A surgery approach to abelian quotients of the level 2 congruence group and the Torelli group

Tudur Lewis

A thesis submitted in fulfilment of the requirements
for the degree of
Doctor of Philosophy
at the
School of Mathematics and Statistics
College of Science and Engineering
University of Glasgow

VIA VERITAS VITA

July 2024

Abstract

We provide a unified framework for studying two families of maps: the Birman–Craggs maps of the Torelli group, and Sato's maps of the level 2 congruence subgroup of the mapping class group. Our framework gives new, elementary proofs that both families of maps are homomorphisms, gives a direct method for evaluating these maps on Dehn twists, and relates the two families when restricted to the Torelli group. Our methods involve 3-manifold techniques that do not depend on results in 4–manifold theory as in the original constructions, giving an answer to a question of Dennis Johnson. We also find a relation between an extension of the Birman–Craggs maps to the level 2 congruence subgroup, and Meyer's signature cocycle.

Contents

List of Figures

Acknowledgements

I thank Prof. Tara Brendle for the guidance and encouragement she has given me throughout my work on this thesis, as well as countless interesting dicussions. I also thank Dr. Brendan Owens, Dr. Vaibhav Gadre, and Prof. Dan Margalit for useful discussions, and for invaluable feedback given.

Author's declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

Background

Let $\Sigma_{g,n}$ denote an oriented surface of genus g with n boundary components. When $n = 0$, we abbreviate $\Sigma_{g,0}$ to Σ_g. These manifolds come up in many different contexts; for example, any degree d curve in the projective plane is a surface of genus $\frac{(d-1)(d-2)}{2}$, Seifert surfaces for links in S^3 produce invariants up to isotopy, and any 3–manifold is diffeomorphic to a Heegaard splitting, that is, two handlebodies glued along their common boundary surface.

Let $\mathrm{Mod}_{g,n} = \pi_0(\mathrm{Diff}^+(\Sigma_{g,n}, \partial\Sigma_{g,n}))$ denote the *mapping class group* of $\Sigma_{g,n}$, the group of orientation–preserving diffeomorphisms of $\Sigma_{g,n}$ fixing the boundary pointwise, modulo isotopy through maps of the same form. This is a natural group to study; gluing two three manifolds with boundary Σ_g along isotopic diffeomorphisms produces diffeomorphic 3–manifolds.

The natural action of $\mathrm{Diff}^+(\Sigma_{g,n}, \partial\Sigma_{g,n})$ on $H_1(\Sigma_{g,n}; \mathbb{Z})$ factors through the mapping class group to give a representation $\mathrm{Mod}_{g,n} \to \mathrm{Aut}(H_1(\Sigma_{g,n}; \mathbb{Z}), Q_{\Sigma_{g,n}})$, where $\mathrm{Aut}(H_1(\Sigma_{g,n}; \mathbb{Z}), Q_{\Sigma_{g,n}})$ denotes the orthogonal group of $H_1(\Sigma_{g,n}; \mathbb{Z})$ equipped with the intersection form; see [FM11, Ch. 6] for more details. We focus on the case of an oriented surface with one boundary component.

Let $\mathcal{I}_{g,1}$ denote the *Torelli group*, that is, the kernel of the action of $\mathrm{Mod}_{g,1}$ on $H_1(\Sigma_{g,1}; \mathbb{Z})$. The Torelli group arises in algebraic geometry as the fundamental group of Torelli space: the moduli space whose points consist of pairs (C, B), where C is a Riemann surface of genus g with one boundary component, and B is a symplectic basis of $H_1(C; \mathbb{Z})$; the abelianization of $\mathcal{I}_{g,1}$ is the first homology of this moduli space, and was calculated by Johnson in a series of papers. Johnson's work is of interest in the theory of 3– and 4–manifolds. For example, Morita showed that the tools developed by Johnson have deep implications for the topology of homology 3–spheres [Mor89, Proposition 2.3]. More recently, Lambert–Cole used Johnson's tools to obstruct intersection forms of smooth 4–manifolds, via trisections [Lam20].

Johnson found that all torsion in the abelianization of $\mathcal{I}_{g,1}$ is characterised by the *Birman–Craggs homomorphisms*: a family of maps $\mu_h : \mathcal{I}_{g,1} \to \mathbb{Z}/2$, indexed by a Heegaard embedding $h : \Sigma_g \to S^3$. These maps are constructed by cutting S^3 along the Heegaard surface $h(\Sigma_g)$, regluing the two handlebodies via an element of $\mathcal{I}_{g,1}$ to get a homology sphere, and then taking the Rochlin invariant of its unique spin structure [Joh85b],[BC78], [Joh80b]; see Chapter 5 and Appendix 8.2.5 for more details.

Let M be an oriented Riemannian manifold, and let $P(M)$ denote the oriented orthonormal frame bundle of M. A *spin structure* on M is an element $\xi \in H^1(P(M); \mathbb{Z}/2) = \mathrm{Hom}(H_1(P(M)), \mathbb{Z}/2)$ that evaluates to 1 on any homotopically non–trivial loop in a fibre of $P(M)$. The *Rochlin invariant* of a spin 3-manifold M is defined as the signature modulo 16 of any spin 4-manifold spin bounding

M; see Chapter 2 and Appendix 8.3. Let $\mathrm{Spin}(M)$ denote the set of spin structures on M; there is a simply–transitive action of $H^1(M;\mathbb{Z}/2)$ on $\mathrm{Spin}(M)$ that gives a bijection of $\mathrm{Spin}(M)$ with $H^1(M;\mathbb{Z}/2)$, hence homology spheres have a unique spin structure.

Let $\mathrm{Mod}_{g,1}[L]$ denote the *level L congruence subgroup* of the mapping class group, that is, the kernel of the action of $\mathrm{Mod}_{g,1}$ on $H_1(\Sigma_{g,1};\mathbb{Z}/L)$, note that $\mathcal{I}_{g,1} < \mathrm{Mod}_{g,1}[L]$; these subgroups are defined in a similar way to the Torelli group, but they produce a family of finite index subgroups of $\mathrm{Mod}_{g,1}$. The subgroup $\mathrm{Mod}_{g,1}[L]$ arises in algebraic geometry as the orbifold fundamental group of the moduli space of genus g Riemann surfaces with one boundary component and a level L structure. As a natural generalisation of Johnson's work, Farb posed the fundamental question of computing the abelianizations of these subgroups [Far06, Problem 5.23].

Our focus is on the *level* 2 *subgroup* $\mathrm{Mod}_{g,1}[2]$; this case often requires separate techniques. The level 2 subgroup is of particular interest in the theory of 3-manifolds due to its connections with rational homology spheres [PR21, Corollary 1.1]. To compute the abelianization of $\mathrm{Mod}_{g,1}[2]$, Sato defines a family of maps to abelian groups, using a construction that is similar to the Birman–Craggs maps [Sat10, Part II]; the natural extension of the Birman–Craggs maps to the subgroup $\mathrm{Mod}_{g,1}[2]$ is no longer a homomorphism [BC78, p. 284]. Sato defines an analogous construction using mapping tori instead. Mapping tori have many spin structures, so Sato constructs a spin structure on the mapping torus M_f of $[f] \in \mathrm{Mod}_{g,1}[2]$ that depends on a fixed spin structure $\sigma \in \mathrm{Spin}(\Sigma_g)$ for the fiber. Sato's maps $\beta_{\sigma,x} : \mathrm{Mod}_{g,1}[2] \to \mathbb{Z}/8$ are given by taking the Rochlin invariant of the spin mapping tori obtained from his construction; see Chapter 2 for more details.

Sato shows that his maps $\beta_{\sigma,x}$ give homomorphisms by using deep results on the signature of 4–manifolds, such as Rochlin's theorem and Novikov additivity. He then took direct summands of a certain subfamily of the $\beta_{\sigma,x}$, and found that those maps alone were enough to compute the abelianization of $\mathrm{Mod}_{g,1}[2]$ [Sat10, Lemma 2.2, Propositions 5.2 and 7.1]. His computation has applications in algebraic geometry. For example, Putman built on Sato's work and computed the Picard groups of moduli spaces of curves with level structures in many cases; see his paper [Put12] for a more algebraic approach to computing abelianizations of congruence subgroups.

Outline and main results

In Chapter 2 we give an overview of spin structures and Sato's definition of his maps $\beta_{\sigma,x}$. The main steps in our reconstruction of both maps are contained in Chapters 3, 4, and 5.

In Chapter 3 we use framed links in S^3, tangle diagrams, and ribbon graphs to describe an algorithm that gives framed link diagrams of mapping tori and Heegaard splittings. This uses the 3–manifold constructions in [RT91, Section 4], and generalises the constructions in [KM94, Appendix] to higher genus. For all the framed links obtained from our algorithm, the fiber surfaces for the mapping tori (respectively Heegaard surfaces for the Heegaard splittings) lie in these surgery diagrams as the standard embedding of a surface into S^3.

In Chapter 4 we give a new definition of Sato's maps $\theta : \mathrm{Spin}(\Sigma_g) \to \mathrm{Spin}(M_f)$, where $[f] \in \mathrm{Mod}_{g,1}[2]$, and M_f denotes the mapping torus of the map $f : \Sigma_g \to \Sigma_g$. Here, we fix a disk $D \subset \Sigma_g$, and think of representatives of elements in $\mathrm{Mod}_{g,1}[2]$ as diffeomorphisms of Σ_g fixing D pointwise, so that the mapping torus M_f is a closed manifold. We begin with a framed link L representing $S^1 \times \Sigma_g$. Using Construction 3.0.1, we obtain a framed link L_f, containing L, that represents M_f. Let M_{L_f} denote the mapping torus obtained by Dehn surgery along L_f in S^3. Then M_{L_f} has a fixed embedding $\Sigma_g \hookrightarrow M_{L_f}$ representing a fiber surface, and a fixed embedding $S^1 \times D \hookrightarrow M_{L_f}$. For a spin structure σ on Σ_g, we use Construction 4.0.5 to obtain a spin structure $\theta_{L_f}(\sigma)$ on M_{L_f}. This spin structure is characterised by the fact that it restricts to σ on the fiber surface, and restricts to a fixed spin structure on the embedding of $S^1 \times D$. This spin structure on M_{L_f} gives an obstruction class $\omega_2(W_{L_f}, M_{L_f})$ (see [KM91, Lemma C.1]) that corresponds to a characteristic sublink C of L_f, characterised by the condition $C \cdot L_i = L_i \cdot L_i \pmod 2$ for all components L_i of L_f.

We give a new definition of Sato's maps $\beta_{\sigma,x}$ in terms of the characteristic sublinks obtained from our map θ_{L_f}, using a combinatorial formula for the Rochlin invariant in [KM91, Appendix C.3]. Our definition involves the Arf invariant; the Arf invariant is a $\mathbb{Z}/2$–valued invariant of knots in S^3, which can be extended to an invariant of proper links in S^3, see Chapter 8.4 for more details. The main result is then the following:

Main Theorem. *For $g \geq 1$ let $\sigma \in \mathrm{Spin}(\Sigma_g)$, and let $x \in H^1(\Sigma_g; \mathbb{Z}/2)$. Then Sato's maps $\beta_{\sigma,x} : \mathrm{Mod}_{g,1}[2] \to \mathbb{Z}/8$ can be evaluated as:*

$$\beta_{\sigma,x}(f) = (\theta_{L_f}(\sigma + x) \cdot \theta_{L_f}(\sigma + x) - \theta_{L_f}(\sigma) \cdot \theta_{L_f}(\sigma) + 8(\mathrm{Arf}(\theta_{L_f}(\sigma)) - \mathrm{Arf}(\theta_{L_f}(\sigma + x))))/2 \pmod 8.$$

Here $\theta_{L_f}(\sigma) \cdot \theta_{L_f}(\sigma)$ and $\mathrm{Arf}(\theta_{L_f}(\sigma))$ denote the total linking number and the Arf invariant of the characteristic sublink specified by $\theta_{L_f}(\sigma)$ in Construction 4.0.5, and we use the affine action of $H^1(\Sigma_g; \mathbb{Z}/2)$ on the set of spin structures.

Using our main result, we give a new proof that Sato's maps are homomorphisms, and we obtain a direct mechanism for evaluating Sato's maps on any product of Dehn twists in $\mathrm{Mod}_{g,1}[2]$; Sato obtained a formula for evaluating his maps on squares of Dehn twists, but it is often difficult to evaluate Sato's maps on other mapping classes using this formula, as it requires the mapping class be factored as a product of squares of Dehn twists. These factorizations can be very intricate, see Chapter 7.0.1 for an example.

Applications to the Birman–Craggs maps and the Torelli group

In Chapter 5 we use the methods of Chapter 3 to give certain framed link presentations of Heegaard splittings of homology spheres; see Construction 5.0.1. We use the combinatorial formula for the Rochlin invariant in [KM91, Appendix C.3] to give a framework for studying the Birman–Craggs maps. It is remarkable that the Birman–Craggs maps are homomorphisms [BC78, Theorem 8]. We give a new proof that the Birman–Craggs maps are homomorphisms in Theorem 5.0.3. The idea of the proof is that gluing along a composition of diffeomorphisms translates to concatenation of tangle diagrams for framed links representing the Heegaard splitting constructions. We then conclude Chapter 5 by calculating Sato's maps on bounding pairs and separating twists in Corollaries 2 and 3. This relates Sato's maps to the Birman–Craggs maps using direct methods. We get the following:

Main Corollary. *Let a, b be a pair of simple closed curves on $\Sigma_{g,1}$ that bound a subsurface. Let η be the spin structure on Σ_g with the characteristic sublink of $\theta_{L_{t_a t_b^{-1}}}(\eta)$ containing none of the components from the link L that represents $S^1 \times \Sigma_g$ (see Construction 4.0.5 and Chapter 5.0.3). If we have that $\sigma = f^*(\eta)$ and $\sigma + x = h^*(\eta)$ for $[f], [h] \in \mathrm{Mod}_{g,1}$, then*

$$\beta_{\sigma,x}(t_a t_b^{-1}) = \mu_\iota(t_{f(a)} t_{f(b)}^{-1}) - \mu_\iota(t_{h(a)} t_{h(b)}^{-1}) \pmod 2,$$

where μ_ι denotes the Birman–Craggs map for the standard embedding $\iota : \Sigma_g \hookrightarrow S^3$. In particular, we have $\beta_{\sigma,x} = \mu_{\iota \circ f} - \mu_{\iota \circ h} \pmod 2$.

In his survey on the Torelli group, Johnson asks if there is a definition of the Birman–Craggs maps that does not involve the implicit construction of a 4-manifold [Joh83a, p.177]. Our definition uses Construction 5.0.1, and a formula computed from framed link diagrams of a 3–manifold [KM91, Appendix C.3 and C.4]. To prove that this formula is well-defined, the fundamental theorem of Kirby calculus is used, and there is no dependence on Rochlin's theorem (see remark under [KM91, Corollary C.5]). Furthermore, there are proofs of the fundamental theorem of Kirby calculus that just use a presentation of the mapping class group [Lu92], [MP94]. We have given a definition of the Birman-Craggs maps that uses 3–manifold topology and knot theory, which removes the logical dependence on Rochlin's theorem. One question, then, is whether it is possible to push the 3–manifold techniques down a dimension and give an inherently 2–dimensional description of the Birman-Craggs maps. This would give a group theoretic description of the Rochlin invariant, as Johnson pointed out.

The Birman–Craggs maps and Meyer's signature cocycle

There is a natural extension of $\mu_h : \mathcal{I}_{g,1} \to \mathbb{Z}/2$ to a map $\mu_h : \mathrm{Mod}_{g,1}[2] \to \mathbb{Z}/16$; if we cut S^3 along the Heegaard surface $h(\Sigma_g)$, and reglue via an element $[f] \in \mathrm{Mod}_{g,1}[2]$, we get a $\mathbb{Z}/2$–homology sphere $S(f)$, so we can take the Rochlin invariant of its unique spin structure. This map is no longer a homomorphism, but our methods imply that the failure of these extensions from being homomorphisms is measured by Meyer's signature cocycle, restricted to $\mathrm{Mod}_{g,1}[2]$; Meyer's signature cocycle $\tau_g : \mathrm{Mod}_{g,1} \times \mathrm{Mod}_{g,1} \to \mathbb{Z}$ computes the signature of surface bundles with prescribed monodromy [Mey73]. More precisely, in Chapter 6, we show the following:

Corollary 4. For the standard embedding $\iota : \Sigma_g \to S^3$, there exists a well–defined map

$$\begin{aligned} \alpha_\iota : \mathrm{Mod}_{g,1}[2] &\to \mathbb{Z}/16 \\ f &\mapsto \mathrm{Sign}(Y_f), \end{aligned}$$

where Y_f is a cobordism between $S(f)$ and M_f, defined in Chapter 6. Then we have $\tau_g \equiv \partial(\alpha_\iota + \mu_\iota)$ (mod 16), where τ_g is Meyer's signature cocycle, and μ_ι is the extension of the Birman–Craggs map described above.

Chapter 6 also contains a formula for the extension $\mu_\iota : \mathrm{Mod}_{g,1}[2] \to \mathbb{Z}/16$ in terms of squares of Dehn twists; see Theorem 6.0.3.

Abelianization of the level 2 congruence group

In Chapter 7 we give an alternative calculation of the image of Sato's homomorphisms when viewed as maps $\beta_\sigma : \mathrm{Mod}_{g,1}[2] \to \mathrm{Map}(H^1(\Sigma_g), \mathbb{Z}/8)$. This approach gives a slightly different description of the abelianization $H_1(\mathrm{Mod}_{g,1}[2]; \mathbb{Z})$. Our key observation here is that we can equip $\mathrm{Map}(H^1(\Sigma_g), \mathbb{Z}/8)$ with its standard algebra structure and repeatedly apply a single relation that holds in the algebra (see Lemma 7.0.2). In Chapter 7.0.1, we use the above observation to analyse the images of the maps β_σ, giving a description of a family of abelian quotients of the Torelli group and the Johnson kernel.

Relations between Rochlin invariants

Our computations in Corollaries 2 and 3 (see Sections 5.0.2 and 5.0.3) of Sato's maps on elements of the Torelli group, give relations between Rochlin invariants of mapping tori and Rochlin invariants of homology spheres. Hence we can ask:

Question 1: Is there a sensible way to enumerate these relations?

Question 2: How do these relations depend on the initial choice of Heegaard embedding in S^3?

Question 3: Is there a formula for evaluating the map $\alpha_\iota : \mathrm{Mod}_{g,1}[2] \to \mathbb{Z}/16$ of Corollary 4?

Chapter 2

Overview of Sato's homomorphisms

In this chapter, we review definitions of spin structures on manifolds, and Sato's construction of his homomorphisms.

We fix an embedded disc $D \subset \Sigma_g$, and think of $\mathrm{Mod}_{g,1}$ as the group of orientation–preserving diffeomorphisms fixing D pointwise, modulo isotopies through maps of the same form. We assume that all homology groups are taken with $\mathbb{Z}/2$ coefficients unless specified otherwise, and use the same notation for a continuous map as its induced homomorphism on homology ($f = f_*$). Sato's idea is to take the mapping torus $M_f = I \times \Sigma_g/(1, x) \sim (0, f(x))$ for $[f] \in \mathrm{Mod}_{g,1}$ and analyze the spin structures on M_f induced by a given spin structure on Σ_g. We begin by recalling definitions of spin structures.

Spin structures on manifolds

Let $\pi : E \to V$ be a smooth oriented real vector bundle of rank $n \geq 2$ equipped with a metric and denote by $SO(n) \xrightarrow{i} P(E) \xrightarrow{p} V$ the oriented orthonormal frame bundle associated to this bundle. When the second Stiefel-Whitney class $\omega_2(E)$ vanishes, we have the short exact sequence

$$0 \to H_1(SO(n)) \xrightarrow{i} H_1(P(E)) \xrightarrow{p} H_1(V) \to 0. \tag{2.1}$$

A *spin structure* τ on E is a homomorphism $\tau : H_1(V) \to H_1(P(E))$ such that $p \circ \tau = \mathrm{id}_{H_1(V)}$. We denote by $\mathrm{Spin}(E)$ the set of spin structures on E.

By the splitting lemma, the existence of τ as above is equivalent to the existence of a homomorphism $\tau' : H_1(P(E)) \to H_1(\mathrm{SO}(n))$ such that $\tau' \circ i = \mathrm{id}_{H_1(\mathrm{SO}(n))}$. So we can equivalently think of a spin structure as a cohomology class $\tau \in H^1(P(E)) = \mathrm{Hom}(H_1(P(E), \mathbb{Z}), \mathbb{Z}/2)$. This class τ can be evaluated on framed curves in V, and the condition $\tau \circ i = \mathrm{id}$ implies that τ evaluates to one on a trivial loop in V with zero framing.

After identifying $H_1(SO(n))$ with $\mathbb{Z}/2$, there is a simply transitive action of $H^1(V) = \mathrm{Hom}(H_1(V), \mathbb{Z}/2)$ on $\mathrm{Spin}(E)$ given by taking a homomorphism $c : H_1(V) \to \mathbb{Z}/2$ and $\tau \in \mathrm{Spin}(E)$ and constructing another right splitting $\tau + i \circ c$. It follows that the number of spin structures for the oriented frame bundle $P(E) \xrightarrow{p} V$ is given by $|H^1(V)|$. We refer to a smooth manifold M as spin if there exists a spin structure on the tangent bundle TM. Denote by $\mathrm{Spin}(M)$ the set of all spin structures on the tangent bundle TM of M, whenever M is a spin manifold.

Note that $\mathrm{Diff}^+(M)$ acts on $\mathrm{Spin}(M)$ via pullback: for a diffeomorphism $g \in \mathrm{Diff}^+(M)$ and spin structure $\sigma : H_1(M) \to H_1(P(TM))$ given as above, we get the spin structure $g^*(\sigma) := dg^{-1} \circ \sigma \circ g$. See Appendix 8.3 for more details on spin structures.

2.0.1 Sato's construction.

To define the homomorphisms $\beta_{\sigma,x}$, we must define a map $\theta : \mathrm{Spin}(\Sigma_g) \to \mathrm{Spin}(M_f)$ for every given $[f] \in \mathrm{Mod}_{g,1}[2]$. We use the homotopy long exact sequence for the fibration $\Sigma_g \to M_f \to S^1$ and the fact that the abelianization functor is a right exact functor as well as a natural transformation between π_1 and H_1. If we combine this with the Wang exact sequence (see [Hat02, Example 2.48]), we get that the following sequence is exact:

$$0 \to H_1(\Sigma_g) \to H_1(M_f) \to H_1(S^1) \to 0, \tag{2.2}$$

where the homomorphisms are induced by the inclusion and projection to S^1 respectively. Since f fixes $D \subset \Sigma_g$ pointwise we have an embedding $l : S^1 \times D \to M_f$, giving a right splitting of the short exact sequence (2.2). This is equivalent to an isomorphism $H_1(M_f) \xrightarrow{h} H_1(S^1) \bigoplus H_1(\Sigma_g)$. We will construct a right splitting of the sequence (2.1) for $V = M_f$ using this splitting isomorphism h.

Choose $p \in \mathrm{int}(D)$ and an arbitrary orthonormal frame $\{b_0, b_1\}$ for $T_p(D)$. Pick a non-zero tangent vector field v of TS^1 and denote by $v_t \in T_tS^1$ the value of v at $t \in S^1$. For $S^1 \times D \subset M_f$ define the framing $\hat{l} : S^1 \to P(S^1 \times D)$ by

$$\hat{l}(t) = (v_t, b_0 \cos(2\pi t) + b_1 \sin(2\pi t), b_1 \cos(2\pi t) - b_0 \sin(2\pi t)).$$

This frames the curve $S^1 \times \{p\}$ with a tangent vector field to the curve, and two transverse vector fields that rotate a total of 2π in one traverse of the curve. This framing induces the homomorphism

$$\hat{l} : H_1(S^1) \xrightarrow{\hat{l}} H_1(P(S^1 \times D)) \xrightarrow{\text{inc}} H_1(P(M_f)), \tag{2.3}$$

where the last map is induced by the differential of the inclusion l.

For the Σ_g factor, consider the smooth map $P((-\epsilon, \epsilon) \times \Sigma_g) \to P(M_f)$ induced by the inclusion of a tubular neighbourhood $(-\epsilon, \epsilon) \times \Sigma_g$ of the fiber into M_f for small $\epsilon > 0$. Think of a spin structure σ of Σ_g as a right splitting of the sequence (2.1) with $V = (-\epsilon, \epsilon) \times \Sigma_g$, and let $\overline{\sigma} : H_1(\Sigma_g) \to H_1(P(M_f))$ denote the following composition

$$\overline{\sigma} : H_1(\Sigma_g) \xrightarrow{\text{inc}} H_1((-\epsilon, \epsilon) \times \Sigma_g) \xrightarrow{\sigma} H_1(P((-\epsilon, \epsilon) \times \Sigma_g)) \xrightarrow{\text{inc}} H_1(P(M_f)). \tag{2.4}$$

Then we can construct the homomorphism $H_1(M_f) \to H_1(P(M_f))$ by combining (2.3) and (2.4), obtaining a map $\theta : \mathrm{Spin}(\Sigma_g) \to \mathrm{Spin}(M_f)$. In summary, the map θ inputs a spin structure σ of $(-\epsilon, \epsilon) \times \Sigma_g$, and outputs the right splitting $(\hat{l} \oplus \overline{\sigma}) \circ h$ as in the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{i} & H_1(P(M_f)) & \xrightarrow{p} & H_1(M_f) & \longrightarrow & 0 \\
 & & & & & \nwarrow^{\hat{l} \oplus \overline{\sigma}} & \downarrow h & & \\
 & & & & & & H_1(S^1) \oplus H_1(\Sigma_g) & &
\end{array}$$

See Appendix 8.3.1 for more details on the spin mapping tori obtained this way.

Now we describe the homomorphisms $\beta_{\sigma,x} : \mathrm{Mod}_{g,1}[2] \to \mathbb{Z}/8$ for $\sigma \in \mathrm{Spin}(\Sigma_g), x \in H^1(\Sigma_g)$. Rochlin's Theorem states that every spin 3-manifold bounds a spin 4-manifold. Fix a spin structure τ on M_f and choose a compact spin 4-manifold V spin bounding (M_f, τ) and define the Rochlin invariant

$$R(M_f, \tau) = \mathrm{Sign}(V) \pmod{16},$$

where $\mathrm{Sign}(V)$ is the signature of the intersection form of V. This is well-defined by Novikov additivity, and Rochlin's result that a closed spin 4-manifold has signature divisible by 16; see Appendix 8.3 for more information.

Let $\sigma \in \mathrm{Spin}(\Sigma_g)$ and $x \in H^1(\Sigma_g)$, define the map $\beta_{\sigma,x} : \mathrm{Mod}_{g,1}[2] \to \mathbb{Z}/8$ to be

$$\beta_{\sigma,x}([f]) = (R(M_f, \theta(\sigma)) - R(M_f, \theta(\sigma + x)))/2 \pmod 8.$$

Sato showed that these maps are homomorphisms and that they have image in $\mathbb{Z}/8$ [Sat10, Lemmas 2.2 and 4.3]. He then examined the Brown invariant of a Pin^- bordism class represented by a surface embedded in M_f to arrive at a formula for the homomorphisms $\beta_{\sigma,x}$ on squares of Dehn twists [Sat10, Prop. 5.2] (see Appendix 8.2.3 for the definition of a Dehn twist). To describe the formula we need the following.

Spin structures and quadratic forms

A *symplectic quadratic form* is a map $q : H_1(\Sigma_g) \to \mathbb{Z}/2$ that satisfies $q(x + y) = q(x) + q(y) + x \cdot y$ for all $x, y \in H_1(\Sigma_g)$, where $x \cdot y$ denotes the pairing given by the intersection form.

Theorem 2.0.1. *[Joh80c, Theorems 3A, 3B] There is a bijection $\sigma \mapsto q_\sigma$ between spin structures $\sigma \in \mathrm{Spin}(\Sigma_g)$ and symplectic quadratic forms $q_\sigma : H_1(\Sigma_g) \to \mathbb{Z}/2$. Let $H^1(\Sigma_g) = \mathrm{Hom}(H_1(\Sigma_g), \mathbb{Z}/2)$ act affinely on $\mathrm{Spin}(\Sigma_g)$ as above, let $f \in \mathrm{Diff}^+(\Sigma_g)$, and let $x \in H^1(\Sigma_g)$. Then $q_{\sigma+x} = q_\sigma + x$, and $q_{f^*(\sigma)} = f^* q_\sigma$.*

Proof. We only sketch the bijection here: let $\sigma \in \mathrm{Spin}(\Sigma_g)$ and let $x \in H_1(\Sigma_g)$. Choose a simple closed curve $\alpha \subset \Sigma_g$ representing x. Let $N(\alpha)$ denote the normal bundle of α in Σ_g. Pick a unit tangent vector field $s : \alpha \to T(\alpha)$ and a nonzero section $X : \alpha \to N(\alpha)$. Viewing $\sigma \in \mathrm{Spin}(\Sigma_g)$ as a left splitting of the short exact sequence (2.1) gives us a homomorphism $k_\sigma : H_1(P(\Sigma_g)) \to \mathbb{Z}/2$. Since $T(\Sigma_g)|_\alpha = N(\alpha) \bigoplus T(\alpha)$, we can define the associated quadratic form q_σ to be

$$q_\sigma(x) = k_\sigma(X \oplus s) + 1.$$

□

Symplectic quadratic forms are determined by their values on a symplectic basis for $H_1(\Sigma_g)$, so we can specify an arbitrary spin structure by choosing the values of q_σ on a fixed symplectic basis.

We need the following function to state Sato's formula for $\beta_{\sigma,x}(t_C^2)$. For a homology class $z \in H_1(\Sigma_{g,r})$, define the map $i_z : H_1(\Sigma_{g,r}) \to \mathbb{Z}/8$ by

$$i_z(y) = \begin{cases} 1, & z \cdot y = 1 \pmod 2 \\ 0, & z \cdot y = 0 \pmod 2 \end{cases}$$

where $\cdot$ denotes the intersection form on $H_1(\Sigma_{g,r})$.

Proposition 2.0.2. *[Sat10, Proposition 5.2] For a non-separating simple closed curve $C \subset \Sigma_g \setminus D$, we have*

$$\beta_{\sigma,x}(t_C^2) = (-1)^{q_\sigma(C)} i_{[C]}(PD(x)),$$

where $q_\sigma : H_1(\Sigma_g) \to \mathbb{Z}/2$ is the quadratic form associated to $\sigma \in \mathrm{Spin}(\Sigma_g)$ and $PD(x)$ is the Poincare dual of $x \in H^1(\Sigma_g)$.

Since $\mathrm{Mod}_{g,1}[2]$ is generated by squares of Dehn twists about non separating simple closed curves this formula is enough to calculate the abelianization of $\mathrm{Mod}_{g,1}[2]$ (see [Hum92, Proposition 2.1]). We give an alternative description of $\beta_{\sigma,x}$ that allows us to directly evaluate these maps. To find this formula we need to write mapping tori as surgery diagrams.

Chapter 3

Surgery diagrams and ribbon graphs

In this chapter, we describe an algorithm that finds a surgery diagram for $M_f, [f] \in \mathrm{Mod}_{g,1}$, where $[f]$ is a product of Dehn twists. We think of f as an orientation–preserving diffeomorphism of the closed surface Σ_g, fixing an embedded disk D pointwise, hence M_f is a closed 3–manifold with a fixed embedding of $S^1 \times D$.

To find surgery diagrams, we define ribbon graphs of Heegaard splittings, and give a procedure that goes from a ribbon graph to a surgery diagram of M_f. For more information about the 3–manifold constructions used here, see [KM94, Appendix], [RT91, Section 4] and [Wri94, Section 2.2]. The terminology in the next paragraph is consistent with [GS99].

An *n-dimensional k-handle* attached to a smooth manifold M will be a copy of $D^k \times D^{n-k}$ attached to ∂M via an embedding $(\partial D^k) \times D^{n-k} \to \partial M$. For a handlebody decomposition of a smooth 4-manifold M, we assume there is one 0-handle, and that any 4-dimensional 1-handles are attached to the boundary of this 0-handle; these 1-handles can be pictured as two copies of D^3 in S^3 identified to each other via a reflection. Any 4-dimensional 2-handles $D^2 \times D^2$ attached to the manifold can be specified by drawing the attaching circle $(\partial D^2) \times \{0\}$, along with a framing of its normal bundle in ∂M. There is a bijection between these framings and the integers, explained below. A collection of framed links in S^3 with embedded pairs of D^3 in S^3 is called a *Kirby diagram* of M.

