Topological Full Groups

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Declaration

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution

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Abstract

We develop our understanding of topological full groups, a way of constructing examples of infinite simple groups with finiteness properties from ample groupoids. Our results are concentrated in three main example classes. Firstly, the topological full groups of purely infinite minimal groupoids share many properties with Thompson's group V. In studying these groups and the associated groupoids in detail, we formalise this phenomenon by relating dynamical properties to group-theoretic properties. Secondly, interval exchange groups are an important concrete example of topological full groups since many are amenable. We classify these groups through computing associated Elliot invariants. Also, we find concrete generating sets and compute the homology of these groups. Thirdly and finally, Stein's groups were introduced by Melanie Stein in 1992 as generalisations of Thompson's group. We show these groups are topological full groups. We then analyse Stein's groups through this framework, showing that the (simple) derived subgroups of Stein's groups are in many cases finitely generated. We study the homology of Stein's groups.

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Contents

Chapter 1

Introduction

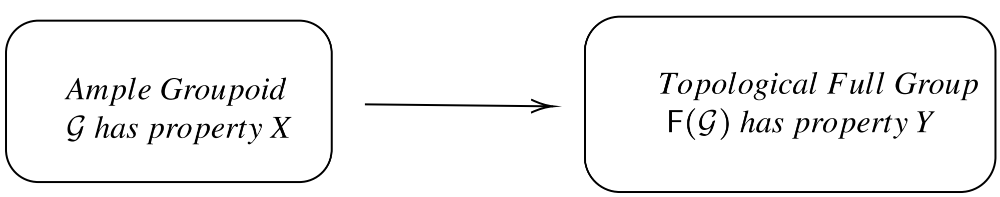
1.1 Overview and summary of main results

The story of topological full groups begins with Giordano-Putnam-Skau [53], who introduced the topological full group of a Cantor minimal system as the group of homeomorphisms of the Cantor set that locally are powers of a homeomorphism T. Later, Matui (in [88]) defined the topological full group $\mathsf{F}(\mathcal{G})$ of an ample groupoid $\mathcal{G}$. The idea is to study the unitary subgroup of the inverse monoid of compact open bisections, in effect piecing together partial symmetries into global symmetries. $\mathsf{D}(\mathcal{G})$ denotes the derived subgroup of this group.

Since then, the framework of topological full groups has lead to the resolution of many existence questions for infinite simple groups with various finiteness properties. For example, they provided the first examples of:

- Infinite simple, finitely generated amenable groups [61].
- Simple finitely generated groups of intermediate growth [92].
- Simple groups separated by finiteness properties [115].

The philosophy of the program of topological full groups is that we would be able to determine information about $\mathsf{F}(\mathcal{G})$ by studying the underlying groupoids $\mathcal{G}$. Many things are known now in this direction [89] [79] [93] but some questions, especially determining amenability or the existence of free subgroups, remain mysterious [60].



This thesis adds to this exciting and growing field through the study of three different types of topological full groups:

- The topological full groups of purely infinite, minimal groupoids (Chapter 5).
- Interval exchange groups (Chapter 6).
- Stein's groups (Chapter 7).

In order to understand the topological full groups of purely infinite minimal groupoids, it is essential to first understand the relationship between topological full groups and Thompson's group V.

Thompson's group V was the first known example of an infinite finitely presented simple group [121]. Since the introduction and dissemination of Thompson's work into the mathematical community, V has impacted mathematics in countless often disconnected and surprising ways, leading Brin to call V "the chameleon group" for its breadth of impact [20]. Combinatorial group theorists have been interested in Thompson's group V because of the connection to interesting open questions such as the Boone-Higman conjecture [6]. Computer scientists have been interested in Thompson's group from the perspective of cryptography, as a platform group [112].

Since its conception, the framework of topological full groups has served as a bridge between Thompson-like groups and the groupoid models of purely infinite C*-algebras, a line of research that was pioneered by Matui [88]. In fact, promising observations by Nekrashevych [94] predate, and even motivate, the definition of topological full groups of étale groupoids by Matui. Because the underlying topological groupoids are often much more accessible to study than the Thompson-like groups themselves, this new dynamical perspective has led to progress in our understanding of the homology [79], finite presentation [78], and subgroup structure of Thompson-like groups [16].

We summarise the correspondence between Thompson-like groups and purely infinite minimal groupoids in the table below, where $\mathcal{E}_k$ is the Deacounu-Renault groupoid associated to full shift on an alphabet with k letters, $\mathcal{R}_r$ is the full equivalence relation on r points, $\mathcal{O}_k$ is the Cuntz algebra with k generators, and $\mathcal{Q}^\lambda$ is the class of Kirchberg algebras considered in [33,76]. Here also, $V_{k,r}$ is the Higman Thompson group and $nV_{k,r}$ is the Brin-Higman-Thompson group.

C^*-algebra	**Groupoid**	**Thompson-like group**
$\mathcal{O}_2$	$\mathcal{E}_2$	V
$M_r(\mathcal{O}_k)$	$\mathcal{R}_r \times \mathcal{E}_k$	$V_{k,r}$
$M_r(\bigotimes_{j=1}^n \mathcal{O}_k)$	$\mathcal{R}_r \times \mathcal{E}_k^n$	$nV_{k,r}$
$\mathcal{Q}^\lambda$	Particular partial actions	Stein's groups with a cyclic group of slopes

Figure 1.1: Realisations of Thompson-like groups as topological full groups

The dynamical realisation of the Brin-Higman-Thompson groups is described in [90], while the realisation of Stein's groups as topological full groups is the subject of Chapter 7.

In light of this correspondence between groupoids and Thompson-like groups, Matui proposed in [89] to regard the topological full groups of other generalisations of the full shift on two generators as generalisations of V, and in particular study them as Thompson-like groups. In turn, analysing the topological full groups led us to the discovery of interesting new groups, many of which exhibit properties enjoyed by Thompson's group V. This vein of research is summarised in the table below:

C*-algebra	**Groupoid $\mathcal{G}$**	Is $\mathsf{D}(\mathcal{G})$ simple and finitely presented?
Cuntz-Krieger algebras	SFT Groupoids	Yes
Tensors of Cuntz-Krieger algebras	Products of SFT groupoids	Yes
Graph algebras	Graph groupoids	Yes
Katsura-Exel-Pardo algebras	Katsura-Exel-Pardo groupoids	Yes

Figure 1.2: New generalisations of Thompson's group V from topological full groups

Very general structural properties have been studied for these examples in [90], and significant progress toward homological properties has been made in [79]. In particular, topological full groups inside this class give rise to many new examples of finitely presented, simple, infinite groups [78, 79, 86, 89, 96, 97].

C*-algebraists have a deep source of such groupoids which have been developed primarily as a way to construct simple, purely infinite, nuclear C*-algebras – also known as Kirchberg algebras. The relevance of these examples stems from their crucial role in the classification theory of simple, nuclear C*-algebras; see [100]. Each of the underlying groupoids is minimal, topological principal and purely infinite in the sense of Matui [89]. In Chapter 5, we study the topological full groups of such groupoids abstractly. We show that this class of groupoids can be characterised in a number of ways intrinsic to the topological full groups. We summarise these characterisations in the following result:

Theorem 1.1.1. *Let $\mathcal{G}$ be an essentially principal, ample groupoid. Then the following are equivalent:*

1. *$\mathcal{G}$ is purely infinite and minimal.*

2. *$\mathsf{D}(\mathcal{G})$ is a vigorous subgroup of* $\mathrm{Homeo}(\mathcal{G}^{(0)})$ *in the sense of Bleak-Elliott-Hyde [11].*

3. *For every $x_0 \in \mathcal{G}^{(0)}$, the subgroup*

$$\mathsf{D}(\mathcal{G})_{x_0} = \{g \in \mathsf{D}(\mathcal{G}) : \textit{there exists a neighbourhood } Y \textit{ of } x_0 \textit{ such that } g|_Y = 1\}$$

acts compressibly on $\mathcal{G}^{(0)} \setminus \{x_0\}$ in the sense of Dudko-Medynets [45].

Moreover, if any of the above holds, then for every nontrivial compact open subset $X \subset \mathcal{G}^{(0)}$, there exists an embedding $\phi_X : V \hookrightarrow \mathsf{D}(\mathcal{G})$ such that $X \subset supp\phi_X(V)$.

Theorem 5.1.21 completely answers Question 6.1 of [22], which asks to determine when a topological full group is vigorous.

Already, the fact that V embeds into every such topological full group is entirely new and in-and-of itself useful for understanding key properties of $\mathsf{F}(\mathcal{G})$. As a consequence, many free products are in $\mathsf{F}(\mathcal{G})$, including the product $\mathbb{F}_2 \times \mathbb{F}_2$ of the free group on two generators with itself. Also, it allows us to conclude that the generalised word problem is not solvable for $\mathsf{F}(\mathcal{G})$. Crucially, this means that many of these topological full groups, for example many groupoids in the Garside category framework introduced by Li in [78] (a broad class whose topological full groups include all Brin-Higman-Thompson groups), have solvable word problem but not solvable generalised word problem. (For the precise definition of the (generalised) word problem, see Definition 3.7.8.)

The fact that conditions (2) and (3) characterise minimality and pure infiniteness of $\mathcal{G}$ is surprising since vigor and compressibility were introduced in a completely different context without having groupoids in mind, and to further develop our understanding of Thompson's group V and its generalisations. We discuss both vigor and compressibility separately, since our results have interesting applications in both settings.

Vigor was introduced in [11] in order to determine when a simple, finitely generated group is two-generated. More specifically, said condition was introduced in order to study the following open question:

Question 1.1.2. *Is every simple, finitely presented group is two-generated?*

Our first corollary answers this for a broad class of derived subgroups in topological full groups; see Theorem 5.1.12.

Corollary 1.1.3. *Let $\mathcal{G}$ be a minimal, expansive, purely infinite, essentially principal Cantor groupoid. Then $\mathsf{D}(\mathcal{G})$ is two-generated.*

On the other hand, compressibility was introduced in [45] in order to understand the representation theory and dynamical properties of the Higman-Thompson groups $V_{k,r}$. This definition is an analogue of compressibility in the realm of measurable dynamics, and the introduction of this property follows the recent trend of ideas in measurable dynamics being imported directly to topological dynamics. This gives us two main facts about $\mathsf{F}(\mathcal{G})$; see Theorem 5.1.18 and Theorem 5.1.20:

Corollary 1.1.4. *Let $\mathcal{G}$ be a minimal, purely infinite, essentially principal Cantor groupoid. Then $\mathsf{D}(\mathcal{G})$ has no proper characters, and there are no nontrivial finite factor representations of $\mathsf{D}(\mathcal{G})$.*

One way to interpret this is that $\mathsf{D}(\mathcal{G})$ is highly nonlinear. This allows us to reduce the question of understanding the representations of $\mathsf{F}(\mathcal{G})$ to understanding the abelianisation $\mathsf{F}(\mathcal{G})_{\mathrm{ab}} = \mathsf{F}(\mathcal{G})/\mathsf{D}(\mathcal{G})$. This abelianisation can often be described via Matui's AH conjecture, which has recently been verified by Xin Li [79] for a general class of ample groupoids. We give a wealth of concrete computations; see for example Theorem 4.4.6. The property of compressibility also gives us dynamical information for actions of Thompson-like groups; see Theorem 5.2.1:

Corollary 1.1.5. *Let $\mathcal{G}$ be an essentially principal, purely infinite, minimal, Cantor groupoid such that $H_1(\mathsf{F}(\mathcal{G}))$ is finite (this is for example the case if $H_0(\mathcal{G})$ and $H_1(\mathcal{G})$ are finite). Then every faithful ergodic measure-preserving action of $\mathsf{F}(\mathcal{G})$ is essentially free.*

The methods we develop in this work allow us to give a complete abstract characterisation of the full and derived subgroups of minimal purely infinite groupoids:

Theorem 1.1.6 (Theorem 5.1.27)**.** *Let $\mathcal{C}$ denote the Cantor space.*

1. *For a subgroup $F \leq \mathrm{Homeo}(\mathcal{C})$, the following are equivalent:*

   (F.1) There exists a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_F$ such that $\mathsf{F}(\mathcal{G}_F)$ realises F as a subgroup of $\mathrm{Homeo}(\mathcal{C})$.

   (F.2) F is vigorous and locally closed.

2. *For a subgroup $D \leq \mathrm{Homeo}(\mathcal{C})$, the following are equivalent:*

   (D.1) There exists a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_D$ such that $\mathsf{D}(\mathcal{G}_D)$ realises D as a subgroup of $\mathrm{Homeo}(\mathcal{C})$.

   (D.2) D is vigorous and simple.

Moreover, the groupoids $\mathcal{G}_F$ and $\mathcal{G}_D$ as in (F.1) and (D.1) above are unique up to groupoid isomorphism.

As an application, we show in Theorem 5.1.29 that if $D \leq F \leq \mathrm{Homeo}(\mathcal{C})$ are nested groups such that D is vigorous and simple, and F is locally closed, then any intermediate group $D \leq H \leq F$ is C*-simple. In particular, any vigorous, simple subgroup of $\mathrm{Homeo}(\mathcal{C})$ is C*-simple, and the same applies to any vigorous subgroup which is locally closed. For Thompson's group V, this had been shown by Le Boudec-Matte Bon in [74].

The second example class we study in great detail is a class of topological full groups which have attracted much attention recently: groups of interval exchanges [87] [36] [60] [16].

Definition 1.1.7 ($IE(\Gamma)$)**.** *Let Γ be a countable dense additive subgroup of $\mathbb{R}$, containing* 1. *Then, let $IE(\Gamma)$ denote the group of right continuous piecewise linear bijections f of $(0,1]$ with finitely many angles, all in Γ. That is, the right continuous piecewise linear bijections such that $\{ft - t \, : t \in (0,1]\} \subset_{fin} \Gamma$.*

One reason for the interest in these groups is the connection to classical dynamics, where dynamical systems coming from interval exchanges have been popular to study for some time [63]. For further information about the dynamical perspective on interval exchanges, we recommend the survey [125] and book [62].

Also, these groups have been studied from the perspective of geometric group theory. The reason for this is that due to results by Juschenko-Monod [61] and Matui [87], whenever $\Gamma = \mathbb{Z} \oplus \lambda\mathbb{Z}$ (for some irrational λ) the derived subgroup of $IE(\Gamma)$ is a rare example of a simple, finitely generated amenable group. The existence of such groups was first shown in [61].
In Chapter 6, we take a different perspective from the above papers. Instead of taking an action of $\Gamma/\mathbb{Z}$ on the Cantor space, we define a partial transformation groupoid $\alpha : \Gamma \curvearrowright X$ based on groupoids considered in [76] which realise $IE(\Gamma)$ as a topological full group. Let $\mathcal{G}_\Gamma$ be the associated partial transformation groupoids. We give a systematic study of the family of groupoids $\mathcal{G}_\Gamma$, their reduced C*-algebras $C_r^*(\mathcal{G}_\Gamma)$, the groupoid homology $H_*(\mathcal{G}_\Gamma)$, and their topological full groups $\mathsf{F}(\mathcal{G}_\Gamma) = IE(\Gamma)$.

Theorem 1.1.8 (Lemma 6.2.11)**.** $C_r^*(\mathcal{G}_\Gamma)$ *is classifiable in the sense of the Elliott classification program.*

This answers a question posed in [76], Section 6, where Li asked if $C_r^*(\mathcal{G}_\Gamma)$ was $\mathcal{Z}$-stable. The Elliott invariant is computed (see Lemma 6.2.12). Through Elliott classification, this identifies $C_r^*(\mathcal{G}_\Gamma)$ with concrete C*-algebras in certain cases (see Corollaries 6.2.15 and 6.2.16). The Elliott invariant recovers Γ as a subset of $\mathbb{R}$, so we obtain the following classification result for the groups $IE(\Gamma)$:

Theorem 1.1.9 (Classification of $IE(\Gamma)$)**.** *(Theorem 6.2.13) Let* Γ, Γ' *be dense additive subgroups of* $\mathbb{R}$*. Then, the following are equivalent:*

- $IE(\Gamma) \cong IE(\Gamma')$ *as abstract groups*
- $\Gamma = \Gamma'$ *as subsets of* $\mathbb{R}$

We remark that this is much stronger than saying $\Gamma \cong \Gamma'$ as abstract groups, for example, we see that $IE(2\pi\mathbb{Z} \oplus \mathbb{Z}) \ncong IE(\pi\mathbb{Z} \oplus \mathbb{Z})$. Note that this classification in the case when Γ is finitely generated can also be recovered as a Corollary of [16, Theorem 10.3].

We study the homology of $\mathcal{G}_\Gamma$. The groupoid homology of $\mathcal{G}_\Gamma$ is a shifted version of the group homology of Γ.

Theorem 1.1.10. *(Lemma 6.4.1) Let* Γ *be a dense countable subgroup of* $\mathbb{R}$*, containing* 1*. Then,* $H_*(\mathcal{G}_\Gamma) = H_{*+1}(\Gamma)$

The key point of inspiration here is the computation of homology for groupoids in work by Li [76]. In Theorem 6.4.3, Matuis HK Conjecture is verified directly for $\mathcal{G}_\Gamma$ i.e. it is shown there are isomorphisms:

$$K_0(C_r^*(\mathcal{G}_\Gamma)) \cong \bigoplus_{i=1}^{\infty} H_{2i-1}(\Gamma) \quad K_1(C_r^*(\mathcal{G}_\Gamma)) \cong \bigoplus_{i=1}^{\infty} H_{2i}(\Gamma)$$

We also use the framework of topological full groups to obtain homological information about $IE(\Gamma)$ in terms of the groupoid homology of $\mathcal{G}_\Gamma$. For example, Matuis AH conjecture was recently confirmed for a broad class of groupoids containing all of the groupoids $\mathcal{G}_\Gamma$ by Li in [[79], Corollary E]. This gives us the first homology group of $IE(\Gamma)$ in terms of the homology of Γ. From this concrete picture of the abelianisation we can say when $IE(\Gamma)$ is finitely generated.

Theorem 1.1.11. *(Theorem 6.4.5) Let Γ be a dense countable subgroup of $\mathbb{R}$ containing 1. The following are equivalent:*

1. *Γ is finitely generated.*
2. *$D(IE(\Gamma))$ is finitely generated.*
3. *$IE(\Gamma)$ is finitely generated.*

1. $\implies$ 2. follows by results of Matui and Nekrashevych, and was observed historically [60] [16]. 2. $\implies$ 1. is shown to be general behavior (Corollary 3.5.6 formalises this). 2. $\implies$ 3. is a consequence of our results in homology, in particular, the long exact sequence seen in Lemma 6.4.4. 3. $\implies$ 1. is an elementary observation presumably known to experts.

We also apply [[79], Corollary C] to describe the rational homology of $IE(\Gamma)$ and $D(IE(\Gamma))$ in terms of the rational homology of Γ (See Lemma 6.4.6). This summarises our general results, but we can be more precise for restricted example classes.

We study the case when $\Gamma \subset \mathbb{Q}$ in Subsection 6.5.1. The corresponding class of groupoids $\mathcal{G}_\Gamma$ are conjugate to the canonical AF groupoid models of UHF algebras in this case (Corollary 6.2.15). From this, we find an explicit infinite presentation of $IE(\Gamma)$ as the inductive limit of finite symmetric groups (Lemma 6.5.1). In this case, we show $IE(\Gamma)$ and $D(IE(\Gamma))$ are rationally acyclic. We also compute the abelianisation to be $\mathbb{Z}_2 \otimes \Gamma$.

In Subsection 6.5.3, we study the case when $\Gamma = \mathbb{Z}[\lambda, \lambda^{-1}]$, taking the viewpoint that $IE(\Gamma)$ is the Lebesgue-measure-preserving subgroup of the Stein-Thompson groups with slopes all powers of λ V_λ on the interval, as studied in [116] [31] [38]. In this case, we obtain a concrete generating set for $IE(\Gamma)$ (Lemma 6.5.4) and study the homology of $IE(\Gamma)$.

The main result of Section 6.3 is to find an explicit finite generating set of such $D(IE(\Gamma))$.

Theorem 1.1.12. *(Theorem 6.3.11 (See [36], Proposition 8)) Let $\Gamma \cong \mathbb{Z}^{d+1}$ be a dense additive subgroup of $\mathbb{R}$ such that $1 \in \Gamma$. Then we have that $\Gamma/\mathbb{Z} \cong \mathbb{Z}^d \oplus \mathbb{Z}_k$. Let $k > 9$ and $d > 1$. Then we describe a concrete generating set S of $D(IE(\Gamma))$ such that $|S| = 2d+4$.*

The proof of this Theorem is to describe an explicit subshift of $\{0,1\}^{\Gamma/\mathbb{Z}}$ realising $IE(\Gamma)$ as a topological full group and then apply the main result of [36]. The specific description of generators can be found in Theorem 6.3.11 and Theorem 6.3.9. In particular, we have a concrete generating set S with four elements for the simple group $D(IE(\mathbb{Z}\oplus\lambda_1\mathbb{Z}\oplus\lambda_2\mathbb{Z}))$ where λ_1,λ_2 are rationally independent (see Example 6.3.10). This group is also known to be amenable by the main result of [60]. We believe this is the first concrete finite generating set of an infinite amenable simple group.

In Subsection 6.5.2, we obtain homological information about $IE(\Gamma)$ for the case when $\Gamma\cong\mathbb{Z}^d$. We show $D(IE(\Gamma))$ is rationally acyclic iff $d=2$, and compute the abelianisation explicitly for the cases $d=2,3$:

$$\Gamma\cong\mathbb{Z}^d\Rightarrow IE(\Gamma)_{ab}=\begin{cases}\mathbb{Z}\oplus\mathbb{Z}_2^2 & d=2\\ \mathbb{Z}\oplus\mathbb{Z}_2^3 & d=3\end{cases}$$

The final example class we study in great detail in Chapter 7 is a family of Thompson-like groups that were first exhibited in a paper by Melanie Stein in 1992 [116], but whose study originated in work by Bieri-Strebel [8]. This definition is sometimes also attributed to Brown, as in [116] Stein mentions that Brown has unpublished work on these groups. However, we choose to call these groups Stein's groups since she was the first mathematician to have a paper published about this class of groups. This class of groups vastly generalises Thompson's group V but still satisfies many of the properties that V satisfies. For example, they provide a source of simple derived subgroups which often are finitely generated. Let us briefly recall her construction.

Definition 1.1.13 (Stein's groups)**.** *Let Λ be a subgroup of $(\mathbb{R}_+,\cdot)$. Let Γ be a subgroup of the group ring $(\mathbb{Z}\cdot\Lambda,+)$. Let $\ell\in\Gamma$. Then, Stein's group $V(\Gamma,\Lambda,\ell)$ associated to the triple (Γ,Λ,ℓ), is the group of right continuous piecewise linear bijections of $(0,\ell]$, with finitely many slopes, all in Λ and finitely many nondifferentiable points, all in Γ.*

This family also encompasses many interesting examples that have been studied in great detail such as:

- Thompson's group V, which corresponds to choosing $\Lambda=\langle 2\rangle,\Gamma=\mathbb{Z}[1/2]$ and $\ell=1$.
- The Higman-Thompson groups $V_{k,r}$ [59] where $k,r\in\mathbb{N}$, which correspond to choosing $\Lambda=\langle k\rangle$, $\Gamma=\mathbb{Z}[1/k]$, and $\ell=r$.
- Stein's integral groups, which corresponds to choosing a finite collection of integers $\{n_1,\ldots,n_k\}$ and a length $r\in\mathbb{N}$, and taking $\Lambda=\langle n_1,\ldots,n_k\rangle$, $\Gamma=\mathbb{Z}\cdot\Lambda$, and $\ell=r$.
- V_τ, Cleary's group [38], [37], [30], also known as the irrational slope Thompson group [31], which corresponds to choosing $\Lambda=\langle\tau\rangle,\Gamma=\mathbb{Z}\cdot\Lambda$, and $\ell=1$ for $\tau=\frac{\sqrt{5}-1}{2}$, and other related Stein-Thompson groups with cyclic slopes (corresponding to different choices of τ).

Stein's groups are the primary object of study in this chapter but we take a different perspective from other authors who have studied Stein's groups previously [116], [31]. We study Stein's groups as topological full groups. This example class is not an exception to the philosophy that ample groupoids are often more accessible to study. These ample groupoids can be described as full corners in the universal groupoid of the inverse hull of $\Gamma_+ \ltimes \Lambda$, which, as shown in work of Li [75], we could equally describe as the partial transformation groupoid of $\Lambda \ltimes \Gamma$ on the Cantor space X, which is the description which we give throughout Section 4. See Lemma 7.2.2 for the precise construction. This perspective is convenient because the partial action $\beta : \Lambda \ltimes \Gamma \curvearrowright X$ is minimal and topologically free. This gives us the simplicity of the derived subgroups of these groups by using standard results in the literature of topological full groups [89].

The partial action β is also expansive, in the sense of classical dynamics. We show that this notion of expansivity for partial actions is closely related to Nekrashevych's notion of expansivity for groupoids [93], even equivalent in the case of compactly generated partial transformation groupoids. We also study the property of compact generation for groupoids, showing that it is preserved under taking full compact open corners. This line of inquiry allows us to show that the derived subgroups are finitely generated under very general circumstances, which is the main result of this Chapter:

Theorem 1.1.14 (Theorem 7.3.12)**.** *Let Λ be a subgroup of $(\mathbb{R}_+, \cdot)$ and Γ be a $\mathbb{Z}\cdot\Lambda$-submodule and $\ell \in \Gamma$. Then, $D(V(\Gamma,\Lambda,\ell))$ is simple. Moreover, the following are equivalent:*

1. $\Lambda \ltimes \Gamma$ *is finitely generated.*
2. $(\Lambda \ltimes \Gamma) \ltimes (0_+, \ell_-]$ *is compactly generated for all* $\ell \in \Gamma$.
3. $D(V(\Gamma,\Lambda,\ell))$ *is finitely generated for all* ℓ.
4. $D(V(\Gamma,\Lambda,\ell))$ *is 2-generated for all* ℓ.

This generalises the known case due to Stein, who showed that $D(V(\Gamma,\Lambda,\ell))$ is simple and finitely generated in the case when Λ is generated by finitely many integers, $\Gamma = \mathbb{Z}\cdot\Lambda$, and $\ell \in \mathbb{N}$ [116] who herself generalised work of Higman [59]. It is important to note here that what Stein shows is a stronger statement, in fact that this particular subclass of groups are of type F_∞. The above result also generalises the result of Burillo-Nucinkis-Reeves who showed $D(V(\Gamma,\Lambda,\ell))$ is simple and finitely generated in the case of Cleary's group, i.e. when $\Lambda = \langle \frac{1+\sqrt{5}}{2} \rangle$, $\Gamma = \mathbb{Z}\cdot\Lambda$ and $\ell = 1$ [31]. In particular, a consequence is that for all choices of irrational number λ, the Stein-Thompson groups with cyclic slopes $V(\mathbb{Z}[\lambda,\lambda^{-1}],\langle\lambda\rangle,1)$ has a simple and finitely generated derived subgroup.

In Section 6. we study the homology of Stein's groups. Since the conception of topological full groups, there has been a deep connection to groupoid homology, most notably in Matui's AH conjecture [88], [89]. This conjecture was recently confirmed by Li [79, Corollary E] under very minor regularity conditions which are satisfied by our groupoid model for Stein's groups. Inspired by Szymik-Wahl's work on the homology of Higman-Thompson groups [118], the framework of Li [79] found other connections between the group homology of a topological full group to the underlying groupoid's homology. In the case of Stein's groups, the groupoid homology is comparatively computable, making the homology of Stein's groups accessible for the first time.

Using this homology computation in combination with [79, Corollary E], we show that the abelianisation of $V(\Gamma,\Lambda,\ell)$ is finite rank for the case of Γ,Λ generated by finitely many algebraic numbers (Lemma 7.4.4). Combining this with Theorem 7.3.12 we prove a finite generation theorem for the V-type groups:

Theorem 1.1.15. *Let Λ be a subgroup of $(\mathbb{R}_+,\cdot)$ generated by finitely many algebraic numbers. Let Γ be a submodule of the group ring $(\mathbb{Z}\cdot\Lambda,+)$. Let $\ell\in\Gamma$. Then, $V(\Gamma,\Lambda,\ell)$ is finitely generated.*

These homology computations are especially interesting from the perspective of homological stability since we show that the natural inclusion of Stein's groups acting upon compact intervals into the noncompact Stein groups (say coming from the inclusion of $(0,\ell]$ in $\mathbb{R}$) induces an isomorphism on the level of homology. We consider this to be a generalisation of [118, Theorem 3.6].

Corollary 1.1.16. *(Corollary 7.4.5) Let Λ be a subgroup of $(\mathbb{R},\cdot)$, Γ be a submodule of $(\mathbb{Z}\cdot\Lambda,+)$. Let U be any closed subset of $\mathbb{R}$ with nonempty interior. Let $V(\Gamma,\Lambda,U)$ be the group of right continuous piecewise linear bijections of $\mathbb{R}$ with finitely many slopes (all in Λ) and finitely many nondifferentiable points (all in Γ) with finite support and that are the identity on U^c. Then for all $*\in\mathbb{N}$:*

$$H_*(V(\Gamma,\Lambda,U))\cong H_*(V(\Gamma,\Lambda,\mathbb{R}))$$

For more specialised homology results, we turn our focus onto two subclasses of Stein's groups. The first subclass is the case where the group of slopes is cyclic, we refer to these groups as Stein-Thompson groups with cyclic slopes.

The homology of Stein-Thompson groups with cyclic slopes is related to the homology of certain ample groupoids which was computed by Li in [76]. This groupoid homology is highly sensitive to the minimal polynomial of the underlying generator λ, which leads the Stein-Thompson groups with cyclic slopes to exhibit diverse homological behaviour in the algebraic case. In Corollary 7.4.8, we compute the abelianisation $V(\mathbb{Z}[\lambda,\lambda^{-1}],\langle\lambda\rangle,\ell))_{ab}$ for low degree algebraic numbers λ explicitly, generalising and unifying the known results due to Higman [59] (who computed the abelianisation for $\lambda\in\mathbb{N}$) and Burillo-Nucinkis-Reeves [31] (who computed the abelianisation to be $\mathbb{Z}_2$ for $\lambda=\frac{\sqrt{5}-1}{2}$).

We also compute the rational homology explicitly for certain Stein-Thompson groups with cyclic slopes in Theorem 7.4.6. Notably, the Higman-Thompson groups were shown to all be rationally acyclic [118, Corollary C] by Szymik-Wahl. The Stein-Thompson groups with cyclic slopes vary dramatically:

- Cleary's group V_τ is rationally acyclic.
- There are Stein-Thompson groups with cyclic slopes which are not rationally acyclic, or even virtually simple. For example, if we take $\lambda = \frac{3+\sqrt{5}}{2}$, then $V(\mathbb{Z}[\lambda,\lambda^{-1}],\langle\lambda\rangle,1)_{ab} = \mathbb{Z}$ by Corollary 7.4.8.

Groupoid homology itself is an invariant for topological full groups via Matui's isomorphism theorem. This allows us to distinguish different examples of Stein's groups. For example, see Corollary 7.4.2, which gives an invariant for Stein's groups in terms of Γ,Λ and ℓ. This invariant is fine enough to give one direction in the classification of Higman-Thompson groups [98] and in combination with our explicit homology computations for Stein-Thompson groups with cyclic slopes enable us to prove an analogue to Higman's result [59] giving one direction on the classification of Higman-Thompson groups [98], but generalised to those whose group of slopes are $\langle\lambda\rangle$, where λ is an algebraic number of degree less than or equal to 2, and more arbitrary lengths ℓ of the compact intervals:

Corollary 1.1.17 (Corollary 7.4.18)**.** *Let $\lambda,\mu < 1$ be algebraic numbers with degree ≤ 2 and let $\ell_1 \in \mathbb{Z}[\lambda,\lambda^{-1}], \ell_2 \in \mathbb{Z}[\mu,\mu^{-1}]$. Suppose that $V(\mathbb{Z}[\lambda,\lambda^{-1}],\langle\lambda\rangle,\ell_1) \cong V(\mathbb{Z}[\mu,\mu^{-1}],\langle\mu\rangle,\ell_2)$.*
Then, $\lambda = \mu$, and $\ell_1 - \ell_2 \in (1-\lambda)\mathbb{Z}[\lambda,\lambda^{-1}]$.

The second subclass we study in detail is the class of examples for which Λ is generated by several integers, which we call Stein's integral groups. To compute the groupoid homology for this case, we rely on the framework of k-graphs, identifying our groupoid with the groupoid of a single vertex k-graph. The observation that we can rephrase the groupoid model of these particular groups in the language of k-graphs is also noted in upcoming work by Conchita Martınez-Pérez, Brita Nucinkis and Alina Vdovina, we include it here for completeness in the literature and for our homology computations. Most notably, this allows us to use the groupoid homology computations of Farsi-Kumjian-Pask-Sims [49] to obtain acyclicity results on the level of Stein's integral groups.

Corollary 1.1.18 (Corollary 7.4.22)**.** *Let $k > 1$ and $n_1,..,n_k$ be a finite collection of natural numbers. Let $\ell \in \mathbb{Z}[\frac{1}{n_1 n_2 \ldots n_k}]$. Then $V(\mathbb{Z}[\frac{1}{n_1 n_2 \ldots n_k}],\langle n_1,n_2,\ldots,n_k\rangle,\ell)$ is rationally acyclic.*
$V(\mathbb{Z}[\frac{1}{n_1 n_2 \ldots n_k}],\langle n_1,n_2,\ldots,n_k\rangle,\ell)$ is integrally acyclic if and only if $gcd(n_1-1,\ldots,n_k-1) = 1$.

This result falls under the context of the result of Szymik-Wahl [118, Corollary C] that the Higman-Thompson groups are rationally acyclic and that Thompson's group V is acyclic. The key tool we use is the two acyclicity results in the paper of Li [79, Corollary C, Corollary D]

1.2 Structure

In Chapter 2. "Ample groupoids and their C*-algebras", we discuss the key preliminaries about ample groupoids needed to understand topological full groups. We comment on the relationship between ample groupoids and groupoid C*-algebras, which is relevant to the research conducted in Chapter 6. This chapter is not based on original research, and has much overlap with classical sources on groupoid C*-algebras such as [114].

In Chapter 3. "Topological full groups of ample groupoids" we discuss the definition of a topological full group and the key results needed. In Section 3.1 we give the definition of a topological full group. In Section 3.2 we discuss the subgroup structure of topological full groups, in particular understanding the alternating and symmetric subgroups, and their relationship with the derived subgroup of the topological full group. In Section 3.2 we discuss the simplicity of the derived subgroup of a topological full group, and prove it is simple in the effective, minimal purely infinite case without invoking the additional assumption of Hausdorffness. In Section 3.4. we discuss Matui's reconstruction theorem, which proves that one may reconstruct an ample groupoid (up to isomorphism) from its topological full group, under the conditions of minimality and effectiveness. In Section 3.5. we discuss the finite generation of the derived subgroup, in particular the relationship to groupoid expansivity. We prove slightly more in the case of expansive actions of finitely generated abelian groups, namely we describe the generators. In Section 3.6. we describe the relationship between the homology of topological full groups and the homology of ample groupoids. The purpose of Section 3.7. is to survey the significant examples of topological full groups which have closed major existence problems in group theory. This section again is not based on original research, but to give an overview of the key results in the literature of topological full groups and generalise/modernise them slightly where needed.

The purpose of Chapter 4. is to survey the research and history of the study of Thompson's group V and many other Thompson-like groups; Stein's groups, the Brin-Higman-Thompson groups. We then connect this research to topological full groups in Section 4.4., describing both V and generalisations as topological full groups. This chapter is not based on original research.

Chapter 5. is the first chapter which is based on original research. This chapter is based on the preprint written by the author and Professor Eusebio Gardella [51] and so the original research is joint work. This chapter is aimed at understanding the topological full groups of purely infinite minimal groupoids, which we consider as generalisations of Thompson's group V. In this paper, we prove Theorem 1.1.1, Theorem 1.1.6 and explore the consequences of these results.

Chapter 6. is the second chapter based on original research. This chapter is heavily based on the solo author preprint [119] which is currently available as a preprint on the ArXiv. In this chapter we study interval exchange groups, and it is in this chapter we prove Theorem 1.1.9, Theorem 1.1.9, Theorem 1.1.11 and Theorem 1.1.12.

Chapter 7. is the third and final chapter based on original research. This chapter is heavily based on the solo author preprint [120], currently available as a preprint on the ArXiv. In this chapter we prove Theorem 1.1.14, Theorem 1.1.15 and explore the consequences of these results in great detail.

Chapter 2

Ample groupoids and their C*-algebras

Let us introduce some notation:

- $\mathcal{C}$ is reserved for the Cantor set.
- Throughout, $\mathbb{N}$ refers to the natural numbers and we take the convention that $0 \notin \mathbb{N}$.
- When coefficients are omitted in homology, this means the underlying module A is taken to be $\mathbb{Z}$.

We begin by introducing groupoids, a relaxed notion of a group that allows for many units. Groupoids are the central models for all the objects we study throughout this text. None of the results in Chapter 2 are original work, we are simply giving a quick introduction to the key preliminaries needed for understanding groupoids. For this reason, one might equally consult several other sources for a deeper understanding of the fundamentals:

- The excellent notes of Aidan Sims [113].
- Books of Paterson [99] or Renault [103].

2.1 Algebraic groupoids

This subsection follows similar content to [113], Section 2.1.

Definition 2.1.1 (Groupoid- Category Theoretic)**.** *A groupoid $\mathcal{G}$ is a small category where every morphism is an isomorphism.*

This generalizes the notion of a group, which is a small category with one object where every morphism is an isomorphism. It is well known that there is an equivalent algebraic definition.

Definition 2.1.2 (Groupoid - Algebraic)**.** *A groupoid is a set $\mathcal{G}$ together with a distinguished subset $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$ called the composable pairs, a composition map from*

$$\mathcal{G}^{(2)} \to \mathcal{G} \quad (g,h) \mapsto gh,$$

and an inverse map

$$\mathcal{G} \to \mathcal{G} \quad g \mapsto g^{-1},$$

such that the following hold:

- *The product is associative. If $(g,f) \in \mathcal{G}^{(2)}$, $((gf),h) \in \mathcal{G}^{(2)}$ then $(f,h) \in \mathcal{G}^{(2)}$ and $(g,fh) \in \mathcal{G}^{(2)}$ such that $(gf)h = g(fh)$.*
- *The inverse map is involutive.*
- *For all $g \in \mathcal{G}, (g,g^{-1}) \in \mathcal{G}^{(2)}$ and moreover for all $(g,h) \in \mathcal{G}^{(2)}$*

$$h = g^{-1}gh, \text{ and } g = ghh^{-1}.$$

We call the set $\mathcal{G}^{(0)} = \{gg^{-1} : g \in \mathcal{G}\}$ the unit space of our groupoid. This terminology relates to the third bullet point– that these elements act as (partial) identities algebraically. It follows from the above axioms that we have two surjective maps, the range map

$$r : \mathcal{G} \to \mathcal{G}^{(0)} \quad g \mapsto g^{-1}g$$

and the source map

$$s : \mathcal{G} \to \mathcal{G}^{(0)} \quad g \mapsto gg^{-1}.$$

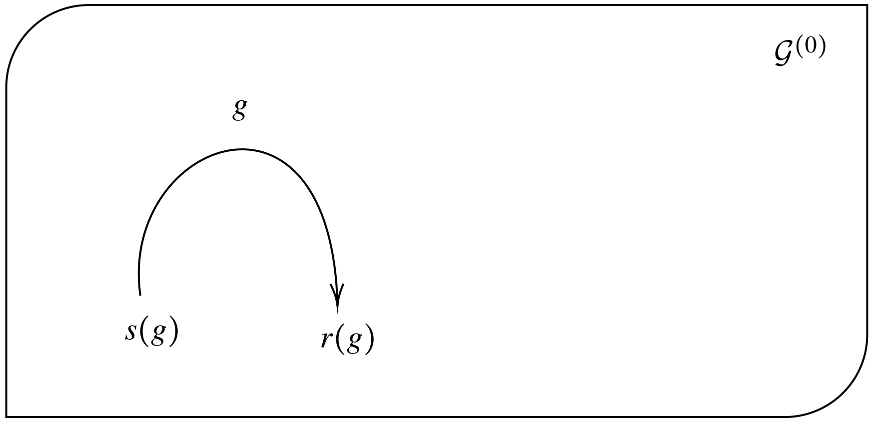
Using these, the following algebraic properties may be deduced:

Lemma 2.1.3. *(Basic Algebraic Properties of Groupoids) Let $\mathcal{G}$ be a groupoid. Then:*

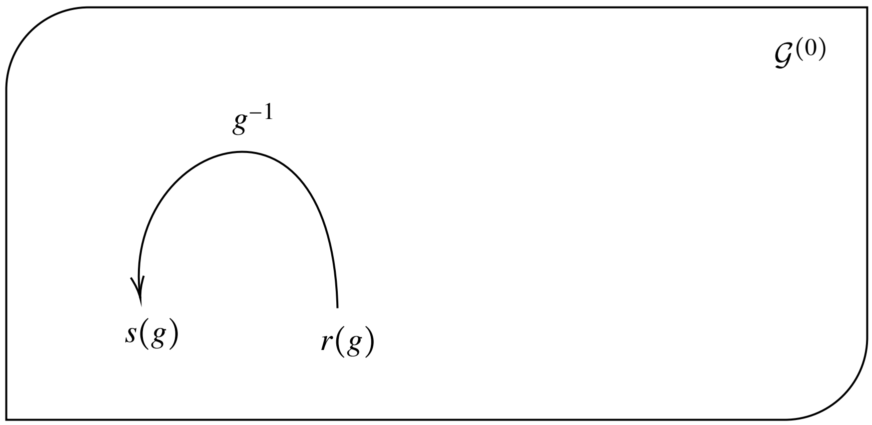
1. *$\mathcal{G}$ is cancellative. If $(g,h),(g,f) \in \mathcal{G}^{(2)}$ and $gh = gf$ then $f = h$.*
2. *If $(g,h) \in \mathcal{G}^{(2)}$ then $(h^{-1},g^{-1}) \in \mathcal{G}^{(2)}$ and $(gh)^{-1} = h^{-1}g^{-1}$.*
3. *For all $g \in \mathcal{G}$, $s(g)g = g = gr(g)$.*
4. *For all $g \in \mathcal{G}$, $s(g^{-1}) = r(g)$ and $r(g^{-1}) = s(g)$.*
5. *For all $u \in \mathcal{G}^{(0)}$, $u = s(u) = r(u) = u^{-1}$.*
6. *For all $(g,h) \in \mathcal{G}^{(2)}$, $s(gh) = s(g)$, and $r(gh) = r(h)$.*
7. *The composable pairs are exactly the pairs of groupoid elements with compatible ranges and sources; $\mathcal{G}^{(2)} = \{(g,h) \in \mathcal{G} : r(g) = s(h)\}$.*

Proof. See [114], Lemma 2.1.2, Lemma 2.1.3, Lemma 2.1.4, and Remark 2.1.5. □

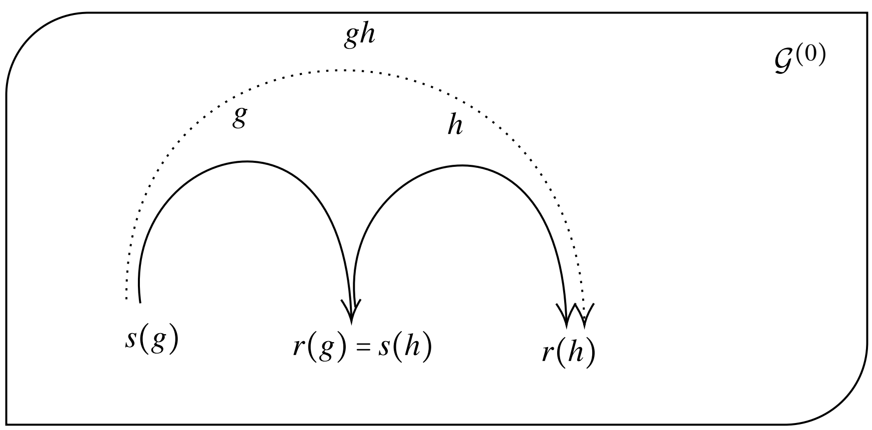
There is an intuitive way to visualise a groupoid. Namely, we think of elements as arrows on our unit space from $r(g)$ to $s(g)$.



In which case, the inverse g^{-1} is an arrow in the opposite direction (4.).



Then, if $s(h) = r(g)$, we are able to compose our arrows g, h. In this case, we get a new arrow gh from $s(g)$ to $r(h)$:



Let us give some examples.

Example 2.1.4 (A set)**.** *We can think of any set X as a groupoid. This is by declaring $X = X^{(0)} = X^{(2)}$. Here then, nothing is composable; we just have a unit space.*

Example 2.1.5 (A group)**.** *Another extreme would be to consider a discrete group G as a groupoid. Here we only have one unit; the unit of the group $\mathcal{G}^{(0)} = \{1\}$. Then, we have ar-*

rows $g : 1 \mapsto 1$ labelled by elements of our group. Therefore, $\mathcal{G}^{(2)} = G \times G$. In fact, the following three properties are equivalent:

1. *$\mathcal{G}$ is a group.*

2. *$\mathcal{G}^{(0)}$ is a singleton.*

3. *$\mathcal{G}^{(2)} = \mathcal{G} \times \mathcal{G}$.*

1. $\implies$ 2. $\iff$ 3. follows directly from Lemma 2.1.3. Suppose we have that $\mathcal{G}^{(2)} = \mathcal{G} \times \mathcal{G}$ and that $\mathcal{G}^{(0)} = \{u\}$ is a singleton. Let us show 1. by verifying the group axioms:

- *The universal multiplication is associative, since groupoid multiplication is associative.*
- *The identity element is $u \in \mathcal{G}$, by 2.1.3, 3. This is since for all $g \in \mathcal{G}$, $s(g) = r(g) = u$.*
- *Given $g \in \mathcal{G}$, the inverse element is the groupoid inverse $g^{-1} \in \mathcal{G}$. This can be verified since for all $g \in \mathcal{G}$, $s(g) = r(g) = u$.*

Therefore, groupoids generalise both sets and groups. We can combine the two previous examples in the following way:

Example 2.1.6 (A group acting on a set)**.** *Suppose we have a group Γ acting on a set X. We can encode this through a groupoid which we call a transformation groupoid and label by $\Gamma \ltimes X$ throughout this text. Our groupoid as a set is the Cartesian product $\Gamma \times X$. For all $x \in X, g \in \Gamma$ we have arrows x to $g(x)$ labelled (g,x). Then, $(g,x)^{-1} = (g^{-1}, g(x))$. Composition is only allowed for pairs $(g,x),(h,y)$ such that $y = g(x)$. In this case, $(g,x)(h,g(x)) = (hg,x)$. This makes the unit space $\mathcal{G}^{(0)} = \{(1,x) : x \in X\}$, which is canonically identifiable with X.*

Another example arises from any equivalence relation on a set.

Example 2.1.7 (Equivalence relation groupoid)**.** *Let X be a set, and $\sim$ be an equivalence relation on X. We can encode this through a groupoid. Let $\mathcal{G}_\sim = \{(x,y) \in X^2 : x \sim y\}$. We say that $(x,y)(z,w)$ is composable iff $y = z$ in which case the product $(x,y)(y,w) = (x,w)$. As before, we can identify $X = \mathcal{G}^{(0)}$ and think of (x,y) as an arrow from x to y.*
A particular example that we shall see later is the groupoid that comes from the full equivalence relation on n points (the equivalence relation on $\{1,\dots,n\}$ that declares each element to be equivalent), we denote this particular groupoid by $\mathcal{R}_n$.

Example 2.1.8 (Groupoids are closed under disjoint unions, Cartesian products, restrictions of unit space)**.** *Because of their relaxed structure, we can do many things to a groupoid to get new groupoids. Say we have two groupoids $\mathcal{G}, \mathcal{H}$.*

- *$\mathcal{G} \sqcup \mathcal{H}$ is a groupoid, taking inverses and multiplication in the constituent groupoids.*

$$g^{-1} = \begin{cases} g^{-1} & g \in \mathcal{G} \\ g^{-1} & g \in \mathcal{H} \end{cases}$$

  The composable pairs are the pairs of the elements which are composable in $\mathcal{G}$ or $\mathcal{H}$, i.e. $(\mathcal{G} \sqcup \mathcal{H})^{(2)} = \mathcal{G}^{(2)} \sqcup \mathcal{H}^{(2)}$. In this case multiplication occurs in $\mathcal{G}$ if $(g,h) \in \mathcal{G}^{(2)}$ and in $\mathcal{H}$ if $(g,h) \in \mathcal{H}^{(2)}$

- *$\mathcal{G} \times \mathcal{H}$ is a groupoid with coordinate wise operations of inversion and composition.*

- *Let $\mathcal{G}$ be a groupoid and $X \subset \mathcal{G}^{(0)}$. Then $\mathcal{G}|_X^X := \{g \in \mathcal{G} : r(g), s(g) \in X\}$ is a groupoid with operations inherited from $\mathcal{G}$.*

It is trivial to show that the above examples are indeed groupoids.

We end with the notion of a groupoid homomorphism.

Definition 2.1.9 (Groupoid Homomorphism)**.** *A map $\Phi : \mathcal{G} \to \mathcal{H}$ between two groupoids is called a homomorphism if for all $(g,h) \in \mathcal{G}^{(2)}, (\Phi(g), \Phi(h)) \in \mathcal{H}^{(2)}$ and $\Phi(gh) = \Phi(g)\Phi(h)$. If Φ is bijective, and its inverse is once again a groupoid homomorphism, it is called a groupoid isomorphism.*

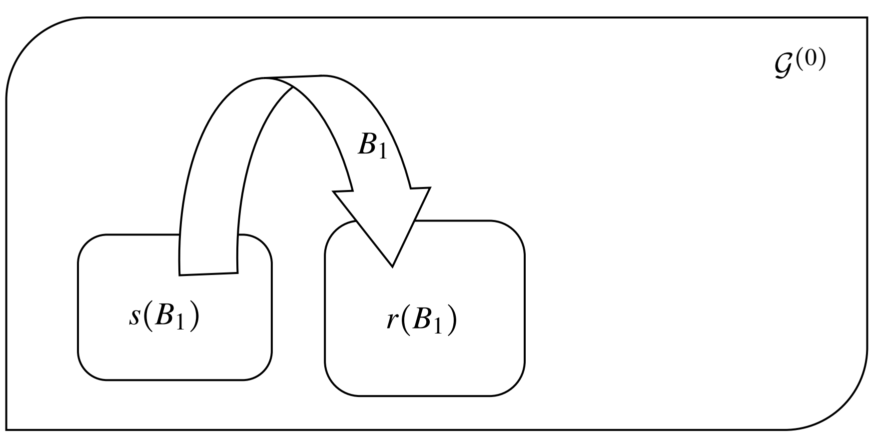
Subsets of a given groupoid can be multiplied together pointwise. Given $B_1, B_2 \subset \mathcal{G}$ we define their product as the pointwise product

$$B_1 B_2 = \{gh \in \mathcal{G} : g \in B_1, h \in B_2 \; s(h) = r(g)\}$$

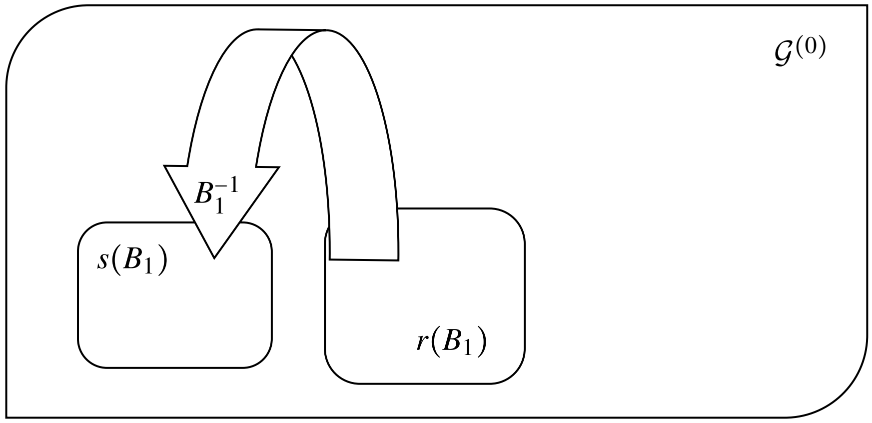
and the inverse of B_1 similarly

$$B_1^{-1} := \{g^{-1} \in \mathcal{G} : g \in B_1\}.$$

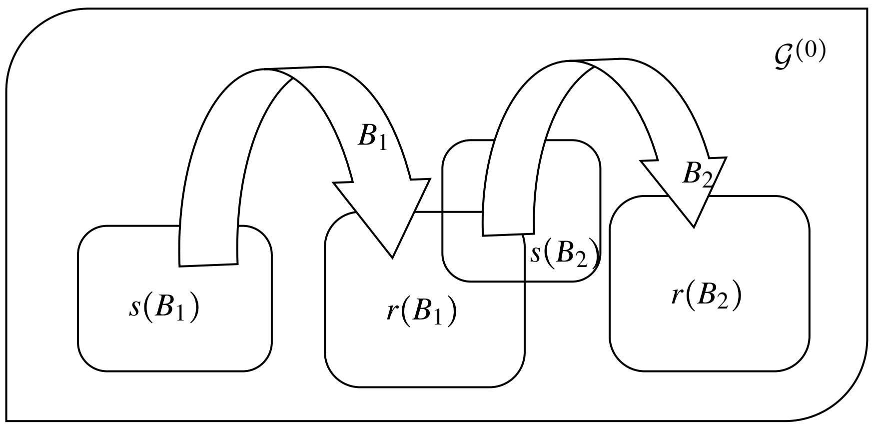
We can visualise the multiplication and inversion of subsets in similar diagrams to groupoid elements. The difference here is that a subset B_1 maps one subset of the unit space to another:



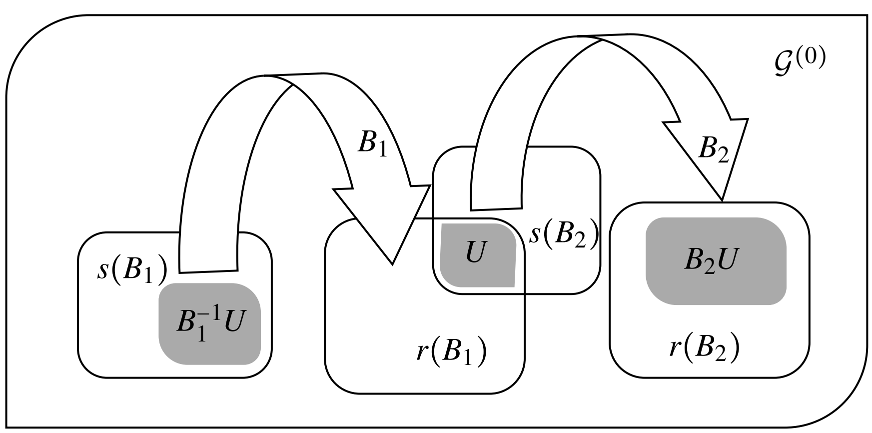
And inversion gives an arrow which takes $r(B_1)$ back to $s(B_1)$. Therefore $B_1 B_1^{-1}, B_1^{-1} B_1$ acts as partial identities on $r(B_1), s(B_1)$ respectively.



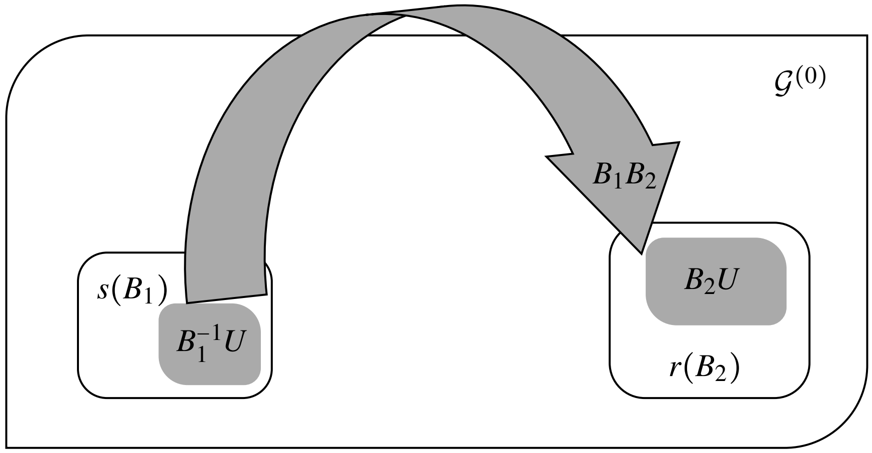
Composition $B_1 B_2$ is given by the composition of collection of arrows gh where $(g,h) \in B_1 \times B_2 \cap \mathcal{G}^{(2)}$.



Let us consider what the composition $B_1 B_2$ may look like for this example. Note that not all of $r(B_1)$is contained in $s(B_2)$. To understand the composition $B_1 B_2$, we are concerned only with the subsets of B_1, B_2 we can compose here; $U = s(B_2) \cap r(B_1)$, it's image under B_2, and preimage under B_1.



Then $B_1 B_2$ is the bisection taking $B_1^{-1} U$ to $B_2 U$.



Note crucially in this example that $r(B_1B_2) \subsetneq r(B_2)$ and $s(B_1B_2) \subsetneq s(B_1)$.

2.2 Étale groupoids

This subsection follows similar content to [113], Section 2.3. and 2.4. In our text, the groupoids we are concerned with have a topology. Just like the notion of a topological group, this topology will need to respect inversion and composition.

Definition 2.2.1. *(Topological Groupoids) A topological groupoid is a groupoid with a topology making the composition and inversion maps continuous.*

Some basic facts follow.

Proposition 2.2.2 (Basic Topological Properties)**.** *Let $\mathcal{G}$ be a topological groupoid. The following hold.*

- *The inversion map is a homeomorphism of $\mathcal{G}$.*
- *The range and source maps are continuous surjections.*

Proof.
- The inversion map is bijective and continuous, and is it's own inverse as a continuous map.
- The range and source maps are each the composition of continuous maps, and therefore continuous. Since each $u \in \mathcal{G}^{(0)}$ satisfies $s(u) = r(u) = u$, it is surjective.

□

Examples in this text tend to have a topology that makes $\mathcal{G}$ Hausdorff. We will signpost if any examples fall outside the scope of this. We also often deal with groupoids which are étale.

Definition 2.2.3 (Étale Groupoids)**.** *A groupoid $\mathcal{G}$ is called étale if the range and source maps are local homeomorphisms. In addition, we have a standing assumption the topology on $\mathcal{G}$ makes our unit space $\mathcal{G}^{(0)}$ second countable and Hausdorff.*

Note that the additional assumption that the unit space is second countable and Hausdorff is not often given within the definition of étaleness, but is often a standing assumption in the literature in general since these restrictions of the unit space give groupoids which can be completed to groupoid C*-algebras.

Étale groupoids are often understood best by studying their open bisections.

Definition 2.2.4 (Open Bisections)**.** *A subset B of a topological groupoid $\mathcal{G}$ is called an open bisection if r, s are injective when restricted to B. Let $\mathcal{B}$ denote the open bisections.*

Since $s|_B$ and $r|_B$ are local homeomorphisms in the étale case, we can think of our bisection B as a local homeomorphism of our unit space taking $s(B)$ to $r(B)$. As mentioned before, étaleness means that open bisections then hold much of the topological information in $\mathcal{G}$. One can make this precise through the following lemma.

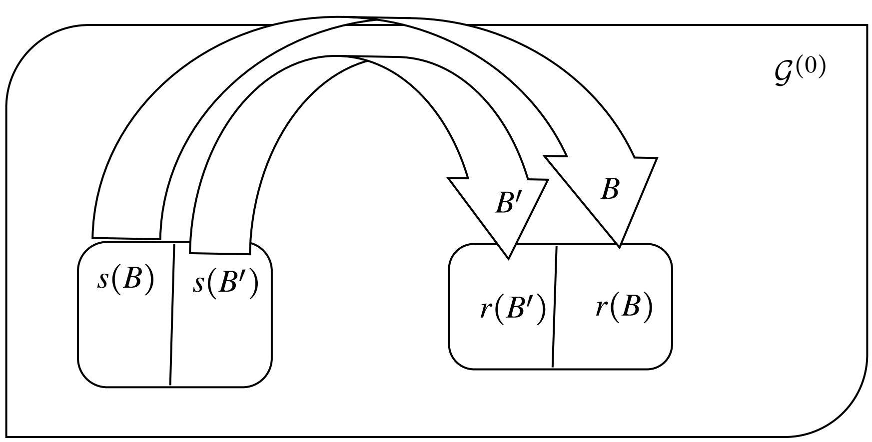
Lemma 2.2.5. *(See [113], Lemma 2.4.9). Let $\mathcal{G}$ be a topological groupoid. The following are equivalent:*

- *$\mathcal{G}$ is étale.*
- *The open bisections generate the topology of $\mathcal{G}$.*

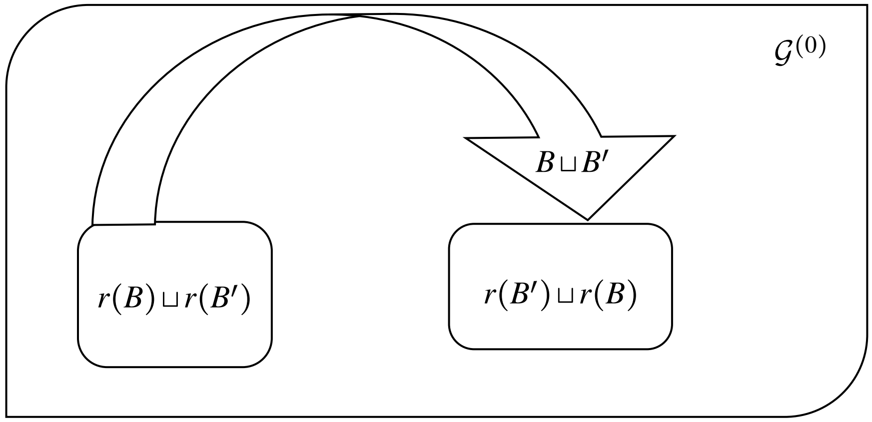
We note that bisections are closed under several operations:

- If B is a bisection, so is B^{-1}.
- If B' is a bisection the compositon BB' is again a bisection (so long as the groupoid is étale).
- If B' is a bisection with $s(B') \cap s(B) = r(B') \cap r(B) = \varnothing$ then $B \sqcup B'$ is a bisection; i.e. we can think of the union of two open bisections.

We will draw a diagram to explain our third bullet point. The hypothesis is that we have two bisections B, B' with disjoint ranges and sources.



This allows us to consider their disjoint union as a single open bisection $B \sqcup B'$:



The compact open bisections play a crucial role in the study of topological full groups.

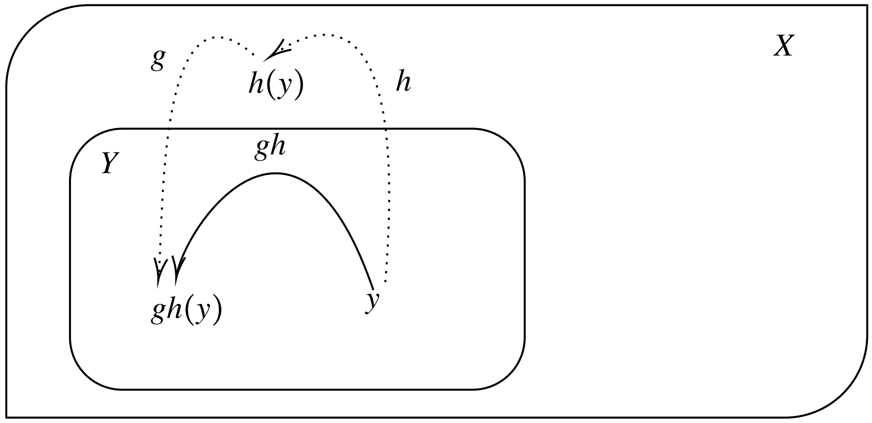
Definition 2.2.6 (Compact Open Bisections)**.** *Let $\mathcal{B}^k$ denote the compact open bisections.*

Let us give some examples of étale groupoids, first by endowing our examples seen previously with a topology.

Example 2.2.7 (Discrete Group acting on a topological space)**.** *Let Γ be a discrete group acting by homeomorphisms on some locally compact Hausdorff space X. Consider the transformation groupoid $\Gamma \ltimes X$, as in example 2.1.6. Then we can topologise by declaring sets of the form (g,U) where U is open in X to be open in our groupoid. This forms the basis of the topology. In turn this makes $\Gamma \ltimes X$ an étale groupoid. . See [113, Example 2.4.5].*

Example 2.2.8 (Partial Transformation Groupoids)**.** *Continuing from the above example, let us perform the third operation from example 2.1.8. Let $Y \subset X$ be an open subset of X. The restriction of $\Gamma \ltimes X$ to $\Gamma \ltimes X|_Y^Y$ is known as a partial transformation groupoid and is written simply as $\Gamma \ltimes Y$. This will again be étale, in fact it is easy to see that étaleness is preserved by each of the operations in example 2.1.8.*

Unlike before, it is not necessarily true that for all $g \in \Gamma, y \in Y$ that $(g,y) \in \Gamma \ltimes Y$ since it might be that $g(y) \notin Y$. In practice this makes working with these groupoids very subtle, since it can be difficult to describe in terms of Γ and Y which elements $(g,y) \in \Gamma \times Y$ are in the partial transformation groupoid. For example, it is a common occurence that $(gh,x) \in \mathcal{G}|_Y^Y$ yet $(h,x),(g,h(x)) \notin \mathcal{G}|_Y^Y$. This fundamental difference can be visualised in the following way:



Lemma 2.2.9. *Let $\mathcal{G}$ be an étale groupoid. Then, the compact open bisections form an inverse semigroup acting upon the unit space by partial homeomorphisms.*

Proof. Subset multiplication is associative, inherited by the associativeness of groupoid notification. The pointwise inverse B^{-1} acts as an inverse in the sense of inverse semigroups, since $BB^{-1}B = r(B)B = B, B^{-1}BB^{-1} = B^{-1}r(B) = B^{-1}s(B^{-1}) = B^{-1}$. As recalled above, open bisections always act by partial homeomorphisms on the unit space for étale groupoids. □

There are many key definitions we use for étale groupoids throughout that are given below..

Definition 2.2.10 (Orbit)**.** *Let $u \in \mathcal{G}^{(0)}$. Then the orbit of u is the subset of $\mathcal{G}^{(0)}$ given by $\mathcal{G}_u := \{r(g) : s(g) = u\}$.*

Definition 2.2.11 (Isotropy Group)**.** *The isotropy group of a unit $u \in \mathcal{G}^{(0)}$ is the group given by $\mathcal{G}_u^u := \{g \in \mathcal{G} : s(g) = r(g) = u\}$.*

Let $\mathcal{G}_{triv}^{(0)} := \{u \in \mathcal{G}^{(0)} : \mathcal{G}_u^u = \{u\}\}$ be the subset of the unit space that has trivial isotropy groups. We let $Iso(\mathcal{G}) = \bigcup_{u \in \mathcal{G}^{(0)}} \mathcal{G}_u^u$, which is referred to as the isotropy groupoid.

