Stratification, costratification, relative tensor-triangular geometry and singularity categories

Charalampos Verasdanis

Abstract. We develop the theory of stratification for a rigidly-compactly generated tensor-triangulated category using the smashing spectrum and the small smashing support. Our first result, outside the stratified context, is that the Hochster dual of the Balmer spectrum is the Kolmogorov quotient of the smashing spectrum equipped with a certain topology. Within the stratified context, we prove that it suffices to check stratification on certain smashing localizations and we investigate connections between big prime ideals, objectwise-prime ideals and homological primes. We give a characterization of the Telescope Conjecture in terms of the homological spectrum and the homological support. Moreover, we study induced maps between smashing spectra and prove a descent theorem for stratification.

We develop the theory of costratification in the setting of relative tensor-triangular geometry and prove that costratification is equivalent to the colocal-to-global principle and cominimality. We also introduce and study prime localizing submodules and prime colocalizing hom-submodules, in the first case, generalizing objectwise-prime localizing tensor-ideals. Further, we prove that it suffices to check costratification on certain localizations with respect to smashing submodules and certain covers of the associated space of supports/cosupports. We apply our results to show that the derived category of quasi-coherent sheaves over a noetherian separated scheme is costratified, generalizing a result of Neeman for noetherian rings.

Finally, we classify the colocalizing subcategories of the singularity category of a locally hypersurface ring and then we generalize this result to schemes with hypersurface singularities.

Contents

ACKNOWLEDGEMENTS. I would like to thank Greg Stevenson for all the enlightening conversations during these past four years. I am also grateful to Paul Balmer, Sira Gratz, Josh Pollitz and Jan Šťovíček for their interest, suggestions and comments. I thank the University of Glasgow, in particular the School of Mathematics and Statistics, for providing an exemplary environment for doing research and I thank EPSRC for their financial support. Lastly, I am indebted to all friends and family members for their support.

Author's declaration. I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Introduction

This thesis consists of three parts developed in Chapters 2, 3, 4, corresponding to the author's papers [**Ver23c, Ver23a, Ver23b**], respectively. Before we explain the main results of each part, let us give a summary of the concepts in tensor-triangular geometry that are related to our work.

* * *

Tensor-triangular geometry started with Balmer's work [**Bal05**] that associates with each essentially small tensor-triangulated category $\mathcal{K} = (\mathcal{K}, \otimes, 1)$ the space $\mathrm{Spc}(\mathcal{K})$ of prime ideals of $\mathcal{K}$, that we call the *Balmer spectrum*, together with a *support* $\mathrm{Supp}(x) \subseteq \mathrm{Spc}(\mathcal{K})$, for each object $x \in \mathcal{K}$. The pair $(\mathrm{Spc}(\mathcal{K}), \mathrm{Supp})$ is the universal support data that classifies the thick tensor-ideals of $\mathcal{K}$, providing a unification of previously known classifications of thick subcategories in stable homotopy theory [**DHS88**], algebraic geometry [**Nee92, Tho97**] and modular representation theory [**BCR97**]. Tensor-triangular geometry has seen great fruition in the past couple decades and besides conceptual clarity, it provides tools that allow the transfer of techniques and ideas between different disciplines, one instance being gluing techniques abstracted from algebraic geometry and then applied to modular representation theory [**BF07**].

Often, $\mathcal{K} = \mathcal{T}^c$ arises as the subcategory of compact objects of a "big" tensor-triangulated category $\mathcal{T}$. In this setting, the analogous problem is the classification of localizing tensor-ideals and colocalizing hom-ideals of $\mathcal{T}$. This is an extremely hard problem to completely solve, but what one can do is use the Balmer spectrum $\mathrm{Spc}(\mathcal{T}^c)$ of the compact objects to study $\mathcal{T}$. In [**BF11**] (assuming that $\mathrm{Spc}(\mathcal{T}^c)$ is noetherian, a condition that was later improved to "weakly noetherian" [**BHS23b**]) Balmer–Favi extended the support of [**Bal05**] to arbitrary not necessarily compact objects of $\mathcal{T}$, while abstracting Rickard's idempotent representations [**Ric96**] and giving algebro-geometric applications.

The classification of localizing tensor-ideals via means of support theory is known as *stratification*. This term was coined by Benson–Iyengar–Krause [**BIK11a, BIK11b**] when they classified the localizing tensor-ideals of the stable module category of a finite group, while the idea first emerged in [**Nee92**] for derived categories of commutative noetherian rings. Inspired by these results, Barthel–Heard–Sanders [**BHS23b**] initiated a systematic treatment in general big tensor-triangulated categories using the Balmer–Favi support and, under a certain topological restriction on the Balmer spectrum, they proved that stratification implies the Telescope Conjecture (meaning that every smashing tensor-ideal is generated by compact objects). The theory of stratification does have limitations. For instance, the presence of non-zero tensor-nilpotent objects (such as the Brown–Comenetz dual of the sphere spectrum in the stable homotopy category) ensures that the category is not stratified.

Dually, the study of colocalizing hom-ideals leads to the theory of cosupport and *costratification*, which was initiated by Bensor–Iyengar–Krause [**BIK12**], inspired by the classification of colocalizing subcategories of the derived category of a commutative noetherian ring by Neeman [**Nee11**]. Their main application was the classification of Hom-closed colocalizing subcategories of the stable module category of a finite group. Compared to the theory of support and stratification, the theory of costratification has not been explored as much. Our aim, as we shall explain in more detail later, is to develop the theory of costratification (in the broader context of relative tensor-triangular geometry [**Ste13**]) and thereby provide a unification of the two aforementioned classifications. Additionally, we provide an application on derived categories of schemes. Work on costratification has also been done independently by Barthel–Castellana–Heard–Sanders [**BCHS23**].

A common theme in tensor-triangular geometry is the approximation of the Balmer spectrum $\mathrm{Spc}(\mathcal{T}^c)$ by other spaces via certain comparison maps. One such construction is the *homological spectrum* $\mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^c)$ [**Bal20a, Bal20b**] consisting of the homological prime ideals of $\mathcal{T}$ (these are subcategories of the abelian category of modules $\mathrm{Mod}(\mathcal{T}^c)$) which gives rise to nilpotence theorems. Further, there is a surjective map $\phi\colon \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^c) \to \mathrm{Spc}(\mathcal{T}^c)$, which is bijective if and only if the homological spectrum is T_0 [**BHS23a**]. This holds in all known examples [**BC21**] and led to Balmer's "*Nerves of steel conjecture*" that ϕ is always bijective. It is interesting to note that the associated support theory (which does not require any topological assumptions on the homological spectrum) satisfies the Tensor Product Formula: $\mathrm{Supp}^{\mathrm{h}}(X \otimes Y) = \mathrm{Supp}^{\mathrm{h}}(X) \cap \mathrm{Supp}^{\mathrm{h}}(Y)$, which is a strong property. However, one cannot expect to have detection of vanishing in general.

More recently, Balchin–Stevenson [**BS23**], building on the work of Krause [**Kra00, Kra05a**] and Balmer–Krause–Stevenson [**BKS20**], based on the hypothesis that the lattice of smashing ideals $\mathrm{S}^{\otimes}(\mathcal{T})$ is a spatial frame, studied the support theory stemming from the smashing spectrum $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ — the space associated with $\mathrm{S}^{\otimes}(\mathcal{T})$ via Stone duality. Notably, there is a surjective continuous map $\psi\colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \to \mathrm{Spc}(\mathcal{T}^c)^{\vee}$ from the smashing spectrum to the Hochster dual of the Balmer spectrum, which is a homeomorphism if and only if $\mathcal{T}$ satisfies the Telescope Conjecture. The hope is that, in examples where the Telescope Conjecture fails, the smashing spectrum (with the accompanying notion of small support) can serve as a better tool than the Balmer spectrum in classifying localizing ideals. The small smashing support is constructed by assuming that the smashing spectrum is T_D — the dual notion of "weakly noetherian". Our intention is to prove results about stratification by using the smashing spectrum instead of the Balmer spectrum. We will also study connections between the smashing spectrum, the homological spectrum and the Telescope Conjecture.

A given triangulated category $\mathcal{K}$ may not be equipped with an obvious tensor product and so one may not be able to access the machinery of tensor-triangular geometry. In some situations, there is a different setup available and that is a tensor-triangulated category $\mathcal{T}$ acting on $\mathcal{K}$. This allows us to construct an induced support and cosupport for objects of $\mathcal{K}$ and use similar methods. This is the realm of *relative tensor-triangular geometry* and has been initiated by Stevenson [**Ste13**], with applications including the classification of localizing subcategories of certain singularity categories [**Ste14b**] and some progress regarding derived categories of representations of small categories [**AS16**]. Our contribution here is the classification of colocalizing subcategories of the singularity category of a scheme with only hypersurface singularities. We achieve this by applying the abstract theory developed for costratification, after dealing with certain technical challenges.

Part I: Stratification and the smashing spectrum

HYPOTHESIS. We denote by $\mathcal{T}$ a rigidly-compactly generated tensor triangulated category (big tt-category) as in Definition 1.1.17 and we assume that the frame $\mathrm{S}^{\otimes}(\mathcal{T})$ of smashing ideals of $\mathcal{T}$ is a spatial frame. We also assume that the smashing spectrum $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D, i.e., all of its points are locally closed; see Definition 1.4.12.

Our first result, which is Proposition 2.1.7, is that $\psi\colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \to \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$ exhibits the Hochster dual of the Balmer spectrum as the Kolmogorov quotient of the smashing spectrum equipped with the *small topology*; namely, the topology with basis of open subsets consisting of the smashing supports of compact objects of $\mathcal{T}$.

Next, we establish results concerned with stratification, using the smashing spectrum and the small smashing support in place of the Balmer spectrum and the Balmer–Favi support. To an extent (specifically in Sections 2.2, 2.3, 2.4) our results are inspired by work of Barthel–Heard–Sanders [**BHS23a, BHS23b**]. In fact, under the presence of the Telescope Conjecture, the two stratification theories are equivalent and we recover many of their results. We expand on this point in detail in Section 2.5.

DEFINITION (2.2.1). The category $\mathcal{T}$ is *stratified by the small smashing support* if the maps

$$\mathcal{P}(\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})) \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \mathrm{Loc}^{\otimes}(\mathcal{T}),$$

between the powerset of the smashing spectrum and the collection of localizing ideals of $\mathcal{T}$, defined by

$$\tau(W) = \{X \in \mathcal{T} \mid \mathrm{Supp}^{\mathrm{s}}(X) \subseteq W\} \quad \& \quad \sigma(\mathcal{L}) = \bigcup_{X \in \mathcal{L}} \mathrm{Supp}^{\mathrm{s}}(X)$$

are mutually inverse bijections.

The motivating factor to our approach to stratification is that (under the assumption that the Balmer spectrum is generically noetherian) the Telescope Conjecture is a consequence, and thus a necessity, for $\mathcal{T}$ to be stratified by the Balmer–Favi support. The proof cannot be reproduced and it is neither obvious nor is it expected that the Telescope Conjecture is a consequence of smashing stratification. In particular, it is still unclear whether the "*Universality Theorem*" of [**BHS23b**] can be applied. Note, however, that an example of a category that is stratified by the small smashing support and fails the Telescope Conjecture is yet to be found. One case under investigation is the derived category of a rank 1 non-noetherian valuation domain.

DEFINITION (2.2.3). Let $\Gamma_P \in \mathcal{T}$ be the Rickard idempotent associated with $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, as in Definition 1.4.15.

(a) $\mathcal{T}$ satisfies the *local-to-global principle* if for every object $X \in \mathcal{T}$, it holds that $\mathrm{loc}^{\otimes}(X) = \mathrm{loc}^{\otimes}(\Gamma_P \otimes X \mid P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}))$.

(b) $\mathcal{T}$ satisfies *minimality* if for every $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, it holds that $\mathrm{loc}^{\otimes}(\Gamma_P)$ is a minimal localizing ideal.

THEOREM (2.2.15). *The category $\mathcal{T}$ is stratified by the small smashing support if and only if $\mathcal{T}$ satisfies the local-to-global principle and minimality.*

A consequence of Theorem 2.2.15 is that in the case where $\mathcal{T}$ is stratified by the small smashing support, a localizing ideal is a big prime if and only if it is objectwise-prime. Moreover, there is a bijective correspondence between the set of meet-prime smashing ideals and the set of big prime localizing ideals; see Corollary 2.2.24 and Corollary 2.2.25.

In Section 2.3, we reduce stratification to smashing localizations and closed covers of the smashing spectrum.

THEOREM (Corollary 2.3.10). *Suppose that $\mathcal{T}$ satisfies the local-to-global principle. Then $\mathcal{T}$ is stratified by the small smashing support if and only if $\mathcal{T}/P$ is stratified by the small smashing support, for every $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.*

THEOREM (Corollary 2.3.14). *Suppose that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) = \bigcup V_{\mathcal{S}_i}$, where $\{\mathcal{S}_i\}$ is a finite set of smashing ideals, and assume that each $\mathcal{T}/\mathcal{S}_i$ is stratified by the small smashing support. Then $\mathcal{T}$ is stratified by the small smashing support. If $\mathcal{T}$ satisfies the local-to-global principle, then the finiteness condition on $\{\mathcal{S}_i\}$ can be dropped.*

In Section 2.4, under the hypothesis that $\mathcal{T}$ is stratified by the small smashing support, we construct an injective comparison map $\xi\colon \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}}) \to \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ from the homological spectrum to the smashing spectrum, which is bijective if and only if the homological support detects vanishing; see Proposition 2.4.4. Then we make a connection between the homological spectrum and the Telescope Conjecture.

THEOREM (2.4.5). *Let $\mathcal{T}$ be a big tt-category whose smashing spectrum is T_D and assume that $\mathcal{T}$ is stratified by the small smashing support. Then $\mathcal{T}$ satisfies the Telescope Conjecture if and only if $\mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}})$ is T_0 and $\mathrm{Supp}^{\mathrm{h}}$ detects vanishing of objects.*

Finally, in Section 2.6, we study the image of the map $\mathrm{Spc}^{\mathrm{s}}(F)\colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{U}) \to \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ induced by a coproduct-preserving tensor-triangulated functor $F\colon \mathcal{T} \to \mathcal{U}$ and we prove the following descent result.

THEOREM (Corollary 2.6.10). *Let $F\colon \mathcal{T} \to \mathcal{U}$ be a coproduct-preserving tt-functor between big tt-categories whose smashing spectra are T_D and let G be the right adjoint to F. Assume that $\mathrm{Spc}^{\mathrm{s}}(F)\colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{U}) \to \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is a homeomorphism. Provided that $\mathrm{loc}(1_{\mathcal{U}}) = \mathcal{U}$ and $\mathrm{loc}(G(1_{\mathcal{U}})) = \mathcal{T}$, if $\mathcal{U}$ is stratified by the small smashing support, then $\mathcal{T}$ is stratified by the small smashing support.*

Part II: Costratification and actions of tensor-triangulated categories

HYPOTHESIS. We denote by $\mathcal{T}$ a big tt-cagegory and by $\mathcal{K}$ a $\mathcal{T}$-*module*, that means a compactly generated triangulated category upon which $\mathcal{T}$ acts via a functor $- * -\colon \mathcal{T} \times \mathcal{K} \to \mathcal{K}$ with relative internal-hom $[-,-]_*\colon \mathcal{T}^{\mathrm{op}} \times \mathcal{K} \to \mathcal{K}$. We denote by $I_{\mathcal{K}}$ the product of the Brown–Comenetz duals of the compact objects of $\mathcal{K}$. See Section 3.1 for details. For simplicity, one could consider $\mathcal{K} = \mathcal{T}$ and $* = \otimes$.

Our goal is to study the class of *colocalizing* hom-*submodules* of $\mathcal{K}$, that is those colocalizing subcategories $\mathcal{C} \subseteq \mathcal{K}$ such that $[X, A]_* \in \mathcal{C}$, $\forall X \in \mathcal{T}$, $\forall A \in \mathcal{C}$. The starting point is to consider a *good support–cosupport pair* $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ on $\mathcal{T}$ with values in a space S (for example the Balmer–Favi support–cosupport or the smashing support–cosupport or the BIK support–cosupport; see Example 3.2.8) and use the *induced support–cosupport* $(\mathrm{s}^*_\Gamma, \mathrm{c}^*_\Gamma)$ on $\mathcal{K}$; see Section 3.2. The reason we do not fix a specific support–cosupport pair is for conceptual clarity. So, for each point $s \in S$, we have a non-zero object $\Gamma_s \in \mathcal{T}$. These objects are pairwise orthogonal idempotents in the sense that $\Gamma_s \otimes \Gamma_s \cong \Gamma_s$ and $\Gamma_s \otimes \Gamma_r = 0$ if $s \neq r$. If $A \in \mathcal{K}$, then $\mathrm{s}^*_\Gamma(A) = \{s \in S \mid \Gamma_s * A \neq 0\}$ and $\mathrm{c}^*_\Gamma(A) = \{s \in S \mid [\Gamma_s, A]_* \neq 0\}$.

Definition (3.2.13). The category $\mathcal{K}$ is *costratified* if the maps

$$\mathcal{P}(\mathrm{c}^*_\Gamma(I_\mathcal{K})) \underset{\sigma_{\mathrm{c}^*_\Gamma}}{\overset{\tau_{\mathrm{c}^*_\Gamma}}{\rightleftarrows}} \mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K}),$$

between the powerset of the subspace $\mathrm{c}^*_\Gamma(I_\mathcal{K}) \subseteq S$ and the collection of colocalizing hom-submodules of $\mathcal{K}$, defined by

$$\tau_{\mathrm{c}^*_\Gamma}(W) = \{A \in \mathcal{K} \mid \mathrm{c}^*_\Gamma(A) \subseteq W\} \quad \& \quad \sigma_{\mathrm{c}^*_\Gamma}(\mathcal{C}) = \bigcup_{A \in \mathcal{C}} \mathrm{c}^*_\Gamma(A)$$

are mutually inverse bijections.

The space $\mathrm{c}^*_\Gamma(I_\mathcal{K})$ consists precisely of those points $s \in S$ such that $\Gamma_s * \mathcal{K} \neq 0$. The inclusion $\mathrm{c}^*_\Gamma(I_\mathcal{K}) \subseteq S$ is not an equality in general. For instance, one could take $\mathcal{K} = 0$, in which case $\mathrm{c}^*_\Gamma(I_\mathcal{K}) = \varnothing$. Apart from the trivial case, if R is a commutative noetherian ring, then for the action of the derived category $\mathcal{T} = \mathrm{D}(R)$ on the singularity category $\mathcal{K} = \mathrm{S}(R)$, it holds that $\mathrm{c}^*_\Gamma(I_\mathcal{K}) = \mathrm{Sing}\, R \subseteq \mathrm{Spec}\, R$ the singular locus of R. This is studied in Chapter 4. Note that if $\mathcal{K} = \mathcal{T}$ and $* = \otimes$, then $\mathrm{c}^*_\Gamma(I_\mathcal{K}) = S$.

Our first main result of this part is that costratification is equivalent to two conditions: the colocal-to-global principle and cominimality, which are in a sense dual to the more well-established local-to-global principle and minimality [**BIK11a, Ste13, Ste17, BHS23b**].

Definition (3.2.15).

(a) $\mathcal{K}$ satisfies the *colocal-to-global principle* if

$$\mathrm{coloc}^{\mathrm{hom}}(A) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid s \in S), \forall A \in \mathcal{K}.$$

(b) $\mathcal{K}$ satisfies *cominimality* if, for all $s \in S$, $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_*)$ is minimal in $\mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K})$ in the sense that it does not contain any non-zero proper colocalizing hom-submodule of $\mathcal{K}$.

Theorem (3.2.21). *The category $\mathcal{K}$ is costratified if and only if $\mathcal{K}$ satisfies the colocal-to-global principle and cominimality.*

The following result, which extends Stevenson's [**Ste13**, Proposition 6.8], will be very useful for the applications.

Proposition (3.2.25). *If $\mathcal{T}$ satisfies the local-to-global principle, then $\mathcal{K}$ satisfies the local-to-global principle and the colocal-to-global principle.*

The next theme of this part concerns two classes of subcategories of $\mathcal{K}$. The class of prime localizing submodules and the class of hom-prime colocalizing hom-submodules. The former generalizes the concept of an objectwise prime localizing tensor-ideal in the context of relative tensor-triangular geometry, while the latter seems to be new even for $\mathcal{K} = \mathcal{T}$. These two notions are intimately related with the Action Formula (which generalizes the Tensor Product Formula; see Remark 3.3.16) and the Internal-Hom Formula, respectively, which are properties that a support–cosupport pair may or may not satisfy. We shall focus more on the results regarding hom-submodules.

DEFINITION (3.3.3).
(a) A proper localizing submodule $\mathcal{L} \subseteq \mathcal{K}$ is called *prime* if $X * A \in \mathcal{L}$ implies $X * \mathcal{K} \subseteq \mathcal{L}$ or $A \in \mathcal{L}$.
(b) A proper colocalizing hom-submodule $\mathcal{C} \subseteq \mathcal{K}$ is called hom-*prime* if $[X, A]_* \in \mathcal{C}$ implies $[X, I_{\mathcal{K}}]_* \in \mathcal{C}$ or $A \in \mathcal{C}$.

If $\mathcal{K}$ is costratified, then we have a complete description of the hom-prime colocalizing hom-submodules in terms of the space $\mathrm{c}^*_\Gamma(I_{\mathcal{K}})$.

THEOREM (3.3.10). *If $\mathcal{K}$ is costratified, then there is a bijective correspondence between* hom-*prime colocalizing* hom-*submodules of $\mathcal{K}$ and points of $\mathrm{c}^*_\Gamma(I_{\mathcal{K}})$. A point $s \in \mathrm{c}^*_\Gamma(I_{\mathcal{K}})$ is associated with* $\mathrm{Ker}[\Gamma_s, -]_* = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_r, I_{\mathcal{K}}]_* \mid r \neq s)$.

DEFINITION (3.3.13).
(a) $\mathcal{K}$ satisfies the *Action Formula* (AF) if

$$\mathrm{s}^*_\Gamma(X * A) = \mathrm{s}_\Gamma(X) \cap \mathrm{s}^*_\Gamma(A),\ \forall X \in \mathcal{T},\ \forall A \in \mathcal{K}.$$

(b) $\mathcal{K}$ satisfies the *Internal-Hom Formula* (IHF) if

$$\mathrm{c}^*_\Gamma([X, A]_*) = \mathrm{c}_\Gamma([X, I_{\mathcal{T}}]) \cap \mathrm{c}^*_\Gamma(A),\ \forall X \in \mathcal{T},\ \forall A \in \mathcal{K}.$$

(It holds that $\mathrm{c}_\Gamma([X, I_{\mathcal{T}}]) = \mathrm{s}_\Gamma(X)$.)

Let $\mathrm{Ann}_{\mathcal{T}}(\mathcal{K}) = \{X \in \mathcal{T} \mid X * \mathcal{K} = 0\}$. If $\mathrm{Ann}_{\mathcal{T}}(\mathcal{K}) = 0$, then we say that $\mathcal{K}$ is a *conservative* $\mathcal{T}$-module.

