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Chiralization of star-products and the isomorphisms of deformations and quantizations of Kleinian singularities

by

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Ai miei genitori,
che mi hanno insegnato
a contare fino alle stelle

Abstract

This thesis contains two directions of research, both related to the quantizations and deformations of Poisson structures.

In the first part, we study the chiralization of star-products, a problem related to the quantization of Poisson vertex algebras. For a Poisson algebra $\mathcal{A}$, a star-product is a new product $\star$ such that the associative algebra $A = (\mathcal{A}, \star)$ is a quantization of $\mathcal{A}$. Famous examples are the Moyal-Weyl and Gutt star-products. Poisson vertex algebras are the chiralization of Poisson algebras and their quantizations are vertex algebras, which are the chiralization of associative algebras. A star-deformation of a Poisson vertex algebra $\mathcal{V}$ is a vertex algebra structure $Y_\star$ such that the vertex algebra $V = (\mathcal{V}, Y_\star)$ is a quantization of $\mathcal{V}$. Star-deformations can thus be seen as the chiral analogue of a star-product, although they are not compatible with the Zhu functors. To solve this, we introduce and study the algebraic structure of $\hbar$-deformed vertex algebras, which is closely related to the Zhu functor. A chiral star-product is then defined as a deformation of a Poisson vertex algebra into an $\hbar$-vertex algebra. We show that chiral star-products commute with the Zhu functor, giving back a star-product on the corresponding Poisson algebra. By putting $\hbar = 0$, we obtain a star-deformation. We study the problem of constructing chiral star-products and we provide explicit formulae in some important examples, including when V is a free-field vertex algebra, the affine vertex algebra, or the Virasoro vertex algebra. In particular, these formulae give the chiralization of the Moyal-Weyl and Gutt star-products. Additionally, we provide a new, more natural proof of the associativity of the Zhu algebra using the formalism of $\hbar$-vertex algebras.

In the second part, we deal with the algebra of functions on Kleinian singularities. It is known that, in this case, the parameter space of filtered Poisson deformations and the parameter space of non-commutative quantizations coincide. We consider all possible isomorphisms between the various deformations (as Poisson algebras) and all isomorphisms between the quantizations (as associative algebras); these form two groupoids, which we denote PIso and Iso. We prove that, for a Kleinian singularity of type A or D, the groupoids Iso and PIso are isomorphic. In particular, the group of automorphisms of the deformation and the quantization corresponding to the same deformation parameter are isomorphic. Furthermore, we describe the groups of automorphisms as abstract groups: for type A they have an amalgamated free product structure, for type D they are subgroups of the group of Dynkin diagram automorphisms. For type D we additionally compute all the possible isomorphisms between deformations as affine varieties.

Contents

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Author’s declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

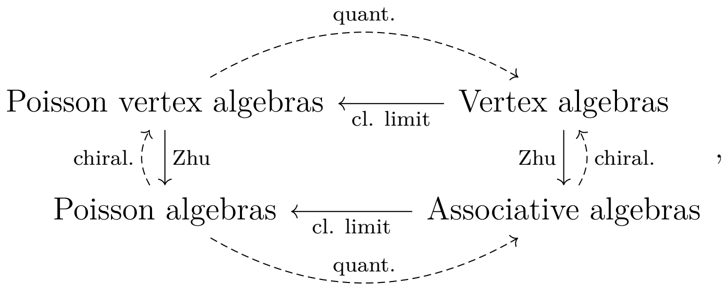
Introduction

Below we sketch the background and the principal motivations behind the problems considered in this thesis. A more precise, and technical, summary of our main results can be found in the introductions to the three main chapters: 3, 4, and 7.

§ 1.1 | Quantization of Poisson structures

A Poisson algebra is an associative, commutative algebra $\mathcal{A}$, together with a Lie bracket $\{\cdot, \cdot\}$, called the Poisson bracket, that satisfies the Leibniz rule. Notably, Poisson algebras are the *classical limit* of associative algebras. Take an associative algebra A: in some cases, we can obtain a commutative product and a Poisson bracket by considering a *first-order* (in a suitable sense) approximation of the associative product and commutator of A, respectively. The inverse process, constructing an associative algebra whose classical limit is the original Poisson algebra, is called *quantization*. While there are canonical ways to perform the classical limit, there are no exact procedures to perform the quantization. Different approaches exist, involving algebraic, analytical, and geometrical methods. Altogether, the quantization of Poisson algebras constitutes an important and active area of research.

While the quantization theory of Poisson algebras is the most widely studied, similar quantization frameworks can be considered for various other algebraic structures. In this thesis, we also consider the case of Poisson vertex algebras and vertex algebras. Informally, they can be thought of as infinitely-many degrees of freedom analogues of Poisson and associative algebras. Since, from a physical point of view, a vertex algebra is the chiral part of a two-dimensional conformal field theory, we say that vertex algebras (respectively Poisson vertex algebras) are the chiralization of associative algebras (Poisson algebras). The situation can be represented by the following diagram:



where Zhu are a pair of functors, and the dotted lines mean that there is no canonical functor that goes in that direction.

There are two fundamental problems regarding the quantization of a Poisson-like structure:

1. Existence, uniqueness and explicit construction of quantizations;

2. Understanding in which cases the properties of the quantization remember the Poisson properties of the classical limit.

In Part I of the thesis, we study Problem 1, in particular the explicit construction of quantizations, in the context of Poisson vertex algebras. The work in Part II is in the spirit of Problem 2. We study the relation between the isomorphisms of the quantizations and the Poisson isomorphisms of the Poisson deformations of conical symplectic singularities. We concentrate in particular on the case of Kleinian singularities.

In the remainder of the introduction, we describe more precisely the main ideas and results of the thesis.

§ 1.2 | Chiralization of star-products

Problem 1 is well-understood for Poisson algebras, where quantizations are constructed explicitly as star-products. We consider the problem in the context of Poisson vertex algebras; in particular, we are interested in the explicit construction of chiral analogues of star-products, which we obtain in an important class of examples.

§ 1.2.1 | Star-products on Poisson algebras

Let us recall the theory of star-products for Poisson algebras, as we use some of its ideas and techniques in our study of chiral star-products.

If we want to construct an associative quantization of a Poisson algebra $\mathcal{A}$, we can try to deform the commutative product of $(\mathcal{A}, \{\cdot,\cdot\})$: this is called deformation quantization, and the new operation obtained is called a *star-product*. Usually, one considers infinitesimal or formal deformations and defines a star-product on $\mathcal{A}$ as a $\mathbb{C}[[\varepsilon]]$-linear, associative, unitary product on the algebra of formal power series $\mathcal{A}[[\varepsilon]]$

$$\star : \mathcal{A}[[\varepsilon]] \otimes \mathcal{A}[[\varepsilon]] \to \mathcal{A}[[\varepsilon]],$$

such that, for $a, b \in \mathcal{A}$

$$\lim_{\varepsilon \to 0} a \star b = ab \qquad \text{and} \qquad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(a \star b - b \star a) = \{a, b\}.$$

If the star-product converges, i.e. if $\star : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}[\varepsilon]$, then $A_\varepsilon := (\mathcal{A}[\varepsilon], \star)$ is a one-parameter family of associative quantizations of $\mathcal{A}$. By putting ε equal to 0, we recover $\mathcal{A}$. Usually, converging star-products are called *strict* deformation quantizations. In this thesis, unless specified differently, we shall consider only strict quantizations. When clear from the context, we will assume that ε is specialized to 1.

As the name suggests, the original motivation for quantization comes from physics. In classical mechanics, observables are real-valued functions on a Poisson manifold (the phase space) and the collection of all observables forms a Poisson algebra. In quantum mechanics, observables are instead operators on a Hilbert space, and these do not commute. Most of the quantum mechanics phenomena, notably Heisenberg's Uncertainty Principle, arise from this non-commutativity. Physicists noticed that it was also possible to formulate quantum mechanics in the classical phase space, by defining a non-commutative product, the *star-product*, on the algebra of functions. The parameter ε here is $\frac{i\hbar}{2}$, where $\hbar$ is the Plank constant, so that, as $\hbar$ approaches 0, we recover classical mechanics (which explains the name classical limit). This is called the phase space formulation of quantum mechanics and it was developed independently by Groenwold and Moyal [Gr; Mo], building on previous results by Weyl and Wigner [We; Wi2]. The physics literature on the topic is extensive, see [CFZ] for a review of the most important results and [CZ] for a historical survey.

From physics comes the most well-known example of a star-product, the Moyal-Weyl star-product (somewhat miscredited, as it was derived by Groenewold in [Gr]). Let f, g be smooth functions on $\mathbb{R}^{2n}$, and let π denote the Poisson bivector corresponding to the standard symplectic structure. Then

$$f \star g := m \circ e^{\frac{i\hbar}{2}\pi}(f \otimes g),$$

where m denotes associative multiplication. For example, in dimension 2 with coordinates x, y, this reads

$$f \star g := m \circ \exp\left(i\hbar \frac{\partial_x \otimes \partial_y - \partial_x \otimes \partial_y}{2}\right)(f \otimes g). \tag{1.2.1}$$

These ideas quickly branched into mathematics; in the 1978 seminal paper [BFF+], deformation quantization was given a proper mathematical formulation, as the problem of existence, uniqueness and construction of star-products on general Poisson algebras. The existence of star-products for symplectic manifolds with trivial third cohomology group was proved in [Ve; BFF+]. The use of cohomological methods for deformations had been pioneered by Gerstenhaber a decade earlier [Ge]. This cohomological restriction turned out to be only technical, as the results were generalised to larger and larger classes of examples. De Wilde and Lecomte proved the existence of star-products on arbitrary symplectic manifolds [DL] and soon after Fedosov provided a geometrical construction for star-products on regular Poisson manifolds [Fe]. The crowning achievement of the theory of star-products is the Formality Theorem, proved by Kontsevich in [Ko], which implies the existence of star-products on arbitrary Poisson manifolds, and a one-to-one correspondence between equivalence classes of formal Poisson structures and equivalence classes of star-products.

§ 1.2.2 | Star-products from filtered quantizations

The chiral star-products that we construct are obtained using filtered quantizations and quantization maps. This is based on a very similar construction for star-products on graded Poisson algebras, that we now explain.

Let $A = \bigcup_{n\geq 0} F_n A$ be a filtered associative algebra. Let us also assume that the commutator of A lowers the degree, so $[F_nA, F_mA] \subset F_{n+m-i}A$, for some $i \geq 1$. We obtain a graded Poisson algebra $\mathcal{A}$ as the associated graded of A:

$$\mathcal{A} := \operatorname{gr} A = \bigoplus_{n\geq 0} F_nA/F_{n-1}A.$$

In this situation, it is easy to construct a (strict) star-product on $\mathcal{A}$. All that we need is a quantization map: this is a linear map $\psi : \mathcal{A} \to A$, such that $\operatorname{gr}\psi = \mathrm{id}$. Then a star-product can be defined as the pull-back

$$a \star_\psi b := \psi^{-1}(\psi(a)\psi(b)), \quad a, b \in \mathcal{A}. \tag{1.2.2}$$

This is how the Moyal-Weyl star-product was originally constructed, using the Weyl quantization map [We]. If we restrict to polynomial functions, this is the symmetrization map

$$x_1 \cdots x_n \mapsto \frac{1}{n!}\sum_{\sigma\in S_n} \hat{x}_{\sigma(1)} \cdots \hat{x}_{\sigma(n)}, \tag{1.2.3}$$

sending a monomial in the commuting variables x_i into the corresponding symmetric polynomial in the non-commutative variables $\hat{x}_i$. From a mathematical point of view, the Weyl quantization map is a vector space isomorphism $\mathcal{S}(U) \to W(U)$, from the symmetric algebra of a symplectic vector space (U, ω) to the corresponding Weyl algebra.

Similarly, we can take $\mathcal{A}$ to be $\mathcal{S}(\mathfrak{g})$, the symmetric algebra of a Lie algebra $\mathfrak{g}$. By the PBW theorem, $\mathcal{S}(\mathfrak{g})$ is the associated graded of $\mathfrak{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. We can take

$$\psi : \mathcal{S}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}),$$

to be defined as in (1.2.3) on monomials in the elements of $\mathfrak{g}$. The corresponding star-product is known as the Gutt star-product [Gu]. Explicitly, for $a, b \in \mathcal{S}(\mathfrak{g})$:

$$a \star b = m \circ \exp(D)(a \otimes b),$$

where D is a certain bidifferential operator whose coefficients are given by the Baker-Campbell-Hausdorff formula.

Formula (1.2.2) gives an implicit definition of the star-product, but it is not particularly useful for computations. What one would like is a closed formula that is written only in terms of the operations of the Poisson algebra, that is, that only depends on the coefficients of the Poisson bivector. Computing (1.2.2) explicitly, we can then perform operations of the non-commutative algebra in the Poisson algebra setting, which can be significantly easier to handle. For example, formula (1.2.1) is a closed formula for the multiplication in the Weyl algebra. Kontsevich's formality theorem [Ko] gives an algorithm to compute these formulae in the smooth case, although expansion after the second term is often impractical.

Notice how in both the Moyal-Weyl and Gutt star-product case, the explicit formula is of the form

$$\exp(\text{bidifferential operator}).$$

This exponential form is quite general, as star-products with this formula appear naturally in Kontsevich's theory (see the discussion in [Ka4]); of course, in general, the bidifferential operator can be incredibly complicated.

§ 1.2.3 | Star-deformations for Poisson vertex algebras

Poisson vertex algebras are the chiralization of Poisson algebras and vertex algebras are the chiralization of associative algebras. A star-product is a deformation of a Poisson algebra into an associative algebra. A deformation of a Poisson vertex algebra into a vertex algebra can thus be considered a chiral analogue of a star-product.

Let us briefly recall the definitions of vertex algebras, Lie vertex algebras, and Poisson vertex algebras. Vertex algebras were introduced by Borcherds in his work on the Monstrous Moonshine [Bo2], and subsequently developed by Frenkel, Lepowsky, and Meurman [FLM]. From a physics point of view, vertex algebras provide a mathematical definition for the chiral part of a 2-dimensional quantum field theory (see [BPZ]). A vertex algebra is essentially the data of a vector space V, a nonzero vector $|0\rangle$, called the vacuum, an endomorphism ∂, called the (infinitesimal) translation operator, and a linear map Y, called the state-field correspondence, which associates to every $a \in V$ an $\mathrm{End}(V)$-valued quantum field $Y(a, z)$. That is,

$$Y : V \to \mathrm{End}(V)[[z, z^{-1}]], \quad a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

such that, for all $b \in V$, $a_{(n)} b = 0$ for $n \gg 0$. The quantum fields $Y(a, z)$, also called vertex operators, satisfy a set of axioms that can be deduced from Wightman's axioms for quantum field theory [Wi1; Ka1].

An important part of the structure of a vertex algebra is the λ-bracket

$$[\cdot\,{}_\lambda\,\cdot] : V \otimes V \to V \otimes \mathbb{C}[\lambda], \qquad [a\,{}_\lambda\,b] := \sum_{n\geq 0} \frac{\lambda^n}{n!} a_{(n)}b.$$

The λ-bracket gives a vertex algebra a natural structure of a *vertex Lie algebra* (also known as a *Lie conformal algebra*), a generalization of Lie algebras [Ka1]. Vertex algebras can equivalently be defined in terms of the λ-bracket, the translation operator, the vacuum, and the (-1) product (which is usually denoted as $::$ and called the normally ordered product) [BK1]. From this point of view, a vertex algebra is a unital differential algebra, with a non-commutative, quasi-associative product; both formulae for the commutator and quasi-associativity are expressed in terms of the λ-bracket. We can thus think of vertex algebras as analogues of associative algebras. The main difference is that the vertex Lie algebra structure is an additional structure, and cannot be defined only in terms of the normally ordered product, while, for an associative algebra, the Lie algebra structure is naturally given by the commutator.

A commutative vertex algebra with an additional compatible vertex Lie algebra structure is called a Poisson vertex algebra [FB]. The classical limit of a vertex algebra is a Poisson vertex algebra. In particular, if a vertex algebra has a *good filtration*, then its associated graded has an induced structure of a graded Poisson vertex algebra [Li1]. As we can see, the framework is very similar to that of the deformation quantization of Poisson algebras. The analogies are not limited to the algebraic picture but can be extended to the physical interpretation. Poisson algebras are related to classical mechanics and ODEs, while Poisson vertex algebras can be applied to classical field theory and PDEs (see [BDK; DKV2]). It is thus natural to consider the problem of deformation quantization for Poisson vertex algebras.

The main difference with Poisson algebras is the following. On a Poisson algebra, once we have a star-product $\star$, we can define a deformation of the Poisson bracket, by the commutator of the star-product. This is not true for Poisson vertex algebras, as the λ-bracket is not completely determined by the normally ordered product. Thus, we need also a "star λ-bracket", i.e. a deformation of the Poisson λ-bracket into a λ-bracket compatible with the star-product. We can avoid writing down two separate formulae by using the integral λ-bracket formalism introduced in [DK2]. On a vertex algebra V, define the following operator, called the integral λ-bracket:

$$I_\lambda(a, b) := {:}ab{:} + \int_0^\lambda [a\,{}_x\,b]\,dx, \quad \text{for } a, b \in V.$$

We can recover the normally ordered product and the λ-bracket:

$$:ab: = I_0(a,b), \quad \text{and} \quad [a\,{}_\lambda\, b] = \frac{d}{d\lambda} I_\lambda(a,b) \quad \text{for all } a,b \in V.$$

Thus, a (strict) deformation quantization of a Poisson vertex algebra $\mathcal{V}$ amounts to a bilinear operator, called a star-deformation,

$$I_{\lambda,\star} : \mathcal{V}[\varepsilon] \otimes \mathcal{V}[\varepsilon] \to \mathcal{V}[\varepsilon,\lambda],$$

such that $I_{\lambda,\star}$ induces a vertex algebra structure on $\mathcal{V}$ (in the integral λ-bracket formalism) and

$$\lim_{\varepsilon\to 0} I_{\lambda,\star}(a,b) = ab, \quad \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\frac{d}{d\lambda} I_{\lambda,\star}(a,b) = \{a\,{}_\lambda\, b\}.$$

As for Poisson algebras, if we have a graded Poisson vertex algebra $\mathcal{V}$, a vertex algebra V with a good filtration such that $\mathrm{gr}(V) \cong \mathcal{V}$, and a quantization map $\phi : \mathcal{V} \to V$, we can define a star-deformation as

$$I_{\lambda,\star}(a,b) := \phi^{-1}\Big(I_\lambda(\phi(a),\phi(b))\Big), \quad \text{for all } a,b \in \mathcal{V}. \tag{1.2.4}$$

Remark. Somewhat surprisingly, there is not much literature on the subject of star-deformations. In [Li1] the technical framework is developed, most notably the definition of a "good filtration". There is also a preprint by Yanagida [Ya], where the problem is considered using chiral algebra methods. In particular, a dg-Lie algebra controlling the chiral deformation quantizations is constructed, and, for some cases, the uniqueness of the chiral deformation quantization (if it exists) is proven [Ya, Corollary 3.5].

As we explain in the section below, a star-deformation is not a proper chiralization of a star-product, as it is not compatible with the Zhu functor. However, explicit formulae for star-deformations have important applications. For example, a formula for a star-deformation allows one to perform vertex algebra computations in the setting of Poisson vertex algebras, which are much more tractable. For these reasons, we study star-deformations as well in Chapter 4.

§ 1.2.4 | Chiralization of star-products

Let us go back to the diagram

$$\begin{array}{ccc} & \overset{\text{quant.}}{\dashrightarrow} & \\ \text{Poisson vertex algebras} & \xleftarrow[\text{cl. limit}]{} & \text{Vertex algebras} \\ \text{chiral.} \uparrow \downarrow \text{Zhu} & & \text{Zhu} \downarrow \uparrow \text{chiral.} \\ \text{Poisson algebras} & \xleftarrow[\text{cl. limit}]{} & \text{Associative algebras} \\ & \underset{\text{quant.}}{\dashrightarrow} & \end{array}, \tag{1.2.5}$$

and concentrate on the vertical lines. The Zhu functors associate a Poisson algebra to a Poisson vertex algebras and an associative algebra to a vertex algebra. On the vertex algebra side, the Zhu functor was introduced by Zhu in his seminal paper [Zh], while the construction on the Poisson vertex algebra side is done in [DK2]. Given a vertex algebra V, its Zhu algebra plays a pivotal role in the study of its representation theory. Informally, chiralization is the inverse of the Zhu functor.

In [DK2], De Sole and Kac considered a large class of examples of algebras that fit into diagram (1.2.5). Let $R = \mathbb{C}[\partial] \otimes \mathfrak{g}$ be a vertex Lie algebra, freely generated as a $\mathbb{C}[\partial]$-module by a vector space $\mathfrak{g}$. There is a canonical Lie algebra structure on $\mathfrak{g}$. We can construct four algebras starting from this setup: the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, the symmetric algebra $\mathcal{S}(\mathfrak{g})$, the universal enveloping vertex algebra $V(R)$ and the symmetric algebra $\mathcal{S}(R)$. They fit in the commutative diagram

$$\begin{array}{ccc} \mathcal{S}(R) & \xleftarrow{\text{cl. limit}} & V(R) \\ {\scriptstyle \text{Zhu}}\downarrow & & \downarrow{\scriptstyle \text{Zhu}} \\ \mathcal{S}(\mathfrak{g}) & \xleftarrow[\text{cl. limit}]{} & \mathfrak{U}(\mathfrak{g}) \end{array} \tag{1.2.6}$$

This setup is more general: in [DK2] the authors proved that this is true also in the case R is a non-linear vertex Lie algebra. In this case, $\mathfrak{g}$ is canonically a non-linear Lie algebra. The definition of a non-linear Lie algebra (respectively, non-linear vertex Lie algebra) is quite technical; it is essentially a vector space $\mathfrak{g}$ (respectively, R) with a bracket (respectively, λ-bracket) with values in its tensor algebra $\mathcal{T}(\mathfrak{g})$ (respectively, $\mathcal{T}(R)[\lambda]$), satisfying some complicated, Lie-like axioms. An important example arising this way are W-algebras (finite and affine). In this thesis, we shall consider only examples of non-linear (vertex) Lie algebras where the (λ)-bracket is allowed to have values in $\mathfrak{g} \oplus \mathbb{C}$ (respectively, $(R \oplus \mathbb{C})[\lambda]$). We call them *sub-linear* (vertex) Lie algebras. The two thematic examples of this thesis are of this form.

Example 1.2.1. Let (U, ω) be a symplectic vector space. We can regard U as a sub-linear Lie algebra with bracket $\omega : U \otimes U \to \mathbb{C}$. Then diagram (1.2.6) becomes

$$\begin{array}{ccc} \mathcal{S}(\mathbb{C}[\partial] \otimes U) & \xleftarrow{\text{cl. limit}} & V^{\beta\gamma} \\ {\scriptstyle \text{Zhu}}\downarrow & & \downarrow{\scriptstyle \text{Zhu}} \\ \mathcal{S}(U) & \xleftarrow[\text{cl. limit}]{} & W(U) \end{array}$$

where $W(U)$ is the Weyl algebra of U and $V^{\beta\gamma}$ is the $\beta\gamma$-system vertex algebra.

Example 1.2.2. Let $\mathfrak{g}$ be a Lie algebra with a symmetric invariant bilinear form $(\cdot|\cdot)$. For every $k \in \mathbb{C}$, consider the sub-linear λ-bracket on $R = \mathbb{C}[\partial] \otimes \mathfrak{g}$ defined by

$$[a\,{}_\lambda\, b] := [a, b] + k(a|b)\lambda.$$

Then diagram (1.2.6) becomes

$$\begin{array}{ccc} \mathcal{S}(\mathbb{C}[\partial]\otimes\mathfrak{g}) & \xleftarrow{\text{cl. limit}} & V^k(\mathfrak{g}) \\ {\scriptstyle\text{Zhu}}\downarrow & & \downarrow{\scriptstyle\text{Zhu}} \\ \mathcal{S}(\mathfrak{g}) & \xleftarrow[\text{cl. limit}]{} & \mathfrak{U}(\mathfrak{g}) \end{array}$$

where $V^k(\mathfrak{g})$ is the affine vertex algebra.

Other examples that fit into this framework are the Virasoro vertex algebra and free-boson vertex algebras.

There is a natural quantization map $\phi : \mathcal{S}(R) \to V(R)$, which sends a monomial in elements of R to its symmetrization

$$u_1 \dots u_n \mapsto \frac{1}{n!}\sum_{\sigma\in S_n} {:}u_{\sigma(1)}\dots u_{\sigma(n)}{:}\ , \tag{1.2.7}$$

where the non-associative normally ordered products are understood from right to left. Using ϕ we obtain a star-deformation as in (1.2.4). This construction looks very similar to that of the Moyal-Weyl and Gutt star-products; to check if it can be regarded as a proper chiralization of them, we need to verify if it is compatible with the Zhu functor.

Definition 1.2.3. Let $\mathcal{V}$ be a Poisson vertex algebra, let $\mathcal{A} := \mathrm{Zhu}(\mathcal{V})$ and $\pi : \mathcal{V} \to \mathcal{A}$ the quotient map. Consider a star-product $\star$ on $\mathcal{A}$. A chiralization of $\star$ is a deformation $\hat{\star}$ of the commutative product of $\mathcal{V}$, such that

$$\pi(a\,\hat{\star}\,b) = \pi(a)\star\pi(b).$$

Unfortunately, the star-deformation discussed before does not in general provide a chiralization of star-products in the sense of Definition 1.2.3. To understand why, let us briefly recall the construction of the Zhu algebras (for more details, see Section 2.5). The Zhu Poisson algebra of a Poisson vertex algebra $\mathcal{V}$ is defined as

$$\mathrm{Zhu}(\mathcal{V}) := \mathcal{V}/(\partial\mathcal{V})\mathcal{V}.$$

For a vertex algebra V, the construction is more complicated. We follow the alternative construction provided by Huang [Hu2]. Take a parameter $\hbar \in \mathbb{C}^\times$ (alternatively, it is possible to take $\hbar$ a formal parameter, see Remark 3.2.2). Consider the formal change of variable $z \mapsto x = \frac{1}{\hbar}\log(1+\hbar z)$ and define the following $\hbar$-deformed vertex operators, for $a \in V$:

$$Y_\hbar(a,z) := Y\left(a, \frac{1}{\hbar}\log(1+\hbar z)\right) = Y(a,x). \tag{1.2.8}$$

It turns out that the $\hbar$-deformed vertex operators $Y_\hbar(a,z)$ are again $\mathrm{End}(V)$-valued quantum fields, that is we can write

$$Y_\hbar(a,z) = \sum_{n\in\mathbb{Z}} a_{(n,\hbar)} z^{-n-1},$$

where the coefficients $a_{(n,\hbar)}$ are in $\mathrm{End}(V)$ and $a_{(n,\hbar)}b = 0$ for $n \gg 0$ for all $a, b \in V$. We call $(n,\hbar)$ the $\hbar$-deformed n-product. Clearly,

$$\lim_{\hbar\to 0} Y_\hbar(a,z) = Y(a,z) \quad \text{and} \quad \lim_{\hbar\to 0} a_{(n,\hbar)}b = a_{(n)}b, \quad \text{for all } a,b \in V \text{ and all } n \in \mathbb{Z}.$$

In particular, we are interested in the $\hbar$-deformations of the (-1) product. We introduce the following notation:

$$a *_\hbar b := a_{(-1,\hbar)}b, \quad \text{for all } a,b \in V.$$

Somewhat miraculously, the vector space $(TV) *_\hbar V$ is a two-sided ideal for the $*_\hbar$ product and the quotient algebra $V/(TV) *_\hbar V$ turns out to be associative. This is the Zhu algebra of V:

$$\mathrm{Zhu}(V) := \left({V} \Big/ {(TV) *_\hbar V}\, ,\ *_\hbar \right)\Big|_{\hbar=1}.$$

Remark. The product in the Poisson Zhu algebra $\mathrm{Zhu}(\mathcal{V})$ is induced from the commutative product of $\mathcal{V}$, but the product in the associative Zhu algebra $\mathrm{Zhu}(V)$ does not come from the normally ordered product of V, but from its $\hbar$-deformation $*_\hbar$.

Suppose we have a star-product $\star$ on $\mathcal{S}(\mathfrak{g})$, induced by the symmetrization quantization map $\psi : \mathcal{S}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$ (1.2.3). Denote by $\hat{\star}$ the deformation of the product on $\mathcal{S}(R)$ induced by the quantization map ϕ defined in (1.2.7). The condition of Definition 1.2.3 requires the following diagram to commute (see Proposition 4.3.3 for more details):

$$\begin{array}{ccc} \mathcal{S}(R) & \xrightarrow[\phi]{} & V(R) \\ \downarrow & & \downarrow \\ \mathcal{S}(\mathfrak{g}) & \xrightarrow{\psi} & \mathfrak{U}(\mathfrak{g}) \end{array}$$

Here the vertical arrows are the quotient projection to the (Poisson) Zhu algebra. It is now clear what the problem is. The quantization map ψ maps monomial in $\mathcal{S}(\mathfrak{g})$ in symmetric polynomials in $\mathfrak{U}(\mathfrak{g})$ and ϕ maps monomial in $\mathcal{S}(R)$ into symmetric polynomials in the normally ordered product in $V(R)$, but the normally ordered product is not compatible with the quotient projection $V(R) \to \mathfrak{U}(\mathfrak{g})$, so the diagram does not commute.

§ 1.3 | Main Results I

Let again $R = \mathbb{C}[\partial] \otimes \mathfrak{g}$ be a sub-linear vertex Lie algebra. Following the discussion in the last section, it makes sense to consider the following quantization map:

$$\phi : \mathcal{S}(R) \to V(R) \quad u_1 \dots u_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(1)} *_\hbar \cdots *_\hbar u_{\sigma(n)}, \tag{1.3.1}$$

where the non-associative $*_\hbar$ products are understood from right to left.
For $a, b \in \mathcal{S}(R)$, let

$$a \star_\hbar b := \phi^{-1}(\phi(a) *_\hbar \phi(b)),$$

the pull-back of the $*_\hbar$-product via ϕ. Our first result is the following.

Theorem 1.3.1. *Any star-product on $\mathcal{S}(\mathfrak{g})$ can be chiralized. In particular, the product $\star_\hbar$ is a chiralization of the star-product defined by the symmetrization map $\mathcal{S}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$, in the sense of Definition 1.2.3.*

Our next objective is to compute explicit formulae for $\star_\hbar$. To do so, we first develop some technical machinery to do computations with $*_\hbar$. In the case of the normally ordered product, the commutator and the associators are expressed in terms of the λ-bracket and this makes computations much easier. We want something analoguous for $\hbar$-vertex algebras.

First, we observe that the $\hbar$-deformed vertex operators satisfy a deformed version of the so-called translation covariance axiom: in a vertex algebra V,

$$[\partial, Y(a, z)] = \partial_z Y(a, z),$$

while the deformed vertex operators satisfy

$$[\partial, Y_\hbar(a, z)] = (1 + \hbar z)\partial_z Y_\hbar(a, z).$$

We introduce the following definition. Here we are assuming $\hbar \in \mathbb{C}^\times$, but it would also be possible to consider $\hbar$ a formal parameter (see Remark 3.2.2).

Definition 1.3.2 (Definition 3.2.1)**.** An $\hbar$-vertex algebra is a vector space V, a nonzero vector $|0\rangle \in V$, and a linear map $Y_\hbar : V \to \mathrm{End}(V)[[z, z^{-1}]]$, denoted by

$$a \mapsto Y_\hbar(a, z) = \sum_{n \in \mathbb{Z}} a_{(n,\hbar)} z^{-n-1},$$

satisfying the following axioms:

(fields) for all $a, b \in V$, $a_{(n,\hbar)} b = 0$ for $n \gg 0$, i.e. $Y_\hbar(a, z)$ is a $\mathrm{End}(V)$-valued field for all $a \in V$;

(vacuum) $Y_\hbar(|0\rangle, z) = \mathrm{id}_V$, $Y(a, z)|0\rangle \in V[[z]]$ and $Y_\hbar(a, z)|0\rangle|_{z=0} = a$, for all $a \in V$;

($\hbar$-translation covariance) $[\partial, Y_\hbar(a,z)] = (1+\hbar z)\partial_z Y_\hbar(a,z)$, where the endomorphism $\partial \in \mathrm{End}(V)$ is defined by $\partial a := a_{(-2,\hbar)}|0\rangle$, for all a in V;

(locality) for all $a, b \in V$, there exists $N > 0$ (dependant on a and b) such that

$$(z-w)^N [Y_\hbar(a,z), Y_\hbar(b,w)] = 0.$$

Except for the translation covariance axiom, this is identical to the definition of a vertex algebra. If we take the formal limit $\hbar \to 0$, we get back the definition of a vertex algebra.

Proposition 1.3.3 (Proposition 3.2.3)**.** *Let V be a vector space. If $(V, |0\rangle, Y)$ is a vertex algebra structure on V, then the $\hbar$-deformed vertex operators $Y_\hbar$ defined as in (1.2.8) give the structure of an $\hbar$-vertex algebra. Conversely, if $(V, |0\rangle, Y_\hbar)$ is an $\hbar$-vertex algebra, we can define a vertex algebra structure via*

$$Y(a,z) := Y_\hbar\left(a, \frac{1}{\hbar}(e^{\hbar z} - 1)\right), \qquad \textit{for all } a \in V.$$

an $\hbar$-vertex algebra can thus be seen either as a deformation of a vertex algebra or as a vertex algebra after the change of variable $z \mapsto \frac{1}{\hbar}\log(1+\hbar z)$. Even if $\hbar$-vertex algebras are in bijection with vertex algebras via the change of variable, it is still convenient to study the $\hbar$-deformed vertex operators from the axiomatic point of view, as some properties are not clearly deduced from the change of variable definition. In Chapter 3, we study the structure of $\hbar$-vertex algebras using only the axioms of Definition 1.3.2. Among others, we prove results analogous to Goddard's Uniqueness Theorem, the Reconstruction Theorem, Borcherds Identity, and the OPE Expansion Formula. Most importantly, we introduce an analogue of the λ-bracket, which we call the $\hbar$-bracket (see Definition 3.3.1). If V is an $\hbar$-vertex algebra, then

$$[\cdot\,{}_\lambda\,\cdot]_\hbar : V \otimes V \to V[\lambda], \quad [a\,{}_\lambda\, b]_\hbar := \sum_{n\geq} \frac{(\lambda)_{n,\hbar}}{n!} a_{(n,\hbar)} b,$$

where $(\lambda)_{n,\hbar} := \lambda(\lambda-\hbar)\ldots(\lambda-(n-1)\hbar)$. An $\hbar$-vertex algebra can then be considered as a differential algebra, with differential ∂, and non-commutative, quasi-associative product $*_\hbar = (-1,\hbar)$, whose commutator and quasi-associativity formulae are written in terms of the $\hbar$-bracket (Theorem 3.3.7). Similarly, we introduce the notion of an $\hbar$-vertex Lie algebra (a vector space with an $\hbar$-bracket) and an $\hbar$-vertex Poisson algebra (a commutative differential algebra with an $\hbar$-bracket $\{\cdot\,{}_\lambda\,\cdot\}_\hbar$). Under the change of variable interpretation, the λ bracket and the $\hbar$ bracket are related by the following formula (see Proposition 3.3.3):

$$[\cdot\,{}_\lambda\,\cdot]_\hbar = [\cdot\,{}_{\lambda+\hbar}\,\cdot]. \tag{1.3.2}$$

The new formalism of $\hbar$-vertex algebras makes the exposition of the theory of Zhu

Vertex algebras	$\hbar$-vertex algebras
$[a_\lambda b] = \mathrm{Res}_z(e^{\lambda z}Y(a,z)b) = \sum_{n\geq 0}\frac{\lambda^n}{n!}a_{(n)}b$	$[a\,{}_\lambda\, b]_\hbar := \mathrm{Res}_z((1+\hbar z)^{\lambda/\hbar}Y_\hbar(a,z)b)$
$:ab: - :ba: = \int_{-\partial}^0 [a\,{}_\lambda\, b]d\lambda$	$a *_\hbar b - b *_\hbar a = \sum_{-\partial-\hbar}^0 [a\,{}_\lambda\, b]_\hbar \delta\lambda$
$[a_\lambda :bc:] = :[a_\lambda b]c: + :b[a_\lambda c]:$	$[a\,{}_\lambda\, b *_\hbar c]_\hbar = b *_\hbar [a\,{}_\lambda\, c]_\hbar + [a\,{}_\lambda\, b]_\hbar *_\hbar c$
$+\int_0^\lambda [[a_\lambda b]_\mu c]d\mu$	$+\sum_0^{\lambda+\hbar}[[a_\lambda b]\,{}_{\hbar}\,{}_{\mu}\, c]_\hbar \delta\mu$

Table 1.1: Formulae comparison (see Section 3.3 for definitions).

algebras simpler and more transparent (see Theorem 3.4.3). In particular, the $\hbar$-bracket simplifies greatly computations involving the $(n,\hbar)$-products. Many formulae for the λ-bracket involve derivatives and integrals in λ. For the $\hbar$-bracket, we move from infinitesimal calculus to discrete calculus (see Table 1.1): the derivatives become finite differences and the integrals become definite sums, the inverse of finite differences (this is denoted by $\sum$, see Section 3.1 for more details). We can also construct an $\hbar$-analogue of the integral λ-bracket (Proposition 3.3.17) by using the definite sum instead of the integral:

$$I_{\lambda,\hbar}(a,b) := a *_\hbar b + \sum_0^\lambda [a\,{}_x\, b]_\hbar \delta x.$$

We describe now the general idea. Consider again our setting with $R = \mathbb{C}[\partial]\otimes\mathfrak{g}$. We can regard R as an $\hbar$-vertex Lie algebra using (1.3.2). Then $\mathcal{S}(R)$ is naturally an $\hbar$-Poisson vertex algebra. Take the quantization map ϕ defined as in (1.3.1) and define

$$I_{\lambda,\hbar,\star}(a,b) := \phi^{-1}\Big(I_{\lambda,\hbar}(\phi(a),\phi(b))\Big).$$

If we get an explicit formula for this "$\hbar$-star-deformation", we solve both the problems we were considering. In fact, putting λ equal to 0 recovers $\star_\hbar$, while putting $\hbar$ equal to 0 gives a formula for the star-deformation of the Poisson vertex algebra $\mathcal{S}(R)$. We call $I_{\lambda,\hbar,\star}$ a *chiral star-product.*

Consider R_L, the Lie algebra structure on R with Lie bracket given by

$$[a,b] := \sum_{-\partial-\hbar}^0 [a\,{}_x\, b]_\hbar \delta x,$$

and its universal enveloping (associative) algebra $\mathfrak{U}(R_L)$. Let $m_\star : V\otimes V \to V$ denote multiplication using the Gutt star-product associated to the quantization of $\mathcal{S}(R)$ into $\mathfrak{U}(R_L)$. Fix an ordered basis $\{u_i\}_{i\in\mathcal{I}}$ of R. Recall that $V(R)$ is called a free-field vertex algebra if the λ-bracket on R takes values in $\mathbb{C}[\lambda]$.

Theorem 1.3.4 (Corollary 4.4.11)**.** *If $V(R)$ is a free-field vertex algebra, then*

$$I_{\lambda,\hbar,\star}(a,b) = m_\star \circ \sigma \circ \exp\Big(\sum_{i\in\mathcal{I}} L_i^\lambda \otimes \partial_{u_i}\Big)(a\otimes b),$$

where σ *is the operator* $x \otimes y \mapsto y \otimes x$*, and*

$$L_i^\lambda(a) = \sum_{j \in \mathcal{I}} \sum\nolimits_{-\partial-\hbar}^{\lambda} \{u_j\, {}_{x+\partial}\, u_i\}_\hbar \underset{\to}{} \frac{\partial a}{\partial u_j}\, \delta x.$$

Now we consider the limit $\hbar \to 0$. The Lie bracket on R_L becomes

$$[a, b] := \int_{-\partial}^{0} [a\, {}_x\, b] dx.$$

Notice that this is 0 in the free-field case, thus $m_\star = m$, the usual multiplication.

Theorem 1.3.5 (Corollary 4.4.15)**.** *Let* $V(R)$ *be a free-field vertex algebra. The following is an explicit formula for the star-deformation of* $\mathcal{S}(R)$ *induced by the quantization map* (1.2.7)*:*

$$I_{\lambda,\star}(a, b) = m \circ \exp\left(\sum_{i \in \mathcal{I}} L_i^\lambda \otimes \partial_{u_i}\right)(a \otimes b),$$

where m *is the multiplication map and*

$$L_i^\lambda(a) = \sum_{j \in \mathcal{I}} \int_{-\partial}^{\lambda} \frac{\partial a}{\partial u_j} \{u_j\, {}_{x+\partial^{(1)}}\, u_i\}\, dx.$$

We also obtain the formulae in the general case, in Theorem 4.4.10 and 4.4.14. They are more complicated, but they still share some similar features with the Moyal-Weyl and Gutt star-product formulae. In particular, they ultimately depend on some bidifferential operator, with coefficients given by the structure constants of the λ-bracket (or the $\hbar$-bracket). We additionally recover the familiar form exp(bidifferential operator).

§ 1.4 | Isomorphisms of deformations and quantizations

Our work in Part II is in the spirit of Problem 2: understanding in which cases the Poisson properties of the classical limit reflect the properties of the quantization. In particular, we are interested in the relation between the isomorphisms of the quantizations and the Poisson isomorphisms of the Poisson deformations of a symplectic quotient singularity. We prove that these two classes of isomorphisms are equivalent in the special case of Kleinian singularities of type **A** and **D**.

Consider a Poisson algebra $\mathcal{B}$ and a non-commutative associative quantization B. There is a rich literature in the direction of Problem 2. For instance, in [Go], it is conjectured that the prime and primitive spectra of the quantized coordinate rings are respectively homeomorphic to the Poisson prime and Poisson primitive spectra of their corresponding semi-classical limits when the base field is algebraically closed and of characteristic zero. This conjecture has since been verified in several examples [Fr; GL; Go]. In [LO2; LO1], the authors studied the derivations of certain deformations $A_{\alpha,\beta}$ of the second Weyl algebra and the Poisson derivations of their semi-classical limit $\mathcal{A}_{\alpha,\beta}$, proving that the first Hochschild cohomology group $HH^1(A_{\alpha,\beta})$ is isomorphic to the first Poisson cohomology group $HP^1(\mathcal{A}_{\alpha,\beta})$. In [CO], it is proved that the endomorphisms of the generalized Weyl algebras are the same as the Poisson endomorphisms of the Poisson generalized Weyl algebras. The connection between the 0-th Poisson homology $HP_0(\mathcal{B})$ and the 0-th Hochschild homology $HH_0(B)$ was studied by Etingof and Schedler in a series of papers [ES1; ES2; ES3; ES4], proving that $HP_0(\mathcal{B}) \cong HH_0(B)$ in many examples, including symmetric powers of isolated quasi-homogeneous surface singularities and finite W-algebras.

A particularly interesting problem concerns the relation between automorphisms of B and Poisson automorphisms of $\mathcal{B}$. Consider a finite-dimensional symplectic vector space (V, ω). The Weyl algebra $W(V, \omega)$ is a quantization of the symmetric algebra $\mathcal{S}(V)$. Belov-Kanel and Kontsevich made the following remarkable conjecture:

Conjecture 1 ([BK2])**.** *The automorphism group of the Weyl algebra $W(V, \omega)$ is isomorphic to the group of Poisson automorphisms of the symmetric algebra $\mathcal{S}(V)$, with the Poisson structure induced by the symplectic form ω.*

The conjecture is verified for $\dim(V) = 2$ and in higher dimensions for the subgroup of tame automorphisms [BK2]. Conjecture 1 is connected to other important open problems in algebra and algebraic geometry, like the Jacobian conjecture and the Dixmier conjecture (see [BK3; BER+]).

The symmetric algebra is a special example, being quite rigid, with the Weyl algebra as its only quantization. In general, a graded Poisson algebra $\mathcal{B}$ may have numerous non-isomorphic non-commutative quantizations. It is also possible to deform the Poisson bracket to obtain filtered Poisson algebras whose associated graded is isomorphic to $\mathcal{B}$. We call such an algebra a *(Poisson) deformation* of $\mathcal{B}$. In the literature, it is common to see the terms deformations and quantizations used interchangeably. To avoid confusion, in Part II we strictly reserve the term quantization for filtered, non-commutative algebras and deformation for filtered Poisson algebras.

If G is a finite subgroup of the symplectic group $Sp(V)$, we can form the quotient of V by the natural action of G, which is the variety $V/G = \operatorname{Spec}(\mathbb{C}[V]^G)$. Since G is not a complex reflection group, V/G is not smooth by the Chevalley-Shepard-Todd Theorem. It turns out that the symplectic quotient V/G is still "nice"; in particular, it has only *conical symplectic singularities* (see [Be2] or Chapter 6 for technical details). In [Na1], Namikawa constructed a Cartan space $\mathfrak{P}$ and Weyl group W associated to a conical symplectic singularity X, with W acting on $\mathfrak{P}$ by crystallographic reflections, and proved the existence of a universal Poisson deformation of $\mathbb{C}[X]$ over $\mathfrak{P}/W$. Recently, Losev proved the existence of a universal filtered quantization of $\mathbb{C}[X]$, which remarkably has the same parameter space [Lo2]. Let $c, c' \in \mathfrak{P}/W$ be two deformation parameters, and denote by $\mathfrak{X}_{1,c}, \mathfrak{X}_{1,c'}$ and $\mathfrak{X}_{0,c}, \mathfrak{X}_{0,c'}$ the corresponding quantizations and deformations, respectively. We can consider either the associative algebra isomorphisms between $\mathfrak{X}_{1,c}$ and $\mathfrak{X}_{1,c'}$ or the Poisson isomorphisms between $\mathfrak{X}_{0,c}$ and $\mathfrak{X}_{0,c'}$. As c, c' vary over the whole parameter space $\mathfrak{P}/W$, we obtain two groupoids, which we denote $\mathrm{Iso}(X)$ and $\mathrm{PIso}(X)$, respectively.

Conjecture 2. *Let V be a finite-dimensional vector space, G a finite subgroup of $Sp(V)$, and consider the symplectic quotient $V/G = \mathrm{Spec}(\mathbb{C}[V]^G)$. There is an isomorphism of groupoids*

$$\mathrm{PIso}(V/G) \cong \mathrm{Iso}(V/G).$$

In particular, Conjecture 2 implies that, for every deformation parameter, the automorphism group of the corresponding quantization and the Poisson automorphism group of the corresponding deformation are isomorphic. Conjecture 2 recovers the BKK Conjecture 1 for $G = \{\mathrm{id}\}$. In that case, $\mathfrak{P} = 0$ and we only have one Poisson deformation (the symmetric algebra itself) and one quantization (the Weyl algebra), so the groupoids become groups of automorphisms.

Notice that the associated graded of each of the quantizations of a symplectic quotient singularity V/G is not the corresponding deformation but the algebra $\mathbb{C}[V]^G$ itself. Nonetheless, there is a different way to realize a deformation as the semi-classical limit of the corresponding quantization. For each symplectic quotient V/G, Etingof and Ginzburg [EG] constructed a symplectic reflection algebra $\mathsf{H}_{t,c}(G)$, for all $t \in \mathbb{C}$ and all c in a certain vector space of dimension $\dim \mathfrak{P}$. The algebra $\mathsf{H}_{t,c}(G)$ is defined as a quotient of the skew-ring $\mathcal{T}(V^*) \rtimes G$, where $\mathcal{T}(V^*)$ is the tensor algebra of V^*. Let $\mathsf{e} := \frac{1}{|G|}\sum_{g\in G} g$ be the canonical idempotent in $\mathbb{C}G$; the algebra $\mathsf{eH}_{t,c}\mathsf{e}$ is called the spherical subalgebra of $\mathsf{H}_{t,c}(G)$. A result by Bellamy [Be3] shows that every filtered Poisson deformation of $\mathbb{C}[V]^G$ is of the form $\mathsf{eH}_{0,c}\mathsf{e}$, for some c. In [Lo2], it is proved that every filtered quantization of $\mathbb{C}[V]^G$ is of the form $\mathsf{eH}_{1,c}\mathsf{e}$, for some c. Since $\mathsf{H}_{t,c} \cong \mathsf{H}_{1,c}$ for all $t \neq 0$, we can regard every deformation as the semi-classical limit of $\mathsf{eH}_{t,c}\mathsf{e}$ as t goes to 0. Unfortunately, this process does not provide a way to clearly relate associative isomorphisms to Poisson isomorphisms.

The reductive part of the group of graded Poisson algebra isomorphisms of $\mathbb{C}[V]^G$ acts on the space of deformations and quantizations as filtered (Poisson) algebra isomorphisms. By a general result of Losev [Lo2, Proposition 3.21 and Corollary 3.22], these two actions coincide and induce all the possible filtered (Poisson) algebra isomorphisms between different deformations or quantizations. The general problem is much harder to approach, as a priori we cannot exclude the existence of exotic, non-filtered isomorphisms.

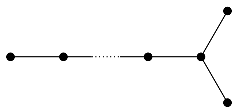
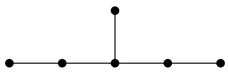
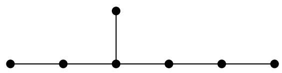
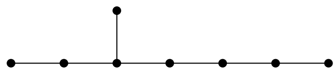
Type	Diagram	Equation
$\mathbf{A}_{n-1}$, $n \geq 2$		$xy - z^n = 0$
$\mathbf{D}_n$, $n \geq 4$		$x^{n-1} + xy^2 + z^2$
$\mathbf{E}_6$		$x^4 + y^3 + z^2$
$\mathbf{E}_7$		$x^3y + y^3 + z^2$
$\mathbf{E}_8$		$x^5 + y^3 + z^2$

Table 1.2: Classification of Kleinian singularities

§ 1.4.1 | Kleinian singularities

We concentrate our study on the case $\dim(V) = 2$, where we know the original BKK Conjecture 1 holds. Recall that, in dimension 2, the symplectic group and the special linear group coincide. Symplectic quotient singularities in dimension 2 are thus exactly the Kleinian singularities, also known as Du Val singularities. Kleinian singularities and their corresponding algebras of functions are classified by simply laced Dynkin diagrams via the McKay correspondence (see [Mc]). Kleinian singularities are isomorphic to surfaces in $\mathbb{A}^3$:

$$\mathbb{C}[V/\Gamma] = \mathbb{C}[V]^\Gamma \cong \mathbb{C}[x, y, z]/(F), \quad \text{with } F \in \mathbb{C}[x, y, z].$$

There is extensive literature about the deformations and quantizations of Kleinian singularities. Most of the results for general conical symplectic singularities were first proved for Kleinian singularities. In this case, the Cartan space $\mathfrak{P}$ and the Namikawa Weyl group W are the actual Cartan space and Weyl group of the simple Lie algebra associated to the corresponding Dynkin diagram. The universal Poisson deformation of a Kleinian singularity coincides with its semi-universal deformation (see [LNS]). This was already well studied by Grothendieck, Brieskorn, Kronheimer, and Slodowy (see [Sl] for reference). The quantizations of a Kleinian singularity were constructed by Crawley-Boevey and Holland in [CH] as a family of algebras that they denote $\mathcal{O}^\lambda$. In [CH] it is also proved that $\mathcal{O}^\lambda$ is isomorphic to $\mathsf{e}_0\Pi^\lambda(Q)\mathsf{e}_0$, the spherical subalgebra of the deformed preprojective algebra of the McKay quiver Q associated to the singularity. Explicit presentation with generators and relations for the quantizations in type **A** appeared in earlier works by Hodges [Ho], Smith [Sm], and Bavula and Jordan [Ba; BJ], who studied them as special cases of generalized Weyl algebras. For type **D**, an explicit construction of the quantizations was given by Levy in [Le] and, using different methods, by Boddington [Bo1]. Explicit presentations are not known in type **E**.

§ 1.5 | Main Results II

The following theorem is the main result of Chapter 7.

Theorem 1.5.1 (Theorem 6.2.6)**.** *Let V/Γ be a Kleinian singularity of type* **A** *or* **D***. Then*

$$\mathrm{PIso}(V/\Gamma) \cong \mathrm{Iso}(V/\Gamma),$$

so Conjecture 2 holds in these cases. In particular, the groups of automorphisms of the quantization and of Poisson automorphisms of the deformation corresponding to the same parameter are isomorphic.

The proof of this result is explicit and is based on a case-by-case study, as detailed in Chapter 7. Even though the results are uniform, the methods used to prove them are quite different: algebraic for type **A**, geometric for type **D**. We briefly summarise the proof strategy in the two cases.

§ 1.5.1 | Type A

The isomorphisms between the quantizations in type **A** were classified in [BJ], while the Poisson side is less studied in the literature. The automorphism groups of the deformations (considered as affine varieties) were computed by Makar-Limanov [Ma1] and further studied by Blanc and Dubouloz [BD]. Naurazbekova and Umirbaev studied in [NU] the groups of Poisson automorphisms of deformed Kleinian singularities in type $\mathbf{A_1}$. They proved that the groups are independent of the deformation parameter and that they have an amalgamated free product structure. In the case $n = 1$, the deformations and quantizations can be identified with quotients of the symmetric algebra and universal enveloping algebra of $\mathfrak{sl}_2$, respectively. The automorphism groups of these quotients of $\mathfrak{U}(\mathfrak{sl}_2)$ are known to have the same amalgamated free product structure [Di; Fl], which implies that the two groups are isomorphic [NU, Theorem 5]. In Chapter 7, we show that these results can be generalized to $n > 1$, even if we no longer have the $\mathfrak{sl}_2$ structure in higher degree. The main difference is that, for $n > 1$, the (Poisson) automorphism groups depend on the deformation parameter.

In Section 7.1.2 we give an explicit presentation for the automorphism groups of the quantizations of type **A** Kleinian singularities and we prove that they have an amalgamated free product structure. Furthermore, we show that all the associative algebra automorphisms descend to Poisson algebra automorphisms of the corresponding deformation. In Section 7.1.3, we show that these are all the Poisson algebra automorphisms. The following theorem summarises these results (for $n = 2$, refer to [NU]).

