Extra formulas for $H_4(3)$

Recall from Section 4.5.3 that the degrees of the $t$ coordinates and $Z$ are $\deg t_1(x)=12$, $\deg t_2(x)=8$, $\deg t_3(x)=6$, $\deg t_4(x)=2$ and $\deg Z(x)=2$. This allows us to deduce which harmonic polynomials of the $t$ coordinates and $Z$ have the same degrees as the basic invariants of $H_4$, which the following Proposition makes precise.

**Proposition A.2.1.** *Let* $V_1=\{p\in\mathbb{C}[t_1,t_2,t_3,t_4,Z]\mid\deg p(x)=30\}$, *let* $V_2=\{p\in\mathbb{C}[t_1,t_2,t_3,t_4,Z]\mid\deg p(x)=20\}$ *and let* $V_3=\{p\in\mathbb{C}[t_1,t_2,t_3,t_4,Z]\mid\deg p(x)=12\}$. *The harmonic elements of* $V_1$ *are proportional to*

$$\begin{aligned}
&273710937500Zt_1t_2^2+166181640625Z^3t_2^3-7725217500000t_1^2t_3-1686059375000Z^2t_1t_2t_3\\
&+2813748437500t_2^3t_3+343343\times10^7Z^3t_1t_3^2+10153143\times10^6Zt_2^2t_3^2\\
&+44607122560000t_1t_3^3+13101968880000Z^2t_2t_3^3-98566359091200Z^3t_3^4\\
&-1067972038189056t_3^5-3695097656250t_1t_2^2t_4-615849609375Z^2t_2^3t_4\\
&-49049\times10^8Zt_1t_2t_3t_4+656906250000Z^3t_2^2t_3t_4+15107092\times10^6Z^2t_1t_3^2t_4\\
&+17289772500000t_2^2t_3^2t_4-75150915840000Zt_2t_3^3t_4-117393641164800Z^2t_3^4t_4\\
&+1290351562500Zt_2^3t_4^2-86080995\times10^6t_1t_2t_3t_4^2-4637758125000Z^2t_2^2t_3t_4^2\\
&+21973952\times10^6Zt_1t_3^2t_4^2+74574099600000Z^3t_2t_3^2t_4^2+1221004616832000t_2t_3^3t_4^2\\
&+2154727058964480Zt_3^4t_4^2+9144135\times10^6t_1^2t_4^3+1686059375000Z^2t_1t_2t_4^3\\
&-7564588281250t_2^3t_4^3-686686\times10^7Z^3t_1t_3t_4^3-17223206\times10^6Zt_2^2t_3t_4^3\\
&-1169830421760000t_1t_3^2t_4^3-40522321840000Z^2t_2t_3^2t_4^3+27564125388800Z^3t_3^3t_4^3\\
&-2719482236190720t_3^4t_4^3-105105\times10^7Zt_1t_2t_4^4-6081075\times10^6Z^3t_2^2t_4^4\\
&-11869858\times10^6Z^2t_1t_3t_4^4-237568831500000t_2^2t_3t_4^4-361502901760000Zt_2t_3^2t_4^4\\
&+1153245738444800Z^2t_3^3t_4^4+35939673\times10^6t_1t_2t_4^5+751587375000Z^2t_2^2t_4^5\\
&+9417408\times10^6Zt_1t_3t_4^5+36133537440000Z^3t_2t_3t_4^5-591423198643200t_2t_3^2t_4^5\\
&-1537761323614208Zt_3^3t_4^5+343343\times10^7Z^3t_1t_4^6+5053609\times10^7Zt_2^2t_4^6\\
&-127320458496000t_1t_3t_4^6+45320481360000Z^2t_2t_3t_4^6-1017307102873600Z^3t_3^2t_4^6\\
&-13578052065492992t_3^3t_4^6-3237234\times10^6Z^2t_1t_4^7+53377236\times10^6t_2^2t_4^7\\
&-652278457600000Zt_2t_3t_4^7+55743687014400Z^2t_3^2t_4^7+1009008\times10^6Zt_1t_4^8\\
&-51692681040000Z^3t_2t_4^8+4742004658406400t_2t_3t_4^8+3739875232006144Zt_3^2t_4^8
\end{aligned}$$

$$
\begin{aligned}
&- 181221189184000t_1t_4^9 + 285365431120000Z^2t_2t_4^9 + 252561088102400Z^3t_3t_4^9 \\
&+ 45179325120061440t_3^2t_4^9 - 315122248320000Zt_2t_4^{10} - 2591627567513600Z^2t_3t_4^{10} \\
&+ 1213103032204800t_2t_4^{11} + 5565686179823616Zt_3t_4^{11} + 621720674713600Z^3t_4^{12} \\
&+ 48281303463264256t_3t_4^{12} - 546313364889600Z^2t_4^{13} - 1453539958751232Zt_4^{14} \\
&- 12719997492494336t_4^{15},
\end{aligned}
$$

*the harmonic elements of $V_2$ are proportional to*

$$
\begin{aligned}
&122718750t_1t_2 + 20453125Z^2t_2^2 - 170170000Z^3t_2t_3 - 3086098400t_2t_3^2 - 3546740736Zt_3^3 \\
&- 93500000Zt_2^2t_4 + 4178328000t_1t_3t_4 + 209440000Z^2t_2t_3t_4 + 351859200Z^3t_3^2t_4 \\
&+ 12413592576t_3^3t_4 + 578531250t_2^2t_4^2 + 1675520000Zt_2t_3t_4^2 - 3518592000Z^2t_3^2t_4^2 \\
&+ 170170000Z^3t_2t_4^3 + 10476188800t_2t_3t_4^3 + 3133892608Zt_3^2t_4^3 + 474012000t_1t_4^4 \\
&- 654500000Z^2t_2t_4^4 + 1147731200Z^3t_3t_4^4 + 114911695360t_3^2t_4^4 + 359040000Zt_2t_4^5 \\
&+ 4758476800Z^2t_3t_4^5 - 2545690400t_2t_4^6 - 13857220608Zt_3t_4^6 - 1499590400Z^3t_4^7 \\
&- 108124341248t_3t_4^7 + 1181241600Z^2t_4^8 + 3202062336Zt_4^9 - 109763075072t_4^{10},
\end{aligned}
$$

*and the harmonic elements of $V_3$ are proportional to*

$$6125t_1 + 60368t_3^2 + 28875t_2t_4^2 + 465696t_3t_4^3 + 114048t_4^6.$$

*Proof.* Using Proposition 4.0.1 we can directly calculate

$$
\begin{aligned}
\Delta(t_1) =& -\frac{22}{175}\left(25Zt_2 - 224Z^2t_3 + 200t_2t_4 - 504Zt_3t_4\right. \\
&\left.+280Z^3t_4^2 + 1232t_3t_4^2 + 384Z^2t_4^3 - 288Zt_4^4 - 3200t_4^5\right), \quad \text{(A.2.1)}\\
\Delta(t_2) =& \frac{112}{75}\left(5Z^3 + 15Z^2t_4 + 15Zt_4^2 - 22t_4^3\right), \quad \text{(A.2.2)}\\
\Delta(t_3) =& -\frac{5}{28}\left(5Z^2 + 10Zt_4 + 32t_4^2\right), \quad \text{(A.2.3)}\\
\Delta(t_4) =& \frac{1}{3}. \quad \text{(A.2.4)}
\end{aligned}
$$

A general element of $V_1$ is of the form

$$\sum_{k=0}^{3} \sum_{\substack{0\le i_1,i_2,i_3,i_4 \\ 2k+\sum_{j=1}^{4} d_j i_j = 30}} a_{(i_1,i_2,i_3,i_4,k)} t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{i_4}Z^k, \quad \text{(A.2.5)}$$

where $a_{(i_1,i_2,i_3,i_4,k)} \in \mathbb{C}$. By calculating the Laplacian of this general element (A.2.5) using Proposition 4.0.1 and formulas (A.2.1)–(A.2.4) we find that the only harmonic elements of $V_1$ are as claimed. A general element of $V_2$ has the form

$$\sum_{k=0}^{3} \sum_{\substack{0 \leq i_1, i_2, i_3, i_4 \\ 2k + \sum_{j=1}^{4} d_j i_j = 20}} b_{(i_1, i_2, i_3, i_4, k)} t_1^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4} Z^k, \tag{A.2.6}$$

where $b_{(i_1, i_2, i_3, i_4, k)} \in \mathbb{C}$. By calculating the Laplacian of this general element (A.2.6) using Propostion 4.0.1 and formulas (A.2.1)–(A.2.4) we find that the only harmonic elements of $V_2$ are as claimed. A general element of $V_3$ has the form

$$\sum_{k=0}^{3} \sum_{\substack{0 \leq i_1, i_2, i_3, i_4 \\ 2k + \sum_{j=1}^{4} d_j i_j = 12}} c_{(i_1, i_2, i_3, i_4, k)} t_1^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4} Z^k, \tag{A.2.7}$$

where $c_{(i_1, i_2, i_3, i_4, k)} \in \mathbb{C}$. By calculating the Laplacian of this general element (A.2.7) using Propostion 4.0.1 and formulas (A.2.1)–(A.2.4) we find that the only harmonic elements of $V_3$ are as claimed. □

Proposition A.2.1 thus allows us to give a proof of Theorem 4.5.15, in which the basic invariants $y_i$ are expressed as polynomials in terms of the $t$ coordinates and the variable $Z$. We now present this proof.

*Proof.* Note that $Y_4 = \frac{1}{8} y_4 = 3t_4$. We now equate $Y_1$, $Y_2$ and $Y_3$ given by relations (4.5.12)–(4.5.14) with general harmonic elements of $V_1$, $V_2$ and $V_3$, respectively, given by Proposition A.2.1. We then rearrange these equations to find $y_i$ in terms of $t_j$ and $Z$. We find

$$\begin{aligned}
y_1 = {} & \frac{7383537763232174309376}{18865} t_4^{15} + \frac{a}{12719997492494336} \big( -273710937500 Z t_1 t_2^2 \\
& - 166181640625 Z^3 t_2^3 + 7725217500000 t_1^2 t_3 + 1686059375000 Z^2 t_1 t_2 t_3 \\
& - 2813748437500 t_2^3 t_3 - 343343 \times 10^7 Z^3 t_1 t_3^2 - 10153143 \times 10^6 Z t_2^2 t_3^2 \\
& - 44607122560000 t_1 t_3^3 - 13101968880000 Z^2 t_2 t_3^3 + 98566359091200 Z^3 t_3^4 \\
& + 1067972038189056 t_3^5 + 3695097656250 t_1 t_2^2 t_4 + 615849609375 Z^2 t_2^3 t_4 \\
& + 49049 \times 10^8 Z t_1 t_2 t_3 t_4 - 656906250000 Z^3 t_2^2 t_3 t_4 - 15107092 \times 10^6 Z^2 t_1 t_3^2 t_4 \\
& - 17289772500000 t_2^2 t_3^2 t_4 + 75150915840000 Z t_2 t_3^3 t_4 + 117393641164800 Z^2 t_3^4 t_4 \\
& - 1290351562500 Z t_2^3 t_4^2 + 86080995 \times 10^6 t_1 t_2 t_3 t_4^2 + 4637758125000 Z^2 t_2^2 t_3 t_4^2 \\
& - 21973952 \times 10^6 Z t_1 t_3^2 t_4^2 - 74574099600000 Z^3 t_2 t_3^2 t_4^2 - 1221004616832000 t_2 t_3^3 t_4^2 \\
& - 2154727058964480 Z t_3^4 t_4^2 - 9144135 \times 10^6 t_1^2 t_4^3 - 1686059375000 Z^2 t_1 t_2 t_4^3 \\
& + 7564588281250 t_2^3 t_4^3 + 686686 \times 10^7 Z^3 t_1 t_3 t_4^3 + 17223206 \times 10^6 Z t_2^2 t_3 t_4^3 \\
& + 1169830421760000 t_1 t_3^2 t_4^3 + 40522321840000 Z^2 t_2 t_3^2 t_4^3 - 27564125388800 Z^3 t_3^3 t_4^3 \\
& + 2719482236190720 t_3^4 t_4^3 + 105105 \times 10^7 Z t_1 t_2 t_4^4 + 6081075 \times 10^6 Z^3 t_2^2 t_4^4
\end{aligned}$$

$$
\begin{aligned}
&+ 11869858 \times 10^6 Z^2 t_1 t_3 t_4^4 + 237568831500000 t_2^2 t_3 t_4^4 + 361502901760000 Z t_2 t_3^2 t_4^4 \\
&- 1153245738444800 Z^2 t_3^3 t_4^4 - 35939673 \times 10^6 t_1 t_2 t_4^5 - 751587375000 Z^2 t_2^2 t_4^5 \\
&- 9417408 \times 10^6 Z t_1 t_3 t_4^5 - 36133537440000 Z^3 t_2 t_3 t_4^5 + 591423198643200 t_2 t_3^2 t_4^5 \\
&+ 1537761323614208 Z t_3^3 t_4^5 - 343343 \times 10^7 Z^3 t_1 t_4^6 - 5053609 \times 10^7 Z t_2^2 t_4^6 \\
&+ 127320458496000 t_1 t_3 t_4^6 - 45320481360000 Z^2 t_2 t_3 t_4^6 + 1017307102873600 Z^3 t_3^2 t_4^6 \\
&+ 13578052065492992 t_3^3 t_4^6 + 3237234 \times 10^6 Z^2 t_1 t_4^7 - 53377236 \times 10^6 t_2^2 t_4^7 \\
&+ 652278457600000 Z t_2 t_3 t_4^7 - 55743687014400 Z^2 t_3^2 t_4^7 - 1009008 \times 10^6 Z t_1 t_4^8 \\
&+ 51692681040000 Z^3 t_2 t_4^8 - 4742004658406400 t_2 t_3 t_4^8 - 3739875232006144 Z t_3^2 t_4^8 \\
&+ 181221189184000 t_1 t_4^9 - 285365431120000 Z^2 t_2 t_4^9 - 252561088102400 Z^3 t_3 t_4^9 \\
&- 45179325120061440 t_3^2 t_4^9 + 315122248320000 Z t_2 t_4^{10} + 2591627567513600 Z^2 t_3 t_4^{10} \\
&- 1213103032204800 t_2 t_4^{11} - 5565686179823616 Z t_3 t_4^{11} - 621720674713600 Z^3 t_4^{12} \\
&- 48281303463264256 t_3 t_4^{12} + 546313364889600 Z^2 t_4^{13} + 1453539958751232 Z t_4^{14} \\
&+ 127199974924943336 t_4^{15}\big) - \frac{830736 b}{48771678865} t_4^5 \big(122718750 t_1 t_2 + 20453125 Z^2 t_2^2 \\
&- 170170000 Z^3 t_2 t_3 - 3086098400 t_2 t_3^2 - 3546740736 Z t_3^3 - 93500000 Z t_2^2 t_4 \\
&+ 4178328000 t_1 t_3 t_4 + 209440000 Z^2 t_2 t_3 t_4 + 351859200 Z^3 t_3^2 t_4 + 12413592576 t_3^3 t_4 \\
&+ 578531250 t_2^2 t_4^2 + 1675520000 Z t_2 t_3 t_4^2 - 3518592000 Z^2 t_3^2 t_4^2 + 170170000 Z^3 t_2 t_4^3 \\
&+ 10476188800 t_2 t_3 t_4^3 + 3133892608 Z t_3^2 t_4^3 + 474012000 t_1 t_4^4 - 654500000 Z^2 t_2 t_4^4 \\
&+ 1147731200 Z^3 t_3 t_4^4 + 114911695360 t_3^2 t_4^4 + 359040000 Z t_2 t_4^5 + 4758476800 Z^2 t_3 t_4^5 \\
&- 2545690400 t_2 t_4^6 - 13857220608 Z t_3 t_4^6 - 1499590400 Z^3 t_4^7 - 108124341248 t_3 t_4^7 \\
&+ 1181241600 Z^2 t_4^8 + 3202062336 Z t_4^9 - 109763075072 t_4^{10}\big) \\
&- \frac{495559064420352 c}{56123375} t_4^9 \left(6125 t_1 + 60368 t_3^2 + 28875 t_2 t_4^2 + 465696 t_3 t_4^3 + 114048 t_4^6\right) \\
&- \frac{2304 c^2}{262609375} t_4^3 \left(6125 t_1 + 60368 t_3^2 + 28875 t_2 t_4^2 + 465696 t_3 t_4^3 + 114048 t_4^6\right)^2 ,
\end{aligned} \tag{A.2.8}
$$

$$
\begin{aligned}
y_2 =& \frac{31701690482688}{77} t_4^{10} + \frac{b}{109763075072} \big(-122718750 t_1 t_2 - 20453125 Z^2 t_2^2 \\
&+ 170170000 Z^3 t_2 t_3 + 3086098400 t_2 t_3^2 + 3546740736 Z t_3^3 + 93500000 Z t_2^2 t_4 \\
&- 4178328000 t_1 t_3 t_4 - 209440000 Z^2 t_2 t_3 t_4 - 351859200 Z^3 t_3^2 t_4 - 12413592576 t_3^3 t_4 \\
&- 578531250 t_2^2 t_4^2 - 1675520000 Z t_2 t_3 t_4^2 + 3518592000 Z^2 t_3^2 t_4^2 - 170170000 Z^3 t_2 t_4^3 \\
&- 10476188800 t_2 t_3 t_4^3 - 3133892608 Z t_3^2 t_4^3 - 474012000 t_1 t_4^4 + 654500000 Z^2 t_2 t_4^4 \\
&- 1147731200 Z^3 t_3 t_4^4 - 114911695360 t_3^2 t_4^4 - 359040000 Z t_2 t_4^5 - 4758476800 Z^2 t_3 t_4^5 \\
&+ 2545690400 t_2 t_4^6 + 13857220608 Z t_3 t_4^6 + 1499590400 Z^3 t_4^7 + 108124341248 t_3 t_4^7 \\
&- 1181241600 Z^2 t_4^8 - 3202062336 Z t_4^9 + 109763075072 t_4^{10}\big) - \frac{165888 c}{20825} t_4^4 \big(6125 t_1 \\
&+ 60368 t_3^2 + 28875 t_2 t_4^2 + 465696 t_3 t_4^3 + 114048 t_4^6\big) ,
\end{aligned} \tag{A.2.9}
$$

$$
y_3 = -\frac{95551488}{7} t_4^6 + \frac{c}{6125} \left(6125 t_1 + 60368 t_3^2 + 28875 t_2 t_4^2 + 465696 t_3 t_4^3 + 114048 t_4^6\right) , \tag{A.2.10}
$$

$$
y_4 = 24 t_4 , \tag{A.2.11}
$$

where $a, b, c \in \mathbb{C}$. In order to find $a, b$ and $c$ we perform steps 5–7 from the introduction of Chapter 4. That is, we transform the intersection form (4.5.75)–(4.5.81) into $y$ coordinates by applying formulas (A.2.8)–(A.2.11) and compare it with the expression given by Lemma 4.5.1. We find that

$$a = -\frac{37891716166732218368}{4095}, \qquad b = -\frac{28099347218432}{561}, \qquad c = 56000,$$

which implies the statement. □

**Proposition A.2.2.** *We have that*

$$\det(g^{ij}(t)) = \frac{c \prod_{\alpha \in R_{H_4}} (\alpha, x)}{Q(t, Z)^2},$$

*where* $c = 2^{68} 3^{10} 5$ *and*

$$\begin{aligned} Q(t,Z) = 5^4 7^2 \big( & 578812500000 t_1^3 + 4134375000000 Z^2 t_1^2 t_2 - 153808593750 t_1 t_2^3 \\ & - 152587890625 Z^2 t_2^4 - 18522 \times 10^8 Z^3 t_1^2 t_3 - 441 \times 10^{10} Z t_1 t_2^2 t_3 \\ & - 3964843750000 Z^3 t_2^3 t_3 - 4926852 \times 10^7 t_1^2 t_3^2 - 135828 \times 10^8 Z^2 t_1 t_2 t_3^2 \\ & - 49566562500000 t_2^3 t_3^2 + 127233792 \times 10^6 Z^3 t_1 t_3^3 - 430808 \times 10^8 Z t_2^2 t_3^3 \\ & + 1397912140800000 t_1 t_3^4 - 218387276800000 Z^2 t_2 t_3^4 - 1532961898496000 Z^3 t_3^5 \\ & - 13221142380544000 t_3^6 + 99225 \times 10^7 Z t_1^2 t_2 t_4 + 259875 \times 10^7 Z^3 t_1 t_2^2 t_4 \\ & + 615234375000 Z t_2^4 t_4 - 703836 \times 10^7 Z^2 t_1^2 t_3 t_4 + 52225425 \times 10^6 t_1 t_2^2 t_3 t_4 \\ & + 8736875 \times 10^6 Z^2 t_2^3 t_3 t_4 + 173267136 \times 10^6 Z t_1 t_2 t_3^2 t_4 + 244608 \times 10^8 Z^3 t_2^2 t_3^2 t_4 \\ & + 199369820160000 Z^2 t_1 t_3^3 t_4 - 1617588 \times 10^8 t_2^2 t_3^3 t_4 - 1263290183680000 Z t_2 t_3^4 t_4 \\ & - 37284541235200 Z^2 t_3^5 t_4 - 9674437500000 t_1^2 t_2 t_4^2 - 31185 \times 10^8 Z^2 t_1 t_2^2 t_4^2 \\ & - 332753906250 t_2^4 t_4^2 - 889056 \times 10^7 Z t_1^2 t_3 t_4^2 - 2878848 \times 10^7 Z^3 t_1 t_2 t_3 t_4^2 \\ & - 134694 \times 10^8 Z t_2^3 t_3 t_4^2 - 369047145600000 t_1 t_2 t_3^2 t_4^2 - 108723552 \times 10^6 Z^2 t_2^2 t_3^2 t_4^2 \\ & - 1026367340544000 Z t_1 t_3^3 t_4^2 + 25341388800000 Z^3 t_2 t_3^3 t_4^2 + 7684836951859200 t_2 t_3^4 t_4^2 \\ & + 9073398532538368 Z t_3^5 t_4^2 + 18522 \times 10^8 Z^3 t_1^2 t_4^3 - 66465 \times 10^8 Z t_1 t_2^2 t_4^3 \\ & - 1887406250000 Z^3 t_2^3 t_4^3 + 407484 \times 10^8 t_1^2 t_3 t_4^3 + 5140296 \times 10^7 Z^2 t_1 t_2 t_3 t_4^3 \\ & + 2955813 \times 10^7 t_2^3 t_3 t_4^3 - 24182323200000 Z^3 t_1 t_3^2 t_4^3 + 415394246400000 Z t_2^2 t_3^2 t_4^3 \\ & + 1831192989696000 t_1 t_3^3 t_4^3 + 2112148383744000 Z^2 t_2 t_3^3 t_4^3 - 3250939776860160 Z^3 t_3^4 t_4^3 \\ & - 5719494463634432 t_3^5 t_4^3 + 254016 \times 10^7 Z^2 t_1^2 t_4^4 + 6695325 \times 10^6 t_1 t_2^2 t_4^4 \\ & + 2444812500000 Z^2 t_2^3 t_4^4 - 217550592 \times 10^6 Z t_1 t_2 t_3 t_4^4 - 1998906 \times 10^7 Z^3 t_2^2 t_3 t_4^4 \\ & + 137633771520000 Z^2 t_1 t_3^2 t_4^4 - 1289929939200000 t_2^2 t_3^2 t_4^4 - 6462823342080000 Z t_2 t_3^3 t_4^4 \\ & - 11570031207383040 Z^2 t_3^4 t_4^4 - 190512 \times 10^7 Z t_1^2 t_4^5 - 2776032 \times 10^7 Z^3 t_1 t_2 t_4^5 \\ & + 1749465 \times 10^7 Z t_2^3 t_4^5 + 245599603200000 t_1 t_2 t_3 t_4^5 - 33817896 \times 10^6 Z^2 t_2^2 t_3 t_4^5 \end{aligned}$$

$$+ 1785711255552000 Z t_1 t_3^2 t_4^5 + 1122404505600000 Z^3 t_2 t_3^2 t_4^5 + 7219945222963200 t_2 t_3^3 t_4^5$$
$$+ 48395369140715520 Z t_3^4 t_4^5 - 3894912 \times 10^7 t_1^2 t_4^6 - 426888 \times 10^8 Z^2 t_1 t_2 t_4^6$$
$$- 52082805 \times 10^6 t_2^3 t_4^6 + 305699788800000 Z^3 t_1 t_3 t_4^6 + 217963603200000 Z t_2^2 t_3 t_4^6$$
$$+ 724142039040000 t_1 t_3^2 t_4^6 - 3395882176512000 Z^2 t_2 t_3^2 t_4^6 + 814851046768640 Z^3 t_3^3 t_4^6$$
$$- 47305520679223296 t_3^4 t_4^6 + 110327616 \times 10^6 Z t_1 t_2 t_4^7 - 420255 \times 10^8 Z^3 t_2^2 t_4^7$$
$$+ 101028372480000 Z^2 t_1 t_3 t_4^7 - 1051192396800000 t_2^2 t_3 t_4^7 + 3304010121216000 Z t_2 t_3^2 t_4^7$$
$$+ 3869587121111040 Z^2 t_3^3 t_4^7 + 597769502400000 t_1 t_2 t_4^8 + 160231608 \times 10^6 Z^2 t_2^2 t_4^8$$
$$+ 190293884928000 Z t_1 t_3 t_4^8 + 989672947200000 Z^3 t_2 t_3 t_4^8 + 10206142416691200 t_2 t_3^2 t_4^8$$
$$- 73495299371827200 Z t_3^3 t_4^8 - 101125785600000 Z^3 t_1 t_4^9 - 186162649600000 Z t_2^2 t_4^9$$
$$- 3191805646848000 t_1 t_3 t_4^9 + 1561948014592000 Z^2 t_2 t_3 t_4^9 - 14989064887336960 Z^3 t_3^2 t_4^9$$
$$- 42460123619393536 t_3^3 t_4^9 - 15910594560000 Z^2 t_1 t_4^{10} + 351501912 \times 10^6 t_2^2 t_4^{10}$$
$$- 296057495552000 Z t_2 t_3 t_4^{10} + 1603542687416320 Z^2 t_3^2 t_4^{10} - 359391707136000 Z t_1 t_4^{11}$$
$$+ 758020723200000 Z^3 t_2 t_4^{11} - 15168913927372800 t_2 t_3 t_4^{11} + 10925704187740160 Z t_3^2 t_4^{11}$$
$$+ 734324032512000 t_1 t_4^{12} - 431227009024000 Z^2 t_2 t_4^{12} - 463196185231360 Z^3 t_3 t_4^{12}$$
$$+ 56795180303253504 t_3^2 t_4^{12} - 998299951104000 Z t_2 t_4^{13} - 6842647714201600 Z^2 t_3 t_4^{13}$$
$$- 10653521299660800 t_2 t_4^{14} - 3339162644643840 Z t_3 t_4^{14} - 98475339612160 Z^3 t_4^{15}$$
$$+ 70986790354550784 t_3 t_4^{15} + 2660695903764480 Z^2 t_4^{16} + 8993140078804992 Z t_4^{17}$$
$$-1510901919514624 t_4^{18}\big) .$$

By Proposition 4.0.3, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 4.5.15, which leads to Proposition A.2.2.

In the next statement we express flat coordinates $t_i$ via basic invariants $y_j$ and $Z$, which is an inversion of the formulas from Theorem 4.5.15.

**Theorem A.2.3.** *We have the following relations:*

$$t_1 = \frac{1}{1161216000}\big(-11444944896 R^2 + 20736 y_3 - 9504000 Z^4 y_4^2 - 85155840 Z R y_4^2 - 9934848 R y_4^3 + 6160 Z y_4^5 + 1313 y_4^6\big), \tag{A.2.12}$$
$$t_2 = \frac{1}{172800}\left(172800 Z^4 + 1548288 Z R + 64512 R y_4 - 112 Z y_4^3 + y_4^4\right), \tag{A.2.13}$$
$$t_3 = R, \tag{A.2.14}$$
$$t_4 = \frac{1}{24} y_4, \tag{A.2.15}$$

*where*

$$R = R(y,Z) = -\frac{1}{96768}\big(986673892041 \times 10^{19} Z^{31} + 40321537568701171875 0 Z^{16} y_1$$
$$- 7588796457909375 \times 10^8 Z^{21} y_2 - 14241744 \times 10^6 Z^6 y_1 y_2 - 44789760 Z y_2^3$$

$$
\begin{aligned}
&- 406639265445 \times 10^6 Z^{11} y_2^2 - 10450944000 Z^3 y_2 y_3^3 + 10450944 \times 10^6 Z^7 y_3^4 \\
&- 258864844669875 \times 10^{13} Z^{25} y_3 + 34862693625 \times 10^6 Z^{10} y_1 y_3 \\
&+ 823747560297890625000 Z^{15} y_2 y_3 - 2208228480000 Z^5 y_2^2 y_3 \\
&- 415448230632890625 \times 10^6 Z^{19} y_3^2 - 111974400000 Z^4 y_1 y_3^2 \\
&- 2032302555 \times 10^6 Z^9 y_2 y_3^2 + 4780994139 \times 10^9 Z^{13} y_3^3 - 2985984 y_2^3 y_4 \\
&- 7247810493424125 \times 10^{15} Z^{30} y_4 + 486495253393359375000 Z^{15} y_1 y_4 \\
&+ 64535886177967968750000 Z^{20} y_2 y_4 - 4884883200000 Z^5 y_1 y_2 y_4 \\
&- 138011277978 \times 10^6 Z^{10} y_2^2 y_4 - 796262400 Z^2 y_2 y_3^3 y_4 + 796262400000 Z^6 y_3^4 y_4 \\
&- 14057685635765625 \times 10^{10} Z^{24} y_3 y_4 + 105583986 \times 10^8 Z^9 y_1 y_3 y_4 \\
&+ 2358994817559375000000 Z^{14} y_2 y_3 y_4 - 572780160000 Z^4 y_2^2 y_3 y_4 \\
&- 574609620279375 \times 10^8 Z^{18} y_3^2 y_4 - 22394880000 Z^3 y_1 y_3^2 y_4 \\
&- 2107347273 \times 10^6 Z^8 y_2 y_3^2 y_4 + 25276431834 \times 10^8 Z^{12} y_3^3 y_4 \\
&+ 14216009216979375 \times 10^{14} Z^{29} y_4^2 - 9134105281875 \times 10^6 Z^{14} y_1 y_4^2 \\
&- 3047517967359375 \times 10^6 Z^{19} y_2 y_4^2 - 811347840000 Z^4 y_1 y_2 y_4^2 \\
&- 23401335213 \times 10^6 Z^9 y_2^2 y_4^2 + 182885430719953125 \times 10^9 Z^{23} y_3 y_4^2 \\
&+ 1284086115 \times 10^6 Z^8 y_1 y_3 y_4^2 + 336483536054062500000 Z^{13} y_2 y_3 y_4^2 \\
&- 71891712000 Z^3 y_2^2 y_3 y_4^2 + 56452283319375 \times 10^7 Z^{17} y_3^2 y_4^2 \\
&- 616024245600000 Z^7 y_2 y_3^2 y_4^2 + 65038545648 \times 10^7 Z^{11} y_3^3 y_4^2 \\
&+ 76308480000 Z^5 y_3^4 y_4^2 + 1430042961628125 \times 10^{13} Z^{28} y_4^3 \\
&- 35543346117187500000 Z^{13} y_1 y_4^3 - 13790325901311035156250 Z^{18} y_2 y_4^3 \\
&- 81596160000 Z^3 y_1 y_2 y_4^3 - 2017629243 \times 10^6 Z^8 y_2^2 y_4^3 - 2239488000 Z^2 y_1 y_3^2 y_4^2 \\
&+ 26211925395636328125 \times 10^6 Z^{22} y_3 y_4^3 - 8627472 \times 10^6 Z^7 y_1 y_3 y_4^3 \\
&+ 385216956646875000 Z^{12} y_2 y_3 y_4^3 - 5647104000 Z^2 y_2^2 y_3 y_4^3 \\
&+ 26103024213954785156 25 Z^{16} y_3^2 y_4^3 - 132710400 Z y_1 y_3^2 y_4^3 \\
&- 109360536480000 Z^6 y_2 y_3^2 y_4^3 + 103524702912 \times 10^6 Z^{10} y_3^3 y_4^3 \\
&+ 8404992000 Z^4 y_3^4 y_4^3 + 738945999701015625 \times 10^{10} Z^{27} y_4^4 \\
&- 4187994966562500000 Z^{12} y_1 y_4^4 - 4642581830481738281250 Z^{17} y_2 y_4^4 \\
&- 5235840000 Z^2 y_1 y_2 y_4^4 - 11725916400000 Z^7 y_2^2 y_4^4 - 76308480 Z y_2 y_3^3 y_4^2 \\
&- 13184511219196875 \times 10^8 Z^{21} y_3 y_4^4 - 30220095600000 Z^6 y_1 y_3 y_4^4 \\
&- 738366917810625000 Z^{11} y_2 y_3 y_4^4 - 265098240 Z y_2^2 y_3 y_4^4 - 663552 y_2 y_3^3 y_4^3 \\
&+ 538650103268671875000 Z^{15} y_3^2 y_4^4 - 13702484880000 Z^5 y_2 y_3^2 y_4^4 \\
&+ 10242993672 \times 10^6 Z^9 y_3^3 y_4^4 + 1032192000 Z^3 y_3^4 y_4^4 - 3317760 y_1 y_3^2 y_4^4 \\
&- 930704737836568359375 \times 10^7 Z^{26} y_4^5 + 14356007915625000 Z^{11} y_1 y_4^5 \\
&- 942160434861621093 75 Z^{16} y_2 y_4^5 + 23015676960000 Z^6 y_2^2 y_4^5 \\
&- 120843182558731640625 0000 Z^{20} y_3 y_4^5 - 5904057600000 Z^5 y_1 y_3 y_4^5
\end{aligned}
$$

$$
\begin{aligned}
&- 1731848046277500000Z^{10}y_2y_3y_4^5 - 7062528y_2^2y_3y_4^5 - 201830400Zy_1y_2y_4^5 \\
&+ 66318052355859375000Z^{14}y_3^2y_4^5 - 1268224560000Z^4y_2y_3^2y_4^5 \\
&+ 358580196 \times 10^6 Z^8y_3^3y_4^5 - 66814128 \times 10^6 Z^7y_3^3y_4^6 + 6881280Zy_3^4y_4^6 \\
&- 15370446742224609375 \times 10^7 Z^{25}y_4^6 + 12729317771250000Z^{10}y_1y_4^6 \\
&+ 130805664862324218750Z^{15}y_2y_4^6 + 137625600Z^2y_3^4y_4^5 - 3517440y_1y_2y_4^6 \\
&+ 3641109840000Z^5y_2^2y_4^6 - 18784112593865625 \times 10^7 Z^{19}y_3y_4^6 \\
&- 686647440000Z^4y_1y_3y_4^6 - 22266008716500000Z^9y_2y_3y_4^6 \\
&+ 438512236439062500 0Z^{13}y_3^2y_4^6 - 89328096000Z^3y_2y_3^2y_4^6 \\
&+ 1094987299592109375 \times 10^8 Z^{24}y_4^7 + 2226816765 \times 10^6 Z^9y_1y_4^7 \\
&+ 36710583274019531250Z^{14}y_2y_4^7 + 331951248000Z^4y_2^2y_4^7 \\
&- 6076501220777343750000Z^{18}y_3y_4^7 - 55390080000Z^3y_1y_3y_4^7 \\
&- 1902754708875000Z^8y_2y_3y_4^7 + 832380298734375 00Z^{12}y_3^2y_4^7 \\
&- 4569888000Z^2y_2y_3^2y_4^7 - 15124053120000Z^6y_3^3y_4^7 + 344064y_3^4y_4^7 \\
&+ 4169983460762109375 \times 10^7 Z^{23}y_4^8 + 174577452187500Z^8y_1y_4^8 \\
&+ 4180427021074218750Z^{13}y_2y_4^8 + 20667384000Z^3y_2^2y_4^8 \\
&+ 309127479700781250000 0Z^{17}y_3y_4^8 - 3102648000Z^2y_1y_3y_4^8 \\
&- 96336212625000Z^7y_2y_3y_4^8 - 17457591268125000Z^{11}y_3^2y_4^8 \\
&- 163296960Zy_2y_3^2y_4^8 - 1701207360000Z^5y_3^3y_4^8 + 862358400Z^2y_2^2y_4^9 \\
&+ 114440645557265625 \times 10^7 Z^{22}y_4^9 - 5626875375000Z^7y_1y_4^9 \\
&+ 424484740898437 50Z^{12}y_2y_4^9 - 3183168y_2y_3^2y_4^9 - 2195520y_1y_3y_4^{10} \\
&+ 749902182726093750000Z^{16}y_3y_4^9 - 115161600Zy_1y_3y_4^9 \\
&- 769723080000Z^6y_2y_3y_4^9 - 2208915336375000Z^{10}y_3^2y_4^9 \\
&- 131454720000Z^4y_3^3y_4^9 - 6105604432246875 \times 10^8 Z^{21}y_4^{10} \\
&- 3710669100000Z^6y_1y_4^{10} - 786250728576562 50Z^{11}y_2y_4^{10} \\
&+ 22438080Zy_2^2y_4^{10} + 92629516156875 \times 10^6 Z^{15}y_3y_4^{10} \\
&+ 341276010000Z^5y_2y_3y_4^{10} - 278541966375000Z^9y_3^2y_4^{10} \\
&- 7297920000Z^3y_3^3y_4^{10} - 151806054114984375 \times 10^6 Z^{20}y_4^{11} \\
&- 584186550000Z^5y_1y_4^{11} - 15875888351906250Z^{10}y_2y_4^{11} \\
&+ 6732539855156250000Z^{14}y_3y_4^{11} + 31459125000Z^4y_2y_3y_4^{11} \\
&- 58152611906250Z^8y_3^2y_4^{11} - 282240000Z^2y_3^3y_4^{11} + 189504y_2^2y_4^{11} \\
&- 984577216528125 \times 10^7 Z^{19}y_4^{12} - 58522260000Z^4y_1y_4^{12} \\
&- 1803047510968750Z^9y_2y_4^{12} + 271825321875000 00Z^{13}y_3y_4^{12} \\
&+ 1346865000Z^3y_2y_3y_4^{12} - 10318713375000Z^7y_3^2y_4^{12} \\
&+ 9617195694375 \times 10^8 Z^{18}y_4^{13} - 4178985000Z^3y_1y_4^{13} \\
&- 120524781375000Z^8y_2y_4^{13} - 65887451468750000Z^{12}y_3y_4^{13}
\end{aligned}
$$

$$+ 12978000Z^2y_2y_3y_4^{13} - 1359785175000Z^6y_3^2y_4^{13} - 80640y_3^3y_4^{13}$$
$$+ 349150732875 \times 10^9Z^{17}y_4^{14} - 212814000Z^2y_1y_4^{14} - 7257600Zy_3^3y_4^{12}$$
$$- 1824399843750Z^7y_2y_4^{14} - 113141698 \times 10^8Z^{11}y_3y_4^{14} - 1555200Zy_2y_3y_4^{14}$$
$$- 129817125000Z^5y_3^2y_4^{14} + 41542453921875 \times 10^6Z^{16}y_4^{15} - 7264800Zy_1y_4^{15}$$
$$+ 666628143750Z^6y_2y_4^{15} - 1196394841250000Z^{10}y_3y_4^{15} - 88200y_2y_3y_4^{15}$$
$$- 9344002500Z^4y_3^2y_4^{15} + 241953303125 \times 10^7Z^{15}y_4^{16} - 135450y_1y_4^{16}$$
$$+ 96705506250Z^5y_2y_4^{16} - 96101132500000Z^9y_3y_4^{16} - 498015000Z^3y_3^2y_4^{16}$$
$$- 8154812875 \times 10^7Z^{14}y_4^{17} + 7982336250Z^4y_2y_4^{17} - 5857818750000Z^8y_3y_4^{17}$$
$$- 19089000Z^2y_3^2y_4^{17} - 320012825 \times 10^8Z^{13}y_4^{18} + 456041250Z^3y_2y_4^{18}$$
$$- 268065 \times 10^6Z^7y_3y_4^{18} - 491400Zy_3^2y_4^{18} - 3700482625 \times 10^6Z^{12}y_4^{19}$$
$$+ 18427500Z^2y_2y_4^{19} - 8350650000Z^6y_3y_4^{19} - 6615y_3^2y_4^{19} + 6615y_2y_4^{21}$$
$$- 26605215 \times 10^7Z^{11}y_4^{20} + 491400Zy_2y_4^{20} - 132300000Z^5y_3y_4^{20}$$
$$-1302357 \times 10^7Z^{10}y_4^{21} - 41349 \times 10^7Z^9y_4^{22} - 6615 \times 10^6Z^8y_4^{23}\big)$$
$$/ \big(13655251938515625 \times 10^{11}Z^{28} + 26985924479443359375 0Z^{18}y_2$$
$$- 62499163359375000Z^{13}y_1 - 103680000Z^3y_1y_2 - 19166139 \times 10^6Z^8y_2^2$$
$$+ 73984856793750000Z^{12}y_2y_3 - 5529600Z^2y_2^2y_3 - 10692 \times 10^8Z^7y_1y_3$$
$$- 468836581285546875 \times 10^6Z^{22}y_3 - 3188133261694335937 5Z^{16}y_3^2$$
$$- 276048 \times 10^6Z^6y_2y_3^2 + 13310568 \times 10^8Z^{10}y_3^3 - 1105920000Z^4y_3^4$$
$$- 98175213390234375 \times 10^{10}Z^{27}y_4 - 13856456109375000Z^{12}y_1y_4$$
$$- 2212828598144531250Z^{17}y_2y_4 - 20736000Z^2y_1y_2y_4 + 368640Zy_2^2y_3y_4$$
$$- 7804188 \times 10^6Z^7y_2^2y_4 + 365484462103125 \times 10^8Z^{21}y_3y_4$$
$$- 471744 \times 10^6Z^6y_1y_3y_4 + 27652969282500000Z^{11}y_2y_3y_4$$
$$- 7753347445078125000Z^{15}y_3^2y_4 - 64229760000Z^5y_2y_3^2y_4$$
$$+ 55954476 \times 10^7Z^9y_3^3y_4 - 147456000Z^3y_3^4y_4 - 1382400Zy_1y_2y_4^2$$
$$+ 19136149325384765625 \times 10^7Z^{26}y_4^2 - 584653443750000Z^{11}y_1y_4^2$$
$$- 7091660792021484375Z^{16}y_2y_4^2 + 1547043426783984375 0000Z^{20}y_3y_4^2$$
$$- 90979200000Z^5y_1y_3y_4^2 + 5006118816 \times 10^6Z^{10}y_2y_3y_4^2 + 49152y_2^2y_3y_4^2$$
$$- 775870064015625000Z^{14}y_3^2y_4^2 - 8214912000Z^4y_2y_3^2y_4^2$$
$$+ 138175956 \times 10^6Z^8y_3^3y_4^2 - 19660800Z^2y_3^4y_4^2 - 1674483840000Z^6y_2^2y_4^2$$
$$- 255831983771484375 \times 10^7Z^{25}y_4^3 + 252618851250000Z^{10}y_1y_4^3$$
$$- 2214749740957031250Z^{15}y_2y_4^3 - 30720y_1y_2y_4^3 - 232299360000Z^5y_2^2y_4^3$$
$$+ 380490084196875 \times 10^7Z^{19}y_3y_4^3 - 10909440000Z^4y_1y_3y_4^3$$
$$+ 405683923500000Z^9y_2y_3y_4^3 + 118895628515625000Z^{13}y_3^2y_4^3$$
$$- 663552000Z^3y_2y_3^2y_4^3 + 22969584 \times 10^6Z^7y_3^3y_4^3 - 983040Zy_3^4y_4^3$$
$$+ 26002745366484375 \times 10^8Z^{24}y_4^4 + 53936229375000Z^9y_1y_4^4$$

$$
\begin{aligned}
&- 126088326949218750Z^{14}y_2y_4^4 - 22513968000Z^4y_2^2y_4^4 - 885120000Z^3y_1y_3y_4^4 \\
&- 1927479388710937500000Z^{18}y_3y_4^4 + 57389849535937500Z^{12}y_3^2y_4^4 \\
&- 27328934250000Z^8y_2y_3y_4^4 - 30336000Z^2y_2y_3^2y_4^4 + 2843756160000Z^6y_3^3y_4^4 \\
&- 49152y_3^4y_4^4 - 129376874141015625 \times 10^7Z^{23}y_4^5 + 4397844375000Z^8y_1y_4^5 \\
&+ 39725982667968750Z^{13}y_2y_4^5 - 1534752000Z^3y_2^2y_4^5 - 49152000Z^2y_1y_3y_4^5 \\
&- 1126590164296875000000Z^{17}y_3y_4^5 - 13616559 \times 10^6Z^7y_2y_3y_4^5 \\
&+ 11619749041875000Z^{11}y_3^2y_4^5 - 721920Zy_2y_3^2y_4^5 + 267079680000Z^5y_3^3y_4^5 \\
&+ 16641302871093750Z^{12}y_2y_4^6 - 3079189554609375 \times 10^7Z^{22}y_4^6 \\
&- 139612500000Z^7y_1y_4^6 - 71539200Z^2y_2^2y_4^6 - 2397493081265625000Z^{16}y_3y_4^6 \\
&- 2289210960000Z^6y_2y_3y_4^6 + 1704364196625000Z^{10}y_3^2y_4^6 + 21504y_2y_3^2y_4^6 \\
&+ 19212480000Z^4y_3^3y_4^6 + 79938205678125 \times 10^8Z^{21}y_4^7 - 1766400Zy_1y_3y_4^6 \\
&- 90848100000Z^6y_1y_4^7 + 3251393076093750Z^{11}y_2y_4^7 - 2039040Zy_2^2y_4^7 \\
&- 1570385323125 \times 10^6Z^{15}y_3y_4^7 - 30720y_1y_3y_4^7 - 252021480000Z^5y_2y_3y_4^7 \\
&+ 5579308017140625 \times 10^6Z^{20}y_4^8 + 205911176625000Z^9y_3^2y_4^7 - 945y_2y_4^{18} \\
&+ 1042560000Z^3y_3^3y_4^7 - 13973025000Z^5y_1y_4^8 + 384779988843750Z^{10}y_2y_4^8 \\
&- 27072y_2^2y_4^8 + 136898416406250000Z^{14}y_3y_4^8 - 20174430000Z^4y_2y_3y_4^8 \\
&+ 23299053843750Z^8y_3^2y_4^8 + 40320000Z^2y_3^3y_4^8 + 29154807646875 \times 10^7Z^{19}y_4^9 \\
&- 1360545000Z^4y_1y_4^9 + 16382317281250Z^9y_2y_4^9 + 7061494968750000Z^{13}y_3y_4^9 \\
&- 1191660000Z^3y_2y_3y_4^9 + 2473352625000Z^7y_3^2y_4^9 - 517821170625 \times 10^8Z^{18}y_4^{10} \\
&+ 1036800Zy_3^3y_4^9 + 1258266528125000Z^{12}y_3y_4^{10} - 93990000Z^3y_1y_4^{10} \\
&- 4035040875000Z^8y_2y_4^{10} - 49968000Z^2y_2y_3y_4^{10} + 242369025000Z^6y_3^2y_4^{10} \\
&+ 11520y_3^3y_4^{10} - 18666838125 \times 10^9Z^{17}y_4^{11} - 4590000Z^2y_1y_4^{11} - 2520y_1y_4^{13} \\
&- 1101247593750Z^7y_2y_4^{11} + 16498648 \times 10^8Z^{11}y_3y_4^{11} - 1360800Zy_2y_3y_4^{11} \\
&+ 20128275000Z^5y_3^2y_4^{11} - 1804807303125 \times 10^6Z^{16}y_4^{12} - 149400Zy_1y_4^{12} \\
&- 157378106250Z^6y_2y_4^{12} + 164754308750000Z^{10}y_3y_4^{12} - 16560y_2y_3y_4^{12} \\
&+ 1364017500Z^4y_3^2y_4^{12} + 250832125 \times 10^7Z^{15}y_4^{13} - 15689643750Z^5y_2y_4^{13} \\
&+ 13254767500000Z^9y_3y_4^{13} + 71145000Z^3y_3^2y_4^{13} + 3150908125 \times 10^7Z^{14}y_4^{14} \\
&- 1169493750Z^4y_2y_4^{14} + 822251250000Z^8y_3y_4^{14} + 2727000Z^2y_3^2y_4^{14} \\
&+ 53112575 \times 10^8Z^{13}y_4^{15} - 65148750Z^3y_2y_4^{15} + 38295 \times 10^6Z^7y_3y_4^{15} \\
&+ 70200Zy_3^2y_4^{15} + 543220375 \times 10^6Z^{12}y_4^{16} - 2632500Z^2y_2y_4^{16} \\
&+ 1192950000Z^6y_3y_4^{16} + 945y_3^2y_4^{16} + 3800745 \times 10^7Z^{11}y_4^{17} + 5907 \times 10^7Z^9y_4^{19} \\
&-70200Zy_2y_4^{17} + 18900000Z^5y_3y_4^{17} + 186051 \times 10^7Z^{10}y_4^{18} + 945 \times 10^6Z^8y_4^{20})
\end{aligned}
$$