THEOREM (Proposition 3.3.14, Proposition 3.3.15).
(a) *If $\mathcal{K}$ satisfies the Action Formula, then* $\mathrm{Ker}(\Gamma_s * -)$ *is a prime localizing submodule, $\forall s \in S$. If $\mathcal{K}$ is a conservative $\mathcal{T}$-module, then the converse holds.*
(b) *If $\mathcal{K}$ satisfies the Internal-Hom Formula, then* $\mathrm{Ker}[\Gamma_s, -]_*$ *is a* hom-*prime colocalizing* hom-*submodule, $\forall s \in S$. If $\mathcal{K}$ is a conservative $\mathcal{T}$-module, then the converse holds.*

(1) *If $\mathcal{T}$ satisfies minimality, then $\mathcal{K}$ satisfies the Action Formula and the Internal-Hom Formula.*
(2) *If $\mathcal{K}$ is a conservative $\mathcal{T}$-module and $\mathcal{K}$ satisfies cominimality, then $\mathcal{K}$ satisfies the Internal-Hom Formula.*
(3) *If $\mathcal{T}$ satisfies the Internal-Hom Formula, then $\mathcal{T}$ satisfies the Action Formula.*

We then reduce costratification to certain smashing localizations. Our result is stated in full generality in Theorem 3.4.4. As a special case, we obtain the following:

COROLLARY (3.4.7). *Suppose that every point of* $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ *is visible. Then:*
(a) *$\mathcal{T}$ satisfies cominimality if and only if $\mathcal{T}/\mathrm{loc}^{\otimes}(\mathfrak{p})$ satisfies cominimality, for all* $\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$.
(b) *Suppose that $\mathcal{T}$ satisfies the colocal-to-global principle. Then $\mathcal{T}$ is costratified if and only if $\mathcal{T}/\mathrm{loc}^{\otimes}(\mathfrak{p})$ is costratified, for all $\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$.*

Moreover, we reduce costratification to covers of the Balmer spectrum. If V is a Thomason subset of $\operatorname{Spc}(\mathcal{T}^c)$ and U is the complement of V, we denote by $\mathcal{T}(U)$ the category $\mathcal{T}/\mathcal{T}_V$, where $\mathcal{T}_V$ is the localizing ideal of $\mathcal{T}$ generated by those compact objects supported on V.

Corollary (3.4.8). *Suppose that every point of* $\operatorname{Spc}(\mathcal{T}^c)$ *is visible and that* $\operatorname{Spc}(\mathcal{T}^c) = \bigcup_{j\in J} U_j$ *is a cover of* $\operatorname{Spc}(\mathcal{T}^c)$ *by complements of Thomason subsets. If* $\mathcal{T}(U_j)$ *satisfies cominimality, for all* $j \in J$*, then* $\mathcal{T}$ *satisfies cominimality. If, moreover,* $\mathcal{T}$ *satisfies the colocal-to-global principle, then* $\mathcal{T}$ *is costratified.*

We also obtain the following generalization, which is the analogous version of [**Ste13**, Theorem 8.11] for hom-submodules.

Theorem (3.4.9). *Suppose that every point of* $\operatorname{Spc}(\mathcal{T}^c)$ *is visible and that* $\operatorname{Spc}(\mathcal{T}^c) = \bigcup_{j\in J} U_j$ *is a cover of* $\operatorname{Spc}(\mathcal{T}^c)$ *by complements of Thomason subsets. If* $\mathcal{K}(U_j)$ *(as a* $\mathcal{T}(U_j)$*-module) satisfies cominimality, for all* $j \in J$*, then* $\mathcal{K}$ *satisfies cominimality. If, moreover,* $\mathcal{K}$ *satisfies the colocal-to-global principle, then* $\mathcal{K}$ *is costratified.*

Finally, using the general machinery developed throughout, we first give in Theorem 3.5.4 a more streamlined proof of Neeman's classification of the colocalizing subcategories of the derived category of a commutative noetherian ring and then we generalize this result to schemes:

Theorem (3.5.8). *The derived category* $\mathrm{D}(X)$ *of quasi-coherent sheaves over a noetherian separated scheme* X *is costratified.*

Part III: Colocalizing subcategories of singularity categories

Let R be a commutative noetherian ring. The *singularity category* $\mathrm{S}(R) = \mathrm{K}_{\mathrm{ac}}(\operatorname{Inj} R)$, i.e., the homotopy category of acyclic complexes of injective R-modules, was introduced and studied in [**Kra05b**] (wherein it was called the *stable derived category*) where it was shown that $\mathrm{S}(R)$ is a compactly generated triangulated category and that there is a *stabilization functor* $I_\lambda Q_\rho \colon \mathrm{D}(R) \to \mathrm{S}(R)$ that can be used to describe the compact objects of $\mathrm{S}(R)$. As we have already discussed, a classic problem regarding such categories is the classification of the localizing and colocalizing subcategories. Since $\mathrm{S}(R)$ is not tensor-triangulated (at least not in any obvious way) the machinery of tensor-triangular geometry is not readily available. However, in [**Ste14b**] Stevenson utilized the action of $\mathrm{D}(R)$ on $\mathrm{S}(R)$ defined by tensoring the objects of $\mathrm{S}(R)$ with K-flat resolutions and classified the localizing subcategories of $\mathrm{S}(R)$, for a locally hypersurface ring R and then generalized this classification result to the singularity category $\mathrm{S}(X)$ of a noetherian separated scheme X with hypersurface singularities.

In this part, we apply the theory of costratification developed in Chapter 3 in order to classify the colocalizing subcategories of $\mathrm{S}(R)$, for a locally hypersurface ring R and then we generalize our result to the singularity category of a scheme with hypersurface singularities. Specifically, for the case of rings, using the action of $\mathrm{D}(R)$ on $\mathrm{S}(R)$, we obtain a notion of cosupport for the objects of $\mathrm{S}(R)$. The cosupport of an object $A \in \mathrm{S}(R)$ is

$$\operatorname{Cosupp}(A) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{Hom}_R(g_\mathfrak{p}, A) \neq 0\} \subseteq \operatorname{Sing} R,$$

where $g_{\mathfrak{p}} = K_\infty(\mathfrak{p}) \otimes_R R_{\mathfrak{p}}$ is the Balmer–Favi idempotent associated with $\mathfrak{p} \in \operatorname{Spec} R$ and $\operatorname{Sing} R$ is the singular locus of R. The assignment of cosupport allows us to define the maps

$$\operatorname{Coloc}(\mathrm{S}(R)) \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \mathcal{P}(\operatorname{Sing} R),$$

where $\sigma(\mathcal{C}) = \bigcup_{A \in \mathcal{C}} \operatorname{Cosupp}(A)$ and $\tau(W) = \{A \in \mathrm{S}(R) \mid \operatorname{Cosupp}(A) \subseteq W\}$. One should note that when the acting category, in this case $\mathrm{D}(R)$, is generated by its tensor-unit as a localizing subcategory, then every colocalizing subcategory of the category acted upon, in this case $\mathrm{S}(R)$, is a hom-submodule and thus our results regarding colocalizing hom-submodules apply to all colocalizing subcategories.

If X is a noetherian separated scheme and $\mathrm{D}(X)$ is the derived category of quasi-coherent sheaves on X, one can use the action of $\mathrm{D}(X)$ on $\mathrm{S}(X)$ in a similar fashion to obtain the notions of cosupport and costratification.

Theorem (4.5.7, 4.6.1). *Let R be a locally hypersurface ring. Then* $\mathrm{S}(R)$ *is costratified, i.e., the maps σ and τ defined above are mutually inverse bijections. Let X be a noetherian separated scheme with hypersurface singularities. Then* $\mathrm{S}(X)$ *is costratified.*

CHAPTER 1

Preliminaries

This chapter consists of preliminary material on the basic structures that we will be discussing throughout this work. Specifically, tensor-triangulated categories and various spectra and support theories associated with them (the Balmer spectrum, the homological spectrum and the smashing spectrum). We assume familiarity with the language and basic concepts of category theory such as functors, natural transformations, adjunctions, limits, colimits, additive categories, abelian categories, monoidal categories etc.; see e.g. [**Rie16, Mac98**]. For the convenience of the reader, we will briefly provide some definitions concerning triangulated categories, mostly to establish terminology and notation. We refer the reader to [**Nee01**] for a more in depth discussion.

1.1. Tensor-triangulated categories

In this section, we recall some standard definitions and concepts about triangulated categories such as Brown representability, purity, modules and Brown–Comenetz duals of compact objects and the main structure that we will be studying in the sequel: a rigidly-compactly generated tensor-triangulated category.

Triangulated categories.

Definition 1.1.1. A *triangulated category* is a triple $\mathcal{T} = (\mathcal{T}, \Sigma, \Delta)$ that consists of an additive category $\mathcal{T}$, an additive automorphism $\Sigma\colon \mathcal{T} \to \mathcal{T}$ called the *suspension functor* and a class Δ of diagrams in $\mathcal{T}$ of the form $X \to Y \to Z \to \Sigma X$, called *triangles*, such that the following properties are satisfied:

TR1(a): The class Δ is closed under isomorphisms in the sense that if

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle f}\downarrow{\scriptstyle \cong} & & {\scriptstyle g}\downarrow{\scriptstyle \cong} & & {\scriptstyle h}\downarrow{\scriptstyle \cong} & & {\scriptstyle \Sigma f}\downarrow{\scriptstyle \cong} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

is a commutative diagram such that $X \to Y \to Z \to \Sigma X \in \Delta$ and f, g, h are isomorphisms, then $X' \to Y' \to Z' \to \Sigma X' \in \Delta$.

TR1(b): For all $X \in \mathcal{T}$, it holds that $X \xrightarrow{\mathrm{Id}_X} X \longrightarrow 0 \longrightarrow \Sigma X \in \Delta$.

TR1(c): For all morphisms $f\colon X \to Y$, there exists a triangle

$$X \xrightarrow{f} Y \to Z \to \Sigma X \in \Delta.$$

TR2: It holds that

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \in \Delta$$

if and only if

$$Y \xrightarrow{-g} Z \xrightarrow{-h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \in \Delta.$$

TR3: For all commutative diagrams of the form

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

where both rows are triangles in Δ, there exists a morphism $w\colon Z \to Z'$ such that $w \circ g = g' \circ v$ and $\Sigma u \circ h = h' \circ w$.

TR4: For all composable pairs of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, there exists a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & X' & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle \mathrm{Id}_X} & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}_{\Sigma X}} \\ X & \xrightarrow{g \circ f} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z' & \xrightarrow{\mathrm{Id}_{Z'}} & Z' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y & \longrightarrow & \Sigma X' & \longrightarrow & \Sigma^2 X \end{array}$$

such that all rows and columns are triangles in Δ.

Definition 1.1.2. Let $\mathcal{T}$ be a triangulated category. An additive subcategory $\mathcal{S}$ of $\mathcal{T}$ is called a *triangulated subcategory* if the following two conditions are satisfied:

(a) $\Sigma X \in \mathcal{S}$ and $\Sigma^{-1} X \in \mathcal{S}$, for all $X \in \mathcal{S}$.
(b) If $X \to Y \to Z \to \Sigma X$ is a triangle in $\mathcal{T}$ such that $X, Y \in \mathcal{S}$, then $Z \in \mathcal{S}$.

If a triangulated subcategory $\mathcal{S}$ is *thick*, i.e., closed under taking summands, then we will refer to $\mathcal{S}$ as a *thick subcategory* of $\mathcal{T}$, skipping the term "triangulated". If, moreover, $\mathcal{S}$ is closed under coproducts (resp. products) then $\mathcal{S}$ is called a *localizing* (resp. *colocalizing*) *subcategory.*

Definition 1.1.3. Let $\mathcal{T}$ be a triangulated category. For each object $X \in \mathcal{T}$, the smallest thick (resp. localizing, resp. colocalizing) subcategory of $\mathcal{T}$ containing X is called the *thick (resp. localizing, resp. colocalizing) subcategory of* $\mathcal{T}$ *generated by* X and is denoted by $\mathrm{thick}(X)$ (resp. $\mathrm{loc}(X)$, resp. $\mathrm{coloc}(X)$).

Definition 1.1.4. An additive functor $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ between triangulated categories is called a *triangulated functor* if $F \circ \Sigma$ and $\Sigma \circ F$ are naturally isomorphic functors and the image of each triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{T}_1$, i.e., the diagram $FX \to FY \to FZ \to F\Sigma X \cong \Sigma FX$, is a triangle in $\mathcal{T}_2$.

Definition 1.1.5. An additive functor $H\colon \mathcal{T} \to \mathcal{A}$, where $\mathcal{T}$ is a triangulated category and $\mathcal{A}$ is an abelian category, is called a *homological functor* if for each triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{T}$, the sequence $HX \to HY \to HZ$ is exact in $\mathcal{A}$.

If H is an additive contravariant functor that sends triangles to exact sequences, then H is called a *cohomological functor*.

Definition 1.1.6. An object X in a triangulated category $\mathcal{T}$ is called *compact* if the functor $\mathrm{Hom}_{\mathcal{T}}(X,-)\colon \mathcal{T} \to \mathrm{Ab}$, where Ab denotes the category of abelian groups, preserves coproducts.

Remark 1.1.7. The subcategory of compact objects of a triangulated category $\mathcal{T}$ is denoted by $\mathcal{T}^{\mathrm{c}}$ and is a thick subcategory of $\mathcal{T}$.

Definition 1.1.8. A triangulated category $\mathcal{T}$ with coproducts is called *compactly generated* if its subcategory of compact objects is essentially small, i.e., there is only a set of isomorphism classes of compact objects, and if $\mathrm{Hom}_{\mathcal{T}}(X,Y)=0$, for all compact objects X, then $Y=0$.

A key feature of compactly generated triangulated categories is the following result, known as Brown representability, which in turn ensures the existence of certain adjunctions.

Theorem 1.1.9 ([**Nee96**, Theorem 3.1])**.** *Let $\mathcal{T}$ be a compactly generated triangulated category and let $H\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ be a cohomological functor that sends coproducts to products. Then H is representable, i.e., there exists an object $X \in \mathcal{T}$ and a natural isomorphism $H \cong \mathrm{Hom}_{\mathcal{T}}(-,X)$.*

Theorem 1.1.10 ([**Nee96**, Theorem 4.1])**.** *Let $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ be a triangulated functor and assume that $\mathcal{T}_1$ is compactly generated. If F preserves coproducts, then F has a right adjoint.*

Purity and modules. Let $\mathcal{T}$ be a compactly generated triangulated category.

Recollection 1.1.11. The category $\mathrm{Mod}(\mathcal{T}^{\mathrm{c}})$ of additive functors $\{\mathcal{T}^{\mathrm{c}}\}^{\mathrm{op}} \to \mathrm{Ab}$ is a Grothendieck abelian category. The functor $h\colon \mathcal{T} \to \mathrm{Mod}(\mathcal{T}^{\mathrm{c}})$ that sends an object $X \in \mathcal{T}$ to $h(X)=\widehat{X}=\mathrm{Hom}_{\mathcal{T}}(-,X)|_{\mathcal{T}^{\mathrm{c}}}$ is called the *restricted Yoneda functor* and is conservative, homological and preserves products and coproducts. A morphism $f\colon X \to Y$ is called a *pure monomorphism* if $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ is a monomorphism. An object $X \in \mathcal{T}$ is called *pure-injective* if, for all objects $Y \in \mathcal{T}$, every pure monomorphism $f\colon X \to Y$ splits (meaning that there exists a morphism $g\colon Y \to X$ such that $g \circ f = \mathrm{Id}_X$). According to [**Kra00**, Theorem 1.8] (see also [**Bel00**, Theorem 8.6]) an object $X \in \mathcal{T}$ is pure-injective if and only if $\widehat{X}$ is an injective object of $\mathrm{Mod}(\mathcal{T}^{\mathrm{c}})$. (In fact, an object $E \in \mathrm{Mod}(\mathcal{T}^{\mathrm{c}})$ is injective if and only if there exists a pure-injective object $X \in \mathcal{T}$ such that $E \cong \widehat{X}$; see [**Kra00**, Corollary 1.9].)

Recollection 1.1.12. Let $\mathcal{T}$ and $\mathcal{U}$ be compactly generated triangulated categories and let $F\colon \mathcal{T} \to \mathcal{U}$ be a coproduct-preserving triangulated functor. By Theorem 1.1.10, F has a right adjoint $G\colon \mathcal{U} \to \mathcal{T}$. If G preserves coproducts, then G preserves pure-injective objects: There is an induced adjunction

$$\mathrm{Mod}(\mathcal{T}^{\mathrm{c}}) \underset{\overline{G}}{\overset{\overline{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Mod}(\mathcal{U}^{\mathrm{c}}),$$

where $\overline{F}$ is an exact functor and $\overline{F}$ and $\overline{G}$ commute with the restricted Yoneda functors. Since $\overline{G}$ is the right adjoint of an exact functor, $\overline{G}$ preserves injective objects. Since pure-injective objects are precisely those objects whose image under the restricted Yoneda functor is injective, it follows that G preserves pure-injective objects; see [**Kra00**, Proposition 2.6].

Recollection 1.1.13. Let x be a compact object of $\mathcal{T}$ and consider the functor $H_x \coloneqq \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathcal{T}}(x,-),\mathbb{Q}/\mathbb{Z})\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$. Since x is compact and $\mathbb{Q}/\mathbb{Z}$ is an injective object of Ab, it follows that H_x is a cohomological functor that sends coproducts to products. So, by Brown representability, H_x is representable. The representing object of H_x is denoted by Ix and is called the *Brown–Comenetz dual* of x. The functor $\mathrm{Hom}_{\mathcal{T}}(-,Ix)|_{\mathcal{T}^{\mathrm{c}}}$ is an injective object of $\mathrm{Mod}(\mathcal{T}^{\mathrm{c}})$; see [**Nee98**]. Hence, Ix is pure-injective. Choosing a skeleton for the subcategory of compact objects, the product of the associated Brown–Comenetz duals is denoted by I. Being a product of pure-injective objects, I is also pure-injective. Using the fact that $\mathcal{T}$ is compactly generated, one can easily check that I is a cogenerator of $\mathcal{T}$ in the sense that $\mathrm{Hom}_{\mathcal{T}}(X,\Sigma^n I) = 0$, $\forall n \in \mathbb{Z}$ implies that $X = 0$. It holds that $\mathcal{T} = \mathrm{coloc}(I)$. This follows from the fact that the Brown–Comenetz duals of the compact objects form a perfect cogenerating set for $\mathcal{T}$; see [**Kra02**]. We use the symbol $I_{\mathcal{T}}$ if there is any possibility for confusion.

Tensor-triangulated categories.

Definition 1.1.14. A *tensor-triangulated category* (*tt-category*) is a triple $(\mathcal{T},\otimes,1)$ that consists of a triangulated category $\mathcal{T}$ and a symmetric monoidal product

$$-\otimes- \colon \mathcal{T}\times\mathcal{T}\to\mathcal{T}$$

that we call the *tensor product*, that is a triangulated functor in both variables. The object $1 \in \mathcal{T}$ denotes the tensor-unit.

Definition 1.1.15. A triangulated subcategory $\mathcal{X}$ of a tt-category $\mathcal{T}$ is called a *tensor-triangulated subcategory* (*tt-subcategory*) if $1 \in \mathcal{X}$ and $X\otimes Y \in \mathcal{X}$, $\forall X, Y \in \mathcal{X}$.

Definition 1.1.16. A triangulated functor $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ between tt-categories is called a *tensor-triangulated functor* (*tt-functor*) if F is a monoidal functor, i.e., if $F(X\otimes Y)\cong FX\otimes FY$.

Let $\mathcal{T} = (\mathcal{T},\otimes,1)$ be a tensor-triangulated category that is compactly generated and suppose that the tensor product preserves coproducts in both variables. As a consequence of Brown representability, for every object $X \in \mathcal{T}$, the functor $X\otimes -$ has a right adjoint $[X,-]$. These right adjoints assemble into a bifunctor $[-,-]\colon \mathcal{T}^{\mathrm{op}}\times\mathcal{T}\to\mathcal{T}$ called the *internal-hom*. The *dual* of an object $X\in\mathcal{T}$, denoted by $X^\vee$, is $[X,1]$. Let X, Y be two objects of $\mathcal{T}$. Tracing the identity morphism Id_Y through the composite

$$\mathcal{T}(Y,Y)\cong\mathcal{T}(1\otimes Y,Y)\xrightarrow{(\epsilon_{X,1}\otimes Y)^*}\mathcal{T}(X^\vee\otimes X\otimes Y,Y)\cong\mathcal{T}(X^\vee\otimes Y,[X,Y])$$

where $\epsilon_{X,1}\colon X^\vee\otimes X\to 1$ is the counit of adjunction, gives rise to a natural *evaluation map* $X^\vee\otimes Y\to[X,Y]$. The object X is called *rigid* if this natural evaluation map is an isomorphism, for all $Y\in\mathcal{T}$.

Definition 1.1.17. Let $\mathcal{T} = (\mathcal{T},\otimes,1)$ be a tensor-triangulated category with coproducts. Then $\mathcal{T}$ is called *rigidly-compactly generated*, henceforth a *big tt-category*, if the following conditions are satisfied:

(a) $\mathcal{T}$ is compactly generated.
(b) $-\otimes-\colon \mathcal{T}\times\mathcal{T}\to\mathcal{T}$ preserves coproducts in both variables.

(c) $\mathcal{T}^c$ is a tensor-triangulated subcategory of $\mathcal{T}$.
(d) The rigid objects of $\mathcal{T}$ coincide with the compact objects.

Convention 1.1.18. From now on, $\mathcal{T}$ will always denote a big tt-category.

Example 1.1.19. The following are examples of big tt-categories. See [**BF11**] for more details.

(a) SH the stable homotopy category of spectra. The tensor product of SH is the smash product and the tensor unit is the sphere spectrum. The subcategory of compact objects of SH is $\mathrm{SH}^{\mathrm{fin}}$ the subcategory of finite spectra.
(b) $\mathrm{D}(R)$ the unbounded derived category of a commutative ring R. The tensor product of $\mathrm{D}(R)$ is the left derived tensor product of complexes and the tensor unit is the image of R in $\mathrm{D}(R)$. The subcategory of compact objects of $\mathrm{D}(R)$ is $\mathrm{D}^{\mathrm{perf}}(R)$ the subcategory of perfect complexes of R-modules, i.e., those complexes that are quasi-isomorphic to bounded complexes of finitely generated projective R-modules.
(c) $\mathrm{D}(X)$ the unbounded derived category of quasi-coherent sheaves of $\mathcal{O}_X$-modules over a quasi-compact separated scheme X. The tensor product of $\mathrm{D}(X)$ is the left derived tensor product of complexes of $\mathcal{O}_X$-modules and the tensor unit is the image of $\mathcal{O}_X$ in $\mathrm{D}(X)$. The subcategory of compact objects of $\mathrm{D}(X)$ is $\mathrm{D}^{\mathrm{perf}}(X)$ the subcategory of perfect complexes of $\mathcal{O}_X$-modules, i.e., those complexes that are locally quasi-isomorphic to bounded complexes of locally free sheaves of $\mathcal{O}_X$-modules.
(d) $\underline{\mathrm{Mod}}(kG)$ the stable module category of the group algebra kG, where G is a finite group. The tensor product of $\underline{\mathrm{Mod}}(kG)$ is over k with diagonal G-action and the tensor unit is k with trivial G-action. The subcategory of compact objects of $\underline{\mathrm{Mod}}(kG)$ is $\underline{\mathrm{mod}}(kG)$ the subcategory of finite dimensional kG-modules.

Definition 1.1.20. A subcategory $\mathcal{I}$ of a tt-category $\mathcal{T}$ is called a *tensor-ideal* if $X \otimes Y \in \mathcal{I}, \forall X \in \mathcal{I}, \forall Y \in \mathcal{T}$. A thick (resp. localizing) subcategory of $\mathcal{T}$ that is a tensor-ideal is called a *thick tensor-ideal* (resp. *localizing tensor-ideal*). For simplicity, a tensor-ideal will be called an ideal, skipping the adjective "tensor".

Definition 1.1.21. For each object $X \in \mathcal{T}$, the smallest thick (resp. localizing) ideal of $\mathcal{T}$ containing X is called the *thick* (resp. *localizing*) *ideal generated by* X and is denoted by $\mathrm{thick}^{\otimes}(X)$ (resp. $\mathrm{loc}^{\otimes}(X)$). The collections of thick ideals and localizing ideals of $\mathcal{T}$ are denoted by $\mathrm{Thick}^{\otimes}(\mathcal{T})$ and $\mathrm{Loc}^{\otimes}(\mathcal{T})$, respectively.