If M has only 4-dimensional 2-handles attached to a 0-handle, we call M a 2-handlebody. Every 2-handle $D^2 \times D^2$ is attached along an embedding $\eta : (\partial D^2) \times D^2 \to S^3$; the $(\partial D^2 \times D^2)$ part of the boundary of $D^2 \times D^2$ is then in the interior of the new manifold, and the $D^2 \times (\partial D^2)$ factor changes the boundary 3-manifold. On the boundary it is equivalent to removing a tubular neighbourhood of the attaching circle and gluing in a solid torus $D^2 \times (\partial D^2)$ by sending the meridian curves $\partial D^2 \times \{pt\}$ to their images under the embedding η. This is referred to as Dehn surgery, and the corresponding Kirby diagram for M is also a surgery diagram for ∂M. The Dehn-Lickorish theorem states that any closed orientable 3-manifold is given by such a surgery diagram [Lic62].

3.0.1 Framings of 2-handles

Suppose we have an embedding $\widetilde{\varphi} : \partial D^2 \to \partial M$ with trivial normal bundle in ∂M. Pick an orthonormal frame $\{s_1, s_2\}$, which gives a global trivialization of this normal bundle, and pick a tubular neighbourhood $N : \nu(\partial D^2) \to \partial M$, where $\nu(\partial D^2)$ is the disc bundle associated to the normal bundle. We can construct a gluing map $\varphi : \partial(D^2) \times D^2 \to \partial M$ for the 2-handle by setting $\varphi(x, a, b) = N(as_1(\widetilde{\varphi}(x)) + bs_2(\widetilde{\varphi}(x)))$. The meridians $\partial(D^2) \times \{pt\}$ are glued to pushoffs of the attaching circle $\widetilde{\varphi}$ along these frames. Note that the *core* $\{0\} \times \partial D^2$ of this added solid torus is sent to a meridian of the attaching circle $\widetilde{\varphi}$ in the Kirby diagram.

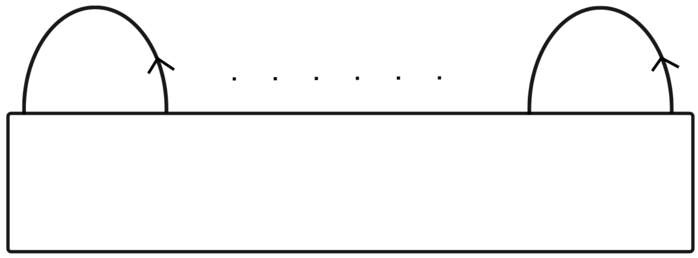

Figure 3.1: The ribbon graph Δ_g, where all tangles are given the 0-framing

Suppose 4-dimensional 2-handles are attached to $\partial D^4 = S^3$; there is a bijection between framings of a 4-dimensional 2-handle $D^2 \times D^2$ and $\pi_1(SO(2)) = \mathbb{Z}$, but this correspondence requires a choice of an arbitrary framing. We define the 0-framing to be the non-zero transverse vector field to the attaching circle $\varphi(\partial D^2 \times \{0\}) = K$ in S^3 induced from the collar of any Seifert surface for K. Then for the pushoff K' of K in the direction of the 0-framing we have that $lk(K, K') = 0$, and the bijection between framings and $\mathbb{Z}$ is realised by linking numbers of pushoffs. The vector field corresponding to $k \in \mathbb{Z}$ is given by a vector field which deviates from the 0-framing by k full twists (right handed twist is $+1$). Note that this framing integer is independent of the orientation chosen for the attaching circle K, since reversing K also reverses the pushoff K' in the direction of the vector field.

3.0.2 Surgery diagrams for Heegaard splittings

We use *tangle diagram* to aid in the constructions below. A tangle is an embedding of an oriented 1-manifold in the unit cube I^3 such that its boundary is contained in $1/2 \times I \times \{0, 1\}$, up to isotopy keeping the endpoints fixed. A tangle is equipped with a framing of its normal bundle that is standard on the boundary, and is specified as above, via the correspondence given by linking number. We allow *coupons* in our tangle diagrams, where a coupon is an embedding of I^2 in I^3. We allow tangles to be connected to the boundary of this coupon, and assume that the framing of the normal bundle of such a tangle is standard on the boundary of a coupon. We call a tangle diagram with coupons a *ribbon graph.* A ribbon graph used often is given in Figure 3.1. From now on we refer to this ribbon graph as Δ_g and denote by $-\Delta_g$ its inversion.

Let H_g be an oriented handlebody of genus g; we assume that H_g has one 0-handle D^3 and g 3-dimensional 1-handles all attached to the 0-handle. Let $-H_g$ denote the handlebody H_g with opposite orientation. We fix our model of $\pm H_g$ to be a regular neighbourhood of $\pm\Delta_g$ in S^3, and we fix $\partial(H_g) = \Sigma_g$ as our model for a surface of genus g. Given an element $[f] \in \mathrm{Mod}_{g,1}$ we can form the following closed 3-manifold

$$S_g(f) = H_g \cup_f -H_g = H_g \sqcup I \times \Sigma_g \sqcup -H_g/\sim,$$

where $(0, x) \in \{0\} \times \Sigma_g \sim x \in \partial(H_g)$ and $(1, x) \in \{1\} \times \Sigma_g \sim f(x) \in \partial(-H_g)$. We refer to the manifold $S_g(f)$ as a *Heegaard splitting* of genus g. If the genus is clear from context then we abbreviate $S_g(f)$ to $S(f)$. There is an embedding of Σ_g in this manifold given by $\partial(H_g)$ that we refer to as a *Heegaard surface.*

Suppose we choose a framed link L in $S^3 - \pm\Delta_g$ such that Dehn surgery along L produces $S(f)$ with $\pm\Delta_g$ embedded in $S^3\backslash L$ as in Figure 3.1. Then we get a ribbon graph $L \cup \Delta_g \cup -\Delta_g$ representing the manifold $S(f)$, which we think of as a Kirby diagram for $S(f)$ with added data. To get a framed link L representing $S(f)$ in this way, note that $\#_g S^1 \times S^2 = S(\mathrm{id})$ has a surgery diagram given by g disjoint unknots with framing 0, and $H_g, -H_g$ are given by tubular neighbourhoods of the copies of $\pm\Delta_g$ in Figure 3.2 .

Take a positive normal to Σ_g in S^3 and take an embedding of $I \times \Sigma_g$ into S^3 specified by this

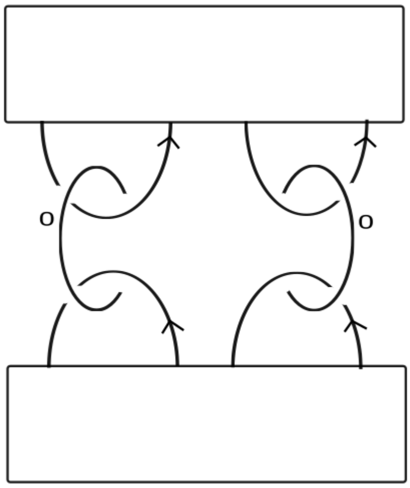

Figure 3.2: $\#_g S^1 \times S^2 = H_g \bigcup_{id} -H_g$ with $\pm\Delta_g$ embedded in them.

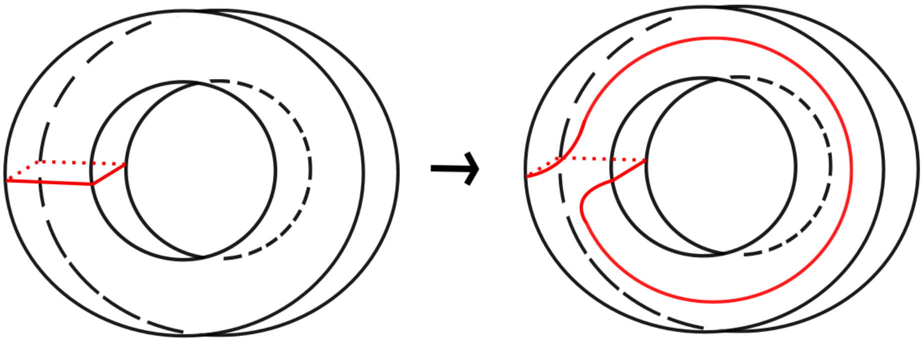

Figure 3.3

framing. Think of $\{0\} \times \Sigma_g$ as $\partial(H_g)$ and $\{t\} \times \Sigma_g$ as a pushoff of Σ_g in the direction of the positive normal(pointing out of the page). Denote the image of this embedding of $I \times \Sigma_g$ into S^3 by $\nu(\Sigma_g)$. We make sure that $\nu(\Sigma_g)$ does not intersect any of the tangles in the diagram. Now we wish to modify the gluing map from $S(id)$ to $S(f)$ using the neighbourhood $\nu(\Sigma_g)$.

Suppose we want to modify $S(id)$ using Dehn surgery to get $S(t_c^i)$ for $i \in \{\pm 1\}$, where c is a simple closed curve in $\Sigma_g \setminus D$. Pick a fiber $\{\text{pt}\} \times \Sigma_g := F$ in $\nu(\Sigma_g)$ and choose an embedding of c in F, then let A be an annular neighbourhood of c in F and write N for the solid torus obtained by thickening A to one side of F in $\nu(\Sigma_g)$. Now we remove N and reglue it by the map of Figure 3.3; by construction this is Dehn surgery. Away from N the fibering is the same, but as we pass across N a Dehn twist about c occurs. This changes the gluing from the identity to a Dehn twist about c. This Dehn surgery corresponds to adding a 4-dimensional 2-handle to the Kirby diagram with framing coefficient specified by the map of Figure 3.3. Alternatively, performing Dehn surgery along c via the map given by Figure 3.3 is equivalent to cutting open the fiber surface F, and then regluing via $t_c^{\pm 1}$.

Next, suppose we have f written as a product of Dehn twists $t_{c_n} \cdots t_{c_1}$. Since $\nu(\Sigma_g) = I \times \Sigma_g$, we pick $t_1 < t_2 < \cdots < t_n \in I$ and place the curve c_i in the Kirby diagram at $\{t_i\} \times \Sigma_g$. Performing Dehn surgery along these curves by the map given by Figure 3.3 will give us $S(f)$. The framings on these curves in the Kirby diagram, along with how these curves link will be captured by Seifert's linking form, which is described below. An example of the tangle diagrams obtained from this method is given in Figure 3.4.

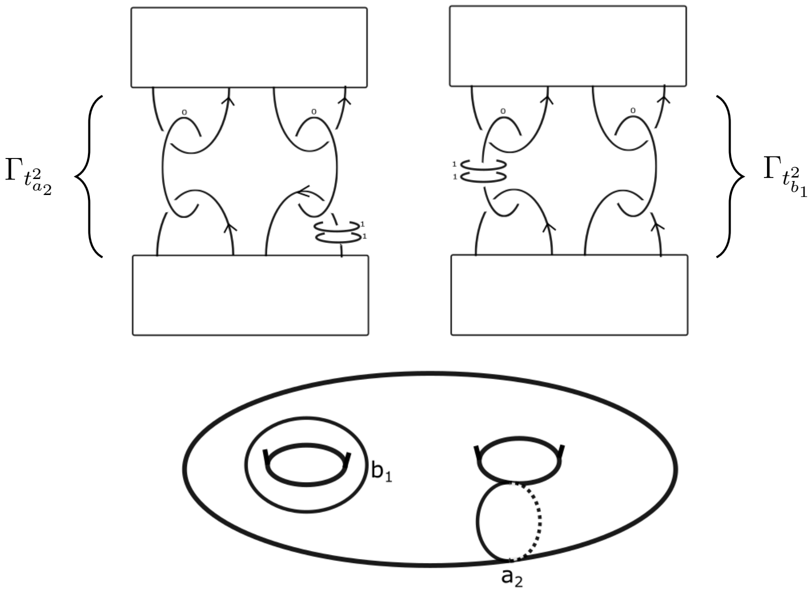

Figure 3.4: Tangle diagrams for $S(t^2_{a_2})$ (left) and $S(t^2_{b_1})$(right).

Seifert pairing

Suppose that an oriented surface F has been embedded in S^3; given a curve a on F representing a cycle, let a^+ denote the pushoff of a in the direction of the positive normal to F. We define Seifert's linking pairing

$$\lambda : H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$$

by the formula $\lambda(a, b) = lk(a, b^+)$. It turns out that λ is a well-defined bilinear pairing and is an invariant of the ambient isotopy class of the embedding of F into S^3. (See [Kau87, Chapter VII] for more details). In our setup we orient S^3 and F so that the positive normal points out of the page, toward the reader.

On the boundary, attaching a 4-dimensional 2-handle with framing n along a knot K is equivalent to removing a solid torus neighbourhood of K and gluing a solid torus back in by sending a meridian to the pushoff K' of K in the direction of the transverse vector field of the framing.

For a curve c on a surface F embedded in S^3, the self-linking form $\lambda(c, c)$ can be computed by $lk(c, c^*)$, where c^* is a parallel copy of c along the surface F. We want to remove a torus neighbourhood of c and reglue by the map of Figure 3.3 and the discussion above implies that this is equivalent to attaching a 4-dimensional 2-handle along $c \subset F$ with framing $\lambda(c, c) \pm 1$, where the ± 1 comes from the gluing being $t_c^{\pm 1}$.

3.0.3 Surgery diagrams of mapping tori

Now we explain how to go from ribbon graphs of $S(f)$ to framed link presentations of the mapping torus M_f. If we remove regular neighbourhoods $\nu(\pm\Delta_g)$ in S^3 of the copies of $\pm\Delta_g$ in our ribbon graph for $S(f)$ we get a manifold diffeomorphic to $I \times \Sigma_g$ cut open along $\{\text{pt}\} \times \Sigma_g$ and reglued via f. So if we identify the remaining boundary surfaces via the identity we get M_f. Suppose we specify the manifold $S(f)$ by a ribbon graph in S^3 as above. Removing the $\nu(\pm\Delta_g)$ and identifying the two boundary surfaces is equivalent to adding a copy of $D^1 \times H_g$ to the 2-handlebody specified by the Kirby diagram of $S(f)$, where we glue $-1 \times H_g$ to $\nu(-\Delta_g)$ and $1 \times H_g$ to $\nu(\Delta_g)$; the $\partial D^1 \times H_g$ part of the boundary of $D^1 \times H_g$ is then in the interior of the new manifold, and the $D^1 \times \partial H_g$ is a new part of the boundary, that has been glued along $\partial D^1 \times \partial H_g$.

We assume that H_g has one 0-handle D^3, so we add one 4-dimensional 1-handle $D^1 \times D^3$ to the

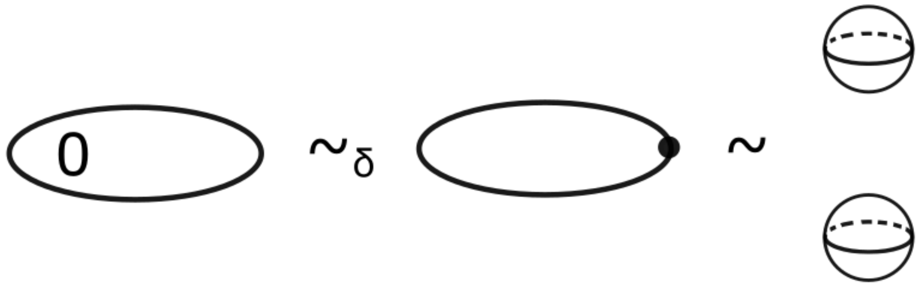

Figure 3.5

Kirby diagram, along with g 4-dimensional 2-handles $D^1 \times D^1 \times D^2$ coming from the g 3-dimensional 1-handles of H_g. These 4-dimensional 2-handles are attached with framing 0 (draw the relevant portion $\partial(D^1 \times D^1) \times \{pt\}$ for $pt \in \partial D^2$ that is visible in the Kirby diagram). This gives us a Kirby diagram for a 4-manifold with boundary M_f.

Now we think of the two 3-balls $\partial D^1 \times D^3$ of the 4-dimensional 1-handle attached in Figures 3.2, 3.4 as tubular neighbourhoods of the coupons of $\pm\Delta_g$ in S^3. The 4-dimensional 2-handles are given by the tangles attached to the coupons (these 2-handles run over the 1-handle). Here the endpoints of the strands of the tangle of the bottom coupon are identified with the endpoint of the strand of the tangle of the top coupon which is directly above it. We call the corresponding 4-manifold given by this Kirby diagram X_f.

Dotted circle notation

We can now find Kirby diagrams of 4-manifolds with boundary M_f, where $f : \Sigma_g \to \Sigma_g$ is a diffeomorphism. Next, we describe dotted circle notation, due to Akbulut [Akb77].

If we smooth corners so that $D^2 \times D^2 = D^4$ we have a diffeomorphism $(D^2 \setminus \nu\{0\}) \times D^2 = (S^1 \times D^1) \times D^2 = S^1 \times D^3$, where $\nu\{0\}$ is a tubular neighbourhood of $0 \in D^2$. From this, we see that adding a 1-handle to D^4 is the same as removing an open tubular neighbourhood of a properly embedded 2-disc $\{0\} \times D^2$, whose boundary $\{0\} \times \partial D^2$ is visible in the Kirby diagram as an unknot in S^3; draw this unknot as a *dotted circle* to indicate that it corresponds to a 1–handle.

The $\nu\{0\} \times \partial D^2$ part of the tubular neighbourhood is visible in $\partial D^4 = S^3$ as a solid torus and the annulus $(D^2 \setminus \nu\{0\}) \times \{pt\}$ allows us to isotope a curve running through the 1- handle to $\partial\nu\{0\} \times pt$, which links once with the removed solid torus in ∂D^4.

Note that $\partial(S^1 \times D^3) = S^1 \times S^2 = \partial(D^2 \times S^2)$, where the Kirby diagram of $D^2 \times S^2$ is given by a 0-framed unknot. Moving from the dotted circle to the 0-framed unknot as in Figure 3.5 is done via surgery in the interior of the 4-manifold, and the symbol $\sim_\delta$ denotes that there is a diffeomorphism between the two boundary 3-manifolds.

Given a Kirby diagram with 1-handles, we can switch to dotted circle notation by isotoping the attaching circles of any 2-handles so that they avoid the regions between the attaching balls of the 1-handles. Then we can push the balls together and switch to dotted circle notation. When doing this, we must remember that curves running through the 4-dimensional 1-handle are linked with the dotted circle, and are joined together by the gluing data of the 4-dimensional 1-handles. We use the convention of drawing dotted lines to indicate the paths taken to push any balls together (For more information see [GS99, Section 5.4]). Applying this operation to the Kirby diagram of the 4-manifold X_f we have found with $\partial X_f = M_f$, then changing the dotted circles to 0-framed unknots gives us a surgery diagram for M_f. The framings are well-defined once we have drawn the dotted lines.

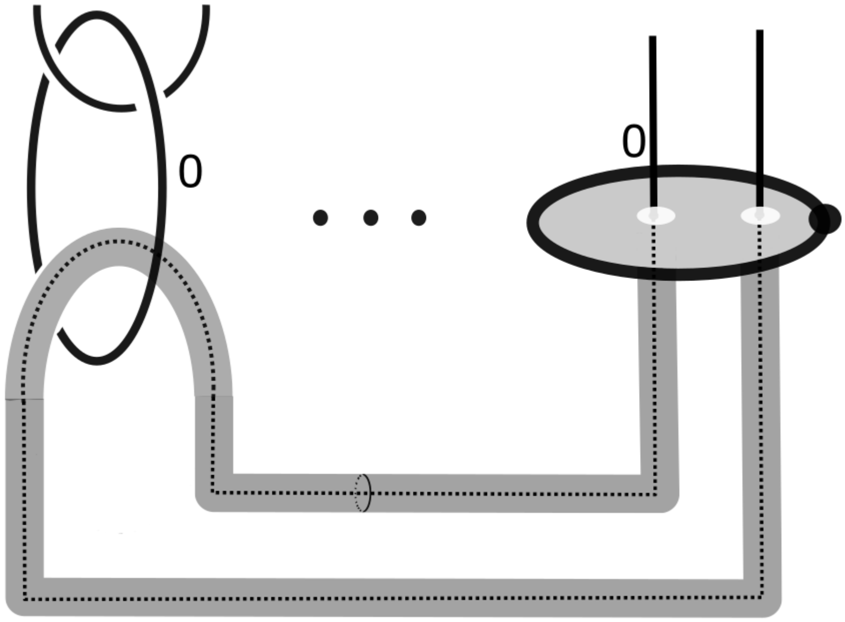

Figure 3.6: A fiber in the surgery diagram for M_f obtained from Construction 3.0.1.

Note that there is a natural way to switch to dotted circle notation using the ribbon graphs above; we simply push the two balls given by the coupons together along a dotted line running to the right. An example is given by Figure 3.7. To summarise:

Construction 3.0.1. *Let $f : \Sigma_g \to \Sigma_g$ be a diffeomorphism with $[f] = t_{c_n}^{k_n} \cdots t_{c_1}^{k_1} \in \mathrm{Mod}_{g,1}$, where $k_i = \pm 1$ for all i. We have the following construction of framed link descriptions of $S(f)$ and M_f:*

1. *Start with a ribbon graph for $S(\mathrm{id}) = H_g \cup_{\mathrm{id}} -H_g$, as in Figure 3.2.*
2. *Pick a collar, $I \times F$, of $F = \partial(\nu_{S^3}(\Delta_g))$ in S^3, and pick $t_1 < \cdots < t_n \in I$. Then place c_i in $\{t_i\} \times F$ with framing $\lambda(c_i, c_i) + k_i$. Here $\lambda : H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$ denotes the Seifert pairing of $F \subset S^3$. The closed components of this diagram give a framed link for $S(f)$, and the $\pm H_g$ are thought of as regular neighbourhoods of $\pm\Delta_g$ in S^3.*
3. *To go from $S(f)$ to M_f, think of the coupons of $\pm\Delta_g$ as 4–dimensional 1–handles, and the tangles of $\pm\Delta_g$ as 4–dimensional 2–handles. Then change to dotted circle notation as in Figure 3.7, and replace the dotted circle by a 0–framed unknot.*

After switching to dotted circle notation in Construction 3.0.1 (3), the surface F gets punctured, and can be visualised in the following way: choose a disk $D \subset S^3$, with boundary the dotted circle component that intersects the components of the framed link transversely in pairs of punctures. Take away small open disks in D around the punctures, and replace with annuli that run along the components of the link intersecting D, to obtain a punctured fiber as in Figure 3.6. The remaining part of the fiber, which is a disk, is in the surgered torus obtained from 0–surgery on the dotted circle component. The complement of the dotted circle in S^3 fibers into disks, and repeating the operation above allows us to see the other fibers of the mapping torus.

3.0.4 Surgery diagrams from composing tangles

We can also construct surgery diagrams for Heegaard splittings and mapping tori by concatenating tangle diagrams; this will be useful later on for studying the Birman–Craggs maps.

The ribbon graphs obtained as in Figure 3.4 give surgery diagrams for $S(f) \coloneqq H_g \bigcup_f -H_g$ if we surger along the closed components of the tangle diagram. The handlebodies $\pm H_g$ are viewed as

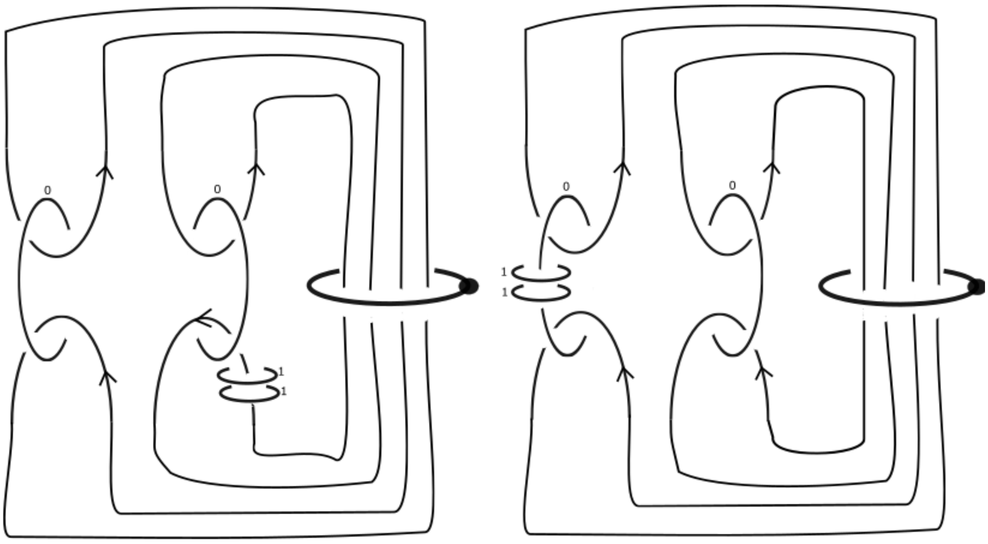

Figure 3.7: Surgery diagrams for $M_{t_{a_2}^2}$ and $M_{t_{b_1}^2}$ obtained from Figure 3.4

tubular neighbourhoods in S^3 of the $\pm\Delta_g$. This breaks $S(f)$ into three pieces, $\nu(\pm\Delta_g)$ and a copy of $I \times \Sigma_g$ cut open along a $\{\text{pt}\} \times \Sigma_g$ and reglued via f. If we remove tubular neighbourhoods in S^3 of the $\pm\Delta_g$ we get tangle diagrams for the mapping cylinder $C(f) = I \times \Sigma_g \bigsqcup \Sigma_g / \sim$, obtained by gluing $(1, x) \in \{1\} \times \Sigma_g$ to $f(x) \in \Sigma_g$. Given such a ribbon graph of $S(f)$ corresponding to $C(f)$, we denote by Γ_f the tangle diagram obtained by deleting the top and bottom coupons. The following useful result is due to Reshetikhin-Turaev [RT91, Lemma 4.4], we include a proof of it here for convenience.

Lemma 3.0.2. *Let $f, h : \Sigma_g \to \Sigma_g$ be diffeomorphisms, and let Γ_f, Γ_h be two tangle diagrams as above representing $C(f), C(h)$. Then the composition $\Gamma_f \circ \Gamma_h$ obtained by stacking Γ_f on top of Γ_h and then putting coupons on the ends gives a ribbon graph for $S(f \circ h)$. Here, the g additional unknotted components obtained from stacking are given the 0–framing.*

Proof. For $\Gamma_f \circ \Gamma_h$, consider the g 0–framed unknots $L_1, .., L_g$ obtained by gluing the top boundary tangles of Γ_h to the bottom boundary tangles of Γ_f. Each of the L_i transversally hits a plane $\mathbb{R}^2 \times \frac{1}{2} \subset \mathbb{R}^3$, along which Γ_h is glued to Γ_f. Complete this plane into a 2–sphere $S^2 = \mathbb{R}^2 \times \frac{1}{2} \cup \{\infty\} \subset S^3$, and take a cylinder, $S^2 \times [0, \epsilon]$, over this 2–sphere, such that each of the L_i meet this cylinder in two vertical segments $\{\text{pt}\} \times [0, \epsilon]$.

Now, surger S^3 along all the closed components of $\Gamma_f \circ \Gamma_h$, using the given framings, to get a closed 3–manifold $\overline{M}$. In doing so, we cut out regular neighbourhoods $U_1, .., U_g$ of $L_1, .., L_g$, and glue in g solid tori $W_1, ..., W_g$. Then

$$N := ((S^2 \times [0, \epsilon]) \setminus \bigcup_{i=1}^{g} U_i) \cup \bigcup_{i=1}^{g} W_i \subset \overline{M}$$

is identified with $[0, 1] \times \partial(\nu(\Delta_g))$; compare with Figure 3.2, the complement of the $\pm H_g = \nu(\pm\Delta_g)$ in $H_g \cup_{\text{id}} -H_g$ is a copy of $I \times \partial(\nu(\Delta_g))$, and can be identified with N by construction. Hence, $\overline{M} \setminus \nu(\pm\Delta_g)$ can be identified with $([0, \frac{1}{3}] \times F) \bigcup_h ([\frac{1}{3}, \frac{2}{3}] \times F) \bigcup_f ([\frac{2}{3}, 1] \times F)$, where $F = \partial(\nu(\Delta_g))$, and $([0, \frac{1}{3}] \times F) \bigcup_h ([\frac{1}{3}, \frac{2}{3}] \times F)$ denotes that $(\frac{1}{3}, x) \in [0, \frac{1}{3}] \times F$ has been glued to $(\frac{1}{3}, h(x)) \in [\frac{1}{3}, \frac{2}{3}] \times F$, and similarly for the other union. This implies $\overline{M} \setminus \nu(\pm\Delta_g)$ is diffeomorphic to $C(f \circ h)$.

After composing the tangle diagrams and putting coupons on the two opposite ends, the resulting copies of $\pm\Delta_g$ in our ribbon graph have regular neighbourhoods in S^3 corresponding to $\pm H_g$.

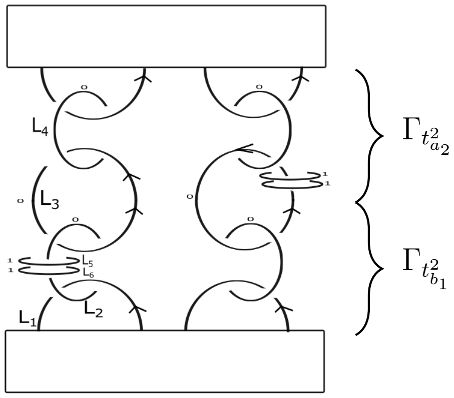

Figure 3.8: Another surgery diagram for $S(t^2_{a_2}t^2_{b_1})$, obtained by concatenating the tangles of Figure 3.4

Removing these copies of $\pm H_g$ gives us the mapping cylinder $C(f \circ h)$ so the ribbon graph obtained gives a surgery diagram for $S(f \circ h)$. □

For an example of the constructions in the proof of Lemma 3.0.2, the tangle diagram for $C(t^2_{a_2}t^2_{b_1})$ given in Figure 3.8 is obtained using the tangle diagrams for $C(t^2_{a_2})$ and $C(t^2_{b_1})$ given in Figure 3.4.

To summarise, we have outlined the following inductive construction:

Construction 3.0.3. *Let $f = f_n \cdots f_1$ be a diffeomorphism, where each f_i is a composition of Dehn twists (or their inverses). Then, by concatenating tangle diagrams, we have the following inductive construction of framed link diagrams for $S(f)$ and M_f.*

1. *Use part* (2) *of Construction 3.0.1 to obtain ribbon graphs of the $S(f_i)$, for $i = 1, .., n$. Let Γ_{f_i} denote the tangle diagrams obtained by deleting any coupons, as above.*
2. *The composition $\Gamma_{f_n} \circ \cdots \circ \Gamma_{f_1}$, obtained using Lemma 3.0.2 inductively, with coupons added to the top and bottom, gives a ribbon graph for $S(f_n \cdots f_1)$; surgering along the closed components of $\Gamma_{f_n} \circ \cdots \circ \Gamma_{f_1}$ gives a framed link for $S(f_n \cdots f_1)$.*
3. *We obtain a framed link for $M_{f_n \cdots f_1}$ by viewing the coupons as* 4*–dimensional* 1*–handles, and the tangles (nonclosed components) as* 4*–dimensional* 2*–handles, then changing to dotted circle notation as in Figure 3.7, and changing the dotted circle into a zero framed unknot.*

Note that the mapping tori obtained from constructions 3.0.1 and 3.0.3 are diffeomorphic. This can be seen using Kirby calculus [MP94, Fig. 3 (K3 move)]; an example for genus 1 is given in Figure 3.9. This argument can be generalised as in Figure 3.9(rightmost); isotope the components corresponding to Dehn twists so that they lie as in Figure 3.9(rightmost), then slide each strand over L'.