Theorem 1.5.2. *Let $n > 2$. The group G of Poisson automorphisms of a deformation of a $\mathbf{A_{n-1}}$ Kleinian singularity is isomorphic to the group of automorphisms of the corresponding quantization. The dependence of G on the deformation parameter splits into two classes, one for generic and one for special parameters (see Theorem 7.1.14* *for more details):*

(i) *for special deformation parameter and n even,* $G \cong (\mathbb{C}[y] \rtimes \mathbb{C}^\times) *_{\mathbb{C}^\times} (\mathbb{C}^\times \rtimes \mathbb{Z}/2\mathbb{Z})$;

(ii) *for special deformation parameter and n odd,* $G \cong (\mathbb{C}[y] \rtimes \mathbb{C}^\times) *_{\mathbb{C}^\times} H$, *where* $H = \langle \mathbb{C}^\times, \Omega \,|\, \Omega^2 = -1,\ \lambda \cdot \Omega = \Omega \cdot \lambda^{-1}\ \forall \lambda \in \mathbb{C}^\times \rangle$;

(iii) *for generic deformation parameter,* $G \cong (\mathbb{C}[y] \rtimes \mathbb{C}^\times) *_{\mathbb{C}^\times} (\mathbb{C}[x] \rtimes \mathbb{C}^\times)$.

The proof of Theorem 1.5.1 is then completed in Section 7.1.4, where the whole isomorphism groupoids are considered.

§ 1.5.2 | Type D

The isomorphisms between the quantizations in type **D** were classified in [Le]. In type **D**, the affine automorphism group of the undeformed Kleinian singularity was computed by Blanc in [Bl]. To our knowledge, the affine isomorphisms between deformations were unknown; we compute them in Theorem 7.2.12, using techniques from [Bl]. Let $\mathcal{D}_c$ be a Poisson deformation of the Kleinian singularity of type **D**. We embed it into a projective normal surface X_c, constructed as a hypersurface of the $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ defined as $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ (see Section 7.2 for more details). The $\mathbb{P}^2$-bundle structure induces a conical bundle structure on X_c. We prove the following technical result (see Proposition 7.2.8).

Proposition 1.5.3. *Let c, c' be two deformation parameters. Every isomorphism $\phi : \mathcal{D}_c \to \mathcal{D}_{c'}$ extends to an isomorphism $X_c \to X_{c'}$.*

We can thus work with the surfaces X_c, which are easier to handle. We first compute all the projective isomorphisms $X_c \to X_{c'}$ that restrict to isomorphisms $\mathcal{D}_c \to \mathcal{D}_{c'}$. By the previous Lemma, these are all the affine isomorphisms $\mathcal{D}_c \to \mathcal{D}_{c'}$. We are then able to check which isomorphisms are Poisson, proving Theorem 1.5.1. In particular, we can give an explicit presentation of the groups of Poisson automorphisms of the deformations.

Theorem 1.5.4. *Let $n \geq 4$. The group G of Poisson automorphisms of a deformation of a $\mathbf{D_n}$ Kleinian singularity is isomorphic to the group of automorphisms of the corresponding quantization. The dependence of G on the deformation parameter splits into three classes, one for generic, one for sub-generic, and one for special parameter (see Corollary 7.2.2 for more details):*

(i) *for generic deformation parameter,* $G = \langle \mathrm{id} \rangle$;

(ii) *for $n > 4$ and special deformation parameter,* $G = \mathbb{Z}/2\mathbb{Z}$;

(iii) *for $n = 4$ and special deformation parameter, $G = S_3$, the symmetric group on 3 elements;*

(iv) for $n = 4$ and sub-generic deformation parameter, $G = \mathbb{Z}/2\mathbb{Z}$.

§ 1.5.3 | Common features

If $X = V/G$ is a symplectic quotient, then the group of graded Poisson automorphisms of $\mathbb{C}[X]$ is equal to $\Theta := N_{Sp(V)}(G)/G$ (see [Lo2, Lemma 3.20]).

From the results of Chapter 7, we know that, for Kleinian singularities of type **A** and **D**, there are no "exotic" non-filtered isomorphisms in Iso and PIso. In fact, in type **D** there exist only filtered isomorphisms, while in type **A** the only non-filtered isomorphisms are inner, that is they are the exponentiation of an inner nilpotent derivation.

For a Kleinian singularity X with Dynkin diagram Δ, the group of diagram automorphisms $\mathrm{Aut}(\Delta)$ acts on $\mathbb{C}[X]$ by graded Poisson automorphisms, and can be identified with a subgroup of Θ. In every type except type **A** the diagram automorphisms coincide with Θ, while in type **A** they are a proper subgroup. In fact, Θ is generated by $\mathrm{Aut}(\Delta)$ and by "hyperbolic rotations". For every $\lambda \in \mathbb{C}^\times$, the corresponding hyperbolic rotation on $\mathbb{C}^3$ is

$$(x, y, z) \mapsto (\lambda x, \lambda^{-1} y, z).$$

It is easy to check that these induce Poisson automorphisms of the **A** Kleinian singularities. Interestingly, the only generators in $\mathrm{PIso}(\mathcal{A})$ that are not automorphisms come from $\mathrm{Aut}(\Delta)$. In other words, the action of the hyperbolic rotations and the inner automorphisms on the moduli space $\mathfrak{P}/W$ is trivial (see Chapter 8 for a precise statement).

§ 1.6 | Thesis structure

This thesis is divided into two parts, reflecting the two directions of research.

Part I

In this part we discuss the problem of chiralization of star-products.

Chapter 2 serves as a preliminary for the remainder of Part I. In Section 2.1 we recall the theory of formal distributions, which is the framework upon which vertex algebras are built. In Section 2.2 we give an overview of notation, definitions and basic facts on vertex algebras. Section 2.3 is dedicated to the λ-bracket formalism and introduces the related structure of a vertex Lie algebra. Poisson vertex algebras and the classical limit of vertex algebras are discussed in Section 2.4. Finally, in Section 2.5 we explain the construction of the associative and Poisson Zhu algebras. The main sources which are used are [Ka1; FB; DK2; Hu2; Zh]. All the results in Chapter 2 are well-known.

In Chapter 3 we introduce the notion of an $\hbar$-vertex algebra, which is a vertex algebra with a deformed translation covariance axiom. As we show, $\hbar$-vertex algebras are connected to the calculus of finite differences, so in Section 3.1 we give a brief overview of the theory and the facts and definitions that are needed for our study. In Section 3.2

we define an $\hbar$-vertex algebra and explain how $\hbar$-vertex algebras and vertex algebras are related via the formal change of variable $x \mapsto \frac{1}{\hbar}\log(1+\hbar x)$. Furthermore, we develop a structure theory for $\hbar$-vertex algebras, which is one of the main results of this thesis. In particular, in Section 3.3 we construct an analogue of the λ-bracket, which we call the $\hbar$-bracket, which has very useful application for the computations in Chapter 4. In Section 3.4 we provide a different, shorter proof of the associativity of the Zhu algebra, using the $\hbar$-vertex algebra formalism. The material in Section 3.1 is well-known, the main reference is [GKP]. The rest of the results are mostly novel, although some of them appeared, with different proofs, in [Hu2; DK2].

In Chapter 4 we discuss the chiralization of star-products. In Section 4.1 we recall the theory of star-products for Poisson algebras. In particular, we discuss the Moyal-Weyl and Gutt star-products and their construction through quantization maps. In Section 4.2 we discuss the similar notion of star-deformations for Poisson vertex algebras. In Section 4.3 we explain that these are not the correct chiral analogues of star-products. We prove that instead the $\hbar$-deformation of a star-deformation has all the desired properties of a chiral star-product. In addition, we prove that in a class of important examples, star-products for Poisson algebras always admit a chiralization. In Section 4.4 we present one of the main results of our thesis, the explicit formulae for a class of chiral star-products, that includes the chiral Moyal-Weyl and Gutt star-products.

The conclusions of Part I are in Chapter 5, where we discuss open problems and future directions. In particular, we explore the possible implications of our results about $\hbar$-vertex algebras and the Zhu algebras on the representation theory of vertex algebras.

Part II

In this part we discuss the isomorphisms of deformations and quantizations of conical symplectic singularities.

Chapter 6 is a technical introduction to Chapter 7, which contains the main results of Part II. After fixing the general notation and conventions, in Section 6.1 we give an overview of the theory of deformations and quantizations of conical symplectic singularities. In Section 6.2 we explain the main problem of this part of the thesis: proving the isomorphism of the groupoids of isomorphisms of deformations and quantizations of the algebra of functions of a conical symplectic singularity. This gives context and motivation to the results of Chapter 7, which concern the special case of Kleinian singularities. In Section 6.3 we recall some basic facts about Kleinian singularities. In particular, we provide the explicit presentations of the deformations and quantizations of Kleinian singularities of type **A** and **D**. The results of this Section are not new. The main references are [Ba; Be3; Ka3; Le; Lo2].

In Chapter 7 we prove the main result of Part II. The Poisson isomorphisms between the deformations of a Kleinian singularity of type **A** and **D** are equivalent to the isomorphisms of associative algebras between the corresponding quantizations. The proof is case by case. In Section 7.1 we prove the theorem in type **A**, while in Section 7.2 we prove it in type **D**.

Finally, in Chapter 8 we discuss the common features we observed between type **A** and **D** and explore open problems and possible future directions.

Part I

Chiralization of star-products

Notation and conventions

If not stated differently, all vector spaces are assumed to be over $\mathbb{C}$.

We often consider non-associative products. In the case of products between three or more elements, unless we specify differently, we always mean that the products should be computed from right to left:

$$a_1 a_2 \cdots a_{n-1} a_n = a_1(a_2(\cdots(a_{n-1}a_n)\cdots)).$$

We provide below a table of commonly used notation for Part I:

Notation	Meaning
U	Vector space
A	Associative algebra
$\mathcal{A}$	Poisson algebra
$\mathfrak{g}$	Lie algebra
V	Vertex algebra
$\mathcal{V}$	Poisson vertex algebra
R	Lie vertex algebra
$\mathbb{N}$	The natural numbers $\{0, 1, 2, \dots\}$
$\mathcal{S}(U)$	Symmetric algebra of U
$W(U, \omega)$	Weyl algebra of a symplectic vector space (U, ω)
$\mathfrak{U}(\mathfrak{g})$	Universal enveloping algebra of $\mathfrak{g}$
$V(R)$	Universal vertex algebra of R

Notation	Meaning	Name
$U[z]$	$\{\sum_{n=0}^{N} a_n z^n \| a_n \in U, N \in \mathbb{N}\}$	Polynomials
$U[z, z^{-1}]$	$\{\sum_{n=-M}^{N} a_n z^n \| a_n \in U, M, N \in \mathbb{N}\}$	Laurent polynomials
$U[[z]]$	$\{\sum_{n\geq 0} a_n z^n \| a_n \in U\}$	Formal power series
$U((z))$	$\{\sum_{n\geq -N} a_n z^n \| a_n \in U, N \in \mathbb{N}\}$	Formal Laurent series
$U[[z, z^{-1}]]$	$\{\sum_{n\in\mathbb{Z}} a_n z^n \| a_n \in U\}$	Bilateral series (formal distributions)

Chapter 2

Vertex algebras

This Chapter is a brief overview of the theory of vertex algebras. It serves as a preliminary to Chapters 3 and 4, which contain the main results of Part I.

We start by recalling some basic facts and definitions about the theory of formal distributions (that is, power series with infinitely many integer powers) in Section 2.1. This sets up the framework and the notation needed to define vertex algebras. The definition is given in Section 2.2, where we also briefly discuss the structure theory of vertex algebras. In Section 2.3 we recall the λ-bracket formalism and the definition of vertex Lie algebras, while Section 2.4 is dedicated to the classical limit of vertex algebras: Poisson vertex algebras. We end by recalling the construction of the Zhu algebra in Section 2.5.

§ 2.1 | Calculus of formal distributions

Definition 2.1.1. Let U be a vector space. A U-valued formal distribution in the variable $z_1, \dots, z_k$ is a bilateral power series $a(z_1, \dots, z_k) \in U[[z_1^{\pm 1}, \dots, z_k^{\pm 1}]]$

$$a(z_1, \dots, z_k) = \sum_{i_1 \dots i_k \in \mathbb{Z}} a_{i_1, \dots, i_k} z_1^{i_1} \dots z_k^{i_k}.$$

For a one-variable formal distribution $a(z) \in U[[z, z^{-1}]]$, we use the notation

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \tag{2.1.1}$$

$$a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}, \qquad a(z)_- = \sum_{n \geq 0} a_{(n)} z^{-n-1}.$$

Remark 2.1.2. As usual, a multi-variable formal distribution is a one-variable formal distribution, where the other variables are thought of as constants. In other words, if R is the vector space $U[[z_1^{\pm 1}, \dots, z_{k-1}^{\pm 1}]]$, then $U[[z_1^{\pm 1}, \dots, z_k^{\pm 1}]] = R[[z_k, z_k^{-1}]]$. In particular, we can define operators on the space of one-variable formal distributions and extend them to multi-variable formal distributions.

Remark 2.1.3. Assume that U, V, W are vector spaces with a product $U \otimes V \to W$. It is always possible to extend it formally to a product on the spaces of distributions

$$U[[z, z^{-1}]] \otimes V[[w, w^{-1}]] \to W[[z^{\pm 1}, w^{\pm 1}]].$$

If $z = w$ though, the product is in general ill-defined, as the coefficient of one of the terms z^i may be an infinite series. There are some cases in which the products are always defined, for example, the product of a formal distribution and a Laurent polynomial or the product of two formal Laurent series.

The formal derivative on $U[[z, z^{-1}]]$ is defined as

$$\partial_z a(z) = -\sum_{n \in \mathbb{Z}} n a_{(n-1)} z^{-n-1}. \tag{2.1.2}$$

The formal residue is, by definition, the coefficient of the z^{-1} term:

$$\operatorname{Res}_z a(z) = a_{(0)}.$$

In particular, this implies:

$$a_{(n)} = \operatorname{Res}_z z^n a(z), \quad \forall n \in \mathbb{Z}.$$

This motivates the seemingly unnatural indexing in (2.1.1). From (2.1.2) we get a short exact sequence:

$$0 \longrightarrow U \longrightarrow U[[z, z^{-1}]] \xrightarrow{\partial_z} U[[z, z^{-1}]] \xrightarrow{\operatorname{Res}_z} U \longrightarrow 0.$$

Integration by parts Let $a(z), b(z) \in A((z))$ be formal Laurent series with coefficients in an algebra A. The product $a(z) \cdot b(z)$ is well-defined and the formal derivative ∂_z acts as a derivation. Since $\operatorname{Res}_z(\partial_z(a(z) \cdot b(z))) = 0$, we must have

$$\operatorname{Res}_z(\partial_z a(z) \cdot b(z)) = -\operatorname{Res}_z(a(z) \cdot \partial_z b(z)),$$

which corresponds to the integration by parts formula of the analytical residue.

Change of variables Let $g(w) \in U((w))$ and $f(z) \in \mathbb{C}[[z]]$ such that $g(w) = \sum_{n \geq N} a_n w^n$ and $f(z) = \sum_{n \geq 1} f_n z^n$, with $f_1 \neq 0$. Then the composition $g(f(z)) \in U((z))$ is well-defined as $g(f(z)) = \sum_{n \geq N} a_n (f(z))^n$. In fact, $f(z)^n \in z^n \mathbb{C}[[z]]$ for all $n \geq 0$, so $\sum_{n \geq 0} a_n (f(z))^n$ converges in the topology of $U[[z]]$. The usual change of variables formula applies:

$$\operatorname{Res}_w g(w) = \operatorname{Res}_z (g(f(z)) \partial_z f(z)). \tag{2.1.3}$$

In fact, we can write $g(w) = a_{-1}w^{-1} + \partial_z A(w)$ for some $A(w) \in U((w))$, so

$$\operatorname{Res}_z(g(f(z))\partial_z f(z)) = a_{-1} \operatorname{Res}_z(f(z)^{-1}\partial_z f(z)) + \operatorname{Res}_z \partial_z(A(f(z)) = a_{-1},$$

because $\operatorname{Res}_z(f(z)^{-1}\partial_z f(z)) = 1$ by our assumptions on $f(z)$.

Example 2.1.4. We shall be interested in particular in the following mutually inverse change of variables:

$$w = \frac{1}{\hbar}\log(1+\hbar z) = \sum_{k\geq 1} \frac{(-1)^{k+1}\hbar^{k-1}z^k}{k}, \tag{2.1.4}$$

$$w = \frac{1}{\hbar}(e^{\hbar z} - 1) = \sum_{k\geq 1} \frac{\hbar^{k-1}z^k}{k!},$$

where $\hbar \in \mathbb{C}^\times$.

Remark 2.1.5. The following motivates the name “distribution”. There is a non-degenerate pairing $U[[z,z^{-1}]] \times \mathbb{C}[z,z^{-1}] \to U$ defined by $\langle a, f\rangle := \operatorname{Res}_z a(z)f(z)$, for $a(z) \in U[[z,z^{-1}]]$ and $f(z) \in \mathbb{C}[z,z^{-1}]$. It is easy to see that all linear functionals on $\mathbb{C}[z,z^{-1}]$ can be realized this way. A formal distribution can be thought of as a linear functional on the space of Laurent polynomials. Formal distributions are then distributions in the analytical sense, with $\mathbb{C}[z,z^{-1}]$ as the space of test functions.

There are natural embeddings of the spaces of Laurent series $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ into $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$. Consider now the element $(z-w)$: it is invertible in both $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$, but the two inverses are different in $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$. This defines two different embeddings:

$$i_{z,w} : \mathbb{C}((z-w)) \hookrightarrow \mathbb{C}((z))((w)) \subset \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]] \quad \frac{1}{(z-w)} \mapsto \sum_{n\geq 0} w^n z^{-n-1}, \tag{2.1.5}$$

$$i_{w,z} : \mathbb{C}((z-w)) \hookrightarrow \mathbb{C}((w))((z)) \subset \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]] \quad \frac{1}{(z-w)} \mapsto -\sum_{n<0} w^n z^{-n-1}. \tag{2.1.6}$$

From an analytical point of view, they correspond to series expansions in the domains $|z| > |w|$ and $|w| > |z|$, respectively.

Definition 2.1.6. The formal delta function is the two variables formal distribution $\delta(z-w) \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$ defined as:

$$\delta(z-w) = (i_{z,w} - i_{w,z})\left(\frac{1}{z-w}\right) = \sum_{n\in\mathbb{Z}} w^n z^{-n-1}.$$

For all $j \in \mathbb{N}$, the j-th derivative of the formal delta function is given by:

$$\frac{1}{j!}\partial_w^j \delta(z-w) = (i_{z,w} - i_{w,z})\left(\frac{1}{(z-w)^{j+1}}\right) = \sum_{n\in\mathbb{Z}} \binom{n}{j} z^{-n-1} w^{n-j}. \tag{2.1.7}$$

Proposition 2.1.7. *For all formal distributions $f(z) \in U[[z, z^{-1}]]$ the product $f(z)\delta(z-w)$ converges and*

$$\mathrm{Res}_z\, f(z)\delta(z-w) = f(w).$$

Proof. This proposition is a formal analogue of the defining property of the analytical delta distribution. It follows from a straightforward check:

$$f(z)\delta(z-w) = \sum_{m,n} f_{(m)} w^n z^{-n-m-2} = \sum_{k\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} f_{(m)} w^{k-m-1} \right) z^{-k-1}.$$

□

A fundamental property in the theory of vertex algebras is locality.

Definition 2.1.8. A formal distribution in two variables $a(z,w) \in U[[z^{\pm1}, w^{\pm1}]]$ is called local if there is a number $N \in \mathbb{N}$ such that $(z-w)^N a(z,w) = 0$.

It is clear by equation (2.1.7) that $\partial_w^n \delta(z-w)$ is local, with $N = n-1$. The following theorem, due to Kac, provides a complete characterization of local formal distributions.

Theorem 2.1.9 (Decomposition theorem)**.** *For all $j \geq 0$, the derivatives of the delta function $\partial_w^j \delta(z-w)$ are local. Any local formal distribution $a(z,w) \in U[[z^{\pm1}, w^{\pm1}]]$ can be uniquely decomposed as a finite sum of derivatives of the formal delta function, with formal distributions in w as coefficients:*

$$a(z,w) = \sum_{j=0}^{N-1} \frac{1}{j!} c^j(w) \partial_w^{(j)} \delta(z-w),$$

where

$$c^j(w) = \mathrm{Res}_z\, a(z,w)(z-w)^j \in U[[w, w^{-1}]].$$

and N is such that $(z-w)^N a(z-w) = 0$.

Proof. See [Ka2, Theorem 1.2]. □

We consider now the case where U has some additional algebraic structure, namely it is a Lie algebra or an associative algebra.

Definition 2.1.10. Let $\mathfrak{g}$ be a Lie algebra, and $a(z), b(z)$ two $\mathfrak{g}$-valued formal distributions. Then $a(z), b(z)$ are called mutually local (or simply local) if the formal distribution $[a(z), b(w)] \in \mathfrak{g}[[z^{\pm1}, w^{\pm1}]]$ is local.

Definition 2.1.11. Let A be an associative algebra, and $a(z), b(z)$ two A-valued formal distributions. The (n)-th product between $a(z), b(z)$ is defined, for all $n \in \mathbb{Z}$, as

$$a(w)_{(n)}b(w) = \mathrm{Res}_z(i_{z,w}(z-w)^n a(z)b(w) - i_{w,z}(z-w)^n b(w)a(z)). \tag{2.1.8}$$

If $n \in \mathbb{N}$, formula (2.1.8) becomes

$$a(w)_{(n)}b(w) := \mathrm{Res}_z(z-w)^n[a(z), b(w)].$$

Remark 2.1.12. The special case of the (-1)-product is denoted by

$$a(z)_{(-1)}b(z) = {:}a(z)b(z){:}\,,$$

and it is known as the normally ordered product. From (2.1.5) and (2.1.6) it follows that the normally ordered product can also be written as

$${:}a(z)b(z){:} = a(z)_+b(z) + b(z)a(z)_-. \tag{2.1.9}$$

Remark 2.1.13. We can compute $(\partial_w a(w))_{(n)}b(w)$ by integrating by part the residue in (2.1.8):

$$(\partial_w a(w))_{(n)}b(w) = -na(w)_{(n-1)}b(w). \tag{2.1.10}$$

By the Leibniz rule and (2.1.8) (2.1.10),

$$\begin{aligned}\partial_w(a(w)_{(n)}b(w)) &= a(w)_{(n)}\partial_w b(w) - na(w)_{(n-1)}b(w)\\ &= a(w)_{(n)}\partial_w b(w) + (\partial_w a(w))_{(n)}b(w),\end{aligned}$$

hence ∂_w is a derivation of all (n)-products, for $n \in \mathbb{Z}$.

Theorem 2.1.14. *Let A be an associative algebra, and $a(z), b(z)$ two A-valued formal distributions, mutually local. Then*

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{a(w)_{(j)}b(w)}{i_{z,w}(z-w)^{j+1}} + {:}a(z)b(z){:}\,. \tag{2.1.11}$$

Proof. See [Ka1, Theorem 2.3]. □

Formula (2.1.11) is known in quantum field theory as operator product expansion (OPE for short). Physicists usually write only the singular part of it, as the more informal:

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{a(w)_{(j)}b(w)}{(z-w)^{j+1}}.$$

Definition 2.1.15. A (quantum) field on a vector space U is an $\mathrm{End}(U)$-valued formal distribution $a(z)$ such that, for all $b \in U$, there exists an $N \in \mathbb{N}$ such that $a_{(n)}b = 0$, for all $n \geq N$, i.e. $a(z)b \in U((z))$ for all $a, b \in U$.

Remark 2.1.16. (i) If $a(z)$ is a field, then by (2.1.2) $\partial_z a(z)$ is still a field.

(ii) If $a(z), b(z)$ are fields, then $a(z)b(w)v$ and $b(w)a(z)v$ are elements of the space $V((z))((w))$ and $V((w))((z))$ respectively, for all $v \in V$. By taking the residue in z, we get an element in $V((w))$. This implies that the (n)-product of fields is again a field for all $n \in \mathbb{Z}$.

Lemma 2.1.17 (Dong's Lemma)**.** *Let $a(z), b(z), c(z)$ be pairwise mutually local fields. Then $a(z)_{(n)}b(z)$ and $c(z)$ are mutually local fields for all $n \in \mathbb{Z}$.*

Proof. See [Ka1, Lemma 3.2]. □

§ 2.2 | Structure theory of vertex algebras

There are various equivalent definitions of a vertex algebra. The following one is the most commonly found in the literature. As the nomenclature suggests, it is in the spirit of quantum field theory; in fact, this definition gives a mathematical formalization to the (right) chiral part of a 2-dimensional conformal field theory and can be derived from the Wightman axioms for quantum field theory (see [Ka1, Section 1.1]).

Definition 2.2.1. A vertex algebra is the data of a vector space V, a nonzero vector $|0\rangle \in V$ (called the vacuum vector), and a linear map $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$ (called the state-field correspondence), denoted by

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

satisfying the following axioms:

(i) (Fields) $Y(a, z)$ is a field for all $a \in V$, i.e. for all $a, b \in V$, $a_{(n)}b = 0$ for $n \gg 0$. The fields $Y(a, z)$ are often called vertex operators.

(ii) (Vacuum) $Y(|0\rangle, z) = \mathrm{id}_V$, $Y(a, z)|0\rangle \in V[[z]]$ and $Y(a, z)|0\rangle|_{z=0} = a$.

(iii) (Translation covariance) The endomorphism $\partial \in \mathrm{End}(V)$ defined as $\partial a := a_{(-2)}|0\rangle$, for all a in V, is called the (infinitesimal) translation operator. Then

$$[\partial, Y(a, z)] = \partial_z Y(a, z).$$

(iv) (Locality) For all $a, b \in V$, there exists $N > 0$ (dependant on a and b) such that

$$(z - w)^N [Y(a, z), Y(b, w)] = 0.$$

This set of axioms imposes strong restrictions on the vertex operators $Y(a, z)$. Many well-known identities can be deduced directly by the axioms; we give here an overview of the most important ones. These classical results (see [Ka1; FB] for reference) are proved in the case of $\hbar$-vertex algebras in Chapter 3. We recall them for comparison with the results in Chapter 3.

Theorem 2.2.2. *Let V be a vertex algebra. The vertex operators $Y(a,z)$ satisfy the following identities:*

1. $Y(a,z)|0\rangle = e^{z\partial}a$, *for all* $a \in V$*;*
2. $(\partial a)_{(n)} = -na_{(-n-1)}$, *for all* $a \in V$ *and* $n \in \mathbb{Z}$*;*
3. *(Skewsymmetry)*
$$Y(a,z)b = e^{z\partial}Y(b,-z)a, \quad \forall a,b \in V;$$
4. *(n-product identity)*
$$Y(a_{(n)}b,z) = Y(a,z)_{(n)}Y(b,z), \quad \forall a,b \in V;$$
5. *(Borcherds identity)*
$$\begin{aligned} &i_{z,w}(z-w)^nY(a,z)Y(b,w) - i_{w,z}(z-w)^nY(b,w)Y(a,z) \\ &\quad = \sum_{j\geq 0}\frac{1}{j!}Y(a_{(n+j)}b,w)\partial_w^j\delta(z-w), \quad \forall a,b \in V, n \in \mathbb{Z}. \end{aligned} \tag{2.2.1}$$

Remark 2.2.3. The Borcherds identity can be considered the fundamental identity of vertex algebras, as the other identities can be deduced from it. In fact; the original definition of vertex algebras by Borcherds in [Bo2] was given in terms of the Borcherds identity, or better in terms of the coefficient version of (2.2.1): for all $k,m,n \in \mathbb{Z}$,

$$\begin{aligned} \sum_{j\geq 0}\binom{n}{j}(-1)^j(a_{(m+n-j)}(b_{(k+j)}c) - (-1)^n b_{(k+n-j)}(a_{(m+j)}c)) \\ = \sum_{j\geq 0}\binom{m}{j}(a_{(n+j)}b)_{(k+m-j)}c. \end{aligned}$$

Remark 2.2.4. The (n)-product identity implies the following formula for the OPEs of vertex operators (see Theorem 2.1.14):

$$Y(a,z)Y(b,w) = \sum_{n\geq 0}\frac{Y(a_{(n)}b,w)}{i_{z,w}(z-w)^{n+1}} + {:}Y(a,z)Y(b,w){:}\,.$$

Thus, the singular part of the OPE is controlled by the non-negative (n)-products.

The constraints the vertex operators need to satisfy force them to be unique, in the following sense.

Theorem 2.2.5 (Goddard's uniqueness theorem)**.** *Let V be a vertex algebra and $B(z)$ an* $\mathrm{End}(V)$*-valued field, which is mutually local with all the fields $Y(a,z)$, $a \in V$.*

Suppose that, for some $b \in V$*:*

$$B(z)|0\rangle = e^{z\partial} b.$$

Then $B(z) = Y(b, z)$*.*

A consequence of Goddard's uniqueness theorem is the following result, which is useful for constructing examples of vertex algebras.

Theorem 2.2.6. *Let* V *be a vector space,* $|0\rangle$ *a nonzero element in* V*, and* $\partial \in \mathrm{End}(V)$*. Let* $\{a^i(z)\}_{i\in I}$*, (*I *an index set) be a collection of* $\mathrm{End}(V)$*-valued fields such that:*

1. $[\partial, a^i(z)] = \partial_z a^i(z)$*;*
2. $\partial|0\rangle = 0$ *and* $a^i(z)|0\rangle \in V[[z]]$*;*
3. *the linear map defined by* $a^i(z) \mapsto a^i := a^i(z)|0\rangle|_{z=0}$ *is injective;*
4. $a^i(z)$ *and* $a^j(z)$ *are mutually local for all* $i, j \in I$*;*
5. *the vectors* $a^{i_1}_{(j_1)} \dots a^{i_n}_{(j_n)}|0\rangle$*, with* $j_s < 0$ *and* $i_s \in I$ *span* V*.*

Then the formula

$$Y(a^{i_1}_{(j_1)} \dots a^{i_n}_{(j_n)}|0\rangle, z) = :\frac{\partial^{-j_1-1}}{(-j_1-1)!} Y(a^{i_1}, z) \dots \frac{\partial^{-j_n-1}}{(-j_n-1)!} Y(a^{i_n}, z):,$$

defines the (unique) structure of vertex algebra on V *such that* $|0\rangle$ *is the vacuum vector,* ∂ *the infinitesimal translation operator, and* $Y(a^i, z) = a^i(z)$*, for all* $i \in I$*.*

The conformal structure

In many important examples (in particular, all of those coming from conformal field theories), the vertex algebra comes with an associated conformal structure. That is, there is a vector $\omega \in V$ such that $Y(\omega, z) = L(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ (note the unusual indexing) satisfies the following:

(i) The operators $\{L_n\}$ span the Virasoro Lie algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3 - m}{12} c\,\mathrm{id}_V,$$

for some $c \in \mathbb{C}$, called the central charge;

(ii) $L_{-1} = \partial$;

(iii) L_0 is a diagonalizable operator.

Definition 2.2.7. A vertex algebra V with a conformal vector ω is called a *conformal vertex algebra* or a *vertex operator algebra* (VOA for short).

Let V be a VOA. The operator L_0 induces a grading $V = \bigoplus V[\Delta]$, called the *conformal grading*, by eigenspace decomposition. For an eigenvector $a \in V$, let Δ_a be the corresponding eigenvalue.

Proposition 2.2.8. *Let $a \in V[\Delta_a]$ and $b \in V[\Delta_b]$. The conformal grading satisfies the following:*

1. $\Delta_{|0\rangle} = 0$;
2. $\Delta_{\partial a} = \Delta_a + 1$;
3. $\Delta_{a_{(n)}b} = \Delta_a + \Delta_b - n - 1$, *for all* $n \in \mathbb{Z}$.

Proof. This follows from a direct computation. See [DK2, Proposition 1.15] for details. □

§ 2.3 | λ-bracket formalism

Let V be a vertex algebra. Since $(\partial a)_{(n)} = -na_{(n-1)}$ for all $a \in V$, $n \in \mathbb{Z}$, all the negative products can be obtained from the (-1)-product via

$$a_{(n)}b = \frac{1}{n!}(\partial^n a)_{(-1)}b, \quad \forall a, b \in V. \tag{2.3.1}$$

The whole vertex algebra is then determined by the (-1) product and by the non-negative products (or by the singular part of the OPEs, from the physical point of view). It turns out that the non-negative products can be packed together to create a Lie-like structure, called the λ-bracket. The following definition is due to Kac [Ka1].

Definition 2.3.1. A vertex Lie algebra is a vector space V together with an endomorphism ∂ and a bilinear operation, called a λ-bracket

$$[\cdot_\lambda \cdot] : V \otimes V \to V \otimes \mathbb{C}[\lambda],$$

that satisfies the axioms:

(i) (skewsymmetry) $[a_\lambda b] = -[b_{-\lambda-\partial} a]$;

(ii) (sesquilinearity) $[\partial a_\lambda b] = -\partial[a_\lambda b], \qquad [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b]$;

(iii) (Jacobi identity) $[[a_\lambda b]_{\lambda+\mu} c] = [a_\lambda [b_\mu c]] - [b_\mu [a_\lambda c]]$.

Note, here we have used the following notation: if $[a_\lambda b] = \sum_{n\geq 0} \frac{\lambda^n}{n!} a_{(n)}b$, then by $[a_{\lambda+\partial} b]$ we mean $\sum_{n\geq 0} \frac{(\lambda+\partial)^n}{n!}(a_{(n)}b)$, with ∂ acting on the coefficients $a_{(n)}b$.

Remark 2.3.2. Every vertex algebra is a vertex Lie algebra, with ∂ given by the infinitesimal translation operator, and the λ-bracket defined as

$$[a\,{}_\lambda\, b] := \mathrm{Res}_z(e^{\lambda z} Y(a, z)b) = \sum_{n\geq 0} \frac{\lambda^n}{n!} a_{(n)}b.$$

The following is an equivalent definition of a vertex algebra, due to Bakalov and Kac [BK1].

Definition 2.3.3. A vertex algebra is a quintuple $(V, |0\rangle, \partial, [\cdot_\lambda \cdot], ::)$ such that

(i) $(V, \partial, [\cdot_\lambda \cdot])$ is a vertex Lie algebra;

(ii) $(V, |0\rangle, \partial, ::)$ is a unital, non-commutative, non-associative differential algebra (i.e. ∂ is a derivation of the product $::$), with unity $|0\rangle$, which satisfies the following two axioms:

$$:ab: - :ba: = \int_{-\partial}^{0} [a_\lambda b] d\lambda,$$

$$:(:ab:)c: - :a(:bc:): = :\left(\int_0^\partial d\lambda\, b\right)[a_\lambda c]: + :\left(\int_0^\partial d\lambda\, a\right)[b_\lambda c]: .$$

(iii) The λ-bracket and the product $::$ are related by the so-called right and left non-commutative Wick formulae

$$[a_\lambda :bc:] = :[a_\lambda b]c: + :b[a_\lambda c]_\hbar: + \int_0^\lambda [[a_\lambda b]_\mu c] d\mu,$$

$$[:ab:_\lambda c] = :a[b_{\lambda+\partial^{(1)}} c]: + :b[a_{\lambda+\partial^{(1)}} c]: + \int_0^\lambda [b_\mu [a_{\lambda-\mu} c]] d\mu.$$

In Definition 2.3.3 we have used the following notation:

$$:\left(\int_0^\partial d\lambda\, b\right)[a_\lambda c]: = \sum_{n\geq 0} :\left(\int_0^\partial \frac{\lambda^n}{n!} d\lambda\, b\right) a_{(n)} c: = \sum_{n\geq 0} :\frac{\partial^{j+1} b}{(j+1)!}(a_{(j)}c): .$$

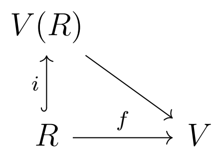
By $\partial^{(1)}$ (respectively $\partial^{(2)}$) we mean that the ∂ acts only the first term of the product (respectively the second), i.e. we have

$$:b[a_{\lambda+\partial^{(1)}} c]: = \sum_{n\geq 0} :\left(\frac{(\lambda+\partial)^n}{n!} b\right) a_{(n)} c: .$$

The product $::$ is called the normally ordered product (it corresponds to the (-1) product of Definition 2.2.1).

Definition 2.3.3 is usually much more convenient for computations and allows us to think of vertex algebras as chiral analogues of associative algebras. To any associative algebra we can associate a Lie algebra using the commutator, and the universal enveloping algebra provides an adjoint to this functor. Similarly, to any vertex algebra, we can associate a vertex Lie algebra (that now controls both the commutator and the associator of the normally ordered product) and there is an adjoint universal construction.

Theorem 2.3.4. *Let R be a vertex Lie algebra. There exists a "universal vertex algebra" $V(R)$ such that $R \hookrightarrow V(R)$ and, for any vertex algebra V and vertex Lie algebra* *morphism* $f : R \to V$*, there is a vertex algebra morphism* $V(R) \to V$ *such that*



is a commutative diagram.

Proof. We define a bracket on R as

$$[a, b] := \int_{-\partial}^{0} [a\,{}_\lambda\, b] d\lambda.$$

By a straightforward check, using properties of the integral, this is a Lie bracket. Consider the universal enveloping algebra $\mathfrak{U}(R_{Lie})$ associated to this Lie algebra structure. We define a vertex algebra structure on $\mathfrak{U}(R_{Lie})$ by constructing a normally ordered product and a λ-bracket satisfying the axioms of Definition 2.3.3. It is sufficient to define them on the PBW basis of $\mathfrak{U}(R_{Lie})$. We proceed by mutual recursion on the degree of the monomials. For $a, b \in R$ the λ-bracket is the one on R and $:ab: = ab$. The formulae of Definition 2.3.3 extend these recursively, in a unique way, to the whole of $\mathfrak{U}(R_{Lie})$. Then $V(R)$ is $\mathfrak{U}(R_{Lie})$ with this vertex algebra structure. Since $:aB: = aB$ for all $a \in R$ and $B \in \mathfrak{U}(R_{Lie})$, the PBW property of $\mathfrak{U}(R_{Lie})$ implies that the PBW monomials with respect to the normally ordered product form a basis of $V(R)$. The map for the universal property is defined on the PBW monomials as $:u_1 \cdots u_n: \mapsto :f(u_1) \cdots f(u_n):$. □

Remark 2.3.5. This construction of $V(R)$ differs from the one most commonly found in the literature (see for example [FB]). A proof of their equivalence can be found in [DK2, Section 1.7].

Theorem 2.3.4 holds in the greater generality of *non-linear vertex Lie algebras* (see [DK1, Theorem 3.9]). The definition of a non-linear vertex Lie algebra is quite technical (see [DK1, Definition 3.1]). Intuitively, it is a vector space R with a λ-bracket valued in $\mathcal{T}(R)[\lambda]$, that satisfies the same axioms of Definition 2.3.1. Some technicalities are needed to give meaning to the axioms on $\mathcal{T}(R)$. We shall consider a special case of non-linear vertex Lie algebras, that we call “sub-linear”.

Definition 2.3.6. A sub-linear vertex Lie algebra is a vector space R, together with an endomorphism ∂ and a λ-bracket

$$[\cdot\,{}_\lambda\,\cdot] : R \otimes R \to (R \oplus \mathbb{C})[\lambda],$$

such that the axioms of Definition 2.3.1 hold on $R \oplus \mathbb{C}$, with the λ-bracket extended by

$$[\mathbb{C}\,{}_\lambda\, -] = [-\,{}_\lambda\, \mathbb{C}] = 0.$$

Alternatively, a sub-linear vertex Lie algebra is the quotient of a central extension of a vertex Lie algebra by putting the central element equal to a constant.

Example 2.3.7. Many important examples of vertex algebras are of the form $V(R)$, where R is a sub-linear vertex Lie algebra.

1. The $\beta\gamma$-system of rank n is the vertex algebra $V(R)$, where R is the free $\mathbb{C}[\partial]$-module generated by $x_1, \dots, x_n, y_1, \dots, y_n$, with bracket induced by

$$[x_i\,{}_\lambda\, y_j] = \delta_{i,j} \quad \forall i, j \in \{1, \dots, n\}.$$

2. Let U be a vector space with a symmetric bilinear form $(\cdot|\cdot)$. The free-boson vertex algebra is $V(R)$, where $R = \mathbb{C}[\partial] \otimes U$ with λ-bracket induced by

$$[u\,{}_\lambda\, v] = \lambda(u|v), \quad \forall u, v \in U.$$

The special case where U is one-dimensional is also known as the Heisenberg vertex algebra.

3. Let $\mathfrak{g}$ be a Lie algebra with a symmetric, invariant bilinear form $(\cdot|\cdot)$. The affine vertex algebra at level $k \in \mathbb{C}$, denoted as $V^k(\mathfrak{g})$, is $V(R)$, where $R = \mathbb{C}[\partial] \otimes \mathfrak{g}$, with λ-bracket induced by

$$[u\,{}_\lambda\, v] = [a, b] + k(u|v)\lambda, \quad \forall u, v \in U.$$

4. The Virasoro vertex algebra Vir_c, with central charge $c \in \mathbb{C}$, is $V(R)$, where R is the free $\mathbb{C}[\partial]$-module generated by a single element L, with λ-bracket

$$[L\,{}_\lambda\, L] = (2\lambda + \partial)L + \frac{c}{12}\lambda^3.$$

Remark 2.3.8. This setting is rather general. Let V be a vertex algebra, PBW generated by a free $\mathbb{C}[\partial]$-module R. Fix a basis of R and let $\mathcal{B}$ be the corresponding PBW basis of V. We assume that the coefficients of the λ-bracket on elements of R, when written in terms of $\mathcal{B}$, are polynomials of degree 1 in R. Then by a result by De Sole and Kac [DK1, Theorem 6.4], R admits the structure of a sub-linear vertex Lie algebra compatible with the vertex Lie algebra structure of V, so that V is canonically isomorphic to $V(R)$.

The integral λ-bracket

There is an additional, equivalent definition of a vertex algebra, introduced by De Sole and Kac in [DK2, Section 1.9]. It rewrites the axioms of Definition 2.3.3 in terms of an operator $I_\lambda(\cdot, \cdot)$ called the integral λ-bracket.

Definition 2.3.9. A vertex algebra is a quadruple $(V, |0\rangle, T, I_\lambda)$, where V is a vector space, $0 \neq |0\rangle \in V$, $\partial \in \mathrm{End}(V)$ and $I_\lambda : V \otimes V \to V \otimes \mathbb{C}[\lambda]$ is a bilinear map that satisfies the following axioms:

(a) (unity)

$$I_\lambda(|0\rangle, a) = I_\lambda(a, |0\rangle) = a \quad \forall a \in V;$$

(b) (sesquilinearity)

$$\frac{d}{d\lambda} I_\lambda(\partial a, b) = -\lambda \frac{d}{d\lambda} I_\lambda(a, b)\,,$$

and

$$\partial\left(I_\lambda(a, b)\right) = I_\lambda(\partial a, b) + I_\lambda(a, \partial b) \quad \forall a, b \in V;$$

(c) (skewsymmetry)

$$I_\lambda(a, b) = I_{-\lambda-\partial}(b, a)\,, \quad \forall a, b \in V;$$

(d) (Jacobi identity)

$$I_\lambda(a, I_\mu(b, c)) - I_\mu(b, I_\lambda(a, c)) = I_{\lambda+\mu}(I_\lambda(a, b) - I_{-\mu-\partial}(a, b), c)\,, \quad \forall a, b, c \in V.$$

Remark 2.3.10. To go from Definition 2.3.3 to Definition 2.3.9, define the integral λ-bracket I_λ as

$$I_\lambda(a, b) := {:}ab{:} + \int_0^\lambda [a_x b] dx.$$

To go the other way, define the λ-bracket and the normally ordered product as

$${:}ab{:} = I_0(a, b), \quad [a_\lambda b] = \frac{d}{d\lambda} I_\lambda(a, b).$$

Remark 2.3.11. Intuitively, $I_\lambda(a, b)$ should be thought as $\int_{-\infty}^{\lambda}[a\,{}_x\,b]dx$, subject to an "integral version" of the vertex Lie algebra axioms. For example, for skew-symmetry,

$$I_\lambda(a, b) = \int_{-\infty}^{\lambda} [a\,{}_x\,b] dx = -\int_{-\infty}^{\lambda} [b\,{}_{-x-\partial}\,a] dx = \int_{-\infty}^{-\lambda-\partial} [b\,{}_y\,a] dy = I_{-\lambda-\partial}(b, a),$$

using a change of variables.

The integral λ-bracket formalism gives a more elegant definition of a vertex algebra, which packs together properties of the normally ordered product and the λ-bracket in the axioms of a vertex Lie algebra written in "integral form". For example, the integral form of the Jacobi identity implies both the Jacobi identity of the λ-bracket, the non-commutative Wick formulae, and the quasi-associativity of the normally ordered product. This will be useful for computations of the chiralization of star-products, as explained in Chapter 4.

§ 2.4 | Classical limit

A Poisson algebra is a commutative algebra with an additional, compatible Lie algebra structure. Similarly, a Poisson vertex algebra is a commutative vertex algebra with a compatible vertex Lie algebra structure. A commutative vertex algebra is a vertex algebra with λ-bracket equal to 0. According to Definition 2.3.3, this is just a commutative differential algebra. In the formalism of Definition 2.2.1, the vertex operators are defined as

$$Y(a,z)b := (e^{z\partial}a)\cdot b.$$

Definition 2.4.1. A Poisson vertex algebra is a unital, associative, commutative differential algebra $(\mathcal{V}, 1, \cdot, \partial)$ with associated a bilinear operator, called the Poisson λ-bracket

$$\{\cdot_\lambda\cdot\} : \mathcal{V}\otimes\mathcal{V} \to \mathcal{V}\otimes\mathbb{C}[\lambda],$$

satisfying the following properties:

1. $(\mathcal{V}, \partial, \{\cdot,\cdot\})$ is a vertex Lie algebra;
2. (right Leibniz rule) $\{a_\lambda bc\} = \{a_\lambda b\}c + \{a_\lambda c\}b$;
3. (left Leibniz rule) $\{ab_\lambda c\} = \{a_{\lambda+\partial}c\}_{\to} b + \{b_{\lambda+\partial}c\}_{\to} a$.

By $\{a_{\lambda+\partial}c\}_{\to} b$ we mean that ∂ acts on the term indicated by the arrow, i.e.

$$\{a_{\lambda+\partial}c\}_{\to} b = \sum_{n\geq 0}\frac{1}{n!}(a_{(n)}c)(\lambda+\partial)^n b.$$

Example 2.4.2. Let R be a (sub-linear) vertex Lie algebra. Then the symmetric algebra $\mathcal{S}(R)$ has a natural structure of Poisson vertex algebra, with Poisson λ-bracket

$$\{a\,{}_\lambda\, b\} = [a\,{}_\lambda\, b],$$

for all $a, b \in R$ and then extended by the left and right Leibniz rule.

Remark 2.4.3. If $\mathfrak{g}$ is a finite-dimensional Lie algebra, then $\mathcal{S}(\mathfrak{g})$ is isomorphic to $\mathbb{C}[\mathfrak{g}^*]$ with the Kirillov–Kostant–Souriau Poisson bracket. Similarly, if $R = \mathbb{C}[\partial]\otimes\mathfrak{g}$, then $\mathcal{S}(R)$ is isomorphic to $\mathbb{C}[J_\infty\mathfrak{g}^*]$, the algebra of functions over the arc space of $\mathfrak{g}^*$. That is, $\mathcal{S}(R)$ is the algebra of differential polynomials on $\mathfrak{g}$.

Definition 2.4.4. A *good filtration* on a vertex algebra V is a filtration $V = \bigcup_{n\geq 0} F_n V$ such that:

1. $\partial F_n V \subset F_n V$;
2. $:(F_n V)(F_m V): \subset F_{n+m} V$;

3. $[F_nV\,{}_\lambda\,F_mV] \subset F_{n+m-1}V[\lambda]$, with λ placed in degree 0.

If V has a good filtration, then $\mathrm{gr}(V)$ has a natural structure of graded Poisson vertex algebra, with derivation $\partial(a + F_{n-1}V) = \partial a + F_{n-1}V$, commutative product

$$(a + F_{n-1}V)(b + F_{m-1}V) = {:}ab{:} + F_{n+m-1}V,$$

and Poisson λ-bracket

$$\{a + F_{n-1}V\,{}_\lambda\,b + F_{m-1}V\} = [a\,{}_\lambda\,b] + F_{n+m-2}V[\lambda],$$

for $a \in F_nV$ and $b \in F_mV$. It is easy to see, from the identities of Definition 2.3.3 and from the fact that the λ-bracket is of negative degree, that all the axioms of Poisson vertex algebras are satisfied.

Example 2.4.5. Let R be a vertex Lie algebra and consider its universal vertex algebra $V(R)$. The PBW basis of $V(R)$ induces a good filtration by the degree of the PBW monomials. This follows easily from the relations in Definition 2.3.3. The associated graded is isomorphic to $\mathcal{S}(R)$ as a Poisson vertex algebra. This can be extended to sub-linear vertex Lie algebras as well. To do so, we need to introduce a formal variable of degree 1 and homogenize the λ-bracket, before computing the associated graded.

§ 2.5 | The Zhu algebra

Let $(V, \partial, |0\rangle, Y)$ be a vertex algebra. Let $\hbar \in \mathbb{C}^\times$ and consider the formal, invertible change of variables $x = \frac{1}{\hbar}\log(1 + \hbar z)$ (2.1.4). Define the following $\hbar$-deformed state-field correspondence:

$$Y_\hbar(a, z) := Y(a, x) = Y\left(a, \frac{1}{\hbar}\log(1 + \hbar z)\right), \quad \forall a \in V. \tag{2.5.1}$$

Proposition 2.5.1. *For all $a \in V$, the $\hbar$-deformed vertex operator $Y_\hbar(a, z)$ is a well-defined* $\mathrm{End}(V)$*-valued quantum field.*

Proof. We need to show that $Y_\hbar(a, z)b \in V((z))$ for all $a, b \in V$. Write $Y_\hbar(a, z) = \sum_{n\in\mathbb{Z}} a_{(n,\hbar)} z^{-n-1}$, where

$$\begin{aligned} a_{(n,\hbar)} &= \mathrm{Res}_z(z^n Y_\hbar(a, z)) = \mathrm{Res}_z\left(\frac{1}{\hbar^n}(e^{\hbar x} - 1)^n Y(a, x)\right) \\ &= \mathrm{Res}_x\left(\frac{1}{\hbar^n}(e^{\hbar x} - 1)^n e^{\hbar x} Y(a, x)\right), \end{aligned}$$

using the change of variables formula (2.1.3). We expand

$$\frac{1}{\hbar^n}(e^{\hbar x} - 1)^n e^{\hbar x} = \sum_{k\geq n} c_n x^n,$$

so that $a_{(n,\hbar)} = \sum_{k\geq n} c_n a_{(n)}$. In particular, $a_{(n,\hbar)}$ is a well-defined endomorphism of V, as $a_{(n,\hbar)}b \in V$ for all n, and $a_{(n,\hbar)}b = 0$ for $n \gg 0$. □

Define the two following products, for $a, b \in V$:

$$a *_\hbar b = a_{(-1,\hbar)}b = \mathrm{Res}_x \left(\frac{\hbar e^{\hbar x}}{e^{\hbar x} - 1} Y(a, x)b \right);$$
$$a \circ_\hbar b = a_{(-2,\hbar)}b = \mathrm{Res}_x \left(\frac{\hbar^2 e^{\hbar x}}{(e^{\hbar x} - 1)^2} Y(a, x)b \right).$$

Theorem 2.5.2. *The space $(V \circ_\hbar V)$ is a two-sided ideal with respect to the $*_\hbar$ product. Moreover, the $*_\hbar$ product induces the structure of a unital associative algebra on the quotient*

$$\mathrm{Zhu}_\hbar(V) = V/(V \circ_\hbar V). \tag{2.5.2}$$

Proof. This is proved in [Hu2, Proposition 6.1] using lengthy computations with the residue. We provide a different, more natural proof, using the formalism of $\hbar$-vertex algebras introduced in Chapter 3 (see Theorem 3.4.4). □

Definition 2.5.3. The Zhu algebra associated to V is $\mathrm{Zhu}(V)$ defined as in (2.5.2), with $\hbar$ set equal to 1.

Remark 2.5.4. This construction of the Zhu algebra is due to Huang [Hu2] and, despite being equivalent, it appears very far from the original construction by Zhu [Zh]. From our point of view, Definition 2.5.3 is more convenient; we briefly explain the other construction and their connection below. We refer to [DK2] and [vEH] for a streamlined exposition of the two original constructions.

Let V be a VOA, with conformal grading $V = \bigoplus V[\Delta]$. Define the following $\hbar$-deformed state-field correspondence:

$$Y_\hbar[a, z] = (1 + \hbar z)^{\Delta_a} Y(a, z) = \sum_{n \in \mathbb{Z}} a_{[n,\hbar]} z^{-n-1}, \tag{2.5.3}$$

for $a \in V[\Delta_a]$ and extended by linearity. Define two products on V:

$$a \bullet_\hbar b := a_{[-1,\hbar]}b = \sum_{k \geq 0} \binom{\Delta_a}{k} \hbar^k a_{(k-1)}b, \tag{2.5.4}$$

$$a \times_\hbar b := a_{[-2,\hbar]}b = \sum_{k \geq 0} \binom{\Delta_a}{k} \hbar^k a_{(k-2)}b, \tag{2.5.5}$$

where the second equality holds only for homogeneous $a, b \in V$. Then again one can prove that $V \times_\hbar V$ is a two-sided ideal for $\bullet_\hbar$ and that the quotient is a unital associative algebra. Then one defines the Zhu algebra as

$$\mathrm{Zhu}'(V) := \left({}^{V}\Big/_{V \times_\hbar V} \right)\Big|_{\hbar=1}. \tag{2.5.6}$$

The equivalence of the two constructions is due to the following fact.