Lemma 1.1.22. *Let $F_1\colon \mathcal{T}_1 \to \mathcal{T}_2$ be a coproduct-preserving tt-functor and let $F_2\colon \mathcal{T}_1 \to \mathcal{T}_1$ be a coproduct-preserving triangulated endofunctor of $\mathcal{T}_1$ such that the following condition holds: $F_2(X \otimes Y) \cong X \otimes F_2 Y, \forall X, Y \in \mathcal{T}_1$. Then for every collection of objects $\mathcal{X} \subseteq \mathcal{T}_1$, if $X \in \mathrm{loc}^{\otimes}(\mathcal{X})$, then $F_i X \in \mathrm{loc}^{\otimes}(F_i(\mathcal{X}))$, $i = 1, 2$.*

Proof. The collection $\{Y \in \mathcal{T}_1 \mid F_i Y \in \mathrm{loc}^{\otimes}(F_i(\mathcal{X}))\}$ is a localizing ideal of $\mathcal{T}_1$ that contains $\mathcal{X}$. Hence, it contains $\mathrm{loc}^{\otimes}(\mathcal{X})$ and, as a result, $F_i X \in \mathrm{loc}^{\otimes}(F_i(\mathcal{X}))$. □

1.2. The Balmer spectrum

In this section, we recall the Balmer spectrum, the classification of thick tensor-ideals of a tensor-triangulated category and the Balmer–Favi support, which extends the support of compact objects to arbitrary objects.

Definition 1.2.1 ([**Bal05**])**.** Let $\mathcal{K}$ be an essentially small tt-category, e.g., $\mathcal{K} = \mathcal{T}^c$ the subcategory of compact objects of a big tt-category $\mathcal{T}$. A proper thick ideal $\mathfrak{p}$ of $\mathcal{K}$ is called a *prime ideal* if $X \otimes Y \in \mathfrak{p}$ implies that $X \in \mathfrak{p}$ or $Y \in \mathfrak{p}$. The *Balmer spectrum* of $\mathcal{K}$, denoted by $\mathrm{Spc}(\mathcal{K})$, is the set of prime ideals of $\mathcal{K}$. Let X be an object of $\mathcal{K}$. The set $\mathrm{Supp}(X) = \{\mathfrak{p} \in \mathrm{Spc}(\mathcal{K}) \mid X \notin \mathfrak{p}\} \subseteq \mathrm{Spc}(\mathcal{K})$ is called the *support* of X. If $\mathcal{T}$ is a big tt-category, when we say the Balmer spectrum of $\mathcal{T}$ we mean $\mathrm{Spc}(\mathcal{T}^c)$.

Lemma 1.2.2 ([**Bal05**, Lemma 2.6])**.** *Let $\mathcal{K}$ be an essentially small tt-category. The support, as defined above, satisfies the following properties:*
(a) $\mathrm{Supp}(0) = \varnothing \quad \& \quad \mathrm{Supp}(1) = \mathrm{Spc}(\mathcal{K})$.
(b) $\mathrm{Supp}(X \oplus Y) = \mathrm{Supp}(X) \cup \mathrm{Supp}(Y)$.
(c) $\mathrm{Supp}(\Sigma X) = \mathrm{Supp}(X)$.
(d) $\mathrm{Supp}(Y) \subseteq \mathrm{Supp}(X) \cup \mathrm{Supp}(Z)$, *for any triangle* $X \to Y \to Z \to \Sigma X$.
(e) $\mathrm{Supp}(X \otimes Y) = \mathrm{Supp}(X) \cap \mathrm{Supp}(Y)$.

Remark 1.2.3. Lemma 1.2.2 implies that the subsets of $\mathrm{Spc}(\mathcal{K})$ of the form $\mathrm{Supp}(X)$ comprise a basis of closed subsets for a topology on $\mathrm{Spc}(\mathcal{K})$. We always consider $\mathrm{Spc}(\mathcal{K})$ as a topological space equipped with the aforementioned topology.

Definition 1.2.4. Let $\mathcal{K}$ be an essentially small tt-category. A thick ideal $\mathcal{I}$ of $\mathcal{K}$ is called *radical* if $\{X \in \mathcal{K} \mid \exists n \geq 1 \colon X^{\otimes n} \in \mathcal{I}\} = \mathcal{I}$.

Definition 1.2.5. Let $\mathcal{K}$ be an essentially small tt-category. A subset V of $\mathrm{Spc}(\mathcal{K})$ is called *Thomason* if V is a union of closed subsets each with quasi-compact complement.

The Thomason subsets of the Balmer spectrum parametrize radical thick ideals in the sense of the following classification theorem:

Theorem 1.2.6 ([**Bal05**, Theorem 4.10])**.** *Let $\mathcal{K}$ be an essentially small tt-category. The map that sends a Thomason subset V of $\mathrm{Spc}(\mathcal{K})$ to the thick ideal $\mathcal{K}_V = \{X \in \mathcal{K} \mid \mathrm{Supp}(X) \subseteq V\}$ is an inclusion-preserving bijection between the collection of Thomason subsets of $\mathrm{Spc}(\mathcal{K})$ and the collection of radical thick ideals of $\mathcal{K}$. Its inverse is given by mapping a radical thick ideal $\mathcal{I}$ to the Thomason subset $\bigcup_{X \in \mathcal{I}} \mathrm{Supp}(X)$.*

Remark 1.2.7. If $\mathcal{K}$ is the subcategory of compact objects of a big tt-category, then all thick ideals of $\mathcal{K}$ are radical and so, Balmer's theorem provides a classification of all thick ideals of $\mathcal{K}$.

* * *

Let $\mathcal{T}$ be a big tt-category.

Definition 1.2.8. A localizing ideal $\mathcal{S} \subseteq \mathcal{T}$ is called a *smashing ideal* if the quotient functor $j_{\mathcal{S}} \colon \mathcal{T} \twoheadrightarrow \mathcal{T}/\mathcal{S}$ admits a right adjoint $k_{\mathcal{S}} \colon \mathcal{T}/\mathcal{S} \hookrightarrow \mathcal{T}$ that preserves coproducts. The collection of smashing ideals of $\mathcal{T}$ is denoted by $\mathsf{S}^{\otimes}(\mathcal{T})$.

Remark 1.2.9. The collection $\mathsf{S}^{\otimes}(\mathcal{T})$ is a set; see [**Kra00**, Theorem 4.9] and also [**BF11**, Remark 2.16].

Recollection 1.2.10. As explained in [**BF11**], every $\mathcal{S} \in \mathsf{S}^{\otimes}(\mathcal{T})$ corresponds to a triangle $(T_{\mathcal{S}}) \colon e_{\mathcal{S}} \to 1 \to f_{\mathcal{S}}$, where $e_{\mathcal{S}}$ is a left idempotent, $f_{\mathcal{S}}$ is a right idempotent and $e_{\mathcal{S}} \otimes f_{\mathcal{S}} = 0$ (with these three conditions being equivalent). Moreover, for any object $X \in \mathcal{T}$, the associated localization triangle $i_{\mathcal{S}} r_{\mathcal{S}}(X) \to X \to k_{\mathcal{S}} j_{\mathcal{S}}(X)$, where $i_{\mathcal{S}} \colon \mathcal{S} \to \mathcal{T}$ is the inclusion functor and $r_{\mathcal{S}} \colon \mathcal{T} \to \mathcal{S}$ is its right adjoint, is isomorphic to $e_{\mathcal{S}} \otimes X \to X \to f_{\mathcal{S}} \otimes X$, i.e., the triangle obtained by tensoring $(T_{\mathcal{S}})$ with X. More succinctly, the localization and acyclization functors corresponding to $\mathcal{S}$ are given by tensoring with $f_{\mathcal{S}}$ and $e_{\mathcal{S}}$, respectively.

A point $\mathfrak{p} \in \operatorname{Spc}(\mathcal{T}^{c})$ is called *visible* (*weakly visible* in [**BHS23b**]) if there exist Thomason subsets V, W of $\operatorname{Spc}(\mathcal{T}^{c})$ such that $\{\mathfrak{p}\} = V \cap (\operatorname{Spc}(\mathcal{T}^{c}) \setminus W)$ [**BF11, Ste13**]. The Balmer spectrum is called *weakly noetherian* if all of its points are visible. In particular, if $\operatorname{Spc}(\mathcal{T}^{c})$ is noetherian, then every point of $\operatorname{Spc}(\mathcal{T}^{c})$ is visible. According to Balmer's classification theorem, the subsets V and W correspond to thick ideals $\mathcal{T}^{c}_{V}$, $\mathcal{T}^{c}_{W}$ of compact objects. Let $\mathcal{T}_V = \operatorname{loc}^{\otimes}(\mathcal{T}^{c}_{V})$ and $\mathcal{T}_W = \operatorname{loc}^{\otimes}(\mathcal{T}^{c}_{W})$. (It should be noted that the localizing subcategories generated by $\mathcal{T}^{c}_{V}$ and $\mathcal{T}^{c}_{W}$ are already tensor-ideals.) Since the ideals $\mathcal{T}_V$ and $\mathcal{T}_W$ are compactly generated, they are smashing ideals [**Mil92**]. Therefore, they have associated left and right idempotents e_V, f_V and e_W, f_W, respectively. Let $g_{\mathfrak{p}} = e_V \otimes f_W$. Then the objects $\{g_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spc}(\mathcal{T}^{c}) \text{ and } \mathfrak{p} \text{ is visible}\}$ are pairwise-orthogonal tensor-idempotents. Assuming that all points of $\operatorname{Spc}(\mathcal{T}^{c})$ are visible, the *Balmer–Favi support* of an object $X \in \mathcal{T}$ is $\operatorname{Supp}(X) = \{\mathfrak{p} \in \operatorname{Spc}(\mathcal{T}^{c}) \mid g_{\mathfrak{p}} \otimes X \neq 0\}$.

Remark 1.2.11. The reason we use the same notation for the Balmer–Favi support and the support of compact objects, as in Definition 1.2.1, is that they coincide on the compact objects.

Proposition 1.2.12 ([**BF11**, Proposition 7.17, Theorem 7.22])**.** *Assuming that all points of* $\operatorname{Spc}(\mathcal{T}^{c})$ *are visible, the Balmer–Favi support satisfies the following properties:*

(a) *If* $x \in \mathcal{T}^{c}$*, then the Balmer–Favi support of* x *is equal to the support of* x*, as in Definition 1.2.1.*
(b) $\operatorname{Supp}(0) = \varnothing \quad \& \quad \operatorname{Supp}(1) = \operatorname{Spc}(\mathcal{T}^{c})$.
(c) $\operatorname{Supp}(\coprod_{i \in I} X_i) = \bigcup_{i \in I} \operatorname{Supp}(X_i)$.
(d) $\operatorname{Supp}(\Sigma X) = \operatorname{Supp}(X)$.
(e) $\operatorname{Supp}(Y) \subseteq \operatorname{Supp}(X) \cup \operatorname{Supp}(Z)$*, for any triangle* $X \to Y \to Z \to \Sigma X$.
(f) $\operatorname{Supp}(X \otimes Y) \subseteq \operatorname{Supp}(X) \cap \operatorname{Supp}(Y)$.
(g) $\operatorname{Supp}(X \otimes y) = \operatorname{Supp}(X) \cap \operatorname{Supp}(y)$, $\forall X \in \mathcal{T}$, $\forall y \in \mathcal{T}^{c}$.

Recollection 1.2.13. Let V be a Thomason subset of $\operatorname{Spc}(\mathcal{T}^{c})$ and let U be its complement. Since $\mathcal{T}_V$ is a smashing ideal, the quotient $\mathcal{T}(U) := \mathcal{T}/\mathcal{T}_V$ is a big tt-category and moreover, $\operatorname{Spc}(\mathcal{T}(U)^{c}) \cong U$; see [**BF07**, Proposition 1.11].

1.3. The homological spectrum

In this section, we recall some standard facts about modules and the homological spectrum. The reader that wishes to learn more about the structure of the module category should consult [**BKS20**, Appendix A], while information about the homological spectrum can be found in [**Bal20a**].

The category $\operatorname{Mod}(\mathcal{T}^{c})$ has a right exact tensor product defined via Day convolution and the restricted Yoneda functor $h \colon \mathcal{T} \to \operatorname{Mod}(\mathcal{T}^{c})$ is monoidal. Since $\operatorname{Mod}(\mathcal{T}^{c})$ is a Grothendieck abelian category, it also admits an internal-hom functor $[-,-] \colon \operatorname{Mod}(\mathcal{T}^{c})^{\mathrm{op}} \times \operatorname{Mod}(\mathcal{T}^{c}) \to \operatorname{Mod}(\mathcal{T}^{c})$. The subcategory of finitely presented objects of $\operatorname{Mod}(\mathcal{T}^{c})$ is denoted by $\operatorname{mod}(\mathcal{T}^{c})$. An additive subcategory of $\operatorname{mod}(\mathcal{T}^{c})$ is called a *Serre subcategory* if it is closed under subobjects, extensions and quotients. A maximal Serre ideal of $\mathrm{mod}(\mathcal{T}^c)$ is called a *homological prime ideal*.

Construction 1.3.1. For each homological prime ideal $\mathcal{B}$, there exists a unique maximal localizing Serre ideal $\mathcal{B}'$ of $\mathrm{Mod}(\mathcal{T}^c)$ that contains $\mathcal{B}$, constructed in the following way: Let $\mathrm{loc}^{\otimes}(\mathcal{B})$ be the localizing Serre ideal of $\mathrm{Mod}(\mathcal{T}^c)$ generated by $\mathcal{B}$. Then the Gabriel quotient $\mathrm{Mod}(\mathcal{T}^c)/\mathrm{loc}^{\otimes}(\mathcal{B})$ remains a Grothendieck category and inherits the monoidal structure of $\mathrm{Mod}(\mathcal{T}^c)$ in such a way that the quotient functor $Q_{\mathcal{B}}\colon \mathrm{Mod}(\mathcal{T}^c) \to \mathrm{Mod}(\mathcal{T}^c)/\mathrm{loc}^{\otimes}(\mathcal{B})$ is monoidal. The injective envelope of the tensor-unit of $\mathrm{Mod}(\mathcal{T}^c)/\mathrm{loc}^{\otimes}(\mathcal{B})$ is of the form $(Q_{\mathcal{B}}\circ h)(I_{\mathcal{B}})$, for some pure-injective object $I_{\mathcal{B}} \in \mathcal{T}$. Then $\mathcal{B}' = \mathrm{Ker}[-, \widehat{I_{\mathcal{B}}}]$ is the unique maximal localizing Serre ideal of $\mathrm{Mod}(\mathcal{T}^c)$ that contains $\mathcal{B}$.

Definition 1.3.2. The *homological spectrum* of $\mathcal{T}$, denoted by $\mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^c)$, is the set of homological prime ideals and the *homological support* of an object $X \in \mathcal{T}$ is the set

$$\mathrm{Supp}^{\mathrm{h}}(X) = \{\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^c) \mid \widehat{X} \notin \mathcal{B}'\}.$$

Remark 1.3.3. Due to the fact that the restricted Yoneda functor is conservative, $\mathrm{Supp}^{\mathrm{h}}(X) = \{\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^c) \mid [X, I_{\mathcal{B}}] \neq 0\}$.

Lemma 1.3.4 ([**Bal20a**, Section 4])**.** *The homological support satisfies the following properties:*

(a) $\mathrm{Supp}^{\mathrm{h}}(0) = \varnothing \quad \& \quad \mathrm{Supp}^{\mathrm{h}}(1) = \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^c)$.
(b) $\mathrm{Supp}^{\mathrm{h}}(\coprod_{i\in I} X_i) = \bigcup_{i\in I} \mathrm{Supp}^{\mathrm{h}}(X_i)$.
(c) $\mathrm{Supp}^{\mathrm{h}}(\Sigma X) = \mathrm{Supp}^{\mathrm{h}}(X)$.
(d) $\mathrm{Supp}^{\mathrm{h}}(Y) \subseteq \mathrm{Supp}^{\mathrm{h}}(X) \cup \mathrm{Supp}^{\mathrm{h}}(Z)$, *for any triangle* $X \to Y \to Z \to \Sigma X$.
(e) $\mathrm{Supp}^{\mathrm{h}}(X \otimes Y) = \mathrm{Supp}^{\mathrm{h}}(X) \cap \mathrm{Supp}^{\mathrm{h}}(Y)$.

Remark 1.3.5 ([**Bal20b**, Remark 3.4 & Corollary 3.9])**.** The map

$$\phi\colon \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^c) \to \mathrm{Spc}(\mathcal{T}^c), \quad \mathcal{B} \mapsto h^{-1}(\mathcal{B}) \cap \mathcal{T}^c$$

is surjective and $\phi^{-1}(\mathrm{Supp}(x)) = \mathrm{Supp}^{\mathrm{h}}(x)$, $\forall x \in \mathcal{T}^c$. The last relation shows that when $\mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^c)$ is equipped with the topology with basis of closed subsets consisting of the homological supports of compact objects, ϕ is continuous.

1.4. The smashing spectrum

This section is devoted to the smashing spectrum of a big tt-category and the associated support theory.

Recollection 1.4.1. The collection $\mathrm{S}^{\otimes}(\mathcal{T})$ of smashing ideals, ordered by inclusion, is a *complete lattice* [**Kra05a**]. That is, every collection of smashing ideals has an infimum (*meet*) and a supremum (*join*). The join of a collection of smashing ideals $\{\mathcal{S}_i\}_{i\in I}$ is $\bigvee_{i\in I} \mathcal{S}_i := \mathrm{loc}^{\otimes}(\bigcup_{i\in I} \mathcal{S}_i)$ and the meet of $\{\mathcal{S}_i\}_{i\in I}$, denoted by $\bigwedge_{i\in I} \mathcal{S}_i$, is the join of all smashing ideals contained in $\bigcap_{i\in I} \mathcal{S}_i$. The meet of finitely many smashing ideals is their intersection. On the contrary, the meet of infinitely many smashing ideals is not necessarily their intersection; see [**BKS20**, Remark 5.12]. By [**BKS20**, Theorem 5.5], $\mathrm{S}^{\otimes}(\mathcal{T})$ is a *frame*. This means that finite meets distribute over arbitrary joins.

Remark 1.4.2. In a previous version of their work [**BS23**], Balchin–Stevenson claimed that $\mathrm{S}^{\otimes}(\mathcal{T})$ is a *spatial frame*, i.e., $\mathrm{S}^{\otimes}(\mathcal{T})$ is isomorphic to the lattice of open subsets of a topological space. Subsequently, it was discovered that there is a mistake in their proposed proof. Details and counterexamples to their arguments can be found in [**BS23**, Appendix A]. Hence, the statement "*the frame of smashing ideals of a big tt-category is a spatial frame*" is still an open problem. Nevertheless, even if counterexamples to this statement are found, assuming that $\mathrm{S}^{\otimes}(\mathcal{T})$ is a spatial frame and building on this hypothesis is still fruitful since examples of interest, notably derived categories of valuation domains satisfy it [**BŠ17**].

Hypothesis 1.4.3. For the rest of this work, we assume that the frame $\mathrm{S}^{\otimes}(\mathcal{T})$ of smashing ideals of $\mathcal{T}$ is a spatial frame.

Definition 1.4.4 ([**BS23**])**.** The space corresponding to the spatial frame $\mathrm{S}^{\otimes}(\mathcal{T})$ via Stone duality is called the *smashing spectrum* of $\mathcal{T}$ and is denoted by $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. The smashing spectrum consists of the *meet-prime* smashing ideals of $\mathcal{T}$. Namely, those smashing ideals P that satisfy the following property: if $\mathcal{S}_1, \mathcal{S}_2$ are two smashing ideals such that $\mathcal{S}_1 \cap \mathcal{S}_2 \subseteq P$, then $\mathcal{S}_1 \subseteq P$ or $\mathcal{S}_2 \subseteq P$.

Remark 1.4.5. For details on Stone duality, see [**Jon82**].

Recollection 1.4.6. The lattice isomorphism $\mathrm{S}^{\otimes}(\mathcal{T}) \to \mathcal{O}(\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}))$, given by Stone duality, maps a smashing ideal $\mathcal{S}$ to the open subset $U_{\mathcal{S}} = \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid \mathcal{S} \nsubseteq P\}$. Accordingly, the closed subsets of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, being the complements of open subsets, are of the form $V_{\mathcal{S}} = \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid \mathcal{S} \subseteq P\}$. Since $\mathrm{S}^{\otimes}(\mathcal{T}) \to \mathcal{O}(\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}))$ is order-preserving, $\mathcal{S} \subseteq \mathcal{R}$ if and only if $U_{\mathcal{S}} \subseteq U_{\mathcal{R}}$. The union of a collection of open subsets $\{U_{\mathcal{S}_i}\}_{i \in I}$ is given by $U_{\bigvee_{i \in I} \mathcal{S}_i}$. Similarly, the intersection of two (or finitely many) open subsets $U_{\mathcal{S}}$ and $U_{\mathcal{R}}$ is $U_{\mathcal{S} \cap \mathcal{R}}$. Lastly, the closure of a point $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is V_P.

Recollection 1.4.7. Since the spatial frame $\mathrm{S}^{\otimes}(\mathcal{T})$ corresponds to $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ via Stone duality, it follows that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is a sober space, i.e., every non-empty irreducible closed subset of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ has a unique generic point. According to [**BS23**, Remark 3.2.12], $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ most likely does not have a basis of quasi-compact open subsets.

Remark 1.4.8. A space X is called *spectral* if X sober, quasi-compact and X has a basis of quasi-compact open subsets that is closed under finite intersections. As mentioned in Recollection 1.4.7, by [**BS23**, Remark 3.2.12], it is likely that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is not a spectral space (equivalently $\mathrm{S}^{\otimes}(\mathcal{T})$ is not a coherent frame).

* * *

Definition 1.4.9. Let X be an object of $\mathcal{T}$. The subset

$$\mathrm{supp}^{\mathrm{s}}(X) = \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid X \notin P\}$$

of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is called the *big smashing support* of X.

Lemma 1.4.10 ([**BS23**, Lemma 3.2.8])**.** *The big smashing support satisfies the following properties:*

(a) $\mathrm{supp}^{\mathrm{s}}(0) = \varnothing \quad \& \quad \mathrm{supp}^{\mathrm{s}}(1) = \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.
(b) $\mathrm{supp}^{\mathrm{s}}(\coprod_{i \in I} X_i) = \bigcup_{i \in I} \mathrm{supp}^{\mathrm{s}}(X_i)$.
(c) $\mathrm{supp}^{\mathrm{s}}(\Sigma X) = \mathrm{supp}^{\mathrm{s}}(X)$.
(d) $\mathrm{supp}^{\mathrm{s}}(Y) \subseteq \mathrm{supp}^{\mathrm{s}}(X) \cup \mathrm{supp}^{\mathrm{s}}(Z)$, *for any triangle* $X \to Y \to Z \to \Sigma X$.

(e) $\mathrm{supp}^{\mathrm{s}}(X \otimes Y) \subseteq \mathrm{supp}^{\mathrm{s}}(X) \cap \mathrm{supp}^{\mathrm{s}}(Y)$.
(f) $\mathrm{supp}^{\mathrm{s}}(x \otimes y) = \mathrm{supp}^{\mathrm{s}}(x) \cap \mathrm{supp}^{\mathrm{s}}(y),\ \forall x, y \in \mathcal{T}^{\mathrm{c}}$.

If $x \in \mathcal{T}^{\mathrm{c}}$, then $\mathrm{Supp}(x)$ is a closed subset of $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$. Analogously, if $X \in \mathcal{T}$ and $\mathrm{loc}^{\otimes}(X) \in \mathrm{S}^{\otimes}(\mathcal{T})$, then $\mathrm{supp}^{\mathrm{s}}(X)$ is an open subset of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. In particular, if $x \in \mathcal{T}^{\mathrm{c}}$, then $\mathrm{loc}^{\otimes}(x) \in \mathrm{S}^{\otimes}(\mathcal{T})$. Thus, $\mathrm{supp}^{\mathrm{s}}(x)$ is an open subset of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.