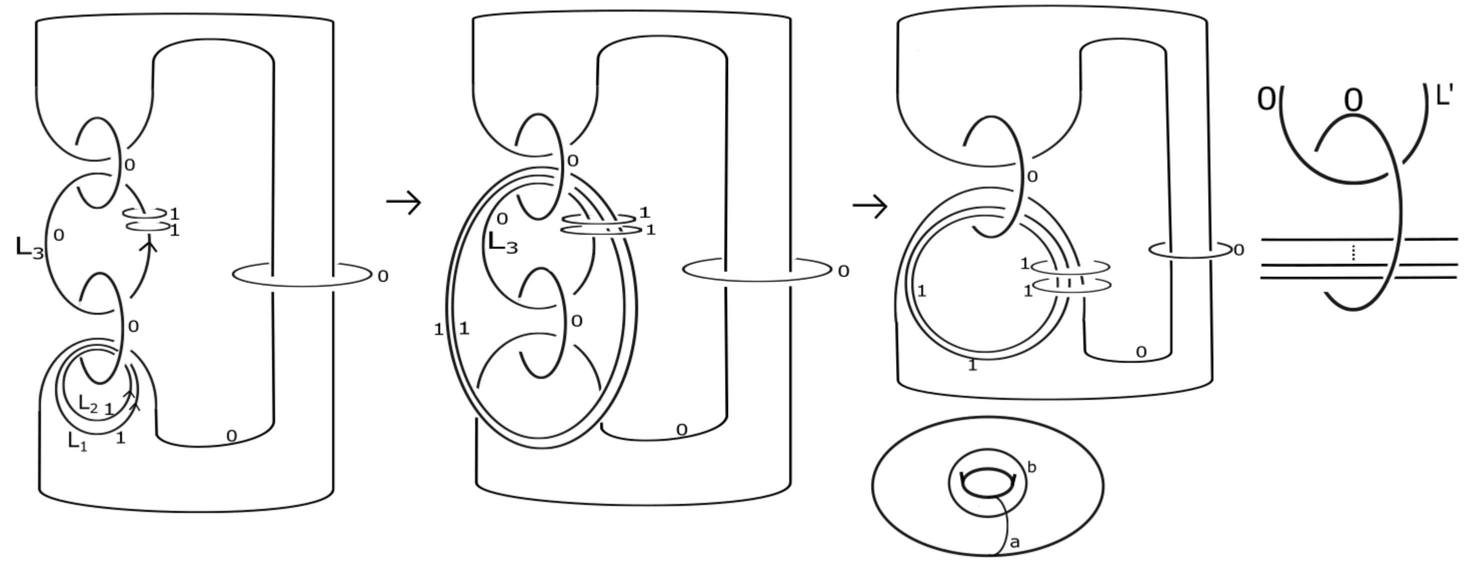

Figure 3.9: Two diffeomorphic surgery diagrams for $M_{t_a^2 t_b^2}$. The first move is to slide the components L_1, L_2 corresponding to t_b^2 over L_3. The second move is [MP94, K3 move]. A similar argument works for diagrams of $M_{t_b^2 t_a^2}$; the order of concatenation matters.

Chapter 4

Rewriting Sato's maps

In this chapter, we define Construction 4.0.5, that outputs a framed link diagram of a mapping torus, along with a characteristic sublink (defined below). This construction plays the role of Sato's map θ in Chapter 2. Construction 4.0.5 allows us to evaluate the Rochlin invariant of a given spin structure from the link diagram.

4.0.1 Spin structures and characteristic sublinks

For an arbitrary framed link L in S^3, write M_L for the 3–manifold obtained via Dehn surgery on L. There is a 2–handlebody W_L with Kirby diagram L, and $\partial W_L = M_L$. The linking matrix of L is the matrix of the intersection form of W_L with respect to a basis of $H_2(W_L)$ obtained from the components L_i of L; use $\cdot$ to denote this intersection form. See Appendix 8.5 for more details.

Definition 4.0.1. Let L be an oriented framed link with components $L_1, .., L_n$, and let C be a sublink of L. Define $(w_i)_{i=1}^n \in (\mathbb{Z}/2)^n$ by $w_i = 1$ if and only if L_i is in C. Then C is characteristic if

$$p_i w_i + \sum_{j \neq i} lk(L_i, L_j) w_j \equiv p_i \pmod 2,$$

for all $1 \leq i \leq n$. Here p_i denotes the integer specifying the framing of L_i. We abbreviate these conditions to $C \cdot L_i \equiv L_i \cdot L_i \pmod 2$.

Lemma 4.0.2. *[KM91, Lemma C.1], [GS99, Prop. 5.7.11] There is a natural bijection between spin structures on M_L and characteristic sublinks of L*

Proof. The correspondence is given by taking a spin structure s of M_L, and defining C to be the union of all components L_i of L such that the spin structure s does not extend over the 2–handle in W_L attached to L_i. □

4.0.2 Rewriting Sato's constructions

Let L_f be a framed link for the mapping torus of f, obtained from Construction 3.0.1, where f is a product of squares of Dehn twists, bounding pairs, or separating twists. By Figure 3.6, we see a punctured fiber F for the mapping torus M_{L_f}, where F lies in $S^3 \setminus L_f$ as the standard embedding of a surface into S^3. Let L be the framed link obtained from Construction 3.0.1 (3) with $f = \text{id}$, then $M_L \cong S^1 \times \Sigma_g$, and L is a sublink of L_f. The framed link L_f is obtained from L by placing the curves appearing in the factorisation of f in pushoffs of the fiber surface, and framing them using the Seifert pairing of $F \subset S^3$.

In Chapter 2.0.1, for a spin structure $\sigma \in \mathrm{Spin}(\Sigma_g)$, we obtained a spin structure $\theta(\sigma)$ as a right splitting of the short exact sequence

$$0 \to H_1(SO(3)) \xrightarrow{i} H_1(P(M_{L_f})) \xrightarrow{p} H_1(M_{L_f}) \to 0. \tag{4.1}$$

Under the splitting of $H_1(M_{L_f})$ obtained from the exact sequence (2.2), it is given by $\hat{l} \oplus \overline{\sigma}$, where $\hat{l}$ is obtained from the fixed embedding of $S^1 \times D \hookrightarrow M_f$ in the notation of Chapter 2.0.1. This construction gives a spin structure on M_f that restricts to $\sigma \in \mathrm{Spin}(\Sigma_g)$ on a fiber, and restricts to a fixed spin structure on $S^1 \times D \subset M_f$.

For the framed link L_f obtained from Construction 3.0.1, we have a fixed embedding of a punctured fiber $F \subset S^3 \setminus L_f$, and a fixed embedding of $S^1 \times D \hookrightarrow S^3 \setminus L_f$ that is identified with a neighbourhood α of a meridian of the dotted circle component; see Figure 3.6. Let $B = \{a_i, b_i\}_{i=1}^g$ denote the standard basis for $H_1(F; \mathbb{Z})$ as in Figure 5.1. When placed in the fiber, the curves in B go to meridians of the components of L, the sublink corresponding to $S^1 \times \Sigma_g$. We can frame these curves using a spin structure $\sigma \in \mathrm{Spin}(\Sigma_g)$ and a positive normal to the fiber F.

To specify $\sigma \in \mathrm{Spin}(\Sigma_g)$, we use Theorem 2.0.1, which says that spin structures on Σ_g are in bijection with symplectic quadratic forms. Let $\sigma \in \mathrm{Spin}(\Sigma_g)$ have associated quadratic form $q_\sigma : H_1(\Sigma_g) \to \mathbb{Z}/2$. For an embedded circle $K \subset \Sigma_g$, the restriction of σ to K defines a spin structure on $T\Sigma_g|_K$, since $H^1(S^1) = \mathbb{Z}/2$, there are only two types up to homotopy. One is the bounding spin structure induced by the product framing on D^2. For a spin structure $\sigma \in \mathrm{Spin}(\Sigma_g)$, and an embedded curve $K \subset \Sigma_g$, we have that $q_\sigma([K]) = 0$ if and only if the spin structure σ restricted to K is spin bounding [BM96, Remark 2.3].

Lemma 4.0.3. *Let $\mu_j \in B \subset F$ be a meridian of a component L_j of L that doesn't come from a dotted circle, and let $\sigma \in \mathrm{Spin}(\Sigma_g)$. If $q_\sigma([\mu_j]) = 0$, then the spin structure on M_{L_f} extends over the 2–handle in W_{L_f} attached along L_j.*

Proof. Since $q_\sigma([\mu_j]) = 0$, the spin structure σ restricted to μ_j is spin bounding. As noted in Chapter 3.0.1, after surgery along L_j, the meridian μ_j can be isotoped to the core $\{0\} \times \partial D^2$ of the 2–handle $h = D^2 \times D^2$ attached along L_j. The spin structure restricted to h has to be the product framing, which agrees with the spin bounding structure on μ_j. □

Following Lemmas 4.0.2 and 4.0.3, we define the characteristic sublink of L_f in the following way: we set the components L_i of L, with meridians $\mu_i \in B$ in the fiber F, satisfying $q_\sigma(\mu_i) = 1$ to be in the characteristic sublink of L_f. We set the dotted circle component of L_f to never be in the characteristic sublink, this corresponds to fixing the spin structure on the embedding of $S^1 \times D$ given by the meridian α. The conditions of definition 4.0.1 determine whether the additional components of L_f are in the characteristic sublink, by the following.

Proposition 4.0.4. *Let $f : \Sigma_g \to \Sigma_g$ be a product of squares of Dehn twists, bounding pairs, or separating twists. Let L_f be the framed link for M_f obtained from Construction 3.0.1, and let L be the sublink corresponding to $S^1 \times \Sigma_g$. Then once we have defined which components of L are in the characteristic sublink, the conditions of definition 4.0.1 uniquely determine if the additional components of L_f are in the characteristic sublink.*

Proof. We prove the case where f is a product of squares. Let L_f have components $L_1, ..., L_n$, and suppose that the components L_i, L_{i+1} correspond to a $t_c^{\pm 2}$ factor of the monodromy f, via construction 3.0.1 (2). Then $p_i = p_{i+1} = \lambda(c, c) \pm 1$, where λ is the Seifert pairing for the punctured fiber surface F that lies in $S^3 \setminus L_f$. Applying the relations of definition 4.0.1, we have

$$p_i w_i + (p_i - 1) w_{i+1} + \sum_{j \neq i, i+1} lk(L_i, L_j) w_j \equiv p_i \pmod 2.$$

The components L_i and L_{i+1} have the same linking number up to signs with all other components, since they are isotopic to each other in the fiber that lies ambiently in S^3, so applying definition 4.0.1 to L_{i+1} we get

$$p_i w_{i+1} + (p_i - 1) w_i + \sum_{j \neq i, i+1} lk(L_{i+1}, L_j) w_j \equiv p_i w_{i+1} + (p_i - 1) w_i + \sum_{j \neq i, i+1} lk(L_i, L_j) w_j \equiv p_{i+1} \equiv p_i \pmod 2.$$

But then

$$w_i \equiv w_{i+1} \equiv p_i - \sum_{j \neq i, i+1} lk(L_i, L_j) w_j \pmod 2.$$

The result follows since the other components corresponding to the monodromy come in pairs, so they contribute zero to the right hand side of the last equation. The general case, where bounding pair and separating twist factors are involved, follows from a similar argument, since the linking number of the components of L_f is determined by the Seifert pairing on homology. $\square$

Note that for the symplectic quadratic form $q_\sigma : H_1(\Sigma_g) \to \mathbb{Z}/2$ associated to $\sigma \in \text{Spin}(\Sigma_g)$, the affine action of $x \in \text{Hom}(H_1(\Sigma_g), \mathbb{Z}/2) = H^1(\Sigma_g)$ on σ satisfies the following identity: $q_{\sigma+x} = q_\sigma + x$ [Joh80c, Theorem 3A]. For example, if $x \in H^1(\Sigma_g)$ satisfies $x([c]) = 1$ for $c \in B$, then $q_{\sigma+x}([c]) = q_\sigma([c]) + 1$, so we switch whether the component of L with meridian c is in the characteristic sublink.

In summary, we have the following construction; recall that any element of $\text{Mod}_{g,1}[2]$ can be written as a product of squares of Dehn twists [Hum92, Prop.2.1].

Construction 4.0.5. *Let $[f] \in \text{Mod}_{g,1}[2]$ be a mapping class, written as a product of squares of Dehn twists.*

1. *We start with a framed link L representing $S^1 \times \Sigma_g$, obtained by applying Construction 3.0.1 (3) to $f = \text{id}$. A punctured fiber surface F for $\{\text{pt}\} \times \Sigma_g$ is visible in $S^3 \setminus L$ as in Figure 3.6. Construction 3.0.1 gives a framed link L_f representing M_f, obtained by placing the curves appearing in the factorisation of f in pushoffs of the fiber surface F in $S^3 \setminus L$, and framing these curves using the Seifert pairing; L_f has L as a sublink.*

2. *For $\sigma \in \text{Spin}(\Sigma_g)$ with associated quadratic form $q_\sigma : H_1(\Sigma_g) \to \mathbb{Z}/2$, define a map $\text{Spin}(\Sigma_g) \to \{\text{Characteristic sublinks of } L\}$ in the following way: a component L_j of L with meridian $\mu_j \in B$ in the fiber F, is in the characteristic sublink if and only if $q_\sigma(\mu_j) = 1$. The dotted circle component of L is never in the characteristic sublink.*

3. *Denote our map $\theta : \text{Spin}(\Sigma_g) \to \{\text{Characteristic sublinks of } L_f\}$ by θ_{L_f} to indicate the dependence on the monodromy f, and our chosen framed link L for $S^1 \times \Sigma_g$. Then $\theta_{L_f}(\sigma)$ is defined by declaring which components of L are in the characteristic sublink as in (2), and the relations of definition 4.0.1 determines if the additional components of L_f are in the characteristic sublink, by Proposition 4.0.4.*

It remains to show that for two different factorizations of $[f] \in \text{Mod}_{g,1}[2]$ into products of squares of Dehn twists, the spin mapping tori obtained from Construction 4.0.5 are spin diffeomorphic.

Theorem 4.0.6. *The map that sends $[f] \in \text{Mod}_{g,1}[2]$ to the spin diffeomorphism class of $(L_f, \theta_{L_f}(\sigma))$, where $(L_f, \theta_{L_f}(\sigma))$ is obtained via Construction 4.0.5, is well–defined.*

Proof. Suppose, for example, that $[f] = t_{c_n}^2 \cdots t_{c_1}^2$, $[h] = t_{d_m}^2 \cdots t_{d_1}^2$, and that $[f] = [h]$ in the mapping class group. In the construction above, the monodromy is modified via Dehn surgery in a single tubular neighbourhood of the punctured fiber surface F obtained from $\partial(\nu(\Delta_g))$ after switching to dotted circle notation (see Figure 3.6). Identify this neighbourhood with $I \times F$, and set $N(f)$ to be the manifold obtained from $I \times F$ by modifying the monodromy by Dehn surgery along the curves $c_i \subset t_i \times F$ as in construction 3.0.1 (2). Define $N(h)$ similarly.

If we isotope the pair of surgery components corresponding to the $t_{c_i}^2$ part of the gluing to lie on the same pushoff of F, we see that $N(f) = ([0, \frac{1}{n+1}]\times F) \cup_{t_{c_1}^2} ([\frac{1}{n+1}, \frac{2}{n+1}]\times F) \cup_{t_{c_2}^2} \cdots \cup_{t_{c_n}^2} ([\frac{n}{n+1}, 1]\times F)$. Here, $([\frac{k-1}{n+1}, \frac{k}{n+1}] \times F) \cup_{t_{c_k}^2} ([\frac{k}{n+1}, \frac{k+1}{n+1}] \times F)$ means that $(\frac{k}{n+1}, x) \in [\frac{k-1}{n+1}, \frac{k}{n+1}] \times F$ has been glued to $(\frac{k}{n+1}, t_{c_k}^2(x)) \in [\frac{k}{n+1}, \frac{k+1}{n+1}] \times F$. To see this, note that performing Dehn surgery along $c_i \subset \{t_i\} \times F$, using the framing $\lambda(c_i, c_i) \pm 1$ coming from the Seifert pairing of $F \subset S^3$, is equivalent to cutting along $\{t_i\} \times F$, and regluing via $t_{c_i}^{\pm 1}$, see Figure 3.3.

We have a map $\psi : N(f) \to C(f) := I \times F \sqcup F/(1, x) \sim f(x)$, defined on the decomposition of $N(f)$ above, as $\psi|_{[0,\frac{1}{n+1}]\times F} = \mathrm{id} \times \mathrm{id}$, and $\psi|_{[\frac{k}{n+1},\frac{k+1}{n+1}]\times F} = \mathrm{id} \times (t_{c_1}^{-2} \cdots t_{c_k}^{-2})$, for $k > 0$. This map has inverse $\psi^{-1} : C(f) \to N(f)$, where $\psi^{-1}|_{[\frac{k}{n+1},\frac{k+1}{n+1}]\times F} = \mathrm{id} \times (t_{c_k}^2 \cdots t_{c_1}^2)$ for $k > 0$, and $\psi^{-1}|_{[0,\frac{1}{n+1}]\times F} = \mathrm{id} \times \mathrm{id}$ on the $I \times F$ part of $C(f)$, and $\psi^{-1}(x) = (1, x)$, for $x \in F \subset C(f)$.

Since $[f] = [h]$, there is a map $H : I \times F \to I \times F$ given as $H(s, x) = (s, H_s(x))$, where H_s is an isotopy with $H_0 = \mathrm{id}_F$, and $H_1 = f^{-1}h$. This defines a map $\phi : C(h) \to C(f)$, by $\phi(s, x) = H(s, x)$ for $(s, x) \in I \times F$, and $\phi(x) = x$, for $x \in F$. The map $G : I \times F \to I \times F$ given by $G(s, x) = (s, H_s^{-1}(x))$ induces an inverse map to ϕ.

The maps of the previous two paragraphs descend to maps of the relevant mapping tori; the 3–manifold specified by L_f via construction 4.0.5 can be identified with $N(f)/(1, x) \sim (0, x)$, and the abstract mapping torus M_f can be identified with $C(f)/(1, x) \sim (0, f(x))$. We have $\psi(1, x) = (1, f^{-1}(x)) \sim (0, f \circ f^{-1}(x)) = (0, x) = \psi(0, x)$. The inverse of ψ also induces a diffeomorphism between the mapping tori by a similar argument. The $\phi^{\pm 1}$ also descend to diffeomorphisms of mapping tori, since $\phi(1, x) = (1, f^{-1}h(x)) \sim (0, ff^{-1}h(x)) = (0, h(x)) = \phi(0, h(x))$ when ϕ is projected to the mapping torus using the identifications above.

Composing the diffeomorphisms obtained this way gives a diffeomorphism between the mapping tori specified by L_f and L_h. Locally, the total derivative of this diffeomorphism is of the form $\mathrm{id}_{\mathbb{R}} \oplus de$, for some $e \in \mathrm{Diff}(F)$ which preserves the spin structure restricted to F, and fixes pointwise a neighbourhood of the meridian of the dotted circle component. Hence, the spin mapping tori $(L_f, \theta_{L_f}(\sigma))$ and $(L_h, \theta_{L_h}(\sigma))$ obtained from construction 4.0.5 are spin diffeomorphic, for fixed $\sigma \in \mathrm{Spin}(\Sigma_g)$. □

4.0.3 Sato's homomorphisms

Denote by $\mathrm{Map}(H_1(\Sigma_g), \mathbb{Z}/8)$ the free $\mathbb{Z}/8$–module consisting of all functions $H_1(\Sigma_g) \to \mathbb{Z}/8$, and recall $H_1(\Sigma_g) = H_1(\Sigma_g; \mathbb{Z}/2)$. Define

$$\beta_\sigma : \mathrm{Mod}_{g,1}[2] \to \mathrm{Map}(H_1(\Sigma_g), \mathbb{Z}/8)$$

by $\beta_\sigma([f])(x) = \beta_{\sigma,PD(x)}([f])$. In this section, we prove that the maps β_σ are homomorphisms using a formula for computing the Rochlin invariant from a surgery diagram with a labelled characteristic sublink. The proof sheds light on why it is a homomorphism on the subgroup generated by squares of Dehn twists, which coincides with $\mathrm{Mod}_{g,1}[2]$ [Hum92, Prop.2.1].

From now on we use construction 4.0.5, and think of θ_{L_f} as a map from $\mathrm{Spin}(\Sigma_g)$ to framed links with characteristic sublink $(L_f, \theta_{L_f}(\sigma))$, such that the corresponding spin manifold is spin diffeomorphic to $(M_f, \theta(\sigma))$. We use the following formula for the Rochlin invariant of $(L_f, \theta_{L_f}(\sigma))$, found by Kirby–Melvin.

Theorem 4.0.7. *[KM91, (C.3), Theorem C.4, Corollary C.5] Let (L, C) denote a framed link L, with a characteristic sublink C as in definition 4.0.1. Suppose L is oriented, then the Rochlin invariant of the corresponding spin manifold is given by*

$$R(L, C) = \Lambda_L - C \cdot C + 8\,\mathrm{Arf}(C) \pmod{16},$$

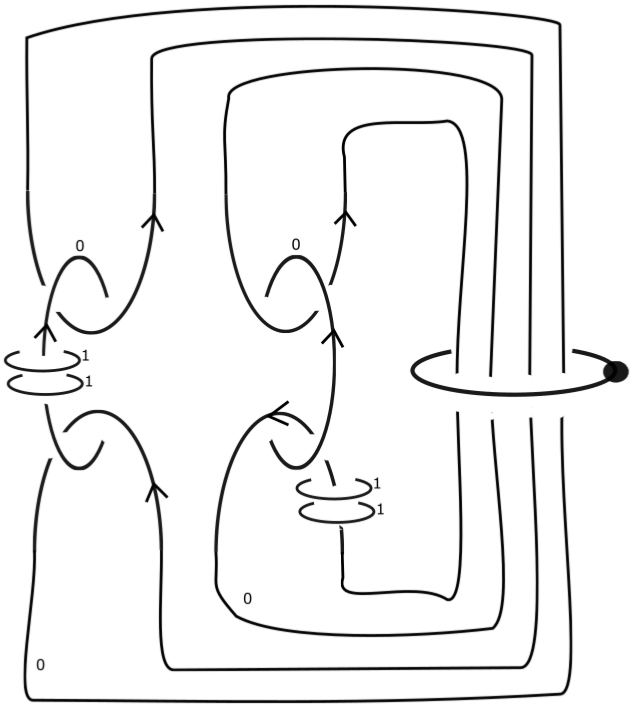

Figure 4.1: A surgery diagram for $M_{t_{a_2}^2 t_{b_1}^2}$ obtained from construction 3.0.1, where a_2 and b_1 are the curves of Figure 3.4

where Λ_L is the signature of the linking matrix of L, $C \cdot C$ is the sum of all entries in the linking matrix of C, and $\operatorname{Arf}(C) \in \mathbb{Z}/2$ is the Arf invariant of C, defined below. The formula above gives a spin diffeomorphism invariant for spin 3–manifolds, and in particular, the formula above is independent of the choice of orientations for the components of L.

The Arf invariant of a proper link

We call a link L proper if $lk(K, L - K)$ is even for every component K of L. Let F be an oriented Seifert surface for L, and let $i : L \to \partial F$ denote the inclusion. Then $\operatorname{im}(i_*)$ is the radical of the intersection form on $H_1(F)$ and the Seifert self–linking form

$$\begin{aligned} \lambda : H_1(F) &\to \mathbb{Z}/2 \\ [a] &\mapsto \operatorname{lk}(a, a^+) \end{aligned}$$

satisfies $\lambda|_{\operatorname{im}(i_*)} = 0$ if and only if L is a proper link. We can define $\operatorname{Arf}(L)$ to be the Arf invariant of the form on $H_1(F)/\operatorname{im}(i_*)$ induced by λ. See Appendix 8.4 for more details.

For a proper link L, we can produce a knot K by band connecting together all the components of L, as long as the bands respect the orientations chosen for the components of L. It is a fact that $\operatorname{Arf}(L) = \operatorname{Arf}(K)$; see [Rob65] or [Hos84] for more details.

Another definition of Sato's maps

Using Theorem 4.0.7, we write the Rochlin invariant of the spin mapping torus $(L_f, \theta_{L_f}(\sigma))$ obtained from construction 4.0.5 as

$$R(L_f, \theta_{L_f}(\sigma)) = \Lambda_{L_f} - \theta_{L_f}(\sigma) \cdot \theta_{L_f}(\sigma) + 8 \operatorname{Arf}(\theta_{L_f}(\sigma)) \pmod{16}. \tag{4.2}$$

Now, substitute formula (4.2) into Sato's definition of the homomorphisms, to get:

Lemma 4.0.8. *Let $\sigma \in \mathrm{Spin}(\Sigma_g)$ be a spin structure, let $x \in H^1(\Sigma_g)$, and let $[f] \in \mathrm{Mod}_{g,1}[2]$. Let $(L_f, \theta_{L_f}(\sigma))$ and $(L_f, \theta_{L_f}(\sigma+x))$ denote the spin mapping tori obtained from construction 4.0.5, then*

$$\beta_{\sigma,x}([f]) = (\theta_{L_f}(\sigma+x)\cdot\theta_{L_f}(\sigma+x) - \theta_{L_f}(\sigma)\cdot\theta_{L_f}(\sigma) + 8(\mathrm{Arf}(\theta_{L_f}(\sigma)) - \mathrm{Arf}(\theta_{L_f}(\sigma+x))))/2 \pmod 8.$$

Here $\theta_{L_f}(\sigma)\cdot\theta_{L_f}(\sigma)$ denotes the sum of all the entries in the linking matrix of the characteristic sublink of $\theta_{L_f}(\sigma)$, and $\mathrm{Arf}(\theta_{L_f}(\sigma)) \in \{0,1\}$ is the Arf invariant of the characteristic sublink of $\theta_{L_f}(\sigma)$. This formula is well–defined by Theorems 4.0.7 and 4.0.6, and is independent of the orientations chosen for the components of L_f.

We often choose orientations for L_f to simplify calculations for the above formula; examples of these orientation choices are given in the proof of Lemma 4.0.9 below. Using our description of θ_{L_f}, a spin structure for these surgery diagrams is given by declaring which curves in the framed link L representing $S^1\times\Sigma_g$ are in the characteristic sublink; the defining relations $C\cdot L_i \equiv L_i\cdot L_i \pmod 2$ of definition 4.0.1 determines whether the additional components of L_f are in the characteristic sublink, by Proposition 4.0.4. For Figure 4.1, giving this diagram a spin structure is equivalent to declaring whether the zero framed components are in the characteristic sublink. The dotted circle component is never in the characteristic sublink for our map θ_{L_f}.

Lemma 4.0.9. *Let $\sigma \in \mathrm{Spin}(\Sigma_g)$ be a spin structure, and let $\beta_\sigma : \mathrm{Mod}_{g,1}[2] \to \mathrm{Map}(H_1(\Sigma_g), \mathbb{Z}/8)$ denote Sato's maps, where $\beta_\sigma([f])(x) = \beta_{\sigma,PD(x)}([f])$, for $PD(x) \in H^1(\Sigma_g)$, and $\beta_{\sigma,PD(x)}([f])$ is given by Lemma 4.0.8. Then the maps β_σ are homomorphisms.*

Proof. The set $S = \{t_c^{\pm 2} \mid c \text{ nonseparating simple closed curve}\}$ is a generating set for $\mathrm{Mod}_{g,1}[2]$. Let $x \in H^1(\Sigma_g)$, we will show that $\beta_{\sigma,x}(f_1\cdots f_n) = \sum_{i=1}^n \beta_{\sigma,x}(f_i)$, for $f_i \in S$. We have a framed link L for $S^1\times\Sigma_g$ described in construction 4.0.5. We then place curves corresponding to $f_1,..,f_n$ in pushoffs of a single (punctured) fiber surface, F that lies in $S^3\setminus L$. Note that the framed link L has linking matrix the zero matrix, hence components of L do not contribute to the $\theta_{L_f}(\sigma)\cdot\theta_{L_f}(\sigma)$ terms in Lemma 4.0.8.

The components corresponding to the monodromy come in pairs: let L_c, L_c' be two components corresponding to a $t_c^{\epsilon_c}$ factor of f, where $\epsilon_c = \pm 2$. By construction 4.0.5, L_c and L_c' can be isotoped to lie in a single fiber surface such that they bound an annulus in this fiber. Now the relations

$$p_i w_i + \sum_{j\neq i} lk(L_i, L_j) w_j \equiv p_i \pmod 2$$

of definition 4.0.1 applied to $L_c, L_c' \subset L_f$ imply that L_c and L_c' are either both in, or both out, of the characteristic sublink for $\theta_{L_f}(\sigma)$ by Proposition 4.0.4.

Orient L_c and L_c' oppositely in the fiber, then L_c and L_c' contribute ϵ_c to the $\theta_{L_f}(\sigma)\cdot\theta_{L_f}(\sigma)$ terms if they are both in the characteristic sublink, and 0 otherwise; let $\lambda : H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$ denote the Seifert linking form for the punctured fiber $F \subset S^3$. In the case $\beta_{\sigma,x}(t_c^{\epsilon_c})$, the linking matrix of the framed link obtained from construction 4.0.5, with the orientation convention given above, is

$$\begin{bmatrix} 0 & \dots & \dots & 0 & 0 & 0 \\ 0 & \dots & \dots & 0 & l_1 & -l_1 \\ 0 & \dots & \dots & 0 & l_2 & -l_2 \\ 0 & \ddots & \dots & 0 & \vdots & \vdots \\ 0 & \dots & \dots & 0 & l_{2g} & -l_{2g} \\ 0 & l_1 & \dots & l_{2g} & m+\epsilon_c/2 & -m \\ 0 & -l_1 & \dots & -l_{2g} & -m & m+\epsilon_c/2 \end{bmatrix},$$

where $m = \lambda(c,c)$, and the l_i are given by the linking numbers between L_c and the components of L. For the general case, pairs of rows and columns are adjoined to this linking matrix, corresponding to modifying the monodromy by squares of Dehn twists. Suppose that $t_c^{\epsilon_c}$ and $t_d^{\epsilon_d}$ are factors of the monodromy, then the relevant block of the linking matrix corresponding to the components L_c, L'_c, L_d, L'_d in the notation/orientation conventions above, has the form

$$\begin{bmatrix} \lambda(c,c)+\epsilon_c/2 & -\lambda(c,c) & l & -l \\ -\lambda(c,c) & \lambda(c,c)+\epsilon_c/2 & -l & l \\ l & -l & \lambda(d,d)+\epsilon_d/2 & -\lambda(d,d) \\ -l & l & -\lambda(d,d) & \lambda(d,d)+\epsilon_d/2 \end{bmatrix},$$

where $l = \lambda(c,d)$ or $\lambda(d,c)$, after fixing orientations. So linking with other components corresponding to the monodromy has no effect on the $\theta_{L_f}(\sigma)\cdot\theta_{L_f}(\sigma)$ terms. The proof of Proposition 4.0.4 also implies that whether L_c and L'_c are in the characteristic sublink only depends on the components of L, and not on other components corresponding to the monodromy. So the $\theta_{L_f}(\sigma)\cdot\theta_{L_f}(\sigma)$ terms are additive with respect to products of squares of Dehn twists.

For the Arf invariant terms in Lemma 4.0.8, note that the Arf invariant is preserved under orientation–preserving band sums. In the convention above, L_c and L'_c bound an annulus, and are oriented oppositely, so after band summing L_c with L'_c, we get an unknot that can be isotoped to be disjoint from the rest of the link. Apply this to all pairs of sublinks corresponding to the monodromy, and note that the sublink of L that is in the characteristic sublink is a disjoint union of unknots, since the dotted circle component is never in the characteristic sublink. Therefore, the Arf invariant terms are always zero in Lemma 4.0.8. □

We have a formula for β_σ on squares of Dehn twists on nonseparating curves, that follows from the proof of Lemma 4.0.9. This formula is analogous to Sato's [Sat10, Proposition 5.2].

Corollary 1. Let c be a nonseparating simple closed curve in $\Sigma_{g,1}$ then we have that

$$\beta_{\sigma,x}(t_c^2) = \begin{cases} 0, & c_1 \cup c_2 \in \theta_{L_{t_c^2}}(\sigma) \text{ and } \in \theta_{L_{t_c^2}}(\sigma+x) \\ 0, & c_1 \cup c_2 \notin \theta_{L_{t_c^2}}(\sigma) \text{ and } \notin \theta_{L_{t_c^2}}(\sigma+x) \\ 1, & c_1 \cup c_2 \in \theta_{L_{t_c^2}}(\sigma+x) \text{ and } \notin \theta_{L_{t_c^2}}(\sigma) \\ -1, & c_1 \cup c_2 \in \theta_{L_{t_c^2}}(\sigma) \text{ and } \notin \theta_{L_{t_c^2}}(\sigma+x), \end{cases}$$

where c_1 and c_2 are pushoffs of c in the fiber surface, framed using the Seifert pairing via construction 4.0.5, and $c_1 \cup c_2 \in \theta_{L_{t_c^2}}(\sigma)$ denotes that the components corresponding to the monodromy are in the characteristic sublink for construction 4.0.5 applied to $\sigma \in \mathrm{Spin}(\Sigma_g)$.