Theorem 2.5.5. *For any $a \in V$, define*

$$Y'(a,z) := Y\left(e^{\hbar z L_0} a, \frac{e^{\hbar z}-1}{\hbar}\right). \tag{2.5.7}$$

Then Y' gives a VOA structure on V and there exists an isomorphism of vertex algebras $T : (V, |0\rangle, Y) \to (V, |0\rangle, Y')$. In particular,

$$Y\left(e^{\hbar z L_0} a, \frac{e^{\hbar z}-1}{\hbar}\right) = TY(T^{-1}a, z)T^{-1}. \tag{2.5.8}$$

Proof. This was originally proved in [Zh, Theorem 4.2.2], and is a special case of Huang's change of variables formula (see [Hu1, Section 7.4]). The isomorphism T is given by $T = \exp\left(\sum_{i\geq 0} l_i L_i\right)$, where the coefficients l_i are the (unique, up to a multiple of $2\pi i$) complex numbers such that

$$\frac{e^{\hbar z}-1}{\hbar} = \exp\left(\sum_{i\geq 0} l_i z^{i+1}\partial_z\right) z.$$

□

Consider now $Y'_\hbar(a,z)$. By (2.5.1) and (2.5.7),

$$Y'_\hbar(a,z) = Y'\left(a, \frac{1}{\hbar}\log(1+\hbar z)\right) = Y\left((1+\hbar z)^{L_0} a, z\right) = (1+\hbar z)^{\Delta_a} Y(a,z) = Y_\hbar[a,z].$$

Then formula (2.5.8) gives

$$Y_\hbar[a,z] = Y'\left(a, \frac{1}{\hbar}\log(1+\hbar z)\right) = TY_\hbar(T^{-1}a, z)T^{-1},$$

or, in other words,

$$T(a_{(n,\hbar)}b) = (Ta)_{[n,\hbar]}(Tb), \quad \forall a, b \in V, \forall n \in \mathbb{Z}.$$

Hence T descends to an algebra isomorphism $T : \mathrm{Zhu}(V) \to \mathrm{Zhu}'(V)$, with $\mathrm{Zhu}'(V)$ defined as in (2.5.6).

Remark 2.5.6. The isomorphism between the two constructions of the Zhu algebra is remarkable, as the second construction depends on the conformal structure, while the first one does not. In particular, this proves that the Zhu algebras associated to different conformal structures are isomorphic. The change of variables construction is more general, as it works also for vertex algebras that do not admit a conformal grading (of course, in those cases, the isomorphism T cannot be defined).

Remark 2.5.7. The Zhu algebra $\mathrm{Zhu}(V)$ plays a crucial role in the representation theory of V. Zhu proved in [Zh] that there is a bijection between irreducible "positive energy" V-modules and irreducible modules of $\mathrm{Zhu}(V)$.

To a Poisson vertex algebra $\mathcal{V}$ with a conformal grading is associated a Poisson algebra, which we again refer to as the Zhu algebra of $\mathcal{V}$ and denote as $\mathrm{Zhu}(\mathcal{V})$. To the best of our knowledge, only the construction via conformal grading has been generalized to the Poisson case (see [DK2, Section 6]). We provide a generalization of the change of variables construction to the Poisson case in Section 3.4.

Let $\mathcal{V} = \bigoplus_\Delta \mathcal{V}[\Delta]$ a Poisson vertex algebra with a conformal grading. That is, the grading satisfies the relations of Proposition 2.2.8 for the non-negative n-products.

Definition 2.5.8. The Zhu algebra associated with $\mathcal{V}$ is

$$\mathrm{Zhu}(\mathcal{V}) := \mathcal{V} \Big/ (\partial\mathcal{V})\mathcal{V} \,.$$

The Poisson structure is induced by

$$\{a, b\} := \sum_{k\geq 0} \binom{\Delta_a - 1}{k} a_{(k)}b,$$

for homogeneous $a, b \in \mathcal{V}$, then extended by linearity.

Theorem 2.5.9. *If $\mathcal{V}$ is the associated graded of a vertex algebra V, then* $\mathrm{Zhu}(\mathcal{V})$ *is the associated graded of* $\mathrm{Zhu}(V)$*. In other words, the following diagram commutes*

$$\begin{array}{ccc} \textit{Poisson vertex algebras} & \xleftarrow[\mathrm{gr}]{} & \textit{Vertex algebras} \\ \Big\downarrow \mathrm{Zhu} & & \mathrm{Zhu} \Big\downarrow \\ \textit{Poisson algebras} & \xleftarrow[\mathrm{gr}]{} & \textit{Associative algebras} \end{array} \tag{2.5.9}$$

Proof. For a proof, see [DK2, Section 6]. □

Definition 2.5.10. Let A be an associative algebra such that $A = \mathrm{Zhu}(V)$, for some vertex algebra V. We say that V is a *chiralization* of A. Similarly, if $\mathcal{A}$ is a Poisson algebra such that $\mathcal{A} = \mathrm{Zhu}(\mathcal{V})$, for some Poisson vertex algebra $\mathcal{V}$, we say that $\mathcal{V}$ is a chiralization of $\mathcal{A}$.

Chapter 3

$\hbar$-Vertex algebras

In this chapter, we introduce the notion of an $\hbar$-vertex algebra. We define it as a vector space with a state-field correspondence satisfying the vertex algebra axioms, with a deformed translation covariance. From this point of view, an $\hbar$-vertex algebra is a deformation of a vertex algebra with respect to the parameter $\hbar$. As we show in Proposition 3.2.3, $\hbar$-vertex algebras are in fact in bijection with vertex algebras, via the change of variables

$$z \mapsto \frac{1}{\hbar} \log(1 + \hbar z), \tag{3.0.1}$$

discussed in Section 2.5. Therefore, $\hbar$-vertex algebras are not a completely new structure. We show however that our axiomatic viewpoint allows us to obtain relations for the deformed $(n, \hbar)$-products that cannot be easily proved using only the change of variables definition. Our main results are the following:

- A complete structure theory of $\hbar$-vertex algebras. We show that all the usual results about the structure theory of vertex algebras admit a generalization to $\hbar$-vertex algebras. Among others, we prove results analogous to Goddard's Uniqueness Theorem, the Reconstruction Theorem, Borcherds Identity, and the OPE Expansion Formula.

- We define an analogue of the λ-bracket, which we call the $\hbar$-bracket (see 3.3.1). We show that the $\hbar$-bracket controls both the commutator and the associator of the $(-1, \hbar)$-product. Our theory of $\hbar$-vertex algebras, and the $\hbar$-bracket in particular, makes computations with the deformed $(n, \hbar)$-products much easier. We believe that this can be useful for computations of higher-level Zhu algebras, which are usually quite complicated to construct explicitly [DLM]. Our main application is to the explicit computation of chiral star-products in Chapter 4.

- The formalism of $\hbar$-vertex algebras allows us to give a much more natural construction of the Zhu algebra. As explained in Section 2.5, the construction of the associative Zhu algebra requires an intermediate step, which is a deformation of the vertex operators $Y(a, z) \mapsto Y_\hbar(a, z)$. These $\hbar$-deformed vertex operators can be constructed using the conformal structure, like in the original paper by Zhu [Zh], or via the change of variable (3.0.1) following Huang [Hu2]. The $\hbar$-deformed vertex operators form an $\hbar$-vertex algebra. The novelty of our approach is to relate the Zhu algebra directly to this $\hbar$-vertex algebra. Using the $\hbar$-bracket formalism, we show that an $\hbar$-vertex algebra has a very natural associative quotient. This is easily seen to be isomorphic to the Zhu algebra of the vertex algebra we get by sending $\hbar \to 0$. This proof avoids the usual complicated computations (see Section 3.4 for more details).

For a more uniform exposition, we prove all our results about $\hbar$-vertex algebras starting from the axioms, even when a different proof would be possible using the change of variables and the properties of vertex algebras. In [DK2], the authors derived some identities of the $\hbar$-deformed vertex operators, but their work relies the conformal grading definition. We recover those results in greater generality, without using the conformal structure and with simpler proofs. Since vertex algebras are $\hbar$-vertex algebras at $\hbar \to 0$, our results imply the structure theory of classical vertex algebras (see Chapter 2 for comparison).

§ 3.1 | Calculus of finite differences

We start this chapter by recalling some basic definitions and facts about the calculus of finite differences, which are needed for our results involving the $\hbar$-bracket (see Section 3.3). The results presented are mostly standard, see [GKP, Section 2.6] for reference. The operators introduced below can be defined on many function spaces. We are mostly interested in polynomial functions, but most of the results still hold for more general function spaces. The limit $\hbar \to 0$ gives back the usual results of infinitesimal calculus.

Let f be a function $\mathbb{C} \to \mathbb{C}$ and $\hbar \in \mathbb{C}^{\times}$. Denote by $S_\hbar$ the shift operator:

$$S_\hbar[f](x) := f(x+\hbar).$$

Definition 3.1.1. The finite difference operator $\Delta_\hbar$ is defined as $(S_\hbar - I)/\hbar$, where I is the identity operator; that is

$$\Delta_\hbar[f](x) := \frac{f(x+\hbar) - f(x)}{\hbar}.$$

It is clear that $\Delta_\hbar$ is a linear operator and that $\Delta_\hbar \to \partial_x$ as $\hbar \to 0$. Denoting by x the operator "multiplication by x", the following commutation relation holds:

$$[\Delta_\hbar, x\, S_\hbar^{-1}] = I. \tag{3.1.1}$$

Under the limit $\hbar \to 0$, this becomes the usual commutation relation $[\partial_x, x] = I$. Because of (3.1.1), a large number of formal relations of standard differential calculus involving functions $f(x)$ map systematically to discrete analogues involving $f(xS_\hbar^{-1}(1))$.

Definition 3.1.2. The falling factorial (also known in the literature as the Pochhammer symbol), is defined, for $n \in \mathbb{N}$, as

$$(x)_n := x(x-1)\dots(x-(n-1)) = \prod_{i=0}^{n-1}(x-i). \tag{3.1.2}$$

If $x \in \mathbb{N}$, the falling factorial is related to the binomial coefficient by

$$\frac{(x)_n}{n!} = \binom{x}{n}. \tag{3.1.3}$$

We can use (3.1.3) as a definition to extend the binomial coefficient to all $x \in \mathbb{C}$ and $n \in \mathbb{N}$.

The k-Pochhammer symbol or k-falling factorial [DP] is a generalization of (3.1.2), depending on a parameter k. To be consistent with our notation, we will use $\hbar$ in place of k and define:

$$(x)_{n,\hbar} := x(x-\hbar)\dots(x-(n-1)\hbar) = \prod_{i=0}^{n-1}(x-i\hbar) = \hbar^n(x/\hbar)_n. \tag{3.1.4}$$

By a slight abuse of notation, we will use the terminology "falling factorial" to refer to both (3.1.2) and (3.1.4).

Since $(x)_{n,\hbar} = (xS_\hbar^{-1})^n(1)$, we can consider the falling factorial a discrete analogue of the monomial x^n. In particular, from (3.1.1) we have

$$\Delta_\hbar[(x)_{n,\hbar}] = \Delta_\hbar[(xS_\hbar^{-1})^n(1)] = n(xS_\hbar^{-1})^{n-1}(1) = n(x)_{n-1,\hbar}.$$

Notice also that

$$\frac{(x)_{n,\hbar}}{n!} = \hbar^n\frac{(x/\hbar)_n}{n!} = \hbar^n\binom{x/\hbar}{n}.$$

The following are some useful formulae when dealing with falling factorials:

$$(x)_{n,\hbar} = x(x-\hbar)_{n-1,\hbar} = (x-\hbar(n-1))(x)_{n-1,\hbar},$$

$$(x)_{n+m,\hbar} = (x)_{n,\hbar}(x-n\hbar)_{m,\hbar},$$

$$(-1)^n(x)_{n,\hbar} = (-x+(n-1)\hbar)_{n,\hbar}. \tag{3.1.5}$$

We consider next the inverse of the finite difference operator, which is a discrete analogue of the integral.

Definition 3.1.3. The indefinite sum of a function g is defined as

$$\sum g(x)\delta x := \{f \mid \Delta_\hbar[f](x) = g(x)\}.$$

Take $f \in \sum g(x)\delta x$ and $a, b \in \mathbb{C}$ such that $a - b \in \hbar\mathbb{Z}$. The definite sum of g from a to b is defined as

$$\sum\nolimits_a^b g(x)\delta x := f(b) - f(a). \tag{3.1.6}$$

The definition of the definite sum is well-posed due to the following theorem

Theorem 3.1.4. *If f and f' are two functions such that $\Delta_\hbar[f] = \Delta_\hbar[f']$, then $f = f' + C$, where C is a periodic function of period $\hbar$.*

Proof. See [GKP, (2.46)]. □

Remark 3.1.5. If g is a polynomial function then the definite sum (3.1.6) is well defined for arbitrary $a, b \in \mathbb{C}$, because the only periodic polynomial functions are the constant functions.

Note that, by definition, definite sums are linear operators and satisfy the following relations:

$$\sum\nolimits_a^c = \sum\nolimits_a^b + \sum\nolimits_b^c, \tag{3.1.7}$$

$$\sum\nolimits_a^b = -\sum\nolimits_b^a. \tag{3.1.8}$$

Theorem 3.1.6. *If the extremes a, b of a definite sum are such that $a - b \in \hbar\mathbb{Z}$, then the definite sum is equal to*

$$\sum\nolimits_a^b g(x)\delta x = \hbar \sum_{k=a}^{b-\hbar} g(k),$$

where by the sum on the right we mean $g(a) + g(a + \hbar) + \cdots + g(b - 2\hbar) + g(b - \hbar)$.

Proof. See [GKP, (2.48)]. □

There are many more similarities between discrete and infinitesimal calculus, for example, there is an analogue of the Leibniz rule and integration by parts (see [GKP]). However, we have no general formula for the discrete change of variables, because there is no analogue for the discrete chain rule. In our case, we will only need a basic change of variables, of the form $x \mapsto c \pm x$.

Proposition 3.1.7. *The following formulae for the change of variables hold, for all $\alpha \in \mathbb{C}$*

$$\sum\nolimits_a^b (x + \alpha)_{n,\hbar}\, \delta x = \sum\nolimits_{a+\alpha}^{b+\alpha} (x)_{n,\hbar}\, \delta x;$$

$$\sum\nolimits_a^b (-x + \alpha)_{n,\hbar}\, \delta x = -\sum\nolimits_{-a+\alpha+\hbar}^{-b+\alpha+\hbar} (x)_{n,\hbar}\, \delta x.$$

Proof. This follows from a straightforward check. □

It makes sense to consider multiple definite sums. As it happens for multiple integrals, in some cases it is convenient to exchange the order of summation. We do not have a general rule for that, but we can do it in some special cases.

Proposition 3.1.8. *Suppose $f(x,y) \in \mathbb{C}[x,y]$. Take a finite number of formal parameters $t, t_1, \dots, t_n$. Take a, a', b, b' in the $\mathbb{Z}$-span of $t_1, \dots, t_n$, and c, c', d, d' in the $\mathbb{Z}$-span of $t, t_1, \dots, t_n$, such that, if we specialize $(t_1, \dots, t_n)$ to any element in $(\hbar\mathbb{Z})^n$, we have*

$$(a \le x \le b-\hbar) \wedge (c(x) \le y \le d(x)-\hbar) \iff (a' \le y \le b'-\hbar) \wedge (c'(y) \le x \le d'(y)-\hbar).$$

Then

$$\sum\nolimits_a^b \delta x \sum\nolimits_{c(x)}^{d(x)} \delta y f(x,y) = \sum\nolimits_{a'}^{b'} \delta y \sum\nolimits_{c'(y)}^{d'(y)} \delta x f(x,y), \tag{3.1.9}$$

where $c(x) := c(x, t_1, \dots, t_n)$ and so on.

Proof. Both sides of the equation (3.1.9) are polynomials in $\mathbb{C}[t_1, \dots, t_n]$. Thus, if they coincide for every value of $(t_1, \dots, t_n) \in (\hbar\mathbb{Z})^n$, they are equal.

Let $(t_1, \dots, t_n) \in (\hbar\mathbb{Z})^n$. Then we can use Theorem 3.1.6 to rewrite (3.1.9) as

$$\sum_{k=a}^{b-\hbar} \sum_{l=c(k)}^{d(k)-\hbar} f(k,l) = \sum_{l=a'}^{b'-\hbar} \sum_{k=c'(l)}^{d'(l)-\hbar} f(k,l),$$

which is just the reordering of ordinary sums. □

§ 3.2 | Structure theory of $\hbar$-vertex algebras

Let $\hbar \in \mathbb{C}^\times$. The definition of an $\hbar$-vertex algebra is the same as that of a vertex algebras (see Definition 2.2.1), with an $\hbar$-deformed translation covariance axiom.

Definition 3.2.1. an $\hbar$-vertex algebra is the data of a vector space V, a nonzero vector $|0\rangle \in V$, and a linear map $Y_\hbar : V \to \mathrm{End}(V)[[z, z^{-1}]]$, denoted by

$$a \mapsto Y_\hbar(a,z) = \sum_{n\in\mathbb{Z}} a_{(n,\hbar)} z^{-n-1},$$

satisfying the following axioms:

(Fields) for all $a, b \in V$, $a_{(n,\hbar)}b = 0$ for $n \gg 0$, i.e. $Y_\hbar(a,z)$ is a $\mathrm{End}(V)$-valued field for all $a \in V$.

(Vacuum) $Y_\hbar(|0\rangle, z) = \mathrm{id}_V$, $Y_\hbar(a,z)|0\rangle \in V[[z]]$ and $Y_\hbar(a,z)|0\rangle|_{z=0} = a$, for all $a \in V$.

($\hbar$-translation covariance) The endomorphism $\partial \in \mathrm{End}(V)$ defined as $\partial a := a_{(-2,\hbar)}|0\rangle$, for all a in V, is called the (infinitesimal) translation operator. Then

$$[\partial, Y_\hbar(a,z)] = (1+\hbar z)\partial_z Y_\hbar(a,z).$$

(Locality) For all $a, b \in V$, there exists $N > 0$ (dependant on a and b) such that

$$(z-w)^N [Y_\hbar(a,z), Y_\hbar(b,w)] = 0.$$

Remark 3.2.2. Alternatively, it is possible to define an $\hbar$-vertex algebra by taking $\hbar$ as a formal parameter. Just change "vector space" with "$\mathbb{C}[\hbar]$-module" and "linear" with "$\mathbb{C}[\hbar]$-linear" in the definition. The case $\hbar = 0$ is more delicate, as in that case the $\hbar$-vertex algebra degenerates to an ordinary vertex algebra. Thus, we can regard vertex algebras as $\hbar$-vertex algebras under the limit $\hbar \to 0$ (see also Remark 3.2.4).

Our first result is that $\hbar$-vertex algebras correspond to vertex algebras under the change of variables $x = \frac{1}{\hbar}\log(1+\hbar z)$ (2.1.4).

Proposition 3.2.3. *Let V be a vector space. If $(V, |0\rangle, \partial, Y)$ is a vertex algebra structure on V, then $(V, |0\rangle, \partial, Y_\hbar)$ is an $\hbar$-vertex algebra, with $\hbar$-deformed vertex operators $Y_\hbar(a,z)$ defined as*

$$Y_\hbar(a,z) = Y\left(a, \frac{1}{\hbar}\log(1+\hbar z)\right), \quad \forall a \in V. \tag{3.2.1}$$

Conversely, if $(V, |0\rangle, \partial, Y_\hbar)$ is an $\hbar$-vertex algebra, then $(V, |0\rangle, \partial, Y)$ is a vertex algebra, where

$$Y(a,z) := Y_\hbar\left(a, \frac{1}{\hbar}(e^{\hbar z}-1)\right), \quad \text{for all } a \in V.$$

Proof. Let $(V, |0\rangle, Y)$ be a vertex algebra and $Y_\hbar$ defined as in (3.2.1). Let $x = x(z) = \frac{1}{\hbar}\log(1+\hbar z)$ and $z = z(x) = \frac{1}{\hbar}(e^{\hbar z}-1)$. Then $Y_\hbar(a,z) = Y(a,x)$. Notice that, from Theorem 2.2.2 and (2.1.3),

$$\begin{aligned} a_{(-2,\hbar)}|0\rangle &= \operatorname{Res}_x\left(\frac{\hbar^2 e^{\hbar x}}{(e^{\hbar x}-1)^2} Y(a,x)|0\rangle\right) \\ &= \operatorname{Res}_x\left(\frac{\hbar^2 e^{\hbar x}}{(e^{\hbar x}-1)^2} e^{x\partial} a\right), \end{aligned}$$

which is equal to ∂a, from the series expansion

$$\frac{\hbar^2 e^{\hbar x}}{(e^{\hbar x}-1)^2} = \frac{1}{x^2} - \frac{\hbar^2}{12} + O(x^2).$$

The $\hbar$-translation covariance follows from:

$$\begin{aligned} [\partial, Y_\hbar(a,z)] &= [\partial, Y(a,z)] = \partial_x Y(a,x) = \partial_x Y_\hbar(a, z(x)) \\ &= e^{\hbar x}\partial_z Y_\hbar(a,z) = (1+\hbar z)\partial_z Y_\hbar(a,z). \end{aligned}$$

Notice that, since $x(z) = z + O(z^2)$, we can write $(x(z) - x(w))^k = (z-w)^k h(z,w)$, for some $h(z,w) \in \mathbb{C}[[z,w]]$ with $h(0,0) = 1$. In particular, this means that $h(z,w)$ is invertible in $\mathbb{C}[[z,w]]$. For all $a,b \in V$, there exists an $N \in \mathbb{N}$ such that

$$(x(z)-x(w))^N [Y(a,x(z)), Y(b,x(w))] = 0,$$

so

$$(z-w)^N h(z,w)^N [Y(a,x(z)), Y(b,x(w))] = 0,$$

which implies locality for the $\hbar$-deformed vertex operators. Thus $(V, |0\rangle, \partial, Y_\hbar)$ is an $\hbar$-vertex algebra. The other implication follows from an almost identical computation.

□

We refer to the $\hbar$-vertex algebra constructed in Proposition 3.2.3 as "the $\hbar$-vertex algebra associated to a given vertex algebra". We can look at $\hbar$-vertex algebras in two ways: as a deformation of vertex algebras, or as vertex algebras under a change of variables. Even if Proposition 3.2.3 proves that $\hbar$-vertex algebras are in bijection with vertex algebras, their independent study still has useful applications as the Zhu algebra is induced by the $\hbar$-deformed structure.

Remark 3.2.4. If we let $\hbar \to 0$, then $\frac{1}{\hbar}\log(1+\hbar z) \to z$, so the change of variable is the identity. This is coherent with our interpretation of vertex algebras being $\hbar$-vertex algebras under the limit $\hbar \to 0$.

Remark 3.2.5. The $\hbar$-deformed vertex operators defined using the conformal grading $Y_\hbar[a,z] = (1+\hbar z)^{\Delta_a} Y(a,z)$ (2.5.3) induce another $\hbar$-vertex algebra structure, which is isomorphic to the one constructed in Proposition 3.2.3, as explained in Section 2.5. Notice that, in this case,

$$a_{[-2,\hbar]}|0\rangle = \sum_{k\geq 0} \binom{\Delta_a}{k} \hbar^k a_{(k-2)}|0\rangle = (\partial + \hbar\Delta_a)a,$$

for an homogeneous $a \in V$. Thus, the translation operator for this $\hbar$-vertex algebra is $\partial + \hbar L_0$, which depends on the conformal structure. In [DK2], the authors study the $\hbar$-vertex operators $Y_\hbar[a,z]$. They focus their attention on the interaction between $Y_\hbar[a,z]$ and ∂. From the $\hbar$-vertex algebras point of view, this is a bit unnatural, as ∂ is not the translation covariance operator of the $\hbar$-vertex algebra they are studying. If instead the correct operator $\partial + \hbar L_0$ is considered, the formulae become simpler, as we show in the rest of this section.

We now develop some results about the structure of an $\hbar$-vertex algebra. As one would expect, the theory is similar to that of ordinary vertex algebras (see Chapter 2 for reference). Notice that all the formulae reduce to the usual one for vertex algebras when we send $\hbar \to 0$. Some of the results in this section were already proved in [DK2] for the $\hbar$-deformed vertex operators $Y_\hbar[a,z]$, using their explicit definition in terms of the conformal grading. In those cases, we give different proofs, that use only the axioms of an $\hbar$-vertex algebra.

We recall the following existence and uniqueness lemma for a formal differential equation.

Lemma 3.2.6. *Let U be a vector space and let $R(z) \in \mathrm{End}(U)[[z]]$. Then the differential equation*

$$\frac{d}{dz} f(z) = R(z) f(z), \tag{3.2.2}$$

with the given initial data f_0 has a unique solution of the form

$$f(z) = \sum_{n \geq 0} f_n z^n, \qquad f_n \in U.$$

Proof. If we write $R(z) = \sum_{n \geq 0} R_n z^n$, equation (3.2.2) becomes

$$n f_n = \sum_{i=0}^{n-1} R_i f_{n-i-1}, \quad \forall n \geq 1,$$

which is a recurrence relation for the coefficients f_n. Thus, once f_0 is fixed, there exists a unique solution $f(z)$. $\square$

Notation. Recall the expansion

$$(1 + \hbar z)^{\alpha} = \sum_{k \geq 0} \frac{(\hbar \alpha)_{k,\hbar}}{k!} z^k.$$

Let U be a vector space and $R \in \mathrm{End}(U)$. We introduce the notation

$$(1 + \hbar z)^{R/\hbar} = \sum_{k \geq 0} \frac{(R)_{k,\hbar}}{k!} z^k.$$

Proposition 3.2.7. *Let V be an $\hbar$-vertex algebra and $a \in V$. Then*

(i)

$$Y_\hbar(a, z)|0\rangle = (1 + \hbar z)^{\partial/\hbar} a;$$

(ii)

$$(1 + \hbar z)^{-\partial/\hbar} Y_\hbar(a, w) (1 + \hbar z)^{\partial/\hbar} = Y_\hbar\left(a, \frac{i_{w,z}(w - z)}{1 + \hbar z}\right),$$

where $i_{w,z}$ is the expansion in the domain $|z| < |w|$ (2.1.6).

Proof. Let U be either V or $\mathrm{End}(V)[[w, w^{-1}]]$; then (i) and (ii) are, respectively, identities in $U[[z]]$. Applying Lemma 3.2.6, we only need to show that both sides of equations (i) and (ii) satisfy the same formal differential equation, with the same initial condition.

Due to the $\hbar$-translation covariance and vacuum axioms, the derivative of left-hand side of (i) is:

$$\partial_z Y_\hbar(a, z)|0\rangle = \frac{[\partial, Y_\hbar(a, z)]|0\rangle}{(1 + \hbar z)} = \frac{\partial}{1 + \hbar z} Y_\hbar(a, z)|0\rangle.$$

Clearly, the right-hand side of (i) satisfies the same differential equation. Setting $z = 0$ and using the vacuum axioms, we immediately get that both sides of (i) satisfy the same initial condition too, hence they are equal.

For (ii), the left-hand side satisfies the formal differential equation

$$\partial_z f(z) = -\frac{\mathrm{ad}(\partial)}{1+\hbar z} f(z),$$

with initial condition $f(0) = Y_\hbar(a, w)$. Using $\hbar$-translation covariance axiom,

$$\partial_z Y_\hbar\left(a, g(z,w)\right) = \frac{\partial_z g(z,w)}{1+\hbar g(z,w)} \left[\partial, Y_\hbar\left(a, g(z,w)\right)\right],$$

for all $g(z, w)$ with $g(0, 0) = 0$. Thus, the right-hand side of (ii) becomes

$$\partial_z Y_\hbar\left(a, \frac{i_{w,z}(w-z)}{1+\hbar z}\right) = -\frac{\mathrm{ad}(\partial)}{1+\hbar z} Y_\hbar\left(a, \frac{i_{w,z}(w-z)}{1+\hbar z}\right).$$

Putting $z = 0$ in the right-hand side, we get $Y_\hbar(a, w)$, so (ii) is proved. □

Proposition 3.2.8 (Skewsymmetry)**.** *For every two elements a and b of an $\hbar$-vertex algebra V the following relation holds:*

$$Y_\hbar(a, z)b = (1+\hbar z)^{\partial/\hbar} Y_\hbar\left(b, -\frac{z}{1+\hbar z}\right) a. \tag{3.2.3}$$

Proof. By locality, for $N \gg 0$:

$$(z-w)^N Y_\hbar(a, z) Y_\hbar(b, w)|0\rangle = (z-w)^N Y_\hbar(b, w) Y_\hbar(a, z)|0\rangle.$$

From part (i) of Proposition 3.2.7:

$$(z-w)^N Y_\hbar(a, z)(1+\hbar w)^{\partial/\hbar} b = (z-w)^N Y_\hbar(b, w)(1+\hbar z)^{\partial/\hbar} a. \tag{3.2.4}$$

Applying part (ii) of Proposition 3.2.7 to the right-hand side of (3.2.4), we get

$$(z-w)^N Y_\hbar(a, z)(1+\hbar w)^{\partial/\hbar} b = (z-w)^N (1+\hbar z)^{\partial/\hbar} Y_\hbar\left(b, \frac{i_{w,z}(w-z)}{1+\hbar z}\right) a.$$

Since the $(n, \hbar)$-products are equal to 0 for $n \gg 0$, we can find a sufficiently large N such that we only have positive powers of z and $w - z$ in the identity above. So we can put $w = 0$ and then divide both sides by z^N, thus completing the proof. □

Remark 3.2.9. The coefficient of z^{-n-1} $(n \in \mathbb{Z})$ of the right-hand side of equation

(3.2.3) is

$$\begin{aligned}
&\mathrm{Res}_z\left(z^n(1+\hbar z)^{\partial/\hbar}Y_\hbar\left(b,\frac{-z}{1+\hbar z}\right)a\right)\\
&=\sum_{k\geq n}(-1)^{k+1}\,\mathrm{Res}_z\left(\frac{(1+\hbar z)^{\partial/\hbar+k+1}}{z^{k+1-n}}\right)b_{(k,\hbar)}a\\
&=\sum_{k\geq n}(-1)^{k+1}\frac{(\partial+\hbar(k+1))_{k-n,\hbar}}{(k-n)!}b_{(k,\hbar)a}.
\end{aligned}$$

We get the following formula for the skewsymmetry of the $(n,\hbar)$ products ($n\in\mathbb{Z}$):

$$a_{(n,\hbar)}b=-\sum_{k\geq 0}(-1)^{k+n}\frac{(\partial+\hbar(k+n+1))_{k,\hbar}}{k!}b_{(k+n,\hbar)}a. \tag{3.2.5}$$

The following theorem is a generalization of Goddard's uniqueness theorem to $\hbar$-vertex algebras:

Theorem 3.2.10. *Let V be an $\hbar$-vertex algebra and $B(z)$ an* End(V)*-valued field, which is mutually local with all the fields $Y_\hbar(a,z)$, $a\in V$. Suppose that, for some $b\in V$:*

$$B(z)|0\rangle=(1+\hbar z)^{\partial/\hbar}b. \tag{3.2.6}$$

Then $B(z)=Y_\hbar(b,z)$.

Proof. For all $a\in V$ and large enough N, we have, using locality, equation (3.2.6), part (a) of Proposition 3.2.7, and then locality again:

$$\begin{aligned}
(z-w)^N B(z)Y_\hbar(a,w)|0\rangle &= (z-w)^N Y_\hbar(a,w)B(z)|0\rangle\\
&=(z-w)^N Y_\hbar(a,w)(1+\hbar z)^{\partial/\hbar}b\\
&=(z-w)^N Y_\hbar(a,w)Y_\hbar(b,z)|0\rangle\\
&=(z-w)^N Y_\hbar(b,z)Y_\hbar(a,w)|0\rangle.
\end{aligned}$$

Since we only have positive powers of w, we can put $w=0$ and divide by z^N. The vacuum axioms imply that $B(z)a=Y_\hbar(b,z)a$. Since this is true for all $a\in V$, $B(z)=Y_\hbar(b,z)$. □

Proposition 3.2.11. *For all $a,b\in V$ and for all $n\in\mathbb{Z}$, the following $(n,\hbar)$-product identity holds:*

$$(1+\hbar w)^{n+1}Y_\hbar(a_{(n,\hbar)}b,w)=Y_\hbar(a,w)_{(n)}Y_\hbar(b,w), \tag{3.2.7}$$

where (n) denotes the n-product of fields (2.1.8).

Proof. Both sides of equation (3.2.7) are local to all $Y_\hbar(a,z)$, $a\in V$, due to Dong's

Lemma 2.1.17. We apply the left-hand side to $|0\rangle$, so

$$(1+\hbar w)^{\partial/\hbar+n+1}a_{(n,\hbar)}b.$$

We only need to prove that the same is true for the right-hand side of (3.2.7), thanks to Goddard's uniqueness theorem. Note that

$$i_{w,z}(z-w)^n Y_\hbar(b,w)Y_\hbar(a,z)|0\rangle$$

only has positive powers of z, so its residue in z is 0. Thus

$$Y_\hbar(a,w)_{(n)}Y_\hbar(b,w)|0\rangle = \mathrm{Res}_z(i_{z,w}(z-w)^n Y_\hbar(a,z)(1+\hbar w)^{\partial/\hbar}b).$$

Applying part (ii) of Proposition 3.2.7,

$$\begin{aligned} Y_\hbar(a,w)_{(n)}Y_\hbar(b,w)|0\rangle &= (1+\hbar w)^{\partial/\hbar}\,\mathrm{Res}_z\left(i_{z,w}(z-w)^n Y_\hbar\left(a,\frac{z-w}{1+\hbar w}\right)b\right) \\ &= (1+\hbar w)^{\partial/\hbar}\sum_{k\in\mathbb{Z}} a_{(k,\hbar)}b\,\mathrm{Res}_z\left((1+\hbar w)^{k+1}i_{z,w}(z-w)^{n-k-1}\right) \\ &= (1+\hbar w)^{\partial/\hbar+n+1}a_{(n,\hbar)}b, \end{aligned}$$

because $i_{z,w}(z-w)^n$ has a z^{-1} term if and only if $n=-1$.

See also [DK2, Proposition 2.2]. □

Corollary 3.2.12. *For every collection of vectors $a^1,\dots,a^n\in V$ and positive integers $j_1,\dots,j_n$,*

$$\frac{1}{(j_1-1)!\cdots(j_n-1)!}{:}\partial_z^{j_1-1}Y_\hbar(a^1,z)\cdots\partial_z^{j_n-1}Y_\hbar(a^n,z){:} = (1+\hbar z)^{\sum_i(-j_i+1)}Y_\hbar(a^1_{(-j_1,\hbar)}\cdots a^n_{(-j_n,\hbar)}|0\rangle,z), \tag{3.2.8}$$

where ${:}a(z)b(z){:} = a(z)_{(-1)}b(z)$ is the normally ordered product of fields (2.1.9).

In particular, for any $a\in V$:

$$Y_\hbar(\partial a,z) = (1+\hbar z)\partial_z Y_\hbar(a,z). \tag{3.2.9}$$

Proof. Let $a(z)$ and $b(z)$ be two fields. Then, by (2.1.10):

$${:}\left(\frac{\partial_z^n}{n!}a(z)\right)b(z){:} = a(z)_{(-n-1)}b(z).$$

Thus (3.2.8) follows from Proposition 3.2.11. Equation (3.2.9) is a special case of (3.2.8), when $n=1$ and $j_1=2$. □

Corollary 3.2.13. *The translation operator ∂ is a derivation of all $(n,\hbar)$-products.*

Proof. This follows at once from $\hbar$-translation covariance and equation (3.2.9). □

We state an $\hbar$-vertex algebra analogue of the reconstruction theorem for vertex algebras.

Theorem 3.2.14. *Let V be a vector space, $|0\rangle$ a nonzero element in V, and $\partial \in \mathrm{End}(V)$. Let $\{a^\alpha(z)\}_{\alpha\in A}$ (A an index set) be a collection of $\mathrm{End}(V)$-valued fields such that:*

1. $[\partial, a^\alpha(z)] = (1+\hbar z)\partial a^\alpha(z)$.
2. $\partial|0\rangle = 0$, $a^\alpha(z)|0\rangle \in V[[z]]$.
3. *The linear map defined by $a^\alpha(z) \mapsto a^\alpha := a^\alpha(z)|0\rangle|_{z=0}$ is injective.*
4. *$a^\alpha(z)$ and $a^\beta(z)$ are mutually local for all $\alpha, \beta \in A$.*
5. *The vectors $a^{\alpha_1}_{(j_1,\hbar)} \dots a^{\alpha_n}_{(j_n,\hbar)}|0\rangle$, with $j_s < 0$ and $\alpha_s \in A$ span V.*

Then the formula

$$Y_\hbar(a^{\alpha_1}_{(j_1,\hbar)} \dots a^{\alpha_n}_{(j_n,\hbar)}|0\rangle, z) = \\ (1+\hbar z)^{-\sum_i (j_i+1)} : \frac{\partial^{-j_1-1}}{(-j_1-1)!} Y_\hbar(a^{\alpha_1}, z) \dots \frac{\partial^{-j_n-1}}{(-j_n-1)!} Y_\hbar(a^{\alpha_n}, z) : ,$$

defines the (unique) structure of an $\hbar$-vertex algebra on V such that $|0\rangle$ is the vacuum vector, ∂ the $\hbar$-infinitesimal translation operator, and $Y_\hbar(a^\alpha, z) = a^\alpha(z)$.

Proof. The proof is identical to the usual one for vertex algebras (see for reference [FB, Theorem 4.4.1]), using the $\hbar$-vertex algebra version of Goddard's theorem and Corollary 3.2.12. □

Theorem 3.2.15. *Let V be an $\hbar$-vertex algebra; the following $\hbar$-Borcherds identity holds, for all $a, b \in V$ and all $n \in \mathbb{Z}$:*

$$\begin{aligned} i_{z,w}(z-w)^n Y_\hbar(a,z)Y_\hbar(b,w) - i_{w,z}(z-w)^n Y_\hbar(b,w)Y_\hbar(a,z) \\ = \sum_{j\geq 0} (1+\hbar w)^{n+j+1} Y_\hbar(a_{(n+j,\hbar)}b, w)\partial_w^j \delta(z-w)/j! \ . \end{aligned} \tag{3.2.10}$$

In particular, for $n = 0$, we obtain the commutator formula:

$$[Y_\hbar(a,z), Y_\hbar(b,w)] = \sum_{n\geq 0} (1+\hbar w)^{n+1} Y_\hbar(a_{(n,\hbar)}b, w)\partial_w^n \delta(z-w)/n!. \tag{3.2.11}$$

Proof. Recall that an element $a(z,w) \in V[[z^{\pm 1}, w^{\pm 1}]]$ is called a local formal distribution if there exists $N \gg 0$ such that $(z-w)^N a(z,w) = 0$. For a local formal distribution, by the Decomposition Theorem 2.1.9, we can write:

$$a(z,w) = \sum_{j\geq 0} c^j(w)\partial_w^j \delta(z-w)/j!,$$

where $c^j(w) = \operatorname{Res}_z(z-w)^j a(z,w)$. The left-hand side of equation (3.2.10) is a local distribution. In fact, if we multiply it by $(z-w)^N$ such that $N+n \geq 0$, it becomes equal to

$$(z-w)^{N+n}[Y_\hbar(a,z), Y_\hbar(b,w)],$$

which is local by the locality axiom. Then (3.2.10) follows from Decomposition Theorem and Proposition 3.2.11.

See also [DK2, Theorem 2.3]. □

Corollary 3.2.16 (OPE expansion)**.** *The Operator Product Expansion of the operators $Y_\hbar(a,z)$, $Y_\hbar(b,w)$ has the following form:*

$$Y_\hbar(a,z)Y_\hbar(b,w) = \sum_{n\geq 0} \frac{(1+\hbar w)^{n+1} Y_\hbar(a_{(n,\hbar)}b, w)}{i_{z,w}(z-w)^{n+1}} + :Y_\hbar(a,z)Y_\hbar(b,w):. \tag{3.2.12}$$

Proof. Due to Proposition 3.2.11, we can rewrite (3.2.12) as

$$Y_\hbar(a,z)Y_\hbar(b,w) = \sum_{n\geq 0} \frac{Y_\hbar(a,w)_{(n)} Y_\hbar(b,w)}{i_{z,w}(z-w)^{n+1}} + :Y_\hbar(a,z)Y_\hbar(b,w):,$$

which holds due to locality and Theorem 2.1.14. □

Remark 3.2.17. At the level of $(n,\hbar)$-products, equation (3.2.9) becomes

$$(\partial a)_{(n,\hbar)}b = -na_{(n-1,\hbar)}b - \hbar(n+1)a_{(n,\hbar)}b \quad a,b\in V, n\in\mathbb{Z},$$

that is

$$((\partial + \hbar(n+1))a)_{(n,\hbar)}b = -na_{(n-1,\hbar)}b \quad a,b\in V, n\in\mathbb{Z}. \tag{3.2.13}$$

If $n>0$, we can iterate equation (3.2.13) to get

$$a_{(-n-1,\hbar)}b = \frac{(\partial)_{n,\hbar}}{n!} a_{(-1,\hbar)}b. \tag{3.2.14}$$

Notice that this is a discrete analogue of the formula (2.3.1) for vertex algebras. We substituted ∂^n with the falling factorial $(\partial)_{n,\hbar}$, which is the discrete analogue of the monomial (see the discussion in Section 3.1). This analogy becomes evident with the $\hbar$-bracket formalism, which we introduce in the next section.

§ 3.3 | $\hbar$-bracket formalism

In this section, we introduce the $\hbar$-analogue of the λ-bracket. We also define the related notions of an $\hbar$-vertex Lie algebra and an $\hbar$-Poisson vertex algebra and study their relation to $\hbar$-vertex algebras.

Definition 3.3.1. An $\hbar$-vertex Lie algebra is a vector space R, together with an endomorphism ∂, and a bilinear operation, called the $\hbar$-bracket

$$[\cdot_\lambda \cdot]_\hbar : V \otimes V \to V \otimes \mathbb{C}[\lambda],$$

that satisfies the axioms:

1. (skewsymmetry) $[a_\lambda b]_\hbar = -[b_{-\lambda-\partial-2\hbar}\, a]_\hbar$;
2. (sesquilinearity) $[\partial a_\lambda b]_\hbar = -(\lambda+\hbar)[a_\lambda b]$, $\quad [a_\lambda \partial b]_\hbar = (\lambda+\partial+\hbar)[a_\lambda b]_\hbar$;
3. (Jacobi identity) $[[a_\lambda b]_\hbar\ {}_{\lambda+\mu+\hbar}\ c]_\hbar = [a_\lambda [b_\mu c]_\hbar\,]_\hbar - [b_\mu [a_\lambda c]_\hbar\,]_\hbar$.

Let V be an $\hbar$-vertex algebra. Define a bracket $[\cdot_\lambda \cdot]_\hbar : V \otimes V \to V \otimes \mathbb{C}[\lambda]$ as

$$[a\,_\lambda\, b]_\hbar := \mathrm{Res}_z((1+\hbar z)^{\lambda/\hbar} Y_\hbar(a,z)b) = \sum_{n\geq 0} \frac{(\lambda)_{n,\hbar}}{n!} a_{(n,\hbar)} b. \tag{3.3.1}$$

Remark 3.3.2. The bracket defined in (3.3.1) can be thought of as a discrete generalization of the usual λ-bracket. In fact, we changed λ^n for the falling factorial $(\lambda)_{n,\hbar}$ (see Remark 3.2.17).

Proposition 3.3.3. *Let R be a vector space, with an endomorphism ∂. Then $(R, \partial, [\cdot\,_\lambda\,\cdot])$ is a vertex Lie algebra if and only if $(R, \partial, [\cdot\,_{\lambda+\hbar}\,\cdot])$ is an $\hbar$-vertex Lie algebra.*

Proof. A direct check shows that $[\cdot\,_{\lambda+\hbar}\,\cdot]$ satisfies the axioms of Definition 3.3.1 if and only if $[\cdot\,_\lambda\,\cdot]$ is a λ-bracket. □

Theorem 3.3.4. *Let V be an $\hbar$-vertex algebra, then the bracket defined as in (3.3.1), together with ∂, induces the structure of an $\hbar$-vertex Lie algebra on V.*

Proof. This follows from Propositions 3.2.3 and 3.3.3. If we write $Y_\hbar(a,z) = Y\left(a, \frac{1}{\hbar}\log(1+\hbar z)\right)$, formula (3.3.1) becomes

$$[a\,_\lambda\, b]_\hbar := \mathrm{Res}_z (1+\hbar z)^{\lambda/\hbar} Y\,(a, x(z))\, b = \mathrm{Res}_x\, e^{x(\lambda+\hbar)} Y(a,x) = [a\,_{\lambda+\hbar}\, b],$$

where $[a\,_\lambda\, b]$ denotes the λ-bracket associated to the vertex operators $Y(a,z)$. In particular, the $\hbar$-bracket defined in (3.3.1) gives an $\hbar$-vertex Lie algebra structure.

We can also prove this directly from the axioms of an $\hbar$-vertex algebra. For completeness, we write here the proof, which is similar to the one for the λ-bracket of a vertex algebra. To prove the skewsymmetry relation, apply $\mathrm{Res}(1+\hbar z)^{\lambda/\hbar}$ to both sides of equation (3.2.3):

$$\begin{aligned}[a\,{}_\lambda\, b]_\hbar &= \mathrm{Res}_z(1+\hbar z)^{(\lambda+\partial)/\hbar} Y_\hbar\left(b, -\frac{z}{1+\hbar z}\right) a \\ &= \sum_{k\ge 0}(-1)^{k+1}\frac{(\lambda+\partial+\hbar(k+1))_{k,\hbar}}{k!} b_{(k,\hbar)a} \\ &= -\sum_{k\ge 0}\frac{(-\lambda-\partial-2\hbar)_{k,\hbar}}{k!} b_{(k,\hbar)a} \\ &= -[a\,{}_{-\lambda-\partial-2\hbar}\, b]_\hbar,\end{aligned}$$

where in the last step we used formula (3.1.5).

The first part of sesquilinearity follows from applying $\mathrm{Res}(1+\hbar z)^{\lambda/\hbar}$ to both sides of (3.2.9), and by integrating by parts when computing the residue on the right-hand side. The second part of sesquilinearity follows from the fact that ∂ is a derivation of all $(n,\hbar)$-products.

Finally, let us prove the Jacobi identity. Take equation (3.2.11), apply both sides to $c \in V$, then apply $\mathrm{Res}_z(1+\hbar z)^{\lambda/\hbar}$ to get

$$\begin{aligned}&\mathrm{Res}_z(1+\hbar z)^{\lambda/\hbar}[Y_\hbar(a,z), Y_\hbar(b,w)]c \\ &\qquad = \mathrm{Res}_z(1+\hbar z)^{\lambda/\hbar}\sum_{n\ge 0}(1+\hbar w)^{n+1} Y_\hbar(a_{(n,\hbar)}b, w)\partial_w^n \delta(z-w)c/n!.\end{aligned} \tag{3.3.2}$$

To compute the right-hand side of (3.3.2), notice that, because of equation (2.1.7),

$$\begin{aligned}\mathrm{Res}_z(1+\hbar z)^{\lambda/\hbar}\partial_w^n\delta(z-w)/n! &= \mathrm{Res}_z(1+\hbar z)^{\lambda/\hbar}\sum_{k\in\mathbb{Z}}\binom{k}{n} w^{k-n}z^{-k-1} \\ &= \sum_{k\ge 0}\frac{(\lambda)_{k+n,\hbar}}{k!n!}w^k \\ &= \frac{(\lambda)_{n,\hbar}}{n!}\sum_{k\ge 0}\frac{(\lambda-n\hbar)_{k,\hbar}}{k!}w^k \\ &= \frac{(\lambda)_{n,\hbar}}{n!}(1+\hbar w)^{\lambda/\hbar - n}.\end{aligned}$$

Using this, we rewrite (3.3.2) as

$$[a\,{}_\lambda\, Y_\hbar(b,w)c]_\hbar - Y_\hbar(b,w)[a\,{}_\lambda\, c]_\hbar = (1+\hbar w)^{\lambda/\hbar+1} Y_\hbar([a\,{}_\lambda\, b]_\hbar, w)c. \tag{3.3.3}$$

The Jacobi identity follows by further applying $\mathrm{Res}_w(1+\hbar w)^{\mu/\hbar}$ to both sides of (3.3.3).

□

Remark 3.3.5. Let (V, Y) be a vertex algebra and $(V_\hbar, Y_\hbar)$ its associated $\hbar$-vertex algebra. Notice that the expression of the $\hbar$-bracket of $V_\hbar$ in terms of the λ-bracket of V is relatively easy, as it involves a simple translation $\lambda \mapsto \lambda+\hbar$. However, the indexing of the products in the $\hbar$-bracket is done in terms of the falling factorials, not of powers. This means that we cannot immediately relate the non-negative (n)-products of V with the non-negative $(n,\hbar)$-products of $V_\hbar$, as we first need to rearrange all the terms in λ. This combinatorial problem involves the Stirling numbers, whose generating series is closely related to the change of variable (3.0.1). See the discussion at the beginning of Section 5 for more details.

Remark 3.3.6. Let V be a VOA and consider the associated $\hbar$-vertex algebra structure induced by the conformal grading: $Y_\hbar(a,z) = (1+\hbar z)^{\Delta_a} Y(a,z)$. In this case, formula (3.3.1) becomes (for homogeneous $a \in V$):

$$[a\,{}_\lambda\, b]_\hbar := \mathrm{Res}_z (1+\hbar z)^{\lambda/\hbar + \Delta_a} Y(a,z) b = \sum_{n\geq 0} \frac{(\lambda + \hbar\Delta_a)_{n,\hbar}}{n!} a_{(n)} b. \tag{3.3.4}$$

To ease the notation, we denote the $(-1,\hbar)$-product by $*_\hbar$, that is

$$a *_\hbar b := a_{(-1,\hbar)} b, \quad \forall a, b \in V.$$

Theorem 3.3.7. *Let V be an $\hbar$-vertex algebra. The $*_\hbar$ product satisfies the following quasi-commutativity and quasi-associativity formulae, expressed in terms of the $\hbar$-bracket for all $a, b, c \in V$:*

$$a *_\hbar b - b *_\hbar a = \sum\nolimits_{-\partial-\hbar}^{0} [a\,{}_\lambda\, b]_\hbar \delta\lambda, \tag{3.3.5}$$

$$(a *_\hbar b) *_\hbar c - a *_\hbar (b *_\hbar c) = \left(\sum\nolimits_0^{\partial} \delta\lambda\, b \right) *_\hbar [a\,{}_\lambda\, c]_\hbar + \left(\sum\nolimits_0^{\partial} \delta\lambda\, a \right) *_\hbar [b\,{}_\lambda\, c]_\hbar. \tag{3.3.6}$$

They also satisfy the following $\hbar$-analogue of the right and left non-commutative Wick formulae:

$$[a\,{}_\lambda\, b *_\hbar c]_\hbar = b *_\hbar [a\,{}_\lambda\, c]_\hbar + [a\,{}_\lambda\, b]_\hbar *_\hbar c + \sum\nolimits_0^{\lambda+\hbar} [[a_\lambda b]_\hbar\, {}_\mu\, c]_\hbar \delta\mu, \tag{3.3.7}$$

$$[a *_\hbar b\,{}_\lambda\, c]_\hbar = a *_\hbar [b\,{}_{\lambda+\partial^{(1)}}\, c]_\hbar + b *_\hbar [a\,{}_{\lambda+\partial^{(1)}}\, c]_\hbar + \sum\nolimits_0^{\lambda} [b\,{}_\mu\, [a\,{}_{\lambda-\mu-\hbar}\, c]_\hbar]_\hbar \delta\mu. \tag{3.3.8}$$

Proof. Equation (3.2.5) for $n=-1$ and the role of a and b swapped becomes:

$$\begin{aligned} b *_\hbar a &= -\sum_{k\geq 0} (-1)^{k+1} \frac{(\partial + \hbar k)_{k,\hbar}}{k!} a_{(k-1,\hbar)} b \\ &= a *_\hbar b + \sum_{k \geq 0} \frac{(-\partial - \hbar)_{k+1,\hbar}}{(k+1)!} a_{(k,\hbar)} b, \end{aligned}$$

thus proving (3.3.5).

Apply $(-1,\hbar)$-product identity (3.2.7) to $c \in V$ and take the constant term:

$$(a *_\hbar b) *_\hbar c = \mathrm{Res}_z\, z^{-1} Y_\hbar(a,z)_{(-1)} Y_\hbar(b,z) c.$$

Now, using formula (2.1.9) and equation (3.2.14):

$$\begin{aligned}\operatorname{Res}_z z^{-1}Y_\hbar(a,z)_{(-1)}Y_\hbar(b,z)c &= \operatorname{Res}_z z^{-1}(Y_\hbar(a,z)_+Y_\hbar(b,z)c + Y_\hbar(b,z)Y_\hbar(a,z)_-c)\\ &= \sum_{k\geq -1} a_{(-k-2,\hbar)}(b_{(k,\hbar)}c) + \sum_{k\geq 0} b_{(-k-2,\hbar)}(a_{(k,\hbar)}c)\\ &= a *_\hbar (b *_\hbar c) + \sum_{k\geq 0}\left(\frac{(\partial)_{k+1,\hbar}}{(k+1)!}a\right) *_\hbar (b_{(k,\hbar)c})\\ &\quad + \sum_{k\geq 0}\left(\frac{(\partial)_{k+1,\hbar}}{(k+1)!}b\right) *_\hbar (a_{(k,\hbar)c}).\end{aligned}$$

This proves (3.3.6).