*where* $Z$ *satisfies the equation*

$$376572715308 \times 10^{30}Z^{50} + 659855787553828125 \times 10^{17}Z^{35}y_1 + 209952 \times 10^8Z^5y_1^3$$

$$
\begin{aligned}
&- 251302523347938537597656 25Z^{20}y_1^2 - 182271654562275 \times 10^{23}Z^{40}y_2\\
&+ 1365357283925844726562 5 \times 10^7Z^{25}y_1y_2 + 12909587625 \times 10^8Z^{10}y_1^2y_2\\
&- 181252526805625078125 \times 10^{12}Z^{30}y_2^2 + 559089218025 \times 10^{10}Z^{15}y_1y_2^2\\
&- 343739706054291210937500 00Z^{20}y_2^3 - 3728747520000Z^4y_2^4y_3\\
&- 258770668896 \times 10^7Z^{10}y_2^4 - 35831808y_2^5 - 1725958278495 \times 10^{27}Z^{44}y_3\\
&- 47075877270474609375 \times 10^{12}Z^{29}y_1y_3 - 2404957359726562500000Z^{14}y_1^2y_3\\
&+ 7230650271837890625 \times 10^6Z^{19}y_1y_2y_3 + 419904 \times 10^7Z^4y_1^2y_2y_3\\
&+ 2784603065148966796875 \times 10^7Z^{24}y_2^2y_3 + 298245977944425 \times 10^{20}Z^{34}y_2y_3\\
&- 1953636165 \times 10^8Z^9y_1y_2^2y_3 - 8957952 \times 10^6Z^5y_1y_3^5 - 33382368 \times 10^6Z^5y_1y_2^3\\
&+ 172450545166125 \times 10^{22}Z^{38}y_3^2 + 1703699162841796875 \times 10^8Z^{23}y_1y_3^2\\
&- 164620750179234375 \times 10^{13}Z^{28}y_2y_3^2 + 3127544124609375000 00Z^{13}y_1y_2y_3^2\\
&- 12622756032623671875 \times 10^6Z^{18}y_2^2y_3^2 - 7665268788 \times 10^7Z^8y_2^3y_3^2\\
&- 5735760339645 \times 10^{20}Z^{32}y_3^3 + 3589892388511962890625 00Z^{17}y_1y_3^3\\
&+ 4817542498353949218 75 \times 10^7Z^{22}y_2y_3^3 - 839808 \times 10^{10}Z^7y_1y_2y_3^3\\
&+ 29541548922525 \times 10^7Z^{12}y_2^2y_3^3 - 15925248000Z^2y_2^3y_3^3 - 2519424 \times 10^{10}Z^8y_1^2y_3^2\\
&+ 16809282 \times 10^{12}Z^{11}y_1y_3^4 - 3080403842640732421875 00Z^{16}y_2y_3^4\\
&- 819652608 \times 10^6Z^6y_2^2y_3^4 + 479602890308232421875 00000Z^{20}y_3^5\\
&+ 852284154843 \times 10^{10}Z^{14}y_2^3y_3 + 342904104 \times 10^{10}Z^{10}y_2y_3^5\\
&- 25448911776 \times 10^{11}Z^{14}y_3^6 - 1791590400000Z^4y_2y_3^6\\
&+ 17915904 \times 10^8Z^8y_3^7 - 156905298045 \times 10^{31}Z^{49}y_4\\
&- 80310602672286328125 \times 10^{15}Z^{34}y_1y_4 - 454394610293625 \times 10^{15}Z^{26}y_3^4\\
&- 2706990453860778808593750Z^{19}y_1^2y_4 + 69984 \times 10^8Z^4y_1^3y_4\\
&- 32475392523518334960937500000Z^{24}y_1y_2y_4 + 478592145 \times 10^9Z^9y_1^2y_2y_4\\
&+ 115167373922216601 5625 \times 10^{11}Z^{29}y_2^2y_4 + 363933121854375 \times 10^{23}Z^{39}y_2y_4\\
&+ 415544014569375 \times 10^7Z^{14}y_1y_2^2y_4 + 2444047647797460937500000Z^{19}y_2^3y_4\\
&- 9363859200000Z^4y_1y_2^3y_4 - 11497785084 \times 10^8Z^9y_2^4y_4 + 5073271303455 \times 10^{27}Z^{43}y_3y_4\\
&+ 2509049177389775390625 \times 10^{10}Z^{28}y_1y_3y_4 - 776786827078125 \times 10^6Z^{13}y_1^2y_3y_4\\
&- 348679820601035156250 0000Z^{18}y_1y_2y_3y_4 + 559872 \times 10^6Z^3y_1^2y_2y_3y_4\\
&- 8737932453217734375 \times 10^9Z^{23}y_2^2y_3y_4 - 2095275033 \times 10^8Z^8y_1y_2^2y_3y_4\\
&+ 24563269450125 \times 10^8Z^{13}y_2^3y_3y_4 - 604661760000Z^3y_2^4y_3y_4\\
&- 33219369829305 \times 10^{23}Z^{37}y_3^2y_4 - 313516802219175 \times 10^{20}Z^{33}y_2y_3y_4\\
&+ 174943923876645996093750 0000Z^{22}y_1y_3^2y_4 - 12807072 \times 10^9Z^7y_1^2y_3^2y_4\\
&+ 36723427465218750 0000Z^{12}y_1y_2y_3^2y_4 - 2120082257012027343750000Z^{17}y_2^2y_3^2y_4\\
&- 59719680000Z^2y_1y_2^2y_3^2y_4 - 34866938592 \times 10^6Z^7y_2^3y_3^2y_4\\
&+ 40202543430928125 \times 10^{16}Z^{31}y_3^3y_4 + 3064483848998882 8125 \times 10^{11}Z^{27}y_2y_3^2y_4
\end{aligned}
$$

$$+ 2020572848535644531250Z^{16}y_1y_3^3y_4 - 27006510283294921875 \times 10^6 Z^{21}y_2y_3^3y_4$$
$$- 329484672 \times 10^7 Z^6 y_1y_2y_3^3y_4 + 1129793472765 \times 10^8 Z^{11}y_2^2y_3^3y_4$$
$$- 1327104000Zy_2^3y_3^3y_4 + 10843777546311796875 \times 10^{10} Z^{25}y_3^4y_4$$
$$+ 907871814 \times 10^{10} Z^{10}y_1y_3^4y_4 - 868578911843765625000000Z^{15}y_2y_3^4y_4$$
$$- 198045388800000Z^5y_2^2y_3^4y_4 + 18982549382765625 \times 10^8 Z^{19}y_3^5y_4$$
$$- 2985984 \times 10^6 Z^4y_1y_3^5y_4 + 11728216152 \times 10^8 Z^9y_2y_3^5y_4$$
$$- 98170370928 \times 10^{10} Z^{13}y_3^6y_4 - 238878720000Z^3y_2y_3^6y_4$$
$$+ 23887872 \times 10^7 Z^7y_3^7y_4 + 296812522135125 \times 10^{28} Z^{48}y_4^2$$
$$+ 576916961684326171875000Z^{18}y_1^2y_4^2 + 93312 \times 10^7 Z^3y_1^3y_4^2$$
$$- 297985679803378125 \times 10^{20} Z^{38}y_2y_4^2 + 3649128971730732421875 \times 10^{13} Z^{33}y_1y_4^2$$
$$- 5323856934658886718755 \times 10^7 Z^{23}y_1y_2y_4^2 + 680933385 \times 10^8 Z^8y_1^2y_2y_4^2$$
$$- 18967816449959809570312 5 \times 10^8 Z^{28}y_2^2y_4^2 + 78256597116093750000 0Z^{13}y_1y_2^2y_4^2$$
$$+ 1579338481260234375 \times 10^6 Z^{18}y_2^3y_4^2 - 1315699200000Z^3y_1y_2^3y_4^2$$
$$- 28795362948 \times 10^7 Z^8y_2^4y_4^2 - 6571281051734625 \times 10^{24} Z^{42}y_3y_4^2$$
$$- 1556955058235625 \times 10^{15} Z^{27}y_1y_3y_4^2 - 73449113554687500000Z^{12}y_1^2y_3y_4^2$$
$$- 95684071506578613281250 0Z^{17}y_1y_2y_3y_4^2 - 3872118693443203125 \times 10^8 Z^{22}y_2^2y_3y_4^2$$
$$- 6722610552 \times 10^7 Z^7y_1y_2^2y_3y_4^2 + 34762445973675 \times 10^7 Z^{12}y_2^3y_3y_4^2$$
$$- 58226688000Z^2y_2^4y_3y_4^2 + 2654587837418596875 \times 10^{18} Z^{36}y_3^2y_4^2$$
$$+ 6759561040400390625 \times 10^6 Z^{21}y_1y_3^2y_4^2 - 31457808 \times 10^8 Z^6y_1^2y_3^2y_4^2$$
$$- 6950364881818042968 75 \times 10^9 Z^{26}y_2y_3^2y_4^2 + 97583602381875 \times 10^6 Z^{11}y_1y_2y_3^2y_4^2$$
$$+ 702845741013433593750 00Z^{16}y_2^2y_3^2y_4^2 - 4976640000Zy_1y_2^2y_3^2y_4^2$$
$$- 7320900268800000Z^6y_2^3y_3^2y_4^2 - 7305373600073015625 \times 10^{13} Z^{30}y_3^3y_4^2$$
$$- 8024785856578125 \times 10^6 Z^{15}y_1y_3^3y_4^2 - 14547474208355484375 \times 10^7 Z^{20}y_2y_3^3y_4^2$$
$$- 611240256 \times 10^6 Z^5y_1y_2y_3^3y_4^2 + 2331226793214 \times 10^7 Z^{10}y_2^2y_3^3y_4^2$$
$$+ 61931520y_2^3y_3^3y_4^2 + 6119410586900625 \times 10^{12} Z^{24}y_3^4y_4^2$$
$$+ 2473501374 \times 10^9 Z^9y_1y_3^4y_4^2 - 111120403748653125000 00Z^{14}y_2y_3^4y_4^2$$
$$- 27712419840000Z^4y_2^2y_3^4y_4^2 - 5216853486346875 \times 10^8 Z^{18}y_3^5y_4^2$$
$$- 398131200000Z^3y_1y_3^5y_4^2 + 26002957608 \times 10^7 Z^8y_2y_3^5y_4^2$$
$$- 237062880144 \times 10^9 Z^{12}y_3^6y_4^2 - 31850496000Z^2y_2y_3^6y_4^2$$
$$+ 31850496 \times 10^6 Z^6y_3^7y_4^2 - 336910542746625 \times 10^{28} Z^{47}y_4^3$$
$$- 72327972908829638671875 \times 10^{11} Z^{32}y_1y_4^3 + 2070331628711791992187 50Z^{17}y_1^2y_4^3$$
$$+ 62208 \times 10^6 Z^2y_1^3y_4^3 + 1264265649686615625 \times 10^{19} Z^{37}y_2y_4^3$$
$$+ 9282537745013671875 \times 10^7 Z^{22}y_1y_2y_4^3 + 24826824 \times 10^8 Z^7y_1^2y_2y_4^3$$
$$+ 7391192058722460937 5 \times 10^9 Z^{27}y_2^2y_4^3 + 47542508879062500000Z^{12}y_1y_2^2y_4^3$$
$$+ 2387400147561445312500 00Z^{17}y_2^3y_4^3 - 106375680000Z^2y_1y_2^3y_4^3$$

$$
\begin{aligned}
&- 48151021536 \times 10^{6} Z^{7} y_2^4 y_4^3 + 49239933462871875 \times 10^{23} Z^{41} y_3 y_4^3 \\
&- 4530969029916035156 25 \times 10^{9} Z^{26} y_1 y_3 y_4^3 + 849437087625 \times 10^{7} Z^{11} y_1^2 y_3 y_4^3 \\
&- 153019945847874375 \times 10^{16} Z^{31} y_2 y_3 y_4^3 + 1152372155455265625 \times 10^{16} Z^{32} y_2 y_3 y_4^2 \\
&- 81599723084874023437500 Z^{16} y_1 y_2 y_3 y_4^3 - 2488320000 Z y_1^2 y_2 y_3 y_4^3 \\
&+ 37883829188576953125 \times 10^{6} Z^{21} y_2^2 y_3 y_4^3 - 12411277488 \times 10^{6} Z^{6} y_1 y_2^2 y_3 y_4^3 \\
&+ 24071935023 \times 10^{8} Z^{11} y_2^3 y_3 y_4^3 - 2687385600 Z y_2^4 y_3 y_4^3 \\
&- 401869701855852539062500000 Z^{20} y_1 y_3^2 y_4^3 - 48615552 \times 10^{7} Z^{5} y_1^2 y_3^2 y_4^3 \\
&- 8599334965472109375 \times 10^{10} Z^{25} y_2 y_3^2 y_4^3 + 12561928600275 \times 10^{6} Z^{10} y_1 y_2 y_3^2 y_4^3 \\
&+ 111789716782528125 \times 10^{6} Z^{15} y_2^2 y_3^2 y_4^3 - 66355200 y_1 y_2^2 y_3^2 y_4^3 \\
&- 998909625600000 Z^{5} y_2^3 y_3^2 y_4^3 + 14600186628795703125 \times 10^{12} Z^{29} y_3^3 y_4^3 \\
&- 219307653849375 \times 10^{7} Z^{14} y_1 y_3^3 y_4^3 - 39946720396770703125 \times 10^{6} Z^{19} y_2 y_3^3 y_4^3 \\
&- 73246809600000 Z^{4} y_1 y_2 y_3^3 y_4^3 + 31250312001 \times 10^{8} Z^{9} y_2^2 y_3^3 y_4^3 \\
&+ 2106545142055078125 \times 10^{9} Z^{23} y_3^4 y_4^3 + 4657962996 \times 10^{8} Z^{8} y_1 y_3^4 y_4^3 \\
&+ 633400090225312500000 Z^{13} y_2 y_3^4 y_4^3 - 2456801280000 Z^{3} y_2^2 y_3^4 y_4^3 \\
&- 3997747687573125 \times 10^{8} Z^{17} y_3^5 y_4^3 - 39813120000 Z^{2} y_1 y_3^5 y_4^3 \\
&+ 40210644672 \times 10^{6} Z^{7} y_2 y_3^5 y_4^3 - 384503075712 \times 10^{8} Z^{11} y_3^6 y_4^3 \\
&- 1592524800 Z y_2 y_3^6 y_4^3 + 1592524800000 Z^{5} y_3^7 y_4^3 - 11496585305197265625 \times 10^{17} Z^{35} y_3^2 y_4^3 \\
&+ 2555969144817628125 \times 10^{24} Z^{46} y_4^4 + 65493209463778125 \times 10^{16} Z^{31} y_1 y_4^4 \\
&+ 16381465339493408203125 Z^{16} y_1^2 y_4^4 + 2073600000 Z y_1^3 y_4^4 \\
&+ 2981420427202119140625000000 Z^{21} y_1 y_2 y_4^4 - 75521484 \times 10^{7} Z^{6} y_1^2 y_2 y_4^4 \\
&- 3523272288924023437 5 \times 10^{10} Z^{26} y_2^2 y_4^4 - 12308805876375 \times 10^{6} Z^{11} y_1 y_2^2 y_4^4 \\
&- 21293055871781835937500 Z^{16} y_2^3 y_4^4 - 5038848000 Z y_1 y_2^3 y_4^4 - 5917861036800000 Z^{6} y_2^4 y_4^4 \\
&- 236440393860320859375 \times 10^{19} Z^{40} y_3 y_4^4 - 287663834469733242187 5 \times 10^{15} Z^{36} y_2 y_4^4 \\
&- 111951333903216796875 \times 10^{9} Z^{25} y_1 y_3 y_4^4 + 3696516197437500000 Z^{10} y_1^2 y_3 y_4^4 \\
&- 1354124887594875 \times 10^{16} Z^{30} y_2 y_3 y_4^4 + 2662102930021875 \times 10^{7} Z^{15} y_1 y_2 y_3 y_4^4 \\
&- 82944000 y_1^2 y_2 y_3 y_4^4 + 389418572016214453125 00000 Z^{20} y_2^2 y_3 y_4^4 \\
&- 15569982 \times 10^{8} Z^{5} y_1 y_2^2 y_3 y_4^4 - 981655730874 \times 10^{7} Z^{10} y_2^3 y_3 y_4^4 \\
&- 89579520 y_2^4 y_3 y_4^4 + 308956915751037890625 \times 10^{15} Z^{34} y_3^2 y_4^4 \\
&- 117962709745253906 25 \times 10^{6} Z^{19} y_1 y_3^2 y_4^4 - 52363584 \times 10^{6} Z^{4} y_1^2 y_3^2 y_4^4 \\
&+ 1739192452706496093 75 \times 10^{8} Z^{24} y_2 y_3^2 y_4^4 + 90287424562500000 Z^{9} y_1 y_2 y_3^2 y_4^4 \\
&+ 239958879646275 \times 10^{8} Z^{14} y_2^2 y_3^2 y_4^4 - 95657397120000 Z^{4} y_2^3 y_3^2 y_4^4 \\
&- 9548061279693046875 \times 10^{12} Z^{28} y_3^3 y_4^4 + 162739397678906250000 Z^{13} y_1 y_3^3 y_4^4 \\
&- 445978940355140625 \times 10^{7} Z^{18} y_2 y_3^3 y_4^4 - 5899392 \times 10^{6} Z^{3} y_1 y_2 y_3^3 y_4^4 \\
&+ 25374935043 \times 10^{7} Z^{8} y_2^2 y_3^3 y_4^4 - 23195369656096875 \times 10^{10} Z^{22} y_3^4 y_4^4 \\
&+ 6495790464 \times 10^{7} Z^{7} y_1 y_3^4 y_4^4 + 522642779167218750000 Z^{12} y_2 y_3^4 y_4^4
\end{aligned}
$$

$$- 134369280000Z^2y_2^2y_3^4y_4^4 - 8338799409001875 \times 10^7 Z^{16}y_3^5y_4^4$$
$$- 1990656000Zy_1y_3^5y_4^4 + 4764636172800000Z^6y_2y_3^5y_4^4 - 470490322176 \times 10^7 Z^{10}y_3^6y_4^4$$
$$- 79626240y_2y_3^6y_4^4 + 79626240000Z^4y_3^7y_4^4 - 13679169747959953125 \times 10^{23} Z^{45}y_4^5$$
$$- 1474894714718830078125 \times 10^{11} Z^{30}y_1y_4^5 - 2287749592599609375000Z^{15}y_1^2y_4^5 + 27648000y_1^3y_4^5$$
$$+ 4105948680167607421875000 0Z^{20}y_1y_2y_4^5 - 16801992 \times 10^7 Z^5y_1^2y_2y_4^5$$
$$+ 988556812415296875 \times 10^{11} Z^{25}y_2^2y_4^5 - 3766984207875 \times 10^6 Z^{10}y_1y_2^2y_4^5$$
$$- 1080139706872031250000 0Z^{15}y_2^3y_4^5 - 111974400y_1y_2^3y_4^5$$
$$- 543691699200000Z^5y_2^4y_4^5 + 770326318614121875 \times 10^{21} Z^{39}y_3y_4^5$$
$$+ 17835843567505078125 \times 10^9 Z^{24}y_1y_3y_4^5 + 534729974625 \times 10^6 Z^9y_1^2y_3y_4^5$$
$$+ 11245454023174453125 \times 10^{12} Z^{29}y_2y_3y_4^5 + 34453146024591328125 \times 10^{16} Z^{35}y_2y_4^5$$
$$+ 913740682833281250000 0Z^{14}y_1y_2y_3y_4^5 + 1609037231168671875 \times 10^6 Z^{19}y_2^2y_3y_4^5$$
$$- 141665112 \times 10^6 Z^4y_1y_2^2y_3y_4^5 - 261934435584 \times 10^7 Z^9y_2^3y_3y_4^5$$
$$- 1180075558461328125 \times 10^6 Z^{18}y_1y_3^2y_4^5 - 4002048 \times 10^6 Z^3y_1^2y_3^2y_4^5$$
$$+ 281347813236796875 \times 10^{10} Z^{23}y_2y_3^2y_4^5 - 303060818587500000Z^8y_1y_2y_3^2y_4^5$$
$$+ 240446304646875 \times 10^7 Z^{13}y_2^2y_3^2y_4^5 - 6604934400000Z^3y_2^3y_3^2y_4^5$$
$$+ 1783964308002328125 \times 10^{12} Z^{27}y_3^3y_4^5 + 147803379593906250000Z^{12}y_1y_3^3y_4^5$$
$$+ 57274806057328125 \times 10^6 Z^{17}y_2y_3^3y_4^5 - 317260800000Z^2y_1y_2y_3^3y_4^5$$
$$+ 4885829928 \times 10^6 Z^7y_2^2y_3^3y_4^5 - 69853708702715625 \times 10^9 Z^{21}y_3^4y_4^5$$
$$+ 7049353536 \times 10^6 Z^6y_1y_3^4y_4^5 + 113673204255037500000Z^{11}y_2y_3^4y_4^5$$
$$- 4777574400Zy_2^2y_3^4y_4^5 - 135404117497575 \times 10^8 Z^{15}y_3^5y_4^5 - 79626240y_1y_3^5y_4^5$$
$$+ 437869670400000Z^5y_2y_3^5y_4^5 - 43693185024 \times 10^7 Z^9y_3^6y_4^5$$
$$+ 5332371316372416796875 \times 10^{20} Z^{44}y_4^6 - 63151976031834375 \times 10^{18} Z^{33}y_3^2y_4^5$$
$$+ 4424377514117167968 75 \times 10^{11} Z^{29}y_1y_4^6 - 875396734910156250000Z^{14}y_1^2y_4^6$$
$$- 1455027089447460937500 00Z^{19}y_1y_2y_4^6 - 19426392 \times 10^6 Z^4y_1^2y_2y_4^6$$
$$+ 9501158788162734375 \times 10^9 Z^{24}y_2^2y_4^6 - 573293099587500000Z^9y_1y_2^2y_4^6$$
$$- 125161068018093750000 0Z^{14}y_2^3y_4^6 - 37770364800000Z^4y_2^4y_4^6$$
$$- 1817583627706509375 \times 10^{20} Z^{38}y_3y_4^6 - 36470290708467421875 \times 10^{15} Z^{34}y_2y_4^6$$
$$+ 3259434148794140625 \times 10^9 Z^{23}y_1y_3y_4^6 + 29159498812500000Z^8y_1^2y_3y_4^6$$
$$+ 1355797778364843750000Z^{13}y_1y_2y_3y_4^6 - 12040319730346875 \times 10^8 Z^{18}y_2^2y_3y_4^6$$
$$- 9453283200000Z^3y_1y_2^2y_3y_4^6 - 43058197647 \times 10^7 Z^8y_2^3y_3y_4^6$$
$$+ 1244368729076111718 75 \times 10^{14} Z^{32}y_3^2y_4^6 - 447452271954140625 \times 10^{12} Z^{28}y_2y_3y_4^6$$
$$+ 50793182470546875 \times 10^6 Z^{17}y_1y_3^2y_4^6 - 214617600000Z^2y_1^2y_3^2y_4^6$$
$$+ 905979904440609375 \times 10^9 Z^{22}y_2y_3^2y_4^6 - 7164953415 \times 10^7 Z^7y_1y_2y_3^2y_4^6$$
$$- 19474415330625000 00Z^{12}y_2^2y_3^2y_4^6 - 318567168000Z^2y_2^3y_3^2y_4^6$$
$$- 155268680645165625 \times 10^{12} Z^{26}y_3^3y_4^6 + 369943869465 \times 10^8 Z^{11}y_1y_3^3y_4^6$$

$$+ 834623336235375 \times 10^8 Z^{16} y_2 y_3^3 y_4^6 - 10824192000 Z y_1 y_2 y_3^3 y_4^6$$
$$- 2097343627200000 Z^6 y_2^2 y_3^3 y_4^6 - 65073391064325 \times 10^{11} Z^{20} y_3^4 y_4^6$$
$$+ 598915296 \times 10^6 Z^5 y_1 y_3^4 y_4^6 + 16133594911492500000 Z^{10} y_2 y_3^4 y_4^6$$
$$- 19906560 y_2^2 y_3^4 y_4^6 - 18480601342125 \times 10^8 Z^{14} y_3^5 y_4^6 - 3125858688 \times 10^7 Z^8 y_3^6 y_4^6$$
$$+ 31119240960000 Z^4 y_2 y_3^5 y_4^6 - 155592739624233515625 \times 10^{21} Z^{43} y_4^7$$
$$- 24577895495302734375 \times 10^{11} Z^{28} y_1 y_4^7 - 12215655150390625000 Z^{13} y_1^2 y_4^7$$
$$- 17015917939892578125000000 Z^{18} y_1 y_2 y_4^7 - 1500768 \times 10^6 Z^3 y_1^2 y_2 y_4^7$$
$$+ 2093736375984375 \times 10^{11} Z^{23} y_2^2 y_4^7 - 59405417212500000 Z^8 y_1 y_2^2 y_4^7$$
$$+ 638539955071875000000 Z^{13} y_2^3 y_4^7 - 1923160320000 Z^3 y_2^4 y_4^7$$
$$+ 35079015080705625 \times 10^{21} Z^{37} y_3 y_4^7 + 781320107381625 \times 10^{19} Z^{33} y_2 y_4^7$$
$$+ 5663874460714453125 \times 10^8 Z^{22} y_1 y_3 y_4^7 - 41598441 \times 10^8 Z^7 y_1^2 y_3 y_4^7$$
$$+ 960507262223437500000 Z^{12} y_1 y_2 y_3 y_4^7 - 229687209875109375 \times 10^6 Z^{17} y_2^2 y_3 y_4^7$$
$$- 449297280000 Z^2 y_1 y_2^2 y_3 y_4^7 - 53287526376 \times 10^6 Z^7 y_2^3 y_3 y_4^7$$
$$- 21544531581547125 \times 10^{17} Z^{31} y_3^2 y_4^7 + 590042952015328125 \times 10^{12} Z^{27} y_2 y_3 y_4^7$$
$$+ 32233844446875 \times 10^9 Z^{16} y_1 y_3^2 y_4^7 - 7464960000 Z y_1^2 y_3^2 y_4^7$$
$$- 102982159605740625 \times 10^9 Z^{21} y_2 y_3^2 y_4^7 - 10269936054 \times 10^6 Z^6 y_1 y_2 y_3^2 y_4^7$$
$$- 41524495814475 \times 10^6 Z^{11} y_2^2 y_3^2 y_4^7 - 9816422400 Z y_2^3 y_3^2 y_4^7$$
$$+ 5570068661116875 \times 10^{13} Z^{25} y_3^3 y_4^7 + 6063368665275 \times 10^6 Z^{10} y_1 y_3^3 y_4^7$$
$$+ 18203572282021875 \times 10^6 Z^{15} y_2 y_3^3 y_4^7 - 124416000 y_1 y_2 y_3^3 y_4^7$$
$$- 371949796800000 Z^5 y_2^2 y_3^3 y_4^7 - 1488869207476875 \times 10^9 Z^{19} y_3^4 y_4^7$$
$$+ 39768019200000 Z^4 y_1 y_3^4 y_4^7 + 16972150589325000000 Z^9 y_2 y_3^4 y_4^7$$
$$- 2394057523725 \times 10^8 Z^{13} y_3^5 y_4^7 + 1688947200000 Z^3 y_2 y_3^5 y_4^7$$
$$- 16889472 \times 10^8 Z^7 y_3^6 y_4^7 + 35417033403912984375 \times 10^{21} Z^{42} y_4^8$$
$$+ 199524887180859375 \times 10^{11} Z^{27} y_1 y_4^8 - 4461015903222656250 Z^{12} y_1^2 y_4^8$$
$$- 26870303807964843750000000 Z^{17} y_1 y_2 y_4^8 - 78732 \times 10^6 Z^2 y_1^2 y_2 y_4^8$$
$$- 422479128632765625 \times 10^9 Z^{22} y_2^2 y_4^8 - 478807929 \times 10^7 Z^7 y_1 y_2^2 y_4^8$$
$$+ 408768817739062500000 Z^{12} y_2^3 y_4^8 - 68864256000 Z^2 y_2^4 y_4^8$$
$$- 633688941387290625 \times 10^{19} Z^{36} y_3 y_4^8 - 605850151108203125 \times 10^{14} Z^{32} y_2 y_4^8$$
$$- 540036038657109375 \times 10^8 Z^{21} y_1 y_3 y_4^8 - 1259283307500000 Z^6 y_1^2 y_3 y_4^8$$
$$- 172207090383271875 \times 10^{12} Z^{26} y_2 y_3 y_4^8 - 4796134102125 \times 10^6 Z^{11} y_1 y_2 y_3 y_4^8$$
$$+ 80372324631375 \times 10^8 Z^{16} y_2^2 y_3 y_4^8 - 13833504000 Z y_1 y_2^2 y_3 y_4^8$$
$$- 5212932033600000 Z^6 y_2^3 y_3 y_4^8 + 289316644260975 \times 10^{18} Z^{30} y_3^2 y_4^8$$
$$+ 6569224247671875 \times 10^6 Z^{15} y_1 y_3^2 y_4^8 - 124416000 y_1^2 y_3^2 y_4^8$$
$$- 318574950446334375 \times 10^8 Z^{20} y_2 y_3^2 y_4^8 - 1062967657500000 Z^5 y_1 y_2 y_3^2 y_4^8$$
$$- 764623257948 \times 10^7 Z^{10} y_2^2 y_3^2 y_4^8 - 153653760 y_2^3 y_3^2 y_4^8$$

$$- 53210815220953125 \times 10^{11} Z^{24} y_3^3 y_4^8 + 768486778846875000 Z^9 y_1 y_3^3 y_4^8$$
$$+ 273941355999375 \times 10^7 Z^{14} y_2 y_3^3 y_4^8 - 37118638800000 Z^4 y_2^2 y_3^3 y_4^8$$
$$- 22489328602125 \times 10^{10} Z^{18} y_3^4 y_4^8 + 2015539200000 Z^3 y_1 y_3^4 y_4^8$$
$$+ 143060501986875000 Z^8 y_2 y_3^4 y_4^8 - 31221346066875 \times 10^6 Z^{12} y_3^5 y_4^8$$
$$+ 65318400000 Z^2 y_2 y_3^5 y_4^8 - 653184 \times 10^8 Z^6 y_3^6 y_4^8 - 6767097596208 \times 10^{27} Z^{41} y_4^9$$
$$- 175997708092265625 \times 10^{12} Z^{26} y_1 y_4^9 + 1675469973867187500 Z^{11} y_1^2 y_4^9$$
$$- 5535194174094375 \times 10^{16} Z^{31} y_2 y_4^9 - 6075862351945312500000 Z^{16} y_1 y_2 y_4^9$$
$$- 2566080000 Z y_1^2 y_2 y_4^9 - 588278804398875 \times 10^{11} Z^{21} y_2^2 y_4^9$$
$$- 340580052 \times 10^6 Z^6 y_1 y_2^2 y_4^9 + 570122058026250000 Z^{11} y_2^3 y_4^9 - 1511654400 Z y_2^4 y_4^9$$
$$- 1573645143268125 \times 10^{10} Z^{20} y_1 y_3 y_4^9 - 174460635 \times 10^6 Z^5 y_1^2 y_3 y_4^9$$
$$+ 1338299662100625 \times 10^{13} Z^{25} y_2 y_3 y_4^9 - 2561757575737500000 Z^{10} y_1 y_2 y_3 y_4^9$$
$$+ 9642596903971875 \times 10^6 Z^{15} y_2^2 y_3 y_4^9 - 216172800 y_1 y_2^2 y_3 y_4^9$$
$$- 409457138400000 Z^5 y_2^3 y_3 y_4^9 - 4119072700660875 \times 10^{16} Z^{29} y_3^2 y_4^9$$
$$+ 95507456248125 \times 10^7 Z^{14} y_1 y_3^2 y_4^9 - 657542824315875 \times 10^{10} Z^{19} y_2 y_3^2 y_4^9$$
$$- 83271264300000 Z^4 y_1 y_2 y_3^2 y_4^9 - 856765207507500000 Z^9 y_2^2 y_3^2 y_4^9$$
$$+ 355506903365625 \times 10^{12} Z^{23} y_3^3 y_4^9 + 82231455909375000 Z^8 y_1 y_3^3 y_4^9$$
$$+ 463787269509375 \times 10^6 Z^{13} y_2 y_3^3 y_4^9 - 2542946400000 Z^3 y_2^2 y_3^3 y_4^9$$
$$- 68469201028125 \times 10^9 Z^{17} y_3^4 y_4^9 + 73716480000 Z^2 y_1 y_3^4 y_4^9$$
$$+ 10118359012500000 Z^7 y_2 y_3^4 y_4^9 - 37059920685 \times 10^8 Z^{11} y_3^5 y_4^9$$
$$+ 1679616000 Z y_2 y_3^5 y_4^9 - 1679616 \times 10^6 Z^5 y_3^6 y_4^9 + 10408370301585 \times 10^{23} Z^{35} y_3 y_4^9$$
$$+ 7894007391375 \times 10^{15} Z^{25} y_1 y_4^{10} + 404742222562500000 Z^{10} y_1^2 y_4^{10}$$
$$- 14996171137275 \times 10^{18} Z^{30} y_2 y_4^{10} + 11782250261129278125 \times 10^{20} Z^{40} y_4^{10}$$
$$+ 5565220450945312500000 Z^{15} y_1 y_2 y_4^{10} - 38880000 y_1^2 y_2 y_4^{10}$$
$$- 115614506496375 \times 10^{10} Z^{20} y_2^2 y_4^{10} - 25020373500000 Z^5 y_1 y_2^2 y_4^{10}$$
$$+ 143871341197500000 Z^{10} y_2^3 y_4^{10} - 16796160 y_2^4 y_4^{10} - 140891691970575 \times 10^{21} Z^{34} y_3 y_4^{10}$$
$$- 291192432958125 \times 10^{10} Z^{19} y_1 y_3 y_4^{10} - 16334338500000 Z^4 y_1^2 y_3 y_4^{10}$$
$$+ 17298250674046875 \times 10^{11} Z^{24} y_2 y_3 y_4^{10} - 475524522534375000 Z^9 y_1 y_2 y_3 y_4^{10}$$
$$+ 1855889992125 \times 10^9 Z^{14} y_2^2 y_3 y_4^{10} - 25545034800000 Z^4 y_2^3 y_3 y_4^{10}$$
$$+ 53969586259640625 \times 10^{14} Z^{28} y_3^2 y_4^{10} + 162071830040625 \times 10^6 Z^{13} y_1 y_3^2 y_4^{10}$$
$$- 3883755099825 \times 10^{11} Z^{18} y_2 y_3^2 y_4^{10} - 4922742600000 Z^3 y_1 y_2 y_3^2 y_4^{10}$$
$$- 72028537908750000 Z^8 y_2^2 y_3^2 y_4^{10} - 1326464762175 \times 10^{14} Z^{22} y_3^3 y_4^{10}$$
$$+ 7694396955 \times 10^6 Z^7 y_1 y_3^3 y_4^{10} + 766965591675 \times 10^8 Z^{12} y_2 y_3^3 y_4^{10}$$
$$- 120022560000 Z^2 y_2^2 y_3^3 y_4^{10} - 11124214218 \times 10^{12} Z^{16} y_3^4 y_4^{10}$$
$$+ 1772928000 Z y_1 y_3^4 y_4^{10} + 650210044500000 Z^6 y_2 y_3^4 y_4^{10}$$
$$- 3862464669 \times 10^8 Z^{10} y_3^5 y_4^{10} + 18662400 y_2 y_3^5 y_4^{10} - 18662400000 Z^4 y_3^6 y_4^{10}$$

$$+\,218182093283671875 \times 10^{10} Z^{24} y_1 y_4^{11} + 45496430789062500 Z^9 y_1^2 y_4^{11}$$
$$+\,248077271433375 \times 10^{16} Z^{29} y_2 y_4^{11} + 1222518361148437500000 Z^{14} y_1 y_2 y_4^{11}$$
$$+\,122485193897625 \times 10^{10} Z^{19} y_2^2 y_4^{11} - 1874477700000 Z^4 y_1 y_2^2 y_4^{11}$$
$$-\,9556456626 \times 10^7 Z^9 y_2^3 y_4^{11} + 1780827790275 \times 10^{22} Z^{33} y_3 y_4^{11}$$
$$-\,1450084256015625 \times 10^8 Z^{18} y_1 y_3 y_4^{11} - 110079 \times 10^7 Z^3 y_1^2 y_3 y_4^{11}$$
$$+\,335614607465625 \times 10^{12} Z^{23} y_2 y_3 y_4^{11} - 67229703590625000 Z^8 y_1 y_2 y_3 y_4^{11}$$
$$+\,148438959496875 \times 10^6 Z^{13} y_2^2 y_3 y_4^{11} - 1227636 \times 10^6 Z^3 y_2^3 y_3 y_4^{11}$$
$$-\,44374058047125 \times 10^{16} Z^{27} y_3^2 y_4^{11} + 2695842486843750000 Z^{12} y_1 y_3^2 y_4^{11}$$
$$+\,55508120791875 \times 10^9 Z^{17} y_2 y_3^2 y_4^{11} - 211264200000 Z^2 y_1 y_2 y_3^2 y_4^{11}$$
$$-\,5154164865 \times 10^6 Z^7 y_2^2 y_3^2 y_4^{11} - 3787241022 \times 10^{15} Z^{21} y_3^3 y_4^{11}$$
$$+\,64209348 \times 10^7 Z^6 y_1 y_3^3 y_4^{11} + 11107464359625 \times 10^6 Z^{11} y_2 y_3^3 y_4^{11}$$
$$-\,3709152000 Z y_2^2 y_3^3 y_4^{11} - 1600788643425 \times 10^9 Z^{15} y_3^4 y_4^{11}$$
$$+\,18662400 y_1 y_3^4 y_4^{11} + 39874477500000 Z^5 y_2 y_3^4 y_4^{11} - 326526471 \times 10^8 Z^9 y_3^5 y_4^{11}$$
$$+\,2647023675465375 \times 10^{22} Z^{38} y_4^{12} - 188821155036065625 \times 10^{21} Z^{39} y_4^{11}$$
$$+\,5342479222359375 \times 10^{11} Z^{23} y_1 y_4^{12} + 1759968316406250 Z^8 y_1^2 y_4^{12}$$
$$+\,2838694254328125 \times 10^{14} Z^{28} y_2 y_4^{12} + 1221155694984375000000 Z^{13} y_1 y_2 y_4^{12}$$
$$+\,220294058281875 \times 10^9 Z^{18} y_2^2 y_4^{12} - 121743 \times 10^6 Z^3 y_1 y_2^2 y_4^{12}$$
$$-\,22899873335625000 Z^8 y_2^3 y_4^{12} - 233651910740625 \times 10^{19} Z^{32} y_3 y_4^{12}$$
$$+\,190711312396875 \times 10^8 Z^{17} y_1 y_3 y_4^{12} - 52706700000 Z^2 y_1^2 y_3 y_4^{12}$$
$$-\,5627670321375 \times 10^{13} Z^{22} y_2 y_3 y_4^{12} - 80857764 \times 10^8 Z^7 y_1 y_2 y_3 y_4^{12}$$
$$-\,7912472799375 \times 10^6 Z^{12} y_2^2 y_3 y_4^{12} - 43040160000 Z^2 y_2^3 y_3 y_4^{12}$$
$$+\,63587785255875 \times 10^{15} Z^{26} y_3^2 y_4^{12} + 4014596044125 \times 10^6 Z^{11} y_1 y_3^2 y_4^{12}$$
$$+\,214048085810625 \times 10^8 Z^{16} y_2 y_3^2 y_4^{12} - 6044868000 Z y_1 y_2 y_3^2 y_4^{12}$$
$$-\,377968032 \times 10^6 Z^6 y_2^2 y_3^2 y_4^{12} - 20230630540875 \times 10^{11} Z^{20} y_3^3 y_4^{12}$$
$$+\,46522775250000 Z^5 y_1 y_3^3 y_4^{12} + 128834526105 \times 10^7 Z^{10} y_2 y_3^3 y_4^{12}$$
$$-\,53654400 y_2^2 y_3^3 y_4^{12} - 16124378805 \times 10^{10} Z^{14} y_3^4 y_4^{12} + 2308852350000 Z^4 y_2 y_3^4 y_4^{12}$$
$$-\,221787315 \times 10^7 Z^8 y_3^5 y_4^{12} - 33307463851425 \times 10^{23} Z^{37} y_4^{13}$$
$$-\,98339742140625 \times 10^{11} Z^{22} y_1 y_4^{13} - 346710121875000 Z^7 y_1^2 y_4^{13}$$
$$+\,549121418775 \times 10^{17} Z^{27} y_2 y_4^{13} + 152991666 \times 10^{10} Z^{12} y_1 y_2 y_4^{13}$$
$$+\,163466664975 \times 10^{11} Z^{17} y_2^2 y_4^{13} - 6036120000 Z^2 y_1 y_2^2 y_4^{13}$$
$$-\,3180712657500000 Z^7 y_2^3 y_4^{13} + 236129569875 \times 10^{21} Z^{31} y_3 y_4^{13}$$
$$+\,7797516688125 \times 10^9 Z^{16} y_1 y_3 y_4^{13} - 1662120000 Z y_1^2 y_3 y_4^{13}$$
$$-\,174095638065 \times 10^{14} Z^{21} y_2 y_3 y_4^{13} - 838511716500000 Z^6 y_1 y_2 y_3 y_4^{13}$$
$$-\,4202606541375 \times 10^6 Z^{11} y_2^2 y_3 y_4^{13} - 979776000 Z y_2^3 y_3 y_4^{13}$$
$$-\,55567655397 \times 10^{17} Z^{25} y_3^2 y_4^{13} + 472527247725 \times 10^6 Z^{10} y_1 y_3^2 y_4^{13}$$