Chapter 5

The Birman–Craggs maps

In this chapter, we use our framework to calculate the Birman–Craggs maps, and to give a proof that these maps are homomorphisms. Then, we calculate Sato's maps on elements of the Torelli group, and find relations between Sato's maps and the Birman–Craggs maps.

5.0.1 The Birman–Craggs homomorphisms.

Birman and Craggs [BC78] associated to every element $k \in \mathcal{I}_{g,1}$ a 3-manifold $M(k)$ defined via a Heegaard splitting. They proved that taking the Rochlin invariant produces a family of homomorphisms from the Torelli group to $\mathbb{Z}/2$. Johnson then reformulated the family of homomorphisms in the following way [Joh80b, Section 5 and 6].

Let $h : \Sigma_g \to S^3$ be a Heegaard embedding. Split S^3 along $h(\Sigma_g)$ into two handlebodies A and B. Take $k \in \mathcal{I}_{g,1}$ and reglue A to B along their boundaries by the map k to get the closed 3-manifold $M(h,k)$; since k acts trivially on the homology of the Heegaard surface, $M(h,k)$ is a homology 3-sphere, we then take the Rochlin invariant of the unique spin structure

$$\mu(h,k) = R(M(h,k)) \pmod 2.$$

This rewrites every Birman–Craggs homomorphism in the form $\mu(h,-) : \mathcal{I} \to \mathbb{Z}/2$. Johnson was able to enumerate all the maps $\mu(h,-)$ using the Seifert pairing induced by the Heegaard embedding h. He collected all these maps into one homomorphism, often referred to as the Birman–Craggs–Johnson map [Joh80b, Section 9]. It turns out that this map is enough to calculate the torsion part of the abelianization of the Torelli group.

We begin our discussion by writing a model for computing the Birman–Craggs maps using the formula in Theorem 4.0.7

$$\mu(L,C) = \Lambda_L - C \cdot C + 8\,\mathrm{Arf}(C) \pmod{16}. \tag{5.1}$$

Here L is a framed link for $M(h,k)$, C is the unique characteristic sublink, Λ_L denotes the signature of the linking matrix of L, $C \cdot C$ denotes the sum of all entries in the linking matrix of C, and $\mathrm{Arf}(C)$ denotes the Arf invariant of C.

A model for calculating the Birman–Craggs maps.

First, we describe a Heegaard splitting of S^3. Take handlebodies $\pm H_g = \nu(\pm\Delta_g)$, and fix $\Sigma_g = \partial(\nu(\Delta_g))$ as in Chapter 3.0.2. Let $\{a_i, b_i\}$ denote the standard symplectic basis for $H_1(\Sigma_g;\mathbb{Z})$, as pictured in Figure 5.1. We fix our Heegaard splitting for S^3 to be

$$S^3 = H_g \bigcup_{i_g} -H_g = S(i_g),$$

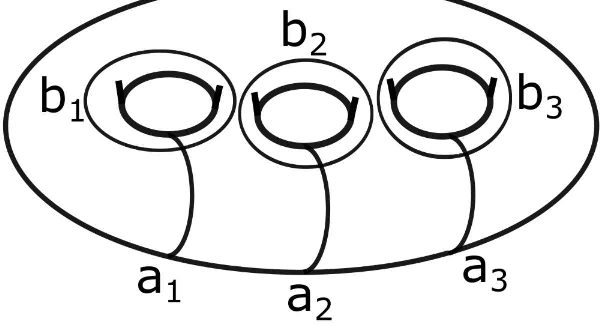

Figure 5.1

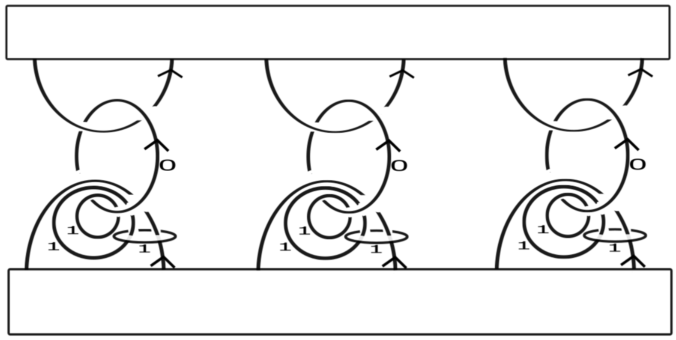

Figure 5.2

where $i_g := \prod_{j=1}^{g} t_{b_j} t_{a_j} t_{b_j}$. Then, for any $k \in \mathcal{I}_{g,1}$, we set

$$M(h,k) = H_g \bigcup_{i_g \circ k} -H_g = S(i_g \circ k),$$

where $h : \Sigma_g \to S^3$ denotes the inclusion map. Since we fix our Heegaard embedding h to be the inclusion from now on, we simplify the notation by letting $V(k)$ denote the 3−manifold $M(h,k)$ described above. If we use construction 3.0.1 (2), we find that our model for S^3 is given by Figure 5.2.

By Johnson, the Torelli group $\mathcal{I}_{g,1}$ is generated by *bounding pair maps* [Joh79, Theorem 1]. Here a bounding pair d_1, d_2 is a pair of simple closed curves on the surface which bound a nontrivial subsurface, and the bounding pair map is given by $t_{d_1} t_{d_2}^{-1}$. Suppose that $k \in \mathcal{I}_{g,1}$ is given as a product of bounding pair maps. We wish to use the method of composing tangles as in construction 3.0.3 (2) to compute $\mu(V(k))$. Here is the construction we use:

Construction 5.0.1. *Suppose $k = f_{n-1} \cdots f_1 : \Sigma_g \to \Sigma_g$ is a diffeomorphism, where each f_i is a bounding pair map, or its inverse.*

1. *Use construction 3.0.3 (2) with $f = f_n f_{n-1} \cdots f_1$, where $f_n = \iota_g = \prod_{j=1}^{g} t_{b_j} t_{a_j} t_{b_j}$, and f_i as above for $i \neq n$, to obtain a tangle diagram for $V(k)$. Let L denote the closed components of this diagram; surgery along L gives $V(k)$.*

2. *Let $\overline{L}_{g,n}$ denote the closed components obtained from construction 3.0.3 (2) applied to $f_n = \iota_g$ and $f_i = \mathrm{id}$ for $i = 1,..,n-1$. Then surgery on $\overline{L}_{g,n}$ gives S^3, and $\overline{L}_{g,n}$ is always a sublink of L. The relations of definition 4.0.1 imply that the unique characteristic sublink of $\overline{L}_{g,n}$ is the union of the components corresponding to the t_{b_j} factors of ι_g in the surgery diagram for $S(\iota_g)$ obtained from construction 3.0.3 (1); see Figure 5.3 for the $n = g = 3$ case, and compare this with Figures 5.1 and 5.2.*

3. *The components of $L - \overline{L}_{g,n}$ come in pairs corresponding to the bounding pairs f_i, $i = 1,..,n-1$ via construction 3.0.3 (1). Let L_i, L_{i+1} be a pair of components of L corresponding to the bounding pair $t_d t_e^{-1}$ in the factorisation of k. Now $[d] = [e] \in H_1(\Sigma_g; \mathbb{Z})$, and the linking numbers of L_i, L_{i+1} with the other components is determined by the Seifert pairing λ for $\Sigma_g = \partial(\nu(\Delta_g)) \subset S^3$. So L_i and L_{i+1} have the same linking number with the other components of L up to sign. Using definition 4.0.1 we have $p_i w_i + (p_i - 1) w_{i+1} + \sum_{j \neq i,i+1} lk(L_i, L_j) w_j \equiv p_i \pmod 2$ and $p_{i+1} w_{i+1} + (p_{i+1} - 1) w_i + \sum_{j \neq i,i+1} lk(L_i, L_j) w_j \equiv p_{i+1} \pmod 2$. By (2), the components of $\overline{L}_{g,n}$ in the characteristic sublink are disjoint from L_i, L_{i+1}, so $\sum_{j \neq i,i+1} lk(L_i, L_j) w_j = 0$. Since $p_i = \lambda([d],[d]) + 1$ and $p_{i+1} = \lambda([d],[d]) - 1$, we conclude that $w_i \equiv w_{i+1} \equiv \lambda([d],[d]) + 1 \pmod 2$. Therefore, L_i and L_{i+1} are in the characteristic sublink if and only if $\lambda([d],[d])$ is even.*

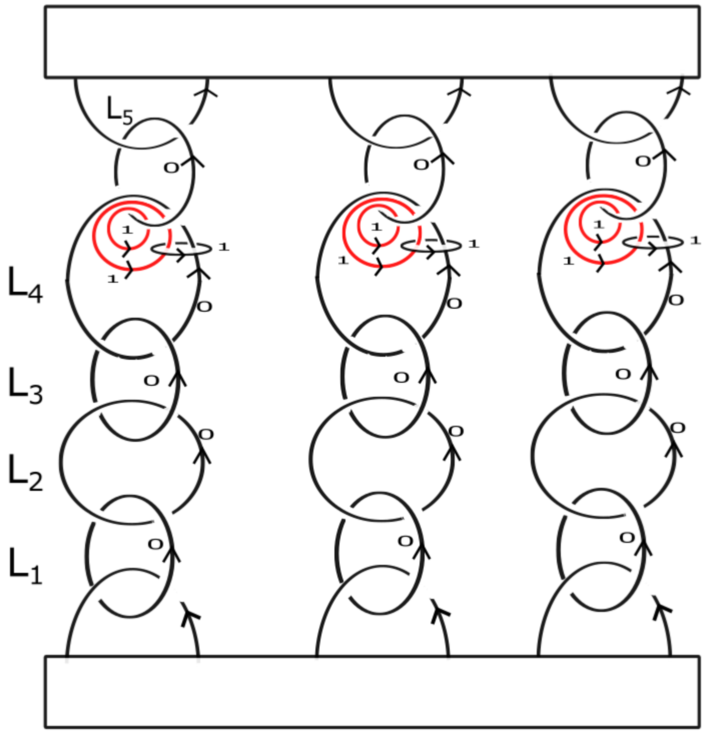

Figure 5.3: The framed link $\overline{L}_{3,3}$, the coloured components give the characteristic sublink.

To show that the Birman–Craggs maps are homomorphisms, we must first analyse the linking matrix of the sublink $\overline{L}_{g,n}$ obtained in construction 5.0.1.

Lemma 5.0.2. *The signature of the linking matrix of $\overline{L}_{g,n}$, using the orientation convention of Figure 5.3, is $2g$.*

Proof. Note that $\overline{L}_{g,n}$ is the disjoint union of g copies of $\overline{L}_{1,n}$; c.f. Figure 5.3. In the orientation convention of Figure 5.3, one can show, using Sylvester's law of inertia, that

$$\Lambda_{\overline{L}_{1,n}} = \text{Sign}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \Lambda_{\overline{L}_{1,n-1}}.$$

For example, add the first row to the third, then add the first column to the third in the linking matrix of $\overline{L}_{1,n}$ in the ordering of Figure 5.3. Since $\Lambda_{\overline{L}_{1,1}} = \Lambda_{\overline{L}_{1,2}} = 2$, and $\text{Sign}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = 0$, we conclude that $\Lambda_{\overline{L}_{1,n}} = 2$, therefore $\Lambda_{\overline{L}_{g,n}} = 2g$. □

Theorem 5.0.3. *[BC78, Theorem 8] Let $\mu : \mathcal{I}_{g,1} \to \mathbb{Z}/2$ be given by $k \mapsto \mu(L, C)$, where (L, C) is obtained from construction 5.0.1 applied to k, and $\mu(L, C)$ is the formula given in Theorem 4.0.7. Then μ is a well–defined homomorphism.*

Proof. To prove that the map is well–defined, note that for any two factorisations of k into bounding pairs, the manifolds obtained from construction 5.0.1 are diffeomorphic by Lemma 3.0.2. There is only one spin structure, so the corresponding values of $\mu(L, C)$ must be equal modulo 16, by Theorem 4.0.7.

To show that the maps are homomorphisms, we analyse the pair (L, C) obtained from construction 5.0.1. Additivity of the $8\,\text{Arf}(C)$ terms in formula (5.1) follows from the components corresponding to the bounding pairs in construction 5.0.1 being disjoint from each other.

To deal with the $\Lambda_L - C \cdot C$ terms in formula (5.1), we orient a pair of components L_i and L_{i+1} corresponding to a bounding pair factor $t_d t_e^{-1}$, as in construction 5.0.1 (3), oppositely on the Heegaard surface. Fix the orientation conventions for the components $\overline{L}_{g,n}$ obtained from construction 5.0.1 (2) as in Figure 5.3. The linking matrix for the closed components in the tangle diagram for $V(k)$ obtained from construction 5.0.1 can be written as

$$R = \begin{pmatrix} \overline{A} & \dots & \dots & 0 & A_{1,1} & A_{2,1} & \vdots & A_{n-1,1} \\ 0 & \ddots & \vdots & 0 & A_{1,2} & A_{2,2} & \vdots & A_{n-1,2} \\ 0 & \dots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \overline{A} & A_{1,g} & A_{2,g} & \dots & A_{n-1,g} \\ A'_{1,1} & A'_{1,2} & \dots & A'_{1,g} & A_1 & 0 & \dots & 0 \\ A'_{2,1} & A'_{2,2} & \dots & A'_{2,g} & 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & \ddots & 0 \\ A'_{n-1,1} & A'_{n-1,2} & \dots & A'_{n-1,g} & 0 & 0 & \dots & A_{n-1} \end{pmatrix}.$$

The top diagonal is the linking matrix of $\overline{L}_{g,n}$, which is the direct sum of g copies of the linking matrix $\overline{A}$ of $\overline{L}_{1,n}$. Note that $\overline{L}_{1,n}$ has $2(n-1)+4$ components. The A_i are the linking matrices of a pair of components corresponding to a bounding pair factor f_i in construction 5.0.1. Suppose that $f_i = t_d t_e^{-1}$, then

$$A_i = \begin{pmatrix} \lambda([d],[d]) + 1 & -\lambda([d],[d]) \\ -\lambda([d],[d]) & \lambda([d],[d]) - 1 \end{pmatrix},$$

in our orientation convention. The $A_{i,j}$ are $(2(n-1)+4) \times 2$ matrices corresponding to how the pair of components coming from f_i link with the components of $\overline{L}_{g,n}$. In our orientation convention, the $A_{i,j}$ have identical columns but with opposite signs. The $A'_{i,j}$ are the transpose of the $A_{i,j}$, so they have identical rows but with opposite signs.

Recall that if E is an elementary matrix for a row operation, then ERE^T is obtained from the matrix R by simultaneous row and column operations. Sylvester's law of inertia states that the signatures of R and ERE^T are equal. We repeatedly apply these operations to get a matrix of the form

$$R' = \begin{bmatrix} \overline{A} & \dots & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \ddots & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & \ddots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \overline{A} & 0 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 & A'_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & A'_2 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & A'_n \end{bmatrix}.$$

Here the signature of the matrices A'_i are the same as the signature of the matrices A_i. Since the signature of any matrix of the form A_i is zero, we get that $\Lambda_L = 2g$ by Lemma 5.0.2. The only components of C that contribute to the $C \cdot C$ terms are the ones contained in $\overline{L}_{g,n}$. So we have $\Lambda_L - C \cdot C = 2g - 2g = 0$. □

5.0.2 Evaluation on separating twists

The aim of the rest of this section is to refine formulas for evaluating Sato's maps on elements of the Torelli group. In some cases, we can relate Sato's maps to the Birman–Craggs maps. See Appendix 8.2.5 for the definition of a separating twist.

We need the following evaluation of $\mu(h, -)$, due to Johnson. Here h is the inclusion, so we think of the Heegaard surface as being in S^3.

Theorem 5.0.4. *[Joh80b, Theorem 1] Let γ be a separating simple closed curve on the Heegaard surface, then*

$$\mu(h, t_\gamma) = \mathrm{Arf}(\gamma).$$

Here $\mathrm{Arf}(\gamma)$ can be evaluated by taking the subsurface $S \subset \Sigma_g$ which γ bounds and calculating the Arf invariant of the quadratic form induced by the Seifert pairing on the Heegaard surface.

We can now begin relating Sato's maps to the Birman–Craggs maps.

Corollary 5.0.5. *Let γ be a separating simple closed curve in Σ_g, then*

$$\beta_{\sigma,x}(t_\gamma) = 4(\mathrm{Arf}(\theta_{L_{t_\gamma}}(\sigma)) - \mathrm{Arf}(\theta_{L_{t_\gamma}}(\sigma + x))).$$

Proof. We can apply construction 3.0.1 to $f = t_\gamma$ to get a framed link L_{t_γ} for M_{t_γ}. Then the method of construction 4.0.5 allows us to define a characteristic sublink $\theta_{L_{t_\gamma}}(\sigma)$. The method of proof of Theorem 4.0.6 implies that the corresponding spin manifold $(L_{t_\gamma}, \theta_{L_{t_\gamma}}(\sigma))$ obtained this way is spin diffeomorphic to construction 4.0.5 applied to any factorisation of t_γ into squares of Dehn twists. We can evaluate $\beta_{\sigma,x}(t_\gamma)$ using Theorem 4.0.7 applied to $(L_{t_\gamma}, \theta_{L_{t_\gamma}}(\sigma))$ and $(L_{t_\gamma}, \theta_{L_{t_\gamma}}(\sigma + x))$ as in Lemma 4.0.8.

Let L be the framed link for $S^1 \times \Sigma_g$ obtained from construction 3.0.1 (3) with $f = \mathrm{id}$, then L is a sublink of L_{t_γ}. Let F be the fiber surface pictured in Figure 3.6. The components of L can all be isotoped in S^3 to the canonical basis for $H_1(F; \mathbb{Z})$ in the fiber. Let L_i be the component of $L_{t_\gamma} - L$ that corresponds to the monodromy. Then the linking number of L_i with any of the components of L is determined by the Seifert pairing λ of the surface $F \subset S^3$. Since $[\gamma] = 0 \in H_1(F; \mathbb{Z})$, L_i has linking number 0 with any component of L. This implies that the linking matrix of L_{t_γ} is zero everywhere except for one entry on the main diagonal, which is the framing of the curve L_i. We compute that this framing is given by $\lambda(\gamma, \gamma) + 1 = \lambda(0, 0) + 1 = 1$. For any spin structure $\sigma \in \mathrm{Spin}(\Sigma)$, using the relations of definition 4.0.1 we have that

$$w_i + 0 \equiv 1 \pmod 2.$$

So L_i is always in the characteristic sublink. This gives us that $\theta_{L_{t_\gamma}}(\sigma + x) \cdot \theta_{L_{t_\gamma}}(\sigma + x) - \theta_{L_{t_\gamma}}(\sigma) \cdot \theta_{L_{t_\gamma}}(\sigma) = 0$ always, so we have that

$$\beta_{\sigma,x}(t_\gamma) = 4(\mathrm{Arf}(\theta_{L_{t_\gamma}}(\sigma)) - \mathrm{Arf}(\theta_{L_{t_\gamma}}(\sigma + x))).$$

□

The previous proof gives the following relation between the Rochlin invariant and the Arf invariant:

$$R(M_{L_{t_\gamma}}, \theta_{L_{t_\gamma}}(\sigma)) = 8\,\mathrm{Arf}(\theta_{L_{t_\gamma}}(\sigma)) \pmod{16},$$

when γ is separating. Let η denote the spin structure on Σ_g with $\theta_{L_{t_\gamma}}(\eta)$ containing none of the components of the link L that represents $S^1 \times \Sigma_g$. We get

$$R(M_{L_{t_\gamma}}, \theta_{L_{t_\gamma}}(\eta)) = 8\,\mathrm{Arf}(\gamma) \pmod{16},$$

the right side can be evaluated using the cut surface of $\gamma \subset F = \Sigma_{g,1}$ and the quadratic form on $H_1(F)$ induced by the Seifert pairing λ of $F \hookrightarrow S^3$.

Note that for $[h] \in \mathrm{Mod}_{g,1}[2]$ and $[f] \in \mathrm{Mod}_{g,1}$, the diffeomorphism $\mathrm{id} \times f$ descends to a diffeomorphism $M_h \to M_{fhf^{-1}}$ of mapping tori. Under this diffeomorphism the spin structure $\theta(\sigma)$ on $M_{fhf^{-1}}$ pulls back to the spin structure $\theta(f^*\sigma)$ on M_h. This implies that

$$R(M_{fhf^{-1}}, \theta(\sigma)) = R(M_h, \theta(f^*\sigma)),$$

for all $[f] \in \mathrm{Mod}_{g,1}$ and $[h] \in \mathrm{Mod}_{g,1}[2]$ (in the notation of Chapter 2).

Johnson proved that $\mathrm{Spin}(\Sigma_g)$ is in bijection with symplectic quadratic forms (see Theorem 2.0.1). Arf showed that two symplectic quadratic forms are in the same orbit under the symplectic group if and only if they have the same Arf invariant. So there are exactly two $\mathrm{Mod}_{g,1}$ orbits in $\mathrm{Spin}(\Sigma_g)$. Suppose we chose $x \in H^1(\Sigma_g)$ such that $\eta + x \in \mathrm{Spin}(\Sigma_g)$ is in the same orbit as η; write $\eta + x = f^*\eta$ for some $[f] \in \mathrm{Mod}_{g,1}$, then

$$\beta_{\sigma,x}(t_\gamma) = (R(M_{L_{t_\gamma}}, \theta_{L_{t_\gamma}}(\eta)) - R(M_{L_{t_\gamma}}, \theta_{L_{t_\gamma}}(f^*\eta)))/2 = (R(M_{L_{t_\gamma}}, \theta_{L_{t_\gamma}}(\eta)) - R(M_{L_{ft_\gamma f^{-1}}}, \theta_{L_{ft_\gamma f^{-1}}}(\eta)))/2,$$

where $ft_\gamma f^{-1} = t_{f(\gamma)}$, and $f(\gamma)$ is also a separating curve of the same genus on the fiber surface. So we have

$$\beta_{\sigma,x}(t_\gamma) = 4(\mathrm{Arf}(\gamma) - \mathrm{Arf}(f(\gamma))) \pmod 8.$$

Note that surgery along $\gamma \subset \Sigma_{g,1}$ with framing 1 is equivalent to cutting S^3 open along a Heegaard surface and regluing by t_γ. Since t_γ acts trivially on homology, this gives a homology sphere. So this relates the Rochlin invariant of mapping tori to the Rochlin invariant of homology spheres, and to Johnson's description of the Birman-Craggs homomorphisms [Joh80b]. In summary, if we combine Theorem 5.0.4 and Corollary 5.0.5 with the discussion above, we get the following.

Corollary 2. Let c be a separating curve on $\Sigma_{g,1}$. Let η be the spin structure on Σ_g with the characteristic sublink of $\theta_{L_{t_c}}(\eta)$ containing none of the components from the link L that represents $S^1 \times \Sigma_g$. Suppose $\sigma = f^*(\eta)$ and $\sigma + x = h^*(\eta)$ for $[f], [h] \in \mathrm{Mod}_{g,1}$, then

$$\beta_{\sigma,x}(t_c) = \mu(t_{f(c)}) - \mu(t_{h(c)}) \pmod 2,$$

where μ denotes the Birman-Craggs homomorphism for the standard embedding $\Sigma_g \hookrightarrow S^3$.

5.0.3 Evaluation on bounding pairs

A bounding pair is a pair of disjoint, homologous, non-separating simple closed curves a, b on $\Sigma_{g,1}$, and the bounding pair map is $f = t_a t_b^{-1}$; see Appendix 8.2.5 for more details.

To calculate $\beta_{\sigma,x}(t_a t_b^{-1})$, use construction 3.0.1 with $f = t_a t_b^{-1}$ to obtain a framed link L_f for M_f. Apply the method of construction 4.0.5 to obtain the pair $(L_f, \theta_{L_f}(\sigma))$ for any $\sigma \in \mathrm{Spin}(\Sigma_g)$. The method of proof of Theorem 4.0.6 implies that the manifold $(L_f, \theta_{L_f}(\sigma))$ obtained this way, is spin diffeomorphic to any manifold obtained from construction 4.0.5 applied to any factorisation of f into squares of Dehn twists. Hence we can calculate $\beta_{\sigma,x}(f)$ by using Theorem 4.0.7 to get

$$\beta_{\sigma,x}(f) = (\theta_{L_f}(\sigma + x) \cdot \theta_{L_f}(\sigma + x) - \theta_{L_f}(\sigma) \cdot \theta_{L_f}(\sigma) + 8(\mathrm{Arf}(\theta_{L_f}(\sigma)) - \mathrm{Arf}(\theta_{L_f}(\sigma + x))))/2 \pmod 8.$$

Let L be the sublink of L_f corresponding to $S^1 \times \Sigma_g$, and let F be a punctured fiber surface pictured as in Figure 3.6. Suppose that $\lambda([a], [a]) = \lambda([b], [b]) = \lambda([a], [b]) = lk(a, b) = m$, where $\lambda(-, -)$ is the Seifert linking form for $F \subset S^3$. The framed link L_f is obtained from L by placing the curves a, b in F, and framing them using the Seifert pairing, to get L_i, L_{i+1}. For any component L_j of L, we have $lk(L_j, L_i) = lk(L_j, L_{i+1})$. To specify $(L_f, \theta_{L_f}(\sigma))$, we fix which components of L are in the characteristic sublink, and use definition 4.0.1 to find the full characteristic sublink. We have

$$(m+1)w_i + \sum_{j \neq i, i+1} lk(L_i, L_j)w_j + mw_{i+1} \equiv (m-1)w_{i+1} + \sum_{j \neq i, i+1} lk(L_i, L_j)w_j + mw_i \equiv m+1 \pmod 2,$$

which simplifies to

$$(m+1)w_i + mw_{i+1} = (m-1)w_{i+1} + mw_i \equiv m + 1 - \sum_{j \neq i, i+1} lk(L_i, L_j)w_j \pmod 2. \tag{5.2}$$

So we see that L_i and L_{i+1} are always either both in, or both out of any characteristic sublink of L_f. Now orient L_i and L_{i+1} oppositely in the fiber surface, this does not change the framings of L_i, L_{i+1}, but changes the sign of the linking number of L_{i+1} with every other component. Then, up to the ordering of the components, the linking matrix for L_f is

$$\begin{bmatrix} 0 & \dots & \dots & 0 & 0 & 0 \\ 0 & \dots & \dots & 0 & l_1 & -l_1 \\ 0 & \dots & \dots & 0 & l_2 & -l_2 \\ 0 & \ddots & \dots & 0 & \vdots & \vdots \\ 0 & \dots & \dots & 0 & l_{2g} & -l_{2g} \\ 0 & l_1 & \dots & l_{2g} & m+1 & -m \\ 0 & -l_1 & \dots & -l_{2g} & -m & m-1 \end{bmatrix}.$$

Suppose that $w_i \equiv w_{i+1} \equiv 1$, then since $\theta_{L_f}(\sigma) \cdot \theta_{L_f}(\sigma)$ is the sum of the entries in the linking matrix of the characteristic sublink specified by $\theta_{L_f}(\sigma)$, we have that

$$\theta_{L_f}(\sigma) \cdot \theta_{L_f}(\sigma) = m + 1 + m - 1 - 2m = 0.$$

If $w_i \equiv w_{i+1} \equiv 0$, then $\theta_{L_f}(\sigma) \cdot \theta_{L_f}(\sigma) = 0$. These are all the cases, so we have

$$\beta_{\sigma,x}(t_a t_b^{-1}) = 4(\mathrm{Arf}(\theta_{L_{t_a t_b^{-1}}}(\sigma)) - \mathrm{Arf}(\theta_{L_{t_a t_b^{-1}}}(\sigma + x))) \pmod 8.$$

Let $\eta \in \mathrm{Spin}(\Sigma_g)$ denote the spin structure with the characteristic sublink of $\theta_{L_{t_a t_b^{-1}}}(\eta)$ containing none of the components from the link L for $S^1 \times \Sigma_g$. Using the relations of (5.2), we have that if $m = lk(a, b)$ is even then

$$w_i \equiv w_{i+1} \equiv 1 \pmod 2,$$

so L_i and L_{i+1} are in the characteristic sublink, and we have

$$R(M_{L_{t_a t_b^{-1}}}, \theta_{L_{t_a t_b^{-1}}}(\eta)) = \Lambda_{L_{t_a t_b^{-1}}} + 8\,\mathrm{Arf}(a \cup b) = 8\,\mathrm{Arf}(a \cup b),$$

where $\Lambda_{L_{t_a t_b^{-1}}}$ is the signature of the linking matrix given above. Here $\mathrm{Arf}(a \cup b)$ can be evaluated using the cut surface of $a \cup b \subset F = \Sigma_{g,1}$ and the quadratic form induced by the Seifert pairing λ.

If m is odd then

$$m w_{i+1} = 0 \pmod 2,$$

so L_i and L_{i+1} are not in the characteristic sublink, and we have

$$R(M_{L_{t_a t_b^{-1}}}, \theta_{L_{t_a t_b^{-1}}}(\eta)) = \Lambda_{L_{t_a t_b^{-1}}} = 0.$$

Note that for $a \cup b \subset \Sigma_{g,1}$, surgery with coefficients $m \pm 1$ is equivalent to cutting S^3 open along a Heegaard surface, and regluing by $t_a t_b^{-1}$. Since $t_a t_b^{-1}$ acts trivially on homology the resulting space is a homology sphere. So we have a relation between the Rochlin invariants of a mapping tori, and Rochlin invariants of homology spheres.

Now, we relate Sato's maps on bounding pairs to the Birman–Craggs maps, using the same method as in the case of separating curves. Suppose we have a spin structure $\sigma \in \mathrm{Spin}(\Sigma_g)$ and a class $x \in H^1(\Sigma_g)$ such that there exists mapping classes $[f], [h] \in \mathrm{Mod}_{g,1}$ with $\sigma = f^*(\eta)$ and $\sigma + x = h^*(\eta)$. We have, in the notation of Chapter 2,

$$\beta_{\sigma,x}(t_a t_b^{-1}) = (R(M_{t_a t_b^{-1}}, \theta(f^*(\eta))) - R(M_{t_a t_b^{-1}}, \theta(h^*(\eta))))/2 = (R(M_{t_{f(a)} t_{f(b)}^{-1}}, \theta(\eta))) - R(M_{t_{h(a)} t_{h(b)}^{-1}}, \theta(\eta)))/2.$$

We also need the following calculation of the Birman–Craggs maps, due to Johnson; this calculation also follows from the proof of Theorem 5.0.3.

Lemma 5.0.6. *[Joh80b, Theorem 1] Let a, b be a pair of simple closed curves on $\Sigma_{g,1}$ that bound a subsurface, and let $\lambda : H_1(\Sigma_g;\mathbb{Z}) \times H_1(\Sigma_g;\mathbb{Z}) \to \mathbb{Z}$ denote the Seifert linking form for the standard embedding $\Sigma_g \hookrightarrow S^3$. Then for the Birman-Craggs map $\mu : \mathcal{I}_{g,1} \to \mathbb{Z}/2$ corresponding to the embedding $\Sigma_g \hookrightarrow S^3$ we have that:*

1. *$\mu(t_a t_b^{-1}) = 0$, if $\lambda(a,a)$ is odd,*
2. *$\mu(t_a t_b^{-1}) = 8\,\mathrm{Arf}(a \cup b)$, if $\lambda(a,a)$ is even.*

In summary, if we combine Lemma 5.0.6 with the calculations of this subsection, we get the following.

Corollary 3. Let a, b be a pair of simple closed curves on $\Sigma_{g,1}$ that bound a subsurface. Let η be the spin structure on Σ_g with the characteristic sublink of $\theta_{L_{t_a t_b^{-1}}}(\eta)$ containing none of the components from the link L that represents $S^1 \times \Sigma_g$. If $\sigma = f^*(\eta)$ and $\sigma + x = h^*(\eta)$ for $[f], [h] \in \mathrm{Mod}_{g,1}$, then

$$\beta_{\sigma,x}(t_a t_b^{-1}) = \mu_\iota(t_{f(a)} t_{f(b)}^{-1}) - \mu_\iota(t_{h(a)} t_{h(b)}^{-1}) \pmod 2,$$

where μ_ι denotes the Birman–Craggs map for the standard embedding $\iota : \Sigma_g \hookrightarrow S^3$. In particular, we have $\beta_{\sigma,x} = \mu_{\iota\circ f} - \mu_{\iota\circ h} \pmod 2$.