To prove (3.3.7), equate the constant term in w in both sides of (3.3.3), getting:

$$\begin{aligned}[a\,{}_\lambda\, b *_\hbar c]_\hbar &= b *_\hbar [a\,{}_\lambda\, c]_\hbar + \sum_{k\geq -1}\frac{(\lambda+\hbar)_{k+1,\hbar}}{(k+1)!}([a\,{}_\lambda\, b]_\hbar)_{(k,\hbar)}c\\ &= b *_\hbar [a\,{}_\lambda\, c]_\hbar + [a\,{}_\lambda\, b]_\hbar *_\hbar c + \textstyle\sum_0^{\lambda+\hbar}[[a_\lambda b]\,{}_\hbar\,{}_\mu\, c]_\hbar\delta\mu.\end{aligned}$$

Finally, the left non-commutative Wick formula (3.3.8) can be derived from (3.3.7). Using the skewsymmetry of the $\hbar$-bracket, write

$$[a *_\hbar b\,{}_\lambda\, c]_\hbar = -[c\,{}_{-\lambda-\partial-2\hbar}\, a *_\hbar b]_\hbar.$$

Then using (3.3.7), this is equal to

$$-a *_\hbar [c\,{}_{-\lambda-\partial-2\hbar}\, b]_\hbar - [c\,{}_{-\lambda-\partial-2\hbar}\, a]_\hbar *_\hbar b - \textstyle\sum_0^{-\lambda-\partial-\hbar}[[c_{-\lambda-\partial-2\hbar}\, a]\,{}_\hbar\,{}_\mu\, b]_\hbar\delta\mu,$$

where each of the ∂ is applied to the result of the $(n,\hbar)$-products between a, b, c. Since ∂ is a derivation of the $(n,\hbar)$-products, we can write $\partial = \partial^{(1)} + \partial^{(2)}$, acting respectively on the left and right factor, i.e. $\partial(a *_\hbar b) = (\partial^{(1)}a) *_\hbar b + a *_\hbar (\partial^{(2)}b)$. Applying skewsymmetry again, we get:

$$a *_\hbar [b\,{}_{\lambda+\partial^{(1)}}\, c]_\hbar + [a\,{}_{\lambda+\partial^{(2)}}\, c]_\hbar *_\hbar b + \textstyle\sum_0^{-\lambda-\partial-\hbar}[[a_{\lambda+\partial^{(2)}}\, c]\,{}_\hbar\,{}_\mu\, b]_\hbar\delta\mu.$$

Now, by the commutator formula (3.3.5) on $[a\,{}_{\lambda+\partial^{(2)}}\, c]_\hbar *_\hbar b$ and by adding and reversing the extremes of summation (3.1.7), (3.1.8):

$$a *_\hbar [b\,{}_{\lambda+\partial^{(1)}}\, c]_\hbar + b *_\hbar [a\,{}_{\lambda+\partial^{(1)}}\, c]_\hbar - \textstyle\sum_{-\lambda-\partial-\hbar}^{-\partial-\hbar}[[a_{\lambda+\partial^{(2)}}\, c]\,{}_\hbar\,{}_\mu\, b]_\hbar\delta\mu.$$

Finally, using skewsymmetry, Proposition 3.1.7 for the change of variables, and the first sesquilinearity relation:

$$\begin{aligned}-\textstyle\sum_{-\lambda-\partial-\hbar}^{-\partial-\hbar}[[a_{\lambda+\partial^{(2)}}\, c]\,{}_\hbar\,{}_\mu\, b]_\hbar\delta\mu &= \textstyle\sum_{-\lambda-\partial-\hbar}^{-\partial-\hbar}[b\,{}_{-\mu-\partial-2\hbar}\,[a_{\lambda+\partial^{(1)}}\, c]_\hbar]_\hbar\delta\mu\\ &= \textstyle\sum_0^{\lambda}[b\,{}_\mu\,[a_{\lambda+\partial^{(1)}}\, c]_\hbar]_\hbar\delta\mu\\ &= \textstyle\sum_0^{\lambda}[b\,{}_\mu\,[a_{\lambda-\mu-\hbar}\, c]_\hbar]_\hbar\delta\mu.\end{aligned}$$

Putting everything together completes the proof. □

Remark 3.3.8. The quasi-associativity formula (3.3.6) is symmetric in b and c. An algebra with this property is called left symmetric or pre-Lie (see [Bu]). Its commutator is always a Lie bracket, and the following formula holds:

$$a *_\hbar (b *_\hbar c) = b *_\hbar (a *_\hbar c) + (a *_\hbar b - b *_\hbar a) *_\hbar c. \tag{3.3.9}$$

In particular, an $\hbar$-vertex Lie algebra R has a natural Lie algebra structure, with bracket

$$[a, b] := \sum\nolimits_{-\partial-\hbar}^{0} [a\,{}_\lambda\, b]_\hbar \delta\lambda, \quad \forall a, b \in R.$$

Remark 3.3.9. In [DK2] the authors consider another bracket, which they also call $\hbar$-bracket, defined as

$$[a, b]_\hbar := \mathrm{Res}_z (1 + \hbar z)^{-1} Y_\hbar(a, z) b = \sum_{n\geq 0} (-\hbar)^j a_{(j,\hbar)} b.$$

It is clear from (3.3.4) that $[a, b]_\hbar = [a_\lambda b]_\hbar\Big|_{\lambda=-\hbar}$. With this in mind, their formulae involving the $\hbar$-bracket become a specialization of ours for $\lambda = -\hbar$.

Definition 3.3.10. An $\hbar$-Poisson vertex algebra is a quadruple $(\mathcal{V}, 1, \cdot, \partial, \{\cdot_\lambda\cdot\}_\hbar)$ where

(i) $(\mathcal{V}, 1, \cdot, \partial)$ is a unital, commutative, differential algebra;

(ii) $(\mathcal{V}, \partial, \{\cdot_\lambda\cdot\}_\hbar)$ is an $\hbar$-vertex Lie algebra;

(iii) the $\hbar$-bracket and the commutative product satisfy the right Leibniz rule, for all $a, b, c \in \mathcal{V}$:

$$\{a\,{}_\lambda\, bc\}_\hbar = b\{a\,{}_\lambda\, c\}_\hbar + c\{a\,{}_\lambda\, b\}_\hbar.$$

Remark 3.3.11. By skewsymmetry of the $\hbar$-bracket, an $\hbar$-Poisson vertex algebra satisfies the following left Leibniz rule:

$$\{ab\,{}_\lambda\, c\}_\hbar = \{b_{\lambda+\partial} c\}_{\underset{\to}{\hbar}} a + \{a_{\lambda+\partial} c\}_{\underset{\to}{\hbar}} b.$$

Here we used the same arrow notation introduced for Poisson vertex algebras (see Definition 2.4.1).

Proposition 3.3.12. *Let $(\mathcal{V}, 1, \cdot, \partial)$ be a unital, commutative, differential algebra. Then $(\mathcal{V}, 1, \partial, \cdot, \{\cdot\,{}_\lambda\,\cdot\})$ is a Poisson vertex algebra if and only if $(\mathcal{V}, 1, \partial, \cdot, \{\cdot\,{}_{\lambda+\hbar}\,\cdot\})$ is an $\hbar$-Poisson vertex algebra.*

Proof. From Proposition 3.3.3 we know that $\{\cdot\,{}_{\lambda+\hbar}\,\cdot\}$ defines an $\hbar$-vertex Lie algebra structure. Clearly, it still satisfies the right Leibniz rule. The other implication follows in the same way. □

Remark 3.3.13. If a Poisson vertex algebra $\mathcal{V}$ admits a conformal grading, then equation (3.3.4) provides another way to construct an $\hbar$-Poisson vertex algebra, isomorphic to the one defined in Proposition 3.3.12.

As with the case of ordinary vertex algebras, the semi-classical limit of an $\hbar$-vertex algebra is an $\hbar$-Poisson vertex algebra.

Definition 3.3.14. A *good filtration* on an $\hbar$-vertex algebra V is a filtration $V = \bigcup_{n\geq 0} F_n V$ such that:

1. $\partial F_n V \subset F_n V$;
2. $(F_n V) *_\hbar (F_m V) \subset F_{n+m} V$;
3. $[F_n V\, {}_\lambda\, F_m V]_\hbar \subset F_{n+m-1} V[\lambda]$, where λ is placed in degree 0.

Proposition 3.3.15. *Let V be an $\hbar$-vertex algebra equipped with a good filtration. Then* $\mathrm{gr}(V)$ *has a natural structure of $\hbar$-Poisson vertex algebra.*

Remark 3.3.16. A good filtration for a vertex algebra V is also a good filtration for the corresponding $\hbar$-vertex algebra given by the change of variables. This is because we can always write

$$a_{(n,\hbar)} b = \sum_{n\geq 0} c_n \hbar^n a_{(n)} b, \quad \forall a, b \in V,$$

for some coefficients $c_n \in \mathbb{C}$. In particular, this implies that the (-1) and $(-1, \hbar)$ products coincide on the associated graded $\mathrm{gr}(V)$.

§ 3.3.1 | The sum $\hbar$-bracket

We introduce an $\hbar$-version of the integral λ-bracket discussed in Section 2.3. Following the analogy between infinitesimal and discrete calculus, the integral is substituted by a definite sum and derivatives by finite difference operators. Again, this formalism allows us to pack together different axioms for the $\hbar$-bracket and the $*_\hbar$ product.

Let V be an $\hbar$-vertex algebra, and define a bilinear map $I_{\lambda,\hbar} : V \otimes V \to V[\lambda]$ as

$$I_{\lambda,\hbar}(a, b) := a *_\hbar b + \sum\nolimits_0^\lambda [a_x b]_\hbar \delta x.$$

We call it the sum $\hbar$-bracket.

Proposition 3.3.17. *The sum $\hbar$-bracket satisfies the following axioms:*

(a) Unity:

$$I_{\lambda,\hbar}(|0\rangle, a) = I_{\lambda,\hbar}(a, |0\rangle) = a\,,$$

for every $a \in V$.

(b) Sesquilinearity:

$$\Delta_\hbar I_{\lambda,\hbar}(\partial a, b) = -(\lambda + \hbar) \Delta_\hbar I_{\lambda,\hbar}(a, b)\,,$$

and

$$\partial\left(I_{\lambda,\hbar}(a,b)\right) = I_{\lambda,\hbar}(\partial a,b) + I_{\lambda,\hbar}(a,\partial b)\,,$$

for every $a,b \in V$.

(c) *Skewsymmetry:*

$$I_{\lambda,\hbar}(a,b) = I_{-\lambda-\partial-\hbar,\hbar}(b,a)\,,$$

for every $a,b \in V$.

(d) *Jacobi identity:*

$$I_{\lambda,\hbar}(a,I_{\mu,\hbar}(b,c)) - I_{\mu,\hbar}(b,I_{\lambda,\hbar}(a,c)) = I_{\lambda+\mu,\hbar}(I_{\lambda,\hbar}(a,b) - I_{-\mu-\partial-\hbar,\hbar}(a,b),c)\,, \quad (3.3.10)$$

for every $a,b,c \in V$.

Proof. The unity axiom follows from definition. Sesquilinearity formulae follow from definition and from the fact that ∂ is a derivation of all the $(n,\hbar)$-products.

Skewsymmetry follows from a direct computation:

$$\begin{aligned} I_{-\lambda-\partial-\hbar,\hbar}(b,a) &= b *_\hbar a + \sum\nolimits_0^{-\lambda-\partial-\hbar} [b\,{}_x\,a]_\hbar \delta x \\ &= a *_\hbar b + \sum\nolimits_{-\partial-\hbar}^{-\lambda-\partial-\hbar} [b\,{}_x\,a]_\hbar \delta x \\ &= a *_\hbar b - \sum\nolimits_{-\partial-\hbar}^{-\lambda-\partial-\hbar} [a\,{}_{-x-\partial-2\hbar}\,b]_\hbar \delta x \\ &= a *_\hbar b + \sum\nolimits_0^{\lambda} [a\,{}_x\,b]_\hbar \delta x, \end{aligned}$$

where we used the commutator formula for $*_\hbar$ (3.3.5), skewsymmetry of the $\hbar$-bracket and Proposition 3.1.7 for the change of variables.

Recall that if two functions have the same finite difference, they coincide modulo an $\hbar$-periodic function. In the case of the Jacobi identity, both sides are polynomial functions, and the only periodic polynomial functions are the constants. Notice also that the Jacobi identity remains unchanged if we swap a,b and λ,μ. Thus, we only need to check that identity (3.3.10) holds when we put $\lambda = \mu = 0$, when we put $\mu = 0$ and compute the finite difference with respect to λ, and finally when we compute the finite difference with respect to both λ and μ.

If $\lambda = \mu = 0$ we get

$$a *_\hbar (b *_\hbar c) - b *_\hbar (a *_\hbar c) = -\left(\sum\nolimits_0^{-\partial-\hbar} [a\,{}_x\,b]_\hbar \delta x\right) *_\hbar c,$$

which is true due to equation (3.3.9). Put now $\mu = 0$ in (3.3.10) and compute the finite difference with respect to λ:

$$[a\,{}_\lambda\,b *_\hbar c]_\hbar = b *_\hbar [a\,{}_\lambda\,c]_\hbar + [a\,{}_\lambda\,b]_\hbar *_\hbar c + \Delta_\hbar \left[\sum\nolimits_0^{\lambda} \delta y \sum\nolimits_y^{\lambda} \delta x [[a_x b]\,{}_{\hbar\,y}\,c]_\hbar\right].$$

To compute the finite difference in the right-hand side above we change the order of summation, using Proposition 3.1.8, so

$$\Delta_\hbar \left[\sum\nolimits_0^\lambda \delta y \sum\nolimits_y^\lambda \delta x [[a_x b]\,{}_{\hbar\, y}\, c]_\hbar\right] = \Delta_\hbar \left[\sum\nolimits_0^\lambda \delta x \sum\nolimits_0^{x+\hbar} \delta y [[a_x b]\,{}_{\hbar\, y}\, c]_\hbar\right] = \sum\nolimits_0^{\lambda+\hbar} [[a_\lambda b]\,{}_{\hbar\, \mu}\, c]_\hbar \delta\mu,$$

recovering the right Wick formula (3.3.7).

Now compute the finite difference of (3.3.10) with respect to both λ and μ. The left-hand side becomes $[a\,{}_\lambda\,[b\,{}_\mu\,c]_\hbar]_\hbar - [b\,{}_\mu\,[a\,{}_\lambda\,c]_\hbar]_\hbar$. The right-hand side becomes

$$\Delta_\hbar^{(\mu)} \Delta_\hbar^{(\lambda)} \sum\nolimits_0^{\lambda+\mu} \delta y \sum\nolimits_{-\mu+y}^{\lambda} \delta x [[a_x b]\,{}_{\hbar\, y}\, c]_\hbar.$$

Changing the order of summation gives

$$\Delta_\hbar^{(\mu)} \Delta_\hbar^{(\lambda)} \sum\nolimits_{-\mu}^{\lambda} \delta x \sum\nolimits_0^{x+\mu+\hbar} \delta y [[a_x b]\,{}_{\hbar\, y}\, c]_\hbar = [[a_\lambda b]\,{}_{\hbar\, \lambda+\mu+\hbar}\, c]_\hbar,$$

recovering the Jacobi identity of the $\hbar$-bracket. □

The following result will be useful for computations in Chapter 4.

Lemma 3.3.18. *Let V be an $\hbar$-vertex algebra. Then the sum $\hbar$-bracket $I_{\lambda,\hbar}$ has the following recursion properties:*

$$I_{\lambda,\hbar}(|0\rangle, A) = A, \tag{3.3.11}$$
$$I_{\lambda,\hbar}(a *_\hbar B, A) = a *_\hbar I_{\lambda+\partial^{(1)}}(B, A) + I_{\lambda,\hbar}\left(B, \sum\nolimits_0^{\lambda+\partial^{(1)}} [a_x A]_\hbar \delta x\right),$$

for all $a, A, B \in V$, and

$$I_{\lambda,\hbar}(A, |0\rangle) = A, \tag{3.3.12}$$
$$I_{\lambda,\hbar}(A, a *_\hbar B) = a *_\hbar I_{\lambda,\hbar}(A, B) + I_{\lambda,\hbar}\left(\sum\nolimits_{-\partial-\hbar}^{\lambda} [A_x a]_\hbar \delta x, B\right), \tag{3.3.13}$$

for all $a, A, B \in V$.

Proof. The two base cases (3.3.11) and (3.3.12) follow from definition.

Notice that $a *_\hbar B = I_{0,\hbar}(a, B)$. Thus we have, using the skewsymmetry and Jacobi identity:

$$\begin{aligned}
I_{\lambda,\hbar}(a *_\hbar B, A) &= I_{\lambda,\hbar}(I_{0,\hbar}(a, B), A) \\
&= I_{-\lambda-\partial-\hbar}(A, I_{0,\hbar}(a, B)) \\
&= I_{0,\hbar}(a, I_{-\lambda-\partial^{(1)}-\partial^{(2)}-\hbar}(A, B)) \\
&\quad + I_{-\lambda-\partial-\hbar,\hbar}(I_{-\lambda-\partial^{(1)}-\partial^{(2)}-\hbar,\hbar}(A, a) - I_{-\partial^{(1)}-\hbar,\hbar}(A, a), B).
\end{aligned}$$

Using skewsymmetry again we get

$$I_{0,\hbar}(a, I_{-\lambda-\partial^{(1)}-\partial^{(2)}-\hbar}(A,B)) = a *_\hbar I_{\lambda+\partial^{(1)}}(B,A),$$

and

$$\begin{aligned}
&I_{-\lambda-\partial-\hbar,\hbar}(I_{-\lambda-\partial^{(1)}-\partial^{(2)}-\hbar,\hbar}(A,a) - I_{-\partial^{(1)}-\hbar,\hbar}(A,a), B) \\
&= I_{\lambda,\hbar}(B, I_{\lambda+\partial^{(1)},\hbar}(a,A) - I_{0,\hbar}(a,A)) \\
&= I_{\lambda,\hbar}\left(B, \sum\nolimits_0^{\lambda+\partial^{(1)}} [a_x A]_\hbar \delta x\right).
\end{aligned}$$

Equation (3.3.13) follows from one application of the Jacobi identity:

$$\begin{aligned}
I_{\lambda,\hbar}(A, I_{0,\hbar}(a,B)) &= I_{0,\hbar}(a, I_{\lambda,\hbar}(A,B)) \\
&\quad + I_{\lambda,\hbar}(I_{\lambda,\hbar}(A,a) - I_{-\partial-\hbar,\hbar}(A,a), B) \\
&= a *_\hbar I_{\lambda,\hbar}(A,B) + I_{\lambda,\hbar}\left(\sum\nolimits_{-\partial-\hbar}^{\lambda} [A_x a]_\hbar \delta x, B\right).
\end{aligned}$$

□

Remark 3.3.19. Lemma 3.3.18 provides two useful recurrence equations for the sum ℏ-bracket, which pack together different recurring relations for the $*_\hbar$-product and the ℏ-bracket. It is stated for an ℏ-vertex algebra, but similar relations hold for the integral λ-bracket of a vertex algebra too. It is sufficient to send $\hbar \to 0$ in Lemma 3.3.18.

§ 3.4 | The Zhu algebra

In this section, we construct an associative algebra associated to an ℏ-vertex algebra, a Poisson algebra associated to an ℏ-Poisson vertex algebra, and a Lie algebra associated to an ℏ-vertex Lie algebra. These algebras are exactly the Zhu algebras of the corresponding vertex algebra, Poisson vertex algebra and vertex Lie algebra. We believe that the ℏ-vertex algebra formalism we developed in the previous sections makes the construction and proofs more natural than the usual ones, which require lengthy, artificial computations.

Let V be an ℏ-vertex algebra and define $J_\hbar$ as the vector space $V_{(-2,\hbar)}V$. Equivalently, $J_\hbar$ can be defined as $(\partial V) *_\hbar V$, by formula (3.2.13). For the rest of the section, we use the notation $[\cdot\,{}_{-\hbar}\,\cdot]_\hbar := [\cdot\,{}_\lambda\,\cdot]_\hbar\big|_{\lambda=-\hbar}$.

Lemma 3.4.1. *The space $J_\hbar$ is a two-sided ideal with respect to both the $*_\hbar$ product and the bracket $[\cdot\,{}_{-\hbar}\,\cdot]_\hbar$. Moreover, it is preserved by the action of ∂.*

Proof. From the quasi-associativity formula (3.3.6), it follows that $(\partial V) *_\hbar V$ is a right ideal. Let now $a, b, c \in V$. Using equation (3.3.9):

$$a *_\hbar ((\partial b) *_\hbar c) = (\partial b) *_\hbar (a *_\hbar c) + (a *_\hbar (\partial b) - (\partial b) *_\hbar a) *_\hbar c.$$

Notice that $a *_\hbar (\partial b) = \partial(a *_\hbar b) - (\partial a) *_\hbar b$, so $a *_\hbar ((\partial b) *_\hbar c) \subset J_\hbar *_\hbar V \subset J_\hbar$. Thus $J_\hbar$ is a left ideal too.

By the right Wick formula (3.3.7),

$$[a\,{}_{-\hbar}\,(\partial b) *_\hbar c]_\hbar = (\partial b) *_\hbar [a\,{}_{-\hbar}\,c]_\hbar + [a\,{}_{-\hbar}\,(\partial b)]_\hbar *_\hbar c,$$

and, by sesquilinearity, $[a\,{}_{-\hbar}\,(\partial b)]_\hbar = \partial[a\,{}_{-\hbar}\,b]_\hbar$. By skewsymmetry, $[a\,{}_{-\hbar}\,b]_\hbar = -[b\,{}_{-\hbar}\,a]_\hbar$ mod $J_\hbar$, so $J_\hbar$ is automatically a two-sided $[\cdot\,{}_{-\hbar}\,\cdot]_\hbar$-ideal.

Since ∂ is a derivation of the $(-2,\hbar)$-product, it clearly preserves the ideal $J_\hbar$. □

Definition 3.4.2. Let V be an $\hbar$-vertex algebra. Define the $\mathrm{Zhu}_\hbar$ algebra associated to V as the vector space quotient $\mathrm{Zhu}_\hbar(V) := V/J_\hbar$.

Theorem 3.4.3. *The vector space* $\mathrm{Zhu}_\hbar(V)$ *is a unital associative algebra, with the product induced by the* $*_\hbar$ *product, and the quotient class of* $|0\rangle$ *as identity. The commutator of* $\mathrm{Zhu}_\hbar(V)$ *is induced by* $\hbar[\cdot\,{}_{-\hbar}\,\cdot]_\hbar$.

Proof. Since $J_\hbar$ is a two-sided ideal of $*_\hbar$, $\mathrm{Zhu}_\hbar(V)$ is a well-defined algebra. By the formula for quasi-associativity (3.3.6), it is clear that $*_\hbar$ is associative in the quotient.

Moreover, from the equation for the commutator (3.3.5),

$$a *_\hbar b - b *_\hbar a \equiv \sum\nolimits_{-\hbar}^{0} [a\,{}_\lambda\,b]_\hbar \delta\lambda \quad mod\ J_\hbar.$$

By Theorem 3.1.6, the definite sum is equal to $\hbar[a\,{}_{-\hbar}\,b]_\hbar$. □

Let now V be an ordinary vertex algebra. Recall Huang's construction of the Zhu algebra $\mathrm{Zhu}(V)$ (see Definition 2.5.3). We can now give a new, easy proof of its associativity.

Theorem 3.4.4. *The algebra* $\mathrm{Zhu}(V)$ *is associative. The commutator on* $\mathrm{Zhu}(V)$ *is induced by* $[\cdot\,{}_\lambda\,\cdot]\big|_{\lambda=0}$.

Proof. The proof is now trivial. In fact, the algebra $\mathrm{Zhu}(V)$ is by definition the $\mathrm{Zhu}_\hbar$ algebra of the $\hbar$-vertex algebra associated to V as in Proposition 3.2.3. This is associative by Theorem 3.4.3. Moreover, by Proposition 3.3.3, $[\cdot\,{}_\lambda\,\cdot]_\hbar = [\cdot\,{}_{\lambda+\hbar}\,\cdot]$, so the commutator on the Zhu algebra is induced by

$$\hbar[a\,{}_{-\hbar}\,b]_\hbar\big|_{\hbar=1} = \hbar[a\,{}_{-\hbar+\hbar}\,b]\big|_{\hbar=1} = [a\,{}_\lambda\,b]\big|_{\lambda=0}.$$

□

Remark 3.4.5. We could consider instead a VOA V and the Zhu algebra $\mathrm{Zhu}'(V)$ constructed using the conformal structure (see (2.5.6)). Then $\mathrm{Zhu}'(V)$ is the $\mathrm{Zhu}_\hbar$

algebra of the $\hbar$-vertex algebra associated to V using the conformal structure (see Remark 3.2.5), after specializing $\hbar$ to 1.

Remark 3.4.6. Using the formalism of $\hbar$-vertex algebras, the proof of Theorem 3.4.4 becomes much for natural (see [Hu2, Proposition 6.1] for comparison). This is because the Zhu algebra is not directly associated to the vertex algebra, but rather to the $\hbar$-vertex algebra associated to the vertex algebra. The usual construction has a "hidden step":

$$\begin{tikzcd} (V,Y) \arrow[r, dotted] \arrow[rr, bend left, "\mathrm{Zhu}"] & (V,Y_\hbar) \arrow[r] & \mathrm{Zhu}_{\hbar=1}(V,Y_\hbar) \end{tikzcd}$$

Now, a vertex algebra can also be seen as an $\hbar$-vertex algebra, after the limit $\hbar \to 0$ (see Remark 3.2.2). It is then natural to ask, what is the $\mathrm{Zhu}_{\hbar\to 0}$ algebra of the $\hbar \to 0$-vertex algebra V? This is simply

$$\mathrm{Zhu}_{\hbar\to 0}(V) = R_V := V/V_{(-2)}V.$$

The algebra R_V is known as the C_2-algebra associated to V. It was introduced by Zhu in [Zh], where it is used to study the modular invariance of characters of a vertex algebra. Zhu proved that the dimension of R_V (namely whether it is finite) plays a crucial role. It is shown in [Zh] that R_V has the structure of a Poisson algebra. This is a special case of Theorem 3.4.3. In fact, $\mathrm{Zhu}_{\hbar\to 0}(V)$ is clearly commutative. Moreover, by the relations in Theorem 2.3.3, $[a\,{}_\lambda\, b]\big|_{\lambda=0}$ induces a Poisson bracket on $\mathrm{Zhu}_{\hbar\to 0}(V)$.

Definition 3.4.7. Let $\mathcal{V}$ be an $\hbar$-Poisson vertex algebra. Define $\mathcal{J}_\hbar$ as the ideal $(\partial\mathcal{V})\mathcal{V}$. The $\mathrm{Zhu}_\hbar$ algebra associated to $\mathcal{V}$ is the quotient $\mathrm{Zhu}_\hbar(\mathcal{V}) := \mathcal{V}/\mathcal{J}_\hbar$.

Theorem 3.4.8. *The ideal $\mathcal{J}_\hbar$ is a two-sided ideal for $\{\cdot\,{}_{-\hbar}\,\cdot\}_\hbar$. The algebra $\mathrm{Zhu}_\hbar(\mathcal{V})$ has the structure of a Poisson algebra, with Poisson bracket induced by $\{\cdot\,{}_{-\hbar}\,\cdot\}_\hbar$.*

Proof. By right Leibniz rule

$$\{a\,{}_{-\hbar}\,(\partial b)c\}_\hbar = (\partial b)\{a\,{}_{-\hbar}\,c\}_\hbar + \{a\,{}_{-\hbar}\,(\partial b)\}_\hbar c,$$

and by sesquilinearity $\{a\,{}_{-\hbar}\,(\partial b)\}_\hbar = \partial(\{a\,{}_{-\hbar}\,b\}_\hbar)$. Since by skewsymmetry $\{a\,{}_{-\hbar}\,b\}_\hbar = -\{b\,{}_{-\hbar}\,a\}_\hbar \mod \mathcal{J}_\hbar$, we have that $\mathcal{J}_\hbar$ is a two-sided $\{\cdot\,{}_{-\hbar}\,\cdot\}_\hbar$-ideal.

Specializing $\lambda = -\hbar$, the axioms of an $\hbar$-vertex Lie algebra recover those of a Lie algebra. So $\{\cdot\,{}_{-\hbar}\,\cdot\}_\hbar$ is a Lie bracket on $\mathrm{Zhu}_\hbar(\mathcal{V})$. Since it satisfies the Leibniz rule, it is a Poisson bracket. □

Remark 3.4.9. Let $\mathcal{V}$ be a Poisson vertex algebra. If $\mathcal{V}$ admits a conformal grading, the Poisson Zhu algebra of $\mathcal{V}$ was constructed by De Sole and Kac in [DK2, Section 6] (see Definition 2.5.8). This corresponds to the $\mathrm{Zhu}_\hbar$ algebra of the $\hbar$-Poisson vertex algebra associated to $\mathcal{V}$ via the conformal structure (see Remark 3.3.13).

Theorem 3.4.8 provides a construction of the Poisson Zhu algebra that does not require the conformal structure. This construction mimics the one introduced by Huang for the associative Zhu algebra.

Definition 3.4.10. Let $\mathcal{V}$ be a Poisson vertex algebra. The Zhu algebra of $\mathcal{V}$ is

$$\mathrm{Zhu}(\mathcal{V}) := \mathcal{V}/(\partial\mathcal{V})\mathcal{V}. \tag{3.4.1}$$

Remark 3.4.11. The algebra defined in (3.4.1) is the $\mathrm{Zhu}_\hbar$ algebra of the $\hbar$-Poisson vertex algebra associated to $\mathcal{V}$ as in Proposition 3.3.12, at $\hbar = 1$. Since by Proposition 3.3.3 we have $\{\cdot\,{}_\lambda\,\cdot\}_\hbar = \{\cdot\,{}_{\lambda+\hbar}\,\cdot\}$, the Poisson bracket on the Zhu algebra is induced by $\{\cdot\,{}_\lambda\,\cdot\}\big|_{\lambda=0}$.

Finally, let us define the $\mathrm{Zhu}_\hbar$ algebra of an $\hbar$-vertex Lie algebra.

Definition 3.4.12. Let R be an $\hbar$-vertex Lie algebra. Define the $\mathrm{Zhu}_\hbar$ algebra associated to R as the vector space quotient $\mathrm{Zhu}_\hbar(R) := R/(\partial R)$.

Theorem 3.4.13. *The vector space ∂R is a two-sided ideal for $[\cdot\,{}_{-\hbar}\,\cdot]_\hbar$. The algebra $\mathrm{Zhu}_\hbar(R)$ has the structure of a Lie algebra, with Lie bracket induced by $[\cdot\,{}_{-\hbar}\,\cdot]_\hbar$.*

Proof. This can be proved by the same computations from the proof of Theorem 3.4.3. □

Remark 3.4.14. In [DK2, Corollary 3.22], the authors also introduce the Zhu algebra of a vertex Lie algebra R, again assuming the existence of a conformal grading. In their definition,

$$\mathrm{Zhu}(R) := R/(\partial + L_0)R, \quad [a, b] := \sum_{n\geq 0} \frac{(\Delta_a - 1)_n}{n!} a_{(n)}b. \tag{3.4.2}$$

This coincides with the $\mathrm{Zhu}_\hbar$ algebra of the $\hbar$-vertex Lie algebra associated to R as in Remark 3.2.5, with $\hbar$ specialized to 1. Again, we can give another equivalent definition that does not rely on the conformal structure.

Definition 3.4.15. Let R be an vertex Lie algebra. The Zhu algebra of R is

$$\mathrm{Zhu}(R) := R/\partial R. \tag{3.4.3}$$

Remark 3.4.16. The Lie algebra defined in (3.4.3) is the $\mathrm{Zhu}_\hbar$ algebra of the $\hbar$-vertex Lie algebra associated to R as in Proposition 3.3.3. Again, the Lie bracket is simply induced by $[\cdot\,{}_\lambda\,\cdot]\big|_{\lambda=0}$. The Lie algebra (3.4.3) was already well-known (it appears already in the original paper of Borcherds [Bo2]). What is new is its interpretation as a Zhu algebra.

Chapter 4

Chiralization of star products

As discussed in Chapter 2, vertex algebras can be seen as analogues of associative algebras. Moreover, there is a corresponding Poisson object called a Poisson vertex algebra. We aim to generalize the theory of star-products for Poisson algebras to Poisson vertex algebras. In particular, we are interested in computing explicit formulae for some chiral analogues of the Moyal-Weyl and Gutt star-products.

As we explain in this chapter, there are two possible candidates for the chiral analogue of a star-product. The "star-deformation" of a Poisson vertex algebra, introduced by Li in [Li1], is the most intuitive one. However, it is not compatible with the Zhu functor. For this reason, we introduce an alternative operation, that we call "chiral star-product", in Section 4.3. A chiral star-product is the deformation of a Poisson vertex algebra into an $\hbar$-vertex algebra. Because of this, it can be considered a deformation, with respect to the parameter $\hbar$, of a star-deformation.

In Section 4.4 we compute explicit formulae for a class of chiral star-products, that include the chiralized Moyal-Weyl and Gutt star-products. By putting $\hbar$ equal to 0, we also obtain explicit formulae for the corresponding star-deformations.

§ 4.1 | Star-products for Poisson algebras

We begin the chapter by recalling some basic facts and definitions about star-products for Poisson algebras. In this section, we follow the Einstein summation convention of summing over repeated indices in the appropriate range.

Let $(\mathcal{A}, 1, \{\cdot,\cdot\})$ be a unitary Poisson algebra.

Definition 4.1.1. A star-product on $\mathcal{A}$ is a bilinear operation

$$\star : \mathcal{A}[[\varepsilon]] \otimes \mathcal{A}[[\varepsilon]] \to \mathcal{A}[[\varepsilon]]$$

defined for $a, b \in \mathcal{A}$, and then extended to $\mathcal{A}[[\varepsilon]]$ by linearity, as

$$a \star b = \sum_{n \geq 0} \varepsilon^n B_n(a, b), \tag{4.1.1}$$

where $B_n : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ are bilinear operators that satisfy the following axioms:

(i) The product $\star$ is associative and unitary, with unit 1.

(ii) $B_0(a, b) = ab$, i.e. $\star$ is a deformation of the commutative product on $\mathcal{A}$.

(iii) $B_1(a, b) - B_1(b, a) = \{a, b\}$, i.e. the star commutator $a \star b - b \star a$ is a deformation of the Poisson bracket.

The star-product is called *strict* if the sum (4.1.1) is convergent for some non-zero values of ε.

Assume now that $\mathcal{A}$ is a positively graded Poisson algebra

$$\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n,$$

with $\mathcal{A}_0 = \mathbb{C}$ and Poisson bracket of degree $-i$ for some $i \geq 1$. That is, $\{\mathcal{A}_n, \mathcal{A}_m\} \subset \mathcal{A}_{n+m-i}$ for all $n, m \in \mathbb{N}$. It is natural to require the star-product to be a graded deformation of the product of $\mathcal{A}$. Namely, this means requiring that

$$B_k : \mathcal{A}_n \otimes \mathcal{A}_m \longrightarrow \mathcal{A}_{n+m-ki} \qquad \forall n, m, k \in \mathbb{N}. \tag{4.1.2}$$

Notice that, for graded star-products, the sum in (4.1.1) is always finite for degree reasons. So the star-product is strict and we can specialize ε to 1.

Remark 4.1.2. The convention of specializing ε to 1 instead of considering it a formal parameter, as done in this thesis, is not the most common. However, once we assume the grading conditions (4.1.2), this becomes a natural choice. In fact, there is no issue with the convergence of the series and the various quantization terms can be identified by the degree instead of by the power of ε. To go back, one can use a procedure similar to the Rees algebra:

$$a \star b := \sum_{n \geq 0} \varepsilon^n \pi_{\deg a + \deg b - n}(a \star b),$$

where π_i is the projection to the i-th degree component. For more references of the use of this convention, see the papers by Etingof et al. [ES5; EKRS].

Let $A = \bigcup_{n \geq 0} F_n A$ be a filtered associative algebra with $F_0 A = \mathbb{C}$. We assume that the commutator is of negative degree, that is $[F_n A, F_m A] \subset F_{n+m-i} A$. Then, the associated graded

$$\mathrm{gr}(A) = \bigoplus_{n \geq 0} F_n A / F_{n-1} A,$$

has a natural structure of graded Poisson algebra. For $a \in F_n A$ and $b \in F_m A$, the product is defined as

$$(a + F_{n-1}A)(b + F_{m-1}A) := (ab + F_{n+m-1}A).$$

It is commutative because the commutator is of negative degree. The Poisson bracket is

$$\{a + F_{n-1}A, b + F_{m-1}A\} = [a, b] + F_{n+m-i-1}A,$$

and it is easy to see that $\mathrm{gr}(A)$ satisfies the Poisson algebra axioms.

Definition 4.1.3. Let A be a filtered, associative algebra. We say that A is a quantization of $\mathcal{A}$ if $\mathrm{gr}\, A \cong \mathcal{A}$ as graded Poisson algebras.

Definition 4.1.4. Let $U = \bigcup_{n\geq 0} F_n U$ be a filtered vector space, and let $\mathcal{U} := \mathrm{gr}(U)$. A quantization map is a vector space isomorphism $\phi : \mathcal{U} \to U$, such that, for all $n \in \mathbb{N}$, $\phi(\mathcal{U}_n) \subset F_n U$ and $\pi_n \circ \phi = \mathrm{id}$, where π_n is the quotient projection

$$\pi_n : F_n U \to F_n U / F_{n-1} U \cong \mathcal{U}_n.$$

Remark 4.1.5. Let $A = \bigcup_{n\geq 0} F_n A$ be a filtered associative algebra with commutator of degree -1 and take $\mathcal{A} := \mathrm{gr}(A)$. If we have a quantization map $\phi : \mathcal{A} \to A$, then we can define a graded star-product as

$$a \star_\phi b := \phi^{-1}(\phi(a)\phi(b)).$$

In fact, $\star_\phi$ is associative by definition and it is unital because $\phi(1) = 1$. Let $a \in A_n$ and $b \in A_m$. Decomposing the star-product into its graded components gives

$$a \star_\phi b = C_0(a, b) + C_1(a, b) + \ldots,$$

with $C_i(a, b) = p_{n+m-i}(a \star_\phi b)$, where p_n is the projection to the n-th graded component.

Since ϕ is a quantization map, $a = \phi(a) + F_{n-1}A$ and $b = \phi(b) + F_{m-1}A$. Hence,

$$C_0(a, b) = p_{n+m}(a \star_\phi b) = \pi_{n+m}(\phi(a)\phi(b)) = \phi(a)\phi(b) + F_{n+m-1}A = ab.$$

Similarly,

$$C_1(a, b) - C_1(b, a) = p_{n+m-1}(a \star_\phi b - b \star_\phi a) = [\phi(a), \phi(b)] + F_{n+m-2} = \{a, b\}.$$

Indeed, all graded star-products can be realized this way. If $\star$ is a graded star-product, take $A := (\mathcal{A}, \star)$. The identity function is a quantization map $\mathcal{A} \to A$ and $\star_{\mathrm{id}} = \star$.

Remark 4.1.6. The construction in Remark 4.1.5 can be generalized to filtered associative algebras of arbitrary negative degrees. In those cases, we need to require some additional conditions, related to the grading, on the algebra A and the quantization map ϕ (see [ES5]). We restrict to the degree -1 case for simplicity, since all our examples can be led back to this case.

Example 4.1.7 (Moyal-Weyl star-product)**.** Let U be a symplectic vector space, with symplectic form ω. Let $\mathcal{S}(U)$ be the symmetric algebra, which is canonically a Poisson algebra with Poisson structure induced by ω. The Weyl algebra

$$W(U) = \mathcal{T}(U) \Big/ \langle u \otimes v - v \otimes u - \omega(u,v) \,|\, \forall u, v \in U \rangle$$

is well-known to be a quantization of $\mathcal{S}(U)$. Let $v_1, \dots, v_n$ be a basis of U. The map $\phi : \mathcal{S}(U) \to W(U)$, given by symmetrization,

$$v_{i_1} \cdots v_{i_k} \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} v_{i_{\sigma(1)}} \cdots v_{i_{\sigma(k)}}, \tag{4.1.3}$$

is a quantization map. Let $\pi \in \Lambda^2 U$ be the Poisson bivector $\pi = \pi^{ij} \partial_i \wedge \partial_j$ induced by ω. The star-product $\star_\phi$ has the following explicit formula:

$$a \star_\phi b = m(\exp(\pi/2)(a \otimes b)), \tag{4.1.4}$$

where m is the multiplication map $a \otimes b \mapsto ab$. For example, if $U = \mathbb{C}x \oplus \mathbb{C}y$, $\pi = \partial_x \wedge \partial_y$

$$a \star_\phi b = m \circ \exp\left(\frac{\partial_x \otimes \partial_y - \partial_y \otimes \partial_x}{2} \right)(a \otimes b).$$

Remark 4.1.8. The commutator on the Weyl algebra is of degree -2. One can check explicitly that the quantization map ϕ defined in (4.1.3) induces a star-product (it satisfies the $\mathbb{Z}/2\mathbb{Z}$-equivariant condition described in [ES5]). Alternatively, we can add a formal variable t, central and of degree 1, and homogenize the defining relations to

$$u \otimes v - v \otimes u - t\omega(u,v), \quad u, v \in U.$$

We extend $\phi : \mathcal{S}(U)[t] \to W(U)[t]$ by sending $t \mapsto t$ and obtain a star-product $\star_\phi$ on $\mathcal{S}(U)[t]$ as in Remark 4.1.5. Setting $t = 1$, this becomes the Moyal-Weyl star-product. We prefer the latter point of view, as it makes the connection between the Moyal-Weyl and Gutt star-product more explicit (see Remark 4.1.11).

Example 4.1.9 (Gutt star-product)**.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then the symmetric algebra $\mathcal{S}(\mathfrak{g})$ is a graded Poisson algebra with Poisson bracket induced by the Lie bracket on $\mathfrak{g}$. From the Poincaré–Birkhoff–Witt theorem, the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is a quantization of $\mathcal{S}(\mathfrak{g})$. Let $v^1, \dots, v^n$ be a basis of $\mathfrak{g}$. The map $\phi : \mathcal{S}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$, given by symmetrization as in (4.1.3), is a quantization map.

For all $x, y \in \mathfrak{g}$, let $\mathrm{BCH}(x,y)$ be the formal series given by the Baker-Campbell-

Hausdorff formula:

$$\begin{aligned}&\exp(x)\exp(y) = \exp(\mathrm{BCH}(x,y)) = \\ &= \exp\Big(x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [[x,y],y]) + \dots\Big),\end{aligned}$$

where the identity above holds in the ring of formal power series in the non-commutative variables x, y.

Define now the following symbol in the commuting variables $\underline{s} = (s_1, \dots, s_n)$ and $\underline{t} = (t_1, \dots, t_n)$:

$$D(\underline{s}, \underline{t}) := \frac{1}{2}s_i t_j [v^i, v^j] + \frac{1}{12}(s_i s_j t_k [v^i[v^j, v^k]] + s_i t_j t_k [[v^i, v^j], v^k]]) + \dots$$

so that, if $\underline{s}, \underline{t} \in \mathbb{C}^n$, $x = s_i v^i$ and $y = t_i v^i$,

$$D(\underline{s}, \underline{t}) = \mathrm{BCH}(x,y) - x - y.$$

The star-product $\star_\phi$ has the following explicit formulation. For $a, b \in \mathcal{S}(\mathfrak{g})$,

$$a \star_\phi b = m \circ \exp\Big(D(\overleftarrow{\underline{\partial}}, \overrightarrow{\underline{\partial}})\Big)(a \otimes b), \tag{4.1.5}$$

where $\overleftarrow{\underline{\partial}} = (\partial_{v_1} \otimes 1, \dots, \partial_{v_n} \otimes 1)$ and $\overrightarrow{\underline{\partial}} = (1 \otimes \partial_{v_1}, \dots, 1 \otimes \partial_{v_n})$.

Remark 4.1.10. The Gutt star-product is usually defined over finite-dimensional Lie algebras, but the construction and formula (4.1.5) work for infinite-dimensional Lie algebras too (see for example [ESW, Section 2.2]). Notice that all the sums appearing in the definition of $D(\underline{s}, \underline{t})$ are now infinite, but only a finite number of terms of $D(\overleftarrow{\underline{\partial}}, \overrightarrow{\underline{\partial}})$ applied to elements in $\mathcal{S}(\mathfrak{g})$ are non zero.

Remark 4.1.11. The Moyal-Weyl star-product can be seen as a special case of the Gutt star-product. In fact, consider the $2n+1$ dimensional Heisenberg Lie algebra $\mathfrak{h}_n$, with basis $x_1, \dots, x_n$, $y_1, \dots, y_n$, Z, and bracket

$$[x_i, y_j] = \delta_{i,j} Z, \qquad [x_i, x_j] = [y_i, y_j] = [Z, x_i] = [Z, y_i] = 0,$$

for all i, j. The quotient $\mathfrak{U}(\mathfrak{h}_n)/(Z-1)$ is isomorphic to the Weyl algebra $W(U)$, where U is a $2n$ dimensional vector space. In the Baker-Campbell-Hausdorff formula for $\mathfrak{h}_n$ only the first two terms may appear, so that

$$D(\overleftarrow{\underline{\partial}}, \overrightarrow{\underline{\partial}}) = Z \sum_{i=1}^{n} \frac{\partial_{x_i} \otimes \partial_{y_i} - \partial_{y_i} \otimes \partial_{x_i}}{2},$$

and, in the quotient, (4.1.5) becomes (4.1.4).

Another way of seeing this fact is to consider a symplectic vector space U as a non-linear Lie algebra, with a Lie bracket with coefficients in $\mathbb{C}$.

Remark 4.1.12. Note how both formulae (4.1.4) and (4.1.5) are of the form

$$\exp(\text{bidifferential operator}),$$

where the bidifferential operator only depends on the structure constants of the Poisson bracket.

§ 4.2 | Star-deformations for Poisson vertex algebras

The definition of star-deformations for Poisson vertex algebras was given by Li in [Li1, Definition 5.20], motivated by the analogy with Poisson algebras. In this definition, a star-deformation is a formal state-field correspondence $Y_\varepsilon(-,z)$ on a Poisson vertex algebra $\mathcal{V}$, such that the non-negative (n)-products give a deformation of the Poisson λ-bracket. We give here a different, equivalent definition, based on the integral λ-bracket formalism (see Definition 2.3.9).

Definition 4.2.1. Let $(\mathcal{V}, 1, \partial, \{\cdot_\lambda\cdot\})$ be a Poisson vertex algebra. Then a star-deformation of $\mathcal{V}$ is a bilinear operation

$$I_{\lambda,\star} : \mathcal{V}[[\varepsilon]] \otimes \mathcal{V}[[\varepsilon]] \to \mathcal{V}[[\varepsilon, \lambda]]$$

defined for $a, b \in \mathcal{V}$, and then extended to $\mathcal{V}[[\varepsilon]]$ by linearity, as

$$I_{\lambda,\star}(a, b) = \sum_{n\geq 0} \varepsilon^n I_{\lambda,n}(a, b), \tag{4.2.1}$$

where $I_{\lambda,n} : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}[\lambda]$ are bilinear operators that satisfy the following axioms:

(i) $(1, \partial, I_{\lambda,\star})$ induce a vertex algebra structure on $\mathcal{V}/\varepsilon^n\mathcal{V}$, for all $n \geq 1$.

(ii) $I_{0,0}(a, b) = ab$ and $\frac{d}{d\lambda} I_{\lambda,0}(a, b) = 0$ for all $a, b \in \mathcal{V}$.

(iii) $\frac{d}{d\lambda} I_{\lambda,1}(a, b) = \{a_\lambda b\}$ for all $a, b \in \mathcal{V}$.

The star-deformation is called *strict* if the sum (4.2.1) is convergent for some non-zero values of ε.

Remark 4.2.2. Due to convergence issues, a star-deformation $I_{\lambda,\star}$ will not define a vertex algebra structure on $\mathcal{V}[[\varepsilon]]$. In fact, even if $I_{\lambda,\star}(a, b) \in \mathcal{V}[[\varepsilon]][\lambda]$ for $a, b \in \mathcal{V}$,

$$I_{\lambda,\star}\left(\sum_{n\geq 0} a_n\varepsilon^n, b\right) = \sum_{n\geq 0} I_{\lambda,\star}(a_n, b)\varepsilon^n \in \mathcal{V}[[\varepsilon, \lambda]].$$

This explains the technicality in condition (i). In the terminology of [Li1], $I_{\lambda,\star}$ induces a ε-adic vertex algebra structure on $\mathcal{V}[[\varepsilon]]$.

Assume now that $\mathcal{V}$ is positively graded with Poisson λ-bracket of degree $-i$, for some

$i \geq 1$, that is

$$\mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}_n, \quad \mathcal{V}_n \cdot \mathcal{V}_m \subset \mathcal{V}_{n+m}, \quad \{\mathcal{V}_n {}_\lambda \mathcal{V}_m\} \subset \mathcal{V}_{n+m-i} \otimes \mathbb{C}[\lambda], \quad \forall m, n \in \mathbb{N}.$$

We again require that the star-deformation is a graded deformation of the Poisson vertex algebra structure on $\mathcal{V}$. Namely, this means imposing that

$$I_{\lambda,k} : \mathcal{V}_n \otimes \mathcal{V}_m \longrightarrow \mathcal{V}_{n+m-ki} \otimes \mathbb{C}[\lambda] \qquad \text{for all } n, m \text{ and } k.$$

Since the sums are finite, the star-deformation is strict and we can specialize ε to 1 and work on $\mathcal{V}$.

Definition 4.2.3. Let V be a vertex algebra with a good filtration $\{F_n V\}_{n \geq 0}$. Then $\mathrm{gr}(V)$ is a Poisson vertex algebra (see Section 2.4). We say that V is a quantization of $\mathcal{V}$ if $\mathrm{gr}\, V \cong \mathcal{V}$ as graded Poisson vertex algebras.

Remark 4.2.4. Let V be a filtered algebra with a good filtration, with λ-bracket of degree -1 and set $\mathcal{V} = \mathrm{gr}(V)$. From a quantization map $\phi : \mathcal{V} \to V$, such that $\phi(\partial a) = \partial \phi(a)$, for all $a \in \mathcal{V}$, we can construct a star-deformation in the following way. Extend $\phi : \mathcal{V}[\lambda] \to V[\lambda]$ by linearity, then define

$$I_{\lambda,\star}(a, b) := \phi^{-1} I_\lambda(\phi(a)\phi(b)).$$

On the other hand, if $I_{\lambda,\star}$ is a star-deformation on $\mathcal{V}$, then $(\mathcal{V}, 1, \partial, I_{\lambda,\star})$ with filtration induced by the grading is a quantization of $\mathcal{V}$, with the identity as quantization map. The proof is basically the same as the one for star-products (see Remark 4.1.5).

Problem. Compute explicit, closed formulae for star-deformations of Poisson vertex algebras, using quantization maps.

As we explain in the next section, a star-deformation is not the “correct” chiral analogue of a star-product. Nonetheless, explicit formulae for star-deformations have important applications. Vertex algebras have very complicated relations and computations, even with the λ-bracket, become complicated very quickly. A formula for a star-deformation allows one to perform vertex algebra computations in the setting of Poisson vertex algebras, which are much more tractable. From the physics point of view, it corresponds to constructing a (very special) quantum field theory inside the formalism of classical field theory.

§ 4.3 | Chiral star-products

Our aim is to compute explicit formulae for some chiral analogues of the Moyal-Weyl and Gutt star-products. First, we need to find a suitable definition of what a chiral star-product should be. Informally, chiralization should be thought of as the inverse of the Zhu functor. Let $\mathcal{A}$ be a Poisson algebra and $\mathcal{V}$ a Poisson vertex algebra, such that $\mathcal{A} = \mathrm{Zhu}(\mathcal{V})$. Let $\star$ be a star-product on $\mathcal{A}$, so $\star$ is a deformation of the Poisson structure on $\mathcal{A}$ such that $(\mathcal{A}, \star)$ is an associative algebra. Ideally, a chiral star-product would satisfy the following two properties:

1. It is a suitable deformation of the Poisson vertex algebra structure on $\mathcal{V}$.

2. Let $\hat{\star}$ denote the deformation of the commutative product on $\mathcal{V}$ given by the chiral star-product and let $p_Z : \mathcal{V} \to \mathcal{V}/\mathcal{J}_\hbar = \mathcal{A}$ be the quotient map. Then,

$$p_Z(a \hat{\star} b) = p_Z(a) \star p_Z(b), \quad \forall a, b \in \mathcal{V}.$$

A strict star-deformation satisfies the first condition. Unfortunately, a star-deformation in general does not satisfy the second condition. In fact, the star-deformation is in general not even well-defined in the Zhu quotient. This is because the space we quotient by is defined in terms of the $\hbar$-Poisson vertex algebra structure, and may not be directly compatible with the vertex algebra structure induced by the star-deformation (see Section 3.4). This suggests the following definition.

Definition 4.3.1. Let $(\mathcal{V}, 1, \partial, \{\cdot\,_\lambda\,\cdot\})$ be a Poisson vertex algebra and $\mathcal{A} = \mathrm{Zhu}(\mathcal{V})$. A chiral star-product is a bilinear operation

$$I_{\lambda,\hbar,\star} : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}[\lambda],$$

satisfying the following:

(i) $(\mathcal{V}, 1, \partial, I_{\lambda,\hbar,\star})$ defines an $\hbar$-vertex algebra on $\mathcal{V}$ (with the $\hbar$-sum bracket formalism, see Section 3.3.1), such that $I_{\lambda,\hbar=0,\star}$ is a star-deformation on $\mathcal{V}$.

(ii) The operation $I_{0,\hbar,\star}$ induces a star-product $\star$ on the Poisson Zhu algebra $\mathcal{A}$ by

$$p_Z(a) \star p_Z(b) := p_Z I_{0,\hbar,\star}(a, b),$$

such that $(\mathcal{A}, \star)$ is isomorphic to the associative Zhu algebra of the $\hbar$-vertex algebra $(\mathcal{V}, I_{\lambda,\hbar,\star})$.

Remark 4.3.2. Since $\hbar$-(Poisson) vertex algebras and (Poisson) vertex algebras structures on $\mathcal{V}$ are in bijection, a deformation of a Poisson vertex algebra induces a deformation of the corresponding $\hbar$-Poisson vertex algebra, and vice-versa. Condition (i) of Definition 4.3.1 can be restated as "$I_{\lambda,\hbar,\star}$ is a deformation of the $\hbar$-Poisson vertex algebra structure on $\mathcal{V}$, such that $(\mathcal{V}, 1, \partial, I_{\lambda,\hbar,\star})$ is an $\hbar$-vertex algebra".