$$+ 356434859025 \times 10^{10} Z^{15} y_2 y_3^2 y_4^{13} - 84564000 y_1 y_2 y_3^2 y_4^{13}$$
$$- 30154356 \times 10^{6} Z^{5} y_2^2 y_3^2 y_4^{13} - 169272165825 \times 10^{12} Z^{19} y_3^3 y_4^{13}$$
$$+ 2834388450000 Z^{4} y_1 y_3^3 y_4^{13} + 117346350375 \times 10^{6} Z^{9} y_2 y_3^3 y_4^{13}$$
$$- 13377366675 \times 10^{9} Z^{13} y_3^4 y_4^{13} + 115254900000 Z^{3} y_2 y_3^4 y_4^{13}$$
$$- 1152549 \times 10^{8} Z^{7} y_3^5 y_4^{13} + 4118195050453125 \times 10^{20} Z^{36} y_4^{14}$$
$$- 9022981746375 \times 10^{12} Z^{21} y_1 y_4^{14} - 83604453750000 Z^{6} y_1^2 y_4^{14}$$
$$- 8851371562125 \times 10^{15} Z^{26} y_2 y_4^{14} - 1615446731812500000 Z^{11} y_1 y_2 y_4^{14}$$
$$- 696872028375 \times 10^{9} Z^{16} y_2^2 y_4^{14} - 201204000 Z y_1 y_2^2 y_4^{14}$$
$$- 320184454500000 Z^{6} y_2^3 y_4^{14} - 207618744375 \times 10^{20} Z^{30} y_3 y_4^{14}$$
$$+ 1263714719625 \times 10^{9} Z^{15} y_1 y_3 y_4^{14} - 26244000 y_1^2 y_3 y_4^{14}$$
$$- 17496461122875 \times 10^{11} Z^{20} y_2 y_3 y_4^{14} - 72734699250000 Z^{5} y_1 y_2 y_3 y_4^{14}$$
$$- 6796946439 \times 10^{8} Z^{10} y_2^2 y_3 y_4^{14} - 11664000 y_2^3 y_3 y_4^{14}$$
$$+ 991738009125 \times 10^{14} Z^{24} y_3^2 y_4^{14} + 43799196825 \times 10^{6} Z^{9} y_1 y_3^2 y_4^{14}$$
$$+ 405803031975 \times 10^{9} Z^{14} y_2 y_3^2 y_4^{14} - 2312752500000 Z^{4} y_2^2 y_3^2 y_4^{14}$$
$$- 29615926725 \times 10^{12} Z^{18} y_3^3 y_4^{14} + 137343600000 Z^{3} y_1 y_3^3 y_4^{14}$$
$$+ 83032371 \times 10^{8} Z^{8} y_2 y_3^3 y_4^{14} - 81628074 \times 10^{10} Z^{12} y_3^4 y_4^{14}$$
$$+ 4417740000 Z^{2} y_2 y_3^4 y_4^{14} - 441774 \times 10^{7} Z^{6} y_3^5 y_4^{14}$$
$$- 156105504196875 \times 10^{10} Z^{20} y_1 y_4^{15} - 10277533125000 Z^{5} y_1^2 y_4^{15}$$
$$- 8578694853 \times 10^{17} Z^{25} y_2 y_4^{15} - 305724135037500000 Z^{10} y_1 y_2 y_4^{15}$$
$$- 3372893217 \times 10^{11} Z^{15} y_2^2 y_4^{15} - 4082400 y_1 y_2^2 y_4^{15} - 24718567500000 Z^{5} y_2^3 y_4^{15}$$
$$+ 2571419925 \times 10^{21} Z^{29} y_3 y_4^{15} - 46287047176875 \times 10^{21} Z^{35} y_4^{15}$$
$$+ 1488937696875 \times 10^{8} Z^{14} y_1 y_3 y_4^{15} + 18558302175 \times 10^{12} Z^{19} y_2 y_3 y_4^{15}$$
$$- 5129717850000 Z^{4} y_1 y_2 y_3 y_4^{15} - 71750231325 \times 10^{6} Z^{9} y_2^2 y_3 y_4^{15}$$
$$- 810266814 \times 10^{17} Z^{23} y_3^2 y_4^{15} + 311964615 \times 10^{7} Z^{8} y_1 y_3^2 y_4^{15}$$
$$+ 34712009325 \times 10^{9} Z^{13} y_2 y_3^2 y_4^{15} - 144196200000 Z^{3} y_2^2 y_3^2 y_4^{15}$$
$$- 279718191 \times 10^{13} Z^{17} y_3^3 y_4^{15} + 4986360000 Z^{2} y_1 y_3^3 y_4^{15}$$
$$+ 445845465 \times 10^{6} Z^{7} y_2 y_3^3 y_4^{15} - 37899009 \times 10^{9} Z^{11} y_3^4 y_4^{15} + 113724000 Z y_2 y_3^4 y_4^{15}$$
$$- 113724 \times 10^{6} Z^{5} y_3^5 y_4^{15} + 4300140909375 \times 10^{21} Z^{34} y_4^{16}$$
$$- 705547789875 \times 10^{11} Z^{19} y_1 y_4^{16} - 876952828125 Z^{4} y_1^2 y_4^{16}$$
$$- 1536460477875 \times 10^{14} Z^{24} y_2 y_4^{16} - 34091592862500000 Z^{9} y_1 y_2 y_4^{16}$$
$$- 46034217675 \times 10^{9} Z^{14} y_2^2 y_4^{16} - 1478849400000 Z^{4} y_2^3 y_4^{16}$$
$$- 16354659375 \times 10^{19} Z^{28} y_3 y_4^{16} + 129930915375 \times 10^{8} Z^{13} y_1 y_3 y_4^{16}$$
$$+ 39427496325 \times 10^{12} Z^{18} y_2 y_3 y_4^{16} - 282560400000 Z^{3} y_1 y_2 y_3 y_4^{16}$$
$$- 550128915 \times 10^{7} Z^{8} y_2^2 y_3 y_4^{16} - 339313725 \times 10^{16} Z^{22} y_3^2 y_4^{16}$$
$$+ 168519285 \times 10^{6} Z^{7} y_1 y_3^2 y_4^{16} + 22125407175 \times 10^{8} Z^{12} y_2 y_3^2 y_4^{16}$$

$$
\begin{aligned}
&- 6561 \times 10^{6} Z^{2} y_2^{2} y_3^{2} y_4^{16} - 2655105075 \times 10^{11} Z^{16} y_3^{3} y_4^{16} \\
&+ 121378500 Z y_1 y_3^{3} y_4^{16} + 17446428 \times 10^{6} Z^{6} y_2 y_3^{3} y_4^{16} - 11563884 \times 10^{8} Z^{10} y_3^{4} y_4^{16} \\
&+ 1530900 y_2 y_3^{4} y_4^{16} - 1530900000 Z^{4} y_3^{5} y_4^{16} - 42071319 \times 10^{25} Z^{33} y_4^{17} \\
&+ 1101385845 \times 10^{13} Z^{18} y_1 y_4^{17} - 54743343750 Z^{3} y_1^{2} y_4^{17} \\
&+ 787001265 \times 10^{16} Z^{23} y_2 y_4^{17} - 268159005 \times 10^{7} Z^{8} y_1 y_2 y_4^{17} \\
&- 40105206 \times 10^{11} Z^{13} y_2^{2} y_4^{17} - 67359600000 Z^{3} y_2^{3} y_4^{17} \\
&+ 856575 \times 10^{22} Z^{27} y_3 y_4^{17} + 86932035 \times 10^{10} Z^{12} y_1 y_3 y_4^{17} \\
&+ 663954003 \times 10^{13} Z^{17} y_2 y_3 y_4^{17} - 11547360000 Z^{2} y_1 y_2 y_3 y_4^{17} \\
&- 311877135 \times 10^{6} Z^{7} y_2^{2} y_3 y_4^{17} - 49379175 \times 10^{16} Z^{21} y_3^{2} y_4^{17} \\
&+ 6573028500000 Z^{6} y_1 y_3^{2} y_4^{17} + 103927698 \times 10^{9} Z^{11} y_2 y_3^{2} y_4^{17} \\
&- 196830000 Z y_2^{2} y_3^{2} y_4^{17} - 174520818 \times 10^{11} Z^{15} y_3^{3} y_4^{17} + 1530900 y_1 y_3^{3} y_4^{17} \\
&+ 454896 \times 10^{6} Z^{5} y_2 y_3^{3} y_4^{17} - 180306 \times 10^{8} Z^{9} y_3^{4} y_4^{17} \\
&+ 403450621875 \times 10^{20} Z^{32} y_4^{18} + 288176859 \times 10^{13} Z^{17} y_1 y_4^{18} \\
&- 2460375000 Z^{2} y_1^{2} y_4^{18} + 141707475 \times 10^{16} Z^{22} y_2 y_4^{18} \\
&- 154484212500000 Z^{7} y_1 y_2 y_4^{18} - 2371661775 \times 10^{8} Z^{12} y_2^{2} y_4^{18} - 2230740000 Z^{2} y_2^{3} y_4^{18} \\
&- 172125 \times 10^{22} Z^{26} y_3 y_4^{18} + 4219452 \times 10^{10} Z^{11} y_1 y_3 y_4^{18} \\
&+ 7119553725 \times 10^{11} Z^{16} y_2 y_3 y_4^{18} - 318208500 Z y_1 y_2 y_3 y_4^{18} \\
&- 12722508 \times 10^{6} Z^{6} y_2^{2} y_3 y_4^{18} - 144311625 \times 10^{14} Z^{20} y_3^{2} y_4^{18} \\
&+ 170586 \times 10^{6} Z^{5} y_1 y_3^{2} y_4^{18} + 32339898 \times 10^{8} Z^{10} y_2 y_3^{2} y_4^{18} - 3061800 y_2^{2} y_3^{2} y_4^{18} \\
&- 891324 \times 10^{12} Z^{14} y_3^{3} y_4^{18} + 6123600000 Z^{4} y_2 y_3^{3} y_4^{18} \\
&- 2358871875 \times 10^{21} Z^{31} y_4^{19} + 33483364875 \times 10^{10} Z^{16} y_1 y_4^{19} \\
&- 73811250 Z y_1^{2} y_4^{19} + 25596675 \times 10^{16} Z^{21} y_2 y_4^{19} - 6343393500000 Z^{6} y_1 y_2 y_4^{19} \\
&- 9769329 \times 10^{9} Z^{11} y_2^{2} y_4^{19} - 48114000 Z y_2^{3} y_4^{19} - 27 \times 10^{24} Z^{25} y_3 y_4^{19} \\
&+ 13818195 \times 10^{8} Z^{10} y_1 y_3 y_4^{19} + 517325454 \times 10^{11} Z^{15} y_2 y_3 y_4^{19} - 4592700 y_1 y_2 y_3 y_4^{19} \\
&- 341172 \times 10^{6} Z^{5} y_2^{2} y_3 y_4^{19} - 80595 \times 10^{16} Z^{19} y_3^{2} y_4^{19} + 2296350000 Z^{4} y_1 y_3^{2} y_4^{19} \\
&+ 513702 \times 10^{8} Z^{9} y_2 y_3^{2} y_4^{19} - 300834 \times 10^{11} Z^{13} y_3^{3} y_4^{19} \\
&+ 1985175 \times 10^{23} Z^{30} y_4^{20} + 257103477 \times 10^{11} Z^{15} y_1 y_4^{20} \\
&- 1148175 y_1^{2} y_4^{20} + 144311625 \times 10^{14} Z^{20} y_2 y_4^{20} - 170586 \times 10^{6} Z^{5} y_1 y_2 y_4^{20} \\
&- 2351754 \times 10^{8} Z^{10} y_2^{2} y_4^{20} - 510300 y_2^{3} y_4^{20} - 63 \times 10^{23} Z^{24} y_3 y_4^{20} \\
&+ 229635 \times 10^{8} Z^{9} y_1 y_3 y_4^{20} + 2673972 \times 10^{12} Z^{14} y_2 y_3 y_4^{20} - 4592700000 Z^{4} y_2^{2} y_3 y_4^{20} \\
&- 945 \times 10^{16} Z^{18} y_3^{2} y_4^{20} - 5103 \times 10^{11} Z^{12} y_3^{3} y_4^{20} \\
&- 17775 \times 10^{24} Z^{29} y_4^{21} + 1336986 \times 10^{12} Z^{14} y_1 y_4^{21} \\
&+ 80595 \times 10^{16} Z^{19} y_2 y_4^{21} - 2296350000 Z^{4} y_1 y_2 y_4^{21} - 27216 \times 10^{8} Z^{9} y_2^{2} y_4^{21} \\
&+ 902502 \times 10^{11} Z^{13} y_2 y_3 y_4^{21} + 15 \times 10^{24} Z^{28} y_4^{22} \\
&+ 451251 \times 10^{11} Z^{13} y_1 y_4^{22} + 945 \times 10^{16} Z^{18} y_2 y_4^{22}
\end{aligned}
$$

$$+ 15309 \times 10^{11} Z^{12} y_2 y_3 y_4^{22} - 7 \times 10^{25} Z^{27} y_4^{23} + 76545 \times 10^{10} Z^{12} y_1 y_4^{23} = 0. \tag{A.2.16}$$

*Proof.* Formula (A.2.15) follows immediately from Theorem 4.5.15. Using relations (4.5.74) and (4.5.84) we get formulas (A.2.12) and (A.2.13). Substituting relations (A.2.12), (A.2.13) and (A.2.15) into relations (4.5.82) and (4.5.83), we get expressions $P_1$ and $P_2$, both equal to zero and polynomial in $t_3$, of degrees 5 and 3, respectively. We can reduce these expressions modulo one another repeatedly, to get a linear expression in $t_3$ which is equal to zero, which we then rearrange to get relation (A.2.14). To find formula (A.2.16) we then compute the resultant of $P_1$ and $P_2$ with respect to $t_3$. □

**Proposition A.2.4.** *The unity vector field $e = \partial_{t_1}$ in the $y$ coordinates has the form*

$$\begin{aligned} e(y) = {} & 56000\partial_{y_3} + 448000\left(125t_2 + 4256t_3t_4 - 5616t_4^4\right)\partial_{y_2} \\ & + \frac{57344000}{9}\big(31250Zt_2^2 - 1764000t_1t_3 - 192500Z^2t_2t_3 \\ & + 392000Z^3t_3^2 + 5092864t_3^3 - 421875t_2^2t_4 - 560000Zt_2t_3t_4 \\ & + 1724800Z^2t_3^2t_4 - 9828000t_2t_3t_4^2 + 2508800Zt_3^2t_4^2 + 1764000t_1t_4^3 \\ & + 192500Z^2t_2t_4^3 - 784000Z^3t_3t_4^3 - 136754688t_3^2t_4^3 - 120000Zt_2t_4^4 \\ & - 1355200Z^2t_3t_4^4 + 19008000t_2t_4^5 + 1075200Zt_3t_4^5 + 392000Z^3t_4^6 \\ & + 520310784t_3t_4^6 - 369600Z^2t_4^7 + 115200Zt_4^8 - 438588160t_4^9\big)\partial_{y_1}. \end{aligned}$$

*Proof.* We have that

$$e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 4.5.15. □

## A.3 Extra formulas for $H_4(4)$

Recall from Section 4.5.4 that the degrees of the $t$ coordinates and $Z$ are $\deg t_1(x) = 10$, $\deg t_2(x) = 10$, $\deg t_3(x) = 2$, $\deg t_4(x) = 2$ and $\deg Z(x) = 2$. This allows us to deduce which harmonic polynomials of the $t$ coordinates and $Z$ have the same degrees as the basic invariants of $H_4$, which the following Proposition makes precise.

**Proposition A.3.1.** *Let $V_1 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 30\}$, let $V_2 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 20\}$ and let $V_3 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 12\}$. The harmonic elements of $V_1$ are proportional to*

$$\begin{aligned} & 6006t_1^2t_2 - 24024t_1t_2^2 + 24024t_2^3 - 30030Z^4t_1t_2t_3 + 60060Z^4t_2^2t_3 + 150150Z^3t_2^2t_3^2 \\ & + 300300Z^2t_1t_2t_3^3 - 600600Z^2t_2^2t_3^3 + 3903900Zt_2^2t_3^4 + 12762750t_1t_2t_3^5 - 25525500t_2^2t_3^5 \\ & + 32692660Z^4t_2t_3^6 + 16336320Z^3t_1t_3^7 - 32672640Z^3t_2t_3^7 + 352592240Z^2t_2t_3^8 \end{aligned}$$

$$- 20180160Zt_1t_3^9 + 40360320Zt_2t_3^9 - 810471662t_2t_3^{10} - 1928006080Z^3t_3^{12}$$
$$+ 6093447360Zt_3^{14} - 60060Z^4t_1t_2t_4 + 150150Z^4t_2^2t_4 - 270270Z^3t_1t_2t_3t_4$$
$$+ 1141140Z^3t_2^2t_3t_4 + 1801800Z^2t_1t_2t_3^2t_4 - 2972970Z^2t_2^2t_3^2t_4 - 14954940Zt_1t_2t_3^3t_4$$
$$+ 61141080Zt_2^2t_3^3t_4 - 23513490t_1^2t_3^4t_4 + 221681460t_1t_2t_3^4t_4 - 333843510t_2^2t_3^4t_4$$
$$- 6666660Z^4t_1t_3^5t_4 + 405645240Z^4t_2t_3^5t_4 + 228708480Z^3t_1t_3^6t_4 - 320420100Z^3t_2t_3^6t_4$$
$$- 2162160Z^2t_1t_3^7t_4 + 5645800160Z^2t_2t_3^7t_4 - 363242880Zt_1t_3^8t_4 + 5537772240Zt_2t_3^8t_4$$
$$+ 4077473400t_1t_3^9t_4 - 24364380040t_2t_3^9t_4 + 2368345980Z^4t_3^{10}t_4 - 46272145920Z^3t_3^{11}t_4$$
$$- 9255486240Z^2t_3^{12}t_4 + 170616526080Zt_3^{13}t_4 - 4188854670t_3^{14}t_4 - 540540Z^3t_1t_2t_4^2$$
$$+ 1261260Z^3t_2^2t_4^2 + 3303300Z^2t_1t_2t_3t_4^2 - 4084080Z^2t_2^2t_3t_4^2 - 89729640Zt_1t_2t_3^2t_4^2$$
$$+ 270570300Zt_2^2t_3^2t_4^2 - 188107920t_1^2t_3^3t_4^2 + 1227686460t_1t_2t_3^3t_4^2 - 1579217640t_2^2t_3^3t_4^2$$
$$- 66666600Z^4t_1t_3^4t_4^2 + 2261859600Z^4t_2t_3^4t_4^2 + 1387836450Z^3t_1t_3^5t_4^2 - 1131710580Z^3t_2t_3^5t_4^2$$
$$- 30270240Z^2t_1t_3^6t_4^2 + 41454553140Z^2t_2t_3^6t_4^2 - 2513330820Zt_1t_3^7t_4^2 + 82007245320Zt_2t_3^7t_4^2$$
$$+ 73394521200t_1t_3^8t_4^2 - 279679880480t_2t_3^8t_4^2 + 47366919600Z^4t_3^9t_4^2 - 490358278410Z^3t_3^{10}t_4^2$$
$$- 222131669760Z^2t_3^{11}t_4^2 + 2105167084020Zt_3^{12}t_4^2 - 117287930760t_3^{13}t_4^2 + 1801800Z^2t_1t_2t_4^3$$
$$- 502645Z^2t_2^2t_4^3 - 178343880Zt_1t_2t_3t_4^3 + 471282240Zt_2^2t_3t_4^3 - 550887480t_1^2t_3^2t_4^3$$
$$+ 3013038600t_1t_2t_3^2t_4^3 - 3495396190t_2^2t_3^2t_4^3 - 253187220Z^4t_1t_3^3t_4^3 + 7072934440Z^4t_2t_3^3t_4^3$$
$$+ 4730025300Z^3t_1t_3^4t_4^3 - 1707576585Z^3t_2t_3^4t_4^3 - 552225960Z^2t_1t_3^5t_4^3 + 181909956800Z^2t_2t_3^5t_4^3$$
$$- 8064496440Zt_1t_3^6t_4^3 + 557051053130Zt_2t_3^6t_4^3 + 571892564100t_1t_3^7t_4^3 - 1713932946440t_2t_3^7t_4^3$$
$$+ 420318640885Z^4t_3^8t_4^3 - 3020584166600Z^3t_3^9t_4^3 - 2485406463540Z^2t_3^{10}t_4^3$$
$$+ 15035772591840Zt_3^{11}t_4^3 - 1003720647215t_3^{12}t_4^3 - 117408720Zt_1t_2t_4^4 + 299467740Zt_2^2t_4^4$$
$$- 698686560t_1^2t_3t_4^4 + 3428323470t_1t_2t_3t_4^4 - 3743165140t_2^2t_3t_4^4 - 452457720Z^4t_1t_3^2t_4^4$$
$$+ 13120844880Z^4t_2t_3^2t_4^4 + 9946304940Z^3t_1t_3^3t_4^4 - 1711812960Z^3t_2t_3^3t_4^4 - 4311450000Z^2t_1t_3^4t_4^4$$
$$+ 517300779710Z^2t_2t_3^4t_4^4 - 5445194040Zt_1t_3^5t_4^4 + 2191042425000Zt_2t_3^5t_4^4$$
$$+ 2526371647800t_1t_3^6t_4^4 - 6387651109840t_2t_3^6t_4^4 + 2177873972560Z^4t_3^7t_4^4$$
$$- 11940681712245Z^3t_3^8t_4^4 - 17128817706000Z^2t_3^9t_4^4 + 67149209560290Zt_3^{10}t_4^4$$
$$+ 306594064920t_3^{11}t_4^4 - 325623078t_1^2t_4^5 + 1500098292t_1t_2t_4^5 - 1666279956t_2^2t_4^5$$
$$- 375151920Z^4t_1t_3t_4^5 + 13820183520Z^4t_2t_3t_4^5 + 13272110280Z^3t_1t_3^2t_4^5$$
$$- 5755392390Z^3t_2t_3^2t_4^5 - 16253780400Z^2t_1t_3^3t_4^5 + 965371252560Z^2t_2t_3^3t_4^5$$
$$+ 51150836880Zt_1t_3^4t_4^5 + 5454942264750Zt_2t_3^4t_4^5 + 7012595618820t_1t_3^5t_4^5$$
$$- 15544189806728t_2t_3^5t_4^5 + 7277652650400Z^4t_3^6t_4^5 - 31377190313040Z^3t_3^7t_4^5$$
$$- 81501909373995Z^2t_3^8t_4^5 + 178725842192040Zt_3^9t_4^5 + 67636787989386t_3^{10}t_4^5$$
$$- 112638240Z^4t_1t_4^6 + 6402120780Z^4t_2t_4^6 + 10642687770Z^3t_1t_3t_4^6 - 13434045900Z^3t_2t_3t_4^6$$
$$- 32414073120Z^2t_1t_3^2t_4^6 + 1140453023160Z^2t_2t_3^2t_4^6 + 200486346060Zt_1t_3^3t_4^6$$
$$+ 8911125490520Zt_2t_3^3t_4^6 + 12912084273000t_1t_3^4t_4^6 - 26251989629580t_2t_3^4t_4^6$$
$$+ 16299801295360Z^4t_3^5t_4^6 - 54123882941260Z^3t_3^6t_4^6 - 285186832252080Z^2t_3^7t_4^6$$

$$+ 159095532391730Zt_3^8t_4^6 + 592155868722440t_3^9t_4^6 + 3951441780Z^3t_1t_4^7 - 9992662845Z^3t_2t_4^7$$
$$- 33273634680Z^2t_1t_3t_4^7 + 769257944400Z^2t_2t_3t_4^7 + 344144366280Zt_1t_3^2t_4^7$$
$$+ 9532029191190Zt_2t_3^2t_4^7 + 16402978515660t_1t_3^3t_4^7 - 33218368469880t_2t_3^3t_4^7$$
$$+ 24477176331850Z^4t_3^4t_4^7 - 55258227121680Z^3t_3^5t_4^7 - 761370806081820Z^2t_3^6t_4^7$$
$$- 824238595771360Zt_3^7t_4^7 + 2816877685314825t_3^8t_4^7 - 13948711920Z^2t_1t_4^8$$
$$+ 222985549710Z^2t_2t_4^8 + 295401780960Zt_1t_3t_4^8 + 6261424455000Zt_2t_3t_4^8$$
$$+ 14784121087560t_1t_3^2t_4^8 - 33394516435170t_2t_3^2t_4^8 + 23584597219920Z^4t_3^3t_4^8$$
$$- 16056960750270Z^3t_3^4t_4^8 - 1574527351650000Z^2t_3^5t_4^8 - 4294385945261940Zt_3^6t_4^8$$
$$+ 8714684199695760t_3^7t_4^8 + 102536628480Zt_1t_4^9 + 1965495743890Zt_2t_4^9$$
$$+ 9101820663060t_1t_3t_4^9 - 24489566932080t_2t_3t_4^9 + 12721193619500Z^4t_3^2t_4^9$$
$$+ 39699237296400Z^3t_3^3t_4^9 - 2503422809509530Z^2t_3^4t_4^9 - 10973606731634800Zt_3^5t_4^9$$
$$+ 18532995357314350t_3^6t_4^9 + 2913828234600t_1t_4^{10} - 9135558521076t_2t_4^{10}$$
$$+ 2175752924720Z^4t_3t_4^{10} + 62265514312950Z^3t_3^2t_4^{10} - 2963428452131280Z^2t_3^3t_4^{10}$$
$$- 18352815805555120Zt_3^4t_4^{10} + 27491356446813864t_3^5t_4^{10} - 670393575975Z^4t_4^{11}$$
$$+ 40062606209880Z^3t_3t_4^{11} - 2453257188097320Z^2t_3^2t_4^{11} - 21135427360218240Zt_3^3t_4^{11}$$
$$+ 28131773838568515t_3^4t_4^{11} + 10470567594995Z^3t_4^{12} - 1262535017225760Z^2t_3t_4^{12}$$
$$- 16440674522572910Zt_3^2t_4^{12} + 18989075206985880t_3^3t_4^{12} - 303171654680595Z^2t_4^{13}$$
$$- 7881409599514040Zt_3t_4^{13} + 7356083542043190t_3^2t_4^{13} - 1769210991401910Zt_4^{14}$$
$$+ 720270845116440t_3t_4^{14} - 439442490606461t_4^{15},$$

*the harmonic elements of $V_2$ are proportional to*

$$187t_1^2 - 748t_1t_2 + 561t_2^2 - 8415Z^3t_2t_3^2 - 151470Zt_2t_3^4 - 191862t_1t_3^5 + 383724t_2t_3^5$$
$$- 75735Z^4t_3^6 - 757350Z^2t_3^8 + 1878228t_3^{10} + 1870Z^4t_2t_4 - 33660Z^3t_2t_3t_4 - 18700Z^2t_2t_3^2t_4$$
$$- 1211760Zt_2t_3^3t_4 - 1918620t_1t_3^4t_4 + 2285140t_2t_3^4t_4 - 908820Z^4t_3^5t_4 - 1698895Z^3t_3^6t_4$$
$$- 12117600Z^2t_3^7t_4 + 6255150Zt_3^8t_4 + 37564560t_3^9t_4 - 42075Z^3t_2t_4^2 - 74800Z^2t_2t_3t_4^2$$
$$- 3006960Zt_2t_3^2t_4^2 - 5755860t_1t_3^3t_4^2 - 905080t_2t_3^3t_4^2 - 4204695Z^4t_3^4t_4^2 - 20386740Z^3t_3^5t_4^2$$
$$- 95594400Z^2t_3^6t_4^2 + 100082400Zt_3^7t_4^2 + 234430680t_3^8t_4^2 - 56100Z^2t_2t_4^3 - 2333760Zt_2t_3t_4^3$$
$$- 3837240t_1t_3^2t_4^3 - 30578240t_2t_3^2t_4^3 - 9402360Z^4t_3^3t_4^3 - 102979965Z^3t_3^4t_4^3 - 468547200Z^2t_3^5t_4^3$$
$$+ 710057700Zt_3^6t_4^3 + 144693120t_3^7t_4^3 + 302170Zt_2t_4^4 + 7843770t_1t_3t_4^4 - 69364020t_2t_3t_4^4$$
$$- 10318825Z^4t_3^2t_4^4 - 280193320Z^3t_3^3t_4^4 - 1507905300Z^2t_3^4t_4^4 + 2916078000Zt_3^5t_4^4$$
$$- 5267590460t_3^6t_4^4 + 9547956t_1t_4^5 - 49353876t_2t_4^5 - 4832740Z^4t_3t_4^5 - 436114965Z^3t_3^2t_4^5$$
$$- 3187773600Z^2t_3^3t_4^5 + 7456862820Zt_3^4t_4^5 - 30924485328t_3^5t_4^5 - 746625Z^4t_4^6$$
$$- 372747540Z^3t_3t_4^6 - 4277277840Z^2t_3^2t_4^6 + 11784099360Zt_3^3t_4^6 - 90377429340t_3^4t_4^6$$
$$- 137914535Z^3t_4^7 - 3325988160Z^2t_3t_4^7 + 10667486500Zt_3^2t_4^7 - 162880356320t_3^3t_4^7$$
$$- 1151667990Z^2t_4^8 + 4493560720Zt_3t_4^8 - 190524418920t_3^2t_4^8 + 378718230Zt_4^9$$

$$-138727401120t_3t_4^9-48850558828t_4^{10},$$

*and the harmonic elements of $V_3$ are proportional to*

$$\begin{aligned}&-7Zt_2-42t_1t_3+84t_2t_3+7Z^4t_3^2-42Z^2t_3^4-2079t_3^6-84t_1t_4+210t_2t_4+28Z^4t_3t_4\\&+84Z^3t_3^2t_4-336Z^2t_3^3t_4+105Zt_3^4t_4-24948t_3^5t_4+21Z^4t_4^2+336Z^3t_3t_4^2-924Z^2t_3^2t_4^2\\&+840Zt_3^3t_4^2-156555t_3^4t_4^2+364Z^3t_4^3-1008Z^2t_3t_4^3+3710Zt_3^2t_4^3-587160t_3^3t_4^3\\&-378Z^2t_4^4+8120Zt_3t_4^4-1165605t_3^2t_4^4+5901Zt_4^5-1029588t_3t_4^5-239377t_4^6.\end{aligned}$$

*Proof.* Using Proposition 4.0.1 we can directly calculate

$$\begin{aligned}\Delta(t_1)=&-36\left(-Z^4t_2^4-4Z^3t_2^4t_3+34Z^2t_2^4t_3^2+18Zt_2^4t_3^3-192t_2^4t_3^4-210Z^4t_2^3t_3^5\right.\\&+1344Z^3t_2^3t_3^6-786Z^2t_2^3t_3^7+4800Zt_2^3t_3^8+7776t_2^3t_3^9-59904Z^4t_2^2t_3^{10}\\&-86688Z^3t_2^2t_3^{11}+133632Z^2t_2^2t_3^{12}-10080Zt_2^2t_3^{13}+147456t_2^2t_3^{14}\\&-48384Z^4t_2t_3^{15}-573696Z^2t_2t_3^{17}+442368Zt_2t_3^{18}+221184t_2t_3^{19}\\&+2211840Z^4t_3^{20}+4423680Z^3t_3^{21}-13271040Z^2t_3^{22}+3317760Zt_3^{23}\\&-16Z^3t_2^4t_4+122Z^2t_2^4t_3t_4+264Zt_2^4t_3^2t_4-1236t_2^4t_3^3t_4-4776Z^4t_2^3t_3^4t_4\\&+11484Z^3t_2^3t_3^5t_4+8268Z^2t_2^3t_3^6t_4+73842Zt_2^3t_3^7t_4+153792t_2^3t_3^8t_4\\&-1197270Z^4t_2^2t_3^9t_4-2149632Z^3t_2^2t_3^{10}t_4+2875626Z^2t_2^2t_3^{11}t_4\\&-1291392Zt_2^2t_3^{12}t_4+4247712t_2^2t_3^{13}t_4+2889216Z^4t_2t_3^{14}t_4\\&+8078400Z^3t_2t_3^{15}t_4-42094080Z^2t_2t_3^{16}t_4+17509824Zt_2t_3^{17}t_4\\&+7520256t_2t_3^{18}t_4+88812288Z^4t_3^{19}t_4+194641920Z^3t_3^{20}t_4\\&-570841344Z^2t_3^{21}t_4+172523520Zt_3^{22}t_4-27205632t_3^{23}t_4+102Z^2t_2^4t_4^2\\&+714Zt_2^4t_3t_4^2-1860t_2^4t_3^2t_4^2-29004Z^4t_2^3t_3^3t_4^2+18744Z^3t_2^3t_3^4t_4^2\\&+153558Z^2t_2^3t_3^5t_4^2+484092Zt_2^3t_3^6t_4^2+1396512t_2^3t_3^7t_4^2-10822914Z^4t_2^2t_3^8t_4^2\\&-23792076Z^3t_2^2t_3^9t_4^2+26602920Z^2t_2^2t_3^{10}t_4^2-28953126Zt_2^2t_3^{11}t_4^2\\&+60384960t_2^2t_3^{12}t_4^2+100169514Z^4t_2t_3^{13}t_4^2+259668288Z^3t_2t_3^{14}t_4^2\\&-1025223318Z^2t_2t_3^{15}t_4^2+371155392Zt_2t_3^{16}t_4^2+86181408t_2t_3^{17}t_4^2\\&+1652410368Z^4t_3^{18}t_4^2+3990066912Z^3t_3^{19}t_4^2-11481108480Z^2t_3^{20}t_4^2\\&+4270685472Zt_3^{21}t_4^2-1329758208t_3^{22}t_4^2+448Zt_2^4t_4^3+976t_2^4t_3t_4^3\\&-72896Z^4t_2^3t_3^2t_4^3-86296Z^3t_2^3t_3^3t_4^3+773268Z^2t_2^3t_3^4t_4^3+1700322Zt_2^3t_3^5t_4^3\\&+7730640t_2^3t_3^6t_4^3-58317912Z^4t_2^2t_3^7t_4^3-155545992Z^3t_2^2t_3^8t_4^3\\&+133880850Z^2t_2^2t_3^9t_4^3-325362348Zt_2^2t_3^{10}t_4^3+544989240t_2^2t_3^{11}t_4^3\\&+1418183856Z^4t_2t_3^{12}t_4^3+3952185156Z^3t_2t_3^{13}t_4^3-13873560012Z^2t_2t_3^{14}t_4^3\\&+5330895174Zt_2t_3^{15}t_4^3-164915136t_2t_3^{16}t_4^3+18922259502Z^4t_3^{17}t_4^3\\&+50595829632Z^3t_3^{18}t_4^3-143299154322Z^2t_3^{19}t_4^3+66549669120Zt_3^{20}t_4^3\end{aligned}$$

$$
\begin{aligned}
&- 30436289568t_3^{21}t_4^3 + 2416t_2^4t_4^4 - 80010Z^4t_2^3t_3t_4^4 - 355664Z^3t_2^3t_3^2t_4^4 \\
&+ 1816674Z^2t_2^3t_3^3t_4^4 + 3147876Zt_2^3t_3^4t_4^4 + 28356624t_2^3t_3^5t_4^4 \\
&- 206603928Z^4t_2^2t_3^6t_4^4 - 666405072Z^3t_2^2t_3^7t_4^4 + 350015040Z^2t_2^2t_3^8t_4^4 \\
&- 2295376290Zt_2^2t_3^9t_4^4 + 3407613768t_2^2t_3^{10}t_4^4 + 12248285940Z^4t_2t_3^{11}t_4^4 \\
&+ 37797166728Z^3t_2t_3^{12}t_4^4 - 124444340118Z^2t_2t_3^{13}t_4^4 + 55895208084Zt_2t_3^{14}t_4^4 \\
&- 16751845440t_2t_3^{15}t_4^4 + 148043564811Z^4t_3^{16}t_4^4 + 441724543344Z^3t_3^{17}t_4^4 \\
&- 1234193348682Z^2t_3^{18}t_4^4 + 728117290188Zt_3^{19}t_4^4 - 433358173440t_3^{20}t_4^4 \\
&- 30840Z^4t_2^3t_4^5 - 448372Z^3t_2^3t_3t_4^5 + 2110500Z^2t_2^3t_3^2t_4^5 + 2005230Zt_2^3t_3^3t_4^5 \\
&+ 69040320t_2^3t_3^4t_4^5 - 499176612Z^4t_2^2t_3^5t_4^5 - 1947842208Z^3t_2^2t_3^6t_4^5 \\
&+ 66947364Z^2t_2^2t_3^7t_4^5 - 11157619356Zt_2^2t_3^8t_4^5 + 15364875312t_2^2t_3^9t_4^5 \\
&+ 73406293200Z^4t_2t_3^{10}t_4^5 + 253909851336Z^3t_2t_3^{11}t_4^5 - 805582960164Z^2t_2t_3^{12}t_4^5 \\
&+ 444122884062Zt_2t_3^{13}t_4^5 - 242312938992t_2t_3^{14}t_4^5 + 820303848000Z^4t_3^{15}t_4^5 \\
&+ 2772771903576Z^3t_3^{16}t_4^5 - 7645233170700Z^2t_3^{17}t_4^5 + 5899857314868Zt_3^{18}t_4^5 \\
&- 4288857872004t_3^{19}t_4^5 - 183432Z^3t_2^3t_4^6 + 1040922Z^2t_2^3t_3t_4^6 - 2557708Zt_2^3t_3^2t_4^6 \\
&+ 107016832t_2^3t_3^3t_4^6 - 827006076Z^4t_2^2t_3^4t_4^6 - 3890997288Z^3t_2^2t_3^5t_4^6 \\
&- 3050723088Z^2t_2^2t_3^6t_4^6 - 39071190924Zt_2^2t_3^7t_4^6 + 50907528288t_2^2t_3^8t_4^6 \\
&+ 324165507678Z^4t_2t_3^9t_4^6 + 1267793785584Z^3t_2t_3^{10}t_4^6 - 3945198687894Z^2t_2t_3^{11}t_4^6 \\
&+ 2741055117084Zt_2t_3^{12}t_4^6 - 2089080278928t_2t_3^{13}t_4^6 + 3136791653928Z^4t_3^{14}t_4^6 \\
&+ 12463828235136Z^3t_3^{15}t_4^6 - 33754252023990Z^2t_3^{16}t_4^6 + 36252281863512Zt_3^{17}t_4^6 \\
&- 31084897849380t_3^{18}t_4^6 + 84636Z^2t_2^3t_4^7 - 4823490Zt_2^3t_3t_4^7 + 97269200t_2^3t_3^2t_4^7 \\
&- 925645464Z^4t_2^2t_3^3t_4^7 - 5130026928Z^3t_2^2t_3^4t_4^7 - 12286343196Z^2t_2^2t_3^5t_4^7 \\
&- 100490508216Zt_2^2t_3^6t_4^7 + 124534746576t_2^2t_3^7t_4^7 + 1087804797984Z^4t_2t_3^8t_4^7 \\
&+ 4853503086444Z^3t_2t_3^9t_4^7 - 15025018608252Z^2t_2t_3^{10}t_4^7 + 13372130414934Zt_2t_3^{11}t_4^7 \\
&- 12734996879808t_2t_3^{12}t_4^7 + 6758811379800Z^4t_3^{13}t_4^7 + 36462537218880Z^3t_3^{14}t_4^7 \\
&- 94458578130360Z^2t_3^{15}t_4^7 + 168423705577692Zt_3^{16}t_4^7 - 167945426241408t_3^{17}t_4^7 \\
&- 2093556Zt_2^3t_4^8 + 43848880t_2^3t_3t_4^8 - 676846008Z^4t_2^2t_3^2t_4^8 - 4056772464Z^3t_2^2t_3^3t_4^8 \\
&- 26294031648Z^2t_2^2t_3^4t_4^8 - 190330344468Zt_2^2t_3^5t_4^8 + 223551438192t_2^2t_3^6t_4^8 \\
&+ 2814565578552Z^4t_2t_3^7t_4^8 + 14481856433592Z^3t_2t_3^8t_4^8 - 45204348324294Z^2t_2t_3^9t_4^8 \\
&+ 52172622104292Zt_2t_3^{10}t_4^8 - 58554722936160t_2t_3^{11}t_4^8 - 6030614169180Z^4t_3^{12}t_4^8 \\
&+ 31780501730496Z^3t_3^{13}t_4^8 - 46054025164872Z^2t_3^{14}t_4^8 + 562679797881456Zt_3^{15}t_4^8 \\
&- 662201660188368t_3^{16}t_4^8 + 6114720t_2^3t_4^9 - 302139126Z^4t_2^2t_3t_4^9 - 1413945504Z^3t_2^2t_3^2t_4^9 \\
&- 34813542510Z^2t_2^2t_3^3t_4^9 - 262436016312Zt_2^2t_3^4t_4^9 + 290536662816t_2^2t_3^5t_4^9 \\
&+ 5634132784416Z^4t_2t_3^6t_4^9 + 33894149886576Z^3t_2t_3^7t_4^9 - 108262063044468Z^2t_2t_3^8t_4^9 \\
&+ 163887061538430Zt_2t_3^9t_4^9 - 209609881026192t_2t_3^{10}t_4^9 - 126459466609824Z^4t_3^{11}t_4^9 \\
&- 341633310630624Z^3t_3^{12}t_4^9 + 1175029899078816Z^2t_3^{13}t_4^9 + 1065315775197456Zt_3^{14}t_4^9
\end{aligned}
$$

$$- 1685181960939312t_3^{15}t_4^{9} - 66625362Z^4t_2^2t_4^{10} + 216602388Z^3t_2^2t_3t_4^{10}$$
$$- 28393260120Z^2t_2^2t_3^2t_4^{10} - 256483120014Zt_2^2t_3^3t_4^{10} + 270490458528t_2^2t_3^4t_4^{10}$$
$$+ 8672982663942Z^4t_2t_3^5t_4^{10} + 62115902197344Z^3t_2t_3^6t_4^{10} - 206673052407282Z^2t_2t_3^7t_4^{10}$$
$$+ 415311387036876Zt_2t_3^8t_4^{10} - 593811454155120t_2t_3^9t_4^{10} - 635145364945800Z^4t_3^{10}t_4^{10}$$
$$- 2415496670521920Z^3t_3^{11}t_4^{10} + 7766438641901352Z^2t_3^{12}t_4^{10} - 1264623321174912Zt_3^{13}t_4^{10}$$
$$- 802208042241840t_3^{14}t_4^{10} + 227476728Z^3t_2^2t_4^{11} - 12989014614Z^2t_2^2t_3t_4^{11}$$
$$- 169099219548Zt_2^2t_3^2t_4^{11} + 184241102904t_2^2t_3^3t_4^{11} + 10102220234352Z^4t_2t_3^4t_4^{11}$$
$$+ 88290087998940Z^3t_2t_3^5t_4^{11} - 313153231112676Z^2t_2t_3^6t_4^{11} + 846802263705570Zt_2t_3^7t_4^{11}$$
$$- 1340858076916320t_2t_3^8t_4^{11} - 2103892611566364Z^4t_3^9t_4^{11} - 9655910137578816Z^3t_3^{10}t_4^{11}$$
$$+ 31053184648866324Z^2t_3^{11}t_4^{11} - 20414903118300432Zt_3^{12}t_4^{11} + 17376993106483104t_3^{13}t_4^{11}$$
$$- 2512447200Z^2t_2^2t_4^{12} - 68189569578Zt_2^2t_3t_4^{12} + 100378152072t_2^2t_3^2t_4^{12}$$
$$+ 8640668849940Z^4t_2t_3^3t_4^{12} + 95533597925496Z^3t_2t_3^4t_4^{12} - 372823053217938Z^2t_2t_3^5t_4^{12}$$
$$+ 1378618176182076Zt_2t_3^6t_4^{12} - 2414536296753024t_2t_3^7t_4^{12} - 5228842101892686Z^4t_3^8t_4^{12}$$
$$- 28124473615972704Z^3t_3^9t_4^{12} + 92357585288556660Z^2t_3^{10}t_4^{12}$$
$$- 98733562102330200Zt_3^{11}t_4^{12} + 107161261746496416t_3^{12}t_4^{12} - 12960439020Zt_2^2t_4^{13}$$
$$+ 45310744656t_2^2t_3t_4^{13} + 5141575064592Z^4t_2t_3^2t_4^{13} + 76240096519176Z^3t_2t_3^3t_4^{13}$$
$$- 342652355682156Z^2t_2t_3^4t_4^{13} + 1767751364936394Zt_2t_3^5t_4^{13} - 3449165671334160t_2t_3^6t_4^{13}$$
$$- 10137828083457024Z^4t_3^7t_4^{13} - 63668976430996080Z^3t_3^8t_4^{13}$$
$$+ 216579276727651656Z^2t_3^9t_4^{13} - 325404069685097256Zt_3^{10}t_4^{13}$$
$$+ 399521317960772136t_3^{11}t_4^{13} + 11797674624t_2^2t_4^{14} + 1911846773682Z^4t_2t_3t_4^{14}$$
$$+ 42480919012464Z^3t_2t_3^2t_4^{14} - 236013724178658Z^2t_2t_3^3t_4^{14} + 1746727825731156Zt_2t_3^4t_4^{14}$$
$$- 3864462628081392t_2t_3^5t_4^{14} - 15496951324333608Z^4t_3^6t_4^{14} - 114463129473201024Z^3t_3^7t_4^{14}$$
$$+ 409112679136988268Z^2t_3^8t_4^{14} - 823358866020487632Zt_3^9t_4^{14}$$
$$+ 1109010438480097512t_3^{10}t_4^{14} + 336506588352Z^4t_2t_4^{15} + 14850919490292Z^3t_2t_3t_4^{15}$$
$$- 115581512008404Z^2t_2t_3^2t_4^{15} + 1284763207472514Zt_2t_3^3t_4^{15} - 3330792797156352t_2t_3^4t_4^{15}$$
$$- 18595241424572040Z^4t_3^5t_4^{15} - 164012600156150592Z^3t_3^6t_4^{15}$$
$$+ 626271650112278088Z^2t_3^7t_4^{15} - 1660736557574787384Zt_3^8t_4^{15}$$
$$+ 2427462924699617568t_3^9t_4^{15} + 2471938417224Z^3t_2t_4^{16}$$
$$- 36193444367634Z^2t_2t_3t_4^{16} + 663776040547404Zt_2t_3^2t_4^{16}$$
$$- 2138341695376608t_2t_3^3t_4^{16} - 17221840385088636Z^4t_3^4t_4^{16}$$
$$- 185849011833304896Z^3t_3^5t_4^{16} + 774301524384595608Z^2t_3^6t_4^{16}$$
$$- 2702313339151434192Zt_3^7t_4^{16} + 4275414050960452032t_3^8t_4^{16}$$
$$- 5487463962108Z^2t_2t_4^{17} + 215653795565850Zt_2t_3t_4^{17}$$
$$- 966014327469168t_2t_3^2t_4^{17} - 11919833559412128Z^4t_3^3t_4^{17}$$
$$- 163432619288075232Z^3t_3^4t_4^{17} + 764356650793358400Z^2t_3^5t_4^{17}$$