Definition 2.2.12 (Minimal)**.** *A topological groupoid is called minimal if for all u, the orbit of u, $\mathcal{G}_u$, is dense in $\mathcal{G}^{(0)}$. This is equivalent to the unit space having no closed invariant subsets.*

Definition 2.2.13 (Effective)**.** *An étale groupoid $\mathcal{G}$ is effective if the interior of the isotropy groupoid, $Iso(\mathcal{G})^\circ$ agrees with the unit space $\mathcal{G}^{(0)}$.*

Definition 2.2.14 (Principal)**.** *An étale groupoid is called principal if $\mathcal{G}_{triv}^{(0)} = \mathcal{G}^{(0)}$.*

Definition 2.2.15 (Topologically Principal)**.** *An étale groupoid is called topologically principal if the points with trivial isotropy is dense in the unit space; $\overline{\mathcal{G}_{triv}^{(0)}} = \mathcal{G}^{(0)}$.*

The relationship between effectiveness and topologically principal is somewhat subtle. However, the following is known:

Remark 2.2.16. *For a Hausdorff second countable groupoid $\mathcal{G}$, topologically principal is equivalent to effective.*

Proof. Note the standing assumption of Hausdorffness in the text of Sims [114]. Therefore, this result is proven in Lemma 4.2.3 of this text (see Remark 4.2.4). □

The following rephrasing of effectiveness for étale groupoids is extremely useful for the study of topological full groups.

Proposition 2.2.17. *Let $\mathcal{G}$ be an étale groupoid. The following are equivalent:*

- *$\mathcal{G}$ is effective.*
- *For all open bisections $B \not\subset \mathcal{G}^{(0)}$, there exists $u \in s(B)$ such that $Bu \neq u$.*

Proof. If $\mathcal{G}$ is étale a basis for the topology is the open bisections.

Suppose the groupoid is effective. Then suppose B is a bisection such that for all $g \in B$, $s(g) = r(g)$. Then B is contained in the interior of the isotropy. Therefore, $B \subset \mathcal{G}^{(0)}$.

Suppose the interior of the isotropy contains something other than $\mathcal{G}^{(0)}$, i.e. the groupoid is not effective. This implies there exists an open set, and therefore an open bisection B such that for all $g \in B, s(g) = r(g)$ but $B \not\subset \mathcal{G}^{(0)}$.

□

One way to interpret this is that for effective groupoids, open bisections will encode much of the algebraic information of $\mathcal{G}$.

Example 2.2.18 (What do these definitions mean for transformation groupoids?)**.** *Étale groupoids are often thought of as generalisations of classical dynamical systems, since transformation groupoids are such an essential source of examples and inspiration. In turn, many of the above definitions are generalisations of notions in classical dynamics, rephrased in terms of groupoids. Let Γ be a discrete group acting by homeomorphisms on a locally compact Hausdorff space X, and let $\Gamma \ltimes X$ be the transformation groupoid:*

- *The orbits of an element $x \in X$ under the action is in classical dynamics is the set $\{g(x) : g \in \Gamma\}$. This agrees with the groupoid notion of orbit of x, $\Gamma \ltimes X_x$.*
- *In classical dynamics, the isotropy group of $x \in X$ is the subgroup $\Gamma_x := \{g \in \Gamma : g(x) = x\}$. This agrees with the isotropy group of the transformation groupoid at x, $\Gamma \ltimes X_x^x$.*
- *A group action is called minimal whenever there are no closed invariant subsets of X. This is equivalent to minimality of the groupoid $\Gamma \ltimes X$.*
- *A group action $\Gamma \curvearrowright X$ is called free if for all $g \in \Gamma, x \in X$, $g(x) = x \implies g = 1$. This property is equivalent to saying that the groupoid $\Gamma \ltimes X$ is principal.*

- *Likewise, an action is called topologically free if the set of points that it acts freely on, $\{x \in X : gx = x \implies g = 1\}$, is dense in X. Because such transformation groupoids are Hausdorff and second countable (assuming the unit space X is second countable), using Remark 2.2.16 it is therefore equivalent to say:*
  - *A group action $\Gamma \curvearrowright X$ is topologically free.*
  - *The transformation groupoid $\{x \in X : gx = x \implies g = 1\}$ is topologically principal.*
  - *The transformation groupoid $\{x \in X : gx = x \implies g = 1\}$ is effective.*

A groupoid of germs is a way of attaching an effective étale groupoid to an arbitrary group action.

Definition 2.2.19 (Groupoid of Germs)**.** *Let G be an infinite discrete group acting by homeomorphisms on a locally compact Hausdorff space X. Let $G \ltimes X$ denote the transformation groupoid. Let $\sim$ be an equivalence relation given by*

$$(g,x) \sim (h,y) \iff x = y \text{ and there exists an open subset } U \subset X \text{ containing } x$$

$$\text{such that } g(z) = h(z)\ \forall z \in U.$$

In other words, the germ at x under g agrees with the germ at x under h. Then, we let $(G \ltimes X)^{germ}$ denote the groupoid of germs of the action $G \curvearrowright X$, which is given by the quotient of $G \ltimes X$ under the equivalence relation $\sim$.

Let us end with the definition of groupoid comparison, a regularity condition that is often convenient to presume. Recall that on a locally compact Hausdorff space, a Radon measure is a measure μ on the σ-algebra of Borel sets such that is finite on all compact sets, outer regular on all Borel sets and inner regular on all open sets.

Definition 2.2.20. *Let $\mathcal{M}(\mathcal{G})$ denote the space of nonzero invariant Radon measures on $\mathcal{G}^{(0)}$, that is Radon measures μ such that for all compact open bisections $B, \mu(s(B)) = \mu(r(B))$.*

An ample groupoid $\mathcal{G}$ is said to have comparison if for every pair of compact open subsets $U,V \subset \mathcal{G}^{(0)}$ such that $\mu(U) < \mu(V), \forall \mu \in \mathcal{M}(\mathcal{G})$, there exists a compact open bisection B such that $s(B) = U$ and $r(B) \subset V$.

Comparison is a somewhat subtle condition, since it is difficult to find an example of an étale groupoid without comparison. See Question 2.7.1 for a relevant open question.

2.3 Ample groupoids

Many groupoids we study have the Cantor space as their unit space. We call these groupoids Cantor groupoids.

Definition 2.3.1 (Cantor Groupoid)**.** *An étale groupoid is called a Cantor groupoid if the unit space is a Cantor space.*

More generally, an ample groupoid is a groupoid whose unit space is totally disconnected.

Definition 2.3.2 (Ample Groupoid)**.** *An étale groupoid is called ample if the unit space is totally disconnected, i.e. such that the only connected subsets are the singletons.*

Remark 2.3.3. *Recall that in our definition of étale we include two topological regularity conditions on the unit space, we ask that $\mathcal{G}^{(0)}$ is locally compact and Hausdorff. Therefore, this is also inherited into the definition of ample. In this case, the unit space of an ample groupoid is locally compact, Hausdorff, and totally disconnected.*

Note that locally compact, Hausdorff and totally disconnected spaces can equivalently be described as the topological spaces that are locally compact, Hausdorff and zero-dimensional; that is the locally compact, Hausdorff spaces who have a basis of compact open subsets (see [3, Proposition 3.1.7]). Furthermore, suppose that an ample groupoid is second countable, and the unit space does not have isolated points. In this case, we have that compact subsets of the unit space are the Cantor space, by Brouwers theorem.

This terminology comes from the fact that there is an ample source of compact open bisections.

Lemma 2.3.4 (Equivalent Definitions)**.** *Let $\mathcal{G}$ be an étale groupoid. The following are equivalent:*

1. *$\mathcal{G}$ is ample.*

2. *$\mathcal{G}$ has a basis of compact open bisections.*

Proof. As discussed above, with the regularity imposed under being ample, a characterisation of our groupoid's unit space being totally disconnected is that it admits a basis of open compact subsets. From these facts, the implication 2. $\implies$ 1. is trivial.

Let us show 1. $\implies$ 2. we have that $\mathcal{G}$ has a basis of open bisections, since it is étale. Let B be an open bisection. Since $\mathcal{G}^{(0)}$ is totally disconnected, it has a basis of open compact subsets. Therefore, since $s(B)$ is open in $\mathcal{G}^{(0)}$, we can write $s(B) = \bigcup_{i\in I} K_i$ where each K_i is compact and open in $\mathcal{G}^{(0)}$. Then $B = \bigcup_{i\in I} B|_{K_i}$ is a decomposition of B into compact open bisections. Therefore any open bisection can be written as a union of compact open bisections and we are done. □

We end with important examples of ample groupoids.

Example 2.3.5 (Cantor Minimal Systems)**.** *Let $T : \mathcal{C} \to \mathcal{C}$ be a minimal homeomorphism of the Cantor set, that is, a homeomorphism such that for all $x \in \mathcal{C}$, the orbit $T^n(x)$ $n \in \mathbb{N}$ is dense in $\mathcal{C}$.*

One can use T to define an action of $\mathbb{Z}$ on the Cantor space by $\alpha(n)(x) = T^n(x)$. Then, the transformation groupoid $\mathbb{Z} \ltimes \mathcal{C}$ as in Example 2.2.7 is an ample, minimal and principal groupoid.

Example 2.3.6 (Transformation Groupoids)**.** *More generally, if we have a discrete group $\Gamma \curvearrowright_\alpha X$ acting by homeomorphisms on a totally disconnected space X, the resulting transformation groupoid $\Gamma \ltimes X$ as in Example 2.2.7 is ample.*

Example 2.3.7 (Partial Transformation Groupoids)**.** *Continuing Example 2.2.8, if we furthermore assume that Y is totally disconnected we can form an ample groupoid from this partial action. Let $\Gamma \curvearrowright X$ be an action of a discrete group on a locally compact Hausdorff space X. Let $Y \subset X$ be a totally disconnected subspace of X. Then the partial action of $\Gamma \curvearrowright Y$ gives rise to an ample partial action groupoid $\Gamma \ltimes X$.*

Example 2.3.8 (AF Groupoids)**.** *Let $\mathcal{G}$ be a second countable étale groupoid whose unit space is totally disconnected and compact.*

- *We say that $K \subset \mathcal{G}$ is an elementary subgroupoid if K is a compact open principal subgroupoid of $\mathcal{G}$ such that $K^{(0)} \cong \mathcal{G}^{(0)}$.*
- *We say that $\mathcal{G}$ is an AF groupoid if it can be written as an increasing union of elementary subgroupoids.*

See [103, Definition III.1.1] [52, Definition 3.7] [88] for further information on AF groupoids.

A more concrete, restricted class of these groupoids are the UHF groupoids, which we refer to later in our chapter on interval exchange groups.

Example 2.3.9 (UHF Groupoids)**.** *Let $\{k_i\}_{i \in \mathbb{N}}$ be a sequence of natural numbers. Let $k(n) = \prod_{i=1}^{n}$. We associate an AF groupoid as follows.*

For each $n \in \mathbb{N}$, let $\{1, \ldots, k(n)\} = X_n$. For each $n \in \mathbb{N}$ let us associate a principal groupoid $\mathcal{K}_n$ as the transformation groupoid $\mathbb{Z}_{k(n)} \ltimes X_n$ of the action of the cyclic group $\alpha : \mathbb{Z}_{k(n)} \curvearrowright X_n$ by:

$$\alpha(1)(n) = n + 1 \mod k(n)$$

Then, there are natural inclusions $\mathcal{K}_n \hookrightarrow \mathcal{K}_{n+1}$, which map the groupoid element $(m, x) \in \mathcal{K}_n$ to the compact open bisection (mk_{n+1}, S_x) where $S_x = \{xk_{n+1}, xk_{n+1} + 1, xk_{n+1} + 2, \ldots, xk_{n+1} + k_{n+1} - 1\}$. Let us take $\mathcal{G}$ to be the inductive limit via these inclusions $\mathcal{G} = \bigcup \mathcal{K}_n$. By construction, this is an AF groupoid, since each $\mathcal{K}_n$ is a compact open principal subgroupoid of $\mathcal{G}$.

This AF groupoid is known as a UHF groupoid. It is ample Cantor, minimal, and principal. It is also the standard groupoid model of the UHF algebra associated with the supernatural number $\prod_{i=1}^{\infty} k_i$, and the Bratelli diagram

$$k(1) \xrightarrow{k(2)/k(1)} k(2) \xrightarrow{k(3)/k(2)} k(3) \xrightarrow{k(4)/k(3)} \cdots \xrightarrow{k(n)/k(n-1)} k(n) \xrightarrow{k(n+1)/k(n)} \cdots$$

See [41] III for more information about UHF algebras, Bratelli diagrams, and supernatural numbers. For the purposes of this text, all one needs to understand is that Bratelli diagrams and *supernatural numbers are a particular invariant one can associate with UHF algebras which we use to identify certain UHF algebras and UHF groupoids in Chapter 6.*

Example 2.3.10 (SFT groupoids)**.** *See [89], Chapter 6 for a detailed introduction and discussion around SFT groupoids, as well as their key properties. Let (V,E) be a finite directed graph, where V is a finite set of vertices, and E is a finite set of edges. For each $e \in E$, let $i(e)$ denote the initial vertex of e and $t(e)$ denote the terminal vertex of e. Let $M = (M(v,v'))_{v,v' \in V}$ be the adjacency matrix of (V,E), that is:*

$$M(v,v') = \#\{e \in E \,:\, i(e) = v \quad \text{and} \quad t(e) = v'\}.$$

Let us assume that for all $v,v' \in V$ there exists $n \in \mathbb{N}$ such that $M^n(v,v') > 0$ (this property is known as irreducibility) and that M is not a permutation matrix. Let us define a Cantor set by endowing:

$$X = \{(x_k)_{k\in\mathbb{N}} \in E^{\mathbb{N}} \,:\, t(x_k) = i(x_{k-1}) \quad \forall k \in \mathbb{N}\}$$

with the product topology. We call this Cantor set the path space. This comes with a natural surjective continuous map

$$\sigma : X \to X \quad \sigma(x)_k = x_{k+1} \forall k \in N, \quad \forall x = (x_k)_k \in X.$$

This is known as the one sided shift on X, and is a local homeomorphism.

The étale groupoid $\mathcal{G}_{(V,E)}$ is (as a set) given by

$$\mathcal{G}_{(V,E)} = \{(x,n,y) \in X \times \mathbb{Z} \times X \,:\, \exists k,l \in \mathbb{N} \text{ s.t. } n = k-l, \sigma^k(x) = \sigma^l(y)\}.$$

The topology of $\mathcal{G}_{(V,E)}$ is generated by the sets

$$\{(x,k-l,y) \in \mathcal{G}_{(V,E)} : x \in A, y \in B,\ \sigma^k(x) = \sigma^l(y)\}$$

where A,B are open subsets of X and $k,l \in \mathbb{N}$. Multiplication is defined only for elements $(x,n,y),(y',n',z)$ such that $y = y'$. In which case the multiplication and inverse are given by

$$(x,n,y)(y,n',z) = (x,n+n'y)\ (x,n,y)^{-1} = (y,-n,x).$$

We identify X with the unit space of $\mathcal{G}_{(V,E)}$ via $x \mapsto (x,0,x)$.

A k-tuple $\mu = (e_1,\ldots,e_k) \in E^k$ such that $t(e_i) = i(e_{i+1})$ for $i = 1,\ldots,k-1$ is called a word. We denote by $|\mu| = k$ the length of μ. We extend i,t to words by saying that $i(\mu) = i(e_1)$, $t(\mu) = t(e_{|\mu|})$. This also allows us to compose words; if $\nu = (f_1,..,f_l)$ and $i(\nu) = t(\mu)$ then we let $\mu\nu = (e_1,\ldots,e_k,f_1,\ldots f_l)$. Moreover, for each word μ we associate an open set in X by

$$C_\mu = \{(x_n)_{n\in\mathbb{N}} \in X \,:\, x_i = e_i \quad i = 1,2,\ldots,k\}.$$

This set is known as the cylinder set of μ. *For any* μ, ν *such that* $t(\mu) = t(\nu)$, *we define a compact open bisection* $U_{\mu,\nu}$ *by*

$$U_{\mu,\nu} = \{(x, |\mu| - |\nu|, y) \in \mathcal{G}_{V,E} \,:\, x \in C_\mu,\, y \in C_\nu, \sigma^{|\mu|}(x) = \sigma^{|\nu|}(y)\}.$$

These subsets form a base for the topology of $\mathcal{G}_{V,E}$. *The resulting groupoid is ample and topologically principal.*

Example 2.3.11 (Products of SFT Groupoids)**.** *Recall in Example 2.1.8, the second operation was to consider the cartesian product two groupoids as a groupoid. We endow the product of two topological groupoids with the product topology. One may also consider products of SFT groupoids to obtain certain interesting topological full groups, following Matui [90].*

Here, we consider a finite collection $M_1, \dots, M_n$ *of irreducible matrices associated to directed graphs and construct the associated SFT groupoids* $\mathcal{G}_1, \dots, \mathcal{G}_n$. *The product of these groupoids* $\mathcal{G}_1 \times \dots \times \mathcal{G}_n$ *is again ample and topologically free.*

2.4 Groupoid homology

Ample groupoids admit a homology theory. Let us introduce this, following Matui [88, Section 3.1.]. We recommend this text, which introduces this notion of groupoid homology for further information.

Let A be a topological abelian group. For a locally compact Hausdorff space X, let $C_c(X,A)$ denote the continuous compactly supported $A-$valued functions. With respect to pointwise addition, this makes $C_c(X,A)$ an abelian group.

Recall that an étale map $\pi : X \to Y$ between topological spaces is a local homeomorphism; a map such that for every neighbourhood U of every point $x \in X$ there exists a homeomorphism given by the restriction $\pi_U : U \to \pi(U)$. Note then, in particular, if $\pi : X \to Y$ is an étale map then every point $y \in Y$ has a finite preimage $\pi^{-1}(y) = \{x \in X \,:\, \pi(x) = y\}$. Therefore, if $\pi : X \to Y$ is an étale map we can induce a map $\pi_* : C_c(X,A) \to C_c(Y,A) \quad f \mapsto \pi_*(f)$, given by

$$\pi_*(f)(y) = \sum_{\pi(x)=y} f(x),$$

since the sum is finite. This is a homomorphism; if $\pi : X \to Y, \pi' : Y \to Z$ are étale maps then $\pi_* \circ \pi'_* = (\pi \circ \pi')_*$.

Let $\mathcal{G}$ be an étale groupoid. Then, we have (by definition) that r and s are étale maps $\mathcal{G} \to \mathcal{G}^{(0)}$ and the multiplication map $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is étale. For $n > 2$ let $\mathcal{G}^{(n)}$ be the n-composable tuples i.e. $\mathcal{G}^{(n)} = \{(g_1, \dots, g_n) \,:\, (g_i, g_{i+1}) \in \mathcal{G}^{(2)},\, i = 1, \dots, n-1\}$. Then, let $\mathcal{G}^{(1)} := \mathcal{G}$. For $i = 0, 1, \dots, n$ let

$d_i : \mathcal{G}^{(n)} \to \mathcal{G}^{(n-1)}$ be the maps given by:

$$d_i(g_1,\dots,g_n) = \begin{cases} (g_2,\dots,g_n) & i=0 \\ (g_1,..,g_ig_{i+1},\dots g_n) & 1 \le i < n \\ (g_1,\dots,g_{n-1}) & i=n \end{cases}$$

The maps d_i are étale, because they are compositions of étale maps, the range and source maps r,s and the multiplication maps. For example, in the case $n=2$ let us verify that d_0,d_1,d_2 are étale. d_0 is just a composition of the source map s on the first component g_1 composed with the multiplication map. In the case d_1 is just the multiplication map between the tuple (g_1,g_2). In the case d_2 is just the range map r on g_1 composed with the multiplication map.

Let A be a topological abelian group. We define $\delta_n : C_c(\mathcal{G}^{(n)},A) \to C_c(\mathcal{G}^{(n-1)},A)$ by:

$$\delta_1 = s_* - r_* \text{ and } \delta_n = \sum_{i=0}^{n} (-1)^i d_{i*} \quad n \ge 1$$

Then,

$$0 \xleftarrow{\delta_0} C_c(\mathcal{G}^{(0)},A) \xleftarrow{\delta_1} C_c(\mathcal{G}^{(1)},A) \xleftarrow{\delta_2} C_c(\mathcal{G}^{(2)},A)\dots$$

is a chain complex. This is straightforward to see, the proof follows entirely analogously to showing the bar resolution for groups is a chain complex (see e.g. [126, Page 178]). We define groupoid homology as the homology of this chain complex.

Definition 2.4.1 (Groupoid Homology)**.** *Let $H_n(\mathcal{G},A) := Ker(\delta_n)/Im(\delta_{n+1})$. In the case where coefficients are committed we take the coefficients to be integral, i.e. $H_n(\mathcal{G}) := H_n(\mathcal{G},\mathbb{Z})$.*

Finally, we define the positive part of H_0 to be $H_0^+(\mathcal{G}) := \{[f] \in H_0(\mathcal{G},\mathbb{Z}) : f(u) \ge 0\ \forall u \in \mathcal{G}^{(0)}\}$.

Let us give the groupoid homology for some key examples of ample groupoids. The prototypical example of an ample groupoid is that of a (partial) transformation groupoid.

Example 2.4.2 ((Partial) Transformation Groupoids)**.** *The discussion in this example is folklore, following the text in [88, Section 3.1.]. The proof is clear– one can concretely identify the two underlying chain complexes.*

Following from example 2.3.7, let $\alpha : \Gamma \curvearrowright X$ be an action of a discrete group on a totally disconnected locally compact Hausdorff space X. Let $Y \subset X$ be a subspace of X. Consider the groupoids $\Gamma \ltimes X, \Gamma \ltimes Y$.

One can consider the abelian groups $C_c(X,\mathbb{Z}),C_c(Y,\mathbb{Z})$ as Γ-modules. Moreover, one may identify the groupoid homology with the group homology of Γ with coefficients in these modules, as in [27], Chapter III.

$$H_*(\Gamma \ltimes X) \cong H_*(\Gamma, C_c(X,\mathbb{Z}))$$

$$H_*(\Gamma \ltimes Y) \cong H_*(\Gamma, C_c(Y,\mathbb{Z}))$$

Furthermore, if Y is Γ-full in X (that is, meets every Γ-orbit in X), the canonical inclusion $Y \rightsquigarrow X$ induces an isomorphism in homology $H_*(\Gamma \ltimes Y) \cong H_*(\Gamma, C_c(X,\mathbb{Z}))$.

Due to the above example, we see that in the cases where $\mathcal{G}$ is a group or $\mathcal{G}$ is a topological space, the groupoid homology simplifies.

Example 2.4.3 ($\mathcal{G}$ a group.)**.** *If $\mathcal{G}$ is a group, the groupoid homology agrees with the regular definition of homology for groups, since then by the above example we can consider this as a transformation groupoid of $\mathcal{G}$ on its unit; $\mathcal{G} = \mathcal{G} \ltimes \mathcal{G}^{(0)}$ and so $H_*(\mathcal{G} \ltimes \mathcal{G}^{(0)}) = H_*(\mathcal{G}, C_c(\mathcal{G}^{(0)},\mathbb{Z})) = H_*(\mathcal{G},\mathbb{Z})$.*

Example 2.4.4 ($\mathcal{G}$ a space.)**.** *The other extreme is if $\mathcal{G} = \mathcal{G}^{(0)}$ is just a topological space that is totally disconnected. Consider this as a transformation groupoid of the trivial group acting on $\mathcal{G}^{(0)}$, i.e. $\{1\} \ltimes \mathcal{G}^{(0)}$. Therefore, the groupoid homology is*

$$H_0(\mathcal{G}) = C_c(\mathcal{G}^{(0)},\mathbb{Z}), \textit{ and } H_*(\mathcal{G}) = 0 \ \forall * > 0.$$

A specific example of a transformation groupoid is a Cantor minimal system, which we can be more precise:

Example 2.4.5 (Cantor Minimal Systems)**.** *Let (X,T) be a Cantor minimal system. Let $\mathcal{G}_T$ be the associated ample groupoid. From the above example we have that this will be the homology of $\mathbb{Z}$ with coefficients. This has been computed, see for example [27] Example III.1.1. The homology is given by*

$$H_0(\mathcal{G}_T) = \frac{C(X,\mathbb{Z})}{\{f - f \circ T \,:\, f \in C(X,\mathbb{Z})\}},$$
$$H_1(\mathcal{G}_T) = \mathbb{Z},$$

and

$$H_*(\mathcal{G}_T) = 0 \quad * \geq 2.$$

Example 2.4.6 (AF Groupoids)**.** *Let $\mathcal{G}$ be an AF groupoid as in example 2.3.8. In in [88, Theorem 4.10, Theorem 4.11], $H_0(\mathcal{G})$ was identified with the usual dimension group of $\mathcal{G}$, i.e. the K_0 group of the associated C^*-algebra, $K_0(C_r^*(\mathcal{G})$, and the higher homology groups vanish.*

Example 2.4.7 (SFT groupoids)**.** *Let $\mathcal{G}_A$ be an irreducible SFT groupoid associated to an $N \times N, \{0,1\}$–Matrix A. Matui computed the homology of SFT groupoids, in [88, Theorem 4.14] to be*

$$H_0(\mathcal{G}_A) = coker(I - A),\ H_1(\mathcal{G}_A) = ker(I - A),\ H_n(\mathcal{G}_A) = 0 \quad n > 1.$$

Similarly to the homology of groups, we also have that the homology of $\mathcal{G}_1 \times \mathcal{G}_2$ may be expressed in terms of the homology of $\mathcal{G}_1$ and $\mathcal{G}_2$ via a Kunneth formula.

Lemma 2.4.8 (Kunneth Formula for Groupoids [90], Theorem 2.4)**.** *Let $\mathcal{G}_1, \mathcal{G}_2$ be ample, effective groupoids. For all $* \in \mathbb{N} \cup \{0\}$, there exists a short exact sequence*

$$0 \to \bigoplus_{i+j=*} H_i(\mathcal{G}_1) \otimes H_j(\mathcal{G}_2) \to H_*(\mathcal{G}_1 \times \mathcal{G}_2) \to \bigoplus_{i+j=*-1} Tor(H_i(\mathcal{G}_1), H_j(\mathcal{G}_2)) \to 0.$$

For example, this can be used to compute the homology of groupoids arising as the product of shifts of finite type groupoids.

Example 2.4.9 (Product of SFT Groupoids)**.** *This example is based on [90, Proposition 5.4.]. Let $A_1, \ldots, A_n$ be irreducible 0-1 matrices associated to irreducible SFT groupoids $\mathcal{G}_{A_1}, \ldots, \mathcal{G}_{A_n}$. Let $\mathcal{G} = \mathcal{G}_{A_1} \times \ldots \times \mathcal{G}_{A_n}$ be the product of the SFT groupoids. By applying the Kunneth formula iteratively we can compute the homology of these groupoids to be*

$$H_k(\mathcal{G}) = (\mathbb{Z}^{{}^{n-1}C_k} \otimes \bigotimes_{i=1}^{n} H_1(\mathcal{G}_{A_i})) \oplus (\mathbb{Z}^{{}^{n-1}C_{k-1}} \otimes \bigotimes_{i=1}^{n} H_0(\mathcal{G}_{A_i})) \quad 0 \leq k \leq n$$

and $H_k(\mathcal{G}) = 0$ for $k > n$.

2.5 Groupoid C* -algebras

Locally compact Hausdorff étale groupoids can be completed to form groupoid C*-algebras. We need to recall some basic facts surrounding this topic. Note that this is not a detailed overview but a very brief tour of what is itself a broad subtopic in the study of C* -algebras. For greater depth and details we recommend the excellent notes of Aidan Sims [113].

For this subsection we assume that $\mathcal{G}$ is a locally compact, Hausdorff and étale groupoid. Let $\mathscr{C}_c(\mathcal{U}_\mathcal{G})$ be the set of $f : \mathcal{G} \to \mathbb{C}$ such that

$$\exists U_f \subset \mathcal{G} \text{ open s.t.} f\restriction_{U_f} \text{ is compactly supported and continuous on } U_f$$

And elsewhere $f = 0$.

We view $\mathscr{C}(\mathcal{G})$ as a convolution algebra. A right Haar system on $\mathcal{G}$ is a family $\nu = \{\nu_u\}_{u \in \mathcal{G}^{(0)}}$ where each ν_u is a positive regular locally finite Borel measure on $\mathcal{G}_u$ such that:

- $\text{supp}(\nu_u) = \mathcal{G}_u$ for all u.
- $\int_{\mathcal{G}_{r(g)}} f(hg) d\nu_{r(g)}(h) = \int_{\mathcal{G}_{s(g)}} f(h) d\nu_{s(g)}(h)$ for all $g \in \mathcal{G}, f \in \mathscr{C}(\mathcal{G})$.
- The maps
$$\alpha_f : \mathcal{G}^{(0)} \to \mathbb{R}^+ \quad u \mapsto \int_{\mathcal{G}_u} f(g) d\nu_u(g)$$
Are such that $\alpha_f \in C_c(\mathcal{G}^{(0)})$ for any $f \in \mathscr{C}(\mathcal{G})$.

If $\nu = \{\nu_u\}$ is a right Haar system, we may naturally assign a left Haar system associated with ν. Define for $A \subset \mathcal{G}^u$ Borel ν^u as the measure $\nu^u(A) = \nu_u(A^{-1}).\nu^{-1} = \{\nu^u\}$ is the left Haar measure induced by ν.

If $\nu = \{\nu_u\}_{u\in\mathcal{G}^{(0)}}$ is a left Haar system, then for $f_i \in \mathscr{C}(\mathcal{G}), g, h \in \mathcal{G}$ we may define an involution by

$$f^*(g) := \overline{f(g^{-1})},$$

a product

$$f_1 * f_2(g) := \int_{\mathcal{G}_{s(g)}} f_1(gh^{-1}) f_2(h) d\nu_{s(g)}(h),$$

and a norm

$$\|f\|_I := \sup_{u\in\mathcal{G}^{(0)}} \{\max\{\int_{\mathcal{G}_u} |f(g)| d\nu_u(g), \int_{\mathcal{G}_u} |f(g^{-1})| d\nu_u(g)\}\}$$

on $\mathscr{C}(\mathcal{G})$. Considering $\mathscr{C}(\mathcal{G})$ with respect to these operations, this becomes what is known as the convolution algebra of $\mathcal{G}$. Note that whilst this looks very similar to the convolution algebra of a group there are many key differences. An important difference is that the Haar measure of a locally compact group exists and is unique up to multiplication by a constant, whereas this may not be true for all étale groupoids $\mathcal{G}$. We use this to build the reduced groupoid C*-algebra of $\mathcal{G}$.

Definition 2.5.1 (Reduced Groupoid C* - algebra)**.** *Let $\mathcal{G}$ be a Hausdorff, locally compact étale groupoid and $\nu = \{\nu_u\}$ be a fixed left Haar system. We may then define for each u a bounded *-representation $\lambda_u : \mathscr{C}(\mathcal{G}) \to \mathcal{B}(L^2(\mathcal{G}_u, \nu_u))$ of the convolution algebra in $\mathcal{B}(L^2(\mathcal{G}_u, \nu_u))$, by setting*

$$\lambda_u(f)\xi(g) = \int_{\mathcal{G}_u} f(gh^{-1})\xi(h) d\nu_u(h).$$

The reduced C-algebra of $\mathcal{G}$ with respect to ν denoted $C_r^*(\mathcal{G}, \nu)$ is the completion of $\mathscr{C}(\mathcal{G})$ with respect to $\|\cdot\|_r$ given by:*

$$\|f\|_r := \sup_u \{\|\lambda_u(f)\|\} \quad f \in \mathscr{C}(\mathcal{G}).$$

In understanding groupoid C*-algebras it would be desirable to translate conditions from the underlying groupoids to their reduced C*-algebras. In particular, in light of the Elliot classification program, we have a good idea of what important regularity conditions there are for C*-algebras, and it is often desirable to show that groupoid C*-algebras are classifiable. When we say classifiable, we mean in the sense of the Elliott classification program see [[123], Corollary D]. Here we substitute finite nuclear dimension for $\mathcal{Z}$-stability by using [[34], Theorem A]

Theorem 2.5.2 (Classification)**.** *Let A, B be simple, unital, separable, nuclear, infinite dimensional, $\mathcal{Z}$-stable C*-algebras satisfying the UCT. Then,*

$$A \cong B \iff Ell(A) \cong Ell(B).$$

Groupoids provide a natural source of classifiable C*-algebras, with many groupoid C*-algebras satisfying some or all of the classifiability adjectives.

For example, it is known that there are the following sufficient conditions:

- An étale groupoid $\mathcal{G}$ is second countable iff $C_r^*(\mathcal{G})$ is separable, since in this case the countable base of the topology gives rise to a countable basis of $C_r^*(\mathcal{G})$. Otherwise, the support which gives a countable basis of $C_r^*(\mathcal{G})$ gives rise to the countable base of the topology of $\mathcal{G}$.
- A groupoid $\mathcal{G}$ is amenable iff $C_r^*(\mathcal{G})$ is nuclear (See [1, Theorem 3.3.7.], or [113, Theorem 4.1.5.]).
- Étale, amenable groupoid C*-algebras satisfy the UCT [124].
- A groupoid C*-algebra $C_r^*(\mathcal{G})$ is unital iff $\mathcal{G}^{(0)}$ is compact, in which one can see that the inclusion of the constant map $1_{\mathcal{G}^{(0)}}$ acts as the identity.
- A groupoid C*-algebra $C_r^*(\mathcal{G})$ is simple if $\mathcal{G}$ is minimal and effective [113, Proposition 4.3.7].
- There are also sufficient conditions for $C_r^*(\mathcal{G})$ to purely infinite, as explored in Chapter 5.

A notable omission above is $\mathcal{Z}$-stability. Understanding how to verify $\mathcal{Z}$-stability on the level of groupoid models is a fascinating active area of research. In fact, verifying $\mathcal{Z}$-stability is even difficult for discrete crossed products by amenable groups. There are many avenues of possible routes including (but not limited to) the notion of dynamic asymptotic dimension [17, 19, 55], almost finiteness [66, 117] and comparison [2, 64]. See the references therein for further discussion. However, for this thesis, we need only the following result due to Kerr-Szabo.

Theorem 2.5.3. *[66, Theorem C] Let Γ be a countably infinite amenable group whose finitely generated subgroups have subexponential growth. Then every free action $\Gamma \curvearrowright X$ of Γ on a compact metrizable finite-dimensional space is almost finite; the associated reduced groupoid C*-algebra $C_r^*(\Gamma \ltimes X)$ is classifiable (in particular $\mathcal{Z}$-stable).*

We end by discussing a conjecture of Matui, the so-called HK Conjecture, which relates the homology of an ample groupoid to the K-theory of its groupoid C*-algebra. It is often desirable to understand the K-theory of groupoid C*-algebras, but to compute this can be very difficult. On the other hand, groupoid homology is relatively computable. Matui's HK conjecture is that the K-Theory of groupoid C*-algebras can be computed in terms of groupoid homology. This was first given in [88].

Conjecture 2.5.4 (Matui's HK Conjecture)**.** *Let $\mathcal{G}$ be a minimal, essentially principal ample groupoid. Then for $* = 0, 1$*

$$K_*(C_r^*(\mathcal{G})) \cong \bigoplus_{i=0}^{\infty} H_{*+2i}(\mathcal{G}).$$

Counterexamples have been found to this conjecture [109], even in the principal setting [43]. However, it is known that the conjecture holds in many interesting cases [17, 49, 76, 90, 129].

2.6 Cartan subalgebras

Recently, the notion of Cartan subalgebras has greatly enhanced our understanding of étale groupoid C*-algebras. Suppose $\mathcal{G}$ is an étale, locally compact Hausdorff effective groupoid. Then we have that the canonical inclusion $C_0(\mathcal{G}^{(0)}) \subset C_r^*(\mathcal{G})$ satisfies a number of regularity properties:

- Since $\mathcal{G}^{(0)}$ is locally compact, $C_0(\mathcal{G}^{(0)})$ contains an approximate unit for $C_r^*(\mathcal{G})$– if $\{U_n\}_{n\in\mathbb{N}}$ is a nested sequence of open subsets in $\mathcal{G}^{(0)}$ such that each U_n is contained in a compact subset V_n of $\mathcal{G}^{(0)}$, and such that $\bigcup_{n\in\mathbb{N}} U_n = \mathcal{G}^{(0)}$
- $C_0(\mathcal{G}^{(0)})$ is a maximal abelian subalgebra; if one took any $x \in C_r^*(\mathcal{G}) \setminus C_0(\mathcal{G}^{(0)})$ and considered the C*-algebra generated by x and $C_0(\mathcal{G}^{(0)})$, this would not be abelian.
- The inclusion is regular, meaning that the set of normalisers:

  $$N_{C_0(\mathcal{G}^{(0)})}(C_r^*(\mathcal{G})) := \{n \in C_r^*(\mathcal{G}) \,:\, nC_0(\mathcal{G}^{(0)})n^* \subseteq C_0(\mathcal{G}^{(0)})\; n^*C_0(\mathcal{G}^{(0)})n \subseteq C_0(\mathcal{G}^{(0)})\}$$

  generate $C_r^*(\mathcal{G})$ as a C* -algebra.

  This is since if the support of an element $n \in C_r^*(\mathcal{G})$ is an open bisection, then n is a normaliser.
- A conditional expectation is an onto positive projection from one C*-algebra A to a subalgebra B of A, i.e. a projection $P : A \to B$ such that

  $$P(b_1ab_2) = b_1P(a)b_2 \;\forall a \in A,\, b_1, b_2 \in B.$$

  Here, there is a (unique) faithful conditional expectation

  $$P : C_r^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$$

  given by $f \mapsto f|_{\mathcal{G}^{(0)}}$.

Such subalgebras are known as Cartan subalgebras, and were studied by Renault [104].

Definition 2.6.1 (Cartan Subalgebra [104]). *A C*-subalgebra B of a C*-algebra A is a Cartan subalgebra if:*

- *B contains an approximate unit for A.*

- *B is a maximal abelian subalgebra of A.*
- *B is regular in A.*
- *There exists a faithful conditional expectation $P : A \to B$.*

In this case, the pair (A,B) is called a Cartan pair.

Suppose $(A_1,B_1),(A_2,B_2)$ are Cartan pairs. A C-isomorphism $\varphi : A_1 \to A_2$ such that $\varphi(B_1) = B_2$ is called a Cartan isomorphism.*

We therefore have in the case when $\mathcal{G}$ is an étale, Hausdorff, locally compact, second countable, topologically principal groupoid that $C_0(\mathcal{G}^{(0)})$ is a Cartan subalgebra of $C_r^*(\mathcal{G})$ suggesting a link between groupoid C*-algebras and Cartan pairs. The below theorem in its original form is known to the community as Renault's reconstruction theorem and sheds light on this relationship.

Theorem 2.6.2. *[104] Let $(\mathcal{G},\Sigma)$ be a twisted étale, Hausdorff, locally compact, second countable, topologically principal groupoid. Then $C_0(\mathcal{G}^{(0)})$ is a Cartan subalgebra of $C_r^*(\mathcal{G},\Sigma)$.*

Conversely, if (A,B) is a Cartan pair where A is separable, then there exists a twisted étale Hausdorff locally compact second countable topologically principal groupoid $(\mathcal{G},\Sigma)$ and a Cartan isomorphism $\varphi : (A,B) \to (C_r^*(\mathcal{G},\Sigma), C_0(\mathcal{G}^{(0)}))$.

Renault's reconstruction theorem has been since generalised in several interesting ways, see for example [47, 70, 101].

A single C*-algebra can have more than one Cartan subalgebra, in fact it can have groupoid models with two unit spaces that are not homeomorphic, as seen in the below remark.

Remark 2.6.3. *Consider the CAR algebra, that is the UHF algebra associated to the supernatural number 2^∞. There are two groupoid models for this C*-algebra.*

- *The standard UHF groupoid model, as in Example 2.3.9. This is a Cantor groupoid.*
- *A transformation groupoid of a locally finite group Γ acting on $\{0,1\}^{\mathbb{N}} \times S^1$ [9].*

Note that for these two groupoid models their unit spaces are not homeomorphic.

The study of the existence and uniqueness of Cartan subalgebras is a fascinating area of active research. See the references [77, 81, 82] for interesting modern developments.

2.7 Outlook

Our first question was regarding comparison.

Question 2.7.1. *Suppose $\mathcal{G}$ is a second countable, minimal, ample groupoid. Does $\mathcal{G}$ have comparison?*

Our other two questions relate to groupoid C*-algebras. A major theme is to understand the following question more completely.

Question 2.7.2. *For which étale groupoids $\mathcal{G}$, is $C_r^*(\mathcal{G})$ $\mathcal{Z}$-stable?*

There are counterexamples to the HK conjecture, and simultaneously broad example classes for which the HK conjecture holds. A natural question arises.

Question 2.7.3. *For which ample effective groupoids $\mathcal{G}$ does the HK conjecture hold?*

Also, in recent years a weakening of the HK conjecture has been suggested where instead of coefficients in $\mathbb{Z}$ we are investigating the HK conjecture with coefficients in $\mathbb{Q}$, notably appearing as [17, Remark 5.12]. This weakened version of the HK conjecture known as the rational HK conjecture remains open, and so we ask this below, for further visibility:

Question 2.7.4 (Rational HK Conjecture)**.** *Let $\mathcal{G}$ be an amenable, ample, second-countable groupoid with torsion-free isotropy, and satisfying the strong Baum-Connes conjecture. Then for $i = 0, 1$ there exist homomorphisms*

$$K_i(C_r^*(\mathcal{G}), \mathbb{Q}) \cong \bigoplus_{k=0}^{\infty} H_{i+2k}(\mathcal{G}, \mathbb{Q}).$$

Chapter 3

Topological full groups of ample groupoids

3.1 Definition of a topological full group

As before, we have a standing assumption that whenever $\mathcal{G}$ is ample, the topology on the unit space is Hausdorff and second countable. However, we do not have a standing assumption of Hausdorffness in this Chapter. There are several interesting examples of topological full groups arising from non-Hausdorff groupoids and so the study of such groupoids is essential. For our definition, we follow Ortega-Nyland whose framework allows us to have non-compact unit spaces.

Definition 3.1.1 (Ortega-Nyland [97])**.** *Let $\mathcal{G}$ be an effective ample groupoid. For each $K \subset \mathcal{G}^{(0)}$ compact and open, let*

$$\mathsf{F}(\mathcal{G})_K := \{B \in \mathcal{B}^k \,:\, s(B) = r(B) = K\}.$$

Consider this group embedded in $Homeo(\mathcal{G}^{(0)})$ via the embedding

$$\iota_K : \mathsf{F}(\mathcal{G})_K \to Homeo(\mathcal{G}^{(0)}) \quad \iota(B)(x) = \begin{cases} B(x) & x \in K \\ x & otherwise \end{cases}.$$

Then, we define the topological full group to be the inductive limit

$$\mathsf{F}(\mathcal{G}) = \bigcup_{K compact} \iota_K(\mathsf{F}(\mathcal{G})_K) \subset Homeo(\mathcal{G}^{(0)}).$$

If $\mathcal{G}^{(0)}$ was already compact we have that $\mathsf{F}(\mathcal{G}) = \mathsf{F}(\mathcal{G})_{\mathcal{G}^{(0)}}$. For example, if $\mathcal{G}$ was a Cantor groupoid, then we could take the definition to be

$$\mathsf{F}(\mathcal{G}) := \{B \in \mathcal{B}^k \,:\, s(B) = r(B) = \mathcal{G}^{(0)}\}.$$

this definition in the case of Cantor groupoid is due to Nekrashevych [93].

Recall that the compact open bisections form an inverse semigroup acting by partial homeomorphisms. Through this association, we have that elements B of topological full groups define homeomorphisms f_B of the unit space in the same way, where

$$f_B = (r_{\restriction_B}) \circ (s_{\restriction_B})^{-1} : (\mathcal{G})^{(0)} \to (\mathcal{G})^{(0)}.$$

The reason we assume our groupoid is effective is the following:

Remark 3.1.2. *Let $\mathcal{G}$ be an ample groupoid. The following are equivalent*

- *$\mathcal{G}$ is effective.*
- *The map $f : \mathsf{F}(\mathcal{G}) \to Homeo(\mathcal{G}^{(0)}) \quad B \mapsto f_B$ is an injection.*

This follows as an immediate consequence of Proposition 2.2.17. This movement between perspectives is routinely used throughout this text, so we often assume that our groupoids are effective. Let us discuss what happens outside the effective case for completeness of the literature.

As we saw in our section on étale groupoids, we can often consider the union of two or more compact open bisections as an compact open bisection. So in some sense we can picture our topological full group as the subgroup of $Homeo(\mathcal{G}^{(0)})$ that locally looks like compact open bisections. In fact, in the effective case, an equivalent definition is the following:

$$\mathsf{F}(\mathcal{G}) = \{g \in Homeo(\mathcal{G}^{(0)}) \ : \forall x \in \mathcal{G}^{(0)}, \ \exists B \in \mathcal{B}^k \text{ such that } x \in s(B), \text{ and } g|_{s(B)} = B\}$$

So our topological full groups are global symmetries (homeomorphisms) built from local symmetries (compact open bisections).

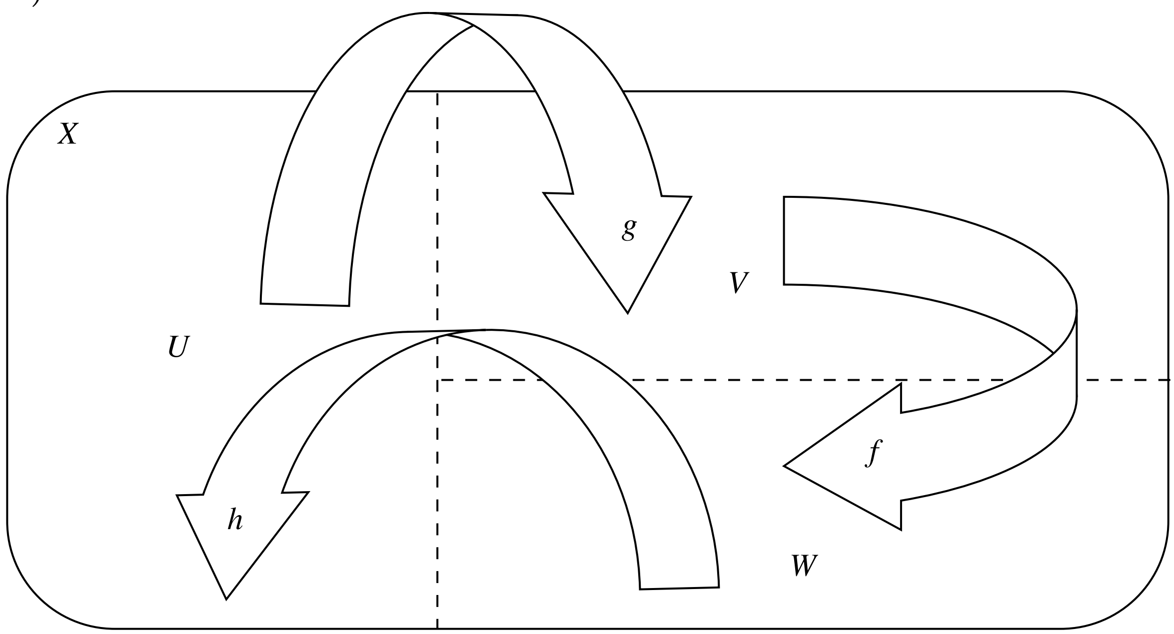
Example 3.1.3 (Topological full group of a transformation groupoid)**.** *Let Γ be a discrete group acting on the Cantor set X. Consider the associated transformation groupoid $\Gamma \ltimes X$. Recall that the topology of this groupoid is generated by collections of arrows of the form (g,U) where $g \in \Gamma$ and U is compact and open in X. Therefore $\mathcal{B}^k$ is disjoint unions of subsets of the form (g,U) where $g \in \Gamma, U$ is clopen in X. Therefore, topological full group elements are of the form:*

$$\bigsqcup_{i=1}^{n} (g_i, U_i) \quad \bigsqcup U_i = \bigsqcup g_i U_i = X,$$

where $g_i \in \Gamma$, and each U_i is a clopen subset in X.

In other words, $\mathsf{F}(\Gamma \ltimes X)$ can be naturally identified with the subgroup of homeomorphisms $g \in \mathrm{Homeo}(X)$ for which there exists a continuous map $f_g : X \to \Gamma$ such that $g(x) = f_g(x) \cdot x$ for all $x \in X$. Such a map f_g is called an orbit cocycle *for g.*

We have below an example of a topological full group element for such a transformation groupoid. Suppose we have a group action $\Gamma \curvearrowright X$*, and a partition of* X *into clopen subsets* U,V,W*. Suppose furthermore there exists some group elements* g,f,h *such that* $g(U)=V, f(V)=W, h(W)=U$*. Then an example element of the topological full group would be* $(g,U)\cup(f,V)\cup(h,W)$*.*



For the associated groupoid of germs, as in Definition 2.2.19 it follows from [95, Proposition 4.6] that $\mathsf{F}((\Gamma \curvearrowright X)^{germ})$ *agrees with* $\mathsf{F}(\Gamma \ltimes X)$*, so it can be computed using orbit cocycles for* $\Gamma \curvearrowright X$ *as in the previous paragraph.*

Similarly, if we were to consider a restriction of the action $\Gamma \curvearrowright X$ *to a partial action* $\Gamma \curvearrowright Y$*, then the topological full group of* $\Gamma \ltimes Y$ *may be identified the subgroup of* $\mathsf{F}(\Gamma \ltimes X)$ *which act as the identity on* $X \setminus Y$*, i.e.*

$$\mathsf{F}(\Gamma \ltimes Y) = \{\gamma \in \mathsf{F}(\Gamma \ltimes X) \;:\; \gamma(x) = x, \; \forall x \in X \setminus Y\}.$$

We end with one more illustrative example, which works well as a guiding example for the next section. One might have noticed that there appears to be a connection with the symmetric group, in the way that the topological full group permutes parts of the unit space of an ample groupoid. In fact, the infinite symmetric group arises as a topological full group of a discrete groupoid:

Example 3.1.4 (The full equivalence relation on $\mathbb{N}$ and S_∞)**.** *Consider* R_n*, the full equivalence relation on n points as in Example 2.1.7. The unit space is n labelled isolated points* $1,\dots,n$ *and there is exactly one arrow from every point to every point. Elements of the topological full group are therefore any bijection from the unit space to itself, i.e. we have that the topological full group* $\mathsf{F}(R_n) = S_n$ *is the symmetric group on n points. Considering the full equivalence relation* *on $\mathbb{N}$ groupoid $R_{\mathbb{N}}$, best described as the inductive limit $R_{\mathbb{N}} = \bigcup_{n\in\mathbb{N}} R_n$ then, we obtain S_∞ the infinite symmetric group as the topological full group of $R_{\mathbb{N}}$. Note also then the derived subgroup of S_∞ is the infinite alternating group, an infinite simple group.*

The main aim of the following section is to define two groups in analogy to the infinite symmetric and alternating group.

3.2 Subgroups of topological full groups

The aim of this subsection is to describe what we understand about the subgroup structure of topological full groups. Let $\mathcal{G}$ be an effective, Cantor groupoid with infinite orbits. We will begin by explaining the key ideas behind Nekrashevych's alternating and symmetric group, before giving the formal definition. Let us first define an analogue of the infinite alternating group in $\mathsf{F}(\mathcal{G})$, which we call the alternating group of $\mathcal{G}$. For compact open bisection B_1, B_2 with $s(B_1), r(B_1) = s(B_2), r(B_2)$ pairwise disjoint, let

$$\gamma_{B_1,B_2} = B_1 \sqcup B_2 \sqcup (B_1B_2)^{-1} \sqcup (\mathcal{G}^{(0)} \setminus s(B_1) \sqcup s(B_2) \sqcup r(B_2)) \in \mathsf{F}(\mathcal{G}).$$

Group elements of the form γ_{B_1,B_2} have order 3, and play an analogous role to three cycles in

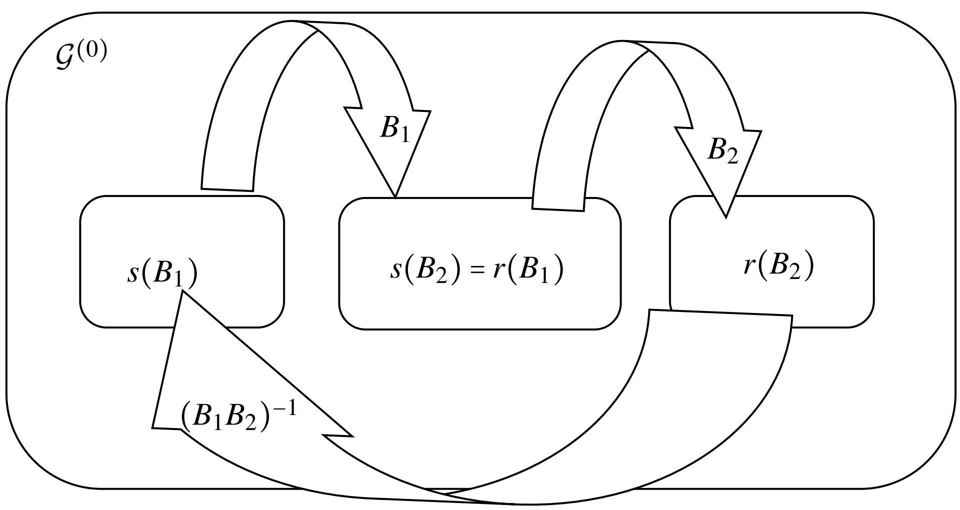


Figure 3.1: A figure to visualise the construction of γ_{B_1,B_2} from B_1 and B_2

the infinite alternating group; they generate the alternating group. We define

$$\mathsf{A}(\mathcal{G}) = \langle \gamma_{B_1,B_2} : B_1, B_2 \in \mathcal{B}^k,\ s(B_1), r(B_1) = s(B_2), r(B_2) \text{ are pairwise disjoint } \rangle.$$

A similarly defined group is Nekrashevych's symmetric group. Here, we take a single compact open bisection $B \in \mathcal{B}^k$ with $s(B) \cap r(B) = \varnothing$, and form an element $\gamma_B \in \mathsf{F}(\mathcal{G})$ from B, B^{-1} and what remains in the unit space,

$$\gamma_B := B \sqcup B^{-1} \sqcup (\mathcal{G}^{(0)} \setminus s(B) \cup r(B)) \in \mathsf{F}(\mathcal{G}).$$

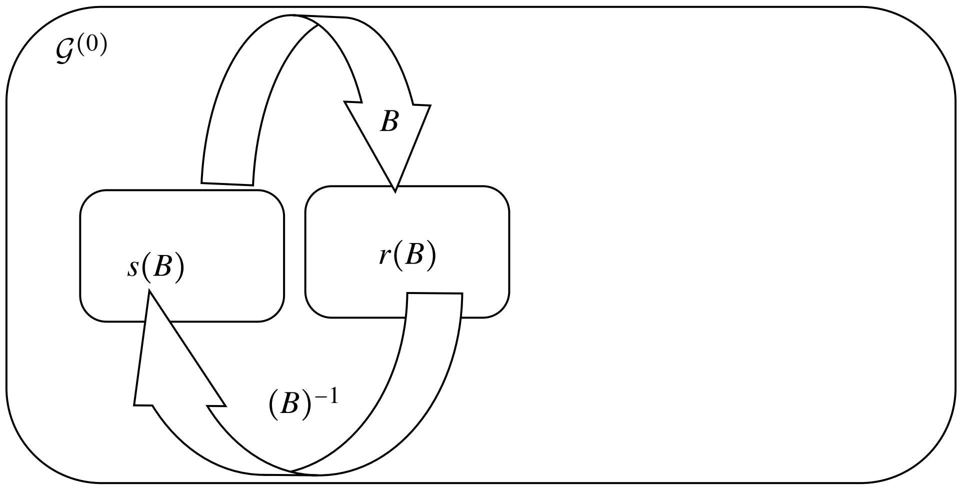


Figure 3.2: A figure to visualise the construction of γ_B, a typical generator of $\mathcal{S}(\mathcal{G})$

Elements of the form γ_B have order 2, and play an analogous role to 2-cycles in the infinite symmetric group; they form a generating set for the symmetric group. Then we define $\mathsf{S}(\mathcal{G})$ to be the subgroup of $\mathsf{F}(\mathcal{G})$ generated by each γ_B;

$$\mathsf{S}(\mathcal{G}) = \langle \gamma_B \,:\, B \in \mathcal{B}^k \, s(B) \cap r(B) = \varnothing \rangle.$$

We consider this to be an analogue of the infinite symmetric group in $\mathsf{F}(\mathcal{G})$, hence we call $\mathsf{S}(\mathcal{G})$ the symmetric subgroup. We need to be more careful when we define the alternating and symmetric group in the case when our orbits are not necessarily infinite. These formal definitions are equivalent to those in the discussion above in the case where the orbits are infinite.

Definition 3.2.1 (Multisection, Alternating and Symmetric Group)**.** *Let $\mathcal{G}$ be an ample effective groupoid. A collection of bisections $M(d) = \{F_{ij}\}_{i,j=1}^{d}$ is called a multisection of degree d if:*

- *$F_{i,j}F_{j,k} = F_{i,k}$ for all $1 \le i, j, k \le d$.*
- *The subsets $F_{i,i}$ are pairwise disjoint subsets of $\mathcal{G}^{(0)}$.*

Given a multisection of degree d there is a canonical inclusion of S_d into $\mathsf{F}(\mathcal{G})$ given by

$$\iota_{M(d)} : S_d \hookrightarrow \mathsf{F}(\mathcal{G}) \quad (i,j) \mapsto F_{i,j}.$$

Let $\mathsf{S}_d(\mathcal{G})$ denote the smallest subgroup of $\mathsf{F}(\mathcal{G})$ containing $\iota_{M(d)}(S(d))$ for all multisections $M(d)$. We take the symmetric subgroup to be the inductive limit $\mathsf{S}(\mathcal{G}) := \bigcup_{d\to\infty} \mathsf{S}_d(\mathcal{G})$.

Similarly, let $\mathsf{A}_d(\mathcal{G})$ denote the smallest subgroup of $\mathsf{F}(\mathcal{G})$ containing $\iota_{M(d)}(A(d))$ for all multisections $M(d)$. We take the alternating subgroup to be the inductive limit $\mathsf{A}(\mathcal{G}) = \bigcup_{d\to\infty} \mathsf{A}_d(\mathcal{G})$.

Comparison, a weak condition will already imply that the alternating and symmetric subgroups nontrivial.

Proposition 3.2.2 (Comparison Implies The Alternating and Symmetric Subgroups are nontrivial)**.** *Suppose that $\mathcal{G}$ has comparison. Then the alternating and symmetric subgroups are nontrivial.*

Proof. It is enough to show that the alternating group is nontrivial, i.e. the existence of γ_{B_1,B_2} or bisections B_1,B_2 such that $s(B_1),s(B_2)=r(B_1),r(B_2)$ are pairwise disjoint. Let us begin with some proper compact open subset of the unit space $U \neq \mathcal{G}^{(0)}$. Then, U^c is nonempty. There exists some proper nonempty subset V_1 of U^c such that for all $\mu \in \mathcal{M}(\mathcal{G}), \mu(V_1) < \mu(U)$. Given V_1, there exists some proper nonempty compact open subset V_2 of $U^c \cap V_1^c$ such that $\mu(V_2) < \mu(U)$.

By comparison, there exists two bisections $B_1, \hat{B_2}$ such that $s(B_1) = U = s(\hat{B_2})$ and $r(B_1) = V_1, r(B_2) = V_2$. Setting $B_2 = \hat{B_2}^{-1}$) we are done. □

Another subgroup of a topological full group we study is the derived subgroup, which we denote by $\mathsf{D}(\mathcal{G})$.

Definition 3.2.3. *Let $\mathcal{G}$ be an effective Cantor groupoid. Let $\mathsf{D}(\mathcal{G})$ denote the derived subgroup of $\mathsf{F}(\mathcal{G})$, that is $\mathsf{D}(\mathcal{G}) = \langle[\gamma_1,\gamma_2] : \gamma_1,\gamma_2 \in \mathsf{F}(\mathcal{G})\rangle$.*

It can be seen that the alternating group is a subgroup of the derived subgroup of an ample groupoid, since $\gamma_{B_1,B_2} = [\gamma_{B_1},\gamma_{B_2}]$. It is currently open whether there is always an inclusion in the other direction or whether there exists an effective, étale, Cantor groupoid $\mathcal{G}$ such that $\mathsf{A}(\mathcal{G}) \neq \mathsf{D}(\mathcal{G})$.

Let us remark that each of the groups we have defined thus far are normal in $\mathsf{F}(\mathcal{G})$. For $\mathsf{A}(\mathcal{G}),\mathsf{S}(\mathcal{G})$ this follows from the observation that for all $\gamma \in \mathsf{F}(\mathcal{G})$, and $\gamma_B \in \mathsf{S}(\mathcal{G})$, $\gamma\gamma_B\gamma^{-1} = \gamma_{\gamma B\gamma^{-1}}$.

$$\mathsf{A}(\mathcal{G}) = D(\mathsf{S}(\mathcal{G})) \subsetneq \mathsf{S}(\mathcal{G}) \subseteq \mathsf{F}(\mathcal{G})$$
$$\mathsf{A}(\mathcal{G}) = D(\mathsf{S}(\mathcal{G})) \subseteq \mathsf{D}(\mathcal{G}) \subsetneq \mathsf{F}(\mathcal{G})$$

Figure 3.3: A figure showing the relationship between the subgroups of topological full groups.

A recommendation to the reader is that when understanding the proofs of the Theorems in Sections 3.3,3.4 and 3.5, one should be drawing the topological full group elements in the proofs. Many of the proofs seem complex, but in the readers opinion, once drawing diagrams such as those in this section, become much more straightforward.

3.3 Simplicity of derived subgroup

In this subsection, we discuss the relationship between minimality of an ample groupoid and the simplicity of the derived subgroup of the topological full group. Let us first recall and prove a related result of Nekrashevych.

Theorem 3.3.1 ([93]). *Let $\mathcal{G}$ be a minimal, effective Cantor groupoid. Then every nonzero subgroup of $\mathsf{F}(\mathcal{G})$ normalised by $\mathsf{A}(\mathcal{G})$ contains $\mathsf{A}(\mathcal{G})$. In particular, $\mathsf{A}(\mathcal{G})$ is simple and contained in any nonzero normal subgroup of $\mathsf{F}(\mathcal{G})$.*

Proof. Let $U \subset \mathcal{G}^{(0)}$ be a clopen subset. Let us identify $\mathsf{A}(\mathcal{G}_U)$ with the natural subgroup of $\mathsf{A}(\mathcal{G})$ by having the elements act as the identity on U^c.

Let $N \subset \mathsf{F}(\mathcal{G})$ be the subgroup normalised by $\mathsf{A}(\mathcal{G})$ and let $g \in N \setminus \{1\}$. By effectiveness, let $\gamma \in g$ be such that $s(\gamma) \neq r(\gamma)$. Then, there exists a clopen neighbourhood U of $s(\gamma)$ such that U and $g(U)$ are disjoint. If $\mathcal{F}$ is any multisection with domain a subset of U, and $h_1, h_2 \in \mathsf{A}(\mathcal{F})$ then we can note that:

- $g\mathcal{F}g^{-1}$ is a multisection with domain in $g(U)$; $gh_1g^{-1} \in \mathsf{A}(g\mathcal{F}g^{-1})$.
- $[g^{-1}, h_1] = gh_1^{-1}g^{-1}h_1$ has support in $U \cup g(U)$, where it acts as h_1 on U and $g_1h_1^{-1}g^{-1}$ on $g(U)$.
- Therefore, $n = [[g^{-1}, h_1], h_2] = [h_1, h_2]$. However $n \in N$, since $g \in N$ and $h_1, h_2 \in \mathsf{A}(\mathcal{G})$.
- Since $\mathsf{A}(\mathcal{G})$ is perfect, we have proved there exists a nonempty clopen subset U of $\mathcal{G}^{(0)}$ such that $\mathsf{A}(\mathcal{G}_U)$ is contained in N.

We would like to show the normal close of $\mathsf{A}(\mathcal{G}_U)$ in $\mathsf{A}(\mathcal{G})$ is all of $\mathsf{A}(\mathcal{G})$. Let $\mathcal{F} = \{F_{i,j}\}_{i,j=1}^d$ be a multisection. Let $x_1 \in F_{1,1}$ be arbitrary. Let $x_i = r(F_{1,i}x_i)$. By minimality let $\gamma_i, \delta_i \in \mathcal{G}$ be such that $s(\gamma_i) = s(\delta_i) = x_i$, $r(\gamma_i) \in U$ and the units $x_1, \ldots, x_d, r(\gamma_1), \ldots, r(\gamma_d), s(\delta_1), \ldots, s(\delta_d)$ all pairwise different.

Let $\mathcal{F}'$ be a multisection such that $F'_{i,j} \subset F_{i,j}$. By ampleness let G_i, D_i be compact open bisections with $\gamma_i \in G_i, \delta_i \in D_i$ with $F'_{ii} = s(G_i) = r(D_i) \subset F_{i,i}, r(G_i) \subset U$, and the sets

$$s(G_1), .., s(G_d), r(G_1), \ldots, r(G_d), s(D_1), \ldots, s(D_d)$$

pairwise disjoint. We have that $\hat{\mathcal{F}} = \{G_jF'_{i,j}G_i^{-1}\}$ is a multisection with domain in U. Consider $C_i = G_i \sqcup D_i^{-1} \sqcup D_iG_i^{-1} \sqcup (\mathcal{G}^{(0)} \setminus s(G_i) \sqcup r(G_i) \sqcup r(D_i))$. Let g_i be the associated element of $\mathsf{A}(\mathcal{G})$. We have that $gF_{i,j}g^{-1} = G_jF'_{i,j}G_i^{-1}$ hence $g\mathsf{A}(\mathsf{F})g^{-1} = \mathsf{A}(\mathcal{H}) \subset \mathsf{A}(\mathcal{G}_U)$. Then $\mathsf{A}(\mathcal{F}') \subset g\mathsf{A}(\mathcal{G}|_U)$. Since we can do this for all $x \in F_{1,1}$ and $F_{1,1}$ is compact we are done. □

Regarding the simplicity of the derived subgroup itself, Matui is the most commonly cited for this result. In particular, [89, Section 4]. Matui proves this result for two cases:

- When $\mathcal{G}$ is a Hausdorff, effective minimal and almost finite Cantor groupoid.
- When $\mathcal{G}$ is Hausdorff, effective minimal and purely infinite.

In fact, Matui does not actually use that $\mathcal{G}$ is Hausdorff in the proof of this result, and so it can be removed; the following Lemma holds:

Theorem 3.3.2 (Matui)**.** *Let $\mathcal{G}$ be an effective minimal Cantor groupoid. Suppose that either:*

- *$\mathcal{G}$ is almost finite.*
- *$\mathcal{G}$ is purely infinite.*

Then, any nonzero subgroup of $\mathsf{F}(\mathcal{G})$ normalised by $\mathsf{D}(\mathcal{G})$ is normalised by $\mathsf{D}(\mathcal{G})$; $\mathsf{D}(\mathcal{G})$ is simple.

This subsection aims to exhibit a proof of this result in the case when $\mathcal{G}$ is purely infinite and effective Cantor groupoid but not necessarily Hausdorff, following [89, Section 4.2.]. First, let us note a useful lemma for understanding the property of being purely infinite and minimal.

Lemma 3.3.3. *([89, Proposition 4.11]). Let $\mathcal{G}$ be an effective, étale, Cantor groupoid. The following are equivalent:*

1. *$\mathcal{G}$ is purely infinite and minimal.*
2. *For all compact and open subsets $X,Y \subseteq \mathcal{G}^{(0)}$, there exists a compact open bisection $B \subseteq \mathcal{G}$ such that $s(B) = X$ and $r(B) \subseteq Y$.*
3. *For all compact and open subsets $X,Y \subseteq \mathcal{G}^{(0)}$ with $X \neq \mathcal{G}^{(0)}$ and $Y \neq \varnothing$, there exists $\alpha \in \mathsf{F}(\mathcal{G})$ such that $\alpha(X) \subseteq Y$.*

Proof.