Lemma 1.4.11. *Let $X \in \mathcal{T}$ and $\mathcal{R} = \bigwedge\{\mathcal{S} \in \mathrm{S}^{\otimes}(\mathcal{T}) \mid X \in \mathcal{S}\}$. Then $U_{\mathcal{R}} \subseteq \mathrm{supp}^{\mathrm{s}}(X)$, with equality when $\mathrm{loc}^{\otimes}(X)$ is a smashing ideal.*

PROOF. Let $\mathcal{I} = \bigcap\{\mathcal{S} \in \mathrm{S}^{\otimes}(\mathcal{T}) \mid X \in \mathcal{S}\}$. Then we have $U_{\mathcal{R}} = \bigcup\{U_{\mathcal{L}} \mid \mathcal{L} \in \mathrm{S}^{\otimes}(\mathcal{T}),\ \mathcal{L} \subseteq \mathcal{I}\}$. Let $\mathcal{L}$ be a smashing ideal such that $\mathcal{L} \subseteq \mathcal{I}$ and assume that $P \in U_{\mathcal{L}}$, in other words $\mathcal{L} \nsubseteq P$. If $X \in P$, then $\mathcal{I} \subseteq P$ (since P is a smashing ideal that contains X). As a result, $\mathcal{L} \subseteq P$, which has been ruled out by assumption. So, $X \notin P$, meaning that $P \in \mathrm{supp}^{\mathrm{s}}(X)$. If $\mathrm{loc}^{\otimes}(X)$ is a smashing ideal, then $\mathrm{loc}^{\otimes}(X) = \mathcal{R}$. Therefore, $U_{\mathcal{R}} = U_{\mathrm{loc}^{\otimes}(X)} = \mathrm{supp}^{\mathrm{s}}(X)$. □

Next we discuss a refinement of the big smashing support, designed to handle non-compact objects more effectively; see [**BS23**, Section 3.3]. This refined support, called the *small smashing support*, is constructed in a way similar to the Balmer–Favi support. In the case of the derived category of a commutative noetherian ring it recovers Foxby's small support [**Fox79**].

Definition 1.4.12. Let X be a topological space. A point $x \in X$ is called *locally closed* if there exist an open subset $U \subseteq X$ and a closed subset $V \subseteq X$ such that $\{x\} = U \cap V$. If every point of X is locally closed, then X is called T_D.

Remark 1.4.13. Since the T_D separation axiom does not appear to be as popular as the rest of the separation axioms, e.g., T_0, T_1, T_2, it might be useful to provide some explanations. To this end, let X be a topological space. The *specialization preorder* on the points of X is defined as follows: $x \le y$ if $x \in \overline{\{y\}}$; equivalently, $\overline{\{x\}} \subseteq \overline{\{y\}}$. The *downward closure* of a point $x \in X$ is $\downarrow x = \{z \in X \mid z \le x\}$. Evidently, $\downarrow x = \overline{\{x\}}$. The space X is T_D if $\downarrow x \setminus \{x\}$ is a closed subset, for every $x \in X$. One can easily check that this definition is equivalent to the one given in Definition 1.4.12. Any T_1 space is T_D and any T_D space is T_0. The original source where the T_D separation axiom was studied is [**AT62**].

Remark 1.4.14. By [**BS23**, Lemma 3.3.10], if $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D, then for every point $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, there exists a smashing ideal $\mathcal{S}$ such that $\{P\} = U_{\mathcal{S}} \cap V_P$.

Definition 1.4.15. Suppose that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D and let P be a point of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ and $\mathcal{S}$ a smashing ideal of $\mathcal{T}$ such that $\{P\} = U_{\mathcal{S}} \cap V_P$. Then the object $\Gamma_P = e_{\mathcal{S}} \otimes f_P$ is called the *Rickard idempotent* corresponding to P.

Recollection 1.4.16. Let $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ be a locally closed point and consider an open subset $U_{\mathcal{S}}$ such that $U_{\mathcal{S}} \cap V_P = \{P\}$. Since $\mathcal{S} \nsubseteq P$, it follows that $\Gamma_P = e_{\mathcal{S}} \otimes f_P \neq 0$. If $U_{\mathcal{R}}$ is another open subset that contains P, then $U_{\mathcal{S} \cap \mathcal{R}} \cap V_P = \{P\}$. It then holds that $e_{\mathcal{S}} \otimes f_P$ and $e_{\mathcal{S} \cap \mathcal{R}} \otimes f_P = e_{\mathcal{S}} \otimes e_{\mathcal{R}} \otimes f_P$ are isomorphic. Therefore, restricting to smaller open neighborhoods of P does not alter the Rickard idempotent Γ_P. More generally, if $U_{\mathcal{S}_1} \cap V_{P_1} = U_{\mathcal{S}_2} \cap V_{P_2}$, then $e_{\mathcal{S}_1} \otimes f_{P_1} \cong e_{\mathcal{S}_2} \otimes f_{P_2}$. This shows that Γ_P does not depend on the choice of open and closed subsets whose intersection is P. See [**BS23**, Lemma 3.3.9] for details.

Definition 1.4.17. Assuming that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D, the *small smashing support* of an object $X \in \mathcal{T}$ is

$$\mathrm{Supp}^{\mathrm{s}}(X) = \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid \Gamma_P \otimes X \neq 0\}.$$

Lemma 1.4.18. *The analogous properties* (a)-(f) *of Lemma 1.4.10 hold for the small smashing support. If $\mathcal{S}$ is a smashing ideal, then $U_{\mathcal{S}} = \mathrm{Supp}^{\mathrm{s}}(e_{\mathcal{S}})$, $V_{\mathcal{S}} = \mathrm{Supp}^{\mathrm{s}}(f_{\mathcal{S}})$. Further, for all $X \in \mathcal{T}$, $\mathrm{Supp}^{\mathrm{s}}(X) \subseteq \mathrm{supp}^{\mathrm{s}}(X)$, with equality when $X \in \mathcal{T}^{\mathrm{c}}$.*

Proof. The claimed properties follow from the definition of the small smashing support. For the rest, see [**BS23**, Lemma 3.3.11 & Lemma 3.3.15]. □

* * *

Definition 1.4.19. We say that $\mathcal{T}$ satisfies the *Telescope Conjecture* if every smashing ideal $\mathcal{S}$ of $\mathcal{T}$ is compactly generated, i.e., there exists a collection of compact objects $\{x_i\}_{i\in I} \subseteq \mathcal{T}^{\mathrm{c}}$ such that $\mathcal{S} = \mathrm{loc}^{\otimes}(\{x_i\}_{i\in I})$.

Let $\mathcal{L}$ be a localizing ideal of $\mathcal{T}$. We denote by $\mathcal{L}^{\mathrm{c}}$ the subcategory of compact objects of $\mathcal{L}$, i.e., those objects $x \in \mathcal{L}$ such that the functor $\mathrm{Hom}_{\mathcal{T}}(x, -)$ preserves coproducts of objects of $\mathcal{L}$. The following lemma, which is a well-known and useful observation regarding compact objects in smashing ideals, shows that the Telescope Conjecture (as stated in Definition 1.4.19) is equivalent to the statement that every smashing ideal $\mathcal{S}$ is generated by compact objects of $\mathcal{S}$, i.e., objects of $\mathcal{S}^{\mathrm{c}}$. We give a tt-flavored proof.

Lemma 1.4.20. *Let $\mathcal{S}$ be a smashing ideal of $\mathcal{T}$. Then $\mathcal{S}^{\mathrm{c}} = \mathcal{S} \cap \mathcal{T}^{\mathrm{c}}$.*

Proof. First, we will prove the following claim: Let $X \in \mathcal{S}$. Then

$$\mathrm{Hom}_{\mathcal{T}}(X, e_{\mathcal{S}} \otimes Y) = \mathrm{Hom}_{\mathcal{T}}(X, Y),\ \forall Y \in \mathcal{T}. \tag{†}$$

Consider the idempotent triangle associated with $\mathcal{S}$ and tensor it with Y. This gives us the triangle $\Sigma^{-1}(f_{\mathcal{S}} \otimes Y) \to e_{\mathcal{S}} \otimes Y \to Y \to f_{\mathcal{S}} \otimes Y$. Applying $\mathrm{Hom}_{\mathcal{T}}(X, -)$, we obtain the exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{-1}(f_{\mathcal{S}} \otimes Y)) \to \mathrm{Hom}_{\mathcal{T}}(X, e_{\mathcal{S}} \otimes Y) \to \mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}}(X, f_{\mathcal{S}} \otimes Y).$$

Since $X \in \mathcal{S}$ and $f_{\mathcal{S}} \otimes Y \in \mathcal{S}^{\perp}$, the first and last terms of the above exact sequence are 0. We infer that $\mathrm{Hom}_{\mathcal{T}}(X, e_{\mathcal{S}} \otimes Y) \to \mathrm{Hom}_{\mathcal{T}}(X, Y)$ is an isomorphism.

Clearly, $\mathcal{S} \cap \mathcal{T}^{\mathrm{c}} \subseteq \mathcal{S}^{\mathrm{c}}$. Let $x \in \mathcal{S}^{\mathrm{c}}$ and let $\{X_i\}_{i\in I} \subseteq \mathcal{T}$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{T}}(x, \coprod_{i\in I} X_i) &= \mathrm{Hom}_{\mathcal{T}}(x, e_{\mathcal{S}} \otimes \coprod_{i\in I} X_i) \\
&= \mathrm{Hom}_{\mathcal{T}}(x, \coprod_{i\in I} (e_{\mathcal{S}} \otimes X_i)) \\
&= \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{T}}(x, e_{\mathcal{S}} \otimes X_i) \\
&= \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{T}}(x, X_i).
\end{aligned}$$

The first and fourth equalities are due to (†) and the third equality holds because $x \in \mathcal{S}^{\mathrm{c}}$ and $e_{\mathcal{S}} \otimes X_i \in \mathcal{S}$. This proves that x is a compact object of $\mathcal{T}$ and so $\mathcal{S}^{\mathrm{c}} \subseteq \mathcal{S} \cap \mathcal{T}^{\mathrm{c}}$, which completes the proof. □

Remark 1.4.21. If $\mathcal{S}$ is a compactly generated localizing ideal of $\mathcal{T}$, then $\mathcal{S}$ is a smashing ideal [**Mil92**]. The Telescope Conjecture is concerned with the converse statement. There are cases where it is known to be true, such as for derived categories of commutative noetherian rings [**Nee92**] (or noetherian schemes more generally [**AJS04**]) and derived categories of absolutely flat rings [**Ste14a, BŠ17**], and there are cases where it fails: derived categories of some non-noetherian valuation domains [**BŠ17**, Example 5.24] (see also [**BS23**, Section 7]) and a construction by Keller [**Kel94**].

The smashing spectrum is related to the Balmer spectrum via the following map that gives a characterization of the Telescope Conjecture. Recall that the *Hochster dual* of a topological space X is the space $X^\vee$ with open subsets the Thomason subsets of X.

Recollection 1.4.22. The map $\psi\colon \operatorname{Spc}^{s}(\mathcal{T}) \to \operatorname{Spc}(\mathcal{T}^{c})^\vee$ that sends a meet-prime smashing ideal P to $P \cap \mathcal{T}^{c}$ is surjective and continuous. Furthermore, the Telescope Conjecture for smashing ideals holds if and only if ψ is a homeomorphism; see [**BS23**, Section 5].

CHAPTER 2

Stratification and the smashing spectrum

In this chapter, we first prove in Section 2.1 that the Hochster dual of the Balmer spectrum is the Kolmogorov quotient of the smashing spectrum equipped with a certain topology. Then in Section 2.2, we prove that stratification is equivalent to the local-to-global principle and minimality and in Section 2.3, we prove that it suffices to check stratification on certain smashing localizations. Further, in Section 2.4, we investigate connections between big prime ideals, objectwise-prime ideals and homological primes and give a characterization of the Telescope Conjecture in terms of the homological spectrum and the homological support. Finally, in Section 2.6, we study induced maps between smashing spectra and prove a descent theorem for stratification. The results of this chapter first appeared in [**Ver23c**].

2.1. The small topology

We prove that the Hochster dual of the Balmer spectrum is the Kolmogorov quotient of the smashing spectrum, when the latter is endowed with the topology with basis consisting of the smashing supports of compact objects. We conclude that the smashing spectrum is T_0 with the aforementioned topology if and only if the meet-prime smashing ideals are compactly generated, i.e., the meet-prime Telescope Conjecture holds.

Definition 2.1.1. The collection $\{\mathrm{Supp}^{\mathrm{s}}(x) \subseteq \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid x \in \mathcal{T}^{\mathrm{c}}\}$ is a basis for a topology on $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ that we denote by T and call the *small topology*. The small topology is coarser than the topology provided by Stone duality (henceforth called the *standard topology*).

Remark 2.1.2. The comparison map $\psi\colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \to \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$ remains continuous when $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is equipped with the small topology, since for all $x \in \mathcal{T}^{\mathrm{c}}$, $\psi^{-1}(\mathrm{Supp}(x)) = \mathrm{Supp}^{\mathrm{s}}(x)$, which is open in the small topology.

Remark 2.1.3. Let $x \in \mathcal{T}^{\mathrm{c}}$. Since $\mathrm{Supp}^{\mathrm{s}}(x)$ is quasi-compact in the standard topology on $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, which is finer than the small topology, it follows that $\mathrm{Supp}^{\mathrm{s}}(x)$ is quasi-compact in the small topology. As discussed in Lemma 1.4.11, $\mathrm{Supp}^{\mathrm{s}}(x) = U_{\mathrm{loc}^{\otimes}(x)}$. Therefore, the subsets $\mathrm{Supp}^{\mathrm{s}}(x)$, where $x \in \mathcal{T}^{\mathrm{c}}$, comprise a basis of quasi-compact open subsets for the small topology on $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. Since $\mathrm{loc}^{\otimes}(x) \cap \mathrm{loc}^{\otimes}(y) = \mathrm{loc}^{\otimes}(x \otimes y)$, $\forall x, y \in \mathcal{T}^{\mathrm{c}}$, it follows that the small topology has a basis of quasi-compact open subsets that is closed under finite intersections. In general, the small topology is not sober; see [**BS23**, Section 7].

Lemma 2.1.4. *Let P be a point of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. The closure of P in the small topology is given by $\overline{\{P\}}^{\mathsf{T}} = V_{\mathrm{loc}^{\otimes}(P^{\mathrm{c}})}$.*

Proof. The basic closed subsets of T are those of the form $V_{\mathrm{loc}^{\otimes}(x)}$, where $x \in \mathcal{T}^{\mathrm{c}}$. Thus,

$$\begin{aligned}\overline{\{P\}}^{\mathsf{T}} &= \bigcap_{P \in V_{\mathrm{loc}^{\otimes}(x)}} V_{\mathrm{loc}^{\otimes}(x)} \\ &= V_{\bigvee(\mathrm{loc}^{\otimes}(x) \mid \mathrm{loc}^{\otimes}(x) \subseteq P)} \\ &= V_{\bigvee(\mathrm{loc}^{\otimes}(x) \mid x \in P^{\mathrm{c}})} \\ &= V_{\mathrm{loc}^{\otimes}(\bigcup_{x \in P^{\mathrm{c}}} \mathrm{loc}^{\otimes}(x))} \\ &= V_{\mathrm{loc}^{\otimes}(P^{\mathrm{c}})}. \end{aligned}$$

□

Remark 2.1.5. The *Kolmogorov quotient* $\mathrm{KQ}(X)$ of a topological space X is the quotient of X with respect to the equivalence relation that identifies two points $x, y \in X$ if $\overline{\{x\}} = \overline{\{y\}}$. The space $\mathrm{KQ}(X)$ is T_0 and the quotient map $X \to \mathrm{KQ}(X)$ is a surjective continuous map that is an initial object in the category of continuous maps out of X into T_0 spaces. The space X is T_0 if and only if the quotient map $X \to \mathrm{KQ}(X)$ is a homeomorphism.

Definition 2.1.6. We say that $\mathcal{T}$ satisfies the *meet-prime Telescope Conjecture* if every meet-prime smashing ideal P of $\mathcal{T}$ is compactly generated, i.e., $P = \mathrm{loc}^{\otimes}(P^{\mathrm{c}})$.

Proposition 2.1.7. *The map* $\psi\colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \to \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$ *exhibits* $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$ *as the Kolmogorov quotient of* $(\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}), \mathsf{T})$. *Moreover, the following are equivalent:*
(a) $(\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}), \mathsf{T})$ *is spectral.*
(b) $(\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}), \mathsf{T})$ *is* T_0.
(c) $\psi\colon (\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}), \mathsf{T}) \to \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$ *is a homeomorphism.*
(d) $\mathcal{T}$ *satisfies the meet-prime Telescope Conjecture.*

Proof. Let $P, Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. By Lemma 2.1.4, $\overline{\{P\}}^{\mathsf{T}} = \overline{\{Q\}}^{\mathsf{T}}$ if and only if $V_{\mathrm{loc}^{\otimes}(P^{\mathrm{c}})} = V_{\mathrm{loc}^{\otimes}(Q^{\mathrm{c}})}$. It follows, by Stone duality, that $\mathrm{loc}^{\otimes}(P^{\mathrm{c}}) = \mathrm{loc}^{\otimes}(Q^{\mathrm{c}})$. Consequently, $P^{\mathrm{c}} = Q^{\mathrm{c}}$, i.e., $\psi(P) = \psi(Q)$. This shows that two points of $(\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}), \mathsf{T})$ have the same closure if and only if they get identified under ψ. Let V be a subset of $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ such that $\psi^{-1}(V)$ is closed in the small topology. Then $\psi^{-1}(V) = \bigcap_{i \in I} V_{\mathrm{loc}^{\otimes}(x_i)}$, for some family of compact objects $\{x_i\}_{i \in I}$. Since

$$\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \setminus \psi^{-1}(\mathrm{Supp}(x_i)) = \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \setminus U_{\mathrm{loc}^{\otimes}(x_i)} = V_{\mathrm{loc}^{\otimes}(x_i)},$$

it follows that $\psi^{-1}(V) = \psi^{-1}(\bigcap_{i \in I}(\mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) \setminus \mathrm{Supp}(x_i)))$. Since ψ is surjective, $V = \bigcap_{i \in I}(\mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) \setminus \mathrm{Supp}(x_i))$, thus V is closed in $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$. This shows that ψ is a quotient map. Therefore, $\mathrm{KQ}((\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}), \mathsf{T})) = \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$.

As an immediate consequence, we infer that $(\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}), \mathsf{T})$ is T_0 if and only if $\psi\colon (\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}), \mathsf{T}) \to \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$ is a homeomorphism. If P is a meet-prime smashing ideal, i.e., a point of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, then one can easily see that ψ is a bijection if and only if $P = \mathrm{loc}^{\otimes}(P^{\mathrm{c}})$ since $P^{\mathrm{c}} = \mathrm{loc}^{\otimes}(P^{\mathrm{c}})^{\mathrm{c}}$. Also, note that when ψ is a bijection, $\psi^{-1}\colon \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee} \to (\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}), \mathsf{T})$ is automatically continuous. The equivalence of conditions (a), (b), (c) and (d) is now clear (since $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$ is a spectral space). □

Remark 2.1.8. By [**BS23**, Corollary 5.1.5], $\psi\colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \to \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee}$ is a homeomorphism if and only if the Telescope Conjecture holds. Let $\mathcal{S}$ be a smashing ideal. Then $U_{\mathcal{S}}$ is open in the small topology if and only if there is a collection of objects $\{x_i\}_{i \in I} \subseteq \mathcal{T}^{\mathrm{c}}$ such that $U_{\mathcal{S}} = \bigcup_{i \in I} \mathrm{Supp}^{\mathrm{s}}(x_i) = U_{\mathrm{loc}^{\otimes}(\coprod_{i \in I} x_i)}$. Equivalently, $\mathcal{S} = \mathrm{loc}^{\otimes}(\coprod_{i \in I} x_i) = \mathrm{loc}^{\otimes}(\{x_i\}_{i \in I})$, so $\mathcal{S}$ is compactly generated. This shows that the Telescope Conjecture holds if and only if the small and standard topologies on $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ coincide. It should be noted that when ψ is a bijection, one can only infer that the meet-prime Telescope Conjecture holds, as we have seen in Proposition 2.1.7. The map $\psi^{-1}\colon \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})^{\vee} \to \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, which is given by $\psi^{-1}(\mathfrak{p}) = \mathrm{loc}^{\otimes}(\mathfrak{p})$, is continuous (in the standard topology) if and only if the Telescope Conjecture holds. However, to the author's current knowledge, there is no example where the meet-prime Telescope Conejcture holds but the Telescope Conjecture fails. For instance, for the derived category of a rank 1 non-noetherian valuation domain, the Telescope Conjecture already fails for meet-prime smashing ideals; see [**BS23**, Section 7].

Remark 2.1.9. Barthel–Heard–Sanders [**BHS23a**] proved an analogous result for the homological spectrum: The map $\phi\colon \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}}) \to \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$, as in Remark 1.3.5, exhibits $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ as the Kolmogorov quotient of $\mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}})$. Thus, $\mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}})$ is T_0 if and only if ϕ is a homeomorphism. In all examples where the map ϕ has been computed, it is known to be a homeomorphism; see [**Bal20b**]. Whether this is always true or not is still under investigation. In contrast, as was shown in Proposition 2.1.7, the smashing spectrum is T_0 with respect to the topology with basis consisting of the supports of compact objects if and only if the meet-prime Telescope Conjecture holds.

2.2. Stratification

The first goal of this section is to prove Theorem 2.2.15, showing that stratification by the small smashing support is equivalent to two conditions — the local-to-global principle and minimality — that make it easier to verify stratification in practice. If the Telescope Conjecture holds, Theorem 2.2.15 recovers [**BHS23b**, Theorem 4.1]. The second goal is to use Theorem 2.2.15 in order to establish a bijective correspondence between the smashing spectrum and the big spectrum of a stratified big tt-category.

Definition 2.2.1. A big tt-category $\mathcal{T}$ such that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D is *stratified by the small smashing support* if the maps $\mathcal{P}(\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})) \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \mathrm{Loc}^{\otimes}(\mathcal{T})$, between the powerset of the smashing spectrum and the collection of localizing ideals of $\mathcal{T}$, defined by

$$\tau(W) = \{X \in \mathcal{T} \mid \mathrm{Supp}^{\mathrm{s}}(X) \subseteq W\} \quad \& \quad \sigma(\mathcal{L}) = \bigcup_{X\in\mathcal{L}} \mathrm{Supp}^{\mathrm{s}}(X)$$

are mutually inverse bijections.

Remark 2.2.2. By the properties of $\mathrm{Supp}^{\mathrm{s}}$, it is clear that τ and σ are well-defined.

Definition 2.2.3. Let $\mathcal{T}$ be a big tt-category such that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D.

(a) $\mathcal{T}$ satisfies the *local-to-global principle* if for every object $X \in \mathcal{T}$, it holds that $\mathrm{loc}^{\otimes}(X) = \mathrm{loc}^{\otimes}(\Gamma_P \otimes X \mid P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}))$.
(b) $\mathcal{T}$ satisfies *minimality* if for every $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, it holds that $\mathrm{loc}^{\otimes}(\Gamma_P)$ is a minimal localizing ideal.

Remark 2.2.4. Suppose that $\mathcal{T}$ satisfies the local-to-global principle. Then we have $\mathcal{T} = \mathrm{loc}^{\otimes}(1) = \mathrm{loc}^{\otimes}(\Gamma_P \otimes 1 \mid P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})) = \mathrm{loc}^{\otimes}(\Gamma_P \mid P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}))$.

Remark 2.2.5. Clearly, if $\operatorname{loc}^{\otimes}(\Gamma_P)$ is minimal, then $\operatorname{loc}^{\otimes}(\Gamma_P) = \operatorname{loc}^{\otimes}(\Gamma_P \otimes X)$, for every object $X \in \mathcal{T}$ such that $\Gamma_P \otimes X \neq 0$. Provided that the local-to-global principle holds, the converse also holds.