$$
\begin{aligned}
&- 3544040633270462640 Z t_3^6 t_4^{17} + 6087468219237708624 t_3^7 t_4^{17} \\
&+ 33296347201284 Z t_2 t_4^{18} - 274362591823920 t_2 t_3 t_4^{18} \\
&- 5820156825009912 Z^4 t_3^2 t_4^{18} - 107866535187657888 Z^3 t_3^3 t_4^{18} \\
&+ 589953573531721224 Z^2 t_3^4 t_4^{18} - 3707284448112032736 Z t_3^5 t_4^{18} \\
&+ 6972457044663230544 t_3^6 t_4^{18} - 36733855300128 t_2 t_4^{19} \\
&- 1792206648026082 Z^4 t_3 t_4^{19} - 50400685550491200 Z^3 t_3^2 t_4^{19} \\
&+ 343817157550915566 Z^2 t_3^3 t_4^{19} - 3030278405119007184 Z t_3^4 t_4^{19} \\
&+ 6339044135536359552 t_3^5 t_4^{19} - 262246379628645 Z^4 t_4^{20} \\
&- 14896784992972176 Z^3 t_3 t_4^{20} + 142547187742292358 Z^2 t_3^2 t_4^{20} \\
&- 1869379249500169236 Z t_3^3 t_4^{20} + 4469092870200509664 t_3^4 t_4^{20} \\
&- 2097971037029160 Z^3 t_4^{21} + 37527452989080804 Z^2 t_3 t_4^{21} \\
&- 820146273740681580 Z t_3^2 t_4^{21} + 2352087809916649020 t_3^3 t_4^{21} \\
&+ 4720434833315610 Z^2 t_4^{22} - 228463761711354696 Z t_3 t_4^{22} \\
&+ 866823797470929948 t_3^2 t_4^{22} - 30420580036922820 Z t_4^{23} \\
&+ 198342119050401888 t_3 t_4^{23} + 20979710370291600 t_4^{24} \big) \\
&/ \big( t_2^4 + 9216 t_2^2 t_3^{10} - 442368 t_3^{20} + 660 t_2^3 t_3^4 t_4 + 184320 t_2^2 t_3^9 t_4 - 829440 t_2 t_3^{14} t_4 \\
&- 17694720 t_3^{19} t_4 + 5280 t_2^3 t_3^3 t_4^2 + 1637910 t_2^2 t_3^8 t_4^2 - 23224320 t_2 t_3^{13} t_4^2 \\
&- 328043520 t_3^{18} t_4^2 + 14680 t_2^3 t_3^2 t_4^3 + 8511840 t_2^2 t_3^7 t_4^3 - 309383820 t_2 t_3^{12} t_4^3 \\
&- 3740774400 t_3^{17} t_4^3 + 16480 t_2^3 t_3 t_4^4 + 28319400 t_2^2 t_3^6 t_4^4 - 2594553120 t_2 t_3^{11} t_4^4 \\
&- 29095415055 t_3^{16} t_4^4 + 5988 t_2^3 t_4^5 + 61350624 t_2^2 t_3^5 t_4^5 - 15266853432 t_2 t_3^{10} t_4^5 \\
&- 159655502304 t_3^{15} t_4^5 + 82990260 t_2^2 t_3^4 t_4^6 - 66501928800 t_2 t_3^9 t_4^6 \\
&- 598208862600 t_3^{14} t_4^6 + 60158880 t_2^2 t_3^3 t_4^7 - 220545317700 t_2 t_3^8 t_4^7 \\
&- 1198429486560 t_3^{13} t_4^7 + 8506440 t_2^2 t_3^2 t_4^8 - 564191645760 t_2 t_3^7 t_4^8 \\
&+ 1858690772460 t_3^{12} t_4^8 - 16489440 t_2^2 t_3 t_4^9 - 1116076067280 t_2 t_3^6 t_4^9 \\
&+ 27559776437280 t_3^{11} t_4^9 - 7677018 t_2^2 t_4^{10} - 1695914739648 t_2 t_3^5 t_4^{10} \\
&+ 133485102850536 t_3^{10} t_4^{10} - 1947316154580 t_2 t_3^4 t_4^{11} + 435800812788000 t_3^9 t_4^{11} \\
&- 1640324334240 t_2 t_3^3 t_4^{12} + 1074133719732150 t_3^8 t_4^{12} - 961586960760 t_2 t_3^2 t_4^{13} \\
&+ 2070469417430880 t_3^7 t_4^{13} - 353240805600 t_2 t_3 t_4^{14} + 3150340236284040 t_3^6 t_4^{14} \\
&- 61807949532 t_2 t_4^{15} + 3765152371411296 t_3^5 t_4^{15} + 3474755640515340 t_3^4 t_4^{16} \\
&+ 2397542098128480 t_3^3 t_4^{17} + 1167641302221720 t_3^2 t_4^{18} + 358861336161120 t_3 t_4^{19} \\
&+ 52449275925729 t_4^{20} \big) \,, \qquad \text{(A.3.1)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta(t_2) = &- 18 \big( -Z^4 t_2^4 + 34 Z^2 t_2^4 t_3^2 - 192 t_2^4 t_3^4 + 1344 Z^3 t_2^3 t_3^6 + 4800 Z t_2^3 t_3^8 - 59904 Z^4 t_2^2 t_3^{10} \\
&+ 133632 Z^2 t_2^2 t_3^{12} + 147456 t_2^2 t_3^{14} + 442368 Z t_2 t_3^{18} + 2211840 Z^4 t_3^{20} \\
&- 13271040 Z^2 t_3^{22} - 8 Z^3 t_2^4 t_4 + 136 Z^2 t_2^4 t_3 t_4 + 156 Z t_2^4 t_3^2 t_4 - 1536 t_2^4 t_3^3 t_4 \\
&- 2676 Z^4 t_2^3 t_3^4 t_4 + 16128 Z^3 t_2^3 t_3^5 t_4 + 19272 Z^2 t_2^3 t_3^6 t_4 + 76800 Z t_2^3 t_3^7 t_4
\end{aligned}
$$

$$
\begin{aligned}
&+ 13824t_2^3t_3^8t_4 - 1198080Z^4t_2^2t_3^9t_4 - 242496Z^3t_2^2t_3^{10}t_4 + 3207168Z^2t_2^2t_3^{11}t_4\\
&- 1029312Zt_2^2t_3^{12}t_4 + 4128768t_2^2t_3^{13}t_4 + 4340736Z^4t_2t_3^{14}t_4 - 22588416Z^2t_2t_3^{16}t_4\\
&+ 15925248Zt_2t_3^{17}t_4 - 884736t_2t_3^{18}t_4 + 88473600Z^4t_3^{19}t_4 + 8847360Z^3t_3^{20}t_4\\
&- 583925760Z^2t_3^{21}t_4 + 19906560Zt_3^{22}t_4 + 130Z^2t_2^4t_4^2 + 624Zt_2^4t_3t_4^2 - 3660t_2^4t_3^2t_4^2\\
&- 21408Z^4t_2^3t_3^3t_4^2 + 65184Z^3t_2^3t_3^4t_4^2 + 231264Z^2t_2^3t_3^5t_4^2 + 525504Zt_2^3t_3^6t_4^2\\
&+ 221184t_2^3t_3^7t_4^2 - 10837494Z^4t_2^2t_3^8t_4^2 - 4849920Z^3t_2^2t_3^9t_4^2 + 33896844Z^2t_2^2t_3^{10}t_4^2\\
&- 24703488Zt_2^2t_3^{11}t_4^2 + 57292416t_2^2t_3^{12}t_4^2 + 121540608Z^4t_2t_3^{13}t_4^2\\
&+ 17316288Z^3t_2t_3^{14}t_4^2 - 722829312Z^2t_2t_3^{15}t_4^2 + 317279808Zt_2t_3^{16}t_4^2\\
&- 31850496t_2t_3^{17}t_4^2 + 1639540224Z^4t_3^{18}t_4^2 + 353894400Z^3t_3^{19}t_4^2\\
&- 12030653952Z^2t_3^{20}t_4^2 + 875888640Zt_3^{21}t_4^2 - 78299136t_3^{22}t_4^2 + 556Zt_2^4t_4^3\\
&- 2352t_2^4t_3t_4^3 - 60920Z^4t_2^3t_3^2t_4^3 + 91392Z^3t_2^3t_3^3t_4^3 + 1110168Z^2t_2^3t_3^4t_4^3\\
&+ 2005248Zt_2^3t_3^5t_4^3 + 1726992t_2^3t_3^6t_4^3 - 58384224Z^4t_2^2t_3^7t_4^3 - 43443504Z^3t_2^2t_3^8t_4^3\\
&+ 207552240Z^2t_2^2t_3^9t_4^3 - 277996392Zt_2^2t_3^{10}t_4^3 + 516234240t_2^2t_3^{11}t_4^3\\
&+ 1621596780Z^4t_2t_3^{12}t_4^3 + 484856064Z^3t_2t_3^{13}t_4^3 - 11043552312Z^2t_2t_3^{14}t_4^3\\
&+ 4377397248Zt_2t_3^{15}t_4^3 - 748763136t_2t_3^{16}t_4^3 + 18679486464Z^4t_3^{17}t_4^3\\
&+ 6557184576Z^3t_3^{18}t_4^3 - 154227732480Z^2t_3^{19}t_4^3 + 17980249536Zt_3^{20}t_4^3\\
&- 3445161984t_3^{21}t_4^3 + 560t_2^4t_4^4 - 72416Z^4t_2^3t_3t_4^4 - 32576Z^3t_2^3t_3^2t_4^4\\
&+ 2714304Z^2t_2^3t_3^3t_4^4 + 4540656Zt_2^3t_3^4t_4^4 + 8337600t_2^3t_3^5t_4^4 - 206443656Z^4t_2^2t_3^6t_4^4\\
&- 229503744Z^3t_2^2t_3^7t_4^4 + 800829180Z^2t_2^2t_3^8t_4^4 - 1936749600Zt_2^2t_3^9t_4^4\\
&+ 3319291512t_2^2t_3^{10}t_4^4 + 13637876256Z^4t_2t_3^{11}t_4^4 + 6457362336Z^3t_2t_3^{12}t_4^4\\
&- 106827257376Z^2t_2t_3^{13}t_4^4 + 44530457184Zt_2t_3^{14}t_4^4 - 12409307136t_2t_3^{15}t_4^4\\
&+ 145040300271Z^4t_3^{16}t_4^4 + 74682798336Z^3t_3^{17}t_4^4 - 1371042946206Z^2t_3^{18}t_4^4\\
&+ 228712343040Zt_3^{19}t_4^4 - 70629603264t_3^{20}t_4^4 - 29668Z^4t_2^3t_4^5 - 173312Z^3t_2^3t_3t_4^5\\
&+ 3514392Z^2t_2^3t_3^2t_4^5 + 5789568Zt_2^3t_3^3t_4^5 + 25388448t_2^3t_3^4t_4^5 - 495982944Z^4t_2^2t_3^5t_4^5\\
&- 783955008Z^3t_2^2t_3^6t_4^5 + 1919636928Z^2t_2^2t_3^7t_4^5 - 9279227736Zt_2^2t_3^8t_4^5\\
&+ 15862464864t_2^2t_3^9t_4^5 + 80575295832Z^4t_2t_3^{10}t_4^5 + 54126634752Z^3t_2t_3^{11}t_4^5\\
&- 731351378376Z^2t_2t_3^{12}t_4^5 + 344612741760Zt_2t_3^{13}t_4^5 - 144676580976t_2t_3^{14}t_4^5\\
&+ 793144532448Z^4t_3^{15}t_4^5 + 579629467512Z^3t_3^{16}t_4^5 - 8851956467256Z^2t_3^{17}t_4^5\\
&+ 2004421113324Zt_3^{18}t_4^5 - 895893419520t_3^{19}t_4^5 - 98592Z^3t_2^3t_4^6\\
&+ 2210400Z^2t_2^3t_3t_4^6 + 3164576Zt_2^3t_3^2t_4^6 + 46831872t_2^3t_3^3t_4^6 - 810189396Z^4t_2^2t_3^4t_4^6\\
&- 1769884416Z^3t_2^2t_3^5t_4^6 + 2296853496Z^2t_2^2t_3^6t_4^6 - 32104704384Zt_2^2t_3^7t_4^6\\
&+ 57399905088t_2^2t_3^8t_4^6 + 352843828704Z^4t_2t_3^9t_4^6 + 318230123328Z^3t_2t_3^{10}t_4^6\\
&- 3751879372992Z^2t_2t_3^{11}t_4^6 + 2074913899056Zt_2t_3^{12}t_4^6 - 1223200609344t_2t_3^{13}t_4^6\\
&+ 2946991640040Z^4t_3^{14}t_4^6 + 3168309489408Z^3t_3^{15}t_4^6 - 41668616311470Z^2t_3^{16}t_4^6\\
&+ 12599716275504Zt_3^{17}t_4^6 - 7816980324492t_3^{18}t_4^6 + 481608Z^2t_2^3t_4^7
\end{aligned}
$$

$$- 529792Zt_2^3t_3t_4^7 + 48328400t_2^3t_3^2t_4^7 - 878053536Z^4t_2^2t_3^3t_4^7 - 2569978656Z^3t_2^2t_3^4t_4^7$$
$$- 1279418976Z^2t_2^2t_3^5t_4^7 - 81949697424Zt_2^2t_3^6t_4^7 + 158364610560t_2^2t_3^7t_4^7$$
$$+ 1177852498308Z^4t_2t_3^8t_4^7 + 1384297834752Z^3t_2t_3^9t_4^7$$
$$- 14896555053912Z^2t_2t_3^{10}t_4^7 + 9885354044544Zt_2t_3^{11}t_4^7 - 7766097211296t_2t_3^{12}t_4^7$$
$$+ 5700652750944Z^4t_3^{13}t_4^7 + 11773138218048Z^3t_3^{14}t_4^7 - 134045466287040Z^2t_3^{15}t_4^7$$
$$+ 56044387120716Zt_3^{16}t_4^7 - 48835205227440t_3^{17}t_4^7 - 919184Zt_2^3t_4^8 + 23401280t_2^3t_3t_4^8$$
$$- 600033384Z^4t_2^2t_3^2t_4^8 - 2177485056Z^3t_2^2t_3^3t_4^8 - 10221103368Z^2t_2^2t_3^4t_4^8$$
$$- 154885738176Zt_2^2t_3^5t_4^8 + 332807241744t_2^2t_3^6t_4^8 + 3036967027776Z^4t_2t_3^7t_4^8$$
$$+ 4580643750048Z^3t_2t_3^8t_4^8 - 46626359993568Z^2t_2t_3^9t_4^8 + 37703460194496Zt_2t_3^{10}t_4^8$$
$$- 38062111064832t_2t_3^{11}t_4^8 - 10826781756300Z^4t_3^{12}t_4^8 + 22868814032640Z^3t_3^{13}t_4^8$$
$$- 197186343219000Z^2t_3^{14}t_4^8 + 153399062480256Zt_3^{15}t_4^8 - 215411492118384t_3^{16}t_4^8$$
$$+ 2966208t_2^3t_4^9 - 235891872Z^4t_2^2t_3t_4^9 - 756861696Z^3t_2^2t_3^2t_4^9 - 18570083904Z^2t_2^2t_3^3t_4^9$$
$$- 214252600752Zt_2^2t_3^4t_4^9 + 529149124800t_2^2t_3^5t_4^9 + 6063292911888Z^4t_2t_3^6t_4^9$$
$$+ 11677233424896Z^3t_2t_3^7t_4^9 - 116126422149672Z^2t_2t_3^8t_4^9 + 115904011574016Zt_2t_3^9t_4^9$$
$$- 146773837693008t_2t_3^{10}t_4^9 - 144324561572640Z^4t_3^{11}t_4^9 - 42034275434592Z^3t_3^{12}t_4^9$$
$$+ 750306356658144Z^2t_3^{13}t_4^9 + 13855350079728Zt_3^{14}t_4^9 - 582213216766464t_3^{15}t_4^9$$
$$- 42943014Z^4t_2^2t_4^{10} + 198614016Z^3t_2^2t_3t_4^{10} - 17610798276Z^2t_2^2t_3^2t_4^{10}$$
$$- 210809616768Zt_2^2t_3^3t_4^{10} + 629231915232t_2^2t_3^4t_4^{10} + 9309632184000Z^4t_2t_3^5t_4^{10}$$
$$+ 22977644113728Z^3t_2t_3^6t_4^{10} - 230704235482752Z^2t_2t_3^7t_4^{10} + 287752792352976Zt_2t_3^8t_4^{10}$$
$$- 450575517478464t_2t_3^9t_4^{10} - 690104345405832Z^4t_3^{10}t_4^{10} - 567819698026752Z^3t_3^{11}t_4^{10}$$
$$+ 7012903693399848Z^2t_3^{12}t_4^{10} - 2489089278347712Zt_3^{13}t_4^{10} - 319111393145 76t_3^{14}t_4^{10}$$
$$+ 171669456Z^3t_2^2t_4^{11} - 8766288144Z^2t_2^2t_3t_4^{11} - 139821503112Zt_2^2t_3^2t_4^{11}$$
$$+ 549643776768t_2^2t_3^3t_4^{11} + 10808439122484Z^4t_2t_3^4t_4^{11} + 34644142851840Z^3t_2t_3^5t_4^{11}$$
$$- 364305876742440Z^2t_2t_3^6t_4^{11} + 575371025706240Zt_2t_3^7t_4^{11} - 1107332760469248t_2t_3^8t_4^{11}$$
$$- 2243564362553760Z^4t_3^9t_4^{11} - 2713192887960000Z^3t_3^{10}t_4^{11} + 30997756176278976Z^2t_3^{11}t_4^{11}$$
$$- 15981413060501136Zt_3^{12}t_4^{11} + 9383822753647680t_3^{13}t_4^{11} - 1773735732Z^2t_2^2t_4^{12}$$
$$- 56265055776Zt_2^2t_3t_4^{12} + 340562860344t_2^2t_3^2t_4^{12} + 9201150573984Z^4t_2t_3^3t_4^{12}$$
$$+ 39341671395552Z^3t_2t_3^4t_4^{12} - 452770329330144Z^2t_2t_3^5t_4^{12} + 919227930922080Zt_2t_3^6t_4^{12}$$
$$- 2178902298433536t_2t_3^7t_4^{12} - 5520891503408022Z^4t_3^8t_4^{12} - 8799020464405248Z^3t_3^9t_4^{12}$$
$$+ 97572632295459132Z^2t_3^{10}t_4^{12} - 64742323876442496Zt_3^{11}t_4^{12} + 59144732802938592t_3^{12}t_4^{12}$$
$$- 10484886456Zt_2^2t_4^{13} + 137037343968t_2^2t_3t_4^{13} + 5437723475736Z^4t_2t_3^2t_4^{13}$$
$$+ 32643295246080Z^3t_2t_3^3t_4^{13} - 434974452288984Z^2t_2t_3^4t_4^{13} + 1156770932374656Zt_2t_3^5t_4^{13}$$
$$- 3414751390622736t_2t_3^6t_4^{13} - 10636000080690528Z^4t_3^7t_4^{13} - 21580466580380592Z^3t_3^8t_4^{13}$$
$$+ 238584568230736560Z^2t_3^9t_4^{13} - 197332741271761560Zt_3^{10}t_4^{13}$$
$$+ 235345232116723968t_3^{11}t_4^{13} + 27281880864t_2^2t_4^{14} + 2003226799968Z^4t_2t_3t_4^{14}$$

$$+ 18767073525696Z^3t_2t_3^2t_4^{14} - 313211596471488Z^2t_2t_3^3t_4^{14} + 1120757485426128Zt_2t_3^4t_4^{14}$$
$$- 4213756834197696t_2t_3^5t_4^{14} - 16180739188814952Z^4t_3^6t_4^{14} - 41411521413815040Z^3t_3^7t_4^{14}$$
$$+ 466944422584225788Z^2t_3^8t_4^{14} - 476899745641134048Zt_3^9t_4^{14}$$
$$+ 703445052147825528t_3^{10}t_4^{14} + 348554325852Z^4t_2t_4^{15} + 6735201027840Z^3t_2t_3t_4^{15}$$
$$- 160042614969672Z^2t_2t_3^2t_4^{15} + 806456128321152Zt_2t_3^3t_4^{15} - 4013868622687776t_2t_3^4t_4^{15}$$
$$- 19335261571229664Z^4t_3^5t_4^{15} - 62724903899429952Z^3t_3^6t_4^{15}$$
$$+ 738389432703466944Z^2t_3^7t_4^{15} - 931782358161342936Zt_3^8t_4^{15}$$
$$+ 1663591001873767776t_3^9t_4^{15} + 1149245014752Z^3t_2t_4^{16} - 52050647564064Z^2t_2t_3t_4^{16}$$
$$+ 405977153459328Zt_2t_3^2t_4^{16} - 2855412992335104t_2t_3^3t_4^{16} - 17834819067545580Z^4t_3^4t_4^{16}$$
$$- 74608113446873856Z^3t_3^5t_4^{16} + 941723568956228904Z^2t_3^6t_4^{16}$$
$$- 1479251209563384192Zt_3^7t_4^{16} + 3173236238341980864t_3^8t_4^{16} - 8139150813240Z^2t_2t_4^{17}$$
$$+ 127759978225152Zt_2t_3t_4^{17} - 1432111817847600t_2t_3^2t_4^{17} - 12291586125618528Z^4t_3^3t_4^{17}$$
$$- 68504391666907488Z^3t_3^4t_4^{17} + 958340754564279264Z^2t_3^5t_4^{17}$$
$$- 1900418688517021776Zt_3^6t_4^{17} + 4905951379405940736t_3^7t_4^{17} + 18965208401136Zt_2t_4^{18}$$
$$- 452707536012480t_2t_3t_4^{18} - 5973960373569144Z^4t_3^2t_4^{18}$$
$$- 47016896590215936Z^3t_3^3t_4^{18} + 762270303367156680Z^2t_3^4t_4^{18}$$
$$- 1952139477717766080Zt_3^5t_4^{18} + 6122092717759533552t_3^6t_4^{18}$$
$$- 67913263059264t_2t_4^{19} - 1830406053002976Z^4t_3t_4^{19}$$
$$- 22775949563831424Z^3t_3^2t_4^{19} + 457679035096460352Z^2t_3^3t_4^{19}$$
$$- 1569422467018854288Zt_3^4t_4^{19} + 6091607522018555712t_3^5t_4^{19}$$
$$- 266446445187681Z^4t_4^{20} - 6964386523200000Z^3t_3t_4^{20}$$
$$+ 195413411081236338Z^2t_3^2t_4^{20} - 953436905745455232Zt_3^3t_4^{20}$$
$$+ 4729217586797049696t_3^4t_4^{20} - 1013307943974408Z^3t_4^{21}$$
$$+ 52946169515125704Z^2t_3t_4^{21} - 412427152757268468Zt_3^2t_4^{21}$$
$$+ 2764149391618428672t_3^3t_4^{21} + 6848395586330562Z^2t_4^{22}$$
$$- 113446118759881680Zt_3t_4^{22} + 1144938892758036660t_3^2t_4^{22}$$
$$- 149478590054625548Zt_4^{23} + 299685884752964304t_3t_4^{23}$$
$$+ 37262803420133424t_4^{24}) \,/\, (t_2^4 + 9216t_2^2t_3^{10} - 442368t_3^{20} + 660t_2^3t_3^4t_4$$
$$+ 184320t_2^2t_3^9t_4 - 829440t_2t_3^{14}t_4 - 17694720t_3^{19}t_4 + 5280t_2^3t_3^3t_4^2$$
$$+ 1637910t_2^2t_3^8t_4^2 - 23224320t_2t_3^{13}t_4^2 - 328043520t_3^{18}t_4^2 + 14680t_2^3t_3^2t_4^3$$
$$+ 8511840t_2^2t_3^7t_4^3 - 309383820t_2t_3^{12}t_4^3 - 3740774400t_3^{17}t_4^3 + 16480t_2^3t_3t_4^4$$
$$+ 28319400t_2^2t_3^6t_4^4 - 2594553120t_2t_3^{11}t_4^4 - 29095415055t_3^{16}t_4^4 + 5988t_2^3t_4^5$$
$$+ 61350624t_2^2t_3^5t_4^5 - 15266853432t_2t_3^{10}t_4^5 - 159655502304t_3^{15}t_4^5$$
$$+ 82990260t_2^2t_3^4t_4^6 - 66501928800t_2t_3^9t_4^6 - 598208862600t_3^{14}t_4^6$$
$$+ 60158880t_2^2t_3^3t_4^7 - 220545317700t_2t_3^8t_4^7 - 1198429486560t_3^{13}t_4^7$$

$$
\begin{aligned}
&+ 8506440t_2^2t_3^2t_4^8 - 564191645760t_2t_3^7t_4^8 + 1858690772460t_3^{12}t_4^8 \\
&- 16489440t_2^2t_3t_4^9 - 1116076067280t_2t_3^6t_4^9 + 27559776437280t_3^{11}t_4^9 \\
&- 7677018t_2^2t_4^{10} - 1695914739648t_2t_3^5t_4^{10} + 133485102850536t_3^{10}t_4^{10} \\
&- 1947316154580t_2t_3^4t_4^{11} + 435800812788000t_3^9t_4^{11} - 1640324334240t_2t_3^3t_4^{12} \\
&+ 1074133719732150t_3^8t_4^{12} - 961586960760t_2t_3^2t_4^{13} + 2070469417430880t_3^7t_4^{13} \\
&- 353240805600t_2t_3t_4^{14} + 3150340236284040t_3^6t_4^{14} - 61807949532t_2t_4^{15} \\
&+ 3765152371411296t_3^5t_4^{15} + 3474755640515340t_3^4t_4^{16} + 2397542098128480t_3^3t_4^{17} \\
&+ 1167641302221720t_3^2t_4^{18} + 358861336161120t_3t_4^{19} + 52449275925729t_4^{20}\big)\,, \qquad \text{(A.3.2)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta(t_3) = &-\frac{4}{5}\big(t_2^4 - 2Z^4t_2^3t_3 - 2Z^2t_2^3t_3^3 + 96t_2^3t_3^5 - 672Z^3t_2^2t_3^7 - 2400Zt_2^2t_3^9 + 9216t_2^2t_3^{10} \\
&+ 6912Z^4t_2t_3^{11} + 71424Z^2t_2t_3^{13} - 73728t_2t_3^{15} - 221184Zt_3^{19} - 442368t_3^{20} \\
&- 4Z^4t_2^3t_4 - 12Z^3t_2^3t_3t_4 - 12Z^2t_2^3t_3^2t_4 - 126Zt_2^3t_3^3t_4 + 1620t_2^3t_3^4t_4 \\
&- 102Z^4t_2^2t_3^5t_4 - 9408Z^3t_2^2t_3^6t_4 + 1050Z^2t_2^2t_3^7t_4 - 43200Zt_2^2t_3^8t_4 + 169632t_2^2t_3^9t_4 \\
&+ 152064Z^4t_2t_3^{10}t_4 + 23616Z^3t_2t_3^{11}t_4 + 1857024Z^2t_2t_3^{12}t_4 + 248256Zt_2t_3^{13}t_4 \\
&- 3041280t_2t_3^{14}t_4 - 48384Z^4t_3^{15}t_4 - 573696Z^2t_3^{17}t_4 - 8404992Zt_3^{18}t_4 \\
&- 17473536t_3^{19}t_4 - 24Z^3t_2^3t_4^2 - 22Z^2t_2^3t_3t_4^2 - 756Zt_2^3t_3^2t_4^2 + 8496t_2^3t_3^3t_4^2 \\
&- 1020Z^4t_2^2t_3^4t_4^2 - 57276Z^3t_2^2t_3^5t_4^2 + 14700Z^2t_2^2t_3^6t_4^2 - 356394Zt_2^2t_3^7t_4^2 \\
&+ 1373526t_2^2t_3^8t_4^2 + 1599642Z^4t_2t_3^9t_4^2 + 519552Z^3t_2t_3^{10}t_4^2 + 22854042Z^2t_2t_3^{11}t_4^2 \\
&+ 6454656Zt_2t_3^{12}t_4^2 - 54735840t_2t_3^{13}t_4^2 - 1451520Z^4t_3^{14}t_4^2 - 147744Z^3t_3^{15}t_4^2 \\
&- 19505664Z^2t_3^{16}t_4^2 - 151266528Zt_3^{17}t_4^2 - 319638528t_3^{18}t_4^2 - 12Z^2t_2^3t_4^3 \\
&- 1522Zt_2^3t_3t_4^3 + 18616t_2^3t_3^2t_4^3 - 3636Z^4t_2^2t_3^3t_4^3 - 196440Z^3t_2^2t_3^4t_4^3 \\
&+ 95658Z^2t_2^2t_3^5t_4^3 - 1763916Zt_2^2t_3^6t_4^3 + 6350832t_2^2t_3^7t_4^3 + 10545876Z^4t_2t_3^8t_4^3 \\
&+ 5382828Z^3t_2t_3^9t_4^3 + 175952700Z^2t_2t_3^{10}t_4^3 + 80751942Zt_2t_3^{11}t_4^3 \\
&- 591943500t_2t_3^{12}t_4^3 - 21386754Z^4t_3^{13}t_4^3 - 4432320Z^3t_3^{14}t_4^3 - 320804226Z^2t_3^{15}t_4^3 \\
&- 1713825216Zt_3^{16}t_4^3 - 3576990240t_3^{17}t_4^3 - 1028Zt_2^3t_4^4 + 16624t_2^3t_3t_4^4 \\
&- 5496Z^4t_2^2t_3^2t_4^4 - 402216Z^3t_2^2t_3^3t_4^4 + 368580Z^2t_2^2t_3^4t_4^4 - 5711250Zt_2^2t_3^5t_4^4 \\
&+ 17805960t_2^2t_3^6t_4^4 + 48182040Z^4t_2t_3^7t_4^4 + 34544664Z^3t_2t_3^8t_4^4 \\
&+ 943405110Z^2t_2t_3^9t_4^4 + 640523268Zt_2t_3^{10}t_4^4 - 4374711216t_2t_3^{11}t_4^4 \\
&- 203820084Z^4t_3^{12}t_4^4 - 64217124Z^3t_3^{13}t_4^4 - 3382314300Z^2t_3^{14}t_4^4 \\
&- 13694628366Zt_3^{15}t_4^4 - 26955990351t_3^{16}t_4^4 + 3972t_2^3t_4^5 - 2526Z^4t_2^2t_3t_4^5 \\
&- 474480Z^3t_2^2t_3^2t_4^5 + 896550Z^2t_2^2t_3^3t_4^5 - 12360660Zt_2^2t_3^4t_4^5 + 28177776t_2^2t_3^5t_4^5 \\
&+ 159621840Z^4t_2t_3^6t_4^5 + 151520976Z^3t_2t_3^7t_4^5 + 3711037356Z^2t_2t_3^8t_4^5 \\
&+ 3578279058Zt_2t_3^9t_4^5 - 23593070232t_2t_3^{10}t_4^5 - 1396034676Z^4t_3^{11}t_4^5 \\
&- 594068904Z^3t_3^{12}t_4^5 - 25531072074Z^2t_3^{13}t_4^5 - 81885932964Zt_3^{14}t_4^5 \\
&- 139036043376t_3^{15}t_4^5 + 900Z^4t_2^2t_4^6 - 281292Z^3t_2^2t_3t_4^6 + 1363620Z^2t_2^2t_3^2t_4^6
\end{aligned}
$$

$$
\begin{aligned}
&- 17576358Zt_2^2t_3^3t_4^6 + 13874004t_2^2t_3^4t_4^6 + 390042540Z^4t_2t_3^5t_4^6 \\
&+ 472995936Z^3t_2t_3^6t_4^6 + 10989802548Z^2t_2t_3^7t_4^6 + 14810697828Zt_2t_3^8t_4^6 \\
&- 96291388656t_2t_3^9t_4^6 - 7239118392Z^4t_3^{10}t_4^6 - 3905719560Z^3t_3^{11}t_4^6 \\
&- 145968902820Z^2t_3^{12}t_4^6 - 379111138830Zt_3^{13}t_4^6 - 444769848936t_3^{14}t_4^6 \\
&- 56088Z^3t_2^2t_4^7 + 1189494Z^2t_2^2t_3t_4^7 - 15504036Zt_2^2t_3^2t_4^7 - 32627952t_2^2t_3^3t_4^7 \\
&+ 700684344Z^4t_2t_3^4t_4^7 + 1062156456Z^3t_2t_3^5t_4^7 + 24733538136Z^2t_2t_3^6t_4^7 \\
&+ 46567070316Zt_2t_3^7t_4^7 - 303373357668t_2t_3^8t_4^7 - 29218818582Z^4t_3^9t_4^7 \\
&- 19229989680Z^3t_3^{10}t_4^7 - 652417739730Z^2t_3^{11}t_4^7 - 1386084888396Zt_3^{12}t_4^7 \\
&- 293853500784t_3^{13}t_4^7 + 454284Z^2t_2^2t_4^8 - 7448958Zt_2^2t_3t_4^8 - 66130200t_2^2t_3^2t_4^8 \\
&+ 903601656Z^4t_2t_3^3t_4^8 + 1695981456Z^3t_2t_3^4t_4^8 + 42153739740Z^2t_2t_3^5t_4^8 \\
&+ 112249281384Zt_2t_3^6t_4^8 - 744414016608t_2t_3^7t_4^8 - 93082608972Z^4t_3^8t_4^8 \\
&- 72841455468Z^3t_3^9t_4^8 - 2320589735244Z^2t_3^{10}t_4^8 - 4040880668622Zt_3^{11}t_4^8 \\
&+ 6153157879884t_3^{12}t_4^8 - 1379484Zt_2^2t_4^9 - 46846224t_2^2t_3t_4^9 + 793483344Z^4t_2t_3^2t_4^9 \\
&+ 1858719312Z^3t_2t_3^3t_4^9 + 53492650200Z^2t_2t_3^4t_4^9 + 206804653524Zt_2t_3^5t_4^9 \\
&- 1423268368272t_2t_3^6t_4^9 - 235063637496Z^4t_3^7t_4^9 - 214827522840Z^3t_3^8t_4^9 \\
&- 6625980740970Z^2t_3^9t_4^9 - 9399924077940Zt_3^{10}t_4^9 + 44156468592144t_3^{11}t_4^9 \\
&- 11396538t_2^2t_4^{10} + 425250378Z^4t_2t_3t_4^{10} + 1304169696Z^3t_2t_3^2t_4^{10} \\
&+ 48837598578Z^2t_2t_3^3t_4^{10} + 286785791496Zt_2t_3^4t_4^{10} - 2104921788480t_2t_3^5t_4^{10} \\
&- 468781690896Z^4t_3^6t_4^{10} - 493784767440Z^3t_3^7t_4^{10} - 15212086507476Z^2t_3^8t_4^{10} \\
&- 17266542374718Zt_3^9t_4^{10} + 186053768129160t_3^{10}t_4^{10} + 105010452Z^4t_2t_4^{11} \\
&+ 510680844Z^3t_2t_3t_4^{11} + 30130294572Z^2t_2t_3^2t_4^{11} + 290191717710Zt_2t_3^3t_4^{11} \\
&- 2370027585492t_2t_3^4t_4^{11} - 729338140494Z^4t_3^5t_4^{11} - 876636139104Z^3t_3^6t_4^{11} \\
&- 27945877328118Z^2t_3^7t_4^{11} - 24287909071356Zt_3^8t_4^{11} + 572730539601744t_3^9t_4^{11} \\
&+ 79603992Z^3t_2t_4^{12} + 11125124910Z^2t_2t_3t_4^{12} + 202419278676Zt_2t_3^2t_4^{12} \\
&- 1974866348016t_2t_3^3t_4^{12} - 865403912844Z^4t_3^4t_4^{12} - 1178011019196Z^3t_3^5t_4^{12} \\
&- 40597701245172Z^2t_3^6t_4^{12} - 23980621193370Zt_3^7t_4^{12} + 1367500843222038t_3^8t_4^{12} \\
&+ 1821515580Z^2t_2t_4^{13} + 87064707546Zt_2t_3t_4^{13} - 1157075741016t_2t_3^2t_4^{13} \\
&- 753941514708Z^4t_3^3t_4^{13} - 1156112680536Z^3t_3^4t_4^{13} - 45682207204590Z^2t_3^5t_4^{13} \\
&- 11118073481964Zt_3^6t_4^{13} + 2585777568129072t_3^7t_4^{13} + 17413109460Zt_2t_4^{14} \\
&- 429536091888t_2t_3t_4^{14} - 452151646776Z^4t_3^2t_4^{14} - 778920697224Z^3t_3^3t_4^{14} \\
&- 38495253696300Z^2t_3^4t_4^{14} + 11683992259782Zt_3^5t_4^{14} + 3887040073039848t_3^6t_4^{14} \\
&- 76676431356t_2t_4^{15} - 165400034778Z^4t_3t_4^{15} - 320224983984Z^3t_3^2t_4^{15} \\
&- 22974798690486Z^2t_3^3t_4^{15} + 32542254016668Zt_3^4t_4^{15} + 4611410853635376t_3^5t_4^{15} \\
&- 27466840692Z^4t_4^{16} - 59750783028Z^3t_3t_4^{16} - 8783439059844Z^2t_3^2t_4^{16} \\
&+ 38715002353782Zt_3^3t_4^{16} + 4239460018547244t_3^4t_4^{16} + 379564056Z^3t_4^{17}
\end{aligned}
$$

$$
\begin{aligned}
&- 1726964402526Z^2t_3t_4^{17} + 28407465464580Zt_3^2t_4^{17}\\
&+ 2922400306199088t_3^3t_4^{17} - 75251495772Z^2t_4^{18} + 12501480733542Zt_3t_4^{18}\\
&+ 1425210330974328t_3^2t_4^{18} + 2552263527276Zt_4^{19} + 439353107929872t_3t_4^{19}\\
&+ 64459951413057t_4^{20}\big) \,/\, \big(t_2^4 + 9216t_2^2t_3^{10} - 442368t_3^{20} + 660t_2^3t_3^4t_4\\
&+ 184320t_2^2t_3^9t_4 - 829440t_2t_3^{14}t_4 - 17694720t_3^{19}t_4 + 5280t_2^3t_3^3t_4^2\\
&+ 1637910t_2^2t_3^8t_4^2 - 23224320t_2t_3^{13}t_4^2 - 328043520t_3^{18}t_4^2 + 14680t_2^3t_3^2t_4^3\\
&+ 8511840t_2^2t_3^7t_4^3 - 309383820t_2t_3^{12}t_4^3 - 3740774400t_3^{17}t_4^3 + 16480t_2^3t_3t_4^4\\
&+ 28319400t_2^2t_3^6t_4^4 - 2594553120t_2t_3^{11}t_4^4 - 29095415055t_3^{16}t_4^4 + 5988t_2^3t_4^5\\
&+ 61350624t_2^2t_3^5t_4^5 - 15266853432t_2t_3^{10}t_4^5 - 159655502304t_3^{15}t_4^5\\
&+ 82990260t_2^2t_3^4t_4^6 - 66501928800t_2t_3^9t_4^6 - 598208862600t_3^{14}t_4^6\\
&+ 60158880t_2^2t_3^3t_4^7 - 220545317700t_2t_3^8t_4^7 - 1198429486560t_3^{13}t_4^7\\
&+ 8506440t_2^2t_3^2t_4^8 - 564191645760t_2t_3^7t_4^8 + 1858690772460t_3^{12}t_4^8\\
&- 16489440t_2^2t_3t_4^9 - 1116076067280t_2t_3^6t_4^9 + 27559776437280t_3^{11}t_4^9\\
&- 7677018t_2^2t_4^{10} - 1695914739648t_2t_3^5t_4^{10} + 133485102850536t_3^{10}t_4^{10}\\
&- 1947316154580t_2t_3^4t_4^{11} + 435800812788000t_3^9t_4^{11} - 1640324334240t_2t_3^3t_4^{12}\\
&+ 1074133719732150t_3^8t_4^{12} - 961586960760t_2t_3^2t_4^{13} + 2070469417430880t_3^7t_4^{13}\\
&- 353240805600t_2t_3t_4^{14} + 3150340236284040t_3^6t_4^{14} - 61807949532t_2t_4^{15}\\
&+ 3765152371411296t_3^5t_4^{15} + 3474755640515340t_3^4t_4^{16}\\
&+ 2397542098128480t_3^3t_4^{17} + 1167641302221720t_3^2t_4^{18}\\
&+ 358861336161120t_3t_4^{19} + 52449275925729t_4^{20}\big)\,, \qquad \text{(A.3.3)}
\end{aligned}
$$

$$\Delta(t_4) = \frac{2}{5}. \qquad \text{(A.3.4)}$$

A general element of $V_1$ is of the form

$$\sum_{k=0}^{4} \sum_{\substack{0\le i_1,\, i_2,\, i_3,\, i_4 \\ 2k+\sum_{j=1}^{4} d_j i_j = 30}} a_{(i_1,\, i_2,\, i_3,\, i_4,\, k)} t_1^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4} Z^k, \qquad \text{(A.3.5)}$$

where $a_{(i_1,\, i_2,\, i_3,\, i_4,\, k)} \in \mathbb{C}$. By calculating the Laplacian of this general element (A.3.5) using Proposition 4.0.1 and formulas (A.3.1)–(A.3.4) we find that the only harmonic elements of $V_1$ are as claimed. A general element of $V_2$ has the form

$$\sum_{k=0}^{4} \sum_{\substack{0\le i_1,\, i_2,\, i_3,\, i_4 \\ 2k+\sum_{j=1}^{4} d_j i_j = 20}} b_{(i_1,\, i_2,\, i_3,\, i_4,\, k)} t_1^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4} Z^k, \qquad \text{(A.3.6)}$$

where $b_{(i_1,\, i_2,\, i_3,\, i_4,\, k)} \in \mathbb{C}$. By calculating the Laplacian of this general element (A.3.6) using

Propostion 4.0.1 and formulas (A.3.1)–(A.3.4) we find that the only harmonic elements of $V_2$ are as claimed. A general element of $V_3$ has the form

$$\sum_{k=0}^{4} \sum_{\substack{0 \leq i_1, i_2, i_3, i_4 \\ 2k + \sum_{j=1}^{4} d_j i_j = 12}} c_{(i_1, i_2, i_3, i_4, k)} t_1^{i_1} t_2^{i_2} t_3^{i_3} t_4^{i_4} Z^k, \tag{A.3.7}$$

where $c_{(i_1, i_2, i_3, i_4, k)} \in \mathbb{C}$. By calculating the Laplacian of this general element (A.3.7) using Propostion 4.0.1 and formulas (A.3.1)–(A.3.4) we find that the only harmonic elements of $V_3$ are as claimed. □

Proposition A.3.1 thus allows us to give a proof of Theorem 4.5.17, in which the basic invariants $y_i$ are expressed as polynomials in terms of the $t$ coordinates and the variable $Z$. We now present this proof.