- 1. $\Rightarrow$ 2. Let X,Y be arbitrary, with $Y \neq \varnothing$. Then, there exists bisections U,V such that $s(U) = s(V) = Y$, $r(U) \cup r(V) \subset Y$ and $r(U) \cap r(V) = \varnothing$, since Y is properly infinite. Let $V_1 = U$, and define V_n inductively by $V_{n+1} = VV_n$. Then we have that for all $n \neq m$, $s(V_n) = Y$, $r(V_n) \subset Y$ and $r(V_n) \cap r(V_m) = \varnothing$. Using minimality, there exists bisections $W_1, W_2, \ldots$ such that $r(W_i) \subset Y$ and $\bigsqcup_{i \in \mathbb{N}} s(W_i) = X$ with disjoint sources. Let B be the bisection given by $B = \bigcup_{i \in \mathbb{N}} V_i W_i$. Then $s(B) = X, r(B) \subset Y$.

- 2. $\Rightarrow$ 3. First assume that $Y \setminus X$ is nonempty. Then there is a bisection U such that $s(B) = X$, $r(B) = Y \setminus X$. Hence we may take $\alpha = U \sqcup U^{-1} \sqcup (\mathcal{G}^{(0)} \setminus s(U) \cup r(U)) \in \mathsf{F}(\mathcal{G})$. Otherwise, $Y \subset X$, in which case by the above argument we can choose $\alpha_1, \alpha_2 \in \mathsf{F}(\mathcal{G})$ with $\alpha_1(X) \subset X^c$, $\alpha_2(X^c) \subset Y$, and $\alpha_1\alpha_2$ satisfies the requirement.

- 3. $\Rightarrow$ 1. It is clear that 3. implies minimality of $\mathcal{G}$. Now let X be a nonempty clopen subset of the unit space. Let Y_0, Y_1, Y_2 be mutually disjoint subsets of X. Then, by 3. there exists $\alpha_1, \alpha_2 \in \mathsf{F}(\mathcal{G})$ such that

$$\alpha_1(X \setminus Y_0) \subset Y_1,\ \alpha_2(B_0 \sqcup B_1) \subset B_2.$$

Let U_i be the bisection associated with α_i. Let

$$V_1 = Y_0 \sqcup U_1|_{(X \setminus Y_0)},\ V_2 = U_2 V_1.$$

We have that $s(V_1) = s(V_2) = X, r(V_1) \subset Y_0 \sqcup Y_1$ and $r(V_2) \subseteq Y_2$.

□

Lemma 3.3.4. *[89, Section 4.2.] Let $\mathcal{G}$ be purely infinite and minimal, effective Cantor groupoid. For all $U,V \subset \mathcal{G}^{(0)}$ clopen with $U \neq \mathcal{G}^{(0)}, V \neq \varnothing$ there exists $\alpha \in \mathsf{D}(\mathcal{G})$ with $\alpha(U) \subset V$.*

Proof. First let us show there exists $\alpha' \in \mathsf{F}(\mathcal{G})$ with $\alpha(U) \subset V$ and $U \cup \operatorname{supp}\alpha \neq \mathcal{G}^{(0)}$. If we take W clopen subset of the unit space, containing U and strictly larger than U, with nonempty intersection with V. The groupoid $\mathcal{G}|_C^C$ is again purely infinite and minimal hence by Lemma 3.3.3 there exists $\alpha'' \in \mathsf{F}(\mathcal{G}|_W^W)$ such that $\alpha''(U) \subset V \cap W$. Extending α identically on W^c we prove the first claim.

Using this, we find an $\alpha' \in \mathsf{F}(\mathcal{G})$ with $\alpha(U) \subset V$ and $U \cup \operatorname{supp}\alpha \neq \mathcal{G}^{(0)}$. By Lemma 3.3.3 there is a $\sigma \in \mathsf{F}(\mathcal{G})$ such that $\sigma(U \cup \operatorname{supp}\alpha) \subset (U \cup \operatorname{supp}\alpha)^c$. Then $[\alpha^{-1}, \sigma^{-1}] = \alpha\sigma\alpha^{-1}\sigma(U) \subset V$. □

Lemma 3.3.5. *[89, Section 4.2.] Let $\mathcal{G}$ be an effective, purely infinite and minimal Cantor groupoid. For any $N \subset \mathsf{F}(\mathcal{G})$ normalised by $\mathsf{D}(\mathcal{G})$, and any $\tau \in N$, there is $\tau_1, \tau_2 \in N$ such that their supports are a proper subset of the unit space ($\operatorname{supp}\tau_i \neq \mathcal{G}^{(0)}$) and $\tau = \tau_1\tau_2$.*

Proof. Let us assume $\operatorname{supp}\tau = \mathcal{G}^{(0)}$. Let U be a nonempty subset of $\mathcal{G}^{(0)}$ moved completely by τ, as in $\tau(U), U$ are disjoint with their union not all of $\mathcal{G}^{(0)}$. Let V be such that $V \cap U = V \cap \tau(U) = \varnothing$, and $V \cup U \cup \tau(U) \neq \mathcal{G}^{(0)}$. Let σ_1 be the element of $\mathsf{S}(\mathcal{G})$ given by σ_{τ_U}. Let σ_0 be the element by Lemma 3.3.3 such that $\sigma_(\tau(U)) \subset V$ σ_2 be the element of $\mathsf{S}(\mathcal{G})$ given by $\sigma_{\sigma_0|_{\tau(U)}}$. We have that $[\sigma_2, \sigma_1]|_A = \tau|_A$, $\operatorname{supp}[\sigma_2, \sigma_1] \subset U \cup \tau(U) \cup V$. By Lemma 3.3.4, let $\alpha \in \mathsf{D}(\mathcal{G})$ be such that $\alpha(U \cup \tau(U) \cup V) \subset U$. By assumption,

$$\tau_1 = \alpha^{-1}[\alpha[\sigma_2, \sigma_1]\alpha^{-1}, \tau]\alpha \in N.$$

Moreover:

- The support of $\tau\alpha[\sigma_2, \sigma_1]\alpha^{-1}\tau^{-1}$ is disjoint from U, since

$$\operatorname{supp}(\tau\alpha[\sigma_2, \sigma_1]\alpha^{-1}\tau^{-1}) = \tau\alpha(\operatorname{supp}[\sigma_2\sigma_1]) \subset \tau(\alpha(U \cup \tau(U) \cup V)) \subset \tau(U).$$

- Therefore, for all $x \in U$, $\alpha(x) \in U$, hence $\tau_1(x) = (\sigma\alpha\alpha^{-1})(x) = \sigma(x) = \tau(x)$.

- The support of τ_1 is a proper subset of the unit space ($\alpha^{-1}(U \cup \tau(U))$).

If we let $\tau_2 = \tau_1^{-1}\tau$ we are done. □

Lemma 3.3.6. *[89, Section 4.2.] Let $\mathcal{G}$ be an effective, purely infinite and minimal Cantor groupoid. For any $N \subset \mathsf{F}(\mathcal{G})$ normalised by $\mathsf{D}(\mathcal{G})$, and any $\tau \in N, \alpha \in \mathsf{F}(\mathcal{G}), \alpha\tau\alpha^{-1} \in N$.*

Proof. Let us reduce the proof to the case when τ, α do not have full support. For $\tau \in N, \alpha \in \mathsf{F}(\mathcal{G})$ arbitrary, we have that by the previous Lemma there exists τ_1, τ_2 without full support such that $\tau = \tau_1\tau_2$. It is straightforward to find a decomposition of $\alpha = \alpha_1\alpha_2$ in $\mathsf{F}(\mathcal{G})$ where neither α_i have full support. Then, $\alpha\tau\alpha^{-1} = (\alpha_1\alpha_2\tau_1\alpha_2^{-1}\alpha_1^{-1})(\alpha_1\alpha_2\tau_1^{-1}\alpha_2^{-1}\alpha_1^{-1})$.

So, let us assume that α, τ do not have full support. By Lemma 3.3.3, there is $\sigma \in \mathsf{F}(\mathcal{G})$ such that $\sigma(\mathrm{supp}\alpha) \subset \mathrm{supp}\tau$. Then since $\sigma\alpha\sigma^{-1}$ commutes with τ, $\alpha\tau\alpha^{-1} = [\alpha, \sigma]\tau[\alpha, \sigma]^{-1} \in N$. □

Theorem 3.3.7. *[89, Section 4.2.] Let $\mathcal{G}$ be an effective, purely infinite and minimal Cantor groupoid. For any $N \subset \mathsf{F}(\mathcal{G})$ normalised by $\mathsf{D}(\mathcal{G})$ contains $\mathsf{D}(\mathcal{G})$.*

Proof. Let $\tau \in N$ be nontrivial, then there exists a clopen subset U of the unit space such that $\tau(U)$ is disjoint from U.

Let $\alpha, \beta \in \mathsf{F}(\mathcal{G})$ without full support. By Lemma 3.3.4, we may find γ such that $\gamma(\mathrm{supp}(\alpha))$ is disjoint from $\mathrm{supp}(\beta)$ and $\mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta) \neq \mathcal{G}^{(0)}$. By Lemma 3.3.3, we can find $\sigma \in \mathsf{F}(\mathcal{G})$ such that $\sigma(\mathrm{supp}(\alpha) \cup \mathrm{supp}(\gamma)) \subset U$. By Lemma 3.3.6, $\hat{\tau} = \sigma^{-1}\tau\sigma$ is in N. It is straightforward to see that $\hat{\tau}$ moves $\mathrm{supp}(\alpha) \cup \mathrm{supp}(\gamma)$ completely. Therefore $\hat{\tau}\gamma\hat{\tau}^{-1}$ is the identity on $\mathrm{supp}(\alpha)$. Consider $[\hat{\gamma} = [\gamma, \hat{\tau}]$, an element of N by Lemma 3.3.6, we have that $\hat{\gamma}\mathrm{supp}(\alpha)$ is disjoint from the support of β.

Therefore:

$$[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1} = \alpha(\hat{\gamma}\alpha\hat{\gamma}^{-1})\beta(\hat{\gamma}\alpha\hat{\gamma}^{-1})^{-1}\alpha^{-1}\beta^{-1} = [[\alpha, \hat{\gamma}], \beta] \in N$$

If $\mathrm{supp}\alpha = \mathcal{G}^{(0)} \neq \mathrm{supp}\beta$ then we can find $\alpha_1, \alpha_2 \in \mathsf{F}(\mathcal{G})$ without full support and such that $\alpha = \alpha_1\alpha_2$. By the above proof $[\alpha_i, \beta] \in N$. Hence $[\alpha, \beta] = \alpha_1\alpha_2\beta\alpha_2^{-1}\alpha_1^{-1}\beta^{-1} = \alpha_1[\alpha_2, \beta]\alpha^{-1}[\alpha_1, \beta] \in N$. A similar algebraic manipulation can be made by taking $\beta = \beta_1\beta_2$. □

Remark 3.3.8. *Matui's proof for almost finite groupoids similarly generalises to the non Hausdorff, effective case.*

Remark 3.3.9. *Since simplicity is preserved by unions, this result naturally extends to the non-compact Cantor space, e.g. for ample, second countable effective groupoids whose unit space does not contain any isolated points.*

3.4 Matui's reconstruction theorem

This section aims to prove the following theorem.

Theorem 3.4.1 (Matui's Isomorphism Theorem)**.** *Let $\mathcal{G}$ and $\mathcal{G}'$ be effective Hausdorff minimal Cantor groupoids. Suppose that there exist isomorphic groups $\mathsf{D}(\mathcal{G}) \subset \Gamma \subset \mathsf{F}(\mathcal{G}), \mathsf{D}(\mathcal{G}') \subset \Gamma' \subset \mathsf{F}(\mathcal{G}')$. Then $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic as topological groupoids.*

This theorem fits into a theme of spatial reconstruction theorems such as Rubin's theorem [107], and can be proven using Rubin's theorem.

Definition 3.4.2 (Rubin Action). *Let $G \curvearrowright X$ be a faithful action of a group on a locally compact Hausdorff space with no isolated points such that for each open set $U \subset X$ and each $p \in U$, the closure of the orbit of p of the action of the subgroup of G that stabilises U contains a neighbourhood of p, i.e. $\overline{G_U \ltimes U_p}$ contains a neighbourhood of p. We call such an action a Rubin action.*

Theorem 3.4.3 (Rubin). *Let $G \curvearrowright X, G \curvearrowright Y$ be two Rubin actions. Then there exists a G-invariant homeomorphism $\phi : X \to Y$. [107]*

Remark 3.4.4. *Let us sketch the proof that Matui's isomorphism theorem is related to Rubins theorem. The proof would be in two parts:*

- *Let $\mathcal{G}$ be an effective, Hausdorff, minimal Cantor groupoid. The action $F(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is a Rubin action. Note here it is enough to show that the orbit of any $p \in \mathcal{G}^{(0)}$ under $F(\mathcal{G})$ contains a neighbourhood of p, since the restriction of an effective, Hausdorff, minimal Cantor groupoid to an open compact subset U is again an effective, Hausdorff, minimal Cantor groupoid. Since every orbit is dense in this case, this follows immediately by minimality.*

- *Suppose $\mathcal{G}_1, \mathcal{G}_2$ are two effective, Hausdorff, minimal Cantor groupoids such that $\mathsf{F}(\mathcal{G}_1) \cong \mathsf{F}(\mathcal{G}_2)$. By Rubin's theorem, it follows that there is a $\mathsf{F}(\mathcal{G}_1)-$equivariant homeomorphism $\varphi : \mathcal{G}_1^{(0)} \to \mathcal{G}_2^{(0)}$. But note, by identifying each $\mathcal{G}_i$ with the groupoid of germs of the transformation groupoid $(\mathsf{F}(\mathcal{G}_i) \ltimes \mathcal{G}_i^{(0)})^{germ} \cong \mathcal{G}_i$, φ extends to an isomorphism of groupoids.*

However, we are aiming to prove this result without invoking Rubin's theorem directly, as in [89, Section 3]. To do this we need to study groups which move the Cantor space in a nontrivial way.

Definition 3.4.5 (Class F). *[89, Section 3] Let X be the Cantor space. A subgroup $\Gamma \subset Homeo(X)$ is said to be of class F if:*

1. *For all $\alpha \in \Gamma$ s.t. $\alpha^2 = 1$, the support of α is clopen.*

2. *For all $x \in X$, and all clopen neighborhood $A \subset X$ of x there exists $\alpha \in \Gamma$ such that $x \in supp\alpha \subset A$ and $\alpha^2 = 1$*

3. *For all $\alpha \neq 1$, such that $\alpha^2 = 1$, and nonempty clopen set $A \subset$ there is $\beta \neq 1$ such that $\beta(x) = \alpha(x)$ for all $x \in supp(\beta) \subset A \cup \alpha(A)$.*

4. *For all nonempty clopen sets $A \subset X$ there exists $\alpha \in \Gamma$ such that $supp(\alpha) \subset A$ and $\alpha^2 \neq 1$.*

For $A \subset X$ we denote $\Gamma_A = \{\alpha \in \Gamma : \operatorname{supp}\alpha \subset A\}$.

Lemma 3.4.6. *Let Γ be a group of class F, for two regular closed sets $A, B \subset X$, we have that $\Gamma(A) \subset \Gamma(B)$ iff $A \subset B$ [89, Section 3].*

Proof. The direction $A \subset B \Rightarrow \Gamma(A) \subset \Gamma(B)$ is trivial. To prove the other direction by contradiction assume $A \setminus B$ is nonempty. By regularity, $A \setminus B$ has a nonempty interior. From the first axiom there is an element α with support contained in $A \setminus B$. □

Let $\tau \in \Gamma \subset Homeo(X)$, with $\tau^2 = 1$ and Γ of class F. Let:

$$C_\tau = \{\alpha \in \Gamma : [\alpha, \tau] = 1\},\ U_\tau = \{\sigma \in C_\tau : \sigma^2 = 1, [\sigma, \alpha\sigma\alpha^{-1}] = 1 \forall \alpha \in C_\tau\}$$

$$S_\tau = \{\alpha^2 : \alpha \in \Gamma, [\alpha, \sigma] = 1\ \forall \sigma \in U_\tau\},\ \text{ and }\ W_\tau = \{\alpha \in \Gamma : [\alpha, \beta] = 1,\ \forall \beta \in S_\tau\}.$$

We can show the following:

1. For any $\sigma \in U_\tau, \operatorname{supp}(\sigma) \subset \operatorname{supp}\tau$.

2. For any A clopen in $\operatorname{supp}(\tau)$, there exists $\sigma \in U_\tau$ whose support is in $A \cup \tau(A)$ and $\sigma|_{\operatorname{supp}(\sigma)} = \tau|_{\operatorname{supp}(\sigma)}$.

3. For any clopen subset A with nonempty intersection with $\operatorname{supp}(\tau)$, there exists $\alpha \in S_\tau$ such that $\varnothing \neq \operatorname{supp}(\alpha) \subset A$.

4. For all $\alpha \in S_\tau$, $\operatorname{supp}\alpha \subset (\operatorname{supp}\tau)^c$.

5. $W_\tau = \Gamma(\operatorname{supp}(\tau))$.

Let us give an outline of the proof of these facts:

1. If $\sigma \in U_\tau$ and the support is outside the support of τ, there is a clopen subset A in the support σ and outside the support of τ. Then using property F we can find $\alpha \in \Gamma(A)$ that permutes A in a nontrivial way such that $[\sigma, \alpha\sigma\alpha^{-1}] = 1$.

2. If A is clopen by property F there is $\alpha \in C_\tau$ agreeing with τ on $A \cup \tau(A)$ and the identity outside. It is easy to verify that $\alpha \in U_\tau$ directly.

3. If A is clopen with nonempty intersection with $\operatorname{supp}\tau$, using property F there is an element α whose support is in A and does not have order 2. Then α^2 is the desired element.

4. If α commutes with elements in U_τ, then for all x in the support of τ, we claim $\alpha(x)$ is either x or $\tau(x)$. If we assume by contradiction that $x, \alpha(x), \tau(x)$ are all different. Then there is a neighbourhood A of x whose images are all different. Then by 2. we can find $\sigma \in U_\tau$ acting like τ on $A \cup \tau(A)$. This element does not commute with α.

5. Using 4. the inclusion $\Gamma(\mathrm{supp}\tau) \subset W_\tau$ is clear. The other inclusion can be shown by contradiction if $\sigma \in W_\tau$ and it moves a clopen subset A, outside of suppτ completely, then by 3. one can find an element of $\beta \in S_\tau$ whose support is in A. Well then, $\beta \neq \alpha\beta\alpha^{-1}$.

See the proof of [89, Lemma 3.3] for further details. We use the above claims to show the following:

Lemma 3.4.7. *Let $\Gamma_i \subset Homeo(X_i)$ $i = 1,2$ be subgroups of class F acting on Cantor sets X_i. Let $\phi : \Gamma_1 \to \Gamma_2$ be an isomorphism. Let τ, σ be elements of Γ_1 of order 2 [89, Section 3].*

- $supp(\tau) \subset supp(\sigma) \iff supp(\phi\tau) \subset supp(\sigma)$.

- *The support of τ, σ are disjoint iff the support of $\phi\tau, \phi\sigma$ are disjoint.*

- Note that $\phi W_\tau = W_{\phi\tau}$. From Lemma 3.4.6 and the above claim 5) we have that:

$$\mathrm{supp}(\tau) \subset \mathrm{supp}(\sigma) \Rightarrow \Gamma_1(\mathrm{supp}\tau) \subset \Gamma_2(\mathrm{supp}\sigma) \Rightarrow W_\tau \subset W_\sigma \Rightarrow \phi(W_\tau) \subset \phi(W_\sigma).$$

  The other direction follows similarly.

- Suppose the supports are not disjoint. By property F, we have that the intersection of their supports have nonempty interior, and therefore there is an element ρ of Γ with order 2 whose support is contained in their intersection. Applying 1) to this element we see that $\phi(\rho) \subset \mathrm{supp}(\phi(\tau)) \cap \mathrm{supp}(\phi(\sigma))$.

We then are ready to prove a spatial realisation result for class F groups.

Theorem 3.4.8. *Let $\Gamma_i \subset Homeo(X_i)$ $i = 1,2$ be subgroups of class F acting on Cantor sets X_i. Let $\phi : \Gamma_1 \to \Gamma_2$ be an isomorphism. Then there is a homeomorphism $\varphi : X_1 \to X_2$ such that $\phi(\alpha) = \varphi\alpha\varphi^{-1}$ [89, Section 3].*

Proof. For $x \in X_i$ let $T(x)$ be the elements of order 2 whose support contains x. We claim that for all $x \in X_1$ the set $P(x) = \bigcap_{\tau \in T(x)} \mathrm{supp}\Phi(\tau)$ is a singleton.

Let $\tau_1, \ldots, \tau_n$ be elements of $T(x)$. We can find a clopen neighbourhood A of x in the support of all of the τ_i. By property F we can find an element of $T(x) \cap \Gamma(A)$. By the above Lemma, supp$\phi\tau$ is contained in all the sup$\phi\tau_i$ and so they have nonempty intersection. By compactness then, $P(x)$ is nonempty.

Assume it contains two distinct points p_1, p_2. By property F there is an element σ of $T(p_1)$ such that p_2 is not in its support. There are two possibilities now:

- $x \notin \mathrm{supp}\phi^{-1}(\sigma)$, in which case due to property F we can find an element $\tau \in T(x)$ whose support is disjoint from the support of the preimage of σ. Considering this under ϕ, through the above lemma we have that $\phi(\tau)$ has disjoint support from σ, a contradiction because p_1 is in both.

- $x \in \text{supp}\phi^{-1}(\sigma)$, then $\phi^{-1}(\sigma)$ is in $T(x)$, so $P(x)$ is in $\text{supp}(\sigma)$, a set which p' is not in, a contradiction.

We can therefore define a map $\varphi : X_1 \to X_2$ which sends x to the unique element of $P(x)$. In the exact same way we can construct a map $\varphi' : X_2 \to X_1$. It is clear that as a set φ' is the inverse of φ, since

$$\{\varphi'\varphi(x)\} = \bigcap_{\sigma \in T(\varphi(x))} \subset \bigcap_{\tau \in T(x)} \text{supp}\phi^{-1}\phi(\tau) = \{x\}.$$

Therefore it is enough to show that φ is continuous. Let $x \in X$ and A be an open neighbourhood of $\varphi(x)$. By definition of $P(x)$ there are $\tau_1, \ldots, \tau_n$ such that $\bigcap_{i \leq n} \text{supp}\phi(\tau_i) \subset A$. As before, we can find $\tau \in T(x)$ whose support is a subset of $\bigcap_{i \leq n} \text{supp}\phi(\tau_i) \subset A$. Then $\text{supp}(\tau)$ is a clopen neighbourhood of x and $\phi(\text{supp}(\tau)) \subset \text{supp}\phi\tau \subset A$. It is straightforward to verify $\phi\alpha = \varphi\alpha\varphi^{-1}$. □

The only thing left to prove is that any subgroup Γ of the derived subgroup of certain topological full groups is indeed of class F.

The most complex part of this is to understand the support of elements of topological full groups to verify the first condition of class F. The following Lemma is useful, appearing as [95, Lemma 4.3].

Lemma 3.4.9. *Let X be a locally compact Hausdorff space, $\Gamma \subset Homeo(X)$. The following are equivalent:*

- *$(\Gamma \ltimes X)^{germ}$ is a Hausdorff groupoid.*
- *Every element of Γ has clopen support.*

Proof. Note by Hausdorffness, it is only plausible that two elements of the form $(g,x),(h,x) \in (\Gamma \ltimes X)^{germ}$ might not be separated by an open set. In fact, one can show algebraically that this reduces further, it is enough to understand elements of the form $(g,x),(1,x)$ which might not be separated by an open set.

Assume all elements have clopen support. If $(g,x) \neq (1,x)$ we have that $(g,\text{supp}g),(1,\text{supp}g)$ are disjoint open neighbourhoods.

Assume there is a g without open support. Let x be on the boundary of its support. Then we aim to show $(g,x),(1,x)$ cannot be separated. Take any basic open neighborhoods $(g,U),(1,V)$ where U,V are open. Well $C = (A \cap B) \setminus \text{supp}g \subset U \cap V$ is a nonempty open set. □

Since we may readily identify $\mathcal{G}$ with $(\mathsf{F}(\mathcal{G}) \ltimes \mathcal{G}^{(0)})^{germ}$, provided $\mathcal{G}$ is effective, then we have the following.

Corollary 3.4.10. *Let $\mathcal{G}$ be effective Cantor groupoid. The following are equivalent.*

- *$\mathcal{G}$ is Hausdorff.*

- $\forall \alpha \in \mathsf{F}(\mathcal{G})$, *supp* α *is clopen.*

Lemma 3.4.11. *[89, Section 3] Let $\mathcal{G}$ be an effective Hausdorff minimal Cantor groupoid. Any subgroup $\mathsf{D}(\mathcal{G}) \subset \Gamma \subset \mathsf{F}(\mathcal{G})$ is of class F.*

Proof.
- 1. Satisfied by the above Lemma.
- 2. Let A be clopen neighborhood of $x \in \mathcal{G}^{(0)}$. Since $\mathcal{G}$ is minimal, there is a compact open bisection U with $x \in s(U), s(U0 \cup r(U) \subset A$ and $s(U) \cap r(U) = \varnothing$. Take γ_U as in the definition of the symmetric group. Similarly we can find V such that $x \in V$, $s(V) \cup r(V) \subset s(U)$ and $s(V), r(V)$ are disjoint. Let γ_V be the element as in the symmetric group. Then $[\gamma_U, \gamma_V] = \gamma_{U,V}$ is an element proving condition 2.
- 3. Let $\alpha = \gamma_U$ be as in 3. Let V be a bisection such that $s(V) \cup r(V) \subset A$, and the four subsets $s(V), r(V), \alpha(s(V)), \alpha(r(V))$ are disjoint. Consider γ_V and $\gamma_{\hat{U}}$ where $\hat{U} = U_{s(V) \cup \alpha s(V)}$. Then, $[\gamma_{\hat{U}}, [\gamma_U, \gamma_V]]$ satisfies the requirement.
- 4. Since $\mathcal{G}$ is minimal, there are nonempty compact open bisections U_1, U_2 such that $s(U_i) \cup r(U_i) \subset A$, $s(U_i) \cap r(U_i) = \varnothing$ $s(U_i) \cap r(U_i) = \varnothing$ and $s(U_i) \cap r(U_i) = \varnothing$ then γ_{U_1,U_2} is well defined and of order 3.

□

This concludes the proof of the main theorem of this section, Lemma 3.4.11 and Lemma 3.4.8 combine to make Matui's isomorphism theorem. This theorem was later generalised by Nyland-Ortega.

Theorem 3.4.12. *[95, Theorem A] Say $\mathcal{G}_1, \mathcal{G}_2$ are minimal effective Hausdorff ample groupoids whose unit space have no isolated points. The following are equivalent:*

- $\mathsf{F}(\mathcal{G}_1) \cong \mathsf{F}(\mathcal{G}_2)$ *as abstract groups.*
- $\mathsf{D}(\mathcal{G}_1) \cong \mathsf{D}(\mathcal{G}_2)$ *as abstract groups.*
- $\mathcal{G}_1 \cong \mathcal{G}_2$ *as ample groupoids.*

Furthermore, they were able to show a version of this theorem which weakens minimality to a nonwandering condition. A subset $A \subset \mathcal{G}^{(0)}$ is called wandering if $|A \cap Orb_{\mathcal{G}}(x)| = 1$ for all $x \in A$.

Definition 3.4.13. *[95] Let $\mathcal{G}$ be an ample groupoid. We say that $\mathcal{G}$ is nonwandering if each $\mathcal{G}$-orbit has length at least 3, and there are no nonempty clopen wandering subsets of $\mathcal{G}^{(0)}$.*

Theorem 3.4.14. *[95, Theorem B] Say $\mathcal{G}_1, \mathcal{G}_2$ are nonwandering effective Hausdorff ample groupoids whose unit space have is a (possibly noncompact) Cantor space. The following are equivalent:*

- $\mathsf{F}(\mathcal{G}_1) \cong \mathsf{F}(\mathcal{G}_2)$ *as abstract groups.*
- $\mathcal{G}_1 \cong \mathcal{G}_2$ *as ample groupoids.*

Let us end this section by noting that the removal of the Hausdorff condition seems to be subtle due to Corollary 3.4.10. This has however been studied by Nekrashevych, in a proof which involves invoking Rubin's theorem to obtain the following stronger result in the (not necessarily Hausdorff case).

Theorem 3.4.15 (Nekrashevych)**.** *Let $\mathcal{G}_1, \mathcal{G}_2$ be minimal effective Cantor groupoids. The following are equivalent:*

- $\mathsf{F}(\mathcal{G}_1) \cong \mathsf{F}(\mathcal{G}_2)$ *as abstract groups.*
- $\mathsf{D}(\mathcal{G}_1) \cong \mathsf{D}(\mathcal{G}_2)$ *as abstract groups.*
- $\mathsf{A}(\mathcal{G}_1) \cong \mathsf{A}(\mathcal{G}_2)$ *as abstract groups.*
- $\mathcal{G}_1 \cong \mathcal{G}_2$ *as ample groupoids.*

3.5 Finite generation

3.5.1 Groupoid expansivity

Nekrashevych showed that the alternating group of a topological full group is finitely generated if the underlying ample groupoid has a technical condition known as expansivity. We recall the definition of expansivity:

Definition 3.5.1 (Expansive [93])**.** *For an étale Cantor groupoid $\mathcal{G}$:*

- *A compact set $K \subset \mathcal{G}$ is called a compact generating set if $\mathcal{G} = \bigcup_{n\in\mathbb{N}} (K \cup K^{-1})^n$.*
- *A finite cover $\mathcal{B} = \{B_i\}_{i=1}^N$ of bisections is called expansive if $\bigcup_{n\in\mathbb{N}} (\mathcal{B} \cup \mathcal{B}^{-1})^n$ forms a basis for the topology of $\mathcal{G}$.*
- *$\mathcal{G}$ is called expansive if there is a compact generating subset K with an expansive cover $\mathcal{B}$.*

In Nekrashevych's paper [93], Nekrashevych showed that this notion of expansivity is related to the finite generation of $\mathsf{A}(\mathcal{G})$.

Theorem 3.5.2. *Let $\mathcal{G}$ be an expansive Cantor groupoid with infinite orbits. Then $\mathsf{A}(\mathcal{G})$ is finitely generated [[93], Theorem 5.6].*

We do not prove this result in this subsection. However, in the subsequent subsection we exhibit the proof for the specific case of certain transformation groupoids. Instead our focus now is on discussing on the verification of expansivity of étale groupoids, which can be a subtle problem.

This notion of expansivity generalises the notion of expansivity for (finitely generated) group actions.

Definition 3.5.3 (Expansive action [93])**.** *Let $\alpha : G \curvearrowright X$ be an action. We say that α is expansive if there exists $\varepsilon > 0$ such that for all $x, y \in X$, $x \neq y$, there exists $g \in G$ such that $d(gx, gy) > \varepsilon$.*

Nekrashevych showed in [93] that a transformation groupoid of a finitely generated group acting on the Cantor space is expansive iff the underlying action is expansive.

Lemma 3.5.4. *Let $G \ltimes X$ be a transformation groupoid of a discrete group on the Cantor space X. Then $G \ltimes X$ is compactly generated iff G is finitely generated.*

Proof. $\Leftarrow$ If G is finitely generated, then it has a generating set $g_1, .., g_n$. Consider the compact set $K = \sqcup_{i \in I}(g_i, X)$. This clearly will then generate the groupoid.
$\Rightarrow$ If $G \ltimes X$ is compactly generated, there exists a compact subset K generating the groupoid. But this groupoid is ample; it's topology is generated by compact open bisections. Therefore there exists a finite set of group elements $\{g_i\}_{i=1}^n$ subsets $\{Y_i\}_{i=1}^n$ such that $K = \sqcup_{i=1}^n (g_i, Y_i)$. Then, $K \subset K' = \sqcup_{i=1}^n (g_i, X)$, so K' is a compact generating set. Hence, $\{g_i\}_{i=1}^n$ is a finite generating set of G. □

Lemma 3.5.5. *Let $\mathcal{G}$ be an essentially principal ample groupoid with infinite orbits. Then $\mathsf{A}(\mathcal{G})$ is finitely generated $\implies$ $\mathcal{G}$ is compactly generated.*

Proof. Let $g \in \mathcal{G}$. It is enough to show that there exists some bisection $B \in \mathsf{A}(\mathcal{G})$ such that $g \in B$, since then the generating set of $\mathsf{A}(\mathcal{G})$ also serves as a compact generating set for $\mathcal{G}$. If $s(g) \neq r(g)$, by ampleness, we have there exists some compact open bisection $\hat{B}_1$ such that $g \in \hat{B}_1$. By restricting the source of $\hat{B}_1$ if necessary, we may assume that $s(\hat{B}_1), r(\hat{B}_1)$ are disjoint with $s(\hat{B}_1) \bigcup r(\hat{B}_1) \neq \mathcal{G}^{(0)}$. By minimality, we have there exists some $h \in \mathcal{G}$ with $s(h) = r(g)$ and $r(h) \in (s(\hat{B}_1) \bigcup r(\hat{B}_1))^c$ Then by ampleness, there exists some compact open bisection B_2 with $h \in B_2$. Again, by restricting $s(B_2)$ if necessary, we can assume $s(B_2) \subset r(\hat{B}_1)$ and $r(B_2) \subset (s(B_2) \bigcup \hat{B}_1^{-1}(s(B_2))^c$. Now let $B_1 = \hat{B}_1|_{\hat{B}_1^{-1} s(B_2)}$. Then

$$\gamma_{B_1,B_2} = B_1 \sqcup B_2 \sqcup (B_1 B_2)^{-1} \cup (\mathcal{G}^{(0)} \setminus s(B_1) \cup s(B_2) \cup r(B_2)) \in \mathsf{A}(\mathcal{G})$$

satisfies $g \in \gamma_{B_1,B_2}$.
Otherwise $s(g) = r(g)$. Then, there exists g_1, g_2 such that $s(g_i) \neq r(g_i)$ but $g_1 g_2 = g$. Hence by the above argument, there exists $\gamma_1, \gamma_2 \in \mathsf{A}(\mathcal{G})$ such that $g_1 \in \gamma_1, \gamma_2 \in B_2$ and hence $g = g_1 g_2 \in \gamma_1 \gamma_2 \in \mathsf{A}(\mathcal{G})$. □

Remark that the converse to Lemma 3.5.5 fails, an explicit counterexample would be any $\mathbb{Z}$-odometer. The above proof is inspired by the proof of [[89], Lemma 3.7]. Combining the previous two Lemmas, we obtain the following Corollary.

Corollary 3.5.6. *Let $\alpha : G \curvearrowright X$ be an expansive, minimal, essentially free action of a discrete countable group on the Cantor set. Then G is finitely generated $\iff$ $\mathsf{A}(G \ltimes X)$ is finitely generated.*

Proof. $\Leftarrow$ Is a combination of Lemma 3.5.5 and Lemma 3.5.4.
$\Rightarrow$ Follows via Theorem 3.5.2. □

Let us also remark a possible application to the algebraic world.

Remark 3.5.7. *Let $\mathcal{G}$ be an ample groupoid such that every orbit has at least two elements. Then the convolution algebra $\mathscr{C}(\mathcal{G})$ is the algebraic closure of the characteristic function on full group elements and subsets of $\mathcal{G}^{(0)}$:*

$$\mathscr{C}(\mathcal{G}) = \overline{\{1_B, 1_U \,:\, U \subset \mathcal{G}^{(0)}, B \in \mathsf{F}(\mathcal{G})\}}$$

Proof. It is enough to show that for all compact open bisections B' and all $u \in s(B')$, there exists a full bisection $\mathsf{F}(\mathcal{G})$ and a compact open neighbourhood U of u such that $B|_U = B'_U$.

- If $B'u \neq u$ then by continuity of B' and Hausdorffness of $\mathcal{G}^{(0)}$ we have there exists a compact open neighbourhood U of u such that U and $r(B'U)$ are disjoint. Then we can take $B = B'|_U \sqcup B'^{-1}|_{r(B'U)} \sqcup \mathcal{G}^{(0)} \setminus U \cup r(B'(U))$.
- Otherwise, there exists $g \in \mathcal{G}$ such that $gu \neq u$. By ampleness, there exists compact open bisections B'_1 such that $g \in B'_1$. By the first bullet point then, there exists full bisections B_1, B_2 and open neighbourhoods U_1, U_2 of $u, g(u)$ such that $B'_1|_{U_1} = B_1|_{U_1}$ and $B_2|_{U_2} = B'B_1'^{-1}|_{U_2}$. Hence $B = B_2 B_1$ agrees with B' on $U_1 \cap B'^{-1}(U_2)$.

□

This might be interesting to be able to write a presentation of associated Steinberg algebras in the case when $\mathsf{F}(\mathcal{G})$ has good finiteness properties.

3.5.2 Expansive actions of finitely generated abelian groups

This subsection aims to describe the generating set of subshifts of finitely generated abelian groups. First let us recall for a subshift the definition of a patch.

Definition 3.5.8. *[36] Let G be an abelian group. Let $\mathcal{A}$ be a finite alphabet and $X \subset \mathcal{A}^G$ be a closed G-shift invariant subset. Consider the subshift (G, X).*

- *A patch is a map* $\pi : S \to \mathcal{A}$ *where* $\{s_1,...,s_n\} = S \subset G$ *is a finite subset.*
- *We say an element* $x \in X$ *contains the patch* π *if* $x|_S = \pi$.
- *With each patch* π *we associate a cylinder set* $W_\pi := \{x \in X : x|_S = \pi\}$.
- *For each patch* π *we say the transformation* T_π *is well defined if* W_π *is nonempty and* $\{sW_\pi\}_{s\in S}$ *are pairwise disjoint.*
- *If* T_π *is well defined, then* $T_\pi \in Homeo(X)$ *is the homeomorphism of order 3 cyclically permuting the* sW_π *in the canonical way i.e.*

$$T_\pi(x) = \begin{cases} (s_{i+1} - s_i)x & x \in s_i W_\pi,\ i \le n-1 \\ (s_1 - s_n)x & x \in s_n W_\pi \\ x & else \end{cases}.$$

- *We say that two patches* $\pi_1 : S_1 \to \mathcal{A}, \pi_2 : \mathcal{S}_2 \to \mathcal{A}$ *are compatible if there exists* $x \in X$ *containing* π_1 *and* π_2*. In other words, if* $W_{\pi_1} \cap W_{\pi_2}$ *is nonempty. In which case, we can define their union,* $\pi_1 \cup \pi_2 : S_1 \cup S_2 \to \mathcal{A}$.

Remark 3.5.9. *If* $\pi : S \to \mathcal{A}$ *is a patch, then* W_π *is equal to the disjoint union of all* $W_{\hat{\pi}}$ *where* $\hat{\pi}|_S = \pi$.

Remark 3.5.10. *If* $\pi : S \to \mathcal{A}$ *is a patch, and* $g \in G$*, then we define* $\pi + g : S + g \to \mathcal{A}$ $(\pi + g)(s) = \pi(s - g)$*. Then we have that* $g(W_\pi) = W_{\pi+g}$.

Definition 3.5.11. *[36] Let* $X \subset \{0,1\}^G$ *a G subshift and a patch* $\pi^i : \{0, g, g'\} \to \{0,1\}$*, we say* T_π *is well defined if* $W_\pi, gW_\pi, g'W_\pi$ *are nonempty and pairwise disjoint. In this case* T_π *is the homeomorphism of X given by*

$$T_\pi(x) = \begin{cases} x+g & x \in W_\pi \\ x+g'-g & x \in gW_\pi \\ x-g' & x \in g'W_\pi \\ x & else \end{cases}.$$

Theorem 3.5.12 (Nekrashevych-Juschenko-Chornyi)**.** *Let* $X \subset \{0,1\}^G$ *be a G-subshift, where G is a finitely generated abelian group with generators* $e_1,...,e_d$*. Then the derived subgroup of the groupoid* $\mathcal{G}_X$*,* $D(\mathcal{G}_X)$ *is generated by* T_{π^i}*, where* $i = 1,...,d$ *and* π^i *ranges over all* π^i *such that* T_{π^i} *is well defined. [[36], Proposition 8]*

Note that as stated in [36], the Theorem is only for the case $G = \mathbb{Z}^d$. However, extending this proof for arbitrary finitely generated abelian groups is relatively straightforward, as explained in the proof of [[36], Proposition 18]. For completeness, this subsection aims to prove the result as stated above.

Lemma 3.5.13. *[36] Let G be a finitely generated abelian group with respect to a finite generating set S. Let $A \subset G$ be a subset not containing the identity and such that not all elements of A have finite order. Then there exists $B \subset G$ such that for all $x \in X$ the patches $(x|_B, B)$ and $((x+g)|_B, B)$ are not compatible.*

Proof. Let us consider the ball metric on X, that is the distance between w, v is equal to 2^r where r is the largest number such that the restrictions of w, v to the ball of radius r (that is, $\prod_{i=1}^{n} S \cup S^{-1}$ in G coincide.

Suppose for contradiction that for all $\varepsilon > 0$ there exists $x \in X, g \in A$ such that $d(g(x), x) < \varepsilon$. Since A is finite, this implies there is a $g \in A$ and a sequence $x_n \in X$, such that $d(g(x_n), x_n) \to 0$, i.e. g has a fixed point. This is a contradiction. □

We need the following result about permutations, appearing as Lemma 5 of [36].

Lemma 3.5.14. *[36] Let $U_1 . U_2, U_3, V_1, V_2, V_3$ be subsets of X such that only U_1, V_1 have nonempty intersections, otherwise each set is pairwise disjoint. Let a be a permutation of X of order 3 acting trivially outside of $\bigcup_{i=1}^{3} A_i$, such that $a(U_1) = U_2, a(U_2) = U_3, a(U_3) = U_1$. Similarly, let b act the same way on the V_i.*

Then

$$[[b^{-1}, a^{-1}], [b, a]] = a|_{(A_1 \cap B_1) \cup a(A_1 \cap B_1) \cup a^2((A_1 \cap B_1))}$$

Proof. This proof is straightforward, and works very similarly to considering $a = (1,2,3)(4,5,6)$ in $b = (1,7,8)$ in A_8. See [36] for a rigorous proof. □

Corollary 3.5.15. *[36] Let π_1, π_2 be patches $g_1, g_2, h_1, h_2 \in \Gamma$ such that $\pi_1, \pi_1 + g_i, \pi_2, \pi_2 + h_i$ are incompatible apart from π_1, π_2. Then:*

$$[[T_{\pi_2}^{-1}, T_{\pi_1}^{-1}], [T_{\pi_2}, T_{\pi_1}]] = T_{\pi_1 \cup \pi_2}$$

Here, π_1 is associated to $(0, g_1, g_2), \pi_2$ to $(0, h_1, h_2)$ and $\pi_1 \cup \pi_2$ to $(0, g_1, g_2)$

Let G be a finitely generated abelian group with respect to a finite generating set S. Let us have the metric on X as in the proof of Lemma 3.5.13. By Lemma 3.5.13 there is r_1 such that for any $x \in \mathcal{A}^G$ the patches $\pi = (B(r_1), x|_{B(r_1)})$ and $\pi + g$ are incompatible, For all $g \in G$ with length less than or equal to 3.

Let T_r be the elements of the topological full group of the form T_π as π runs through all patches of the form $\pi = (B(r), x|_{B(r)})$ for $x \in \mathcal{A}^G$.

Lemma 3.5.16. *[36] Let $r > r_1 + 2$. Then the group generated by T_r contains T_{r+1}*

Proof. Let S be the symmetric generating set of G. We have that $B(r+1) = \cup_{h\in S} B(r+h)$. It is therefore enough to show that the group generated by:

$$T_{(\pi,0,g,-g)} \quad g \in S,\ (\pi = x|_B, B),\ x \in \mathcal{A}^G$$

Contains all elements of the form

$$T_{(\pi,0,g,-g)} \quad g \in S,\ \pi = (x|_{B\cup B+h}, B \cup B+h)\ x \in \mathcal{A}^G, h \in S.$$

We do this by applying Corollary 3.5.15 twice:

- First for $\pi_1 = (x|_B, B), \pi_2 = (x|_{B+h}, B+h)$ $g_1 = g, g_2 = -g$, $h_1 = h, h_2 = 2h$ with $g \neq h$. This obtains $T_{\rho,(0,g,-g)}$ where $\rho = \pi_1 \cup \pi_2$.

- Secondly we apply to $\pi_1' = (x|_{B-g}, B-g), \pi_2' = (x|_{B-2g}, B-2g)$ $g_1' = -2g, g_2' = -g$, $h_1' = h, h_2' = -h$. This obtains $T_{\pi_1'\cup\pi_2',(0,-2g,g)}$. Note that this is the same as $T_{\rho,(0,g,-g)}$ where $\rho = \pi_1 \cup \pi_2$ as before.

This shows the desired claim. □

But note that the above Lemma shows that T_{r_1+2}, a finitely generated group, contains T_r for all $r > r_1 + 2$. Therefore, T_{r_1+2} is finished by the proposition below.

Lemma 3.5.17. *[36] The derived subgroup of the full group of the action of a finitely generated abelian group $G = \langle S \rangle$ on X is generated by the set of elements of the form $T_{\pi,(0,h,-h)}$ where $h \in S$.*

Proof. We know already that $\mathsf{D}(G \ltimes X) \cong \mathsf{A}(G \ltimes X)$ is simple in this case (see Theorem 3.3.2). Therefore $\mathsf{D}(G \ltimes X)$ is generated by elements of order three, σ, cyclically permuting three disjoint clopen subsets U_1, U_2, U_3 in X.

Given such a σ, there exists partitions of each U_i into cylindrical sets such that T maps each piece of the partitions equal the restriction of an element of G. In other words, σ is a finite product of elements of the form $T_{\pi(g_1,g_2,g_3)}$.

We remain to show that any element of the form $T_{\pi,(g_1,g_2,g_3)}$ is generated by elements of the form $T_{\pi,(0,h,-h)}, h \in S$. We may assume that g_1, g_2, g_3 are in the same direct factor in the decomposition of G into cyclic groups, since A_n is generated by the three cycles $(k, k+1, k+2)$. In other words, this proof reduces to the cyclic case. In fact, one can see that this proof reduces to a nontrivial fact about the classical symmetric groups.

Claim: Let $X^d = \{x_1, \ldots, x_d\ : x_i \in \{0,1,2\}\}$ be the 3^d element set of d letter words over the alphabet $\{0,1,2\}$. Let S_{X_d}, A_{X_d} be the symmetric and alternating group on X_d. Let B_d be the set of all elements of the form $(\mu 0 \nu, \mu 1 \nu, \mu 2 \nu)$ where μ, ν are (possibly empty) words such that $|\mu| + |\nu| = d - 1$. Then A_{X_d} is generated by B_d.

Proof of Claim Let us prove this by induction. If $d = 1$, we are simply discussing A_3 which is trivially generated by $(0,1,2)$. Assume true for d, then for $d+1$ it is enough to show that every 3 cycle is generated by B_{d+1}. Let us have a cycle $(\mu x, \nu y, \alpha z)$ where $\mu, \nu, \alpha \in X_d$ are pairwise distinct and $x, y, z \in \{0,1,2\}$. We know that:

- Elements of the form $(\mu x, \nu x, \alpha x), (\mu y, \nu y, \alpha y)(\mu z, \nu z, \alpha z)$ are generated by B_{k+1} (e.g. by taking the elements of B_k generating (μ, ν, α) and appending x.
- Elements of the form $(\mu x, \mu y, \mu z)$ (and similarly for ν, α) are in by definition of B_{k+1}

Therefore $(\mu x, \nu y, \alpha z)$ is in the group generated by B_{k+1}. The case where μ, ν, α are not pairwise disjoint can be deduced similarly, however with the subtlety that if say $\mu = \nu$ then $x \neq y$. This completes the proof of the claim and therefore of the proof. □

The above Lemma completes the proof as required.

3.6 Homology of topological full groups

Recally as in Section 2.4. the notion of homology for ample groupoids. In this section, as throughout, where coefficients are ommitted we mean to take coefficients in the integers, $\mathbb{Z}$. In this context we use the same notation H_* for the homology of groups and of groupoids, since the homology theory of groupoids generalises that of groupoids, as seen in Section 2.4.

In 2022, Li launched an enquiry into the question:

Question 3.6.1. *What homological information moves from $\mathcal{G}$ to $\mathsf{F}(\mathcal{G})$?*

His approach was to look to generalise techniques used by Szymik and Wahl [118] in studying the homology of the Higman-Thompson groups.

First, Li was able to phrase the integral homology of an ample groupoid with that of certain invariants arising from algebraic topology. Specifically, in [79, Theorem A], Li establishes that the groupoid homology of an ample groupoid with locally compact Hausdorff unit space (with no isolated points) can be identified with the reduced homology of the algebraic K-theory spectrum $\mathbb{K}(\mathscr{B}_{\mathcal{G}})$ of a small permutative category of bisections $\mathscr{B}_{\mathcal{G}}$ constructed in section 3 of the paper, which allows us to describe groupoid homology in terms of the algebraic K-Theory spectrum;

$$H_*(\mathcal{G}) = \hat{H}_*(\mathbb{K}(\mathscr{B}_{\mathcal{G}})).$$

If one assumes a little more, namely that our groupoid is minimal and has comparison, Li establishes a connection here with the group homology of the topological full group, here $\Omega_0^\infty \mathbb{K}(\mathscr{B}_{\mathcal{G}})$ refers to the connected component of the base point in the infinite loop space associated to $\mathbb{K}(\mathscr{B}_{\mathcal{G}})$ giving rise to an identification

$$H_*(\mathsf{F}(\mathcal{G})) = H_*(\Omega_0^\infty \mathbb{K}(\mathscr{B}_{\mathcal{G}})).$$

This forms the content of [79, Theorem B]. The combination of these two theorems establishes a deep relationship between groupoid homology and the group homology of a topological full group. Groupoid homology is relatively computable compared to the group homology of topological full groups, which is somewhat mysterious.

Because of this relationship between groupoid homology and homology of topological full groups, Li was able to establish several interesting transfers of homological information which will be highly applicable in this text. In most areas of homology, rational homology is often more accessible than integral homology. This is no exception, as demonstrated by the below Corollary appearing as part of [79, Corollary C].

Corollary 3.6.2. *[79] Let $\mathcal{G}$ be an ample groupoid. Let us define*

$$H_*^{odd}(\mathcal{G},\mathbb{Q}) = \begin{cases} H_*(\mathcal{G},\mathbb{Q}) & * > 0 \text{ odd} \\ 0 & \text{else} \end{cases}, \quad H_{*>1}^{odd}(\mathcal{G},\mathbb{Q}) = \begin{cases} H_*(\mathcal{G},\mathbb{Q}) & * > 1 \text{ odd} \\ 0 & \text{else} \end{cases},$$

and,

$$H_*^{even}(\mathcal{G},\mathbb{Q}) = \begin{cases} H_*(\mathcal{G},\mathbb{Q}) & * > 0 \text{ even} \\ 0 & \text{else} \end{cases}.$$

Let $\mathcal{G}$ be a minimal ample groupoid with locally compact Hausdorff unit space (with no isolated points) and comparison. Then,

$$H_*(\mathsf{F}(\mathcal{G}),\mathbb{Q}) \cong Ext(H_*^{odd}(\mathcal{G},\mathbb{Q})) \otimes Sym(H_*^{even}(\mathcal{G},\mathbb{Q})),$$

and,

$$H_*(\mathsf{D}(\mathcal{G}),\mathbb{Q}) \cong Ext(H_{*>1}^{odd}(\mathcal{G},\mathbb{Q})) \otimes Sym(H_*^{even}(\mathcal{G},\mathbb{Q}))$$

as graded vector spaces over $\mathbb{Q}$.

Here Ext, Sym denote respectively the Exterior and Symmetric algebras in the sense of Multilinear Algebra [54].

Another transfer of homological information shown in [79] was noticing that if the groupoid homology vanishes, so does the group homology of the topological full group. This works both integrally and rationally.

Corollary 3.6.3 (Vanishing Integral Homology)**.** *[79] Let $\mathcal{G}$ be a minimal ample groupoid with locally compact Hausdorff unit space (with no isolated points) and comparison. Suppose that for some $k \in \mathbb{N}$ we have that $H_*(\mathcal{G}) = 0, * = 0,1,\dots,k-1$. Then, $H_*(\mathsf{F}(\mathcal{G})) = 0, * = 1,\dots,k-1$ and $H_k(\mathsf{F}(\mathcal{G})) = H_k(\mathcal{G})$.*

The above result appears as [79, Corollary D].

Corollary 3.6.4 (Vanishing Rational Homology)**.** *[79] Let $\mathcal{G}$ be a minimal ample groupoid with locally compact Hausdorff unit space (with no isolated points) and comparison. Suppose that $\mathcal{G}$ is rationally acyclic i.e. $H_*(\mathcal{G},\mathbb{Q}) = 0$ in all degrees. Then, $\mathsf{F}(\mathcal{G})$ is rationally acyclic i.e. $H_*(\mathsf{F}(\mathcal{G}),\mathbb{Q}) = 0$ in all degrees.*

The above result appears as part of [79, Corollary C]. Finally, Li was able to show an important stability result for the homology of topological full groups. The below result appears as [79, Theorem F]

Theorem 3.6.5 (Homological Stability)**.** *[79] Let $\mathcal{G}$ be a minimal ample groupoid with locally compact Hausdorff unit space (with no isolated points) and comparison. Then for all nonempty compact open subspaces $U \subset V \subset \mathcal{G}^{(0)}$, the canonical inclusions,*

$$\mathsf{F}(\mathcal{G}_U^U) \hookrightarrow \mathsf{F}(\mathcal{G}_V^V), \quad \mathsf{D}(\mathcal{G}_U^U) \hookrightarrow \mathsf{D}(\mathcal{G}_V^V),$$

induce an isomorphism in homology in all degrees.

The first homology group of a topological full group of a minimal ample groupoid is especially of interest, since it agrees with the abelianisation and therefore tells us what the quotient $\mathsf{F}(\mathcal{G})/\mathsf{D}(\mathcal{G})$ is. In light of this, Matui conjectured that there was a concrete relationship between $H_1(\mathsf{F}(\mathcal{G}))$ and the first two homology groups of $\mathcal{G}$.

Let us define two maps, I, j: Note that if $B \in \mathsf{F}(\mathcal{G})$, it will be in the kernel of δ_1 since

$$\delta_1(\chi_B)(u) = s_*(\chi_B)(u) - r_*(\chi_B)(u) = \sum_{s(g)=u} \chi_B(g) - \sum_{r(g)=u} \chi_B(g) = 1 - 1 = 0.$$

Therefore, we may define a map

$$I : \mathsf{F}(\mathcal{G}) \to H_1(\mathcal{G},\mathbb{Z}) \quad B \mapsto [B],$$

where $[\cdot]$ denotes the equivalence class in $H_1(\mathcal{G},\mathbb{Z})$. Note that since $Ker(I) = K(\mathcal{G})$, which contains $D(\mathcal{G})$, we have that we can actually define I more precisely on the abelianisation

$$I : \mathsf{F}(\mathcal{G})/D(\mathcal{G}) \to H_1(\mathcal{G},\mathbb{Z}) \quad B \mapsto [B].$$

Another map of interest is

$$j : H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \to \mathsf{F}(\mathcal{G})/D(\mathcal{G}).$$

Suppose U is a bisection satisfying $r(U) \cap s(U) = \varnothing$. Then we can map U to an element of the full group with order two

$$U \mapsto \tau, \quad \tau(x) := \begin{cases} Ux & x \in s(U) \\ U^{-1}x & x \in r(U) \\ x & \text{else} \end{cases},$$

then, j is the extension of this map to $H_0(\mathcal{G})$. This is well defined epimorphism onto $S(\mathcal{G})/D(\mathcal{G})$ whenever $\mathcal{G}$ has orbits that always have size bigger than 3 by [93], Theorem 7.2.

Conjecture 3.6.6 (Matui's AH Conjecture)**.** *Let $\mathcal{G}$ be a minimal, étale Cantor groupoid. The sequence*

$$H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{j} \mathsf{F}(\mathcal{G})/D(\mathcal{G}) \xrightarrow{I} H_1(\mathcal{G}) \to 0$$

is exact. If it is short exact, we say the strong AH property holds for $\mathcal{G}$, i.e. if the sequence

$$0 \to H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{j} \mathsf{F}(\mathcal{G})/D(\mathcal{G}) \xrightarrow{I} H_1(\mathcal{G}) \to 0$$

is exact. (See e.g. [88]).

The AH conjecture is an acronym – *A* stands for abelianisation– the abelianisation of the topological full group, and H stands for groupoid homology. The conjecture gives one in terms of the other. There are counterexamples to the strong AH conjecture [90, Section 5.5] [93, Example 7.1], in the authors view the simplest counterexample is that of a $\mathbb{Z}^3$-Cantor minimal system, (for example the interval exchange group as in Chapter 6 associated to $\Gamma = \mathbb{Z} \oplus \pi\mathbb{Z} \oplus e\mathbb{Z} \oplus \sqrt{2}\mathbb{Z}$ would be a concrete example). As of yet, no counterexamples for the AH conjecture have been found. Much progress has been made since the conjecture was first made for particular examples and cases [96] [97] [90]. Matui was also able to confirm the conjecture in general case when $\mathcal{G}$ is minimal, ample, principal and with comparison [88].
A further breakthrough was made recently. Li was recently able to confirm the above conjecture under very mild assumptions (losing principality) and provide a greater understanding of when the strong AH property holds.

Theorem 3.6.7 (Li [79] Cor E)**.** *Let $\mathcal{G}$ be minimal, ample, with comparison. Suppose $\mathcal{G}^{(0)}$ is locally compact Hausdorff with no isolated points. Then, there exists a long exact sequence*

$$H_2(D(\mathcal{G})) \xrightarrow{f} H_2(\mathcal{G}) \to H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{j} \mathsf{F}(\mathcal{G})/D(\mathcal{G}) \xrightarrow{I} H_1(\mathcal{G}) \to 0.$$

In particular, the AH conjecture is verified for this class of groupoids, and the strong AH conjecture is verified for this class of groupoids iff f is surjective (e.g. if $H_2(\mathcal{G}) = 0$).

This convenient theorem allows us to compute the abelianisation of many topological full groups, which a priori would be very difficult to compute.

3.7 Significant examples

Because topological full groups provide a source of infinite simple groups with finiteness properties, they have resolved many existence problems for infinite simple groups with finiteness properties. The first of these major results is due to Juschenko-Monod [61], who established that the topological full group of a Cantor minimal system is always amenable.

Theorem 3.7.1 (Juschenko-Monod)**.** *Let $\mathcal{G}$ be the transformation groupoid of a Cantor minimal system. Then, $\mathsf{F}(\mathcal{G})$ is amenable. [61]*

A corollary, since there are expansive Cantor minimal systems (for example, Sturmian subshifts), is that there are amenable, finitely generated infinite simple groups.

Corollary 3.7.2 (Juschenko-Monod)**.** *Let $\mathcal{G}$ be the transformation groupoid of a Cantor minimal system that is expansive. Then, $\mathsf{D}(\mathcal{G})$ is an amenable, finitely generated infinite simple group. [61]*

In particular then, $\mathsf{D}(\mathcal{G})$ is not elementary amenable. We note that the problem of establishing amenability of topological full groups remains a very difficult problem in general.

Even for Cantor minimal $\mathbb{Z}^2$ systems, the situation is very subtle and poorly understood in general. There are examples where the topological full group fails to be amenable, in fact spectacularly so by admitting an embedding of the free group on two generators F_2 [46]. On the other hand, if one takes any $\mathbb{Z}^2$ odometer, one will obtain a topological full group that is locally finite [39], and therefore elementary amenable. In fact the topological full groups of an equicontinuous system is always locally finite [39]. It is also known that certain interval exchange groups arising from expansive Cantor minimal $\mathbb{Z}^2$ systems are amenable, which will be explored further in Chapter 6 [60].

One might ask if it is possible to find simple groups of intermediate growth. Nekrashevych constructed the first examples of such groups via the topological full groups of certain Cantor minimal dihedral systems [92].

Theorem 3.7.3 (Nekrashevych)**.** *There are simple groups of intermediate growth [92].*

Some groups are finitely presented, which is a strengthening of finite generation. In fact, these properties fit into a family of topological finiteness properties:

Definition 3.7.4 (Type F_n)**.** *Given $n \in \mathbb{N}$, a group Γ is said to be type F_n if there exists an aspherical CW-complex whose fundamental group is isomorphic to Γ, whose $n-$skeleton is finite.*

Γ is said to be type F_∞ if it is type F_n for all n.

Remark 3.7.5. *A group is type F_1 iff it is finitely generated.*

A group is type F_2 iff it is finitely presented.

For all n, type F_{n+1} is stronger than type F_n.

Remark 3.7.6. *Type F_n can be rephrased dynamically. A group Γ has type F_n iff it acts freely, properly discontinuously and cocompactly on a CW-complex whose homotopy groups $\pi_0, ..., \pi_{n-1}$ vanish [115].*

One might ask: if type F_{n+1} is stronger than type F_n then can we construct examples of simple groups that are type F_n but not F_{n+1}? This question was resolved by Skipper, Witzel and Zaremsky by considering certain topological full groups.

Theorem 3.7.7 (Skipper-Witzel-Zaremsky)**.** *[115] For all n, there exists a simple group of type F_n but not F_{n+1}.*

Note another construction was later discovered by Belk-Zaremsky [7]. There are many other interesting examples of topological full groups that the literature has seen. We highlight the above three examples due to their influence and prominence in the field, but note the other notable examples of groups with a combination of surprising properties unseen before:

- Examples of simple, finitely generated nonamenable groups with generalised forms of amenability such as property Gamma, inner amenability [67], or more recently, whose group von Neumann algebras have the McDuff property [65].
- Topological full groups of minimal subshifts which contain Grigorchuk's group as a subgroup [15].

Another important point to emphasise is that certain topological full groups, have served recently as container groups for a longstanding conjecture of Boone and Higman, concerning a rephrasing of the word problem for groups in terms of certain embeddings.

Definition 3.7.8. *Let Γ be a finitely generated group with finite symmetric generating set S. We say that Γ has*

1. Solvable word problem*, if given a finite word $w \in S^n$ there exists an algorithm that determines, in finitely many steps, whether or not w is the identity.*
2. Solvable generalised word problem *with respect to a fixed finitely generated subgroup $\Lambda \subseteq \Gamma$. Which asks, given a finite word $w \in S^n$, there exists an algorithm that determines, in finitely many steps, whether w belongs to Λ.*

Conjecture 3.7.9 (Boone-Higman Conjecture)**.** *A finitely generated group Γ has solvable word problem iff it embeds into a simple finitely presented group. [6]*

The use of the framework of topological full groups is due to the abundance of examples of (derived subgroups of) topological full groups that are simple and finitely presented. See [6] for a survey on recent progress on the Boone-Higman conjecture. See [5] for the resolution of the Boone-Higman conjecture for hyperbolic groups, which relies heavily on the framework of topological full groups, in particular twisted Brin Thompson groups [7].

3.8 Outlook

One open question has already been explained in the subgroup structure section:

Question 3.8.1. *Let $\mathcal{G}$ be an ample groupoid. Is $\mathsf{D}(\mathcal{G}) = \mathsf{A}(\mathcal{G})$?*

As explained in Section 3.5, understanding the finiteness properties of $\mathsf{D}(\mathcal{G})$ is important. Finite generation of this group is relatively well understood due to the notion of groupoid expansivity. However, many such $\mathsf{D}(\mathcal{G})$ are finitely presented, and understanding when this happens seems altogether more complex. Therefore we ask:

Question 3.8.2. *Is there a dynamical property, intrinsic to the ample groupoid $\mathcal{G}$ that guarantees that $\mathsf{D}(\mathcal{G})$ is finitely presented?*

Similarly, some topological full groups are amenable, and some are not. Determining whether a topological full group is amenable or not remains a subtle question, so we ask:

Question 3.8.3. *Is there a characterisation of between the amenability of topological full groups and a dynamical property intrinsic to the ample groupoid $\mathcal{G}$?*

You could also ask the same question about growth:

Question 3.8.4. *Is there a relationship between the growth of topological full groups and a dynamical property intrinsic to the ample groupoid $\mathcal{G}$?*

Given the significant examples seen above, we can ask natural open existence questions for simple groups with finiteness properties in this context, such as:

Question 3.8.5. *Are there finitely presented amenable infinite simple groups?*

Finally let us mention the major guiding open problem which is the Boone-Higman conjecture.

Question 3.8.6. *Does every finitely generated group with solvable word problem embed into a finitely presented simple group?*

Chapter 4

Thompson's group and generalisations

4.1 Thompson's group

Thompson's group V can be described as a group of homeomorphisms of the Cantor space, or certain piecewise linear maps on the unit interval.

Definition 4.1.1 (Thompson's Group V)**.** *Thompson's group V is the group of right continuous piecewise linear bijections of $(0,1]$ with finitely many slopes, all in $\langle 2\rangle = \{2^n : n \in \mathbb{Z}\}$, and finitely many nondifferentiable points, all dyadic (i.e. all in $\mathbb{Z}[1/2] = \{a/2^n : a \in \mathbb{Z}, n \in \mathbb{N}\}$).*

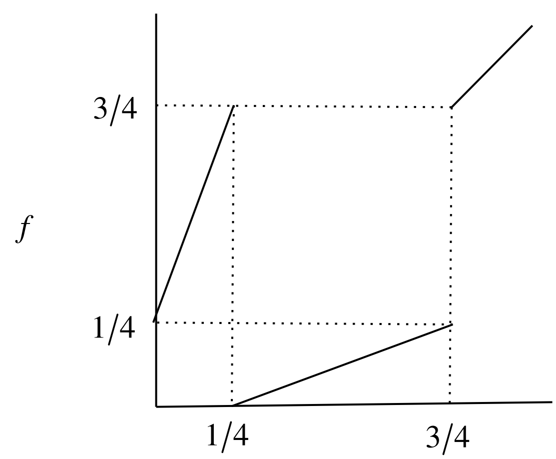


Figure 4.1: An example element of Thompson's group V

This group was the first example of an infinite finitely presented simple group [121]. Since then, it has been shown that V is type F_∞ [25] (see [122] for a very general approach) and is acyclic [118].

Thompson's group V was defined alongside two notable subgroups called Thompson's group F, T. These groups nest as subgroups in alphabetical order, i.e. $F \subset T \subset V$.

Definition 4.1.2 (Thompson's Group T, F)**.** *Let $\sim$ be the equivalence relation on $[0,1]$ which identifies $0 \sim 1$ only, so that $[0,1]/\sim$ is homeomorphic to a circle.*

Then, $T \subset V$ is the group of continuous piecewise linear homeomorphisms of $[0,1]/\sim$ with finitely many nondifferentiable points in $\mathbb{Z}[1/2]$ and slopes in $\langle 2\rangle$.