Remark 2.2.6. If $\mathcal{T}$ satisfies the local-to-global principle, then $\operatorname{Supp}^{\mathrm{s}}$ detects vanishing of objects, i.e., $\operatorname{Supp}^{\mathrm{s}}(X) = \varnothing \Leftrightarrow X = 0$. Another simple observation is that

$$\operatorname{loc}^{\otimes}(\Gamma_P \otimes X \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})) = \operatorname{loc}^{\otimes}(\Gamma_P \otimes X \mid P \in \operatorname{Supp}^{\mathrm{s}}(X)). \tag{2.2.7}$$

Remark 2.2.8. If $\mathcal{S} \in \mathrm{S}^{\otimes}(\mathcal{T})$, then $\sigma(\mathcal{S}) = \bigcup_{X \in \mathcal{S}} \operatorname{Supp}^{\mathrm{s}}(X) \subseteq U_{\mathcal{S}} = \operatorname{Supp}^{\mathrm{s}}(e_{\mathcal{S}})$. Since $e_{\mathcal{S}} \in \mathcal{S}$, it follows that $\sigma(\mathcal{S}) = U_{\mathcal{S}}$. In other words, the map σ extends the bijection $\mathrm{S}^{\otimes}(\mathcal{T}) \to \mathcal{O}(\operatorname{Spc}^{\mathrm{s}}(\mathcal{T}))$ that takes a smashing ideal $\mathcal{S}$ to the open subset $U_{\mathcal{S}}$. The maps σ, τ, and their respective restrictions, assemble into the diagram

$$\begin{array}{ccc}
\mathcal{O}(\operatorname{Spc}^{\mathrm{s}}(\mathcal{T})) & \underset{\longleftarrow}{\overset{\longrightarrow}{\cong}} & \mathrm{S}^{\otimes}(\mathcal{T}) \\
\big\downarrow & & \big\downarrow \\
\mathcal{P}(\operatorname{Spc}^{\mathrm{s}}(\mathcal{T})) & \underset{\sigma}{\overset{\tau}{\rightleftarrows}} & \operatorname{Loc}^{\otimes}(\mathcal{T})
\end{array}$$

where the two obvious squares commute.

Classification of localizing ideals.

Lemma 2.2.9. *It holds that* $\sigma \circ \tau = \mathrm{Id}$ *(therefore,* τ *is injective and* σ *is surjective).*

Proof. If $W \in \mathcal{P}(\operatorname{Spc}^{\mathrm{s}}(\mathcal{T}))$, then $\sigma(\tau(W)) = \bigcup_{\operatorname{Supp}^{\mathrm{s}}(X) \subseteq W} \operatorname{Supp}^{\mathrm{s}}(X) \subseteq W$. In addition, $\operatorname{Supp}^{\mathrm{s}}(\Gamma_P) = \{P\} \subseteq W$, $\forall P \in W$. Thus, $W \subseteq \sigma(\tau(W))$, showing that $\sigma \circ \tau = \mathrm{Id}$, which proves the statement. □

Lemma 2.2.10. *Suppose that* $\mathcal{T}$ *satisfies minimality. Then the following hold:*

(a) $\operatorname{loc}^{\otimes}(\Gamma_P \otimes X \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})) = \operatorname{loc}^{\otimes}(\Gamma_P \mid P \in \operatorname{Supp}^{\mathrm{s}}(X))$, $\forall X \in \mathcal{T}$.
(b) $\operatorname{loc}^{\otimes}(\Gamma_P \mid P \in \sigma(\mathcal{L})) \subseteq \mathcal{L}$, $\forall \mathcal{L} \in \operatorname{Loc}^{\otimes}(\mathcal{T})$.

Proof.

(a) If $P \in \operatorname{Supp}^{\mathrm{s}}(X)$, i.e., $\Gamma_P \otimes X \neq 0$, then $\operatorname{loc}^{\otimes}(\Gamma_P) = \operatorname{loc}^{\otimes}(\Gamma_P \otimes X)$ due to minimality of $\operatorname{loc}^{\otimes}(\Gamma_P)$. In conjunction with (2.2.7):

$$\operatorname{loc}^{\otimes}(\Gamma_P \mid P \in \operatorname{Supp}^{\mathrm{s}}(X)) = \operatorname{loc}^{\otimes}(\Gamma_P \otimes X \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})).$$

(b) Let $P \in \sigma(\mathcal{L})$. Then there exists an object $X \in \mathcal{L}$ such that $\Gamma_P \otimes X \neq 0$. Since $X \in \mathcal{L}$, it holds that $\Gamma_P \otimes X \in \mathcal{L}$. So, $\operatorname{loc}^{\otimes}(\Gamma_P) = \operatorname{loc}^{\otimes}(\Gamma_P \otimes X) \subseteq \mathcal{L}$. This proves that $\Gamma_P \in \mathcal{L}$, $\forall P \in \sigma(\mathcal{L})$. Hence, $\operatorname{loc}^{\otimes}(\Gamma_P \mid P \in \sigma(\mathcal{L})) \subseteq \mathcal{L}$. □

Lemma 2.2.11. *Let* $\mathcal{E}$ *be a set of objects of* $\mathcal{T}$*. Then* $\sigma(\operatorname{loc}^{\otimes}(\mathcal{E})) = \bigcup_{X \in \mathcal{E}} \operatorname{Supp}^{\mathrm{s}}(X)$.

Proof. The result is deduced by the following host of equivalences making use, in the second one, of the fact that $\operatorname{Ker}(\Gamma_P \otimes -)$ is a localizing ideal:

$$\begin{aligned}
P \notin \bigcup_{X \in \mathcal{E}} \operatorname{Supp}^{\mathrm{s}}(X) &\Leftrightarrow \mathcal{E} \subseteq \operatorname{Ker}(\Gamma_P \otimes -) \\
&\Leftrightarrow \operatorname{loc}^{\otimes}(\mathcal{E}) \subseteq \operatorname{Ker}(\Gamma_P \otimes -) \\
&\Leftrightarrow P \notin \bigcup_{X \in \operatorname{loc}^{\otimes}(\mathcal{E})} \operatorname{Supp}^{\mathrm{s}}(X) = \sigma(\operatorname{loc}^{\otimes}(\mathcal{E})).
\end{aligned}$$ □

Remark 2.2.12. Suppose that $\mathcal{T}$ is stratified by the small smashing support. Invoking Lemma 2.2.11, we obtain: $\mathrm{Supp}^{\mathrm{s}}(X) = \mathrm{Supp}^{\mathrm{s}}(Y) \Leftrightarrow \mathrm{loc}^{\otimes}(X) = \mathrm{loc}^{\otimes}(Y)$ (which implies that $\mathrm{Supp}^{\mathrm{s}}(X) = \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \Leftrightarrow \mathrm{loc}^{\otimes}(X) = \mathcal{T}$). Consequently, the small smashing support distinguishes between objects that generate different localizing ideals. More precisely, we can define an equivalence relation on the class of objects of $\mathcal{T}$ by declaring two objects X, Y to be equivalent if $\mathrm{loc}^{\otimes}(X) = \mathrm{loc}^{\otimes}(Y)$. In case $\mathcal{T}$ is stratified by the small smashing support, X is equivalent to Y if and only if $\mathrm{Supp}^{\mathrm{s}}(X) = \mathrm{Supp}^{\mathrm{s}}(Y)$.

Lemma 2.2.13. *Suppose that $\mathcal{T}$ satisfies the local-to-global principle and let $X \in \mathcal{T}$ be a non-zero object and $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. If $X \in \mathrm{loc}^{\otimes}(\Gamma_P)$, then $\Gamma_P \otimes X \neq 0$.*

Proof. Since $X \in \mathrm{loc}^{\otimes}(\Gamma_P)$, it holds that $\mathrm{loc}^{\otimes}(X) \subseteq \mathrm{loc}^{\otimes}(\Gamma_P)$. Lemma 2.2.11 and the fact that σ is order-preserving lead to:

$$\mathrm{Supp}^{\mathrm{s}}(X) = \sigma(\mathrm{loc}^{\otimes}(X)) \subseteq \sigma(\mathrm{loc}^{\otimes}(\Gamma_P)) = \mathrm{Supp}^{\mathrm{s}}(\Gamma_P) = \{P\}.$$

As a result, $\mathrm{Supp}^{\mathrm{s}}(X)$ is either empty or equal to $\{P\}$. Remark 2.2.6 implies that $\mathrm{Supp}^{\mathrm{s}}(X) \neq \varnothing$. Consequently, $\mathrm{Supp}^{\mathrm{s}}(X) = \{P\}$. Hence, $\Gamma_P \otimes X \neq 0$. □

Remark 2.2.14. If $X \in \mathrm{loc}^{\otimes}(\Gamma_P)$ is a non-zero object and $\mathrm{loc}^{\otimes}(\Gamma_P)$ is minimal, then $\mathrm{Ker}(\Gamma_P \otimes -) \cap \mathrm{loc}^{\otimes}(\Gamma_P)$, being a localizing ideal, is either zero or $\mathrm{loc}^{\otimes}(\Gamma_P)$. The latter cannot hold, since $\Gamma_P \neq 0$. It follows that $\Gamma_P \otimes X \neq 0$.

Theorem 2.2.15. *The category $\mathcal{T}$ is stratified by the small smashing support if and only if $\mathcal{T}$ satisfies the local-to-global principle and minimality.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{T}$ satisfies the local-to-global principle and minimality. Since $\sigma \circ \tau = \mathrm{Id}$, it suffices to show that $\tau \circ \sigma = \mathrm{Id}$. Let $\mathcal{L}$ be a localizing ideal of $\mathcal{T}$. The relation $\mathcal{L} \subseteq (\tau \circ \sigma)(\mathcal{L})$ follows from the definition of τ and σ. Let $X \in (\tau \circ \sigma)(\mathcal{L})$, i.e., $\mathrm{Supp}^{\mathrm{s}}(X) \subseteq \sigma(\mathcal{L})$. Then

$$\begin{aligned}
\mathrm{loc}^{\otimes}(X) &= \mathrm{loc}^{\otimes}(\Gamma_P \otimes X \mid P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})) && \text{(local-to-global principle)}\\
&= \mathrm{loc}^{\otimes}(\Gamma_P \otimes X \mid P \in \mathrm{Supp}^{\mathrm{s}}(X)) && \text{(2.2.7)}\\
&= \mathrm{loc}^{\otimes}(\Gamma_P \mid P \in \mathrm{Supp}^{\mathrm{s}}(X)) && \text{(Lemma 2.2.10)}\\
&\subseteq \mathrm{loc}^{\otimes}(\Gamma_P \mid P \in \sigma(\mathcal{L})) && (\mathrm{Supp}^{\mathrm{s}}(X) \subseteq \sigma(\mathcal{L}))\\
&\subseteq \mathcal{L}. && \text{(Lemma 2.2.10)}
\end{aligned}$$

As a result, $X \in \mathcal{L}$. So, $(\tau \circ \sigma)(\mathcal{L}) \subseteq \mathcal{L}$ implying that $\tau \circ \sigma = \mathrm{Id}$.

($\Leftarrow$) Let X be an object of $\mathcal{T}$ such that $\Gamma_P \otimes X \neq 0$, i.e., $P \in \mathrm{Supp}^{\mathrm{s}}(X)$. Then $\mathrm{Supp}^{\mathrm{s}}(\Gamma_P \otimes X) \subseteq \mathrm{Supp}^{\mathrm{s}}(\Gamma_P) \cap \mathrm{Supp}^{\mathrm{s}}(X) = \{P\} \cap \mathrm{Supp}^{\mathrm{s}}(X) = \{P\}$. Since Γ_P is an idempotent, $P \in \mathrm{Supp}^{\mathrm{s}}(\Gamma_P \otimes X)$. Therefore, $\mathrm{Supp}^{\mathrm{s}}(\Gamma_P \otimes X) = \{P\} = \mathrm{Supp}^{\mathrm{s}}(\Gamma_P)$. According to Lemma 2.2.11, $\sigma(\mathrm{loc}^{\otimes}(\Gamma_P \otimes X)) = \mathrm{Supp}^{\mathrm{s}}(\Gamma_P \otimes X) = \mathrm{Supp}^{\mathrm{s}}(\Gamma_P) = \sigma(\mathrm{loc}^{\otimes}(\Gamma_P))$. Since σ is injective, $\mathrm{loc}^{\otimes}(\Gamma_P \otimes X) = \mathrm{loc}^{\otimes}(\Gamma_P)$. This establishes minimality. Next, use the relation $\mathrm{Supp}^{\mathrm{s}}(\Gamma_P \otimes X) = \{P\}$, when $P \in \mathrm{Supp}^{\mathrm{s}}(X)$, and Lemma 2.2.11 to deduce that

$$\begin{aligned}\sigma(\mathrm{loc}^{\otimes}(\Gamma_P \otimes X \mid P \in \mathrm{Supp}^{\mathrm{s}}(X))) &= \bigcup_{P \in \mathrm{Supp}^{\mathrm{s}}(X)} \mathrm{Supp}^{\mathrm{s}}(\Gamma_P \otimes X) \\ &= \bigcup_{P \in \mathrm{Supp}^{\mathrm{s}}(X)} \{P\} \\ &= \mathrm{Supp}^{\mathrm{s}}(X) \\ &= \sigma(\mathrm{loc}^{\otimes}(X)).\end{aligned}$$

Since σ is injective, $\mathrm{loc}^{\otimes}(X) = \mathrm{loc}^{\otimes}(\Gamma_P \otimes X \mid P \in \mathrm{Supp}^{\mathrm{s}}(X))$. Consequently, $\mathcal{T}$ satisfies the local-to-global-principle. □

Corollary 2.2.16. *If $\mathcal{T}$ satisfies the local-to-global principle and minimality, then the collection $\mathrm{Loc}^{\otimes}(\mathcal{T})$ of localizing ideals of $\mathcal{T}$ is a set and every localizing ideal of $\mathcal{T}$ is generated by a set of objects, hence by a single object.*

Proof. The first half of the statement is immediate from Theorem 2.2.15. The second half stems from [**KS17**, Lemma 3.3.1] by specializing the arguments to localizing ideals instead of general localizing subcategories, as noted in [**BHS23b**, Proposition 3.5]. □

Objectwise and big primes.

Definition 2.2.17. Let $\mathcal{L}$ be a proper localizing ideal of $\mathcal{T}$.

(a) $\mathcal{L}$ is called *objectwise-prime* if $X \otimes Y \in \mathcal{L}$ implies $X \in \mathcal{L}$ or $Y \in \mathcal{L}$.
(b) $\mathcal{L}$ is called *radical* if $X^{\otimes n} \in \mathcal{L}$ implies $X \in \mathcal{L}$, for all $n \geq 1$.
(c) $\mathcal{L}$ is called a *big prime* if $\mathcal{L}$ is radical and $\mathcal{I}_1 \cap \mathcal{I}_2 \subseteq \mathcal{L}$ implies $\mathcal{I}_1 \subseteq \mathcal{L}$ or $\mathcal{I}_2 \subseteq \mathcal{L}$, for all radical localizing ideals $\mathcal{I}_1, \mathcal{I}_2$.

The collection of big prime localizing ideals of $\mathcal{T}$, introduced in [**BS23**] and called the *big spectrum* of $\mathcal{T}$, is denoted by $\mathrm{SPC}(\mathcal{T})$. Evidently, if $\mathcal{L}$ is objectwise-prime, then $\mathcal{L}$ is radical.

The big prime ideals are precisely the meet-prime elements of the lattice of radical localizing ideals partially ordered by inclusion. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are radical localizing ideals, then $\mathcal{I}_1 \cap \mathcal{I}_2 = \sqrt{\mathcal{I}_1 \otimes \mathcal{I}_2}$, where $\mathcal{I}_1 \otimes \mathcal{I}_2$ is the smallest localizing ideal containing the collection $\{X \otimes Y \mid X \in \mathcal{I}_1,\ Y \in \mathcal{I}_2\}$ and $\sqrt{\mathcal{I}}$ is the smallest radical localizing ideal containing an ideal $\mathcal{I}$; see [**BS23**, Lemma 4.1.3]. Hence, in case all localizing ideals are radical, it holds that $\mathcal{I}_1 \cap \mathcal{I}_2 = \mathcal{I}_1 \otimes \mathcal{I}_2$, resembling the more familiar definition of prime ideals in a (not necessarily commutative) ring.

Proposition 2.2.18. *Suppose that $\mathcal{T}$ satisfies minimality. Then the following hold:*

(a) $\mathrm{Supp}^{\mathrm{s}}(X \otimes Y) = \mathrm{Supp}^{\mathrm{s}}(X) \cap \mathrm{Supp}^{\mathrm{s}}(Y)$, $\forall X, Y \in \mathcal{T}$. (*Tensor Product Formula*)
(b) $\mathrm{Ker}(\Gamma_P \otimes -)$ *is an objectwise-prime localizing ideal of* $\mathcal{T}$, $\forall P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.

Proof. First of all, the statements (a) and (b) are equivalent, since they both state: $\forall X, Y \in \mathcal{T}$, $\forall P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$: $\Gamma_P \otimes X \otimes Y = 0$ if and only if $\Gamma_P \otimes X = 0$ or $\Gamma_P \otimes Y = 0$. Let $X, Y \in \mathcal{T}$ such that $\Gamma_P \otimes X \neq 0$ and $\Gamma_P \otimes Y \neq 0$. If $Y \in \mathrm{Ker}(\Gamma_P \otimes X \otimes -)$, then $\mathrm{loc}^{\otimes}(\Gamma_P) = \mathrm{loc}^{\otimes}(\Gamma_P \otimes Y) \subseteq \mathrm{loc}^{\otimes}(Y) \subseteq \mathrm{Ker}(\Gamma_P \otimes X \otimes -)$. Therefore, $\Gamma_P \in \mathrm{Ker}(\Gamma_P \otimes X \otimes -)$, which is a contradiction, since we assumed that $\Gamma_P \otimes X \neq 0$. This proves that $\Gamma_P \otimes X \otimes Y \neq 0$ and the proof is complete. □

Remark 2.2.19. From Remark 2.2.6 and Proposition 2.2.18, we learn that if $\mathcal{T}$ is stratified, then $\mathrm{Supp}^{\mathrm{s}}$ detects vanishing of objects and satisfies the Tensor Product Formula. As a consequence, $\mathcal{T}$ cannot have any non-zero $\otimes$-nilpotent objects.

Lemma 2.2.20. *Let $\mathcal{L}$ be a localizing ideal of $\mathcal{T}$. Then*

$$\tau(\sigma(\mathcal{L})) = \bigcap_{\mathcal{L}\subseteq \mathrm{Ker}(\Gamma_P\otimes -)} \mathrm{Ker}(\Gamma_P \otimes -).$$

Proof. Let X be an object of $\mathcal{T}$. Then $X \notin \bigcap_{\mathcal{L}\subseteq \mathrm{Ker}(\Gamma_P\otimes -)} \mathrm{Ker}(\Gamma_P \otimes -)$ if and only if there exists $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ such that $\mathcal{L} \subseteq \mathrm{Ker}(\Gamma_P \otimes -)$ and $\Gamma_P \otimes X \neq 0$. Equivalently, $P \in \mathrm{Supp}^{\mathrm{s}}(X)$ and $P \notin \bigcup_{Y\in\mathcal{L}} \mathrm{Supp}^{\mathrm{s}}(Y) = \sigma(\mathcal{L})$. In other words, $\mathrm{Supp}^{\mathrm{s}}(X) \nsubseteq \sigma(\mathcal{L})$. By definition of τ, the latter happens if and only if $X \notin \tau(\sigma(\mathcal{L}))$. ☐

Corollary 2.2.21. *If $\mathcal{T}$ is stratified by the small smashing support, then all localizing ideals of $\mathcal{T}$ are radical.*

Proof. Let $\mathcal{L}$ be a localizing ideal of $\mathcal{T}$. Since $\mathcal{T}$ is stratified, it holds that $\tau \circ \sigma = \mathrm{Id}$. Further, Lemma 2.2.20 implies that $\mathcal{L} = \bigcap_{\mathcal{L}\subseteq \mathrm{Ker}(\Gamma_P\otimes -)} \mathrm{Ker}(\Gamma_P \otimes -)$. By Theorem 2.2.15, $\mathcal{T}$ satisfies minimality. According to Proposition 2.2.18, each $\mathrm{Ker}(\Gamma_P \otimes -)$ is objectwise-prime, hence radical. Since radical ideals are closed under intersections, $\mathcal{L}$ is radical. ☐

Lemma 2.2.22. *Suppose that $\mathcal{T}$ satisfies the local-to-global principle. Then for all $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, it holds that $\mathrm{Ker}(\Gamma_P \otimes -) = \mathrm{loc}^{\otimes}(\Gamma_Q \mid Q \neq P)$.*

Proof. Let P and Q be distinct meet-prime smashing ideals. Since $\Gamma_P \otimes \Gamma_Q = 0$, it follows that $\Gamma_Q \in \mathrm{Ker}(\Gamma_P \otimes -)$. Therefore, $\mathrm{loc}^{\otimes}(\Gamma_Q \mid Q \neq P) \subseteq \mathrm{Ker}(\Gamma_P \otimes -)$. If $X \in \mathrm{Ker}(\Gamma_P \otimes -)$, then $\mathrm{loc}^{\otimes}(X) = \mathrm{loc}^{\otimes}(\Gamma_Q \otimes X \mid Q \neq P) \subseteq \mathrm{loc}^{\otimes}(\Gamma_Q \mid Q \neq P)$. Hence, $\mathrm{Ker}(\Gamma_P \otimes -) \subseteq \mathrm{loc}^{\otimes}(\Gamma_Q \mid Q \neq P)$, proving the claimed equality. ☐

Proposition 2.2.23. *Suppose that $\mathcal{T}$ is stratified by the small smashing support. Then every objectwise-prime localizing ideal $\mathcal{L}$ is of the form $\mathrm{Ker}(\Gamma_P \otimes -)$, for a unique $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.*

Proof. If $P \neq Q$ are meet-prime smashing ideals, then $\Gamma_P \otimes \Gamma_Q = 0 \in \mathcal{L}$. Therefore, $\Gamma_P \in \mathcal{L}$ or $\Gamma_Q \in \mathcal{L}$. Since $\mathcal{T} = \mathrm{loc}^{\otimes}(\Gamma_P \mid P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}))$ and $\mathcal{L}$ is proper by definition, $\mathcal{L}$ contains all Rickard idempotents except one. So, there exists a meet-prime smashing ideal P such that $\mathrm{Ker}(\Gamma_P \otimes -) = \mathrm{loc}^{\otimes}(\Gamma_Q \mid Q \neq P) \subseteq \mathcal{L}$, where the equality is by Lemma 2.2.22. Suppose that this containment relation is proper. Then there exists an object $X \in \mathcal{L}$ such that $\Gamma_P \otimes X \neq 0$. Since $\mathrm{loc}^{\otimes}(\Gamma_P)$ is minimal, $\mathrm{loc}^{\otimes}(\Gamma_P \otimes X) = \mathrm{loc}^{\otimes}(\Gamma_P)$. Moreover, $\Gamma_P \otimes X \in \mathcal{L}$ implies $\Gamma_P \in \mathcal{L}$. This forces $\mathcal{L} = \mathcal{T}$, leading to a contradiction. Uniqueness follows from the fact that $\Gamma_P \otimes \Gamma_Q = 0$ if and only if $P \neq Q$. ☐

Corollary 2.2.24. *Suppose that $\mathcal{T}$ is stratified by the small smashing support. Then the big prime localizing ideals of $\mathcal{T}$ coincide with the objectwise-prime localizing ideals of $\mathcal{T}$ and there is only a set of such. In particular,*

$$\mathrm{SPC}(\mathcal{T}) = \{\mathrm{Ker}(\Gamma_P \otimes -) \mid P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})\}.$$

Proof. This follows from Corollary 2.2.21, Proposition 2.2.23 and [**BS23**, Lemma 4.1.7]. ☐

Corollary 2.2.25. *Suppose that* $\mathcal{T}$ *is stratified by the small smashing support. Then the map* $\mathrm{Spc^s}(\mathcal{T}) \to \mathrm{SPC}(\mathcal{T})$, $P \mapsto \mathrm{Ker}(\Gamma_P \otimes -)$ *is bijective.*

Remark 2.2.26. Let P be a meet-prime smashing ideal of $\mathcal{T}$. It is straightforward to verify that $\sigma(\mathrm{Ker}(\Gamma_P \otimes -)) = \mathrm{Spc^s}(\mathcal{T}) \setminus \{P\}$. Utilizing this relation and Remark 2.2.8 leads to the following series of equivalences:

$$\mathrm{Ker}(\Gamma_P \otimes -) \in \mathrm{S}^{\otimes}(\mathcal{T}) \Leftrightarrow \mathrm{Ker}(\Gamma_P \otimes -) = P \Leftrightarrow V_P = \{P\}.$$

The first equivalence holds because if $\mathrm{Ker}(\Gamma_P \otimes -)$ is smashing, then $\mathrm{Ker}(\Gamma_P \otimes -)$ is contained in some $Q \in \mathrm{Spc^s}(\mathcal{T})$. Since $\Gamma_Q \notin Q$, it holds that $\Gamma_P \otimes \Gamma_Q \neq 0$. Therefore, $P = Q$. Combined with the inclusion $P \subseteq \mathrm{Ker}(\Gamma_P \otimes -)$, we obtain $\mathrm{Ker}(\Gamma_P \otimes -) = P$. This also explains why $\mathrm{Ker}(\Gamma_P \otimes -) = P \Rightarrow V_P = \{P\}$. If $V_P = \{P\}$, then $\Gamma_P = f_P$. Thus, $\mathrm{Ker}(\Gamma_P \otimes -) = \mathrm{Ker}(f_P \otimes -) = P$.