*Proof.* Note that $Y_4 = \frac{1}{8} y_4 = \frac{5}{2} t_4$. We now equate $Y_1$, $Y_2$ and $Y_3$ given by relations (4.5.12)–(4.5.14) with general harmonic elements of $V_1$, $V_2$ and $V_3$, respectively, given by Proposition A.3.1. We then rearrange these equations to find $y_i$ in terms of $t_j$ and $Z$. We find

$$\begin{aligned}
y_1 = {} & \frac{958464 \times 10^{14}}{3773} t_4^{15} + \frac{a}{6666660} \big( -6006 t_1^2 t_2 + 24024 t_1 t_2^2 - 24024 t_2^3 \\
& + 30030 Z^4 t_1 t_2 t_3 - 60060 Z^4 t_2^2 t_3 - 150150 Z^3 t_2^2 t_3^2 - 300300 Z^2 t_1 t_2 t_3^3 + 600600 Z^2 t_2^2 t_3^3 \\
& - 3903900 Z t_2^2 t_3^4 - 12762750 t_1 t_2 t_3^5 + 25525500 t_2^2 t_3^5 - 32692660 Z^4 t_2 t_3^6 - 16336320 Z^3 t_1 t_3^7 \\
& + 32672640 Z^3 t_2 t_3^7 - 352592240 Z^2 t_2 t_3^8 + 20180160 Z t_1 t_3^9 - 40360320 Z t_2 t_3^9 \\
& + 810471662 t_2 t_3^{10} + 1928006080 Z^3 t_3^{12} - 6093447360 Z t_3^{14} + 60060 Z^4 t_1 t_2 t_4 \\
& - 150150 Z^4 t_2^2 t_4 + 270270 Z^3 t_1 t_2 t_3 t_4 - 1141140 Z^3 t_2^2 t_3 t_4 - 1801800 Z^2 t_1 t_2 t_3^2 t_4 \\
& + 2972970 Z^2 t_2^2 t_3^2 t_4 + 14954940 Z t_1 t_2 t_3^3 t_4 - 61141080 Z t_2^2 t_3^3 t_4 + 23513490 t_1^2 t_3^4 t_4 \\
& - 221681460 t_1 t_2 t_3^4 t_4 + 333843510 t_2^2 t_3^4 t_4 + 6666660 Z^4 t_1 t_3^5 t_4 - 405645240 Z^4 t_2 t_3^5 t_4 \\
& - 228708480 Z^3 t_1 t_3^6 t_4 + 320420100 Z^3 t_2 t_3^6 t_4 + 2162160 Z^2 t_1 t_3^7 t_4 - 5645800160 Z^2 t_2 t_3^7 t_4 \\
& + 363242880 Z t_1 t_3^8 t_4 - 5537772240 Z t_2 t_3^8 t_4 - 4077473400 t_1 t_3^9 t_4 + 24364380040 t_2 t_3^9 t_4 \\
& - 2368345980 Z^4 t_3^{10} t_4 + 46272145920 Z^3 t_3^{11} t_4 + 9255486240 Z^2 t_3^{12} t_4 \\
& - 170616526080 Z t_3^{13} t_4 + 4188854670 t_3^{14} t_4 + 540540 Z^3 t_1 t_2 t_4^2 - 1261260 Z^3 t_2^2 t_4^2 \\
& - 3303300 Z^2 t_1 t_2 t_3 t_4^2 + 4084080 Z^2 t_2^2 t_3 t_4^2 + 89729640 Z t_1 t_2 t_3^2 t_4^2 - 270570300 Z t_2^2 t_3^2 t_4^2 \\
& + 188107920 t_1^2 t_3^3 t_4^2 - 1227686460 t_1 t_2 t_3^3 t_4^2 + 1579217640 t_2^2 t_3^3 t_4^2 + 66666600 Z^4 t_1 t_3^4 t_4^2 \\
& - 2261859600 Z^4 t_2 t_3^4 t_4^2 - 1387836450 Z^3 t_1 t_3^5 t_4^2 + 1131710580 Z^3 t_2 t_3^5 t_4^2 + 30270240 Z^2 t_1 t_3^6 t_4^2 \\
& - 41454553140 Z^2 t_2 t_3^6 t_4^2 + 2513330820 Z t_1 t_3^7 t_4^2 - 82007245320 Z t_2 t_3^7 t_4^2 - 73394521200 t_1 t_3^8 t_4^2 \\
& + 279679880480 t_2 t_3^8 t_4^2 - 47366919600 Z^4 t_3^9 t_4^2 + 490358278410 Z^3 t_3^{10} t_4^2 \\
& + 222131669760 Z^2 t_3^{11} t_4^2 - 2105167084020 Z t_3^{12} t_4^2 + 117287930760 t_3^{13} t_4^2 - 1801800 Z^2 t_1 t_2 t_4^3 \\
& + 502645 Z^2 t_2^2 t_4^3 + 178343880 Z t_1 t_2 t_3 t_4^3 - 471282240 Z t_2^2 t_3 t_4^3 + 550887480 t_1^2 t_3^2 t_4^3
\end{aligned}$$

$$
\begin{aligned}
&- 3013038600 t_1 t_2 t_3^2 t_4^3 + 3495396190 t_2^2 t_3^2 t_4^3 + 253187220 Z^4 t_1 t_3^3 t_4^3 - 7072934440 Z^4 t_2 t_3^3 t_4^3 \\
&- 4730025300 Z^3 t_1 t_3^4 t_4^3 + 1707576585 Z^3 t_2 t_3^4 t_4^3 + 552225960 Z^2 t_1 t_3^5 t_4^3 \\
&- 181909956800 Z^2 t_2 t_3^5 t_4^3 + 8064496440 Z t_1 t_3^6 t_4^3 - 557051053130 Z t_2 t_3^6 t_4^3 \\
&- 571892564100 t_1 t_3^7 t_4^3 + 1713932946440 t_2 t_3^7 t_4^3 - 420318640885 Z^4 t_3^8 t_4^3 \\
&+ 3020584166600 Z^3 t_3^9 t_4^3 + 2485406463540 Z^2 t_3^{10} t_4^3 - 15035772591840 Z t_3^{11} t_4^3 \\
&+ 1003720647215 t_3^{12} t_4^3 + 117408720 Z t_1 t_2 t_4^4 - 299467740 Z t_2^2 t_4^4 + 698686560 t_1^2 t_3 t_4^4 \\
&- 3428323470 t_1 t_2 t_3 t_4^4 + 3743165140 t_2^2 t_3 t_4^4 + 452457720 Z^4 t_1 t_3^2 t_4^4 - 13120844880 Z^4 t_2 t_3^2 t_4^4 \\
&- 9946304940 Z^3 t_1 t_3^3 t_4^4 + 1711812960 Z^3 t_2 t_3^3 t_4^4 + 4311450000 Z^2 t_1 t_3^4 t_4^4 \\
&- 517300779710 Z^2 t_2 t_3^4 t_4^4 + 5445194040 Z t_1 t_3^5 t_4^4 - 2191042425000 Z t_2 t_3^5 t_4^4 \\
&- 2526371647800 t_1 t_3^6 t_4^4 + 6387651109840 t_2 t_3^6 t_4^4 - 2177873972560 Z^4 t_3^7 t_4^4 \\
&+ 11940681712245 Z^3 t_3^8 t_4^4 + 17128817706000 Z^2 t_3^9 t_4^4 - 67149209560290 Z t_3^{10} t_4^4 \\
&- 306594064920 t_3^{11} t_4^4 + 325623078 t_1^2 t_4^5 - 1500098292 t_1 t_2 t_4^5 + 1666279956 t_2^2 t_4^5 \\
&+ 375151920 Z^4 t_1 t_3 t_4^5 - 13820183520 Z^4 t_2 t_3 t_4^5 - 13272110280 Z^3 t_1 t_3^2 t_4^5 \\
&+ 5755392390 Z^3 t_2 t_3^2 t_4^5 + 16253780400 Z^2 t_1 t_3^3 t_4^5 - 965371252560 Z^2 t_2 t_3^3 t_4^5 \\
&- 51150836880 Z t_1 t_3^4 t_4^5 - 5454942264750 Z t_2 t_3^4 t_4^5 - 7012595618820 t_1 t_3^5 t_4^5 \\
&+ 15544189806728 t_2 t_3^5 t_4^5 - 7277652650400 Z^4 t_3^6 t_4^5 + 31377190313040 Z^3 t_3^7 t_4^5 \\
&+ 81501909373995 Z^2 t_3^8 t_4^5 - 178725842192040 Z t_3^9 t_4^5 - 67636787989386 t_3^{10} t_4^5 \\
&+ 112638240 Z^4 t_1 t_4^6 - 6402120780 Z^4 t_2 t_4^6 - 10642687770 Z^3 t_1 t_3 t_4^6 \\
&+ 13434045900 Z^3 t_2 t_3 t_4^6 + 32414073120 Z^2 t_1 t_3^2 t_4^6 - 1140453023160 Z^2 t_2 t_3^2 t_4^6 \\
&- 200486346060 Z t_1 t_3^3 t_4^6 - 8911125490520 Z t_2 t_3^3 t_4^6 - 12912084273000 t_1 t_3^4 t_4^6 \\
&+ 26251989629580 t_2 t_3^4 t_4^6 - 16299801295360 Z^4 t_3^5 t_4^6 + 54123882941260 Z^3 t_3^6 t_4^6 \\
&+ 285186832252080 Z^2 t_3^7 t_4^6 - 159095532391730 Z t_3^8 t_4^6 - 592155868722440 t_3^9 t_4^6 \\
&- 3951441780 Z^3 t_1 t_4^7 + 9992662845 Z^3 t_2 t_4^7 + 33273634680 Z^2 t_1 t_3 t_4^7 \\
&- 769257944400 Z^2 t_2 t_3 t_4^7 - 344144366280 Z t_1 t_3^2 t_4^7 - 9532029191190 Z t_2 t_3^2 t_4^7 \\
&- 16402978515660 t_1 t_3^3 t_4^7 + 33218368469880 t_2 t_3^3 t_4^7 - 24477176331850 Z^4 t_3^4 t_4^7 \\
&+ 55258227121680 Z^3 t_3^5 t_4^7 + 761370806081820 Z^2 t_3^6 t_4^7 + 824238595771360 Z t_3^7 t_4^7 \\
&- 2816877685314825 t_3^8 t_4^7 + 13948711920 Z^2 t_1 t_4^8 - 222985549710 Z^2 t_2 t_4^8 \\
&- 295401780960 Z t_1 t_3 t_4^8 - 6261424455000 Z t_2 t_3 t_4^8 - 14784121087560 t_1 t_3^2 t_4^8 \\
&+ 33394516435170 t_2 t_3^2 t_4^8 - 23584597219920 Z^4 t_3^3 t_4^8 + 16056960750270 Z^3 t_3^4 t_4^8 \\
&+ 1574527351650000 Z^2 t_3^5 t_4^8 + 4294385945261940 Z t_3^6 t_4^8 - 8714684199695760 t_3^7 t_4^8 \\
&- 102536628480 Z t_1 t_4^9 - 1965495743890 Z t_2 t_4^9 - 9101820663060 t_1 t_3 t_4^9 \\
&+ 24489566932080 t_2 t_3 t_4^9 - 12721193619500 Z^4 t_3^2 t_4^9 - 39699237296400 Z^3 t_3^3 t_4^9 \\
&+ 2503422809509530 Z^2 t_3^4 t_4^9 + 10973606731634800 Z t_3^5 t_4^9 - 18532995357314350 t_3^6 t_4^9 \\
&- 2913828234600 t_1 t_4^{10} + 9135558521076 t_2 t_4^{10} - 2175752924720 Z^4 t_3 t_4^{10} \\
&- 62265514312950 Z^3 t_3^2 t_4^{10} + 2963428452131280 Z^2 t_3^3 t_4^{10} + 18352815805555120 Z t_3^4 t_4^{10}
\end{aligned}
$$

$$
\begin{aligned}
&- 27491356446813864t_3^5t_4^{10} + 670393575975Z^4t_4^{11} - 40062606209880Z^3t_3t_4^{11}\\
&+ 2453257188097320Z^2t_3^2t_4^{11} + 21135427360218240Zt_3^3t_4^{11} - 28131773838568515t_3^4t_4^{11}\\
&- 10470567594995Z^3t_4^{12} + 1262535017225760Z^2t_3t_4^{12} + 16440674522572910Zt_3^2t_4^{12}\\
&- 18989075206985880t_3^3t_4^{12} + 303171654680595Z^2t_4^{13} + 7881409599514040Zt_3t_4^{13}\\
&- 7356083542043190t_3^2t_4^{13} + 1769210991401910Zt_4^{14} - 720270845116440t_3t_4^{14}\\
&+ 439442490606461t_4^{15}\big) - \frac{51280000b}{3334050640011}t_4^5\big(187t_1^2 - 748t_1t_2 + 561t_2^2 - 8415Z^3t_2t_3^2\\
&- 151470Zt_2t_3^4 - 191862t_1t_3^5 + 383724t_2t_3^5 - 75735Z^4t_3^6 - 757350Z^2t_3^8 + 1878228t_3^{10}\\
&+ 1870Z^4t_2t_4 - 33660Z^3t_2t_3t_4 - 18700Z^2t_2t_3^2t_4 - 1211760Zt_2t_3^3t_4 - 1918620t_1t_3^4t_4\\
&+ 2285140t_2t_3^4t_4 - 908820Z^4t_3^5t_4 - 1698895Z^3t_3^6t_4 - 12117600Z^2t_3^7t_4 + 6255150Zt_3^8t_4\\
&+ 37564560t_3^9t_4 - 42075Z^3t_2t_4^2 - 74800Z^2t_2t_3t_4^2 - 3006960Zt_2t_3^2t_4^2 - 5755860t_1t_3^3t_4^2\\
&- 905080t_2t_3^3t_4^2 - 4204695Z^4t_3^4t_4^2 - 20386740Z^3t_3^5t_4^2 - 95594400Z^2t_3^6t_4^2\\
&+ 100082400Zt_3^7t_4^2 + 234430680t_3^8t_4^2 - 56100Z^2t_2t_4^3 - 2333760Zt_2t_3t_4^3 - 3837240t_1t_3^2t_4^3\\
&- 30578240t_2t_3^2t_4^3 - 9402360Z^4t_3^3t_4^3 - 102979965Z^3t_3^4t_4^3 - 468547200Z^2t_3^5t_4^3\\
&+ 710057700Zt_3^6t_4^3 + 144693120t_3^7t_4^3 + 302170Zt_2t_4^4 + 7843770t_1t_3t_4^4 - 69364020t_2t_3t_4^4\\
&- 10318825Z^4t_3^2t_4^4 - 280193320Z^3t_3^3t_4^4 - 1507905300Z^2t_3^4t_4^4 + 2916078000Zt_3^5t_4^4\\
&- 5267590460t_3^6t_4^4 + 9547956t_1t_4^5 - 49353876t_2t_4^5 - 4832740Z^4t_3t_4^5 - 436114965Z^3t_3^2t_4^5\\
&- 3187773600Z^2t_3^3t_4^5 + 7456862820Zt_3^4t_4^5 - 30924485328t_3^5t_4^5 - 746625Z^4t_4^6\\
&- 372747540Z^3t_3t_4^6 - 4277277840Z^2t_3^2t_4^6 + 11784099360Zt_3^3t_4^6 - 90377429340t_3^4t_4^6\\
&- 137914535Z^3t_4^7 - 3325988160Z^2t_3t_4^7 + 10667486500Zt_3^2t_4^7 - 162880356320t_3^3t_4^7\\
&- 1151667990Z^2t_4^8 + 4493560720Zt_3t_4^8 - 190524418920t_3^2t_4^8 + 378718230Zt_4^9\\
&- 138727401120t_3t_4^9 - 48850558828t_4^{10}\big) + \frac{288128\times 10^9c}{6580234353}t_4^9\big(-7Zt_2 - 42t_1t_3\\
&+ 84t_2t_3 + 7Z^4t_3^2 - 42Z^2t_3^4 - 2079t_3^6 - 84t_1t_4 + 210t_2t_4 + 28Z^4t_3t_4 + 84Z^3t_3^2t_4\\
&- 336Z^2t_3^3t_4 + 105Zt_3^4t_4 - 24948t_3^5t_4 + 21Z^4t_4^2 + 336Z^3t_3t_4^2 - 924Z^2t_3^2t_4^2 + 840Zt_3^3t_4^2\\
&- 156555t_3^4t_4^2 + 364Z^3t_4^3 - 1008Z^2t_3t_4^3 + 3710Zt_3^2t_4^3 - 587160t_3^3t_4^3 - 378Z^2t_4^4\\
&+ 8120Zt_3t_4^4 - 1165605t_3^2t_4^4 + 5901Zt_4^5 - 1029588t_3t_4^5 - 239377t_4^6\big)\\
&- \frac{4000c^2}{1203328310709}t_4^3\big(-7Zt_2 - 42t_1t_3 + 84t_2t_3 + 7Z^4t_3^2 - 42Z^2t_3^4 - 2079t_3^6 - 84t_1t_4\\
&+ 210t_2t_4 + 28Z^4t_3t_4 + 84Z^3t_3^2t_4 - 336Z^2t_3^3t_4 + 105Zt_3^4t_4 - 24948t_3^5t_4 + 21Z^4t_4^2\\
&+ 336Z^3t_3t_4^2 - 924Z^2t_3^2t_4^2 + 840Zt_3^3t_4^2 - 156555t_3^4t_4^2 + 364Z^3t_4^3 - 1008Z^2t_3t_4^3\\
&+ 3710Zt_3^2t_4^3 - 587160t_3^3t_4^3 - 378Z^2t_4^4 + 8120Zt_3t_4^4 - 1165605t_3^2t_4^4 + 5901Zt_4^5\\
&- 1029588t_3t_4^5 - 239377t_4^6\big)^2, \qquad \text{(A.3.8)}
\end{aligned}
$$

$$
\begin{aligned}
y_2 = &\frac{512\times 10^{10}}{77}t_4^{10} + \frac{b}{48850558828}\big(-187t_1^2 + 748t_1t_2 - 561t_2^2 + 8415Z^3t_2t_3^2\\
&+ 151470Zt_2t_3^4 + 191862t_1t_3^5 - 383724t_2t_3^5 + 75735Z^4t_3^6 + 757350Z^2t_3^8\\
&- 1878228t_3^{10} - 1870Z^4t_2t_4 + 33660Z^3t_2t_3t_4 + 18700Z^2t_2t_3^2t_4 + 1211760Zt_2t_3^3t_4
\end{aligned}
$$

$$
\begin{aligned}
&+ 1918620 t_1 t_3^4 t_4 - 2285140 t_2 t_3^4 t_4 + 908820 Z^4 t_3^5 t_4 + 1698895 Z^3 t_3^6 t_4 \\
&+ 12117600 Z^2 t_3^7 t_4 - 6255150 Z t_3^8 t_4 - 37564560 t_3^9 t_4 + 42075 Z^3 t_2 t_4^2 \\
&+ 74800 Z^2 t_2 t_3 t_4^2 + 3006960 Z t_2 t_3^2 t_4^2 + 5755860 t_1 t_3^3 t_4^2 + 905080 t_2 t_3^3 t_4^2 \\
&+ 4204695 Z^4 t_3^4 t_4^2 + 20386740 Z^3 t_3^5 t_4^2 + 95594400 Z^2 t_3^6 t_4^2 - 100082400 Z t_3^7 t_4^2 \\
&- 234430680 t_3^8 t_4^2 + 56100 Z^2 t_2 t_4^3 + 2333760 Z t_2 t_3 t_4^3 + 3837240 t_1 t_3^2 t_4^3 \\
&+ 30578240 t_2 t_3^2 t_4^3 + 9402360 Z^4 t_3^3 t_4^3 + 102979965 Z^3 t_3^4 t_4^3 + 468547200 Z^2 t_3^5 t_4^3 \\
&- 710057700 Z t_3^6 t_4^3 - 144693120 t_3^7 t_4^3 - 302170 Z t_2 t_4^4 - 7843770 t_1 t_3 t_4^4 \\
&+ 69364020 t_2 t_3 t_4^4 + 10318825 Z^4 t_3^2 t_4^4 + 280193320 Z^3 t_3^3 t_4^4 + 1507905300 Z^2 t_3^4 t_4^4 \\
&- 2916078000 Z t_3^5 t_4^4 + 5267590460 t_3^6 t_4^4 - 9547956 t_1 t_4^5 + 49353876 t_2 t_4^5 \\
&+ 4832740 Z^4 t_3 t_4^5 + 436114965 Z^3 t_3^2 t_4^5 + 3187773600 Z^2 t_3^3 t_4^5 - 7456862820 Z t_3^4 t_4^5 \\
&+ 30924485328 t_3^5 t_4^5 + 746625 Z^4 t_4^6 + 372747540 Z^3 t_3 t_4^6 + 4277277840 Z^2 t_3^2 t_4^6 \\
&- 11784099360 Z t_3^3 t_4^6 + 90377429340 t_3^4 t_4^6 + 137914535 Z^3 t_4^7 + 3325988160 Z^2 t_3 t_4^7 \\
&- 10667486500 Z t_3^2 t_4^7 + 162880356320 t_3^3 t_4^7 + 1151667990 Z^2 t_4^8 - 4493560720 Z t_3 t_4^8 \\
&+ 190524418920 t_3^2 t_4^8 - 378718230 Z t_4^9 + 138727401120 t_3 t_4^9 + 48850558828 t_4^{10} \\
&- 187 t_1^2 + 748 t_1 t_2 - 561 t_2^2 + 8415 Z^3 t_2 t_3^2 + 151470 Z t_2 t_3^4 + 191862 t_1 t_3^5 \\
&- 383724 t_2 t_3^5 + 75735 Z^4 t_3^6 + 757350 Z^2 t_3^8 - 1878228 t_3^{10} - 1870 Z^4 t_2 t_4 \\
&+ 33660 Z^3 t_2 t_3 t_4 + 18700 Z^2 t_2 t_3^2 t_4 + 1211760 Z t_2 t_3^3 t_4 + 1918620 t_1 t_3^4 t_4 \\
&- 2285140 t_2 t_3^4 t_4 + 908820 Z^4 t_3^5 t_4 + 1698895 Z^3 t_3^6 t_4 + 12117600 Z^2 t_3^7 t_4 \\
&- 6255150 Z t_3^8 t_4 - 37564560 t_3^9 t_4 + 42075 Z^3 t_2 t_4^2 + 74800 Z^2 t_2 t_3 t_4^2 \\
&+ 3006960 Z t_2 t_3^2 t_4^2 + 5755860 t_1 t_3^3 t_4^2 + 905080 t_2 t_3^3 t_4^2 + 4204695 Z^4 t_3^4 t_4^2 \\
&+ 20386740 Z^3 t_3^5 t_4^2 + 95594400 Z^2 t_3^6 t_4^2 - 100082400 Z t_3^7 t_4^2 - 234430680 t_3^8 t_4^2 \\
&+ 56100 Z^2 t_2 t_4^3 + 2333760 Z t_2 t_3 t_4^3 + 3837240 t_1 t_3^2 t_4^3 + 30578240 t_2 t_3^2 t_4^3 \\
&+ 9402360 Z^4 t_3^3 t_4^3 + 102979965 Z^3 t_3^4 t_4^3 + 468547200 Z^2 t_3^5 t_4^3 - 710057700 Z t_3^6 t_4^3 \\
&- 144693120 t_3^7 t_4^3 - 302170 Z t_2 t_4^4 - 7843770 t_1 t_3 t_4^4 + 69364020 t_2 t_3 t_4^4 \\
&+ 10318825 Z^4 t_3^2 t_4^4 + 280193320 Z^3 t_3^3 t_4^4 + 1507905300 Z^2 t_3^4 t_4^4 - 2916078000 Z t_3^5 t_4^4 \\
&+ 5267590460 t_3^6 t_4^4 - 9547956 t_1 t_4^5 + 49353876 t_2 t_4^5 + 4832740 Z^4 t_3 t_4^5 \\
&+ 436114965 Z^3 t_3^2 t_4^5 + 3187773600 Z^2 t_3^3 t_4^5 - 7456862820 Z t_3^4 t_4^5 \\
&+ 30924485328 t_3^5 t_4^5 + 746625 Z^4 t_4^6 + 372747540 Z^3 t_3 t_4^6 + 4277277840 Z^2 t_3^2 t_4^6 \\
&- 11784099360 Z t_3^3 t_4^6 + 90377429340 t_3^4 t_4^6 + 137914535 Z^3 t_4^7 + 3325988160 Z^2 t_3 t_4^7 \\
&- 10667486500 Z t_3^2 t_4^7 + 162880356320 t_3^3 t_4^7 + 1151667990 Z^2 t_4^8 - 4493560720 Z t_3 t_4^8 \\
&+ 190524418920 t_3^2 t_4^8 - 378718230 Z t_4^9 + 138727401120 t_3 t_4^9 + 48850558828 t_4^{10} \Big) \\
&+ \frac{400000c}{4069409} t_4^4 \Big( -7 Z t_2 - 42 t_1 t_3 + 84 t_2 t_3 + 7 Z^4 t_3^2 - 42 Z^2 t_3^4 - 2079 t_3^6 - 84 t_1 t_4 \\
&+ 210 t_2 t_4 + 28 Z^4 t_3 t_4 + 84 Z^3 t_3^2 t_4 - 336 Z^2 t_3^3 t_4 + 105 Z t_3^4 t_4 - 24948 t_3^5 t_4 + 21 Z^4 t_4^2 \\
&+ 336 Z^3 t_3 t_4^2 - 924 Z^2 t_3^2 t_4^2 + 840 Z t_3^3 t_4^2 - 156555 t_3^4 t_4^2 + 364 Z^3 t_4^3 - 1008 Z^2 t_3 t_4^3
\end{aligned}
$$

$$
\begin{aligned}
&+ 3710Zt_3^2t_4^3 - 587160t_3^3t_4^3 - 378Z^2t_4^4 + 8120Zt_3t_4^4 - 1165605t_3^2t_4^4 + 5901Zt_4^5 \\
&- 1029588t_3t_4^5 - 239377t_4^6\big), \qquad \text{(A.3.9)}\\
y_3 =& -\frac{32\times10^6}{7}t_4^6 + \frac{c}{239377}\big(7Zt_2 + 42t_1t_3 - 84t_2t_3 - 7Z^4t_3^2 + 42Z^2t_3^4 \\
&+ 2079t_3^6 + 84t_1t_4 - 210t_2t_4 - 28Z^4t_3t_4 - 84Z^3t_3^2t_4 + 336Z^2t_3^3t_4 \\
&- 105Zt_3^4t_4 + 24948t_3^5t_4 - 21Z^4t_4^2 - 336Z^3t_3t_4^2 + 924Z^2t_3^2t_4^2 \\
&- 840Zt_3^3t_4^2 + 156555t_3^4t_4^2 - 364Z^3t_4^3 + 1008Z^2t_3t_4^3 - 3710Zt_3^2t_4^3 \\
&+ 587160t_3^3t_4^3 + 378Z^2t_4^4 - 8120Zt_3t_4^4 + 1165605t_3^2t_4^4 - 5901Zt_4^5 \\
&+ 1029588t_3t_4^5 + 239377t_4^6\big), \qquad \text{(A.3.10)}\\
y_4 =& 20t_4, \qquad \text{(A.3.11)}
\end{aligned}
$$

where $a, b, c \in \mathbb{C}$. In order to find $a, b$ and $c$ we perform steps 5–7 from the introduction of Chapter 4. That is, we transform the intersection form (4.5.86)–(4.5.92) into $y$ coordinates by applying formulas (A.3.8)–(A.3.11) and compare it with the expression given by Lemma 4.5.1. We find that

$$
a = \frac{75776\times10^{10}}{177147}, \qquad b = \frac{183907986176\times10^6}{72171}, \qquad c = \frac{19150160000}{1701},
$$

which implies the statement. □

**Proposition A.3.2.** *We have that*

$$
\det(g^{ij}(t)) = \frac{c\prod_{\alpha\in R_{H_4}}(\alpha,\, x)}{Q(t,Z)^2},
$$

*where* $c = 2^{48}3^{54}5$ *and*

$$
\begin{aligned}
Q(t,Z) =& 5^{18}\big(t_1^4 - 8t_1^3t_2 + 27t_1^2t_2^2 - 44t_1t_2^3 + 28t_2^4 - 10Z^4t_1^3t_3 + 60Z^4t_1^2t_2t_3 - 150Z^4t_1t_2^2t_3 \\
&+ 140Z^4t_2^3t_3 + 135Z^3t_1^2t_2t_3^2 - 540Z^3t_1t_2^2t_3^2 + 540Z^3t_2^3t_3^2 + 60Z^2t_1^3t_3^3 - 360Z^2t_1^2t_2t_3^3 \\
&- 990Z^2t_1t_2^2t_3^3 + 2940Z^2t_2^3t_3^3 + 810Zt_1^2t_2t_3^4 - 3240Zt_1t_2^2t_3^4 + 2565Zt_2^3t_3^4 + 4698t_1^3t_3^5 \\
&- 28188t_1^2t_2t_3^5 + 31752t_1t_2^2t_3^5 + 11664t_2^3t_3^5 - 14175Z^4t_1^2t_3^6 + 56700Z^4t_1t_2t_3^6 \\
&- 90720Z^4t_2^2t_3^6 + 30240Z^3t_1t_2t_3^7 - 60480Z^3t_2^2t_3^7 + 260010Z^2t_1^2t_3^8 \\
&- 1040040Z^2t_1t_2t_3^8 + 372195Z^2t_2^2t_3^8 + 2264760Zt_1t_2t_3^9 - 4529520Zt_2^2t_3^9 \\
&+ 5112639t_1^2t_3^{10} - 20450556t_1t_2t_3^{10} + 21171537t_2^2t_3^{10} + 28331370Z^4t_1t_3^{11} \\
&- 56662740Z^4t_2t_3^{11} + 59435370Z^3t_2t_3^{12} + 261113220Z^2t_1t_3^{13} - 522226440Z^2t_2t_3^{13} \\
&+ 577980360Zt_2t_3^{14} - 951725538t_1t_3^{15} + 1903451076t_2t_3^{15} - 2250685440Z^4t_3^{16} \\
&- 24907655520Z^2t_3^{18} + 41039061561t_3^{20} - 20Z^4t_1^3t_4 + 120Z^4t_1^2t_2t_4 - 300Z^4t_1t_2^2t_4 \\
&+ 280Z^4t_2^3t_4 - 80Z^3t_1^3t_3t_4 + 1020Z^3t_1^2t_2t_3t_4 - 3090Z^3t_1t_2^2t_3t_4 + 2740Z^3t_2^3t_3t_4 \\
&+ 360Z^2t_1^3t_3^2t_4 - 14580Z^2t_1t_2^2t_3^2t_4 + 27630Z^2t_2^3t_3^2t_4 - 120Zt_1^3t_3^3t_4 + 7200Zt_1^2t_2t_3^3t_4
\end{aligned}
$$

$$- 30420Zt_1t_2^2t_3^3t_4 + 27600Zt_2^3t_3^3t_4 + 46980t_1^3t_3^4t_4 - 228150t_1^2t_2t_3^4t_4 + 102600t_1t_2^2t_3^4t_4$$
$$+ 409860t_2^3t_3^4t_4 - 170100Z^4t_1^2t_3^5t_4 + 679050Z^4t_1t_2t_3^5t_4 - 1085940Z^4t_2^2t_3^5t_4$$
$$+ 175095Z^3t_1^2t_3^6t_4 - 277020Z^3t_1t_2t_3^6t_4 + 878040Z^3t_2^2t_3^6t_4 + 4160160Z^2t_1^2t_3^7t_4$$
$$- 15176160Z^2t_1t_2t_3^7t_4 + 3026160Z^2t_2^2t_3^7t_4 - 1180170Zt_1^2t_3^8t_4 + 45486360Zt_1t_2t_3^8t_4$$
$$- 83506545Zt_2^2t_3^8t_4 + 102252780t_1^2t_3^9t_4 - 404041770t_1t_2t_3^9t_4 + 413492040t_2^2t_3^9t_4$$
$$+ 623290140Z^4t_1t_3^{10}t_4 - 1409886000Z^4t_2t_3^{10}t_4 - 212848560Z^3t_1t_3^{11}t_4$$
$$+ 1852146000Z^3t_2t_3^{11}t_4 + 6788943720Z^2t_1t_3^{12}t_4 - 15367035690Z^2t_2t_3^{12}t_4$$
$$- 696749040Zt_1t_3^{13}t_4 + 17576948160Zt_2t_3^{13}t_4 - 28551766140t_1t_3^{14}t_4$$
$$+ 63192905730t_2t_3^{14}t_4 - 72021934080Z^4t_3^{15}t_4 + 1767161610Z^3t_3^{16}t_4$$
$$- 896675598720Z^2t_3^{17}t_4 - 161702143560Zt_3^{18}t_4 + 1641562462440t_3^{19}t_4$$
$$- 160Z^3t_1^3t_4^2 + 1365Z^3t_1^2t_2t_4^2 - 3480Z^3t_1t_2^2t_4^2 + 2780Z^3t_2^3t_4^2 + 660Z^2t_1^3t_3t_4^2$$
$$+ 4680Z^2t_1^2t_2t_3t_4^2 - 45180Z^2t_1t_2^2t_3t_4^2 + 71760Z^2t_2^3t_3t_4^2 - 720Zt_1^3t_3^2t_4^2$$
$$+ 18360Zt_1^2t_2t_3^2t_4^2 - 83160Zt_1t_2^2t_3^2t_4^2 + 72990Zt_2^3t_3^2t_4^2 + 157740t_1^3t_3^3t_4^2$$
$$- 516600t_1^2t_2t_3^3t_4^2 - 826380t_1t_2^2t_3^3t_4^2 + 3083640t_2^3t_3^3t_4^2 - 808245Z^4t_1^2t_3^4t_4^2$$
$$+ 3219480Z^4t_1t_2t_3^4t_4^2 - 5432130Z^4t_2^2t_3^4t_4^2 + 2101140Z^3t_1^2t_3^5t_4^2 - 7904790Z^3t_1t_2t_3^5t_4^2$$
$$+ 19697580Z^3t_2^2t_3^5t_4^2 + 29586060Z^2t_1^2t_3^6t_4^2 - 97841520Z^2t_1t_2t_3^6t_4^2 + 3676590Z^2t_2^2t_3^6t_4^2$$
$$- 18882720Zt_1^2t_3^7t_4^2 + 370281780Zt_1t_2t_3^7t_4^2 - 621104760Zt_2^2t_3^7t_4^2 + 922901040t_1^2t_3^8t_4^2$$
$$- 3602155860t_1t_2t_3^8t_4^2 + 3755785320t_2^2t_3^8t_4^2 + 6355298070Z^4t_1t_3^9t_4^2$$
$$- 15976710540Z^4t_2t_3^9t_4^2 - 4682668320Z^3t_1t_3^{10}t_4^2 + 24567864975Z^3t_2t_3^{10}t_4^2$$
$$+ 81725441670Z^2t_1t_3^{11}t_4^2 - 206390441340Z^2t_2t_3^{11}t_4^2 - 18115475040Zt_1t_3^{12}t_4^2$$
$$+ 230079285405Zt_2t_3^{12}t_4^2 - 424595173320t_1t_3^{13}t_4^2 + 1019692803240t_2t_3^{13}t_4^2$$
$$- 1069824008205Z^4t_3^{14}t_4^2 + 56549171520Z^3t_3^{15}t_4^2 - 15055905236895Z^2t_3^{16}t_4^2$$
$$- 5821277168160Zt_3^{17}t_4^2 + 31230655376220t_3^{18}t_4^2 + 360Z^2t_1^3t_4^3 + 6480Z^2t_1^2t_2t_4^3$$
$$- 39960Z^2t_1t_2^2t_4^3 + 56520Z^2t_2^3t_4^3 - 2120Zt_1^3t_3t_4^3 + 17040Zt_1^2t_2t_3t_4^3 - 85260Zt_1t_2^2t_3t_4^3$$
$$+ 59920Zt_2^3t_3t_4^3 + 194760t_1^3t_3^2t_4^3 + 70740t_1^2t_2t_3^2t_4^3 - 4679640t_1t_2^2t_3^2t_4^3 + 9329940t_2^3t_3^2t_4^3$$
$$- 1929960Z^4t_1^2t_3^3t_4^3 + 7822980Z^4t_1t_2t_3^3t_4^3 - 14848920Z^4t_2^2t_3^3t_4^3 + 11327175Z^3t_1^2t_3^4t_4^3$$
$$- 57245400Z^3t_1t_2t_3^4t_4^3 + 127791000Z^3t_2^2t_3^4t_4^3 + 122063760Z^2t_1^2t_3^5t_4^3$$
$$- 367832340Z^2t_1t_2t_3^5t_4^3 - 38147760Z^2t_2^2t_3^5t_4^3 - 129925620Zt_1^2t_3^6t_4^3$$
$$+ 1602377640Zt_1t_2t_3^6t_4^3 - 2447251380Zt_2^2t_3^6t_4^3 + 4950149760t_1^2t_3^7t_4^3$$
$$- 19302997680t_1t_2t_3^7t_4^3 + 21310356960t_2^2t_3^7t_4^3 + 39600548460Z^4t_1t_3^8t_4^3$$
$$- 108433972620Z^4t_2t_3^8t_4^3 - 48264927300Z^3t_1t_3^9t_4^3 + 191367918900Z^3t_2t_3^9t_4^3$$
$$+ 603105622020Z^2t_1t_3^{10}t_4^3 - 1672532110170Z^2t_2t_3^{10}t_4^3 - 219064276440Zt_1t_3^{11}t_4^3$$
$$+ 1724330984040Zt_2t_3^{11}t_4^3 - 4110912589680t_1t_3^{12}t_4^3 + 10433350736820t_2t_3^{12}t_4^3$$
$$- 9788930687340Z^4t_3^{13}t_4^3 + 991283684175Z^3t_3^{14}t_4^3 - 156594617111520Z^2t_3^{15}t_4^3$$
$$- 96402712770585Zt_3^{16}t_4^3 + 375751110671280t_3^{17}t_4^3 - 2320Zt_1^3t_4^4 + 7710Zt_1^2t_2t_4^4$$

$$
\begin{aligned}
&- 39120Zt_1t_2^2t_4^4 + 17360Zt_2^3t_4^4 + 19370t_1^3t_3t_4^4 + 1402260t_1^2t_2t_3t_4^4 - 8028420t_1t_2^2t_3t_4^4 \\
&+ 12528440t_2^3t_3t_4^4 - 2232225Z^4t_1^2t_3^2t_4^4 + 9763740Z^4t_1t_2t_3^2t_4^4 - 22496130Z^4t_2^2t_3^2t_4^4 \\
&+ 34587000Z^3t_1^2t_3^3t_4^4 - 201876300Z^3t_1t_2t_3^3t_4^4 + 406474200Z^3t_2^2t_3^3t_4^4 \\
&+ 318524400Z^2t_1^2t_3^4t_4^4 - 889979400Z^2t_1t_2t_3^4t_4^4 - 155756925Z^2t_2^2t_3^4t_4^4 \\
&- 501675120Zt_1^2t_3^5t_4^4 + 3978937620Zt_1t_2t_3^5t_4^4 - 5557521240Zt_2^2t_3^5t_4^4 \\
&+ 17586874170t_1^2t_3^6t_4^4 - 70059884040t_1t_2t_3^6t_4^4 + 84824190630t_2^2t_3^6t_4^4 \\
&+ 166936990500Z^4t_1t_3^7t_4^4 - 488053009800Z^4t_2t_3^7t_4^4 - 306848493000Z^3t_1t_3^8t_4^4 \\
&+ 996284423700Z^3t_2t_3^8t_4^4 + 3040272265500Z^2t_1t_3^9t_4^4 - 9109160013600Z^2t_2t_3^9t_4^4 \\
&- 1631090474640Zt_1t_3^{10}t_4^4 + 8182348108920Zt_2t_3^{10}t_4^4 - 28618601994810t_1t_3^{11}t_4^4 \\
&+ 74849306395860t_2t_3^{11}t_4^4 - 61371221262300Z^4t_3^{12}t_4^4 + 11922175131300Z^3t_3^{13}t_4^4 \\
&- 1125923468008500Z^2t_3^{14}t_4^4 - 973703622339360Zt_3^{15}t_4^4 + 3213742938357960t_3^{16}t_4^4 \\
&- 79724t_1^3t_4^5 + 1135578t_1^2t_2t_4^5 - 4568592t_1t_2^2t_4^5 + 6215464t_2^3t_4^5 - 746820Z^4t_1^2t_3t_4^5 \\
&+ 4965570Z^4t_1t_2t_3t_4^5 - 16569900Z^4t_2^2t_3t_4^5 + 61896825Z^3t_1^2t_3^2t_4^5 \\
&- 383579820Z^3t_1t_2t_3^2t_4^5 + 696924540Z^3t_2^2t_3^2t_4^5 + 535662720Z^2t_1^2t_3^3t_4^5 \\
&- 1433327400Z^2t_1t_2t_3^3t_4^5 - 145485720Z^2t_2^2t_3^3t_4^5 - 1168171200Zt_1^2t_3^4t_4^5 \\
&+ 5438749320Zt_1t_2t_3^4t_4^5 - 7202077065Zt_2^2t_3^4t_4^5 + 43776292536t_1^2t_3^5t_4^5 \\
&- 183313806372t_1t_2t_3^5t_4^5 + 247371652944t_2^2t_3^5t_4^5 + 497212846200Z^4t_1t_3^6t_4^5 \\
&- 1523271884940Z^4t_2t_3^6t_4^5 - 1332748420200Z^3t_1t_3^7t_4^5 + 3699321716880Z^3t_2t_3^7t_4^5 \\
&+ 11028724409880Z^2t_1t_3^8t_4^5 - 35113428144900Z^2t_2t_3^8t_4^5 - 8356962971160Zt_1t_3^9t_4^5 \\
&+ 25214869319760Zt_2t_3^9t_4^5 - 149974007567868t_1t_3^{10}t_4^5 + 396226024262118t_2t_3^{10}t_4^5 \\
&- 275723215492320Z^4t_3^{11}t_4^5 + 104262336075060Z^3t_3^{12}t_4^5 \\
&- 5890247939283120Z^2t_3^{13}t_4^5 - 6662287485691110Zt_3^{14}t_4^5 \\
&+ 20862377981699328t_3^{15}t_4^5 + 390285Z^4t_1^2t_4^6 + 54000Z^4t_1t_2t_4^6 - 3981420Z^4t_2^2t_4^6 \\
&+ 60539940Z^3t_1^2t_3t_4^6 - 377909550Z^3t_1t_2t_3t_4^6 + 618993900Z^3t_2^2t_3t_4^6 \\
&+ 562951620Z^2t_1^2t_3^2t_4^6 - 1517408640Z^2t_1t_2t_3^2t_4^6 + 274416660Z^2t_2^2t_3^2t_4^6 \\
&- 1607005440Zt_1^2t_3^3t_4^6 + 2709098460Zt_1t_2t_3^3t_4^6 - 4345902720Zt_2^2t_3^3t_4^6 \\
&+ 78521584140t_1^2t_3^4t_4^6 - 354246753240t_1t_2t_3^4t_4^6 + 527101019460t_2^2t_3^4t_4^6 \\
&+ 1059949743300Z^4t_1t_3^5t_4^6 - 3343754387160Z^4t_2t_3^5t_4^6 - 4138465368240Z^3t_1t_3^6t_4^6 \\
&+ 10124434831410Z^3t_2t_3^6t_4^6 + 29508760554660Z^2t_1t_3^7t_4^6 \\
&- 98083899309960Z^2t_2t_3^7t_4^6 - 31214243093040Zt_1t_3^8t_4^6 + 47845380249990Zt_2t_3^8t_4^6 \\
&- 605682309980880t_1t_3^9t_4^6 + 1594108771443000t_2t_3^9t_4^6 - 892824183131805Z^4t_3^{10}t_4^6 \\
&+ 681168388356000Z^3t_3^{11}t_4^6 - 22841532966872925Z^2t_3^{12}t_4^6 - 32133293778214440Zt_3^{13}t_4^6 \\
&+ 107334366449160420t_3^{14}t_4^6 + 24978105Z^3t_1^2t_4^7 - 151912800Z^3t_1t_2t_4^7 + 223789500Z^3t_2^2t_4^7 \\
&+ 334555920Z^2t_1^2t_3t_4^7 - 977146740Z^2t_1t_2t_3t_4^7 + 648476280Z^2t_2^2t_3t_4^7 - 1134888300Zt_1^2t_3^2t_4^7 \\
&- 2766081960Zt_1t_2t_3^2t_4^7 + 939876210Zt_2^2t_3^2t_4^7 + 101594826720t_1^2t_3^3t_4^7 - 498246279840t_1t_2t_3^3t_4^7
\end{aligned}
$$