Thompson's group F is the subgroup of T consisting of all piecewise linear maps with finitely many nondifferentiable points in $\mathbb{Z}[1/2]$ and slopes in $\langle 2\rangle$ that are continuous on $(0,1]$

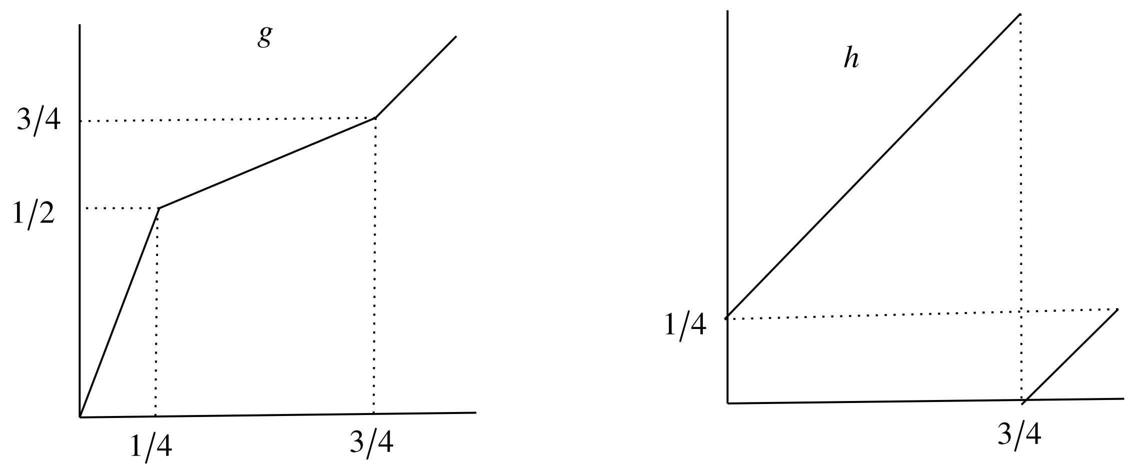


Figure 4.2: Examples of elements of Thompson's group F and T. g is an element of F, and therefore also g is an element of T. h is an element of T, but not an element of F.

The primary interest in Thompson's group V is that it was the first known example of an infinite, finitely presented simple group [121]. Let us give an outline of why these facts are true. These results are due to Thompson, but since his notes are not publically available, the author is unsure exactly how Thompson proved these results. A common method of proof can be found in Section 6 of Cannon-Floyd-Parry's excellent introductory notes [32]. Here we prefer to follow the ideas of Bleak-Quick [14].

Let us begin with notation. Much the same as Nekrashevych, Bleak-Quick took the perspective that Thompson's group V should be viewed simultaneously as a generalisation of the infinite symmetric and alternating group.

Definition 4.1.3 (V as permutations)**.** *Let $X = \{0,1\}^{\mathbb{N}}$ be the Cantor set. Let X^* denote the finite words in $\{0,1\}$. Given $\mu \in X^*$, let $\mu + X$ denote the words in X with μ as a prefix. Given $\mu = (\mu_1, \ldots, \mu_{l_\mu}), \nu = (\nu_1, \ldots, \nu_{l_\nu})$ be such that such that $\mu + X, \nu + X$ have empty intersection. Let (μ,ν) denote the homeomorphism $X \to X$ of order 2 where:*

$$(\mu,\nu)(x) = \begin{cases} (\mu_1, \ldots, \mu_{l_\mu}, x_1, \ldots,) & x = (\nu_1, \ldots, \nu_{l_\nu}, x_1, \ldots,) \in \nu + X \\ (\nu_1, \ldots, \nu_{l_\nu}, x_1, \ldots,) & x = (\mu_1, \ldots, \mu_{l_\mu}, x_1, \ldots,) \in \mu + X \\ x & \textit{otherwise} \end{cases}$$

Let V be the group generated by all such homeomorphisms.

Remark 4.1.4 (Identifying the definitions). *One can identify this definition of V via the surjection*

$$\varphi : X \to (0,1] \quad (x_1,\ldots,) \mapsto \sum_{i=1}^{\infty} x_i 2^{-i}.$$

Then, the isomorphism Φ can be defined by sending (μ,ν) to the unique PL map $f_{\mu,\nu} \in V$ such that $f_{\mu_i}\varphi = \varphi(\mu,\nu)$. The fact that this is injective is clear, and surjective takes an argument.

It's sufficient to show that V as a group of PL homeomorphisms is generated by elements of the form $f_{\mu,\nu}$.

First note the measure-preserving subgroup $IE(\mathbb{Z}[1/2])$ of Thompson's group V where each PL section has slope 1, is clearly generated by elements of this form. This corresponds to the group generated by two cycles (μ,ν) with the length of μ equal to the length of ν. Then $S_{2^n} \cong IE(2^{-n}\mathbb{Z})$ is generated by the two cycles permuting the 2^n intervals of length 2^{-n}.

Since V is the Zappa-Szép product between its measure preserving subgroup $IE(\mathbb{Z}[1/2])$ and Thompson's group F (this is well known, however see Lemma 7.1.2 for a rigorous proof) it is therefore enough to show that we may obtain any element of Thompson's group F from such permutations.

Let $f \in F$ be arbitrary. Let $t = \inf supp f$ and assume (by taking inverses if necessary) the right derivative is less than 1 at t. Let n be large enough that $f(t) = 2^{-m}t$ for some $m \in \mathbb{N}$ on $[t, t+2^{-n}] \subset (0,1]$. Then, there is $\mu_t, \mu_{t+3\cdot 2^{-n+2}} \in X^*$ such that $\varphi(\mu_t) = t, \varphi(\mu_{t+3\cdot 2^{-n+2}}) = t + 3\cdot 2^{-n+2}$ and the length of $\mu_{t+3\cdot 2^{-n+2}}$ is one more than the length of μ_t, which is $n+1$. Therefore, $f_{\mu_t,\mu_{t+3\cdot 2^{-n+2}}}$ is an element of V with

- *Support in $[t, t+2^n]$*
- *Right derivative $1/2$ at t*

There exists a unique element of $f' \in IE(\mathbb{Z}[1/2])$ such that $\hat{f}_1 = f' f_{\mu_t,\mu_{t+3}} \in F$ with support in $[t, t+2^n]$. Hence, ${\hat{f}_1}^{-m} f \in F$ has strictly smaller support than f. Repeating this argument to obtain $\hat{f}_2, \hat{f}_3$ will eventually terminate as the support gets closer and closer to $\sup supp f$, hence all elements of F can be obtained as a finite string of two cycles.

Given μ,ν such that $\mu+X, \nu+X$ have empty intersection. Given a finite word $\gamma \in X^*$ we define $(\mu,\nu)(\gamma) :=$

$$\begin{cases} (\mu_1,\ldots,\mu_{l_\mu},\gamma_1,\ldots,\gamma_n) & \gamma = (\nu_1,\ldots,\nu_{l_\nu},\gamma_1,\ldots,\gamma_n) \\ (\nu_1,\ldots,\nu_{l_\nu},,\gamma_1,\ldots,\gamma_n) & \gamma = (\mu_1,\ldots,\mu_{l_\mu},\gamma_1,\ldots,\gamma_n) \\ \gamma & (\gamma+X)\cap(\mu+X) = (\gamma+X)\cap(\nu+X) = \varnothing \\ \text{undefined} & \text{otherwise} \end{cases}.$$

The above remark gives an idea of how we can prove:

Theorem 4.1.5 ([14], Theorem 1.1.)**.** *Let μ,ν be all symbols such that $\mu+X,\nu+X$ have empty intersection. Let (μ,ν) denote the homeomorphism as above. There is an infinite presentation for V given by*

$$(\mu,\nu)^2 = 1,$$

$$(\mu,\nu) = (\mu 0,\nu 0)(\mu 1,\nu 1),$$

$$(\mu,\nu)^{(\alpha,\beta)} = ((\alpha,\beta)(\mu),(\alpha,\beta)(\nu)),$$

where μ,ν,α,β range over all sequences such that they are defined.

Given $\mu^1,\dots,\mu^n$ such that μ^j+X,μ^i+X have nonempty intersection for $i\neq j$, one may similarly define the homeomorphism $(\mu^1,\dots,\mu^n)$ in the analogous way to the two cycles

$$(\mu^1,\dots,\mu^n)(x) = \begin{cases} (\mu^1_1,\dots,\mu^1_{l^1_\mu},x_1,\dots,) & x = (\mu^n_1,\dots,\mu^n_{l^n_\mu},x_1,\dots,) \in \mu^n+X \\ (\mu^i_1,\dots,\mu^i_{l^i_\mu},x_1,\dots,) & x = (\mu^{i-1}_1,\dots,\mu^{i-1}_{l^{i-1}_\mu},x_1,\dots,) \in \mu^{i-1}+X \quad i=2,\dots,n \\ x & \text{otherwise} \end{cases}.$$

Using this notation one can show the following relation for the elements $(00,01).(01,10,11)$, and $(1,00)$ of V:

$$R_1:\ (00,01)^2 = (01,10,11)^3 = ((00,01)(01,10,11))^4 = 1,$$

$$R_2:\ ((1,00),(00,01))^{(1,00)} = (00,01),$$

$$R_3:\ (1,00) = (10,000)(11,001),$$

$$R_4:\ [(00,010),(10,111)] = [(00,011),(10,111)] = 1,$$

$$R_5: [(000,010),(10,110)].$$

These are straightforward algebra to verify. Note that R_1 follows since these elements generated the copy of S_4, $IE(1/4\mathbb{Z})$ in $IE(\mathbb{Z}[1/2])$. R_5 simply describes the commuting nature of certain elements of V with disjoint supports.

The remainder of the proof strategy is to show two nontrivial facts:

- The three elements $(00,01).(01,10,11),(1,00)$ can be used to generate any two cycle, (μ,ν) and therefore generate V.

- Let V' be a group generated by a,b,c such that a,b generate the symmetric group S_4 (R_1) and c is related as in $R_2,\dots,R_5$. Then $V' = \langle a,b,c\rangle$ is simple.

Since there is a canonical quotient map $V'\to V$, this then shows that:

Theorem 4.1.6 (Thompson 1967)**.** *Thompson's group V is finitely presented and simple. It is generated by the elements $(00,01).(01,10,11),(1,00)$ for which a finite presentation is given by the relations $R_1,...,R_5$*

It is important to appreciate that infinite, simple groups with finiteness properties are difficult to construct because a-priori examples of infinite groups do not have good finiteness properties and the sorts of operations that preserve simplicity (direct limits) tend not to preserve finiteness properties. Higman had only constructed the first example of an infinite finitely generated simple group in 1951 [58].

In fact, V has stronger finiteness properties than finite presentation. In 1987, Brown showed V was of type F_∞, a fact the proof of which is beyond the scope of this thesis [25].

Theorem 4.1.7 (Brown 1987)**.** *Thompson's group V is of type F_∞.*

Let us give an outline of the history of the homology of Thompson's group V. In 1992 [26], Brown showed that V was rationally acyclic.

Theorem 4.1.8 (Brown 1992)**.** *Thompson's group V is rationally acyclic.*

In the same paper, Brown conjectures that V is integrally acyclic. This remained an open problem for 36 years until 2018 when Szymik-Wahl showed that Thompson's group V was integrally acyclic [118].

Theorem 4.1.9 (Szymik-Wahl)**.** *Thompson's group V is integrally acyclic.*

In 1974, Higman introduced a generalised class of Thompson's groups which have become known as Higman-Thompson groups [59].

Definition 4.1.10 (Higman-Thompson groups)**.** *For $k > 1, k \in \mathbb{N}$, $r \in \mathbb{N}$, the Higman-Thompson group, denoted $V_{k,r}$ is the group of piecewise linear, right continuous bijections of $(0,r]$ with finitely many nondifferentiable points, all in $\mathbb{Z}[1/k]$ and slopes in $\langle k \rangle$.*

A key difference here is that Higman-Thompson groups are not always simple. Higman shows already in [59]:

$$(V_{k,r})_{ab} = \begin{cases} \mathbb{Z}_2 & k \text{ odd} \\ 0 & k \text{ even} \end{cases}.$$

Therefore, whenever k is even, $V_{k,r}$ is it's own derived subgroup, otherwise if k is odd, $V_{k,r}$ is not simple. Higman was able to show that in either case their derived subgroups are simple and finitely presented:

Theorem 4.1.11 (Higman)**.** *Let $k, r \in \mathbb{N}$ with $k > 2$. Then, the derived subgroup of $V_{k,r}$ is simple and finitely presented [59].*

A full classification of Higman-Thompson groups has been obtained. In this notation, the main result of [98] states that:

$$V_{k,r} \cong V_{k',r'} \iff k = k',\ \gcd(k-1,r) = \gcd(k-1,r')$$

Here, we can see the isomorphism class of the Higman-Thompson group $V_{k,r}$ depends on the length r of the underlying interval. It is notable that the direction $\Rightarrow$ was already known to Higman, and the direction $\Leftarrow$, considered the hard direction by experts, used tools from Leavitt path algebras [98].

The key difference that Higman-Thompson groups sometimes have nontrivial abelianisation implies of course that they cannot always be acyclic. The homology of Higman-Thompson groups is still yet to be fully understood. Brown's proof that V is rationally acyclic is noted in his paper to generalise to the Higman-Thompson groups. However, this result was first written down by Szymik-Wahl [118].

Theorem 4.1.12. *For all $k > 1, r \in \mathbb{N}$ the Higman-Thompson groups $V_{k,r}$ are rationally acyclic.*

They were able to show was a homological stability result, that the homology of $V_{k,r}$ does not depend on r.

Theorem 4.1.13. *Let $k, r, r' \in \mathbb{N}$ with $k > 1$, then we have that the homology of $V_{k,r}$ is the same as the homology of $V_{k,r'}$, i.e. $H_*(V_{k,r}) \cong H_*(V_{k,r'})$. [118]*

It still remains to understand the integral homology of $V_{k,r}$ for $k > 1$ in totality. In fact, it is open to compute the homology of $V_{3,1}$.

4.2 Stein's groups

We now turn to the piecewise linear homeomorphisms perspective of Thompson-like groups, following Stein [116]. Note similar generalisations of Thompson's group F have been studied in detail by Bieri-Streibel [8].

Definition 4.2.1 (Stein's Groups)**.** *Let $\Lambda \subset \mathbb{R}$ be a multiplicative subgroup of $(0, +\infty)$, Γ be a $\mathbb{Z}\cdot\Lambda$ submodule (i.e. such that $\Lambda \cdot \Gamma = \Gamma$) let $1 \leq \ell \in \Gamma$. Then, $V(\Gamma, \Lambda, \ell)$ denotes the right continuous, piecewise linear bijections of $(0, \ell]$ with slopes in Λ and finitely many discontinuities in Γ. In addition, let us define two nested subgroups in analogy of Thompson's groups $F \subset T \subset V$:*

- *Let $\sim$ be the equivalence relation on $[0, \ell]$ which identifies only the endpoints $0 \sim \ell$ in order that $[0, \ell]/\sim$ is homeomorphic to a circle. Then, $T(\Gamma, \Lambda, \ell) \subset V(\Gamma, \Lambda, \ell)$ is the group of piecewise linear homeomorphismns of $[0, \ell]/\sim$ with finitely many nondifferentiable points, all in Γ and slopes in Λ.*

- *$F(\Gamma,\Lambda,\ell) \subset T(\Gamma,\Lambda,\ell) \subset V(\Gamma,\Lambda,\ell)$ is the group of piecewise linear homeomorphisms of $(0,\ell]$ with finitely many nondifferentiable points in Γ and slopes in Λ.*

Note that the restriction of $\ell \geq 1$ does not limit the diversity of these groups. This definition due to Stein encompasses many generalisations of Higman-Thompson groups of interest to geometric group theory. We list some specific examples below:

Example 4.2.2 (Thompson's group V)**.** *Let $\Gamma = \mathbb{Z}[1/2], \Lambda = \langle 2 \rangle$ and $\ell = 1$. Then, $V(\mathbb{Z}[1/2], \langle 2 \rangle, 1)$ is Thompson's group V.*

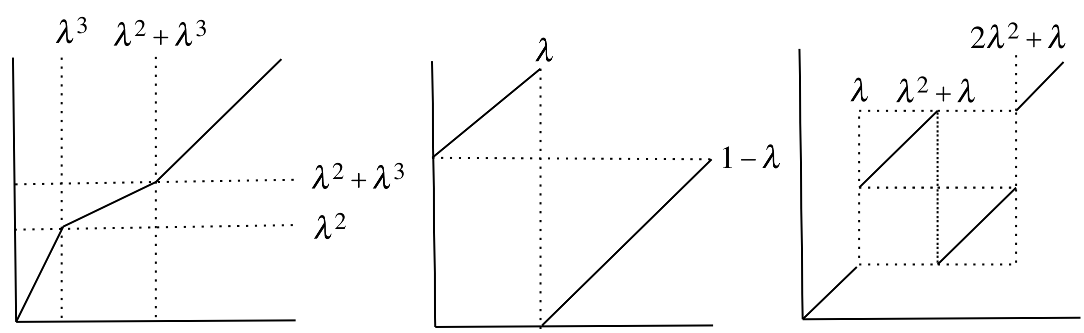
These groups were generalised by Higman.

Example 4.2.3 (The Higman-Thompson groups)**.** *Let $n, r \in \mathbb{N}$, with $n > 2$. Then, $V(\mathbb{Z}[1/n], \langle n \rangle, r) \cong V_{n,r}$, the Higman-Thompson group.*

But the class of Stein's group studied in Chapter 7 is much more general than any of the above example classes. We should therefore verify that whichever group Λ and module Γ we choose, we obtain nontrivial groups.

Lemma 4.2.4. *Let Λ be a multiplicative subgroup of $\mathbb{R} \cap (0,+\infty)$. Let Γ be a $\mathbb{Z}\cdot\Lambda$ submodule and $\ell \in \Gamma$. Then, all three of the groups $F(\Gamma,\Lambda,\ell) \subset T(\Gamma,\Lambda,\ell) \subset V(\Gamma,\Lambda,\ell)$ are nontrivial, properly nested groups.*

Proof. This proof is entirely constructive. We show that for $\ell = 1$, since the other cases are analogous. For any choice of Λ, Γ, consider $\lambda \in \Lambda, \lambda < 1/2$ then consider the following piecewise linear functions:



$f \in F$ $\qquad$ $b \in T$ $\qquad$ $g \in V$

$$f(t) := \begin{cases} \lambda^{-1}t & t \in (0,\lambda^2] \\ \lambda t + \lambda^2 - \lambda^3 & t \in (\lambda^2, \lambda^2+\lambda^3] \\ t & \text{otherwise} \end{cases}$$

$$b(t) := \begin{cases} t+1-\lambda & t \in (0,\lambda] \\ t-\lambda & t \in (\lambda,1] \end{cases} \qquad g(t) := \begin{cases} t+\lambda^2 & t \in (\lambda, \lambda+\lambda^2] \\ t-\lambda^2 & t \in (\lambda+\lambda^2, \lambda+2\lambda^2] \\ t & \text{otherwise} \end{cases}$$

These clearly define nontrivial elements of $V(\Gamma,\Lambda,1)$. Furthermore, we have that $f \in F(\Gamma,\Lambda,1)$, $b \in T(\Gamma,\Lambda,1) \setminus F(\Gamma,\Lambda,1)$, and $g \in V(\Gamma,\Lambda,1) \setminus T(\Gamma,\Lambda,1)$.

□

Remark that the existence of nontrivial elements of F also follows from [[8], Theorem 1].

It is important to consider when Γ,Λ are generated by more than one number, i.e. when $\Lambda \ncong \mathbb{Z}$. The case where Λ is generated by multiple integers were the main object of study by Stein, and so we accordingly name them Stein's integral groups:

Example 4.2.5 (Stein's Integral Groups)**.** *Let $N = \{n_1, n_2, \ldots n_k\}$ be a finite set of algebraically independent integers, such that $n_i > 2$ for all i. Let $\ell \in \mathbb{N}$. Then, we have that Stein's integral group is $V(\mathbb{Z}[\prod_{i=1}^k 1/n_i], \langle n_i \rangle_{i=1}^k, \ell)$.*

Stein defined the groups V as generalisations of Thompson's group V, or Higman-Thompson groups. She was able to show a few key ways they are similar:

Theorem 4.2.6 (Stein [116], Theorem 5.2)**.** *Let Λ be a submultiplicative group of $(\mathbb{R}_+, \cdot)$. Let Γ be a $\mathbb{Z}\Lambda$ submodule and $\ell \in \Gamma \cap (0,\infty)$. Then, $D(V(\Gamma,\Lambda,\ell))$ is simple.*

These derived subgroups also enjoy a number of finiteness properties. In the same paper [116], Stein wrote a proof, credited to Brown, which shows that the integral groups are of type F_∞.

Theorem 4.2.7. *Let $\{n_1, n_2, \ldots n_k\}$ be a finite set of algebraically independent integers. Let $\ell \in \mathbb{N}$. Then, $V(\mathbb{Z}[\prod_{i=1}^k 1/n_i], \langle n_i \rangle_{i=1}^k, \ell)$ is of type F_∞.*

Finally, Stein was able to compute the abelianisation of her integral groups [116].

Lemma 4.2.8. *Let $\{n_1, n_2, \ldots n_k\}$ be a finite set of algebraically independent integers. Let $d = gcd(n_1 - 1, \ldots, n_k - 1)$ Let $\ell \in \mathbb{N}$. Then,*

$$V(\mathbb{Z}[\prod_{i=1}^k 1/n_i], \langle n_i \rangle_{i=1}^k, \ell)_{ab} \cong \begin{cases} 0 & d \text{ even} \\ \mathbb{Z}_2 & d \text{ odd} \end{cases}.$$

4.2.1 Cleary's group

As well as including integers, one can consider arbitrary irrational numbers to generated the group of slopes. The most studied group of this form is Cleary's group [38] [37] [31] [30].

Example 4.2.9 (Cleary's irrational slope Thompson Group)**.** *Let $\tau = \frac{1+\sqrt{5}}{2}$ denote the golden ratio. The Stein's group $V(\mathbb{Z}[\tau,\tau^{-1}], \langle \tau \rangle, 1)$, is known as Cleary's group, denoted by V_τ [38]. This group is otherwise known as the irrational slope Thompsons group [31].*

Researchers have extensively studied the F-type subgroup $F_\tau = F(\mathbb{Z}[\tau,\tau^{-1}],\langle\tau\rangle,1)$. Like Thompson's group, it admits a presentation through carets [30].

Little is known about Cleary's group V_τ, but there have been exciting developments in recent years. For example, explicit finite generating sets have been found [31].

Theorem 4.2.10 (Burrilo-Nucinkis-Reeves)**.** *V_τ is finitely generated.*

The abelianisation of this group was shown to be $\mathbb{Z}_2$ [31].

Theorem 4.2.11 (Burrilo-Nucinkis-Reeves)**.** $(V_\tau)_{ab} \cong \mathbb{Z}_2$.

V_τ also has very different dynamical properties to that Thompson's group V.

Remark 4.2.12. *From a dynamical perspective, V_τ contains minimal homeomorphisms of the Cantor space, a concrete example being the map*

$$f_\tau : (0,1] \to (0,1] \quad t \mapsto t+\tau \ (\ mod\ \mathbb{Z})$$

Thompson's group V does not contain minimal homeomorphisms, since if we examine the orbit structure of an element of Thompson's group V, we will find that there are finite orbits and the underlying homeomorphism of the Cantor space is therefore not minimal [108].

4.3 Brin-Higman-Thompson groups

In 2004, Brin introduced a class of groups nV [21] which have since become known as Brin-Thompson groups. The variation in which we write below is the natural enveloping class of groups including the Brin-Thompson groups and the Higman-Thompson groups.

Let $n,k,r \in \mathbb{N}$ with $k > 1$. Let $\mathcal{C} = \{1,\dots,k\}^{\mathbb{N}}$ be the Cantor space, and consider the n-product $\mathcal{C}^n$ of this Cantor space. Let $\{1,\dots,k\}^*$ denote the set of all finite strings in $\{1,\dots,k\}$, including the empty string. Given an $n-$tuple $\phi = (\phi_1,\dots,\phi_n) \in (\{1,\dots,k\}^*)^n$ we associate the cone $B(\phi) = \{x = (\phi_1 x_1,\dots,\phi_n x_n) \in \mathcal{C}^n\}$. Note each $B(\phi)$ is canonically homeomorphic to $\mathcal{C}^n$. Given two n-tuples ϕ,ϕ', let

$$B(\phi,\phi') : B(\phi) \to B(\phi') \quad (\phi_1 x_1,\dots,\phi_n x_n) \mapsto (\phi_1 x_1,\dots,\phi_n x_n)$$

Definition 4.3.1 (Brin-Higman-Thompson Groups)**.** *The Brin-Higman-Thompson $nV_{k,r}$ groups are the groups of homeomorphisms f of the Cantor space $\mathcal{C}^n$ such that for all $x \in \mathcal{C}^n$, there exists $\phi,\phi' \in (\{1,\dots,k\}^*)^n$ such that $f|_{B(\phi)} = B(\phi,\phi')$.*

Equivalently, one could describe the Brin-Higman-Thompson group as the group of piecewise linear bijections g on $[0,r]^n$, such that for all $i,j \in 1,\dots,n$ and all $x_1,\dots,x_{n-1} \in [0,r]^{n-1}$ the map:

$$g_{i,j} : [0,r] \to [0,r] \quad t \mapsto (g(x_1,..,x_{i-1},t,x_{i+1},\dots,x_{n-1}))_j$$

Is an element of the Higman-Thompson group $V_{k,r}$.

These groups have now been classified, with the final pieces of this jigsaw resolved by Dicks Martinez-Perez [44] giving the following classification

$$nV_{k,r} \cong n'V_{k',r'} \iff n' = n, k' = k \text{ and } gcd(k, r-1) = gcd(k, r'-1).$$

Much like the Higman-Thompson groups, these groups were defined as a new class of infinite, finitely presented simple groups. Not unlike the Higman-Thompson groups, the derived subgroups of $nV_{k,r}$ are usually the simple groups, although $1V_{2k,r}$ is perfect for all k, r, and so is $nV_{2,r}$ for all n, r. Brin-Thompson groups are of type F_∞.

Theorem 4.3.2. *Let $n, k, r \in \mathbb{N}$, with $k > 2$. Then, the Brin-Higman-Thompson group $nV_{k,r}$ is of type F_∞.*

Let us explain the history that led to this result. In Brin's original paper, he shows that $2V$ is simple and finitely presented [21]. Later, Hennig-Mattuci [57] show that nV is simple and finitely presented for all n. Kochloukova, Martinez-Perez, and Nucinkis proved that the Brin–Thompson groups 2V and 3V are of type F_∞ [68], and Fluch, Marschler, Witzel, and Zaremsky extended this result to all of the Brin–Thompson groups [50]. To the best of the authors knowledge, the Brin-Higman-Thompson groups were not then dealt with separately, although it is clear that the techniques of Fluch, Marschler, Witzel, and Zaremsky extend to the Brin-Higman-Thompson groups. For written proof, the result above can be viewed as a Corollary of work by Martinez-Perez, Mattuci and Nucinkis [83], or of Thumann [122].

4.4 Thompson-like groups as topological full groups

4.4.1 V as a topological full group

Let us first explain two perspectives on the groupoid model of Thompson's group V. The classic perspective is that V is This observation goes back to historic work of Nekrashevych [94] but was first formalised by Matui [89]. Nekrashevych observed that Thompson's group V can be described inside the Cuntz algebra, which is the reduced groupoid C^*-algebra of $\mathcal{E}_2$; $V_2 \cong \{S \in U(\mathcal{O}_2) : S = \sum_{i=1}^{n} S_{\mu_i} S_{\nu_i}^*\}$.

Let us recall the canonical groupoid model $\mathcal{E}_2$ for $\mathcal{O}_2$, the SFT groupoid associated with the directed graph E_2, as in Example 2.4.7.

Let $X = \{0,1\}^{\mathbb{N}}$ be cantor space. Consider the graph groupoid; for $k \in \mathbb{Z}$ we say

$$(x_n)_n \sim_k (y_n)_n \iff \exists N \in \mathbb{N}\, s.t.\, n > N \implies x_n = y_{n+k}.$$

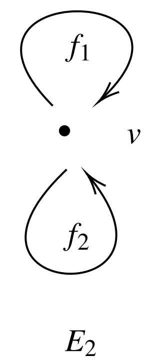


Figure 4.3: The directed graph associated to $\mathcal{O}_2$

Then, the groupoid elements are given by:

$$((x_n)_n, k, (y_n)_n)) \quad (x_n)_n \sim_k (y_n)_n.$$

Composable pairs are of the form

$$((x_n)_n, k, (y_n)_n))((\hat{y}_n)_n, \hat{k}, (z_n)_n)) \; s.t. \; (y_n)_n \sim_0 (\hat{y}_n)_n.$$

Inverses are given by

$$((x_n)_n, k, (y_n)_n))^{-1} = ((y_n)_n, -k, (x_n)_n)).$$

A basis for the topology on $\{0,1\}^{\mathbb{N}}$ is given by:

$$C_\mu = \mu + \{0,1\}^{\mathbb{N}}$$

Where μ is some finite word- i.e. arbitrary sequences that start with μ. Then, a basis for the topology on $\mathcal{E}_2$ is given by

$$C_{\mu,\nu} = \{((x_n)_n, k, (y_n)_n)) : (x_n)_n \in C_\mu, \; (y_n)_n \in C_\nu, \; k = |\nu| - |\mu|\}.$$

This groupoid is ample and the above form a basis for the compact open bisections. It is not hard to see then that since $s(C_{\mu,\nu}) = C_\mu$ $r(C_{\mu,\nu}) = C_\nu$ elements of our topological full group are of the form

$$B = \bigsqcup_{i=1}^{N} C_{\mu_i,\nu_i} \text{ s.t. } X = \bigsqcup_i C_{\mu_i} = \bigsqcup_i C_{\nu_i},$$

i.e. so that every sequence starts with exactly one μ_i and exactly one ν_i. Now let us notice something subtle. Any finite word in $\{0,1\}$ corresponds uniquely to a dyadic number in $(0,1)$ via it's binary expansion, for an explicit example:

$$0110 \mapsto 0/2 + 1/4 + 1/8 + 0/16 = 3/8.$$

This lets us define a map:

$$f : \bigcup_N \{0,1\}^N \to \mathbb{Z}[1/2] \cap (0,1).$$

Furthermore, since every number has an infinite binary expansion, we can extend such a map to a map $\{0,1\}^{\mathbb{N}} \to (0,1)$. It is in this way that we see such an identification:

$$f(C_\mu) = [f(\mu), f(\mu) + 2^{-|\mu|}].$$

Therefore, each $C_{\mu,\nu}$ gives us a pair of intervals with length in $\langle 2 \rangle$ and end points in $\mathbb{Z}[1/2] \cap (0,1]$, allowing us to define the map

$$\Phi : \mathcal{B}^k_{\mathcal{E}_2} \to \{\text{partial homeomorphisms of } (0,1) \text{ by linear maps }\} \quad C_{\mu,\nu} \to f_{\mu,\nu},$$

where

$$f_{\mu,\nu} : [f(\mu), f(\mu) + 2^{-|\mu|}) \to [f(\nu), f(\nu) + 2^{-|\nu|}) \quad t \mapsto f(\nu) - f(\mu) + 2^{|\mu|-|\nu|} t.$$

Considering Φ on the topological full group we get therefore an isomorphism

$$\Phi : \mathsf{F}(\mathcal{E}_2) \to V_2.$$

It is easy to see that $ker(\Phi) = \mathcal{E}_2^0 = (X, 0, X)$. To see surjectivity, consider an arbitrary $f \in V_2$. Since f is bijective, it is enough for each piecewise linear component to find a corresponding cylinder set $C_{\mu,\nu}$. Let

$$f_i : [a, a + 2^{-k}) \to [c, c + 2^{-m}) \quad t \mapsto c + 2^{k-m}(t - a)$$

be some piecewise linear component of f. We may assume without loss of generality that $2^k a, 2^m c \in \mathbb{N}$ since otherwise, we can just rewrite f_i as the union of 2^n functions of the form

$$f_l : [a + l2^{-n}, a + l2^{-(n+k)}) \to [c + l2^{-n}, c + l2^{-(n+m)})) \quad t \mapsto c + 2^{k-m}(t - a)) \quad l \in \{0, 1, 2, 3..., 2^n\}$$

for suitably large n.

Then let μ be the binary expansion of $2^k a$, ν be the binary expansion of $2^m c$. We see that $f_i = \Phi(C_{\mu,\nu})$ so that we are done.

The next observation we give is that the groupoid $\mathcal{E}_2$ is isomorphic to a transformation groupoid. Recall [[35], Proposition 8.5.]:

Lemma 4.4.1. *There exists an amenable, topologically free action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on the Cantor space X such that $\mathbb{Z}_2 * \mathbb{Z}_3 \ltimes X \cong \mathcal{E}_2$ where $\mathcal{E}_2$ is the full shift on two generators.*

Proof. Let F, E_2 be as in the below diagram.

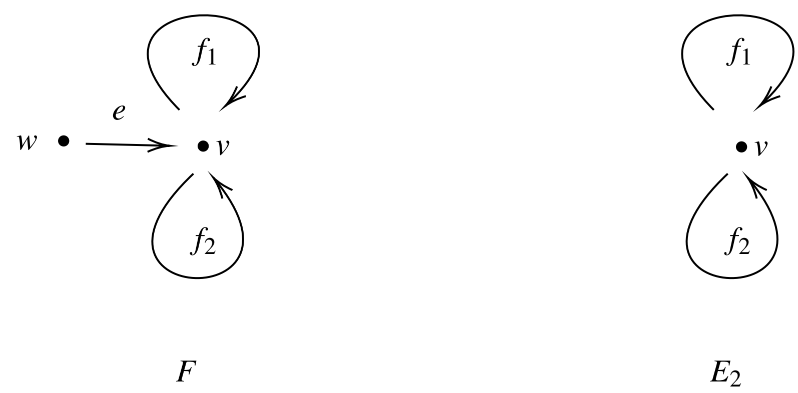


Figure 4.4: The graphs F and E_2

Note that $E_2 \subset F$.
We first show that $\mathcal{G}_F \cong \mathcal{G}_{E_2}$. Define

$$\varphi : F^\infty \to E_2^\infty \quad \varphi(x_1, x_2, x_3, \ldots) = \begin{cases} (f_1, x_2, x_3, \ldots) & x \notin E_2 \\ (f_2, x_1, x_2 \ldots) & x \in E_2 \end{cases},$$

and let χ_{E_2} be the indicator function of E_2^∞ on F^∞. It is clear that φ is a homeomorphism. Note that this behaves well with respect to shifts;

$$\sigma_{E_2}^{2-\chi_{E_2}(x)}(\varphi(x)) = \sigma_F(x)$$

This insures that the map

$$\Phi : \mathcal{G}_F \to \mathcal{G}_{E_2} \quad (x, k, y) \mapsto (\varphi(x), k + \chi_{E_2}(x) - \chi_{E_2}(y), \varphi(y))$$

is well defined. Say $k = m - n$ and $\sigma_F^m(x) = \sigma_F^n(y)$, then

$$\sigma_{E_2}^{m+2-\chi_{E_2}(x)}(\varphi(x)) = \sigma_F^{m+1}(x) = \sigma_F^{n+1}(y) = \sigma_{E_2}^{n+2-\chi_{E_2}(y)}(\varphi(y)).$$

Showing this is surjective and compatible with the binary maps $\cdot, r, s$ is straightforward.
Our next claim is that there is an action $\mathbb{Z}_2 * \mathbb{Z}_3 \curvearrowright F^\infty$ such that $\mathbb{Z}_2 * \mathbb{Z}_3 \ltimes X \cong \mathcal{G}_F$. Say $a \in \mathbb{Z}_2, b \in \mathbb{Z}_3$ are the generators. Set

$$a \cdot (x_1, x_2, x_3, \ldots) = \begin{cases} (e, x_1, ..) & x \in E_2^\infty \\ (x_2, x_3, ..) & x \in eE_2^\infty \end{cases}$$

and

$$b\cdot(x_1,x_2,x_3,\dots)=\begin{cases}(e,f_1,x_1,x_2,x_3) & x\in E_2^\infty\\(e,f_2,x_3,..) & x\in ef_1E_2^\infty\\(x_3,x_4,..) & x\in ef_2E_2^\infty\end{cases}.$$

It is clear this is an action by homeomorphisms on F^∞. Let $\varepsilon_a=1-2\chi_{eE_2^\infty}$, $\varepsilon_b=2\chi_{E_2^\infty}-2\chi_{ef_2E_2^\infty}$. Then for $g=\prod_{i=1}^n g_i$ where each $g_i\in\{a,b\}$, let $\varepsilon_g\colon F^\infty\to\mathbb{Z}$ be given by

$$\varepsilon_g(x):=\sum_{j=1}^n\varepsilon_{g_j}(\prod_{i=j+1}^n g_i x).$$

This is well-defined, and normalised by the conditions $gx=x$, $\varepsilon_g(x)=0 \iff g=1$. Let $g\in\mathbb{Z}_2*\mathbb{Z}_3$, we can define an action $\Psi\colon\mathbb{Z}_2*\mathbb{Z}_3\curvearrowright\mathcal{G}_F$ by

$$\Psi_g(x,k,y)=(gx,k+\varepsilon_g(x),y).$$

Ψ is well-defined, bijective and $\Psi_{gh}=\Psi_g\circ\Psi_h$. We claim the map

$$\Omega\colon(\mathbb{Z}_2*\mathbb{Z}_3)\ltimes F^\infty\to\mathcal{G}_F$$

is a groupoid isomorphism. Note that this is injective–

$$\Omega(g,x)=\Omega(h,y)\iff x=y,\ gh^{-1}x=x,\ \varepsilon_{gh^{-1}}(x)=0.$$

The last two equalities together imply $g=h$. This finishes the second claim.
Combining both claims together we obtain that $\mathbb{Z}_2*\mathbb{Z}_3\ltimes X\cong E_2\cong\mathcal{G}_{E_2}$. It follows that the action is topologically free, and amenable, since E_2 is amenable and effective. Furthermore, the action is minimal since $\mathcal{O}_2$ is simple (and a crossed product). □

Theorem 4.4.2. *The topological full group of $\mathbb{Z}_2*\mathbb{Z}_3\curvearrowright X$ is isomorphic to V.*

Proof. This follows via the above result combined with Matui's isomorphism theorem (Theorem 3.4.12). □

We may now employ results in Chapter 3, to reprove many of the key results of Thompson's group V using the literature surrounding topological full groups.

Example 4.4.3 (V is simple and finitely generated)**.** *One can observe that the derived subgroup of V is simple and finitely generated as a consequence of Theorem 7.2.9, Theorem 3.5.2 since the action of $\mathbb{Z}_2*\mathbb{Z}_3$ on X is expansive and minimal.*

Recall also that we computed the homology of the full shift on two generators in Example 2.4.7. We have that the groupoid is acyclic, i.e. the homology vanishes. Therefore, via Theorem 3.6.7, the abelianisation is 0*. Hence, we recover that V is perfect.*

We also see here that Szymik-Wahl's Theorem that V is acyclic is recovered through our understanding of the homology of topological full groups.

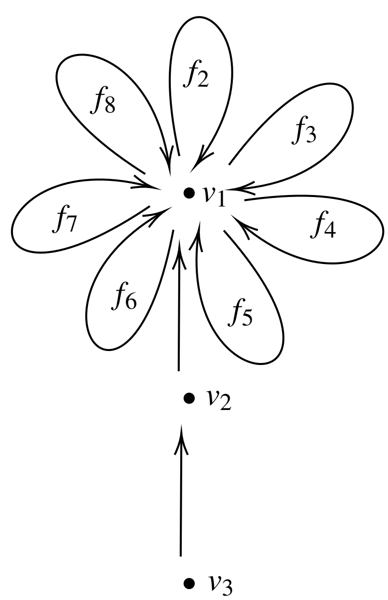
Example 4.4.4. *We are also able to explain why many of the results in Section 3.6. are a generalisation of the work of Szymik-Wahl [118]. Note that the fact that $\mathcal{G}$ is acyclic in combination with Corollary 3.6.3 recovers that V is acyclic.*

4.4.2 Generalisations of V as topological full groups

Example 4.4.5. *The Higman-Thompson groups $V_{k,r}$ also arise naturally from SFT groupoids, as explained in [89, Remark 6.3]. Namely, $V_{k,r}$ is associated to the graph groupoid $\mathcal{R}_r \times \mathcal{E}_k$ consisting of the r-stabilisation of the of the shift of finite type groupoid $\mathcal{E}_k$ associated to the single vertex graph with k loops. Then $\mathsf{F}(\mathcal{R}_r \times \mathcal{E}_k)$ has been explicitly identified with theHigman-Thompson group $V_{k,r}$ in [89, Remark 6.3]. The groupoid homology is independent of r, and it is computable via the Künneth formula as in Example 2.4.9, yielding*

$$H_i(\mathcal{R}_r \times \mathcal{E}_k) \cong \begin{cases} \mathbb{Z}_{k-1} & i = 0 \\ 0 & i > 0 \end{cases}.$$

One way to think about this in terms of a directed graph is the "flower with k petals and a stem of length $r-1$". For example, we draw the directed graph associated with $V_{7,3}$ below.



$E_{7,3}$, the directed graph whose SFT groupoid is $\mathcal{R}_3 \times \mathcal{E}_7$.

Example 4.4.6. *Let $n,k,r \in \mathbb{N}$, with $k > 2$ and consider the groupoid $\mathcal{R}_r \times \mathcal{E}_k^n$ consisting of the r-stabilisation of the n-fold product of the shift of finite type groupoid $\mathcal{E}_k$ associated to the single* *vertex graph with k loops. Then* $\mathsf{F}(\mathcal{R}_r \times \mathcal{E}_k^n)$ *can been explicitly identified with the Brin-Higman-Thompson group* $nV_{k,r}$ *(see, e.g. [90]). The groupoid homology of* $\mathcal{R}_r \times \mathcal{E}_k^n$ *groupoid homology is independent of r, and it is computable via the Künneth formula, as in Example 2.4.9 yielding:*

$$H_i(\mathcal{R}_r \times \mathcal{E}_k^n) \cong \mathbb{Z}_{k-1}^{\,^{n-1}C_i}$$

for all $i \geq 0$*, where* ${}^nC_i$ *is the binomial coefficient* ${}^nC_i = \frac{n!}{(n-i)!i!}$.

Recall that we computed the homology of products of SFT groupoids in Example 2.4.9. In light of our section on the homology of topological full groups, we can consider the homology of the Brin-Higman-Thompson groups $nV_{k,r}$. First we can use the vanishing results (Corollary 3.6.3 and Corollary 3.6.4) to understand the acyclity of Brin-Higman-Thompson groups.

Proposition 4.4.7. $nV_{2,r}$ *is acyclic for all* $n, r \in \mathbb{N}$. $nV_{k,r}$ *is rationally acyclic for all* $n, r \in \mathbb{N}$ *and* $k \in \mathbb{N}$, $k > 2$.

We can also apply Theorem 3.6.5 to understand the homological stability of the Brin-Higman-Thompson groups:

Proposition 4.4.8. *Let* $n, k, r, r' \in \mathbb{N}$ *with* $n > 0, r > r'$, *and* $k > 2$. *The inclusion* $nV_{k,r'} \hookrightarrow nV_{k,r}$ *induces an isomorphism in homology.*

Note that the above result is a generalisation of the work of Szymik-Wahl, who proved the above theorem restricted to the case $n = 1$ (in other words, the case of the Higman-Thompson groups).

Whilst realising known groups that generalise V in various ways proves fertile, it loses out on a lot of the creativity and enjoyment that one gets from studying topological full groups. This is because as well as using topological full groups to study interesting groups, one can use topological full groups to generate new examples of interesting groups.

Therefore, from the identification of V as a topological full group, a new philosophy emerged. Let $\mathcal{G}$ be a groupoid that generalises the full shift on two generators in some kind of way, for example by coming from one of these broader families of groupoids which the full shift comes from. Then, one can study the topological full group of this groupoid as a new generalisation of V, which might be otherwise hard to have found a-priori.

From C*-algebras there is a whole wealth of examples that generalise the Cuntz algebra in all kinds of interesting ways. Many of these have groupoid models, and these groupoid models give rise to V-like groups. We summarise some of this reasearch through examples below.

Example 4.4.9 (Matui-Matsumoto)**.** *Following on from Example 4.4.5, one might consider other SFT groupoids as generalised Higman-Thompson groups. This was a philosophy taken by [89]*

Matui, and later in a collaboration of Matui-Matsumoto [86], a philosophy which proved fruitful. In summary, they were able to show:

- *The topological full groups are type F_∞ (see [89], subsection 6.5).*
- *The derived subgroups are simple (see [89], subsection 4.2) but also remark this follows from Theorem 7.2.9.*
- *Proof that these groups inherit the Haagerup property from embeddings into V (see [89], subsection 6.3).*
- *Develop our understanding of the abelianisations via the AH conjecture (see [89], section 5).*
- *Representations of these groups in terms of certain PL-bijections on the unit interval $(0,1]$ (see subsection 6, [86]).*

Example 4.4.10 (Matui-Matsumoto)**.** *The groupoids in example 4.4.5 also fall into the class of groupoids arising from so-called β-expansions. Matui-Matsumoto studied the topological full groups arising from other kinds of β-expansions in [85], with this observation in mind. In this paper, they were able to show amongst other things:*

- *Such groups can be represented as groups of piecewise linear right continuous bijections on $(0,1]$ (Theorem 1.2. [85]).*
- *The derived subgroups are often simple (Theorem 1.1. [85]) and isomorphic to certain Higman-Thompson groups (Theorem 1.3 [85]).*

Example 4.4.11 (Matui)**.** *Following on from example 4.4.6, one might consider other products of SFT groupoids as generalised Brin-Higman-Thompson groups. This was a philosophy taken by Matui [90]. In this paper, he was able to show amongst other things:*

- *Classification and rigidity results ([90], Theorem 5.12).*
- *A confirmation of the AH conjecture for this class of groupoids ([90], Theorem 2.8).*

He asks if they are, like the Brin-Higman-Thompson groups, of type F_∞ but leaves this open. It was confirmed to be true in a paper of Li [78].

Example 4.4.12 (Nyland-Ortega)**.** *A third perspective of the groupoids in the example 4.4.5 is that they are the groupoids of directed graphs. One might consider other graph groupoids as generalised Higman-Thompson groups. This was the approach of Nyland-Ortega [97] [95]. Here they were able to show:*

- *Generalisations of the classification theorem of Matui-Matsumoto in several directions ([95] Theorem A, B, C and D).*

- *A confirmation of the AH conjecture for such groupoids ([97] Theorem A).*

- *A computation of groupoid homology for such groupoids ([97], Theorem B).*

Example 4.4.13. *A final example class of interest is to observe that $\mathcal{E}_2$ may be viewed as the tight groupoid of the bicyclic inverse monoid, or the universal groupoid of certain cancellative semigroup. These groupoids have natural generalisations, which have been studied extensively in work of Lawson [71], Lawson-Vdovina [73] and Lawson-Vdovina-Sims [72].*

Example 4.4.14 (Belk-Zaremsky)**.** *In [7], Belk-Zaremsky introduced a very interesting new class of groups called Twisted Brin-Thompson Groups. These are generalised Brin-Higman-Thompson groups, which allow for "switching of cordinates" by permutations of some abstract group G.*

This class of groups constructed through topological full groups have proven incredibly useful for embedding results, showing amongst other things:

- *Every finitely generated group embeds quasi-isometrically as a subgroup of a two-generated simple group.*

- *Further examples of simple groups of type F_n but not F_{n+1}.*

- *A resolution of the Boone-Higman conjecture for hyperbolic groups makes heavy use of these groups [5].*

4.5 Outlook

We note in addition to the examples listed above, there are other classes of groupoids which yet to be studied in greater detail. We could, for instance, study the following as generalisations of Thompson's group V.

- The topological full groups of purely infinite crossed products of Rordram-Sierakowski.

- The tight groupoids of other classes of inverse monoids.

In studying Thompson-like groups through the framework of topological full groups, open questions are abundant. The first is to notice a theme that we can understand many things about topological full groups using techniques once used to understand Thompson's group V. For example, we generalised pre-existing literature on V as the basis to understand many things about topological full groups in general:

- Matui's proofs and theorems concerning simplicity and finite generation of derived subgroups are generalisations of well known arguments made for Thompson's group V.

- The research concerning C*-simplicity of topological full groups and the connection to amenability [24] [110] is largely inspired by the work of Le Boudec-Matte Bon [74] on Thompson's group V.
- In homology the work of Li [79] was largely inspired by the work of Szymik-Wahl [118] on Thompson groups V.
- The standard proof of finiteness properties for topological full groups, e.g. type F_∞ proofs contained in [95] [78] [89] still are generalisations of Brown's original proof for V, and our understanding of the Haagerup property for topological full groups remain based largely on generalisations of Farley's techniques [48].

Clearly, many of the techniques used to understand Thompson's group V are useful to understand topological full groups. There are many other interesting lines of inquiry about Thompson's group V which could give us insight into the properties of topological full groups in general. For example, consider two veins of research around Thompson-like groups:

- Embedding and nonembedding results such as [13] [29] [111].
- Classification of Higman-Thompson groups, and of the Brin-Higman-Thompson groups, via the associated Steinberg algebras [98] [44].

Such papers contain techniques that may generalise to other topological full groups. There are of course also many interesting open questions about Thompson-like groups in their own right. We list some well-known problems below:

Question 4.5.1. *Is Thompson's group F amenable?*

This major open problem, open for decades can be rephrased in terms of C*-simplicity of T, see [56] [12].

Question 4.5.2. *Is Thompson's group T C*-simple?*

There are also some less well-known open questions about Thompson-like groups.

Question 4.5.3. *Does $2V_{2,1}$ have the Haagerup property?*

This is interesting because of the abundance of groups which embed into the group $2V_{2,1}$, and that the Haagerup property is hereditary. We also do not understand fully the integral homology of the Higman-Thompson groups, which would be interesting to understand better.

Question 4.5.4. *What is the homology of $V_{3,1}$?*

For Stein's groups, we answer some questions in Chapter 7. What is left open is explored in the outlook section of Chapter 7.

Chapter 5

Topological full groups of purely infinite groupoids

5.1 Characterisations of pure infiniteness and minimality

We begin by recalling the notion of pure infiniteness for étale groupoids due to Matui [89]. All groupoids in this section are effective and ample.

Definition 5.1.1. *A groupoid $\mathcal{G}$ is said to be* purely infinite *if for every compact and open set $X \subseteq \mathcal{G}^{(0)}$ there exist compact open bisections $B, B' \subseteq \mathcal{G}$ such that $s(B) = s(B') = X$, and $r(B), r(B')$ are disjoint and contained in X.*

The terminology above is justified by the fact that the reduced groupoid C*-algebra of a minimal, essentially principal, purely infinite étale groupoid is purely infinite and simple. The following result is known to the experts [89], but we provide a proof nonetheless for the convenience of the reader. Recall that a projection in a C*-algebra is said to be *properly infinite* if it is Murray-von Neumann equivalent to two pairwise orthogonal subprojections of itself.

Proposition 5.1.2. *Let $\mathcal{G}$ be a minimal, essentially principal, purely infinite, ample groupoid. Then $C_r^*(\mathcal{G})$ is purely infinite and simple.*

Proof. Simplicity of $C_r^*(\mathcal{G})$ is well-known, so we only prove pure infiniteness. By [18, Theorem B], it suffices to show that every projection in $C_0(\mathcal{G}^{(0)})$ is properly infinite in $C_r^*(\mathcal{G})$. Let $p \in C_0(\mathcal{G}^{(0)})$ and let X denote its support, which is compact and open. Use pure infiniteness of $\mathcal{G}$ to obtain compact open bisections $B, B' \subseteq \mathcal{G}$ satisfying $s(B) = s(B') = X$ and $r(B) \sqcup r(B') \subseteq X$. Set $v = 1_B$ and $w = 1_{B'}$, which belong to $C_c(\mathcal{G})$ since B and B' are compact. The conditions on B and B' easily give

$$v^*v = w^*w = p, \quad vv^* \perp ww^*, \text{ and } vv^*, ww^* \leq p.$$

Thus vv^* and ww^* are pairwise orthogonal subprojections of p which are Murray-von Neumann equivalent to p. We conclude that p is properly infinite, and hence that $C_r^*(\mathcal{G})$ is purely infinite.

□

Note that many ample groupoids are known to be purely infinite and minimal, for example:

1. Transformation groupoids of certain nonamenable groups acting amenably, such as those studied in [46, 105].

2. Shift of finite type groupoids [89], or more generally, large classes of graph groupoids [95].

3. Groupoids arising from Beta expansions [85].

4. Certain groupoids arising from left regular representations of Garside categories [78]; see also [80, Theorem A].

Next, we recall the fact, due to Matui, that the subgroups $\mathsf{A}(\mathcal{G})$ and $\mathsf{D}(\mathcal{G})$ agree for the class of purely infinite, minimal groupoids.

Proposition 5.1.3. *Let $\mathcal{G}$ be an ample, effective, purely infinite and minimal groupoid. Then $\mathsf{D}(\mathcal{G})$ is simple and thus $\mathsf{A}(\mathcal{G}) = \mathsf{D}(\mathcal{G})$.*

Proof. Simplicity of $\mathsf{D}(\mathcal{G})$ follows from Theorem 3.3.2. Thus $\mathsf{A}(\mathcal{G}) = \mathsf{D}(\mathcal{G})$ since $\mathsf{A}(\mathcal{G})$ is normal in $\mathsf{D}(\mathcal{G})$.

□

For the remainder of this Chapter, we prefer the notation $\mathsf{A}(\mathcal{G})$ over $\mathsf{D}(\mathcal{G})$ for consistency with the paper. Let us remark that as in the introduction theorems, we often use the notation $\mathsf{D}(\mathcal{G})$ since the definition of $\mathsf{A}(\mathcal{G})$ is technical. We apologise to the reader for this discrepancy.

5.1.1 Embeddings of Thompson's group V

In [89, Proposition 4.10], Matui proved that if $\mathcal{G}$ is purely infinite, then $\mathsf{F}(\mathcal{G})$ contains $\mathbb{Z}_2 * \mathbb{Z}_3$ as a subgroup (in fact, for this it suffices for $\mathcal{G}$ to be *properly infinite*). In particular, $\mathsf{F}(\mathcal{G})$ is nonamenable. We will strengthen this result by showing that the Thompson group V *always* embeds into $\mathsf{F}(\mathcal{G})$. In fact, we give a characterisation of pure infiniteness of $\mathcal{G}$ in terms of the existence of certain embeddings of V into $\mathsf{F}(\mathcal{G})$; see Theorem 5.1.6.

We need some preparation first. The following notion is due to Bleak, Elliott and Hyde [11].

Definition 5.1.4. *([11, Definition 1.1]). Let $\Gamma \leq \mathrm{Homeo}(\mathcal{C})$ be a subgroup of homeomorphisms of the Cantor space $\mathcal{C}$. We say that Γ is* vigorous[1] *if whenever $X, Y_1, Y_2 \subseteq \mathcal{C}$ are compact and open with $Y_1 \neq X$ and $Y_2 \neq \varnothing$, and satisfy $Y_1, Y_2 \subseteq X$, then there exists $g \in \Gamma$ such that g is the identity on $\mathcal{C} \setminus X$, and $g(Y_1) \subseteq Y_2$.*

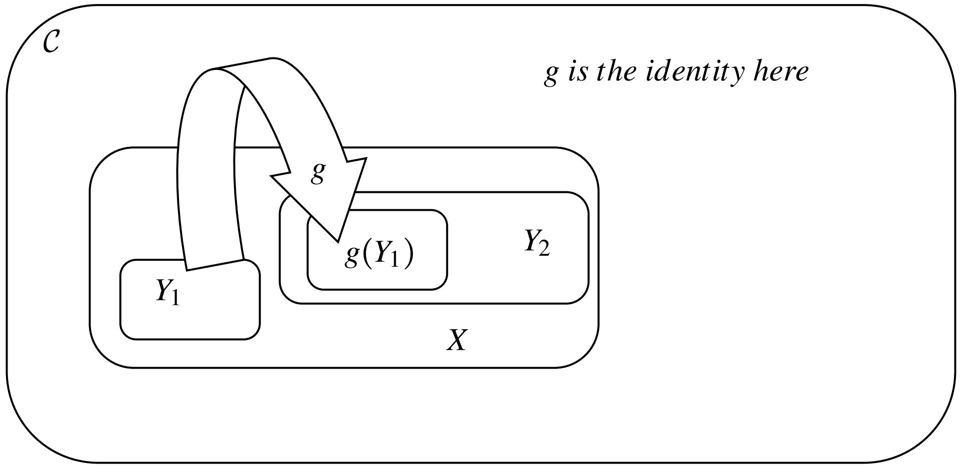


Figure 5.1: A diagram to visualise the property of vigor. Given $Y_1, Y_2 \subset X$ compact open subsets such that $X \neq \mathcal{C}$, $Y_2 \neq \varnothing$ there exists a group element g which maps Y_1 inside Y_2, whilst acting as the identity outside of A.

We will need the following observations.

Proposition 5.1.5. *Let $\Gamma \leq \mathrm{Homeo}(\mathcal{C})$ be a vigorous group. Then:*

1. *There is no Γ-invariant Borel probability measure on $\mathcal{C}$. In particular, Γ is not amenable.*
2. *Γ has infinite conjugacy classes (ICC).*

Proof. (1). Let μ be a Borel probability measure on $\mathcal{C}$. Choose nontrivial compact and open sets $Y_1, Y_2 \subseteq \mathcal{C}$ with $\mu(Y_1) < \mu(Y_2)$. By vigor, taking $X = \mathcal{C}$, there is $g \in \Gamma$ with $g(Y_1) \subseteq Y_2$. Then $\mu(gY_2) \leq \mu(Y_1) < \mu(Y_2)$, so $g \cdot \mu \neq \mu$ and hence μ is not Γ-invariant.

The last claim follows since a discrete group is amenable if and only if every action on a compact Hausdorff space admits an invariant Borel probability measure.

(2). Let $g \in \Gamma \setminus \{1\}$; we will show that the conjugacy class of g is infinite. Find a compact and open subset $Y_1 \subseteq \mathcal{C}$ such that $g(Y_1) \cap Y_1 = \varnothing$, and find an infinite family $\{Y_2^{(n)} : n \in \mathbb{N}\}$ of pairwise disjoint compact and open subsets of Y_1. Use vigor, with $X = \mathcal{C}$ of Γ to find, for every $n \in \mathbb{N}$, a group element $h_n \in \Gamma$ with support contained in $\mathcal{C} \setminus Y_1$, such that $h_n(g(Y_1)) \subseteq Y_2^{(n)}$. One readily checks that $h_n^{-1} g h_n(Y_1) \subseteq Y_2^{(n)}$ for all $n \in \mathbb{N}$; in particular, this implies that $h_n^{-1} g h_n \neq h_m^{-1} g h_m$ if $n \neq m$, as desired. □

Note that in the following theorem, we must exclude $X = \mathcal{G}^{(0)}$, since the existence of an embedding $V \hookrightarrow \mathsf{F}(\mathcal{G})$ with full support implies that $\mathcal{G}^{(0)}$ has trivial homology class.

Theorem 5.1.6. *Let $\mathcal{G}$ be an essentially principal, étale, Cantor, purely infinite and minimal groupoid. Then, for every nontrivial compact and open subset $X \subseteq \mathcal{G}^{(0)}$, there exists an embedding $\varphi_X : V \hookrightarrow \mathsf{A}(\mathcal{G})$ such that $\mathrm{supp}(\varphi_X(V))$ contains X.*

[1]It would be more technically correct to say that the action of Γ on $\mathcal{C}$ is vigorous, since vigor is not a property of Γ, but rather of the way it sits in $\mathrm{Homeo}(\mathcal{C})$.

Proof. Let $X \subseteq \mathcal{G}^{(0)}$ be nontrivial, compact and open. Use Lemma 3.3.3 to find a compact and open bisection B such that $s(B) = \mathcal{G}^{(0)}$ and $r(B) \subsetneq \mathcal{G}^{(0)} \setminus X$. Set $Y = \mathcal{G}^{(0)} \setminus r(B)$, which is a compact and open subset of $\mathcal{G}^{(0)}$ containing X. Moreover,

$$[Y] = [\mathcal{G}^{(0)}] - [r(B)] = [s(B)] - [r(B)] = 0.$$

Denote by $\mathcal{G}|_Y$ the *reduction* of $\mathcal{G}$ to Y, namely

$$\mathcal{G}|_Y = r^{-1}(Y) \cap s^{-1}(Y),$$

which is an étale subgroupoid of $\mathcal{G}$ with unit space equal to Y. Since Y is itself nonempty and compact and open, there is an embedding $\mathsf{F}(\mathcal{G}_Y) \hookrightarrow \mathsf{F}(\mathcal{G})$ whose support is exactly Y. Denote by $\mathcal{E}_2$ the groupoid associated to the shift of finite type corresponding to the single vertex graph with 2 loops. (This groupoid is also called the *Cuntz groupoid*, since its C*-algebra is canonically isomorphic to the Cuntz algebra $\mathcal{O}_2$.) Recall that $\mathsf{F}(\mathcal{E}_2) \cong V$ and that $H_0(\mathcal{E}_2) = \{0\}$. Since the class of the unit space of $\mathcal{G}|_Y$ is trivial in homology, by [90, Proposition 5.14] there exists a unital homomorphism $\pi\colon C_r^*(\mathcal{E}_2) \to C_r^*(\mathcal{G}_Y)$ satisfying

(a) $\pi(C(\mathcal{E}_2^{(0)})) \subseteq C(Y) = C(\mathcal{G}_Y^{(0)})$, and

(b) for every compact, open bisection $B \subseteq \mathcal{E}_2$, there exists a compact, open bisection $B' \subseteq \mathcal{G}_Y$ such that $\pi(1_B) = 1_{B'}$.

Since $C_r^*(\mathcal{E}_2) \cong \mathcal{O}_2$ is simple, the map π is injective. In particular, π induces an embedding

$$\varphi_X\colon V \cong \mathsf{F}(\mathcal{E}_2) \hookrightarrow \mathsf{F}(\mathcal{G}_Y) \subseteq \mathsf{F}(\mathcal{G}).$$

It remains to show that the image of the embedding φ_X is actually contained in $\mathsf{A}(\mathcal{G}_X)$. By Theorem 5.1.3, it suffices to show that the image of this embedding is contained in $\mathsf{D}(\mathcal{G}_X) = \left[\mathsf{F}(\mathcal{G}_X), \mathsf{F}(\mathcal{G}_X)\right]$. Since group homomorphisms map commutators to commutators, and since V is equal to its own commutator (because it is simple), it follows that the image of the embedding $V \hookrightarrow \mathsf{F}(\mathcal{G}|_X)$ is contained in $\mathsf{D}(\mathcal{G}_X) = \mathsf{A}(\mathcal{G}_X)$. □

The last statement in the above proposition is not equivalent to the remaining ones. That is, having sufficiently many embeddings of V into the topological full group does not imply pure infiniteness of the groupoid. For example, if V was to act trivially on the Cantor space, then the associated transformation groupoid admits an embedding of V on any subset of the unit space, however, it is not purely infinite.

We also do not seem to be able to obtain much information of the dynamics of these embeddings of V. The next example, shows that V does not always act vigorously.

Example 5.1.7. *Consider the natural embeddings $nV \hookrightarrow (n+1)V$ given by the subgroup of $(n+1)V$ which stabilises the 1st coordinate. Consider the inductive limit $\mathbb{N}V = \bigcup_{n\in\mathbb{N}} nV$ This group agrees with the higher dimensional twisted Brin Thompson group $\mathbb{N}V_1$ taking the trivial action.*

Let $\mathcal{G}$ denote the groupoid of germs for the action of $\mathbb{N}V \curvearrowright \mathcal{C}$. Then $\mathcal{G}$ is purely infinite, minimal, ample, Cantor groupoid and $\mathbb{N}V$ is the topological full group of this groupoid.

Then, let G be any finitely generated subgroup of $\mathbb{N}V$. We have that G is contained in nV for some $n \in \mathbb{N}$. Therefore, the action of G on $\mathcal{C}$ is not vigorous.

In particular, if G was isomorphic to V, we have that the action of V on $\mathcal{C}$ is not purely infinite.

We now derive some conclusions regarding the (generalised) word problem for topological full groups. Knowing that topological full groups contain V allows us to deduce that they have unsolvable generalised word problem:

Corollary 5.1.8. *Let $\mathcal{G}$ be a minimal, essentially principal, ample, purely infinite groupoid. Then $\mathsf{F}(\mathcal{G})$ and $\mathsf{A}(\mathcal{G})$ do not have a solvable generalised word problem.*

Proof. Note that unsolvable generalised word problems are inherited by containing groups. Thus, it suffices to show the statement for the alternating group. In turn, since $\mathsf{A}(\mathcal{G})$ contains V by Theorem 5.1.6, it suffices to argue that V has an unsolvable generalized word problem; this follows from [91]. □

In contrast to the above corollary, in many cases $\mathsf{F}(\mathcal{G})$ *does* have a solvable word problem, for example the topological full groups of many groupoids that arise as left regular representations of Garside Categories (see [78, Corollary C]) are known to have solvable word problem. Therefore, we obtain infinitely many nonisomorphic groups spanning many families that demonstrate this boundary between the word problem and the generalised word problem.

5.1.2 Vigor

In fact, vigor of the derived subgroup is *equivalent* to the groupoid being purely infinite and minimal even in the effective case. Vigor was introduced by Bleak-Elliot-Hyde to investigate the property of 2-generation for simple grous. Their main theorem is recalled below:

Theorem 5.1.9. *[11, Theorem 1.12] Let Γ be simple, finitely generated and vigorous. Let $n \in \mathbb{N}$ with $n > 2$. Then there exists group elements g_1, g_2 of finite order, with g_2 of order n such that Γ is generated by g_1 and g_2.*

Proposition 5.1.10. *Let $\mathcal{G}$ be an étale, effective Cantor groupoid. Then the following are equivalent:*

1. *$\mathcal{G}$ is purely infinite and minimal.*

2. *One (equivalently, both) of* $\mathsf{A}(\mathcal{G})$ *or* $\mathsf{F}(\mathcal{G})$ *is vigorous.*

Proof. Note that vigor of $\mathsf{A}(\mathcal{G})$ implies vigor of $\mathsf{F}(\mathcal{G})$. Thus, it suffices to show that (1) is equivalent to vigor of $\mathsf{A}(\mathcal{G}) \leq \mathrm{Homeo}(\mathcal{G}^{(0)})$.

(1) implies (2). This proof is entirely constructive. Let $X, Y_1, Y_2 \subseteq \mathcal{G}^{(0)}$ be compact and open sets with $Y_1, Y_2 \subseteq X$, $Y_1 \neq X$ and $Y_2 \neq \varnothing$. We aim to find a multisection $\alpha \in \mathsf{F}(\mathcal{G})$ of order 3 such that α is the identity outside of X and takes Y_1 inside Y_2. Write $Y_2 \setminus Y_1$ as a nontrivial disjoint union $Y_2 \setminus Y_1 = Z_{2,1} \sqcup Z_{2,2}$ of compact and open sets. Since $\mathcal{G}$ is minimal and purely infinite, part (2) of Lemma 3.3.3 provides us with a compact open bisection $B_1 \subseteq \mathcal{G}$ with $s(B_1) = Y_1$ and $r(B_1) \subseteq Z_{2,1}$. Use part (2) of Lemma 3.3.3 again to find a compact open bisection B_2 with $s(B_2) = r(B_1) \subseteq Z_{2,1}$ and $r(B_2) \subseteq Z_{2,2}$. Note that $(B_2B_1)^{-1}$ is also a bisection. Set

$$\alpha = \big((B_2B_1)^{-1} \cup B_1 \cup B_2\big) \cup \big(\mathcal{G}^{(0)} \setminus s(B_1) \cup r(B_1) \cup r(B_2)\big).$$

One readily checks that α is a full bisection, so it defines an element of $\mathsf{F}(\mathcal{G})$. It is also clear that it defines a multisection in the sense of Definition 3.2.1, since it satisfies

$$\alpha(Y_1) = Z_{2,1}, \quad \alpha(Z_{2,1}) = Z_{2,2}, \quad \text{and} \quad \alpha(Z_{2,2}) = Y_1,$$

while α acts trivially on the rest of $\mathcal{G}^{(0)}$. Since α has order 3, it follows that $\alpha \in \mathsf{A}(\mathcal{G})$. Note that α is the identity outside of X, and that $\alpha(Y_1) = r(B_1) \subseteq Z_{2,1} \subseteq Y_2$. This shows that $\mathsf{A}(\mathcal{G})$ is vigorous.