Corollary 2.2.27. *Suppose that* $\mathcal{T}$ *is stratified by the small smashing support. Then the smashing objectwise-prime localizing ideals of* $\mathcal{T}$ *correspond to the closed points, with respect to the standard topology, of* $\mathrm{Spc^s}(\mathcal{T})$.

Proof. The claim follows from Corollary 2.2.25 and Remark 2.2.26. □

We conclude with an observation about the small and big smashing supports.

Proposition 2.2.28. *The small and big smashing supports coincide, meaning that* $\mathrm{Supp^s}(X) = \mathrm{supp^s}(X)$, $\forall X \in \mathcal{T}$, *if and only if every point of* $\mathrm{Spc^s}(\mathcal{T})$, *with respect to the standard topology, is a closed point, i.e.,* $\mathrm{Spc^s}(\mathcal{T})$ *is* T_1.

Proof. It holds that

$$\mathrm{Supp^s} = \mathrm{supp^s} \Leftrightarrow (\forall X \in \mathcal{T}, \forall P \in \mathrm{Spc^s}(\mathcal{T})\colon \Gamma_P \otimes X \neq 0 \Leftrightarrow X \notin P).$$

Equivalently, $P = \mathrm{Ker}(\Gamma_P \otimes -)$, $\forall P \in \mathrm{Spc^s}(\mathcal{T})$. By Remark 2.2.26, this statement holds if and only if every point of $\mathrm{Spc^s}(\mathcal{T})$ is a closed point. □

Example 2.2.29. Let R be a commutative absolutely flat ring, i.e., every R-module is flat. Then $\mathrm{D}(R)$ satisfies the Telescope Conjecture; see [**Ste14a, BŠ17**]. Therefore, $\mathrm{Spc^s}(\mathrm{D}(R)) \cong \mathrm{Spec}(R)^\vee$ is T_1, since in this particular case $\mathrm{Spec}(R)$ is Hausdorff (and $\mathrm{Spec}(R) \cong \mathrm{Spec}(R)^\vee$). Hence, the small and big smashing supports on $\mathrm{D}(R)$ coincide.

2.3. Smashing localizations

The aim of this section is to prove that if a big tt-category $\mathcal{T}$ satisfies the local-to-global principle and each meet-prime smashing localization $\mathcal{T}/P$ is stratified by the small smashing support, then $\mathcal{T}$ is stratified by the small smashing support. If $\mathcal{T}$ satisfies the Telescope Conjecture, this recovers [**BHS23b**, Corollary 5.3]. The local-to-global principle hypothesis cannot be dropped. Further, we show that if $\mathrm{Spc^s}(\mathcal{T})$ admits a cover by finitely many closed subsets such that each corresponding smashing localization is stratified, then $\mathcal{T}$ is stratified.

Meet-prime smashing localizations.

Recollection 2.3.1. Let $\mathcal{L}$ be a localizing ideal of $\mathcal{T}$. Then the collection of localizing ideals of $\mathcal{T}/\mathcal{L}$ stands in bijection with the collection of localizing ideals of $\mathcal{T}$ that contain $\mathcal{L}$. More precisely, if $j_{\mathcal{L}}\colon \mathcal{T} \to \mathcal{T}/\mathcal{L}$ is the quotient functor, then the map that takes a localizing ideal $\mathcal{R} \subseteq \mathcal{T}/\mathcal{L}$ to $j_{\mathcal{L}}^{-1}(\mathcal{R})$ is a bijection with inverse given by taking direct images of localizing ideals of $\mathcal{T}$ under $j_{\mathcal{L}}$. In case $j_{\mathcal{L}}$ has a right adjoint $k_{\mathcal{L}}$, the following property (known as the *projection formula*) holds: $k_{\mathcal{L}}(j_{\mathcal{L}}(X) \otimes Y) \cong X \otimes k_{\mathcal{L}}(Y)$, $\forall X \in \mathcal{T}$, $\forall Y \in \mathcal{T}/\mathcal{L}$ [**BDS16**].

Remark 2.3.2. In keeping with the notation of Recollection 2.3.1, if $\mathcal{L}$ is a smashing ideal, then $k_{\mathcal{L}} j_{\mathcal{L}} = - \otimes f_{\mathcal{L}}$. Further, if $\mathcal{R}$ is a localizing ideal that contains $\mathcal{L}$, then $(- \otimes f_{\mathcal{L}})^{-1}(\mathcal{R}) = \mathcal{R}$. Indeed, if $X \otimes f_{\mathcal{L}} \in \mathcal{R}$, then tensoring the idempotent triangle corresponding to $\mathcal{L}$ with X yields $e_{\mathcal{L}} \otimes X \to X \to f_{\mathcal{L}} \otimes X$. Since $\mathcal{L} \subseteq \mathcal{R}$, it follows that $e_{\mathcal{L}} \otimes X \in \mathcal{R}$, thus $X \in \mathcal{R}$. Since $\mathcal{R}$ is an ideal, the converse inclusion also holds.

Let P be a smashing ideal of $\mathcal{T}$ (not necessarily meet-prime). Then $\mathcal{T}/P$ is a big tt-category and the quotient functor $j_P \colon \mathcal{T} \to \mathcal{T}/P$ is an essentially surjective coproduct-preserving tt-functor with a fully faithful right adjoint $k_P \colon \mathcal{T}/P \to \mathcal{T}$ that preserves coproducts, since j_P preserves rigid=compact objects. Therefore, j_P induces an injective continuous map $f \colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}/P) \to \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. By identifying $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}/P)$ with V_P, the induced map f is identified with the inclusion $V_P \hookrightarrow \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. If $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D, then $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}/P)$ is T_D, since being T_D is a hereditary topological property. If $Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}/P)$, i.e., $Q \supseteq P$, then the corresponding Rickard idempotent is $j_P(\Gamma_Q)$.

Lemma 2.3.3. *Let $P \in \mathrm{S}^{\otimes}(\mathcal{T})$ and let $Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ such that $P \nsubseteq Q$. Then $j_P(\Gamma_Q) = 0$.*

Proof. Let $\mathcal{S}$ be a smashing ideal such that $\{Q\} = U_{\mathcal{S}} \cap V_Q$ (recall that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is assumed T_D). Since $P \nsubseteq Q$, we have $U_{P\cap\mathcal{S}} \cap V_Q = U_P \cap U_{\mathcal{S}} \cap V_Q = \{Q\}$. This means $\Gamma_Q = e_{P\cap\mathcal{S}} \otimes f_Q = e_P \otimes e_{\mathcal{S}} \otimes f_Q$. So, $j_P(\Gamma_Q) = j_P(e_P) \otimes j_P(e_{\mathcal{S}}) \otimes j_P(f_Q) = 0$, due to the fact that $e_P \in P$. ☐

Proposition 2.3.4. *Suppose that $\mathcal{T}$ satisfies the local-to-global principle. Then $\mathcal{T}/P$ satisfies the local-to-global-principle, for every $P \in \mathrm{S}^{\otimes}(\mathcal{T})$.*

Proof. By assumption, $\mathcal{T}$ satisfies the local-to-global principle and so we have $1_{\mathcal{T}} \in \mathrm{loc}^{\otimes}(\Gamma_Q \mid Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}))$. So, $1_{\mathcal{T}/P} = j_P(1_{\mathcal{T}}) \in \mathrm{loc}^{\otimes}(j_P(\Gamma_Q) \mid Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})) = \mathrm{loc}^{\otimes}(j_P(\Gamma_Q) \mid Q \in V_P)$. Here we used Lemma 1.1.22 and Lemma 2.3.3. Thus, $\mathcal{T}/P = \mathrm{loc}^{\otimes}(j_P(\Gamma_Q) \mid Q \in V_P)$, meaning that $\mathcal{T}/P$ satisfies the local-to-global principle. ☐

Lemma 2.3.5. *Let P be a smashing ideal of $\mathcal{T}$. Then*

$$j_P^{-1}(\mathrm{loc}^{\otimes}(j_P(X))) = \mathrm{loc}^{\otimes}(e_P, X),$$

where $j_P \colon \mathcal{T} \to \mathcal{T}/P$ is the quotient functor.

Proof. Clearly, $\mathrm{loc}^{\otimes}(e_P, X) \subseteq j_P^{-1}(\mathrm{loc}^{\otimes}(j_P(X)))$. Since $k_P j_P(X) \cong f_P \otimes X$, it follows that $j_P(X) \in k_P^{-1}(\mathrm{loc}^{\otimes}(e_P, X))$. Thus, $\mathrm{loc}^{\otimes}(j_P(X)) \subseteq k_P^{-1}(\mathrm{loc}^{\otimes}(e_P, X))$. Therefore, $j_P^{-1}(\mathrm{loc}^{\otimes}(j_P(X))) \subseteq (k_P j_P)^{-1}(\mathrm{loc}^{\otimes}(e_P, X)) = \mathrm{loc}^{\otimes}(e_P, X)$, with the last equality by Remark 2.3.2. ☐

Remark 2.3.6. Let P be a meet-prime smashing ideal of $\mathcal{T}$ and let $\mathcal{S}$ be a smashing ideal of $\mathcal{T}$ such that $U_{\mathcal{S}} \cap V_P = \{P\}$. Applying j_P to the idempotent triangle corresponding to P yields $j_P(e_P) \to 1_{\mathcal{T}/P} \to j_P(f_P)$. Since $j_P(e_P) \cong 0$, it follows that $j_P(f_P) \cong 1_{\mathcal{T}/P}$, so $j_P(\Gamma_P) \cong j_P(e_{\mathcal{S}})$. Conclusion: $j_P(\Gamma_P)$ is a left idempotent.

Proposition 2.3.7. *Let P be a meet-prime smashing ideal of $\mathcal{T}$. Then* $\mathrm{loc}^{\otimes}(\Gamma_P)$ *is minimal in* $\mathrm{Loc}^{\otimes}(\mathcal{T})$ *if and only if* $\mathrm{loc}^{\otimes}(j_P(\Gamma_P))$ *is minimal in* $\mathrm{Loc}^{\otimes}(\mathcal{T}/P)$.

Proof. ($\Rightarrow$) Suppose that $\mathrm{loc}^{\otimes}(\Gamma_P)$ is minimal and let $X \in \mathrm{loc}^{\otimes}(j_P(\Gamma_P))$ be a non-zero object. Since $j_P(\Gamma_P)$ is a left idempotent, $j_P(\Gamma_P) \otimes X \neq 0$. Write $X = j_P(Y)$, for some $Y \in \mathcal{T}$. Then $j_P(\Gamma_P \otimes Y) = j_P(\Gamma_P) \otimes X \neq 0$, thus $\Gamma_P \otimes Y \neq 0$. It follows, by minimality of $\mathrm{loc}^{\otimes}(\Gamma_P)$, that $\mathrm{loc}^{\otimes}(\Gamma_P \otimes Y) = \mathrm{loc}^{\otimes}(\Gamma_P)$. Hence, $\Gamma_P \in \mathrm{loc}^{\otimes}(Y)$. Invoking Lemma 1.1.22 for the functor j_P results in $j_P(\Gamma_P) \in \mathrm{loc}^{\otimes}(j_P(Y)) = \mathrm{loc}^{\otimes}(X)$. Conclusion: $\mathrm{loc}^{\otimes}(j_P(\Gamma_P))$ is minimal.

($\Leftarrow$) Suppose that $\mathrm{loc}^{\otimes}(j_P(\Gamma_P))$ is minimal. Then $P \subsetneq j_P^{-1}(\mathrm{loc}^{\otimes}(j_P(\Gamma_P)))$, which is minimal over P. By Lemma 2.3.5, this reads $P \subsetneq \mathrm{loc}^{\otimes}(e_P, \Gamma_P)$. Pick a non-zero object $X \in \mathrm{loc}^{\otimes}(\Gamma_P)$. Since $\mathrm{loc}^{\otimes}(\Gamma_P) \subseteq \mathrm{loc}^{\otimes}(f_P) = \mathrm{Im}(- \otimes f_P)$, it follows that $X \otimes f_P \cong X \neq 0$, i.e., $X \notin P$. This shows that $\mathrm{loc}^{\otimes}(\Gamma_P) \cap P = 0$. Therefore, $P \subsetneq \mathrm{loc}^{\otimes}(e_P, X) \subseteq \mathrm{loc}^{\otimes}(e_P, \Gamma_P)$. Since $\mathrm{loc}^{\otimes}(e_P, \Gamma_P)$ is minimal over P, it follows that $\mathrm{loc}^{\otimes}(e_P, X) = \mathrm{loc}^{\otimes}(e_P, \Gamma_P)$, so $\Gamma_P \in \mathrm{loc}^{\otimes}(e_P, X)$. By Lemma 1.1.22, $\Gamma_P \cong \Gamma_P \otimes \Gamma_P \in \mathrm{loc}^{\otimes}(\Gamma_P \otimes e_P, \Gamma_P \otimes X) = \mathrm{loc}^{\otimes}(\Gamma_P \otimes X)$. As a result, $\mathrm{loc}^{\otimes}(\Gamma_P) = \mathrm{loc}^{\otimes}(\Gamma_P \otimes X) \subseteq \mathrm{loc}^{\otimes}(X)$, so $\Gamma_P \in \mathrm{loc}^{\otimes}(X)$. Conclusion: $\mathrm{loc}^{\otimes}(\Gamma_P)$ is minimal. □

Proposition 2.3.8. *The category $\mathcal{T}$ satisfies minimality if and only if $\mathcal{T}/P$ satisfies minimality, for every $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.*

Proof. Suppose that $\mathcal{T}$ satisfies minimality. Let $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ and $Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}/P)$, i.e., $Q \in V_P$ and consider a non-zero object $j_P(X) \in \mathrm{loc}^{\otimes}(j_P(\Gamma_Q))$. The fact that $\mathrm{loc}^{\otimes}(j_P(\Gamma_Q)) \cap j_P(Q) = 0$ leads to $X \notin Q$. Now ponder the quotient $j_Q \colon \mathcal{T} \to \mathcal{T}/Q$. The ideal $\mathrm{loc}^{\otimes}(\Gamma_Q)$ is minimal by assumption, so by Proposition 2.3.7, $\mathrm{loc}^{\otimes}(j_Q(\Gamma_Q))$ is minimal. Equivalently, $\mathrm{loc}^{\otimes}(e_Q, \Gamma_Q)$ is minimal over Q. Since $X \notin Q$, it follows that $\mathrm{loc}^{\otimes}(e_Q, X) = \mathrm{loc}^{\otimes}(e_Q, \Gamma_Q)$. Invoking Lemma 2.2.11 yields $\mathrm{Supp}^{\mathrm{s}}(e_Q) \cup \mathrm{Supp}^{\mathrm{s}}(X) = \mathrm{Supp}^{\mathrm{s}}(e_Q) \cup \mathrm{Supp}^{\mathrm{s}}(\Gamma_Q)$. Thus, $U_Q \cup \mathrm{Supp}^{\mathrm{s}}(X) = U_Q \cup \{Q\}$. Since $Q \notin U_Q$, we infer that $Q \in \mathrm{Supp}^{\mathrm{s}}(X)$, which means that $\Gamma_Q \otimes X \neq 0$. Further, minimality of $\mathrm{loc}^{\otimes}(\Gamma_Q)$ implies that $\Gamma_Q \in \mathrm{loc}^{\otimes}(X)$. In consequence, $j_P(\Gamma_Q) \in \mathrm{loc}^{\otimes}(j_P(X))$, proving that $\mathrm{loc}^{\otimes}(j_P(\Gamma_Q))$ is minimal. The converse implication is given by Proposition 2.3.7. □

Remark 2.3.9. The right-hand implication in Proposition 2.3.8 holds without assuming that P is necessarily meet-prime.

Corollary 2.3.10. *Suppose that $\mathcal{T}$ satisfies the local-to-global principle. Then $\mathcal{T}$ is stratified by the small smashing support if and only if $\mathcal{T}/P$ is stratified by the small smashing support, for every $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.*

Proof. Combine Proposition 2.3.4 and Proposition 2.3.8. □

Remark 2.3.11. If $\mathcal{T}$ is not assumed to a-priori satisfy the local-to-global principle, the converse of Corollary 2.3.10 does not hold in general. For instance, if R is an absolutely flat ring that is not semi-artinian (semi-artinian means that every non-zero homomorphic image of R, in the category of R-modules, contains a simple submodule) then $\mathrm{D}(R)$ does not satisfy the local-to-global principle, even though its localizations $\mathrm{D}(k(\mathfrak{p}))$ (where $k(\mathfrak{p})$ is the residue field at $\mathfrak{p} \in \mathrm{Spec}(R)$) are stratified; see [**Ste14a**, Theorem 4.7].

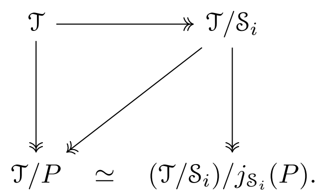
Stratification and closed covers.

Proposition 2.3.12. *Suppose that* $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T}) = \bigcup V_{\mathcal{S}_i}$*, where* $\{\mathcal{S}_i\}$ *is a finite set of smashing ideals, and assume that each* $\mathcal{T}/\mathcal{S}_i$ *satisfies the local-to-global principle. Then* $\mathcal{T}$ *satisfies the local-to-global principle.*

Proof. By an easy induction argument, it suffices to prove the statement in the case $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T}) = V_{\mathcal{S}_1} \cup V_{\mathcal{S}_2}$. Consider the quotient functor $j_{\mathcal{S}_i} \colon \mathcal{T} \to \mathcal{T}/\mathcal{S}_i$, where $i = 1, 2$. Since $\mathcal{T}/\mathcal{S}_i$ satisfies the local-to-global principle and $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T}/\mathcal{S}_i) \cong V_{\mathcal{S}_i}$, it holds that $j_{\mathcal{S}_i}(1) = 1 \in \operatorname{loc}^{\otimes}(j_{\mathcal{S}_i}(\Gamma_P) \mid P \in V_{\mathcal{S}_i})$. By Lemma 2.3.5, it follows that $1 \in \operatorname{loc}^{\otimes}(e_{\mathcal{S}_i}, \{\Gamma_P\}_{P \in V_{\mathcal{S}_i}})$. If $P \in V_{\mathcal{S}_i}$, i.e., $P \supseteq \mathcal{S}_i$, then $f_P \otimes f_{\mathcal{S}_i} \cong f_P$. Thus, $\Gamma_P \otimes f_{\mathcal{S}_i} \cong \Gamma_P$. Invoking Lemma 1.1.22, we have $f_{\mathcal{S}_i} \in \operatorname{loc}^{\otimes}(\Gamma_P \mid P \in V_{\mathcal{S}_i})$. Therefore, $f_{\mathcal{S}_i} \in \operatorname{loc}^{\otimes}(\Gamma_P \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T}))$. So, $f_{\mathcal{S}_1} \oplus f_{\mathcal{S}_2}$, $f_{\mathcal{S}_1} \otimes f_{\mathcal{S}_2} \in \operatorname{loc}^{\otimes}(\Gamma_P \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T}))$. From the Mayer–Vietoris triangle $f_{\mathcal{S}_1 \cap \mathcal{S}_2} \to f_{\mathcal{S}_1} \oplus f_{\mathcal{S}_2} \to f_{\mathcal{S}_1} \otimes f_{\mathcal{S}_2}$, see [**BF11**, Theorem 3.13], it follows that $f_{\mathcal{S}_1 \cap \mathcal{S}_2} \in \operatorname{loc}^{\otimes}(\Gamma_P \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T}))$. Since $V_0 = \operatorname{Spc}^{\mathrm{s}}(\mathcal{T}) = V_{\mathcal{S}_1} \cup V_{\mathcal{S}_2} = V_{\mathcal{S}_1 \cap \mathcal{S}_2}$, we have $\mathcal{S}_1 \cap \mathcal{S}_2 = 0$. Consequently, $f_{\mathcal{S}_1 \cap \mathcal{S}_2} = 1$. In conclusion, $1 \in \operatorname{loc}^{\otimes}(\Gamma_P \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T}))$, which proves that $\mathcal{T}$ satisfies the local-to-global principle. □

Proposition 2.3.13. *Suppose that* $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T}) = \bigcup V_{\mathcal{S}_i}$*, where* $\{\mathcal{S}_i\}$ *is a set of smashing ideals, and assume that each* $\mathcal{T}/\mathcal{S}_i$ *satisfies minimality. Then* $\mathcal{T}$ *satisfies minimality.*

Proof. By assumption, if $P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$, then P lies in some $V_{\mathcal{S}_i}$. In other words, there exists some $\mathcal{S}_i$ such that $\mathcal{S}_i \subseteq P$. The category $\mathcal{T}/P$ can be realized as a localization of $\mathcal{T}/\mathcal{S}_i$, as in the following commutative diagram:



Since $\mathcal{T}/\mathcal{S}_i$ satisfies minimality, it follows by Proposition 2.3.8 that $\mathcal{T}/P$ satisfies minimality. In conclusion, $\mathcal{T}/P$ satisfies minimality, for every $P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$. So, again by Proposition 2.3.8, $\mathcal{T}$ satisfies minimality. □

Corollary 2.3.14. *Suppose that* $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T}) = \bigcup V_{\mathcal{S}_i}$*, where* $\{\mathcal{S}_i\}$ *is a finite set of smashing ideals, and assume that each* $\mathcal{T}/\mathcal{S}_i$ *is stratified by the small smashing support. Then* $\mathcal{T}$ *is stratified by the small smashing support. If* $\mathcal{T}$ *satisfies the local-to-global principle, then the finiteness condition on* $\{\mathcal{S}_i\}$ *can be dropped.*

Remark 2.3.15. The case of the trivial cover $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T}) = \bigcup_{P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})} V_P$ in Proposition 2.3.13 recovers the statement of Proposition 2.3.8. In Proposition 2.3.12, if $\{\mathcal{S}_i\}$ is allowed to be an infinite set, then the most we can deduce is that $\mathcal{T}/P$ satisfies the local-to-global principle, for every $P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$. As we have already seen in Remark 2.3.11, this is not enough to guarantee that $\mathcal{T}$ satisfies the local-to-global principle.