Let V be a vertex algebra with a good filtration, $\mathcal{V}$ a Poisson vertex algebra, A an associative algebra, and $\mathcal{A}$ a Poisson algebra that fit in the following diagram:

$$\begin{array}{ccc} \mathcal{V} & \xleftarrow{\text{gr}} & V \\ {\scriptstyle p_Z}\downarrow & & \downarrow{\scriptstyle q_Z} \\ \mathcal{A} & \xleftarrow{\text{gr}} & A \end{array} \tag{4.3.1}$$

where p_Z, q_Z denote the Zhu quotient maps. Let $\mathcal{J}_\hbar$ and $J_\hbar$ be the respective kernels and let $\tau_n : F_nV \to F_nV/F_{n-1}V$ and $\pi_n : F_nA \to F_nA/F_{n-1}A$ denote the projection maps to the associated graded, for all $n \in \mathbb{N}$. Then

$$\pi_n \circ q_Z = p_Z \circ \tau_n, \quad \forall n \in \mathbb{N}. \tag{4.3.2}$$

Proposition 4.3.3. *Assume that the λ-bracket on V is of degree -1. Let ϕ be a quantization map $\mathcal{V} \to V$, such that $\phi \circ \partial = \partial \circ \phi$ and $\phi(\mathcal{J}_\hbar) = J_\hbar$. Then ϕ induces a quantization map $\hat{\phi} : \mathcal{A} \to A$, defined by*

$$\hat{\phi}(a + \mathcal{J}_\hbar) = \phi(a) + J_\hbar.$$

Moreover, the operation $I_{\lambda,\hbar,\star}$ defined by

$$I_{\lambda,\hbar,\star}(a, b) := \phi^{-1}\left(I_{\lambda,\hbar}(\phi(a), \phi(b))\right), \quad \forall a, b \in \mathcal{V},$$

is a chiral star-product, where $I_{\lambda,\hbar}$ is the sum $\hbar$-bracket of the $\hbar$-vertex algebra associated to V. It is the chiralization of the star-product $\star_{\hat{\phi}}$.

Proof. Since $\mathrm{Zhu}(\mathcal{V}) = \mathcal{V}/\mathcal{J}_\hbar$ and $\mathrm{Zhu}(V) = V/J_\hbar$, the map $\hat{\phi}$ is a well defined vector space isomorphism. We are left to prove that $\pi_n \circ \hat{\phi} = \mathrm{id}$ for all $n \in \mathbb{N}$. Let a be an homogeneous element of degree n in $\mathcal{A}$. Then $a = p_Z(x)$, for some $x \in \mathcal{V}$. By (4.3.2), $\hat{\phi}$ inherits the quantization map property from ϕ:

$$\pi_n \circ \hat{\phi}(a) = \pi_n \circ q_z(\phi(x)) = p_Z(\tau_n \circ \phi(x)) = p_Z(x) = a.$$

Since ϕ is an isomorphism of vector spaces commuting with ∂, $(\mathcal{V}, 1, \partial, I_{\lambda,\hbar,\star})$ defines an $\hbar$-vertex algebra structure on $\mathcal{V}$. Moreover, $I_{\lambda,0,\star}$ is a star-deformation on $\mathcal{V}$ by Remark 4.2.4. Now, let $\star_{\hat{\phi}}$ be the star-product on $\mathcal{A}$ defined by $\hat{\phi}$. From a direct computation:

$$\begin{aligned} p_Z(I_{0,\hbar,\star}(a,b)) &= p_Z[\phi^{-1}(\phi(a) *_\hbar \phi(b))] = \hat{\phi}^{-1} \circ q_Z(\phi(a) *_\hbar \phi(b)) \\ &= \hat{\phi}^{-1}[q_Z(\phi(a)) \cdot q_Z(\phi(b))] = \hat{\phi}^{-1}[\hat{\phi}(p_Z(a)) \cdot \hat{\phi}(p_Z(b))] \\ &= p_Z(a) \star_{\hat{\phi}} p_Z(b). \end{aligned}$$

Thus $I_{\lambda,\hbar,\star}$ is a chiral star-product. □

We now have a suitable definition of chiral star-products and a way to construct them. We would like to construct chiral star-products that are the chiralization of known star-products, in particular the Moyal-Weyl and Gutt star-products. We put ourselves in the following setting.

Let R be a sub-linear vertex Lie algebra. Assume that $R = \mathbb{C}[\partial] \otimes \mathfrak{g}$ is a free $\mathbb{C}[\partial]$-module, generated by a finite dimensional vector space $\mathfrak{g}$. Consider the universal enveloping vertex algebra $V(R)$ (see Theorem 2.3.4). Recall the definition of the Lie Zhu algebra of a vertex Lie algebra R as in (3.4.3):

$$\operatorname{Zhu}(R) = R/\partial R.$$

This is naturally a Lie algebra, with bracket induced by $[\cdot\,{}_\lambda\,\cdot]\big|_{\lambda=0}$.

Theorem 4.3.4. *Let $R = \mathbb{C}[\partial] \otimes \mathfrak{g}$ be as above. Then*

$$\operatorname{Zhu}(V(R)) \cong \mathfrak{U}(\mathfrak{g}).$$

Proof. This is essentially [DK2, Corollary 3.26], rewritten by considering the change of variables Zhu algebra construction. In particular, we consider the Lie Zhu algebra $\operatorname{Zhu}(R)$ defined as in (3.4.3), while De Sole and Kac consider the one defined in (3.4.2). □

Fix $\overline{\mathcal{B}} = \{u_1, \ldots, u_n\}$ a basis of $\mathfrak{g}$ and extend it to a basis of R by setting $\mathcal{B} = \{u_\alpha\}_{\alpha \in I}$, where $I = \{1, \ldots, n\} \times \mathbb{N}$ and $u_{(i,k)} := \partial^k u_i$. Order $\mathcal{B}$ by lexicographic order. Since $V(R)$ is PBW generated by R, ordered monomials in elements of $\mathcal{B}$ with respect to the normally ordered product form a basis of $V(R)$. Consider the good filtration $\{F_n V(R)\}_{n \geq 0}$ induced by the grade of the PBW monomials.

Remark 4.3.5. As proved in [DK2, Lemma 3.18], monomials with respect to the $*_\hbar$ product give an alternative PBW basis of $V(R)$. Since $a *_\hbar b$ and $:ab:$ differ only by non-negative (n)-products, which are in lower terms of the filtration, the filtration induced by this second PBW basis coincide with $\{F_n V(R)\}_{n \geq 0}$.

Consider $\mathcal{S}(R)$, the symmetric algebra on R, with its natural structure of Poisson vertex algebra, induced by R. The Poisson vertex algebra $\mathcal{S}(R)$ can also be thought of as the algebra of differential polynomials on $\mathfrak{g}$:

$$\mathcal{S}(R) \cong \mathbb{C}[\partial^k u_i]_{i=1,\ldots,n}^{k \geq 0} = \mathbb{C}[u_\alpha]_{\alpha \in I},$$

with Poisson λ-bracket

$$\{\partial^k u_i\,{}_\lambda\,\partial^m u_j\} = (-\lambda)^k (\lambda + \partial)^m [u_i\,{}_\lambda\,u_j].$$

The Zhu algebra associated to $\mathcal{S}(R)$ is

$$\operatorname{Zhu}(\mathcal{S}(R)) = \mathcal{S}(R)/(\mathcal{S}(R)\partial\mathcal{S}(R)) \cong \mathcal{S}(\operatorname{Zhu}(R)) = \mathcal{S}(\mathfrak{g}).$$

Thus, diagram (4.3.1) in this new setting becomes:

$$\begin{array}{ccc} \mathcal{S}(R) & \xleftarrow{\text{gr}} & V(R) \\ {\scriptstyle p_Z}\downarrow & & \downarrow{\scriptstyle q_Z} \\ \mathcal{S}(\mathfrak{g}) & \xleftarrow[\text{gr}]{} & \mathfrak{U}(\mathfrak{g}) \end{array}$$

Theorem 4.3.6. *Any quantization map* $\hat{\phi} : \mathcal{S}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$ *can be lifted to a quantization map* $\phi : \mathcal{S}(R) \to V(R)$ *such that* $\hat{\phi}(a + \mathcal{J}_\hbar) = \phi(a) + J_\hbar$. *The chiral star-product defined by* ϕ *as in Proposition 4.3.3 is the chiralization of the star-product* $\star_{\hat{\phi}}$, *defined as in Remark 4.1.5.*

To prove the theorem, we need a technical result.

Lemma 4.3.7.

(a) $B_1 = \{u_{i_1} \dots (\partial u_{i_k}) \dots u_{i_n} \,|\, u_{i_j} \in \mathcal{B}, n \geq 1, i_1 < i_2 < \dots < i_n\}$ *is a basis of* $\mathcal{J}_\hbar$.

(b) $B_2 = \{u_{i_1} *_\hbar \dots *_\hbar (\partial u_{i_k}) *_\hbar \dots *_\hbar u_{i_n} \,|\, u_{i_j} \in \mathcal{B}, n \geq 1, i_1 < i_2 < \dots < i_n\}$ *is a basis of* $J_\hbar$.

In particular, as vector spaces, $J_\hbar \cong \mathcal{J}_\hbar$.

Proof. A generic element in $\mathcal{J}_\hbar$ is of the form $(\partial a)b$, with $a, b \in \mathcal{S}(R)$. Writing a, b as polynomials in elements of $\mathcal{B}$ and since ∂ is a derivation, $(\partial a)b$ is spanned by elements in B_1. Since it is also a linearly independent set, B_1 is a basis of $\mathcal{J}_\hbar$.

For part (b), we first show that $J_\hbar \subset \text{span}_{\mathbb{C}}(B_2)$. Let $x \in J_\hbar$. We proceed by induction on n, the lowest natural number such that $x \in F_n V(R)$. We can assume that $x = (\partial a) *_\hbar b$ for some $a, b \in V(R)$. Write a, b in the PBW basis with respect to the product $*_\hbar$. Since ∂ is a derivation of $*_\hbar$, x is a linear combination of elements

$$x_\alpha = (u_{i_1} *_\hbar \dots *_\hbar (\partial u_{i_k}) *_\hbar \dots *_\hbar u_{i_m}) *_\hbar (u_{j_1} *_\hbar \dots *_\hbar u_{j_t}),$$

with $u_{i_s}, u_{j_q} \in \mathcal{B}$, $i_1 < \dots < i_m$, $j_1 < \dots < j_t$, and $m + t \leq n$. By the quasi-associativity axiom of $*_\hbar$ (3.3.6), we can rearrange the parenthesis so that $x_\alpha \in B_2$, modulo elements in $F_{n-1}V(R)$. By Theorem 3.4.3, all the associators are elements of $J_\hbar$. So $x_\alpha \in B_2$ modulo some terms in $F_{n-1}V(R) \cap J_\hbar$. By induction hypothesis, we can write x as a linear combination of elements in B_2.

Since it is a subset of the PBW basis, B_2 is a linearly independent set. We are left to prove that $B_2 \subset J_\hbar$. Let $B_3 := \{a_1 *_\hbar \dots *_\hbar (\partial a_k) *_\hbar \dots *_\hbar a_n | a_i \in R, n \geq 1\}$. Clearly, $B_2 \subset B_3$. We show that $B_3 \subset J_\hbar$. Take $y = a_1 *_\hbar \dots *_\hbar (\partial a_k) *_\hbar \dots *_\hbar a_n \in B_3$. We proceed by induction on n. Using equation (3.3.9) multiple times, we can move (∂a_k)

to the first position on the left, modulo terms of the form

$$(\dagger)\ a_1 *_\hbar \cdots *_\hbar a_{l-1} *_\hbar (a_l *_\hbar (\partial a_k) - (\partial a_k) *_\hbar a_l) *_\hbar a_{l+1} *_\hbar \cdots *_\hbar a_{k-1} *_\hbar a_{k+1} *_\hbar \cdots *_\hbar a_n.$$

By the proof of Lemma 3.4.1, and since the commutator goes down in the filtration,

$$a_l *_\hbar (\partial a_k) - (\partial a_k) *_\hbar a_l \in J_\hbar \cap F_1 V(R) = \partial R.$$

Hence $(\dagger)$ is an element in $B_3 \cap F_{n-1}V(R)$. By induction, it follows that $y \in J_\hbar$. □

Proof (of Theorem 4.3.6). Recall that $R = \mathbb{C}[\partial] \otimes \mathfrak{g}$, so, as vector spaces,

$$\mathcal{S}(R) \cong V(R) = \mathcal{S}(\mathfrak{g}) \oplus \partial \mathcal{S}(R), \tag{4.3.3}$$

where $\partial\mathcal{S}(R)$ is the span of the ordered monomials in $\mathcal{B}$, where at least one factor is of the form ∂u_i, for $u_i \in \mathcal{B}$. By Lemma 4.3.7, $\partial\mathcal{S}(R) = \mathcal{J}_\hbar \cong J_\hbar$, as vector spaces. We extend $\hat{\phi}$ to an isomorphism of vector spaces $\phi : \mathcal{S}(R) \to V(R)$ using decomposition (4.3.3) and by identifying the bases of $J_\hbar$ and $\mathcal{J}_\hbar$. By Remark 4.3.5, the PBW basis of $V(R)$ with respect to the $*_\hbar$-product induces the good filtration. Thus ϕ is a quantization map.

We now want to apply Proposition 4.3.3. First, we need to make sure that the λ-bracket is of degree -1 (recall that R is a sub-linear vertex Lie algebra, so this is not guaranteed). Consider a central extension of $\mathfrak{g}$ by a new variable t. Let $R' = \mathbb{C}[\partial] \otimes (\mathfrak{g} \oplus \mathbb{C}t)$. It is a vertex Lie algebra, with λ-bracket obtained by homogenizing the λ-bracket of R. The vertex algebra $V(R')$ has a good filtration and λ-bracket of degree -1. Extend $\phi : \mathcal{S}(R') \to V(R')$ by sending $t \mapsto t$. We are now in the hypothesis of Proposition 4.3.3, so ϕ induces a chiral star-product on $\mathcal{S}(R')$. To get a chiral star-product on $\mathcal{S}(R)$, it is sufficient to put $t = 1$. □

Example 4.3.8. Let $\hat{\phi} : \mathcal{S}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$ be the symmetrization map

$$v_{i_1} \cdots v_{i_k} \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} v_{i_{\sigma(1)}} \cdots v_{i_{\sigma(k)}}, \quad v_{i_j} \in \bar{\mathcal{B}}.$$

Its chiralization is $\phi : \mathcal{S}(R) \to V(R)$ given by

$$u_{i_1} \ldots u_{i_n} \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} u_{i_{\sigma(1)}} *_\hbar \cdots *_\hbar u_{\sigma(i_n)}, \quad u_{i_j} \in \mathcal{B}. \tag{4.3.4}$$

Example 4.3.9. Both the Moyal-Weyl and Gutt star-products are constructed using the symmetrization map $\hat{\phi}$, where $\mathfrak{g}$ is either a proper, finite-dimensional Lie algebra (Gutt) or a symplectic vector space (Moyal-Weyl), that can be thought of as a sub-linear Lie algebra with coefficients in $\mathbb{C}$. Their chiralizations are then induced by the quantization map ϕ (4.3.4).

§ 4.4 | Explicit formulae for chiralized star-products

We keep the same notation and conventions of Section 4.3. In this section we compute an explicit formula for the chiral star-product induced by the quantization map ϕ (4.3.4).

Remark 4.4.1. The map $\phi : \mathcal{S}(R) \to V(R)$ can be written as the following composition:

$$\mathcal{S}(R) \xrightarrow{\gamma} \mathfrak{U}(R_L) \xrightarrow{\psi} V(R), \qquad \underbrace{}_{\phi} \tag{4.4.1}$$

where

- R_L is the Lie algebra associated to R, with Lie bracket given by

$$[a, b] := \textstyle\sum_{-\partial-\hbar}^{0} [a\,{}_x\, b]_\hbar \delta x,$$

  and $\mathfrak{U}(R_L)$ is its universal enveloping algebra;
- γ is the symmetrization map $\mathcal{S}(R) \to \mathfrak{U}(R_L)$ given by (4.1.3);
- ψ is the map defined inductively by $\psi(a) := a$, for all $a \in R$ and $\psi(aB) := a *_\hbar B$, for all $a \in R$ and for all $B \in \mathfrak{U}(R_L)$.

Both γ and ψ are vector spaces isomorphisms; we are thus performing the quantization $\mathcal{S}(R) \to V(R)$ in two steps: the first is a non-commutative deformation, the second is a non-associative deformation.

Let us first consider the non-associative deformation induced by ψ. Define

$$\hat{I}_{\lambda,\hbar} : \mathfrak{U}(R_L) \otimes \mathfrak{U}(R_L) \to \mathfrak{U}(R_L)[\lambda]$$

as, for $a, b \in \mathfrak{U}(R_L)$,

$$\hat{I}_{\lambda,\hbar}(a, b) := \psi^{-1}(I_{\lambda,\hbar}(\psi(a), \psi(b))).$$

Define the following operators:

(a) $L_a^\lambda : \mathfrak{U}(R_L) \to \mathfrak{U}(R_L)[\lambda]$, for every $a \in R_L$, is

$$L_a^\lambda(x) := \psi^{-1}\left(\textstyle\sum_{-\partial-\hbar}^{\lambda} [\psi(x)\,{}_z\, a]_\hbar \delta z\right);$$

(b) let $x \in \mathfrak{U}(R_L)$ and $a = a_1 \otimes \cdots \otimes a_n$, with $a_i \in R$ for all i; then $L_\lambda : \mathfrak{U}(R_L) \otimes \mathcal{T}(R) \to \mathfrak{U}(R_L)[\lambda]$ is defined as

$$L_\lambda(x, a) := \sum_{k=0}^{n} \sum_{1 \le i_1 < \cdots < i_k \le n} (a_1 \cdots a_n)_{i_1,\ldots,i_k} L_{a_{i_k}}^\lambda \circ \cdots \circ L_{a_{i_1}}^\lambda(x), \tag{4.4.2}$$

where the subscript $i_1, \dots, i_k$ means that the terms $a_{i_1}, \dots, a_{i_k}$ are removed from the product.

Proposition 4.4.2. *The operator L_λ induces a well-defined operator (that we still denote by L_λ)*

$$L_\lambda : \mathfrak{U}(R_L) \otimes \mathfrak{U}(R_L) \to \mathfrak{U}(R_L)[\lambda].$$

Moreover, for all $a, b \in \mathfrak{U}(R_L)$,

$$\hat{I}_{\lambda,\hbar}(a, b) = L_\lambda(a, b).$$

Proof. We want to prove that L_λ satisfies the following recursive relations, for all $x \in \mathfrak{U}(R_L)$, $a_0 \in R$ and $B \in \mathcal{T}(R)$:

(i) $L_\lambda(x, 1) = x$;

(ii) $L_\lambda(x, (a_0 \otimes B)) = a_0 L_\lambda(x, B) + L_\lambda(L^\lambda_{a_0}(x), B)$.

The base case is true by definition. Let now $x \in \mathfrak{U}(R_L)$, $a_0 \in R$ and $B \in \mathcal{T}(R)$, with $B = a_1 \otimes \cdots \otimes a_n$. Then

$$L_\lambda(x, a_0 \otimes B) = \sum_{k=0}^{n+1} \sum_{0 \le i_1 < \cdots < i_k \le n} (a_0 a_1 \cdots a_n)_{i_1,\dots,i_k} L^\lambda_{a_{i_k}} \dots L^\lambda_{a_{i_1}}(x).$$

We can split the inner sum into the two cases $i_1 \neq 0$ and $i_1 = 0$. In the first case, we can take a_0 out of the sum, getting $a_0 L_\lambda(x, B)$. In the case $i_1 = 0$ we have instead

$$\sum_{k=1}^{n+1} \sum_{1 \le i_2 < \cdots < i_k \le n} (a_1 \cdots a_n)_{i_2,\dots,i_k} L^\lambda_{a_{i_k}} \dots L^\lambda_{a_{i_2}}(L^\lambda_{a_0}(x)) = L_\lambda(L^\lambda_{a_0}(x), B).$$

On the other hand, from Lemma 3.3.18, for all $x, B \in \mathfrak{U}(R_L)$ and $a \in R$,

(i) $\hat{I}_{\lambda,\hbar}(x, 1) = x$,

(ii) $\hat{I}_{\lambda,\hbar}(x, aB) = a\hat{I}_{\lambda,\hbar}(x, B) + \hat{I}_{\lambda,\hbar}(L^\lambda_a(x), B)$.

It follows that $L_\lambda(x, a_1 \otimes \cdots \otimes a_n) = \hat{I}_{\lambda,\hbar}(x, a_1 \cdots a_n)$, for all $x \in \mathfrak{U}(R_L)$, $a_i \in R$. Since L_λ depends only on the projection of the second factor to $\mathfrak{U}(R_L)$, it induces a well-defined operator on $\mathfrak{U}(R_L) \otimes \mathfrak{U}(R_L)$. □

Corollary 4.4.3. *For all $a, b \in R$,*

$$L^\lambda_b L^\lambda_a - L^\lambda_a L^\lambda_b = L^\lambda_{[a,b]}.$$

In particular, $L^\lambda_{(-)}$ defines an action of the opposite Lie algebra R_L^{op} on $\mathfrak{U}(R_L)$.

Proof. Let $a, b \in R$ and $x \in \mathfrak{U}(R_L)$. By direct computation:

$$\begin{aligned} 0 = L_\lambda(x, ab - ba - [a,b]) &= abx + aL_b^\lambda(x) + bL_a^\lambda(x) + L_b^\lambda(x)L_a^\lambda(x) \\ &- bax - bL_a^\lambda(x) - aL_b^\lambda(x) - L_a^\lambda(x)L_b^\lambda(x) \\ &- [a,b]x - L_{[a,b]}^\lambda \\ &= L_b^\lambda L_a^\lambda - L_a^\lambda L_b^\lambda - L_{[a,b]}^\lambda. \end{aligned}$$

□

For what follows, it is convenient to rewrite formula (4.4.2) in a different way. We embed $\mathcal{T}(R)$ in $\mathcal{T}(R \oplus \mathbb{C})$, and define operators $\frac{\delta}{\delta i} : \mathcal{T}(R \oplus \mathbb{C}) \to \mathcal{T}(R \oplus \mathbb{C})$, as

$$\frac{\delta}{\delta i}(a_1 \otimes \cdots \otimes a_n) = \begin{cases} 0 & \text{if } i > n \\ a_1 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n & \text{if } a_i \in R \\ 0 & \text{if } a_i \in \mathbb{C} \end{cases}$$

It is easy to check that, for all $a, x \in \mathfrak{U}(R_L)$,

$$L_\lambda(x, a) = \sum_{k=0}^{n} \sum_{1 \le i_1 < \cdots < i_k \le n} \frac{\delta^k a}{\delta i_1 \ldots \delta i_k} L_{a_{i_k}}^\lambda \circ \cdots \circ L_{a_{i_1}}^\lambda(x). \tag{4.4.3}$$

Remark 4.4.4. There is an abuse of notation in formula (4.4.3), because the operators $\frac{\delta}{\delta i}$ are defined on the tensor algebra. What we mean is "take a lift of a to the tensor algebra, apply $\frac{\delta}{\delta i}$ and project back to $\mathfrak{U}(R_L)$". By Proposition 4.4.2, we know that L_λ does not depend on the choice of the lift.

Notice that $(\frac{\delta}{\delta i})^n = 0$ for all i and $n \ge 2$. In particular,

$$\exp\left(\sum_{i=1}^{n} \frac{\delta}{\delta i}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\delta}{\delta 1} + \cdots + \frac{\delta}{\delta n}\right)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{1 \le i_1 < \cdots < i_k \le n} k! \frac{\delta^k}{\delta i_1 \cdots \delta i_k}.$$

If the operators $L_{a_i}^\lambda$ were commutative, (4.4.3) could be rewritten in exponential form.

Notation Consider the free algebra generated by an ordered set of elements $\{a_i\}_{i \in I}$. The "normal order" of a monomial $a = a_{i_1} \cdots a_{i_k}$ is

$$:a: = :a_{i_1} \cdots a_{i_k}: = a_{i_{\sigma(1)}} \cdots a_{i_{\sigma(k)}},$$

where σ is the permutation such that $i_{\sigma(1)} \ge i_{\sigma(2)} \ge \cdots \ge i_{\sigma(k)}$. On linear combinations of monomials, the normal order is extended by linearity. We also use the notation $:\exp:$, which means that the powers in the expansion of the exponential are normally ordered. That is, for any P in the free algebra,

$$:\exp:(P) = 1 + :P: + \frac{1}{2}:PP: + \frac{1}{6}:PPP: + \cdots$$

If we consider the ordered set $\{\frac{\delta}{\delta i}\otimes L_{a_i}\}_{i\geq 1}$, then (4.4.3) becomes

$$L_\lambda(x,a) = m\circ :\exp:\left(\sum_{i\geq 1}\frac{\delta}{\delta i}\otimes L^\lambda_{a_i}\right)(a\otimes x),$$

where m is the associative multiplication.

Proposition 4.4.5. *Let $a\in\mathfrak{U}(R_L)$ and suppose that there are distinct elements $a_1,\ldots,a_s\in R$ such that $a=\pi(a_1^{\otimes n_1}\otimes\cdots\otimes a_s^{\otimes n_s})$, for some $n_1,\ldots,n_s\geq 1$. Then, for all $x\in\mathfrak{U}(R_L)$,*

$$\begin{aligned}L_\lambda(x,a) &= \sum_{k=0}^{n}\frac{1}{k!}\sum_{l_1+\cdots+l_s=k}\binom{k}{l_1\cdots l_s}\frac{\partial^k a}{\partial^{l_1}a_1\cdots\partial^{l_s}a_s}(L^\lambda_{a_s})^{l_s}\cdots(L^\lambda_{a_1})^{l_1}(x)\\ &= m\circ :\exp:\left(\sum_{i=1}^{s}\frac{\partial}{\partial a_i}\otimes L^\lambda_{a_i}\right)(a\otimes x).\end{aligned}\tag{4.4.4}$$

where $n=n_1+\cdots+n_s$ and the normal order is given by $1<2<\cdots<s$.

Again, there is a slight abuse of notation in (4.4.4), as the derivatives are computed on the lift $a_1^{\otimes n_1}\otimes\cdots\otimes a_s^{\otimes n_s}$ of a to the tensor algebra.

Proof. Notice that, by Leibniz rule,

$$\frac{\partial}{\partial a_1}(a_1^{\otimes n_1}\otimes\cdots\otimes a_s^{\otimes n_s}) = \left(\frac{\delta}{\delta 1}+\frac{\delta}{\delta 2}+\cdots+\frac{\delta}{\delta n_1}\right)(a_1^{\otimes n_1}\otimes\cdots\otimes a_s^{\otimes n_s}),$$

and similarly for all the other partial derivatives. Hence,

$$\sum_{i\geq 1}\frac{\delta}{\delta i}\otimes L^\lambda_{a_i} = \sum_{j=1}^{s}\frac{\partial}{\partial a_i}\otimes L^\lambda_{a_i}.$$

This implies that

$$\begin{aligned}L_\lambda(x,a) &= m\circ :\exp:\left(\sum_{i\geq 1}\frac{\delta}{\delta i}\otimes L^\lambda_{a_i}\right)(a\otimes x)\\ &= m\circ :\exp:\left(\sum_{i=1}^{s}\frac{\partial}{\partial a_i}\otimes L^\lambda_{a_i}\right)(a\otimes x).\end{aligned}$$

Notice that this is only true because we considered a lift of the form $a_1^{\otimes n_1}\otimes\cdots\otimes a_s^{\otimes n_s}$, with a_i distinct. This ensures that the two normal orders are compatible with each others. □

Remark 4.4.6. If we take the convention of writing b in the PBW basis, formula

(4.4.4) becomes, for $a, b \in \mathfrak{U}(R_L)$,

$$\begin{aligned} L_\lambda(a,b) &= m \circ :\exp\colon \left(\sum_{i\in\mathcal{I}} \partial_{u_i} \otimes L_i^\lambda\right)(b\otimes a) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{i_1<\dots<i_s\in\mathcal{I} \\ l_{i_1}+\dots+l_{i_s}=k}} \binom{k}{l_{i_1}\ \cdots\ l_{i_s}} \frac{\partial^k b}{\partial^{l_{i_1}}u_{i_1}\dots\partial^{l_{i_s}}u_{i_s}} (L_{i_k}^\lambda)^{l_{i_k}}\dots(L_{i_1}^\lambda)^{l_{i_1}}(a), \end{aligned} \tag{4.4.5}$$

where $L_i^\lambda := L_{u_i}^\lambda$. When applied to $b\otimes a$, only finite terms of the infinite sum inside the exponential are nonzero.

If we compute $\hat{I}_{\lambda,\hbar}(a,b) = L_\lambda(a,b)$ for $a, b \in \mathfrak{U}(R_L)$ using (4.4.5), we obtain an expression that depends on $L_i^\lambda(a)$, with $i \in \mathcal{I}$. But notice that, by definition of the $\hbar$-sum bracket and skew-symmetry,

$$\begin{aligned} L_i^\lambda(x) &= \hat{I}_{\lambda,\hbar}(x,u_i) - \hat{I}_{-\partial-\hbar,\hbar}(x,u_i) \\ &= \hat{I}_{-\lambda-\partial-\hbar,\hbar}(u_i,x) - \hat{I}_0(u_i,x) = \hat{I}_{-\lambda-\partial-\hbar,\hbar}(u_i,x) - u_i x. \end{aligned} \tag{4.4.6}$$

Computing $\hat{I}_{-\lambda-\partial-\hbar,\hbar}(u_i,x)$ explicitly, we get

$$\hat{I}_{-\lambda-\partial-\hbar,\hbar}(u_i,x) = m \circ :\exp\colon \left(\sum_{j\in\mathcal{I}} \partial_{u_j} \otimes L_j^{-\lambda-\partial-\hbar}\right)(x\otimes u_i),$$

which only depends on the $\hbar$-brackets of elements of $\mathcal{B}$:

$$L_j^{-\lambda-\partial-\hbar}(u_i) = \textstyle\sum_{-\partial-\hbar}^{-\lambda-\partial-\hbar} [u_j\ {}_x\ u_i]_\hbar \delta x.$$

Putting everything together, we obtain the following formula:

$$\begin{aligned} \hat{I}_{\lambda,\hbar}(a,b) &= m \circ :\exp\colon \left(\sum_{i\in\mathcal{I}} \partial_{u_i} \otimes L_i^\lambda\right)(b\otimes a) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{i_1<\dots<i_s\in\mathcal{I} \\ l_{i_1}+\dots+l_{i_s}=k}} \binom{k}{l_{i_1}\ \cdots\ l_{i_s}} \frac{\partial^k b}{\partial^{l_{i_1}}u_{i_1}\dots\partial^{l_{i_s}}u_{i_s}} (L_{i_k}^\lambda)^{l_{i_k}}\dots(L_{i_1}^\lambda)^{l_{i_1}}(a), \end{aligned} \tag{4.4.7}$$

where every $L_{i_1}^\lambda$ is a (rather complicated) differential operator, defined by (4.4.6). So $\hat{I}_{\lambda,\hbar}(a,b)$ is a "normally ordered exponential" of a bidifferential operator, with coefficients that depend only on the $\hbar$-bracket within the basis elements of R.

Definition 4.4.7. A vertex algebra V is a "free-field" vertex algebra if $V = V(R)$, and the λ-bracket on R takes values in $\mathbb{C}[\lambda]$. Many algebras fall into this class, for example, the $\beta\gamma$-systems and free boson vertex algebras (see Example 2.3.7).

Proposition 4.4.8. *If $V(R)$ is a free-field vertex algebra, formula (4.4.7) reduces to*

$$\hat{I}_{\lambda,\hbar}(a,b) = m\circ\sigma\circ\exp\left(\sum_{i\in\mathcal{I}} L_i^\lambda \otimes \partial_{u_i}\right)(a\otimes b), \tag{4.4.8}$$

where σ is the operator $x \otimes y \mapsto y \otimes x$, and

$$L_i^\lambda(a) = \sum_{j\in\mathcal{I}} \sum\nolimits_{-\partial-\hbar}^{\lambda} \frac{\partial a}{\partial u_j} [u_j \,{}_{x+\partial^{(1)}}\, u_i]_\hbar \, \delta x.$$

Proof. Note that $[a,b] = \sum_{-\partial-\hbar}^{0}[a\,{}_\lambda\, b]_\hbar \delta\lambda = \sum_{-\hbar}^{0}[a\,{}_\lambda\, b]_\hbar\delta\lambda \in \mathbb{C}[\lambda]$, so $L_{[a,b]}^\lambda = 0$. From Corollary 4.4.3, the operators L_a^λ, for $a \in R$, commute with each other. We can thus rewrite (4.4.7) as

$$\hat{I}_{\lambda,\hbar}(a,b) = m \circ \sigma \circ \exp\Big(\sum_{i\in\mathcal{I}} L_i^\lambda \otimes \partial_{u_i}\Big)(a\otimes b).$$

By the left Wick formula (3.3.8), for all $x, y \in R$ and $Y \in V(R)$,

$$[x *_\hbar Y\,{}_\lambda\, y]_\hbar = x[Y\,{}_{\lambda+\partial^{(1)}}\, y]_\hbar + Y[x\,{}_{\lambda+\partial^{(1)}}\, y]_\hbar + \sum\nolimits_0^\lambda [Y\,{}_\mu\,[x\,{}_{\lambda-\mu-\hbar}\, y]_\hbar]_\hbar \delta\mu.$$

The definite sum is 0 because $[x\,{}_\lambda\, y]_\hbar \in \mathbb{C}[\lambda]$. Thus the $\hbar$-bracket $[x *_\hbar Y\,{}_\lambda\, y]_\hbar$ can be expanded by the left Leibniz rule. This implies that

$$L_i^\lambda(a) = \psi^{-1} \sum\nolimits_{-\partial-\hbar}^{\lambda} [\psi(a)\,{}_x\, u_i]_\hbar \delta x = \sum_{j\in\mathcal{I}} \sum\nolimits_{-\partial-\hbar}^{\lambda} \frac{\partial a}{\partial u_j}[u_j \,{}_{x+\partial^{(1)}}\, u_i]_\hbar \, \delta x.$$

□

We consider now the whole quantization. Take the chiral star-product $I_{\lambda,\hbar,\star}$ induced by ϕ, that is, for $a, b \in \mathcal{V}$,

$$I_{\lambda,\hbar,\star}(a,b) := \phi^{-1}(I_{\lambda,\hbar}(\phi(a),\phi(b))).$$

Remark 4.4.9. As defined in (4.4.1), the quantization map ϕ is equal to $\gamma \circ \psi$. Thus, for all $a, b \in \mathcal{S}(R)$:

$$\begin{aligned} I_{\lambda,\hbar,\star}(a,b) &= \gamma^{-1}(\hat{I}_{\lambda,\hbar}(\gamma(a),\gamma(b))) \\ &= \gamma^{-1} \circ m \circ :\exp: \Big(\sum_{i\in\mathcal{I}} \partial_{u_i} \otimes L_i^\lambda\Big) (\gamma(b)\otimes\gamma(a)). \end{aligned}$$

So, if we can find some $x, y \in \mathcal{S}(R)$ such that

$$:\exp: \Big(\sum_{i\in\mathcal{I}} \partial_{u_i} \otimes L_i^\lambda\Big) (\gamma(b)\otimes\gamma(a)) = \gamma(x)\otimes\gamma(y),$$

then $I_{\lambda,\hbar,\star}(a,b) = \gamma^{-1}(\gamma(x)\gamma(y)) = x \star_\gamma y$, where $\star_\gamma$ is the Gutt star-product defined in Example 4.1.9.

Theorem 4.4.10. *For all $a, b \in \mathcal{S}(R)$, the chiral star-product $I_{\lambda,\hbar,\star}$ has the following* *expression:*

$$I_{\lambda,\hbar,\star}(a,b) = m_{\star_\gamma} \circ \exp\left(\sum_{i\in\mathcal{I}} \partial_{u_i} \otimes (D_i^{-\lambda-\partial-\hbar} - \mathcal{L}_i^\star)\right)(b\otimes a)$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1\dots i_k\in\mathcal{I}} \frac{\partial^k b}{\partial u_{i_1}\dots\partial u_{i_k}} \star_\gamma (D_{i_k}^{-\lambda-\partial-\hbar} - \mathcal{L}_{i_k}^\star)\cdots(D_{i_1}^{-\lambda-\partial-\hbar} - \mathcal{L}_{i_1}^\star)(a), \tag{4.4.9}$$

where $\star_\gamma$ denotes the Gutt star-product, $\mathcal{L}_i^\star$ the left multiplication operator $u_i \star_\gamma -$, and D_j^λ for $j\in\mathcal{I}$ is the operator

$$D_j^\lambda(a) := \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1\dots i_k\in\mathcal{I}} \left(\frac{\partial^k a}{\partial u_{i_1}\dots\partial u_{i_k}}\right) \star_\gamma \left(\sum\nolimits_{-\partial-\hbar}^{\lambda} \delta\lambda_1 \sum\nolimits_{-\partial-\hbar}^{\lambda_1+\hbar} \delta\lambda_2 \dots \right.$$
$$\left. \cdots \sum\nolimits_{-\partial-\hbar}^{\lambda_{k-1}+\hbar} \delta\lambda_k \{\cdots\{\{u_j\, {}_{\lambda_1} u_{i_1}\}_\hbar\, {}_{\lambda_2} u_{i_2}\}_\hbar \dots {}_{\lambda_k} u_{i_k}\}_\hbar \right).$$

Proof. It is sufficient to prove (4.4.9) when a and b are PBW monomials, say of degree m and n respectively. Denote $b = b_1 \cdots b_n$, with $b_i \in \mathcal{B}$, and

$$\gamma(b) = \frac{1}{n!} \sum_{\sigma\in S_n} b_\sigma,$$

with $b_\sigma := b_{\sigma(1)} \cdots b_{\sigma(n)} \in \mathfrak{U}(R_L)$. Then

$$I_{\lambda,\hbar,\star}(a,b) = \gamma^{-1}\left(\frac{1}{n!}\sum_{\sigma\in S_n} \hat{I}_{\lambda,\hbar}(\gamma(a), b_\sigma)\right).$$

We compute $\hat{I}_{\lambda,\hbar}(\gamma(a), b_\sigma)$ using (4.4.3):

$$\hat{I}_{\lambda,\hbar}(\gamma(a), b_\sigma) = \sum_{k=0}^{\infty} \sum_{1\le i_1<\cdots<i_k\le n} \frac{\delta^k b_\sigma}{\delta i_1 \dots \delta i_k} L^\lambda_{b_{\sigma(i_k)}} \circ \cdots \circ L^\lambda_{b_{\sigma(i_1)}}(\gamma(a)).$$

Thus

$$I_{\lambda,\hbar,\star}(a,b) = \gamma^{-1} \sum_{k=0}^{\infty} \sum_{\substack{i_1\neq\cdots\neq i_k \\ i_1,\dots,i_k\in\{1,\dots,n\}}} \left(\frac{1}{n!} \sum_{\substack{\sigma\in S_n \\ \sigma(i_1)<\cdots<\sigma(i_k)}} \frac{\delta^k b_{\sigma^{-1}}}{\delta\sigma(i_1)\dots\delta\sigma(i_k)}\right) L^\lambda_{b_{i_k}} \circ \cdots \circ L^\lambda_{b_{i_1}}(\gamma(a)).$$

Notice that

$$\frac{1}{n!} \sum_{\substack{\sigma\in S_n \\ \sigma(i_1)<\cdots<\sigma(i_k)}} \frac{\delta^k b_{\sigma^{-1}}}{\delta\sigma(i_1)\dots\delta\sigma(i_k)} = \frac{1}{k!}\gamma\left(\frac{\delta^k b}{\delta i_1 \dots \delta i_k}\right),$$

so we get

$$\begin{aligned} I_{\lambda,\hbar,\star}(a,b) &= \gamma^{-1}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\substack{i_1\neq\cdots\neq i_k\\ i_1,\dots,i_k\in\{1,\dots,n\}}}\gamma\left(\frac{\delta^k b}{\delta i_1\dots\delta i_k}\right)L^{\lambda}_{b_{i_k}}\circ\dots\circ L^{\lambda}_{b_{i_1}}(\gamma(a))\\ &= \gamma^{-1}\circ(\gamma\otimes 1)\circ\exp\left(\sum_{i=1}^{n}\frac{\delta}{\delta i}\otimes L^{\lambda}_{b_i}\right)(b\otimes\gamma(a))\\ &= \gamma^{-1}\circ(\gamma\otimes 1)\circ\exp\left(\sum_{i\in\mathcal{I}}\frac{\partial}{\partial u_i}\otimes L^{\lambda}_{i}\right)(b\otimes\gamma(a))\\ &= \gamma^{-1}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i_1\dots i_k\in\mathcal{I}}\gamma\left(\frac{\partial^k b}{\partial u_{i_1}\dots\partial u_{i_k}}\right)L^{\lambda}_{i_k}\circ\dots\circ L^{\lambda}_{i_1}(\gamma(a)). \end{aligned} \tag{4.4.10}$$

The exponentials here are the usual exponentials of non-commutative variables. By (4.4.6), for every $x\in\mathcal{S}(R)$ and $j\in\mathcal{I}$,

$$\gamma^{-1}L^{\lambda}_{j}(\gamma(x)) = I_{-\lambda-\partial-\hbar,\hbar,\star}(u_j,x) - u_j\star_{\gamma}x.$$

As explained in Remark 4.4.9, the only thing left to prove is that $I_{\lambda,\hbar,\star}(u_j,a)=D^{\lambda}_{j}(a)$. By (4.4.10) it follows that

$$\hat{I}_{\lambda,\hbar}(u_j,\gamma(a)) = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i_1\dots i_k\in\mathcal{I}}\gamma\left(\frac{\partial^k a}{\partial u_{i_1}\dots\partial u_{i_k}}\right)L^{\lambda}_{i_k}\dots L^{\lambda}_{i_1}(u_j),$$

where $L^{\lambda}_{i_k}\dots L^{\lambda}_{i_1}(u_j)\in R[\lambda]$. So, by Remark 4.4.9,

$$I_{\lambda,\hbar,\star}(u_j,a) = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i_1\dots i_k\in\mathcal{I}}\frac{\partial^k a}{\partial u_{i_1}\dots\partial u_{i_k}}\star_{\gamma}L^{\lambda}_{i_k}\dots L^{\lambda}_{i_1}(u_j). \tag{4.4.11}$$

The $\hbar$-bracket of elements of R is the same in $\mathcal{S}(R)$ and $V(R)$, so, for all $i\in\mathcal{I}$

$$L^{\lambda}_{i}(u_j) = \psi^{-1}\textstyle\sum_{-\partial-\hbar}^{\lambda}[u_j\,{}_x\,u_i]_{\hbar}\delta x = \textstyle\sum_{-\partial-\hbar}^{\lambda}\{u_j\,{}_x\,u_i\}_{\hbar}\delta x.$$

We can expand the composition of two operators $L^{\lambda}_{i_2}$ and $L^{\lambda}_{i_1}$, using the sesquilinearity property of the sum $\hbar$-bracket and the change of order of summation (Proposition 3.1.8):

$$\begin{aligned} L^{\lambda}_{i_2}L^{\lambda}_{i_1}(u_j) &= \textstyle\sum_{-\partial-\hbar}^{\lambda}\delta\lambda_1\{\left(\textstyle\sum_{-\partial-\hbar}^{\lambda}\delta\lambda_2\{u_j\,{}_{\lambda_2}\,u_{i_1}\}_{\hbar}\right)\,{}_{\lambda_1}\,u_{i_2}\}_{\hbar}\\ &= \textstyle\sum_{-\partial-\hbar}^{\lambda}\delta\lambda_1\textstyle\sum_{\lambda_1}^{\lambda}\delta\lambda_2\{\{u_j\,{}_{\lambda_2}\,u_{i_1}\}_{\hbar}\,{}_{\lambda_1}\,u_{i_2}\}_{\hbar}\\ &= \textstyle\sum_{-\partial-\hbar}^{\lambda}\delta\lambda_2\textstyle\sum_{-\partial-\hbar}^{\lambda_1+\hbar}\delta\lambda_1\{\{u_j\,{}_{\lambda_2}\,u_{i_1}\}_{\hbar}\,{}_{\lambda_1}\,u_{i_2}\}_{\hbar}. \end{aligned} \tag{4.4.12}$$

Putting together (4.4.11) and (4.4.12) we get $I_{\lambda,\hbar,\star}(u_j,a)=D^{\lambda}_{j}(a)$, completing the proof. □

Corollary 4.4.11. *If $V(R)$ is a free-field vertex algebra, then*

$$I_{\lambda,\hbar,\star}(a,b) = m_{\star_\gamma} \circ \sigma \circ \exp\Big(\sum_{i\in\mathcal{I}} L_i^\lambda \otimes \partial_{u_i}\Big)(a\otimes b), \tag{4.4.13}$$

where σ is the operator $x\otimes y \mapsto y \otimes x$, and

$$L_i^\lambda(a) = \sum_{j\in\mathcal{I}} \sum\nolimits_{-\partial-\hbar}^{\lambda} \{u_j\,{}_{x+\partial}\, u_i\}_{\hbar_{\to}} \frac{\partial a}{\partial u_j}\,\delta x. \tag{4.4.14}$$

More explicitly:

$$I_{\lambda,\hbar,\star}(a,b) = \sum_{k=0}^{\infty} \sum_{i_1,j_1,\dots,i_k,j_k\in\mathcal{I}} \frac{\partial^k b}{\partial u_{j_1}\dots\partial u_{j_k}} \star_\gamma$$
$$\Big(\sum\nolimits_{\lambda_{k+1}-\hbar}^{\lambda} \delta\lambda_1 \sum\nolimits_{\lambda_{k+1}-\hbar}^{\lambda_1} \delta\lambda_2 \cdots \sum\nolimits_{\lambda_{k+1}-\hbar}^{\lambda_{k-1}} \delta\lambda_k \prod_{s=1}^{k} \{u_{i_s}\,{}_{\lambda_s-\lambda_{s+1}}\, u_{j_s}\}_\hbar\Big)\Big(\Big|_{\lambda_{k+1}=-\partial} \frac{\partial^k a}{\partial u_{i_1}\dots\partial u_{i_k}}\Big). \tag{4.4.15}$$

Proof. Equations (4.4.13) and (4.4.14) follow from putting together Theorem 4.4.10 and Proposition 4.4.8. Formula (4.4.15) follows from iterating (4.4.14) and by the fact that $\{u_i\,{}_\lambda\, u_j\}_\hbar \in \mathbb{C}[\lambda]$ for all $i,j\in\mathcal{I}$. □

Remark 4.4.12. Formula (4.4.13) for the free-field case consists of an exponential of a bidifferential operator, which is similar to the Moyal-Weyl and Gutt star-products (see Examples 4.1.7 and 4.1.9).

Example 4.4.13. In the case of $\beta\gamma$-systems, we can see explicitly how the chiral star-product $I_{\lambda,\hbar,\star}$ reduces to the Moyal-Weyl star-product in the Zhu algebra. First of all, notice that the Gutt star-product $\star_\gamma$ in the Zhu algebra reduces itself to the Moyal-Weyl star-product. Denote by π the Poisson bivector of the Zhu algebra and let $a\star_\hbar b := I_{0,\hbar,\star}(a,b)$; from equation (4.4.13) and by expanding the formula for the Moyal-Weyl star-product (4.1.4), it follows that

$$a\star_\hbar b \equiv m\circ \exp(\pi/2)\circ\sigma\circ\exp\Big(\sum_{i\in\mathcal{I}} L_i^0\otimes\partial_{u_i}\Big)(a\otimes b) \mod \mathcal{J}_\hbar.$$

Moreover, by (4.4.14) and Theorem 3.1.6,

$$\begin{aligned} L_i^0(a) &\equiv \sum_{j\in\mathcal{I}} \frac{\partial a}{\partial u_j} \sum\nolimits_{-\hbar}^{0} \{u_j\,{}_x\, u_i\}_\hbar\,\delta x \mod \mathcal{J}_\hbar. \\ &= \sum_{j\in\mathcal{I}} \frac{\partial a}{\partial u_j}\{u_j\,{}_{-\hbar}\, u_i\}_\hbar \mod \mathcal{J}_\hbar. \end{aligned}$$

Since the Poisson bracket in the Zhu algebra is induced by $\{\cdot\,{}_{-\hbar}\,\cdot\}_\hbar$,

$$\begin{aligned} a\star_\hbar b &\equiv m\circ\exp(\pi/2)\circ\sigma\circ\exp(\pi)(a\otimes b) \mod \mathcal{J}_\hbar \\ &= m\circ\sigma\circ\exp(\pi/2)(a\otimes b) \mod \mathcal{J}_\hbar. \end{aligned}$$

Since m is commutative, we can get rid of σ, getting back the formula for the Moyal-Weyl star-product.

§ 4.4.1 | Star-deformations of Poisson vertex algebras

A (Poisson) vertex algebra is an $\hbar$-(Poisson) vertex algebra under the limit $\hbar \to 0$. So, our formulae for $I_{\lambda,\hbar,\star}$ can be specialized to obtain formulae for the star-deformation of the corresponding Poisson vertex algebras.

Theorem 4.4.14. *The following formulae describe the star-deformation of $\mathcal{S}(R)$ induced by the symmetrization map:*

$$\begin{aligned} I_{\lambda,\star}(a,b) &= m_{\star_\gamma} \circ \exp\left(\sum_{i\in\mathcal{I}} \partial_{u_i} \otimes (D_i^{-\lambda-\partial} - \mathcal{L}_i^\star)\right)(b\otimes a) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1\ldots i_k\in\mathcal{I}} \frac{\partial^k b}{\partial u_{i_1}\ldots\partial u_{i_k}} \star_\gamma (D_{i_k}^{\lambda-\partial} - \mathcal{L}_{i_k}^\star)\ldots(D_{i_1}^{\lambda-\partial} - \mathcal{L}_{i_1}^\star)(a), \end{aligned} \tag{4.4.16}$$

for all $a, b \in \mathcal{S}(R)$, where $\star_\gamma$ denotes the Gutt star-product (4.1.5), $\mathcal{L}_i^\star$ the left multiplication operator by u_i with respect to $\star_\gamma$, and D_j^λ for $j \in \mathcal{I}$ is the operator:

$$\begin{aligned} D_j^\lambda(a) := \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1\ldots i_k\in\mathcal{I}} \left(\frac{\partial^k a}{\partial u_{i_1}\ldots\partial u_{i_k}}\right) \star_\gamma \Bigg(\int_{-\partial}^{\lambda} d\lambda_1 \int_{-\partial}^{\lambda_1} d\lambda_2 \ldots \\ \cdots \int_{-\partial}^{\lambda_{k-1}} d\lambda_k \{\cdots\{\{u_j\,{}_{\lambda_1} u_{i_1}\}_{\lambda_2} u_{i_2}\}\ldots{}_{\lambda_k}\, u_{i_k}\}\Bigg). \end{aligned}$$

Proof. It follows directly from Theorem 4.4.10, by sending $\hbar \to 0$. If R_L is abelian, then the map γ in (4.4.1) is the identity. This implies that $I_{\lambda,\star} = \hat{I}_{\lambda,\hbar|_{\hbar\to 0}}$. All the properties used in the proof of Proposition 4.4.8 are true in the case where R_L is abelian. So (4.4.17) follows from (4.4.8). □

Corollary 4.4.15. *If R_L is an abelian Lie algebra, then*

$$I_{\lambda,\star}(a,b) = m \circ \sigma \circ \exp\left(\sum_{i\in\mathcal{I}} L_i^\lambda \otimes \partial_{u_i}\right)(a\otimes b), \tag{4.4.17}$$

where m is the multiplication map, σ is the operator $x\otimes y \mapsto y \otimes x$, and

$$L_i^\lambda(a) = \sum_{j\in\mathcal{I}} \int_{-\partial}^{\lambda} \frac{\partial a}{\partial u_j} \{u_j\,{}_{x+\partial^{(1)}}\, u_i\}\, dx.$$

More explicitly:

$$\begin{aligned} I_{\lambda,\star}(a,b) = \sum_{k=0}^{\infty} \sum_{i_1,j_1,\ldots,i_k,j_k\in\mathcal{I}} \frac{\partial^k b}{\partial u_{j_1}\ldots\partial u_{j_k}} \cdot \\ \left(\int_{\lambda_{k+1}}^{\lambda} d\lambda_1 \int_{\lambda_{k+1}}^{\lambda_1} d\lambda_2 \cdots \int_{\lambda_{k+1}}^{\lambda_{k-1}} d\lambda_k \prod_{s=1}^{k} \{u_{i_s}\,{}_{\lambda_s-\lambda_{s+1}} u_{j_s}\}\right)\left(\Big|_{\lambda_{k+1}=-\partial} \frac{\partial^k a}{\partial u_{i_1}\ldots\partial u_{i_k}}\right). \end{aligned}$$

Proof. All the properties used in the proof of Proposition 4.4.8 are true in the case where R_L is abelian. So (4.4.17) follows from (4.4.8). □

Remark 4.4.16. If $V(R)$ is a free field vertex algebra, then R_L is an abelian Lie algebra. This is not true if $\hbar \neq 0$, because

$$[a, b] = \sum\nolimits_{-\partial-\hbar}^{0} [a\,{}_x\, b]_\hbar \delta x = \sum\nolimits_{-\hbar}^{0} [a\,{}_x\, b]_\hbar \delta x \neq 0.$$

Remark 4.4.17. In [BDK], the authors derive an explicit, closed formula for a Poisson vertex algebra structure on an algebra of differential polynomials, which they call the "Master Formula". The formulae in Theorem 4.4.14 are a quantization of the Master Formula.