(2) implies (1). We check condition (2) of Lemma 3.3.3. Let Y_1, Y_2 be nonempty compact and open subsets of $\mathcal{G}^{(0)}$. Use vigor with $X = \mathcal{C}$ to find $g \in \mathsf{A}(\mathcal{G})$ with $\mathrm{supp}(g) = Y_1 \cup Y_2$ and $g(Y_1) \subseteq Y_2$. If $B \subseteq \mathcal{G}$ denotes the compact open full bisection determining g, then $B|_{Y_1}$ is a compact open bisection satisfying $s(B|_{Y_1}) = Y_1$ and $r(B|_{Y_1}) \subseteq Y_2$, as desired. □

As an immediate consequence, we show that $\mathsf{F}(\mathcal{G})$ enjoys a strengthening of the ICC property with respect to $\mathsf{D}(\mathcal{G})$; this will be needed later.

Corollary 5.1.11. *Let $\mathcal{G}$ be a minimal, purely infinite, étale Cantor groupoid. Then the* $\mathsf{D}(\mathcal{G})$-*conjugacy class of every nontrivial element of* $\mathsf{F}(\mathcal{G})$ *is infinite.*

Proof. This proof is similar to that of Lemma 5.1.5. Let $g \in \mathsf{F}(\mathcal{G}) \setminus \{1\}$. Find a compact and open subset $Y_1 \subseteq \mathcal{G}^{(0)}$ such that $g(Y_1) \cap Y_1 = \varnothing$, and find an infinite family $\{Y_2^{(n)} : n \in \mathbb{N}\}$ of pairwise disjoint compact and open subsets of Y_1. Use vigor of $\mathsf{A}(\mathcal{G})$ to find, for every $n \in \mathbb{N}$, a group element $h_n \in \mathsf{A}(\mathcal{G})$ with support contained in $\mathcal{G}^{(0)} \setminus Y_1$, such that $h_n(g(Y_1)) \subseteq Y_2^{(n)}$. One readily checks that $h_n^{-1} g h_n \neq h_m^{-1} g h_m$ if $n \neq m$, as they map Y_1 to different subsets (namely $Y_2^{(n)}$ and $Y_2^{(m)}$, respectively). This finishes the proof. □

We obtain the following consequence of our results in combination with [11] and results from Chapter 3.

Corollary 5.1.12. *Let $\mathcal{G}$ be an expansive, ample, essentially principal, minimal, purely infinite groupoid. Given $n > 2$, there exist $g_1, g_2 \in \mathsf{A}(\mathcal{G})$ such that g_1 has order n, the order of g_2 is finite, and $\mathsf{A}(\mathcal{G})$ is generated by $\{g_1, g_2\}$.*

Proof. Since $\mathcal{G}$ is minimal and essentially principal, it follows from Theorem 5.1.3 that $\mathsf{A}(\mathcal{G})$ is simple. Moreover, since $\mathcal{G}$ is expansive, Theorem 3.5.12 implies that $\mathsf{A}(\mathcal{G}) = \mathsf{D}(\mathcal{G})$ is finitely generated. By Lemma 5.1.10, $\mathsf{A}(\mathcal{G})$ is vigorous, so the result follows from Theorem 5.1.9. □

5.1.3 Compressibility

The existence of compressible actions was shown to have relevance to the representation theory of Thompson-like groups in [45]. We recall the definition below:

Definition 5.1.13. *An action of a discrete group Γ on a locally compact Hausdorff space X is said to be* compressible*, if there exists a subbase $\mathcal{U}$ for the topology on X such that:*

1. *for all $g \in \Gamma$, there exists $U \in \mathcal{U}$ such that* $\operatorname{supp}(g) \subseteq U$;
2. *for all $U_1, U_2 \in \mathcal{U}$, there exists $g \in \Gamma$ such that $g(U_1) \subseteq U_2$;*
3. *for all $U_1, U_2, U_3 \in \mathcal{U}$ with $\overline{U}_1 \cap \overline{U}_2 = \varnothing$, there exists $g \in \Gamma$ such that $g(U_1) \cap U_3 = \varnothing$ and* $\operatorname{supp}(g) \cap U_2 = \varnothing$;
4. *for all $U_1, U_2 \in \mathcal{U}$, there exists $U_3 \in \mathcal{U}$ such that $U_1 \cup U_2 \subseteq U_3$.*

Remark 5.1.14. *Note that if $\Gamma \curvearrowright X$ is compressible, then X cannot be compact. This is because X cannot belong to $\mathcal{U}$ by (2), and at the same time X cannot be written as a finite union of elements of $\mathcal{U}$ by (4). Therefore often when dealing with a group acting by homeomorphisms on the Cantor space, we consider point-stabiliser subgroups.*

Compressibility and vigor are closely related notions. For example, the following is essentially a generalisation of the discussion in [45, Subection 3.2].

Lemma 5.1.15. *Let $\mathcal{C}$ be a Cantor space and let $D \leq \operatorname{Homeo}(\mathcal{C})$ be a vigorous subgroup. For $x_0 \in \mathcal{C}$, set*

$$D_{x_0} := \{g \in D\colon \text{there exists a neighbourhood } Y \text{ of } x_0 \text{ such that } g|_Y = Id_Y\}.$$

Then $D_{x_0} \curvearrowright \mathcal{C} \setminus \{x_0\}$ is compressible.

Proof. For an open set $Y \subseteq \mathcal{C}$, we set $D_Y = \{g \in D\colon g|_Y = \mathrm{Id}_Y\}$. Fix $x_0 \in \mathcal{C}$, and let $(Y_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of compact and open subsets of $\mathcal{C}$ such that $\bigcap_{n=1}^{\infty} Y_n = \{x_0\}$. Then $D_{x_0} = \bigcup_{n=1}^{\infty} D_{Y_n}$.

Let $\mathcal{U}$ be any basis of compact open subsets for the topology on $\mathcal{C} \setminus \{x_0\}$ which is closed under finite unions. We verify properties (1) through (4) in Definition 5.1.13 for $D_{x_0} \curvearrowright \mathcal{C} \setminus \{x_0\}$ below.

(1). Let $g \in D_{x_0}$, and find a neighbourhood Y of x_0 such that $g|_Y = \mathrm{Id}_Y$. Find a basic open set $U \in \mathcal{U}$ such that $U \subseteq Y \setminus \{x_0\}$. Since $\mathcal{C} \setminus Y$ is compact, there exist $U_1, \dots, U_n \in \mathcal{U}$ such that the set $U = U_1 \cup \dots \cup U_n$ contains $\mathcal{C} \setminus Y$. Since $\mathcal{U}$ is closed under finite unions, it follows that U belongs to $\mathcal{U}$. Since g is supported on U, this shows (1).

(2). Let $U_1, U_2 \in \mathcal{U}$ be given. Since $x_0 \notin U_1, U_2$, this follows immediately by using vigor of $D \curvearrowright \mathcal{C}$.

(3). Let $U_1, U_2, U_3 \in \mathcal{U}$ satisfy $\overline{U}_1 \cap \overline{U}_2 = \varnothing$. Since $U_1 \cup U_2 \cup U_3 \subseteq \mathcal{C} \setminus \{x_0\}$, we may find $n \in \mathbb{N}$ large enough so that $U_j \cap Y_n = \varnothing$ for $j = 1, 2, 3$. Set $X = \overline{U_1} \cup Y_n \setminus Y_{n+1}$. Noting that $U_2 \cap X = \varnothing$ and that X^c contains the neighbourhood Y_{n+1} of x_0, we use vigor to find an element $g \in D_{x_0}$ such that

$$g(U_1) \subseteq g(\overline{U_1}) \subseteq Y_n \setminus Y_{n+1} \subseteq Y_n \subseteq (U_1 \cup U_2 \cup U_3)^c \subseteq U_3^c.$$

Moreover, the support of g, which is a subset of X, contains a neighbourhood of x_0 and is contained in U_2^c. Thus $g \in D_{x_0}$ satisfies the required properties.

(4). This follows by construction, as $\mathcal{U}$ is closed under finite unions. □

We now obtain further characterisations of pure infiniteness for minimal groupoids:

Lemma 5.1.16. *Let $\mathcal{G}$ be an essentially principal, étale Cantor groupoid. Then, the following are equivalent:*

1. *$\mathcal{G}$ is purely infinite and minimal.*
2. *For all $x_0 \in \mathcal{G}^{(0)}$, the action $\mathsf{A}(\mathcal{G})_{x_0} \curvearrowright \mathcal{G}^{(0)} \setminus \{x_0\}$ is compressible*

Proof. If $\mathcal{G}$ is purely infinite and minimal, then $\mathsf{A}(\mathcal{G})$ is vigorous by Lemma 5.1.10, and thus $\mathsf{A}(\mathcal{G})_{x_0} \curvearrowright \mathcal{G}^{(0)} \setminus \{x_0\}$ is compressible for all $x_0 \in \mathcal{G}^{(0)}$ by Lemma 5.1.15. Conversely, suppose that $\mathsf{A}(\mathcal{G})_{x_0} \curvearrowright \mathcal{G}^{(0)} \setminus \{x_0\}$ is compressible for all $x_0 \in \mathcal{G}^{(0)}$. Let $X, Y \subseteq \mathcal{G}^{(0)}$ be compact and open subsets. Shrinking Y if necessary, we may assume that $X \cup Y$ is strictly smaller than $\mathcal{G}^{(0)}$. Fix $x_0 \in \mathcal{G}^{(0)} \setminus (X \cup Y)$, and use compressibility of $\mathsf{A}(\mathcal{G})_{x_0} \curvearrowright \mathcal{G}^{(0)} \setminus \{x_0\}$ to find a subbasis $\mathcal{U}$ for the topology of $\mathcal{G}^{(0)} \setminus \{x_0\}$ satisfying the conditions in Definition 5.1.13. Given $U_1, U_2 \in \mathcal{U}$ with $U \subseteq X$ and $U_2 \subseteq Y$, use (2) in Definition 5.1.13 to find $g \in \mathsf{A}(\mathcal{G})_{x_0}$ such that $g(X) \subseteq g(U_1) \subseteq U_2 \subseteq Y$. Restricting the source of the bisection corresponding to g to X, this shows that there is a compact and open bisection $B \subseteq \mathcal{G}^{(0)}$ satisfying $\alpha_B(X) \subseteq Y$. Since X and Y are arbitrary, Lemma 3.3.3 shows that $\mathcal{G}^{(0)}$ is purely infinite and minimal, as desired. □

We now turn to the definition of proper characters in a group. We warn the reader that these are not characters in the sense of Pontryagin duality, that is, they are not group homomorphisms to the unit circle, and they are only assumed to be invariant under conjugation.

Definition 5.1.17. *Let Γ be a group. A* character *on Γ is a map $\chi:\Gamma\to\mathbb{C}$ satisfying the following conditions:*

(a) $\chi(gh)=\chi(hg)$ for all $g,h\in\Gamma$;

(b) $\chi(1)=1$;

(c) for every finite collection $\{g_1,\ldots,g_n\}$ of elements in Γ, the $n\times n$ matrix with (i,j)-th entry $(\chi(g_ig_j^{-1}))$, is non-negatively definite.

A character χ is called decomposable if there exist characters χ_1,χ_2 and a real number $\lambda\in(0,1)$ such that $\chi=\lambda\chi_1+(1-\lambda)\chi_2$. Otherwise, χ is called indecomposable.

The regular character *is the indecomposable character given by $\chi(g)=0$ whenever $g\neq 1$. The* identity character *is the indecomposable character given by $\chi(g)=1$ for all $g\in\Gamma$. We say that Γ* has no proper characters *if the only indecomposable characters are the identity character and the regular character.*

Using this definition, we deduce the following useful fact about derived subgroups in purely infinite groupoids.

Corollary 5.1.18. *Let $\mathcal{G}$ be a purely infinite and minimal Cantor groupoid. Then $\mathsf{A}(\mathcal{G})$ has no proper characters.*

Proof. Note that $\mathsf{A}(\mathcal{G})$ is vigorous by Proposition 5.1.10. Fix $x_0\in\mathcal{G}^{(0)}$. It follows from Lemma 5.1.15 that $\mathsf{A}(\mathcal{G})_{x_0}\curvearrowright\mathcal{G}^{(0)}\setminus\{x_0\}$ is compressible.

We claim that $\mathsf{A}(\mathcal{G})_{x_0}$ is simple. Let $(Z_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact and open subsets of $\mathcal{G}^{(0)}\setminus\{x_0\}$ such that $\bigcup_{n\in\mathbb{N}}Z_n=\mathcal{G}^{(0)}\setminus\{x_0\}$. Note that $\mathcal{G}|_{Z_n}$ is a purely infinite minimal Cantor groupoid for all $n\in\mathbb{N}$, and thus $\mathsf{A}(\mathcal{G}|_{Z_n})$ is simple by Theorem 5.1.3. Therefore $\mathsf{A}(\mathcal{G})_{x_0}=\bigcup_{n\in\mathbb{N}}\mathsf{A}(\mathcal{G}|_{Z_n})$ is simple as well.

Applying [45, Theorem 2.9], we deduce that $\mathsf{A}(\mathcal{G})_{x_0}$ has no proper characters. This implies that $\mathsf{A}(\mathcal{G})$ has no proper characters: if χ was a character for $\mathsf{A}(\mathcal{G})$, then it restricts to a character on $\mathsf{A}(\mathcal{G})_{x_0}$, therefore $\mathsf{A}(\mathcal{G})_{x_0}\subseteq\ker(\chi)$, and by simplicity we then get $\ker(\chi)=\mathsf{A}(\mathcal{G})$, as desired. □

We can also obtain some information about the finite factor representations of topological full groups. Before we define these representations, we recall that for a subset $S\subseteq\mathsf{B}(H)$ of the bounded operators on a Hilbert space H, the commutant S' is the weak-$*$ closed subalgebra of $\mathsf{B}(H)$ given by

$$S'=\{a\in\mathsf{B}(H): as=sa \text{ for all } s\in S\}.$$

Definition 5.1.19. *Let $\pi:\Gamma\to\mathsf{B}(H)$ be a unitary representation of a group Γ on a Hilbert space H. We say that π is a* finite factor representation *if $\mathcal{M}_\pi=\pi(\Gamma)''$ is a factor, that is, $\mathcal{M}_\pi'\cap\mathcal{M}_\pi=\mathbb{C}id_H$.*

It is well known that finite factor representations are in one-to-one correspondence with proper characters. For a groupoid $\mathcal{G}$, we write

$$\pi_{\mathrm{ab}}^{\mathcal{G}}\colon \mathsf{F}(\mathcal{G}) \to \mathsf{F}(\mathcal{G})_{\mathrm{ab}} \cong \mathsf{F}(\mathcal{G})/\mathsf{D}(\mathcal{G})$$

for the canonical quotient map (the abelianisation).

Proposition 5.1.20. *Let $\mathcal{G}$ be a purely infinite, minimal Cantor groupoid, and let $\chi\colon \mathsf{F}(\mathcal{G}) \to \mathbb{T}$ be an indecomposable character. Then χ is either regular, or has the form $\chi(g) = \rho \circ \pi_{\mathrm{ab}}^{\mathcal{G}}$ for some group homomorphism $\rho\colon \mathsf{F}(\mathcal{G})_{\mathrm{ab}} \to \mathbb{T}$. In particular, the finite factor representations of $\mathsf{F}(\mathcal{G})$ are all of the form $g \mapsto \rho(\pi_{\mathrm{ab}}^{\mathcal{G}}(g))Id_H$, where ρ is a character on $\mathsf{F}(\mathcal{G})_{\mathrm{ab}}$ and H is a Hilbert space.*

Proof. We verify the assumptions of [45, Theorem 2.11] for $G = \mathsf{F}(\mathcal{G})$ and $R = \mathsf{A}(\mathcal{G})$. Note that $\mathsf{A}(\mathcal{G})$ is ICC by the combination of Theorem 5.1.10 and part (2) of Lemma 5.1.5; has no proper characters by Corollary 5.1.18; and is normal in $\mathsf{F}(\mathcal{G})$.

Let $g \in \mathsf{F}(\mathcal{G}) \setminus \{1\}$. Find a compact and open subset $Y_1 \subseteq \mathcal{G}^{(0)}$ such that $g(Y_1) \cap Y_1 = \varnothing$. Find an infinite family $\{Y_2^{(n)}\colon n \in \mathbb{N}\}$ of nonempty pairwise disjoint compact and open subsets of Y_1. Since the canonical action of $\mathsf{A}(\mathcal{G})$ on $\mathcal{G}^{(0)}$ is vigorous by Theorem 5.1.10, for every $n \in \mathbb{N}$ there exists $h_n \in \mathsf{A}(\mathcal{G})$ with support contained in $\mathcal{G}^{(0)} \setminus Y_1$, such that $h_n(g(Y_1)) \subseteq Y_2^{(n)}$. For $n, m \in \mathbb{N}$ distinct, it follows that $h_n^{-1} g h_n$ is different from $h_m^{-1} g h_m$. Moreover, we have

$$(h_n g^{-1} h_n^{-1})(h_m g h_m^{-1}) = (h_n g^{-1} h_n^{-1} g)(g^{-1} h_m g h_m^{-1}) = \left[h_n, g^{-1}\right]\left[g^{-1}, h_m\right],$$

which therefore belongs to $\mathsf{D}(\mathcal{G}) = \mathsf{A}(\mathcal{G})$. The result now follows from [45, Theorem 2.11]. □

5.1.4 Proof of Main Theorems

In this subsection, we aim to complete the proofs of Theorems 1.1.1 and Theorem 1.1.6. We begin with Theorem 1.1.1, whose statement we reproduce:

Theorem 5.1.21. *Let $\mathcal{G}$ be an essentially principal, ample groupoid whose unit space has no isolated points. Then the following are equivalent:*

1. *$\mathcal{G}$ is purely infinite and minimal.*
2. *$\mathsf{A}(\mathcal{G}) \leq \mathrm{Homeo}(\mathcal{G}^{(0)})$ is vigorous.*
3. *For every $x_0 \in \mathcal{G}^{(0)}$, the subgroup*

$$\mathsf{A}(\mathcal{G})_{x_0} = \{g \in \mathsf{A}(\mathcal{G})\colon \textit{there is a neighbourhood } Y \textit{ of } x_0 \textit{ such that } g|_Y = Id_Y\}$$

acts compressibly on $\mathcal{G}^{(0)} \setminus \{x_0\}$.

Moreover, if one (or equivalently, all) of the above conditions hold, then for every compact open subset $X \subsetneq \mathcal{G}^{(0)}$, there exists an embedding

$$\phi_X : V \to \mathsf{A}(\mathcal{G})$$

such that $X \subseteq supp(\phi_X(V))$.

Proof. What we have shown in the previous sections proves the result assuming that $\mathcal{G}^{(0)}$ is a Cantor set. Indeed, the equivalence between (1) and (2) is 5.1.10; the equivalence between (1) and (3) is Lemma 5.1.16; and the fact that (1) implies the final part of the statement is Theorem 5.1.6.

Let us recall a fact from point set topology. In our hypothesis we have assumed that our unit space is Hausdorff, locally compact, totally disconnected (as part of being ample). Therefore, compact subsets of this unit space are Cantor spaces, by Brouwer's theorem.

We now explain how to obtain the result in full generality from the Cantor case. Note that the equivalence between (1) and (2) is automatic, since being purely infinite and minimal is invariant under restrictions to compact subsets $X \subseteq \mathcal{G}^{(0)}$. The same is true about (1) implying the last part, since both of these are local properties.

It thus remains to show (1) is equivalent to (3). Let us begin by showing that (1) implies (3). Let $x_0 \in \mathcal{G}^{(0)}$ be given, and find a compact open cover $(X_i)_{i \in I}$ of $\mathcal{G}^{(0)} \setminus \{x_0\}$. For all $i \in I$, it follows from the compact case that the action

$$\mathsf{A}(\mathcal{G})_{X_i \cup \{x_0\}} \curvearrowright X_i \cup \{x_0\} \setminus \{x_0\}$$

is compressible; compressibility of the overall action then follows easily.

Now let us show that (3) implies (1). Let $X, Y \subseteq \mathcal{G}^{(0)}$ be nonempty compact open subsets. Let us assume without loss of generality that $X \cup Y \neq \mathcal{G}^{(0)}$. Let $x_0 \notin X \cup Y$, and let $\mathcal{U}$ be a compressible cover associated by the action of $\mathsf{F}(\mathcal{G})_{\{x_0\}} \curvearrowright \mathcal{G}^{(0)} \setminus \{x_0\}$. Let $U_1, U_2 \in \mathcal{U}$ with $X \subseteq U_1$, and $Y \subseteq U_2$ be given (such U_i exist via condition (4) of compressibility). Then, using condition (2) of compressibility there exists $g \in \mathsf{F}(\mathcal{G})_{\{x_0\}} \subset \mathsf{F}(\mathcal{G})$ such that $g(U_1) \subseteq U_2$. Then, $g(X) \subseteq g(U_1) \subseteq U_2 \subseteq Y$, and so the restriction $g|_X$ gives a compact open bisection B with $BX \subseteq Y$. □

We stress the fact that for the groupoids covered by the above theorem, the alternating and derived subgroups always agree by Theorem 5.1.3. In particular, the alternating group is always perfect (meaning that it equals its commutator subgroup).

Remark 5.1.22. *For Hausdorff groupoids, essential principality is equivalent to effectiveness. On the other hand, effectiveness is the right notion to consider in the non-Hausdorff setting. We expect the above theorem to hold in this case; however, this involves obtaining (mostly routine) generalisations of many results from the Hausdorff to the non-Hausdorff case. Since for most applications the Hausdorff case is sufficient, we focus on essentially principal groupoids.*

We now aim to complete our proof of Theorem 1.1.6. For this, we need to use that the canonical action $\mathsf{F}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is rigid in the sense of Rubin [107], so we define this notion first.

Definition 5.1.23. *Let Γ be a discrete group and let C be a locally compact Hausdorff space without isolated points. We say that an action $\Gamma \curvearrowright C$ is a* Rubin action *if for every open set $U \subseteq C$ and every $x \in U$, the closure of*

$$\{g \cdot x : g \in \Gamma \text{ and } \operatorname{supp}(g) \subseteq U\}$$

contains an open neighbourhood of x.

Rubin's reconstruction Theorem from [107] is extremely useful when working with Rubin actions.

Theorem 5.1.24 (Rubin)**.** *Let $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$ be Rubin actions of discrete groups on locally compact Hausdorff spaces X_1 and X_2 without isolated points. If $\rho : \Gamma_1 \to \Gamma_2$ is a group isomorphism, then there exists a homeomorphism $\Phi : X_1 \to X_2$ satisfying $\Phi(g \cdot x) = \rho(g) \cdot \Phi(x)$ for all $x \in X_1$ and all $g \in \Gamma_1$.*

We will the notion of a *locally closed* subgroup of homeomorphisms, due to Nyland-Ortega; see [95, Definition 4.4].

Definition 5.1.25. *Let X be a locally compact Hausdorff space and let $\Gamma \leq Homeo(X)$ let be a subgroup. We say that a homeomorphism $\phi \in Homeo(X)$ is locally in Γ if for all $x \in X$, there exist a neighbourhood U of x and an element $g \in \Gamma$ such that $\phi|_U = g|_U$. We say that Γ is locally closed if any homeomorphism of X which is locally in Γ is automatically in Γ.*

In other words, Γ is locally closed if homeomorphisms from it can be "glued" to get another homeomorphism of Γ. It is not difficult to see that topological full groups satisfy this property, essentially by construction; see [95, Theorem 4.5].

Proposition 5.1.26. *Let $\mathcal{G}$ be an ample essentially principal groupoid. Then $\mathsf{F}(\mathcal{G})$ is locally closed.*

Next, we completely characterise the subgroups of Homeo$(\mathcal{C})$ that arise as full groups of minimal, purely infinite groupoids: they are precisely the vigorous groups which are also locally closed. We can also abstractly characterise the derived subgroups of such groupoids: they are the simple, vigorous groups.

The following is an existence and uniqueness theorem. While the existence part is completely new, the uniqueness part can be deduced from Matui's isomorphism theorem. We nevertheless give a shorter proof using Rubin's Theorem and Theorem 5.1.10, since it illustrates the scope of our results. It should be pointed out that isomorphism arguments of this nature have also been used in [97].

Theorem 5.1.27. *Let $\mathcal{C}$ denote the Cantor space, and let $F \leq \mathrm{Homeo}(\mathcal{C})$ be a subgroup. Then the following are equivalent: Let $\mathcal{C}$ denote the Cantor space.*

1. *For a subgroup $F \leq \mathrm{Homeo}(\mathcal{C})$, the following are equivalent:*

   (F.1) There exists a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_F$ such that $\mathsf{F}(\mathcal{G}_F)$ realises F as a subgroup of $\mathrm{Homeo}(\mathcal{C})$.

   (F.2) F is vigorous and locally closed.

2. *For a subgroup $A \leq \mathrm{Homeo}(\mathcal{C})$, the following are equivalent:*

   (A.1) There exists a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_A$ such that $\mathsf{A}(\mathcal{G}_A)$ realises A as a subgroup of $\mathrm{Homeo}(\mathcal{C})$.

   (A.2) A is vigorous and simple.

Moreover, the groupoids $\mathcal{G}_F$ and $\mathcal{G}_A$ as in (F.1) and (A.1) above are unique up to groupoid isomorphism.

Proof. (F.1) implies (F.2). Let $\mathcal{G}$ be a minimal, purely infinite, essentially principal, Cantor étale groupoid. Then $\mathsf{F}(\mathcal{G})$ is locally closed by Proposition 5.1.26, and it is vigorous by Theorem 5.1.10.

(F.2) implies (F.1). Let $\mathcal{G}$ be the groupoid of germs of the canonical action $F \curvearrowright \mathcal{C}$. We claim that $F = \mathsf{F}(\mathcal{G})$. It is immediate that $F \subseteq \mathsf{F}(\mathcal{G})$, so we only prove the converse inclusion. Given $g \in \mathsf{F}(\mathcal{G})$, we use the description of the topological full group of a germ groupoid given in Example 3.1.3 to find a partition $\mathcal{C} = X_1 \sqcup \cdots \sqcup X_n$ of $\mathcal{C}$ into compact and open sets, and group elements $f_1, \ldots, f_n \in F$ such that $g|_{X_j} = f_j|_{X_j}$ for all $j = 1, \ldots, n$. Since F is locally closed, it follows immediately that g belongs to F, as desired.

The fact that $\mathcal{G}_F = (F \curvearrowright \mathcal{C})^{germ}$ is minimal and purely infinite follows from Theorem 5.1.10, since its topological full group is vigorous.

(A.1) implies (A.2). Similarly to the first part of this proof, vigor of A follows from Theorem 5.1.10, while simplicity follows from Theorem 5.1.3.

(A.2) implies (A.1). Let $F_A \leq \mathrm{Homeo}(\mathcal{C})$ denote the smallest subgroup of $\mathrm{Homeo}(\mathcal{C})$ containing A which is locally closed. Since A is vigorous, it is immediate to see that so is F_A. By the first part of this theorem, there is a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_A$ satisfying $\mathsf{F}(\mathcal{G}_A) \cong F_A$. By the equivalence between (1) and (6) in [11, Theorem 1.11], it follows that A is the commutator subgroup of F_A, and in combination with Theorem 5.1.3 we get $A = [F_A, F_A] \cong \mathsf{D}(\mathcal{G}_A) = \mathsf{A}(\mathcal{G}_A)$, as desired.

We now show the uniqueness part of the statement. We claim that if $\mathcal{G}$ is a purely infinite, minimal Cantor groupoid, then $\mathsf{A}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is a Rubin action. (This implies that $\mathsf{F}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is also a Rubin action, since $\mathsf{F}(\mathcal{G})$ contains $\mathsf{A}(\mathcal{G})$.) To show this, let $U \subseteq \mathcal{G}^{(0)}$ be a compact and open set and let $x,y \in U$. Write U as a disjoint union $U = X \sqcup Y$ such that $x \in X$ and $y \in Y$. Let $Y_n \subseteq Y$ be a decreasing sequence of compact and open neighbourhoods of y such that $\bigcap_{n\in\mathbb{N}} Y_n = \{y\}$. Since $\mathsf{A}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is vigorous by Lemma 5.1.10, for every $n \in \mathbb{N}$ there exists $g_n \in \mathsf{A}(\mathcal{G})$ with $\mathrm{supp}(g_n) \subseteq U$ such that $g_n(X) \subseteq Y_n$. Therefore $g_n \cdot x \in Y_n$ for all $n \in \mathbb{N}$, and thus $y = \lim_{n\to I} g_n \cdot x$ belongs to the closure of

$$\left\{g \cdot x : g \in \mathsf{A}(\mathcal{G}) \text{ and } g|_{\mathcal{G}^{(0)} \setminus U} = \mathrm{id}\right\}.$$

Since $y \in U$ is arbitrary, this shows that

$$U = \overline{\left\{g \cdot x : g \in \mathsf{A}(\mathcal{G}) \text{ and } g|_{\mathcal{G}^{(0)} \setminus U} = \mathrm{id}\right\}},$$

and hence $\mathsf{A}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is a Rubin action. This proves the claim.

We now show uniqueness of $\mathcal{G}_F$ in (F.1). Note that, since $\mathcal{G}_F$ is essentially principal, it is isomorphic to the groupoid of germs of its canonical action $\mathsf{F}(\mathcal{G}_F) \curvearrowright \mathcal{G}_F^{(0)}$. Now, assume that $\mathcal{H}$ is another minimal, purely infinite, essentially principal, étale groupoid with $\mathsf{F}(\mathcal{H}) \cong F$. Since $\mathsf{F}(\mathcal{G}_F)$ and $\mathsf{F}(\mathcal{H})$ act in a Rubin manner on the Cantor space by the previous paragraph, Theorem 5.1.24 implies that $\mathsf{F}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is conjugate to $\mathsf{F}(\mathcal{H}) \curvearrowright \mathcal{H}^{(0)}$. In particular, their groupoids of germs are clearly isomorphic, and thus $\mathcal{G} \cong \mathcal{H}$.

Finally, we turn to uniqueness of $\mathcal{G}_A$ in (A.1), so let $\mathcal{H}$ be a minimal, purely infinite, essentially principal, étale groupoid with $\mathsf{A}(\mathcal{H}) \cong A$. Since $\mathsf{A}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ and $\mathsf{A}(\mathcal{H}) \curvearrowright \mathcal{H}^{(0)}$ are Rubin actions, it follows from Theorem 5.1.24 that they are conjugate. Thus, if we denote by $F_{\mathsf{A}(\mathcal{H})}$ the smallest subgroup of $\mathrm{Homeo}(\mathcal{H}^{(0)})$ which is locally closed and contains $\mathsf{A}(\mathcal{H})$, it follows that $F_A \cong F_{\mathsf{A}(\mathcal{H})}$. On the other hand, it is routine to check that $F_{\mathsf{A}(\mathcal{H})}$ is precisely $\mathsf{F}(\mathcal{H})$. Putting these things together, we conclude that

$$\mathsf{F}(\mathcal{G}_A) \cong F_A \cong F_{\mathsf{A}(\mathcal{H})} \cong \mathsf{F}(\mathcal{H}).$$

Thus $\mathcal{H} \cong \mathcal{G}_A$ by uniqueness in (F.1). □

Remark 5.1.28. *The proof above also shows that if $A \leq F \leq \mathrm{Homeo}(\mathcal{C})$ are nested groups such that A is vigorous and simple and F is locally closed, then there exists a unique minimal, purely infinite, topologically principal, étale Cantor groupoid $\mathcal{G}$ such that $A \cong \mathsf{A}(\mathcal{G})$ and $F \cong \mathsf{F}(\mathcal{G})$. In particular, normality of A in F follows from the remaining assumptions.*

As a corollary, we can deduce that simple, vigorous subgroups of $\mathrm{Homeo}(\mathcal{C})$ are C*-simple. More precisely:

Corollary 5.1.29. *Let $A \leq F \leq \mathrm{Homeo}(\mathcal{C})$ be nested groups such that A is vigorous and simple, and F is locally closed. Then any intermediate group $A \leq H \leq F$ is C*-simple.*

Proof. By Theorem 5.1.27 and the remark after it, there exists a minimal, purely infinite, topologically principal, étale Cantor groupoid $\mathcal{G}$ such that $A \cong \mathsf{A}(\mathcal{G})$ and $F \cong \mathsf{F}(\mathcal{G})$. Using vigor of

$\mathsf{A}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$, and in particular the absence of invariant probability measures (Lemma 5.1.5), the arguments in [110] can be adapted in a routine way to show that both $\mathsf{A}(\mathcal{G})$ and $\mathsf{F}(\mathcal{G})$ are C*-simple. Instead of doing this, we give a more direct proof using the recent results in [4]; we thank Eduardo Scarparo for sharing this argument with us and for allowing us to include it here.

By Corollary 5.1.11, the $\mathsf{D}(\mathcal{G})$-conjugacy classes of $\mathsf{F}(\mathcal{G})$ are infinite. Hence, it follows from [4, Theorem 6.2], that every intermediate C*-algebra $C_r^*(\mathsf{A}(\mathcal{G})) \leq A \leq C_r^*(\mathsf{F}(\mathcal{G}))$ is simple. Since $C_r^*(H)$ is of that form, the result follows. □

Here is another direct application of Theorem 5.1.27.

Corollary 5.1.30. *Vigorous simple groups have no proper characters.*

One can combineTheorem 5.1.27 with Theorem 5.1.21 in order to obtain further equivalent conditions for a group to be realized as the alternating group of a purely infinite, minimal groupoid. More specifically:

Remark 5.1.31. *Let $\mathcal{C}$ denote the Cantor space and let $A \leq \mathrm{Homeo}(\mathcal{C})$ be a subgroup. Then the following are equivalent:*

(A.1) There exists a minimal, purely infinite, essentially principal, Cantor étale groupoid $\mathcal{G}_A$ such that $\mathsf{A}(\mathcal{G}_A)$, realises A as a subgroup of $\mathrm{Homeo}(\mathcal{C})$.

(A.2) A is vigorous and simple.

(A.3) A is simple and for all $x \in \mathcal{C}$, the subgroup

$$A_x = \{g \in A : \textit{there exists a neighbourhood } Y \textit{ of } x \textit{ such that } g|_Y = Id_Y\}$$

acts compressibly on $\mathcal{C} \setminus \{x\}$

We close this section by pointing out that Theorem 5.1.27 can be generalized easily to non-compact unit spaces, since Rubin's theorem is also valid for locally compact spaces. Another way to deal with non-compact spaces is to use the generalisation of Matui's spatial isomorphism proved by Nyland-Ortega in [95, Theorem 7.2] in the setting where we replace $\mathcal{C}$ with an arbitrary totally disconnected locally compact Hausdorff space X. In this context, one must also replace $\mathrm{Homeo}(\mathcal{C})$ with the group of compactly supported homeomorphisms:

$$\mathrm{Homeo_c}(X) = \{f \in \mathrm{Homeo}(X) : \mathrm{supp}(f) \text{ is compact}\}.$$

We omit the details, but stress the fact that these arguments also allow one to deal with non-Hausdorff groupoids since they only depend on effectiveness.

5.2 Examples and applications

For the purposes of this section, we use groupoid homology, primarily as a way to compute the quotients $\mathsf{F}(\mathcal{G})/\mathsf{D}(\mathcal{G})$ for key examples. Fortunately, groupoid homology has already been computed in many interesting examples. For more on groupoid homology, we refer the reader to [88].

Our first application giving us restrictions on what kinds of actions are possible for $\mathsf{F}(\mathcal{G})$.

Proposition 5.2.1. *Let $\mathcal{G}$ be a purely infinite minimal groupoid such that $\mathsf{F}(\mathcal{G})_{\text{ab}}$ is finite (for example, if $H_1(\mathcal{G})$ is finite and $H_0(\mathcal{G})$ has finite rank). Then every faithful ergodic measure-preserving action of $\mathsf{F}(\mathcal{G})$ is essentially free.*

Proof. We saw in Theorem 5.1.20 that every finite factor representation factors through the abelianisation $\mathsf{F}(\mathcal{G})_{\text{ab}}$. Therefore, whenever $\mathsf{F}(\mathcal{G})_{\text{ab}}$ is finite, there are finitely many factor representations. We may then apply [45, Theorem 2.11] to get the conclusion. □

We can be even more concrete in some cases where homology has been computed.

Corollary 5.2.2. *For each of the following groups, every faithful ergodic measure-preserving action is essentially free:*

1. *The Brin-Higman-Thompson groups $nV_{k,r}$.*

2. *More generally, topological full groups arising from products of shift of finite type groupoids.*

Proof. It is shown in [90] that the abelianisations of the topological full groups of such groupoids are finite. Therefore the result follows from Theorem 5.2.1. □

As another application, we confirm Conjecture 1.1.2 for many natural examples including the class of groups above.

Example 5.2.3. *The following groups, which are already known to be finitely presented and simple, are 2-generated.*

1. *Derived subgroups of the topological full groups of Graph groupoids .*

2. *Derived subgroups of the topological full groups of Katsura-Exel-Pardo groupoids.*

3. *Simple subgroups arising as subgroups of topological full groups arising from products of shifts of finite type.*

4. *Derived subgroups of the topological full groups of Beta expansion groupoids.*

5. *Certain simple groups arising from groupoids that are left regular representations of Garside categories.*

In particular, Conjecture 1.1.2 holds true for these examples.

Proof. The groups listed above are known to be the topological full groups of purely infinite, minimal, ample, and effective groupoids: for (1), this is shown in [95]; for (2), this is shown in [96]; for (3), this is shown in [90]; for (4), this is shown in [85]; and for (5) this is shown in [78, Theorem C] and [80, Theorem A]. By Theorem 1.1.1, these groups act vigorously on the Cantor set. It thus follows from [11, Theorem 1.12] that these groups are all 2-generated. □

Next, we show C*-simplicity for many examples of Thompson-like groups.

Example 5.2.4. *The following Thompson-like groups are C*-simple:*

1. *The Higman-Thompson groups $V_{k,r}$, for $k \geq 2 \in \mathbb{N}$ and $r \geq 1$.*
2. *More generally, the Brin-Higman-Thompson groups $nV_{k,r}$, for $r, n \in \mathbb{N}$ and $k \geq 2$.*
3. *Stein's groups $V(\Lambda, \Gamma, \ell)$ for all subgroups Λ of $(\mathbb{R}, \cdot)$, all $\mathbb{Z} \cdot \Lambda$ submodules Γ and all $\ell \in \Gamma \cap (0, +\infty)$.*

Proof. All these groups act vigorously on the Cantor set and are locally closed; in other words, they are topological full groups of minimal, purely infinite Cantor groupoids by 5.1.27. The result then follows from 5.1.29.

□

Using Theorem 1.1.6, we can bring new groups into our framework. As an example, we discuss the twisted Brin-Higman-Thompson groups introduced by Belk and Zaremsky [7], which were previously not known to be topological full groups of purely infinite, minimal groupoids. We thank James Belk for bringing this example class to our attention and for an outline of this proof.

Example 5.2.5 (Twisted Brin-Higman-Thompson groups)**.** *The twisted Brin-Higman-Thompson groups SV were introduced in [7]. Already in their construction, we see that they are topological full groups, and hence they are locally closed. In the case where S is finite, say $|S| = n$, it follows that nV embeds into SV with full support, and therefore SV is vigorous. Similarly, if S is countable, we have that nV embeds into SV for all n in a way that $\bigcup_{n\in\mathbb{N}} \mathrm{supp}(nV)$ covers the whole of the Cantor space. Therefore, SV is again vigorous. We conclude that SV is always a locally closed, vigorous subgroup of* $\mathrm{Homeo}(\mathcal{C})$*. Applying Theorem 1.1.6, we deduce that SV is the topological full group of an essentially principal, purely infinite, minimal Cantor étale groupoid.*

We end with an example that generalises Brin-Thompson groups, advertising the flexibility that comes with working with topological full groups of étale groupoids, by explaining a new construction of interesting groups.

Example 5.2.6 (Perfect Brin-Higman-Thompson like groups). *Consider the variation of the Brin-Higman-Thompson groups on r^n cubes such that on each dimension $j = 1,\dots,n$ we have potentially different (integrally generated) slope sets generated by integers $k_j \geq 2$. Write $\overline{k}$ for the n-tuple $\overline{k} = (k_1,\dots,k_n)$, and denote the group described above by $V_{\overline{k},r}$. Note that, if $k_1 = \cdots = k_n =: k$, then $V_{\overline{k},r}$ is just the Brin-Higman-Thompson group $nV_{k,r}$.*

It is not difficult to see that $V_{\overline{k},r}$ is the topological full group of the groupoid:

$$\mathcal{G}_{\overline{k},r} := \mathcal{R}_r \times \prod_{j=1}^{n} \mathcal{E}_{k_j}.$$

This subclass of products of shifts of finite type groupoids fits into the framework of [90], and it is not difficult to see that if we set $g = \gcd(k_1 - 1,\dots,k_n - 1)$, then for $j \geq 0$ we have:

$$H_j\big(\mathcal{G}_{\overline{k},r}\big) = (\mathbb{Z}/g\mathbb{Z})^{{}^{(n-1)}C_j}.$$

In particular, if in the discussion above we have $g = 1$, for example if $k_j = 2$ for some j, then the homology vanishes.

Proposition 5.2.7. *Let $k_1,\dots,k_n \geq 2$ satisfy $\gcd(k_1 - 1,\dots,k_n - 1) = 1$. Then $V_{\overline{k},r}$ has the following properties:*

1. *It is acyclic, so in particular perfect and simple.*
2. *It is F_∞; in particular it is finitely presented.*
3. *It is 2-generated and C*-simple,*
4. *It has no proper characters.*
5. *Every faithful ergodic measure preserving action of $V_{\overline{k},r}$ is essentially free.*

Proof. We observed above that $\gcd(k_1 - 1,\dots,k_n - 1) = 1$ implies that the homology of $\mathcal{G}_{\overline{k},r}$ vanishes. Inserting this computation into the long exact sequence in Corollary 3.6.7, it is easily seen that $V_{\overline{k},r} = \mathsf{F}(\mathcal{G}_{\overline{k},r})$ is perfect in this case. Moreover, by Corollary 3.6.3, it follows that $V_{\overline{k},r}$ is acyclic. The fact this group is of type F_∞ follows due to [78], Theorem C. The fact this group is C*-simple and two generated follows via Corollary 5.1.29 and Corollary 5.1.12. The fact that this group has no proper characters follows via Corollary 5.1.30. The final claim follows from Proposition 5.2.1. □

Example 5.2.8 (V-absorption). *An interesting property now well-documented operator algebraists is that many Kirchberg algebras are absorbed by $\mathcal{O}_2$. Using the framework of topological full groups, we can capture this phenomenon group theoretically. Let $\hat{V}$ be a generalisation of V coming from a purely infinite minimal expansive groupoid $\mathcal{G}$ which can be described as* *piecewise linear maps on* $(0,1]$ *(e.g.* $\hat{V}$ *could be a Stein group, Cleary's group or a Higman-Thompson group* $V_{k,1}$*). Then form the groupoid product of this with* V*, that is, the piecewise linear bijections of* $(0,1]^2$ *that in the x direction look like* V *(have finitely many slopes in* 2^n*) and in the y direction look like* $\hat{V}$*. Then* $V \times_{Gpd} \hat{V}$ *is the topological full group of* $\mathcal{E}_2 \times \mathcal{G}$*. Consequently, this group is acyclic; perfect. It has no proper characters and* $V \times_{Gpd} \hat{V}$ *is 2-generated.*

5.3 Outlook

A class of generalisations of Thompson's group V which has recently been of interest is certain full automorphism groups $V_r(\Sigma)$ of Cantor algebras, as described in [84], which have been shown to be type F_∞ whenever the underlying Cantor algebras $U_r(\Sigma)$ are *valid, bounded* and *complete*. We ask:

Question 5.3.1. *Can the family of groups* $V_r(\Sigma)$*, for* $U_r(\Sigma)$ *valid, bounded and complete, be described as the topological full groups of purely infinite minimal groupoids?*

Given Theorem 1.1.6, Question 5.3.1 is equivalent to asking whether the (simple) derived subgroups of the groups described admit vigorous actions on the Cantor space. The difficulty in both cases above is that the groups in question are not constructed as subgroups of homeomorphisms of the Cantor set.

The next question relates to Conjecture 1.1.2 in the introduction. Topological full groups provide interesting examples of simple, finitely generated groups outside of the purely infinite setting, but it is unknown if any example of a topological full group outside of the purely infinite setting is finitely presented. We therefore ask:

Question 5.3.2. *Are there finitely presented, simple derived subgroups of topological full groups outside of the purely infinite setting?*

Our final question addresses a potential generalisation of Corollary 5.1.18. In recent work in preparation by Dudko-Medynets, they show that for AF-groupoids, there is a one-to-one correspondence between proper characters of the topological full group and invariant probability measures on the unit space. Corollary 5.1.18 shows that the same is true for purely infinite minimal groupoids (since these do not admit invariant probability measures). It seems reasonable to believe that this correspondence may exist in full generality, so we ask:

Question 5.3.3. *Let* $\mathcal{G}$ *be an étale, essentially principal Cantor groupoid. Is there a one-to-one correspondence between invariant probability measures* $\mathcal{M}(\mathcal{G})$ *on* $\mathcal{G}$ *and the proper characters on* $\mathsf{A}(\mathcal{G})$*?*

Recall Question 2.7.1, which asks whether comparison is automatic for certain groupoids. Restricting to the case of no measures, this is similar to the question asked by Bruce-Li [28, Question 4.21] "Let $\mathcal{G}$ be a second countable, topologically free, minimal ample étale groupoid with compact unit space. If $\mathcal{G}$ has no invariant measures, is $\mathcal{G}$ purely infinite?". Through the results of this chapter, we can ask a related group theoretic question.

Question 5.3.4. *Let $\Gamma \subset Homeo(X)$ be a subgroup of the homeomorphisms on the Cantor set such that:*

- Γ *is locally closed, (in other words,* Γ *is a topological full group).*
- Γ *acts minimally on* X.
- *There are no invariant measures on X with respect to the action on* Γ.

Is Γ *vigorous?*

Chapter 6

Interval exchange groups

6.1 Interval exchange groups

Let us begin with the definition of interval exchange groups.

Definition 6.1.1 (Interval Exchange Groups)**.** *Let Γ be a countable additive subgroup of $\mathbb{R}$, containing* 1. *Then, $IE(\Gamma)$ is the group of right continuous piecewise linear bijections f of $(0,1]$ with finitely many angles $\{ft - t \,:\, t \in (0,1]\}$ all in Γ.*

These are so-called because we think of these as cutting the interval $(0,1]$ at finitely many points, all in Γ and exchanging them. A typical element is drawn below.

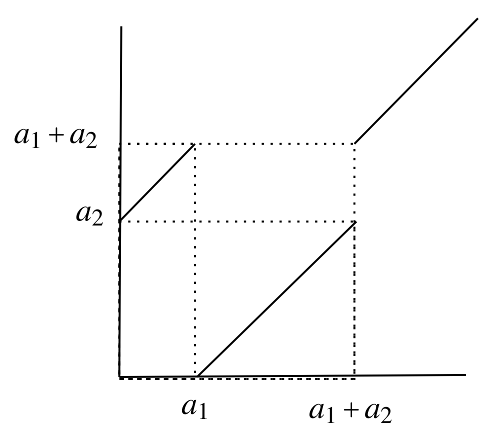


Figure 6.1: An example element of $IE(\Gamma)$, where $a_1, a_2 \in \Gamma$

These groups have attracted much attention recently in the context of topological full groups [87] [36] [60] [16]. However, the interest in interval exchange groups predates these papers.

One reason for the interest in these groups is the connection to classical dynamics, where dynamical systems coming from interval exchanges have been popular to study for some time [63]. For further information about the dynamical perspective on interval exchanges, we recommend the survey [125] and book [62].

Also, these groups have been studied from the perspective of geometric group theory. The reason for this is that due to results by Juschenko-Monod [61] and Matui [87], whenever $\Gamma = \mathbb{Z} \oplus \lambda\mathbb{Z}$ (for some irrational λ) the derived subgroup of $IE(\Gamma)$ is a rare example of a simple, finitely generated amenable group. The existence of such groups was first shown in [61].

Another reason for the interest in $IE(\Gamma)$ when Γ is finitely generated is to understand the group of interval exchange transformations.

Definition 6.1.2 (IET). *Let IET be the group of right continuous permutations g of $(0,1]$ such that the set $\{gt - t,\ t \in (0,1]\}$ is finite.*

Remark 6.1.3. $IE(\Gamma) \cong V(\langle 1 \rangle, \Gamma, 1)$

Remark 6.1.4 (IET Locally Embeds into $IE(\Gamma)$, where $\Gamma \cong \mathbb{Z}^n$)**.** *For any finite set of elements $\gamma_1, \gamma_2, \dots, \gamma_n \in IET$. Then $S = \{\gamma_i t - t\ : t \in (0,1], i = 1, \dots, n\}$ is finite. Note $\mathbb{R}$ is locally finitely generated abelian. Therefore, there exists some finitely generated abelian subgroup of $\mathbb{R}$, $\Gamma(\cong \mathbb{Z}^n)$ such that $S \subset_{fin} \Gamma$. Then, for all i, $\gamma_i \in IE(\Gamma)$.*

There are outstanding open questions about IET.

Question 6.1.5. *Does IET contain any nonamenable free groups? Is IET amenable?*

Through Remark 6.1.4, one approach to the above question is to study the same question about $IE(\Gamma)$ where Γ is finitely generated abelian. Note that the main result of [61] establishes that $IE(\Gamma)$ is amenable whenever $\Gamma \cong \mathbb{Z}^2$ and the main result of [60] establishes that $IE(\Gamma)$ is amenable whenever $\Gamma \cong \mathbb{Z}^3$.

We take a different perspective from the above papers. Instead of taking an action of $\Gamma/\mathbb{Z}$ on the Cantor space, we define a partial transformation groupoid $\alpha : \Gamma \curvearrowright X$ based on groupoids considered in [76] which realise $IE(\Gamma)$ as a topological full group. There are two reasons we prefer to think about these groupoids this way:

- The homology of these groupoids is easier to understand than of the transformation groupoid of $\Gamma/\mathbb{Z}$.
- The related C*-algebras fit into the class of semigroup C*-algebras, a perspective which proves useful for analyzing them.

6.2 Interval exchange groups as topological full groups

Let us first adapt $\mathbb{R}$ so that we can allow discontinuities at some subset Γ, by including two points τ_+, τ_-, separated in the topology, at each point $\tau \in \Gamma$. This definition follows the notation of [36].

Definition 6.2.1. *Let $\Gamma \subset \mathbb{R}$. Let be the space obtained by splitting the real line at each point $a \in \Gamma$ into two points a_+, a_-, i.e. take $\mathbb{R}_\Gamma := \{t, a_+, a_- : t \in \mathbb{R} \setminus \Gamma, a \in \Gamma\}$. Consider this with the canonical quotient map q onto $\mathbb{R}$ identifying a_+, a_- for all $a \in \mathbb{R}$.*

$$q : \mathbb{R}_\Gamma \to \mathbb{R} \quad t \mapsto \begin{cases} t & t \in \mathbb{R} \\ a & t = a_+ \notin \mathbb{R} \\ a & t = a_+ \notin \mathbb{R} \end{cases}.$$

Let us define a total order on $\mathbb{R}_\Gamma$ by setting

$$x < y \iff q(x) < q(y) \text{ or } q(x) = q(y) = a,\ x = a_- \text{ and } y = a_+.$$

We may then topologise $\mathbb{R}_\Gamma$ by the order topology, i.e. the topology generated by open intervals:

$$(x, y) = \{z \in \mathbb{R}_\Gamma : x < z < y\},\ x, y \in \mathbb{R}_\Gamma$$

Let $\Gamma_\pm := \{a_+, a_- : x \in \Gamma\} \subset \mathbb{R}_\Gamma$.

Lemma 6.2.2. *Let $\Gamma \subset \mathbb{R}$ be dense and countable. Then a (countable) basis for the topology on $\mathbb{R}_\Gamma$ is given by:*

$$(a_-, b_+)\ a < b,\ a, b \in \Gamma$$

Moreover, each set of the form (a_-, b_+) with $a < b$ is a Cantor set.

Proof. First let us remark that for all $a < b$ that $(a_-, b_+) = [a_+, b_-]$, so that each of these sets are clopen. By density of Γ, these form a basis for the topology on $\mathbb{R}_\Gamma$. Thus we establish that $\mathbb{R}_\Gamma$ is second countable, with a basis of clopen sets. Note moreover that the basis elements clearly separate points in $\mathbb{R}_\Gamma$. Note that q is continuous, indeed the preimage of $(a, b) \subset \mathbb{R}$ is of the form (x, y) for some $x, y \in \mathbb{R}_\Gamma$. But since $q(a_-, b_+) = q([a_+, b_-]) = [a, b]$, compactness follows. In total then we have that for all $a, b \in \Gamma$ (a_-, b_+) is compact, and has a countable basis of compact open subsets; by Brouwer's theorem, we are done. □

This makes the topology of $\mathbb{R}_\Gamma$, when Γ is countable, identifiable with the disjoint union of countably many Cantor spaces.

In [76], Section 2.3 Li constructs an analogous space as follows. Let Γ be an additive subgroup of $\mathbb{R}$. Let $D(\Gamma^+)$ be the (abelian) semigroup C*-algebra of $\Gamma \cap (0, \infty)$. This group has the basis of idempotents $\{1_{a+\Gamma^+}\ a \in \Gamma \cap (0, +\infty)\}$ He then considers the Gelfand dual space $\Omega(D(\Gamma^+))$, and removes the trivial character χ_∞ such that for all $a \in \Gamma$ $\chi_\infty(1_{a+\Gamma^+}) = 1$. This space is denoted $O_{\Gamma^+ \subseteq \Gamma}$ Concretely, this is the space of nonzero, nontrivial, characters $\chi : D(\Gamma^+) \to \{0, 1\}$ that are strictly decreasing with on the basis (with respect to the partial order $1_{a+\Gamma^+} \leq 1_{b+\Gamma^+} \iff a \leq b$). This space is topologised in the weak operator topology, which is the topology generated by the basic compact open sets $U_{a,b} := \{\chi \in_{\Gamma^+ \subseteq \Gamma} : \chi(a) = 1, \chi(b) = 0\}, a, b \in \Gamma$. There is a canonical homeomorphism:

$$f : \mathbb{R}_\Gamma \to O_{\Gamma^+ \subseteq \Gamma} \quad a_+ \mapsto \chi_a^+,\ a_- \mapsto \chi_a^-,\ t \mapsto \chi_t \quad a \in \Gamma, t \in \mathbb{R}_\Gamma \setminus \Gamma_\pm$$

Where for all $a, b \in \Gamma$, $t \in \mathbb{R}_\Gamma \setminus \Gamma_\pm$, $\chi_a^+(1_{b+\Gamma^+}) = 1 \iff b \leq a$, $\chi_a^-(1_{b+\Gamma^+}) = 1 \iff b < a$, $\chi_t(1_{b+\Gamma^+}) = 1 \iff b < t$. For all $a, b \in \Gamma, a < b$ $f[a_+, b_-]) = U_{a,b}$.

Let $\ell \in \Gamma \cap (0, +\infty)$. Then, let:

$$q^* : (0, \ell] \hookrightarrow (0_+, \ell_-] \quad t \mapsto \begin{cases} t & t \notin \Gamma \\ t_- & t \in \Gamma \setminus \{0\} \end{cases}.$$

Note then, for all $a, b \in \Gamma$ with $a < b$, $q^*((a, b]) \subset (a_+, b_-]$. We are now ready to describe $IE(\Gamma)$ as the topological full group of a partial action. Recall the definition of a partial transformation groupoid as in Example 2.3.7.

Lemma 6.2.3 ($IE(\Gamma)$ as a topological full group of a partial action)**.** *Let Γ be a countable, dense additive subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Let $\alpha : \Gamma \curvearrowright \mathbb{R}_\Gamma$ be the canonical additive action of Γ on $\mathbb{R}_\Gamma$ (i.e. given by for each $c \in \Gamma_+$ $\alpha_c(a_\pm) = (a + c)_\pm$, $\alpha_c(t) = t + c$ $\forall a, c \in \Gamma$, $\forall t \notin \Gamma$) and consider the restriction of α to a partial action on $(0_+, 1_-]$. Then we have that*

$$\mathsf{F}(\Gamma \ltimes_\alpha (0_+, 1_-]) \cong IE(\Gamma)$$

Proof. A basis of the compact open bisections of the groupoid $\Gamma \ltimes_\alpha (0_+, 1_-]$ is given by:

$$(c, [a_+, b_-])\ c \in \Gamma \quad a, b \in \Gamma, \quad \max\{-c, 0\} \leq a < b \leq \min\{1 - c, 1\}$$

Elements of $\mathsf{F}(\Gamma \ltimes_\alpha (0_+, 1_-])$ are therefore homeomorphisms f of $(0_+, 1_-]$ for which there exists a finite subset $\{x_i\}_{i=1}^n \subset \Gamma$, with $0 = x_1 < x_2 < \ldots < x_n = 1$, and elements $\{c_i\}_{i=1}^n \subset \Gamma$ such that

$$f|_{[(x_i)_+, (x_{i+1})_-]} = \alpha(c_i)|_{[(x_i)_+, (x_{i+1})_-]}$$

Hence, making use of q, q^* we can obtain the isomorphism:

$$\varphi : \mathsf{F}(\Gamma \ltimes_\alpha (0_+, 1_-]) \to IE(\Gamma) \quad g \mapsto qgq^*$$

For any element $f \in \mathsf{F}(\Gamma \ltimes_\alpha (0_+, 1_-])$, $\varphi(f)$ is a right continuous piecewise linear bijection of $(0, 1]$ with finitely many angles, $c_1, \ldots, c_n$, since on each interval $(x_i, x_i + 1]$, we have that $q^*(x_{i-1}, x_i] \subset ((x_{i-1})_+, (x_i)_-]$, therefore $gq^*(x_{i-1}, x_i] \subset ((x_{i-1} + c_i)_+, (x_i + c_i)_-]$ and so $qgq^*((x_{i-1}, x_i]) = (x_{i-1} + c_i, x_i + c_i]$.

Moreover, if $t \in (x_{i-1}, x_i] \cap \Gamma$, then

$$qgq^*(t) = qg(t_-) = q((t + c_i)_-) = t + c_i.$$

And, if $t \in (x_{i-1}, x_i] \cap \Gamma^c$, then

$$qgq^*(t) = qg(t_-) = q((t + c_i)) = t + c_i.$$

We note also then, that there are only finitely many $c_i \in \Gamma$ describing f. Note that φ is indeed a group homomorphism since $qq^* = Id_{(0,1]}$. If $\varphi(f) = 1$, then $c_i = 0$ for all i, hence $f \in \mathsf{F}(\Gamma \ltimes_\alpha (0_+, 1_-])$ is the identity, hence φ is an isomorphism. □

Let us also remark that for α, we have that $(0_+, \ell_-]$ is $\Gamma \ltimes \mathbb{R}_\Gamma$ full (by density, for all $x \in \mathbb{R}_\Gamma$, we can choose $a \in \Gamma$ such that $0 < q(x) - a < \ell$).

Remark 6.2.4. *The above partial action is conjugate (via the homeomorphism f) to the groupoid model $\Gamma \ltimes O_{\Gamma^+ \subseteq \Gamma}|_{N(\Gamma^+)}^{N(\Gamma^+)}$ of $\mathcal{F}^\lambda$ in [76], Section 2.3.*

It is also relatively straightforward to see an alternative, equivalent description of $IE(\Gamma)$ in terms of a group action.

Lemma 6.2.5 ($IE(\Gamma)$ as the topological full group of an action)**.** *Let Γ be a countable, dense additive subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Then*

$$\hat{\alpha} : \Gamma/\mathbb{Z} \curvearrowright (0_+, 1_-]$$

Let $[t]$ denote the equivalence class of $t \in \mathbb{R}$ (mod $\mathbb{Z}$). Then let

$$\hat{\alpha}([c])(t) = [t + c] \quad \hat{\alpha}([c])(a_\pm) = [a + c]_\pm.$$

We have that $IE(\Gamma) = \mathsf{F}(\Gamma \ltimes_{\hat{\alpha}} (0_+, 1_-])$

We omit this proof since this perspective agrees with the perspective as in [16] [60] [36], and the proof of this follows the same proof as that of Lemma 6.2.3.

It is time to remark on some basic facts for these partial action groupoids:

Lemma 6.2.6 (Regularity of α)**.** *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Then $\alpha : \Gamma \curvearrowright \mathbb{R}_\Gamma$ is a free, amenable, minimal action.*

Proof. The action is amenable since Γ is an amenable group. Suppose for some $c \in \Gamma$, and some $x \in \mathbb{R}_\Gamma$, $\alpha(c)(x) = x$. Then in particular in $\mathbb{R}$, $q(\alpha(c)x) = q(x) - c = q(x) \Rightarrow c = 0$. This establishes that α is free.
Now let us establish minimality. First let us remark the following convergence rules in $\mathbb{R}_\Gamma$:

$$\lim_{n\to\infty} x_n = x_+ \iff \lim_{n\to\infty} q(x_n) = q(x) \text{ from above}$$

$$\lim_{n\to\infty} x_n = x_- \iff lim_{n\to\infty} q(x_n) = q(x) \text{ from below}$$

$$\lim_{n\to\infty} x_n = x \text{ s.t. } x \notin \Gamma_\pm \iff \lim_{n\to\infty} q(x_n) = x$$

For all $x \in \mathbb{R}_\Gamma$, the image of the orbit Γx under q, $q(\Gamma x) = q(x) + \Gamma \subset \mathbb{R}$ is dense in $\mathbb{R}$, so we can find sequences in $q(x) + \Gamma$ tending to any $x' \in \Gamma$ from above or below, and sequences approximating any $x' \notin \Gamma$ in $q(\Gamma x)$. Minimality follows. □

Corollary 6.2.7. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. As a groupoid, $\Gamma \ltimes_\alpha (0_+, 1_-]$ is minimal, principal, and amenable.*

Note in particular as a Corollary of Matui's isomorphism Theorem (Theorem 3.4.12), and the regularity established for α, $\hat{\alpha}$ the identification of the topological full groups implies the groupoids coming from α and $\hat{\alpha}$ are conjugate.

Corollary 6.2.8. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. As groupoids, $\Gamma/\mathbb{Z} \ltimes_{\hat{\alpha}} (0_+, 1_-] \cong \Gamma \ltimes_\alpha (0_+, 1_-]$*

This identification establishes that the groupoid is almost finite.

Lemma 6.2.9. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Then the groupoid $\Gamma \ltimes_\alpha (0_+, 1_-]$ is an almost finite groupoid.*

Proof. Using that $\Gamma \ltimes_\alpha (0_+, 1_-] \cong \Gamma/\mathbb{Z} \ltimes_{\hat{\alpha}} (0_+, 1_-]$ can be written as the transformation groupoid of a free minimal action of $\Gamma/\mathbb{Z}$ (a countable amenable group) on the Cantor space. Moreover, since $\Gamma/\mathbb{Z}$ is abelian, all finitely generated subgroups have polynomial growth. Free minimal actions of such groups were shown to be almost finite in [[66], Theorem C], and so the underlying groupoid is almost finite. □

We then fit into the scope of Theorem 3.3.2, identifying $\mathsf{A}(\Gamma \ltimes_\alpha (0_+, 1_-])$ with $\mathsf{D}(\Gamma \ltimes_\alpha (0_+, 1_-])$ and establishing simplicity.

Corollary 6.2.10. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Then $D(IE(\Gamma)) = \mathsf{A}(\Gamma \ltimes_\alpha (0_+, 1_-])$ is simple.*

Another corollary we have as a consequence of almost finiteness is that the associated crossed product is $\mathcal{Z}$-stable.

Lemma 6.2.11. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Then the C^* algebra given by $C((0_+, 1_-]) \ltimes_\alpha \Gamma$ is classifiable.*

Proof. $\mathcal{Z}$-stability is established by almost finiteness via [[64], Theorem 12.4]. Let us note that the group $\Gamma/\mathbb{Z}$ is always abelian, so, in particular, has polynomial growth. By construction, this is the groupoid C*-algebra of an amenable groupoid hence by [[124], Proposition 10.7] they satisfy the UCT. The other classifiability conditions can be read off from Corollary 6.2.7; the groupoid is minimal and principal (implying simplicity), with a compact unit space (implying unital), second countable (implying separable), and amenable (implying nuclearity). □

We can also compute the Elliott invariant in this case.

Lemma 6.2.12 (Li). *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. The Elliott invariant for $C((0_+,1_-]) \ltimes_\alpha \Gamma$ is as follows.*

$$(K_0(C((0_+,1_-]) \ltimes_\alpha \Gamma),[1]_0,K_1(C((0_+,1_-]) \ltimes_\alpha \Gamma) \cong (K_1(C_r^*(\Gamma),[U_1]_1,K_0(C_r^*(\Gamma)/\mathbb{Z}[1]_0)$$

Where $[U_1]_1$ here is a unique trace τ on K_0 satisfying $\tau(K_0(C((0_+,1_-]) \ltimes_\alpha \Gamma))) = \Gamma$. In particular, $C((0_+,1_-]) \ltimes_\alpha \Gamma \cong C((0_+,1_-]) \ltimes_\alpha \Gamma' \implies \Gamma = \Gamma'$ (as subsets of $\mathbb{R}$).

Proof. This proof directly generalises via Corollary 3.3 and Proposition 3.6 of [76]. □

Since $Ell(C_r^*(\mathcal{G}_\Gamma))$ recovers Γ as a subset of $\mathbb{R}$, each of the C*-algebras are pairwise nonisomorphic. In turn then, so are the groups $IE(\Gamma)$:

Theorem 6.2.13. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Then, the following are equivalent:*

1. $\Gamma = \Gamma'$ *(as subsets of $\mathbb{R}$)*
2. $IE(\Gamma) \cong IE(\Gamma')$ *(as groups)*
3. $\mathsf{D}(IE(\Gamma)) \cong \mathsf{D}(IE(\Gamma'))$ *(as groups)*
4. $\Gamma \ltimes_\alpha (0_+,1_-] \cong \Gamma \ltimes_\alpha (0_+,1_-]'$ *(as groupoids)*
5. $C_r^*(\Gamma \ltimes_\alpha (0_+,1_-]) \cong C_r^*(\Gamma \ltimes_\alpha (0_+,1_-]')$ *(as C^* -algebras)*
6. $Ell(C_r^*(\Gamma \ltimes_\alpha (0_+,1_-])) \cong Ell(\Gamma \ltimes_\alpha (0_+,1_-]'))$

Proof. The implications 1. ⇒ 2. ⇒ 3., 4. ⇒ 5. ⇒ 6. are straightforward. 6 ⇒ 1. follows from Lemma 6.2.12, since part of the Elliott invariant is the unique trace that recovers Γ. 3. ⇒ 4. is the Matui's isomorphism Theorem (Theorem 3.4.12). □

Remark 6.2.14. *This classification result can already be seen for the case of Γ finitely generated as a corollary of work by Matte Bon [[16], Theorem 10.3].*

Note that this also identifies, via Theorem 2.5.2 many of the associated crossed products with concrete C*-algebras. The K-theory and tracial data of UHF algebras is computed in [[41], Chapter III]. Note also that it was recently shown that UHF algebras have unique AF Cartan subalgebras [[102], Theorem D], and hence the identification here is actually an identification on the groupoid level, due to Renault's reconstruction theorem [[101], Theorem 1.1].

Corollary 6.2.15. *Let $\Gamma \subset \mathbb{Q}$. Let $\{k(n)\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers such that $k(n)|k(n+1)$ and $\{1/(k(n))\}$ is a generating set of Γ. Then, $\Gamma \ltimes (0_+, 1_-]$ is conjugate to the UHF groupoid associated with the Bratelli diagram*

$$k(1) \xrightarrow{k(2)/k(1)} k(2) \xrightarrow{k(3)/k(2)} k(3) \xrightarrow{k(4)/k(3)} \cdots \xrightarrow{k(n)/k(n-1)} k(n) \xrightarrow{k(n+1)/k(n)} \cdots$$

As in Example 2.3.9. This limit is independent of the choice of generating set, therefore such inductive limits of symmetric/alternating groups are classified by their supernatural numbers $\prod_{i=1}^{\infty} k_i$.