Remark 2.3.16. Results of similar flavor appear in [**BHS23b**], where $\operatorname{Spc}(\mathcal{T}^{\mathrm{c}})$ is covered by complements of Thomason subsets and the notion of support considered is the Balmer–Favi support. Note that the cover $\{V_{\mathcal{S}_i}\}$ of $\operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$ in the above results consists of closed subsets, which are complements of open subsets. This should not come as a surprise since, by Stone duality, smashing ideals of $\mathcal{T}$ correspond to open subsets of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, while thick ideals of $\mathcal{T}^{\mathrm{c}}$ correspond to Thomason subsets of $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$.

2.4. Comparison maps

Let $\mathcal{T}$ be a big tt-category such that Hypothesis 1.4.3 holds and whose smashing spectrum is T_D and assume that $\mathcal{T}$ is stratified by the small smashing support. Let $\mathcal{B}$ be a homological prime. By [**BS23**, Lemma 5.2.1], $\chi(\mathcal{B}) \coloneqq h^{-1}(\mathcal{B}') = \mathrm{Ker}[-, I_{\mathcal{B}}]$ is a big prime localizing ideal of $\mathcal{T}$, where $\mathcal{B}'$ is the unique maximal localizing ideal of $\mathrm{Mod}(\mathcal{T}^{\mathrm{c}})$ that contains $\mathcal{B}$ and $I_{\mathcal{B}}$ is the associated pure-injective object; see Construction 1.3.1. Corollary 2.2.24 asserts that $\chi(\mathcal{B}) = \mathrm{Ker}(\Gamma_P \otimes -)$, for a unique $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. This produces a well-defined map $\xi\colon \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}}) \to \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ that associates each $\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}})$ with the unique $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ such that $\chi(\mathcal{B}) = \mathrm{Ker}(\Gamma_P \otimes -)$.

By construction, ξ is the composite $\mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}}) \xrightarrow{\chi} \mathrm{SPC}(\mathcal{T}) \xrightarrow{\cong} \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, where the second map is the inverse of the map that takes $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ to $\mathrm{Ker}(\Gamma_P \otimes -)$; see Corollary 2.2.25.

Lemma 2.4.1. *The map* $\xi\colon \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}}) \to \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ *is injective.*

Proof. It suffices to show that $\chi\colon \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}}) \to \mathrm{SPC}(\mathcal{T})$ is injective. To this end, let $\mathcal{B}$ and $\mathcal{C}$ be two distinct homological primes. According to [**Bal20a**, Corollary 4.9], $[I_{\mathcal{B}}, I_{\mathcal{C}}] = 0$. Thus, $I_{\mathcal{B}} \in \chi(\mathcal{C})$. Since $I_{\mathcal{B}} \notin \chi(\mathcal{B})$, it follows that $\chi(\mathcal{B}) \neq \chi(\mathcal{C})$. $\square$

Lemma 2.4.2. *Let X be an object of $\mathcal{T}$. Then* $\xi^{-1}(\mathrm{Supp}^{\mathrm{s}}(X)) = \mathrm{Supp}^{\mathrm{h}}(X)$.

Proof. Let $\mathcal{B}$ be a homological prime and let $I_{\mathcal{B}} \in \mathcal{T}$ be the associated pure-injective object. Then $\chi(\mathcal{B}) = \mathrm{Ker}[-, I_{\mathcal{B}}] = \mathrm{Ker}(\Gamma_{\xi(\mathcal{B})} \otimes -)$. It follows from the definition of $\mathrm{Supp}^{\mathrm{s}}$ and $\mathrm{Supp}^{\mathrm{h}}$ that $\xi^{-1}(\mathrm{Supp}^{\mathrm{s}}(X)) = \mathrm{Supp}^{\mathrm{h}}(X)$. $\square$

Remark 2.4.3. If $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ is weakly noetherian and $\mathcal{T}$ is stratified by the Balmer–Favi support, then the comparison map $\phi\colon \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}}) \to \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ is a homeomorphism; see [**BHS23a**, Theorem 4.7]. In our case, we had to assume that $\mathcal{T}$ is stratified by the small smashing support (with the analogous topological hypothesis being that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D) to even obtain the map $\xi\colon \mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}}) \to \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. This, however, is not enough to guarantee that ξ is bijective. The failure of surjectivity of ξ is measured by the "kernel" of the homological support.

Proposition 2.4.4 (cf. [**BHS23a**, Proposition 3.14])**.** *Let $\mathcal{T}$ be a big tt-category whose smashing spectrum is T_D and assume that $\mathcal{T}$ is stratified by the small smashing support. The following are equivalent:*

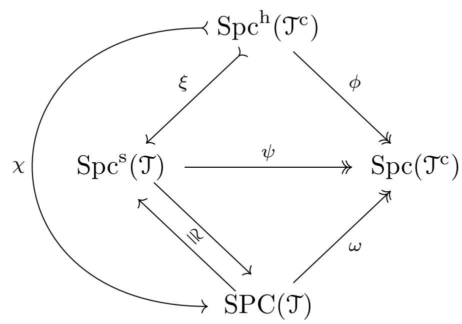
(a) ξ *is surjective.*
(b) $\xi(\mathrm{Supp}^{\mathrm{h}}(X)) = \mathrm{Supp}^{\mathrm{s}}(X)$, $\forall X \in \mathcal{T}$.
(c) $\mathrm{Supp}^{\mathrm{h}}$ *detects vanishing of objects.*

Proof. (a) $\Rightarrow$ (b) Suppose that ξ is surjective. Lemma 2.4.1 implies that ξ is bijective. The statement now follows by applying ξ to the relation obtained in Lemma 2.4.2.

(b) $\Rightarrow$ (c) Suppose that $\mathrm{Supp}^{\mathrm{h}}(X) = \varnothing$. It follows, by the assumption we made in (b), that $\mathrm{Supp}^{\mathrm{s}}(X) = \varnothing$. Since $\mathcal{T}$ is stratified, by Theorem 2.2.15, $\mathcal{T}$ satisfies the local-to-global principle; so, $\mathrm{Supp}^{\mathrm{s}}$ detects vanishing. As a result, $X = 0$.

(c) $\Rightarrow$ (a) As a special case of Lemma 2.4.2, for $X = \Gamma_P$: $\xi^{-1}(\{P\}) = \mathrm{Supp}^{\mathrm{h}}(\Gamma_P)$. Hence, ξ is surjective if and only if $\mathrm{Supp}^{\mathrm{h}}(\Gamma_P) \neq \varnothing$, $\forall P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. Since $\Gamma_P \neq 0$ and $\mathrm{Supp}^{\mathrm{h}}$ is assumed to detect vanishing, $\mathrm{Supp}^{\mathrm{h}}(\Gamma_P) \neq \varnothing$; proving that ξ is surjective. $\square$

The four spectra. We continue to assume that $\mathcal{T}$ is a big tt-category whose smashing spectrum is T_D and that $\mathcal{T}$ is stratified by the small smashing support. The spaces $\mathrm{Spc}(\mathcal{T})$, $\mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}})$, $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, $\mathrm{SPC}(\mathcal{T})$ are related via the following commutative diagram:



To begin with, let $\mathcal{B}$ be a homological prime such that $\chi(\mathcal{B}) = \mathrm{Ker}(\Gamma_P \otimes -)$. Using the relation $\mathrm{Ker}(\Gamma_P \otimes -) \cap \mathcal{T}^{\mathrm{c}} = P \cap \mathcal{T}^{\mathrm{c}}$ yields $\phi(\mathcal{B}) = h^{-1}(\mathcal{B}) \cap \mathcal{T}^{\mathrm{c}} = h^{-1}(\mathcal{B}') \cap \mathcal{T}^{\mathrm{c}} = \chi(\mathcal{B}) \cap \mathcal{T}^{\mathrm{c}} = \xi(\mathcal{B}) \cap \mathcal{T}^{\mathrm{c}} = (\psi \circ \xi)(\mathcal{B})$. This shows that $\psi \circ \xi = \phi$ and also that ψ is equal to the composite $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \xrightarrow{\cong} \mathrm{SPC}(\mathcal{T}) \xrightarrow{\omega} \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$, where ω maps an objectwise-prime localizing ideal to its compact part. Lastly, the triangle on the left commutes by construction of ξ.

Theorem 2.4.5. *Let $\mathcal{T}$ be a big tt-category whose smashing spectrum is T_D and assume that $\mathcal{T}$ is stratified by the small smashing support. Then $\mathcal{T}$ satisfies the Telescope Conjecture if and only if* $\mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}})$ *is T_0 and* $\mathrm{Supp}^{\mathrm{h}}$ *detects vanishing of objects.*

Proof. By [**BS23**, Corollary 5.1.6], $\mathcal{T}$ satisfies the Telescope Conjecture if and only if ψ is bijective. Since ξ is injective and ϕ is surjective and $\psi \circ \xi = \phi$, it holds that ψ is bijective if and only if ξ and ϕ are bijective. According to Lemma 2.4.1 and Proposition 2.4.4, ξ is bijective if and only if $\mathrm{Supp}^{\mathrm{h}}$ detects vanishing of objects. By [**BHS23a**, Proposition 4.5], ϕ is bijective if and only if $\mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}})$ is T_0. We conclude that $\mathcal{T}$ satisfies the Telescope Conjecture if and only if $\mathrm{Spc}^{\mathrm{h}}(\mathcal{T}^{\mathrm{c}})$ is T_0 and $\mathrm{Supp}^{\mathrm{h}}$ detects vanishing of objects. $\square$

2.5. Smashing stratification vs Balmer–Favi stratification

Let $\mathcal{T}$ be a big tt-category. The *generalization closure* of a point $\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ is $\mathrm{gen}(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) \mid \mathfrak{p} \in \overline{\{\mathfrak{q}\}}\} = \{\mathfrak{q} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) \mid \mathfrak{p} \subseteq \mathfrak{q}\}$. The Balmer spectrum is called *generically noetherian* if $\mathrm{gen}(\mathfrak{p})$ is a noetherian space, for all $\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$. If $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ is generically noetherian, then $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ is weakly noetherian. For more details see [**BHS23b**].

Remark 2.5.1. If X is a spectral space, then X is weakly noetherian if and only if the Hochster dual space $X^{\vee}$ (whose open subsets are the Thomason subsets of X)

is T_D. In particular, this applies to $\mathrm{Spc}(\mathcal{T}^c)$. Note that $\mathrm{Spc}^s(\mathcal{T})$ is likely not spectral; see [**BS23**, Remark 3.2.12].

Lemma 2.5.2. *Suppose that* $\mathrm{Spc}^s(\mathcal{T})$ *is* T_D *and that* $\mathcal{T}$ *satisfies the Telescope Conjecture. Then* $\Gamma_P = g_{P^c}$, $\forall P \in \mathrm{Spc}^s(\mathcal{T})$.

Proof. Let $P \in \mathrm{Spc}^s(\mathcal{T})$ and $\mathcal{S} \in \mathrm{S}^{\otimes}(\mathcal{T})$ such that $\{P\} = U_{\mathcal{S}} \cap V_P$. Since the Telescope Conjecture holds and $\mathrm{Spc}^s(\mathcal{T})$ is T_D, $\psi\colon \mathrm{Spc}^s(\mathcal{T}) \to \mathrm{Spc}(\mathcal{T}^c)^\vee$ is a homeomorphism and $\mathrm{Spc}(\mathcal{T}^c)$ is weakly noetherian. It holds that $\{P^c\} = \psi(U_{\mathcal{S}}) \cap \psi(V_P) = \psi(U_{\mathcal{S}}) \cap (\mathrm{Spc}(\mathcal{T}^c) \setminus \psi(U_P))$, with $\psi(U_{\mathcal{S}})$ and $\psi(U_P)$ being Thomason subsets of $\mathrm{Spc}(\mathcal{T}^c)$. The thick ideal corresponding to $\psi(U_{\mathcal{S}})$ is $\mathcal{T}^c_{\psi(U_{\mathcal{S}})} = \{x \in \mathcal{T}^c \mid \mathrm{Supp}(x) \subseteq \psi(U_{\mathcal{S}})\}$. For all $x \in \mathcal{T}^c$, $\psi^{-1}(\mathrm{Supp}(x)) = U_{\mathrm{loc}^{\otimes}(x)}$. So, $\mathrm{Supp}(x) \subseteq \psi(U_{\mathcal{S}})$ if and only if $U_{\mathrm{loc}^{\otimes}(x)} \subseteq U_{\mathcal{S}}$, with the latter being equivalent to $x \in \mathcal{S}^c$. Therefore, $\mathcal{T}^c_{\psi(U_{\mathcal{S}})} = \mathcal{S}^c$. Similarly, $\mathcal{T}^c_{\psi(U_P)} = P^c$. Consequently, $\mathrm{loc}^{\otimes}(\mathcal{T}^c_{\psi(U_{\mathcal{S}})}) = \mathcal{S}$ and $\mathrm{loc}^{\otimes}(\mathcal{T}^c_{\psi(U_P)}) = P$. We infer that $e_{\psi(U_{\mathcal{S}})} = e_{\mathcal{S}}$ and $f_{\psi(U_P)} = f_P$ and as a result, $\Gamma_P = g_{P^c}$. ☐

Corollary 2.5.3. *Suppose that* $\mathrm{Spc}^s(\mathcal{T})$ *is* T_D *and that* $\mathcal{T}$ *satisfies the Telescope Conjecture. Then* $\psi^{-1}(\mathrm{Supp}(X)) = \mathrm{Supp}^s(X)$, *for all* $X \in \mathcal{T}$.

Theorem 2.5.4. *Let* $\mathcal{T}$ *be a big tt-category.*

(a) *If* $\mathrm{Spc}(\mathcal{T}^c)$ *is generically noetherian and the Balmer–Favi support stratifies* $\mathcal{T}$, *then* $\mathcal{T}$ *satisfies the Telescope Conjecture,* $\mathrm{Spc}^s(\mathcal{T})$ *is* T_D *and the small smashing support stratifies* $\mathcal{T}$.
(b) *If* $\mathrm{Spc}^s(\mathcal{T})$ *is* T_D *and the small smashing support stratifies* $\mathcal{T}$ *and* $\mathcal{T}$ *satisfies the Telescope Conjecture, then the Balmer–Favi support stratifies* $\mathcal{T}$.

Proof. If $\mathrm{Spc}(\mathcal{T}^c)$ is generically noetherian and $\mathcal{T}$ is stratified by the Balmer–Favi support, then, by [**BHS23b**, Theorem 9.11], $\mathcal{T}$ satisfies the Telescope Conjecture. Thus, $\psi\colon \mathrm{Spc}^s(\mathcal{T}) \to \mathrm{Spc}(\mathcal{T}^c)^\vee$ is a homeomorphism. Since $\mathrm{Spc}(\mathcal{T}^c)$ is weakly noetherian, $\mathrm{Spc}(\mathcal{T}^c)^\vee$ is T_D. Therefore, $\mathrm{Spc}^s(\mathcal{T})$ is T_D, so the small smashing support is defined. Invoking Lemma 2.5.2 completes the proof of (a).

If $\mathcal{T}$ is stratified by the small smashing support and $\mathcal{T}$ satisfies the Telescope Conjecture, then $\mathrm{Spc}^s(\mathcal{T})$ is a spectral space and, by Corollary 2.5.3, the hypotheses of [**BHS23b**, Theorem 7.6] are satisfied. We deduce that the Balmer–Favi support stratifies $\mathcal{T}$. This proves (b). ☐

Remark 2.5.5. Let us emphasize once more that if the Telescope Conjecture holds, the theory of stratification developed here recovers the theory of [**BHS23b**]. More explicitly, if $\mathcal{T}$ satisfies the Telescope Conjecture, then the following hold:

(a) The comparison map $\psi\colon \mathrm{Spc}^s(\mathcal{T}) \to \mathrm{Spc}(\mathcal{T}^c)^\vee$ is a homeomorphism.
(b) $\mathrm{Spc}^s(\mathcal{T})$ is T_D if and only if $\mathrm{Spc}(\mathcal{T}^c)$ is weakly noetherian.

Assuming that $\mathrm{Spc}^s(\mathcal{T})$ is T_D:

(c) The small smashing support coincides with the Balmer–Favi support (under the identification of the two spectra via ψ).
(d) The formulation of the local-to-global principle and of minimality in Definition 2.2.3 coincides with those given in [**BHS23b**].
(e) $\mathcal{T}$ is stratified by the small smashing support if and only if $\mathcal{T}$ is stratified by the Balmer–Favi support.

Remark 2.5.6. According to [**BHS23b**, Theorem 7.6], any stratifying support theory $\sigma\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(X)$ (where X is a weakly noetherian spectral space) that satisfies three equivalent conditions must be "isomorphic" to the Balmer–Favi support theory and the latter has to stratify $\mathcal{T}$ as well. If $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D and $\sigma = \mathrm{Supp}^{\mathrm{s}}$ stratifies $\mathcal{T}$, then the three equivalent conditions we alluded to are equivalent to the Telescope Conjecture. In [**BHS23b**, Theorem 9.11], under the hypothesis that $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ is generically noetherian, it is proved that if $\mathcal{T}$ is stratified by the Balmer–Favi support, then the Telescope Conjecture holds. In the case of the small smashing support, that proof can neither be reproduced nor is it expected that a different proof exists. Conclusion: The theory of smashing stratification has the potential to encompass a wider range of categories than the theory of Balmer–Favi stratification, since the Telescope Conjecture is necessary for a category to be stratified by the Balmer–Favi support, but it is probably not necessary for smashing stratification. To be clear, at the time of this writing, an example of a category that is stratified by the small smashing support and fails the Telescope Conjecture is not known.

2.6. Induced maps and descent

In the first part of this section, we probe the image of the map between smashing spectra induced by a tensor-triangulated functor; see also [**Bal20a**] for analogous results concerning homological spectra. In the second part, we present conditions under which stratification descends along tensor-triangulated functors. All big tt-categories involved are assumed to satisfy Hypothesis 1.4.3.

Let $F\colon \mathcal{T} \to \mathcal{U}$ be a coproduct-preserving tt-functor between big tt-categories. Then F induces a map of frames $\mathrm{S}^{\otimes}(\mathcal{T}) \to \mathrm{S}^{\otimes}(\mathcal{U})$, $\mathcal{S} \mapsto \mathrm{loc}^{\otimes}(F\mathcal{S})$, which, via Stone duality, gives rise to a continuous map $\mathrm{Spc}^{\mathrm{s}}(F)\colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{U}) \to \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. Explicitly, $\mathrm{Spc}^{\mathrm{s}}(F)$ acts by sending $Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{U})$ to $\bigvee\{\mathcal{S} \in \mathrm{S}^{\otimes}(\mathcal{T}) \mid \mathcal{S} \subseteq F^{-1}(Q)\}$. Additionally, since F preserves rigid=compact objects, there is an induced continuous map $\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{U}^{\mathrm{c}}) \to \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ that takes $\mathfrak{q} \in \mathrm{Spc}(\mathcal{U}^{\mathrm{c}})$ to $F^{-1}(\mathfrak{q}) \cap \mathcal{T}^{\mathrm{c}}$.

Remark 2.6.1. The map $\mathrm{Spc}^{\mathrm{s}}(F)$ does not behave in a way similar to the more classical $\mathrm{Spc}(F)$, namely by taking inverse images. For one, the formula for $\mathrm{Spc}^{\mathrm{s}}(F)$ is given by Stone duality, as explained above. More concretely, there are cases where $F^{-1}(Q)$, for $Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{U})$, is not a smashing ideal. An example is the derived base change functor $\pi\colon \mathrm{D}(\mathbb{Z}) \to \mathrm{D}(\mathbb{F}_p)$ for a prime number p, as demonstrated in [**BS23**, Example 3.4.5].

Lemma 2.6.2. *The following square is commutative:*

$$\begin{array}{ccc} \mathrm{Spc}^{\mathrm{s}}(\mathcal{U}) & \xrightarrow{\mathrm{Spc}^{\mathrm{s}}(F)} & \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \\ {\scriptstyle \psi_{\mathcal{U}}}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{\mathcal{T}}} \\ \mathrm{Spc}(\mathcal{U}^{\mathrm{c}}) & \xrightarrow{\mathrm{Spc}(F)} & \mathrm{Spc}(\mathcal{T}^{\mathrm{c}}). \end{array}$$

Proof. Let $Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{U})$. Then

$$\mathfrak{p}_1 := (\psi_{\mathcal{T}} \circ \mathrm{Spc}^{\mathrm{s}}(F))(Q) = \bigvee\{\mathcal{S} \in \mathrm{S}^{\otimes}(\mathcal{T}) \mid \mathcal{S} \subseteq F^{-1}(Q)\} \cap \mathcal{T}^{\mathrm{c}},$$
$$\mathfrak{p}_2 := (\mathrm{Spc}(F) \circ \psi_{\mathcal{U}})(Q) = F^{-1}(Q \cap \mathcal{U}^{\mathrm{c}}) \cap \mathcal{T}^{\mathrm{c}} = F^{-1}(Q) \cap \mathcal{T}^{\mathrm{c}}.$$

Clearly, $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. For any $x \in \mathfrak{p}_2$, it holds that $\mathrm{loc}^{\otimes}(x)$ is a smashing ideal of $\mathcal{T}$ and $\mathrm{loc}^{\otimes}(x) \subseteq F^{-1}(Q)$. This shows that $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$, thus $\mathfrak{p}_1 = \mathfrak{p}_2$. □

Corollary 2.6.3. *Let $F\colon \mathcal{T} \to \mathcal{U}$ be a coproduct-preserving tt-functor between big tt-categories such that the induced map $\mathrm{Spc}^{\mathrm{s}}(F)\colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{U}) \to \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is a homeomorphism. If $\mathcal{T}$ satisfies the Telescope Conjecture, then $\mathcal{U}$ satisfies the Telescope Conjecture and the induced map $\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{U}^{\mathrm{c}}) \to \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ is a homeomorphism.*

Proof. By [**BS23**, Corollary 5.1.6], the map $\psi_{\mathcal{T}}$ is a homeomorphism. Therefore, by Lemma 2.6.2, $\mathrm{Spc}(F) \circ \psi_{\mathcal{U}}$ is a homeomorphism. This implies that $\psi_{\mathcal{U}}$ is injective. By [**BS23**, Proposition 5.1.5], $\psi_{\mathcal{U}}$ is surjective. Hence, $\psi_{\mathcal{U}}$ is a homeomorphism and, again by [**BS23**, Corollary 5.1.6], $\mathcal{U}$ satisfies the Telescope Conjecture. Since $\psi_{\mathcal{U}}$ and $\mathsf{Spc}(F) \circ \psi_{\mathcal{U}}$ are homeomorphisms, $\mathsf{Spc}(F)$ is a homeomorphism. □

The image of $\mathrm{Spc}^{\mathrm{s}}(F)$. Let $F\colon \mathcal{T} \to \mathcal{U}$ be a coproduct-preserving tt-functor between big tt-categories. By Brown representability, $F\colon \mathcal{T} \to \mathcal{U}$ has a right adjoint $G\colon \mathcal{U} \to \mathcal{T}$. Since F is monoidal, hence preserves compact objects, G is lax-monoidal and preserves coproducts. Further, F and G are related by the *projection formula*: $G(FX \otimes Y) = X \otimes GY$; see [**BDS16**, Proposition 2.15]. For $Y = 1$, we obtain the equality $GF(-) = G(1) \otimes -$. Moreover, $G(1) \neq 0$, since $\mathrm{Hom}_{\mathcal{U}}(1 = F1, 1) \cong \mathrm{Hom}_{\mathcal{T}}(1, G(1))$.