$$
\begin{aligned}
&+ 792374490720t_2^2t_3^3t_4^7 + 1600729131240Z^4t_1t_3^4t_4^7 - 5113751063580Z^4t_2t_3^4t_4^7 \\
&- 9343072192680Z^3t_1t_3^5t_4^7 + 20661806439720Z^3t_2t_3^5t_4^7 + 58644536657400Z^2t_1t_3^6t_4^7 \\
&- 199376069372820Z^2t_2t_3^6t_4^7 - 87890851336320Zt_1t_3^7t_4^7 + 39993247190880Zt_2t_3^7t_4^7 \\
&- 1908111011495040t_1t_3^8t_4^7 + 4959742322181600t_2t_3^8t_4^7 - 1988221653444420Z^4t_3^9t_4^7 \\
&+ 3389816075638725Z^3t_3^{10}t_4^7 - 6479252158781484 0Z^2t_3^{11}t_4^7 - 107938134187599195Zt_3^{12}t_4^7 \\
&+ 451550442078290160t_3^{13}t_4^7 + 84679830Z^2t_1^2t_4^8 - 296654400Z^2t_1t_2t_4^8 + 363518280Z^2t_2^2t_4^8 \\
&- 197711280Zt_1^2t_3t_4^8 - 4715582940Zt_1t_2t_3t_4^8 + 3348224640Zt_2^2t_3t_4^8 + 91241494335t_1^2t_3^2t_4^8 \\
&- 481040221500t_1t_2t_3^2t_4^8 + 787491606465t_2^2t_3^2t_4^8 + 1652169478050Z^4t_1t_3^3t_4^8 \\
&- 5258678120100Z^4t_2t_3^3t_4^8 - 15281893155600Z^3t_1t_3^4t_4^8 + 31231958611950Z^3t_2t_3^4t_4^8 \\
&+ 85875404004900Z^2t_1t_3^5t_4^8 - 290760223113720Z^2t_2t_3^5t_4^8 - 190300878178560Zt_1t_3^6t_4^8 \\
&- 44139772494900Zt_2t_3^6t_4^8 - 4711370651790390t_1t_3^7t_4^8 + 12037575667445100t_2t_3^7t_4^8 \\
&- 2416263783387660Z^4t_3^8t_4^8 + 13030761474944100Z^3t_3^9t_4^8 - 124274418518095110Z^2t_3^{10}t_4^8 \\
&- 222472263571812360Zt_3^{11}t_4^8 + 1589231062583667810t_3^{12}t_4^8 + 125799210Zt_1^2t_4^9 \\
&- 2029894560Zt_1t_2t_4^9 + 1744323120Zt_2^2t_4^9 + 51065065980t_1^2t_3t_4^9 - 282159168930t_1t_2t_3t_4^9 \\
&+ 459824509800t_2^2t_3t_4^9 + 1080611343180Z^4t_1t_3^2t_4^9 - 3370338514260Z^4t_2t_3^2t_4^9 \\
&- 17683516733400Z^3t_1t_3^3t_4^9 + 34044343351200Z^3t_2t_3^3t_4^9 + 90377706911400Z^2t_1t_3^4t_4^9 \\
&- 292586368091850Z^2t_2t_3^4t_4^9 - 319880244841200Zt_1t_3^5t_4^9 - 164227495885200Zt_2t_3^5t_4^9 \\
&- 9101737976823540t_1t_3^6t_4^9 + 22812488913077550t_2t_3^6t_4^9 + 1512102693009600Z^4t_3^7t_4^9 \\
&+ 39010320692781750Z^3t_3^8t_4^9 - 104933849342326200Z^2t_3^9t_4^9 - 576479130448716 00Zt_3^{10}t_4^9 \\
&+ 4750149485088794160t_3^{11}t_4^9 + 13333272732t_1^2t_4^{10} - 74948092884t_1t_2t_4^{10} \\
&+ 118479230604t_2^2t_4^{10} + 381343372470Z^4t_1t_3t_4^{10} - 1143784169820Z^4t_2t_3t_4^{10} \\
&- 13738452022800Z^3t_1t_3^2t_4^{10} + 25325013726855Z^3t_2t_3^2t_4^{10} + 65057899482630Z^2t_1t_3^3t_4^{10} \\
&- 186063754795740Z^2t_2t_3^3t_4^{10} - 416402516538720Zt_1t_3^4t_4^{10} - 164901377328795Zt_2t_3^4t_4^{10} \\
&- 13634662952546136t_1t_3^5t_4^{10} + 33509084616078840t_2t_3^5t_4^{10} + 14774294669675625Z^4t_3^6t_4^{10} \\
&+ 91134379135657440Z^3t_3^7t_4^{10} + 245429506237473135Z^2t_3^8t_4^{10} \\
&+ 1546823706363365760Zt_3^9t_4^{10} + 12139695071909773884t_3^{10}t_4^{10} + 46581258060Z^4t_1t_4^{11} \\
&- 130448066760Z^4t_2t_4^{11} - 6428631402180Z^3t_1t_3t_4^{11} + 11510518356900Z^3t_2t_3t_4^{11} \\
&+ 29112472888740Z^2t_1t_3^2t_4^{11} - 58230892791810Z^2t_2t_3^2t_4^{11} - 410911248018600Zt_1t_3^3t_4^{11} \\
&+ 19555275943800Zt_2t_3^3t_4^{11} - 15547240859288400t_1t_3^4t_4^{11} + 37476852009476580t_2t_3^4t_4^{11} \\
&+ 37246528816981740Z^4t_3^5t_4^{11} + 165385801859444505Z^3t_3^6t_4^{11} \\
&+ 1234278774817248240Z^2t_3^7t_4^{11} + 6876401514451407585Zt_3^8t_4^{11} \\
&+ 26487620179073694000t_3^9t_4^{11} - 1368912535560Z^3t_1t_4^{12} + 2415363453900Z^3t_2t_4^{12} \\
&+ 6548631363180Z^2t_1t_3t_4^{12} + 2196763934400Z^2t_2t_3t_4^{12} - 291302732248560Zt_1t_3^2t_4^{12} \\
&+ 196139559167100Zt_2t_3^2t_4^{12} - 13055209242804270t_1t_3^3t_4^{12} + 30876359195017980t_2t_3^3t_4^{12} \\
&+ 57575289947662980Z^4t_3^4t_4^{12} + 230321163043311660Z^3t_3^5t_4^{12}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2840269371197001480Z^2t_3^6t_4^{12} + 18070887886075610640Zt_3^7t_4^{12}\\
&+ 48911702641818454800t_3^8t_4^{12} + 342214149240Z^2t_1t_4^{13} + 4989372957000Z^2t_2t_4^{13}\\
&- 132094704189960Zt_1t_3t_4^{13} + 173586060037440Zt_2t_3t_4^{13} - 7613871428584980t_1t_3^2t_4^{13}\\
&+ 17666917064237610t_2t_3^2t_4^{13} + 59702848965923040Z^4t_3^3t_4^{13}\\
&+ 240750778794019200Z^3t_3^4t_4^{13} + 4362778849373560800Z^2t_3^5t_4^{13}\\
&+ 33833237099751795330Zt_3^6t_4^{13} + 75333095064339828480t_3^7t_4^{13} - 28601385359760Zt_1t_4^{14}\\
&+ 52696455586560Zt_2t_4^{14} - 2753498167526160t_1t_3t_4^{14} + 6265112215573800t_2t_3t_4^{14}\\
&+ 40849057642622625Z^4t_3^2t_4^{14} + 182004077683166400Z^3t_3^3t_4^{14}\\
&+ 4763567155988589525Z^2t_3^4t_4^{14} + 47161494206638893720Zt_3^5t_4^{14}\\
&+ 94896799829443418220t_3^6t_4^{14} - 464574810325824t_1t_4^{15} + 1035825991392456t_2t_4^{15}\\
&+ 16792617868854660Z^4t_3t_4^{15} + 93394202064082515Z^3t_3^2t_4^{15}\\
&+ 3683303041183491240Z^2t_3^3t_4^{15} + 49078149204484853235Zt_3^4t_4^{15}\\
&+ 95330482737696148752t_3^5t_4^{15} + 3156310738979460Z^4t_4^{16} + 28839900599723340Z^3t_3t_4^{16}\\
&+ 1927299710744410770Z^2t_3^2t_4^{16} + 37269058468813061400Zt_3^3t_4^{16}\\
&+ 73834767529239730365t_3^4t_4^{16} + 3982485947837580Z^3t_4^{17} + 613367713823364360Z^2t_3t_4^{17}\\
&+ 19593274872184389180Zt_3^2t_4^{17} + 42001392050838322920t_3^3t_4^{17}\\
&+ 89638642917188280Z^2t_4^{18} + 6393261915974095920Zt_3t_4^{18}\\
&+ 16246061129946251160t_3^2t_4^{18} + 977495449215058560Zt_4^{19} + 3707034900420091680t_3t_4^{19}\\
&+ 353368519727182416t_4^{20}\Big)\,.
\end{aligned}
$$

By Proposition 4.0.3, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 4.5.17, which leads to Proposition A.3.2.

**Proposition A.3.3.** *The unity vector field $e = \partial_{t_1}$ in the $y$ coordinates has the form*

$$
\begin{aligned}
e(y) = {}& \frac{160000}{81}\left(t_3 + 2t_4\right)\partial_{y_3} - \frac{128\times 10^6}{6561}\big(t_1 - 2t_2 - 513t_3^5 - 5130t_3^4t_4 - 15390t_3^3t_4^2\\
&- 10260t_3^2t_4^3 + 23355t_3t_4^4 + 30294t_4^5\big)\partial_{y_2} - \frac{2048\times 10^9}{531441}\big(2t_1t_2 - 4t_2^2\\
&- 5Z^4t_2t_3 + 50Z^2t_2t_3^3 + 2125t_2t_3^5 + 2720Z^3t_3^7 - 3360Zt_3^9 - 10Z^4t_2t_4\\
&- 45Z^3t_2t_3t_4 + 300Z^2t_2t_3^2t_4 - 2490Zt_2t_3^3t_4 - 7830t_1t_3^4t_4 + 36910t_2t_3^4t_4\\
&- 1110Z^4t_3^5t_4 + 38080Z^3t_3^6t_4 - 360Z^2t_3^7t_4 - 60480Zt_3^8t_4 + 678900t_3^9t_4\\
&- 90Z^3t_2t_4^2 + 550Z^2t_2t_3t_4^2 - 14940Zt_2t_3^2t_4^2 - 62640t_1t_3^3t_4^2 + 204410t_2t_3^3t_4^2\\
&- 11100Z^4t_3^4t_4^2 + 231075Z^3t_3^5t_4^2 - 5040Z^2t_3^6t_4^2 - 418470Zt_3^7t_4^2 + 12220200t_3^8t_4^2\\
&+ 300Z^2t_2t_4^3 - 29630Zt_2t_3t_4^3 - 183060t_1t_3^2t_4^3 + 500900t_2t_3^2t_4^3 - 42220Z^4t_3^3t_4^3\\
&+ 787550Z^3t_3^4t_4^3 - 91560Z^2t_3^5t_4^3 - 1342740Zt_3^6t_4^3 + 95239300t_3^7t_4^3 - 19420Zt_2t_4^4\\
&- 231120t_1t_3t_4^4 + 567345t_2t_3t_4^4 - 75720Z^4t_3^2t_4^4 + 1655290Z^3t_3^3t_4^4 - 714000Z^2t_3^4t_4^4
\end{aligned}
$$

$$
\begin{aligned}
&- 907590Zt_3^5t_4^4 + 420908600t_3^6t_4^4 - 103086t_1t_4^5 + 238302t_2t_4^5 - 63170Z^4t_3t_4^5\\
&+ 2205180Z^3t_3^2t_4^5 - 2691600Z^2t_3^3t_4^5 + 8506980Zt_3^4t_4^5 + 1167543000t_3^5t_4^5\\
&- 19140Z^4t_4^6 + 1762495Z^3t_3t_4^6 - 5370720Z^2t_3^2t_4^6 + 33331510Zt_3^3t_4^6\\
&+ 2138618800t_3^4t_4^6 + 651230Z^3t_4^7 - 5518080Z^2t_3t_4^7 + 57157380Zt_3^2t_4^7\\
&+ 2694048260t_3^3t_4^7 - 2315520Z^2t_4^8 + 48981110Zt_3t_4^8 + 2453396760t_3^2t_4^8\\
&+ 16963980Zt_4^9 + 1621710860t_3t_4^9 + 597401560t_4^{10}\Big)\,\partial_{y_1}.
\end{aligned}
$$

*Proof.* We have that

$$e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 4.5.17. □

## A.4 Extra formulas for $H_4(7)$

Recall from Section 4.5.5 that the degrees of the $t$ coordinates and $Z$ are $\deg t_1(x) = \frac{20}{3}$, $\deg t_2(x) = \frac{16}{3}$, $\deg t_3(x) = \frac{10}{3}$, $\deg t_4(x) = 2$ and $\deg Z(x) = \frac{2}{3}$. This allows us to deduce which harmonic polynomials of the $t$ coordinates and $Z$ have the same degrees as the basic invariants of $H_4$, which the following Proposition makes precise.

**Proposition A.4.1.** *Let $V_1 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 30\}$, let $V_2 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 20\}$ and let $V_3 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 12\}$. The harmonic elements of $V_1$ are proportional to*

$$
\begin{aligned}
&24743957519531250000Z^7t_1^3t_2 - 534469482421875000Zt_1^2t_2^3 + 56251263427734375Z^3t_1t_2^4\\
&- 2237342529296875Z^5t_2^5 - 11134780883789062500000t_1^4t_3 + 242490783691406250000Z^2t_1^3t_2t_3\\
&+ 14356444152832031250Z^4t_1^2t_2^2t_3 - 235859403076171875 0Z^6t_1t_2^3t_3 + 93457622070312500t_2^5t_3\\
&- 5542646484375 \times 10^8Z^5t_1^3t_3^2 - 403346302734375 \times 10^6Z^7t_1^2t_2t_3^2\\
&+ 740339208984375000000Zt_1t_2^3t_3^2 - 7401221953125 \times 10^6Z^3t_2^4t_3^2\\
&- 231840984375 \times 10^{12}t_1^3t_3^3 - 7814181375 \times 10^{12}Z^2t_1^2t_2t_3^3\\
&- 8355988265625 \times 10^8Z^4t_1t_2^2t_3^3 - 5423496203125 \times 10^7Z^6t_2^3t_3^3\\
&- 1402295895 \times 10^{14}Z^5t_1^2t_3^4 - 826809984 \times 10^{14}Z^7t_1t_2t_3^4\\
&+ 85186776675 \times 10^{11}Zt_2^3t_3^4 - 48988167228 \times 10^{15}t_1^2t_3^5\\
&+ 16002470484 \times 10^{14}Z^2t_1t_2t_3^5 - 47794802055 \times 10^{13}Z^4t_2^2t_3^5\\
&- 85084455456 \times 10^{15}Z^5t_1t_3^6 + 50056134128 \times 10^{14}Z^7t_2t_3^6\\
&- 4315200873984 \times 10^{15}t_1t_3^7 + 1286995021312 \times 10^{14}Z^2t_2t_3^7\\
&- 1617096312832 \times 10^{15}Z^5t_3^8 - 117523421503488 \times 10^{15}t_3^9\\
&+ 18186808776855468750Z^4t_1^3t_2t_4 + 3580945532226562500Z^6t_1^2t_2^2t_4\\
&- 4058009033203125t_1t_2^4t_4 - 11903981933593750Z^2t_2^5t_4 - 360272021484375 \times 10^6Z^7t_1^3t_3t_4
\end{aligned}
$$

$$+ 217445939648437500000 Z t_1^2 t_2^2 t_3 t_4 - 2879536816406250000 Z^3 t_1 t_2^3 t_3 t_4$$
$$- 1341535507812500000 Z^5 t_2^4 t_3 t_4 - 310388203125 \times 10^{11} Z^2 t_1^3 t_3^2 t_4$$
$$+ 579735484453125 \times 10^7 Z^4 t_1^2 t_2 t_3^2 t_4 - 26925226453125 \times 10^7 Z^6 t_1 t_2^2 t_3^2 t_4$$
$$+ 66825086453125 \times 10^6 t_2^4 t_3^2 t_4 - 82574917425 \times 10^{12} Z^7 t_1^2 t_3^3 t_4$$
$$- 4307698395 \times 10^{12} Z t_1 t_2^2 t_3^3 t_4 - 480846441075 \times 10^{10} Z^3 t_2^3 t_3^3 t_4$$
$$+ 1000215216 \times 10^{15} Z^2 t_1^2 t_3^4 t_4 - 50918678811 \times 10^{13} Z^4 t_1 t_2 t_3^4 t_4$$
$$+ 4648352709 \times 10^{13} Z^6 t_2^2 t_3^4 t_4 - 8489072592 \times 10^{14} Z^7 t_1 t_3^5 t_4$$
$$- 179154031056 \times 10^{13} Z t_2^2 t_3^5 t_4 - 2048316221952 \times 10^{14} Z^2 t_1 t_3^6 t_4$$
$$+ 4215229450432 \times 10^{13} Z^4 t_2 t_3^6 t_4 - 4452830333952 \times 10^{14} Z^7 t_3^7 t_4$$
$$- 164735362727936 \times 10^{14} Z^2 t_3^8 t_4 - 748257275390625000000 Z t_1^3 t_2 t_4^2$$
$$+ 31384048007812500000 Z^3 t_1^2 t_2^2 t_4^2 + 6648008554687500000 Z^5 t_1 t_2^3 t_4^2$$
$$- 151272214843750000 Z^7 t_2^4 t_4^2 - 191775568359375 \times 10^7 Z^4 t_1^3 t_3 t_4^2$$
$$- 617780209921875 \times 10^6 Z^6 t_1^2 t_2 t_3 t_4^2 + 258266422476562500000 t_1 t_2^3 t_3 t_4^2$$
$$- 42629813789062500000 Z^2 t_2^4 t_3 t_4^2 - 360370585575 \times 10^{11} Z t_1^2 t_2 t_3^2 t_4^2$$
$$+ 29867919456375 \times 10^9 Z^3 t_1 t_2^2 t_3^2 t_4^2 + 62631944375 \times 10^9 Z^5 t_2^3 t_3^2 t_4^2$$
$$- 33209723547 \times 10^{13} Z^4 t_1^2 t_3^3 t_4^2 - 46125845766 \times 10^{13} Z^6 t_1 t_2 t_3^3 t_4^2$$
$$+ 227825292695 \times 10^{11} t_2^3 t_3^3 t_4^2 + 18199765584 \times 10^{13} Z t_1 t_2 t_3^4 t_4^2$$
$$- 358305251304 \times 10^{12} Z^3 t_2^2 t_3^4 t_4^2 - 8327725293888 \times 10^{13} Z^4 t_1 t_3^5 t_4^2$$
$$- 931697518752 \times 10^{13} Z^6 t_2 t_3^5 t_4^2 + 38054757092352 \times 10^{13} Z t_2 t_3^6 t_4^2$$
$$- 2102716656488448 \times 10^{13} Z^4 t_3^7 t_4^2 + 18844998046875 \times 10^6 Z^6 t_1^3 t_4^3$$
$$- 188841811640625000000 t_1^2 t_2^2 t_4^3 + 62360081824218750000 Z^2 t_1 t_2^3 t_4^3$$
$$+ 3124451575781250000 Z^4 t_2^4 t_4^3 + 9577693125 \times 10^{12} Z t_1^3 t_3 t_4^3$$
$$- 1266402111975 \times 10^{10} Z^3 t_1^2 t_2 t_3 t_4^3 - 3589243515 \times 10^{11} Z^5 t_1 t_2^2 t_3 t_4^3$$
$$- 2530416156125 \times 10^8 Z^7 t_2^3 t_3 t_4^3 + 172574827425 \times 10^{12} Z^6 t_1^2 t_3^2 t_4^3$$
$$+ 2216213826255 \times 10^{11} t_1 t_2^2 t_3^2 t_4^3 - 1497137884385 \times 10^{10} Z^2 t_2^3 t_3^2 t_4^3$$
$$+ 217097296416 \times 10^{13} Z t_1^2 t_3^3 t_4^3 - 1094727183264 \times 10^{12} Z^3 t_1 t_2 t_3^3 t_4^3$$
$$+ 646279471552 \times 10^{11} Z^5 t_2^2 t_3^3 t_4^3 - 979621939296 \times 10^{13} Z^6 t_1 t_3^4 t_4^3$$
$$+ 558196422784 \times 10^{13} t_2^2 t_3^4 t_4^3 - 10797875472384 \times 10^{13} Z t_1 t_3^5 t_4^3$$
$$- 291362895842304 \times 10^{12} Z^3 t_2 t_3^5 t_4^3 + 5708325689470976 \times 10^{12} Z^6 t_3^6 t_4^3$$
$$- 3722245453774848 \times 10^{13} Z t_3^7 t_4^3 - 70945875 \times 10^{12} Z^3 t_1^3 t_4^4$$
$$+ 66496555125 \times 10^9 Z^5 t_1^2 t_2 t_4^4 + 22349415391875 \times 10^7 Z^7 t_1 t_2^2 t_4^4$$
$$+ 412606126075 \times 10^8 Z t_2^4 t_4^4 - 14330651789025 \times 10^{10} t_1^2 t_2 t_3 t_4^4$$
$$+ 9461764959075 \times 10^9 Z^2 t_1 t_2^2 t_3 t_4^4 - 96525280709 \times 10^{10} Z^4 t_2^3 t_3 t_4^4$$
$$+ 77294841624 \times 10^{13} Z^3 t_1^2 t_3^2 t_4^4 + 485360966376 \times 10^{11} Z^5 t_1 t_2 t_3^2 t_4^4$$
$$- 10195366909728 \times 10^{10} Z^7 t_2^2 t_3^2 t_4^4 + 3805352982288 \times 10^{13} t_1 t_2 t_3^3 t_4^4$$

$$
\begin{aligned}
&- 317368025278304 \times 10^{10} Z^2 t_2^2 t_3^3 t_4^4 - 68296287780864 \times 10^{12} Z^3 t_1 t_3^4 t_4^4 \\
&- 3161300456664064 \times 10^{10} Z^5 t_2 t_3^4 t_4^4 - 17052515541428224 \times 10^{11} t_2 t_3^5 t_4^4 \\
&+ 552628557034684416 \times 10^{11} Z^3 t_3^6 t_4^4 + 354595725 \times 10^{13} t_1^3 t_4^5 \\
&+ 54793161423 \times 10^{10} Z^2 t_1^2 t_2 t_4^5 + 12932934695505 \times 10^8 Z^4 t_1 t_2^2 t_4^5 \\
&+ 1091166926817 \times 10^8 Z^6 t_2^3 t_4^5 - 4859154492192 \times 10^{10} Z^5 t_1^2 t_3 t_4^5 \\
&+ 2315774569284 \times 10^{10} Z^7 t_1 t_2 t_3 t_4^5 + 195716977379968 \times 10^8 Z t_2^3 t_3 t_4^5 \\
&- 21730238693856 \times 10^{12} t_1^2 t_3^2 t_4^5 - 696017451241728 \times 10^9 Z^2 t_1 t_2 t_3^2 t_4^5 \\
&- 9935298677625024 \times 10^8 Z^4 t_2^2 t_3^2 t_4^5 + 28759221078331392 \times 10^9 Z^5 t_1 t_3^3 t_4^5 \\
&+ 5772016647536128 \times 10^9 Z^7 t_2 t_3^3 t_4^5 + 170652869646336 \times 10^{13} t_1 t_3^4 t_4^5 \\
&- 1894186506405101568 \times 10^8 Z^2 t_2 t_3^4 t_4^5 - 16989997031015579648 \times 10^8 Z^5 t_3^5 t_4^5 \\
&- 3360518734381580288 \times 10^{11} t_3^6 t_4^5 + 153208063008 \times 10^9 Z^7 t_1^2 t_4^6 \\
&- 70760044544904 \times 10^8 Z t_1 t_2^2 t_4^6 + 76447239934736 \times 10^7 Z^3 t_2^3 t_4^6 \\
&- 2795574285504 \times 10^{10} Z^2 t_1^2 t_3 t_4^6 + 3330184278285216 \times 10^8 Z^4 t_1 t_2 t_3 t_4^6 \\
&+ 178678609003488 \times 10^8 Z^6 t_2^2 t_3 t_4^6 - 87607113825927168 \times 10^8 Z^7 t_1 t_3^2 t_4^6 \\
&- 59889460453075968 \times 10^7 Z t_2^2 t_3^2 t_4^6 - 247545167551266816 \times 10^9 Z^2 t_1 t_3^3 t_4^6 \\
&+ 6591131182148513792 \times 10^7 Z^4 t_2 t_3^3 t_4^6 + 162937474372515397632 \times 10^7 Z^7 t_3^4 t_4^6 \\
&+ 1204378237145373999104 \times 10^8 Z^2 t_3^5 t_4^6 + 1225664504064 \times 10^9 Z^4 t_1^2 t_4^7 \\
&- 389504664385056 \times 10^7 Z^6 t_1 t_2 t_4^7 - 115438056621888 \times 10^6 t_2^3 t_4^7 \\
&- 10315284491077632 \times 10^7 Z t_1 t_2 t_3 t_4^7 + 26410170374218752 \times 10^6 Z^3 t_2^2 t_3 t_4^7 \\
&- 5804292202406387712 \times 10^7 Z^4 t_1 t_3^2 t_4^7 - 6326814318017605632 \times 10^6 Z^6 t_2 t_3^2 t_4^7 \\
&- 480243239071727222784 \times 10^6 Z t_2 t_3^3 t_4^7 + 10117147629150400413696 \times 10^6 Z^4 t_3^4 t_4^7 \\
&- 9510192649469952 \times 10^6 Z^3 t_1 t_2 t_4^8 + 8269657493374464 \times 10^6 Z^5 t_2^2 t_4^8 \\
&+ 1418313416502951936 \times 10^6 Z^6 t_1 t_3 t_4^8 - 716171194134379315200000 t_2^2 t_3 t_4^8 \\
&+ 12913682113095008256 \times 10^7 Z t_1 t_3^2 t_4^8 - 3527367747316702248960000 Z^3 t_2 t_3^2 t_4^8 \\
&+ 2855268783596211535872000000 Z^6 t_3^3 t_4^8 + 19162787387043974479872 \times 10^6 Z t_3^4 t_4^8 \\
&- 666622767231885312000000 t_1 t_2 t_4^9 + 641177421488207872000000 Z^2 t_2^2 t_4^9 \\
&+ 1227921547908140236800000 Z^3 t_1 t_3 t_4^9 - 2251531199576460492800000 Z^5 t_2 t_3 t_4^9 \\
&- 258127966510865312645120000 t_2 t_3^2 t_4^9 + 1640085863520276662190080000 Z^3 t_3^3 t_4^9 \\
&+ 170103990886418350080000 Z^5 t_1 t_4^{10} + 203619049381414961152000 Z^7 t_2 t_4^{10} \\
&- 543003016413198680064000000 t_1 t_3 t_4^{10} - 3514146554715324809216000 Z^2 t_2 t_3 t_4^{10} \\
&+ 79884996846105722355712000 Z^5 t_3^2 t_4^{10} - 8407144392489103618211840000 t_3^3 t_4^{10} \\
&- 333488371957613199360000 Z^2 t_1 t_4^{11} + 1160253521214272385843200 Z^4 t_2 t_4^{11} \\
&- 27703358182969454257766400 Z^7 t_3 t_4^{11} - 430963596901043402702848000 Z^2 t_3^2 t_4^{11} \\
&- 2942876266346822112378880 Z t_2 t_4^{12} - 149803364286801119365038080 Z^4 t_3 t_4^{12} \\
&- 182224039730083379806208 Z^6 t_4^{13} + 376688162092393230384496640 Z t_3 t_4^{13}
\end{aligned}
$$

$$+ 5461535231270129615503362Z^3t_4^{14} + 1603156672774144 \times 10^{10}t_4^{15},$$

*the harmonic elements of $V_2$ are proportional to*

$$\begin{aligned}
&24653320312500t_1^3 - 279404296875Z^4t_1t_2^2 - 3067968750Z^6t_2^3 - 17355937500000Z^7t_1t_2t_3 \\
&+ 1956487500000Zt_2^3t_3 + 179870625 \times 10^8t_1^2t_3^2 - 2970495 \times 10^9Z^2t_1t_2t_3^2 \\
&- 15441525 \times 10^6Z^4t_2^2t_3^2 + 1561824 \times 10^{10}Z^5t_1t_3^3 - 19419576 \times 10^8Z^7t_2t_3^3 \\
&+ 106554096 \times 10^{10}t_1t_3^4 - 1072308864 \times 10^8Z^2t_2t_3^4 - 6319104 \times 10^{11}Z^5t_3^5 \\
&+ 167169024 \times 10^{10}t_3^6 + 7520906250000Zt_1t_2^2t_4 - 1174126250000Z^3t_2^3t_4 \\
&- 30925125 \times 10^7Z^4t_1t_2t_3t_4 + 2702337 \times 10^7Z^6t_2^2t_3t_4 + 5708736 \times 10^9Z^7t_1t_3^2t_4 \\
&- 23162568 \times 10^7Zt_2^2t_3^2t_4 + 38022336 \times 10^{10}Z^2t_1t_3^3t_4 - 2955413824 \times 10^7Z^4t_2t_3^3t_4 \\
&+ 7258544128 \times 10^8Z^7t_3^4t_4 + 137255534592 \times 10^8Z^2t_3^5t_4 - 11507512500000Z^6t_1t_2t_4^2 \\
&- 6394137750000t_2^3t_4^2 - 7627356 \times 10^8Zt_1t_2t_3t_4^2 - 122742312 \times 10^6Z^3t_2^2t_3t_4^2 \\
&+ 189788544 \times 10^8Z^4t_1t_3^2t_4^2 + 858545424 \times 10^7Z^6t_2t_3^2t_4^2 - 948966656 \times 10^8Zt_2t_3^3t_4^2 \\
&+ 1012447608832 \times 10^7Z^4t_3^4t_4^2 + 5540436 \times 10^7Z^3t_1t_2t_4^3 - 30803388 \times 10^6Z^5t_2^2t_4^3 \\
&- 19980576 \times 10^8Z^6t_1t_3t_4^3 - 4919601984 \times 10^6t_2^2t_3t_4^3 - 255923712 \times 10^8Zt_1t_3^2t_4^3 \\
&+ 6138722304 \times 10^6Z^3t_2t_3^2t_4^3 - 39768993792 \times 10^7Z^6t_3^3t_4^3 + 1183867666432 \times 10^7Zt_3^4t_4^3 \\
&+ 1638755316 \times 10^6t_1t_2t_4^4 - 179721308800000Z^2t_2^2t_4^4 + 3626447616 \times 10^6Z^3t_1t_3t_4^4 \\
&- 715204019200000Z^5t_2t_3t_4^4 - 1150847936972800000t_2t_3^2t_4^4 - 5190275337420800000Z^3t_3^3t_4^4 \\
&+ 827098905600000Z^5t_1t_4^5 - 397722248704000Z^7t_2t_4^5 + 567184983552 \times 10^6t_1t_3t_4^5 \\
&+ 33260070711296000Z^2t_2t_3t_4^5 - 416096931610624000Z^5t_3^2t_4^5 - 5228574927093760000t_3^3t_4^5 \\
&- 1235413555200000Z^2t_1t_4^6 - 3964235422822400Z^4t_2t_4^6 + 57643463914291200Z^7t_3t_4^6 \\
&+ 225458408783872000Z^2t_3^2t_4^6 + 10920108609372160Zt_2t_4^7 + 405391937841397760Z^4t_3t_4^7 \\
&- 16985438513004544Z^6t_4^8 - 1397773901999636480Zt_3t_4^8 + 26417935991963648Z^3t_4^9 \\
&+ 87358963712 \times 10^7t_4^{10},
\end{aligned}$$

*and the harmonic elements of $V_3$ are proportional to*

$$\begin{aligned}
&2953125t_1t_2 - 227500Z^2t_2^2 + 6440000Z^5t_2t_3 + 164360000t_2t_3^2 + 448 \times 10^7Z^3t_3^3 \\
&- 28000Z^7t_2t_4 + 831600000t_1t_3t_4 + 23744000Z^2t_2t_3t_4 + 89600000Z^5t_3^2t_4 \\
&+ 107985920000t_3^3t_4 - 4264400Z^4t_2t_4^2 + 98112000Z^7t_3t_4^2 - 207872000Z^2t_3^2t_4^2 \\
&+ 11388160Zt_2t_4^3 + 246364160Z^4t_3t_4^3 - 27026944Z^6t_4^4 - 1457684480Zt_3t_4^4 \\
&+ 42549248Z^3t_4^5 + 4259840000t_4^6.
\end{aligned}$$

*Proof.* Using Proposition 4.0.1 we can directly calculate

$$\Delta(t_1) = -\frac{272Z}{75\,(t_2 - 128t_3t_4)}\left(Z^6t_2 + 80Zt_2t_3 + 4Z^3t_2t_4 - 208Z^6t_3t_4 - 15360Zt_3^2t_4\right.$$

$$- 32t_2t_4^2 - 1024Z^3t_3t_4^2 + 5120t_3t_4^3\Big), \tag{A.4.1}$$

$$\Delta(t_2) = -\frac{832}{75\,(t_2 - 128t_3t_4)}\Big(Z^5t_2 + 120t_2t_3 + 10Z^2t_2t_4 - 128Z^5t_3t_4 - 15360t_3^2t_4$$

$$- 20Z^7t_4^2 - 2560Z^2t_3t_4^2 - 128Z^4t_4^3 + 256Zt_4^4\Big), \tag{A.4.2}$$

$$\Delta(t_3) = -\frac{7Z}{150\,(t_2 - 128t_3t_4)}\left(-5Zt_2 + 20Z^6t_4 + 1920Zt_3t_4 + 128Z^3t_4^2 - 256t_4^3\right), \tag{A.4.3}$$

$$\Delta(t_4) = \frac{3}{5}. \tag{A.4.4}$$

A general element of $V_1$ is of the form

$$\sum_{k=0}^{7} \sum_{\substack{0\le i_1,\,i_2,\,i_3,\,i_4\\ \frac{2}{3}k+\sum_{j=1}^{4} d_j i_j=30}} a_{(i_1,\,i_2,\,i_3,\,i_4,\,k)} t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{i_4}Z^k, \tag{A.4.5}$$

where $a_{(i_1,\,i_2,\,i_3,\,i_4,\,k)} \in \mathbb{C}$. By calculating the Laplacian of this general element (A.4.5) using Proposition 4.0.1 and formulas (A.4.1)–(A.4.4) we find that the only harmonic elements of $V_1$ are as claimed. A general element of $V_2$ has the form

$$\sum_{k=0}^{7} \sum_{\substack{0\le i_1,\,i_2,\,i_3,\,i_4\\ \frac{2}{3}k+\sum_{j=1}^{4} d_j i_j=20}} b_{(i_1,\,i_2,\,i_3,\,i_4,\,k)} t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{i_4}Z^k, \tag{A.4.6}$$

where $b_{(i_1,\,i_2,\,i_3,\,i_4,\,k)} \in \mathbb{C}$. By calculating the Laplacian of this general element (A.4.6) using Propostion 4.0.1 and formulas (A.4.1)–(A.4.4) we find that the only harmonic elements of $V_2$ are as claimed. A general element of $V_3$ has the form

$$\sum_{k=0}^{7} \sum_{\substack{0\le i_1,\,i_2,\,i_3,\,i_4\\ \frac{2}{3}k+\sum_{j=1}^{4} d_j i_j=12}} c_{(i_1,\,i_2,\,i_3,\,i_4,\,k)} t_1^{i_1}t_2^{i_2}t_3^{i_3}t_4^{i_4}Z^k, \tag{A.4.7}$$

where $c_{(i_1,\,i_2,\,i_3,\,i_4,\,k)} \in \mathbb{C}$. By calculating the Laplacian of this general element (A.4.7) using Propostion 4.0.1 and formulas (A.4.1)–(A.4.4) we find that the only harmonic elements of $V_3$ are as claimed. □

Proposition A.4.1 thus allows us to give a proof of Theorem 4.5.19, in which the basic invariants $y_i$ are expressed as polynomials in terms of the $t$ coordinates and the variable $Z$. We now present this proof.

*Proof.* Note that $Y_4 = \frac{1}{8}y_4 = \frac{5}{3}t_4$. We now equate $Y_1$, $Y_2$ and $Y_3$ given by relations (4.5.12)–(4.5.14) with general harmonic elements of $V_1$, $V_2$ and $V_3$, respectively, given by Proposition

A.4.1. We then rearrange these equations to find $y_i$ in terms of $t_j$ and $Z$. We find

$$
\begin{aligned}
y_1 = {} & \frac{3489660928\times 10^{14}}{6015380679}t_4^{15} + \frac{a}{2474395751953125000}\big(2474395751953125000Z^7t_1^3t_2 \\
& - 534469482421875000Zt_1^2t_2^3 + 562512634277343 75Z^3t_1t_2^4 \\
& - 2237342529296875Z^5t_2^5 - 11134780883789062 50000t_1^4t_3 \\
& + 242490783691406250000Z^2t_1^3t_2t_3 + 14356444152832031250Z^4t_1^2t_2^2t_3 \\
& - 235859403076171875 0Z^6t_1t_2^3t_3 + 93457622070312500t_2^5t_3 \\
& - 5542646484375\times 10^8Z^5t_1^3t_3^2 - 403346302734375\times 10^6Z^7t_1^2t_2t_3^2 \\
& + 740339208984375000 00Zt_1t_2^3t_3^2 - 7401221953125\times 10^6Z^3t_2^4t_3^2 \\
& - 231840984375\times 10^{12}t_1^3t_3^3 - 7814181375\times 10^{12}Z^2t_1^2t_2t_3^3 \\
& - 8355988265625\times 10^8Z^4t_1t_2^2t_3^3 - 5423496203125\times 10^7Z^6t_2^3t_3^3 \\
& - 1402295895\times 10^{14}Z^5t_1^2t_3^4 - 826809984\times 10^{14}Z^7t_1t_2t_3^4 \\
& + 85186776675\times 10^{11}Zt_2^3t_3^4 - 48988167228\times 10^{15}t_1^2t_3^5 \\
& + 16002470484\times 10^{14}Z^2t_1t_2t_3^5 - 47794802055\times 10^{13}Z^4t_2^2t_3^5 \\
& - 85084455456\times 10^{15}Z^5t_1t_3^6 + 50056134128\times 10^{14}Z^7t_2t_3^6 \\
& - 4315200873984\times 10^{15}t_1t_3^7 + 1286995021312\times 10^{14}Z^2t_2t_3^7 \\
& - 1617096312832\times 10^{15}Z^5t_3^8 - 117523421503488\times 10^{15}t_3^9 \\
& + 18186808776855468750Z^4t_1^3t_2t_4 + 35809455322265625 00Z^6t_1^2t_2^2t_4 \\
& - 4058009033203125t_1t_2^4t_4 - 11903981933593750Z^2t_2^5t_4 \\
& - 360272021484375\times 10^6Z^7t_1^3t_3t_4 + 217445939648437500000Zt_1^2t_2^2t_3t_4 \\
& - 28795368164062500 00Z^3t_1t_2^3t_3t_4 - 1341535507812500000Z^5t_2^4t_3t_4 \\
& - 310388203125\times 10^{11}Z^2t_1^3t_3^2t_4 + 579735484453125\times 10^7Z^4t_1^2t_2t_3^2t_4 \\
& - 26925226453125\times 10^7Z^6t_1t_2^2t_3^2t_4 + 66825086453125\times 10^6t_2^4t_3^2t_4 \\
& - 82574917425\times 10^{12}Z^7t_1^2t_3^3t_4 - 4307698395\times 10^{12}Zt_1t_2^2t_3^3t_4 \\
& - 480846441075\times 10^{10}Z^3t_2^3t_3^3t_4 + 1000215216\times 10^{15}Z^2t_1^2t_3^4t_4 \\
& - 50918678811\times 10^{13}Z^4t_1t_2t_3^4t_4 + 4648352709\times 10^{13}Z^6t_2^2t_3^4t_4 \\
& - 8489072592\times 10^{14}Z^7t_1t_3^5t_4 - 179154031056\times 10^{13}Zt_2^2t_3^5t_4 \\
& - 2048316221952\times 10^{14}Z^2t_1t_3^6t_4 + 4215229450432\times 10^{13}Z^4t_2t_3^6t_4 \\
& - 4452830333952\times 10^{14}Z^7t_3^7t_4 - 164735362727936\times 10^{14}Z^2t_3^8t_4 \\
& - 74825727539062500000Zt_1^3t_2t_4^2 + 313840480078125000 00Z^3t_1^2t_2^2t_4^2 \\
& + 6648008554687500000Z^5t_1t_2^3t_4^2 - 151272214843750000Z^7t_2^4t_4^2 \\
& - 191775568359375\times 10^7Z^4t_1^3t_3t_4^2 - 617780209921875\times 10^6Z^6t_1^2t_2t_3t_4^2 \\
& + 258266422476562500000t_1t_2^3t_3t_4^2 - 42629813789062500000Z^2t_2^4t_3t_4^2 \\
& - 360370585575\times 10^{11}Zt_1^2t_2t_3^2t_4^2 + 29867919456375\times 10^9Z^3t_1t_2^2t_3^2t_4^2 \\
& + 62631944375\times 10^9Z^5t_2^3t_3^2t_4^2 - 33209723547\times 10^{13}Z^4t_1^2t_3^3t_4^2
\end{aligned}
$$

$$
\begin{aligned}
&- 46125845766 \times 10^{13} Z^6 t_1 t_2 t_3^3 t_4^2 + 227825292695 \times 10^{11} t_2^3 t_3^3 t_4^2 \\
&+ 18199765584 \times 10^{13} Z t_1 t_2 t_3^4 t_4^2 - 358305251304 \times 10^{12} Z^3 t_2^2 t_3^4 t_4^2 \\
&- 8327725293888 \times 10^{13} Z^4 t_1 t_3^5 t_4^2 - 931697518752 \times 10^{13} Z^6 t_2 t_3^5 t_4^2 \\
&+ 38054757092352 \times 10^{13} Z t_2 t_3^6 t_4^2 - 2102716656488448 \times 10^{13} Z^4 t_3^7 t_4^2 \\
&+ 18844998046875 \times 10^6 Z^6 t_1^3 t_4^3 - 18884181164062500000 t_1^2 t_2^2 t_4^3 \\
&+ 6236008182421875000 Z^2 t_1 t_2^3 t_4^3 + 3124451575781250000 Z^4 t_2^4 t_4^3 \\
&+ 9577693125 \times 10^{12} Z t_1^3 t_3 t_4^3 - 1266402111975 \times 10^{10} Z^3 t_1^2 t_2 t_3 t_4^3 \\
&- 3589243515 \times 10^{11} Z^5 t_1 t_2^2 t_3 t_4^3 - 2530416156125 \times 10^8 Z^7 t_2^3 t_3 t_4^3 \\
&+ 172574827425 \times 10^{12} Z^6 t_1^2 t_3^2 t_4^3 + 2216213826255 \times 10^{11} t_1 t_2^2 t_3^2 t_4^3 \\
&- 1497137884385 \times 10^{10} Z^2 t_2^3 t_3^2 t_4^3 + 217097296416 \times 10^{13} Z t_1^2 t_3^3 t_4^3 \\
&- 1094727183264 \times 10^{12} Z^3 t_1 t_2 t_3^3 t_4^3 + 646279471552 \times 10^{11} Z^5 t_2^2 t_3^3 t_4^3 \\
&- 979621939296 \times 10^{13} Z^6 t_1 t_3^4 t_4^3 + 558196422784 \times 10^{13} t_2^2 t_3^4 t_4^3 \\
&- 10797875472384 \times 10^{13} Z t_1 t_3^5 t_4^3 - 291362895842304 \times 10^{12} Z^3 t_2 t_3^5 t_4^3 \\
&+ 5708325689470976 \times 10^{12} Z^6 t_3^6 t_4^3 - 3722245453774848 \times 10^{13} Z t_3^7 t_4^3 \\
&- 70945875 \times 10^{12} Z^3 t_1^3 t_4^4 + 66496555125 \times 10^9 Z^5 t_1^2 t_2 t_4^4 \\
&+ 22349415391875 \times 10^7 Z^7 t_1 t_2^2 t_4^4 + 412606126075 \times 10^8 Z t_2^4 t_4^4 \\
&- 14330651789025 \times 10^{10} t_1^2 t_2 t_3 t_4^4 + 9461764959075 \times 10^9 Z^2 t_1 t_2^2 t_3 t_4^4 \\
&- 96525280709 \times 10^{10} Z^4 t_2^3 t_3 t_4^4 + 77294841624 \times 10^{13} Z^3 t_1^2 t_3^2 t_4^4 \\
&+ 485360966376 \times 10^{11} Z^5 t_1 t_2 t_3^2 t_4^4 - 10195366909728 \times 10^{10} Z^7 t_2^2 t_3^2 t_4^4 \\
&+ 3805352982288 \times 10^{13} t_1 t_2 t_3^3 t_4^4 - 317368025278304 \times 10^{10} Z^2 t_2^2 t_3^3 t_4^4 \\
&- 68296287780864 \times 10^{12} Z^3 t_1 t_3^4 t_4^4 - 3161300456664064 \times 10^{10} Z^5 t_2 t_3^4 t_4^4 \\
&- 17052515541428224 \times 10^{11} t_2 t_3^5 t_4^4 + 552628557034684416 \times 10^{11} Z^3 t_3^6 t_4^4 \\
&+ 354595725 \times 10^{13} t_1^3 t_4^5 + 54793161423 \times 10^{10} Z^2 t_1^2 t_2 t_4^5 \\
&+ 12932934695505 \times 10^8 Z^4 t_1 t_2^2 t_4^5 + 1091166926817 \times 10^8 Z^6 t_2^3 t_4^5 \\
&- 4859154492192 \times 10^{10} Z^5 t_1^2 t_3 t_4^5 + 2315774569284 \times 10^{10} Z^7 t_1 t_2 t_3 t_4^5 \\
&+ 195716977379968 \times 10^8 Z t_2^3 t_3 t_4^5 - 21730238693856 \times 10^{12} t_1^2 t_3^2 t_4^5 \\
&- 696017451241728 \times 10^9 Z^2 t_1 t_2 t_3^2 t_4^5 - 9935298677625024 \times 10^8 Z^4 t_2^2 t_3^2 t_4^5 \\
&+ 28759221078331392 \times 10^9 Z^5 t_1 t_3^3 t_4^5 + 5772016647536128 \times 10^9 Z^7 t_2 t_3^3 t_4^5 \\
&+ 170652869646336 \times 10^{13} t_1 t_3^4 t_4^5 - 1894186506405101568 \times 10^8 Z^2 t_2 t_3^4 t_4^5 \\
&- 1698997031015579648 \times 10^8 Z^5 t_3^5 t_4^5 - 3360518734381580288 \times 10^{11} t_3^6 t_4^5 \\
&+ 153208063008 \times 10^9 Z^7 t_1^2 t_4^6 - 70760044544904 \times 10^8 Z t_1 t_2^2 t_4^6 \\
&+ 76447239934736 \times 10^7 Z^3 t_2^3 t_4^6 - 2795574285504 \times 10^{10} Z^2 t_1^2 t_3 t_4^6 \\
&+ 3330184278285216 \times 10^8 Z^4 t_1 t_2 t_3 t_4^6 + 178678609003488 \times 10^8 Z^6 t_2^2 t_3 t_4^6 \\
&- 87607113825927168 \times 10^8 Z^7 t_1 t_3^2 t_4^6 - 59889460453075968 \times 10^7 Z t_2^2 t_3^2 t_4^6 \\
&- 247545167551266816 \times 10^9 Z^2 t_1 t_3^3 t_4^6 + 6591131182148513792 \times 10^7 Z^4 t_2 t_3^3 t_4^6
\end{aligned}
$$