Chapter 6

Relation to Meyer's signature cocycle

In this chapter, we apply the methods developed above to study Meyer's signature cocycle restricted to $\mathrm{Mod}_{g,1}[2]$. The extension of the Birman–Craggs maps to $\mathrm{Mod}_{g,1}[2]$ no longer give homomorphisms, but we can find a relation between these extensions and Meyer's signature cocycle.

Let P denote a pair of pants, that is, a sphere with three boundary components. Pick two based loops that run around one distinct boundary component each as in Figure 6.1, call them α, β. Identify $\pi_1(P)$ with the free group generated by α, β. Let $[f], [h] \in \mathrm{Mod}_{g,1}$ be mapping classes, and consider the Σ_g–bundle $E_{f,h}$ over P with monodromy $\rho : \pi_1(P) \to \mathrm{Mod}_{g,1}$ given by $\alpha \mapsto [f], \beta \mapsto [h]$. The diffeomorphism type of $E_{f,h}$ does not depend on the choice of representatives for the mapping classes $[f], [h]$. Furthermore, $E_{f,h}$ has a natural orientation coming from that of Σ_g and P.

Meyer's signature cocycle is defined by

$$\begin{aligned} \tau_g : \mathrm{Mod}_{g,1} \times \mathrm{Mod}_{g,1} &\to \mathbb{Z} \\ ([f], [h]) &\mapsto \mathrm{Sign}(E_{f,h}), \end{aligned}$$

where $\mathrm{Sign}(E_{f,h})$ denotes the signature of the 4–manifold $E_{f,h}$ [Mey73]. Note that $\partial E_{f,h} = M_f \sqcup M_h \sqcup M_{(f\circ h)^{-1}}$. Now, construction 4.0.5 and Theorem 4.0.6 give us a well–defined map

$$\begin{aligned} R_\sigma : \mathrm{Mod}_{g,1}[2] &\to \mathbb{Z}/16 \\ [f] &\mapsto R(M_{L_f}, \theta_{L_f}(\sigma)), \end{aligned}$$

where $R(M_{L_f}, \theta_{L_f}(\sigma))$ denotes the Rochlin invariant of the spin 3–manifold $(M_{L_f}, \theta_{L_f}(\sigma))$ obtained from construction 4.0.5. Sato shows in [Sat10, Lemma 2.2] that for any $\sigma \in \mathrm{Spin}(\Sigma_g)$, there exists

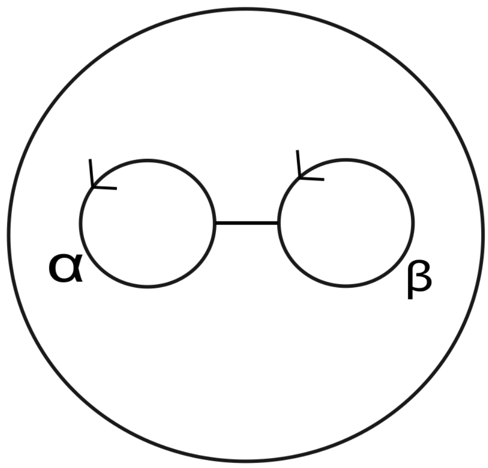

Figure 6.1

a spin structure on $E_{f,h}$ that spin bounds the mapping tori $M_f \sqcup M_h \sqcup M_{(f\circ h)^{-1}}$ with spin structure $\theta(\sigma)$ (in the notation of Chapter 2). Then we have

$$R(M_f, \theta(\sigma)) + R(M_h, \theta(\sigma)) - R(M_{f\circ h}, \theta(\sigma)) \equiv \mathrm{Sign}(E_{f,h}) \pmod{16},$$

or, in the language of group cohomology $\tau_g \equiv \partial(R_\sigma) \pmod{16}$, for any $\sigma \in \mathrm{Spin}(\Sigma_g)$.

Recall that in defining R_σ using construction 4.0.5, we start with a framed link L representing $S^1 \times \Sigma_g$. We have the monodromy f written as a product of Dehn twists, and our method gives a framed link L_f representing M_f, obtained by placing the curves appearing in the factorisation in pushoffs of the fiber surface in $S^3 \setminus L$, framed using the Seifert pairing; L_f has L as a sublink. Let η denote the spin structure on Σ_g such that the characteristic sublink of $\theta_{L_f}(\eta)$ contains none of the components from the framed link L. For any component added to our diagram of $S^1 \times \Sigma_g$ to modify the monodromy, the relations of definition 4.0.1 determine whether this component is added to the characteristic sublink.

We wish to relate the above construction to the Birman–Craggs map for the standard inclusion $\iota : \Sigma_g \hookrightarrow S^3$. Recall Johnson's definition of this map

$$\begin{aligned} \mu_\iota : \mathrm{Mod}_{g,1}[2] &\to \mathbb{Z}/16 \\ [f] &\mapsto R(M(\iota, f)), \end{aligned}$$

where $R(M(\iota, f))$ is the Rochlin invariant of the manifold $M(\iota, f)$, obtained by cutting S^3 along $\iota(\Sigma_g)$ into two handlebodies, and regluing them along their boundaries by the map f. Since $[f] \in \mathrm{Mod}_{g,1}[2]$, the manifold $M(\iota, f)$ is a $\mathbb{Z}/2$–homology sphere, so it has a unique spin structure.

Let $\widetilde{L}_f = L_f \setminus L$ denote the framed link given by the union of the components corresponding to the monodromy f. The fiber surface in our diagram for $S^1 \times \Sigma_g$ lies in S^3 as the standard embedding $\iota : \Sigma_{g,1} \hookrightarrow S^3$ by Figure 3.6. Surgery along the framed link $\widetilde{L}_f$ is equivalent to cutting S^3 along this Heegaard surface, and regluing via $[f] \in \mathrm{Mod}_{g,1}[2]$, which gives the $\mathbb{Z}/2$–homology sphere $M(\iota, f)$. We begin relating R_η and μ_ι with the following Lemma.

Lemma 6.0.1. *The characteristic sublink of $\widetilde{L}_f$ for $M(\iota, f)$ coincides with the characteristic sublink for $(L_f, \theta_{L_f}(\eta))$ obtained from construction 4.0.5. For a pair of components L_i and L_{i+1} in $\widetilde{L}_f$ that corresponds to a factor of the form $t_a^{\pm 2}$ appearing in f, we have that L_i and L_{i+1} are in the characteristic sublink of $\widetilde{L}_f$ if and only if $\lambda([a], [a])$ is even. Here, λ denotes the Seifert pairing for $\iota : \Sigma_g \hookrightarrow S^3$.*

Proof. Take a pair of components L_i, L_{i+1} in $\widetilde{L}_f$, that corresponds to factor of the form $t_a^{\pm 2}$, or $t_a t_b^{-1}$ for a bounding pair a, b, in the monodromy f. Using construction 4.0.5 to find the characteristic sublink for our chosen $\eta \in \mathrm{Spin}(\Sigma_g)$, we set $w_j = 0$ if L_j is a component from the link L representing $S^1 \times \Sigma_g$. Using definition 4.0.1, and that L_i and L_{i+1} have the same linking number with other components up to sign, we get

$$\begin{aligned} (m+1)w_i + m w_{i+1} &\equiv m w_i + (m+1) w_{i+1} \\ &\equiv m + 1 - \sum_{j \neq i, i+1} lk(L_i, L_j) w_j \pmod{2}, \end{aligned}$$

where $m = \lambda([a], [a])$ and λ is the Seifert pairing for the punctured fiber surface in S^3. This implies that $w_i \equiv w_{i+1} \equiv m + 1 - \sum_{j \neq i,i+1} lk(L_i, L_j) w_j \pmod{2}$. The other components of $\widetilde{L}_f$ come in pairs corresponding to factors of the form $t_c^{\pm 2}$, $t_{d_1} t_{d_2}^{-1}$ appearing in f. Since any such pair is either both in, or both out of the characteristic sublink by above, we get that $\sum_{j \neq i,i+1} lk(L_i, L_j) w_j = 0 \pmod{2}$, hence $w_i \equiv w_{i+1} \equiv m+1$. So L_i and L_{i+1} are in the characteristic sublink if and only if m is even. □

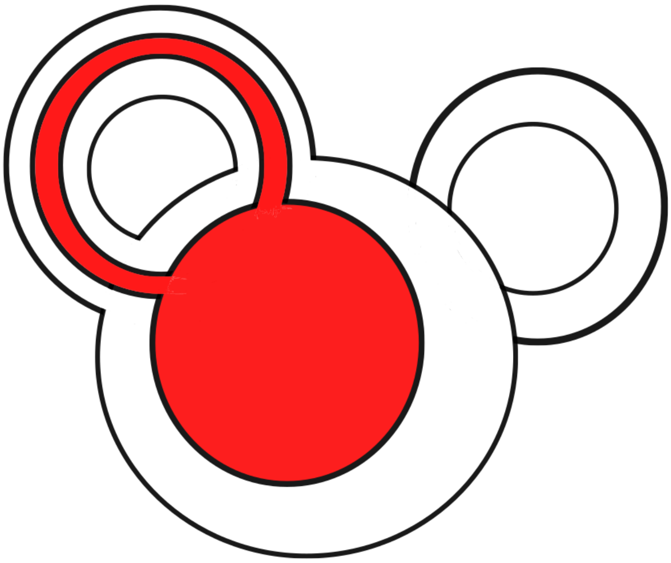

Figure 6.2: The handlebody $W_{\widetilde{L}_f}$ is shaded red, the rest of the handlebody is given by the cobordism Y_f.

There exists a cobordism Y_f between $M_{\widetilde{L}_f}$ and M_{L_f}, given in the following way: take $M_{\widetilde{L}_f} \times D^1$ and attach 2–handles to $M_{\widetilde{L}_f} \times \{1\}$ along the framed link L to obtain Y_f. Then $\partial Y_f = M_{\widetilde{L}_f} \sqcup M_{L_f}$. So the boundary of Y_f is the union of $M(\iota, f)$ and the mapping torus M_f. Let W_{L_f} be the 2–handlebody specified by L_f, with boundary M_{L_f}, and define $W_{\widetilde{L}_f}$ similarly. Then W_{L_f} is the union of $W_{\widetilde{L}_f}$ and Y_f along $M_{\widetilde{L}_f}$, since we can identify $W_{\widetilde{L}_f} \cup_{M_{\widetilde{L}_f}} (M_{\widetilde{L}_f} \times D^1)$ with $W_{\widetilde{L}_f}$ by thinking of $M_{\widetilde{L}_f} \times D^1$ as a collar of the boundary. See figure 6.2.

Recall that for the framed link L_f, the term Λ_{L_f} in the formula of Theorem 4.0.7 is also the signature of the intersection form of the 2–handlebody W_{L_f} specified by L_f. Then, using Novikov additivity (see Theorem 8.3.2 in Appendix 8.3) and Lemma 6.0.1, we have

$$\begin{aligned} R(M_{L_f}, \theta_{L_f}(\eta)) &= \mathrm{Sign}(W_{L_f}) - \theta_{L_f}(\eta) \cdot \theta_{L_f}(\eta) + 8\,\mathrm{Arf}(\theta_{L_f}(\eta)) \\ &= \mathrm{Sign}(Y_f) + \mathrm{Sign}(W_{\widetilde{L}_f}) - \theta_{L_f}(\eta) \cdot \theta_{L_f}(\eta) + 8\,\mathrm{Arf}(\theta_{L_f}(\eta)) \\ &= \mathrm{Sign}(Y_f) + R(M(\iota, f)). \end{aligned}$$

In summary, we have shown:

Corollary 4. Define the map

$$\begin{aligned} \alpha_\iota : \mathrm{Mod}_{g,1}[2] &\to \mathbb{Z}/16 \\ f &\mapsto \mathrm{Sign}(Y_f), \end{aligned}$$

where Y_f is the cobordism described above. Then we have $R_\eta(f) = \alpha_\iota(f) + \mu_\iota(f)$, where $\mu_\iota : \mathrm{Mod}_{g,1}[2] \to \mathbb{Z}/16$ denotes the extension of the Birman–Craggs map for the standard embedding $\iota : \Sigma_g \hookrightarrow S^3$. Consequently, α_ι is well–defined, and $\tau_g \equiv \partial(\alpha_\iota + \mu_\iota) \pmod{16}$.

We finish with a formula for evaluating μ_ι on an element of $\mathrm{Mod}_{g,1}[2]$. We use the following construction to evaluate μ_ι:

Construction 6.0.2. *Let $[f] = t_{c_n}^{\epsilon_n} \cdots t_{c_1}^{\epsilon_1} \in \mathrm{Mod}_{g,1}[2]$, where $\epsilon_i = \pm 2$ for all i. We evaluate $\mu(M(\iota, f))$ in the following way:*

1. *Start with the standard Heegaard embedding $\Sigma_g = \partial(\nu(\Delta_g))$ as in Chapter 3. The unit normal to Σ_g in S^3 that points out of the page, defines an embedding of $I \times \Sigma_g$ in S^3. Pick $t_1 < \cdots < t_n \in I$*

and place two parallel copies of c_i in $\{t_i\} \times \Sigma_g$, both with framing $\lambda([c_i],[c_i]) + \epsilon_i/2$ to obtain $\widetilde{L}_f$. Here, λ denotes the Seifert pairing for Σ_g.

2. *Use Lemma 6.0.1 to find the unique characteristic sublink C of $\widetilde{L}_f$. Then orient each pair of components corresponding to $t_{c_i}^{\epsilon_i}$ oppositely on the Heegaard surface.*

3. *Evaluate $\mu(\widetilde{L}_f, C)$ using Theorem 4.0.7.*

Example: Let a, b, c be the curves given in Figure 6.3.

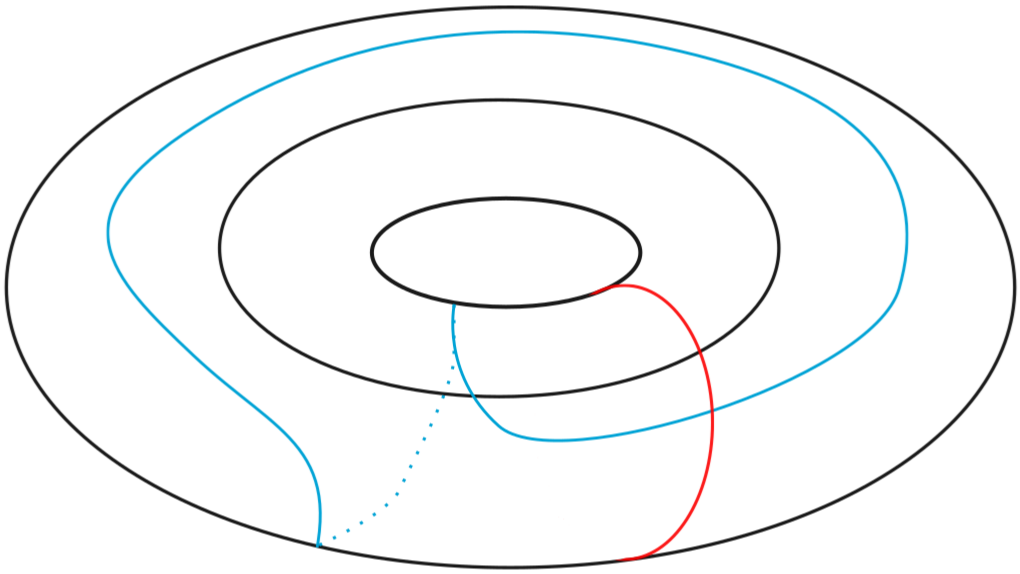

Figure 6.3: Let a denote the black curve, b denote the blue curve, and c denote the red curve. Here, a, b, and c are curves on the torus, standardly embedded in S^3

Then using construction 6.0.2, we obtain the following surgery diagram $\widetilde{L}_f$, where $f = t_c^2 t_b^2 t_a^2$.

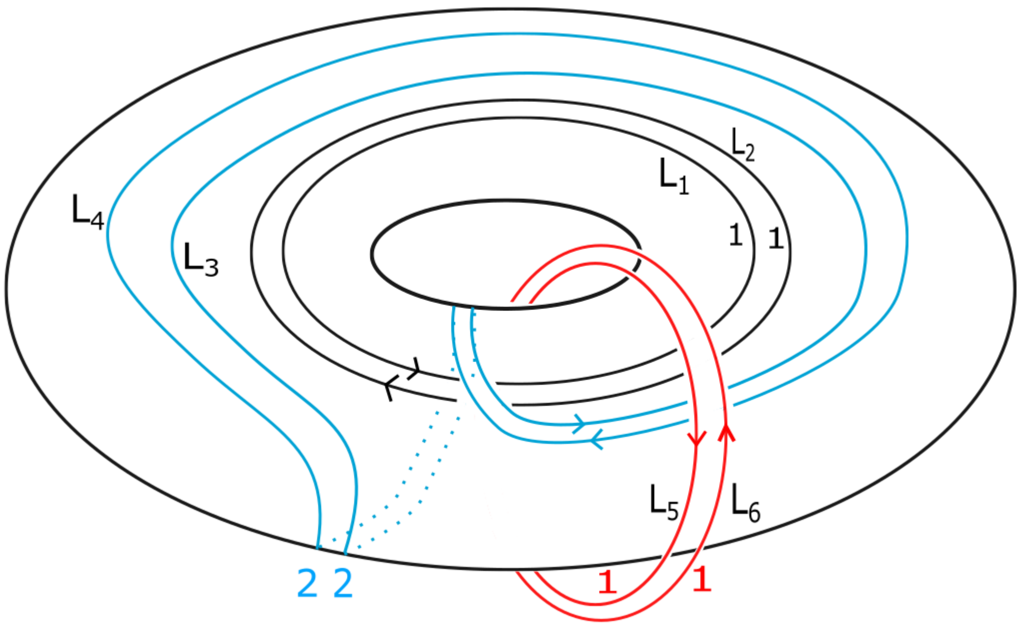

Figure 6.4: A surgery diagram for $M(\iota, t_c^2 t_b^2 t_a^2)$ obtained from construction 6.0.2, where $t_c^2 t_b^2 t_a^2 \in \mathrm{Mod}_{1,1}[2]$, and a, b, c are the simple closed curves given in Figure 6.3.

In the ordering of Figure 6.4, the linking matrix of $\widetilde{L}_f$ is given by

$$\begin{pmatrix} 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 2 & 1 & 1 & -1 \\ -1 & 1 & 1 & 2 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 1 \end{pmatrix}.$$

Using Lemma 6.0.1, we see that the characteristic sublink C in Figure 6.4 is given by the union of the red and black curves. We can then evaluate $\mu(\widetilde{L}_f, C)$ using Theorem 4.0.7.

Theorem 6.0.3. *Let $[f] = t_{c_n}^{\epsilon_n} \cdots t_{c_1}^{\epsilon_1} \in \mathrm{Mod}_{g,1}[2]$, where each c_i is a simple closed curve, and $\epsilon_i = \pm 2$ for all $i = 1, .., n$. Then*

$$\mu_\iota([f]) = \Lambda_{\widetilde{L}_f} - \sum_{\lambda([c_i],[c_i]) \equiv 0 \pmod 2} \epsilon_i \pmod{16},$$

where $\widetilde{L}_f$ is obtained using construction 6.0.2, and $\lambda : H_1(\Sigma_g; \mathbb{Z}) \times H_1(\Sigma_g; \mathbb{Z}) \to \mathbb{Z}$ denotes the Seifert pairing for the standard inclusion $\iota : \Sigma_g \hookrightarrow S^3$.

Proof. Using Lemma 6.0.1, we find the unique characteristic sublink of $\widetilde{L}_f$ to be the union of each pair of components corresponding to $t_{c_i}^{\epsilon_i}$ with $\lambda([c_i], [c_i])$ even. For a pair of components L_i and L_{i+1} of this form, note that L_i is isotopic to L_{i+1} ambiently in the Heegaard surface in S^3. Since L_i and L_{i+1} are oriented oppositely, a similar argument to the proof of Lemma 4.0.9 implies that L_i and L_{i+1} contribute ϵ_i to the $C \cdot C$ terms in Theorem 4.0.7, and that $\mathrm{Arf}(C) = 0$ always. □

Chapter 7

The abelianization of the level 2 congruence group

In this chapter, we give an alternative description of $H_1(\mathrm{Mod}_{g,1}[2];\mathbb{Z})$ by defining a family of polynomial algebras indexed by a spin structure $\sigma \in \mathrm{Spin}(\Sigma_g)$. We rewrite Sato's homomorphisms as maps from $\mathrm{Mod}_{g,1}[2]$ to this algebra. We can then use a single relation in this polynomial algebra repeatedly to calculate the image of Sato's maps on certain subgroups. Let us begin by recalling some of the results in Sato's paper [Sat10].

For $z \in H_1(\Sigma_g)$, define $i_z : H_1(\Sigma_g) \to \mathbb{Z}/8$ (where $H_1(\Sigma_g)$ is always assumed with $\mathbb{Z}/2$ coefficients, unless specified otherwise) by

$$i_z(y) = \begin{cases} 1, & z \cdot y = 1 \mod 2 \\ 0, & z \cdot y = 0 \mod 2. \end{cases}$$

This function is not a homomorphism, but we have the following two identities

$$i_a(x) + i_b(x) - 2i_a(x)i_b(x) = i_{a+b}(x), \tag{7.1}$$

$$((-1)^{q_\sigma(C)} i_C(x))^2 = (-1)^{q_\sigma(C)}((-1)^{q_\sigma(C)} i_C(x)) \tag{7.2}$$

where q_σ is the symplectic quadratic form associated to $\sigma \in \mathrm{Spin}(\Sigma_g)$ as in Theorem 2.0.1. Both identities are checked by comparing both sides of the equation elementwise.

We need to make use of the following results of Sato.

Proposition 7.0.1. *[Sat10, Lemma 2.2, Propositions 5.2 and 7.1] Fix a spin structure $\sigma \in Spin(\Sigma_g)$ and let q_σ be the associated quadratic form on $H_1(\Sigma_g)$. The map $\beta_\sigma : \mathrm{Mod}_{g,1}[2] \to \mathrm{Map}(H_1(\Sigma_g), \mathbb{Z}/8)$ given by $\beta_\sigma(t_C^2) = (-1)^{q_\sigma(C)} i_{[C]}$ is a homomorphism and $|H_1(\mathrm{Mod}_{g,1}[2];\mathbb{Z})| \leq |\mathbb{Z}/8^{2g} \bigoplus \mathbb{Z}/4^{\binom{2g}{2}} \bigoplus \mathbb{Z}/2^{\binom{2g}{3}}|$.*

Recall that $\mathrm{Map}(H_1(\Sigma_g), \mathbb{Z}/8)$ is the free $\mathbb{Z}/8$-module consisting of all functions $H_1(\Sigma_g) \to \mathbb{Z}/8$. If we add the operation of function multiplication, this turns $\mathrm{Map}(H_1(\Sigma_g), \mathbb{Z}/8)$ into a $\mathbb{Z}/8$-algebra. Lemma 7.0.2 gives the two relations we use to analyse the image of Sato's maps.

Lemma 7.0.2. *Let $\sigma \in Spin(\Sigma_g)$ and denote by q_σ the associated quadratic form. Then in the subalgebra W_σ of $\mathrm{Map}(H_1(\Sigma_g), \mathbb{Z}/8)$ generated by $\overline{C} = (-1)^{q_\sigma(C)} i_C$ for all $C \in H_1(\Sigma_g)$, the following relations hold:*

1. $\overline{C_1 + C_2} = (-1)^{C_1 \cdot C_2}((-1)^{q_\sigma(C_2)}\overline{C_1} + (-1)^{q_\sigma(C_1)}\overline{C_2} - 2\overline{C_1 C_2})$, *for all* $C_1 \neq C_2 \in H_1(\Sigma_g)$.
2. $\overline{C}^2 = (-1)^{q_\sigma(C)}\overline{C}$, *for all* $C \in H_1(\Sigma_g)$.

Proof. Using the identity (7.1), we have

$$\begin{aligned}(-1)^{q_\sigma(C_1+C_2)} i_{C_1+C_2}(x) &= (-1)^{q_\sigma(C_1+C_2)}(i_{C_1}(x) + i_{C_2}(x) - 2i_{C_1}(x)i_{C_2}(x)) \\ &= (-1)^{q_\sigma(C_2)+C_1\cdot C_2}((-1)^{q_\sigma(C_1)} i_{C_1}(x)) + (-1)^{q_\sigma(C_1)+C_1\cdot C_2}((-1)^{q_\sigma(C_2)} i_{C_2}(x)) \\ &\quad - 2(-1)^{q_\sigma(C_1+C_2)} i_{C_1}(x) i_{C_2}(x). \end{aligned} \tag{7.3}$$

We also have that

$$(-1)^{q_\sigma(C_1)} i_{C_1}(x)(-1)^{q_\sigma(C_2)} i_{C_2}(x) = (-1)^{q_\sigma(C_1+C_2)-C_1\cdot C_2} i_{C_1}(x) i_{C_2}(x). \tag{7.4}$$

Set $\overline{C} = (-1)^{q_\sigma(C)} i_C$ and substitute (7.4) into the last line of (7.3) to get,

$$\begin{aligned}\overline{C_1 + C_2} &= (-1)^{q_\sigma(C_2)+C_1\cdot C_2}\overline{C_1} + (-1)^{q_\sigma(C_1)+C_1\cdot C_2}\overline{C_2} - 2(-1)^{C_1\cdot C_2}\overline{C_1 C_2} \\ &= (-1)^{C_1\cdot C_2}((-1)^{q_\sigma(C_2)}\overline{C_1} + (-1)^{q_\sigma(C_1)}\overline{C_2} - 2\overline{C_1 C_2}).\end{aligned}$$

□

Fix a symplectic basis $B = \{A_1, B_1, ..., A_g, B_g\}$ for $H_1(\Sigma_g)$ with $A_i \cdot B_j = \delta_{ij}$, $A_i \cdot A_j = 0$ and $B_i \cdot B_j = 0$. Any $C \in H_1(\Sigma_g)$ can be written uniquely as a linear combination of the A_i, B_i's, so iteratively applying the relations of Lemma 7.0.2, we see that any element in the $\mathbb{Z}/8$-submodule $\operatorname{im}\beta_\sigma$ spanned by the $\overline{C_i}$'s can be written as a linear combination of elements of the form

$$\overline{X_i}, 2\overline{X_i X_j}, 4\overline{X_i X_j X_k}, \tag{7.5}$$

where X_i, X_j and X_k are distinct elements from B. We wish to prove that

$$\operatorname{im}\beta_\sigma = \mathbb{Z}/8^{2g} \bigoplus 2\mathbb{Z}/8^{\binom{2g}{2}} \bigoplus 4\mathbb{Z}/8^{\binom{2g}{3}}$$

as a submodule, this will follow from the following result.

Proposition 7.0.3. *If a* $\mathbb{Z}/8$*-linear combination of elements of the form (7.5) is equal to zero, then the coefficients of the monomial terms must be* 0 *and the coefficients of the* $2\overline{X_i X_j}, 4\overline{X_i X_j X_k}$ *terms must be either* 0 *or zero divisors.*

Proof. Suppose that we had such a linear combination $f \in \operatorname{Map}(H_1(\Sigma_g), \mathbb{Z}/8)$ that was equal to the constant zero function. Since B is a symplectic basis, the definition of the functions $i_{[C]}$ gives us the following.

(i) If $X_i \in B$ then $i_{X_i}(X_j) = 1$ for the unique $X_j \in B$ with $X_i \cdot X_j = \pm 1$, and it is 0 on every other element of B. This implies that for distinct X_i, X_j, X_k in B we have that $i_{X_i}(X_l)i_{X_j}(X_l) = i_{X_i}(X_l)i_{X_j}(X_l)i_{X_k}(X_l) = 0$ for any $X_l \in B$.

Using (i), if we evaluate f on all elements of B we see that all the monomial coefficients are zero. So f has no monomial terms.

(ii) Similarly we have that if $X_i, X_j, X_k, X_l \in B$ with X_i, X_j distinct, then $i_{X_i}(X_l + X_k)i_{X_j}(X_l + X_k)$ can only be nonzero if up to reordering indices we have that $X_i \cdot X_l = \pm 1$ and $X_j \cdot X_k = \pm 1$. Since B is a symplectic basis this tuple (X_l, X_k) is unique, so for $X_i, X_j, X_k \in B$ distinct, we have that $i_{X_i}(X_l + X_n)i_{X_j}(X_l + X_n)i_{X_k}(X_l + X_n)$ is zero for all $X_l, X_n \in B$ and $i_{X_i}(X_l + X_k)i_{X_j}(X_l + X_k)$ is only non-zero for one element $X_r + X_s$.

Using (ii), evaluate f on all elements of the form $X_l + X_k$ to get that all the coefficients of the quadratic terms of f must be either 0 or zero divisors, and so f can only contain cubic terms.

(iii) If $X_i, X_j, X_k \in B$ are distinct then $i_{X_i}(X_a+X_b+X_c)i_{X_j}(X_a+X_b+X_c)i_{X_k}(X_a+X_b+X_c)$ can only be nonzero if, up to reordering, we have that $X_i \cdot X_a = \pm 1$, $X_j \cdot X_b = \pm 1$, $X_k \cdot X_c = \pm 1$. Since B is a symplectic basis, this sum is unique, so evaluating f on all elements of the form $X_a + X_b + X_c$ gives the result. □

Combining this result with the last statement of Proposition 7.0.1 gives us that

$$H_1(\mathrm{Mod}_{g,1}[2];\mathbb{Z}) = \mathbb{Z}/8^{2g} \bigoplus 2\mathbb{Z}/8^{\binom{2g}{2}} \bigoplus 4\mathbb{Z}/8^{\binom{2g}{3}}.$$

7.0.1 Abelian quotients of the Torelli group and Johnson kernel

Write Sato's maps in the form $\beta_\sigma : \mathrm{Mod}_{g,1} \to W_\sigma$, where W_σ is the subalgebra of $\mathrm{Map}(H_1(\Sigma_g), \mathbb{Z}/8)$ generated by $\overline{C} = (-1)^{q_\sigma(C)} i_C$, for all $C \in H_1(\Sigma_g)$. In Chapter 5.0.3, we saw that these maps have image in a $\mathbb{Z}/2$–vector space when restricted to the Torelli group. The aim of this subsection is to compute the image of the Torelli group, and Johnson kernel under β_σ in W_σ, using the relations of Lemma 7.0.2. To do this, we need a few results about bounding pair maps, and separating curves.