Remark 4.4.18. In [Ya], Yanagida studies the deformation quantization of Poisson vertex algebras from a chiral algebraic point of view. According to [Ya, Corollary 3.15], if a chiral deformation of a Poisson vertex algebra exists, it is unique. It is not immediately clear if the definition of a chiral deformation is equivalent to that of a star-deformation. If that is the case, then Theorem 4.4.14 gives the explicit formula for the *unique* star deformation of $\mathcal{S}(R)$.

Chapter 5

Future directions

In this Chapter, we discuss some open problems and future directions stemming from Part I of this thesis.

Affine W-algebras

We have concentrated our study on vertex algebras of the form $V(R)$, where R is a non-linear vertex Lie algebra. Let $a, b \in R$, and consider the degree of $[a\,{}_\lambda\, b]$ in $\mathcal{T}(R)[\lambda]$ (with $\deg \lambda = 0$). In this thesis, we have considered the case when $\deg[a\,{}_\lambda\, b] = 0$, i.e. $V(R)$ is a free-field vertex algebra, for example, a $\beta\gamma$-system (see formulae (4.4.13) and (4.4.17)), and the case when $\deg[a\,{}_\lambda\, b] = 1$, which includes the affine vertex algebras (see (4.4.9) and (4.4.16)).

What is left to consider is the case $\deg[a\,{}_\lambda\, b] > 1$. This means that the λ-bracket between elements of R contains some "genuine" non-linearities. There are some technical difficulties in doing so. The main one is that, in this case, the PBW filtration on $V(R)$ is no longer a good filtration. It would be interesting to obtain a formula for chiral star-products in this case, which notably includes affine W-algebras.

Affine W-algebras are an important class of vertex algebras; they were introduced in physics as "extended conformal algebras" [Za], and the theory was subsequently developed by mathematicians (see [FF; KRW]). They are related to other important mathematical structures. To any choice of a semi-simple Lie algebra $\mathfrak{g}$ and of a nilpotent element $f \in \mathfrak{g}$ can be associated an object in each of the four categories of diagram (2.5.9):

$$\begin{array}{ccc} \mathcal{W}(\mathfrak{g}, f) & \longleftarrow & \mathsf{W}^k(\mathfrak{g}, f) \\ \downarrow & & \downarrow \\ \mathbb{C}[\mathcal{S}(\mathfrak{g}, f)] & \longleftarrow & \mathsf{W}^{fin}(\mathfrak{g}, f) \end{array}$$

where $\mathcal{S}(\mathfrak{g}, f)$ is the algebra of functions on the Slodowy slice [Sl, Section 7.4], $\mathsf{W}^{fin}(\mathfrak{g}, f)$

is Premet's *finite* W-algebra [Pr], $\mathcal{W}(\mathfrak{g}, f)$ is the *classical* W-algebra [DKV1], and $\mathsf{W}^k(\mathfrak{g}, f)$ is the *affine* W-algebra at level k.

Even if our results do not hold in the case of affine W-algebras, they can still have useful applications. In fact, it is possible to realise an affine W-algebra $\mathsf{W}^k(\mathfrak{g}, f)$ as a subalgebra of an affine vertex algebra $V^{\psi_k}(\mathfrak{g})$ (notice the different level), see [KRW; KW; DK2] for reference. Since our results hold for affine vertex algebras, they can be applied to affine W-algebras via the embedding mentioned above.

What makes the theory of affine W-algebras complicated is that, in general, there is no presentation by generators and relations. Recall that a nilpotent element $f \in \mathfrak{g}$ is called short if the corresponding Dynkin grading on $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. In [DKV1], De Sole, Kac, and Valeri computed the generators and relations for the classical W-algebra associated to a short nilpotent element. Some of the formulae in this thesis can be used to compute a quantization of this presentation. I intend to study the problem in a following paper, which is a collaboration with Daniele Valeri.

Further work on $\hbar$-vertex algebras

Other future directions regard $\hbar$-vertex algebras. First, it would be interesting to find a combinatorial explanation for the appearance of finite difference calculus in the formulae of the $\hbar$-bracket (see Section 3.3). This should follow in some way from the change of variables $x = \frac{1}{\hbar}\log(1 + \hbar z)$. In fact,

$$\log(1 + z) = \sum_{k\geq 1} s(n, k)\frac{z^k}{k!}.$$

This is the generating function of the Stirling numbers of the first kind, which are the coefficients $s(n, k)$ of the expansion of the falling factorial

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1) = \sum_{k\geq 0} s(n, k)x^k.$$

In Section 3.4, we show how the construction of the Zhu algebra becomes more natural in the formalism of $\hbar$-vertex algebras. Another possible application of the $\hbar$-vertex algebra formalism is to higher-level Zhu algebras, which are usually quite complicated to construct explicitly. These are a sequence of associative algebras $\mathrm{Zhu}_n(V)$, for $n \in \mathbb{N}$, associated to a vertex algebra V [DLM]. The first term of the sequence, $\mathrm{Zhu}_0(V)$, is the usual Zhu algebra. A vertex algebra V is rational if and only if $\mathrm{Zhu}_n(V)$ is finite-dimensional and semi-simple for all $n \in \mathbb{N}$. The construction is similar to that of the Zhu algebra: one defines two products, $*_n$ and $\circ_n$, and then shows that $*_n$ is an associative product on $\mathrm{Zhu}_n(V) = V/(\partial V + V \circ_n V)$. The most common definition of $*_n$ and $\circ_n$ requires the conformal grading again, but it is possible to give an equivalent definition using a change of variables (see [Li3]). It would be interesting to see if there exist higher level $\hbar$-vertex algebras that control the higher level Zhu algebras.

Representation theory

Zhu algebras are important because of their relation to representation theory; this connection is not investigated in this thesis. In [Zh], the Zhu algebra was originally constructed using (implicitly) the $\hbar$-vertex algebra induced by the conformal grading (see Remark 3.2.5). It is shown in [Zh] that the Zhu algebra acts on "positive energy modules". The definition of a positive energy module involves the conformal grading. In a nutshell, a positive energy module for a VOA V is a graded vector space $M = \bigoplus_{n\geq 0} M_n$, together with an action of V via quantum fields:

$$Y^M : V \to \operatorname{End}(M)[[z, z^{-1}]], \quad Y^M(a, z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1},$$

satisfying some axioms, including a version of the Borcherds identity (2.2.1). The grading condition is $a^M_{(n)} M_k \subset M_{k+\Delta_a-n-1}$. In particular, $\mathsf{o}(a) := a^M_{(\Delta_a-1)}$ preserves the degree. Recall the definition of the $\bullet$ and $\times$ product (2.5.4) (2.5.5) (here we are specializing $\hbar = 1$). It turns out that $\mathsf{o}(a \bullet b) = \mathsf{o}(a)\mathsf{o}(b)$, and that the restriction of o to degree piece M_0 gives a module for the Zhu algebra $\operatorname{Zhu}'(V) = V/(V \times V)$ (2.5.6). This defines a reduction functor, which has a right inverse, called the induction functor, from $\operatorname{Zhu}'(V)$-modules to positive energy V-modules. Moreover, these two functors give a bijection on equivalence classes of simple modules [Zh, Theorem 2.2.2].

In [Hu2], the Zhu algebra $\operatorname{Zhu}(V)$ is constructed using (implicitly) the $\hbar$-vertex algebra structure coming from the change of variables (see Proposition 3.2.3). This construction does not require the existence of a conformal grading. However, the connection with representation theory proved in [Hu2] still involves positive energy modules, and thus a conformal grading. In fact, the action on positive energy modules is the action of the Zhu algebra constructed via the conformal grading, induced via the isomorphism $T : \operatorname{Zhu}(V) \to \operatorname{Zhu}'(V)$ of Theorem 2.5.5. That is, if $a \in \operatorname{Zhu}(V)$, $m \in M_0$, the action of a on m is defined as $\mathsf{o}(T(a))$.

It would be interesting to get a representation-theoretic interpretation of the Zhu algebra that does not require the conformal grading. It is not difficult to get an restriction functor. In fact, if M is a module (not graded) of a vertex algebra V with action given by $Y^M(\cdot, z)$, then $Y^M\left(a, \frac{1}{\hbar}\log(1+\hbar z)\right) = \sum_{n\in\mathbb{Z}} a^M_{(n,\hbar)} z^{-n-1}$ gives an action of the $\hbar$-vertex algebra associated to V. The quotient of M by the span of $\{a^M_{(-2,\hbar)} m \,|\, a \in V, m \in M\}$ becomes naturally a module of $\operatorname{Zhu}_\hbar(V)$, with the action induced by $a^M_{(-1,\hbar)}$. The proof is similar to that of the associativity of $\operatorname{Zhu}_\hbar(V)$ (see Theorem 3.4.3) and also appears (without the $\hbar$-vertex algebra formalism) in Huang's paper [Hu2, Proposition 6.7]. It is not clear if a right inverse of this functor exists.

Another possible approach is the following. Recall that, in the construction of the $\hbar$-vertex algebra associated to a vertex algebra (V, Y), the $\hbar$-vertex operators are given by $Y_\hbar(a, z) = Y\left(a, \frac{1}{\hbar}\log(1+\hbar z)\right)$. The change of variable construction, when applied to the isomorphic vertex algebra (V, Y') (2.5.7), gives the $\hbar$-deformed vertex operators $Y_\hbar[a, z] = Y((1 + \hbar z)^{L_0}, z)$. So the conformal grading appears via the isomorphism $T : (V, Y) \to (V, Y')$ (see Section 3.4).

Now, consider a V-module $(M = \bigoplus M_n, Y^M)$, graded but not conformally graded. That is, $a^M_{(n)} M_k \subset M_{k-n-1}$, for all $a \in V$. We can deform Y^M as follows:

$$Y^M[a, z] := Y^M(z^{L_0} a, z).$$

Clearly, $\mathrm{Res}_z(z^{-1} Y^M[a, z]) = a^M_{\Delta_a - 1} = \mathsf{o}(a)$. So the (-1) action with respect to $Y^M[\cdot, z]$ is the one that induces the action of $\mathrm{Zhu}'(V)$. This suggests the existence of another change of variable $z \mapsto f(z)$, depending on $\hbar$, such that $Y^M_\hbar(a, z) := Y^M(a, f(z))$ is naturally a module for the $\hbar$-vertex algebra $Y_\hbar(a, z)$. Ideally, when applying this change of variable to the action of the isomorphic vertex algebra (V, Y'), we should get $Y^M(z^{L_0} a, z)$. This is probably too naive, as the change of variable needed seems to be degenerate, but it is possible this could be made precise. Some of the results of Li [Li2] are in this direction. Li defines a class of modules, that he calls "ϕ-coordinated modules". He proves that $Y^M(z^{L_0} a, z)$ is a ϕ-coordinated module for the vertex algebra $(V, Y[\cdot, z])$ [Li2, Proposition 5.8]. It would be interesting to rephrase these results in the formalism of $\hbar$-vertex algebras and to further investigate the problem.

Part II

Isomorphisms of deformations and quantizations of Kleinian singularities

Notation and conventions

If not stated differently, all vector spaces are assumed to be over $\mathbb{C}$.

To avoid confusion, we strictly reserve the term quantization for filtered, non-commutative algebras and deformation for filtered Poisson algebras.

We provide below a table of commonly used notation for Part II:

Notation	Meaning
V	Vector space
$A, D, \ldots$	Associative algebras
$\mathcal{A}, \mathcal{D}, \ldots$	Poisson algebras
$\mathcal{S}(V)$	Symmetric algebra of V
$\mathcal{T}(V)$	Tensor algebra of V
Aut	Group of associative algebra automorphisms
PAut	Group of Poisson algebra automorphisms
Iso	Groupoid of associative algebra isomorphisms
PIso	Groupoid of Poisson algebra isomorphisms
$\mathbb{P}^n$	Complex projective space
$\mathbb{A}^n$	Complex affine space
$\mathbb{C}G$	Group algebra of the group G
Spec(R)	Spectrum of the ring R
$\mathbb{C}[X]$	Algebra of polynomial functions of the affine variety X
$\mathbb{C}[X]^G$	Subalgebra of G-invariant functions
$\mathbf{A_n}, \mathbf{D_n}, \mathbf{E_n}$	ADE Dynkin types
$A \rtimes G$	Skew ring (Definition 6.1.6)

Chapter 6

Preliminaries and motivation

This Chapter is a technical introduction to Chapter 7, which contains the main results of Part II. It also aims to provide motivation and broader context to the main results. Section 6.1 is an introduction to the most important aspects of the theory of quantization and deformation of conical symplectic singularities. In Section 6.2 we discuss the main problem about equivalence of groupoids of isomorphisms. Our results focus on the special case of Kleinian singularities, which are discussed in detail in Section 6.3.

Let us fix some notation that will be used throughout the rest of the thesis. When we say that an algebra A is graded (respectively filtered), we always assume the grading (filtration) to be non-negative, that is $A = \bigoplus_{i\geq 0} A_i$ (respectively $A = \bigcup_{i\geq 0} F_i A$). Moreover, we assume each graded (filtered) component to be finite-dimensional, and $A_0 = \mathbb{C}$ (respectively $F_0 A = \mathbb{C}$). If, in addition, A is a Poisson algebra, we assume the Poisson bracket to be of negative degree, that is, there exists an integer $i \geq 1$ such that

$$\{\cdot,\cdot\} : A_n \otimes A_m \to A_{n+m-i} \quad \text{or} \quad \{\cdot,\cdot\} : F_n A \otimes F A_m \to F_{n+m-i} A,$$

for all n, m. Similarly, if A is a filtered associative algebra, we require the commutator to be of negative degree:

$$[\cdot,\cdot] : F_n A \otimes F_m A \to F_{n+m-i} A.$$

Note that this implies that the associated graded $\operatorname{gr} A$ is commutative and has a natural structure of graded Poisson algebra (see Section 4.1). Let A and B be graded (respectively filtered) algebras. A map $\phi : A \to B$ is a graded (filtered) algebra isomorphism if ϕ is an algebra isomorphism such that $\phi(A_i) = B_i$ (respectively, if $\phi(F_i A) = F_i B$), for all $i \geq 0$.

Let $\mathcal{A}$ be a graded Poisson algebra.

Definition 6.0.1. A *filtered deformation* of $\mathcal{A}$ is a pair $(\mathcal{D}, i)$, where

- $\mathcal{D}$ is a filtered Poisson algebra
- $i : \operatorname{gr}(\mathcal{D}) \to \mathcal{A}$ is a graded Poisson algebra isomorphism.

Definition 6.0.2. A *filtered quantization* of $\mathcal{A}$ is a pair (Q, i), where

- Q is a filtered associative algebra
- $i : \operatorname{gr}(Q) \to \mathcal{A}$ is a graded Poisson algebra isomorphism.

Filtered deformations and quantizations and their interplay are the main objects of study in Part II of this thesis. First, we define a suitable notion of isomorphic deformations and isomorphic quantizations.

Definition 6.0.3. Two deformations $(\mathcal{D}, i)$ and $(\mathcal{D}', j)$ of $\mathcal{A}$ are isomorphic if there exists a filtered Poisson algebra isomorphism $\phi : \mathcal{D} \to \mathcal{D}'$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathcal{A} & & \\ & \overset{i}{\nearrow} & & \overset{j}{\nwarrow} & \\ \operatorname{gr}(\mathcal{D}) & & \xrightarrow{\operatorname{gr}(\phi)} & & \operatorname{gr}(\mathcal{D}') \end{array}$$

Similarly, two quantizations (Q, i) and (Q', j) of $\mathcal{A}$ are isomorphic if there exists a filtered algebra isomorphism $\psi : Q \to Q'$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathcal{A} & & \\ & \overset{i}{\nearrow} & & \overset{j}{\nwarrow} & \\ \operatorname{gr}(Q) & & \xrightarrow{\operatorname{gr}(\psi)} & & \operatorname{gr}(Q') \end{array}$$

When it does not create ambiguity, we write "the algebra A is a deformation/quantization of $\mathcal{A}$", making the choice of the isomorphism implicit.

§ 6.1 | Conical symplectic singularities

The definition of a variety with symplectic singularities was introduced by Beauville in [Be2], as a generalization of the notion of symplectic manifolds to singular algebraic varieties.

Definition 6.1.1. Let X be a normal Poisson algebraic variety over $\mathbb{C}$ such that the smooth locus X^{reg} is a symplectic variety. Let ω_{reg} denote the symplectic form on X^{reg}. We say that X has symplectic singularities if there is a resolution of singularities $\rho : \hat{X} \to X$ such that the pull-back $\rho^*(\omega_{reg})$ extends to a regular (but not necessarily symplectic) 2-form on $\hat{X}$.

By slight abuse of terminology, we will call a variety X with symplectic singularities a symplectic singularity. Recall that an affine algebraic Poisson variety X is called conical if its algebra of functions $\mathbb{C}[X]$ is a graded Poisson algebra. We are interested in the class of symplectic singularities that are conical, which are called *conical symplectic singularities*.

Example 6.1.2. Let (V, ω) be a finite-dimensional symplectic vector space. Consider a finite subgroup G of the symplectic group $Sp(V)$. The group G acts on the polynomial ring $\mathbb{C}[V]$ by

$$(g \cdot f)(v) = f^g(v) := f(g^{-1}v), \quad \forall f \in \mathbb{C}[V],\, g \in G,\, v \in V.$$

Then the symplectic quotient $V/G := \mathrm{Spec}(\mathbb{C}[V]^G)$ is a conical symplectic singularity (see [Be2]).

Theorem 6.1.3. *Let X be a conical symplectic singularity. There exists an affine space that parameterizes both the deformations and the quantizations of $\mathbb{C}[X]$, up to isomorphisms.*

Proof. This theorem is the sum of the results by Namikawa [Na1; Na2], for the deformations part, and Losev [Lo2], for the quantizations part. □

For the results of this thesis, the exact nature of this parameter space in the full generality of conical symplectic singularities is not needed. We briefly sketch the construction, for completeness' sake. First, we need some preliminary facts.

Theorem 6.1.4. *Every symplectic singularity has a finite stratification by symplectic leaves.*

Proof. See [Ka3, Theorem 2.3]. □

In particular, we are interested in the codimension 2 symplectic leaves. Any symplectic singularity admits a crepant, partial resolution, called the $\mathbb{Q}$-factorial terminalization, that has no codimension 2 symplectic leaves (see [Lo1, Proposition 2.1]). Let Y be a $\mathbb{Q}$-factorial terminalization of X. We can construct all the deformations of $\mathbb{C}[X]$ starting from the deformations of $\mathbb{C}[Y]$. The parameter space for the latter deformations is given by $\mathfrak{P} = H^2(Y^{reg}, \mathbb{C})$, which is sometimes referred to as the "Namikawa Cartan space".

Let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be the codimension 2 symplectic leaves of X. For every $i = 1, \dots, k$, take a point $x_i \in \mathcal{L}_i$ and consider (Σ_i, x_i), the transverse slice of $\mathcal{L}_i$ in X at the point x_i. These transverse slices have now dimension 2, so they are Kleinian singularities $(\Sigma_i, x_i) = (V/\Gamma_i, 0)$ of type $\hat{\Delta}_i$ (see Section 6.3). In particular, they have associated a Weyl group $\hat{W}_i$ of the corresponding Dynkin type.

Remark 6.1.5. Although it is not always possible to take an algebraic transverse slice to each codimension two symplectic leaf, we can always take either a formal slice or a transverse slice in the complex analytic category and proceed similarly.

Consider the $\mathbb{Q}$-terminalization map $\pi : Y \to X$ and the preimages $\pi^{-1}(x_i)$. Since x_i corresponds to the point $0 \in V/\Gamma_i$ on the transverse slice Σ_i, its preimage is a union of $\mathbb{P}^1$ in the shape of the corresponding Dynkin diagram (see Theorem 6.3.2). We obtain a monodromy action of the fundamental group $\pi_1(\mathcal{L}_i, x_i)$ on $\hat{\Delta}_i$ by graph automorphism. This induces an action of $\pi_1(\mathcal{L}_i, x_i)$ on $\hat{W}_i$ and we define

$$W_i := \hat{W}_i^{\pi_1(\mathcal{L}_i, x_i)},$$

the subgroup of fixed points. The groups W_i are the Weyl groups associated to the folded Dynkin diagram $\Delta_i := \hat{\Delta}_i/\pi_1(\mathcal{L}_i, x_i)$. Finally, let

$$W := \prod_{i=1}^{k} W_i.$$

The group W is called the "Namikawa Weyl group" and it acts on $\mathfrak{P}$ by crystallographic reflections. Two deformations of $\mathbb{C}[X]$ are isomorphic (as deformations) if and only if their corresponding parameters are in the same W-orbit. So the parameter space for the deformations is the quotient $\mathfrak{P}/W$, which is an affine space since W is a reflection group.

In [Lo2], Losev proves that it is also possible to construct all the filtered quantizations of $\mathbb{C}[X]$ starting from the $\mathbb{Q}$-factorial terminalization. Again, two quantizations are isomorphic if and only if the corresponding parameters are in the same W-orbit. So, $\mathfrak{P}/W$ is also the parameter space of the filtered quantizations.

§ 6.1.1 | The symplectic reflection algebra

Consider a finite-dimensional, symplectic, vector space (V, ω) and a finite subgroup G of $Sp(V)$. For symplectic quotients V/G, there is an algebraic construction of the deformations and quantizations of $\mathbb{C}[V/G]$, using symplectic reflection algebras.

Definition 6.1.6. Let A be an algebra and H a finite group acting on A. The skew-ring $A \rtimes H$ is, as a vector space, the tensor product $A \otimes \mathbb{C}H$. It is an algebra with multiplication given by

$$h \cdot a = a^h \cdot h \quad \forall a \in A,\, h \in H,$$

where a^h is the action of h on a.

The symplectic reflection algebra is defined as a quotient of the skew-ring $\mathcal{T}(V^*) \rtimes G$, where $\mathcal{T}(V^*)$ is the tensor algebra of V^*.

Definition 6.1.7. An element $s \in G$ is called a symplectic reflection if $\operatorname{rk}(\operatorname{id} - s) = 2$. Let $\mathcal{S}$ denote the set of symplectic reflections in G.

Remark 6.1.8. The name symplectic reflection is due to the analogy with complex reflections. Namely, let s act on a vector space $\mathfrak{h}$ as a complex reflection, that is $\mathrm{rk}(\mathrm{id} - s) = 1$. Consider the action of s on $V := \mathfrak{h} \oplus \mathfrak{h}^*$: if we give V the symplectic structure coming from the dual pairing, then $s \in Sp(V)$ and

$$\mathrm{rk}(\mathrm{id} - s)\Big|_V = \mathrm{rk}(\mathrm{id} - s)\Big|_{\mathfrak{h}} + \mathrm{rk}(\mathrm{id} - s)\Big|_{\mathfrak{h}^*} = 2.$$

Every symplectic reflection admits such a splitting, but the whole of G need not.

Let $s \in \mathcal{S}$. The vector space V splits as $V = \mathrm{Im}(\mathrm{id} - s) \oplus \mathrm{Ker}(\mathrm{id} - s)$, with $\mathrm{Im}(\mathrm{id} - s)$ of dimension 2. Let ω_s be the 2-form on V whose restriction to $\mathrm{Im}(\mathrm{id} - s)$ is ω and which is equal to 0 on $\mathrm{Ker}(\mathrm{id} - s)$. Let $c \in \mathbb{C}[\mathcal{S}]^G$. This means that c is a function $c : \mathcal{S} \to \mathbb{C}$, invariant under the action of G by conjugation. Take an additional parameter $t \in \mathbb{C}$.

Definition 6.1.9 ([EG])**.** The symplectic reflection algebra $\mathsf{H}_{t,c}(G)$ is the quotient

$$\mathsf{H}_{t,c}(G) = \mathcal{T}(V^*) \rtimes G \Big/ \Big\langle u \otimes v - v \otimes u - t\omega(u,v) + 2\sum_{s \in \mathcal{S}} c(s)\omega_s(u,v)s \mid u, v \in V^* \Big\rangle .$$

Remark 6.1.10. There is an obvious isomorphism $\mathsf{H}_{t,c}(G) \cong \mathsf{H}_{\lambda t, \lambda c}$ for any $\lambda \in \mathbb{C}^\times$ by rescaling generators. Thus, we can reduce to considering only the cases where $t = 0, 1$.

Symplectic reflection algebras admit a version of the Poincaré-Birkhoff-Witt (PBW) Theorem. Take the filtration on $\mathsf{H}_{t,c}(G)$ with $\mathbb{C}G$ in degree 0 and V^* in degree 1.

Theorem 6.1.11 (PBW Theorem)**.** *The symplectic reflection algebra $\mathsf{H}_{t,c}(G)$ is isomorphic, as a vector space, to $\mathbb{C}[V] \otimes \mathbb{C}G$. This upgrades to an isomorphism of algebras after taking the associated graded*

$$\mathrm{gr}(\mathsf{H}_{t,c}(G)) \cong \mathbb{C}[V] \rtimes G.$$

Proof. The proof uses techniques from non-homogeneous Koszul duality. Namely, if we consider only the quadratic part of the relation in Definition 6.1.9, we get

$$\mathcal{T}(V^*) \rtimes G / \langle u \otimes v - v \otimes u \,|\, u, v \in V^* \rangle \cong \mathbb{C}[V] \rtimes G.$$

Since $\mathbb{C}[V] \rtimes G$ is a Koszul algebra, a result by Braverman and Gaitsgory [BG, Lemma 3.3] gives sufficient conditions on the defining relation of $\mathsf{H}_{t,c}(G)$ to get the PBW property, which hold in the case of symplectic reflection algebras. To check the details, see the original paper by Etingof and Ginzburg [EG, Theorem 1.3]. □

Take t as a formal variable, and define the $\mathbb{C}[\mathsf{t}]$-algebra $\mathsf{H}_{\mathsf{t},c}(G)$ as in Definition 6.1.9. The PBW Theorem implies that the algebra $\mathsf{H}_{\mathsf{t},c}(G)$ is a free $\mathbb{C}[\mathsf{t}]$-module. Hence $\lim_{\mathsf{t}\to 0} \mathsf{H}_{\mathsf{t},c}(G) = \mathsf{H}_{0,c}(G)$. That is, $\mathsf{H}_{\mathsf{t},c}(G)$ is a one-parameter deformation of $\mathsf{H}_{0,c}(G)$.

Example 6.1.12. Let G be the trivial group. Then

$$\mathsf{H}_{0,c} = \mathcal{S}(V) \quad \text{and} \quad \mathsf{H}_{1,c} = W(V, \omega),$$

and we recover our standard example of the symmetric algebra and Weyl algebra. In fact, if we consider $\mathsf{c}(s)$, for $s \in \mathcal{S}$, to be the formal parameters, the symplectic reflection algebra can be seen as a multi-parameter deformation of the skew-ring $W(V, \omega) \rtimes G$.

Let

$$\mathsf{e} := \frac{1}{|G|} \sum_{g \in G} g$$

be the symmetrizer idempotent, seen as an element of $\mathbb{C}G \subset \mathsf{H}_{t,c}(G)$.

Definition 6.1.13 ([EG]). The *spherical subalgebra* of the symplectic reflection algebra is $\mathsf{eH}_{t,c}(G)\mathsf{e}$.

Multiplying on both sides by the idempotent e is a standard technique to obtain information about the G-invariants. This is due to the relation $g \cdot \mathsf{e} = \mathsf{e}$ in $\mathbb{C}G$. For example, the algebra $\mathsf{e}(\mathbb{C}[V] \rtimes G)\mathsf{e}$ can be identified with $\mathbb{C}[V]^G$ by the relation

$$\mathsf{e} \cdot f \cdot \mathsf{e} = \frac{1}{|G|} \sum_{g \in G} f^g \cdot \mathsf{e}^2 = \frac{1}{|G|} \sum_{g \in G} f^g \cdot \mathsf{e}, \quad \forall f \in \mathbb{C}[V].$$

Theorem 6.1.14.

(i) The spherical subalgebra $\mathsf{eH}_{0,c}(G)\mathsf{e}$ *is a filtered Poisson algebra.*

(ii) For any choice of t, c,

$$\operatorname{gr}(\mathsf{eH}_{t,c}(G)\mathsf{e}) \cong \mathbb{C}[V]^G.$$

Proof. The spherical subalgebra is commutative for $t = 0$ and any c (see [EG, Theorem 1.6]). Since $\mathsf{eH}_{0,c}(G)\mathsf{e}$ is the classical limit of $\mathsf{eH}_{\mathsf{t},c}(G)\mathsf{e}$, for $\mathsf{t} \to 0$, we must have

$$[\mathsf{eH}_{\mathsf{t},c}(G)\mathsf{e}, \mathsf{eH}_{\mathsf{t},c}(G)\mathsf{e}] \subset \mathsf{t}(\mathsf{eH}_{\mathsf{t},c}(G)\mathsf{e}).$$

This means that $\frac{1}{t}[\cdot,\cdot]$ is well defined at $t = 0$ and defines a Poisson bracket on $\mathsf{eH}_{0,c}(G)\mathsf{e}$. Since $\mathsf{eH}_{t,c}(G)\mathsf{e}$ is an associative filtered algebra, with the filtration coming by restriction from the filtration on $\mathsf{H}_{t,c}(G)$, the induced Poisson bracket on $\mathsf{eH}_{0,c}(G)\mathsf{e}$ must be filtered too.

Part (ii) is a direct consequence of Theorem 6.1.11. □

In particular, $(\mathsf{eH}_{0,c}(G)\mathsf{e}, i)$ is a deformation of $\mathbb{C}[V]^G$, where i is the canonical isomorphism $i : \operatorname{gr}(\mathsf{eH}_{0,c}(G)\mathsf{e}) \cong \mathbb{C}[V]^G$. Similarly, the spherical subalgebra $\mathsf{eH}_{1,c}(G)\mathsf{e}$ gives a quantization of $\mathbb{C}[V]^G$.

Notice that $\mathcal{S}$ is a finite set, so $\mathbb{C}[\mathcal{S}]^G$ is a finite dimensional vector space, hence an affine variety.

Theorem 6.1.15. *There is an isomorphism $j : \mathbb{C}[\mathcal{S}]^G \to \mathfrak{P}$ of affine varieties, such that (i) and (ii) below hold. For $c \in \mathfrak{P}$, let $\mathfrak{X}_{1,c}$ and $\mathfrak{X}_{0,c}$ be the corresponding quantization and deformation, respectively. Then, for all $c \in \mathbb{C}[\mathcal{S}]^G$:*

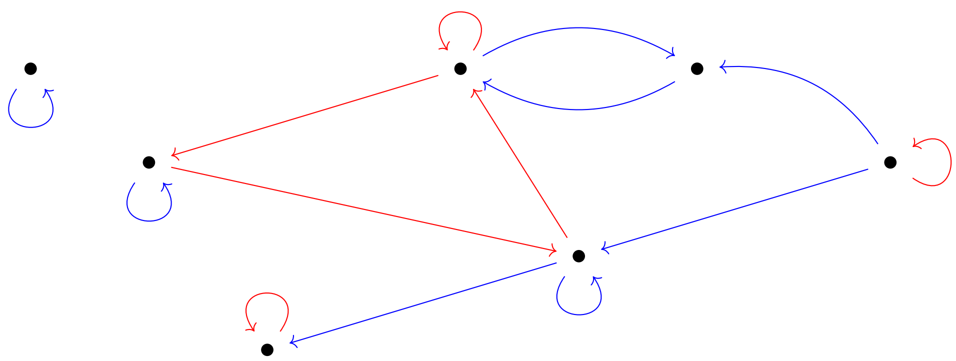
(i) $\mathsf{e}\mathsf{H}_{0,c}(G)\mathsf{e} \cong \mathfrak{X}_{0,j(c)}$ *as filtered deformations of* $\mathbb{C}[V]^G$*;*

(ii) $\mathsf{e}\mathsf{H}_{1,c}(G)\mathsf{e} \cong \mathfrak{X}_{1,j(c)}$ *as filtered quantizations of* $\mathbb{C}[V]^G$*.*

Proof. Part (i) was proved by Bellamy in [Be3, Theorem 1.4], while (ii) was proved by Losev in [Lo2, Proposition 3.17]. □

§ 6.2 | Statement of the main problem

Let $\mathcal{A}$ be a graded Poisson algebra, such that its deformations and its quantizations have the same parameter space $\mathcal{P}$. A conical symplectic singularity X offers a natural example: let $\mathcal{A} = \mathbb{C}[X]$, then, by Theorem 6.1.3, $\mathcal{P} = \mathfrak{P}/W$. Let c, c' be two deformation parameters. They correspond to two deformations $\mathfrak{X}_{0,c}$ and $\mathfrak{X}_{0,c'}$. These are distinct as filtered deformations, but the underlying algebras may still be isomorphic as Poisson algebras. Let us consider a diagram where vertices correspond to points in $\mathcal{P}$ and arrows are Poisson isomorphisms between the corresponding deformations. Alternatively, we could consider a different diagram, where the arrows between c and c' correspond to the associative algebra isomorphisms between the quantizations $\mathfrak{X}_{1,c}$ and $\mathfrak{X}_{1,c'}$. We get a picture like this,



where the red arrows correspond to Poisson isomorphisms of the deformations and the blue arrows to associative algebra isomorphisms between the quantizations. Since the arrows are all invertible, we have defined two groupoids: let us denote them $\mathrm{PIso}(\mathcal{A})$, for the Poisson isomorphisms, and $\mathrm{Iso}(\mathcal{A})$, for the associative algebra isomorphisms. It is natural to ask the following question.

Question. What is the relation between the groupoids $\mathrm{PIso}(\mathcal{A})$ and $\mathrm{Iso}(\mathcal{A})$? In which cases are they isomorphic?

The question is in general very hard to solve. In the case of conical symplectic singularities, there is some positive evidence for the two groupoids to be isomorphic.

Let X be a conical symplectic singularity and consider $\mathcal{G}$, the group of *graded* Poisson automorphisms of $\mathbb{C}[X]$. Let $c \in \mathfrak{P}/W$ and consider the corresponding deformation $\mathfrak{X}_{0,c}$. This comes with the choice of an isomorphism $i : \mathrm{gr}(\mathfrak{X}_{0,c}) \cong \mathbb{C}[X]$. Let $g \in \mathcal{G}$ and consider the following diagram:

$$\begin{array}{ccc} \mathrm{gr}(\mathfrak{X}_{0,c}) & & \\ \big\downarrow{\scriptstyle i} & \overset{g\circ i}{\searrow} & \\ \mathbb{C}[X] & \xrightarrow{\ g\ } & \mathbb{C}[X] \end{array}$$

Since g is graded Poisson, the couple $(\mathfrak{X}_{0,c}, g \circ i)$ is another deformation of $\mathbb{C}[X]$. This means that there exists some parameter $c' \in \mathfrak{P}/W$ such that $(\mathfrak{X}_{0,c}, g \circ i) \cong (\mathfrak{X}_{0,c'}, i')$. In particular, there exists an isomorphism of filtered algebras $\mathfrak{X}_{0,c} \cong \mathfrak{X}_{0,c'}$. In other words, the group $\mathcal{G}$ acts on the parameter space $\mathfrak{P}/W$. We have a second action of $\mathcal{G}$ on $\mathfrak{P}/W$, constructed similarly, by looking at the induced isomorphisms on the quantizations. Up to this point, we have only used the properties of the parameter space. In general, the two actions may have no connection with each other. However, for conical symplectic singularities we have:

Theorem 6.2.1. *The Poisson and quantum actions of $\mathcal{G}$ on $\mathfrak{P}/W$ coincide.*

Proof. See [Lo2, Section 3.7]. □

Corollary 6.2.2. *Two deformations $\mathfrak{X}_{0,c}$ and $\mathfrak{X}_{0,c'}$ of $\mathbb{C}[X]$ are isomorphic as filtered Poisson algebras if and only if the two corresponding quantizations $\mathfrak{X}_{1,c}$ and $\mathfrak{X}_{1,c'}$ are isomorphic as filtered associative algebras.*

Proof. Let $\phi : \mathfrak{X}_{1,c} \to \mathfrak{X}_{1,c'}$ a filtered algebra isomorphism. Consider the following diagram:

$$\begin{array}{ccc} \mathrm{gr}(\mathfrak{X}_{1,c}) & \xrightarrow{\mathrm{gr}(\phi)} & \mathrm{gr}(\mathfrak{X}_{1,c'}) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i'} \\ \mathbb{C}[X] & \xrightarrow{\ g\ } & \mathbb{C}[X] \end{array}$$

Notice that $\mathrm{gr}(\phi)$ is well defined since ϕ is filtered, and it is a graded Poisson algebra isomorphism. Thus $g = i' \circ \mathrm{gr}(\phi) \circ i^{-1}$ is an element of $\mathcal{G}$. We get the commuting diagram

$$\begin{array}{ccc} \mathrm{gr}(\mathfrak{X}_{1,c}) & \xrightarrow{\mathrm{gr}(\phi)} & \mathrm{gr}(\mathfrak{X}_{1,c'}) \\ \big\downarrow{\scriptstyle i} & \overset{g\circ i}{\searrow} & \big\downarrow{\scriptstyle i'} \\ \mathbb{C}[X] & \xrightarrow{\ g\ } & \mathbb{C}[X] \end{array}$$

thus $(\mathfrak{X}_{1,c}, g \circ i)$ is isomorphic to $(\mathfrak{X}_{1,c'}, i')$ as quantizations. This means that $c' = g(c)$ under the $\mathcal{G}$-action on $\mathfrak{P}/W$. In particular, two quantizations are isomorphic as filtered associative algebras if and only if the corresponding parameters are in the same $\mathcal{G}$-orbit.

We can reason analogously for the deformations. Since, by Theorem 6.2.1, the Poisson and quantum actions coincide, this completes the proof. □

This result is of course not enough to answer Question 6.2. It says nothing about non-filtered isomorphisms and, a priori, we could also have multiple filtered isomorphisms with the same associated graded.

Remark 6.2.3. Ideally, we would like to regard the deformation $\mathfrak{X}_{0,c}$ as the classical limit of the corresponding quantization $\mathfrak{X}_{1,c}$. The associated graded of $\mathbb{C}[\mathfrak{X}_{1,c}]$ though is $\mathbb{C}[X]$ and not $\mathbb{C}[\mathfrak{X}_{0,c}]$. In the case of symplectic quotient singularities, the spherical subalgebra realization of deformations and quantizations (see Section 6.1.1) allows us to make this connection explicit, as $\mathsf{e}\mathsf{H}_{0,c}\mathsf{e}$ is the classical limit of $\mathsf{e}\mathsf{H}_{1,c}\mathsf{e}$ with respect to the auxiliary parameter $\mathtt{t}$. This can be useful, and we employ a similar argument as part of the proof of Theorem 6.2.6 in type **A**. Unfortunately, it does not provide a general way to directly relate Poisson and associative algebra isomorphisms.

Question 6.2 is motivated by the Belov-Kanel-Kontsevich Conjecture [BK2].

Conjecture 6.2.4. Let (V, ω) be a finite-dimensional symplectic vector space. The automorphism group of the Weyl algebra $W(V, \omega)$ is isomorphic to the group of Poisson automorphisms of the symmetric algebra $\mathcal{S}(V)$, with the standard Poisson structure induced by the symplectic structure.

The symmetric algebra $\mathcal{S}(V)$ is quite rigid, in fact it admits exactly one (up to isomorphisms) filtered quantization, the Weyl algebra, and exactly one filtered deformation, the trivial deformation, that is $\mathcal{S}(V)$ itself. Since the parameter space is only a point, the groupoids $\mathrm{PIso}(\mathcal{S}(V))$ and $\mathrm{Iso}(\mathcal{S}(V))$ coincide with the groups of (Poisson) isomorphisms. In particular, we can regard $\mathcal{S}(V) \cong \mathbb{C}[V^*]$ as the algebra of function of the symplectic quotient associated to the trivial group $G = 1$. Then the following is a direct generalization of Conjecture 6.2.4.

Conjecture 6.2.5. Let V be a finite-dimensional vector space and G a finite subgroup of $Sp(V)$. There is an isomorphism of groupoids

$$\mathrm{PIso}(V/G) \cong \mathrm{Iso}(V/G).$$

The Belov-Kanel Kontsevich Conjecture is known to be true only in dimension 2. We want to check if Conjecture 6.2.5 holds in dimension 2 as well. Symplectic quotient singularities in dimension 2 are exactly the Kleinian singularities. In Chapter 7, we concentrate our study on Kleinian singularities of type **A** and **D**, confirming Conjecture 6.2.5 in those cases.

Theorem 6.2.6. *Let V/Γ be a Kleinian singularity of type* **A** *or* **D**. *Then*

$$\mathrm{PIso}(V/\Gamma) \cong \mathrm{Iso}(V/\Gamma).$$

§ 6.3 | Kleinian singularities

Let us briefly recall some basic facts and definitions about Kleinian singularities.

Definition 6.3.1. Let V be a two-dimensional vector space and Γ a finite subgroup of $\mathrm{SL}(V)$. The quotient $V/\Gamma := \mathrm{Spec}(\mathbb{C}[V]^\Gamma)$ is called a *Kleinian singularity.*

Since, in dimension 2, we have $\mathrm{SL}(V) = \mathrm{Sp}(V)$, Kleinian singularities are symplectic quotients. The invariant ring $\mathbb{C}[V]^\Gamma$ is generated by three elements, subject to one polynomial relation [PV, Section 0.3]. The Kleinian singularity V/Γ embeds into the three-dimensional affine space $\mathbb{A}^3$ as a hypersurface with a singularity at the origin. These hypersurfaces are also known as Du Val singularities or rational double points.

Theorem 6.3.2 (McKay Correspondence)**.** *Finite subgroups of* $\mathrm{SL}_2(\mathbb{C})$ *up to conjugation (and thus Kleinian singularities) are classified by the simply-laced Dynkin diagrams* $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$.

Proof. This is a well-known result. The Dynkin diagram can be associated in two ways. From the geometric side, if we look at the configuration of the exceptional divisors of the minimal resolution of V/Γ, we recover the corresponding Dynkin diagram Δ (see [Sl, Chapter 6.1]). On the group-theoretic side, consider $V_0, \dots, V_k$, the simple representations of Γ. The group Γ acts on $\mathbb{C}^2$ as a subgroup of SL_2. Compute the coefficients in the decomposition

$$V_i \otimes \mathbb{C}^2 = \bigoplus_{j=0}^{k} m_{i,j} V_j,$$

and draw the graph with $k+1$ vertices and $m_{i,j}$ edges between the vertices i, j (notice that $\mathbb{C}^2$ is self-dual as a representation of Γ, so $m_{i,j} = m_{j,i}$). This graph, known as the McKay graph of Γ is the extended Dynkin diagram $\hat{\Delta}$ of Δ (see [Sl, Appendix III]). □

Since Kleinian singularities are symplectic quotient singularities, the results of Theorem 6.1.3 and 6.1.15 apply. Deformations and quantizations of Kleinian singularities have been studied in much more detail, so we can provide a more precise description of their construction. First of all, the Namikawa Cartan space $\mathfrak{P}$ and the Namikawa Weyl group W are isomorphic to the actual Cartan space $\mathfrak{h}$ and the actual Weyl group of

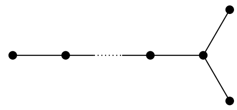
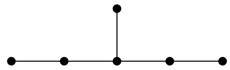
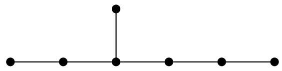
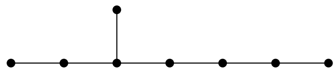
Type	Group	Diagram	Equation
$\mathbf{A}_{n-1}$, $n \geq 2$	Cyclic $\mathbb{Z}_n$		$xy - z^n = 0$
$\mathbf{D}_n$, $n \geq 4$	Binary Dihedral $\mathbb{BD}_{4(n-2)}$		$x^{n-1} + xy^2 + z^2$
$\mathbf{E}_6$	Binary Tetrahedral $\mathbb{BT}$		$x^4 + y^3 + z^2$
$\mathbf{E}_7$	Binary Octahedral $\mathbb{BO}$		$x^3y + y^3 + z^2$
$\mathbf{E}_8$	Binary Icosahedral $\mathbb{BI}$		$x^5 + y^3 + z^2$

Table 6.1: Classification of Kleinian singularities

corresponding Dynkin type.

The filtered deformations of $\mathbb{C}[V]^\Gamma$ can be constructed explicitly by considering the semi-universal deformation of V/Γ (seen as a hypersurface in $\mathbb{A}^3$) and specializing the parameters. Explicitly, every deformation of $\mathbb{C}[V]^\Gamma$ is the algebra of functions of a hypersurface in $\mathbb{A}^3$ with equation depending on $\dim(\mathfrak{h}/W)$ parameters.

The explicit presentation by generators and relations of the filtered quantizations of $\mathbb{C}[V]^G$ is known in type **A** and type **D**. We describe them in detail in the rest of this section, as they are crucial to the proof of Theorem 6.2.6. Unfortunately, no explicit presentation is known in type **E**. This was the main obstruction to extending Theorem 6.2.6 to the exceptional types.

Remark 6.3.3. The fact that the families of deformations and quantizations that we consider coincide with the universal families coming from the theory of Namikawa and Losev needs some explanation. For the deformations, the fact that the semi-universal deformation coincides with the universal Poisson deformation was proved in [LNS, Theorem 1.3]. The analogous result for the quantization is proved in Remarks 6.3.8 and Remarks 6.3.9. We use the fact that, to prove that an algebra is the universal quantization of $\mathbb{C}[V]^\Gamma$, it is sufficient to show that its associated graded coincides the universal Poisson deformation. A proof of this result can be found in [ACET, Lemma 2.14]. It is worth pointing out that this is not true for general algebras, but holds for the quantizations and deformations of the algebra of functions of a symplectic quotient singularity, so in particular for Kleinian singularities.

The term "universal" used here for deformations and quantizations has a precise category theoretic meaning. For our purposes, it just means that all the possible deformations and quantizations are obtained from the universal one, by specialization of parameters. For a more precise definition, we refer to [ACET].

§ 6.3.1 | Type A

Let V be a complex vector space of dimension 2. Choose a basis for V, and let X, Y be the corresponding coordinate functions. We then identify $\mathrm{SL}(V)$ with $\mathrm{SL}_2(\mathbb{C})$. Take $\Gamma = C_n \subset \mathrm{SL}_2(\mathbb{C})$ the cyclic group of order n, with $n \geq 2$, acting on V via matrices of the form

$$\begin{pmatrix} e^{2k\pi i/n} & 0 \\ 0 & e^{-2k\pi i/n} \end{pmatrix}, \quad \text{with } 1 \leq k \leq n\,.$$

The quotient V/Γ is the Kleinian singularity of type $\mathbf{A_{n-1}}$. The algebra of functions $\mathbb{C}[V/\Gamma] = \mathbb{C}[V]^\Gamma$ is generated by the monomials X^n, Y^n and XY, so

$$\mathbb{C}[\mathbf{A_{n-1}}] = \mathbb{C}[x, y, z]/(xy - z^n)\,,$$

for all $n \geq 2$. Considering X, Y to be of degree 1, we have a grading on $\mathbb{C}[\mathbf{A_{n-1}}]$ given by $\deg x = \deg y = n$ and $\deg z = 2$. The symplectic structure on V induces a Poisson structure on $\mathbb{C}[\mathbf{A_{n-1}}]$; we choose here the normalization such that $\{Y, X\} = 1/n$. The induced Poisson structure on $\mathbb{C}[\mathbf{A_{n-1}}]$ is then

$$\{x, y\} = -nz^{n-1}\,, \quad \{z, x\} = x \quad \{z, y\} = -y\,,$$

with Poisson bracket of degree -2.

Remark 6.3.4. Let ψ be a polynomial in x, y, z. It induces a Poisson structure on $\mathbb{C}[x, y, z]$, defined by

$$\{x, y\} = \frac{\partial \psi}{\partial z}, \quad \{x, z\} = -\frac{\partial \psi}{\partial y}, \quad \{y, z\} = \frac{\partial \psi}{\partial x}.$$

Since the ideal $(\psi) \subset \mathbb{C}[x, y, z]$ is Poisson, ψ induces a Poisson structure on the quotient $\mathbb{C}[x, y, z]/(\psi)$. If, additionally, ψ is homogeneous with respect to some grading of $\mathbb{C}[x, y, z]$, then $\mathbb{C}[x, y, z]$ and $\mathbb{C}[x, y, z]/(\psi)$ are graded Poisson algebras, and the Poisson bracket has degree $(\deg \psi - \deg x - \deg y - \deg z)$. The Poisson structure on $\mathbb{C}[\mathbf{A_{n-1}}]$ is of this type, where $\psi = xy - z^n$.

In type $\mathbf{A}_{n-1}$, the Cartan algebra is $\mathfrak{h} = \mathbb{C}^{n-1}$ and the Weyl group is the symmetric group $\mathcal{S}_n$. Thus $\mathbb{C}[\mathfrak{h}]^W \cong \mathbb{C}[s_2, \dots, s_n]$, where s_i is the elementary symmetric polynomial of degree i. The parameter space is thus $\mathfrak{h}/W \cong \mathbb{C}^{n-1}$. The deformation parameters can be arranged as coefficients of a monic polynomial $P \in \mathbb{C}[z]$ of degree n, with no degree $n-1$ term, $P(z) = z^n + \sum_{i=2}^{n} s_i z^{n-i}$ (see [KM, Table 3]).

Denote by $\mathcal{A}(P)$ the deformation associated to the parameter P of the algebra $\mathbb{C}[\mathbf{A_{n-1}}]$. Explicitly, we have

$$\mathcal{A}(P) = \mathbb{C}[x, y, z]/(xy - P(z)).$$

We consider on $\mathcal{A}(P)$ the Poisson algebra structure of Remark 6.3.4:

$$\{x,y\} = \frac{\partial\psi}{\partial z} = -P'(z), \quad \{z,x\} = \frac{\partial\psi}{\partial y} = x, \quad \{z,y\} = -\frac{\partial\psi}{\partial x} = -y.$$

The algebra $\mathcal{A}(P)$ is a filtered Poisson algebra, with filtration induced by the degree on the generators $\deg x = \deg y = n$ and $\deg z = 2$, with Poisson bracket of degree -2.

Quantizations of $\mathbb{C}[\mathbf{A_{n-1}}]$ are examples of generalized Weyl algebras [BJ]. We recall here the general definition.

Definition 6.3.5. Let D be a ring, σ an automorphism of D, and a a central element of D. The generalized Weyl algebra $D(\sigma, a)$ is the ring extension of D generated by two indeterminates x, y subject to the relations

1. $xd = \sigma(d)x$ and $yd = \sigma^{-1}(d)y$ for all $d \in D$,
2. $xy = \sigma(a)$ and $yx = a$.

In our case, the quantization associated to the parameter P is the generalized Weyl algebra

$$A(P) := \mathbb{C}[z](\sigma, P),$$

with σ defined by $z \mapsto z-1$. Explicitly, $A(P)$ is the $\mathbb{C}$-algebra generated by x, y, z subject to the relations

$$xz = (z-1)x, \quad yz = (z+1)y, \quad xy = P(z-1), \quad yx = P(z).$$

The algebra $A(P)$ is a filtered associative algebra, with filtration induced by $\deg x = \deg y = n$ and $\deg z = 2$. The commutator is of degree -2.

We give an explicit construction of the algebra $\mathcal{A}(P)$ as the semi-classical limit of $A(P)$. Define

$$A_t(P) := \mathbb{C}[t,z](\sigma_t, P),$$

with σ_t induced by $z \mapsto z-t$, $t \mapsto t$. Explicitly, $A_t(P)$ is the $\mathbb{C}[t]$-algebra generated by x, y, z subject to the following relations:

$$xz = (z-t)x, \quad yz = (z+t)y, \quad xy = P(z-t), \quad yx = P(z). \tag{6.3.1}$$

Remark 6.3.6. The defining relations of $A(P)$ and $A_t(P)$ imply respectively that

$$[x,y] = (\sigma - 1)(P), \qquad [x,y] = (\sigma_t - 1)(P).$$

Denote by $A_1(P) := A_t(P)/((t-1)A_t(P))$ and by $A_0(P) := A_t(P)/(tA_t(P))$.

Proposition 6.3.7. *The algebras $A_1(P)$ and $A(P)$ are isomorphic as filtered asso-*

ciative algebras. The algebra A_0 has the structure of a filtered Poisson algebra, with Poisson bracket given by $\{\cdot,\cdot\} = \frac{1}{t}[\cdot,\cdot] \mod (t)$. Moreover, the algebras $A_0(P)$ and $\mathcal{A}(P)$ are isomorphic as filtered Poisson algebras.

Proof. The isomorphism $A_1(P) \cong A(P)$ is trivial.

Notice that $[A_t(P), A_t(P)] \subset tA_t(P)$ and $A_t(P)$ is free over $\mathbb{C}[t]$, so $A_0(P)$ is commutative and $\{\cdot,\cdot\} = \frac{1}{t}[\cdot,\cdot] \mod (t)$ defines a Poisson bracket. We can see that it coincides with the one of $\mathcal{A}(P)$: the only non-trivial check is

$$\frac{1}{t}[x,y] \mod (t) = \frac{P(z-t)-P(z)}{t} \mod (t) = -P'(z) \mod (t).$$

□

Remark 6.3.8. Consider the deformation parameters $s_2,\dots,s_n$ to be formal. The ring $\mathbb{C}[\mathfrak{h}/W] = \mathbb{C}[s_2,\dots,s_n]$ is graded, with $\deg s_i = i$, for all i. We consider the $\mathbb{C}[\mathfrak{h}/W]$-algebra generated by three elements x,y,z, subject to the same relations of $A(P)$. It is a filtered algebra, with x,y in degree n, z in degree 2, and s_i in degree i. Its associated graded is easily seen to be commutative, subject to a single relation $xy = P(z)$. This coincides with the universal Poisson deformation. This proves that the family of quantizations considered here is indeed a universal family (see Remark 6.3.3).