$$+ 16293747437251539763 2 \times 10^{7} Z^{7} t_3^{4} t_4^{6} + 120437823714537399910 4 \times 10^{8} Z^{2} t_3^{5} t_4^{6}$$

$$+ 1225664504064 \times 10^{9} Z^{4} t_1^{2} t_4^{7} - 389504664385056 \times 10^{7} Z^{6} t_1 t_2 t_4^{7}$$

$$- 115438056621888 \times 10^{6} t_2^{3} t_4^{7} - 10315284491077632 \times 10^{7} Z t_1 t_2 t_3 t_4^{7}$$

$$+ 26410170374218752 \times 10^{6} Z^{3} t_2^{2} t_3 t_4^{7} - 5804292202406387712 \times 10^{7} Z^{4} t_1 t_3^{2} t_4^{7}$$

$$- 6326814318017605632 \times 10^{6} Z^{6} t_2 t_3^{2} t_4^{7} - 480243239071727222784 \times 10^{6} Z t_2 t_3^{3} t_4^{7}$$

$$+ 1011714762915040041369 6 \times 10^{6} Z^{4} t_3^{4} t_4^{7} - 9510192649469952 \times 10^{6} Z^{3} t_1 t_2 t_4^{8}$$

$$+ 8269657493374464 \times 10^{6} Z^{5} t_2^{2} t_4^{8} + 1418313416502951936 \times 10^{6} Z^{6} t_1 t_3 t_4^{8}$$

$$- 716171194134379315200000 t_2^{2} t_3 t_4^{8} + 12913682113095008256 \times 10^{7} Z t_1 t_3^{2} t_4^{8}$$

$$- 35273677473167022489600000 Z^{3} t_2 t_3^{2} t_4^{8} + 28552687835962115358720000 0 Z^{6} t_3^{3} t_4^{8}$$

$$+ 191627873870439744798 72 \times 10^{6} Z t_3^{4} t_4^{8} - 666622767231885312000000 t_1 t_2 t_4^{9}$$

$$+ 641177421488207872000000 Z^{2} t_2^{2} t_4^{9} + 12279215479081402368000000 Z^{3} t_1 t_3 t_4^{9}$$

$$- 22515311995764604928000000 Z^{5} t_2 t_3 t_4^{9} - 258127966510865312645120000 t_2 t_3^{2} t_4^{9}$$

$$+ 164008586352027666219008000 0 Z^{3} t_3^{3} t_4^{9} + 170103990886418350080000 Z^{5} t_1 t_4^{10}$$

$$+ 2036190493814149611520000 Z^{7} t_2 t_4^{10} - 543003016413198680064000000 t_1 t_3 t_4^{10}$$

$$- 351414655471532480921600 0 Z^{2} t_2 t_3 t_4^{10} + 798849968461057223557120 00 Z^{5} t_3^{2} t_4^{10}$$

$$- 84071443924891036182118400 00 t_3^{3} t_4^{10} - 333488371957613199360000 Z^{2} t_1 t_4^{11}$$

$$+ 116025352121427238584320 0 Z^{4} t_2 t_4^{11} - 27703358182969454257766400 Z^{7} t_3 t_4^{11}$$

$$- 43096359690104340270284800 0 Z^{2} t_3^{2} t_4^{11} - 2942876266346822112378880 Z t_2 t_4^{12}$$

$$- 14980336428680111936503808 0 Z^{4} t_3 t_4^{12} - 18222403973008337980620 8 Z^{6} t_4^{13}$$

$$+ 37668816209239323038449664 0 Z t_3 t_4^{13} + 54615352312701296155033 6 Z^{3} t_4^{14}$$

$$+ 1603156672774144 \times 10^{10} t_4^{15} \Big) - \frac{641 b}{338728937586399360} t_4^{5} \Big( 24653320312500 t_1^{3}$$

$$- 279404296875 Z^{4} t_1 t_2^{2} - 3067968750 Z^{6} t_2^{3} - 17355937500000 Z^{7} t_1 t_2 t_3$$

$$+ 1956487500000 Z t_2^{3} t_3 + 179870625 \times 10^{8} t_1^{2} t_3^{2} - 2970495 \times 10^{9} Z^{2} t_1 t_2 t_3^{2}$$

$$- 15441525 \times 10^{6} Z^{4} t_2^{2} t_3^{2} + 1561824 \times 10^{10} Z^{5} t_1 t_3^{3} - 19419576 \times 10^{8} Z^{7} t_2 t_3^{3}$$

$$+ 106554096 \times 10^{10} t_1 t_3^{4} - 1072308864 \times 10^{8} Z^{2} t_2 t_3^{4}$$

$$- 6319104 \times 10^{11} Z^{5} t_3^{5} + 167169024 \times 10^{10} t_3^{6} + 7520906250000 Z t_1 t_2^{2} t_4$$

$$- 1174126250000 Z^{3} t_2^{3} t_4 - 30925125 \times 10^{7} Z^{4} t_1 t_2 t_3 t_4 + 2702337 \times 10^{7} Z^{6} t_2^{2} t_3 t_4$$

$$+ 5708736 \times 10^{9} Z^{7} t_1 t_3^{2} t_4 - 23162568 \times 10^{7} Z t_2^{2} t_3^{2} t_4$$

$$+ 38022336 \times 10^{10} Z^{2} t_1 t_3^{3} t_4 - 2955413824 \times 10^{7} Z^{4} t_2 t_3^{3} t_4$$

$$+ 7258544128 \times 10^{8} Z^{7} t_3^{4} t_4 + 137255534592 \times 10^{8} Z^{2} t_3^{5} t_4$$

$$- 11507512500000 Z^{6} t_1 t_2 t_4^{2} - 6394137750000 t_2^{3} t_4^{2} - 7627356 \times 10^{8} Z t_1 t_2 t_3 t_4^{2}$$

$$- 122742312 \times 10^{6} Z^{3} t_2^{2} t_3 t_4^{2} + 189788544 \times 10^{8} Z^{4} t_1 t_3^{2} t_4^{2}$$

$$+ 858545424 \times 10^{7} Z^{6} t_2 t_3^{2} t_4^{2} - 948966656 \times 10^{8} Z t_2 t_3^{3} t_4^{2}$$

$$+ 1012447608832 \times 10^{7} Z^{4} t_3^{4} t_4^{2} + 5540436 \times 10^{7} Z^{3} t_1 t_2 t_4^{3}$$

$$
\begin{aligned}
&- 30803388 \times 10^6 Z^5 t_2^2 t_4^3 - 19980576 \times 10^8 Z^6 t_1 t_3 t_4^3 \\
&- 4919601984 \times 10^6 t_2^2 t_3 t_4^3 - 255923712 \times 10^8 Z t_1 t_3^2 t_4^3 \\
&+ 6138722304 \times 10^6 Z^3 t_2 t_3^2 t_4^3 - 39768993792 \times 10^7 Z^6 t_3^3 t_4^3 \\
&+ 1183867666432 \times 10^7 Z t_3^4 t_4^3 + 1638755316 \times 10^6 t_1 t_2 t_4^4 \\
&- 179721308800000 Z^2 t_2^2 t_4^4 + 3626447616 \times 10^6 Z^3 t_1 t_3 t_4^4 \\
&- 715204019200000 Z^5 t_2 t_3 t_4^4 - 1150847936972800000 t_2 t_3^2 t_4^4 \\
&- 519027533742080000 Z^3 t_3^3 t_4^4 + 827098905600000 Z^5 t_1 t_4^5 \\
&- 397722248704000 Z^7 t_2 t_4^5 + 567184983552 \times 10^6 t_1 t_3 t_4^5 \\
&+ 33260070711296000 Z^2 t_2 t_3 t_4^5 - 416096931610624000 Z^5 t_3^2 t_4^5 \\
&- 5228574927093760000 t_3^3 t_4^5 - 1235413555200000 Z^2 t_1 t_4^6 \\
&- 3964235422822400 Z^4 t_2 t_4^6 + 57643463914291200 Z^7 t_3 t_4^6 \\
&+ 225458408783872000 Z^2 t_3^2 t_4^6 + 10920108609372160 Z t_2 t_4^7 \\
&+ 405391937841397760 Z^4 t_3 t_4^7 - 16985438513004544 Z^6 t_4^8 \\
&- 1397773901999636480 Z t_3 t_4^8 + 26417935991963648 Z^3 t_4^9 \\
&+ 87358963712 \times 10^7 t_4^{10}\big) + \frac{36880384 \times 10^6 c}{3787461909} t_4^9 \,(2953125 t_1 t_2 \\
&- 227500 Z^2 t_2^2 + 6440000 Z^5 t_2 t_3 + 164360000 t_2 t_3^2 + 448 \times 10^7 Z^3 t_3^3 \\
&- 28000 Z^7 t_2 t_4 + 831600000 t_1 t_3 t_4 + 23744000 Z^2 t_2 t_3 t_4 + 89600000 Z^5 t_3^2 t_4 \\
&+ 107985920000 t_3^3 t_4 - 4264400 Z^4 t_2 t_4^2 + 98112000 Z^7 t_3 t_4^2 \\
&- 207872000 Z^2 t_3^2 t_4^2 + 11388160 Z t_2 t_4^3 + 246364160 Z^4 t_3 t_4^3 \\
&- 27026944 Z^6 t_4^4 - 1457684480 Z t_3 t_4^4 + 42549248 Z^3 t_4^5 + 4259840000 t_4^6\big) \\
&- \frac{c^2}{13891500} t_4^3 \big(2953125 t_1 t_2 - 227500 Z^2 t_2^2 + 6440000 Z^5 t_2 t_3 \\
&+ 164360000 t_2 t_3^2 + 448 \times 10^7 Z^3 t_3^3 - 28000 Z^7 t_2 t_4 + 831600000 t_1 t_3 t_4 \\
&+ 23744000 Z^2 t_2 t_3 t_4 + 89600000 Z^5 t_3^2 t_4 + 107985920000 t_3^3 t_4 \\
&- 4264400 Z^4 t_2 t_4^2 + 98112000 Z^7 t_3 t_4^2 - 207872000 Z^2 t_3^2 t_4^2 \\
&+ 11388160 Z t_2 t_4^3 + 246364160 Z^4 t_3 t_4^3 - 27026944 Z^6 t_4^4 \\
&- 1457684480 Z t_3 t_4^4 + 42549248 Z^3 t_4^5 + 4259840000 t_4^6\big)^2, \qquad \text{(A.4.8)}
\end{aligned}
$$

$$
\begin{aligned}
y_2 = &\frac{524288 \times 10^{10}}{4546773} t_4^{10} + \frac{b}{5228574927093760000} \big(-24653320312500 t_1^3 \\
&+ 279404296875 Z^4 t_1 t_2^2 + 3067968750 Z^6 t_2^3 + 17355937500000 Z^7 t_1 t_2 t_3 \\
&- 1956487500000 Z t_2^3 t_3 - 179870625 \times 10^8 t_1^2 t_3^2 \\
&+ 2970495 \times 10^9 Z^2 t_1 t_2 t_3^2 + 15441525 \times 10^6 Z^4 t_2^2 t_3^2 \\
&- 1561824 \times 10^{10} Z^5 t_1 t_3^3 + 19419576 \times 10^8 Z^7 t_2 t_3^3 \\
&- 106554096 \times 10^{10} t_1 t_3^4 + 1072308864 \times 10^8 Z^2 t_2 t_3^4 \\
&+ 6319104 \times 10^{11} Z^5 t_3^5 - 167169024 \times 10^{10} t_3^6
\end{aligned}
$$

$$
\begin{aligned}
&- 7520906250000 Z t_1 t_2^2 t_4 + 1174126250000 Z^3 t_2^3 t_4 \\
&+ 30925125 \times 10^7 Z^4 t_1 t_2 t_3 t_4 - 2702337 \times 10^7 Z^6 t_2^2 t_3 t_4 \\
&- 5708736 \times 10^9 Z^7 t_1 t_3^2 t_4 + 23162568 \times 10^7 Z t_2^2 t_3^2 t_4 \\
&- 38022336 \times 10^{10} Z^2 t_1 t_3^3 t_4 + 2955413824 \times 10^7 Z^4 t_2 t_3^3 t_4 \\
&- 7258544128 \times 10^8 Z^7 t_3^4 t_4 - 137255534592 \times 10^8 Z^2 t_3^5 t_4 \\
&+ 11507512500000 Z^6 t_1 t_2 t_4^2 + 6394137750000 t_2^3 t_4^2 \\
&+ 7627356 \times 10^8 Z t_1 t_2 t_3 t_4^2 + 122742312 \times 10^6 Z^3 t_2^2 t_3 t_4^2 \\
&- 189788544 \times 10^8 Z^4 t_1 t_3^2 t_4^2 - 858545424 \times 10^7 Z^6 t_2 t_3^2 t_4^2 \\
&+ 948966656 \times 10^8 Z t_2 t_3^3 t_4^2 - 1012447608832 \times 10^7 Z^4 t_3^4 t_4^2 \\
&- 5540436 \times 10^7 Z^3 t_1 t_2 t_4^3 + 30803388 \times 10^6 Z^5 t_2^2 t_4^3 \\
&+ 19980576 \times 10^8 Z^6 t_1 t_3 t_4^3 + 4919601984 \times 10^6 t_2^2 t_3 t_4^3 \\
&+ 255923712 \times 10^8 Z t_1 t_3^2 t_4^3 - 6138722304 \times 10^6 Z^3 t_2 t_3^2 t_4^3 \\
&+ 39768993792 \times 10^7 Z^6 t_3^3 t_4^3 - 1183867666432 \times 10^7 Z t_3^4 t_4^3 \\
&- 1638755316 \times 10^6 t_1 t_2 t_4^4 + 179721308800000 Z^2 t_2^2 t_4^4 \\
&- 3626447616 \times 10^6 Z^3 t_1 t_3 t_4^4 + 715204019200000 Z^5 t_2 t_3 t_4^4 \\
&+ 1150847936972800000 t_2 t_3^2 t_4^4 + 5190275337420800000 Z^3 t_3^3 t_4^4 \\
&- 827098905600000 Z^5 t_1 t_4^5 + 397722248704000 Z^7 t_2 t_4^5 \\
&- 567184983552 \times 10^6 t_1 t_3 t_4^5 - 33260070711296000 Z^2 t_2 t_3 t_4^5 \\
&+ 416096931610624000 Z^5 t_3^2 t_4^5 + 5228574927093760000 t_3^3 t_4^5 \\
&+ 1235413555200000 Z^2 t_1 t_4^6 + 3964235422822400 Z^4 t_2 t_4^6 \\
&- 576434639142912 00 Z^7 t_3 t_4^6 - 225458408783872000 Z^2 t_3^2 t_4^6 \\
&- 10920108609372160 Z t_2 t_4^7 - 405391937841397760 Z^4 t_3 t_4^7 \\
&+ 16985438513004544 Z^6 t_4^8 + 1397773901999636480 Z t_3 t_4^8 \\
&- 26417935991963648 Z^3 t_4^9 - 87358963712 \times 10^7 t_4^{10} \big) \\
&+ \frac{1600c}{9639} t_4^4 \big(2953125 t_1 t_2 - 227500 Z^2 t_2^2 + 6440000 Z^5 t_2 t_3 \\
&+ 164360000 t_2 t_3^2 + 448 \times 10^7 Z^3 t_3^3 - 28000 Z^7 t_2 t_4 \\
&+ 831600000 t_1 t_3 t_4 + 23744000 Z^2 t_2 t_3 t_4 + 89600000 Z^5 t_3^2 t_4 \\
&+ 107985920000 t_3^3 t_4 - 4264400 Z^4 t_2 t_4^2 + 98112000 Z^7 t_3 t_4^2 \\
&- 207872000 Z^2 t_3^2 t_4^2 + 11388160 Z t_2 t_4^3 + 246364160 Z^4 t_3 t_4^3 \\
&- 27026944 Z^6 t_4^4 - 1457684480 Z t_3 t_4^4 + 42549248 Z^3 t_4^5 + 4259840000 t_4^6 \big) ,
\end{aligned} \tag{A.4.9}
$$

$$
\begin{aligned}
y_3 = &- \frac{2048 \times 10^6}{5103} t_4^6 + \frac{c}{28000} \big(-2953125 t_1 t_2 + 227500 Z^2 t_2^2 - 6440000 Z^5 t_2 t_3 \\
&- 164360000 t_2 t_3^2 - 448 \times 10^7 Z^3 t_3^3 + 28000 Z^7 t_2 t_4 - 831600000 t_1 t_3 t_4 \\
&- 23744000 Z^2 t_2 t_3 t_4 - 89600000 Z^5 t_3^2 t_4 - 107985920000 t_3^3 t_4 \\
&+ 4264400 Z^4 t_2 t_4^2 - 98112000 Z^7 t_3 t_4^2 + 207872000 Z^2 t_3^2 t_4^2
\end{aligned}
$$

$$
\begin{aligned}
&- 11388160Zt_2t_4^3 - 246364160Z^4t_3t_4^3 + 27026944Z^6t_4^4 \\
&+ 1457684480Zt_3t_4^4 - 42549248Z^3t_4^5 - 4259840000t_4^6\big)\,, \qquad \text{(A.4.10)}
\end{aligned}
$$

$$
y_4 = \frac{40}{3}t_4, \qquad \text{(A.4.11)}
$$

where $a, b, c \in \mathbb{C}$. In order to find $a, b$ and $c$ we perform steps 5–7 from the introduction of Chapter 4. That is, we transform the intersection form (4.5.93)–(4.5.99) into $y$ coordinates by applying formulas (A.4.8)–(A.4.11) and compare it with the expression given by Lemma 4.5.1. We find that

$$
a = -\frac{19073486328125000}{282429536481}, \qquad b = -\frac{742695302144 \times 10^8}{6586148313}, \qquad c = \frac{125000}{177147},
$$

which implies the statement. □

**Proposition A.4.2.** *We have that*

$$
\det(g^{ij}(t)) = \frac{c \prod\limits_{\alpha \in R_{H_4}} (\alpha,\, x)}{Q(t,Z)^2},
$$

*where* $c = 2^{138}3^{108}5$ *and*

$$
\begin{aligned}
Q(t,Z) = 5^{15}\big(&543147325515747070312 5 \times 10^6 t_1^7 - 1448392868041992187 5 \times 10^8 Z^2 t_1^6 t_2 \\
&+ 1050084829330444335937 50000 Z^4 t_1^5 t_2^2 + 6018877029418945312 500000 Z^6 t_1^4 t_2^3 \\
&+ 78573971986770629882 8125 t_1^3 t_2^5 - 16236848831176757812 5000 Z^2 t_1^2 t_2^6 \\
&- 20065855979919433593 75 Z^4 t_1 t_2^7 + 60743045806884765625 0 Z^6 t_2^8 \\
&+ 23174285888671875 \times 10^{12} Z^5 t_1^6 t_3 + 16337871551513671875 \times 10^9 Z^7 t_1^5 t_2 t_3 \\
&- 69167518615722656 25 \times 10^9 Z t_1^4 t_2^3 t_3 + 637371826171875 \times 10^{12} Z^3 t_1^3 t_2^4 t_3 \\
&+ 5317822265625 \times 10^{12} Z^5 t_1^2 t_2^5 t_3 - 20401897430419921875 00000 Z^7 t_1 t_2^6 t_3 \\
&+ 25745285034179687500000 Z t_2^8 t_3 - 41018486022949218 75 \times 10^{12} t_1^6 t_3^2 \\
&+ 311647796630859375 \times 10^{13} Z^2 t_1^5 t_2 t_3^2 - 19632946014404296875 \times 10^{10} Z^4 t_1^4 t_2^2 t_3^2 \\
&+ 124543212890625 \times 10^{14} Z^6 t_1^3 t_2^3 t_3^2 - 62532590103149414062 5 \times 10^6 t_1^2 t_2^5 t_3^2 \\
&+ 650950927734375 \times 10^9 Z^2 t_1 t_2^6 t_3^2 + 261167083740234375 \times 10^6 Z^4 t_2^7 t_3^2 \\
&- 246796875 \times 10^{23} Z^5 t_1^5 t_3^3 + 15082459716796875 \times 10^{15} Z^7 t_1^4 t_2 t_3^3 \\
&- 2783021484375 \times 10^{18} Z t_1^3 t_2^3 t_3^3 + 19154161669921875 \times 10^{13} Z^3 t_1^2 t_2^4 t_3^3 \\
&+ 2682505546875 \times 10^{15} Z^5 t_1 t_2^5 t_3^3 - 4846960498046875 \times 10^{11} Z^7 t_2^6 t_3^3 \\
&- 338693115234375 \times 10^{18} t_1^5 t_3^4 + 57663193359375 \times 10^{19} Z^2 t_1^4 t_2 t_3^4 \\
&+ 583245615234375 \times 10^{17} Z^4 t_1^3 t_2^2 t_3^4 + 3034130203125 \times 10^{18} Z^6 t_1^2 t_2^3 t_3^4 \\
&- 8960614123535156 25 \times 10^{12} t_1 t_2^5 t_3^4 + 4133317434375 \times 10^{16} Z^2 t_2^6 t_3^4 \\
&+ 2622391875 \times 10^{24} Z^5 t_1^4 t_3^5 + 142607469375 \times 10^{22} Z^7 t_1^3 t_2 t_3^5 \\
&- 50716822905 \times 10^{22} Z t_1^2 t_2^3 t_3^5 + 1048898512575 \times 10^{20} Z^3 t_1 t_2^4 t_3^5 \\
&- 4199407065 \times 10^{21} Z^5 t_2^5 t_3^5 + 348123470625 \times 10^{24} t_1^4 t_3^6
\end{aligned}
$$

$$
\begin{aligned}
&- 186144057 \times 10^{26} Z^2 t_1^3 t_2 t_3^6 + 317676118761 \times 10^{23} Z^4 t_1^2 t_2^2 t_3^6 \\
&- 295039790256 \times 10^{22} Z^6 t_1 t_2^3 t_3^6 + 97994102241125 \times 10^{18} t_2^5 t_3^6 \\
&+ 20364160896 \times 10^{26} Z^5 t_1^3 t_3^7 - 3758190357024 \times 10^{23} Z^7 t_1^2 t_2 t_3^7 \\
&+ 184087037952 \times 10^{23} Z t_1 t_2^3 t_3^7 - 6512748914128 \times 10^{21} Z^3 t_2^4 t_3^7 \\
&+ 832959727344 \times 10^{26} t_1^3 t_3^8 - 81812500144128 \times 10^{23} Z^2 t_1^2 t_2 t_3^8 \\
&- 87451533751104 \times 10^{22} Z^4 t_1 t_2^2 t_3^8 + 8785218353408 \times 10^{21} Z^6 t_2^3 t_3^8 \\
&+ 1460644653146112 \times 10^{23} Z^5 t_1^2 t_3^9 - 2248483147524096 \times 10^{22} Z^7 t_1 t_2 t_3^9 \\
&+ 957200788467712 \times 10^{21} Z t_2^3 t_3^9 + 77829079650951168 \times 10^{23} t_1^2 t_3^{10} \\
&- 56060138840162304 \times 10^{22} Z^2 t_1 t_2 t_3^{10} - 552059861342208 \times 10^{23} Z^4 t_2^2 t_3^{10} \\
&+ 30372186581630976 \times 10^{23} Z^5 t_1 t_3^{11} - 18461907124224 \times 10^{24} Z^7 t_2 t_3^{11} \\
&+ 3376913958412222464 \times 10^{23} t_1 t_3^{12} - 142174508261834752 \times 10^{23} Z^2 t_2 t_3^{12} \\
&+ 1034390633722150912 \times 10^{23} Z^5 t_3^{13} + 56724530236508602368 \times 10^{23} t_3^{14} \\
&+ 926971435546875 \times 10^{13} Z^7 t_1^6 t_4 - 3371858596801757812 5 \times 10^8 Z t_1^5 t_2^2 t_4 \\
&+ 7689228057861328125 \times 10^8 Z^3 t_1^4 t_2^3 t_4 - 7305908203125 \times 10^{12} Z^5 t_1^3 t_2^4 t_4 \\
&- 2044879074096679687500000 Z^7 t_1^2 t_2^5 t_4 - 1609062423706054687500000 Z t_1 t_2^7 t_4 \\
&+ 2280996856689453125 0000 Z^3 t_2^8 t_4 + 88062286376953125 \times 10^{13} Z^2 t_1^6 t_3 t_4 \\
&+ 93624114990234375 \times 10^{12} Z^4 t_1^5 t_2 t_3 t_4 - 86678009033203125 \times 10^{11} Z^6 t_1^4 t_2^2 t_3 t_4 \\
&- 749266307830810546875 \times 10^7 t_1^3 t_2^4 t_3 t_4 + 1364630424499511718 75 \times 10^7 Z^2 t_1^2 t_2^5 t_3 t_4 \\
&- 818241315307617187 5 \times 10^7 Z^4 t_1 t_2^6 t_3 t_4 - 9604083251953125 \times 10^7 Z^6 t_2^7 t_3 t_4 \\
&- 347947998046875 \times 10^{16} Z^7 t_1^5 t_3^2 t_4 + 30459144287109375 \times 10^{14} Z t_1^4 t_2^2 t_3^2 t_4 \\
&+ 730480078125 \times 10^{17} Z^3 t_1^3 t_2^3 t_3^2 t_4 + 106572603515625 \times 10^{14} Z^5 t_1^2 t_2^4 t_3^2 t_4 \\
&- 2923664044921875 \times 10^{12} Z^7 t_1 t_2^5 t_3^2 t_4 + 18390234111328125 \times 10^{10} Z t_2^7 t_3^2 t_4 \\
&- 523494140625 \times 10^{21} Z^2 t_1^5 t_3^3 t_4 + 103649994140625 \times 10^{18} Z^4 t_1^4 t_2 t_3^3 t_4 \\
&- 505056515625 \times 10^{19} Z^6 t_1^3 t_2^2 t_3^3 t_4 - 252471276298828125 \times 10^{13} t_1^2 t_2^4 t_3^3 t_4 \\
&+ 3577314146484375 \times 10^{14} Z^2 t_1 t_2^5 t_3^3 t_4 - 2874886225703125 \times 10^{13} Z^4 t_2^6 t_3^3 t_4 \\
&- 328566121875 \times 10^{22} Z^7 t_1^4 t_3^4 t_4 + 1308419409375 \times 10^{21} Z t_1^3 t_2^2 t_3^4 t_4 \\
&+ 166187875275 \times 10^{21} Z^3 t_1^2 t_2^3 t_3^4 t_4 - 2381730669 \times 10^{22} Z^5 t_1 t_2^4 t_3^4 t_4 \\
&+ 11573238760125 \times 10^{17} Z^7 t_2^5 t_3^4 t_4 - 18892254375 \times 10^{25} Z^2 t_1^4 t_3^5 t_4 \\
&+ 188951751 \times 10^{25} Z^4 t_1^3 t_2 t_3^5 t_4 - 357425755164 \times 10^{22} Z^6 t_1^2 t_2^2 t_3^5 t_4 \\
&+ 42089870435175 \times 10^{19} t_1 t_2^4 t_3^5 t_4 - 2358024415725 \times 10^{19} Z^2 t_2^5 t_3^5 t_4 \\
&- 57359300976 \times 10^{25} Z^7 t_1^3 t_3^6 t_4 + 20531142806112 \times 10^{22} Z t_1^2 t_2^2 t_3^6 t_4 \\
&- 23294113824 \times 10^{24} Z^3 t_1 t_2^3 t_3^6 t_4 + 391956045664 \times 10^{21} Z^5 t_2^4 t_3^6 t_4 \\
&- 193630178304 \times 10^{26} Z^2 t_1^3 t_3^7 t_4 - 418733354368512 \times 10^{22} Z^4 t_1^2 t_2 t_3^7 t_4 \\
&- 490484693910528 \times 10^{21} Z^6 t_1 t_2^2 t_3^7 t_4 + 126362427684768 \times 10^{20} t_2^4 t_3^7 t_4 \\
&- 4209978473275392 \times 10^{22} Z^7 t_1^2 t_3^8 t_4 + 11097323707226112 \times 10^{21} Z t_1 t_2^2 t_3^8 t_4 \\
&- 21221067387875328 \times 10^{20} Z^3 t_2^3 t_3^8 t_4 - 8164166619119616 \times 10^{23} Z^2 t_1^2 t_3^9 t_4 \\
&- 736568851756253184 \times 10^{21} Z^4 t_1 t_2 t_3^9 t_4 + 31450885434032128 \times 10^{21} Z^6 t_2^2 t_3^9 t_4 \\
&+ 80574635760156672 \times 10^{22} Z^7 t_1 t_3^{10} t_4 - 184960240258777088 \times 10^{21} Z t_2^2 t_3^{10} t_4 \\
&- 547400107633410048 \times 10^{22} Z^2 t_1 t_3^{11} t_4 + 607061380420337664 \times 10^{22} Z^4 t_2 t_3^{11} t_4
\end{aligned}
$$

$$
\begin{aligned}
&- 398024503902339072 \times 10^{22} Z^7 t_3^{12} t_4 + 5592885893846269952 \times 10^{23} Z^2 t_3^{13} t_4 \\
&+ 50983428955078125 \times 10^{12} Z^4 t_1^6 t_4^2 + 20574131011962890625 \times 10^9 Z^6 t_1^5 t_2 t_4^2 \\
&- 6109668731689453125 \times 10^8 t_1^4 t_2^3 t_4^2 + 12561712646484375 \times 10^{11} Z^2 t_1^3 t_2^4 t_4^2 \\
&- 6257706722259521484375000 Z^4 t_1^2 t_2^5 t_4^2 - 5222287902832031250000 Z^6 t_1 t_2^6 t_4^2 \\
&+ 357140063781738281250000 t_2^8 t_4^2 + 18852374267578125 \times 10^{14} Z t_1^5 t_2 t_3 t_4^2 \\
&- 4854116821289062 5 \times 10^{13} Z^3 t_1^4 t_2^2 t_3 t_4^2 - 33512431640625 \times 10^{15} Z^5 t_1^3 t_2^3 t_3 t_4^2 \\
&- 816352218017578125 \times 10^9 Z^7 t_1^2 t_2^4 t_3 t_4^2 - 22586107939453125 \times 10^{10} Z t_1 t_2^6 t_3 t_4^2 \\
&- 2966604723632812 5 \times 10^9 Z^3 t_2^7 t_3 t_4^2 - 8255830078125 \times 10^{18} Z^4 t_1^5 t_3^2 t_4^2 \\
&+ 5090529638671875 \times 10^{15} Z^6 t_1^4 t_2 t_3^2 t_4^2 + 1939176755859375 \times 10^{15} t_1^3 t_2^3 t_3^2 t_4^2 \\
&+ 95024757638671875 \times 10^{13} Z^2 t_1^2 t_2^4 t_3^2 t_4^2 - 704855374294921875 \times 10^{11} Z^4 t_1 t_2^5 t_3^2 t_4^2 \\
&+ 2962940173945312 5 \times 10^{11} Z^6 t_2^6 t_3^2 t_4^2 + 3199858171875 \times 10^{20} Z t_1^4 t_2 t_3^3 t_4^2 \\
&- 39199016475 \times 10^{22} Z^3 t_1^3 t_2^2 t_3^3 t_4^2 - 33723505575 \times 10^{20} Z^5 t_1^2 t_2^3 t_3^3 t_4^2 \\
&- 35129316853125 \times 10^{16} Z^7 t_1 t_2^4 t_3^3 t_4^2 - 14272245088375 \times 10^{16} Z t_2^6 t_3^3 t_4^2 \\
&- 2737857105 \times 10^{25} Z^4 t_1^4 t_3^4 t_4^2 + 730292907825 \times 10^{22} Z^6 t_1^3 t_2 t_3^4 t_4^2 \\
&+ 69880786884 \times 10^{22} t_1^2 t_2^3 t_3^4 t_4^2 - 110473683705 \times 10^{20} Z^2 t_1 t_2^4 t_3^4 t_4^2 \\
&+ 137692971892162 5 \times 10^{16} Z^4 t_2^5 t_3^4 t_4^2 - 90694034424 \times 10^{24} Z t_1^3 t_2 t_3^5 t_4^2 \\
&+ 1422434450016 \times 10^{21} Z^3 t_1^2 t_2^2 t_3^5 t_4^2 - 9952671439872 \times 10^{21} Z^5 t_1 t_2^3 t_3^5 t_4^2 \\
&- 954672182322 \times 10^{20} Z^7 t_2^4 t_3^5 t_4^2 - 1977266392704 \times 10^{24} Z^4 t_1^3 t_3^6 t_4^2 \\
&+ 118604931364224 \times 10^{21} Z^6 t_1^2 t_2 t_3^6 t_4^2 + 2319000845076096 \times 10^{20} t_1 t_2^3 t_3^6 t_4^2 \\
&- 23472393008672 \times 10^{20} Z^2 t_2^4 t_3^6 t_4^2 - 17134185862729728 \times 10^{21} Z t_1^2 t_2 t_3^7 t_4^2 \\
&- 141704475250655232 \times 10^{20} Z^3 t_1 t_2^2 t_3^7 t_4^2 - 234898433601536 \times 10^{20} Z^5 t_2^3 t_3^7 t_4^2 \\
&- 42768022852435968 \times 10^{21} Z^4 t_1^2 t_3^8 t_4^2 + 48315506863067136 \times 10^{21} Z^6 t_1 t_2 t_3^8 t_4^2 \\
&- 7219966326124544 \times 10^{20} t_2^3 t_3^8 t_4^2 - 1540138656127647744 \times 10^{21} Z t_1 t_2 t_3^9 t_4^2 \\
&+ 278042326821896192 \times 10^{20} Z^3 t_2^2 t_3^9 t_4^2 - 2457926286276820992 \times 10^{21} Z^4 t_1 t_3^{10} t_4^2 \\
&+ 346237316825088 \times 10^{23} Z^6 t_2 t_3^{10} t_4^2 + 13003065004985942016 \times 10^{21} Z t_2 t_3^{11} t_4^2 \\
&- 1350467358223635578 88 \times 10^{21} Z^4 t_3^{12} t_4^2 - 185394287109375 \times 10^{15} Z t_1^6 t_4^3 \\
&+ 18760418701171875 \times 10^{13} Z^3 t_1^5 t_2 t_4^3 + 34904718017578125 \times 10^{12} Z^5 t_1^4 t_2^2 t_4^3 \\
&+ 46673701171875 \times 10^{14} Z^7 t_1^3 t_2^3 t_4^3 + 10628650074462890625 \times 10^8 Z t_1^2 t_2^5 t_4^3 \\
&+ 4919396455078125 \times 10^9 Z^3 t_1 t_2^6 t_4^3 + 52035744775390625 \times 10^8 Z^5 t_2^7 t_4^3 \\
&- 395408935546875 \times 10^{16} Z^6 t_1^5 t_3 t_4^3 + 798601728515625 \times 10^{15} t_1^4 t_2^2 t_3 t_4^3 \\
&- 39300525 \times 10^{22} Z^2 t_1^3 t_2^3 t_3 t_4^3 - 125573664990234375 \times 10^{12} Z^4 t_1^2 t_2^4 t_3 t_4^3 \\
&- 7654266191953125 \times 10^{12} Z^6 t_1 t_2^5 t_3 t_4^3 + 15598880734375 \times 10^{12} t_2^7 t_3 t_4^3 \\
&- 152843203125 \times 10^{21} Z t_1^5 t_3^2 t_4^3 + 8849096859375 \times 10^{19} Z^3 t_1^4 t_2 t_3^2 t_4^3 \\
&+ 23086009125 \times 10^{21} Z^5 t_1^3 t_2^2 t_3^2 t_4^3 + 6524691042375 \times 10^{18} Z^7 t_1^2 t_2^3 t_3^2 t_4^3 \\
&+ 403118568365625 \times 10^{15} Z t_1 t_2^5 t_3^2 t_4^3 + 19688741888375 \times 10^{15} Z^3 t_2^6 t_3^2 t_4^3 \\
&+ 4965384375 \times 10^{22} Z^6 t_1^4 t_3^3 t_4^3 - 228539226285 \times 10^{22} t_1^3 t_2^2 t_3^3 t_4^3 \\
&+ 229499895708 \times 10^{21} Z^2 t_1^2 t_2^3 t_3^3 t_4^3 - 5439081019545 \times 10^{19} Z^4 t_1 t_2^4 t_3^3 t_4^3 \\
&- 2027902787493 \times 10^{18} Z^6 t_2^5 t_3^3 t_4^3 - 456599916 \times 10^{26} Z t_1^4 t_3^4 t_4^3 \\
&+ 932964178968 \times 10^{23} Z^3 t_1^3 t_2 t_3^4 t_4^3 + 2813129296848 \times 10^{21} Z^5 t_1^2 t_2^2 t_3^4 t_4^3
\end{aligned}
$$

$$+ 16083834488352 \times 10^{20} Z^7 t_1 t_2^3 t_3^4 t_4^3 + 1622731889337 \times 10^{20} Z t_2^5 t_3^4 t_4^3$$
$$- 46416819888 \times 10^{25} Z^6 t_1^3 t_3^5 t_4^3 + 1878198602300928 \times 10^{20} t_1^2 t_2^2 t_3^5 t_4^3$$
$$- 6849694628352 \times 10^{22} Z^2 t_1 t_2^3 t_3^5 t_4^3 - 1131975640532448 \times 10^{19} Z^4 t_2^4 t_3^5 t_4^3$$
$$+ 1806955011072 \times 10^{24} Z t_1^3 t_3^6 t_4^3 + 138201969168838656 \times 10^{20} Z^3 t_1^2 t_2 t_3^6 t_4^3$$
$$+ 3261226796138496 \times 10^{20} Z^5 t_1 t_2^2 t_3^6 t_4^3 + 2085877638413312 \times 10^{20} Z^7 t_2^3 t_3^6 t_4^3$$
$$- 13868230277799936 \times 10^{22} Z^6 t_1^2 t_3^7 t_4^3 + 343528947079790592 \times 10^{20} t_1 t_2^2 t_3^7 t_4^3$$
$$- 403062224228122624 \times 10^{19} Z^2 t_2^3 t_3^7 t_4^3 + 389926662633160704 \times 10^{21} Z t_1^2 t_3^8 t_4^3$$
$$- 17720220757662498816 \times 10^{20} Z^3 t_1 t_2 t_3^8 t_4^3 + 1355650761464807424 \times 10^{20} Z^5 t_2^2 t_3^8 t_4^3$$
$$+ 918733541121196032 \times 10^{22} Z^6 t_1 t_3^9 t_4^3 + 8261448724811087872 \times 10^{20} t_2^2 t_3^9 t_4^3$$
$$- 2691725938567151616 \times 10^{21} Z t_1 t_3^{10} t_4^3 - 32852298085060050944 \times 10^{21} Z^3 t_2 t_3^{10} t_4^3$$
$$+ 95574453458802049024 \times 10^{21} Z^6 t_3^{11} t_4^3 - 30525392781054050304 \times 10^{22} Z t_3^{12} t_4^3$$
$$+ 600038250732421875 \times 10^{12} t_1^5 t_2 t_4^4 + 2552843408203125 \times 10^{14} Z^2 t_1^4 t_2^2 t_4^4$$
$$+ 661412414794921875 \times 10^{11} Z^4 t_1^3 t_2^3 t_4^4 + 12832379721826171875 \times 10^9 Z^6 t_1^2 t_2^4 t_4^4$$
$$+ 133679592091992187 5 \times 10^9 t_1 t_2^6 t_4^4 + 881530794140625 \times 10^{11} Z^2 t_2^7 t_4^4$$
$$- 28450838671875 \times 10^{18} Z^3 t_1^5 t_3 t_4^4 - 69850044140625 \times 10^{17} Z^5 t_1^4 t_2 t_3 t_4^4$$
$$+ 180666789609375 \times 10^{16} Z^7 t_1^3 t_2^2 t_3 t_4^4 - 837972479053125 \times 10^{15} Z t_1^2 t_2^4 t_3 t_4^4$$
$$- 2675209396235625 \times 10^{14} Z^3 t_1 t_2^5 t_3 t_4^4 + 114872307615625 \times 10^{13} Z^5 t_2^6 t_3 t_4^4$$
$$+ 97475971640625 \times 10^{18} t_1^4 t_2 t_3^2 t_4^4 + 2860466248125 \times 10^{20} Z^2 t_1^3 t_2^2 t_3^2 t_4^4$$
$$+ 768977526391875 \times 10^{17} Z^4 t_1^2 t_2^3 t_3^2 t_4^4 + 725066574546375 \times 10^{16} Z^6 t_1 t_2^4 t_3^2 t_4^4$$
$$+ 5627701200733 \times 10^{17} t_2^6 t_3^2 t_4^4 - 311641425 \times 10^{24} Z^3 t_1^4 t_3^3 t_4^4$$
$$- 38193328044 \times 10^{23} Z^5 t_1^3 t_2 t_3^3 t_4^4 + 61061675508 \times 10^{20} Z^7 t_1^2 t_2^2 t_3^3 t_4^4$$
$$- 8149680001656 \times 10^{19} Z t_1 t_2^4 t_3^3 t_4^4 - 786937204533432 \times 10^{17} Z^3 t_2^5 t_3^3 t_4^4$$
$$- 989637588468 \times 10^{23} t_1^3 t_2 t_3^4 t_4^4 + 238044643268544 \times 10^{20} Z^2 t_1^2 t_2^2 t_3^4 t_4^4$$
$$+ 2274814361965968 \times 10^{19} Z^4 t_1 t_2^3 t_3^4 t_4^4 + 286577277964524 \times 10^{19} Z^6 t_2^4 t_3^4 t_4^4$$
$$- 806555742912 \times 10^{25} Z^3 t_1^3 t_3^5 t_4^4 - 767939187022848 \times 10^{20} Z^5 t_1^2 t_2 t_3^5 t_4^4$$
$$- 142730464853376 \times 10^{21} Z^7 t_1 t_2^2 t_3^5 t_4^4 - 5042158261618944 \times 10^{19} Z t_2^4 t_3^5 t_4^4$$
$$+ 1165017845341292544 \times 10^{20} t_1^2 t_2 t_3^6 t_4^4 - 1756588864246358016 \times 10^{19} Z^2 t_1 t_2^2 t_3^6 t_4^4$$
$$+ 332601981466846208 \times 10^{19} Z^4 t_2^3 t_3^6 t_4^4 - 78526582831742976 \times 10^{22} Z^3 t_1^2 t_3^7 t_4^4$$
$$+ 5414830600493334528 \times 10^{20} Z^5 t_1 t_2 t_3^7 t_4^4 + 394466988093308928 \times 10^{19} Z^7 t_2^2 t_3^7 t_4^4$$
$$+ 22741807909911330816 \times 10^{20} t_1 t_2 t_3^8 t_4^4 - 65669546186240950272 \times 10^{19} Z^2 t_2^2 t_3^8 t_4^4$$
$$+ 11428830995112630681 6 \times 10^{21} Z^3 t_1 t_3^9 t_4^4 + 28205590967418355712 \times 10^{21} Z^5 t_2 t_3^9 t_4^4$$
$$+ 32278876630079137382 4 \times 10^{20} t_2 t_3^{10} t_4^4 - 259381484657410729574 4 \times 10^{20} Z^3 t_3^{11} t_4^4$$
$$+ 4851551953125 \times 10^{17} Z^5 t_1^5 t_4^5 + 59155791328125 \times 10^{16} Z^7 t_1^4 t_2 t_4^5$$
$$+ 158703119859375 \times 10^{15} Z t_1^3 t_2^3 t_4^5 + 1499374897408125 \times 10^{14} Z^3 t_1^2 t_2^4 t_4^5$$
$$+ 1726813992125625 \times 10^{13} Z^5 t_1 t_2^5 t_4^5 + 974784459268125 \times 10^{12} Z^7 t_2^6 t_4^5$$
$$+ 2525510390625 \times 10^{19} t_1^5 t_3 t_4^5 - 738152746875 \times 10^{19} Z^2 t_1^4 t_2 t_3 t_4^5$$
$$+ 15646126606875 \times 10^{18} Z^4 t_1^3 t_2^2 t_3 t_4^5 - 138598565238 \times 10^{18} Z^6 t_1^2 t_2^3 t_3 t_4^5$$
$$- 14072751115986 \times 10^{17} t_1 t_2^5 t_3 t_4^5 + 2559318032198 \times 10^{16} Z^2 t_2^6 t_3 t_4^5$$
$$+ 792314325 \times 10^{23} Z^5 t_1^4 t_3^2 t_4^5 + 25321424904 \times 10^{22} Z^7 t_1^3 t_2 t_3^2 t_4^5$$