Factoring bounding pairs into squares

We can use the result under Proposition 4.12 of [FM11] to factor a genus 1 bounding pair into a product of squares of Dehn twists. We define a chain of simple closed curves c_1, c_2, c_3 to be a triple such that $i(c_1, c_2) = i(c_2, c_3) = 1$ and all other pairwise geometric intersection numbers are zero. Let c_1, c_2, c_3 be a chain of simple closed curves and let d_1, d_2 be the boundary curves of a regular neighbourhood of $c_1 \cup c_2 \cup c_3$, then the chain relation gives that

$$(t_{c_1}^2 t_{c_2} t_{c_3})^3 = t_{d_1} t_{d_2}.$$

So we have that

$$\begin{aligned}(t_{c_1}^2 t_{c_2} t_{c_3})^3 &= t_{c_1}^2 (t_{c_2} t_{c_3} t_{c_1}^2 t_{c_3}^{-1} t_{c_2}^{-1})(t_{c_2} t_{c_3})^2 (t_{c_1}^2 t_{c_2} t_{c_3}) \\ &= t_{c_1}^2 (t_{c_2} t_{c_3} t_{c_1}^2 t_{c_3}^{-1} t_{c_2}^{-1})(t_{c_2} t_{c_3})^2 t_{c_1}^2 (t_{c_2} t_{c_3})^{-2} (t_{c_2} t_{c_3})^3.\end{aligned}$$

Using the braid relation we also have that the rightmost term in the last equality can be written as

$$\begin{aligned}(t_{c_2} t_{c_3})^3 &= t_{c_3} t_{c_2} t_{c_3}^2 t_{c_2} t_{c_3} \\ &= (t_{c_3} t_{c_2} t_{c_3}^2 t_{c_2}^{-1} t_{c_3}^{-1})(t_{c_3} t_{c_2}^2 t_{c_3}) \\ &= (t_{c_3} t_{c_2} t_{c_3}^2 t_{c_2}^{-1} t_{c_3}^{-1})(t_{c_3} t_{c_2}^2 t_{c_3}^{-1}) t_{c_3}^2,\end{aligned}$$

and so we have the following factorisation

$$t_{d_1} t_{d_2}^{-1} = t_{c_1}^2 t_{t_{c_2} t_{c_3}(c_1)}^2 t_{(t_{c_2} t_{c_3})^2(c_1)}^2 t_{t_{c_3} t_{c_2}(c_3)}^2 t_{t_{c_3}(c_2)}^2 t_{c_3}^2 t_{d_2}^{-2}.$$

Image of the Torelli group

To calculate the image of the Torelli group, we use a result of Johnson [Joh79, Theorem 1], that says that the maps $t_{d_1} t_{d_2}^{-1}$ (running over all d_1, d_2 bounding pairs of genus 1) generate $\mathcal{I}_{g,1}$; if we evaluate β_σ on one such bounding pair map, and use the fact that $\beta_\sigma(\varphi t_c^2 \varphi^{-1}) = \beta_\sigma(t_{\varphi(c)}^2) = (-1)^{q_\sigma(\varphi_*[c])} i_{\varphi_*[c]}$, we describe the image, since bounding pairs of genus 1 are all in the same orbit under the conjugation action $\mathrm{Mod}_{g,1}$ on $\mathcal{I}_{g,1}$. Note that $\beta_{\sigma,x}(t_c^2)$ only depends on the homology class $C \in H_1(\Sigma_g)$ of the curve c. From now on we will denote C_i by the homology class of the curve c_i. Motivated by the formula for bounding pairs written above, we compute that in $H_1(\Sigma_g)$ we have that

$$t_{c_2}t_{c_3}(C_1) = C_1 + C_2,$$

$$(t_{c_2}t_{c_3})^2(C_1) = C_1 + C_2 + C_3,$$
$$t_{c_3}t_{c_2}(C_3) = C_2,$$
$$t_{c_3}(C_2) = C_2 + C_3.$$

Note that $C_1, C_2, D_1 = D_2$ can be completed into a symplectic basis for $H_1(\Sigma_g)$. We compute $\beta_\sigma(t_{d_1}t_{d_2}^{-1})$ using the relations of Lemma 7.0.2. Substituting in the factorisation of a bounding pair we get that

$$\beta_\sigma(t_{d_1}t_{d_2}^{-1}) = \overline{C_1} + \overline{C_1 + C_2} + \overline{C_1 + C_2 + C_3} + \overline{C_2} + \overline{C_2 + C_3} + \overline{C_3} - \overline{D_1}.$$

So $\operatorname{im} \beta_\sigma|_{\mathcal{I}_{g,1}}$ is the submodule generated by $\{\overline{C_1} + \overline{C_1 + C_2} + \overline{C_1 + C_2 + C_3} + \overline{C_2} + \overline{C_2 + C_3} + \overline{C_3} - \overline{D_1}\}$ running over all chains c_1, c_2, c_3 of simple closed curves with $d_1 \cup d_2 = \partial(\nu(c_1 \cup c_2 \cup c_3))$. The computations of Chapter 5.0.3 imply that this submodule is a $\mathbb{Z}/2$-vector space.

Let us choose a spin structure σ with $q_\sigma(C_1) = q_\sigma(C_2) = q_\sigma(D_1) = 0$, then we have that $q_\sigma(C_3) = q_\sigma(C_1 + D_1) = 0$. For any such spin structure, the formula above simplifies and we get that

$$\beta_\sigma(t_{d_1}t_{d_2}^{-1}) = 4\overline{C_1C_2} + 4\overline{C_1C_2D_1}.$$

Image of the Johnson kernel

Recall that the Johnson kernel $\mathcal{K}_{g,1} \subset \mathcal{I}_{g,1}$ is the subgroup of the mapping class group generated by all separating twists. Suppose we have a chain $c_1, .., c_k$ of simple closed curves in $\Sigma_{g,1}$, so $i(c_i, c_{i+1}) = 1$, and $i(c_i, c_j) = 0$ for $|i - j| > 1$. When k is even, the boundary of a regular neighbourhood of the union of the c_i is a separating curve d. The following relation holds [FM11, Proposition 4.12]

$$t_d = (t_{c_1}^2 t_{c_2} \cdots t_{c_k})^{2k}.$$

For $k = 2$, we have

$$(t_{c_1}^2 t_{c_2})^4 = t_{c_1}^2 (t_{c_2} t_{c_1}^2 t_{c_2}^{-1}) t_{c_2}^2 (t_{c_1}^2 t_{c_2})^2 = t_{c_1}^2 t_{t_{c_2}(c_1)}^2 (t_{c_2}^2 t_{c_1}^2 t_{c_2}^{-2}) t_{c_2}^3 (t_{c_1}^2 t_{c_2}) = t_{c_1}^2 t_{t_{c_2}(c_1)}^2 t_{t_{c_2}^2(c_1)}^2 t_{t_{c_2}^3(c_1)}^2 t_{c_2}^4.$$

For even k, set $t_i = t_{c_i}$, $f_{2k} = t_2 \cdots t_k$, and $\prod_{j=1}^n x_j = x_1 \cdots x_n$, then the previous computation generalises to

$$t_d = (t_1^2 f_{2k})^{2k} = (\prod_{j=0}^{2k-1} t^2_{f_{2k}^j(c_1)}) \cdot (f_{2k})^{2k}.$$

Note that for the $k = 2$ case, the $C_1, C_2 \in H_1(\Sigma_g)$ can be completed to a symplectic basis, and we have $t_2(C_1) = C_1 + C_2$, $t_2^2(C_1) = C_1$, and $t_2^3(C_1) = C_1 + C_2$. Therefore

$$\begin{aligned} \beta_\sigma(t_d) &= 2\overline{C_1} + 2\overline{C_1 + C_2} + 2\overline{C_2} \\ &= 2(1 + (-1)^{C_1 \cdot C_2 + q_\sigma(C_2)})\overline{C_1} + 2(1 + (-1)^{C_1 \cdot C_2 + q_\sigma(C_1)})\overline{C_2} - 4(-1)^{C_1 \cdot C_2}\overline{C_1C_2}, \end{aligned}$$

implying that $\beta_\sigma(\mathcal{K}_{g,1})$ is non–trivial. It would be of interest to calculate the isomorphism type of the image of $\mathcal{K}_{g,1}$ explicitly.

Chapter 8

Appendix

In the appendix, we describe many preliminary notions used throughout the text. We begin with a summary of the relation between cohomology of subgroups of the mapping class group, and characteristic classes of surface bundles.

8.1 Characteristic classes of surface bundles

A *fiber bundle* is a continuous surjection $\pi : E \to B$ such that, for all $x \in B$, there exists a neighbourhood $U \subset B$ of x such that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_1} & \\ U & & \end{array}$$

commutes, where φ is a homeomorphism. The space of all open sets of the form above will be denoted by $\{(U_i, \varphi_i)\}_{i\in I}$, and any such (U_i, φ_i) is called a local trivialization. The space E is called the *total space*, B the *base space*, and F the *fiber*.

Transition functions: Given two local trivializations $(U_i, \varphi_i), (U_j, \varphi_j)$, with $x \in U_i \cap U_j$, we have the following commutative diagram

$$\begin{array}{ccccc} (U_i \cap U_j) \times F & \xrightarrow{\varphi_i^{-1}} & \pi^{-1}(U_i \cap U_j) & \xrightarrow{\varphi_j} & (U_i \cap U_j) \times F \\ & {\scriptstyle \mathrm{pr}_1} \searrow & {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & & U_i \cap U_j & & \end{array}$$

so $\mathrm{pr}_1 \circ \varphi_j \circ \varphi_i^{-1}(x, f) = \pi \circ \varphi_i^{-1}(x, f) = \mathrm{pr}_1(x, f) = x$, implying that $\varphi_j \circ \varphi_i^{-1}(x, f) = (x, t_{ij}(f))$. Here, the maps $t_{ij} : U_i \cap U_j \to \mathrm{Homeo}(F)$ are continuous, where $\mathrm{Homeo}(F)$ has been equipped with the compact–open topology. We call the maps t_{ij} *transition functions*.

Pullback of a bundle: Given a continuous map $f : B \to X$, and a fiber bundle $\pi : E \to X$ with fiber F, we can form the pullback bundle

$$f^*(E) := \{(b, e) \mid f(b) = \pi(e)\} \subset B \times E,$$

here $f^*(E)$ has the subspace topology, and $B \times E$ has the product topology. The two projection maps of $B \times E$ induce maps in the following commutative diagram

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \pi \\ B & \xrightarrow[f]{} & X \end{array}$$

where the left vertical arrow $f^*(E) \to B$ is a fiber bundle over B with fiber F.

Fact: If $f, g : B \to X$ are homotopic maps, then $f^*(E)$ is isomorphic to $g^*(E)$ as a bundle over B.

Classifying spaces

Let G be a topological group, then a classifying space for G is a principal G–bundle $\pi : EG \to BG$ that satisfies the following property: given any principal G–bundle $\gamma : Y \to Z$, there exists a classifying map $\varphi : Z \to BG$, such that the bundle γ is isomorphic to the pullback of π along φ. This universal property uniquely determines BG up to homotopy equivalence, and such a map π always exists.

When M is a compact manifold, we can turn $\mathrm{Diff}(M)$ into a topological group via the Whitney topology, or via the compact–open topology; both topologies coincide when M is compact [Hir76, Ch.2]. Note that an arbitrary M–bundle $\pi : E \to B$ need not be a principal $\mathrm{Diff}(M)$–bundle. There is, however, a bijection between homotopy classes of maps $B \to B\,\mathrm{Diff}(M)$, and isomorphism classes of M–bundles over B. The bijection is given by sending a map $B \to B\,\mathrm{Diff}(M)$ to the bundle $E \to B$ defined by pullback:

$$\begin{array}{ccc} E & \longrightarrow & (E\,\mathrm{Diff}(M) \times M)/\,\mathrm{Diff}(M) \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & B\,\mathrm{Diff}(M) \end{array}$$

where $(E\,\mathrm{Diff}(M) \times M)/\,\mathrm{Diff}(M)$ is the quotient by the diagonal action, with bundle map induced by the universal bundle $E\,\mathrm{Diff}(M) \to B\,\mathrm{Diff}(M)$.

One important property of classifying spaces is functoriality: let $\phi : H \to G$ be a homomorphism of topological groups, then this induces a map $B(\phi) : BH \to BG$; construct the following G–bundle over BH:

$$EH \times_H G := EH \times G/(a, b) \sim (h \cdot a, \phi(h^{-1})b),$$

with bundle projection $EH \times_H G \to BH$ induced by $EH \to BH$. This is a principal G–bundle over BH, so it is the pullback of a classifying map $B(\phi) : BH \to BG$.

Reduction of structure groups

Suppose we have a fiber bundle $F \hookrightarrow E \xrightarrow{\pi} X$, and a topological subgroup $H \subset \mathrm{Diff}(F)$, then a *reduction of the structure group* to H is a choice of local trivializations $(U_i, \varphi_i : \pi^{-1}(U_i) \to U_i \times F)_{i \in I}$ such that the transition functions $t_{ij} : U_i \cap U_j \to \mathrm{Diff}(F)$ have image in H, for all $i, j \in I$. This family of transition functions defines an isomorphism between π and

$$P \times_\rho F := P \times F/(a, f) \sim (h \cdot a, \rho(h^{-1})f) \to X$$

where $P \to X$ is a principal H–bundle obtained from $\bigsqcup_{i \in I} H \times U_i$ by identifying $(h, x) \in H \times U_i$ with $(t_{ij}(x)h, x) \in H \times U_j$ for all $x \in U_i \cap U_j$. Here $\rho : H \hookrightarrow \mathrm{Diff}(F)$ is the inclusion, and $P \times_\rho F \to X$ is induced by the bundle map $P \to X$.

Note that $P \to X$ is the pullback of $EH \to BH$ along some map $X \to BH$; the constructions above imply that isomorphism classes of fiber bundles $F \hookrightarrow E \xrightarrow{\pi} X$ with reduction of structure group to $H \subset \mathrm{Diff}(F)$ are in bijection with homotopy classes of maps $f : X \to BH$ such that

$$\begin{array}{ccc} & & BH \\ & \overset{f}{\nearrow} & \big\downarrow {\scriptstyle B(\rho)} \\ X & \xrightarrow[g]{} & B\,\mathrm{Diff}(F) \end{array}$$

commutes up to homotopy, where $g : X \to B\,\mathrm{Diff}(M)$ is the classifying map corresponding to $\pi : E \to X$.

For any $c \in H^*(BH; \mathbb{Z})$, we get a characteristic class $f^*(c) \in H^*(X; \mathbb{Z})$ for the bundle $\pi : E \to X$ with a reduction of the structure group to $H \subset \mathrm{Diff}(F)$.

We introduce the *monodromy representation* here: let $\pi : E \to X$ be a fiber bundle with fiber F, and a reduction of the structure group to $H \subset \mathrm{Diff}(F)$. There exists two classifying maps $f : X \to BH$, and $g : X \to B\,\mathrm{Diff}(F)$ such that $B(\rho) \circ f = g$, up to homotopy, as above. Applying the π_1 functor, we get a homomorphism

$$g_* : \pi_1(X) \to \pi_1(B\,\mathrm{Diff}(F)) \cong \pi_0(\mathrm{Diff}(F))$$

where the isomorphism in the target comes from applying the long exact sequence for homotopy groups to the universal bundle $\mathrm{Diff}(F) \hookrightarrow E\,\mathrm{Diff}(F) \to B\,\mathrm{Diff}(F)$, and using that $E\,\mathrm{Diff}(F)$ is contractible. The group $\pi_0(\mathrm{Diff}(F)) := \mathrm{Mod}(F)$ is the *mapping class group* of F. The commutative diagram above imples g_* has image in $B(\rho)_*(\pi_1(BH))$, which we identify with a subgroup of $\mathrm{Mod}(F)$.

The case of surface bundles.

Consider a fiber bundle with fiber $F = \Sigma_g$ a 2–manifold of genus $g \geq 2$. In this case, work of Earle–Eells implies that the path components of $\mathrm{Diff}(F)$ are contractible [EE67], so the natural map $\mathrm{Diff}(F) \to \mathrm{Mod}(F)$ that sends a diffeomorphism to its mapping class, is a homotopy equivalence, where $\mathrm{Mod}(F)$ is thought of as a discrete group. This implies that $B\,\mathrm{Diff}(F)$ is homotopy equivalent to $B\,\mathrm{Mod}(F)$. The remark above implies that cohomology classes in $H^*(\mathrm{Mod}(F); \mathbb{Z}) = H^*(B\,\mathrm{Mod}(F); \mathbb{Z}) \cong H^*(B\,\mathrm{Diff}(F); \mathbb{Z})$ give characteristic classes of surface bundles. This seems much more tractable since $\mathrm{Mod}(F)$ is a discrete group, so we have $K(\pi, 1)$ theory at our disposal.

We single out subgroups of $\mathrm{Diff}(F)$ which have added importance with regard to F–bundles with reduction of structure to H:

Let R be a ring, and define $I_R \subset \mathrm{Diff}(F)$ to be the kernel of the action of $\mathrm{Diff}(F)$ on $H_1(F; R)$. Let P be a path component of $\mathrm{Diff}(F)$, and suppose there is an $f \in P$ with $f \in I_R$, then for any other $g \in P$, there is an isotopy between f and g, and since $H_*(-; R)$ is a homotopy invariant functor, g must also be in I_R. Therefore $P \subset I_R$. Let $\mathrm{Mod}(F)[R]$ denote the subgroup of $\mathrm{Mod}(F)$ that is the kernel of the action of $\mathrm{Mod}(F)$ on $H_1(F; R)$. We get a surjection $I_R \to \mathrm{Mod}(F)[R]$ sending $f \in I_R$ to its mapping class. We know that the path components of $\mathrm{Diff}(F)$ are contractible, and I_R contains a subset of those path components, therefore the surjection $I_R \to \mathrm{Mod}(F)[R]$ is also a homotopy equivalence, implying that $BI_R \cong B\,\mathrm{Mod}(F)[R]$.

The discussion above implies that elements of $H^*(B\,\mathrm{Mod}(F)[R]; \mathbb{Z}) \cong H^*(BI_R; \mathbb{Z})$ give cohomology classes for F–bundles with a reduction of the structure group to I_R. The groups $\mathrm{Mod}(F)[R]$ are of particular importance because for an F–bundle with monodromy lying in $\mathrm{Mod}(F)[R]$, the Serre spectral sequence can be applied to understand the cohomology of the total space, with coefficients in R, in terms of the cohomology of the base and fiber.

Now, let $D \subset F$ be an embedded 2–disk in our 2–manifold F. Let $\mathrm{Diff}^+(F, D)$ denote the subgroup of $\mathrm{Diff}^+(F)$ given by orientation–preserving diffeomorphisms that fix D pointwise. Let $\pi : E \to X$ be an F–bundle with a reduction of the structure group to $\mathrm{Diff}^+(F, D)$, then we have the following.

Lemma 8.1.1. *There exists a D–family of sections $s_r : X \to E$, $r \in D$, of $\pi : E \to X$.*

Proof. Let $r \in D$, $x \in X$, and pick a local trivialization $(U_i, \varphi_i : \pi^{-1}(U_i) \to U_i \times F)$ with $x \in U_i$. Define

$$s_r(x) = \varphi_i^{-1}(x, r).$$

We claim that s_r is independent of choice of trivialization: if $(U_j, \varphi_j : \pi^{-1}(U_j) \to U_j \times F)$ is another choice, with $x \in U_j$, then

$$\varphi_i^{-1}(x, r) = \varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1}(x, r) = \varphi_j^{-1}(x, t_{ij}(r)) = (x, r)$$

since the transition function $t_{ij} : U_i \cap U_j \to \mathrm{Diff}(F)$ has image in $\mathrm{Diff}^+(F, D)$. □

Hence, $B\,\mathrm{Diff}^+(F, D)$ classifies isomorphism classes of oriented F–bundles $E \to X$ with a *distinguished D–family of sections*. These sections often give a splitting of $H^*(E)$ in terms of $H^*(X)$ and $H^*(F)$; see Chapter 2 on Sato's homomorphisms, for example. Since these spaces have geometric meaning, we should study their cohomology.

Corollary 8.1.2. $\mathrm{Diff}^+(F, D)$ *with the compact–open topology, is isomorphic as a topological group to* $\mathrm{Diff}^+(W, \partial W)$. *Here $W = F \setminus \mathrm{int}(D)$, and* $\mathrm{Diff}^+(W, \partial W)$ *is the group of orientation–preserving diffeomorphisms of W that fix the boundary pointwise.*

Proof. The restriction map

$$\psi : \mathrm{Diff}^+(F, D) \to \mathrm{Diff}^+(W, \partial W)$$
$$f \mapsto f|_W$$

is a homomorphism. ψ is invertible because we can extend $g \in \mathrm{Diff}^+(W, \partial W)$ over D via the identity. Furthermore, ψ pulls back subbases of the compact–open topology to subbases, similarly for ψ^{-1}, so ψ and ψ^{-1} are also continuous. □

Theorem 8.1.3. *When F is a surface of genus $g \geq 2$, and $D \subset F$ is an embedded 2–disk, then the path components of* $\mathrm{Diff}^+(F, D)$ *are contractible.*

Proof. This follows from the corresponding result for $\mathrm{Diff}^+(W, \partial W)$ (see [EE69] and [ES70, Thm.C]), and Corollary 8.1.2. □

The arguments above imply that $B\,\mathrm{Diff}^+(F, D) \cong B\pi_0(\mathrm{Diff}^+(F, D))$; set $\mathrm{Mod}_{g,1}(F) := \pi_0(\mathrm{Diff}^+(F, D))$, when F is a closed surface of genus g. We can define analogues of the groups I_R, and $\mathrm{Mod}_{g,1}(F)[R]$ defined above, where R is a ring. The arguments given above imply that cohomology classes of $\mathrm{Mod}_{g,1}(F)[R]$ give characteristic classes of bundles with a reduction of the structure group to I_R, and a D–family of sections.

Surface bundles in algebraic geometry.

We end this section with a construction of surface bundles that naturally appears in complex geometry.

Let $P(x, y, z), Q(x, y, z)$ be homogeneous polynomials with complex coefficients in three variables. Suppose both P and Q have degree $d \geq 1$, then we obtain a rational map

$$f : \mathbb{C}P^2 \dashrightarrow \mathbb{C}P^1$$
$$[x : y : z] \mapsto [Q(x, y, z) : -P(x, y, z)],$$

called a *Lefschetz pencil*. This map is undefined at the common zeroes of P and Q, and if P and Q have no common factor, Bezout's theorem implies there are d^2 common zeroes in $\mathbb{C}P^2$, counting multiplicity. Let $B = \{p_1, .., p_{d^2}\}$ be the set of common zeroes of P and Q, and let $\epsilon : S \to \mathbb{C}P^2$ be the blowup of $\mathbb{C}P^2$ at $p_1, .., p_{d^2}$.

For a generic point $[a : b] \in \mathbb{C}P^1$, the fiber

$$f^{-1}([a : b]) = \{[x : y : z] \mid aP(x, y, z) + bQ(x, y, z) = 0\} := F_{[a:b]}$$

is a surface of genus $\frac{(d-1)(d-2)}{2}$ by the degree–genus formula. The union of the fibers $F_{[a:b]}$ is $\mathbb{C}P^2$, and they all intersect at the points of B. Take the *proper transforms*

$$F'_{[a:b]} = \mathrm{cl}(\epsilon^{-1}(F_{[a:b]} - B)),$$

where cl denotes the Zariski closure in S. Each of the $F_{[a:b]}$ correspond to different complex lines through the points of B, hence the proper transforms $F'_{[a:b]}$ are disjoint in S. We get a regular map $\pi : S \to \mathbb{C}P^1$ given by $\pi(F'_{[a:b]}) = [a : b]$.

Recall that for a smooth map $f : X \to Y$ between smooth manifolds X and Y, a point $y \in Y$ is a *regular value* if df_x is surjective for all $x \in f^{-1}(y)$. *Sard's Theorem* states that the complement in Y of the set of regular values has measure zero; generic points of Y are regular values [Lee00, Ch.6].

In our situation, we have the regular map $\pi : S \to \mathbb{C}P^1$ which is smooth when viewed as a map between smooth 4–manifolds. Let D denote the complement of the set of regular values of π. Since π is regular, D forms a projective subvariety with nonzero codimension, and $\mathbb{C}P^1$ has dimension 1, so D must be a finite set of points. Hence, the restriction of π to $\pi^{-1}(\mathbb{C}P^1 - D)$ is a fiber bundle by the *Ehresmann fibration lemma*, which states that a smooth proper submersion is a fiber bundle [Ehr50]. We obtain in this way a Σ_{d^2}–bundle over $\mathbb{C}P^1 - D$.

8.2 Mapping class groups

To begin our discussion of mapping class groups, we recall some results from differential topology.

8.2.1 Isotopy extension theorem

Let M be a smooth n–manifold, and let $X \in \Gamma(TM)$ be a smooth vector field. A curve $\gamma : (-\epsilon, \epsilon) \to M$ is an *integral curve* for X if $\gamma'(t) = X(\gamma(t))$ for all $t \in (-\epsilon, \epsilon)$. Suppose the vector field X satisfies the following property: for all $p \in M$, X has a unique integral curve starting at p, and defined for all $t \in \mathbb{R}$. Then we get the *flow*

$$F : M \times \mathbb{R} \to M$$
$$(m, t) \mapsto \gamma_m(t),$$

where $\gamma_m : \mathbb{R} \to M$ is the unique integral curve for X with $\gamma_m(0) = m$. For fixed $t \in \mathbb{R}$, the flow map $F(-, t) : M \to M$ slides points of M along the integral curves until time t.

Pick a chart $\varphi = (x^1, ..., x^n) : U \to \mathbb{R}^n$, and let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve with image in U, then write $\varphi \circ \gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$. Write the vector field X locally as $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x^i}$ with respect to the local frame $\{\frac{\partial}{\partial x^i}\}$ for TM, where the functions $X_i : U \to \mathbb{R}$ are smooth. The condition $\gamma'(t) = X(\gamma(t))$ is then written as

$$\sum_{i=1}^{n} \gamma_i'(t) \frac{\partial}{\partial x^i}|_{\gamma(t)} = \sum_{i=1}^{n} X_i(\gamma(t)) \frac{\partial}{\partial x^i}|_{\gamma(t)}.$$

Comparing coefficients gives a system of ordinary differential equations

$$\gamma_1'(t) = X_1(\gamma_1(t), ..., \gamma_n(t)), ..., \gamma_n'(t) = X_n(\gamma_1(t), ..., \gamma_n(t)),$$

this system must be solved in order to find the integral curves.

The notion of integral curves and flow is used to prove an important theorem in differential topology, the isotopy extension theorem. To prove the Isotopy extension theorem, we adapt the following argument: let V be compact, and let $H : V \times I \to M$ be an isotopy, so H is a smooth map, and for $t \in I$, the map $H(-,t) : V \to I$ is an embedding. Construct a vector field on an open neighbourhood of $H(V \times I)$ that has integral curves $H(\{x\} \times I)$ for all $x \in V$. Then take the flow of this vector field to obtain the extended isotopy.

The problem with this argument is the following: $H : V \times I \to M$ itself might not be injective, we only have that $H(-,t) : V \to M$ is injective for all $t \in I$. So there might be multiple curves of the form $H(pt \times I)$ going through an $m \in H(V \times I) \subset M$. So we need to pass to an intermediate notion of a *time–dependent vector field*, to deal with this complication.

The idea is: if $H : X \times I \to X$ is smooth, and $H(-,t) : X \to X$ is injective for all $t \in I$, then the map

$$\begin{aligned} \hat{H} : X \times I &\to X \times I \\ (x,t) &\mapsto (H(x,t),t) \end{aligned}$$

is injective. To see this, if $\hat{H}(x,t) = \hat{H}(y,s)$, then $(H(x,t),t) = (H(y,s),s)$, so $t = s$, but $H(-,t)$ is injective, so $x = y$ too.

A *time–dependent vector field* is a smooth map $G : M \times I \to TM$ such that $G(x,t) \in T_xM$, and $G(\partial M \times I) \subset T(\partial M)$. A time–dependent vector field $G : M \times I \to TM$ has *bounded velocity* if M has a complete Riemannian metric such that $|G(x,t)| < K$ for some constant K.

Theorem 8.2.1. *[Hir76, Thm. 1.1., p.179] Let G be a time–dependent vector field on M with bounded velocity. Then G generates an isotopy of M, that is, there exists a unique isotopy $F : M \times I \to M$ such that*

$$\frac{\partial F}{\partial t}(x,t) = G(F(x,t),t).$$

Proof. Form the vector field

$$\begin{aligned} X : M \times I &\to T(M \times I) \\ (x,t) &\mapsto (G(x,t), \frac{\partial}{\partial s}|_t) \end{aligned}$$

where s is a coordinate function on I centred at t.

The assumptions imply that for $x \in M$, there is an integral curve of X having the form $t \mapsto (F(x,t),t)$, defined for all $t \in I$, since the vector field X is constant in the I direction. Then

$$\begin{aligned} F : M \times I &\to M \\ (x,t) &\mapsto F(x,t), \end{aligned}$$

is the required isotopy. □

The *support* of a time–dependent vector field $G : M \times I \to TM$ is

$$\text{Supp}(G) = \{x \in M \mid G(x,t) \neq 0 \textit{ for some } t \in I\}.$$

If the support of G is compact, then G has bounded velocity, hence generates an isotopy by Theorem 8.2.1. So every time–dependent vector field on a compact manifold generates an isotopy.

Theorem 8.2.2. *[Hir76, Thm. 1.3] Let $V \subset M$ be a compact submanifold, and $F : V \times I \to M$ be an isotopy. If either $F(V \times I) \subset \partial M$, or $F(V \times I) \subset M - \partial M$, then F extends to an isotopy of M.*

Proof. Let

$$\begin{aligned} \hat{F} : V \times I &\to M \times I \\ (x,t) &\mapsto (F(x,t),t) \end{aligned}$$

be the *track* of F. Construct a vector field X on $\hat{F}(V \times I)$ in the following way: fix $x \in V$, and take the curve

$$\begin{aligned} \gamma_x : I &\to M \times I \\ t &\mapsto \hat{F}(x,t), \end{aligned}$$

and define the vector field

$$X(\hat{F}(x,t)) = d\gamma_x(\frac{\partial}{\partial s}|_t).$$

In other words, X is the vector field with integral curves $t \mapsto \hat{F}(x,t)$. Extend X to a vector field Y on a neighbourhood of $\hat{F}(V \times I)$ in $M \times I$.

The hypotheses on F allow us to assume that $Y_{(x,t)}$ is tangent to $\partial M \times I$ whenever $x \in \partial M$. After restricting to a smaller neighbourhood, $d\text{pr}_1(Y)$ is extended to a compactly–supported time–dependent vector field on M, where $\text{pr}_1 : M \times I \to M$ is the canonical projection. The isotopy is then constructed from Theorem 8.2.1. □

Corollary 8.2.3. *Let $V \subset M$ be a compact submanifold, and let $f, g : V \hookrightarrow M$ be embeddings such that f is isotopic to g, then there exists a diffeomorphism $F : M \to M$, isotopic to id_M, such that $F \circ f = g$.*

Proof. Let $H : V \times I \to M$ be the isotopy with $H(x,0) = f(x)$, and $H(x,1) = g(x)$, for all $x \in V$. Use Theorem 8.2.2 to obtain an isotopy $\hat{H} : M \times I \to M$ extending H. The diffeomorphism $F = \hat{H}_1 \circ \hat{H}_0^{-1}$ is isotopic to the identity, where $\hat{H}_t = \hat{H}(-,t) : M \to M$. Then

$$F \circ f(x) = \hat{H}_1 \circ \hat{H}_0^{-1}(f(x)) = \hat{H}_1(x) = g(x).$$

□

8.2.2 Tubular Neighbourhood Theorem

Let M be a smooth manifold, and let $i : A \hookrightarrow M$ be a smooth, closed submanifold of M. Then

$$TM|A = i^*(TM) \cong TA \oplus \nu_M(A),$$

where $\nu_M(A)$ is the normal bundle to A in M. If we pick a Riemannian metric on M, then $\nu_M(A)$ can be identified with $T(A)^\perp$.

An *open tubular neighbourhood of A in M* is a diffeomorphism $\phi : \nu_M(A) \to M$ onto some neighbourhood of A in M, such that the restriction of ϕ to the 0–section of $\nu_M(A)$ is the inclusion $i : A \hookrightarrow M$.

The restriction of ϕ to the unit disc bundle of a metric on $\nu_M(A)$ is called a *closed tubular neighbourhood of A.*

Theorem 8.2.4. *[Bre72, Ch.VI, Thm.2.2] Open tubular neighbourhoods of a closed submanifold always exist.*

Now we discuss uniqueness: two tubular neighbourhoods $\phi_1 : \nu_M(A) \to M$ and $\phi_0 : \nu_M(A) \to M$ of a closed submanifold $A \subset M$ are *isotopic* if there exists tubular neighbourhoods $\phi_t : \nu_M(A) \to M$, for all $t \in I$, such that the map

$$\begin{aligned} \nu_M(A) \times I &\to M \\ (v,t) &\mapsto \phi_t(v) \end{aligned}$$

is smooth.

Theorem 8.2.5. *[Bre72, Ch.VI, Thm.2.6] Suppose $A \subset M$ is a closed submanifold, then any two (open or closed) tubular neighbourhoods of A are isotopic.*

Example. Let $\gamma : S^1 \to \mathbb{R}^3$ be a knot K in $\mathbb{R}^3$, and suppose $\gamma'(t) \neq 0$ for all t. Since $\mathbb{R}^3$ has a global coordinates $(x^1, .., x^n)$, we get a global frame $\frac{\partial}{\partial x^i}$ for $T\mathbb{R}^3$, giving a vector bundle isomorphism $T(\mathbb{R}^3) \cong \mathbb{R}^3 \times \mathbb{R}^3$; identify these two vector bundles. The tangent curve $\gamma' : S^1 \to T(\mathbb{R}^3)$ has the description

$$\gamma'(t) = (\gamma(t), \gamma'(t)) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

and $\{(\gamma(t), c\gamma'(t)) : c \in \mathbb{R}\} \subset \{\gamma(t)\} \times \mathbb{R}^3$ is the tangent space to K at $\gamma(t)$.