§ 6.3.2 | Type D

Once again, let V be a complex vector space of dimension 2. Choose a basis for V, and let X,Y be the corresponding coordinate functions. We then identify $\mathrm{SL}(V)$ with $\mathrm{SL}_2(\mathbb{C})$. Take $\Gamma = BD_{n-2} \subset \mathrm{SL}_2(\mathbb{C})$ the binary dihedral group of order $4(n-2)$, with $n \geq 4$. The group Γ is generated by

$$\sigma := \begin{pmatrix} e^{\pi i/(n-2)} & 0 \\ 0 & e^{-\pi i/(n-2)} \end{pmatrix}, \quad \tau := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The quotient V/Γ is the Kleinian singularity of type $\mathbf{D_n}$. The algebra of functions $\mathbb{C}[V/\Gamma] = \mathbb{C}[V]^\Gamma$ is generated by the polynomials

$$x = X^2Y^2 \quad y = (X^{2(n-2)} + Y^{2(n-2)}), \quad z = XY(X^{2(n-2)} - Y^{2(n-2)}).$$

Thus

$$\mathbb{C}[\mathbf{D_n}] \cong \mathbb{C}[x,y,z]/(x^{n-1} + xy^2 + z^2),$$

for all $n \geq 4$. Considering X,Y to be of degree 1, we have a grading on $\mathbb{C}[\mathbf{D_n}]$ given by $\deg x = 4$, $\deg y = 2(n-2)$ and $\deg z = 2(n-1)$. We give $\mathbb{C}[\mathbf{D_n}]$ the structure of a graded Poisson algebra as described in Remark 6.3.4, with Poisson bracket of degree -2. This structure is equivalent to the one induced by the standard symplectic structure on V, after a suitable renormalization. Explicitly, we have

$$\{x, y\} = 2z\,, \quad \{x, z\} = -2xy, \quad \{y, z\} = (n-1)x^{n-2} + y^2\,.$$

The Cartan space of $\mathbf{D}_n$ is $\mathfrak{h} = \mathbb{C}^n$ and the Weyl group is $W = (\mathbb{Z}/2)^{n-1} \rtimes S_n$, with $(\mathbb{Z}/2)^{n-1}$ acting as even number of sign changing. Thus $\mathbb{C}[\mathfrak{h}]^W \cong \mathbb{C}[\gamma, \delta_2, \delta_4, \dots, \delta_{2(n-1)}]$, where

$$\gamma := s_n(t_1, \dots, t_n),$$
$$\delta_{2i} := s_i(t_1^2, \dots, t_n^2),$$

where $t_1, \dots, t_n$ are coordinates for $\mathfrak{h}$ and s_i is the i-th symmetric polynomial. We arrange the deformation parameters δ_{2i} as coefficients of $Q \in \mathbb{C}[x]$, a degree $n-1$ monic polynomial

$$Q(x) = x^{n-1} + \sum_{i=1}^{n-1} \delta_{2i} x^{n-i-1}.$$

The deformation parameter is then a pair (Q, γ), with Q as above and $\gamma \in \mathbb{C}$.

Fix $n \geq 4$ and denote by $\mathcal{D}_n(Q, \gamma)$ the deformation of the algebra $\mathbb{C}[\mathbf{D_n}]$ associated to the parameter (Q, γ) . Explicitly:

$$\mathcal{D}_n(Q, \gamma) = \mathbb{C}[x, y, z]/(Q(x) + xy^2 + z^2 - \gamma y).$$

The deformation $\mathcal{D}_n(Q, \gamma)$ is a Poisson algebra as in Remark 6.3.4,

$$\{x, y\} = \frac{\partial \psi}{\partial z} = 2z, \quad \{x, z\} = -\frac{\partial \psi}{\partial y} = -2xy + \gamma, \quad \{y, z\} = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} Q(x) + y^2.$$

The algebra $\mathcal{D}_n(Q, \gamma)$ is a filtered Poisson algebra, with filtration induced by $\deg x = 4$, $\deg y = 2(n-2)$ and $\deg z = 2(n-1)$, with Poisson bracket of degree -2.

We denote by $D_n(Q, \gamma)$ the quantization associated to the parameter (Q, γ). The algebras $D_n(Q, \gamma)$ were studied by Levy in [Le]. They are the $\mathbb{C}$-algebras generated by x, y, z, subject to the relations

$$[x, y] = 2z, \quad [x, z] = -2xy + 2z + \gamma, \quad [y, z] = y^2 + P(x) - n,$$

$$Q(x) + x(y^2 - n) + z^2 - 2yz - \gamma y = 0,$$

where $P(x)$ is the unique degree $n-2$ polynomial satisfying

$$Q(-x(x-1)) - Q(-x(x+1)) = (x-1)P(-x(x-1)) + (x+1)P(-x(x+1)).$$

Remark 6.3.9. Our parametrization of the quantizations $D_n(Q, \gamma)$ differs slightly from Levy's. The algebra defined in [Le, Definition 1.5] coincides with our $D_n(Q + nx, \gamma)$. We prove that the family of quantizations we considered is a universal family.

Consider the deformation parameters $\gamma, \delta_2, \delta_4, \ldots, \delta_{2(n-1)}$ to be formal. The ring $\mathbb{C}[\mathfrak{h}/W] = \mathbb{C}[\gamma, \delta_2, \delta_4, \ldots, \delta_{2(n-1)}]$ is graded, with $\deg \gamma = 2n$, $\deg \delta_{2i} = 4i$ for all i (to be coherent with our grading on x, y, z, we are doubling all the degrees). We consider the $\mathbb{C}[\mathfrak{h}/W]$-algebra generated by three elements x, y, z, subject to the same relations of $D(Q, \gamma)$. It is a filtered algebra, with x in degree 4, y in degree $2(n-2)$, z in degree $2(n-1)$, γ in degree $2n$, and δ_{2i} in degree $4i$. Its associated graded is easily seen to be commutative, subject to a single relation $Q(x) + xy^2 + z^2 - \gamma y = 0$. This coincides with the universal Poisson deformation. This proves that the family of quantizations considered here is indeed a universal family (see Remark 6.3.3).

Chapter 7

Proof of the Main Result II

In this Chapter we prove Theorem 6.2.6. The proof is case by case, so we have different proofs for type **A** and type **D**.

We introduce here the following notation, that will be used throughout the rest of the chapter. If ϕ is an endomorphism of an algebra with three generators x, y, z, we will identify ϕ with the triple $(\phi(x), \phi(y), \phi(z))$ of its value on the generators (x, y, z).

§ 7.1 | Type A

The proof is structured as follows. Let P be a deformation parameter. Bavula computed the generators of the group of automorphisms of the quantizations $A(P)$. First, we prove that they all descend to Poisson automorphisms of the corresponding deformation $\mathcal{A}(P)$. We prove that the map $\mathrm{Aut}(A(P)) \to \mathrm{PAut}(\mathcal{A}(P))$ is injective. We do so by expressing both $\mathrm{Aut}(A(P))$ and its image as abstract groups, and proving that they are isomorphic. In particular, we show that they admit an amalgamated free product structure. Then we show that the map $\mathrm{Aut}(A(P)) \to \mathrm{PAut}(\mathcal{A}(P))$ is surjective too. Finally, we prove that the isomorphism groupoids are generated by the automorphism groups and one single isomorphism. We show that these satisfy the same relations in both the Poisson and quantum case, thus proving the isomorphism of groupoids.

§ 7.1.1 | Automorphisms of the quantizations

Let P be a deformation parameter. Define, for all $m \in \mathbb{N}$, the operators $\delta_m := (\sigma^m - 1)$ and $\delta_{m,t} := (\sigma_t^m - 1)$, so that

$$[x, y] = \delta_1(P) \qquad [x, y] = \delta_{1,t}(P),$$

respectively in $A(P)$ and $A_t(P)$.

Theorem 7.1.1 ([BJ, Theorem 3.29]). *The group of automorphisms of the algebra $A(P)$ has the following generators:*

(a)

$$\Phi_{\lambda,m} = \left(x + \sum_{i=1}^{n} \frac{(-\lambda)^i}{i!} y^{im-1} \delta_m^i(P), y, z + m\lambda y^m \right),$$

for every $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$;

(b)

$$\Psi_{\lambda,m} = \left(x, y + \sum_{i=1}^{n} \frac{\lambda^i}{i!} \delta_m^i(P) x^{im-1}, z - m\lambda x^m \right),$$

for every $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$;

(c)

$$\Theta_\nu = (\nu x, \nu^{-1} y, z),$$

for all $\nu \in \mathbb{C}^\times$;

(d) There is an isomorphism

$$\Omega : A(P) \to A((-1)^n P(-z)), \quad \Omega := (y, (-1)^n x, 1 - z).$$

This induces an automorphism of $A(P)$ if and only if the polynomial P is either odd or even, i.e. if $P(-z) = \pm P(z)$. In these cases, Ω needs to be added to the generators.

Remark 7.1.2. A polynomial of degree n is called reflective if there exists a $\rho \in \mathbb{C}$ such that $P(\rho - x) = (-1)^n P(x)$. If P is reflective, then $A(P)$ has an automorphism of the form $\Omega_\rho = (y, (-1)^n x, 1 + \rho - z)$ [BJ, Lemma 3.8]. Notice though that if we restrict ourselves to polynomials with no term of degree $n-1$, P can be reflective only if $\rho = 0$. In that case, the polynomial P is either odd or even, and $\Omega_\rho = \Omega$. For brevity and to stick with the original notation of Bavula and Jordan, we write "P is reflective" instead of "P is either odd or even".

Remark 7.1.3. We can define the automorphisms of type Φ and Ψ in a different way. For all $m \geq 0$, consider $\mathrm{ad}(x^m)$ and $\mathrm{ad}(y^m)$, the adjoint actions of x^m and y^m. These are locally nilpotent derivations of $A(P)$ (see [BJ, Lemma 3.4]). If we exponentiate them, we get automorphisms of $A(P)$. In fact,

$$e^{\lambda \,\mathrm{ad}(x^m)} = \Phi_{\lambda,m} \text{ and } e^{\lambda \,\mathrm{ad}(y^m)} = \Psi_{\lambda,m},$$

(see [BJ, Lemma 3.4]).

We can lift the generators of $\mathrm{Aut}(A(P))$ to the algebra $A_t(P)$. Define the following $\mathbb{C}[t]$-linear automorphisms of $A_t(P)$:

(a)

$$\Phi_{\lambda,m,t} := e^{\lambda/t \,\mathrm{ad}(x^m)} = \left(x + \sum_{i=1}^{n} \frac{(-\lambda)^i}{t^i i!} y^{im-1} \delta_{m,t}^i(P), y, z + m\lambda y^m \right),$$

for all $\lambda \in \mathbb{C}$ and $m \geq 0$;

(b)

$$\Psi_{\lambda,m,t} := e^{\lambda/t\,\mathrm{ad}(y^m)} = \left(x, y + \sum_{i=1}^{n} \frac{\lambda^i}{t^i i!}\delta^i_{m,t}(P)x^{im-1}, z - m\lambda x^m\right),$$

for all $\lambda \in \mathbb{C}$ and $m \geq 0$;

(c) $\Theta_{\nu,t} = (\nu x, \nu^{-1} y, z)$ for all $\nu \in \mathbb{C}^\times$;

We also lift the isomorphism Ω:

(d)

$$\Omega_t : A_t(P) \to A_t((-1)^n P(-z)), \quad \Omega_t := (y, (-1)^n x, t - z).$$

This is an automorphism if and only if P is reflective.

We can also consider how they act on the quotient A_0. Using that $\{\cdot,\cdot\} = \frac{1}{t}[\cdot,\cdot] \mod (t)$, we have

$$e^{\lambda\,\mathrm{ad}(x^m)} = \Phi_{\lambda,m,0} \text{ and } e^{\lambda\,\mathrm{ad}(y^m)} = \Psi_{\lambda,m.0},$$

where now "ad" denotes the adjoint action with respect to the Poisson bracket.

Denote by G_t the group generated by the automorphisms (a)-(d) of $A_t(P)$. Since the automorphisms in G_t are $\mathbb{C}[t]$-linear, G_t acts on the algebras $A_0(P)$ and $A_1(P)$. Denote the images of G_t in these representations as G_0 and G_1 respectively. By Theorem 7.1.1,

$$G_1 = \mathrm{Aut}(A(P)).$$

We also have

$$G_0 \leq \mathrm{PAut}(\mathcal{A}(P)). \tag{7.1.1}$$

In fact, let $\phi \in G_0$ and $a, b \in \mathcal{A}(P) \cong A_0(P)$. Then

$$\phi(\{a,b\}) = \frac{1}{t}\phi([\hat{a},\hat{b}]) \mod (t) = \frac{1}{t}[\phi(\hat{a}),\phi(\hat{b})] \mod (t) = \{\phi(a),\phi(b)\},$$

where $\hat{a}, \hat{b}$ are some lifts of a, b in $A_t(P)$.

§ 7.1.2 | Amalgamated free product group structure

The goal of this section is to prove Theorem 7.1.14, which gives us the explicit group structure of $G_t(P)$ as an abstract group.

The following proposition gives us some useful identities.

Lemma 7.1.4. *The following relations hold in G_t:*

$$\Theta_{\nu,t} \circ \Theta_{\mu,t} = \Theta_{\mu+\nu,t}, \tag{7.1.2}$$

$$\Theta_{\nu,t} \circ \Psi_{\lambda,m,t} = \Psi_{\lambda\nu^m,m,t} \circ \Theta_{\nu,t}, \tag{7.1.3}$$

$$\Theta_{\nu,t} \circ \Phi_{\lambda,m,t} = \Phi_{\lambda\nu^{-m},m,t} \circ \Theta_{\nu,t}, \tag{7.1.4}$$

for all $\nu \in \mathbb{C}^\times$, $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$. We also have the following relations between automorphisms in G_t and the isomorphism Ω_t:

$$\Omega_t^2 = \Theta_{(-1)^n,t}, \tag{7.1.5}$$

$$\Omega_t \circ \Theta_{\nu,t} = \Theta_{\nu^{-1},t} \circ \Omega_t \tag{7.1.6}$$

$$\Theta_{(-1)^n,t} \circ \Omega_t \circ \Phi_{\lambda,m,t} \circ \Omega_t = \Psi_{\lambda,m,t}, \tag{7.1.7}$$

for all $\nu \in \mathbb{C}^\times$, $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$. Notice that, if P is reflective, these are all relations in G_t. In particular, if P is reflective, G_t can be generated by just the automorphisms $\Theta_{\nu,t}$, $\Phi_{\lambda,m,t}$ and Ω_t.

Proof. Equations (7.1.2) and (7.1.5) follow from a direct check.

To prove (7.1.3), compute

$$\begin{aligned}\Theta_{\nu,t} \circ \Psi_{\lambda,m,t} &= \left(\nu x, \nu^{-1} y + \sum_{i=1}^{n} \frac{\lambda^i}{t^i i!} \nu^{im-1} \delta^i_{m,t}(P) x^{im-1}, z - \nu^m m \lambda x^m\right) \\ &= \Psi_{\lambda\nu^m,m,t} \circ \Theta_{\nu,t}.\end{aligned}$$

Similarly, for equation (7.1.4)

$$\begin{aligned}\Theta_{\nu,t} \circ \Phi_{\lambda,m,t} &= \left(\nu x + \sum_{i=1}^{n} \frac{(-\lambda)^i}{t^i i!} \nu^{1-im} y^{im-1} \delta^i_{m,t}(P), \nu^{-1} y, z + \nu^{-m} m \lambda y^m\right) \\ &= \Phi_{\lambda\nu^{-m},m,t} \circ \Theta_{\nu,t}.\end{aligned}$$

Equation (7.1.6) follows from a direct check

$$\Omega_t \circ \Theta_{\nu,t} = (\nu y, (-1)^n \nu^{-1} x, t - z) = \Theta_{\nu^{-1},t} \circ \Omega_t.$$

Equation (7.1.7) is more involved. Let us check it separately on each of the three generators. For x and z we have

$$x \mapsto y \mapsto y \mapsto (-1)^n x \mapsto x,$$

$$z \mapsto t - z \mapsto t - z - m\lambda y^m \mapsto z - (-1)^{n \cdot m} \lambda m x^m \mapsto z - m\lambda x^m.$$

Let us now check it for y:

$$\begin{aligned}y &\mapsto (-1)^n x \mapsto (-1)^n \left[x + \sum_{i=1}^{n} \frac{(-\lambda)^i}{t^i i!} y^{im-1} \delta^i_{m,t}(P)\right] \\ &\mapsto (-1)^n \left[y + \sum_{i=1}^{n} \frac{(-\lambda)^i}{t^i i!} (-1)^{(im-1)n} x^{im-1} (\delta^i_{m,t} P)(t - z)\right] \\ &\mapsto y + (-1)^n \sum_{i=1}^{n} \frac{(-\lambda)^i}{t^i i!} x^{im-1} (\delta^i_{m,t} P)(t - z).\end{aligned} \tag{7.1.8}$$

The final term in (7.1.8) is similar to $\Psi_{\lambda,m,t}(y)$, but we need to move x^{im-1} to the right of $(\delta^i_{m,t}P)(t-z)$. By relations (6.3.1), we can do that by applying σ_t^{im-1} to $(\delta^i_{m,t}P)(t-z)$. Notice that

$$(\delta^i_{m,t}P)(t-z) = \sigma_t \circ \gamma \circ (\sigma_t^m - 1)^i(P),$$

where γ is the map $z \mapsto -z$. Hence

$$(\delta^i_{m,t}P)(t-z) = (-1)^n \sigma_t \circ (\sigma_t^{-m} - 1)^i(P),$$

because P is reflective. It follows that

$$\begin{aligned}\sigma_t^{im-1}((\delta^i_{m,t}P)(t-z)) &= (-1)^n \sigma_t^{im} \circ (\sigma_t^{-m} - 1)^i(P)\\ &= (-1)^n (1-\sigma_t^m)^i(P)\\ &= (-1)^{n+i}\delta^i_{m,t}(P).\end{aligned}$$

Putting this together with (7.1.8) we get

$$\Theta_{(-1)^n,t} \circ \Omega_t \circ \Phi_{\lambda,m,t} \circ \Omega_t(y) = y + \sum_{i=1}^{n} \frac{\lambda^i}{t^i i!}\delta^i_{m,t}(P)x^{im-1} = \Psi_{\lambda,m,t}(y).$$

This completes the proof. □

Let us recall the definition of the amalgamated free product. Let G be a group and H, K two subgroups of G, and let $L := H \cap K$. The group G is the free product of the subgroups H and K with the amalgamated subgroup L, and is denoted by $G = H *_L K$, if:

(a) G is generated by the subgroups H and K;

(b) the defining relations of G consist only of the defining relations of the subgroups H and K.

Theorem 7.1.5. *Let H, K be subgroups of G, and $L = H \cap K$. If S_1 is a set of left coset representatives for L in H and S_2 is a set of left coset representatives for L in K, then $G = H *_L K$ if and only if every element $g \in G$ can be written uniquely as*

$$g = g_1 \dots g_k \alpha,$$

where $\alpha \in L$, $g_i \in S_1 \cup S_2$ and g_i, g_{i+1} do not belong in S_1 and S_2 at the same time, for all $i = 1, \dots, k$.

A proof of Theorem 7.1.5 can be found in [SMK, Corollary 4.4.1].

We want to show that G_t, G_0 and G_1 have an amalgamated free product structure. We introduce the following notation.

(i) $\Phi := \langle \Phi_{\lambda,m,t} \mid \lambda \in \mathbb{C}, m \in \mathbb{N} \rangle$;

(ii) $\Psi := \langle \Psi_{\lambda,m,t} \,|\, \lambda \in \mathbb{C}, m \in \mathbb{N} \rangle$;

(iii) $\Theta := \langle \Theta_{\nu,t} \,|\, \nu \in \mathbb{C}^{\times} \rangle$.

Let us first consider the case when P is reflective. Define

(a) $T := \langle \Phi, \Theta \rangle$;

(b) $J := \langle \Omega_t, \Theta \rangle$.

Lemma 7.1.4 implies that G_t is generated by T and J.

Remark 7.1.6. We have $T = \Phi \rtimes \Theta$. This follows at once from relation (7.1.3). Thus, the elements of Φ form a set of left coset representatives for Θ in T. Notice that, due to the properties of the exponential, each automorphism in Φ has the form $\exp\left(\frac{1}{t}\,\mathrm{ad}(g(y))\right)$, for some $g \in \mathbb{C}[y]$. For consistency with the notation used in Section 7.1.3, we define

$$\Phi_{g,t} := \exp\left(\frac{1}{t}\,\mathrm{ad}(\hat{g}(y))\right), \tag{7.1.9}$$

where $\hat{g} := \int_0^y g(t)dt$ denotes the antiderivative of g. From the properties of the exponential,

$$\Phi_{g,t} \circ \Phi_{h,t} = \Phi_{g+h,t} \qquad \Phi_{0,t} = \mathrm{id},$$

so $\Phi \cong \mathbb{C}[y]$ as an additive group, via the identification $g \mapsto \Phi_{g,t}$.

From relation (7.1.6) we can also see that $\{\Omega_t, \mathrm{id}\}$ is a set of left coset representatives for Θ in J. In particular, this shows that $T \cap J = \Theta$, so we are in the framework of Theorem 7.1.5.

Proposition 7.1.7. *If P is reflective, all $\phi \in G_t$ can be written in the form*

$$\phi = \Phi_{g_1,t} \circ \Omega_t \circ \cdots \circ \Omega_t \circ \Phi_{g_s,t} \circ \Theta_{\nu,t},$$

where $s \geq 0$, $g_i \in \mathbb{C}[y]$ for all i and $0 \neq g_i$ for $1 < i < s$, and $\nu \in \mathbb{C}^{\times}$.

Proof. By definition of G_t we have

$$\phi = \phi_1 \circ \cdots \circ \phi_k, \tag{7.1.10}$$

with ϕ_i either $\Phi_{g,t}, \Omega_t$ or $\Theta_{\nu,t}$. From relations (7.1.3) and (7.1.6), we can take every automorphism of type $\Theta_{\nu,t}$ to the right. Since $\Theta_{\nu,t} \circ \Theta_{\mu,t} = \Theta_{\nu\mu,t}$, $\Omega_t^2 = \Theta_{(-1)^n,t}$ and $\Phi_g \circ \Phi_h = \Phi_{g+h}$, we can always rewrite (7.1.10) as

$$\phi = \Phi_{g_1,t} \circ \Omega_t \circ \cdots \circ \Omega_t \circ \Phi_{g_s,t} \circ \Theta_{\nu,t},$$

with $s \geq 0$, $g_i \in \mathbb{C}[y]$ for all i and $0 \neq g_i$ for $1 < i < s$, and $\nu \in \mathbb{C}^{\times}$. ∎

Let us now consider the case where P is not reflective. Define

(a) $Q_1 = \langle \Phi, \Theta \rangle$;

(b) $Q_2 = \langle \Psi, \Theta \rangle$.

By definition, G_t is generated by Q_1 and Q_2.

Remark 7.1.8. We have $Q_1 = \Phi \rtimes \Theta$ and $Q_2 = \Psi \rtimes \Theta$. This follows at once from relation (7.1.4). Thus, the elements of Φ (respectively Ψ) form a set of left coset representatives for Θ in Q_1 (respectively for Θ in Q_2). In particular, $Q_1 \cap Q_2 = \Theta$. As in (7.1.9) we can define

$$\Psi_{g,t} := \exp\Big(\frac{1}{t}\,\mathrm{ad}(\hat{g}(x))\Big).$$

This defines an isomorphism $\Psi \cong \mathbb{C}[x]$ as additive groups via the map $g \mapsto \Psi_{g,t}$.

Proposition 7.1.9. *If P is not reflective, every $\phi \in G_t$ can be written in the form*

$$\phi = \Phi_{g_1,t} \circ \Psi_{h_1,t} \circ \cdots \circ \Phi_{g_{s-1},t} \circ \Psi_{h_{s-1},t} \circ \Phi_{g_s,t} \circ \Theta_{\nu,t},$$

with $s \geq 1$, $g_i \in \mathbb{C}[y]$ for all i with $0 \neq g_i$ for $i = 2, \dots, s-1$, $0 \neq h_i \in \mathbb{C}[x]$ for all i, and $\nu \in \mathbb{C}^\times$.

Proof. The proof is analogous to that of Proposition 7.1.7, and it is a simple application of the relations (7.1.2), (7.1.3) and (7.1.4). □

We are left to prove that the decompositions of Propositions 7.1.7 and 7.1.9 are unique. To do that, we introduce the notion of multidegree of an automorphism. For all $a \in A_t(P)$, we define $\deg(a)$ as the smallest natural number i such that $a \in F_i$, with $\deg(0) = -\infty$. For every automorphism ϕ in G_t, we define its multidegree to be:

$$\mathrm{mdeg}(\phi) = (\deg(\phi(x)), \deg(\phi(y)), \deg(\phi(z))).$$

Lemma 7.1.10. *For all $a, b \in A_t(P)$, $\deg(ab) = \deg(a) + \deg(b)$.*

Proof. For $a \in A_t(P)$, let $\mathrm{gr}(a) := a + F_{i-1} \in \mathrm{gr}(A_t(P))$, where $i = \deg(a)$. Clearly, $i = \deg(a) = \deg(\mathrm{gr}(a))$. Since $\mathrm{gr}(A_t(P)) \cong \mathbb{C}[t, x, y, z]/(xy - z^n)$ is a domain,

$$\deg(\mathrm{gr}(a)\,\mathrm{gr}(b)) = \deg(\mathrm{gr}(a)) + \deg(\mathrm{gr}(b)),$$

for all $a, b \in A_t(P)$. Thus

$$\deg(ab) = \deg(\mathrm{gr}(ab)) = \deg(\mathrm{gr}(a)\,\mathrm{gr}(b)) = \deg(a) + \deg(b).$$

□

Lemma 7.1.11. *Let $g \in \mathbb{C}[x]$ be a polynomial of degree k, and $\phi = (v_1, v_2, v_3)$ be an automorphism in G_t such that $\deg(v_2) > \deg(v_3)$ and $\deg(v_2) \geq \deg(v_1)$. Then $\phi \circ \Phi_{g,t}$*

has multidegree

$$((nk+n-1)\deg(v_2), \deg(v_2), (k+1)\deg(v_2)).$$

Similarly, if $\deg(v_1) > \deg(v_3)$ *and* $\deg(v_1) \geq \deg(v_2)$*, then* $\phi \circ \Psi_{g,t}$ *has multidegree*

$$(\deg(v_1), (nk+n-1)\deg(v_1), (k+1)\deg(v_1)).$$

Proof. Recall that $\Phi_{g,t} = e^{1/t\,\mathrm{ad}(\hat{g}(y))}$ (7.1.9). We have

$$\mathrm{ad}(y^m) : x \mapsto -y^{m-1}\delta_{m,t}(P),\ \ f(z) \mapsto -y^m\delta_{m,t}(f),\ \ y \mapsto 0, \tag{7.1.11}$$

for all $f \in \mathbb{C}[z]$ (see [BJ, Equation 8]). This implies that $\Phi_{g,t}(y) = y$, hence $\deg(\phi \circ \Phi_{g,t}(y)) = \deg(v_2)$.

Let ay^{k+1} be the leading term of $\hat{g}$. From (7.1.11)

$$\phi \circ \Phi_{g,t}(z) = v_3 + (k+1)av_2^{k+1} + \text{lower terms in } v_2.$$

Since $\deg(v_2) \geq \deg(v_i)$ for $i = 1, 2, 3$, from Lemma 7.1.10

$$\deg(\phi \circ \Phi_{g,t}(z)) = (k+1)\deg(v_2).$$

From (7.1.11) it follows that, for all $m, r \in \mathbb{N}$

$$[\hat{g}(y), y^m z^r] = \alpha t y^{k+m+1} z^{r-1} + \text{monomials proportional to } y^i z^j, \tag{7.1.12}$$

with $i \leq k+m+1$, $j \leq r-1$ and $(i,j) \neq (k+m+1, r-1)$, for some $\alpha \in \mathbb{C}^\times$. It also follows that

$$[\hat{g}(y), x] = \beta t y^k z^{n-1} + \text{monomials proportional to } y^i z^j, \tag{7.1.13}$$

with $i \leq k$, $j \leq n-1$ and $(i,j) \neq (k, n-1)$, for some $\beta \in \mathbb{C}^\times$.

We now want to prove, by induction on s, that

$$\mathrm{ad}^s(\hat{g}(y))(x) = \alpha t^s y^{s(k+1)-1} z^{n-s} + \text{monomials proportional to } y^i z^j, \tag{7.1.14}$$

with $i \leq s(k+1)-1$, $j \leq n-s$ and $(i,j) \neq (s(k+1)-1, n-s)$, for some $\alpha \in \mathbb{C}^\times$. The base step is (7.1.13). Consider now

$$\mathrm{ad}^{s+1}(\hat{g}(y))(x) = [\hat{g}(y), \alpha t^s y^{s(k+1)-1} z^{n-s} + \text{lower terms}],$$

for some $\alpha \in \mathbb{C}^\times$, true by the inductive hypothesis. Applying (7.1.12) we get (7.1.14). It is then clear that

$$\deg\left(\phi \circ \mathrm{ad}^s(\hat{g}(y))(x)\right) = \deg\left(\alpha t^s v_2^{s(k+1)-1} v_3^{n-s}\right).$$

Since $\deg(v_2) > \deg(v_3)$, we get that $\deg(\phi \circ \Phi_{g,t}(x)) = \deg(\alpha v_2^{nk+n-1})$, hence from Lemma 7.1.10

$$\deg(\phi \circ \Phi_{g,t}) = (nk + n - 1)\deg(v_2).$$

The proof for $\phi \circ \Psi_{g,t}$ is analogous, using

$$\mathrm{ad}(x^m) : x \mapsto 0,\ f(z) \mapsto \delta_{m,t}(f)x^m,\ y \mapsto \delta_{m,t}(P)x^{m-1}, \tag{7.1.15}$$

which is [BJ, Equation 6], instead of (7.1.11).

□

Proposition 7.1.12. *Let $n > 2$. An automorphism of the form*

$$\phi = \Phi_{g_1,t} \circ \Omega_t \circ \Phi_{g_2,t} \cdots \circ \Omega_t \circ \Phi_{g_s,t},$$

with $s \geq 1$, $0 \neq g_i \in \mathbb{C}[y]$ has multidegree

$$\left(n\prod_{i=1}^{s}(nk_i + n - 1), n\prod_{i=1}^{s-1}(nk_i + n - 1), n(k_s + 1)\prod_{i=1}^{s-1}(nk_i + n - 1)\right),$$

where $k_i = \deg(g_i)$.

Proof. We will prove this by induction on s. If $s = 1$, then we have $\phi = \Phi_{g_1,t}$; since $\deg(y) = \deg(x) > \deg(z)$, we can apply Lemma 7.1.11 on $\mathrm{id} \circ \Phi_{g_1,t}$. Thus

$$\mathrm{mdeg}(\phi) = (n(nk + n - 1), n, n(k + 1)).$$

When $s > 1$, we have

$$\phi = \psi \circ \Omega_t \circ \Phi_{g_s,t},$$

with

$$\psi = \Phi_{g_1,t} \circ \Omega_t \circ \Phi_{g_2,t} \cdots \circ \Omega_t \circ \Phi_{g_{s-1},t}.$$

We know by the induction hypothesis that $\psi = (u_1, u_2, u_3)$ has multidegree

$$\left(n\prod_{i=1}^{s-1}(nk_i + n - 1), n\prod_{i=1}^{s-2}(nk_i + n - 1), n(k_{s-1} + 1)\prod_{i=1}^{s-2}(nk_i + n - 1)\right).$$

Notice that $\deg(u_2) \leq \deg(u_3) < \deg(u_1)$, because $(nk + n - 1) > (k + 1) \geq 1$ for all $k \geq 0$, when $n > 2$. But now $\psi \circ \Omega_t = (u_2, (-1)^n u_1, t - u_3)$, hence we can once again apply lemma (7.1.11) to $(\psi \circ \Omega_t) \circ \Phi_{g_s,t}$ and complete the proof. □

Proposition 7.1.13. *Let $n > 2$. An automorphism of the form*

$$\phi = \Phi_{g_1,t} \circ \Psi_{h_1,t} \circ \cdots \circ \Psi_{h_{s-1},t} \circ \Phi_{g_s,t},$$

with $s \geq 1$, $0 \neq g_i \in \mathbb{C}[y]$ and $0 \neq h_i \in \mathbb{C}[x]$ has multidegree

$$\begin{pmatrix} n(nk_s + n - 1) \prod_{i=1}^{s-1}(nk_i + n - 1)(nl_i + n - 1) \\ \\ n \prod_{i=1}^{s-1}(nk_i + n - 1)(nl_i + n - 1) \\ \\ n(k_s + 1) \prod_{i=1}^{s-1}(nk_i + n - 1)(nl_i + n - 1) \end{pmatrix}$$

where $k_i = \deg(g_i)$ and $l_i = \deg(h_i)$.

Proof. We will prove this by induction on s. When $s = 1$ we have $\phi = \Phi_{g_1,t}$, which has multidegree $(n(nk_1 + n - 1), n, n(k_1 + 1))$.

Let us now consider the case $s > 1$. Take

$$\psi = \Phi_{g_1,t} \circ \Psi_{h_1,t} \circ \cdots \circ \Psi_{h_{s-2},t} \circ \Phi_{g_{s-1},t} = (v_1, v_2, v_3),$$

so that $\phi = \psi \circ \Psi_{h_{s-1},t} \circ \Phi_{h_s,t}$. By the induction hypothesis, we know that ψ has multidegree

$$\begin{pmatrix} n(nk_{s-1} + n - 1) \prod_{i=1}^{s-2}(nk_i + n - 1)(nl_i + n - 1) \\ \\ n \prod_{i=1}^{s-2}(nk_i + n - 1)(nl_i + n - 1) \\ \\ n(k_{s-1} + 1) \prod_{i=1}^{s-2}(nk_i + n - 1)(nl_i + n - 1) \end{pmatrix}$$

Since $n > 2$, we have $nk_{s-1} + n - 1 > k_{s-1} + 1 \geq 1$, which implies that $\deg(v_2) \leq \deg(v_3) < \deg(v_1)$. We can apply again Lemma 7.1.11 to $\psi \circ \Psi_{h_{s-1},t}$ to get $\operatorname{mdeg}(\psi \circ \Psi_{h_{s-1},t}) =$

$$\begin{pmatrix} n(nk_{s-1} + n - 1) \prod_{i=1}^{s-2}(nk_i + n - 1)(nl_i + n - 1) \\ \\ n \prod_{i=1}^{s-1}(nk_i + n - 1)(nl_i + n - 1) \\ \\ n(l_{s-1} + 1)(nk_{s-1} + n - 1) \prod_{i=1}^{s-2}(nk_i + n - 1)(nl_i + n - 1) \end{pmatrix}$$

Let $\psi \circ \Psi_{h_{s-1},t} = (u_1, u_2, u_3)$. Then $\deg(u_2) > \deg(u_3) \geq \deg(u_1)$, and we can apply Lemma 7.1.11 to $(\psi \circ \Psi_{h_{s-1},t}) \circ \Phi_{g_s,t}$ to complete the proof. □

Theorem 7.1.14. *Let $n > 2$.*

(i) If P is reflective, the group G_t is a free product with amalgamation

$$G_t = T *_\Theta J.$$

As an abstract group, if n is even,

$$G_t \cong (\mathbb{C}[y] \rtimes \mathbb{C}^\times) *_{\mathbb{C}^\times} (\mathbb{C}^\times \rtimes \mathbb{Z}/2\mathbb{Z}).$$

If n is odd, then

$$G_t \cong (\mathbb{C}[y] \rtimes \mathbb{C}^\times) *_{\mathbb{C}^\times} H,$$

where $H = \langle \mathbb{C}^\times, \Omega \,|\, \Omega^2 = -1,\ \lambda \cdot \Omega = \Omega \cdot \lambda^{-1}\ \forall \lambda \in \mathbb{C}^\times \rangle$.

(ii) If P is not reflective, the group G_t is a free product with amalgamation

$$G_t = Q_1 *_\Theta Q_2.$$

As an abstract group,

$$G_t \cong (\mathbb{C}[y] \rtimes \mathbb{C}^\times) *_{\mathbb{C}^\times} (\mathbb{C}[x] \rtimes \mathbb{C}^\times).$$

Proof. Thanks to Theorem 7.1.5, we only need to prove that the decompositions in Propositions 7.1.7 and 7.1.9 are unique.

Consider first the case where P is reflective. Assume, for contradiction, that

$$\phi = \Phi_{g_1,t} \circ \Omega_t \circ \cdots \circ \Omega_t \circ \Phi_{g_s,t} = \Theta_{\nu,t},$$

for some $g_i \in \mathbb{C}[y]$ such that $0 \neq g_i$ for $1 < i < s$, with $s \geq 1$, and for some $\nu \in \mathbb{C}^\times$. In particular, ϕ is a linear automorphism, since $\Theta_{\nu,t}$ is linear. Assume first that $g_1, g_s \neq 0$. By Proposition 7.1.12,

$$\deg(\phi(x)) = n \prod_{i=1}^{s} (nk_i + n - 1),$$

where $k_i = \deg(g_i)$. So if $n > 2$ then $\deg(\phi(x)) > n$, which leads to a contradiction. If g_1 is 0, we can move Ω_t to the right to get

$$\phi' = \Phi_{g_2,t} \circ \Omega_t \circ \cdots \circ \Omega_t \circ \Phi_{g_s,t} = \Omega_t^{-1} \circ \Theta_{\nu,t}.$$

The right-hand side is still a linear automorphism, so we still get a contradiction. We can reason similarly if $g_s = 0$.

Let us assume now that P is not reflective. Assume, by contradiction, that

$$\phi = \Phi_{g_1,t} \circ \Psi_{h_1,t} \circ \cdots \circ \Phi_{g_{s-1},t} \circ \Psi_{h_{s-1},t} \circ \Phi_{g_s,t} = \Theta_{\nu,t}, \tag{7.1.16}$$

for some $g_i \in \mathbb{C}[y]$ such that $0 \neq g_i$ for $i = 2, \ldots, s-1$, $0 \neq h_i \in \mathbb{C}[x]$, with $s \geq 1$, and for some $\nu \in \mathbb{C}^\times$. Assume first that $g_1, g_s \neq 0$. Then by Proposition 7.1.13 we have

$$\deg(\phi(x)) = n(nk_s + n - 1)\prod_{i=1}^{s-1}(nk_i + n - 1)(nl_i + n - 1),$$

where $k_i = \deg(g_i)$ and $l_i = \deg(h_i)$. So if $n > 2$ then $\deg(\phi(x)) > n$, which leads to a contradiction, because $\Theta_{\nu^{-1},t}$ is linear.

If one or both of g_1 and g_s are zero, then we can multiply both sides of (7.1.16) on the left and/or on the right by some $\Phi_{s_i,t}$, for some $0 \neq s_i \in \mathbb{C}[y]$. We get an equation of the form

$$\phi' = \Phi_{g_1,t} \circ \Psi_{h_1,t} \circ \cdots \circ \Phi_{g_{s-1},t} \circ \Psi_{h_{s-1},t} \circ \Phi_{g_s,t} = \Theta_{\nu,t} \circ \Phi_{q,t},$$

where q has the degree of either g_1, g_s or $g_1 + g_s$, and $0 \neq g_i$ for all i. We can now apply Proposition 7.1.13 and get that $\deg(\phi'(y)) = n\prod_{i=1}^{s-1}(nk_i + n - 1)(nl_i + n - 1)$, while $\deg(\Theta_{\nu^{-1},t} \circ \Phi_{q,t}(y)) = n$. This leads to a contradiction, unless $s = 1$. In that case though, equation (7.1.16) is either of the form $\mathrm{id} = \Theta_{\nu,t}$ or $\Phi_{g_1,t} = \Theta_{\nu^{-1},t}$, which again leads to contradiction. The decomposition of Proposition 7.1.9 is thus unique.

The abstract group structures follow from the isomorphisms

$$\begin{aligned}
\mathbb{C}[y] &\cong \Phi, & g(y) &\mapsto \Phi_{g,t},\\
\mathbb{C}[x] &\cong \Psi, & g(x) &\mapsto \Psi_{g,t},\\
\mathbb{C}^\times &\cong \Theta, & \nu &\mapsto \Theta_{\nu,t},
\end{aligned}$$

and from the relations in Lemma 7.1.4.

□

Corollary 7.1.15. *Let $n > 2$. The groups G_0 and G_1 have the same amalgamated free product structure described in Theorem 7.1.14. In particular,*

$$G_0 \cong G_1 = \mathrm{Aut}(A(P)).$$

Proof. Notice that the relations in Lemma 7.1.4 still hold in G_0 and G_1, so we get decompositions analogous to those of Propositions 7.1.7 and 7.1.9. We can define a notion of degree for basis monomials in $A(P)$ and $\mathcal{A}(P)$ just as for $A_t(P)$. Notice also that in the proof of Lemma 7.1.11, no t appears in the coefficients of the leading words, so the result holds in G_0 and G_1 too. These two lemmas were the only ingredients used in the proofs of Propositions 7.1.12 and 7.1.13, and subsequently of Theorem 7.1.14. So, G_0 and G_1 have the same amalgamated free product structure as of G_t. In particular, they are isomorphic. □

§ 7.1.3 | The group of Poisson automorphisms of $\mathcal{A}(P)$

We know that $G_0 \leq \mathrm{PAut}(\mathcal{A}(P))$ by (7.1.1). In this section, we show that this inclusion is an equality.

The group of automorphisms of $\mathcal{A}(P)$ as an affine variety is well known. Makar-Limanov computed its generators [Ma1], and Blanc and Dubouloz [BD] proved that it has an amalgamated free product structure.

Theorem 7.1.16 ([Ma1, Theorem 1])**.** *Let R be the quotient algebra*

$$R = \mathbb{C}[x, y, z]/(xy - P(z)),$$

with $P(z) \in \mathbb{C}[z]$. Then the group $Aut(R)$ is generated by the following automorphisms:

(a) Hyperbolic rotations: $\Theta_{\nu,0} = (\nu x, \nu^{-1} y, z)$, for all $\nu \in \mathbb{C}^\times$.

(b) Involution: $V = (y, x, z)$.

(c) Triangular automorphisms:

$$\Delta_g = (x + [P(z + yg(y)) - P(z)]y^{-1}, y, z + yg(y))$$

for all $g(y) \in \mathbb{C}[y]$.

(d) (If $P(z) = c(z + a)^n$) *Rescalings $R_\nu = (\nu^n x, y, \nu z + (\nu - 1)a)$, for all $\nu \in \mathbb{C}^\times$.*

(e) (If $P(z) = (z + a)^i Q((z + a)^d)$) *Symmetries: $S_\mu = (\mu^i x, y, \mu z + (\mu - 1)a)$, for all $\mu^d = 1$.*

Lemma 7.1.17. *If P is a monic polynomial of degree n with no term of degree $n - 1$, we can substitute (d) and (e) from the list of generators in Theorem 7.1.16 with*

(d) (If $P(z) = z^n$) *Rescalings $R_\nu = (\nu^n x, y, \nu z)$, for all $\nu \in \mathbb{C}^\times$.*

(e) (If $P(z) = z^i Q(z^d)$) *Symmetries: $S_\mu = (\mu^i x, y, \mu z)$, for all $\mu^d = 1$.*

Proof. For (d), $P(z) = c(z + a)^n$ only if $c = 1$ and $a = 0$, since P is monic and has no term of degree $n - 1$.

For (e), first notice that the polynomial Q must be monic since P is. For degree reasons, $n = i + dk$, where k is the degree of Q. We can ignore the case $d = 1$, since the only corresponding automorphism is $S_1 = \mathrm{id}$. Expanding $P(z)$ we get

$$(z + a)^{i+dk} + \alpha(z + a)^{i+d(k-1)} + \text{terms of lower degree},$$

for some $\alpha \in \mathbb{C}$. Since $d > 1$, the term of degree $n - 1 = i + dk - 1$ of $P(z)$ comes only from the expansion of $(z + a)^{i+dk}$. Since $P(z)$ has no term of degree $n - 1$, we must have $a = 0$. □

Remark 7.1.18. Notice that if $P(z) = z^iQ(z^d)$ is of degree n, and $\mu^d = 1$, then $\mu^i = \mu^n$. So, if $P(z) = z^n$, the automorphisms of type (e) are a special type of automorphisms of type (d), i.e. $R_\mu = S_\mu$ for all μ with $\mu^d = 1$.

Let us introduce the affine automorphism

$$\begin{aligned}\nabla_g &= V \circ \Delta_{-g} \circ V \\ &= (x, y + [P(z - xg(x)) - P(z)]x^{-1}, z - xg(x)).\end{aligned}$$

Proposition 7.1.19. *Let $g \in \mathbb{C}[t]$. Then*

$$\Delta_g = e^{\mathrm{ad}(\hat{g}(y))} = \Phi_{g,0},$$

$$\nabla_g = e^{\mathrm{ad}(\hat{g}(x))} = \Psi_{g,0},$$

where $\hat{g}(x) := \int_0^x g(t)dt$ denotes the antiderivative of g. In particular, Δ_g and ∇_g are Poisson automorphisms and elements of G_0.

Proof. We have that

$$\{y^k, f(z)\} = kf'(z)y^k, \quad \{y^k, x\} = kP'(z)y^{k-1}, \tag{7.1.17}$$

and

$$\{x^k, f(z)\} = kx^kf'(z), \quad \{x^k, y\} = -kx^{k-1}P'(x), \tag{7.1.18}$$

for all $k \geq 0$. These relations can be checked directly or derived from (7.1.11) and (7.1.15) using the isomorphism $\mathcal{A}(P) \cong A_0(P)$. By (7.1.17):

$$\begin{aligned}&\{\hat{g}(y), z\} = yg(y), \\ &\{\hat{g}(y), x\} = P'(z)g(y), \\ &\{\hat{g}(y), P^{(i)}(z)y^{i-1}g(y)^i\} = P^{(i+1)}(z)y^ig(y)^{i+1},\end{aligned}$$

where $P^{(i)}$ denotes the i-th derivative of P. Hence,

$$e^{\mathrm{ad}(\hat{g}(y))} = (x + \sum_{i=1}^{n} \frac{1}{i!}P^{(i)}(z)y^{i-1}g(y)^i, y, z + yg(y)).$$

To complete the proof, we need to show that $\Phi_{g,0}$ acts on x in the same way as Δ_g. Let $P(z) = \sum_{i=0}^n a_iz^i$,

$$\begin{aligned}P(z + yg(y)) - P(z) &= \sum_{k=0}^{n} a_k(z + yg(y))^k - \sum_{k=0}^{n} a_kz^k \\ &= \sum_{k=0}^{n} a_k \sum_{i=1}^{k} \binom{k}{i} z^{k-i}y^ig(y)^i = \sum_{i=1}^{n} y^ig(y)^i \sum_{k=i}^{n} \binom{k}{i} a_kz^{k-i} \\ &= \sum_{i=1}^{n} \frac{1}{i!}P^{(i)}(z)y^ig(y)^i.\end{aligned}$$

A similar computation using (7.1.18) shows that

$$\nabla_g = e^{\mathrm{ad}(\hat{g}(x))} = (x, y + \sum_{j=1}^{n} \frac{(-1)^j}{j!} P^{(j)}(z) x^{j-1} g(A)^j, z - xg(x)).$$

□

Remark 7.1.20. Consider the automorphism $S_{-1} = ((-1)^n x, y, -z)$. This is in $\mathrm{Aut}(\mathcal{A})$ if and only if $P(z)$ is of the form $z^i Q(z^2)$, by Proposition 7.1.17, i.e. if and only if $P(z)$ is reflective. If P is reflective, then

$$\Omega_0 = S_{-1} \circ V = (y, (-1)^n x, -z).$$

Theorem 7.1.21. *The group* $\mathrm{PAut}(\mathcal{A}(P))$ *of Poisson automorphisms of* $\mathcal{A}(P)$ *is equal to* G_0.

Proof. Because of Theorem 7.1.14 and (7.1.1), it is sufficient to prove that $\mathrm{PAut}(\mathcal{A}(P))$ is generated by $\Theta_{\nu,0}$, Δ_g and Ω_0 when P is reflective, and by $\Theta_{\nu,0}$, Δ_g and ∇_h when P is not reflective, for all $\nu \in \mathbb{C}^\times$ and $g, h \in \mathbb{C}[x]$.

Let us compute some relations between the generators in $\mathrm{Aut}(\mathcal{A}(P))$. The following relations hold whenever the automorphisms are defined.

1. $V^2 = \mathrm{id}$,
2. $\Theta_{\nu,0} \circ V = V \circ \Theta_{\nu^{-1},0}$,
3. $\Delta_g \circ V = V \circ \nabla_{-g}$ and $\nabla_g \circ V = V \circ \Delta_{-g}$,
4. $R_\lambda \circ R_\mu = R_{\lambda \cdot \mu}$ and $S_\lambda \circ S_\mu = S_{\lambda \cdot \mu}$,
5. $\Theta_{\nu,0} \circ R_\mu = R_\mu \circ \Theta_{\nu,0}$ and $\Theta_{\nu,0} \circ S_\mu = S_\mu \circ \Theta_{\nu,0}$,
6. $V \circ R_\nu = R_\nu \circ V \circ \Theta_{\nu^n,0}$ and $V \circ S_\mu = S_\mu \circ V \circ \Theta_{\mu^n,0}$.

Suppose now that P is of the form $P(z) = z^i Q(z^d)$.

$$\Delta_g \circ S_\mu = \left(\mu^i x + \mu^i \sum_{j=1}^{n} \frac{1}{j!} P^{(j)}(z) y^{j-1} g^j(y), y, \mu z + \mu y g(y) \right),$$
$$S_\mu \circ \Delta_{\mu g} = \left(\mu^i x + \sum_{j=1}^{n} \frac{\mu^j}{j!} P^{(j)}(\mu z) y^{j-1} g^j(y), y, \mu z + \mu y g(y) \right). \tag{7.1.19}$$

Since $P(z) = z^i Q(z^d)$, the j-th derivative $P^{(j)}$ has only terms of degree equal to $d(k-l) + i - j$, where k is the degree of Q, for some $l \geq 0$. Hence each term of the sum in (7.1.19) has a factor of $\mu^{d(k-l)+i} = \mu^i$. Thus

7. $\Delta_g \circ S_\mu = S_\mu \circ \Delta_{\mu g}$.

Similarly, we get

$$\nabla_g \circ S_\mu = (\mu^i x, y + \sum_{j=1}^{n} \frac{(-1)^j}{j!} P^{(j)}(z) x^{j-1} g^j(x), \mu z - \mu x g(x)),$$
$$S_\mu \circ \nabla_{\mu^{1-i} g(\mu^{-i} x)} = (\mu^i x, y + \sum_{j=1}^{n} \frac{(-1)^j}{j!} P^{(j)}(\mu z) \mu^{j-i} x^{j-1} g^j(x), \mu z - \mu x g(x)).$$

Again, from $P^{(j)}(\mu z)$ we can take out μ, getting factors of the form $\mu^{d(k-l)+i-j}$, which is equal to μ^{i-j}. Thus

8. $\nabla_g \circ S_\mu = S_\mu \circ \nabla_{\mu^{1-i} g(\mu^{-i} x)}$.

Suppose now that $P(z) = z^n$. We have:

9. $\Delta_g \circ R_\nu = R_\nu \circ \Delta_{\nu g}$, since they are both equal to

$$(\nu^n x + \nu^n \sum_{i=1}^{n} \binom{n}{i} z^{n-i} y^{i-1} g^i(y), y, \nu z + \nu y g(y)),$$

10. $\nabla_g \circ R_\nu = R_\nu \circ \nabla_{\nu^{1-n} g(\nu^{-n} x)}$, since they are both equal to

$$(\nu^n x, y + \sum_{j=1}^{n} (-1)^j \binom{n}{j} z^{n-j} x^{j-1} g^j(x), \nu z - \nu x g(x)).$$

Take any automorphism $\psi \in \mathrm{Aut}(\mathcal{A}(P))$. From Theorem 7.1.16 and Proposition 7.1.17 we know that we can write it as a composition of automorphisms of the form $V, \Theta_{\nu,0}, \Delta_g, S_\mu$ or R_μ. Using relations (1) to (10) we can rewrite ψ as $\psi = \omega \circ \phi$, where ϕ is in G_0 and ω is one of the following automorphisms: $\mathrm{id}, V, R_\nu, S_\mu, R_\nu \circ V$ or $S_\mu \circ V$ (we can ignore the case $R_\nu \circ S_\mu$ due to Remark 7.1.18). If $\omega = \mathrm{id}$ then $\phi \in G_0$. If ν or μ are equal to (-1), then $R_{-1} \circ V = S_{-1} \circ V = \Omega_0$. Since $\Omega_0 \in G_0$ then $\psi \in G_0$. We want to show that in all the other cases, ω is not a Poisson automorphism. It follows from a direct check:

$$\{V(z), V(x)\} = -y \neq y = V(\{z, x\}),$$
$$\{R_\nu(z), R_\nu(y)\} = -\nu y \neq -y = R_\nu(\{z, y\}), \quad \forall \nu \neq 1,$$
$$\{S_\mu(z), S_\mu(y)\} = -\mu y \neq -y = S_\mu(\{z, y\}), \quad \forall \mu \neq 1,$$
$$\{S_\mu \circ V(z), S_\mu \circ V(x)\} = -\mu y \neq y = R_\mu \circ V(\{z, x\}), \quad \forall \mu \neq -1,$$
$$\{R_\nu \circ V(z), R_\nu \circ V(x)\} = -\nu y \neq y = R_\nu \circ V(\{z, x\}), \quad \forall \nu \neq -1.$$

Hence, for all $\psi \in \mathrm{Aut}(\mathcal{A}(P))$, either $\psi \in G_0$ or ψ is a composition of a Poisson automorphism and a non-Poisson automorphism, hence ψ is non-Poisson.

□

From the relations in the proof of Theorem 7.1.21 we also get:

Corollary 7.1.22. *The subgroup G of Poisson automorphisms is normal in Aut$(\mathcal{A}(P))$. The group* $\mathrm{Aut}(\mathcal{A}(P))$ *has the following structure:*

- *For generic $P(z)$, it is the semidirect product*

$$\mathrm{Aut}(\mathcal{A}(P)) = \mathrm{PAut}(\mathcal{A}(P)) \rtimes \mathbb{Z}/2\mathbb{Z},$$

with $\mathbb{Z}/2\mathbb{Z}$ generated by V.