In [33], Li constructs F^{λ} as in [76] and [33] as particular examples of our groupoids in the case where $\Gamma = \mathbb{Z}[\lambda, \lambda^{-1}]$.

Corollary 6.2.16. *Let $\Gamma = \mathbb{Z}[\lambda, \lambda^{-1}]$, where $\lambda \in \mathbb{R}$. Then, $C((0_+, 1_-]) \ltimes_{\alpha} \Gamma \cong F^{\lambda}$ as in [76] and [33].*

6.3 Concrete generating sets in the finitely generated abelian case

When Γ is finitely generated, so is $D(IE(\Gamma))$. The aim of this section is to find a concrete generating set for $D(IE(\Gamma))$ in the case when Γ is finitely generated abelian. Let us first show that indeed the derived subgroup is finitely generated.

Lemma 6.3.1. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. The action $\hat{\alpha} : \Gamma/\mathbb{Z} \curvearrowright (0_+, 1_-]$ as in Theorem 6.2.5 is an expansive action in the sense of Definition 3.5.3.*

Proof. Let $\lambda \in \Gamma \cap (0,1)$. Let $x, x' \in (0_+, 1_-]$ be distinct. Let $\varepsilon = d((0_+, (\lambda)_-], [(\lambda)_-, 1_+])$ We separate into two cases:

- If $q(x) \neq q(x')$, suppose wlog $q(x) < q(x')$. By density, suppose the difference of $q(x') - q(x) > c > 0$ for some $c \in \Gamma$. Also there exists some $c' \in \Gamma$ such that $\lambda - c < q(x) - c' < \lambda$. Hence, we have that $q(c'x) = q(x) - c' < \lambda$ and $q(c'x') = q(x') - c' > q(x) - c' + c > \lambda - c + c = \lambda$.
- If $q(x) = q(x) \in \Gamma$, then $c' = q(x) - \lambda$ will separate the two characters into $(0_+, \lambda_-], [\lambda_+, 1_-]$.

□

Let us remark that this was observed already in [60]. Also, it is clear that $\Gamma/\mathbb{Z}$ is finitely generated iff Γ is finitely generated. Hence by Theorem 3.5.6 we obtain:

Corollary 6.3.2. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. The derived subgroup of $IE(\Gamma)$ is finitely generated iff Γ is finitely generated as a group.*

The fact that Γ finitely generated implies that $D(IE(\Gamma))$ is finitely generated was observed in [60]. But recall that any finitely generated subgroups of $\mathbb{R}$ in particular embed into finitely generated abelian subgroups of $\mathbb{R}$. So we look to focus on the case where $\Gamma \cong \mathbb{Z}^n$ and obtain a finite generating set in this case. Abstractly speaking, Lemma 6.3.1 establishes that the dynamical system $\alpha : \Gamma/\mathbb{Z} \curvearrowright (0_+, 1_-]$ is a subshift in the case when Γ is finitely generated. This is a classical result in dynamics, reproven by Nekrashevych in ([93], Proposition 5.5). Let us describe now how to obtain this picture concretely.

We are inspired by so-called Sturmian subshifts, a classical object in dynamics. See [10], Chapter 1. for a discussion of Sturmian subshifts and the paper [23] in which Brix studied the associated C* -algebras. Let Γ be a dense additive subgroup of $\mathbb{R}$ containing 1. Let $\lambda \in \Gamma \cap (0,1)$ be arbitrary.

For $t \in (0,1)$, set:

$$x_t : \Gamma/\mathbb{Z} \to \{0,1\} \quad b \mapsto \begin{cases} 1 & 0 \le t+b < \lambda < 1 \\ 0 & \text{else} \end{cases}$$

Where $t+b$ is taken modulo $\mathbb{Z}$. Let

$$X_{\Gamma,\lambda} = \overline{\{x_t \; : \; t \in (0,1)\}} \subset \{0,1\}^{\Gamma/\mathbb{Z}}$$

Let $\mathcal{E}_{\Gamma,a} = (X_{\Gamma,a}, \sigma)$. The shift for $c \in \Gamma/\mathbb{Z}$ is given by:

$$\sigma_c : X_\Gamma \to X_\Gamma \quad \sigma_c(x)(b) = x(b+c)$$

Note that $\sigma_c(x_t) = x_{t+c}$ (where addition is taken mod $\mathbb{Z}$) since

$$\sigma_c(x_t)(b) = x_t(b+c) = \begin{cases} 1 & 0 \le t+b+c < \lambda < 1 \\ 0 & \text{else} \end{cases}$$

Let us examine further X_Γ.

Lemma 6.3.3. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Let $\lambda \in \Gamma \cap (0,1)$ be arbitrary. Then $X_\Gamma = \{x_t \; : \; t \in (0,1)\} \sqcup \{\hat{x}_t := \lim_{n\to\infty} x_{t-1/n} \; : \; t \in (0,1] \cap \Gamma\}$. The topology is generated by basic open sets of the form:*

$$\overline{\{x_t \; : \; t \in [a,b)\}} = \{x_t \; : \; t \in [a,b)\} \sqcup \{\hat{x}_t \; : \; t \in (a,b]\}$$

Proof. First let us show that $\{x_t \; : \; t \in (0,1)\} \sqcup \{\hat{x}_t \; : \; t \in (0,1]\} \subset X_\Gamma$. Consider first $\lim_{n\to\infty} x_{1-1/n}$. One has that this is convergent (for all $t' \in \Gamma/\mathbb{Z}$ there exists some $N_{t'} \in \mathbb{N}$ such that $n > N_{t'} \implies x_{1-1/n}(t') = x_{1-1/N_{t'}}(t')$, hence $(\lim_{n\to\infty} x_{1-1/n})(t') = x_{1-1/N_g}(t')$). At the same time, this is different from $x_{t'}$ for any t'.

Set $\hat{x}_1 := \lim_{n\to\infty} x_{1-1/n}$. Suppose for contradiction that $\hat{x}_1 = x_t$ for some t. Note first that for $n > 1/(1-\lambda)$, $x_{1-1/n}(0) = 0$. Therefore, $\hat{x}_1(0) = 0$, hence $t \in [\lambda, 1)$. Let $c \in (0, 1-t) \cap (0, \lambda)$, then $\sigma_c(x_t)(0) = x_{t+c}(0) = 0$. However, for any $c \in (0, \lambda)$ we have that $\sigma_c(\hat{x}_1)(0) = \sigma_c(\lim_{n\to\infty} x_{t-1/n})(0) = \lim_{n\to\infty} x_{c-1/n}(0) = 1$. This is a contradiction hence $\hat{x}_1$ is distinct from all the $x_t, t \in (0,1)$.
Let $t \in (0,1) \cap \Gamma$. Consider $\hat{x}_t = \lim_{n\to\infty} x_{t-1/n}$. Since σ_t is continuous, it is enough to remark that $\sigma_t(\hat{x}_1) = \sigma_t(\lim_{n\to\infty} x_{1-1/n}) = \lim_{n\to\infty} \sigma_t(x_{1-1/n}) = \lim_{n\to\infty} x_{t-1/n} = \hat{x}_t$. Therefore $\hat{x}_t$ exists and is distinct from any $x_{t'}$ since we saw that $\{x_{t'} : t' \in (0,1)\}$ is invariant under σ_t.

Now let us show that $X_\Gamma \subset \{x_t : t \in (0,1)\} \sqcup \{\hat{x}_t : t \in (0,1]\}$. Let $(t_n)_{n\in\mathbb{N}}$ be sequence in $(0,1)$ such that $\lim_{n\to\infty} x_{t_n}$ exists. Then, for all $t \in \Gamma/\mathbb{Z}$, there exists some $N \in \mathbb{N}$ such that $x_{t_{n+N}}(t)$ is constant $(n \in \mathbb{N})$. This implies that t_n is a convergent sequence in $(0,1)$ that is eventually monotone. If t_n is eventually monotone increasing to $t \in \Gamma$, $\lim_n x_{t_n} = \hat{x}_t$. If t_n is eventually monotone decreasing to $t \in \Gamma$, $\lim_n x_{t_n} = x_t$, finally if the limit of t_n is some $t \notin \Gamma$, regardless of the direction, $\lim_n x_{t_n} = x_t$.

Now let us describe the topology on X_Γ. The topology is generated by cylinder sets. These sets come in two forms $C(a,0) := \{x \in X_\Gamma : x(a) = 0\}, C(b,1) = \{x \in X_\Gamma x(b) = 1\}$ where $a, b \in \Gamma \cap (0,1)$ are arbitrary. It is clear that $C(a,0) = \overline{\{x_t : t \in [1-a, 1+\lambda-a) \cup (0, \lambda - a)\}}, C(b,1) = \overline{\{x_t : t \in [\lambda - b, 1-b) \cup [1+\lambda-b, 1)\}}$. Therefore, for $a < b$, we have that $\overline{\{x_t : t \in [a,b)\}}$ is open, and that cylinder sets can be written $C(a,0) := \overline{\{x_t : t \in [1-a, 1+\lambda-a)\}} \cup \overline{\{x_t : t \in (0, \lambda-a)\}}$, $C(b,1) := \overline{\{x_t : t \in [\lambda-b, 1-b)\}} \cup \overline{\{x_t : t \in [1+\lambda-b, 1)\}}$. Therefore the topology is generated by the compact open subsets of the form $\overline{\{x_t : t \in [a,b)\}}$ □

Lemma 6.3.4. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Let $\lambda \in \Gamma \cap (0,1)$. Then, $\mathcal{E}_{\Gamma,\lambda} \cong \Gamma/\mathbb{Z} \ltimes (0_+, 1_-]$ (they are conjugate as groupoids). In particular, the groupoid conjugacy class is independent of our choice of $\lambda \in \Gamma \cap (0,1)$.*

Proof. Our groupoid conjugacy is concrete.

$$\phi : \mathcal{E}_{\Gamma,\lambda} \to \Gamma/\mathbb{Z} \ltimes U_{0,1} \quad (c,x) \mapsto \begin{cases} (c, t_+) & x = x_t,\ t \in \Gamma \\ (c,t) & x = x_t,\ t \notin \Gamma \\ (c, t_-) & x = \hat{x}_t,\ t \in \Gamma \end{cases}$$

Note that in particular, $\phi(\overline{\{x_t\ t \in [c,d)\}}) = [c_+, d_-]$ so that ϕ restricts to a homeomorphism of the unit spaces. □

Our motivation for this description of the groupoid is the ability to apply results from [36], specifically Theorem 3.5.12 to obtain a concrete generating set. To understand how to apply this, we need to understand this groupoid as a subshift and analyse its patches. We first look to obtain a generating set of $D(IE(\Gamma))$ whenever $\Gamma/\mathbb{Z} \cong \mathbb{Z}^d$, $d > 2$. Our first step is to notice we may assume our generators are in a prescribed subset $(a, 1/2)$:

Lemma 6.3.5. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Suppose $\Gamma/\mathbb{Z} \cong \mathbb{Z}^d$, with $d > 1$. Then for any $a \in (0, 1/2) \cap \mathbb{Q}$, $\Gamma/\mathbb{Z}$ is generated by a set $\{\lambda_i\}_{i=1}^d$, where $a < \lambda_1 < \lambda_2 < \ldots < \lambda_d < 1/2$.*

Proof. Let us prove this by induction on d. If $d = 2$, suppose we have an algebraically independent generating set $\hat{\lambda}_1, \hat{\lambda}_2$. Then, since $\mathbb{Z} \oplus \lambda_1\mathbb{Z}$ is dense in $\mathbb{R}$, there exists $n, m \in \mathbb{Z}$ such that $\lambda_2 = \hat{\lambda}_2 + n + m\hat{\lambda}_1 \in (a, 1/2)$. Then it is clear that $\hat{\lambda}_1, \lambda_2$ form an algebraically independent generating set for $\Gamma/\mathbb{Z}$. Since $\mathbb{Z} \oplus \lambda_2\mathbb{Z}$ is dense in $\mathbb{R}$, there exists $n, m \in \mathbb{Z}$ such that $\lambda_1 = n\lambda_2 + m + \hat{\lambda}_1 \in (a, \lambda_2)$. Then again, it is clear that λ_1, λ_2 form an algebraically independent generating set of $\Gamma/\mathbb{Z}$ of the required form.
Assume the statement is true for d and let us suppose that $\Gamma/\mathbb{Z} \cong \mathbb{Z}^{d+1}$. Then let us assume by the inductive hypothesis that our generators of $\Gamma/\mathbb{Z}$ are of the form $1/3 < \lambda_1 < \ldots < \lambda_d < 1/2$ and $\hat{\lambda}_{d+1}$. Since $\mathbb{Z} \oplus \bigoplus_{i=1}^d \lambda_i\mathbb{Z}$ is dense in $\mathbb{R}$, we may find some $\gamma \in \mathbb{Z} \oplus \bigoplus_{i=1}^d \lambda_i\mathbb{Z}$ such that $\lambda_{d+1} = \gamma + \hat{\lambda}_{d+1} \in (\lambda_d, 1/2)$. Then $\{\lambda_i\}_{i=1}^n$ is a generating set of the required form. □

Let us fix notation for our patch maps, and look to remove patches π such that W_π is nonempty. For $(0, e_i, -e_i)$ let the patch $\pi^i_{a,b,c}$ with $a, b, c \in \{0, 1\}$ be the patch such that $\pi^i_{a,b,c}(0) = a$, $\pi^i_{a,b,c}(e_i) = b, \pi^i_{a,b,c}(-e_i) = c$.

Lemma 6.3.6. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Suppose further that $\Gamma/\mathbb{Z} \cong \mathbb{Z}^d, d > 1$ is generated by λ_i with $1/3 < \lambda_1 < \lambda_2 < \ldots < \lambda_d < 1/2$. Take $\lambda = \lambda_1$ in our construction of the subshift (Lemma 6.3.4). Then for all i we have that,*

$$W_{\pi^i_{1,1,0}} = W_{\pi^i_{1,0,1}} = W_{\pi^i_{1,1,1}} = W_{\pi^i_{1,1,0}} = W_{\pi^i_{0,0,0}} = \varnothing$$

Proof. Let us begin with i arbitrary. We show that $W_{\pi^i_{a,b,c}} = \varnothing$ by showing that there exists no x_t such that $x_t \in W_{\pi^i_{a,b,c}}$. Suppose first $x_t \in W_{\pi^i_{1,1,0}} \cup W_{\pi^i_{1,1,1}}$. Then in particular $x_t(0) = x_t(\lambda_i) = 1$. Hence $t \in (0, \lambda_1)$ and $t + \lambda_i \in (0, \lambda_1)$. This is a contradiction since then $\lambda_1 \leq \lambda_i \leq t + \lambda_i < \lambda_1$. Therefore $W_{\pi^i_{1,1,0}} \cup W_{\pi^i_{1,1,1}} = \varnothing$. Now suppose $x_t \in W_{\pi^i_{1,0,1}}$. Then in particular $x_t(0) = x_t(-\lambda_i) = 1$. It follows that $t, t - \lambda_i \in (0, \lambda_1)$. But then $\lambda_1 \leq (t - \lambda_i) + \lambda_1 < t + (-\lambda_i + \lambda_1) \leq t < \lambda_1$. This is a contradiction, hence $W_{\pi^i_{1,0,1}}$ is empty. Now let us suppose $x_t \in W_{\pi^i_{0,0,0}}$. Then we have that $x_t(0) = x_t(\lambda_i) = x_t(-\lambda_i) = 0$. Then $t, t \pm \lambda_i \in [\lambda_1, 1)$. Note in particular then, $\lambda_1 < t - \lambda_i < t < t + \lambda_i < 1$. This is impossible since $1 - \lambda_1 < 2/3$ but $2\lambda_i > 2/3$. □

Lemma 6.3.7. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Suppose further that $\Gamma/\mathbb{Z} \cong \mathbb{Z}^d, d > 1$ is generated by λ_i with $1/3 < \lambda_1 < \lambda_2 < \ldots < \lambda_d < 1/2$. Take $\lambda = \lambda_1$ in our construction of the subshift (Lemma 6.3.4). Then for all i we have:*

1. $x_t \in W_{\pi^i_{1,0,0}} \iff t \in (0, \lambda_1)$

2. $x_t \in W_{\pi^i_{0,1,0}} \iff t \in [\lambda_1+\lambda_i, 1+\lambda_1-\lambda_i)$

3. $x_t \in W_{\pi^i_{0,1,1}} \iff t \in [1-\lambda_i, \lambda_i+\lambda_1)$

4. $x_t \in W_{\pi^i_{0,0,1}} \iff t \in [\lambda_i, 1-\lambda_i)$

Proof. 1. If $x_t \in W_{\pi^i_{1,0,0}}$, $x_t(0) = 1 \implies t \in (0,\lambda_1)$. Moreover, if $t \in (0,\lambda_1)$ then $x_t(0) = 0$, $t+\lambda_i \in [\lambda_i, \lambda_1+\lambda_i) \subset [\lambda_1, 1)$ hence $x_t(\lambda_i) = 0$. $t-\lambda_i \in [1-\lambda_i, 1+\lambda_1-\lambda_i) \subset [\lambda_1, 1)$ hence $x_t(-\lambda_i) = 0$.

2. $x_t(0) \iff t \in [\lambda_1, 1)$. $x_t(\lambda_i) = 1 \iff t \in [1-\lambda_i, 1+\lambda_1-\lambda_i)$, $x_t(-\lambda_i) = 0 \iff t \in [\lambda_1+\lambda_i, 1)$. Note that $1-\lambda_i < 2/3 < 2\lambda_1 < \lambda_1+\lambda_i$, hence the intersection of these sets is $[\lambda_1+\lambda_i, 1+\lambda_1-\lambda_i)$.

3. $x_t(0) \iff t \in [\lambda_1, 1)$. $x_t(\lambda_i) = 1 \iff t \in [1-\lambda_i, 1+\lambda_1-\lambda_i)$, $x_t(-\lambda_i) = 1 \iff t \in [\lambda_i, \lambda_1+\lambda_i)$. $\lambda_i < 1/2 < 1-\lambda_i$. Also note that $\lambda_i+\lambda_1 < 1+\lambda_1-\lambda_i$. Hence, the intersection of these sets is $[1-\lambda_i, \lambda_i+\lambda_1)$.

4. $x_t(0) \iff t \in [\lambda_1, 1)$. $x_t(\lambda_i) = 0 \iff t \in [\lambda_1, 1-\lambda_i)$, $x_t(-\lambda_i) = 1 \iff t \in [\lambda_i, \lambda_1+\lambda_i)$. Noting that $1-\lambda_i < 2/3 < 2\lambda_1 \le \lambda_1+\lambda_i$, it is clear the intersection of these sets is $[\lambda_i, 1-\lambda_i)$. □

Lemma 6.3.8. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Suppose further that $\Gamma/\mathbb{Z} \cong \mathbb{Z}^d, d > 1$ is generated by λ_i with $2/5 < \lambda_1 < \lambda_2 < \ldots < \lambda_d < 1/2$. Take $\lambda = \lambda_1$ in our construction of the subshift (Lemma 6.3.4). Then for all i the only a,b,c such that $W_{\pi^i_{a,b,c}}$ is nonempty and the sets $W_{\pi^i_{a,b,c}}, \lambda_i W_{\pi^i_{a,b,c}}, -\lambda_i W_{\pi^i_{a,b,c}}$ are pairwise disjoint are $W_{\pi^i_{0,1,0}}$ and $W_{\pi^i_{0,0,1}}$.*

Proof. Note the following equivalences, since intersections of cylinder sets are cylinder sets:

- $W_{\pi^i_{a,b,c}}, \pm\lambda_i W_{\pi^i_{a,b,c}}$ are pairwise disjoint.
- There exists no t such that x_t is in the intersection of one of the above sets.
- By Lemma 6.3.7, $x_t \in W_{\pi^i_{a,b,c}}$ iff $t \in [d,f)$ for some $d,f \in \Gamma$. It follows that $[d,f), [d+\lambda_i, f+\lambda_i), [d-\lambda_i, f-\lambda_i)$ are disjoint intervals in $(0,1] \mod \mathbb{Z}$.

Note that if the interval in Lemma 6.3.7 is greater than 1/3, these three sets cannot be disjoint, since this would imply there are 3 disjoint subintervals of $(0,1]$ all of have Lebesgue measure greater than 1/3. Therefore $W_{\pi^i_{1,0,0}}$ can be excluded for all i, since we assumed $\lambda_1 > 1/3$.
The other intervals are all less than 1/3 in length. In particular then, when we translate by $1/2 > \lambda_i > 1/3$, we have that $\lambda_i W_{\pi^i_{a,b,c}} \cap W_{\pi^i_{a,b,c}} = \emptyset, -\lambda_i W_{\pi^i_{a,b,c}} \cap W_{\pi^i_{a,b,c}} = \emptyset$. It is for this reason enough to check if $\lambda_i W_{\pi^i_{a,b,c}} \cap -\lambda_i W_{\pi^i_{a,b,c}} = \emptyset$.

- If $(a,b,c) = (0,1,0)$, the sets to compare are $[\lambda_1, 1+\lambda_1-\lambda_i)$ and $[\lambda_1+2\lambda_i-1, \lambda_1)$. These sets are disjoint clearly.

- If $(a,b,c) = (0,1,1)$, the sets to compare are $(1-2\lambda_i, \lambda_1]$ and $(0, 2\lambda_i + \lambda_1 - 1]$. These have nonempty intersection $\iff 1-2\lambda_i > 2\lambda_i + \lambda_1 - 1 \iff 2 > 4\lambda_i + \lambda_1 > 5\lambda_1$. We assumed that $\lambda_1 > 2/5$, hence this never occurs and these sets always have nonempty intersection.

- If $(a,b,c) = (0,0,1)$, the sets to compare are $(0, 1-2\lambda_i]$ and $(2\lambda_i, 1]$. These sets are disjoint since $1-2\lambda_i < 1/3 < 2/3 < 2\lambda_i$.

□

We are now ready to apply Theorem 3.5.12 to obtain a generating set.

Theorem 6.3.9. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Suppose further that $\Gamma/\mathbb{Z} \cong \mathbb{Z}^d, d > 1$. Then there exists a generating set $\{\lambda_i\}_{i=1}^d$ of $\Gamma/\mathbb{Z}$ with $2/5 < \lambda_1 < \lambda_i < \lambda_{i+1} < \lambda_d < 1/2$. Then, $D(IE(\Gamma))$ is generated by*

$$\sigma_i(t) = \begin{cases} t+\lambda_i & t \in (\lambda_1, 1-2\lambda_i+\lambda_1] \\ t+\lambda_i - 1 & t \in (\lambda_1+\lambda_i, 1+\lambda_1-\lambda_i] \\ t+1-2\lambda_i & t \in (\lambda_1+2\lambda_i-1, \lambda_1] \\ t & else \end{cases}, \quad \hat{\sigma}_i(t) = \begin{cases} t+\lambda_i & t \in (0, 1-2\lambda_i] \sqcup (\lambda_i, 1-\lambda_i] \\ t-2\lambda_i & t \in (2\lambda_i, 1] \\ t & else \end{cases}$$

Where $i = 1,..,d$.

Proof. First, note that the generating set λ_i of $\Gamma/\mathbb{Z}$ is indeed a generating set due to Lemma 6.3.5. We have for all i, the only patches $\pi^i : \{0, \lambda_i, -\lambda_i\} \to \{0,1\}$ such that T_π is defined are $\pi^i_{0,1,0}$ and $\pi^i_{0,0,1}$ by Lemma 6.3.8. Hence, the result follows by Theorem 3.5.12 that $D(IE(\Gamma))$ is generated by $T_{\pi^i_{0,1,0}}, T_{\pi^i_{0,0,1}}$. It is a straightforward computation to check that $T_{1,i} = \varphi \circ \phi(T_{\pi^i_{0,0,1}})$, $T_{1,i} = \varphi \circ \phi(T_{\pi^i_{0,1,0}})$.

□

Example 6.3.10. *Let $\Gamma \cong \mathbb{Z}^3$, and suppose $\Gamma \cong \mathbb{Z} \oplus \lambda_1\mathbb{Z} \oplus \lambda_2\mathbb{Z}$ (where wlog we assume $2/5 < \lambda_i < 1/2$ and that λ_i are rationally independent). Then the main result of [60] says that $D(IE(\Gamma))$ is a simple finitely generated amenable group. Moreover, we have an explicit generating set given by:*

$$\sigma_1(t) = \begin{cases} t+\lambda_1 & t \in (\lambda_1, 1-\lambda_1] \\ t+\lambda_1 - 1 & t \in (2\lambda_1, 1] \\ t+1-2\lambda_1 & t \in (3\lambda_1 - 1, \lambda_1] \\ t & else \end{cases}, \quad \hat{\sigma}_1(t) = \begin{cases} t+\lambda_1 & t \in (0, 1-2\lambda_1] \sqcup (\lambda_1, 1-\lambda_1] \\ t-2\lambda_1 & t \in (2\lambda_1, 1] \\ t & else \end{cases}$$

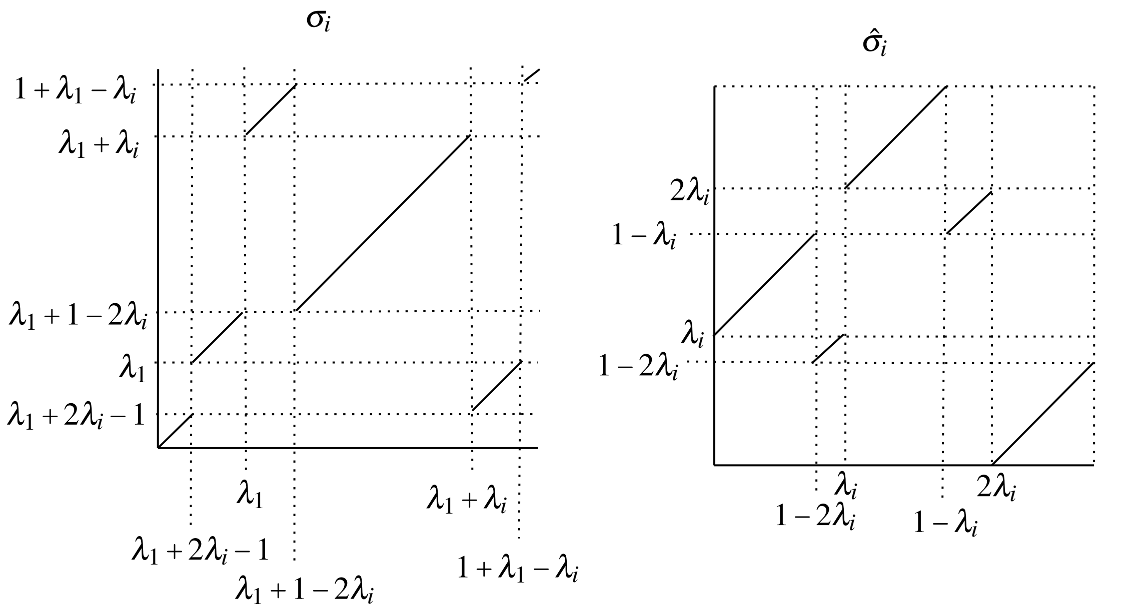


Figure 6.2: A pictoral representation of $\sigma_i, \hat{\sigma}_i$ as defined in Theorem 6.3.9

$$\sigma_2(t) = \begin{cases} t+\lambda_2 & t \in (\lambda_1, 1-2\lambda_2+\lambda_1] \\ t+\lambda_2-1 & t \in (\lambda_1+\lambda_2, 1+\lambda_1-\lambda_2] \\ t+1-2\lambda_2 & t \in (\lambda_1+2\lambda_2-1, \lambda_1] \\ t & else \end{cases}, \quad \hat{\sigma}_2(t) = \begin{cases} t+\lambda_2 & t \in (0, 1-2\lambda_2] \sqcup (\lambda_2, 1-\lambda_2] \\ t-2\lambda_2 & t \in (2\lambda_2, 1] \\ t & else \end{cases}$$

It would be interesting to find independent proof that this group is amenable by using this concrete picture of its generators.

Theorem 6.3.11. *Let Γ be a dense countable subgroup of $\mathbb{R}$ with $1 \in \Gamma$. Suppose $\Gamma \cong \mathbb{Z}^{d+1}$, $d > 1$. Suppose further that $\Gamma/\mathbb{Z} \cong \mathbb{Z}^d \oplus \mathbb{Z}_k, d > 1$, where $k > 9$. Using Lemma 6.3.5, we may take the generators of $\Gamma/\mathbb{Z}$ to be $1/k$ and irrational numbers λ_i, with $2/5 < \lambda_1 < \ldots < \lambda_d < 1/2$. Take $\lambda = \lambda_1$ in our construction of the subshift (Lemma 6.3.4). Then a generating set for $D(IE(\Gamma))$ is given by:*

$$S = \{\sigma_i, \hat{\sigma}_i, r_{k,a} \,:\, i \in \{1,2,..,d\} \;\; a \in \{\lambda_1 - 1/k, \lambda_1 - 2/k, 1-2/k, 1-1/k\}\}$$

Where $\sigma_i, \hat{\sigma}_i$ are as in Theorem 6.3.9, and $r_{k,a}$ is given by:

$$r_{k,a}(t) = \begin{cases} t+1/k \mod \mathbb{Z} & t \in (a, a+2/k] \\ t-2/k \mod \mathbb{Z} & t \in (a+2/k, a+3/k] \\ t & else \end{cases}$$

Proof. Let $\pi_{a,b,c} : \{0, 1/k, -1/k\} \to \{0,1\}$ be the patch such that $\pi_{a,b,c}(0) = a, \pi_{a,b,c}(1/k) = b, \pi_{a,b,c}(-1/k) = c$. As before we fix the picture of this dynamical as a subshift where $x_t(0) = 1 \iff t \in (0, \lambda_1)$.

- $x_t \in \sqcup_{b,c\in\{0,1\}} W_{\pi_{1,b,c}} \iff t \in (0,\lambda_1)$
- $x_t \in \sqcup_{a,c\in\{0,1\}} W_{\pi_{a,1,c}} \iff t \in (0,\lambda_1 - 1/k) \sqcup [1-1/k,1)$
- $x_t \in \sqcup_{a,b\in\{0,1\}} W_{\pi_{a,b,1}} \iff t \in [1/k, 1/k+\lambda_1)$

Therefore $\varnothing = W_{\pi_{1,0,0}} = W_{\pi_{0,1,1}}$. Note some of the translated cylinder sets are also not disjoint– $x_{1/k} \in W_{\pi_{1,1,1}} \cap -1/kW_{\pi_{1,1,1}}$ and $x_{1-2/k} \in W_{\pi_{0,0,0}} \cap +1/kW_{\pi_{0,0,0}}$ The other corresponding $T_{\pi_{a,b,c}}$ are well defined. It is straightforward to verify that:

$$\varphi\circ\phi(T_{\pi_{0,1,0}}) = r_{k,1-2/k}, \quad \varphi\circ\phi(T_{\pi_{1,1,0}}) = r_{k,1-1/k},$$

$$\varphi\circ\phi(T_{\pi_{1,0,1}}) = r_{k,\lambda_1-2/k}, \quad \varphi\circ\phi(T_{\pi_{0,0,1}}) = r_{k,\lambda_1-1/k}.$$

Then, the proof follows via Theorem 6.3.9 and Theorem 3.5.12. □

Let us remark how these generating sets behave with respect to inclusions $\Gamma' \subset \Gamma$:

Corollary 6.3.12. *Let $\Gamma' \subset \Gamma$ be dense, nested cyclic subgroups, finitely generated subgroups of $\mathbb{R}$ containing 1. Given a subset $P \subset \mathbb{R}$, Let $Rank_{\mathbb{Q}}(P)$ denote the number of rationally independent numbers in P. Then there exists generating sets S, S' of Γ, Γ' such that:*

- $S' \subset S$
- $|S \setminus S'| \le 2Rank_{\mathbb{Q}}(\Gamma' \setminus \Gamma) + 4.$
- $|S| \le 2Rank_{\mathbb{Q}}(\Gamma) + 4$

Let us remark that IET locally embeds into groups of the form $\Gamma = 1/k\mathbb{Z} \oplus \bigoplus_{i=1}^{d} \lambda_i\mathbb{Z}$, where $d > 1, k > 9$. This is because finitely generated subgroups of $\mathbb{R}$ are finitely generated abelian, hence if we let k being arbitrarily large enough to contain all the rational angles of a given finite subset of IET. Note also that $D(IE(\Gamma))$ is amenable iff $IE(\Gamma)$ is amenable. Hence, we have the following reductions of Question 6.1.5:

Corollary 6.3.13. *The following are equivalent:*

- *IET is nonamenable.*
- *There exists $k > 9$, and a finite collection of irrational numbers $\lambda_1, \ldots, \lambda_d \in [2/5, 1/2)$ such that the group generated by S as in Theorem 6.3.11 is nonamenable.*

And similarly for the existence of a nonamenable free subgroup, the question will reduce to the finitely generated abelian case.

6.4 Homology of interval exchange groups

We look to apply the results of Section 3.6 to further our understanding of the interval exchange groups. Let us first compute the homology of the underlying groupoids. Let us remark that as with all globalisable partial actions we saw in Example 2.4.2 that our groupoid homology will reduce to group homology with certain coefficients:

$$H_*(\Gamma \ltimes_\alpha (0_+,1_-]) \cong H_*(\Gamma \ltimes \mathbb{R}_\Gamma|_{(0_+1_-]}^{(0_+1_-]}) \cong H_*(\Gamma \ltimes \mathbb{R}_\Gamma) \cong H_*(\Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z}))$$

In other words, the groupoid homology with coefficients in $\mathbb{Z}$ reduces to the group homology with coefficients in $C_c(\mathbb{R}_\Gamma, \mathbb{Z})$. It is for this reason we did not introduce groupoid homology. In fact, in our case the groupoid homology is just a shifted version of the homology of Γ:

Lemma 6.4.1. *Let Γ be a dense additive subgroup of $\mathbb{R}$ containing 1. Then, $k \geq 0$ $H_k(\Gamma \ltimes_\alpha C((0_+,1_-])) \cong H_{k+1}(\Gamma)$.*

Proof. We follow a similar technique to that used in ([76], Proposition 5.8). Let $\hat{\mathbb{R}}_\Gamma = \mathbb{R}_\Gamma \bigcup \{+\infty\}$ be the one sided compactification of $\mathbb{R}_\Gamma$, i.e. the basic open sets in $\hat{\mathbb{R}}_\Gamma$ are $[a_+,b_-], [a_+,\infty], a,b \in \Gamma$. Let $\Gamma \curvearrowright \hat{\mathbb{R}}_\Gamma$ by acting as α on $\mathbb{R}_\Gamma$ and by fixing ∞.

The set $\{1_{[a_+,\infty]} : a \in \Gamma\}$ is a $\mathbb{Z}$-basis of $C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})$ that Γ acts freely and transitively on. Hence, $C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z}) \cong \mathbb{Z}\Gamma$ as $\mathbb{Z}\Gamma$ modules.

$$H_*(\Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}.$$

Then note that

$$0 \to C_c(\mathbb{R}_\Gamma, \mathbb{Z}) \to C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z}) \to \mathbb{Z} \to 0$$

Is a short exact sequence of $\mathbb{Z}\Gamma$ modules. By Proposition 6.1, Chapter III of Brown [27], we get that there is a long exact sequence:

$$\ldots \to H_1(\Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})) \to H_1(\Gamma) \to H_0(\Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z})) \to H_0(\Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})) \to H_0(\Gamma) \to 0$$

But we already know that $H_1(\Gamma) = \Gamma_{ab} = \Gamma, H_0(\Gamma) = \mathbb{Z}$, $H_0(\Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})) = \mathbb{Z}$ and for $H_i(\Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})) = 0$ for $i \geq 1$. Hence we get that there is a long exact sequence of the form:

$$\to 0 \to H_{i+1}(\Gamma) \to H_i(\Gamma \ltimes_\alpha C((0_+,1_-])) \to 0 \to \ldots$$

$$\ldots \to 0 \to \Gamma \to H_0(\Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z})) \to \mathbb{Z} \to \mathbb{Z} \to 0$$

The section around $H_i(\Gamma \ltimes_\alpha C((0_+,1_-]))$ looks like this for all $i \geq 0$, forcing an isomorphism $H_{i+1}(\Gamma) \cong H_i(\Gamma \ltimes_\alpha C((0_+,1_-]))$. Around $H_0(\Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z}))$, exactness forces the map from

$H_0(\Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z}) \to \mathbb{Z}$ to be 0:

$$H_0(\Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z}) \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} 0$$

hence, there is also an isomorphism $\Gamma \cong H_0(\Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z})$. The conclusion follows. □

Note in particular that $H_0(\Gamma \ltimes_\alpha (0_+, 1_-]) = H_1(\Gamma) = \Gamma_{ab} = \Gamma$.

Corollary 6.4.2. *Let* Γ *be a dense additive subgroup of* $\mathbb{R}$ *containing* 1*. Then,* $H_0(\Gamma \ltimes_\alpha (0_+, 1_-]) \cong \Gamma$

For this class of groupoids, the HK conjecture (Conjecture 2.5.4) holds.

Theorem 6.4.3. *Let* Γ *be a dense countable subgroup of* $\mathbb{R}$ *containing 1. Then, the HK conjecture holds for* $\Gamma \ltimes_\alpha (0_+, 1_-]$*, i.e.*

$$K_0(C_r^*(\Gamma \ltimes_\alpha C((0_+, 1_-]))) \cong K_1(C_r^*(\Gamma)) \cong \bigoplus_{i=0}^{\infty} H_{2i}(\Gamma \ltimes_\alpha (0_+, 1_-]) \cong \bigoplus_{i=0}^{\infty} H_{2i+1}(\Gamma)$$

$$K_1(C_r^*(\Gamma \ltimes_\alpha C((0_+, 1_-]))) \cong K_0(C_r^*(\Gamma))/\mathbb{Z}[1]_0 \cong \bigoplus_{i=0}^{\infty} H_{2i+1}(\Gamma \ltimes_\alpha (0_+, 1_-]) \cong \bigoplus_{i=1}^{\infty} H_{2i}(\Gamma)$$

Proof. This was observed in the case where Γ is a ring in [76], Remark 5.8. One can verify the case where Γ is finitely generated abelian by direct computation. This is in fact enough, since we can write any Γ as an inductive limit of finitely generated abelian groups. Set $\Gamma = \lim_{n\to}\Gamma_n$, where Γ_n is finitely generated abelian. It follows that $K_i(C_r^*(\Gamma \ltimes_\alpha (0_+, 1_-])) \cong \lim_{n\to} K_i(\Gamma_n \ltimes_\alpha C_r^*((0_+, 1_-])), H_i(\Gamma \ltimes_\alpha (0_+, 1_-]) \cong \lim_{n\to} H_i(\Gamma_n \ltimes_\alpha (0_+, 1_-])$; both the K-theory and groupoid homology decompose as direct limits of this case. □

We would also like to understand the homology of $IE(\Gamma)$ in terms of the homology of Γ. Let us construct certain maps I, j appearing in the AH conjecture. Let B be a compact open bisection of $\Gamma \ltimes_\alpha (0_+, 1_-]$ with $s(B) \cap r(B) = \varnothing$.

$$j : \Gamma \otimes \mathbb{Z}_2 \cong H_0(\Gamma \ltimes_\alpha (0_+, 1_-]) \otimes \mathbb{Z}_2 \to IE(\Gamma)_{ab} \quad [1_{s(B)}]_{H_0} \to \gamma_B$$

Where γ_B is the generator of the symmetric group, as in 3.2.1.

$$I : IE(\Gamma)_{ab} \to H_1(\Gamma \ltimes \alpha) \cong H_2(\Gamma) \quad \hat{B} \mapsto [1_B]_{H_1}$$

Let us now some information about the homology of $IE(\Gamma)$ in terms of the homology of $\Gamma \ltimes_\alpha (0_+, 1_-]$ using I, j. First we note that since $\Gamma \ltimes_\alpha (0_+, 1_-]$ is a minimal, almost finite groupoid, with a Cantor unit space, we fall into the scope of Theorem 3.6.7 allows us to describe the abelianization of these groups in terms of groupoid homology. Substituting what we know, we have that:

Lemma 6.4.4. *Let* Γ *be a dense additive subgroup of* $\mathbb{R}$ *containing* 1. *Then, I and j are well defined and moreover there exists a long exact sequence:*

$$H_2(\mathsf{D}(IE(\Gamma))) \to H_3(\Gamma) \to \Gamma \otimes \mathbb{Z}_2 \xrightarrow{j} IE(\Gamma)_{ab} \xrightarrow{I} H_2(\Gamma) \to 0$$

Theorem 6.4.5. *Let* Γ *be a dense countable subgroup of* $\mathbb{R}$ *containing* 1. *The following are equivalent:*

- Γ *is finitely generated.*
- $D(IE(\Gamma))$ *is finitely generated.*
- $IE(\Gamma)$ *is finitely generated.*

Proof. We also showed already that Γ is finitely generated iff $D(IE(\Gamma))$ is finitely generated (Corollary 3.5.6). If Γ is finitely generated, then $H_2(\Gamma)$ is also finitely generated. By extension, $IE(\Gamma)_{ab}$ is finitely generated by Theorem 3.6.7. It follows that $IE(\Gamma)$, the extension of $IE(\Gamma)_{ab}$ by $D(IE(\Gamma))$ is finitely generated whenever Γ is finitely generated.

Now suppose $IE(\Gamma)$ is finitely generated. Let $f_1,\ldots,f_n \in IE(\Gamma)$ be the generators. Let A be the union of all the angles in each f_i. We then have that $\mathbb{Z}A \subset \Gamma$ since each f_i in $IE(\Gamma)$, conversely, every $g \in IE(\Gamma)$ is a finite string of f_i, hence $\Gamma \subset \mathbb{Z}A$. It follows that A is a finite generating set of Γ. □

Notice that we can describe the rational homology of $IE(\Gamma)$ in terms of the rational homology of Γ by applying Corollary 3.6.2.

Lemma 6.4.6. *Let* Γ *be a dense additive subgroup of* $\mathbb{R}$ *containing* 1. *For a group G, let*

$$H_*^{even}(G) = \begin{cases} H_*(G) & * \text{ even} \\ \{0\} & \text{else} \end{cases} \qquad H_*^{odd}(G) = \begin{cases} H_*(G) & * \text{ odd} \\ \{0\} & \text{else} \end{cases}$$

and, let

$$H_{*>2}^{even}(G) = \begin{cases} H_*(G) & * > 2 \text{ even} \\ \{0\} & \text{else} \end{cases}$$

Then,

$$H_*(IE(\Gamma),\mathbb{Q}) \cong Ext(H_{*+1}^{even}(\Gamma,\mathbb{Q})) \otimes Sym(H_{*+1}^{odd}(\Gamma,\mathbb{Q}))$$

$$H_*(D(IE(\Gamma)),\mathbb{Q}) \cong Ext(H_{*+1>2}^{even}(\Gamma,\mathbb{Q})) \otimes Sym(H_{*+1}^{odd}(\Gamma,\mathbb{Q}))$$

Corollary 6.4.7. *Let* Γ,Γ' *be dense additive subgroups of* $\mathbb{R}$ *containing 1. Then,* $H_*(\Gamma,\mathbb{Q}) \cong H_*(\Gamma',\mathbb{Q}) \implies H_*(IE(\Gamma),\mathbb{Q}) \cong H_*(IE(\Gamma'),\mathbb{Q})$

We can also say the following about integral homological stability by applying Theorem 3.6.5.

Lemma 6.4.8. *Let Γ be a dense countable subgroup of $\mathbb{R}$ containing 1 and some $x \neq 0$. Then, the isomorphism $\Gamma \to 1/x\Gamma \quad t \mapsto t/x$ induces an isomorphism in homology $H_*(IE(\Gamma)) \cong H_*(IE(1/x\Gamma))$*

Proof. Assume wlog that $x > 1$. Then, it is clear that $IE(\Gamma)$ is isomorphic to the subgroup $IE(1/x\Gamma)_{(0,1/x]} = \{f \in IE(1/x\Gamma) : t > 1/x \implies f(t) = t\}$ by conjugating via the homothety $t \mapsto t/x$. Also, $IE(1/x\Gamma)_{(0,1/x]} = \mathsf{F}(1/x\Gamma \ltimes_\alpha (0_+, 1_-]|^{(0_+,1/x_-]}_{(0_+,1/x_-]})$. Therefore, the result follows Theorem 3.6.5. □

6.5 Concrete examples of interval exchange groups

6.5.1 Rational acting group and UHF groupoids

An easy case for computing an explicit generating set is when $\Gamma \subset \mathbb{Q}$, by identifying $IE(\Gamma)$ with a certain increasing sequence of symmetric groups.

Lemma 6.5.1. *Let $\Gamma \subset \mathbb{Q}$ be dense with $1 \in \Gamma$. Then Γ is generated by a strictly decreasing sequence of rational numbers $\{1/k_i\}_{i=1}^{\infty} \subset \mathbb{Q}$. Moreover, set $k(n) = \prod_{i=1}^{n} k_i$. Then, $IE(\Gamma) \cong \lim_{n\to\infty} S_{k(n)}$. The connecting map is given by (letting $\sigma^n_{i,j}$ be the element of $S_{k(n)}$ permuting the i and jth element):*

$$\iota_n : S_{k(n)} \hookrightarrow S_{k(n+1)} \quad \sigma^n_{i,i+1} \mapsto \prod_{k=1}^{k_n-1} \sigma^{n+1}_{ik_n+k,(i+1)k_n+k}$$

An explicit generating set is given by:

$$\{\sigma^n_{i,i+1} \; : 0 < i < k(n), n \in \mathbb{N}\}$$

Subject to the (infinitely many) relations:

- $\sigma^n_{i,i+1} = \prod_{k=1}^{k_n-1} \sigma^{n+1}_{ik_n+k,(i+1)k_n+k}$, $1 \leq i < k(n)$
- $(\sigma^n_{i,i+1})^2 = 1$, $1 \leq i < k(n)$
- $1 = [\sigma_{i,i+1}, \sigma_{j,j+1}]$ $1 \leq i < i+2 \leq j < k(n)$
- $\sigma^n_{i,i+1}\sigma^n_{i+1,i+2}\sigma^n_{i,i+1} = \sigma^n_{i+1,i+2}\sigma^n_{i,i+1}\sigma^n_{i+1,i+2}$, , $1 \leq i < k(n) - 1$

In terms of piecewise linear bijections, one can take $\sigma^n_{i,j}(t) = \begin{cases} t + \frac{j-i}{k(n)} & t \in (\frac{i}{k(n)}, \frac{i+1}{k(n)}] \\ t + \frac{i-j}{k(n)} & t \in (\frac{j}{k(n)}, \frac{j+1}{k(n)}] \\ t & \text{else} \end{cases}$

Proof. If $\Gamma \subset \mathbb{Q}$, it is clear that Γ is generated by the (strictly decreasing) sequence of rational numbers $\{1/k_i\}_{i=1}^{\infty}$. Let $\Gamma_n = 1/(k(n))\mathbb{Z}$. It is clear that $\Gamma = \bigcup \Gamma_n$. Then note we have:

$$IE(\Gamma) = \mathsf{F}(\Gamma/\mathbb{Z} \ltimes U_{0,1}) = \bigcup_{n\in\mathbb{N}} \mathsf{F}(\Gamma_n/\mathbb{Z} \ltimes U_{0,1}) \cong \bigcup_{n\in\mathbb{N}} S_{k(n)}$$

And that the generators of $\mathsf{F}(\Gamma_n/\mathbb{Z} \ltimes U_{0,1})$ are $\sigma_{i,j}^n$ □

Implicitly in the above proof we also realize independently $IE(\Gamma)$ for $\Gamma \subset \mathbb{Q}$ cannot be finitely generated. We also verify that $IE(\Gamma)$ is amenable in this case. Via Theorem 3.4.12, we also obtain an alternative proof of Corollary 6.2.15.

This Corollary establishes that $\Gamma \ltimes_\alpha (0_+, 1_-]$ in this case is the unique AF groupoid associated with the following Bratelli diagram:

$$1 \xrightarrow{k_1} k_1 \xrightarrow{k_2} k(2) \xrightarrow{k_3} k(3) \xrightarrow{k_4} \dots \xrightarrow{k_n} k(n) \xrightarrow{k_{n+1}} \dots$$

With each $k_i > 1$. The supernatural number associated with this AF algebra is always infinite and given by $\prod_{i=1}^{\infty} k_i = \lim_{n\to\infty} k(n)$.

If Γ is generated by a strictly decreasing sequence of rational numbers $\{1/k_i\}_{i=1}^{\infty}$, then Γ is the direct limit of the sequence: $\mathbb{Z} \xrightarrow{k_1} \mathbb{Z} \xrightarrow{k_2} \mathbb{Z} \xrightarrow{k_3} \dots$. From this one can observe $H_0(\Gamma) = \Gamma$ and $H_*(\Gamma) = 0$ for all $* \geq 1$. Therefore, via Lemma 6.4.1 we obtain that

$$H_*(\Gamma \ltimes (0_+, 1_-]) \cong H_{*+1}(\Gamma) = \begin{cases} \Gamma & * = 0 \\ 0 & \text{else} \end{cases}$$

We have then that the 5-term exact sequence in Theorem 3.6.7 collapses to an isomorphism

$$IE(\Gamma)_{ab} \cong \Gamma \otimes \mathbb{Z}_2$$

Therefore $IE(\Gamma) \cong \mathbb{Z}_2$ iff Γ is 2-divisible. This demonstrates that $IE(\Gamma) \cong \mathsf{S}(\Gamma \ltimes (0_-, 1_+])$ since the map j in Theorem 3.6.7 is the zero map. This can also be verified via the generating set in Lemma 6.5.1, since it is clear that each $\sigma_{i,j}^n \in \mathsf{S}(\Gamma \ltimes (0_-, 1_+])$. There is also only one nontrivial quotient homomorphism, given by the sign homomorphism:

$$sgn : IE(\Gamma) \to (\mathbb{Z}_2, +) \quad \sigma_{i,j}^n \mapsto 1$$

As with the classical symmetric group on n-generators. Let us also apply Lemma 3.6.4. We have that $D(IE(\Gamma))$ is rationally acyclic immediately. Noting then $IE(\Gamma)$ is the extension of a rationally acyclic group by a rationally acyclic group:

$$0 \to D(IE(\Gamma)) \xrightarrow{\iota} IE(\Gamma) \to \Gamma \otimes \mathbb{Z}_2 \to 0$$

If Γ is rationally acyclic, $IE(\Gamma)$ is also rationally acyclic.

Example 6.5.2 (Measure Preserving V and the CAR algebra)**.** *Let $n \in \mathbb{N}$, $n \geq 2$ and $\Gamma = \mathbb{Z}[1/n] \subset \mathbb{Q}$ be the ring generated by $1/n$. Then, $IE(\mathbb{Z}[1/n])$ is canonically isomorphic to the group of the measure-preserving subgroup of the Higman-Thompson group $V_{n,1}$ (that is, the subgroup elements which preserve the Lebesgue measure on $(0,1]$).. The associated Bratelli diagram is given by:*

$$1 \xrightarrow{n} n \xrightarrow{n} n^2 \xrightarrow{n} n^3 \xrightarrow{n} \ldots$$

And the corresponding UHF algebra is $\bigcup_{k=1}^{\infty} M_{n^k}(\mathbb{C})$, where the inclusion maps are given by block diagonal inclusions:

$$A_{n^k} \mapsto \begin{pmatrix} A_{n^k}^{(1)} & 0_{n^k} & \ldots & 0_{n^k} \\ 0_{n^k} & A_{n^k}^{(2)} & \ldots & 0_{n^k} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n^k} & 0_{n^k} & \ldots & A_{n^k}^{(n)} \end{pmatrix}$$

Moreover, if $n = 2$, the groupoid $\mathbb{Z}[1/2] \ltimes (0_+, 1_-]$ is isomorphic to the standard groupoid model of the CAR algebra with supernatural number 2^∞.

Example 6.5.3 ($\mathbb{Q}$ and the universal UHF algebra $\mathcal{Q}$)**.** *A final example to highlight is that of $\mathbb{Q}$ and the universal UHF algebra $\mathcal{Q}$. Here, we are discussing the groupoid associated with the following Bratelli diagram:*

$$1 \xrightarrow{2} 2 \xrightarrow{3} 6 \xrightarrow{4} \ldots \xrightarrow{n} n! \xrightarrow{n+1} \ldots$$

This groupoid is the standard model of the universal UHF algebra $\mathcal{Q}$ and $IE(\mathbb{Q}) = \bigcup_{n \in \mathbb{N}} S_{n!}$.

6.5.2 Finitely generated abelian acting group

Now let us suppose that we are in the case where Γ is finitely generated abelian and generated by irrational numbers that is, let $\Gamma = \mathbb{Z} \oplus \bigoplus_{i=1}^{d} \lambda_i \mathbb{Z}$ where λ_i are $\mathbb{Q}$-independent numbers and $d \geq 1$. As an abstract group $\Gamma \cong \mathbb{Z}^{d+1}$. Therefore, we have that:

$$H_*(\Gamma \ltimes (0_+, 1_-]) \cong H_{*+1}(\mathbb{Z}^{d+1}) \cong \mathbb{Z}^{{}^{d+1}C_{*+1}}$$

By ${}^{d+1}C_{*+1}$ we mean the combinatorial "choose" function. For example, if $d = 1$ we have $H_0(\mathbb{Z} \oplus \lambda_1 \mathbb{Z} \ltimes (0_+, 1_-]) \cong \mathbb{Z}^2$, $H_1(\mathbb{Z} \oplus \lambda_1 \mathbb{Z} \ltimes (0_+, 1_-]) \cong \mathbb{Z}$. Otherwise, the homology vanishes. Plugging this into Theorem 3.6.7 we obtain the short exact sequence:

$$0 \to \mathbb{Z}^2 \otimes \mathbb{Z}_2 = \mathbb{Z}_2^2 \xrightarrow{j} IE((\mathbb{Z} \oplus \lambda_1 \mathbb{Z})_{ab} \xrightarrow{I} \mathbb{Z} \to 0$$

It follows that $IE(\mathbb{Z}\oplus\lambda_1\mathbb{Z})_{ab}\cong\mathbb{Z}\oplus\mathbb{Z}_2^2$.

Rationally then, $H_1(IE((\mathbb{Z}\oplus\lambda_1\mathbb{Z}),\mathbb{Q})\cong H_0(IE((\mathbb{Z}\oplus\lambda_1\mathbb{Z}),\mathbb{Q})\cong\mathbb{Q}$, and for $*\geq 1, H_*(IE((\mathbb{Z}\oplus\lambda_1\mathbb{Z}),\mathbb{Q})=0$ by Lemma 6.4.6. In other words, when $d=1$, we can see that $D(IE((\mathbb{Z}\oplus\lambda_1\mathbb{Z}))$ is rationally acyclic through Lemma 6.4.6.

This is contrast to the case $d>1$. If $d>1$ one has that $H_d(\mathbb{Z}\oplus\bigoplus_{i=1}^d\lambda_i\mathbb{Z}\ltimes(0_-,1_+])\cong\mathbb{Z}$ and for all $*>d, H_d(\mathbb{Z}\oplus\bigoplus_{i=1}^d\lambda_i\mathbb{Z}\ltimes(0_-,1_+])\cong 0$. This implies $H_d(D(IE(\mathbb{Z}\oplus\bigoplus_{i=1}^d\lambda_i\mathbb{Z}),\mathbb{Q})\neq 0$ by Lemma 6.4.6.

Suppose $d=2$. Then $H_2(\mathbb{Z}\oplus\lambda_1\mathbb{Z}\oplus\lambda_2\mathbb{Z}\ltimes(0_+,1_-])\cong\mathbb{Z}\ H_0(\mathbb{Z}\oplus\lambda_1\mathbb{Z}\oplus\lambda_2\mathbb{Z}\ltimes(0_+,1_-])\cong H_1(\mathbb{Z}\oplus\lambda_1\mathbb{Z}\oplus\lambda_2\mathbb{Z}\ltimes(0_+,1_-])\cong\mathbb{Z}^3$ and all other homology terms vanish. Via Theorem 3.6.7 we obtain a short exact sequence:

$$0\to\mathbb{Z}_2^3\xrightarrow{j}IE(\mathbb{Z}\oplus\lambda_1\mathbb{Z}\oplus\lambda_2\mathbb{Z})_{ab}\xrightarrow{I}\mathbb{Z}\to 0$$

We again have that this short exact sequence splits and that $IE(\mathbb{Z}\oplus\lambda_1\mathbb{Z}\oplus\lambda_2\mathbb{Z})_{ab}\cong\mathbb{Z}\oplus\mathbb{Z}_2^3$.

If $d>2$, $H_3(\mathbb{Z}\oplus\bigoplus_{i=1}^d\lambda_i\mathbb{Z})\neq 0$, and so we do not get a short exact sequence in Theorem 3.6.7. However, we may still obtain some information from the long exact sequence, which shall look like:

$$H_2(D(IE(\mathbb{Z}\oplus\bigoplus_{i=1}^d\lambda_i\mathbb{Z}))\to\mathbb{Z}^{{}^{d+1}C_4}\to\mathbb{Z}^{d+1}\otimes\mathbb{Z}_2\xrightarrow{j}IE(\mathbb{Z}\oplus\bigoplus_{i=1}^d\lambda_i\mathbb{Z})_{ab}\xrightarrow{I}\mathbb{Z}^{{}^{d+1}C_3}\to 0$$

We need a more concrete description of I, j. We have established already that $\mathsf{D}(IE(\Gamma))\cong\mathsf{A}(\Gamma\ltimes(0_+,1_-])$ (Corollary 6.2.10). Then, [79], Corollary 6.17 identifies $Ker(I)=Im(j)$ with $\mathsf{S}(\Gamma\ltimes(0_+,1_-])$. This can also be verified explicitly, if γ_B is a generator of $\mathsf{S}(\Gamma\ltimes(0_+,1_-])$ then, $j([1_{s(B)}])=\gamma_B$. This demonstrates that $\mathsf{S}(\Gamma\ltimes(0_+,1_-])$ is at most $d+1$ generated.

Note that we have only considered the case where $\Gamma/\mathbb{Z}$ is torsion free and Γ is finitely generated abelian. The reason this is enough is that for any Γ finitely generated abelian with $\Gamma/\mathbb{Z}$ having torsion, there exists some $k\in\mathbb{N}$ such that $k\Gamma/\mathbb{Z}$ is torsion free. Applying Lemma 3.6.5 thus reduces the torsion free to the torsion case.

6.5.3 Acting group the ring generated by an algebraic number

Let $\lambda\in\mathbb{R}$ be algebraic with degree d. Consider the ring generated by λ, i.e. $\Gamma=\mathbb{Z}[\lambda,\lambda^{-1}]$. In this case, we have that $IE(\mathbb{Z}[\lambda,\lambda^{-1}])$ is canonically isomorphic to the measure-preserving subgroup of the Stein-Thompson groups with cyclic slopes V_λ with slope $\langle\lambda\rangle$ and breakpoints

$\mathbb{Z}[\lambda,\lambda^{-1}]$ as studied in [[116], [38], [31]].
Let us describe a concrete generating set for $D(IE(\mathbb{Z}[\lambda,\lambda^{-1}]))$.

Lemma 6.5.4. *Let λ be irrational and $\Gamma = \mathbb{Z}[\lambda,\lambda^{-1}]$. Then there exists a countable, generating set $\{1,\lambda_i\}_{i=1}^{n}$ for Γ such that $4/5 < \lambda_i < \lambda_{i+1} < 1/2$ for all i. Then, let $\sigma_i, \hat{\sigma}_i$ be as in Theorem 6.3.9. $IE(\mathbb{Z}[\lambda,\lambda^{-1}])$ is generated by $S = \{\sigma_i, \hat{\sigma}_i\}_{i\in\mathbb{N}}$.*

Proof. Let $\Gamma_n = \bigoplus_{i=-n}^{n} \lambda^n\mathbb{Z}$. For each n, $\Gamma_n \cong \mathbb{Z}^{2n+1}$ hence by Theorem 6.3.9 and Lemma 6.3.5 we have that for all n there exists $2n+1$ generators of Γ_n where one of them is 1, and the others are a collection of rationally independent irrational numbers $\{\lambda_i\}_{i=1}^{2n}$ in the interval $(4/5,1/2)$. Then, by iteratively applying the argument in the proof of Lemma 6.3.5, we may assume, that for all n there is a generating set of Γ_n given by 1 and a collecton of rationally independent irrational numbers $\{\lambda_i\}_{i=1}^{2n}$ such that $4/5 < \lambda_i < \lambda_{i+1} < 1/2$ for all i. Since $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$, the proof follows. By Theorem 6.3.9 we also have that $D(IE(\Gamma_n)) = \langle \sigma_i, \hat{\sigma}_i\ i = 1,\ldots,n\rangle$. The result then follows from the observation that $D(IE(\Gamma)) = \cup_{n=1}^{\infty} D(IE(\Gamma_n))$. □

Remark 6.5.5. *Note that in certain cases, (for a concrete example if $\lambda = \sqrt[n]{a}$ for $a \in \mathbb{Q}$), we have that $Rank_{\mathbb{Q}}(\Gamma)$ is finite. In this case, Γ is finitely generated abelian, with generating set $\{\lambda^i\}_{i=1}^{Rank_{\mathbb{Q}}(\Gamma)}$. In this case, we may apply Theorem 6.3.11 to find a finite generating set.*

Let us compute homological information for examples of the form $\mathbb{Z}[\lambda,\lambda^{-1}]$. Let the minimal polynomial of λ be given by: $\lambda^d + \sum_{i=0}^{d-1} a_i\lambda^i = 0$ be the minimal polynomial of λ. Consider the map ϕ_λ given by:

$$\phi_\lambda : \mathbb{Z}^d \cong \bigoplus_{i=0}^{d-1} \lambda^i\mathbb{Z} \to \bigoplus_{i=0}^{d-1} \lambda^i\mathbb{Z} \quad t \mapsto \lambda t$$

Moreover, Γ is the direct limit of the sequence:

$$\mathbb{Z}^d \xrightarrow{\phi_\lambda} \mathbb{Z}^d \xrightarrow{\phi_\lambda} \mathbb{Z}^d \xrightarrow{\phi_\lambda} \cdots$$

It follows that $H_*(\Gamma) = \lim_{\phi_\lambda} H_*(\mathbb{Z}^d)$. Then, $H_*(\Gamma) = 0$ whenever $* > d$, and $H_d(\Gamma) = 0$. Therefore, $IE(\Gamma)$ is not rationally acyclic by 3.6.2.

Example 6.5.6 ($d = 2$)**.** *If $d = 2$, one obtains that $H_3(\mathbb{Z}[\lambda,\lambda^{-1}]) = 0$ through the above computation. Hence, Theorem 3.6.7 reduces to a (split) short exact sequence of abelian groups:*

$$0 \to \mathbb{Z}[\lambda,\lambda^{-1}] \otimes \mathbb{Z}_2 \xrightarrow{I} IE(\mathbb{Z}[\lambda,\lambda^{-1}])_{ab} \xrightarrow{j} H_2(\mathbb{Z}[\lambda,\lambda^{-1}])$$

$(\mathbb{Z}[\lambda,\lambda^{-1}] \otimes \mathbb{Z}_2) \oplus H_2(\mathbb{Z}[\lambda,\lambda^{-1}]) \cong IE(\mathbb{Z}[\lambda,\lambda^{-1}])_{ab}$

6.6 Outlook

We finish with the main open question in this area, which is whether the interval exchange groups are always amenable, and the related question of whether IET contains F_2. See for [42], [60].

Question 6.6.1. *Is IET amenable? Equivalently, for all* Γ *finitely generated abelian, is* $IE(\Gamma)$ *amenable?*

Similarly, does IET contain the free group on two generators? Equivalently, for all Γ *finitely generated abelian, does* $IE(\Gamma)$ *contain the free group on two generators?*

Given our analysis, we have reduced this to groups generated by concrete elements of $IE(\Gamma)$. Therefore, it would be interesting to understand these groups more concretely:

Question 6.6.2. *Let* Γ *be a dense countable subgroup of* $\mathbb{R}$ *such that* $\Gamma \cong \mathbb{Z}^{d+1}$, $\Gamma/\mathbb{Z} \cong \mathbb{Z}^d \oplus \mathbb{Z}_k$ *(with* $k > 9$*). Let S be as in Theorem 6.3.11. What are the relations between the elements in S?*

Chapter 7

Stein's groups

7.1 Relationship with interval exchange groups

There is a relationship between interval exchange groups and Stein's groups. In Chapter 6, the author explored groups of so-called interval exchanges on $(0,1]$. We need to slightly adapt the definition to account for the different lengths of intervals as experienced by Stein's groups.

Definition 7.1.1. *Let Γ be a countable subadditive group of $\mathbb{R}$, and $\ell \in \Gamma$. Then, $IE(\Gamma,\ell)$ is the group of right continuous piecewise linear bijections f of $(0,\ell]$ with finitely many angles $\{ft - t \,:\, t \in (0,\ell]\}$ all in Γ.*

Let us explain the relationship between $V(\Gamma,\Lambda,\ell)$ and $IE(\Gamma,\ell)$. One perspective on these groups is that you could consider these groups to be the measure-preserving subgroups of $V(\Gamma,\Lambda,\ell)$ for their actions on $(0,\ell]$. It is notable also, that on a groupoid level, there is an action of Λ on the groupoid model for $IE(\Gamma,\ell)$ which gives us a groupoid model for $V(\Gamma,\Lambda,\ell)$. The algebraic relationship between these V-like groups and their measure-preserving subgroups is formalised through a Zappa-Szép product decomposition.

Lemma 7.1.2. *Let Λ be a multiplicative subgroup of $\mathbb{R}\cap(0,+\infty)$. Let Γ be a $\mathbb{Z}\cdot\Lambda$ submodule and $\ell \in \Gamma$. Then:*

$$V(\Gamma,\Lambda,\ell) = F(\Gamma,\Lambda,\ell) \bowtie IE(\Gamma,\ell)$$

Proof. We show that $\cdot : IE(\Gamma,\ell)\times F(\Gamma,\Lambda,\ell) \to V(\Gamma,\Lambda,\ell) \quad (g,h)\mapsto g\cdot h$ is bijective. Let $f \in V(\Gamma,\Lambda,\ell)$. Then there exists a finite partition of $(0,\ell]$, $[x_0,x_1],(x_1,x_2],\ldots(x_{n-1},x_n]$ such that on each $I_i = (x_i,x_{i+1}]$, $f(t) = \mu_i t + c_i$ where $\mu_i \in \Lambda, c_i \in \Gamma$. Let us construct an element of $\hat{f} \in F(\Gamma,\Lambda,\ell)$ as follows. On I_0 let $\hat{f}(t) = \mu_1 t$ then on $I_i, i>0$, let $\hat{f}(t) = \mu_i t - \mu_i x_i + \sum_{k=0}^{i-1}\mu_k(x_{k+1}-x_k)$, then let the element of $IE(\Gamma,\ell)$ be $\tilde{f}(t) = t + \mu_i x_i - \sum_{k=0}^{i-1}\mu_k(x_{k+1}-x_k) + c_i$ for $t \in I_i$. It is clear that $f = \tilde{f}\hat{f}$ and that this decomposition is unique- one would need an element of $F(\Gamma,\Lambda,\ell)$ with the same derivative on each I_i as f, and $\hat{f}$ is the unique group element with this property. □

Remark 7.1.3 (Juschenko-Monod groups in Cleary's group)**.** *One can see that in Cleary's group V_τ, we have a canonical embedding the interval exchange group $IE(\mathbb{Z}\oplus\tau\mathbb{Z}) \hookrightarrow V_\tau$. The derived subgroup of $IE(\mathbb{Z}\oplus\tau\mathbb{Z})$ is known, due to results by Juschenko-Monod [61] to be a (rare) example of an infinite, finitely generated, amenable simple group as in Corollary 3.7.2.*

This means that Cleary's group exhibits dynamical behaviours not seen in Thompson's group V. Thompson's group V does embed into Cleary's group V_τ, as shown in [31], (this can also be seen as a consequence of Theorem 1.1.1). It would be interesting to understand whether there are groups that embed into V_τ but not V. Throughout this text, we study $V(\Gamma,\Lambda,\ell)$, with the perspective that this group is the topological full group of a certain ample groupoid.