Give $\mathbb{R}^3 \times \mathbb{R}^3$ the flat Riemannian metric given by the standard dot product on $\{pt\} \times \mathbb{R}^3$. Then write

$$\nu_{S^3}(K) = \{(\gamma(t), v) \mid \gamma'(t) \cdot v = 0\}.$$

A transverse vector field to the knot K in $\mathbb{R}^3$ is given by a map $P : S^1 \to T(\mathbb{R}^3)$, where $P(t) = (\gamma(t), p(t))$ with $p(t) \cdot \gamma'(t) = 0$ in $\gamma(t) \times \mathbb{R}^3$. We obtain another vector field $Q : S^1 \to T(\mathbb{R}^3)$ given by $Q(t) = (\gamma(t), \gamma'(t) \times p(t))$, where $\gamma'(t) \times p(t)$ is the cross product of the vectors $\gamma'(t)$ and $p(t)$ in $\{\gamma(t)\} \times \mathbb{R}^3$. The $P, Q : S^1 \to T(\mathbb{R}^3)$ give an orthonormal frame for $\nu_{S^3}(K)$.

We obtain a vector bundle isomorphism

$$\begin{aligned} \eta_{P,Q} : S^1 \times \mathbb{R}^2 &\to \nu_{S^3}(K) \\ (t, a, b) &\mapsto (\gamma(t), aP(t) + bQ(t)), \end{aligned}$$

that depends on the smooth isotopy class of the orthonormal frame $\{P, Q\}$.

We construct a tubular neighbourhood of K in S^3 as a map

$$\begin{aligned} \alpha : \nu_{S^3}(K) &\to \mathbb{R}^3 \\ (\gamma(t), v) &\mapsto \gamma(t) + v \end{aligned}$$

where $v \cdot v << 1$. The composition $\alpha \circ \eta_{P,Q} : S^1 \times D^2 \to \mathbb{R}^3$ defines us an embedding.

For an example a tubular neighbourhood of a simple closed curve on a surface, see Figure 8.1

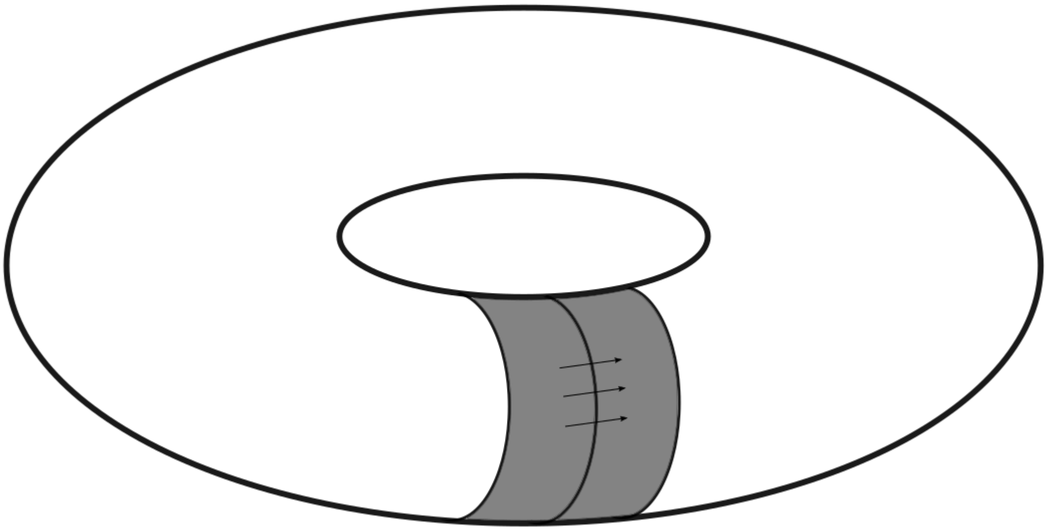

Figure 8.1: Closed tubular neighbourhood of a simple closed curve on a torus.

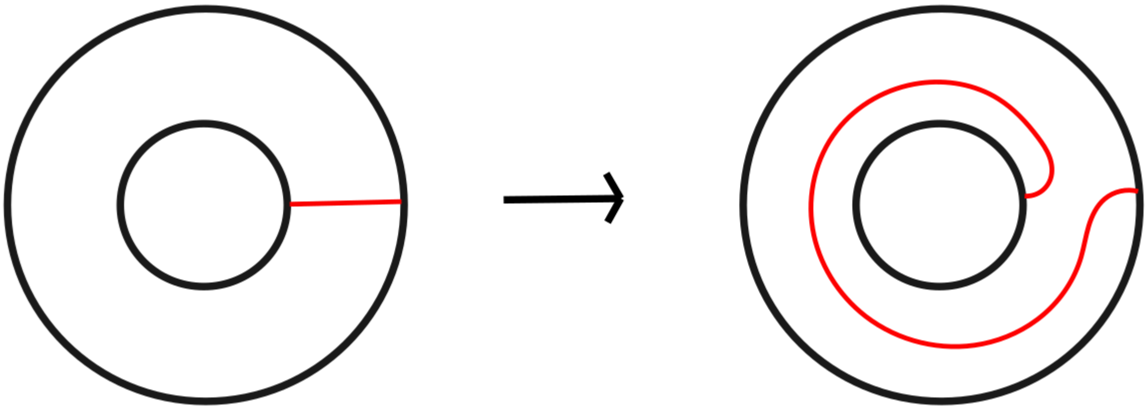

Figure 8.2: Action of T on the red arc $I \times \{1\} \subset A$.

8.2.3 Dehn twists

Let Σ be an oriented surface, in this section, we analyse certain elements of $\pi_0(\mathrm{Diff}^+(\Sigma, \partial\Sigma))$ supported on tubular neighbourhoods of curves, called Dehn twists.

First, consider the annulus $A = I \times S^1$. Embed A in $\mathbb{C}$ via the map

$$\begin{aligned} A &\to \mathbb{C} \\ (x, e^{2\pi i y}) &\mapsto (x+1)e^{2\pi i y}, \end{aligned}$$

and orient A using the standard orientation of the plane. Let

$$\begin{aligned} T : A &\to A \\ (x, e^{2\pi i y}) &\mapsto (x, e^{2\pi i (x+y)}) \end{aligned}$$

denote the Dehn twist about an annulus. See Figure 8.2 for a picture of how this map acts on arcs in A.

We can use T to define infinite order mapping classes with representatives supported in annular neighbourhoods of curves in Σ. To do this, let $\alpha : S^1 \to \Sigma$ denote an oriented simple closed curve in Σ. Let $\phi : \nu_\Sigma(\alpha) \to \Sigma$ be a tubular neighbourhood of α in Σ. Now α is an oriented codimension 1 submanifold of Σ, so there exists a nowhere–zero section $s : \alpha \to \nu_\Sigma(\alpha)$ that agrees with the chosen orientation. This section induces a diffeomorphism

$$\begin{aligned}\psi : \mathbb{R} \times S^1 &\to \nu_\Sigma(\alpha)\\ (r,t) &\mapsto rs(\alpha(t)).\end{aligned}$$

So we obtain an embedding

$$\eta = \phi \circ \psi : I \times S^1 \to \Sigma.$$

Pull–back the orientation on Σ to an orientation on A, and consider the following diffeomorphism of Σ,

$$\tau_\alpha(x) = \begin{cases} \eta \circ T \circ \eta^{-1}(x)\ ,\ \text{if } x \in \eta(A) \\ x\ ,\ \text{else} \end{cases}.$$

Define a *Dehn twist* about α to be

$$t_\alpha = [\tau_\alpha] \in \pi_0(\mathrm{Diff}^+(\Sigma, \partial\Sigma)).$$

Lemma 8.2.6. *The mapping class t_α is independent of*

1. *The choice of tubular neighbourhood $\phi : \nu_\Sigma(\alpha) \to \Sigma$.*
2. *The choice of section $s : \alpha \to \nu_\Sigma(\alpha)$.*
3. *The representative of the isotopy class of $\alpha : S^1 \to \Sigma$.*

Proof. (1) Given an isotopy $\phi_t : \nu_\Sigma(\alpha) \to \Sigma$ of tubular neighbourhoods of α, as above, we obtain the isotopy

$$\begin{aligned} I \times \Sigma &\to \Sigma \\ (t,x) &\mapsto \begin{cases} \eta_t \circ T \circ \eta_t^{-1}(x)\ ,\ x \in \eta_t(A) \\ x\ ,\ \text{else} \end{cases},\end{aligned}$$

where $\eta_t = \phi_t \circ \psi$. This isotopy is a smooth map because

$$\begin{aligned}\phi : I \times \nu_\Sigma(\alpha) &\to \Sigma \\ (t,v) &\mapsto \phi_t(v)\end{aligned}$$

is smooth, implying that

$$\phi \circ (\mathrm{id} \times \psi) : I \times A \to \Sigma$$

is smooth. Then the track

$$\begin{aligned} F : I \times A &\to I \times \Sigma \\ (t,x) &\mapsto (t, \phi \circ (\mathrm{id} \times \psi)(x))\end{aligned}$$

is injective, and restricts to η_t on $\{t\} \times A$. The isotopy above can be written as $\mathrm{pr}_2 \circ F \circ (\mathrm{id} \times \tau) \circ F^{-1}$ on $F(I \times A)$, and the identity elsewhere, where $\mathrm{pr}_2 : I \times \Sigma \to \Sigma$ is the canonical projection. Hence, if τ_α is defined using either identifications $\phi_0 \circ \psi$ or $\phi_1 \circ \psi$ of A with a neighbourhood of α, then the resulting maps are isotopic.

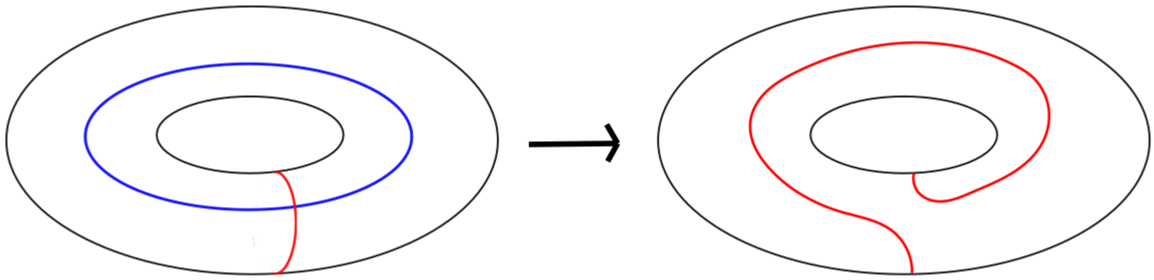

Figure 8.3: A left Dehn twist of the red curve about the blue curve.

(2) If we choose two non–zero sections $s_1, s_2 : \alpha \to \nu_\Sigma(\alpha)$ that agree with the orientation, then we get a map

$$S^1 \to \mathbb{R}_{>0}$$
$$t \mapsto s_1(\alpha(t))/s_2(\alpha(t)).$$

Since $\pi_1(\mathbb{R}_{>0}) = 0$, this choice doesn't matter up to isotopy.

(3) Suppose α is isotopic to $\beta : S^1 \to \Sigma$. Then by Corollary 8.2.3 there exists a diffeomorphism $f : \Sigma \to \Sigma$, isotopic to the identity, such that $f \circ \alpha = \beta$. But then

$$f\tau_\alpha f^{-1} = \tau_{f(\alpha)} = \tau_\beta;$$

f^{-1} maps a neighbourhood of β to a neighbourhood of α, does the Dehn twist there, then maps back to a tubular neighbourhood of β.

Given an isotopy F_t with $F_0 = \mathrm{id}$ and $F_1 = f$, we get the isotopy $F_t \circ \tau_\alpha \circ F_t^{-1}$ from τ_α to τ_β, since $\mathrm{Diff}(\Sigma, \partial\Sigma)$ is a topological group. □

The proof of the third claim in Lemma 8.2.6 gives the following fact about Dehn twists.

Corollary 8.2.7. *Let α be a simple closed curve on Σ, and let $[f] \in \mathrm{Mod}(\Sigma)$ be a mapping class, then*

$$[f]t_\alpha[f]^{-1} = t_{f(\alpha)}.$$

See Figure 8.3 for an example of a Dehn twist on a torus.

8.2.4 Action on homology

Since Dehn twists have small support, we can calculate their action on homology using functoriality. Let α denote a simple closed curve embedded in Σ. If we pick an oriented curve $\gamma : S^1 \to \Sigma$ such that $\gamma_*([S^1]) = v \in H_1(\Sigma; \mathbb{Z})$ represents a basis vector. Then by functoriality

$$t_\alpha(v) = (\tau_\alpha \circ \gamma)_*([S^1]).$$

Since homotopic maps induce the same morphism on homology, we calculate $t_\alpha(v)$ by drawing $\tau_\alpha \circ \gamma$ and homotoping to a concatenation of curves whose homology class is understood.

To describe the action of t_α on homology, recall the intersection pairing

$$H_1(\Sigma; \mathbb{Z}) \times H_1(\Sigma; \mathbb{Z}) \to \mathbb{Z}$$
$$(a, b) \mapsto \langle PD(a) \cup PD(b), [\Sigma, \partial\Sigma] \rangle = a \cdot b.$$

Proposition 8.2.8. *[FM11, Prop. 6.3] Let a and b be isotopy classes of oriented simple closed curves in Σ. For any $k \geq 0$, we have*

$$t_b^k([a]) = [a] + (k[a] \cdot [b])[b].$$

8.2.5 Torelli groups

We fix our 2–manifold Σ to be an oriented surface of genus g with one boundary component, and denote its mapping class group by $\mathrm{Mod}_{g,1}$. The Torelli group is the kernel of the action of $\mathrm{Mod}_{g,1}$ on $H_1(\Sigma;\mathbb{Z})$, and is denoted by $\mathcal{I}_{g,1}$. In this subsection, we discuss the abelianization of $\mathcal{I}_{g,1}$ in more detail. This was calculated by Johnson in a series of papers [Joh80b], [Joh80a], [Joh83b], [Joh85a], [Joh85b]; see Putman's notes [Put11] for a more thorough discussion.

We begin by discussing elements of $\mathcal{I}_{g,1}$. Suppose we have two non–isotopic, oriented, non–separating simple closed curves d, e on Σ such that $[d] = [e] \in H_1(\Sigma;\mathbb{Z})$. Then by Proposition 8.2.8, the mapping classes t_d and t_e have the same action on homology, hence $t_d t_e^{-1}$ is in the Torelli group. For another example, let c denote a separating simple closed curve on Σ, so $[c] = 0 \in H_1(\Sigma;\mathbb{Z})$, then by Proposition 8.2.8 the mapping class t_c is in the Torelli group. Elements of the form $t_d t_e^{-1}$ are called bounding pair maps, and elements of the form t_c are called separating twists. These elements form the building blocks of all mapping classes in the Torelli group, by the following.

Theorem 8.2.9. *For $g \geq 1$, the group $\mathcal{I}_{g,1}$ is generated by bounding pair maps and separating twists [Pow78]. For $g \geq 3$, bounding pair maps generate $\mathcal{I}_{g,1}$ [Joh79], and there exists a generating set with only finitely many bounding pair maps [Joh83b].*

Now we outline the construction of the *Birman–Craggs–Johnson homomorphism*, which determines all torsion in the abelianization of the Torelli group. We begin by sketching the original construction of the Birman–Craggs maps: let H_g be an oriented handlebody of genus g, and let $-H_g$ denote the same handlebody with reversed–orientation. Identify $\Sigma_g = \partial(H_g)$ as in Chapter 5, but denote by Mod_g the group $\pi_0(\mathrm{Diff}^+(\Sigma_g))$.

For $h \in \mathrm{Mod}_g$, let $M(h)$ denote the 3–manifold obtained from the disjoint union of H_g and $-H_g$ with the indentification $ih(x) = x$, where i denotes the identity function, thought of as a map $i : \partial(H_g) \to \partial(-H_g)$. Note that $M(\mathrm{id}) = \#_g S^1 \times S^2$. Now suppose that $h_1, h_2 \in \mathrm{Mod}_g$ are a pair of mapping classes with $M(h_2 h_1)$ a $\mathbb{Z}/2$–homology sphere. By taking the Rochlin invariant, Birman and Craggs construct a homomorphism

$$\begin{aligned} \rho_{h_2,h_1} &: \mathcal{I}_g \to \mathbb{Z}/2 \\ k &\mapsto R(M(h_2 k h_1)) - R(M(h_2 h_1)), \end{aligned}$$

where $\mathcal{I}_g$ is the kernel of the action of Mod_g on $H_1(\Sigma_g;\mathbb{Z})$.

To show that the maps ρ_{h_2,h_1} are homomorphisms, they constructed a *fundamental triple*, defined for any pair $(h_2, h_3) \in \mathrm{Mod}_g \times \mathrm{Mod}_g$, as the triple of Heegaard splittings $(M(h_3), M(h_2), M(h_3 h_2^{-1}))$. To a fundamental triple, they associate a 4–manifold $N(h_2, h_3)$ with boundary the disjoint union $-M(h_3) \cup M(h_2) \cup M(h_3 h_2^{-1})$. By analysing the signature of the 4–manifolds $N(h_2, h_3)$, they show that the ρ_{h_2,h_1} are homomorphisms. Interestingly, the 4–manifolds $N(h_2, h_3)$ were among the first examples of trisections; see [Gay22].

Johnson gave an alternative description of the Birman–Craggs maps and studies their extension to the groups $\mathcal{I}_{g,1}$: define a *Heegaard embedding* $h : \Sigma \to S^3$ to be an embedding such that $h(\Sigma)$ is contained in some closed Heegaard surface Σ' for S^3. Let $k \in \mathcal{I}_{g,1}$. Form the 3–manifold $M(h,k)$ by cutting S^3 along Σ' into two handlebodies A and B, and regluing A to B along their boundaries by the map k, where k has been extended along the rest of Σ' by the identity. Since $k \in \mathcal{I}_{g,1}$, the 3–manifold $M(h,k)$ is a $\mathbb{Z}$–homology 3–sphere, and we obtain a map

$$\begin{aligned} R(h,-) &: \mathcal{I}_{g,1} \to \mathbb{Z}/2 \\ k &\mapsto R(M(h,k)). \end{aligned}$$

Let $\overline{h} : \Sigma' \to S^3$ denote the inclusion map of the Heegaard surface containing $h(\Sigma)$. Let $\Sigma'' =$

$\mathrm{cl}(\Sigma' - h(\Sigma))$. The embedding $h : \Sigma \to \Sigma'$ induces a splitting of $H_1(\Sigma')$ into $H_1(\Sigma) \oplus H_1(\Sigma'')$, where $x \cdot x'' = 0$ for all $x \in H_1(\Sigma)$ and $x'' \in H_1(\Sigma'')$. Now any symplectic quadratic form ω' on $H_1(\Sigma')$ can be restricted to forms ω, ω'' on $H_1(\Sigma), H_1(\Sigma'')$ respectively. The condition above implies that $\omega' = \omega \oplus \omega''$.

Johnson shows that every Birman–Craggs homomorphism is of the form $R(\overline{h}, -)$ for some Heegaard embedding $\overline{h} : \Sigma' \to S^3$ that contains $h(\Sigma)$. Here, any element of $\mathrm{Mod}_{g,1}$ is extended over $\Sigma' - h(\Sigma)$ by the identity [Joh80b, Lemma 7].

For a Heegaard embedding $h : \Sigma \to S^3$, let $\omega_h : H_1(\Sigma; \mathbb{Z}/2) \to \mathbb{Z}/2$ denote the symplectic quadratic form induced by the Seifert linking pairing. Johnson shows that $R(h, k)$ only depends on ω_h and k [Joh80b, Cor. 1, Cor. 1']. He then introduces the notation

$$\rho_\omega(k) = R(\omega, k) = R(M(h, k))$$

which is justified by [Joh80b, Lemmas 5 and 16], stating that any symplectic quadratic form on $H_1(\Sigma; \mathbb{Z}/2)$ is induced by some Heegaard embedding $h : \Sigma \to S^3$.

Let Ψ denote the set of all symplectic quadratic forms on $H_1(\Sigma; \mathbb{Z}/2)$. Then in [Joh80b, Thm.2], he obtains the result that the ρ_ω, ranging over all $\omega \in \Psi$, is the set of all Birman–Craggs homomorphisms, and that $\rho_\omega \neq \rho_{\omega'}$ if $\omega, \omega' \in \Psi$ are distinct.

In [Joh80b, Section 9], he introduces the following map

$$\begin{aligned} \zeta_k : \Psi &\to \mathbb{Z}/2 \\ \omega &\mapsto R(\omega, k), \end{aligned}$$

for every $k \in \mathcal{I}_{g,1}$, and showed that the map

$$\begin{aligned} \zeta : \mathcal{I}_{g,1} &\to \mathrm{Map}(\Psi, \mathbb{Z}/2) \\ k &\mapsto \zeta_k, \end{aligned}$$

is a homomorphism, where $\mathrm{Map}(\Psi, \mathbb{Z}/2)$ is the $\mathbb{Z}/2$–vector space of functions $\Psi \to \mathbb{Z}/2$. Let W denote the image of $\mathcal{I}_{g,1}$ under ζ. Then in [Joh80b, Theorem 6], he shows that W has dimension $\sum_{i=0}^{3} \binom{2g}{i}$ as a $\mathbb{Z}/2$ vector space, hence

$$W \cong (\mathbb{Z}/2)^{\sum_{i=0}^{3} \binom{2g}{i}},$$

whenever $g \geq 2$. This completely determines torsion in the abelianization of $\mathcal{I}_{g,1}$ by the following result of Johnson.

Theorem 8.2.10. *[Joh85b] For $g \geq 3$, we have $H_1(\mathcal{I}_{g,1}; \mathbb{Z}) = W \oplus \bigwedge^3 H_1(\Sigma; \mathbb{Z})$. The torsion part is given by the image of the homomorphism ζ above. The torsion free part is given by the Johnson homomorphism $\mathcal{I}_{g,1} \to \bigwedge^3 H_1(\Sigma; \mathbb{Z})$; see [Joh80a], [CF12], [Joh83a] for constructions of the Johnson homomorphism.*

Let $\mathcal{K}_{g,1} \subset \mathrm{Mod}_{g,1}$ denote the normal subgroup generated by Dehn twists about separating curves. We end by noting that the kernel of the Johnson homomorphism is precisely $\mathcal{K}_{g,1}$ [Joh85a]. In [Joh80b, Thm. 6], Johnson also shows that the image of $\mathcal{K}_{g,1}$ under the Birman–Craggs–Johnson homomorphism ζ has dimension $\sum_{i=0}^{2} \binom{2g}{i}$ as a $\mathbb{Z}/2$–vector space.

8.3 Spin structures and spin bordism

Now we explain spin structures, and what it means for a spin $n+1$–manifold to spin bound a spin n–manifold.

Let M be a smooth n–manifold. Then M is *orientable* if M admits an atlas $\{(U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{R}^n)\}_{\alpha \in I}$ such that, for every $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta$ non–empty, the determinant of the Jacobian matrix $J(\varphi_\beta \circ \varphi_\alpha^{-1})$ is positive.

Let $\varphi_\alpha = (x^1, .., x^n)$, and let $\varphi_\beta = (y^1, .., y^n)$, then, by abuse–of–notation, write

$$\varphi_\beta \circ \varphi_\alpha^{-1}(x^1, .., x^n) = (y^1(x^1, ..., x^n), ..., y^n(x^1, .., x^n)).$$

We have that $J(\varphi_\beta \circ \varphi_\alpha^{-1})|_{\varphi_\alpha(p)} = (\frac{\partial y^i}{\partial x^j})^n_{i,j=1}|_{\varphi_\alpha(p)}$ is the change–of–basis matrix between the bases of partials $\{\frac{\partial}{\partial x^i}|_p\}$ and $\{\frac{\partial}{\partial y^i}|_p\}$ for T_pM. The transition function for the tangent bundle TM on $U_\alpha \cap U_\beta$ is also induced by $d(\varphi_\beta \circ \varphi_\alpha^{-1})$. Hence, the orientability condition implies there is a reduction of the structure group of TM from $GL(n, \mathbb{R})$ to $GL^+(n, \mathbb{R})$.

For an orientable n–manifold M, an *orientation* of M is a consistent choice of orientation of the tangent space T_pM, at every point $p \in M$. It can be shown that an n–manifold M is orientable if $H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$, and an orientation of M is given by a choice of generator $[M, \partial M]$ for $H_n(M, \partial M; \mathbb{Z})$, called a *fundamental class*.

Let (M, g) be an oriented Riemannian manifold with $\omega_2(M) = 0$. Let $SO(n) \xrightarrow{i} P(M) \xrightarrow{\pi} M$ denote the oriented orthonormal frame bundle associated to TM equipped with the metric g. A *spin structure* is a class $\xi \in H^1(P(M); \mathbb{Z}/2)$ such that $i^*(\xi) \in H^1(SO(n); \mathbb{Z}/2) = \mathbb{Z}/2$ is a generator [Mil63].

By the universal coefficient theorem, we have $H^1(P(M); \mathbb{Z}/2) = \text{Hom}(H_1(P(M)), \mathbb{Z}/2)$. Let $\gamma : S^1 \to M$ be a curve in M, after framing this curve, we obtain a map $\widetilde{\gamma} : S^1 \to P(M)$, and $\widetilde{\gamma}_*([S^1])$ represents a homology class of $H_1(P(M))$. Hence, a spin structure $\xi \in H^1(P(M))$ can be evaluated on a framed curve. The condition that $i^*(\xi) \in H^1(SO(n); \mathbb{Z}/2)$ is a generator can be interpreted as saying that ξ evaluates to 1 on a homotopically non–trivial loop in the fiber, $SO(n)$, of $P(M)$.

Given an orientation–preserving diffeomorphism $f : M \to M$, the total derivative df induces a diffeomorphism $df : P(M) \to P(M)$, and hence an isomorphism

$$df^* : H^1(P(M); \mathbb{Z}/2) \to H^1(P(M); \mathbb{Z}/2).$$

We say that f *preserves the spin structure* $\xi \in H^1(P(M); \mathbb{Z}/2)$ if $df^*(\xi) = \xi$.

Let W be an oriented $(n + 1)$–manifold, and suppose $\partial W = M$. Let $\iota : M \to W$ denote the inclusion of M as the boundary of W. The outward normal to ∂W induces a map $d\iota : P(M) \to P(W)$; take a basis $T_p(M)$ and complete to a basis of $T_p(W)$ using the outward normal. Using this, we say that a spin $(n + 1)$–manifold (W, ψ) *spin bounds* (M, ξ) if $\partial W = M$ and $d\iota^*(\psi) = \xi$.

We mostly focus on the case when W is a spin 4–manifold, with $\partial(W, \tau) = (M, \sigma)$, where (M, σ) is a spin 3–manifold. Let R be a commutative ring, then we have a symmetric R–bilinear form

$$\begin{aligned} Q_W : H^2(W, \partial W; R) \times H^2(W, \partial W; R) &\to R \\ (a, b) &\mapsto \langle a \cup b, [W, \partial W] \rangle, \end{aligned}$$

called the *intersection form* of W. Here, $a \cup b$ denotes the cup product of $a, b \in H^2(W, \partial W; R)$, and $\langle a \cup b, [W, \partial W] \rangle$ denotes the evaluation of $a \cup b$ on the relative fundamental class $[W, \partial W] \in H_4(W, \partial W; R)$ via the universal coefficient theorem. By Poincare duality, $H_2(W; R) \cong H^2(W, \partial W; R)$, hence Q_W gives a symmetric R–bilinear form on $H_2(W; R)$ as well.

For a symmetric $\mathbb{Z}$–bilinear form $Q : A \times A \to \mathbb{Z}$ on a finitely generated free abelian group A, the *signature* of Q is defined in the following way: extend Q to an $\mathbb{R}$–bilinear form on $A \otimes_{\mathbb{Z}} \mathbb{R}$, let b_2^+ denote the number of positive eigenvalues for any matrix representing Q, and let b_2^- denote the number of negative eigenvalues for the same matrix representing Q. The difference $b_2^+ - b_2^-$ is the *signature* of Q. Sylvester's law of inertia states that the signature is independent of the choice of basis used to represent Q as a matrix.

We deal with the case where W is an oriented 4–manifold, and $(A, Q) = (H_2(W; \mathbb{Z})/\text{torsion}, Q_W)$. Firstly, we collect a few facts about the intersection form. The following result of Rochlin has many applications; for example, it implies that the simply–connected topological 4–manifold with intersection form the $\mathbb{E}_8$ lattice, obtained by Freedman, has no smooth structure.

Theorem 8.3.1. *[Roc52] If X is a smooth, closed spin 4–manifold, then the signature of $(H_2(X), Q_X)$ is divisible by 16.*

The following result is also also very useful, and is referred to as *Novikov additivity.*

Theorem 8.3.2. *[Kir89] Let X_1 and X_2 be two compact, oriented 4–manifolds, and denote by $X = X_1 \cup_N X_2$ the 4–manifold obtained by gluing X_1 and X_2 along a common connected component N of the boundaries of X_1 and X_2. Then the signature of X is the sum of the signatures of X_1 and X_2.*

Signatures play an important role in spin cobordism theory as well. Let (M_1, s_1) and (M_2, s_2) denote two closed spin n–manifolds. Then (M_1, s_1) and (M_2, s_2) are spin cobordant if there is a compact spin manifold (W, s) such that its boundary ∂W, with its induced spin structure, is $(\overline{M_1}, s_1) \sqcup (M_2, s_2)$. Spin cobordism forms an equivalence relation, and the cobordism classes of spin n–manifolds forms an abelian group Ω_n^{Spin} under disjoint union; the cobordism class of the empty set is the identity element, and the inverse of (M, s) is $\overline{M}$ with the spin structure induced by s, where $\overline{M}$ denotes the n–manifold M with opposite orientation.

Theorem 8.3.3. *[Mil63] [Kir89] We have $\Omega_3^{\text{Spin}} = 0$; any spin 3–manifold spin bounds a spin 4–manifold. The map*

$$\begin{aligned} \Omega_4^{\text{Spin}} &\to \mathbb{Z} \\ [(X, s)] &\mapsto \text{Sign}(X)/16 \end{aligned}$$

is a group isomorphism, where Sign(X) *denotes the signature of X. The class of the K3 surface generates Ω_4^{Spin}.*

Theorems 8.3.1, 8.3.3, and 8.3.2 imply that the *Rochlin invariant* of a spin 3–manifold (M, σ) is well–defined. The Rochlin invariant of (M, σ) is the signature $\pmod{16}$ of any spin 4–manifold spin bounding (M, σ). To show well–definedness: suppose we have two spin 4–manifolds (W_1, ψ_1) and (W_2, ψ_2) spin bounding (M, σ), then we glue W_1 and W_2 along M to get a closed spin 4–manifold $(W = W_1 \cup_M \overline{W_2}, \psi)$. Then Novikov additivity, and Rochlin's theorem, imply that

$$\text{Sign}(W) = \text{Sign}(W_1) + \text{Sign}(\overline{W_2}) = \text{Sign}(W_1) - \text{Sign}(W_2) \equiv 0 \pmod{16},$$

hence $\text{Sign}(W_1) \equiv \text{Sign}(W_2) \pmod{16}$.

8.3.1 Spin structures on mapping tori

We prove some facts about mapping tori, since they feature heavily throughout the text.

Lemma 8.3.4. *Let Σ_g be an oriented surface, and let $f : \Sigma_g \to \Sigma_g$ be an orientation–preserving diffeomorphism, then the mapping torus is given by*

$$M_f = I \times \Sigma_g/(1, x) \sim (0, f(x)).$$

Then

1. *M_f is an orientable 3–manifold, and is a Σ_g–bundle over S^1, hence it inherits a natural orientation from that of Σ_g and S^1.*

2. *$M_{f^{-1}} = \overline{M_f}$.*

3. *For $[h] \in \mathrm{Mod}_{g,1}[2]$ and $[f] \in \mathrm{Mod}_{g,1}$, the diffeomorphism $\mathrm{id} \times f : M_h \to M_{fhf-1}$ pulls back the spin structure $(M_{fhf-1}, \theta(\sigma))$ to the spin structure $(M_h, \theta(f^*(\sigma)))$, in the notation of Chapter 2.*

Proof. 1. Let $(U_i, \varphi_i : U_i \to \mathbb{R}^2)_{i\in I}$ be an orientable atlas for Σ_g. We use this atlas to build an atlas for M_f. for $(t,x) \in M_f$ with $t \neq 0,1$, let $\varphi : U \to \mathbb{R}^2$ be a chart of Σ_g at x, then

$$\begin{aligned} \psi : (t-\epsilon, t+\epsilon) \times U &\to \mathbb{R}^3 \\ (t,x) &\mapsto (t, \varphi(x)) \end{aligned}$$

gives a chart at $(t,x) \in M_f$.