- *For $P(z) = z^i Q(z^d)$, it is the extension of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ by* $\mathrm{PAut}(\mathcal{A}(P))$.

- *For $P(z) = z^n$, it is the extension of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{C}^\times$ by* $\mathrm{PAut}(\mathcal{A}(P))$.

Theorem 7.1.23. *Let $n > 2$. Then* $\mathrm{Aut}(A(P))$ *and* $\mathrm{PAut}(\mathcal{A}(P))$ *admit the amalgamated free product structure described in Theorem 7.1.14 and*

$$\mathrm{Aut}(A(P)) \cong \mathrm{PAut}(\mathcal{A}(P)),$$

for every deformation parameter P.

Proof. It follows directly from Theorem 7.1.14 and Theorem 7.1.21. □

The case $n = 2$ is summarised in the following theorem, due to the work of Dixmier [Di], Fleury [Fl], and Naurazbekova and Umirbaev [NU].

Theorem 7.1.24. *For $n = 2$ and for every deformation parameter P, we have a group isomorphism*

$$\mathrm{Aut}(A(P)) \cong \mathrm{PAut}(\mathcal{A}(P)),$$

and they have an amalgamated free product structure given by

$$L *_{L\cap T} T,$$

where L is the subgroup of linear (Poisson) automorphisms and T is the subgroup generated by triangular (Poisson) automorphisms and hyperbolic rotations.

Remark 7.1.25. The main difference for $n = 2$ is that triangular automorphisms Δ_g can be linear (if and only if g is a constant). This is a consequence of the fact that the case $n = 2$ is the only one where the Poisson algebra structure is linear on the generators x, y, z, inducing a Lie algebra isomorphic to $\mathfrak{sl}_2$.

§ 7.1.4 | Isomorphism groupoids

In this section we compute $\mathrm{Iso}(\mathbf{A_{n-1}})$ and $\mathrm{PIso}(\mathbf{A_{n-1}})$, the groupoid of (Poisson) isomorphisms between quantizations and deformations of $\mathbb{C}[\mathbf{A_{n-1}}]$, respectively, and prove Theorem 6.2.6 for type $\mathbf{A}$.

Let us first look at the quantization case.

Theorem 7.1.26. *Let $P_1, P_2 \in \mathbb{C}[z]$. Then the generalized Weyl algebras $A(P_1), A(P_2)$ are isomorphic as associative algebras if and only if there exists some $\eta, \alpha \in \mathbb{C}$, with $\eta \neq 0$, such that $P_2(z) = \eta P_1(\pm z + \alpha)$.*

Proof. This is [BJ, Theorem 3.28]. □

Corollary 7.1.27. *For all $n \geq 2$, the groupoid $\mathrm{Iso}(\mathbf{A_{n-1}})$ is generated by the groups $\mathrm{Aut}(A(P))$, for all deformation parameters P, and by the isomorphism*

$$\Omega : A(P(z)) \to A((-1)^n P(-z)), \quad \Omega := (y, (-1)^n x, 1 - z).$$

Proof. Let P_1 be a deformation parameter, and $\eta, \alpha \in \mathbb{C}$, with $\eta \neq 0$. Suppose that $P_2 = \eta P_1(\pm z + \alpha)$ is a deformation parameter. Then $\alpha = 0$ because P_1 and P_2 have no term of degree $n - 1$, and $\eta = (\pm 1)^n$ because P_2 and P_1 are monic. The claim now follows from Theorem 7.1.26. □

Let us consider the deformation case. We first compute a set of generators for $\mathrm{Iso}_{aff}(\mathbf{A_{n-1}})$, the groupoid of affine isomorphisms between deformations.

In [BD] the authors studied the isomorphisms between $\mathbb{A}^1$-fibered surfaces. An affine surface X is $\mathbb{A}^1$-fibered if there exists a surjective morphism of affine varieties $\pi_X : X \to \mathbb{A}^1$, with general fibres isomorphic to $\mathbb{A}^1$. Two $\mathbb{A}^1$-fibered surfaces (X, π) and (X', π') are isomorphic as $\mathbb{A}^1$-fibered surfaces if there are isomorphisms $f : X \to X'$ and $g : \mathbb{A}^1 \to \mathbb{A}^1$ such that

$$\pi' \circ f = g \circ \pi.$$

Let $P \in \mathbb{C}[z]$ be a polynomial of degree n. The algebra $\mathcal{A}(P)$ is $\mathbb{A}^1$-fibered by $\pi_P : \mathcal{A}(P) \to \mathbb{C}[x]$, the projection on the variable x.

Lemma 7.1.28. *Let $P, Q \in \mathbb{C}[z]$ be of degree n.*

(i) Every fibration $\pi : \mathcal{A}(P) \to \mathbb{A}^1$ is of the form $\pi_P \circ \gamma$, where γ is an affine automorphism of $\mathcal{A}(P)$;

(ii) $(\mathcal{A}(P), \pi_P) \cong (\mathcal{A}(Q), \pi_Q)$ if and only if $Q(z) = \eta P(\alpha z + \beta)$, for some $\eta, \alpha, \beta \in \mathbb{C}$, with $\eta \neq 0$.

Proof. A proof can be found in [BD, Theorem 5.4.5]. Point (i) was first proved by Daigle in [Da]. □

Proposition 7.1.29. *For all $n \geq 2$, the groupoid* $\mathrm{Iso}_{aff}(\mathbf{A}_n)$ *is generated by the groups* $\mathrm{Aut}(\mathcal{A}(P))$, *for all deformation parameters P, and by the isomorphisms*

$$R_\alpha : \mathcal{A}(P(z)) \to \mathcal{A}(\alpha^{-n}P(\alpha z)), \quad R_\alpha := (\alpha^n x, y, \alpha z), \quad \forall \alpha \in \mathbb{C}^\times.$$

Proof. Let P, Q be two deformation parameters, and $\phi : \mathcal{A}(P) \to \mathcal{A}(Q)$ be an affine isomorphism. Then ϕ is a fibered isomorphism between

$$\phi : (\mathcal{A}(P), \pi_Q \circ \phi) \to (\mathcal{A}(Q), \pi_Q).$$

From Lemma 7.1.28 there exists $\gamma \in \mathrm{Aut}(\mathcal{A}(P))$ such that $\pi_Q \circ \phi = \pi_P \circ \gamma$. Thus $\psi := \phi \circ \gamma^{-1} : (\mathcal{A}(P), \pi_P) \to (\mathcal{A}(Q), \pi_Q)$ is a fibered isomorphism. By Lemma 7.1.28, there exist $\eta, \alpha, \beta \in \mathbb{C}$, with $\eta \neq 0$, such that $Q(z) = \eta P(\alpha z + \beta)$. Since P and Q are deformation parameters, $\beta = 0$ and $\alpha^n = \eta$.

Consider the fibered isomorphism

$$R_\alpha : (\mathcal{A}(P), \pi_P) \to (\mathcal{A}(Q), \pi_Q), \quad R_\alpha := (\alpha^n x, y, \alpha z).$$

Clearly $R_\alpha^{-1} \circ \psi \in \mathrm{Aut}(\mathcal{A}(P))$, so ϕ can be written as composition of R_α and an automorphism of $\mathcal{A}(P)$. □

Theorem 7.1.30. *For all $n \geq 2$, the groupoid* $\mathrm{PIso}(\mathcal{A})$ *is generated by the groups* $\mathrm{PAut}(\mathcal{A}(P))$, *for all deformation parameters P, and by the isomorphism*

$$\tau : \mathcal{A}(P(z)) \to \mathcal{A}((-1)^n P(-z)) \quad \tau := (y, (-1)^n x, -z).$$

Proof. Let P, Q be two deformation parameters, and $\phi : \mathcal{A}(P) \to \mathcal{A}(Q)$ be a Poisson isomorphism. In particular, ϕ is an affine isomorphism, so, by Proposition 7.1.29, $\phi = R_\alpha \circ \gamma$, for some $\gamma \in \mathrm{Aut}(\mathcal{A}(P))$. By the proof of Theorem 7.1.21, γ can be written as $\omega \circ \xi$, where $\xi \in \mathrm{PAut}(\mathcal{A}(P))$ and ω is one of the following affine automorphisms: $\mathrm{id}, V, R_\nu, S_\mu, R_\nu \circ V$ or $S_\mu \circ V$. Since ϕ is Poisson, $R_\alpha \circ \omega$ needs to be too. With the same computations as in the proof of Theorem 7.1.21, we can check that the only case where $R_\alpha \circ \omega$ is Poisson is $\alpha = -1$ and $\omega = V$. In that case, $R_{-1} \circ V = \tau$. □

Putting together Corollary 7.1.27 and Theorem 7.1.30 we have proved Theorem 6.2.6 for type **A**.

Theorem 7.1.31. *Let $n \geq 2$. We have an isomorphism of groupoids*

$$\mathrm{Iso}(\mathbf{A_{n-1}}) \cong \mathrm{PIso}(\mathbf{A_{n-1}}).$$

§ 7.2 | Type D

The plan of the proof of Theorem 6.2.6 in type **D** is as follows. The generators and relations for the isomorphism groupoid for the quantizations are known. For the Poisson case, we first compute the groupoid of the isomorphisms as affine varieties of the deformations. We are then able to check which isomorphisms are Poisson, proving Theorem 6.2.6. To compute the affine isomorphisms, we first embed each deformation into a projective normal surface. We prove that every isomorphism between deformations extends to an isomorphism between these projective surfaces. These projective surfaces have some additional structure, namely they are conical bundles over $\mathbb{P}^1$, so their isomorphisms are more tractable.

§ 7.2.1 | Isomorphisms of the quantizations

Let (Q, γ) be a deformation parameter. For all $n \geq 4$, there is an isomorphism

$$\Sigma : D_n(Q, \gamma) \to D_n(Q, -\gamma),$$

given by

$$\Sigma := (x, -y, -z).$$

If $n = 4$, write $Q(x) = x^3 + ax^2 + bx + c$. Set

$$\begin{aligned} b' &= \frac{1}{8}(3a^2 - 4b - 12i\gamma), \\ c' &= c + \frac{1}{16}(a^3 - 4ab - 4ia\gamma), \\ \gamma' &= \frac{i}{8}(a^2 - 4b + 4i\gamma), \end{aligned}$$

and let $Q'(x) = x^3 + ax^2 + b'x + c'$. There is an isomorphism $T : D_n(Q, \gamma) \to D_n(Q', \gamma')$ given by

$$T := \left(-\frac{1}{2}x + \frac{i}{2}y - \left(1 + \frac{1}{4}a\right), \frac{3i}{2}x - \frac{1}{2}y + i\left(1 + \frac{1}{4}a\right), z\right).$$

Theorem 7.2.1. *The isomorphisms* Σ *and* T *satisfy the* S_3 *relations, whenever the compositions make sense. In particular* $\Sigma^2 = \mathrm{id}$ *and* $T^3 = \mathrm{id}$. *The only possible isomorphisms between quantizations* $D(Q,\gamma)$ *are* $\mathrm{id}, \Sigma, T, T^{-1}, (\Sigma \circ T)$ *and* $(\Sigma \circ T^{-1})$.

Proof. This is [Le, Theorem 2.22 & 3.6]. □

In particular, this gives an explicit description of all the automorphism groups.

Corollary 7.2.2. *The group* $\mathrm{Aut}(D_n(Q,\gamma))$ *has the following description. If* $n > 4$:

- *if* $\gamma \neq 0$, *then* $\mathrm{Aut}(D_n(Q,\gamma)) = \langle \mathrm{id} \rangle$;
- *if* $\gamma = 0$, *then* $\mathrm{Aut}(D_n(Q,\gamma)) = \mathbb{Z}/2\mathbb{Z}$, *generated by* Σ.

If $n = 4$:

- *if* $\gamma = 0$ *and* $b = a^2/4$, *then* $\mathrm{Aut}(D_4(Q,\gamma)) = S_3$, *generated by* T *and* Σ;
- *if* $\gamma \neq 0$ *and* $b = a^2/4 - i\gamma$ *(respectively* $b = a^2/4 + i\gamma$*), then* $\mathrm{Aut}(D_4) = \mathbb{Z}/2\mathbb{Z}$, *generated by* $\Sigma \circ T$ *(respectively* $\Sigma \circ T^{-1}$*)*;
- *if* $\gamma = 0$ *and* $b \neq a^2/4$, *then* $\mathrm{Aut}(D_4(Q,\gamma)) = \mathbb{Z}/2\mathbb{Z}$, *but this time it is generated by* Σ;
- *if* $\gamma \neq 0$ *and* $b \neq a^2/4 \pm i\gamma$, *then* $\mathrm{Aut}(D_4(Q,\gamma)) = \langle \mathrm{id} \rangle$.

§ 7.2.2 | Isomorphisms of the deformations

In this section we compute all the Poisson isomorphisms between the deformations $\mathcal{D}_n(Q,\gamma)$ and prove Theorem 6.2.6 in type **D**. We use some basic techniques from algebraic geometry, like self-intersection numbers and blow-ups. The uninitiated reader can refer to the first two chapters of [Be1].

Let us first compute the groupoid $\mathrm{Iso}_{aff}(\mathcal{D}_n)$ of isomorphisms between the deformations $\mathcal{D}_n(Q,\gamma)$ as affine varieties. We follow a method similar to [Bl]. We embed each $\mathcal{D}_n(Q,\gamma)$ into a projective normal surface $X_n(Q,\gamma)$, such that every point in $X_n(Q,\gamma)\setminus \mathcal{D}_n(Q,\gamma)$ is smooth in $X_n(Q,\gamma)$. We then prove that every isomorphism $\mathcal{D}_n(Q,\gamma) \to \mathcal{D}_n(Q',\gamma')$ extends to an isomorphism of $X_n(Q,\gamma) \to X_n(Q',\gamma')$. This way we reduce to studying the groupoid $\mathrm{Iso}(X_n,\mathcal{D}_n)$ of isomorphisms $X_n(Q,\gamma) \to X_n(Q',\gamma')$ that restrict to affine isomorphisms $\mathcal{D}_n(Q,\gamma) \to \mathcal{D}_n(Q',\gamma')$, which is simpler to compute.

We construct $X_n(Q,\gamma)$ as a hypersurface of $F_{s,t} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(s) \oplus \mathcal{O}_{\mathbb{P}^1}(t))$, $s,t \in \mathbb{Z}$, a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. It can be viewed as the gluing of $U_{s,t,0} = \mathbb{P}^2 \times \mathbb{C}$ and $U_{s,t,\infty} = \mathbb{P}^2 \times \mathbb{C}$ along $\mathbb{P}^2 \times \mathbb{C}^\times$, where the identification map is given by the involution

$$((w:y:z),x) \longrightarrow \left((w : x^{-s}y : x^{-t}z),\ \frac{1}{x}\right).$$

The $\mathbb{P}^2$-bundle is given by the map $F_{s,t} \to \mathbb{P}^1$ corresponding to $((w:y:z),x) \mapsto (x:1)$ in the first chart and $((w:y:z),x) \mapsto (1:x)$ in the second one. If $n = 2k$, we take the $\mathbb{P}^2$-bundle $F_{k-1,k-1}$, and denote by $X_n = X_n(Q,\gamma)$ the projective surface that restricts to the following surfaces on each chart:

$$\{((w:y:z),x) \in U_{k-1,k-1,0} \,|\, Q(x)w^2 + xy^2 + z^2 - \gamma yw = 0\},$$
$$\{((w:y:z),x) \in U_{k-1,k-1,\infty} \,|\, Q^r(x)w^2 + y^2 + xz^2 - \gamma x^k yw = 0\}.$$

Here by $Q^r(x)$ we mean the reciprocal polynomial of Q, i.e. $Q^r = x^{n-1}Q(x^{-1})$.

If $n = 2k+1$, we take the $\mathbb{P}^2$-bundle $F_{k-1,k}$, and denote by $X_n = X_n(Q,\gamma)$ the projective surface that restricts to the following surfaces on each chart:

$$\{((w:y:z),x) \in U_{k-1,k,0} \,|\, Q(x)w^2 + xy^2 + z^2 - \gamma yw = 0\},$$
$$\{((w:y:z),x) \in U_{k-1,k,\infty} \,|\, Q^r(x)w^2 + xy^2 + z^2 - \gamma x^{k+1} yw = 0\}.$$

In both cases, we embed the surface $\mathcal{D}_n$ in the first affine chart of X_n, via the embedding $(x,y,z) \mapsto ((1:y:z),x)$. Geometrically the situation is similar to the case considered in [Bl] and we can make similar remarks. All the singular points of X_n are in the image, under the above embedding, of a singular point in $\mathcal{D}_n$; in particular, for generic Q and γ, the surface $\mathcal{D}_n$ is smooth and so is X_n.

The $\mathbb{P}^2$-bundles $F_{k-1,k-1} \to \mathbb{P}^1$ and $F_{k-1,k} \to \mathbb{P}^1$ restrict to a morphism $\rho : X_n \to \mathbb{P}^1$. The fibres are conics in $\mathbb{P}^2$ (curves defined by a polynomial of degree 2), which are smooth for generic values of x. The generic fibre is thus isomorphic to $\mathbb{P}^1$. The fibre F_∞ over $(1:0)$ is always degenerate. It is a union of two transversal lines in the second chart, given by the equations $x = 0$, $w = \pm iy$ (if n is even) or $x = 0$, $w = \pm iz$ (if n is odd). Denote the two lines in F_∞ by F_+ and F_-, respectively. The other degenerate fibres are the ones lying over the solutions of the equation

$$\det\begin{pmatrix} Q(x) & -\gamma/2 & 0 \\ -\gamma/2 & x & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0.$$

Counting with multiplicity, we have $n+1$ points in $\mathbb{P}^1$ with degenerate fibre.

For every Q and γ, the complement $X_n(Q,\gamma) \setminus \mathcal{D}_n(Q,\gamma)$ consists of the curve C_n, given by the equation $w = 0$ in each chart, and the curve $F_\infty = F_+ \cup F_-$, with equation $x = 0$ in the second chart, corresponding to the fibre over $(1:0)$. We do not specify the deformation parameters when referring to the curves C_n, F_+, F_-, since they are defined by the same equation in $F_{k-1,k-1}$ or $F_{k-1,k}$.

The geometric description of the boundary is the same for every deformation parameter. We have the following (compare with [Bl, Lemma 4.3]).

Lemma 7.2.3. *For every monic polynomial Q of degree $n-1$ and for all $\gamma \in \mathbb{C}$, the complement of $\mathcal{D}_n(Q,\gamma)$ in $X_n(Q,\gamma)$ is the union of the three curves C_n, F_+, F_-, all isomorphic to $\mathbb{P}^1$. Any two of them intersect transversally, in exactly one point, which is $C_n \cap F_+ \cap F_-$. Moreover, $C_n^2 = 3-n$, and $F_+^2 = F_-^2 = -1$.*

Proof. The only thing that does not follow from the discussion above is the self-intersection numbers. Let P_n be the curve given by the equation $y=0$ in both charts. If $n=2k$, P_n and C_n intersect only in the point $((0:0:1),0)$ of the second chart. This happens along the distinct directions $w=0$ and $y=0$, so $C_n \cdot P_n = 1$. If $n=2k+1$, the two curves are disjoint, so $C_n \cdot P_n = 0$. In both cases, $C_n \cdot P_n = 2k+1-n$.

Consider the rational function $g \in \mathbb{C}(X_n)^*$ given by w/y on the second chart, and by wx^{k-1}/y on the first chart. The associated principal divisor is $C_n + (k-1)F_0 - P_n$, where F_0 is given by the equation $x=0$ in the first chart, i.e. it is the fibre over $(0:1)$. Computing the principal divisor associated to the rational function x, it is clear that F_0 is linearly equivalent to F_∞. The intersection of C_n and F_∞ has multiplicity 2, since

$$C_n \cdot F_\infty = C_n \cdot (F_+ + F_-) = 2.$$

Thus

$$C_n^2 = C_n \cdot (P_n - (k-1)F_\infty) = (2k+1-n) - (2k-2) = 3-n.$$

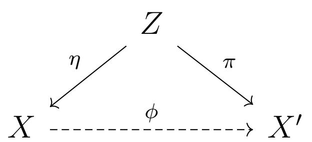
Clearly, F_0 and F_∞ are disjoint, so $F_0 \cdot F_\infty = 0$. Since they are linearly equivalent, this means

$$0 = F_0 \cdot F_\infty = F_\infty^2 = (F_+ + F_-)^2 = F_+^2 + F_-^2 + 2F_+ \cdot F_- = F_+^2 + F_-^2 + 2.$$

The linear equivalence of F_0 and F_∞ implies $F_+ = F_0 - F_-$, so $F_+^2 = F_-^2 + F_0^2$. Since $F_0^2 = F_0 \cdot F_\infty = 0$, this implies that $F_+^2 = F_-^2 = -1$. □

Consider now $\phi \in \mathrm{Iso}_{aff}(\mathcal{D}_n)$, an isomorphism $\mathcal{D}_n(Q,\gamma) \to \mathcal{D}_n(Q',\gamma')$. This extends to a birational map $\phi : X_n(Q,\gamma) \dashrightarrow X_n(Q',\gamma')$, that is biregular between $\mathcal{D}_n(Q,\gamma) \subset X_n(Q,\gamma)$ and $\mathcal{D}_n(Q',\gamma') \subset X_n(Q',\gamma')$. We recall the following results about birational maps and blow-ups.

Lemma 7.2.4. *Let X, X' be projective complex surfaces and $\phi : X \dashrightarrow X'$ a birational map. Assume that X and X' are smooth outside of the open sets S and S', and that $\phi : S \to S'$ is biregular. Then there exists a surface Z and a commutative diagram*



where the morphisms η, π are composite of blow-ups.

Proof. If X, X' are smooth, this is Theorem II.11 of [Be1]. If X or X' are not smooth, ϕ induces a birational map $\hat{\phi}$ between resolutions $\hat{X}$ and $\hat{X}'$. We are now in the smooth case, so we get the following commutative diagram

$$\begin{array}{ccccc} & & Z' & & \\ & \swarrow^{\eta'} & & \searrow^{\pi'} & \\ \hat{X} & & \overset{\hat{\phi}}{\dashrightarrow} & & \hat{X}' \\ \downarrow f & & & & \downarrow g \\ X & & \overset{\phi}{\dashrightarrow} & & X' \end{array}$$

where both f and g are composition of blow-ups. Now, the singularities of X and X' are inside S and S' respectively, and ϕ is biregular between S and S'. We can thus contract all the curves blown up when resolving the singularities. Call Z the image of Z' under these contractions. We end up with the following diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{\eta} & & \searrow^{\pi} & \\ X & & \overset{\phi}{\dashrightarrow} & & X' \end{array}$$

with $\eta = f \circ \eta' \circ f^{-1}$ and $\pi = g \circ \pi' \circ g^{-1}$. □

Lemma 7.2.5. *Let S be a smooth surface, $\pi : \hat{S} \to S$ the blow-up of a point $p \in S$ and $E \subset \hat{S}$ the exceptional divisor.*

(i) Let $C \subset S$ be an irreducible curve. Then $\pi^*C = \overline{C} + mE$, where m is the multiplicity of p in C, and the curve $\overline{C}$ is the strict transform of C;

(ii) Let D, D' be divisors on S. Then $(\pi^*D) \cdot (\pi^*D') = D \cdot D'$, $E \cdot \pi^*D = 0$ and $E^2 = -1$.

Proof. This is standard. See, for example, [Be1, Proposition II.2-3] □

Corollary 7.2.6. *In the setting of Lemma 7.2.5, let $C \subset S$ be an irreducible curve.*

(i) The intersection number $E \cdot \overline{C}$ is the multiplicity of p in C.

(ii) $C^2 \geq \overline{C}^2$, and $C^2 = \overline{C}^2$ if and only if $p \notin C$;

Proof. From point (i) of Lemma 7.2.5, $\pi^*C = \overline{C} + mE$. Multiply both sides by E. Using point (ii) of Lemma 7.2.5, this equation becomes $0 = \overline{C} \cdot E - m$.

We compute the self-intersection of C. Using again point (ii) of Lemma 7.2.5,

$$C^2 = (\pi^*C)^2 = (\overline{C} + mE)^2 = \overline{C}^2 + 2m\overline{C} \cdot E - m^2E^2 = \overline{C}^2 + m^2 \geq \overline{C}^2.$$

□

We can apply Lemma 7.2.4 to our birational map ϕ and get the diagram

$$\begin{array}{ccc} & Z' & \\ {}^{\eta}\swarrow & & \searrow^{\pi} \\ X_n(Q,\gamma) & \overset{\phi}{\dashrightarrow} & X_n(Q',\gamma') \end{array} \tag{7.2.1}$$

where the morphisms η, π are compositions of blow-ups of points of $X_n(Q,\gamma)\setminus\mathcal{D}_n(Q,\gamma)$ and $X_n(Q',\gamma')\setminus\mathcal{D}_n(Q',\gamma')$ respectively. Denote by $\overline{F_+}, \overline{F_-}$ and $\overline{C_n}$ the strict transforms via η^{-1} of F_+, F_- and C_n. Note that since we are blowing-up away from the singularities, we can assume to be in the setting of Lemma 7.2.5. Since ϕ is a biregular morphism between $\mathcal{D}_n(Q,\gamma)$ and $\mathcal{D}_n(Q',\gamma')$, it sends the boundary $F_+ \cup F_- \cup C_n$ into itself. Suppose now that ϕ is not an isomorphism $X_n(Q,\gamma) \xrightarrow{\sim} X_n(Q',\gamma')$. Then π must contract one of the curves $\overline{F_+}, \overline{F_-}$ and $\overline{C_n}$ to a point.

Remark 7.2.7. Without loss of generality, we can assume that π decomposes into a sequence of blow-ups of points $\pi_s \circ \cdots \circ \pi_1$, where π_1 contracts one of the curves $\overline{F_+}, \overline{F_-}$ or $\overline{C_n}$. To see why, let us decompose $\eta = \eta_1 \circ \ldots \circ \eta_k$ into blow-ups of points. Each η_i adds an exceptional divisor, which we denote E_i. Denote by $\overline{E_i}$ their strict transforms. If π_1 contracts E_k, then we can remove π_1 and η_k and get another diagram of the form (7.2.1). Suppose now that π_1 contracts $\overline{E_i}$ for some $i < k$. This means that $\overline{E_i}^2 = -1$, so the self-intersection number of E_i does not change under the strict transform. By point (ii) of Corollary 7.2.6, the maps η_j with $j > i$ blow-up points that are not in E_i. So η_i commutes with all η_j with $j > i$ and we reduce ourselves to the case E_k.

We have the following proposition (compare with [Bl, Proposition 4.4]).

Proposition 7.2.8. *Let $n \geq 4$, Q, Q' monic polynomials of degree $n-1$ and $\gamma, \gamma' \in \mathbb{C}$. Every affine isomorphism $\phi : \mathcal{D}_n(Q,\gamma) \to \mathcal{D}_n(Q',\gamma')$ extends to an isomorphism $X_n(Q,\gamma) \to X_n(Q',\gamma')$.*

Proof. Suppose that $\phi : \mathcal{D}_n(Q,\gamma) \to \mathcal{D}_n(Q',\gamma')$ does not extend to an isomorphism $X_n(Q,\gamma) \to X_n(Q',\gamma')$. We are in the situation described in diagram (7.2.1). By Remark 7.2.7, we can assume that the first curve contracted by π is either $\overline{F_+}, \overline{F_-}$ or $\overline{C_n}$. Since this curve is a (-1)-curve, it is either $\overline{F_+}$ or $\overline{F_-}$ (or $\overline{C_n}$ if $n = 4$), by (ii) of Corollary 7.2.6; say it is $\overline{F_+}$.

Since taking the strict transform via η^{-1} does not change the self-intersection number of F_+, η does not blow-up any point of F_+ by (ii) of Corollary 7.2.6. In particular, it does not blow-up the triple intersection point of the components of the boundary. So $\overline{F_+}, \overline{F_-}$ and $\overline{C_n}$ still intersect transversely in one point inside Z. Denote $F^{(1)} := \pi_1(\overline{F_-})$ and $C^{(1)} = \pi_1(\overline{C_n})$. Then

$$F^{(1)} \cdot C^{(1)} = (\overline{F_-} + \overline{F_+}) \cdot (\overline{C_n} + \overline{F_+}) = 1 + 1 + 1 - 1 = 2,$$

by applying both points of Lemma 7.2.5 and the fact that $m = \overline{F_-} \cdot \overline{F_+} = 1$ (by (i) of Corollary 7.2.6). Thus $F^{(1)}$ and $C^{(1)}$ are tangent. This leads to a contradiction because it is not possible to recover the original boundary via other contractions. In fact, if π contracts $F^{(1)}$ or $C^{(1)}$, say $F^{(1)}$, then $\pi(C^{(1)})$ is a curve whose strict transform $C^{(1)}$ intersects the exceptional divisor with multiplicity 2. This means, by (i) of Corollary 7.2.6, that it contains a point with multiplicity 2, so it is not smooth. This is a contradiction, because the curves of the boundary are all smooth, and contracting does not resolve singularities. If instead π never contracts $F^{(1)}$ or $C^{(1)}$, then it does not modify their intersection point, and it is impossible to recover the original boundary. The same reasoning works if the curve contracted by π_1 is $\overline{F_-}$ (or $\overline{C_n}$ if $n = 4$). □

Using Proposition 7.2.8, we can compute the groupoid $\mathrm{Iso}_{aff}(\mathcal{D}_n)$ for all $n \geq 4$ (compare with [Bl, Corollary 4.5]). First, let us recall some facts that will be used in the proof.

Lemma 7.2.9. *Every isomorphism between two non-degenerate conics in $\mathbb{P}^2$ can be extended to a projective transformation of $\mathbb{P}^2$.*

Proof. Since two nondegenerate conics are always isomorphic via a projective transformation of $\mathbb{P}^2$, it is sufficient to show that every automorphism of a specific nondegenerate conic extends to a projective transformation of $\mathbb{P}^2$.

Consider the conic $C : xy - z^2 = 0$. Since it is nondegenerate, it is isomorphic to $\mathbb{P}^1$. Explicitly, this is given by

$$(s : t) \mapsto (s^2 : st : t^2).$$

Thus, every automorphism of C will be induced by an automorphism of $\mathbb{P}^1$, which has the form $(s : t) \mapsto (as + bt : cs + dt)$, with $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. The induced automorphism on the conic is

$$(x : y : z) \mapsto (a^2x + 2aby + b^2z : acx + (ad + bc)y + bdz : c^2x + 2cdy + d^2z),$$

which extends to an element of $PGL(3, \mathbb{C})$, because the determinant of the associated matrix is $(ad - bc)^3$. □

Proposition 7.2.10. *Consider the algebra $A := \mathbb{C}[x, y, z]/(xy^2 - z^2)$, graded with* $\deg x = 0, \deg y = 1, \deg z = 1$. *The group of graded automorphisms of A is:*

$$\{(\alpha^2\beta^{-2}x, \beta y, \alpha z) \,|\, \alpha, \beta \in \mathbb{C}^\times\}.$$

Proof. It is shown in [Ma2, Theorem 1] that the group of automorphisms of A is generated by:

1. Hyperbolic rotations: $H_\beta = (\beta^{-2}x, \beta y, z)$, for all $\beta \in \mathbb{C}^\times$.

2. Rescalings: $R_\alpha = (\alpha^2 x, y, \alpha z)$, for all $\alpha \in \mathbb{C}^\times$.

3. Triangular automorphisms:

$$\Delta_g = (x + [(z + y^2 g(y))^2 - z^2]y^{-2}, y, z + y^2 g(y)),$$

   for all $g(y) \in \mathbb{C}[y]$.

The triangular automorphisms form a normal subgroup Δ isomorphic to the additive group $\mathbb{C}[x]$ via $g \mapsto \Delta_g$, and the group of automorphisms is the semidirect product of Δ and of the subgroup generated by the automorphisms of type (1) and (2) ([Ma2, Final Remark]). The proposition follows by noticing that automorphisms in Δ do not preserve the grading (except for the identity) and that $H_\mu H_\nu = H_{\mu\nu}$, $R_\mu R_\nu = R_{\mu\nu}$ and $R_\mu H_\nu = H_\nu R_\mu$. Thus, the group of graded automorphisms is the subgroup generated by all H_ν and R_μ. □

Lemma 7.2.11. *Suppose $\rho_1 : X_1 \to \mathbb{P}^1$ and $\rho_2 : X_2 \to \mathbb{P}^1$ are proper, surjective morphisms with irreducible fibres, and let ϕ be an isomorphism $X_1 \to X_2$. If there exists a fibre in X_1 that ϕ sends to a fibre in X_2, then ϕ sends all fibres to fibres.*

Proof. Suppose there exists $x_0 \in \mathbb{P}^1$ such that $\phi(\rho_1^{-1}(x_0)) = \rho_2^{-1}(x_1)$ for some $x_1 \in \mathbb{P}^1$. Take any other fibre $\rho_1^{-1}(z)$, with $z \in \mathbb{P}^1$. Assume $\phi(\rho_1^{-1}(z))$ is not a fibre. Then $\rho_2(\phi(\rho_1^{-1}(z))) = \mathbb{P}^1$ since ρ_2 is proper, which means that there is a point in $\phi(\rho_1^{-1}(z))$ that gets mapped to x_1. So, $\phi(\rho_1^{-1}(x_0)) \cap \phi(\rho_1^{-1}(z)) \neq \emptyset$, which is absurd because ϕ is an isomorphism. □

Define the following morphisms in $\mathrm{Iso}_{aff}(\mathcal{D}_n)$, for $n \geq 4$.

1. $$R_\lambda^\pm : \mathcal{D}_n(Q, \gamma) \to \mathcal{D}_n(Q', \gamma'),$$
   $$R_\lambda^\pm = (\lambda^2 x, \mu\lambda^{-2} y, \pm\mu\lambda^{-1} z),$$
   with $\mu = \gamma/\gamma'$, for all Q, Q' such that $Q(\lambda^2 x) = \lambda^{2(n-1)} Q'(x)$ and for all $\gamma, \gamma' \neq 0$ such that $\lambda^n = \pm\mu$;

2. $$P_\lambda^\pm : \mathcal{D}_n(Q, 0) \to \mathcal{D}_n(Q', 0),$$
   $$P_\lambda^\pm = (\lambda^2 x, \pm\lambda^{n-2} y, \pm\lambda^{n-1} z),$$
   for all Q, Q' such that $Q(\lambda^2 x) = \lambda^{2(n-1)} Q'(x)$ and for all $\lambda \in \mathbb{C}^\times$.

Theorem 7.2.12. *Let Q, Q' be monic polynomials of degree $n - 1$ and $\gamma, \gamma' \in \mathbb{C}$.*

(i) If $n > 4$, the only isomorphisms in $\mathrm{Iso}_{aff}(\mathcal{D}_n)$ are of the form $R_\lambda^\pm$ and $P_\lambda^\pm$.

(ii) If $n = 4$, let $Q(x) = x^3 + ax^2 + bx + c$ and $Q'(x) = x^3 + a'x^2 + b'x + c'$. Then we have an isomorphism $\tau : \mathcal{D}_n(Q, \gamma) \to \mathcal{D}_n(Q', \gamma')$ given by

$$\tau = \left(-\frac{1}{2}x + \frac{i}{2}y - \frac{1}{4}a, \frac{3i}{2}x - \frac{1}{2}y + \frac{i}{4}a, z\right), \tag{7.2.2}$$

every time (Q', γ') satisfies the following condition:

$$\begin{aligned} a' &= a, \\ b' &= \frac{1}{8}(3a^2 - 4b - 12i\gamma), \\ c' &= c + \frac{1}{16}(a^3 - 4ab - 4ia\gamma), \\ \gamma' &= \frac{i}{8}(a^2 - 4b + 4i\gamma). \end{aligned} \tag{7.2.3}$$

The groupoid $\mathrm{Iso}_{aff}(\mathcal{D}_4)$ is generated by all isomorphisms of the form $R^{\pm}_{\lambda}$, $P^{\pm}_{\lambda}$ and τ.

Proof. It is easy to check that the listed actions are well-defined isomorphisms. Thus, we only need to prove that they generate the whole of $\mathrm{Iso}_{aff}(\mathcal{D}_n)$. We know from Proposition 7.2.8 that every element of $\mathrm{Iso}_{aff}(\mathcal{D}_n)$ lifts to an element of $\mathrm{Iso}(X_n)$. We can thus consider the groupoid $K := \mathrm{Iso}(X_n, \mathcal{D}_n)$ of isomorphisms $X_n(Q, \gamma) \to X_n(Q', \gamma')$ that restrict to affine isomorphisms $\mathcal{D}_n(Q, \gamma) \to \mathcal{D}_n(Q', \gamma')$.

First, let us compute the subgroupoid $K_0 \leq K$ of isomorphisms that preserve the conic bundle structure, i.e. that send a fibre to another fibre. Every isomorphism $g \in K_0$ induces an automorphism of $\mathbb{P}^1$. In addition, it needs to preserve the boundary $F_\infty \cup C_n$, so it must preserve the fibre over $(1 : 0)$. Thus, the action on $\mathbb{P}^1$ is given by a degree one polynomial $x \mapsto ax + b$. Since g sends conics to isomorphic conics, by Lemma 7.2.9, it extends to a projective transformation of the projective plane that contains them (which is the fibre of the $\mathbb{P}^2$ bundle $F_{s,t}$). Since it also preserves the intersection of the conics with C_n (i.e. with $w = 0$) we can write g (thought of as an isomorphism $\mathcal{D}_n(Q, \gamma) \to \mathcal{D}_n(Q', \gamma')$) as

$$(x, y, z) \mapsto (ax + b, cy + dz + e, fy + hz + k),$$

with $a, b \in \mathbb{C}$ and $c, d, e, f, h, k \in \mathbb{C}[x]$ satisfying $ch - df \neq 0$.

Consider on $\mathcal{D}_n(Q, \gamma)$ the filtration $\mathcal{F}$ induced by $\deg x = 0$ and $\deg y = \deg z = 1$. Then

$$\mathrm{gr}_{\mathcal{F}}(\mathcal{D}_n(Q, \gamma)) \cong A = \mathbb{C}[x, y, z]/(xy^2 - z^2),$$

for every deformation parameter (Q, γ). Since g preserves the filtration $\mathcal{F}$, it induces a graded automorphism $\mathrm{gr}(g)$ of A. By Proposition 7.2.10, $\mathrm{gr}(g) = (\alpha^2\beta^{-2}x, \beta y, \alpha z)$ for some $\alpha, \beta \in \mathbb{C}^\times$, so

$$g = (\alpha^2\beta^{-2}x + b, \beta y + c, \alpha z + d), \tag{7.2.4}$$

with $b \in \mathbb{C}$ and $c, d \in \mathbb{C}[x]$. Plugging (7.2.4) into the defining equation of $\mathcal{D}_n(Q, \gamma)$ we get the following condition:

$$Q(\alpha^2\beta^{-2}x+b)+(\alpha^2\beta^{-2}x+b)(\beta y+c)^2+(\alpha z+d)^2-\gamma(\beta y+c) = k(Q'(x)+xy^2+z^2-\gamma' y),$$

for some $k \in \mathbb{C}[x, y, z]$. Comparing the coefficients of z^2, z, y^2, y and the constant term we get, respectively

$$\alpha^2 = k, \tag{7.2.5}$$

$$2\alpha d = 0, \tag{7.2.6}$$

$$(\alpha^2\beta^{-2}x + b)\beta^2 = kx, \tag{7.2.7}$$

$$2(\alpha^2\beta^{-2}x + b)\beta c - \gamma\beta = -k\gamma', \tag{7.2.8}$$

$$Q(\alpha^2\beta^{-2}x + b) + (\alpha^2\beta^{-2}x + b)c^2 + d^2 - \gamma c = kQ'(x). \tag{7.2.9}$$

From (7.2.5), (7.2.6) and (7.2.7) we get that $k \in \mathbb{C}^\times$, $d = 0$ and $b = 0$. From (7.2.8), since k is a constant, the coefficient of x is 0, so $c = 0$. Combining (7.2.5),(7.2.7), and (7.2.8), we also get $\gamma = \gamma' = 0$ or $\gamma\beta = \alpha^2\gamma'$. From (7.2.5) and (7.2.9) we have $Q(\alpha^2\beta^{-2}x) = \alpha^2 Q'(x)$. In particular, from comparing the x^{n-1} coefficients, we get $\alpha^{2n-4} = \beta^{2n-2}$, so

$$\alpha^{n-2} = \pm\beta^{n-1}. \tag{7.2.10}$$

Consider $\lambda := \alpha\beta^{-1}$. Then (7.2.10) implies $\lambda^{n-1} = \pm\alpha$ and $\lambda^{n-2} = \pm\beta$, with the same sign. We can thus rewrite the relation on Q as

$$Q(\lambda^2 x) = \lambda^{2(n-1)}Q'(x). \tag{7.2.11}$$

If we write $Q(x) = x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$, then by (7.2.11) we have $Q'(x) = x^{n-1} + \lambda^{-2}a_{n-2}x^{n-2} + \cdots + \lambda^{-2(n-1)}a_0$.

Let us first consider the case with $\gamma, \gamma' \neq 0$. Denote by $\mu := \gamma/\gamma'$. We have the condition $\mu\beta = \alpha^2$. So $\pm\mu\lambda^{n-2} = \lambda^{2n-2}$, i.e. $\lambda^n = \pm\mu$. Thus, K_0 contains an isomorphism $R^\pm_\lambda : \mathcal{D}_n(Q, \gamma) \to \mathcal{D}_n(Q', \gamma')$ of the form

$$R^\pm_\lambda = (\lambda^2 x, \lambda^{n-2}y, \pm\lambda^{n-1}z) = (\lambda^2 x, \mu\lambda^{-2}y, \pm\mu\lambda^{-1}z),$$

for all λ such that $\lambda^n = \pm\mu$ and for all Q, Q' such that (7.2.11) holds.

When $\gamma = \gamma' = 0$ we have no extra condition. Thus, K_0 contains an isomorphism $P^\pm_\lambda : \mathcal{D}_n(Q, 0) \to \mathcal{D}_n(Q', 0)$ of the form

$$P^\pm_\lambda = (\lambda^2 x, \pm\lambda^{n-2}y, \pm\lambda^{n-1}z),$$

for all $\lambda \in \mathbb{C}^\times$ and for all Q, Q' such that (7.2.11) holds. By equations (7.2.5) to (7.2.9), the only elements in K_0 are of the form R_λ, P_λ.

As we noted before, the automorphisms in K_0 preserve the fibre F_∞. On the other hand, if an isomorphism in K preserves F_∞, then it is in K_0, by Lemma 7.2.11. Thus an isomorphism in $K \setminus K_0$ does not preserve F_∞ or, equivalently, C_n. Since the self intersection numbers for C_n, F_+ and F_- are $3-n, -1, -1$ we see that $\mathrm{Iso}_{aff}(\mathcal{D}_n) = K_0$ for $n > 4$. For $n = 4$, one directly checks that we have an isomorphism $\tau : \mathcal{D}_4(Q, \gamma) \to \mathcal{D}_4(Q', \gamma')$ of order 3 defined by (7.2.2), with Q', γ' as in (7.2.3). The isomorphism τ does not preserve the conic bundle, so it cyclically permutes the curves F_+, F_- and C_4. Together with the isomorphism $\sigma_y : \mathcal{D}_4(Q, \gamma) \to \mathcal{D}_4(Q, -\gamma)$ defined by $\sigma_y = (x, -y, z)$, we have a full action of S_3 on set of the three curves F_+, F_-, C_4.

Take now any isomorphism $\phi : \mathcal{D}_4(Q, \gamma) \to \mathcal{D}_4(Q', \gamma')$. If ϕ fixes the curves F_+, F_- and C_4, then it is in K_0. Otherwise, we can compose ϕ with the appropriate permutation generated by τ and σ_y to get an isomorphism that fixes the three curves and thus is again in K_0. Since $\sigma_y \in K_0$, it follows that τ and K_0 generate the whole groupoid K.

□

Theorem 7.2.13. *Let Q, Q' monic polynomials of degree $n-1$ and $\gamma, \gamma' \in \mathbb{C}$.*

- *If $n > 4$ there are only two classes of Poisson isomorphisms,* id *and σ, where*

$$\sigma : \mathcal{D}_n(Q, \gamma) \to \mathcal{D}_n(Q, -\gamma), \qquad \sigma = (x, -y, -z). \tag{7.2.12}$$

- *If $n = 4$ the groupoid $\mathrm{PIso}(\mathcal{D}_4)$ is generated by the isomorphism σ defined in (7.2.12) and by the isomorphism τ defined in (7.2.2). Moreover, every time the composition makes sense, σ and τ satisfy the S_3 relations $\tau^3 = \mathrm{id}$, $\sigma^2 = \mathrm{id}$ and $\sigma \circ \tau \circ \sigma = \tau^2$.*

Proof. It is sufficient to check when the isomorphisms of Theorem 7.2.12 satisfy the Poisson relations. In particular, by imposing $\{x, y\} = 2z$, we get that $R_\lambda^\pm$ and $P_\lambda^\pm$ are Poisson isomorphisms only if $\lambda = 1$, in which case $R_1^+ = P_1^+ = \mathrm{id}$ and $R_1^- = P_1^- = \sigma$. It is then straightforward to check that σ is a Poisson isomorphism. For the case $n = 4$, one can directly check that τ is a Poisson isomorphism and that τ and σ satisfy the S_3 relations. □

Putting together Theorem 7.2.1 and 7.2.13 we can prove Theorem 6.2.6 for Kleinian singularities of type **D**.

Theorem 7.2.14. *Let $n \geq 4$. We have an isomorphism of groupoids*

$$\mathrm{Iso}(\mathbf{D_n}) \cong \mathrm{PIso}(\mathbf{D_n}).$$

In particular, for every deformation parameter (Q,γ)*,*

$$\mathrm{Aut}(D(Q,\gamma)) \cong \mathrm{PAut}(\mathcal{D}(Q,\gamma)).$$

We can also describe the groups of affine automorphisms of $\mathcal{D}_n(Q,\gamma)$ by checking for which deformation parameters the isomorphisms in Theorem 7.2.12 are automorphisms.

Theorem 7.2.15. *Let* $n \geq 4$*,* Q *monic of degree* $n-1$ *and* $\gamma \in \mathbb{C}$*. Write* $Q(x) = x^d P(x^m)$*, with* $d \in \mathbb{Z}_{\geq 0}$ *and* $m \in \mathbb{Z}_{\geq 1}$*, with* m *maximal possible.*

- *If* $\gamma \neq 0$*, the group*

  $$G = \{(\lambda^2 x, \lambda^{-2} y, \pm\lambda^{-1} z) \,|\, \lambda^{2n} = 1,\ \lambda^{2m} = 1\}$$

  acts on $\mathcal{D}_n(Q,\gamma)$*.*

- *If* $\gamma = 0$*, the group*

  $$G = \{(\lambda^2 x, \pm\lambda^{n-2} y, \pm\lambda^{n-1} z) \,|\, \lambda^{2m} = 1\}$$

  acts on $\mathcal{D}_n(Q,\gamma)$*.*

If $n > 4$*, then* $\mathrm{Aut}(\mathcal{D}_n(Q,\gamma)) = G$*.*

In particular, for generic (Q,γ)*,*

$$\mathrm{Aut}(\mathcal{D}_n(Q,\gamma)) = \mathbb{Z}/2\mathbb{Z},$$

generated by $\sigma_z = (x, y, -z)$*.*

For generic Q *and* $\gamma = 0$*,*

$$\mathrm{Aut}(\mathcal{D}_n(Q,\gamma)) = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}),$$

generated by σ_z *and* $\sigma_y = (x, -y, z)$*.*

If $n = 4$*, write* $Q(x) = x^3 + ax^2 + bx + c$*. Then:*

- *if* $\gamma = 0$ *and* $b = a^2/4$*, then* $\mathrm{Aut}(\mathcal{D}_4(Q,\gamma))$ *is generated by* G *and* τ*;*
- *if* $\gamma \neq 0$ *and* $b = a^2/4 - i\gamma$ *(respectively* $b = a^2/4 + i\gamma$*), then Aut*$(\mathcal{D}_4(Q,\gamma))$ *is generated by* G *and* $\sigma \circ \tau$ *(respectively* $\sigma \circ \tau^{-1}$*);*
- *otherwise, Aut*$(\mathcal{D}_4(Q,\gamma)) = G$*.*

Proof. Theorem 7.2.15 follows at once from Theorem 7.2.12, by noticing that $Q(\lambda^2 x) = \lambda^{2(n-1)} Q(x)$ if and only if $Q(x) = x^d P(x^m)$, with $\lambda^{2m} = 1$. □

Remark 7.2.16. Notice that we recover from Theorem 7.2.15 the result of [Bl, Corollary 4.5] for the undeformed case, i.e. when $Q(x) = x^{n-1}$ and $\gamma = 0$.

Chapter 8

Conclusion

The results of Chapter 7 provide positive evidence for Conjecture 6.2.5 and, consequently, for the Belov-Kanel Kontsevich Conjecture 6.2.4. Moreover, Conjecture 6.2.5 suggests new, non-trivial examples where to test the BKK Conjecture.

Even if the proof is different for Type **A** and **D**, a posteriori we can appreciate some common features that are worth highlighting, as they could be useful when investigating other cases.

Thanks to Theorem 6.2.1, the cases of filtered isomorphisms are well understood. It is thus important to determine how many non-filtered isomorphisms there are. In the cases examined, we know that there are no "exotic", non-filtered isomorphisms. Indeed, in type **D** all the isomorphisms are filtered. In type **A**, the only non-filtered generators of Iso are inner automorphisms. That is, they are all of the form $\exp(\operatorname{ad} a)$, for some ad-nilpotent element a. Inner automorphisms are easy to lift from a quantization to the corresponding deformation, by considering the exponentiation of the corresponding Poisson-adjoint action. It would be interesting to understand if this is a general fact, that could be used to prove Conjecture 6.2.5 in more general cases, or if it is simply a coincidence.

If $X = V/G$ is a symplectic quotient, then the group of graded Poisson automorphisms of $\mathbb{C}[X]$ is equal to $\Theta := N_{Sp(V)}(G)/G$ (see [Lo2, Lemma 3.20]). For a Kleinian singularity X with Dynkin diagram Δ, the group of diagram automorphisms $\operatorname{Aut}(\Delta)$ acts on $\mathbb{C}[X]$ by graded Poisson automorphisms, and can be identified with a subgroup of Θ. The action of Θ on the Δ arises via the action on the exceptional fibre of the minimal resolution of the singularity, which is a union of projective lines in the shape of Δ. In every type except type **A**, the diagram automorphisms coincide with Θ, while in type **A** they are a proper subgroup. In fact, Θ is generated by $\operatorname{Aut}(\Delta)$ and by hyperbolic rotations. The hyperbolic rotations act non-trivially on each projective line in the exceptional fibre, but stabilise them. Interestingly, the only isomorphisms in $\operatorname{PIso}(\mathcal{A})$ that are not automorphisms come from $\operatorname{Aut}(\Delta)$. If c_1, c_2 are two deformation parameters such that $\mathfrak{X}_{c_1} \cong \mathfrak{X}_{c_2}$, then there is an isomorphism ϕ induced by a Dynkin diagram automorphism between them. All the isomorphisms $\mathfrak{X}_{c_1} \cong \mathfrak{X}_{c_2}$ are the composition of ϕ with some automorphism. In other words, the action of the hyperbolic rotations and the inner automorphisms on the moduli space $\mathfrak{P}/W$ is trivial. Again, it would be interesting to investigate this further. If it is a general feature, then it suggests a way to approach Conjecture 6.2.5.

Future directions

The proof of Theorem 6.2.6 is case by case and it is not clear if the result can be generalised directly. Hopefully, the same techniques can be used in other cases.

- Theorem 6.2.6 leaves open the problem of type **E**. I believe this to hold almost trivially: if what we observed for type **A** and **D** holds, then only in type $\mathbf{E}_6$ should we have a non-trivial isomorphism, coming from the involution of the Dynkin diagram. The geometric techniques of Section 7.2 can be applied to Kleinian singularities of type **E** as well (see [Bl]), so the deformation side is approachable. The problem is the quantizations: it is hard to show that there exist no "exotic" isomorphisms. The first issue is that we do not have a presentation by generator and relations of the quantization. Levy's paper [Le] suggests a way to obtain such a presentation, but even then finding all the possible isomorphisms is not trivial, as the relations are expected to be complicated. A different approach is likely needed.

- In [Kl], Klyuev considered inclusions of pairs of Kleinian singularities and proved that the deformations and quantizations of these inclusions are parameterized by $\mathfrak{h}'/W'$, where $\mathfrak{h}'$ and W' are, respectively, the Cartan subalgebra and the Weyl group of a folded root system. We can interpret these inclusions as a non-simply laced analogue of Kleinian singularities. It would be interesting to see if a result similar to Theorem 6.2.6 holds when considering isomorphisms of deformations/quantizations of pairs of Kleinian singularities.

- It is likely very hard to get a full proof of Conjecture 6.2.5, but some other special cases may be easier to handle. It would be very interesting to test the conjecture on a symplectic quotient singularity, not of Kleinian type. An algebraic approach is possible in this case, using the symplectic reflection algebra (see Definition 6.1.9). In particular, the realization of $\mathsf{eH}_{0,c}\mathsf{e}$ as the classical limit of $\mathsf{eH}_{t,c}\mathsf{e}$, for $t \neq 0$ may be useful.

- Finally, the most general case of conical symplectic singularities may still be approached using the results of this thesis. In fact, Kleinian singularities play a crucial role in the construction of the deformation and quantizations of a conical symplectic singularity (see Section 6.1). Kleinian singularities appear as transverse slices to the codimension 2 symplectic leaves. A possible approach would be to study the way the isomorphisms of conical symplectic singularities act on the stratification by symplectic leaves. Isomorphic conical symplectic singularities will admit the same stratification. The key point is to prove that we can choose the transverse slices such that they are stable under the action of any given isomorphism. This should be possible, at least in some cases, although no general result is known. If this is true, then the classification of the isomorphisms should follow from the classification of the slices and of all the possible isomorphisms between them. The latter one can be deduced from the results of this thesis (if the slices are not Kleinian singularities of type **E**).

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