7.2 Construction of Stein's groups as topological full groups

We aim to construct a Stein's groups $V(\Gamma,\Lambda,\ell)$ as topological full groups where Λ is a subgroup of the positive reals, Γ is a $\mathbb{Z}\cdot\Lambda$ submodule and $\ell\in\Gamma$. The construction of Stein's groups as topological full groups mirrors the construction of the interval exchange groups as in the previous chapter. To construct $V(\Gamma,\Lambda,\ell)$ as a topological full group, we make use of the construction of $\mathbb{R}_\Gamma$ as in Definition 6.2.1. Noting that since Λ is nontrivial, Γ is dense in $\mathbb{R}$. Therefore, using Lemma 6.2.2, we have that a basis for the topology on $\mathbb{R}_\Gamma$. As in the previous chapter, these groupoids can be studied from a number of perspectives. Another perspective given by Li in [76], is that they are the groupoids associated to the inverse semigroups $\Gamma\cap(0,+\infty)\rtimes\Lambda$, i.e. the positive cone of $\Gamma\rtimes\Lambda$. Let us explain the relationship between the space $\mathbb{R}_\Gamma$ and the space of characters here.

Remark 7.2.1 (Relationship between $\mathbb{R}_\Gamma, O_{\Gamma^+\subseteq\Gamma}$)**.** *In [76], Section 2.3 Li constructs an analogous space to $\mathbb{R}_\Gamma$ as follows. Let Γ be an additive subgroup of $\mathbb{R}$. Let $D(\Gamma^+)$ be the (abelian) semigroup C*-algebra of $\Gamma\cap(0,\infty)$. This group has the basis of idempotents $\{1_{a+\Gamma^+}\ a\in\Gamma\cap(0,+\infty)\}$ He then considers the Gelfand dual space $\Omega(D(\Gamma^+))$, and removes the trivial character χ_∞ such that for all $a\in\Gamma$ $\chi_\infty(1_{a+\Gamma^+})=1$. This space is denoted $O_{\Gamma^+\subseteq\Gamma}$ Concretely, this is the space of nonzero, nontrivial, characters $\chi: D(\Gamma^+)\to\{0,1\}$ that are strictly decreasing with on the basis (with respect to the partial order $1_{a+\Gamma^+}\leq 1_{b+\Gamma^+}\iff a\leq b$). This space is topologised in the weak operator topology, which is the topology generated by the basic compact open sets $U_{a,b}:=\{\chi\in O_{\Gamma^+\subseteq\Gamma}:\chi(1_{a+\Gamma^+})=1,\chi(1_{b+\Gamma^+})=0\}, a,b\in\Gamma$. There is a canonical homeomorphism:*

$$f:\mathbb{R}_\Gamma\to O_{\Gamma^+\subseteq\Gamma}\quad a_+\mapsto\chi_a^+,\ a_-\mapsto\chi_a^-,\ t\mapsto\chi_t\quad a\in\Gamma, t\in\mathbb{R}_\Gamma\smallsetminus\Gamma_\pm$$

Where for all $a,b\in\Gamma$, $t\in\mathbb{R}_\Gamma\smallsetminus\Gamma_\pm$, $\chi_a^+(1_{b+\Gamma^+})=1\iff b\leq a$, $\chi_a^-(1_{b+\Gamma^+})=1\iff b<a$, $\chi_t(1_{b+\Gamma^+})=1\iff b<t$. For all $a,b\in\Gamma, a<b$ $f[a_+,b_-])=U_{a,b}$.

Recall the map q^* :

$$q^* : (0,\ell] \hookrightarrow (0_+,\ell_-] \quad t \mapsto \begin{cases} t & t \notin \Gamma \\ t_- & t \in \Gamma \setminus \{0\} \end{cases}.$$

Note then, for all $a,b \in \Gamma$ with $a < b$, $q^*((a,b]) \subset (a_+, b_-]$. Let Λ be a multiplicative subgroup of $(0,+\infty)$ and Γ a $\mathbb{Z}\cdot\Lambda$-submodule, let us form the semidirect product in the following way. Let $\Lambda \curvearrowright \Gamma$ by multiplication, and for us to form the semidirect product:

$$(a,\mu)(b,\lambda) = (\lambda^{-1}a + b, \mu\lambda) \quad \forall (a,\mu),(b,\mu) \in \Lambda \ltimes \Gamma$$

Note $(a,\mu)^{-1} = (-\mu a, \mu^{-1})$, $\forall (a,\mu) \in \Lambda \ltimes \Gamma$. Then, using the canonical action of $\Lambda \ltimes \Gamma$ on $\mathbb{R}_\Gamma$, we can define a partial action that realises $V(\Gamma,\Lambda,\ell)$ as a topological full group.

Lemma 7.2.2 ($V(\Gamma,\Lambda,\ell)$ as the topological full group of a partial action)**.** *Let Λ be a countable submultiplicative group of $(0,+\infty)$, and Γ be a $\mathbb{Z}\cdot\Lambda$-submodule. Let $1 < \ell \in \Gamma$. Consider the following canonical action of $\Lambda \ltimes \Gamma$ on $\mathbb{R}_\Gamma$:*

$$\beta : \Lambda \ltimes \Gamma \curvearrowright \mathbb{R}_\Gamma$$

Where for all $a_\pm \in q^{-1}(\Gamma) \cap \mathbb{R}_\Gamma, t \in q^{-1}(\Gamma^c) \cap \mathbb{R}_\Gamma$ and all $(c,\mu) \in \Lambda \ltimes \Gamma$

$$(c,\mu)(a_\pm) = (\mu(a+c))_\pm \quad (c,\mu)(t) = \mu(t+c)$$

Consider the restriction of β to a partial action on $(0_+,\ell_-]$. We have that:

$$\mathsf{F}((\Lambda \ltimes \Gamma) \ltimes (0_+,\ell_-]) \cong V(\Gamma,\Lambda,\ell)$$

.

Proof. First let us note that β does define an action on $\mathbb{R}_\Gamma$. For all $t \in \mathbb{R}_\Gamma \cap q^{-1}(\Gamma^c)$

$$(c,\mu)((b,\nu)(t)) = (c,\mu)(\nu t + b) = \mu((\nu t + b) + c) = \mu\nu(t + b + \nu^{-1}c) = ((c,\mu)(b,\nu))(t)$$

And similarly for $a_\pm \in \mathbb{R}_\Gamma \cap q^{-1}(\Gamma)$. Now let us follow Example 3.1.3 for understanding the ample groupoid. From here, we essentially follow as with the proof of Lemma 6.2.3. A basis of the compact open bisections of $(\Lambda \ltimes \Gamma) \ltimes (0_+,\ell_-]$ is given by

$$((c,\mu),[a_+,b_-]) \quad c \in \Gamma \quad a,b \in \Gamma \quad \max\{-c,0\} \le a < b \le \min\{\mu^{-1} - c, 1\}$$

Hence elements of $\mathsf{F}((\Lambda \ltimes \Gamma) \ltimes (0_+,\ell_-])$ are homeomorphisms f of $(0_+,\ell_-]$ for which there exists a finite subset $\{x_i\}_{i=1}^n \subset \Gamma$ with $0 = x_1 < \ldots < x_n = 1$ and elements $\{(c_i, \mu_i)\}_{i=1}^n$ such that

$$f|_{[(x_i)_+,(x_{i+1})_-]} = \beta(c_i,\mu_i)[(x_i)_+,(x_{i+1})_-] = [(\mu_i x_i + c_i)_+, (\mu_i x_{i+1} + c_i)_-]$$

From here, the proof follows since

$$\varphi : \mathsf{F}((\Lambda \ltimes \Gamma) \ltimes_\alpha (0_+, 1_-]) \to V(\Gamma, \Lambda, \ell) \quad g \mapsto qgq^*$$

provides an explicit isomorphism. For any element $f \in \mathsf{F}(\Gamma \ltimes_\alpha (0_+, 1_-])$, $\varphi(f)$ is a right continuous piecewise linear bijection of $(0,1]$ with slopes μ_i and translations c_i on each interval $(x_i, x_i + 1]$, this is since $q^*(x_{i-1}, x_i] \subset ((x_{i-1})_+, (x_i)_-]$, therefore $gq^*(x_{i-1}, x_i] \subset ((\mu_i x_{i-1} + c_i)_+, (\mu_i x_i + c_i)_-]$ and so $qgq^*((x_{i-1}, x_i]) = (\mu_i x_{i-1} + c_i, \mu_i x_i + c_i]$.

Moreover, if $t \in (x_{i-1}, x_i] \cap \Gamma$, then

$$qgq^*(t) = qg(t_-) = q((\mu_i t + c_i)_-) = \mu_i t + c_i.$$

And, if $t \in (x_{i-1}, x_i] \cap \Gamma^c$, then

$$qgq^*(t) = qg(t_-) = q((\mu_i t + c_i)) = \mu_i t + c_i.$$

We note also then, that there are only finitely many $\mu_i \in \Lambda, c_i \in \Gamma$ describing g, that there are only finitely many such linear segments, and they map all of $(0,\ell]$ onto all of $(0,\ell]$. □

Remark 7.2.3. *There is a canonical embedding of groupoids describing the interval exchange groups and Stein's groups, following from the canonical embedding of the larger groupoids:* $\Gamma \ltimes \mathbb{R}_\Gamma \hookrightarrow (\Lambda \ltimes \Gamma) \ltimes \mathbb{R}_\Gamma$

Let us also remark that $(0_+, \ell_-]$ is always $\Gamma \ltimes \mathbb{R}_\Gamma$ full (in fact, it meets every Γ orbit in $\mathbb{R}_\Gamma$, by density, since for all $x \in \mathbb{R}_\Gamma$, we can choose $a \in \Gamma$ such that $0 < q(x) - a < \ell$). It is time to remark on some basic facts for these partial action groupoids:

Lemma 7.2.4. $\beta : \Lambda \ltimes \Gamma \curvearrowright \mathbb{R}_\Gamma$ *is a topologically free, amenable, minimal action. In other words, as a groupoid,* $\Lambda \ltimes \Gamma \ltimes_\beta (0_+, \ell_-]$ *is topologically principal, amenable and minimal.*

Proof. The action β is amenable since $\Lambda \ltimes \Gamma$ is an amenable group.

Now let us establish minimality. First let us remark the following convergence rules in $\mathbb{R}_\Gamma$:

$$\lim_{n\to\infty} x_n = x_+ \iff \lim_{n\to\infty} q(x_n) = q(x) \text{ from above}$$

$$\lim_{n\to\infty} x_n = x_- \iff lim_{n\to\infty} q(x_n) = q(x) \text{ from below}$$

$$\lim_{n\to\infty} x_n = x \text{ s.t. } x \notin \Gamma \iff \lim_{n\to\infty} q(x_n) = x$$

For all $x \in \mathbb{R}_\Gamma$, the image of the orbit Γx under q, $q(\Gamma x) = q(x) + \Gamma \subset \mathbb{R}$ is both left-dense and right-dense in $\mathbb{R}$, so we can find sequences in $q(x)+\Gamma$ tending to any $x' \in \Gamma$ from above or below, and sequences approximating any $x' \notin \Gamma$ in $q(\Gamma x)$. Minimality follows.

It remains to show that β is topologically free. Let $(c,\mu) \in \Lambda \ltimes \Gamma$, with $(c,\mu) \neq (0,1)$. Suppose that $(c,\mu)x = x$ for $x \in \mathbb{R}_\Gamma$. This occurs $\iff q((c,\mu)x) = q(x) \iff q(x)(1-\mu) = c\mu$. Note then $\mu \neq 1$, otherwise this forces $c = 0$. Otherwise, we have that (c,μ) fixes only $q^{-1}(c\mu/(1-\mu)) = \{c\mu/(1-\mu)_\pm\}$. In particular, (c,μ) does not fix x such that $q(x) \notin \Gamma$, so $\mathbb{R}_\Gamma \setminus \{x_\pm : x \in \Gamma\}$ is a dense set which $\Lambda \ltimes \Gamma$ acts freely on. □

Note in particular as a Corollary of Matui's isomorphism Theorem (Theorem 3.4.12), and the regularity established for β are conjugate to some known groupoids. Let $\mathcal{E}_k$ denote the full shift on k generators, and $\mathcal{R}_r$ be the full equivalence relation on r points. We have the following:

Corollary 7.2.5. *Up to groupoid conjugacy, $\Lambda \ltimes \Gamma \ltimes_\beta (0_+, \ell_-]$ is the unique ample groupoid of germs $\mathcal{G}$ such that $\mathsf{F}(\mathcal{G}) \cong V(\Gamma,\Lambda,\ell)$. In particular, for all $2 < k \in \mathbb{N}$, $r \in \mathbb{N}$, $\mathbb{Z}[1/k] \ltimes \langle k \rangle \ltimes_\beta (0_+, r_-] \cong \mathcal{R}_r \times \mathcal{E}_k$.*

Remark 7.2.6. *Following on from Remark 7.2.1, let us make the point that there are equivalent but nonetheless alternative ways to construct this groupoid (up to groupoid conjugacy). One such description arises from cancellative semigroups. Let Γ^+ denote the positive cone of Γ. Consider the subsemigroup $\Gamma^+ \rtimes \Lambda$ of $\Lambda \ltimes \Gamma$, and the universal groupoid (in the sense of Li) $\mathcal{G}(\Gamma^+ \rtimes \Lambda)$ of this semigroup. This groupoid is the semigroup transformation groupoid of the canonical action of the semigroup on the dual of the semilattice of idempotents, which in this case is just the dual of Γ^+. If we take the full corner of this groupoid restricted to the subset U of characters χ such that $\chi(\ell) = 1$, this describes a conjugate groupoid. See [[76], [40], Chapter 5] for more information on the universal groupoids of semigroups that embed into groups.*

We next verify that the groupoid $(\Lambda \ltimes \Gamma) \ltimes_\beta (0_+, \ell_-]$ is purely infinite, in the sense of Definition 5.1.1.

Lemma 7.2.7. *Let Λ a nontrivial multiplicative subgroup of $\mathbb{R}$, Γ a $\mathbb{Z}\cdot\Lambda$-submodule and $\ell \in \Gamma$. Then $(\Lambda \ltimes \Gamma) \ltimes_\beta (0_+, \ell_-]$ is a purely infinite groupoid.*

Proof. By compactness, any clopen subset can be written as a finite disjoint union of sets of the form $[a_+, b_-]$. Therefore, let $A = \cup_{i=1}^n [(a_i)_+, (b_i)_-]$ where each $a_i, b_i \in \Gamma$. Then let $\lambda \in \Lambda$ with $0 \leq \lambda \leq 1/2$, set

$$U = \bigsqcup_{i=1}^n ((a_i(1-\lambda^{-1}),\lambda), [(a_i)_+, (b_i)_-]),\ V = \bigsqcup_{i=1}^n (b_i(1-\lambda^{-1}),\lambda), [(a_i)_+, (b_i)_-]).$$

It is straightforward to verify that U, V are indeed bisections.

We compute

$$U^{-1} = \bigsqcup_{i=1}^{n} ((-\lambda a_i(1-\lambda^{-1}),\lambda),[(a_i)_+,(a_i+\lambda(b_i-a_i))_-])$$

$$V^{-1} = \bigsqcup_{i=1}^{n} (-\lambda b_i(1-\lambda^{-1}),\lambda),[(b_i-\lambda(a_i-b_i))_+,(b_i)_-])$$

This gives us exactly what we need. $s(U) = s(V) = \bigsqcup_{i=1}^{n}[(a_i)_+,(b_i)_-] = A$, $r(U) = \bigsqcup_{i=1}^{n}[(a_i)_+,(a_i+\lambda(a_i-b_i)_-]$, $r(V) = \bigsqcup_{i=1}^{n}[(b_i-\lambda(a_i-b_i))_+,(b_i)_-]$ and so $r(V)\cap r(U) = \varnothing$ whilst simultaneously $r(U)\cup r(V)\subset A$. □

Remark 7.2.8. *An alternative, (group theoretic) proof that $(\Lambda\ltimes\Gamma)\ltimes_\beta(0_+,\ell_-]$ is a purely infinite, minimal groupoid is to observe that $V(\Gamma,\Lambda,\ell)$ is vigorous for all choices of Γ,Λ,ℓ and apply Theorem 5.1.21 (proven in Chapter 5).*

Hence, by Theorem 3.3.2 we obtain that $D(V(\Gamma,\Lambda,\ell))$ is always simple. We also have via Corollary 5.1.18 that $D(V(\Gamma,\Lambda,\ell))$ has no proper characters.

Corollary 7.2.9 (Simple, Vigorous Derived Subgroup)**.** *Let Γ,Λ,ℓ be arbitrary. Then, $D(V(\Gamma,\Lambda,\ell))$ is simple and vigorous. Moreover, it is identified with the alternating subgroup $A(\Lambda\ltimes\Gamma\ltimes(0_+,1_-])$. Finally, $D(V(\Gamma,\Lambda,\ell))$ has no proper characters.*

7.3 Finite generation of the derived subgroup

7.3.1 Expansivity of partial transformation groupoids

There is also a natural notion of expansivity for partial actions, which generalises the natural notion of expansivity of global actions.

Definition 7.3.1 (Expansive partial action)**.** *Let $\alpha : G \curvearrowright X$ be a partial action. We say that α is expansive if there exists $\varepsilon > 0$ such that for all $x,y\in X, x\neq y$, there exists $g\in G$ such that $d(gx,gy) > \varepsilon$.*

Nekrashevych showed that a transformation groupoid of a finitely generated group acting on the Cantor space is expansive if and only if the underlying action is expansive. We give an analogous result for partial actions.

Lemma 7.3.2. *Let $\alpha : G \curvearrowright X$ be an expansive partial action, of a discrete group on the Cantor space and suppose $G\ltimes_\alpha X$ is compactly generated. Then, the partial transformation groupoid $G\ltimes_\alpha X$ is expansive in the sense of Definition 3.5.1.*

Proof. Let $\varepsilon > 0$ be as in definition 7.3.1. Let K be a compact generating set for the groupoid $G\ltimes_\alpha X$. By ampleness, there exists a finite cover of K by compact open bisections. Moreover, we may refine this cover to a cover $\mathcal{S}$ of K by bisections such that for each $S \in \mathcal{S}$, both $r(S)$ and $s(S)$ have diameter less than ε. We claim that this cover is expansive. It follows that for all $A \in \bigcup_{n=1}^{\infty}(\mathcal{S} \cup \mathcal{S}^{-1})^n$, $r(A)$ and $s(A)$ have diameter less than ε with respect to the metric, since $r(A'A) \subset r(A')$ and $s(AA') \subset s(A')$. Note that also for all $A \in \bigcup_{n=1}^{\infty}(\mathcal{S} \cup \mathcal{S}^{-1})^n$, we can assume $A = (g, U)$ where $U \subset \mathcal{G}^{(0)}$ and $g \in G$.

Recall [[93], 5.3]. We have that $\mathcal{S}$ is expansive if and only if for every $x \neq y$, there exists a subset of the form $s(A)$ such that $x \in s(A)$, $y \notin s(A)$ and $A \in (\mathcal{S} \cup \mathcal{S}^{-1})^n$ for some n. (This is condition (4)).

Let g be such that $d(gx, gy) > \varepsilon$. For n large enough, by compact generation we have that there exists A such that $(g,x) \in A \in (\mathcal{S} \cup \mathcal{S})^n$. By construction then, $(1,x) \in s(A)$. Suppose $(1,y) \in s(A)$. Then $(g,y) \in A$, $(1,gy) \in r(A)$. But A must have diameter with length less than ε by construction. Therefore, we have $d(gx,gy) < \varepsilon$, a contradiction. Therefore $(g,y) \notin A$, and so the underlying groupoid is expansive in the sense of Nekrashevych. □

In contrast to global actions, understanding when a partial transformation groupoid is compactly generated is a subtle question. For transformation groupoids on compact spaces, $G \ltimes_\alpha X$, compact generation of the groupoid is equivalent to the finite generation of G. One does not have an analogous result in the case of partial actions. However, we below show that restricting compactly generated groupoids to certain subsets of the unit space preserves compact generation:

Theorem 7.3.3. *Let $\mathcal{G}$ be an ample compactly generated groupoid. Let U be a full clopen subset of $\mathcal{G}^{(0)}$ (i.e. U meets every $\mathcal{G}$ orbit). Then $\mathcal{G}|_U^U$ is compactly generated.*

Proof. Let $\mathcal{S}_1 = \{B_1, \ldots, B_k\}$ be the compact generating set of $\mathcal{G}$. For $i = 1, \ldots, k$ let:

$$S_i = B_i|_U^U \quad E_i = B_i|_{U^c}^{U^c} \quad T_i = B_i|_{U^c}^U \sqcup B_i^{-1}|_{U^c}^U$$

Then $\mathcal{S}_2 = \{S_i, E_i, T_i\}_{i=1}^k$ is a compact generating set of $\mathcal{G}$, since for all $0 < i \leq k$, $B_i = S_i \sqcup E_i \sqcup T_i|_{U^c} \sqcup T_i^{-1}|_U$.

By fullness, for all $x \in U^c$ there exists a groupoid element $g \in \mathcal{G}$ such that $s(g) = x, r(g) \in U$. By ampleness then, there exists a compact open bisection F_x containing g, such that $x \in s(F_x) \subset U^c$ and $r(F_x) \subset U$. Then $\{s(F_x)\}_{x \in U^c}$ is an open cover of U^c. Since $\mathcal{G}$ is compactly generated, $\mathcal{G}^{(0)}$ must be compact. Then U^c, a closed subset of $\mathcal{G}^{(0)}$ must be compact. By compactness, we can refine our open cover $s(F_x), x \in U^c$ of U^c to a finite cover. Therefore, there exists a finite subset $F_j, j = 1, \ldots, m$ of the F_x such that $\cup_{j=1}^m s(F_j) = U^c, \cup_{j=1}^m r(F_j) \subset U$. Our claim is that the finite collection of compact open bisections given by:

$$\mathcal{S}_U = \{S_i, T_i F_j^{-1}, F_j E_i^{\pm 1} F_l^{-1}\}_{\substack{i=1,\ldots,k \\ j,l=1,\ldots,m}}$$

forms a compact generating set for $\mathcal{G}|_U^U$.

Let $g \in \mathcal{G}|_U^U$. Then there exists a finite word W in $\mathcal{S}_2$ such that $g \in W$. Let us examine the form of a word in S_i, E_i, T_i with source and range in U. We have that after each T_i, we will be followed by a (possibly empty) word of $E_j^{\pm 1}$ followed by a T_l. Therefore, let $2n$ be the number of letters in W of the form $T_i, T_i^{-1} i = 1, \dots, k$. By induction on n, we will show that $g \in \bigcup_{k \in \mathbb{N}} (\mathcal{S}_U \cup \mathcal{S}_U^{-1})^k$.

Base case: If $n = 0$, we have that W is a finite word in $\{S_i, S_i^{-1}\}_{i=1}^k$, and so we are done, since $S_i \in \mathcal{S}_U$ for all i.

Inductive step: assume true for n, then for $n+1$ we have that we may rewrite W in the form:

$$W_0 T_j (E_{k_1}^{\pm 1} E_{k_2}^{\pm 1} \dots E_{k_n}^{\pm 1}) T_i^{-1} W_n$$

Where W_0 is a word in $\{S_i, S_i^{-1}\}_{i=1}^k$, and W_n is a word in $\mathcal{S}_U$ such that there are $2n$ letters in $\{T_i, T_i^{-1}\}_{i=1}^k$. By inductive hypothesis we have that the groupoid element $(W_n)_{s(g)}$ is in some finite word $\tilde{W}_n$ in $\mathcal{S}_U \cup \mathcal{S}_U^{-1}$. We have that $r((T_i^{-1} W_n)_s(g)) \in U^c$, therefore there exists l_n such that $r((T_i^* W_n)_s(g)) \in s(F_{l_n})$. Similarly, for $j = 0, \dots, n-1$ there exists l_j such that $r(E_{k_j}^{\pm 1} E_{k_{j+1}}^{\pm 1} \dots E_{k_n^{\pm 1}}) T_i^{-1} W_n)_{s(g)}) \in s(F_{l_{n-j}})$ Then we have that:

$$g \in W_0 T_j (F_{l_1}^{-1} F_{l_1}) E_{k_1}^{\pm 1} (F_{l_2}^{-1} F_{l_2}) E_{k_2}^{\pm 1} \dots (F_{l_{n-1}}^{-1} F_{l_{n-1}}) E_{k_n}^{\pm 1} ((F_{l_n}^{-1} F_{l_n}) T_i^{-1} \tilde{W}_n$$

Which we may rewrite in the form:

$$W_0 T_j (F_{l_1}^{-1})(F_{l_1} E_{k_1}^{\pm 1} F_{l_2}^{-1}) \dots (F_{l_{n-1}} E_{k_n}^{\pm 1} F_{l_n}^{-1})(T_i F_{l_n}^{-1})^{-1} \tilde{W}_n,$$

a finite word in $\mathcal{S}_U$. □

Combining this with our expansivity result for partial actions we obtain:

Corollary 7.3.4. *Let $\alpha : G \curvearrowright X$ be an expansive action of a finitely generated group on a compact space X. Let U be α-full open subset of X. Then the partial transformation groupoid $G \ltimes U$ is expansive in the sense of Nekrahsevych.*

7.3.2 Finite generation of Stein's groups

In the case of our action, the partial action β is indeed expansive, as shown below.

Lemma 7.3.5. *Let Γ, Λ, ℓ be arbitrary. Then β is expansive in the sense of Definition 7.3.1*

Proof. Let $\lambda \in \Gamma \cap (0,1)$. Let $x, x' \in (0_+, \ell_-]$ be distinct. Let $\varepsilon = d((0_+, (\lambda)_-], [(\lambda)_-, 1_+]) > 0$. We separate into two cases:

- If $q(x) \neq q(x')$, suppose without loss of generality that $q(x) < q(x')$. By density, suppose the difference of $q(x') - q(x) > c > 0$ for some $c \in \Gamma$. Also there exists some $c' \in \Gamma$ such that $\lambda - c < q(x) - c' < \lambda$. Therefore we have that $q(c'x) = q(x) - c' < \lambda$ and $q(c'x') = q(x') - c' > q(x) - c' + c > \lambda - c + c = \lambda$. Therefore, $(c', 1) \in \Gamma \rtimes \Lambda$ is a group element separating x, x'.

- If $q(x) = q(x) \in \Gamma$, then for $c' = q(x) - \lambda$, $(c', 1) \in \Gamma \rtimes \Lambda$ is a group element that will separate the two characters into $(0_+, \lambda_-], [\lambda_+, 1_-]$.

□

Applying Theorem 7.3.3, Lemma 7.3.2 and Theorem 3.5.2 from the previous subsection, it is enough for us to consider compact generation in the case when $\ell = 1$.

Corollary 7.3.6. *Let Λ be a submultiplicative group of $(0, +\infty)$, Γ be a $\mathbb{Z} \cdot \Lambda$-submodule. The following are equivalent:*

1. *The groupoid $\Lambda \ltimes \Gamma \ltimes (0_+, 1_-]$ is compactly generated.*
2. *The groupoid $(\Lambda \ltimes \Gamma) \ltimes (0_+, \ell_-]$ is compactly generated for all $\ell \in \Gamma$.*
3. *$D(V(\Gamma, \Lambda, \ell))$ is finitely generated for all $\ell \in \Gamma$.*
4. *$D(V(\Gamma, \Lambda, \ell))$ is 2-generated for all $\ell \in \Gamma$.*

Proof.

- 1. $\Longrightarrow$ 2. Suppose $\Lambda \ltimes \Gamma \ltimes (0_+, 1_-]$ is compactly generated and let ℓ be arbitrary. Let $\mu \in \Lambda$ be such that $\mu > \ell$. Note that $\Lambda \ltimes \Gamma \ltimes (0_+, 1_-] \cong \Lambda \ltimes \Gamma \ltimes (0_+, \mu_-]$, therefore $\Lambda \ltimes \Gamma \ltimes (0_+, \mu_-]$ is compactly generated. $(0_+, \ell_-]$ is full and open as a subset of $(0_+, \mu_-]$. Applying Theorem 7.3.3, we have that $(\Lambda \ltimes \Gamma) \ltimes (0_+, \ell_-]$ is compactly generated.
- 2. $\Longrightarrow$ 3. follows directly as a combination of Lemma 7.3.2 and Theorem 3.5.2.
- 3. $\Longrightarrow$ 1. Note then in particular, $D(V(\Gamma, \Lambda, 1)$ is finitely generated. Hence by Lemma 3.5.5, $\Lambda \ltimes \Gamma \ltimes (0_+, 1_-]$ is compactly generated.
- 4. $\Longleftrightarrow$ 3. follows from noting that $D(V(\Gamma, \Lambda, \ell)$ is simple and vigorous (Corollary 3.3.2), and applying [11, Theorem 5.15].

□

These groupoids are not *always* compactly generated.

Lemma 7.3.7. *If $\Lambda \ltimes \Gamma$ is not finitely generated (for example, if Λ is of infinite rank) then $\Lambda \ltimes \Gamma \ltimes (0_+, 1_-]$ is not compactly generated and thus $D(V(\Gamma, \Lambda, \ell))$ is not finitely generated.*

Proof. Suppose there is a compact generating set for $\Lambda \ltimes \Gamma \ltimes (0_+, 1_-]$. Then, this compact generating set may be written as a finite collection of our basis elements for the topology $((c_i, \mu_i), [(a_i)_+, (b_i)_-]), i = 1, \ldots, n$ and therefore there is a finite string in (c_i, μ_i) that gives every group element in the set $M = \{(c, 1)\ (0, \mu)\ \mu \in \Lambda \cap (0, 1], c \in (0, 1]\}$, since there are bisections $((c, 1), (0_+, (1-c)_-]), ((0, \mu), (0_+, 1_-])$. But note that M generates $\Lambda \ltimes \Gamma$ as a group. Hence (c_i, μ_i) also forms a finite generating set of the group $\Lambda \ltimes \Gamma$. □

For this reason, the question of compact generation is only relevant when Λ is finitely generated. Let us begin with the simplest case, namely the case when Λ is cyclic. Here we show explicitly that the groupoid is always compactly generated. Set $\Lambda = \langle\lambda\rangle$, $\Gamma = \mathbb{Z}[\lambda,\lambda^{-1}]$, where $\lambda < 1$. Let $\mathcal{G}_\lambda = \langle\lambda\rangle \ltimes \mathbb{Z}[\lambda,\lambda^{-1}] \ltimes (0_+, 1_-]$. Let $K \in \mathbb{N}$ be large enough that $\lambda + \lambda^K < 1$. Let $\mu_1 = \lambda^{-1} - \lfloor\lambda^{-K-1}\rfloor\lambda^K$:

$$c = ((0,\lambda),(0_+,1_-])$$

$$f_i = ((\lambda^i,1),(0_+,(1-\lambda^i)_-]) \sqcup ((\lambda^i-1,1),[(1-\lambda^i)_+,1_-])$$

$$g_1 = ((\mu_1,1),(0_+,(1-\mu_1)_-]) \sqcup ((\mu_1-1,1),[(1-\mu_1)_+,1_-])$$

Let $\mathcal{S} = \{c, f_1, \ldots, f_K, g_1\}$. Note that $\mathcal{S}$ is compact. Let $\mathcal{G}_\lambda^{\mathcal{S}}$ be the subgroupoid of $\mathcal{G}_\lambda$ generated algebraically by $\mathcal{S}$. We aim to show that $\mathcal{G}_\lambda = \mathcal{G}_\lambda^{\mathcal{S}}$. Our first claim is that for all $i \in \mathbb{N}$ we have the bisection f_i which adds $\lambda^i (\mathrm{mod}\mathbb{Z})$.

Lemma 7.3.8. *Let $\lambda < 1$ be arbitrary, let $\mathcal{S} = \{c, f_1, \ldots, f_K, g_1\}$, and let $\mathcal{G}_\lambda^{\mathcal{S}}$ be the subgroupoid of $\mathcal{G}_\lambda = \langle\lambda\rangle \ltimes \mathbb{Z}[\lambda,\lambda^{-1}] \ltimes (0_+,1_-]$ generated by $\mathcal{S}$. Then, for all $i \in \mathbb{N}$, the bisection*

$$f_i = ((\lambda^i,1),(0_+,(1-\lambda^i)_-]) \sqcup ((\lambda^i-1,1),[(1-\lambda^i)_+,1_-]) \in \mathcal{G}_\lambda^{\mathcal{S}}.$$

Proof. We prove this by induction. We know it is true for $i = 1, \ldots, K$. Assume true for $i \geq K$, for $i+1$ we have that:

$$f_{i+1}|_{(0_+,(\lambda-\lambda^{i+1})_-]} = c f_i c^{-1}|_{(0_+,(\lambda-\lambda^{i+1})_-]}.$$

Let us verify this on each domains:

$$(0_+,(\lambda-\lambda^{i+1})_-] \xrightarrow{c^{-1}} (0_+,(1-\lambda^i)_-] \xrightarrow{f_i} [\lambda^i_+, 1_-] \xrightarrow{c} [\lambda^{i+1}_+, \lambda_-].$$

Similarly,

$$f_{i+1}|_{[(\lambda-\lambda^{i+1})_+,\lambda_-]} = f_1 c f_i c^{-1}|_{[(\lambda-\lambda^{i+1})_+,\lambda_-]}.$$

Let us verify this on each domains:

$$[(\lambda-\lambda^{i+1})_+,\lambda_-] \xrightarrow{c^{-1}} [(1-\lambda^i)_+,1_-] \xrightarrow{f_i} (0_+,\lambda^i_-] \xrightarrow{c} (0_+,\lambda^{i+1}_-] \xrightarrow{f_1} [\lambda_+,(\lambda+\lambda^{i+1})_-].$$

Using that $\lambda^{i+1} < \lambda^i < \lambda^K < 1-\lambda$. Taking the union of these two bisections, we have $f_{i+1}|_{(0_+,\lambda_-]}$. In order to obtain $f_{i+1}|_{(0_+,1_-]}$, note that for all $n = 0, \ldots, \lfloor\lambda^{-1}\rfloor - 1$:

$$f_{i+1}|_{[(n\lambda)_+,((n+1)\lambda)_-]} = f_1^n f_{i+1}|_{(0_+,\lambda_-]} f_1^{-n}.$$

Finally, we have that for $n = \lfloor \lambda^{-1} \rfloor$,

$$f_{i+1}|_{[(n\lambda)_+,1_-]} = f_1^n f_{i+1}|_{(0_+,\lambda_-]} f_1^{-n}|_{[(n\lambda)_+,1_-]}.$$

This covers f_{i+1} on $(0_+, 1_-]$, completing our inductive step. $\square$

Our second claim is that for all $i \in \mathbb{N}$ we have the bisection f_{-i} which adds $\lambda^{-i}(\text{mod}\mathbb{Z})$.

Lemma 7.3.9. *Let $\lambda < 1$ be arbitrary, let $\mathcal{S} = \{c, f_1, \ldots, f_K, g_1\}$, and let $\mathcal{G}_\lambda^{\mathcal{S}}$ be the subgroupoid of $\mathcal{G}_\lambda = \langle\lambda\rangle \ltimes \mathbb{Z}[\lambda,\lambda^{-1}] \ltimes (0_+,1_-]$ generated by $\mathcal{S}$. Then, for all $i \in \mathbb{N}$ we have that the bisection $f_{-i} = ((\lambda^{-i} - \lfloor\lambda^{-i}\rfloor, 1), (0_+, (1-(\lambda^{-i} - \lfloor\lambda^{-i}\rfloor))_-]) \sqcup ((\lambda^{-i} - \lfloor\lambda^{-i}\rfloor - 1, 1), [(1-(\lambda^{-i} - \lfloor\lambda^{-i}\rfloor))_+, 1_-]) \in \mathcal{G}_\lambda^{\mathcal{S}}$.*

Proof. For all $i \in \mathbb{N}$ let (as before) K be large enough that $\lambda + \lambda^K < 1$, and set $\mu_{i+1} = \lambda^{-1}\mu_i - \lfloor\lambda^{-1}\mu_i\lambda^{-K}\rfloor\lambda^K$. Our first aim is to show that for all i, we have that the bisection $g_i = ((\mu_i, 1), (0_+, (1-\mu_i)_-]) \sqcup ((\mu_i - 1, 1), [(1-\mu_i)_+, 1_-]) \in \mathcal{G}_\lambda^{\mathcal{S}}$. Let us prove the statement by induction. We have that the statement is true by assumption for $i = 1$, ($g_1 \in \mathcal{S}$) so let us proceed with the inductive step, assuming true for i and aiming to show it is true for $i+1$. Let $m_i = \lfloor\lambda^{-1}\mu_i\lambda^{-K}\lfloor$. We claim that:

$$g_{i+1}|_{(0_+,\lambda_-]} = f_K^{-m_i} c^{-1} g_i c|_{(0_+,\lambda_-]}.$$

Let us verify on our domains, note $\mu_i < \lambda^K$. Therefore,

$$\lambda^2 + \mu_i < \lambda^2 + \lambda^K = \lambda(\lambda + \lambda^K) < \lambda.$$

Therefore, $[(\mu_i)_+, (\lambda^2 + \mu_i)_-] \subset (0_+, \lambda_-]$. Using this we can verify our domains:

$$(0_+,\lambda_-] \xrightarrow{c} (0_+,\lambda_-^2] \xrightarrow{g_i} [(\mu_i)_+,(\lambda^2+\mu_i)_-] \xrightarrow{c^{-1}} [(\lambda^{-1}\mu_i)_+,(\lambda+\lambda^{-1}\mu_i)_-]$$
$$\xrightarrow{f_K^{-m_i}} [(\mu_{i+1})_+,(\lambda+\mu_{i+1})_-].$$

Note that for all $n = 0, \ldots, \lfloor\lambda^{-1}\rfloor - 1$:

$$g_{i+1}|_{[(n\lambda)_+,((n+1)\lambda)_-]} = f_1^n g_{i+1}|_{(0_+,\lambda_-]} f_1^{-n}.$$

Finally, we have that for $n = \lfloor\lambda^{-1}\rfloor$

$$g_{i+1}|_{[(n\lambda)_+,1_-]} = f_1^n g_{i+1}|_{(0_+,\lambda_-]} f_1^{-n}|_{[(n\lambda)_+,1_-]}.$$

This covers g_{i+1} on $(0_+, 1_-]$.

Let us finish by observing that for all i we can recover f_{-i} as a finite word in $g_1, \ldots., g_i$ and $f_j, j \in \mathbb{N}$ which are themselves elements of $\mathcal{G}_\lambda^{\mathcal{S}}$ by the above argument and Lemma 7.3.8. This is because μ_i are polynomials in λ of the form $\mu_i = \lambda^{-i} + \sum_{k=1}^N a_k\lambda^{-i+k}$ where $N \in \mathbb{N}$ and $a_k \in \mathbb{Z}$. $\square$

Using the above Lemmas, we have all additive germs in $\mathcal{G}_\lambda$ are also in $\mathcal{G}_\lambda^{\mathcal{S}}$. We use this to conclude that $\mathcal{G}_\lambda = \mathcal{G}_\lambda^{\mathcal{S}}$.

Lemma 7.3.10. *Let $\lambda < 1$ be arbitrary, let $\mathcal{S} = \{c, f_1, ..., f_K, g_1\}$, then, $\mathcal{G}_\lambda = \langle\lambda\rangle \ltimes \mathbb{Z}[\lambda,\lambda^{-1}] \ltimes (0_+, 1_-]$ is generated by $\mathcal{S}$; $\mathcal{G}_\lambda$ is compactly generated.*

Proof. Lemma 7.3.8 and 7.3.9 say that the (full) bisections:

$$f_i, f_{-i} \in \mathcal{G}_\lambda^{\mathcal{S}},$$

for all $i \in \mathbb{N}$, where $\mathcal{G}_\lambda^{\mathcal{S}}$ is the subgroupoid of $\mathcal{G}_\lambda$ generated by $\mathcal{S}$. It is clear that f_i, f_{-i} span a subgroup of $V(\mathbb{Z}[\lambda,\lambda^{-1}], \langle\lambda\rangle, 1)$ isomorphic to $\mathbb{Z}[\lambda,\lambda^{-1}]/\mathbb{Z}$. Therefore, $\mathcal{G}_\lambda^{\mathcal{S}}$ has all germs of the form:

$$((\alpha,1),x) \in \mathcal{G}_\lambda^{\mathcal{S}}.$$

Let $g = ((\hat{\alpha},\lambda^{-n}),x) \in \mathcal{G}_\lambda$ be arbitrary, with $n > 0$. Then $0 < q(r(g)) = \lambda^{-n}(q(x)+\hat{\alpha}) < 1$, so in particular, $0 < q(x)+\hat{\alpha} < \lambda^n$. Then, $((\hat{\alpha},1),x) \in \mathcal{G}_\lambda^{\mathcal{S}}$. Then $q(r((\hat{\alpha},1),x) < \lambda^n \Rightarrow r((\hat{\alpha},1),x) \in s(c^{-n})$. Therefore $g = c^{-n}((\hat{\alpha},1),x) \in \mathcal{G}_\lambda^{\mathcal{S}}$. □

It remains to understand the case when Λ is finitely but not singly generated. In fact, we can reduce the case when Λ is finitely generated abelian to the case when Λ is cyclic by using the following Lemma.

Let us first explain the inclusion of our groupoids in one another. Let $\mathcal{G} = \langle\lambda_1, ..., \lambda_n\rangle \ltimes \mathbb{Z}[\lambda_1^{\pm 1}, \lambda_n^{\pm 1}] \ltimes (0_+, 1_-]$ be a groupoid where Λ is finitely generated abelian. Let $N = \{\lambda_{k_1}, .., \lambda_{k_K}\}$ be a finite collection of λ_i. Let B be a compact open bisection in $\mathcal{G}_N = \langle\lambda_{k_1}, ..., \lambda_{k_K}\rangle \ltimes \mathbb{Z}[\lambda_{k_1}^{\pm 1}, \lambda_{k_K}^{\pm 1}] \ltimes (0_+, 1_-]$. We can consider B as a compact open bisection in $\mathcal{G}$ via the canonical inclusion. Moreover, the canonical inclusion map is an inclusion on the level of the inverse semigroup of compact open bisections, so that if K was a compact generating set for $\mathcal{G}_N$ then K is a compact set in $\mathcal{G}$ which generates a subgroupoid containing all compact bisections of the form $((c,\mu),[a_+,b_-])$ where $(c,\mu) \in \langle\lambda_{k_1}, ..., \lambda_{k_K}\rangle \ltimes \mathbb{Z}[\lambda_{k_1}^{\pm 1}, \lambda_{k_K}^{\pm 1}]$ and $a, b \in \mathbb{Z}[\lambda_1^{\pm 1}, \lambda_n^{\pm 1}]$ are arbitrary.

Lemma 7.3.11. *Let $\lambda_1, ..., \lambda_N \in (0,1]$ be a collection of real numbers. Then, the groupoid $\langle\lambda_1, ..., \lambda_N\rangle \ltimes \mathbb{Z}[\lambda_1^{\pm 1}, ..., \lambda_N^{\pm 1}] \ltimes (0_+, 1_-]$ is compactly generated, and therefore the derived subgroup $D(V(\langle\lambda_1, ..., \lambda_N\rangle, \mathbb{Z}[\lambda_1^{\pm 1}, ..., \lambda_N^{\pm 1}], 1))$ is finitely generated.*

Proof. Let us prove this by induction. The base case $N = 1$ is covered by Lemma 7.3.10 and Corollary 7.3.6. Assume true for N, let us show for $N+1$.

By inductive hypothesis there is a compact generating set $\mathcal{S}_N$ for the groupoid $\langle\lambda_1, ..., \lambda_N\rangle \ltimes \mathbb{Z}[\lambda_1^{\pm 1}, ..., \lambda_N^{\pm 1}] \ltimes (0_+, 1_-]$ By Lemma 7.3.10 and Corollary 7.3.6 there is a compact generating set $\mathcal{S}_{N+1}$ which generates the groupoid $\langle\lambda_{N+1}\rangle \ltimes \mathbb{Z}[\lambda_{N+1}, \lambda_{N+1}^{-1}] \ltimes (0_+, 1_-]$. Let $\mathcal{S} = \mathcal{S}_N \sqcup \mathcal{S}_{N+1}$, considering the bisections in the subgroupoids as bisections in the enveloping groupoid. Let

$\mathcal{G}^{\mathcal{S}} \subset \mathcal{G} := \langle \lambda_1, ..., \lambda_{N+1} \rangle \ltimes \mathbb{Z}[\lambda_1^{\pm 1}, ..., \lambda_{N+1}^{\pm 1}] \ltimes (0_+, 1_-]$ be the subgroupoid generated by $\mathcal{S}$. By construction then, we have that

$$(c, \mu) \in \mathbb{Z}[\lambda_{N+1}, \lambda_{N+1}^{-1}], \ltimes \langle \lambda_{N+1} \rangle, a, b \in \mathcal{Z}[\lambda_1^{\pm 1} ..., \lambda_{N+1}^{\pm 1}] \Rightarrow ((c, \mu), [a_+, b_-]) \in \mathcal{G}^{\mathcal{S}} \tag{7.1}$$

and,

$$(c, \mu) \in \mathbb{Z}[\lambda_1^{\pm 1}, ..., \lambda_N^{\pm 1}] \ltimes \langle \lambda_1, ..., \lambda_N \rangle, a, b \in \mathcal{Z}[\lambda_1^{\pm 1} ..., \lambda_{N+1}^{\pm 1}] \Rightarrow ((c, \mu), [a_+, b_-]) \in \mathcal{G}^{\mathcal{S}}. \tag{7.2}$$

Let us proceed to prove two claims, in analogy to Lemma 7.3.8 and Lemma 7.3.9.

1. Our first claim is that for all (suitable) additive germs in $\mathcal{G}^{\mathcal{S}}$ we may multiply them by positive powers of λ_N, we proceed in analogy to Lemma 7.3.8. More precisely, let us show that for all $\alpha \in (0,1) \cap \mathbb{Z}[\lambda_1^{\pm 1}, ..., \lambda_{N+1}^{\pm 1}]$ such that the bisection $f_\alpha = ((\alpha, 1), (0_+, (1-\alpha)_+]) \sqcup ((\alpha - 1, 1), [(1-\alpha)_-, 1_+] \in \mathcal{G}_\lambda^{\mathcal{S}}$, we have that $f_{\lambda_{N+1}\alpha} = ((\lambda_{N+1}\alpha, 1), (0_+, (1-\lambda_{N+1}\alpha)_-] \sqcup ((\lambda_{N+1}\alpha - 1, 1), [(1-\lambda_{N+1}\alpha)_-, 1_+] \in \mathcal{G}_\lambda^{\mathcal{S}}$.

   Let $c_{N+1} = ((0, \lambda_{N+1}), (0_+, 1_-]) \in \mathcal{G}^{\mathcal{S}}$ by (7.1). We have that $f_{\lambda_{N+1}\alpha}|_{(0_+, (\lambda_{N+1}(1-\alpha))_-]} = c_{N+1} f_\alpha c_{N+1}^{-1}|_{(0_+, (\lambda_{N+1}(1-\alpha))_-]} \in \mathcal{G}^{\mathcal{S}}$. Note $f_{\lambda_{N+1}} = ((\lambda_{N+1}, 1), (0_+, (1-\lambda_{N+1})_-] \sqcup ((\lambda_{N+1} - 1, 1), [(1-\lambda_{N+1})_-, 1_+] \in \mathcal{G}^{\mathcal{S}}$ by (7.1). We have that:

   $$f_{\lambda_N \alpha}|_{[(\lambda_{N+1}(1-\alpha))_+, (\lambda_{N+1})_-]} = f_{\lambda_{N+1}}^{-1} c_{N+1} f_\alpha c_{N+1}^{-1}|_{[(\lambda_{N+1}(1-\alpha))_+, (\lambda_{N+1})_-]}$$

   Therefore, $f_{\lambda_{N+1}}|_{(0_+, (\lambda_{N+1})_-]} = f_{\lambda_{N+1}\alpha}|_{(0_+, (\lambda_{N+1}(1-\alpha))_-]} \sqcup f_{\lambda_N \alpha}|_{[(\lambda_{N+1}(1-\alpha))_+, (\lambda_{N+1})_-]} \in \mathcal{G}^{\mathcal{S}}$. We have that for all $n \in 0, 1, ..., \lfloor \lambda_{N+1}^{-1} \rfloor - 1$

   $$f_{\lambda_{N+1}\alpha}|_{[(n\lambda_{N+1})_+, ((n+1)\lambda_{N+1})_-]} = f_{\lambda_{N+1}}^n f_{\lambda_N \alpha}|_{(0_+, (\lambda_N)_-]} f_{\lambda_{N+1}}^{-n} \in \mathcal{G}^{\mathcal{S}}$$

   And for $n = \lfloor \lambda_{N+1}^{-1} \rfloor$,

   $$f_{\lambda_N \alpha}|_{[(n\lambda_{N+1})_+, 1_-]} = f_{\lambda_{N+1}}^n f_{\lambda_N \alpha}|_{(0_+, (\lambda_N)_-]} f_{\lambda_{N+1}}^{-n}|_{[(n\lambda_{N+1})_+, 1_-]} \in \mathcal{G}^{\mathcal{S}}.$$

2. Our second claim is that for all (suitable) additive germs in $\mathcal{G}^{\mathcal{S}}$ we may multiply them by negative powers of λ_N, proceeding in analogy to Lemma 7.3.9. More precisely, let us show that for all $\alpha \in (0, \lambda_{N+1}) \cap \mathbb{Z}[\lambda_1^{\pm 1}, ..., \lambda_{N+1}^{\pm 1}]$ such that the bisection

   $$f_\alpha = ((\alpha, 1), (0_+, (1-\alpha)_+]) \sqcup ((\alpha - 1, 1), [(1-\alpha)_-, 1_+] \in \mathcal{G}^{\mathcal{S}},$$

   we have that $f_{\lambda_{N+1}^{-1}\alpha} = ((\lambda_{N+1}^{-1}\alpha, 1), (0_+, (1-\lambda_{N+1}^{-1}\alpha)_-] \sqcup ((\lambda_{N+1}^{-1}\alpha - 1, 1), [(1-\lambda_{N+1}^{-1}\alpha)_-, 1_+]) \in \mathcal{G}^{\mathcal{S}}$.

Let $c_{N+1} = ((0, \lambda_{N+1}), (0_+, 1_-]) \in \mathcal{G}^{\mathcal{S}}$ by (7.1). We have that:

$$f_{\lambda_{N+1}^{-1}\alpha}|_{(0_+,(1-\lambda_{N+1}^{-1}\alpha)_-]} = c_{N+1}^{-1} f_\alpha c_{N+1}|_{(0_+,(1-\lambda_{N+1}^{-1}\alpha)_-]}$$

Let K be large enough that $\lambda_{N+1}^K < 1 - \lambda_{N+1}^{-1}\alpha$. We have that

$$f_{\lambda_{N+1}^K} = ((\lambda_{N+1}^K, 1), (0_+, (1-\lambda_{N+1}^K)_-] \sqcup ((\lambda_{N+1}^K - 1, 1), [(1-\lambda_{N+1}^K)_-, 1_+] \in \mathcal{G}^{\mathcal{S}}$$

by (7.1). Then, conjugating $f_{\Lambda_{N+1}^{-1}\alpha}|_{(0_+,(1-\lambda_{N+1}^{-1}\alpha)_-]}$ by $(f_{\lambda_{N+1}^K})^n$, as before, will cover $f_{\lambda_{N+1}^{-1}\alpha}$ on $(0_+, 1_-]$.

Using the above claims we next establish that for all $m_1, ..., m_{N+1} \in \mathbb{Z}$, for $\mu = \prod_{i=1}^{N+1} \lambda_i^{m_i}$ we have that the bisection:

$$f_{\mu-\lfloor\mu\rfloor} = ((\mu - \lfloor\mu\rfloor, 1), (0_+, (1-\mu+\lfloor\mu\rfloor)_-] \sqcup ((\mu-\lfloor\mu\rfloor - 1, 1), [(1-\mu+\lfloor\mu\rfloor)_-, 1_+] \in \mathcal{G}^{\mathcal{S}}$$

Case where $|m_{N+1}| = 0$ follows by (7.2), since $\mu - \lfloor\mu\rfloor \in \mathbb{Z}[\lambda_1^{\pm 1}, ..., \lambda_N^{\pm 1}]$. Let us prove the inductive step separately for the cases $m_{N+1} > 0$ and $m_{N+1} < 0$

- Let us do the case when $m_{N+1} > 0$ by induction. Assuming we have for $\nu = \lambda_{N+1}^{-1}\mu$, $f_{\nu-\lfloor\nu\rfloor} \in \mathcal{G}^{\mathcal{S}}$.

  Then use claim 1) from above, we have that $f_{\lambda_{N+1}\mu - \lfloor\lambda_{N+1}^{-1}\mu\rfloor\lambda_{N+1}} \in \mathcal{G}^{\mathcal{S}}$. But we have already established that $f_{\lambda_{N+1}} \in \mathcal{G}^{\mathcal{S}}$, hence

  $f_{\mu-\lfloor\mu\rfloor} = f_{\lambda_{N+1}}^{\lfloor\lambda_{N+1}^{-1}\mu\rfloor} f_{\mu-\lfloor\lambda_{N+1}^{-1}\mu\rfloor\lambda_{N+1}} \in \mathcal{G}^{\mathcal{S}}$

- The case when $m_{N+1} < 0$ is completely analogous, assume that we have $f_{\lambda_{N+1}\mu-\lfloor\lambda^{N+1}\mu\rfloor} \in \mathcal{G}^{\mathcal{S}}$ we have that $\lambda_{N+1}\mu - \lfloor\lambda^{N+1}\mu\rfloor \in [n\lambda_{N+1}, (n+1)\lambda_{N+1}]$, for some n. Hence,

  $$f_{\lambda_{N+1}\mu-\lfloor\lambda^{N+1}\mu\rfloor-n\lambda_{N+1}} = f_{\lambda_{N+1}}^{-n} f_{\lambda_{N+1}\mu-\lfloor\lambda_{N+1}\mu\rfloor} \in \mathcal{G}^{\mathcal{S}}$$

  Now using claim 2) we have that

  $$f_{\mu-\lambda_{N+1}^{-1}\lfloor\lambda_{N+1}\mu\rfloor-n} \in \mathcal{G}^{\mathcal{S}}$$

  But note that $f_{\lambda_{N+1}^{-1}-\lfloor\lambda_{N+1}\rfloor} \in \mathcal{G}^{\mathcal{S}}$, it is generated by elements in $\mathcal{S}_{N+1}$. Hence, we have that:

  $$f_{\mu-\lfloor\mu\rfloor} = f_{\lambda_{N+1}^{-1}-\lfloor\lambda_{N+1}\rfloor}^{\lfloor\lambda_{N+1}\mu\rfloor} f_{\mu-\lambda_{N+1}^{-1}\lfloor\lambda_{N+1}\mu\rfloor-n} \in \mathcal{G}^{\mathcal{S}}$$

Since for all μ as above, we have that $f_{\mu-\lfloor\mu\rfloor} \in \mathcal{G}^{\mathcal{S}}$. These elements span a subgroup of the topological full group isomorphic to $\mathbb{Z}[\lambda_1^{\pm 1}, ..., \lambda_{N+1}^{\pm 1}]/\mathbb{Z}$

Let $\mu \in \langle\lambda_1,\ldots,\lambda_{N+1}\rangle$. Then we may rewrite μ as $\mu_N\lambda_{N+1}^k$ where $k\in\mathbb{Z}$ and $\mu_N\in\langle\lambda_1,\ldots,\lambda_N\rangle$. Then, $c_{\mu_N} = ((0,\mu_N),(0_+,1_-]\cap(0_+,\mu_N^{-1}])\in\mathcal{G}^{\mathcal{S}}$ by (7.2) and $c_{N+1} = ((0,\lambda_{N+1}),(0_+,1_-])\in\mathcal{G}^{\mathcal{S}}$ by (7.1). If $\lambda^k > \mu_N$, then $c_\mu = ((0,\mu),(0_+,1_-]\cap(0_+,\mu_-^{-1}]) = c_{N+1}^k c_{\mu_N}\in\mathcal{G}^{\mathcal{S}}$. Otherwise $c_\mu = c_{\mu_N}c_{N+1}^k\in\mathcal{G}^{\mathcal{S}}$.

Therefore to obtain any line segment corresponding to any $(a,\mu)\in\mathbb{Z}[\lambda_1,\ldots,\lambda_{N+1}]\ltimes\langle\lambda_1,\ldots,\lambda_{N+1}\rangle$, where $\mu<1$ one may write in the form $f_{\mu^{-1}a}c_\mu|_{D(a,\mu)}$ where $D(a,\mu)$ is the maximal domain. However, up to taking inverses, this is all possible slopes, hence, $\mathcal{G}^{\mathcal{S}} = \mathcal{G} = \langle\lambda_1,\ldots,\lambda_{N+1}\rangle\ltimes\mathbb{Z}[\lambda_1^{\pm1},\ldots,\lambda_{N+1}^{\pm1}]\ltimes(0_+,1_-]$, completing our inductive step and our proof. □

We summarise our discussion in the below theorem, by combining the above Lemma 7.3.11 and Corollary 7.3.6.

Theorem 7.3.12. *Let Λ be a subgroup of $(\mathbb{R}_+,\cdot)$ and Γ be a $\mathbb{Z}\cdot\Lambda$ submodule. The following are equivalent:*

- $\Lambda\ltimes\Gamma$ *is finitely generated.*
- $(\Lambda\ltimes\Gamma)\ltimes(0_+,\ell_-]$ *is compactly generated for all* $\ell\in\Gamma$.
- $D(V(\Gamma,\Lambda,\ell))$ *is finitely generated for all* $\ell\in\Gamma$.
- $D(V(\Gamma,\Lambda,\ell))$ *is 2 generated for all* $\ell\in\Gamma$.

Note that our discussion does not investigate the higher finiteness properties of Stein's groups, but these have been partially studied. Notably, Stein showed the derived subgroup is type F_∞ in the case when Λ is generated by finitely many integers, $\Gamma=\mathbb{Z}\cdot\Lambda$, and $\ell\in\mathbb{N}$ [116]. We therefore ask:

Question 7.3.13. *Suppose Λ is a finitely generated subgroup of $(\mathbb{R}_+,\cdot)$ let $\Gamma=\Lambda\cdot\mathbb{Z}$ $\ell\in\Gamma\cap\mathbb{R}_+$. Under what conditions is $D(V(\Gamma,\Lambda,\ell))$ finitely presented?*

7.4 Homology of Stein's groups

7.4.1 Initial observations

Let us again exploit the results of Section 3.6 to understand the homology of Stein's groups. Recall as in Example 2.4.2 that we already understand the homology of partial transformation groupoids. Therefore the homology of the groupoids $\Lambda\ltimes\Gamma\ltimes_\beta(0_+,\ell_-]$ is related to the homology of the groups $\Lambda\ltimes\Gamma$:

$$H_*(\Lambda\ltimes\Gamma\ltimes_\beta(0_+,\ell_-]) = H_*(\Lambda_\lambda\ltimes, C_c(\mathbb{R}_\Gamma,\mathbb{Z}))$$

Let us try to compute these homology groups. We begin with our analogy of Lemma 5.5. [76], wherein we compute $H_0(\Lambda\ltimes\Gamma\ltimes_\beta(0_+,\ell_-])$.

Lemma 7.4.1. *Let* $\Lambda = \langle \lambda_1, \ldots, \lambda_k \rangle$, $\Gamma = \mathbb{Z}[\lambda_1^{\pm 1}, \ldots., \lambda_k^{\pm 1}]$ *be the canonical groups and submodules generated by finitely many algebraic numbers, assuming without loss of generality that the finite collection of algebraic numbers* $\{\lambda_i\}_{k=1}^{N}$ *are pairwise algebraically independent. Then,*

$$H_0(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z})) \cong \Gamma / N_\Lambda$$

Where N_Λ *is the normal subgroup given by* $N_\Lambda = \sum_i (1 - \lambda_i)\Gamma$. *Considering* H_0 *as an ordered group, the order unit is given by the equivalence class of* ℓ, $[\ell] \in \Gamma / N_\Lambda$.

Moreover, $H_1(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z}))$ *is finitely generated.*

Proof. Let $\hat{\mathbb{R}}_\Gamma = \mathbb{R}_\Gamma \cup \{\infty\}$, and let us extend β to an action of $\Lambda \ltimes \Gamma \curvearrowright \hat{\mathbb{R}}_\Gamma$ by setting ∞ to be a fixed point. Let us topologise $\hat{\mathbb{R}}_\Gamma$ with the order topology, where we say that $\infty > a$, $a \in \mathbb{R}_\Gamma$. Then, a $\mathbb{Z}$-basis of $C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})$ which Γ will always act on freely and transitively is given by

$$\chi_{(a_+, \infty]}(t) = \begin{cases} 1 & a_+ < t \leq \infty \\ 0 & \text{else} \end{cases} \qquad a \in \Gamma.$$

For this reason, $C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z}) \cong \mathbb{Z}\Gamma$. Consequently,

$$H_q(\Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & \text{else} \end{cases}$$

Now note by construction of $\Lambda \ltimes \Gamma$ we have the split short exact sequence:

$$1 \to \Gamma \to \Lambda \ltimes \Gamma \to \mathbb{Z}^n \to 1$$

It follows by the Hochschild-Serre spectral sequence (Chapter VII, Theorem 6.3) that

$$H_i(\Lambda \ltimes \Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})) \cong H_i(\Lambda) = H_i(\mathbb{Z}^n) = \mathbb{Z}^{{}^iC_n}$$

The final equality follows by the Kunneth formula, and this is where n is the number of generators of Λ. Now let us apply the Hochschild-Serre spectral sequence again, this time choosing as the module $M = \mathbb{Z} = M_\Gamma$. Then all coefficients are in $\mathbb{Z}$. We obtain the following exact sequence:

$$H_2(\Lambda \ltimes \Gamma) \to H_2(\mathbb{Z}^n) \to H_1(\Gamma)_{\mathbb{Z}^n} \to H_1(\Lambda \ltimes \Gamma) \to H_1(\mathbb{Z}^n) \to 0$$

But since the SES of groups with $\Lambda \ltimes \Gamma$ in the middle splits, the map from $H_2(\Lambda \ltimes \Gamma) \to H_2(\mathbb{Z}^n)$ is a surjection. Therefore, the next map must be the zero map, revealing the short exact sequence:

$$0 \to H_1(\Gamma)_{\mathbb{Z}^n} = \Gamma \to H_1(\Lambda \ltimes \Gamma) \to \mathbb{Z}^n \to 0.$$

Therefore, $H_1(\Lambda \ltimes \Gamma) = \mathbb{Z}^n \oplus \Gamma/N$ where $N = \sum_i (1-\lambda_i)\Gamma$ Then,

$$0 \to C_c(\mathbb{R}_\Gamma, \mathbb{Z}) \to C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z}) \to \mathbb{Z} \to 0$$

Is a short exact sequence of $\mathbb{Z}(\Lambda \ltimes \Gamma)$ modules. By proposition 6.1, Chapter III of Brown, we get that there is a long exact sequence:

$$\dots \to H_1(\Lambda \ltimes \Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})) \to H_1(\Lambda \ltimes \Gamma) \to H_0(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z}))$$

$$\to H_0(\Lambda \ltimes \Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})) \to H_0(\Lambda \ltimes \Gamma) \to 0$$

Plugging in what we know:

$$\to \mathbb{Z}^n \to \mathbb{Z}^n \oplus \Gamma/N \xrightarrow{f} H_0(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z})) \xrightarrow{g} \mathbb{Z} \to \mathbb{Z} \to 0$$

Is an exact sequence. The map $\mathbb{Z}^n \to \mathbb{Z}^n \oplus \Gamma/N$ is just the canonical inclusion $z \mapsto (z, 0)$. Then the map into $H_0(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma \mathbb{Z}))$ is 0 exactly on $\mathbb{Z}^n \oplus 0$; the kernel is Γ/N. Also, the map $\mathbb{Z} \to \mathbb{Z}$ must be an isomorphism, so one has that g is the zero map. But also, $im(f) = ker(g)$. Therefore, by the first isomorphism theorem, we get that $H_0(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z}) \cong Im(f)/ker(g) = Im(f) = \Gamma/N$, as required.

Let us show the first homology group is finite rank. Let us reconsider the long exact sequence, this time around $H_1(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z}))$:

$$\dots H_2(\Lambda \ltimes \Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})) \to H_2(\Lambda \ltimes \Gamma) \to H_1(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z}) \to H_1(\Lambda \ltimes \Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z}) \to \dots$$

$H_2(\Lambda \ltimes \Gamma)$ and $H_1(\Lambda \ltimes \Gamma, C_c(\hat{\mathbb{R}}_\Gamma, \mathbb{Z})) \cong \mathbb{Z}^n$ are both finite rank. Therefore it follows that $H_1(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z})$ is finite rank.

It remains to determine the position of the unit in $H_0((\Lambda \ltimes \Gamma) \ltimes (0_+, \ell_-])$. This is given by considering $(0_+, \ell_-]$ as a bisection. It follows from the above computation that the position of the unit is exactly the equivalence class of ℓ $[\ell] \in \Gamma / \sum_i (1-\lambda_i)\Gamma$. □

It would be interesting to compute the higher groupoid homology groups in this much generality, but the methods of Lemma 5.5. [76] do not generalise directly here. Therefore, one would need a new methodology for computing the group homology $H_*(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma, \mathbb{Z}))$ in general.

The good news is that even from just the computation of H_0, one may already distinguish many of these groups.

Corollary 7.4.2. *Suppose* $\Lambda = \langle \lambda_1, \dots, \lambda_n \rangle$, $\hat{\Lambda} = \langle \hat{\lambda}_1, \dots, \hat{\lambda}_m \rangle$ *be multiplicative subgroups of* $\mathbb{R}_+$, *generated by algebraic numbers. Let* $\Gamma, \hat{\Gamma}$ *be (respectively)* $\mathbb{Z} \cdot \Lambda, \mathbb{Z}\hat{\Lambda}$ *submodules and* $\ell \in \Gamma, \hat{\ell} \in \hat{\Gamma}$. *Let* $N_\Lambda = \sum_{i=1}^n (1-\lambda_i)\Gamma$, *and* $N_{\hat{\Lambda}} = \sum_{i=1}^n (1-\lambda_i)\hat{\Gamma}$. *Suppose* $V(\Gamma, \Lambda, \ell) \cong V(\hat{\Gamma}, \hat{\Lambda}, \hat{\ell})$. *Then* $\Gamma/N_\Lambda \cong \hat{\Gamma}/N_{\hat{\Lambda}}$ *and* $\ell - \hat{\ell} \in N_\Lambda$.

The above Corollary recovers for example, the fact observed by Higman [59] that $V_{k,r} \cong V_{k',r'} \Rightarrow k = k'$ and $r - r' \cong 0 \mod k\mathbb{Z}$. We remark upon some transfers of homological information from $\Lambda \ltimes \Gamma \ltimes (0_+, 1_-]$ to $V(\Gamma,\Lambda,\ell)$. This is Theorem 3.6.7 applied to this particular circumstance.