$$
\begin{aligned}
&+ 244567076423688 \times 10^{18} Z t_1^2 t_2^3 t_3^2 t_4^5 + 32013861725232 \times 10^{18} Z^3 t_1 t_2^4 t_3^2 t_4^5 \\
&+ 799190880072264 \times 10^{16} Z^5 t_2^5 t_3^2 t_4^5 - 1669469589 \times 10^{25} t_1^4 t_3^3 t_4^5 \\
&+ 586317603744 \times 10^{22} Z^2 t_1^3 t_2 t_3^3 t_4^5 + 565927801165152 \times 10^{19} Z^4 t_1^2 t_2^2 t_3^3 t_4^5 \\
&- 742926575637504 \times 10^{18} Z^6 t_1 t_2^3 t_3^3 t_4^5 - 3219680377344384 \times 10^{17} t_2^5 t_3^3 t_4^5 \\
&+ 2443780727424 \times 10^{23} Z^5 t_1^3 t_3^4 t_4^5 - 51708049909632 \times 10^{22} Z^7 t_1^2 t_2 t_3^4 t_4^5 \\
&+ 157602368069466624 \times 10^{18} Z t_1 t_2^3 t_3^4 t_4^5 + 32271555448818432 \times 10^{18} Z^3 t_2^4 t_3^4 t_4^5 \\
&- 47765117581056 \times 10^{24} t_1^3 t_3^5 t_4^5 - 2315374251350925312 \times 10^{19} Z^2 t_1^2 t_2 t_3^5 t_4^5 \\
&- 19692419764091904 \times 10^{20} Z^4 t_1 t_2^2 t_3^5 t_4^5 + 241598255470977024 \times 10^{18} Z^6 t_2^3 t_3^5 t_4^5 \\
&- 203712682667507712 \times 10^{21} Z^5 t_1^2 t_3^6 t_4^5 - 8579106622951391232 \times 10^{19} Z^7 t_1 t_2 t_3^6 t_4^5 \\
&- 4665867550987681792 \times 10^{18} Z t_2^3 t_3^6 t_4^5 + 101651072541376905216 \times 10^{20} t_1^2 t_3^7 t_4^5 \\
&- 40562976075002413056 \times 10^{19} Z^2 t_1 t_2 t_3^7 t_4^5 + 81006473337705411379 2 \times 10^{18} Z^4 t_2^2 t_3^7 t_4^5 \\
&+ 23557433425570824192 \times 10^{21} Z^5 t_1 t_3^8 t_4^5 - 45474925683284115456 \times 10^{20} Z^7 t_2 t_3^8 t_4^5 \\
&+ 972035317825090564915 2 \times 10^{20} t_1 t_3^9 t_4^5 + 280150434736717483212 8 \times 10^{19} Z^2 t_2 t_3^9 t_4^5 \\
&+ 162677917526695199375 36 \times 10^{20} Z^5 t_3^{10} t_4^5 + 321055959761984631603 2 \times 10^{21} t_3^{11} t_4^5 \\
&+ 997989609375 \times 10^{18} Z^2 t_1^5 t_4^6 + 262194861515625 \times 10^{16} Z^4 t_1^4 t_2 t_4^6 \\
&+ 45006813298125 \times 10^{16} Z^6 t_1^3 t_2^2 t_4^6 + 375727778261325 \times 10^{15} t_1^2 t_2^4 t_4^6 \\
&+ 1824853412590125 \times 10^{14} Z^2 t_1 t_2^5 t_4^6 + 1465493900776525 \times 10^{13} Z^4 t_2^6 t_4^6 \\
&+ 13215985623 \times 10^{22} Z^7 t_1^4 t_3 t_4^6 + 27155757159 \times 10^{21} Z t_1^3 t_2^2 t_3 t_4^6 \\
&+ 50824157937408 \times 10^{17} Z^3 t_1^2 t_2^3 t_3 t_4^6 + 13812697468992 \times 10^{17} Z^5 t_1 t_2^4 t_3 t_4^6 \\
&+ 203462956436892 \times 10^{15} Z^7 t_2^5 t_3 t_4^6 + 9410324337 \times 10^{24} Z^2 t_1^4 t_3^2 t_4^6 \\
&- 789840495576 \times 10^{21} Z^4 t_1^3 t_2 t_3^2 t_4^6 + 275412538879632 \times 10^{18} Z^6 t_1^2 t_2^2 t_3^2 t_4^6 \\
&- 2799233533366944 \times 10^{17} t_1 t_2^4 t_3^2 t_4^6 + 6398274405500112 \times 10^{16} Z^2 t_2^5 t_3^2 t_4^6 \\
&+ 32218832369664 \times 10^{21} Z^7 t_1^3 t_3^3 t_4^6 + 116284503201228288 \times 10^{18} Z t_1^2 t_2^2 t_3^3 t_4^6 \\
&- 208575692695489536 \times 10^{17} Z^3 t_1 t_2^3 t_3^3 t_4^6 + 4916148267574272 \times 10^{16} Z^5 t_2^4 t_3^3 t_4^6 \\
&+ 397022975940096 \times 10^{22} Z^2 t_1^3 t_3^4 t_4^6 - 457899703632 \times 10^{25} Z^4 t_1^2 t_2 t_3^4 t_4^6 \\
&+ 29913242409027072 \times 10^{19} Z^6 t_1 t_2^2 t_3^4 t_4^6 + 635186496452957184 \times 10^{17} t_2^4 t_3^4 t_4^6 \\
&- 3277400514086633472 \times 10^{19} Z^7 t_1^2 t_3^5 t_4^6 + 9903680881358045184 \times 10^{18} Z t_1 t_2^2 t_3^5 t_4^6 \\
&+ 7344620692912806297 6 \times 10^{17} Z^3 t_2^3 t_3^5 t_4^6 - 58603861279192566988 8 \times 10^{19} Z^2 t_1^2 t_3^6 t_4^6 \\
&- 102611728150268923084 8 \times 10^{18} Z^4 t_1 t_2 t_3^6 t_4^6 - 98130699848666801766 4 \times 10^{17} Z^6 t_2^2 t_3^6 t_4^6 \\
&- 509960344369014767616 \times 10^{19} Z^7 t_1 t_3^7 t_4^6 + 459422133190922338304 \times 10^{18} Z t_2^2 t_3^7 t_4^6 \\
&+ 921770170069089779712 \times 10^{19} Z^2 t_1 t_3^8 t_4^6 + 4456154199249676507545 6 \times 10^{18} Z^4 t_2 t_3^8 t_4^6 \\
&- 2425519576234643391774 72 \times 10^{18} Z^7 t_3^9 t_4^6 + 1312868703673558780346368 \times 10^{19} Z^2 t_3^{10} t_4^6 \\
&- 26787726972 \times 10^{21} Z t_1^4 t_2 t_4^7 + 487227313422 \times 10^{19} Z^3 t_1^3 t_2^2 t_4^7 \\
&+ 5138002291752 \times 10^{17} Z^5 t_1^2 t_2^3 t_4^7 + 29525531608824 \times 10^{16} Z^7 t_1 t_2^4 t_4^7 \\
&+ 703282402281004 \times 10^{14} Z t_2^6 t_4^7 + 967786684668 \times 10^{21} Z^4 t_1^4 t_3 t_4^7 \\
&+ 14716118944512 \times 10^{19} Z^6 t_1^3 t_2 t_3 t_4^7 + 703166957985024 \times 10^{17} t_1^2 t_2^3 t_3 t_4^7 \\
&+ 4827712014245664 \times 10^{16} Z^2 t_1 t_2^4 t_3 t_4^7 + 69384024108574944 \times 10^{14} Z^4 t_2^5 t_3 t_4^7 \\
&- 82843008592896 \times 10^{20} Z t_1^3 t_2 t_3^2 t_4^7 + 32392003709495808 \times 10^{17} Z^3 t_1^2 t_2^2 t_3^2 t_4^7 \\
&- 10432861998468096 \times 10^{16} Z^5 t_1 t_2^3 t_3^2 t_4^7 + 62030381760797184 \times 10^{15} Z^7 t_2^4 t_3^2 t_4^7
\end{aligned}
$$

$$
\begin{aligned}
&+ 2891283315554304 \times 10^{20} Z^4 t_1^3 t_3^3 t_4^7 + 76283928853364736 \times 10^{18} Z^6 t_1^2 t_2 t_3^3 t_4^7 \\
&- 1225857696127131648 \times 10^{17} t_1 t_2^3 t_3^3 t_4^7 - 6861500442758393856 \times 10^{15} Z^2 t_2^4 t_3^3 t_4^7 \\
&+ 3252562962636865536 \times 10^{18} Z t_1^2 t_2 t_3^4 t_4^7 + 10872645903813869568 \times 10^{17} Z^3 t_1 t_2^2 t_3^4 t_4^7 \\
&- 51445644065662304256 \times 10^{15} Z^5 t_2^3 t_3^4 t_4^7 - 16367794813221470208 \times 10^{18} Z^4 t_1^2 t_3^5 t_4^7 \\
&+ 620871039145564176384 \times 10^{17} Z^6 t_1 t_2 t_3^5 t_4^7 + 492199709804405981184 \times 10^{17} t_2^3 t_3^5 t_4^7 \\
&- 3112476099737265635328 \times 10^{18} Z t_1 t_2 t_3^6 t_4^7 - 11876990735856051421184 \times 10^{16} Z^3 t_2^2 t_3^6 t_4^7 \\
&- 60027798326235283587072 \times 10^{18} Z^4 t_1 t_3^7 t_4^7 - 2315929925784424975171584 \times 10^{16} Z^6 t_2 t_3^7 t_4^7 \\
&+ 540855932736493288685568 \times 10^{17} Z t_2 t_3^8 t_4^7 + 1896590220191587954065408 \times 10^{17} Z^4 t_3^9 t_4^7 \\
&- 86582819052 \times 10^{20} Z^6 t_1^4 t_4^8 - 201986756469 \times 10^{18} t_1^3 t_2^2 t_4^8 \\
&+ 56056162173552 \times 10^{17} Z^2 t_1^2 t_2^3 t_4^8 + 1420259889220062 \times 10^{15} Z^4 t_1 t_2^4 t_4^8 \\
&+ 7634732170153504 \times 10^{13} Z^6 t_2^5 t_4^8 - 54818947584 \times 10^{21} Z t_1^4 t_3 t_4^8 \\
&+ 54960629235264 \times 10^{19} Z^3 t_1^3 t_2 t_3 t_4^8 - 5164317883574784 \times 10^{16} Z^5 t_1^2 t_2^2 t_3 t_4^8 \\
&+ 16342571529271392 \times 10^{16} Z^7 t_1 t_2^3 t_3 t_4^8 + 679259413994126464 \times 10^{14} Z t_2^5 t_3 t_4^8 \\
&- 26629554065780736 \times 10^{18} Z^6 t_1^3 t_3^2 t_4^8 + 417623077645252992 \times 10^{17} t_1^2 t_2^2 t_3^2 t_4^8 \\
&+ 10953828991071611904 \times 10^{15} Z^2 t_1 t_2^3 t_3^2 t_4^8 + 22186694081228414208 \times 10^{14} Z^4 t_2^4 t_3^2 t_4^8 \\
&- 19828443715633152 \times 10^{20} Z t_1^3 t_3^3 t_4^8 + 5245133155204939776 \times 10^{17} Z^3 t_1^2 t_2 t_3^3 t_4^8 \\
&- 220713204349807558656 \times 10^{15} Z^5 t_1 t_2^2 t_3^3 t_4^8 - 72261316961982861312 \times 10^{15} Z^7 t_2^3 t_3^3 t_4^8 \\
&- 330801727814182895616 \times 10^{17} Z^6 t_1^2 t_3^4 t_4^8 - 2190429175409157193728 \times 10^{16} t_1 t_2^2 t_3^4 t_4^8 \\
&- 66115987243256701648896 \times 10^{14} Z^2 t_2^3 t_3^4 t_4^8 - 681163422305328562176 \times 10^{18} Z t_1^2 t_3^5 t_4^8 \\
&+ 4171177691662198505472 \times 10^{17} Z^3 t_1 t_2 t_3^5 t_4^8 + 514776851109737517809664 \times 10^{14} Z^5 t_2^2 t_3^5 t_4^8 \\
&+ 593870732612785498226688 \times 10^{16} Z^6 t_1 t_3^6 t_4^8 + 9924234758718594459107328 \times 10^{15} t_2^2 t_3^6 t_4^8 \\
&+ 5931187240820312899584 \times 10^{19} Z t_1 t_3^7 t_4^8 - 19497764296979195306704896 \times 10^{16} Z^3 t_2 t_3^7 t_4^8 \\
&- 1149123126890652460004671488 \times 10^{15} Z^6 t_3^8 t_4^8 + 53645129828797212342091776 \times 10^{17} Z t_3^9 t_4^8 \\
&- 440080252416 \times 10^{20} Z^3 t_1^4 t_4^9 + 47536282237248 \times 10^{18} Z^5 t_1^3 t_2 t_4^9 \\
&+ 2246426387852544 \times 10^{15} Z^7 t_1^2 t_2^2 t_4^9 - 81220080453294144 \times 10^{14} Z t_1 t_2^4 t_4^9 \\
&+ 2979280891864 \times 10^{17} Z^3 t_2^5 t_4^9 + 18975480880992768 \times 10^{17} t_1^3 t_2 t_3 t_4^9 \\
&- 1612266108589642752 \times 10^{15} Z^2 t_1^2 t_2^2 t_3 t_4^9 + 8554123745651429376 \times 10^{14} Z^4 t_1 t_2^3 t_3 t_4^9 \\
&+ 1434185788453952 \times 10^{17} Z^6 t_2^4 t_3 t_4^9 - 1305514732390612992 \times 10^{17} Z^3 t_1^3 t_3^2 t_4^9 \\
&+ 50610268799232933888 \times 10^{15} Z^5 t_1^2 t_2 t_3^2 t_4^9 + 555540699217702404096 \times 10^{14} Z^7 t_1 t_2^2 t_3^2 t_4^9 \\
&+ 2854758668626254651392 \times 10^{13} Z t_2^4 t_3^2 t_4^9 + 501651292554077995008 \times 10^{16} t_1^2 t_2 t_3^3 t_4^9 \\
&+ 16406395156945066524672 \times 10^{14} Z^2 t_1 t_2^2 t_3^3 t_4^9 - 38336382685278479843328 \times 10^{13} Z^4 t_2^3 t_3^3 t_4^9 \\
&- 23867257668946322522112 \times 10^{16} Z^3 t_1^2 t_3^4 t_4^9 - 78065177451668321599488 \times 10^{14} Z^5 t_1 t_2 t_3^4 t_4^9 \\
&- 3357820116874590582472704 \times 10^{13} Z^7 t_2^2 t_3^4 t_4^9 - 6022362993784961466630144 \times 10^{15} t_1 t_2 t_3^5 t_4^9 \\
&- 151998661765072452707155968 \times 10^{13} Z^2 t_2^2 t_3^5 t_4^9 + 48831688427024771539009536 \times 10^{15} Z^3 t_1 t_3^6 t_4^9 \\
&+ 729251608263710322271977472 \times 10^{13} Z^5 t_2 t_3^6 t_4^9 + 9653091533927461316433281024 \times 10^{14} t_2 t_3^7 t_4^9 \\
&- 114897612177989845172413792256 \times 10^{14} Z^3 t_3^8 t_4^9 + 2717908992 \times 10^{23} t_1^4 t_4^{10} \\
&+ 32281461801414144 \times 10^{16} Z^2 t_1^3 t_2 t_4^{10} + 3600173188424475648 \times 10^{13} Z^4 t_1^2 t_2^2 t_4^{10} \\
&- 5796032773675064832 \times 10^{12} Z^6 t_1 t_2^3 t_4^{10} - 14123562831052364288 \times 10^{11} t_2^5 t_4^{10} \\
&- 559925260477267968 \times 10^{16} Z^5 t_1^3 t_3 t_4^{10} - 125671784435007750144 \times 10^{13} Z^7 t_1^2 t_2 t_3 t_4^{10}
\end{aligned}
$$

$$
\begin{aligned}
&- 580904853826083033907 2 \times 10^{12} Z t_1 t_2^3 t_3 t_4^{10} + 1080241385793505678131 2 \times 10^{11} Z^3 t_2^4 t_3 t_4^{10} \\
&- 2961093112923193344 \times 10^{17} t_1^3 t_3^2 t_4^{10} + 3614012780872093846732 8 \times 10^{13} Z^2 t_1^2 t_2 t_3^2 t_4^{10} \\
&+ 333640284199212515328 \times 10^{15} Z^4 t_1 t_2^2 t_3^2 t_4^{10} + 57214443198876314573209 6 \times 10^{11} Z^6 t_2^3 t_3^2 t_4^{10} \\
&- 10031686119133533123379 2 \times 10^{13} Z^5 t_1^2 t_3^3 t_4^{10} + 91143336171739360882851 84 \times 10^{12} Z^7 t_1 t_2 t_3^3 t_4^{10} \\
&+ 5666576977654071069088153 6 \times 10^{11} Z t_2^3 t_3^3 t_4^{10} - 943192297267937044070 4 \times 10^{16} t_1^2 t_3^4 t_4^{10} \\
&+ 792691861424652971327619072 \times 10^{12} Z^2 t_1 t_2 t_3^4 t_4^{10} - 2562359415392924410904051712 \times 10^{11} Z^4 t_2^2 t_3^4 t_4^{10} \\
&- 257300888407494813213838540 8 \times 10^{12} Z^5 t_1 t_3^5 t_4^{10} - 4798792395626759409406417305 6 \times 10^{11} Z^7 t_2 t_3^5 t_4^{10} \\
&- 157862234175379577921077248 \times 10^{15} t_1 t_3^6 t_4^{10} - 2843902853417474551549859463168 \times 10^{11} Z^2 t_2 t_3^6 t_4^{10} \\
&+ 5238874597991936687838089707 52 \times 10^{11} Z^5 t_3^7 t_4^{10} + 4934896492869381003241670574 08 \times 10^{14} t_3^8 t_4^{10} \\
&+ 607307379035111424 \times 10^{15} Z^7 t_1^3 t_4^{11} - 21851476179569639424 \times 10^{12} Z t_1^2 t_2^2 t_4^{11} \\
&- 91560924114995920896 \times 10^{11} Z^3 t_1 t_2^3 t_4^{11} + 80869030215849074688 \times 10^{11} Z^5 t_2^4 t_4^{11} \\
&- 51634095229698048 \times 10^{16} Z^2 t_1^3 t_3 t_4^{11} - 109130418446774572154 88 \times 10^{12} Z^4 t_1^2 t_2 t_3 t_4^{11} \\
&- 297177275436349025157 12 \times 10^{11} Z^6 t_1 t_2^2 t_3 t_4^{11} + 507629102655557807636 48 \times 10^{10} t_2^4 t_3 t_4^{11} \\
&+ 337833011193596916793344 \times 10^{12} Z^7 t_1^2 t_3^2 t_4^{11} - 14162049270750790759219 2 \times 10^{13} Z t_1 t_2^2 t_3^2 t_4^{11} \\
&+ 42635273712468053695922176 \times 10^{10} Z^3 t_2^3 t_3^2 t_4^{11} + 4981556818011699648921 6 \times 10^{13} Z^2 t_1^2 t_3^3 t_4^{11} \\
&+ 51117837350009309726402150 4 \times 10^{11} Z^4 t_1 t_2 t_3^3 t_4^{11} + 527695629661743007605456896 \times 10^{10} Z^6 t_2^2 t_3^3 t_4^{11} \\
&+ 667097216656493527359396249 6 \times 10^{11} Z^7 t_1 t_3^4 t_4^{11} + 564322011173317794419242434 56 \times 10^{10} Z t_2^2 t_3^4 t_4^{11} \\
&+ 170730720179289446636417187 84 \times 10^{12} Z^2 t_1 t_3^5 t_4^{11} \\
&- 294087451041083102728802009 088 \times 10^{11} Z^4 t_2 t_3^5 t_4^{11} \\
&- 271884470233727551531695042723 84 \times 10^{10} Z^7 t_3^6 t_4^{11} \\
&- 2235405208848545877601098357800 96 \times 10^{11} Z^2 t_3^7 t_4^{11} \\
&+ 4127371884760891392 \times 10^{15} Z^4 t_1^3 t_4^{12} - 33789621905653078425 6 \times 10^{11} Z^6 t_1^2 t_2 t_4^{12} \\
&+ 942478436446431407308 8 \times 10^{10} t_1 t_2^3 t_4^{12} + 894196497454561532313 6 \times 10^{10} Z^2 t_2^4 t_4^{12} \\
&+ 971551410571376527933 44 \times 10^{11} Z t_1^2 t_2 t_3 t_4^{12} - 794702821938348162023424 \times 10^{10} Z^3 t_1 t_2^2 t_3 t_4^{12} \\
&+ 61843071996207438338457 6 \times 10^{10} Z^5 t_2^3 t_3 t_4^{12} + 239364844582422851081994 24 \times 10^{11} Z^4 t_1^2 t_3^2 t_4^{12} \\
&- 223032099425911384540446 72 \times 10^{10} Z^6 t_1 t_2 t_3^2 t_4^{12} + 397351567459146497762787328 \times 10^{9} t_2^3 t_3^2 t_4^{12} \\
&- 998213399946818588790856089 6 \times 10^{10} Z t_1 t_2 t_3^3 t_4^{12} + 193876153230314611968667811 84 \times 10^{9} Z^3 t_2^2 t_3^3 t_4^{12} \\
&+ 516338223741156378549612969 984 \times 10^{10} Z^4 t_1 t_3^4 t_4^{12} \\
&- 793567166772590187732386447 36 \times 10^{10} Z^6 t_2 t_3^4 t_4^{12} \\
&+ 132205612055116282496361378 6112 \times 10^{9} Z t_2 t_3^5 t_4^{12} \\
&- 161538019503296585369640055 3934848 \times 10^{9} Z^4 t_3^6 t_4^{12} \\
&- 927712935936 \times 10^{22} Z t_1^3 t_4^{13} + 125040458427828724039 68 \times 10^{10} Z^3 t_1^2 t_2 t_4^{13} \\
&- 111956252529564010414 08 \times 10^{10} Z^5 t_1 t_2^2 t_4^{13} + 193610298034189452705792 \times 10^{9} Z^7 t_2^3 t_4^{13} \\
&+ 101751234584480796848947 2 \times 10^{10} Z^6 t_1^2 t_3 t_4^{13} + 495367657064839087977922 56 \times 10^{9} t_1 t_2^2 t_3 t_4^{13} \\
&+ 699373260199305572285153 28 \times 10^{9} Z^2 t_2^3 t_3 t_4^{13} - 429955482404977298343198 72 \times 10^{11} Z t_1^2 t_3^2 t_4^{13} \\
&- 234313409887086098099522764 8 \times 10^{9} Z^3 t_1 t_2 t_3^2 t_4^{13} + 152657424477514377300698726 4 \times 10^{9} Z^5 t_2^2 t_3^2 t_4^{13} \\
&- 575078433752979752024958566 4 \times 10^{10} Z^6 t_1 t_3^3 t_4^{13} + 253179028159776671802589184 \times 10^{11} t_2^2 t_3^3 t_4^{13} \\
&- 215286110606730023164981542 912 \times 10^{10} Z t_1 t_3^4 t_4^{13} \\
&- 671641740349146074931193393 9712 \times 10^{9} Z^3 t_2 t_3^4 t_4^{13}
\end{aligned}
$$

$$- 607377370370862097868099072032768 \times 10^8 Z^6 t_3^5 t_4^{13}$$
$$- 510284694342397195104181957532057 6 \times 10^9 Z t_3^6 t_4^{13}$$
$$- 2011410056346878261329 92 \times 10^9 t_1^2 t_2 t_4^{14} + 2887315447582082350448 64 \times 10^9 Z^2 t_1 t_2^2 t_4^{14}$$
$$+ 120631282618072738289418 24 \times 10^8 Z^4 t_2^3 t_4^{14} - 1420314090261645469365043 2 \times 10^9 Z^3 t_1^2 t_3 t_4^{14}$$
$$+ 569264807732818463969771 52 \times 10^9 Z^5 t_1 t_2 t_3 t_4^{14} + 1280228101029937272871452672 \times 10^8 Z^7 t_2^2 t_3 t_4^{14}$$
$$+ 1627429051491697304496871833 6 \times 10^9 t_1 t_2 t_3^2 t_4^{14} + 1651982695101568571388670771 2 \times 10^9 Z^2 t_2^2 t_3^2 t_4^{14}$$
$$- 184644996554724796320586924032 \times 10^9 Z^3 t_1 t_3^3 t_4^{14}$$
$$- 915100746058983370992880975872 \times 10^8 Z^5 t_2 t_3^3 t_4^{14}$$
$$- 171061205549780844406726335660032 \times 10^8 t_2 t_3^4 t_4^{14}$$
$$- 235554814308541800324018111879577 6 \times 10^8 Z^3 t_3^5 t_4^{14}$$
$$+ 152882249027563547818721 28 \times 10^8 Z^5 t_1^2 t_4^{15} - 1117172036781428663302225 92 \times 10^7 Z^7 t_1 t_2 t_4^{15}$$
$$- 271429074048832168763641036 8 \times 10^6 Z t_2^3 t_4^{15} + 2574604872124004174502297 6 \times 10^9 t_1^2 t_3 t_4^{15}$$
$$- 137041844505719487891536609 28 \times 10^7 Z^2 t_1 t_2 t_3 t_4^{15}$$
$$+ 892015412190888101392278355968 \times 10^6 Z^4 t_2^2 t_3 t_4^{15}$$
$$- 931050336162228145925639897088 \times 10^7 Z^5 t_1 t_3^2 t_4^{15}$$
$$+ 348805837941658001301978564526 08 \times 10^6 Z^7 t_2 t_3^2 t_4^{15}$$
$$+ 960868055593990836630929276928 \times 10^9 t_1 t_3^3 t_4^{15}$$
$$+ 153422863765366710001624312656691 2 \times 10^6 Z^2 t_2 t_3^3 t_4^{15}$$
$$- 229664341220550359636852616336506 88 \times 10^6 Z^5 t_3^4 t_4^{15}$$
$$- 148093522720905102187439812394352 64 \times 10^8 t_3^5 t_4^{15}$$
$$- 257460487212400417450229 76 \times 10^8 Z^2 t_1^2 t_4^{16}$$
$$+ 157686704534975098605325516 8 \times 10^6 Z^4 t_1 t_2 t_4^{16}$$
$$- 9170032818302852789283323904 00000 Z^6 t_2^2 t_4^{16}$$
$$+ 130516336497402585140995031 04 \times 10^7 Z^7 t_1 t_3 t_4^{16}$$
$$- 2076367675443771752050042758758400000 Z t_2^2 t_3 t_4^{16}$$
$$+ 171492222773606616123608295014 4 \times 10^7 Z^2 t_1 t_3^2 t_4^{16}$$
$$+ 2077731398546251687856063637356544 00000 Z^4 t_2 t_3^2 t_4^{16}$$
$$+ 245460746734740724114902978291302 4 \times 10^6 Z^7 t_3^3 t_4^{16}$$
$$- 200588103092159633347019199700008 96 \times 10^6 Z^2 t_3^4 t_4^{16}$$
$$+ 2030879592689935607722200268800000 Z t_1 t_2 t_4^{17}$$
$$+ 1848171433285662882966703964160000 Z^3 t_2^2 t_4^{17}$$
$$- 28984489922036435905737255813120 0000 Z^4 t_1 t_3 t_4^{17}$$
$$+ 3595044650552048410338036547584 00000 Z^6 t_2 t_3 t_4^{17}$$
$$- 465938357168762673484515615430410240000 Z t_2 t_3^2 t_4^{17}$$
$$+ 16742853032212084296488061597237903360000 Z^4 t_3^3 t_4^{17}$$
$$- 16250104332031975027644479569920000 Z^6 t_1 t_4^{18}$$
$$- 2959961733728970613296539369472000 t_2^2 t_4^{18}$$
$$- 259952587864311757788441634406400000 Z t_1 t_3 t_4^{18}$$
$$- 43076116565753744311050290017075 2000 Z^3 t_2 t_3 t_4^{18}$$

$$
\begin{aligned}
&- 6051260244720079268826417352566374400\,0Z^6t_3^2t_4^{18} \\
&- 38998982585752693724524027389337927680000Zt_3^3t_4^{18} \\
&+ 2599525878643117577884416344064 0000Z^3t_1t_4^{19} \\
&+ 226101767396691185885864169308160 00Z^5t_2t_4^{19} \\
&+ 4253773886874547642912599202332672 00t_2t_3t_4^{19} \\
&+ 97403159503619800376825787377201971200Z^3t_3^2t_4^{19} \\
&- 2765416326381908932481125667831808 0Z^2t_2t_4^{20} \\
&- 7822021662339635349431421245168025 60Z^5t_3t_4^{20} \\
&- 595229270657875530103076876042895360 0t_3^2t_4^{20} \\
&- 8186674472180777476514488439852236 8Z^7t_4^{21} \\
&- 714639136114826489146132372075315200Z^2t_3t_4^{21} \\
&+ 13098684067727020192492092481182105 6Z^4t_4^{22}\big)\,.
\end{aligned}
$$

By Proposition 4.0.3, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 4.5.19, which leads to Proposition A.4.2.

**Proposition A.4.3.** *The unity vector field $e = \partial_{t_1}$ in the $y$ coordinates has the form*

$$
\begin{aligned}
e(y) = &-\frac{390625}{26244}\left(5t_2 + 1408t_3t_4\right)\partial_{y_3} + \frac{390625}{344373768}\big(14062500t_1^2 - 53125Z^4t_2^2 \\
&- 3300000Z^7t_2t_3 + 684\times 10^7t_1t_3^2 - 564800000Z^2t_2t_3^2 + 2969600000Z^5t_3^3 \\
&+ 202598400000t_3^4 + 1430000Zt_2^2t_4 - 58800000Z^4t_2t_3t_4 + 1085440000Z^7t_3^2t_4 \\
&+ 72294400000Z^2t_3^3t_4 - 2188000Z^6t_2t_4^2 - 145024000Zt_2t_3t_4^2 + 3608576000Z^4t_3^2t_4^2 \\
&+ 10534400Z^3t_2t_4^3 - 379904000Z^6t_3t_4^3 - 4866048000Zt_3^2t_4^3 + 616528640t_2t_4^4 \\
&+ 689520640Z^3t_3t_4^4 + 157261824Z^5t_4^5 + 193714094080t_3t_4^5 - 234897408Z^2t_4^6\big)\,\partial_{y_2} \\
&- \frac{390625}{847288609443}\big(439453125000Z^7t_1^2t_2 - 63281250000Zt_1t_2^3 + 3330078125Z^3t_2^4 \\
&- 263671875\times 10^6t_1^3t_3 + 43066406250000Z^2t_1^2t_2t_3 + 1699804687500Z^4t_1t_2^2t_3 \\
&- 139628906250Z^6t_2^3t_3 - 984375\times 10^8Z^5t_1^2t_3^2 - 4775625\times 10^7Z^7t_1t_2t_3^2 \\
&+ 4382812500000Zt_2^3t_3^2 - 41175\times 10^{12}t_1^2t_3^3 - 9252\times 10^{11}Z^2t_1t_2t_3^3 \\
&- 494675\times 10^8Z^4t_2^2t_3^3 - 166032\times 10^{11}Z^5t_1t_3^4 \\
&- 489472\times 10^{10}Z^7t_2t_3^4 - 58002048\times 10^{11}t_1t_3^5 \\
&+ 9473472\times 10^{10}Z^2t_2t_3^5 - 50370048\times 10^{11}Z^5t_3^6 \\
&- 2554601472\times 10^{11}t_3^7 + 3229980468750Z^4t_1^2t_2t_4 + 423984375000Z^6t_1t_2^2t_4 \\
&- 240234375t_2^4t_4 - 63984375\times 10^6Z^7t_1^2t_3t_4 + 25745625\times 10^6Zt_1t_2^2t_3t_4 \\
&- 170468750000Z^3t_2^3t_3t_4 - 55125\times 10^{11}Z^2t_1^2t_3^2t_4 + 6864075\times 10^8Z^4t_1t_2t_3^2t_4 \\
&- 1593975\times 10^7Z^6t_2^2t_3^2t_4 - 977688\times 10^{10}Z^7t_1t_3^3t_4 - 255016\times 10^9Zt_2^2t_3^3t_4 \\
&+ 1184256\times 10^{11}Z^2t_1t_3^4t_4 - 30143888\times 10^9Z^4t_2t_3^4t_4 \\
&- 5025536\times 10^{10}Z^7t_3^5t_4 - 1212604416\times 10^{10}Z^2t_3^6t_4
\end{aligned}
$$

$$
\begin{aligned}
&- 13289062500000 Z t_1^2 t_2 t_4^2 + 3715875 \times 10^6 Z^3 t_1 t_2^2 t_4^2 + 393562500000 Z^5 t_2^3 t_4^2 \\
&- 34059375 \times 10^7 Z^4 t_1^2 t_3 t_4^2 - 7314525 \times 10^7 Z^6 t_1 t_2 t_3 t_4^2 + 15289387500000 t_2^3 t_3 t_4^2 \\
&- 4266792 \times 10^9 Z t_1 t_2 t_3^2 t_4^2 + 17681826 \times 10^8 Z^3 t_2^2 t_3^2 t_4^2 \\
&- 39320352 \times 10^9 Z^4 t_1 t_3^3 t_4^2 - 27306528 \times 10^9 Z^6 t_2 t_3^3 t_4^2 \\
&+ 10774272 \times 10^9 Z t_2 t_3^4 t_4^2 - 4930018304 \times 10^9 Z^4 t_3^5 t_4^2 \\
&+ 3346875 \times 10^6 Z^6 t_1^2 t_4^3 - 879862500000 t_1 t_2^2 t_4^3 + 3587256250000 Z^2 t_2^3 t_4^3 \\
&+ 1701 \times 10^{12} Z t_1^2 t_3 t_4^3 - 14994216 \times 10^8 Z^3 t_1 t_2 t_3 t_4^3 \\
&- 182912 \times 10^8 Z^5 t_2^2 t_3 t_4^3 + 2043288 \times 10^{10} Z^6 t_1 t_3^2 t_4^3 \\
&+ 131954704 \times 10^8 t_2^2 t_3^2 t_4^3 + 257043456 \times 10^9 Z t_1 t_3^3 t_4^3 \\
&- 627507712 \times 10^8 Z^3 t_2 t_3^3 t_4^3 - 579936768 \times 10^9 Z^6 t_3^4 t_4^3 \\
&- 6392348672 \times 10^9 Z t_3^5 t_4^3 - 126 \times 10^{11} Z^3 t_1^2 t_4^4 + 78732 \times 10^8 Z^5 t_1 t_2 t_4^4 \\
&+ 1321801 \times 10^7 Z^7 t_2^2 t_4^4 - 162037944 \times 10^8 t_1 t_2 t_3 t_4^4 + 54162276 \times 10^7 Z^2 t_2^2 t_3 t_4^4 \\
&+ 91517184 \times 10^9 Z^3 t_1 t_3^2 t_4^4 + 374721408 \times 10^7 Z^5 t_2 t_3^2 t_4^4 \\
&+ 2323609344 \times 10^9 t_2 t_3^3 t_4^4 - 34638528512 \times 10^8 Z^3 t_3^4 t_4^4 \\
&- 27936 \times 10^{11} t_1^2 t_4^5 + 64875168 \times 10^6 Z^2 t_1 t_2 t_4^5 + 87537610400000 Z^4 t_2^2 t_4^5 \\
&- 5753244672 \times 10^6 Z^5 t_1 t_3 t_4^5 + 2215719872 \times 10^6 Z^7 t_2 t_3 t_4^5 \\
&- 41304554496 \times 10^8 t_1 t_3^2 t_4^5 + 99264934297600000 Z^2 t_2 t_3^2 t_4^5 \\
&+ 991216605593600000 Z^5 t_3^3 t_4^5 + 65669464064 \times 10^9 t_3^4 t_4^5 \\
&+ 18139852800000 Z^7 t_1 t_4^6 - 761788568320000 Z t_2^2 t_4^6 - 3309963264 \times 10^6 Z^2 t_1 t_3 t_4^6 \\
&+ 33590644321280000 Z^4 t_2 t_3 t_4^6 - 770186709565440000 Z^7 t_3^2 t_4^6 \\
&- 3228085287649280000 Z^2 t_3^3 t_4^6 + 145118822400000 Z^4 t_1 t_4^7 \\
&+ 289647473152000 Z^6 t_2 t_4^7 + 30001074438144000 Z t_2 t_3 t_4^7 \\
&- 4282756594794496000 Z^4 t_3^2 t_4^7 - 3107951122841600 Z^3 t_2 t_4^8 \\
&+ 172952984669388800 Z^6 t_3 t_4^8 + 8641006375272448000 Z t_3^2 t_4^8 \\
&- 121855490317352960 t_2 t_4^9 - 89661391828418560 Z^3 t_3 t_4^9 - 28213513851764736 Z^5 t_4^{10} \\
&- 413108139113788211120 t_3 t_4^{10} + 37440687137882112 Z^2 t_4^{11} \big) \, \partial_{y_1}.
\end{aligned}
$$

*Proof.* We have that

$$
e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1} \partial_{y_\alpha},
$$

which gives the statement by applying the relations from Theorem 4.5.19. □

# Bibliography


[1] D. Abriani, *Frobenius manifolds associated to Coxeter groups of type $E_7$ and $E_8$*, arXiv preprint arXiv:0910.5453 (2009).

[2] A. Brini and K. Van Gemst, *Mirror symmetry for extended affine Weyl groups*, Journal de l'École polytechnique—Mathématiques **9** (2022), 907–957.

[3] R. W. Carter, *Conjugacy classes in the Weyl group*, Compositio Mathematica **25** (1972), no. 1, 1–59.

[4] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebra: an introduction*, CRC Press, 1993.

[5] F. Delduc and L. Fehér, *Regular conjugacy classes in the Weyl group and integrable hierarchies*, Journal of Physics A: Mathematical and General **28** (1995), no. 20, 5843.

[6] R. Dijkgraaf, H. Verlinde, and E. Verlinde, *Notes on topological string theory and 2D quantum gravity*, String theory and quantum gravity (Trieste, 1990) (1990), 91–156.

[7] Y. I. Dinar, *On classification and construction of algebraic Frobenius manifolds*, Journal of Geometry and Physics **58** (2008), no. 9, 1171–1185.

[8] ———, *The quadratic WDVV solution $E_8(a_1)$*, arXiv:1110.2003 preprint (2011).

[9] ———, *Frobenius manifolds from subregular classical W-algebras*, International Mathematics Research Notices **2013** (2013), no. 12, 2822–2861.

[10] ———, *Algebraic classical W-algebras and Frobenius manifolds*, Letters in Mathematical Physics **111** (2021), no. 5, 115.

[11] Y. I. Dinar and J. Sekiguchi, *The WDVV solution $E_8(a_1)$*, Journal of Geometry and Physics **170** (2021), 104388.

[12] T. Douvropoulos, J. B. Lewis, and A. H. Morales, *Hurwitz numbers for reflection groups II: Parabolic quasi-Coxeter elements*, arXiv preprint arXiv:2209.00066 (2022).

[13] B. Dubrovin, *Geometry of 2D topological field theories*, Integrable Systems and Quantum Groups: Lectures given at the 1st Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Montecatini Terme, Italy, June 14–22, 1993, 1996, pp. 120–348, DOI 10.1007/BFb0094793.

[14] ———, *Differential geometry of the space of orbits of a Coxeter group, 181–211*, Surveys in Differential Geometry IV: Integrable Systems, Boston: International Press (1998).

[15] ———, *Flat pencils of metrics and Frobenius manifolds*, Proceedings of 1997 Taniguchi symposium "Integrable systems and algebraic geometry" (1998).

[16] ———, *Painlevé transcendents in two-dimensional topological field theory*, The Painlevé Property: One Century Later, 1999, pp. 287–412.

[17] ———, *On almost duality for Frobenius manifolds*, Geometry, topology, and mathematical physics **212** (2004), no. 55, 75-132.

[18] B. Dubrovin and M. Mazzocco, *Monodromy of certain Painlevé–VI transcendents and reflection groups*, Inventiones mathematicae **141** (2000), no. 1, 55–147.

[19] B. Dubrovin and S. P. Novikov, *Hamiltonian formalism of one-dimensional systems of hydrodynamic type, and the bogolyubov-whitman averaging method*, 30 Years Of The Landau Institute—Selected Papers, 1996, pp. 382–386.

[20] B. Dubrovin, S. Liu, and Y. Zhang, *Frobenius manifolds and central invariants for the Drinfeld–Sokolov bihamiltonian structures*, Advances in Mathematics **219** (2008), no. 3, 780–837.

[21] B. Dubrovin and Y. Zhang, *Extended affine Weyl groups and Frobenius manifolds*, Compositio Mathematica **111** (1998), no. 2, 167–219.

[22] ———, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants*, arXiv preprint math/0108160 (2001).

[23] C. F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables*, Cambridge University Press, 2014.

[24] P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Inventiones mathematicae **147** (2002), 243–348.

[25] M. Feigin and A. Silantyev, *Singular polynomials from orbit spaces*, Compositio Mathematica **148** (2012), no. 6, 1867–1879.

[26] M. Feigin, D. Valeri, and J. Wright, *Flat coordinates of algebraic Frobenius manifolds in small dimensions*, Journal of Geometry and Physics (2024), 105151.

[27] ———, *Flat coordinates of algebraic Frobenius manifolds in small dimensions*, The Notebook Archive (2024). `https://notebookarchive.org/2024-03-2sleam2`.

[28] M. V. Feigin and A. P. Veselov, *Logarithmic Frobenius structures and Coxeter discriminants*, Advances in mathematics **212** (2007), no. 1, 143–162.

[29] P. K. H. Gragert and R. Martini, *Solutions of WDVV equations in Seiberg-Witten theory from root systems*, Journal of nonlinear mathematical physics **6** (1999), no. 1, 1–4.

[30] D. Guzzetti, *Inverse Problem for semisimple Frobenius Manifolds, Monodromy Data and the Painleve'VI Equation*, arXiv preprint math/0010235 (2000).

[31] C. Hertling, *Frobenius manifolds and moduli spaces for singularities*, Vol. 151, Cambridge University Press, 2002.

[32] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge university press, 1992.

[33] E. L. Ince, *Ordinary differential equations*, Courier Corporation, 1956.

[34] V. G. Kac and D. H. Peterson, *112 constructions of the basic representation of the loop group of $E_8$*, Proceedings of a symposium on anomalies geometry topology, 1985.

[35] M. Kato, T. Mano, and J. Sekiguchi, *Flat structure on the space of isomonodromic deformations*, SIGMA. Symmetry, Integrability and Geometry: Methods and Applications **16** (2020), 110.

[36] M. Kato and S. Watanabe, *The flat coordinate system of the rational double point of E8 type*, University of the Ryukyus, 1981.

[37] J. Kock, *Frobenius algebras and 2D topological quantum field theories*, Cambridge University Press, 2004.

[38] F. Magri, *A simple model of the integrable Hamiltonian equation*, Journal of Mathematical Physics **19** (1978), no. 5, 1156–1162.

[39] Y.I. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, Vol. 47, American Mathematical Soc., 1999.

[40] A. Marshakov, A. Mironov, and A. Morozov, *More evidence for the WDVV equations in SUSY Yang–Mills theories*, International Journal of Modern Physics A **15** (2000), no. 08, 1157–1206.

[41] J. Michel, *Hurwitz action on tuples of Euclidean reflections*, Journal of Algebra **295** (2006), no. 1, 289-292, DOI https://doi.org/10.1016/j.jalgebra.2004.12.020.

[42] E. K. Morrison and I. A. B. Strachan, *Modular Frobenius manifolds and their invariant flows*, International Mathematics Research Notices **2011** (2011), no. 17, 3957–3982.

[43] O. Pavlyk, *Solutions to WDVV from generalized Drinfeld-Sokolov hierarchies*, arXiv:0003020 preprint (2000).

[44] K. Saito, *Period mapping associated to a primitive form*, Publications of the Research Institute for Mathematical Sciences **19** (1983), no. 3, 1231–1264.

[45] ______, *Primitive forms for a universal unfolding of a function with an isolated critical point*, J. Fac. Sci. Univ. Tokyo Sect. IA Math **28** (1981), no. 3, 775–792.

[46] ______, *On a linear structure of the quotient variety by a finite reflexion group*, Publications of the Research Institute for Mathematical Sciences **29** (1993), no. 4, 535–579.

[47] K. Saito, T. Yano, and J. Sekiguchi, *On a certain generator system of the ring of invariants of a finite reflection group*, Communications in Algebra **8** (1980), no. 4, 373–408.

[48] J. Sekiguchi, *The Construction Problem of Algebraic Potentials and Reflection Groups*, Proceedings of the Symposium on Representation Theory **2019** (2019), 151-165, DOI 10.34508/repsympo.2019.0_151. Extended version (2023): arXiv:2312.15888.

[49] T. A. Springer, *Regular elements of finite reflection groups*, Inventiones mathematicae **25** (1974), no. 2, 159–198.

[50] R. Stedman and I. A. B. Strachan, *Generalized Legendre transformations and symmetries of the WDVV equations*, Journal of Physics A: Mathematical and Theoretical **50** (2017), no. 9, 095202.

[51] V. Talamini, *Flat bases of invariant polynomials and $\widehat{P}$-matrices of $E_7$ and $E_8$*, Journal of mathematical physics **51** (2010), no. 2.

[52] A. P. Veselov, *Deformations of the root systems and new solutions to generalised WDVV equations*, Physics Letters A **261** (1999), no. 5-6, 297–302.

[53] E. Witten, *On the structure of the topological phase of two-dimensional gravity*, Nuclear Physics B **340** (1990), no. 2-3, 281–332.

[54] D. Yang, *Analytic theory of Legendre-type transformations for a Frobenius manifold*, arXiv preprint arXiv:2311.04200 (2023).