For $(1,x) \sim (0, f(x)) \in M_f$, let $\varphi_i : U_i \to \mathbb{R}^2$ be a chart for Σ_g at x, and let

$$\begin{aligned} V_1 &= (1-\epsilon, 1] \times U_i \\ V_2 &= [0, \epsilon) \times f(U_i). \end{aligned}$$

Then $V_1 \cup V_2 \subset M_f$ is open, and define

$$\begin{aligned} \psi_i : V_1 \cup V_2 &\to \mathbb{R}^3 \\ (t,x) &\mapsto \begin{cases} (t, \varphi_i(x)), \text{ if } (t,x) \in V_1 \\ (t+1, \varphi_i \circ f^{-1}(x)), \text{ if } (t,x) \in V_2 \end{cases}. \end{aligned}$$

Then ψ_i is well–defined: if $(1,x) \in V_1$, then $\psi_i(1,x) = (1, \varphi_i(x))$, and if $(0, f(x)) \in V_2$, then $\psi_i(0, f(x)) = (1, \varphi_i \circ f^{-1} \circ f(x)) = (1, \varphi_i(x))$. These charts are also orientable: suppose φ_i, φ_j are two charts for Σ_g at x. Then for $(t,x) \in [1, 1+\epsilon) \times \mathbb{R}^2$, we have $\psi_j^{-1}(t,x) = (t-1, f \circ \varphi_j^{-1}(x))$, and

$$\psi_i \circ \psi_j^{-1}(t,x) = (t, \varphi_i \circ \varphi_j^{-1}(x)),$$

which is orientation–preserving. Hence M_f is an orientable 3–manifold.

Now we show that the map

$$\begin{aligned} p : M_f &\to I/0 \sim 1 = S^1 \\ (t,x) &\mapsto t \end{aligned}$$

turns M_f into a Σ_g–bundle over S^1. For $t \in S^1$ with $t \neq 0,1$, take an interval of the form $(t-\epsilon, t+\epsilon)$, that does not contain $0,1$. Then $p^{-1}((t-\epsilon, t+\epsilon)) = (t-\epsilon, t+\epsilon) \times \Sigma_g$. For $0 \sim 1 \in S^1$, write $S^1 = \mathbb{R}/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{R}$ by translation. Take the open neighbourhood $U_0 = (1-\epsilon, 1+\epsilon) \subset S^1$ of 0. Then

$$p^{-1}(U_0) = \{(t,x) \in M_f \mid 0 \leq t \leq \epsilon \text{ or } 1-\epsilon \leq t \leq 1\}.$$

Define a homeomorphism

$$\begin{aligned} \Psi : p^{-1}(U_0) &\to U_0 \times \Sigma_g \\ (t,x) &\mapsto \begin{cases} (t,x)\ , \text{ if } 1-\epsilon \leq t \leq 1 \\ (t+1, f^{-1}(x))\ , \text{ if } 0 \leq t \leq \epsilon \end{cases}. \end{aligned}$$

This map is well–defined because $\Psi(1,x)=(1,x)$, and $\Psi(0,f(x))=(1,f^{-1}f(x))=(1,x)$. Furthermore $\mathrm{pr}_1\circ\Psi(t,x)=p(t,x)$, since $t+1\sim t\in S^1=\mathbb{R}/\mathbb{Z}$. Hence we have described a family of local trivializations for $p:M_f\to S^1$, so the claim follows.

2. The map

$$\begin{aligned}M_f &\to M_{f^{-1}}\\ (t,x)&\mapsto(1-t,x)\end{aligned}$$

is well–defined, and is an orientation–reversing diffeomorphism due to the I factor.

3. The diffeomorphism $\mathrm{id}\times f:M_h\to M_{fhf^{-1}}$ is well defined, since $\mathrm{id}\times f(1,x)=(1,f(x))\sim(0,fh(x))$, and $\mathrm{id}\times f(0,h(x))=(0,fh(x))$. Then, in the language of Chapter 2, the pullback satisfies

$$(\mathrm{id}\times f)^*(\theta(\sigma))=d(\mathrm{id}\times f^{-1})\circ\theta(\sigma)\circ(\mathrm{id}\times f)=\theta(df^{-1}\circ\sigma\circ f)=\theta(f^*(\sigma)),$$

for any spin structure $\sigma\in\mathrm{Spin}(\Sigma_g)$. □

8.4 The Arf invariant

Our exposition of the Arf invariant follows [Sav11] very closely.

8.4.1 The Arf invariant of a quadratic form

Let V be a finite dimensional vector space over $\mathbb{Z}/2$. A function $q:V\to\mathbb{Z}/2$ is a *quadratic form* if

$$I(x,y):=q(x+y)-q(x)-q(y)$$

is a bilinear form over $\mathbb{Z}/2$. We have that $I(x,y)=I(y,x), I(x,x)=0$, and $q(0)=0$.

We call a quadratic form q *non–degenerate* if the associated bilinear form I is non–degenerate.

Given two quadratic forms $q_1,q_2:V\to\mathbb{Z}/2$, both with associated bilinear form $I:V\times V\to\mathbb{Z}/2$. We say that q_1 and q_2 are *equivalent* if there exists a $\mathbb{Z}/2$–linear isomorphism $\alpha:V\to V$ such that $q_1(x,y)=q_2(\alpha(x),\alpha(y))$ for all $x,y\in V$.

The quadratic forms given by the two examples below are very important for the theory that follows. Let $U=(\mathbb{Z}/2)^2$ with basis a,b, and let $I:U\times U\to\mathbb{Z}/2$ denote the non–degenerate symmetric bilinear form given by $I(a,a)=I(b,b)=0$, and $I(a,b)=1$. The following two examples both have I as their associated bilinear form.

Example 1: Define the quadratic form $q_0:U\to\mathbb{Z}/2$ by $q_0(a)=q_0(b)=0$, and $q_0(a+b)=1$.

Example 2: Define the quadratic form $q_1:U\to\mathbb{Z}/2$ by $q_1(a)=q_1(b)=q_1(a+b)=1$.

Lemma 8.4.1. *The quadratic forms $q_0,q_1:U\to\mathbb{Z}/2$ are not equivalent. Furthermore, any other non–degenerate quadratic form q on U is equivalent to either q_0 or q_1.*

Proof. If q_0 and q_1 were equivalent, then the number of vectors in $U=(\mathbb{Z}/2)^2$ sent to zero by the q_i would be equal, this gives a contradiction.

For the second claim, note that to define a quadratic form $q:U\to\mathbb{Z}/2$, we need only specify its values of the four elements of U. The only case left to consider then is the case $q(a)=0$ and $q(b)=1$. There exists a change of basis $a'=a$ and $b'=a+b$ with $q(a')=0$ and $q(b')=0$. So q is equivalent to q_0. □

The following lemma is used to define the Arf invariant of a quadratic form. For a proof, see [Sav11, Lemma 9.1].

Lemma 8.4.2. *For any non–degenerate quadratic form $q : V \to \mathbb{Z}/2$, there exists a symplectic basis $\{a_i, b_i\}_{i=1}^n$ for V such that $I(a_i, a_j) = I(b_i, b_j) = 0$ and $I(a_i, b_j) = \delta_{ij}$. In particular,* $\dim(V)$ *is always even.*

Let $q : V \to \mathbb{Z}/2$ be a non–degenerate quadratic form, and let $\{a_i, b_i\}_{i=1}^{\dim(V)/2}$ be a symplectic basis for V. Define the *Arf invariant* of q by

$$\mathrm{Arf}(q) := \sum_{i=1}^{\dim(V)/2} q(a_i)q(b_i) \in \mathbb{Z}/2.$$

We sketch below why $\mathrm{Arf}(q)$ is independent of the choice of symplectic basis for V.

Example: We have $\mathrm{Arf}(q_0) = 0$, and $\mathrm{Arf}(q_1) = 1$ for the two examples above. This implies that non–degenerate quadratic forms on U are classified up to equivalence by the Arf invariant.

The following Lemma can be shown by finding an appropriate change of basis; see [Sav11, Lemma 9.2].

Lemma 8.4.3. *On $U \oplus U$, the forms $q_0 + q_0$ and $q_1 + q_1$ are equivalent.*

Lemma 8.4.4. *Let $q : V \to \mathbb{Z}/2$ be a non–degenerate quadratic form, where* $\dim(V) = 2m$*. Then q is equivalent to $q_1 + (m-1)q_0$ if, with respect to some basis,* $\mathrm{Arf}(q) = 1$*. The form q is equivalent to mq_0 if* $\mathrm{Arf}(q_0) = 0$.

Proof. Let $\{a_i, b_i\}_{i=1}^m$ be a symplectic basis for V, and let $V_i = \mathbb{Z}/2\{a_i, b_i\}$. Let ψ_i denote the restriction of q to V_i, then $q = \sum_{i=1}^m \psi_i$, and each ψ_i is equivalent in V_i to either q_0 or q_1 by Lemma 8.4.1. By Lemma 8.4.3 $2q_0 = 2q_1$, so q must be equivalent to either mq_0 or $q_1 + (m-1)q_0$. We can compute that $\mathrm{Arf}(q_1 + (m-1)q_0) = 1$, and $\mathrm{Arf}(mq_0) = 0$. □

To complete the classification of non–degenerate quadratic forms $q : V \to \mathbb{Z}/2$, it remains to show that $q_1 + (m-1)q_0$ and mq_0 are not equivalent. We introduce the following notation to deal with this problem: Let V be a vector space over $\mathbb{Z}/2$, and let Ω denote the space of quadratic forms $\varphi : V \to \mathbb{Z}/2$. Then the group $GL(V)$ of all $\mathbb{Z}/2$–linear isomorphisms of V acts on Ω via precomposition, and two forms in Ω are equivalent if they are in the same orbit under this action.

Define $p : \Omega \to \mathbb{Z}$ by $p(\varphi) =$ the number of elements of V sent to 1 by φ. Define $n : \Omega \to \mathbb{Z}$ by $n(\varphi) =$ the number of elements of V sent to 0 by φ. These two functions play an important role in the following Lemma.

Lemma 8.4.5. *The form $q_1 + (m-1)q_0$ sends a majority of elements of V to 1, while mq_0 sends a majority of elements of V to 0.*

Proof. Proceed by induction on m. The case $m = 1$ was discussed above. For the inductive step, note that the functions $p, n : \Omega \to \mathbb{Z}$ defined above satisfy the following identities:

$$p(\varphi + q_0) = 3p(\varphi) + n(\varphi)$$

$$n(\varphi + q_0) = 3n(\varphi) + p(\varphi),$$

where $\varphi + q_0$ is a quadratic form on $V \oplus U$.

Set $r(\varphi) = p(\varphi) - n(\varphi)$, then

$$\begin{aligned} r(\varphi + q_0) &= 3p(\varphi) + n(\varphi) - 3n(\varphi) - p(\varphi) \\ &= 2(p(\varphi) - n(\varphi)) \\ &= 2r(\varphi). \end{aligned}$$

This identity implies that if $r(\varphi) > 0$, then $r(\varphi + q_0) > 0$. Similarly, if $r(\varphi) < 0$, then $r(\varphi + q_0) < 0$. We have $r(q_1) = 2$, and $r(q_0) = -2$, implying that $r(q_1 + (m-1)q_0) > 0$, and $r(mq_0) < 0$. □

Using the above, we arrive at the following result of Arf.

Theorem 8.4.6. *[Arf41] Two non–degenerate quadratic forms on a $\mathbb{Z}/2$–vector space V of finite dimension are equivalent if and only if they have the same Arf invariant.*

Proof. Let $R : \Omega \to \mathbb{Z}/2$ be given by $R(\varphi) = \text{sign}(r(\varphi))$, where $r : \Omega \to \mathbb{Z}$ is the function given in the proof of Lemma 8.4.5. Then R is invariant under the action of $GL(V)$ on Ω. Since R takes different values on $q_1 + (m-1)q_0$ and mq_0, these two forms are not equivalent. □

Arf's result is very useful, for example, if we combine Theorem 8.4.6 with Theorem 2.0.1, we get that there are only two spin structures on a Riemann surface Σ_g, up to spin–diffeomorphism.

8.4.2 The Arf invariant of a knot

To define the Arf invariant of a knot, recall the following notions from knot theory.

A *Seifert surface* for an oriented link in S^3 is a connected, compact, oriented surface smoothly embedded in S^3, with oriented boundary equal to the link. *Seifert's algorithm* implies that every oriented link in S^3 bounds a Seifert surface. How are two Seifert surfaces for the same link related? To answer this, recall the following two operations on Seifert surfaces:

1. Cut out two 2–discs D_1 and D_2 in the Seifert surface.
2. Take a copy of $S^1 \times I$ embedded in S^3 disjointly from the surface, but with $S^1 \times \partial I$ attached to ∂D_1 and ∂D_2.

Call this process 1*–surgery* on a Seifert surface. The other process is the following:

1. Find a curve α on the Seifert surface F such that α bounds a disc in $S^3 - F$.
2. Cut out a copy of $\alpha \times I$ in F, and cap off the two new boundary circles with discs.

Call this process 0*–surgery.*

If F and F' are two oriented Seifert surfaces for the same link in S^3, we say that F and F' are *S–equivalent* if F' can be obtained from F by combinations of 0–surgery, 1–surgery, and ambient isotopy. It turns out that S–equivalence is enough to describe all Seifert surfaces for a fixed link.

Theorem 8.4.7. *[Kau87, Theorem 7.7] Let F and F' be Seifert surfaces for ambient isotopic links L and L' in S^3. Then F and F' are S–equivalent.*

Seifert's linking form. Let F be a Seifert surface for an oriented link in S^3. Given a curve $a \subset F$, let a^* denote the pushoff of a into $S^3 - F$ using the positive normal to F in S^3. The *Seifert pairing* of F is given by

$$\begin{aligned} \lambda : H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) &\to \mathbb{Z} \\ ([a],[b]) &\mapsto lk(a^*, b). \end{aligned}$$

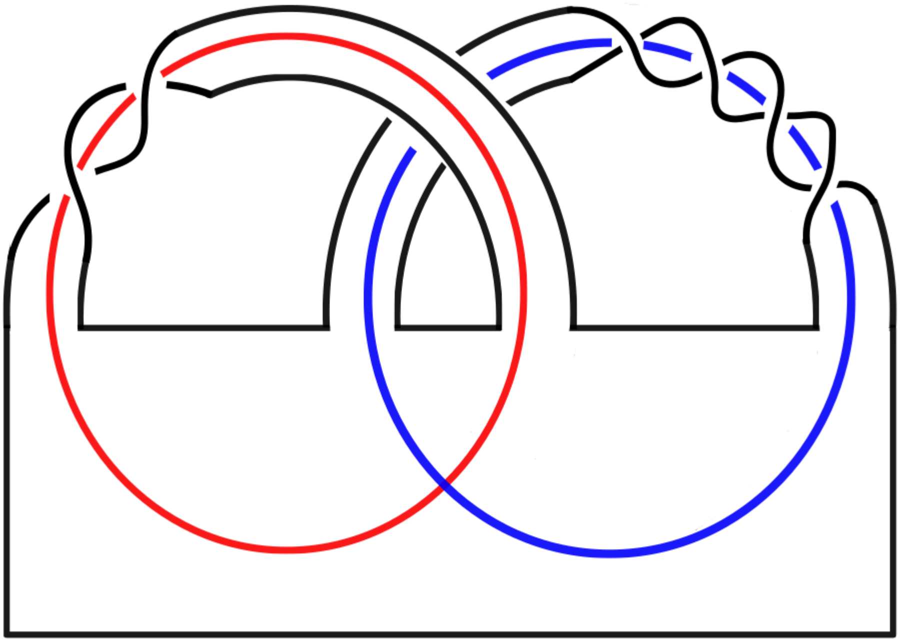

Figure 8.4: Let γ_1 denote the red curve, and let γ_2 denote the blue curve. Then $[\gamma_1], [\gamma_2]$ gives a symplectic basis for $H_1(F; \mathbb{Z}/2)$, where $q_F(\gamma_1) = 1$ and $q_F(\gamma_2) = 0$.

It is a well–defined bilinear pairing that satisfies the identity:

$$\lambda([a], [b]) = \lambda([b], [a]) + [a] \cdot [b],$$

where $[a] \cdot [b]$ denotes the intersection form on $H_1(F; \mathbb{Z})$; see [Kau87, Ch. VII].

Given a Seifert surface $F \subset S^3$ for a knot, define the quadratic form

$$\begin{aligned} q_F : H_1(F; \mathbb{Z}/2) &\to \mathbb{Z}/2 \\ [a] &\mapsto \lambda([a], [a]). \end{aligned}$$

Its associated bilinear form is the intersection from on $H_1(F; \mathbb{Z}/2)$.

To calculate q_F, pick a collection of curves in F that represents a basis for $H_1(F; \mathbb{Z}/2)$. Let x be one of these curves, then $q_F([x])$ is the number of full twists (mod 2) in a neighbourhood of x in $F \subset S^3$. To see this, note that the pushoff x^* of x can be isotoped to a curve in F that is parallel to x. This curve is isotopic to x, hence we get an embedding of an annulus in F with boundary x and x^*. The isotopy class of this annulus need not be trivial, and it can have an even number of twists in S^3; see Figure 8.4 for an example.

Lemma 8.4.8. *For an oriented knot $K \subset S^3$, and a Seifert surface F for K, the Arf invariant* $\mathrm{Arf}(q_F)$ *only depends on the knot K.*

Proof. We need only show that $\mathrm{Arf}(q_F)$ doesn't change under 1–surgery applied to F, since 0–surgery is an inverse operation to 1–surgery, and λ is an invariant of the ambient isotopy class of the Seifert surface. So let F' be the result of 1–surgery applied to F.

Fix a symplectic basis B for $H_1(F; \mathbb{Z})$, and let P denote the matrix representing the Seifert pairing $\lambda : H_1(F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \to \mathbb{Z}$. Then 1–surgery replaces P with the matrix

$$\begin{pmatrix} P & a & 0 \\ b & c & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where $a = (a_1, .., a_{2g})$ is a column vector with $g = g(F)$ and entry a_i in the ith row, $b = (b_1, ..., b_{2g})$ is a row vector with entry b_i in the ith column, and the bottom right is a 2×2 matrix, with c an integer.

Using elementary row and column operations, we can make $c = 0$ and $a_i + b_i = 0$, for all $i = 1, .., g$. This corresponds to a completion of the symplectic basis B for $H_1(F; \mathbb{Z})$ to a symplectic basis for $H_1(F'; \mathbb{Z})$, with

$$\mathrm{Arf}(q_{F'}) = \mathrm{Arf}(q_F) + \mathrm{Arf}(q_0) = \mathrm{Arf}(q_F) \pmod 2.$$

□

Using Lemma 8.4.8, we define the *Arf invariant* of a knot $K \subset S^3$ to be the Arf invariant of the Seifert self linking pairing q_F, for any Seifert surface F for K in S^3.

8.5 The intersection form of a 2–handlebody

For an oriented, framed link $L = (L_i)_{i=1}^n$ in S^3, a 2*–handlebody* W_L is the compact 4–manifold obtained in the following way: start with the 4–disc D^4, and attach a 2–handle $h_i = D^2 \times D^2$ along an embedding $\partial D^2 \times D^2 \hookrightarrow \partial D^4$ specified by the framed knot L_i. This embedding is determined by sending $\partial D^2 \times \{0\}$ to L_i, and sending $\partial D^2 \times \{pt\}$ to a pushoff of L_i along the framing. In this section, we explain a nice relation between the intersection form of W_L, and the linking matrix of L. We begin by recalling results about intersection forms of 4–manifolds.

Let X be a compact, oriented 4–manifold, and let $a, b \in H^2(X, \partial X; \mathbb{Z})$. Take surfaces Σ_a and Σ_b representing the Poincare duals of a and b respectively. Suppose Σ_a and Σ_b intersect *transversely* at every $p \in \Sigma_a \cap \Sigma_b$, so that $T_p\Sigma_a$ and $T_p\Sigma_b$ span T_pX. Attach a sign to every point of $\Sigma_a \cap \Sigma_b$ in the following way: concatenate a positively oriented basis of $T_p\Sigma_a$ and $T_p\Sigma_b$ to get a basis for T_pX. Assign 1 if this basis is positively oriented in T_pX, and -1 otherwise.

Proposition 8.5.1. *[GS99, Prop. 1.2.5] For $a, b \in H^2(X; \partial X; \mathbb{Z})$, and Σ_a, Σ_b as above, we have that $Q_X(a, b)$ is the number of points in $\Sigma_a \cap \Sigma_b$, counted with sign as above.*

To compute the intersection form of W_L, recall that the *linking matrix* of L is the symmetric $n \times n$ matrix (a_{ij}), where $a_{ij} = lk(L_i, L_j)$ if $i \neq j$, and a_{ii} is the framing coefficient of L_i.

Up to homotopy, attaching a 2–handle to D^4 is the same as attaching a 2–cell to a point, so W_L is homotopy equivalent to a wedge sum of n two spheres. Hence W_L is simply–connected, and $H_2(W_L; \mathbb{Z}) \cong \mathbb{Z}^n$. We obtain a basis for $H_2(W_L; \mathbb{Z})$ by constructing surfaces in W_L as follows:

Identify $I \times S^3$ with a collar of ∂D^4 in D^4, with $\{1\} \times S^3 = \partial D^4$. Take a Seifert surface F_i for L_i in S^3, and place it at $\{t\} \times S^3$ for some $t < 1$. Then cap off this surface with the union of $D^2 \times \{0\}$ in the 2–handle $h_i = D^2 \times D^2$, and the annulus $[t, 1] \times (\partial D^2 \times \{0\}) = [t, 1] \times L_i$. We obtain a closed surface $\widetilde{F}_i$ in this way, with an orientation induced by that of F_i. The classes $[\widetilde{F}_i]_{i=1}^n$ give a basis for $H_2(W_L; \mathbb{Z})$.

Proposition 8.5.2. *[GS99, Prop. 4.5.11] The matrix of Q_{W_L} with respect to the basis $[\widetilde{F}_i]_{i=1}^n$ obtained above is given by the linking matrix of L.*

Proof. Fix $i \neq j$, and assume that $\widetilde{F_j}$ was obtained from $\{s\} \times F_j \subset I \times S^3$, and that $\widetilde{F}_i$ was obtained from $\{t\} \times F_i \subset I \times S^3$, where $s < t$. Then

$$[\widetilde{F}_i] \cdot [\widetilde{F_j}] = (\{t\} \times F_i) \cdot (\{t\} \times L_j) = lk(L_i, L_j).$$

To compute $[\widetilde{F}_i]^2$, construct a surface $\widetilde{F}_i'$, isotopic to $\widetilde{F}_i$, in the following way: begin with a disc $D^2 \times \{p\} \subset h_i = D^2 \times D^2$, parallel to $D^2 \times \{0\}$. Then $\partial D^2 \times \{p\}$ is a pushoff L_i' of L_i in the direction of the framing. Take a Seifert surface F_i' for L_i' in $\{r\} \times S^3$, and cap off with $([r,1] \times L_i') \cup (D^2 \times \{p\})$, to get a closed surface $\widetilde{F}_i'$. The $D^2 \times \{p\}$ part of $\widetilde{F}_i'$ intersects ∂D^4 in L_i', so

$$[\widetilde{F}_i] \cdot [\widetilde{F}_i'] = F_i \cdot L_i' = lk(L_i, L_i').$$

$\square$

Bibliography

[Akb77] S. Akbulut. "On 2-dimensional homology classes of 4-manifolds". In: *Math. Proc. Camb. Phil. Soc., 82* (1977), pp. 99–106 (cit. on p. 16).

[Arf41] C. Arf. "Untersuchungen uber quadratische Formen in Korpern der Charakteristik 2". In: *J. Reine Angew. Math. 183* (1941) (cit. on p. 63).

[BC78] J. S. Birman and R. Craggs. "The mu-invariant of 3−manifolds and certain structural properties of the group of homemorphisms of a closed, oriented 2−manifold". In: *Transactions of the American Mathematical Society, Volume 237* (1978), pp. 283–309 (cit. on pp. 5–7, 28, 30).

[BM96] C. Blanchet and G. Masbaum. "Topological Quantum Field Theories for surfaces with spin structure". In: *Duke Math. J., Vol. 82, No. 2* (1996), pp. 229–267 (cit. on p. 22).

[Bre72] G. Bredon. *Introduction to compact transformation groups*. Elsevier, 1972 (cit. on p. 52).

[CF12] T. Church and B. Farb. "Parametrised Abel–Jacobi maps and abelian cycles in the Torelli group". In: *J. Topol., 5, no.1* (2012), pp. 15–38 (cit. on p. 57).

[EE67] C.J. Earle and J. Eells. "The diffeomorphism group of a compact Riemann surface". In: *Bull.Am.Math.Soc. 73* (1967) (cit. on p. 47).

[EE69] C.J. Earle and J. Eells. "A fiber bundle description of Teichmuller theory". In: *J. Diff. Geom. 3* (1969) (cit. on p. 48).

[Ehr50] C. Ehresmann. "Les connexions infinitésimales dans un espace fibré différentiable". In: *Colloque de topologie (espaces fibrés), Bruxelles* (1950) (cit. on p. 49).

[ES70] C.J. Earle and A. Schatz. "Teichmuller theory for surfaces with boundary". In: *J. Diff. Geom. 4* (1970) (cit. on p. 48).

[Far06] B. Farb. *Problems on Mapping Class Groups and Related Topics*. Proceedings of Symposia in Pure Mathematics Volume: 74, 2006 (cit. on p. 6).

[FM11] B. Farb and D. Margalit. *A Primer on Mapping Class Groups*. Princeton Mathematical Series - 49, 2011 (cit. on pp. 5, 43, 44, 55).

[Gay22] D.T. Gay. "From near–symplectic constructions to trisections of 4–manifolds". In: *Celebratio Mathematica* (2022) (cit. on p. 56).

[GS99] R. Gompf and A. Stipsicz. *4-manifolds and Kirby calculus*. A.M.S. Graduate Studies in Mathematics, Volume: 20, 1999 (cit. on pp. 12, 16, 21, 65).

[Hat02] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002 (cit. on p. 10).

[Hir76] M.W. Hirsch. *Differential Topology*. Springer, 1976 (cit. on pp. 46, 50, 51).

[Hos84] J. Hoste. "The Arf Invariant of a Totally Proper Link". In: *Topology and its Applications 18* (1984), pp. 163–177 (cit. on p. 25).

[Hum92] S. P. Humphries. "Normal closures of powers of Dehn twists in mapping class groups." In: *Glasgow Math. J. 34* (1992), pp. 313–317 (cit. on pp. 11, 23, 24).

[Joh79] D. Johnson. "Homeomorphisms of a Surface which act trivially on homology". In: *Proc. Amer. Math. Soc., Vol. 75, No. 1* (1979) (cit. on pp. 29, 43, 56).

[Joh80a] D. Johnson. "An abelian quotient of the mapping class group I_g". In: *Math. Ann., Vol. 249, No. 3* (1980), pp. 225–242 (cit. on pp. 56, 57).

[Joh80b] D. Johnson. “Quadratic forms and the Birman-Craggs homomorphisms”. In: *Transactions of the American Mathematical Society, Vol. 261* (1980), pp. 235–254 (cit. on pp. 5, 28, 32, 33, 35, 56, 57).

[Joh80c] D. Johnson. “Spin Structures and Quadratic forms on Surfaces”. In: *J. London Math. Soc.* (1980) (cit. on pp. 11, 23).

[Joh83a] D. Johnson. “A survey of the Torelli group”. In: *Contemporary Mathematics, Volume 20* (1983), pp. 165–179 (cit. on pp. 7, 57).

[Joh83b] D. Johnson. “The structure of the Torelli group I: A finite set of generators for I”. In: *Annals of Mathematics, 118* (1983), pp. 423–442 (cit. on p. 56).

[Joh85a] D. Johnson. “The structure of the Torelli group-II: A characterization of the group generated by twists on bounding curves.” In: *Topology, Vol. 24, No. 2* (1985), pp. 113–126 (cit. on pp. 56, 57).

[Joh85b] D. Johnson. “The structure of the Torelli group-III: The abelianization of I”. In: *Topology, Vol. 24, No. 2* (1985), pp. 127–144 (cit. on pp. 5, 56, 57).

[Kau87] L. Kauffman. *On Knots*. Princeton University Press AM-115, 1987 (cit. on pp. 15, 63, 64).

[Kir89] R. Kirby. *The Topology of 4-Manifolds*. Springer Lecture Notes in Mathematics, volume 1374, 1989 (cit. on p. 59).

[KM91] R. Kirby and P. Melvin. “The 3-manifold invariants of Witten and Reshetikhin- Turaev for sl(2, C)”. In: *Invent. Math. 105* (1991), pp. 473–545 (cit. on pp. 7, 21, 24).

[KM94] R. Kirby and P. Melvin. “Dedekind sums, mu-invariants and the signature cocycle”. In: *Math. Ann. 299* (1994), pp. 231–267 (cit. on pp. 6, 12).

[Lam20] P. Lambert-Cole. “Trisections, intersection forms and the Torelli group”. In: *Algebraic and Geometric Topology 20* (2020), pp. 1015–1040 (cit. on p. 5).

[Lee00] J.M. Lee. *Introduction to Smooth Manifolds, Second Edition*. Springer, 2000 (cit. on p. 49).

[Lic62] W.B.R Lickorish. “A Representation of Orientable Combinatorial 3-Manifolds”. In: *Annals of Mathematics, Second Series, Vol. 76, No. 3* (1962), pp. 531–540 (cit. on p. 12).

[Lu92] N. Lu. “A simple proof of the fundamental theorem of Kirby calculus on links”. In: *Transactions of the American Mathematical Society, Vol. 331, Number 1* (1992), pp. 143–156 (cit. on p. 7).

[Mey73] W. Meyer. “Die Signatur von Flachenbundeln”. In: *Math. Ann., Vol. 201* (1973), pp. 239–264 (cit. on pp. 8, 36).

[Mil63] J. Milnor. “Spin structures on manifolds”. In: *L’Ensignement Math. 9* (1963) (cit. on pp. 58, 59).

[Mor89] S. Morita. “Casson’s invariant for homology 3-spheres and characteristic classes of surface bundles I”. In: *Topology, Vol. 28 No. 3* (1989), pp. 305–323 (cit. on p. 5).

[MP94] S. Matveev and M. Polyak. “A Geometrical Presentation of the Surface Mapping Class Group and Surgery”. In: *Commun. Math. Phys. 160* (1994), pp. 537–556 (cit. on pp. 7, 19, 20).

[Pow78] J. Powell. “Two theorems on the mapping class group of a surface”. In: *Proc. Amer. Math. Soc. 68, no.3* (1978), pp. 347–350 (cit. on p. 56).

[PR21] W. Pitsch and R. Riba. “Invariants of rational homology 3-spheres from the abelianization of the mod-p Torelli group”. In: *arXiv preprint arXiv:2103.15519* (2021) (cit. on p. 6).

[Put11] A. Putman. “The Torelli group and congruence subgroups of the mapping class group”. In: *Notes* (2011) (cit. on p. 56).

[Put12] A. Putman. “The Picard group of the moduli space of curves with level structures”. In: *Duke Mathematical Journal, Vol.161, No.4* (2012), pp. 623–674 (cit. on p. 6).

[Rob65] R. A. Robertello. “An Invariant of Knot Cobordism”. In: *Communications on Pure and Applied Mathematics, Vol XVIII* (1965), pp. 543–555 (cit. on p. 25).

[Roc52] V. Rochlin. “New results in the theory of four dimensional manifolds”. In: *Dokl. Akad. Nauk. 84* (1952) (cit. on p. 59).

[RT91] N. Reshetikhin and V.G. Turaev. “Invariants of 3-manifolds via link polynomials and quantum groups”. In: *Inventiones mathematicae* (1991), pp. 547–597 (cit. on pp. 6, 12, 18).

[Sat10] M. Sato. “The abelianization of the level d mapping class group”. In: *Journal of Topology, Volume 3, Issue 4* (2010), pp. 847–882 (cit. on pp. 6, 11, 27, 36, 41).

[Sav11] N. Saveliev. *Lectures on the Topology of 3–Manifolds.* De Gruyter, 2011 (cit. on pp. 61, 62).

[Wri94] G. Wright. “The Reshetikhin-Turaev representation of the mapping class group”. In: *Journal of Knot Theory and its Ramifications, Vol. 3 No. 4* (1994), pp. 547–574 (cit. on p. 12).