Theorem 7.4.3 (AH Exact Sequence)**.** *Let Γ,Λ,ℓ be arbitrary. Then, there is a 5-term exact sequence:*

$$H_2(D(V(\Gamma,\Lambda,\ell)) \to H_2(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma,\mathbb{Z})) \to H_0(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma,\mathbb{Z})) \otimes \mathbb{Z}_2 \to H_1(V(\Gamma,\Lambda,\ell)) =$$

$$= V(\Gamma,\Lambda,\ell)_{ab} \to H_1(\Lambda \ltimes \Gamma, C_c(\mathbb{R}_\Gamma,\mathbb{Z})) \to 0$$

We can use this to determine that many of the V-type groups are finitely generated. This is Theorem 1.1.15, our second main result.

Theorem 7.4.4. *Let Λ be a finitely generated multiplicative subgroup of the positive algebraic numbers. Let Γ be a $\mathbb{Z}\cdot\Lambda$ submodule, and let $\ell \in \Gamma$. Then $V(\Gamma,\Lambda,\ell)$ is finitely generated.*

Proof. By Lemma 7.4.1, we have that whenever Γ and Λ are generated by finitely many algebraic numbers H_0 and H_1 of the underlying groupoids are finite rank. Plugging this into the 5-term exact sequence in Theorem 3.6.7, it follows that the abelianisation is finite rank. Therefore $V(\Gamma,\Lambda,\ell)$, the extension of $D(V(\Gamma,\Lambda,\ell))$ (finitely generated by Theorem 7.3.12) by $V(\Gamma,\Lambda,\ell)_{ab}$ must be finitely generated. □

Using Theorem 3.6.5, we can obtain a homological stability result that generalizes [118, Theorem 3.6] to this more general setting.

Lemma 7.4.5 (Homological Stability)**.** *Let $\Gamma,\Lambda,\ell,\hat{\ell}$ be arbitrary then, there are isomorphisms $H_*(V(\Gamma,\Lambda,\ell)) \cong H_*(V(\Gamma,\Lambda,\hat{\ell}))$*

Proof. Suppose without loss of generality that $\hat{\ell} < \ell$. Then we have that there is an inclusion as in Theorem 3.6.5:

$$\Lambda \ltimes \Gamma \ltimes_\beta (0_+, \ell_-]|^{(0_+,\hat{\ell}_-]}_{(0_+,\hat{\ell}_-]} = \Lambda \ltimes \Gamma \ltimes_\beta (0_+, \hat{\ell}_-]$$

Then Theorem 3.6.5 implies the above theorem. □

Finally, let us apply Corollary 3.6.2, to obtain an expression for the rational homology of Stein's groups.

Theorem 7.4.6 (Rational Homology Computation)**.** *Let Γ,λ,ℓ be arbitrary. For a group G, let*

$$H_*^{even}(G) = \begin{cases} H_*(G) & * \text{ even} \\ \{0\} & \text{else} \end{cases} \qquad H_*^{odd}(G) = \begin{cases} H_*(G) & * \text{ odd} \\ \{0\} & \text{else} \end{cases}$$

and, let

$$H^{odd}_{*>1}(G) = \begin{cases} H_*(G) & * > 1 \text{ odd} \\ \{0\} & \text{else} \end{cases}$$

Then,

$$H_*(V(\Gamma,\Lambda,\ell),\mathbb{Q}) \cong Ext(H^{odd}_*(\Lambda \ltimes \Gamma, C_c(\mathbb{Z}_\Gamma,\mathbb{Q})) \otimes Sym(H^{even}_*(\Lambda \ltimes \Gamma, C_c(\mathbb{Z}_\Gamma,\mathbb{Q}))$$

and

$$H_*(D(V(\Gamma,\Lambda,\ell)),\mathbb{Q}) \cong Ext(H^{odd}_{*>1}(\Lambda \ltimes \Gamma, C_c(\mathbb{Z}_\Gamma,\mathbb{Q})) \otimes Sym(H^{even}_*(\Lambda \ltimes \Gamma, C_c(\mathbb{Z}_\Gamma,\mathbb{Q}))$$

7.4.2 Homology for Stein-Thompson groups with cyclic slopes

Let $\lambda \in (0,1)$. Consider $\Lambda_\lambda = \langle\lambda\rangle, \Gamma_\lambda = \mathbb{Z}[\lambda, 1/\lambda]$. Let $\ell \in \Gamma_\lambda$ Consider the group $V_{\lambda,\ell} := V(\Lambda_\lambda, \Gamma_\lambda, \ell)$. Such groups are natural generalisations of Thompson's group V or Cleary's group V_τ, and the analogous F-type subgroups, also known as Bieri-Strebel groups have been studied for example in [31], [127], [8], [37]. The associated groupoids and their associated C*-algebras have also been studied extensively by Li in [76]. In particular, there were many concrete homology computations by Li, which we recall now.

Lemma 7.4.7 ([76], Prop 5.5). *Let $\Gamma_\lambda, \Lambda_\lambda, r$ be as above. Then,*

$$H_0(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes_\beta (0_+, \ell_-]) \cong \Gamma_\lambda/(1-\lambda)\Gamma_\lambda$$

with distinguished order unit $[\ell]$.

$$H_k(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes_\beta (0_+, \ell_-]) \cong H_{k+1}(\Lambda_\lambda \ltimes \Gamma_\lambda) \quad \forall k \geq 1$$

In particular, if λ is transcendental, we have that:

$$H_k(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes_\beta (0_+, 1_-]) = \bigoplus_{i=1}^{\infty} \mathbb{Z} \quad \forall k$$

Let us be more explicit. If λ be an algebraic number that is the root of the minimal polynomial $f(t) = t^d + a_{d-1}t^{d-1} + \ldots + a_1 t + a_0$. Let $\Gamma = \mathbb{Z}[\lambda, \lambda^{-1}]$, $\Lambda = \langle\lambda\rangle$ and let ℓ be arbitrary. If $d < 4$, homology has been computed explicitly see the table of [76], Page 19. Using these results, we may compute many of the abelianisations of Stein-Thompson groups with cyclic slopes concretely, generalising results of [59], [31], [38].

Corollary 7.4.8. *Let λ have the minimal polynomial $t^d + a_{d-1}t^{d-1} + \ldots. + a_0 = 0$. Let $\Lambda_\lambda =$*

$\langle\lambda\rangle, \Gamma_\lambda = \mathbb{Z}[\lambda, \lambda^{-1}]$ *and* $\ell \in \Gamma_\lambda$. *Let* $V(\Gamma_\lambda, \Lambda_\lambda, \ell)$ *be the Stein-Thompson groups with cyclic slopes with slopes* $\langle\lambda\rangle$.

- *If* $d = 1$, *then* $V(\Gamma_\lambda, \Lambda_\lambda, \ell)_{ab} = \begin{cases} \mathbb{Z}_2 & f(1) \text{ even} \\ 0 & f(1) \text{ odd} \end{cases}$
- *If* $d = 2, a_0 = 1$, *and* a_1 *is odd (i.e. the minimal polynomial is of the form* $t^2 + (1-2n)t + 1$ *where* $n \in \mathbb{Z}$*) then* $V(\Gamma_\lambda, \Lambda_\lambda, \ell)_{ab} = \mathbb{Z}$
- *If* $d = 2, a_0 \neq 1$, *then* $V(\Gamma_\lambda, \Lambda_\lambda, \ell)_{ab} = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/(1+a_0)\mathbb{Z} & f(1) \text{ even} \\ \mathbb{Z}/(1+a_0)\mathbb{Z} & f(1) \text{ odd} \end{cases}$
- *If* $d = 3, a_0 = -1$ *and* $a_2 + a_1$ *is odd, (i.e. the minimal polynomial is of the form* $t^3 + (m)t^2 + (m+2n+1)t - 1$*) then* $V(\Gamma_\lambda, \Lambda_\lambda, \ell)_{ab} = \mathbb{Z}/(a_1 + a_2)\mathbb{Z}$

Proof. These results follow from the computations as in the table of [76], Page 19, and the AH long exact sequence (Theorem 3.6.7). If $d = 1$, then the groupoid homology is

$$H_*(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, 1_-]) = \begin{cases} \mathbb{Z}/f(1)\mathbb{Z} & * = 0 \\ 0 & * > 0 \end{cases}$$

Therefore the exact sequence reduces to an isomorphism: $\mathbb{Z}/f(1)\mathbb{Z} \otimes \mathbb{Z}_2 = (V(\Gamma_\lambda, \Lambda_\lambda, \ell))_{ab}$. If $d = 2$, $a_0 = 1$ then the groupoid homology is

$$H_*(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, 1_-]) = \begin{cases} \mathbb{Z}/f(1)\mathbb{Z} = \mathbb{Z}/(2+a_1)\mathbb{Z} & * = 0 \\ \mathbb{Z} & * = 1, 2 \\ 0 & * > 2 \end{cases}$$

Therefore if a_1 is odd, then the exact sequence reduces to an isomorphism $(V(\Gamma_\lambda, \Lambda_\lambda, \ell))_{ab} = \mathbb{Z}$. If $d = 2, a_0 \neq 1$ then the groupoid homology is given by

$$H_*(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, 1_-]) = \begin{cases} \mathbb{Z}/f(1)\mathbb{Z} & * = 0 \\ \mathbb{Z}/(1-a_0)\mathbb{Z} & * = 1 \\ 0 & * > 1 \end{cases}$$

Therefore the exact sequence reduces to a short exact sequence that splits, and $(V(\Gamma_\lambda, \Lambda_\lambda, \ell))_{ab} = (\mathbb{Z}_2 \otimes \mathbb{Z}/f(1)\mathbb{Z}) \oplus \mathbb{Z}/(1-a_0)\mathbb{Z}$. Finally if $d = 3, a_0 = -1$. We have the groupoid homology is given by:

$$H_*(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, 1_-]) = \begin{cases} \mathbb{Z}/f(1) = \mathbb{Z}/(a_2+a_1)\mathbb{Z} & * = 0, 1 \\ \mathbb{Z} & * = 2, 3 \\ 0 & * > 3 \end{cases}$$

Therefore if a_2+a_1 is odd, the exact sequence reduces to an isomorphism $(V(\Gamma_\lambda,\Lambda_\lambda,\ell))_{ab} = \mathbb{Z}/(a_2+a_1)\mathbb{Z}$

□

Remark 7.4.9. *The above Corollary recovers the computation of the abelianisation of Cleary's group seen in [31], and of the Higman-Thompson groups seen in [59].*

We see that in contrast to the Higman-Thompson groups, or Brin-Higman-Thompson groups that are either perfect or have an index 2 derived subgroup, theStein-Thompson groups with cyclic slopes can have a variety of abelianisations. In particular, not all Stein-Thompson groups with cyclic slopes are virtually simple:

Example 7.4.10. *For $n \in \mathbb{N}\cup\{0\}\cup\{\infty\}$ there exists an algebraic number λ_n such that the first homology group, (or in other words the abelianisation) $V(\Gamma_{\lambda_n},\Lambda_{\lambda_n},\ell)_{ab}$ is a cyclic group of order n. For $n=0$, we can take $\lambda_0=2$, for $n=2$, we can take $\lambda_2=3$. λ_3 must be irrational, we can take for example λ_3 the smallest positive root of t^3+3t-1. We could take $\lambda_4=\frac{7-\sqrt{37}}{2}$ the smallest positive root of t^2-7t+3. We can take λ_5 to be the smallest positive root of t^3+5t^2-1. For $n\geq 6$, we can take λ_n to be the smallest root of $t^2-(n-1)(1-t)$, $\frac{n-1+\sqrt{n^2-6n+5}}{2}$. For the infinite cyclic group we can take λ_∞ to be smallest real root of t^2-3t+1.*

Example 7.4.11. *Suppose n isnt a square number and $\ell\in\mathbb{Z}[\sqrt{n},1/\sqrt{n}]$. Then*

$$V(\Gamma_{\sqrt{n}},\Lambda_{\sqrt{n}}\ell)_{ab} = \begin{cases} \mathbb{Z}_2\oplus\mathbb{Z}_{n+1} & n \text{ odd} \\ \mathbb{Z}_{n+1} & n \text{ even} \end{cases}$$

Whenever k is even, the Brin-Higman-Thompson group $nV_{k,r}$ is perfect. There are actually no examples of "true" irrational slope Thompson groups (that is Stein-Thompson groups with cyclic slopes and an irrational generator of the slope group) that are perfect. This is because if $d>1$, then $H_1(\Lambda_\lambda\ltimes\Gamma_\lambda\ltimes\mathbb{R}_\Gamma)\neq 0$, and so the abelianisation admits a nontrivial quotient.

Corollary 7.4.12. *Let λ be an irrational number and let $\ell\in\mathbb{Z}[\lambda,\lambda^{-1}]$. Then $V(\Gamma_\lambda,\Lambda_\lambda,\ell)$ is not perfect.*

A final corollary is that we may use this abelianisation in conjunction with Corollary 7.2.9 to describe all finite factor presentations, or proper characters of $V(\Gamma_\lambda,\Lambda_\lambda,\ell)$:

Lemma 7.4.13 (Proper Characters of $V(\Gamma_\lambda,\Lambda_\lambda,\ell)$)**.** *Let λ have the minimal polynomial $t^d+a_{d-1}t^{d-1}+\ldots+a_0=0$. Then every proper character of $V(\Gamma_\lambda,\Lambda_\lambda,\ell)$ factors through the abelianisation, as computed in Theorem 7.4.8. In particular, Cleary's group V_τ has one proper character.*

The other advantage of having a concrete picture of groupoid homology is that this simplifies also the rational homology computation of the underlying groups.

Corollary 7.4.14 (Rational Homology Computation Stein-Thompson groups with cyclic slopes)**.** *Suppose λ is transcendental. Then,*

$$H_*(V(\Gamma_\lambda,\Lambda_\lambda,\ell),\mathbb{Q}) = \bigoplus_{n\in\mathbb{N}}\mathbb{Q}$$

Suppose λ is algebraic. Then, let

$$H_*^{even}(\Lambda_\lambda \ltimes \Gamma_\lambda,\mathbb{Q}) = \begin{cases} H_*(\Lambda_\lambda \ltimes \Gamma_\lambda,\mathbb{Q}) & *even \\ 0 & otherwise \end{cases},$$

$$H_{*>2}^{even}(\Lambda_\lambda \ltimes \Gamma_\lambda,\mathbb{Q}) = \begin{cases} H_*(\Lambda_\lambda \ltimes \Gamma_\lambda,\mathbb{Q}) & * > 2,\ even \\ 0 & otherwise \end{cases}$$

and

$$\hat{H}_*^{odd}(\Lambda_\lambda \ltimes \Gamma_\lambda,\mathbb{Q}) = \begin{cases} \Gamma_\lambda/(1-\lambda)\Gamma_\lambda \otimes \mathbb{Q} & * = 1 \\ H_*(\Lambda_\lambda \ltimes \Gamma_\lambda,\mathbb{Q}) & * > 1,\ odd \\ 0 & otherwise \end{cases}$$

then,

$$H_*(V(\Gamma_\lambda,\Lambda_\lambda,\ell),\mathbb{Q}) = Ext(H_{*+1}^{even}(\Lambda_\lambda \ltimes \Gamma_\lambda)) \otimes Sym(\hat{H}_{*+1}^{odd}(\Lambda_\lambda \ltimes \Gamma_\lambda))$$

$$H_*(D(V(\Gamma_\lambda,\Lambda_\lambda,\ell),\mathbb{Q})) = Ext(H_{*+1>2}^{even}(\Lambda_\lambda \ltimes \Gamma_\lambda)) \otimes Sym(\hat{H}_{*+1}^{odd}(\Lambda_\lambda \ltimes \Gamma_\lambda))$$

Where Ext, Sym denote respectively the Exterior and Symmetric algebras in the sense of Multilinear Algebra [54].

Like the Higman-Thompson groups $V_{k,r}$, certain low degree Stein-Thompson groups with cyclic slopes are rationally acyclic.

Example 7.4.15. *Let λ be the root of the minimal polynomial $t^2 + a_1 t + a_0$. The following are equivalent:*

- $a_0 \neq 1$.
- $V(\Gamma_\lambda,\Lambda_\lambda,\ell)$ *and* $D(V(\Gamma_\lambda,\Lambda_\lambda,\ell))$ *are rationally acyclic.*

Similarly, let λ be the root of a minimal polynomial $t^3 + a_2 t^2 + a_1 t + a_0$. The following are equivalent:

- $a_0 \neq -1$.
- $V(\Gamma_\lambda,\Lambda_\lambda,\ell)$ *and* $D(V(\Gamma_\lambda,\Lambda_\lambda,\ell))$ *are rationally acyclic.*

In particular, Cleary's group is rationally acyclic, being associated to the minimal polynomial $t^2 + t - 1$.

However, in contrast to behaviour that is seen for the rational homology of the Higman-Thompson groups [118] or more generally the Brin-Higman-Thompson groups Proposition 4.4.7, Stein-Thompson groups with cyclic slopes are not rationally acyclic in general. Concrete calculations are shown below:

- *If λ is transcendental,*

$$H_*(V(\Gamma_\lambda,\Lambda_\lambda,\ell),\mathbb{Q}) = \begin{cases} \bigoplus_{i=1}^{\infty}\mathbb{Q} & * > 0 \\ \mathbb{Q} & * = 0 \end{cases},\ H_*(D(V(\Gamma_\lambda,\Lambda_\lambda,\ell)),\mathbb{Q}) \cong \begin{cases} \bigoplus_{i=1}^{\infty}\mathbb{Q} & * > 1 \\ 0 & * = 1 \\ \mathbb{Q} & * = 0 \end{cases}$$

- *If λ has minimal polynomial of the form $t^2 + a_1 t + 1$ we have that:*

$$H_*(V(\Gamma_\lambda,\Lambda_\lambda,\ell),\mathbb{Q}) = \mathbb{Q},\ H_*(D(V(\Gamma_\lambda,\Lambda_\lambda,\ell)),\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & * \neq 1 \\ 0 & * = 1 \end{cases}$$

- *If λ has minimal polynomial of the form $t^3 + a_2 t + a_1 t - 1$*

$$H_*(V(\Gamma_\lambda,\Lambda_\lambda,\ell),\mathbb{Q}) = H_*(D(V(\Gamma_\lambda,\Lambda_\lambda,\ell)),\mathbb{Q}) = \begin{cases} \mathbb{Q} & * \neq 1 \\ 0 & * = 1 \end{cases}$$

Theorem 7.4.16 ([76], Theorem 1.3 and 1.4)**.** *It is clear that for all choices of λ, $C_r^*(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes_\beta (0_+, 1_-]) := \mathcal{Q}^\lambda$ is a UCT Kirchberg algebra. For more information on these C^*-algebras, we refer the reader to the references [76], [33]. The (ordered) K-theory was computed by Li in [76], Theorem 1.3.*

$$(K_0(\mathcal{Q}^\lambda), [1]_0, K_1(\mathcal{Q}^\lambda)) \cong K_1(C^*(\Gamma_\lambda)/\mathbb{Z}[U_{0,\lambda}]_1, [U_{1,1}]_1^{\cdot}, K_0(C^*(\Gamma_\lambda)/\mathbb{Z}[1]_0)$$

Here, by $[U_{1,1}]_1^{\cdot}$ we mean the class of $[U_{1,1}]_1$ in the quotient $K_1(C^*(\Gamma_\lambda)/\mathbb{Z}[U_{0,\lambda}]_1$, and by $\mathbb{Z}[1]_0$ we mean the group generated by the unit in $K_0(C^*(\Gamma_\lambda)$.

Let λ be algebraic. Then the ordered K-Theory simplifies: $(K_0(\mathcal{Q}^\lambda), [1]_0, K_1(\mathcal{Q}^\lambda))$

$$\cong (\Gamma_\lambda)/(1-\lambda\Gamma_\lambda) \oplus \bigoplus_{j=1}^{\infty} H_{2j+1}(\Lambda_\lambda \ltimes \Gamma_\lambda), (1,0), \bigoplus_{j=1}^{\infty} H_{2j}(\Lambda_\lambda \ltimes \Gamma_\lambda))$$

*By the Kirchberg-Phillips classification Theorem [[106], Chapter 8], this is a complete *-isomorphism invariant of the C*-algebras $\mathcal{Q}^\lambda$.*

It is not hard to show that if we introduce flexibility on the length ℓ of our interval, this would induce morita equivalent C*-algebras with (possibly) different order units in K_0.

Now that we have seen that we can compute many homological or topological invariants that we associate to the groupoids $\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes_\beta (0_+, \ell_-]$, let us describe what we know about distinguishing groups up to isomorphism for this class. The groupoid homology recovers the initial choice of algebraic number λ. First note that we can *see* that λ is algebraic through our homology results.

Corollary 7.4.17. *Let λ be a real number and $\ell_\lambda \in \Gamma_\lambda$ be arbitrary. The following are equivalent:*

- *λ is transcendental.*
- *$V(\Gamma_\lambda, \Lambda_\lambda, \ell_\lambda)_{ab} \cong \mathbb{Z}^\infty$*

The algebraic case is a complex problem in general to classify, but due to the groupoid homology computations we can say something if we restrict to algebraic numbers with degree ≤ 2.

Corollary 7.4.18. *Let $\lambda, \mu < 1$ be algebraic numbers with degree ≤ 2 and let $\ell_1 \in \Gamma_\lambda, \ell_2 \in \Gamma_\mu$. If*

$$V(\Gamma_\lambda, \Lambda_\lambda, \ell_1) \cong V(\Gamma_\mu, \Lambda_\mu, \ell_2)$$

Then $\lambda = \mu$, and $\ell_1 - \ell_2 \in (1-\lambda)\Gamma_\lambda$.

Proof. Our first step is to recover λ from the abstract group $V(\Gamma_\lambda, \Lambda_\lambda, \ell_1)$. Let us examine the groupoid $\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, (\ell_1)_-]$, and attempt to recover the minimal polynomial from the groupoid homology. $H_*(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, (\ell_1)_-])$. Let us consider two cases:

1. If $\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, (\ell_1)_-]$ is rationally acyclic. Then:
   - If $H_1(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, (\ell_1)_-]) = 0$, $d = 1$ and a_0 is determined by $H_0(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, (\ell_1)_-]) \cong \mathbb{Z}/(1+a_0)\mathbb{Z}$.
   - Otherwise, $d = 2$ and $a_0 \neq -1$. Then a_0 is determined by $H_1(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, (\ell_1)_-]) \cong \mathbb{Z}/(1-a_0)\mathbb{Z}$ and a_1 is determined by $H_0(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, (\ell_1)_-]) \cong \mathbb{Z}/(1+a_1+a_0)\mathbb{Z}$
2. If $\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, (\ell_1)_-]$ is not rationally acyclic then we have that $H_1(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, (\ell_1)_-]) = H_2(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, (\ell_1)_-]) = \mathbb{Z}$, then $d = 2$, $a_0 = 1$. Moreover, $H_0(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, (\ell_1)_-]) \cong \mathbb{Z}/(2+a_1)\mathbb{Z}$, which uniquely determines a_1.

This shows we may recover λ from the groupoid homology. Groupoid homology is an invariant for groupoids. By Theorem 3.4.12, these groupoids are uniquely determined by the abstract groups $V(\Gamma, \Lambda, \ell)$ and so for any isomorphism of the abstract groups, we must necessarily have that $\lambda = \mu$. It remains to understand how the length of the underlying interval affects the isomorphism class of these groups. One may be precise here, considering $H_0(\Lambda_\lambda \ltimes \Gamma_\lambda \ltimes (0_+, \ell_-]) \cong$

$\Gamma/(1-\lambda)\Gamma$ as an ordered group, with the class of $(0_+, \ell_-]$, $[\ell] \in \Gamma/(1-\lambda)\Gamma_\lambda$ determining the positive cone. We therefore have an isomorphism as ordered groups, this occurs if and only if $\ell - \ell' \in (1-\lambda)\Gamma_\lambda$. □

This Corollary showcases the diversity of cyclic slope Stein's group, even for the case when λ has degree 2. We can also distinguish some cyclic slope Stein's groups of higher degrees using groupoid homology. However, in general, this classification remains open for degree 3 algebraic numbers; our methods do not generalise. For a concrete example we cannot distinguish using groupoid homology, consider λ, μ to be the algebraic numbers with respective minimal polynomials:

$$t^3 + t^2 - 1, \quad t^3 + t - 1$$

The resulting groupoids have the same homology. We, therefore, ask if one can determine whether $V(\Gamma_\lambda, \Lambda_\lambda, 1) \cong V(\Gamma_\mu, \Lambda_\mu, 1)$ for this specific choice of λ, μ? Another similarly subtle question is to try to distinguish groups with different transcendental slopes, e.g. is $V(\Gamma_\pi, \Lambda_\pi, 1) \cong V(\Gamma_e, \Lambda_e, 1)$?

7.4.3 Homology for Stein's integral groups

Let $N = \{n_1, ... n_k\}$ be a finite collection of integers. Let $r \in \mathbb{N}$, let $\Lambda_N = \langle n_1, .., n_k \rangle$ and $\Gamma_N = \mathbb{Z}[1/(n_1 n_2 ... n_k)]$. Consider the group $V(\Lambda_N, \Gamma_N, r)$. These groups have been studied in detail by Stein [116], and are known to fit into many other frameworks of generalized Thompson's groups, for example, those explored in [84]. Because they have been more heavily studied, much more is known about the finiteness properties and presentations of these groups, for example, it was shown already by Stein in [116] that they are all of type F_∞. However, the homology of these groups remains mysterious. For us to better understand this, it is useful to change perspectives on the underlying groupoids.

The approach we take here is related to work involving the topological full groups of k-graphs, which has notably also been explored (though not for this reason) in work by Lawson-Sims-Vdovina [72], Lawson-Vdovina [73], and Dilian Yang [128]. The observation that we can rephrase the groupoid model of these exact groups in the language of k-graphs is also noted in Conchita Martınez-Pérez, Brita Nucinkis and Alina Vdovina, we include it here for completeness in the literature and for our homology computations. Let $N = \{n_1, ..., n_k\}$ be a collection of integers where $k > 1, n_i > 1, \forall k$. Consider the single vertex k-graph that has n_i loops for each $i = 1, ..., k$. Let us label these loops $a_0^{(n_i)}, ..., a_{n_i - 1}^{(n_i)}$ and let them be colored in the i-th color. We then include commutation relations as follows. Between color i and color j there are $n_i n_j$ commutation relations, given by:

$$a_k^{(n_i)} a_l^{(n_j)} = a_{k'}^{(j)} a_{l'}^{(i)} \iff k/n_i + l/n_i n_j = k'/n_i n_j + l'/n_i$$

For all $0 < k < n_i, 0 < l < n_j$. These commutation relations induce a description of the factorisation map d. Notice there is a relationship already between the edges $a_k^{(n_i)}$ of our k-graph and $(k, 1/n_i) \in \langle n_1, .., n_k \rangle \ltimes \mathbb{Z}[\prod_{i=1}^k 1/n_i]$: these commutation relations are the same as saying, for $(k, 1/n_i), (k', 1/n_i), (l, 1/n_j), (l', 1/n_j) \in \langle n_1, .., n_k \rangle \ltimes \mathbb{Z}[\prod_{i=1}^k 1/n_i]$:

$$a_k^{(n_i)} a_l^{(n_j)} = a_{k'}^{(j)} a_{l'}^{(i)} \iff (k, 1/n_i)(l, 1/n_j) = (l', 1/n_j)(k', 1/n_i)$$

Since it is single-vertex, the paths in this k-graph form a left-cancellative monoid with respect to concatenation, and the identifications from our equivalence relation. This monoid considered as a left-cancellative category fits directly into Kumjian-Pask's framework of k-graphs [69]. Let us denote the above k-graph by $(\Lambda(n_1, .., n_k), d)$ For more information on k-graphs and their groupoids, we refer the reader to this source.
Let us describe the associated path groupoid, introduced in [69]. For $k > 1$, Let Ω_k be the small category with objects in $\mathbb{N}^k$ and morphisms $\Omega := \{(m,n) : \mathbb{N}^k \times \mathbb{N}^k, m \le n\}$, the range and source maps of this morphism being $r(m,n) = n, s(m,n) = n$. Let $d(m,n) = n - m$.

The path space $\Lambda^\infty(n_1, .., n_k)$ refers to all k-graph morphisms (that is, structure-preserving morphisms):

$$x : (\Omega_k, d) \to (\Lambda(n_1, ..., n_k), d)$$

The topology on this is the natural cylinder sets (paths that begin with a finite path λ):

$$Z(\lambda) = \{\lambda x \in \Lambda^\infty(n_1, .., n_k)\}$$

For each $m \in \mathbb{N}^k$ we associate a shift map $\sigma^m : \Lambda^\infty(n_1, .., n_k) \to \Lambda^\infty(n_1, ..., n_k)\ x \mapsto \sigma^m x$, where $\sigma^m x$ is the function given by

$$\sigma^m x(n_1, n_2) = x(n_1 + m, n_2 + m)$$

Notice then that this is a semigroup homomorphism: $\sigma^{m_1 + m_2} = \sigma^{m_1} \circ \sigma^{m_2}$. Our groupoid is of the form:

$$\mathcal{G}(n_1, ..., n_k) := \{(x, m_1 - m_2, y)\ x, y \in \Lambda^\infty(n_1, ..., n_k)\ m_1, m_2 \in \mathbb{N}^k,\ \sigma^{m_1}(x) = \sigma^{m_2}(y)\}$$

The source of $(x, m_1 - m_2, y) \in \mathcal{G}(n_1, ..., n_k)$ is $(x, 0 - 0, x)$. The range of $(x, m_1 - m_2, y) \in \mathcal{G}(n_1, ..., n_k)$ is $(y, 0 - 0, y)$. The inverse of $(x, m_1 - m_2, y) \in \mathcal{G}(n_1, ..., n_k)$ is $(y, (-m_2) - (-m_1), x) \in \mathcal{G}(n_1, ..., n_k)$. Our unit space is identified with the path space by the canonical identification, as with Deacounu-Renault groupoids of graphs ($(x, 0, x) \mapsto x, \forall x \in \Lambda^\infty$).

A basis for the topology is coming from the open bisections (the groupoid is étale), which are the inverse semigroup generated by partial bijections of the form:

$$\mathbb{Z}(\lambda, \mu) = \{(x, m_1 - m_2, y)\ x \in Z(\lambda)\ y \in Z(\mu),\ m_1, m_2 \in \mathbb{N}^k,\ \sigma^{m_1}(x) = \sigma^{m_2}(y)\}$$

Now let us describe why this groupoid agrees with our description of the same groupoid, namely $\mathbb{Z}[\prod_{i=1}^{k} 1/n_i] \ltimes \langle n_1,...,n_k\rangle \ltimes (0_+, 1_-]$. The first step is to identify the unit spaces. To do this, let us notice the following distinguished elements of Λ^∞:

$$x_0 : \mathbb{N}^k \mapsto \Lambda(n_1,..,n_k) \quad (l_1,l_2..,l_k) \mapsto (a_0^{(n_1)})^{l_1}(a_0^{(n_2)})^{l_2}...(a_0^{(n_k)})^{l_k}$$

$$x_1 : \mathbb{N}^k \mapsto \Lambda(n_1,..,n_k) \quad (l_1,l_2..,l_k) \mapsto (a_{n_1-1}^{(n_1)})^{l_1}(a_{n_2-1}^{(n_2)})^{l_2}...(a_{n_k-1}^{(n_k)})^{l_k}$$

Note that since for all i,j $a_0^{(n_i)}a_0^{(n_j)} = a_0^{(n_j)}a_0^{(n_i)}$ and $a_{n_i-1}^{(n_i)}a_{n_j-1}^{(n_j)} = a_{n_j-1}^{(n_j)}a_{n_i-1}^{(n_i)}$ so that these are well defined paths. x_0 will map to 0_+ and x_1 to 1_-. Let us first define a map on finite paths

$$\phi_0 : \Lambda(n_1,...,n_k) \to \mathbb{Z}[\prod_{i=1}^{k} n_i] \quad (a_{l_1}^{(n_{i_1})},...,a_{l_m}^{(n_{i_m})})) \mapsto \sum_{i=1}^{m} n_i \prod_{j=1}^{i} \frac{1}{n_j}$$

Due to the commutation relations, this map is well-defined and bijective. From here, the map is extended to become φ_0.

$$\varphi_0 : \Lambda^\infty(n_1,...,n_k) \to (0_+, 1_-] \subset \mathbb{R}_{\mathbb{Z}[\prod_{i=1}^{k} 1/n_i]}$$

$$x = [(a_{l_1}^{(n_{i_1})}, a_{l_2}^{(n_{i_2})}, ...)] \mapsto \begin{cases} \phi_0(a_{l_1}^{(n_{i_1})},..,a_{l_m}^{(n_{i_m})})_+ & x = (a_{l_1}^{(n_{i_1})},..,a_{l_m}^{(n_{i_m})})x_0 \\ (\prod_{i=1}^{m} \frac{1}{n_i} + \phi_0(a_{l_1}^{(n_{i_1})},..,a_{l_m}^{(n_{i_m})})_- & x = (a_{l_1}^{(n_{i_1})},..,a_{l_m}^{(n_{i_m})})x_1 \\ \sum_{m=1}^{\infty} \phi_0(a_{l_1}^{(n_{i_1})},..,a_{l_m}^{(n_{i_m})}) & otherwise \end{cases}$$

Remark that this does not depend on the choice of representative for x, due to our canonical commutation relations. Note also this map is a homeomorphism, since it is a continuous bijection between compact Hausdorff spaces:

$$\varphi_0(Z(a_{l_1}^{(n_{i_1})},..,a_{l_m}^{(n_{i_m})})) = [\phi_0(a_{l_1}^{(n_{i_1})},..,a_{l_m}^{(n_{i_m})})_+, (\prod_{i=1}^{m} \frac{1}{n_i} + \phi_0(a_{l_1}^{(n_{i_1})},..,a_{l_m}^{(n_{i_m})})_-]$$

Every open set in $(0_+, 1_-]$ can be written as a finite union of sets of the above form since ϕ_0 is bijective. The next step is to identify the arrow spaces. Consider the map:

$$\mu : \mathbb{Z}^k \to \langle n_1,...,n_k\rangle \quad (m_1,..,m_k) \mapsto \prod_{i=1}^{k} n_i^{m_i}$$

$$\varphi : \mathcal{G}(n_1,...,n_k) \mapsto \mathbb{Z}[\prod_{i=1}^{k} 1/n_i] \rtimes \langle n_1,...,n_k\rangle \ltimes (0_+, 1_-]$$

$$(x, m_1 - m_2, y) \mapsto ((-\varphi_0(x) + \frac{\varphi_0(y)}{\mu(m_1 - m_2)}, \mu(m_1 - m_2)), \varphi_0(x))$$

Notice then in particular:

$$\varphi(Z(\varnothing, a_0^{n_i})) = ((0,1/n_i),(0_+,1_-])$$

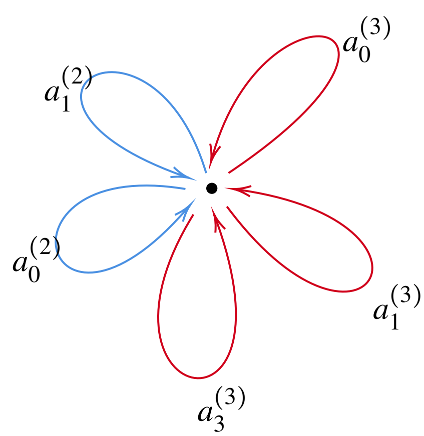
$$\varphi(Z(a_{n_i-1}^{(n_i)}, a_{n_i-1}^{(n_i)}) \bigsqcup \sqcup_{l=0}^{n_i-2} Z(a_l^{(n_i)}, a_{l+1}^{(n_i)}))$$
$$= ((1/n_i - 1, 1),[(1-1/n_i)_+, 1_-]) \sqcup ((1/n_i, 1),(0_+,(1-1/n_i)_-])$$

As shown in the previous section (Theorem 7.3.12), these bisections form a compact generating set of the groupoid $\mathbb{Z}[\prod_{i=1}^{k} 1/n_i] \rtimes \langle n_1,\dots,n_k\rangle \ltimes (0_+,1_-]$, hence φ is a groupoid conjugacy. The discussion above can be summarised in the form of the Lemma below.

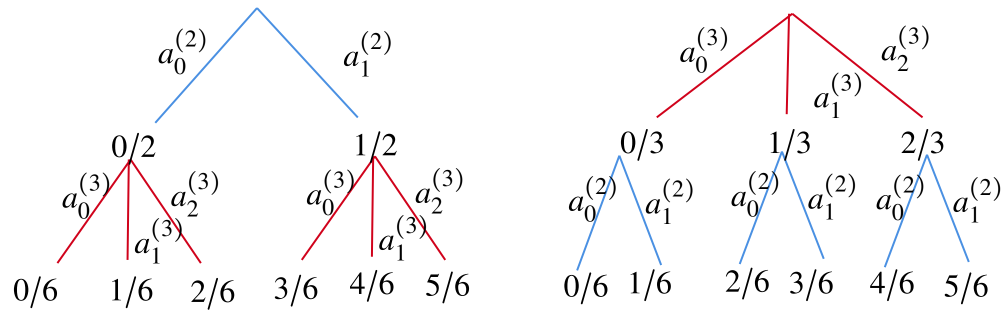
Lemma 7.4.19. *Let $k>1$, and $N=\{n_1,\dots,n_k\}$ be a collection of algebraically independent integers. Then Stein's integral group $V(\Gamma_N,\Lambda_N,1)$ the topological full group of the path groupoid model of the k-graph above, $\mathcal{G}(n_1,\dots,n_k)$.*

This identification will give us many interesting new facts about Stein's groups. Notably, this identification is an identification of the underlying left cancellative monoids- the paths space of the k-graph may be identified with the underlying monoid seen in the universal groupoid description. Let us first explore in greater detail the simplest nontrivial case, namely $k=2, n_1=2, n_2=3$ for exposition purposes.

Example 7.4.20 ($V(2,3)$)**.** *Let us construct $V(2,3)=V(\mathbb{Z}[\frac{1}{6}],\langle 2,3\rangle,1)$ using k-graphs. Here, $k=2$. We consider the single vertex 2-graph with 2-loops in the first color (in the case of the diagram below, blue) labelled $a_1^{(2)}, a_1^{(2)}$ and 3-loops in the second color (in the case of the diagram below, red) labelled $a_0^{(3)} a_1^{(3)}, a_2^{(3)}$.*



By drawing the related diagram below:



We can read the 6 *necessary commutation relations:*

$$a_0^{(2)}a_0^{(3)} = a_0^{(3)}a_0^{(2)},\ a_0^{(2)}a_1^{(3)} = a_0^{(3)}a_1^{(2)},\ a_0^{(2)}a_2^{(3)} = a_1^{(3)}a_0^{(2)},$$

$$a_1^{(2)}a_0^{(3)} = a_1^{(3)}a_1^{(2)},\ a_1^{(2)}a_1^{(3)} = a_2^{(3)}a_0^{(2)},\ a_1^{(2)}a_2^{(3)} = a_2^{(3)}a_1^{(2)}$$

This describes the relevant k-graph to $V(2,3)$*. So let us describe the associated path groupoid* $\mathcal{G}(2,3)$.

- *The unit space is formed of infinite paths* $x \in \Lambda^\infty(2,3)$ *alternating between the two colours, up to the equivalence relation induced by the commutation relations.*

- *Elements of the groupoid are triples:*

$$(x, m_1 - m_2, y) \in \Lambda^\infty(2,3) \times \mathbb{Z}^2 \times \Lambda^\infty(2,3)$$

  Let $m_1 - m_2 = (m_{1,1}, m_{1,2}) - (m_{2,1}, m_{2,2})$*. The arrow takes the infinite path x, removes the first* $m_{2,1}$ *sections of blue path in x, replacing them with the first* $m_{1,1}$ *sections of blue path in y. Similarly, it removes the first* $m_{2,2}$ *sections of red path in x, replacing them with the first* $m_{2,2}$ *sections of red path in y. In doing this, x is mapped to y (they are necessarily tail equivalent).*

- *The topology (topologically) and the inverse semigroup of bisections (algebraically) are generated by the cylinder sets of the form* $Z(\lambda,\mu)$*, is a homeomorphism* $Z(\lambda) \to Z(\mu), \lambda x \mapsto \mu x$.

Elements of the topological full group are therefore disjoint unions of the form $\sqcup_{i=1}^{l} Z(\lambda_i,\mu_i)$ *where* $\sqcup_{i=1}^{l} Z(\lambda_i), \sqcup_{i=1}^{l} Z(\mu_i)$ *form a partition of the entire path space.*

Let us sketch a direct isomorphism:

$$f : F(\mathcal{G}(2,3)) \to V(2,3)$$

f *takes an element* $\sqcup_{i=1}^{l} Z(\lambda_i,\mu_i)$ *of the topological full group to the piecewise linear map* $f(\sqcup_{i=1}^{l} Z(\lambda_i,\mu_i))$ *in Stein's group formed of l linear sections, corresponding to each* $Z(\lambda_i,\mu_i)$.

These linear sections for each i are the unique linear maps that map the interval $(\phi_0(\lambda_i), \varphi_0(\lambda_i x_1)]$ *associated with* λ_i *to the interval* $(\phi_0(\mu_i), \varphi_0(\mu_i x_1)]$ *associated with* μ_i*. The endpoints of these* *intervals are in $\mathbb{Z}[1/6]\cap(0,1]$ and the ratio of the lengths is in $\langle 2,3\rangle$. Their union therefore defines a unique element of $V(2,3)$.*

Lemma 7.4.21 (Homology of groupoids with integral slope sets)**.** *Let $N=\{n_1,\dots,n_k\}$ be a finite collection of integers, where $k>1$. Let $d=gcd(n_1-1,\dots,n_k-1)$. Let $\ell\in\Gamma_N$ be arbitrary.*

$$H_*(\Lambda_N\ltimes\Gamma_N\ltimes(0_+,\ell_-])\cong\begin{cases}(\mathbb{Z}/d\mathbb{Z})^{{}^kC_*} & *=0,\dots,k-1\\ 0 & *\geq k\end{cases}$$

Proof. First let us apply Lemma 3.6.5. We have that:

$$H_*(\Lambda_N\ltimes\Gamma_N\ltimes(0_+,\ell_-])\cong H_*(\Lambda_N\ltimes\Gamma_N\ltimes(0_+,1_-])$$

Then by combining Lemma 7.4.19 and Corollary 7.2.5, we have that:

$$H_*(\Lambda_N\ltimes\Gamma_N\ltimes(0_+,\ell_-])\cong H_*(\mathcal{G}(n_1,n_2,\dots,n_k))$$

The homology of the groupoids of single vertex k-graphs was computed in [49], and was shown to not depend on the underlying commutation relations. Therefore, let $d=gcd(n_1-1,\dots,n_k-1)$. We have that:

$$H_*(\mathcal{G}(n_1,..,n_k)):=\begin{cases}(\mathbb{Z}/d\mathbb{Z})^{{}^{k-1}C_i} & i=0,\dots,k-1\\ 0 & i\geq k\end{cases}$$

□

Corollary 7.4.22. *Let $N=\{n_1,..,n_k\}$ be a finite collection of integers with $k>1$. Let $\ell\in\Gamma_N$. Then $V(\Gamma_N,\Lambda_N,\ell)$ is rationally acyclic. Moreover, $V(\Gamma_N,\Lambda_N,\ell)$ is integrally acyclic if and only if $d=gcd(n_1-1,\dots,n_k-1)=1$.*

Proof. Observe that in Lemma 7.4.21, we have that the groupoid $\Lambda_N\ltimes\Gamma_N\ltimes(0_+,\ell_-]$ is always rationally acyclic, and is acyclic if and only if $d=gcd(n_1-1,\dots,n_k-1)=1$. Then, Theorem 3.6.4 confirms that the topological full group is always rationally acyclic. Similarly, Corollary D confirms that the topological full group is acyclic if $d=1$. Finally, if $d\neq 1$, the original abelianization computation of Stein [116], shows that in this case, the group is not acyclic. □

Example 7.4.23. *Stein's integral group $V(2,3)$ is integrally acyclic. Stein's integral group $V(3,5)$ is rationally acyclic, but not integrally acyclic.*

We end this section by remarking that this realisation of Stein's groups gives a different perspective on two of Stein's key results in [116], namely the higher finiteness properties of Stein's integral groups and the computation of their abelianisation.

Remark 7.4.24 (Alternative Proof of Abelianisation)**.** *Let $N = \{n_1,..,n_k\}$ with $k > 1$. We can recover Stein's computation of the abelianization of $V(\Gamma_N,\Lambda_N,\ell)$ in [116] by examining the exact sequence in Theorem 3.6.7. Let $d = gcd(n_1,...,n_k)$ It is clear the abelianisation surjects onto $H_1(\mathcal{G}(n_1,..,n_k) = (\mathbb{Z}/d\mathbb{Z})^{k-1}$. Also, we have that the map $H_2(\mathcal{G}(n_1,..,n_k)) = \mathbb{Z}_d^{{}^{k-1}C_2} \to H_0(\mathcal{G}(n_1,..,n_k)) \otimes \mathbb{Z}_2 = \mathbb{Z}_d \otimes \mathbb{Z}_2$ is the zero map whenever d is odd, ensuring the surjection is an isomorphism. If d is even, the image of the above map is $\mathbb{Z}_2 = \mathbb{Z}_d \otimes \mathbb{Z}_2$. In this case, we get a split exact sequence that recovers Stein's computation of the abelianisation.*

Remark 7.4.25 (Left regular representations of Garside categories, Type F_∞ simple groups)**.** *In Stein's original paper [116], she was able to show that the integral groups are of type F_∞ and are virtually simple. It is interesting to remark that the realisation of Stein's groups as the topological full groups arising from certain k-graphs also gives rise to this fact, due to the Garside framework of Li [78], one may alternatively apply [Theorem C, [78]] to reach the same conclusion.*

Corollary 7.4.26. *Let $N_1 = \{n_1,...,n_k\}, N_2 = \{m_1,...,m_j\}$ be two collections of integers. Let $\ell_1 \in \Gamma_{N_1}$, $\ell_2 \in \Gamma_{N_2}$. Then,*

$$\Gamma_{N_1}/N_{\Lambda_{N_1}} \cong \Gamma_{N_2}/N_{\Lambda_{N_2}} \cong \mathbb{Z}_{gcd(n_1-1,...,n_k-1)} \cong \mathbb{Z}_{gcd(m_1-1,...,m_j-1)},$$

and $[\ell_1] = [\ell_2] \in \mathbb{Z}_{gcd(n_1-1,...,n_k-1)}$.

The converse is unknown.

7.5 Further generalisations

7.5.1 Thompson-like groups acting on noncompact intervals

A modest generalization of our groupoid model allows us to have noncompact unit spaces–taking for example piecewise linear bijections on $\mathbb{R}$, or on $(0,+\infty)$ as in [8], [116].

Definition 7.5.1 (Noncompact $V(\Gamma,\Lambda,U)$)**.** *Let Λ be a multiplicative subgroup of $\mathbb{R} \cap (0,\infty)$, and Γ be a $\mathbb{Z}\cdot\Lambda$ submodule. Let U be some closed, not necessarily compact, interval in Γ (for example $(0,\ell]$, $(-\infty,0]$, $(0,\infty)$, $\mathbb{R}$). Then, let $V(\Gamma,\Lambda,U)$ denote the group of right continuous piecewise linear bijections of U with compact support, finitely many slopes (all in Λ) and finitely many nondifferentiable points, (all in Γ).*

Using the definition of the topological full group as in Chapter 3, we can realize these generalisations:

Lemma 7.5.2. *Let Λ be a multiplicative subgroup of $(0,+\infty)$. Let Γ be a $\mathbb{Z}\cdot\Lambda$ submodule. Let U be a closed interval of $\mathbb{R}$ with endpoints in Γ. Then, $V(\Gamma,\Lambda,U) \cong F(\Lambda \ltimes \Gamma \ltimes U)$. The derived subgroup, $D(V(\Gamma,\Lambda,U))$ is simple and vigorous.*

Such groups cannot be finitely generated because they act with full support on noncompact intervals; the underlying groupoids are not compactly generated.

Remark 7.5.3. *Suppose $\mathcal{G}$ is a minimal ample groupoid such that $\mathsf{A}(\mathcal{G})$ is finitely generated. Then $\mathcal{G}$ is compactly generated by Lemma 3.5.5. A consequence of this is that $\mathcal{G}^{(0)}$ is compact, since then if K was our compact generating set of our groupoid, K only moves a compact subset of $\mathcal{G}^{(0)}$, contradicting minimality. The same argument holds for any minimal ample groupoid.*

For this reason, the generalisations do not behave well algebraically with respect to the natural inclusions. However, these changes are stable under group homology, giving rise to our most general homological stability result, which appears as Corollary 1.1.16 in the introduction.

Corollary 7.5.4. *Let Γ, Λ be arbitrary. Let $U_1, U_2 \subset \mathbb{R}$ be closed intervals with endpoints in $\Gamma \cup \{-\infty, +\infty\}$. Then for all $*$, $H_*(V(\Gamma,\Lambda,U_1)) \cong H_*(V(\Gamma,\Lambda,U_2))$ and $H_*(D(V(\Gamma,\Lambda,U_1))) \cong H_*(D(V(\Gamma,\Lambda,U_2)))$*

Proof. Let U_1 be arbitrary let $\hat{U}_1 \subset \mathbb{R}_\Gamma$ be the unique cylinder set such that $q(\hat{U}_1) = U_1$. Then the canonical inclusion:

$$\Lambda \ltimes \Gamma \ltimes \hat{U}_1 \hookrightarrow \Lambda \ltimes \Gamma \ltimes \mathbb{R}_\Gamma$$

is inducing an isomorphism in groupoid homology. Hence by Theorem 3.6.5 this induces an isomorphism in group homology:

$$H_*(V(\Gamma,\Lambda,U_1)) \cong H_*(V(\Gamma,\Lambda,\mathbb{R}))$$

$$H_*(D(V(\Gamma,\Lambda,U_1))) \cong H_*(D(V(\Gamma,\Lambda,\mathbb{R})))$$

But since U_1 was arbitrary, we are done. □

We see how to apply this result in practice, by computing the homology of the analogy of Thompson's group V that acts on $\mathbb{R}$ rather than $(0,1]$.

Example 7.5.5. $V(\mathbb{Z}[1/2],\langle 2\rangle,\mathbb{R})$ *is integrally acyclic and simple.*

7.5.2 Brin-Stein groups

Another modest generalization of our groupoid model of Stein's groups gives rise to a broad class of groups including Brin's higher-dimensional Thompson groups. We call these groups Brin-Stein groups, being a cross between Brin's and Stein's generalisations of Thompson's group V

Definition 7.5.6 (Brin-Stein Groups)**.** *Let $n \in \mathbb{N}$. For $1 \le i \le n$, let Λ_i be a multiplicative subgroup of $\mathbb{R}$, Γ_i be a $\mathbb{Z}\cdot\Lambda_i$-submodule. Let $\ell \in \bigcap_{i=1}^n \Gamma_i$. Then let $V(n,\{\Lambda_i\}_{i=1}^n,\{\Gamma_i\}_{i=1}^n,\ell)$ denote the* *bijections $f:(0,\ell]^n \to (0,\ell]^n$ such that, for all $i=1,..,n$ and all $x=(x_1,..,x_{n-1})\in(0,\ell]^{n-1}$, if we consider the map:*

$$f_{i,x}:(0,\ell]\to(0,\ell] \quad t\mapsto f(x_1,..,x_{i-1},t,x_i,..,x_{n-1})_i$$

This map describes a right continuous piecewise linear bijection of $(0,\ell]$ with finitely many slopes, all in Λ_i and nondifferentiable points all in Γ_i.

This definition encompasses the classical Brin-Higman-Thompson groups.

Example 7.5.7 (Brin-Higman-Thompson Groups)**.** *Let $n,r\in\mathbb{N},k>2$. For all $1\le i\le n$, let $\Lambda_i=\langle k\rangle$, $\Gamma_i=\mathbb{Z}[1/k]$. Then,*

$$V(n,\{\langle k\rangle\}_{i=1}^n,\{\mathbb{Z}[1/k]\}_{i=1}^n,r)\cong nV_{k,r}$$

Where $nV_{k,r}$ are the regular Brin-Higman-Thompson groups, as in [21], [90]

We also describe interesting new examples in two main veins. The first is higher dimensional Thompson groups with irrational slopes:

Example 7.5.8 (nV_τ)**.** *For example, if we take τ to be the small golden ratio, $n\in\mathbb{N}$ then we define nV_τ to be $V(n,\{\langle\tau\rangle\}_{i=1}^n,\{\mathbb{Z}[\tau,\tau^{-1}]\}_{i=1}^n,1)$- a higher dimensional analogue of Cleary's group V_τ.*

The second class of examples is higher dimensional Thompson groups with mixed slopes:

Example 7.5.9 ($V_{3\times 2}$)**.** *Let $n=2$, $\Lambda_1=\langle 2\rangle,\Lambda_2=\langle 3\rangle$, $\Gamma_1=\mathbb{Z}[1/2],\Gamma_2=\mathbb{Z}[1/3]$. Then $V(2,\{\langle i+1\rangle\}_{i=1}^2,\{\mathbb{Z}[1/(i+1)]\}_{i=1}^2,1)$ describes a higher dimensional Thompson group whose slopes are in $\langle 2\rangle$ parallel to the x-axis and in $\langle 3\rangle$ parallel to the y-axis. Concrete examples in the above class remain mysterious. For example, we cannot determine if the group $V_{3\times 2}$ described above is isomorphic to $2V$.*

We realize these groups as topological full groups by considering Cartesian products of groupoids, inspired by Matui [90]:

Lemma 7.5.10 (Brin-Stein groups as Topological Full Groups)**.** *Let $n\in\mathbb{N}$. For $1\le i\le n$, let Λ_i be a multiplicative subgroup of $\mathbb{R}$, Γ_i be a $\mathbb{Z}\cdot\Lambda_i$-submodule. Let $\ell\in\bigcap_{i=1}^n\Gamma_i$. For each i, let $\mathcal{G}_i=\Lambda_i\ltimes\Gamma_i\ltimes(0_+,\ell_-]$ be the groupoid as in Lemma 7.2.2.*

Then let $V(n,\{\Lambda_i\}_{i=1}^n,\{\Gamma_i\}_{i=1}^n,\ell)$ be the higher dimensional Stein groups as in Definition 7.5.6. We have that:

$$V(n,\{\Lambda_i\}_{i=1}^n,\{\Gamma_i\}_{i=1}^n,\ell)\cong F(\mathcal{G}_1\times....\times\mathcal{G}_n)$$

Much of the study of these groups as topological full groups reduces to studying the component of groupoids $\mathcal{G}_i$ in the Cartesian product. This is because many groupoid properties are preserved by Cartesian products.

Lemma 7.5.11 (Groupoid Properties Preserved by Cartesian products)**.** *Let $\mathcal{G}_1, \mathcal{G}_2$ be effective, ample groupoids with $\mathcal{G}_1^{(0)}$ homeomorphic to the Cantor space.*

1. *If $\mathcal{G}_1, \mathcal{G}_2$ are minimal, $\mathcal{G}_1 \times \mathcal{G}_2$ is minimal.*

2. *If $\mathcal{G}_1$ is purely infinite, $\mathcal{G}_1 \times \mathcal{G}_2$ is purely infinite.*

3. *If $\mathcal{G}_1, \mathcal{G}_2$ are expansive, $\mathcal{G}_1 \times \mathcal{G}_2$ is expansive.*

Proof. Let us prove these properties one by one.

1. If $\mathcal{G}_1, \mathcal{G}_2$ are minimal, let $(x_1,x_2),(y_1,y_2) \in (\mathcal{G}_1 \times \mathcal{G}_2)^{(0)}$ be arbitrary. For $x_1, y_1 \in \mathcal{G}_1^{(0)}$, by minimality, there exists $g \in \mathcal{G}_1$ such that $s(g) = x_1, r(g) = y_1$. Simultaneously, for all $x_2, y_2 \in \mathcal{G}_2^{(0)}$, by minimality, there exists $h \in \mathcal{G}_2$ such that $s(h) = x_2, r(h) = y_2$. Therefore for all $(x_1,x_2),(y_1,y_2) \in (\mathcal{G}_1 \times \mathcal{G}_2)^{(0)}$, (g,h) satisfies $s(g,h) = (x_1,x_2)$, $r(g,h) = (y_1,y_2)$.

2. This is straightforward. Let $A_1 \times A_2$ be an arbitrary compact open subset of $(\mathcal{G}_1 \times \mathcal{G}_2)^{(0)}$. Then A_1 is a compact open subset of $\mathcal{G}_1^{(0)}$. Appplying the definition of pure infiniteness, we have there exists compact open bisections B, B' $\mathcal{G}_1$ such that $s(B) = r(B') = A_1$, $r(B) \cap r(B') = \varnothing$, $r(B) \sqcup r(B') \subset A$. Therefore for the compact open bisections $B \times A_2, B' \times A_2$ in $\mathcal{G}_1 \times \mathcal{G}_2$, it satisfies the definition of pure infiniteness.

3. To show this, one just considers the Cartesian product of the compact generating sets K_1, K_2 of the groupoids $\mathcal{G}_1, \mathcal{G}_2$, to obtain a compact generating set $K_1 \times K_2$ of $\mathcal{G}_1 \times \mathcal{G}_2$. Moreover, one can consider the (pairwise) Cartesian product of the expansive cover, and this will be an expansive cover for $\mathcal{G}_1 \times \mathcal{G}_2$.

□

It would be interesting to understand the other properties of topological full groups that are preserved under Cartesian products of the underlying groupoids.

We obtain via the Kunneth formula the following results for higher-dimensional analogues of Stein's groups:

Theorem 7.5.12. *Let $n \in \mathbb{N}$. For $1 \le i \le n$, let Λ_i be a multiplicative subgroup of $\mathbb{R}$, Γ_i be a $\mathbb{Z}\cdot\Lambda_i$-submodule. Let $\ell \in \bigcap_{i=1}^n \Gamma_i$. Then let $V(n, \{\Lambda_i\}_{i=1}^n, \{\Gamma_i\}_{i=1}^n, \ell)$ be the higher dimensional Stein groups as in Definition 7.5.6. The derived subgroups $D(V(n, \{\Lambda_i\}_{i=1}^n, \{\Gamma_i\}_{i=1}^n, \ell))$ are simple, vigorous. Moreover, if the groups $D(V(\Lambda_i, \Gamma_i, \ell))$ are finitely generated for $i = 1, \ldots, n$, then $D(V(n, \{\Lambda_i\}_{i=1}^n, \{\Gamma_i\}_{i=1}^n, \ell))$ is finitely generated.*

Proof. The first statement follows by observing that the groupoids forming the product are always minimal and purely infinite, by Lemma 7.2.4 and Lemma 7.2.7. Then by Lemma 7.5.11 the product of the groupoids is purely infinite minimal and so by Theorem 5.1.21, the topological full group is simple and vigorous. The final statement follows from Lemma 7.5.11 and Corollary 7.3.6, in conjunction with Theorem 3.5.2. □

Corollary 7.5.13. *Let $n \in \mathbb{N}$. For $1 \le i \le n$, let Λ_i be a multiplicative subgroup of $\mathbb{R}_+$, generated by finitely many numbers and Γ_i be a Λ_i-submodule. Then $D(V(n, \{\lambda_i\}_{i=1}^n, \{\Gamma_i\}_{i=1}^n, \ell))$ is a simple finitely generated vigorous group.*

Remark 7.5.14. *One can also "twist" these groupoids in an entirely analogous way to the topological full groups introduced by Belk-Zaremsky [7], seen in Example 4.4.14 and used subsequently in the resolution of the Boone-Higman conjecture for hyperbolic groups [5]. It is notable that these twisted Brin-Stein groups have interesting embeddings which do not obviously appear in the usual twisted Brin-Thompson groups, such as Juschenko-Monod groups, as demonstrated in Remark 7.1.3. The study of such groups could therefore be interesting in studying the Boone-Higman conjecture.*

Also via the Kunneth formula (Lemma 2.4.8), we can study the homology of these groups.

Corollary 7.5.15. *Let $n \in \mathbb{N}$. For $1 \le i \le n$, let Λ_i be a multiplicative subgroup of $\mathbb{R}$, Γ_i be a $\mathbb{Z} \cdot \Lambda_i$-submodule. Let $\ell \in \bigcap_{i=1}^n \Gamma_i$. Then let $V(n, \{\Lambda_i\}_{i=1}^n, \{\Gamma_i\}_{i=1}^n, \ell)$ be the higher dimensional Stein groups as in Definition 7.5.6.*

- *Suppose that for some i, the groupoid $\Lambda_i \ltimes \Gamma_i \ltimes (0_+, \ell_-]$ is acyclic.*

  Then, $V(n, \{\Lambda_i\}_{i=1}^n, \{\Gamma_i\}_{i=1}^n, \ell)$ is acyclic.

- *Suppose that for some i, the groupoid $\Lambda_i \ltimes \Gamma_i \ltimes (0_+, \ell_-]$ is rationally acyclic. Then, the group $V(n, \{\Lambda_i\}_{i=1}^n, \{\Gamma_i\}_{i=1}^n, \ell)$ is rationally acyclic.*

Proof. This proof is simple application of Lemma 2.4.8, combined with Corollary 3.6.3 and Corollary 3.6.4. □

Note that the above result confirms the result of Li that the Brin-Thompson groups nV are integrally acyclic, but also that mixed groups such as $V_{2\times 3}$ from Example 7.5.9 are integrally acyclic.

Corollary 7.5.16 (Embedding into simple finitely generated groups)**.** *Let $V(\Gamma, \Lambda, \ell)$ be finitely generated. Consider the 2 dimensional analogue that combines $V(\Gamma, \Lambda, \ell)$ with $V(\mathbb{Z}[1/2], \langle 2\rangle, \ell)$, i.e.*

$$V(2, \{\Gamma, \mathbb{Z}[1/2]\}, \{\Lambda, \langle 2\rangle\}, \ell)$$

This is a simple, finitely generated group which $V(\Gamma, \Lambda, \ell)$ embeds into.

Similar ideas can be used to show certain facts about the homology of the higher dimensional analogues of Stein's groups. For example, there are ways to describe the abelianization of these higher-dimensional analogues of Steins groups in terms of the constituent one-dimensional groups. Let us show by example how one can compute the abelianisation of one of these Brin-Stein groups:

Example 7.5.17 (nV_τ is perfect simple and finitely generated for $n>1$.)**.** *Following on from Example 7.5.8, let us first compute $(2V_\tau)_{ab}$. Recall that the groupoid homology of $\langle\tau\rangle\ltimes\mathbb{Z}[\tau,\tau^{-1}]\ltimes(0_+,1_+]$ has been computed previously:*

$$H_*(\langle\tau\rangle\ltimes\mathbb{Z}[\tau,\tau^{-1}]\ltimes(0_+,1_+]) = \begin{cases}\mathbb{Z}_2 & *=1\\ 0 & otherwise\end{cases}$$

Let $\mathcal{G}_1=\mathcal{G}_2=\langle\tau\rangle\ltimes\mathbb{Z}[\tau,\tau^{-1}]\ltimes(0_+,1_+]$. By the Kunneth formula for $*=0$, we have.

$$0\to H_0(\mathcal{G}_1)\otimes H_0(\mathcal{G}_2)\to H_0(\mathcal{G}_1\times\mathcal{G}_2)\to 0\to 0$$

So $H_0(\mathcal{G}_1\times\mathcal{G}_2)=0$. Similary, if we apply the Kunneth formula for $*=1$

$$0\to H_1(\mathcal{G}_1)\otimes H_0(\mathcal{G}_2)\oplus H_0(\mathcal{G}_1)\otimes H_1(\mathcal{G}_2)\to H_1(\mathcal{G}_1\times\mathcal{G}_2)\to Tor(H_0(\mathcal{G}_1),H_0(\mathcal{G}_2))\to 0$$

So $H_1(\mathcal{G}_1\times\mathcal{G}_2)=0$. Hence by Theorem 3.6.7, $(2V_\tau)_{ab}=0$. So assume that $H_0(\mathcal{G}_1^{\times n})=H_1(\mathcal{G}_1^{\times n})=0$ is true for $n\geq 2$. For $n+1$ we have by the Kunneth formula around $*$=0, that there is a short exact sequence:

$$0\to H_0(\mathcal{G}_1)\otimes H_0(\mathcal{G}_1^{\times n})\to H_0(\mathcal{G}_1^{\times n+1})\to 0$$

so that $H_0(\mathcal{G}_1^{\times n+1})=0$ by inductive hypothesis. Similarly, for $*=1$, and $n+1$ we have by the Kunneth formula:

$$0\to H_1(\mathcal{G}_1)\otimes H_0(\mathcal{G}_1^{\times n})\oplus H_0(\mathcal{G}_1)\otimes H_1(\mathcal{G}_1^{\times n})\to H_1(\mathcal{G}_1\times\mathcal{G}_1^{\times n})\to Tor(H_0(\mathcal{G}_1),H_0(\mathcal{G}_1^{\times n}))\to 0$$

Which reveals that $H_1(\mathcal{G}_1^{\times n+1})=0$ by inductive hypothesis. Therefore, due to the AH long exact sequence (Theorem 3.6.7) $((n+1)V_\tau)_{ab}=0$, concluding the proof by induction.

We claim this group is, in addition, simple and finitely generated. This follows as a consequence of Corollary 7.5.13 in combination with Theorem 7.3.12.

7.6 Outlook

Many things remain to be understood about Stein's groups. Notably, their classification remains open in general:

Question 7.6.1. *Let Λ_0,Λ_1 be nontrivial multiplicative subgroups of $\mathbb{R}_+$, Γ_i be $\mathbb{Z}\cdot\Lambda_i$ submodules for $i=1,2$ and $\ell_i\in\Gamma_i$ be arbitrary. When is $V(\Gamma_1,\Lambda_1,\ell_1)\cong V(\Gamma_2,\Lambda_2,\ell_2)$?*

A smaller goal would be to ask if the converse to Corollaries 7.4.18 and 7.4.26 holds, i.e. there are isomorphism when the (pointed) homology of the groupoids are isomorphic (as in the Higman Thompson case). An immediate counterexample would be to show that the integral groups $V(2,3) \ncong V(2,5)$ which is already unclear. One step towards understanding their classification would be to compute various invariants associated to the groupoids. We do not yet fully understand the groupoid homology:

Question 7.6.2. *Let* Λ *be a multiplicative subgroup of* $\mathbb{R}_+$*,* Γ *be a* $\mathbb{Z}\cdot\Lambda$*-submodule* $\ell \in \Gamma$ *be arbitrary. What is the homology of* $(\Lambda \ltimes \Gamma) \ltimes (0_+, \ell_-]$*?*

We also do not understand the Elliot invariant (K-theory) of the associated C*-algebras, which also may be thought of as an invariant of $V(\Gamma, \Lambda, \ell)$.

Question 7.6.3. *Let* Λ *be a multiplicative subgroup of* $\mathbb{R}_+$*,* Γ *be a* $\mathbb{Z}\cdot\Lambda$*-submodule* $\ell \in \Gamma$ *be arbitrary. What is the K-theory of* $\Lambda \ltimes \Gamma \ltimes C((0_+, \ell_-])$*?*

We also do not completely understand the higher finiteness properties of Stein's groups. We ask:

Question 7.6.4. *Let* Λ *be a multiplicative subgroup of* $\mathbb{R}_+$*,* Γ *be a* $\mathbb{Z}\cdot\Lambda$*-submodule* $\ell \in \Gamma$ *be arbitrary. Is* $D(V(\Gamma, \Lambda, \ell))$ *finitely presented?*

A final thing to notice is that, due to Li's perspective of the underlying groupoids, that they are full corners in the universal groupoids of cancellative semigroups, it would be interesting to make a systemic study into the topological full groups of such groupoids in general, since they are a natural class of groupoids generalising the groupoid model for Stein's groups.

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