Proposition 2.6.4. *Let $F\colon \mathcal{T} \to \mathcal{U}$ be a coproduct-preserving tt-functor between big tt-categories with right adjoint G. Then $\mathrm{Im}\,\mathrm{Spc}^{\mathrm{s}}(F) \subseteq \mathrm{supp}^{\mathrm{s}}(G(1))$. Assuming that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D, if G is conservative, i.e., $\mathrm{Ker}\, G = 0$, then $\mathrm{Im}\,\mathrm{Spc}^{\mathrm{s}}(F) \subseteq \mathrm{Supp}^{\mathrm{s}}(G(1))$.*

Proof. Let $Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{U})$ and $P = \mathrm{Spc}^{\mathrm{s}}(F)(Q)$. Since P and Q are smashing ideals, the localizations $\mathcal{T}/P$ and $\mathcal{U}/Q$ are big tt-categories. Let j_P and j_Q denote the corresponding quotient functors with right adjoints k_P and k_Q, respectively. It holds that $P \subseteq F^{-1}(Q) = \mathrm{Ker}(j_Q \circ F)$. Therefore, there exists a unique triangulated functor $\widetilde{F}\colon \mathcal{T}/P \to \mathcal{U}/Q$ such that $\widetilde{F} \circ j_P \cong j_Q \circ F$. Moreover, $\widetilde{F}$ is monoidal and preserves coproducts. Hence, $\widetilde{F}$ has a right adjoint $\widetilde{G}$. Since $\widetilde{F} \circ j_P \cong j_Q \circ F \dashv G \circ k_Q$ and $\widetilde{F} \circ j_P \dashv k_P \circ \widetilde{G}$, we infer that $G \circ k_Q \cong k_P \circ \widetilde{G}$. Our discussion so far is recorded in the following diagram:

$$\begin{array}{ccc} \mathcal{T} & \underset{k_P}{\overset{j_P}{\rightleftarrows}} \ (\bot) & \mathcal{T}/P \\ F \downarrow \dashv \uparrow G & & \widetilde{F} \downarrow \dashv \uparrow \widetilde{G} \\ \mathcal{U} & \underset{k_Q}{\overset{j_Q}{\rightleftarrows}} \ (\bot) & \mathcal{U}/Q. \end{array}$$

Applying j_P to both sides of $G \circ k_Q \cong k_P \circ \widetilde{G}$, we obtain the relation $\widetilde{G} \cong j_P \circ G \circ k_Q$. As a result, $j_P(G(f_Q)) \cong j_P(G(k_Q(1))) \cong \widetilde{G}(1) \neq 0$. This reads $G(f_Q) \notin P$. In particular, $G(f_Q) \neq 0$. The next piece of information we need is that the morphism adjoint to the right idempotent $1 \to f_Q$, i.e., $1 \xrightarrow{\eta} G(1) \to G(f_Q)$, where η is the unit of adjunction, is a weak ring. Tensoring this composite with $G(f_Q)$ results in a split monic $G(f_Q) \to G(1) \otimes G(f_Q) \to G(f_Q) \otimes G(f_Q)$. It follows that $G(f_Q) \to G(1) \otimes G(f_Q)$ is split monic. So, $G(1) \otimes G(f_Q) \neq 0$ since it admits the non-zero object $G(f_Q)$ as a summand. Finally, suppose that $G(1) \in P$. Then $G(1) \in F^{-1}(Q)$. This implies that $FG(1) \otimes f_Q = 0$. By the projection formula,

$G(1) \otimes G(f_Q) = 0$, which leads to a contradiction. We conclude that $G(1) \notin P$, i.e., $P \in \mathrm{supp}^{\mathrm{s}}(G(1))$.

Now assume that $\operatorname{Ker} G = 0$. Claim: $\operatorname{Ker} \widetilde{G} = 0$. If $X \in \mathcal{U}$ and $\widetilde{G}(j_Q(X)) = 0$, then $G(f_Q \otimes X) = G(k_Q(j_Q(X))) = k_P(\widetilde{G}(j_Q(X))) = 0$. Therefore, $f_Q \otimes X = 0$, which means that $X \in Q$, so $j_Q(X) = 0$. This proves the claim, which implies that $\operatorname{Ker}(\widetilde{G}(1) \otimes -) = \operatorname{Ker} \widetilde{G}\widetilde{F} = \operatorname{Ker} \widetilde{F}$. Now let $\{P\} = U_{\mathcal{S}} \cap V_P$, so that $\Gamma_P = e_{\mathcal{S}} \otimes f_P$. Since $e_{\mathcal{S}} \notin P$ and $P = \mathrm{Spc}^{\mathrm{s}}(F)(Q)$, it follows that $F(e_{\mathcal{S}}) \notin Q$. Thus, $\widetilde{F}(j_P(e_{\mathcal{S}})) = j_Q(F(e_{\mathcal{S}})) \neq 0$. In other words, $j_P(e_{\mathcal{S}}) \notin \operatorname{Ker} \widetilde{F}$. As a result, $\Gamma_P \otimes G(f_Q) = \Gamma_P \otimes G(k_Q(1)) = k_P(j_P(e_{\mathcal{S}}) \otimes \widetilde{G}(1)) \neq 0$, with the second equality by using the relation $j_P(\Gamma_P) = j_P(e_{\mathcal{S}})$ and the projection formula for $j_P \dashv k_P$. Tensoring the split monic $G(f_Q) \to G(1) \otimes G(f_Q)$ with Γ_P, we conclude that $\Gamma_P \otimes G(1) \neq 0$, so $P \in \mathrm{Supp}^{\mathrm{s}}(G(1))$. □

Remark 2.6.5. The inclusion $\operatorname{Im} \mathrm{Spc}^{\mathrm{s}}(F) \subseteq \mathrm{supp}^{\mathrm{s}}(G(1))$ in Proposition 2.6.4 is not an equality in general. For instance, let $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ and $\mathcal{S} \in \mathrm{S}^{\otimes}(\mathcal{T}) \setminus \{\mathcal{T}\}$ such that $P \subsetneq \mathcal{S}$. Then $\mathrm{Supp}^{\mathrm{s}}(f_{\mathcal{S}}) = V_{\mathcal{S}} \neq \mathrm{supp}^{\mathrm{s}}(f_{\mathcal{S}})$, since the former does not contain P ($\Gamma_P \otimes f_{\mathcal{S}} = f_{\mathcal{S}} \otimes e_{\mathcal{S}} \otimes f_P = 0$) while the latter does ($f_{\mathcal{S}} \notin P$). Let $j_{\mathcal{S}} \colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ be the quotient functor and $k_{\mathcal{S}}$ its right adjoint. Then $\operatorname{Im} \mathrm{Spc}^{\mathrm{s}}(j_{\mathcal{S}}) = V_{\mathcal{S}} = \mathrm{Supp}^{\mathrm{s}}(f_{\mathcal{S}}) = \mathrm{Supp}^{\mathrm{s}}(k_{\mathcal{S}}(1)) \neq \mathrm{supp}^{\mathrm{s}}(k_{\mathcal{S}}(1)) = \mathrm{supp}^{\mathrm{s}}(f_{\mathcal{S}})$. A more concrete incarnation: Consider the derived category of a rank 1 non-noetherian valuation domain $(A, \mathfrak{m})$, e.g., the perfection of $\mathbb{F}_p[[x]]$, with field of fractions Q and let $P = 0$ and $\mathcal{S} = \mathrm{loc}(Q/\mathfrak{m})$. In this case, $\mathrm{Supp}^{\mathrm{s}}(f_{\mathcal{S}}) = \{\mathrm{loc}(\mathfrak{m}), \mathrm{D}_{\{\mathfrak{m}\}}(A)\}$ and $\mathrm{supp}^{\mathrm{s}}(f_{\mathcal{S}}) = \{0, \mathrm{loc}(\mathfrak{m}), \mathrm{D}_{\{\mathfrak{m}\}}(A)\} = \mathrm{Spc}^{\mathrm{s}}(\mathrm{D}(A))$; see [**BS23**, Section 7].

Stratification and descent. The results that follow are inspired by the article [**SW21**], in which appear descent theorems about tt-functors between R-linear big tt-categories within the context of stratification in the sense of [**BIK11a**] (with subsequent applications in the theory of *DG*-rings). Contrasted with our setup, there are two vital differences. First, the local-to-global principle is a property that holds automatically in their setting. Second, the categories involved have the same spectrum, namely $\mathrm{Spec}(R)$, by assumption. We step closer to the spirit of the alluded configuration by requiring the induced map on smashing spectra to be a homeomorphism.

Let $F \colon \mathcal{T} \to \mathcal{U}$ be a coproduct-preserving tt-functor between big tt-categories whose smashing spectra are T_D and assume that $f := \mathrm{Spc}^{\mathrm{s}}(F) \colon \mathrm{Spc}^{\mathrm{s}}(\mathcal{U}) \to \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is a homeomorphism. By Stone duality, the map $\mathrm{S}^{\otimes}(\mathcal{T}) \to \mathrm{S}^{\otimes}(\mathcal{U})$ that carries a smashing ideal $\mathcal{S}$ to $\mathrm{loc}^{\otimes}(F\mathcal{S})$ is a lattice isomorphism. It follows that $f^{-1}(P) = \mathrm{loc}^{\otimes}(FP) \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{U})$, $\forall P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. Therefore, if $\{P\} = U_{\mathcal{S}} \cap V_P$, then $\{f^{-1}(P)\} = U_{\mathrm{loc}^{\otimes}(F\mathcal{S})} \cap V_{\mathrm{loc}^{\otimes}(FP)}$. Hence, the Rickard idempotent corresponding to $\mathrm{loc}^{\otimes}(FP)$ is $F(e_{\mathcal{S}}) \otimes F(f_P) = F(\Gamma_P)$. Since every smashing-prime of $\mathcal{U}$ is realized as $\mathrm{loc}^{\otimes}(FP)$, for a unique $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, we see that the Rickard idempotents of $\mathcal{U}$ are precisely the images, under F, of the Rickard idempotents of $\mathcal{T}$.

Lemma 2.6.6. *Let $H \colon \mathcal{C}_1 \to \mathcal{C}_2$ be a coproduct-preserving triangulated functor (e.g., $H = X \otimes - \colon \mathcal{T} \to \mathcal{T}$ for a big tt-category $\mathcal{T}$, or H could be the right adjoint of a coproduct-preserving tt-functor between big tt-categories). Let A be an object of $\mathcal{C}_1$. Then, for all $B \in \mathrm{loc}(A)$, it holds that $H(B) \in \mathrm{loc}(H(A))$.*

Proof. Identical to the proof of Lemma 1.1.22; replace "localizing ideal" with "localizing subcategory". □

Lemma 2.6.7. *Let A be an object of $\mathcal{T}$ such that $\operatorname{loc}(A) = \mathcal{T}$. Then $\operatorname{loc}(A \otimes X)$ is a tensor-ideal and $\operatorname{loc}^{\otimes}(X) = \operatorname{loc}(A \otimes X)$, for all $X \in \mathcal{T}$.*

Proof. Let $Z \in \operatorname{loc}(A \otimes X)$ and $\mathcal{Y} = \{Y \in \mathcal{T} \mid Y \otimes Z \in \operatorname{loc}(A \otimes X)\}$. Then $\mathcal{Y}$ is a localizing subcategory of $\mathcal{T}$ and we claim that $A \in \mathcal{Y}$, i.e., $A \otimes Z \in \operatorname{loc}(A \otimes X)$. This is deduced by the following two series of implications, where we invoke Lemma 2.6.6 for the tensor product:

$$A \otimes A \in \operatorname{loc}(A) \Rightarrow A \otimes A \otimes X \in \operatorname{loc}(A \otimes X) \Rightarrow \operatorname{loc}(A \otimes A \otimes X) \subseteq \operatorname{loc}(A \otimes X),$$
$$Z \in \operatorname{loc}(A \otimes X) \Rightarrow A \otimes Z \in \operatorname{loc}(A \otimes A \otimes X) \Rightarrow A \otimes Z \in \operatorname{loc}(A \otimes X).$$

This shows that $\mathcal{Y} = \mathcal{T}$, proving the first part of the statement. For the second part, since $\operatorname{loc}(A \otimes X)$ is a tensor-ideal, $\operatorname{loc}(A \otimes X) = \operatorname{loc}^{\otimes}(A \otimes X) \subseteq \operatorname{loc}^{\otimes}(X)$. Finally, $1 \in \operatorname{loc}^{\otimes}(A)$ implies $X \in \operatorname{loc}^{\otimes}(A \otimes X)$. We infer that $\operatorname{loc}^{\otimes}(X) = \operatorname{loc}(A \otimes X)$. □

Remark 2.6.8. Lemma 2.6.7 can be generalized: If $\mathcal{A}$ is a set of objects of $\mathcal{T}$ such that $\operatorname{loc}(\mathcal{A}) = \mathcal{T}$, then $\operatorname{loc}(A \otimes X \mid A \in \mathcal{A}) \in \operatorname{Loc}^{\otimes}(\mathcal{T})$ and $\operatorname{loc}^{\otimes}(X) = \operatorname{loc}(A \otimes X \mid A \in \mathcal{A})$, for all $X \in \mathcal{T}$.

In the proof of the following theorem, Lemma 1.1.22 and Lemma 2.6.7 will be used without explicit reference.

Theorem 2.6.9. *Let $F\colon \mathcal{T} \to \mathcal{U}$ be a coproduct-preserving tt-functor between big tt-categories whose smashing spectra are T_D and let G be the right adjoint to F. Assume that $\operatorname{Spc}^{\mathrm{s}}(F)\colon \operatorname{Spc}^{\mathrm{s}}(\mathcal{U}) \to \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$ is a homeomorphism. Then:*

(a) *If $\mathcal{T}$ satisfies the local-to-global principle, then $\mathcal{U}$ satisfies the local-to-global principle.*
(b) *Suppose that there exists a collection of objects $\mathcal{X} \subseteq \mathcal{U}$ such that $\operatorname{loc}(\mathcal{X}) = \mathcal{U}$ and $\operatorname{loc}(G(\mathcal{X})) = \mathcal{T}$. Then: if $\mathcal{U}$ satisfies minimality, then $\mathcal{T}$ satisfies minimality.*
(c) *Suppose that $\operatorname{loc}(1_{\mathcal{U}}) = \mathcal{U}$ and $\operatorname{loc}(G(1_{\mathcal{U}})) = \mathcal{T}$. Then: if $\mathcal{U}$ satisfies the local-to-global principle, then $\mathcal{T}$ satisfies the local-to-global principle.*

Proof.

(a) If $\mathcal{T}$ satisfies the local-to-global principle, then $1_{\mathcal{T}} \in \operatorname{loc}^{\otimes}(\Gamma_P \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T}))$. Thus, $1_{\mathcal{U}} = F1_{\mathcal{T}} \in \operatorname{loc}^{\otimes}(F(\Gamma_P) \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})) = \operatorname{loc}^{\otimes}(\Gamma_Q \mid Q \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{U}))$ and the conclusion follows.
(b) Let $X \in \mathcal{T}$ and assume that $FX = 0$. Since $\operatorname{loc}(G(\mathcal{X})) = \mathcal{T}$, it holds that $\operatorname{loc}^{\otimes}(X) = \operatorname{loc}(G(\mathcal{X}) \otimes X) = \operatorname{loc}(G(\mathcal{X} \otimes FX)) = 0$. Thus, $X = 0$, proving that F is conservative. Now consider a non-zero object $X \in \operatorname{loc}^{\otimes}(\Gamma_P)$. Then the object $FX \in \operatorname{loc}^{\otimes}(F(\Gamma_P))$ must also be non-zero. Therefore, $\operatorname{loc}(\mathcal{X} \otimes FX) = \operatorname{loc}^{\otimes}(FX) = \operatorname{loc}^{\otimes}(F(\Gamma_P)) = \operatorname{loc}(\mathcal{X} \otimes F(\Gamma_P))$, with the second equality by minimality of $\mathcal{U}$. As a result, $\operatorname{loc}^{\otimes}(X) = \operatorname{loc}(G(\mathcal{X}) \otimes X)) = \operatorname{loc}(G(\mathcal{X} \otimes FX)) = \operatorname{loc}(G(\mathcal{X} \otimes F(\Gamma_P))) = \operatorname{loc}^{\otimes}(\Gamma_P)$. Consequently, $\operatorname{loc}^{\otimes}(\Gamma_P)$ is minimal.
(c) By assumption, $\mathcal{U}$ satisfies the local-to-global principle and $\operatorname{loc}(1_{\mathcal{U}}) = \mathcal{U}$. So, every localizing subcategory of $\mathcal{U}$ is an ideal and $1_{\mathcal{U}} \in \operatorname{loc}^{\otimes}(\Gamma_Q \mid Q \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{U}))$. So, $G(1_{\mathcal{U}}) \in \operatorname{loc}(GF(\Gamma_P) \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})) = \operatorname{loc}(G(1_{\mathcal{U}}) \otimes \Gamma_P \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})) = \operatorname{loc}^{\otimes}(\Gamma_P \mid P \in \operatorname{Spc}^{\mathrm{s}}(\mathcal{T}))$. Since $G(1_{\mathcal{U}})$ generates $\mathcal{T}$, the proof is complete. □

Corollary 2.6.10. *Let $F\colon \mathcal{T} \to \mathcal{U}$ be a coproduct-preserving tt-functor between big tt-categories whose smashing spectra are T_D and let G be the right adjoint to F. Assume that $\operatorname{Spc}^{\mathrm{s}}(F)\colon \operatorname{Spc}^{\mathrm{s}}(\mathcal{U}) \to \operatorname{Spc}^{\mathrm{s}}(\mathcal{T})$ is a homeomorphism. Provided that $\operatorname{loc}(1_{\mathcal{U}}) = \mathcal{U}$ and $\operatorname{loc}(G(1_{\mathcal{U}})) = \mathcal{T}$, if $\mathcal{U}$ is stratified by the small smashing support, then $\mathcal{T}$ is stratified by the small smashing support.*

CHAPTER 3

Costratification and actions of tensor-triangulated categories

The theme of this chapter is the theory of costratification in the context of relative tensor-triangular geometry. In Section 3.2, we show that costratification is equivalent to the colocal-to-global principle and cominimality and also that the local-to-global principle implies the colocal-to-global principle. In Section 3.3, we introduce and study the classes of prime localizing submodules and prime colocalizing hom-submodules, the former generalizing objectwise-prime localizing tensor-ideals and we relate these two classes of subcategories with the Action Formula and the Internal-Hom Formula. In Section 3.4, we reduce costratification to localizations with respect to smashing submodules and certain covers of the associated space of supports/cosupports. As an application, in Section 3.5, we prove that the derived category of quasi-coherent sheaves over a notherian separated scheme is costratified, generalizing Neeman's result for derived categories of commutative noetherian rings. The results of this chapter first appeared in [**Ver23a**].

3.1. Actions and basic lemmas

Throughout, $\mathcal{T}$ will denote a big tt-category. Let $\mathcal{K}$ be a compactly generated triangulated category and let $*\colon \mathcal{T}\times\mathcal{K}\to\mathcal{K}$ be an action of $\mathcal{T}$ on $\mathcal{K}$, in the sense of [**Ste13**]. In short, $*\colon \mathcal{T}\times\mathcal{K}\to\mathcal{K}$ is a coproduct-preserving triangulated functor in each variable such that there exist natural (in all variables) isomorphisms $\alpha_{X,Y,A}\colon X*(Y*A)\xrightarrow{\cong}(X\otimes Y)*A$ and $l_A\colon 1*A\xrightarrow{\cong}A$, $\forall X,Y\in\mathcal{T}$, $\forall A\in\mathcal{K}$. The natural isomorphism α is called the *associator* and the natural isomorphism l is called the *unitor*. There is also a host of coherence conditions that need to be satisfied; we refer the reader to the aforementioned source for details. We call $\mathcal{K}=(\mathcal{K},*)$ a $\mathcal{T}$-*module*.

By definition, for every object $X\in\mathcal{T}$, the functor $X*-\colon\mathcal{K}\to\mathcal{K}$ is a coproduct-preserving triangulated functor. Hence, by Brown representability, $X*-$ admits a right adjoint $[X,-]_*\colon\mathcal{K}\to\mathcal{K}$. Assembling these right adjoints yields a functor $[-,-]_*\colon\mathcal{T}^{\mathrm{op}}\times\mathcal{K}\to\mathcal{K}$ that we call the *relative internal-hom*. Since $[1,-]_*$ is the right adjoint of $1*-\cong\mathrm{Id}_{\mathcal{K}}$, it holds that $[1,-]_*\cong\mathrm{Id}_{\mathcal{K}}$. Specifically, the composite $m:=\mathrm{Id}_{\mathcal{K}}\to[1,1*-]_*\xrightarrow{[1,l]_*}[1,-]_*$, where the first map is the unit of adjunction, is a natural isomorphism (which we call the *hom-unitor*).

Hypothesis 3.1.1. We further assume that the relative internal-hom of $\mathcal{K}$ is a triangulated functor in the first variable, i.e., $[-,A]_*\colon\mathcal{T}^{\mathrm{op}}\to\mathcal{K}$ preserves triangles, for all $A\in\mathcal{K}$. This is true, e.g., if $\mathcal{K}$ satisfies a formulation of May's TC3 axiom ([**May01**]) replacing the tensor product of $\mathcal{T}$ with the action of $\mathcal{T}$ on $\mathcal{K}$.

The proof of [**Mur07**, Theorem C.1] goes through verbatim. Our assumption is satisfied by all known examples.

One could, of course, incorporate Hypothesis 3.1.1 in the definition of a $\mathcal{T}$-module. We decided to state it as an extra hypothesis because the abundance of examples satisfying it, i.e., all known examples, indicate that it is a property satisfied by every $\mathcal{T}$-module (even though a proof has not been discovered yet).

Lemma 3.1.2. *Let $X, Y \in \mathcal{T}$ and $A \in \mathcal{K}$. Then there exists a natural (in all variables) isomorphism $\beta_{X,Y,A} \colon [X \otimes Y, A]_* \xrightarrow{\cong} [X, [Y, A]_*]_*$ called the hom-associator.*

Proof. Let $B \in \mathcal{K}$. By the adjunction between the action and the relative internal-hom and the relation $(X \otimes Y) * B \cong (Y \otimes X) * B \cong Y * (X * B)$, we have:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}}(B, [X \otimes Y, A]_*) &\cong \mathrm{Hom}_{\mathcal{K}}((X \otimes Y) * B, A) \\ &\cong \mathrm{Hom}_{\mathcal{K}}((Y \otimes X) * B, A) \\ &\cong \mathrm{Hom}_{\mathcal{K}}(Y * (X * B), A) \\ &\cong \mathrm{Hom}_{\mathcal{K}}(X * B, [Y, A]_*) \\ &\cong \mathrm{Hom}_{\mathcal{K}}(B, [X, [Y, A]_*]_*). \end{aligned}$$

Consequently, for $B = [X \otimes Y, A]_*$, the image of the identity morphism on $[X \otimes Y, A]_*$ under the above series of isomorphisms gives a natural (in all variables) isomorphism $\beta_{X,Y,A} \colon [X \otimes Y, A]_* \xrightarrow{\cong} [X, [Y, A]_*]_*$. ∎

Notation 3.1.3. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two $\mathcal{T}$-modules. For $i = 1, 2$, the relative internal-hom of $\mathcal{K}_i$ will be denoted by $[-,-]_i$. Let $X, Y \in \mathcal{T}$ and $A \in \mathcal{K}_i$. The associator and unitor natural isomorphisms will be denoted by $\alpha^i_{X,Y,A}$ and l^i_A, respectively. The hom-associator and hom-unitor natural isomorphisms will be denoted by $\beta^i_{X,Y,A}$ and m^i_A, respectively. The unit and the counit of the action-hom adjunction will be denoted by $u^i_{X,A} \colon A \to [X, X *_i A]_i$ and $c^i_{X,A} \colon X *_i [X, A]_i \to A$, respectively. We denote by $\sigma_{X,Y} \colon X \otimes Y \xrightarrow{\cong} Y \otimes X$ the symmetry natural isomorphism. We denote by $c_{X,-} \colon X \otimes [X, -] \to \mathrm{Id}_{\mathcal{T}}$ the counit of the adjunction $X \otimes - \dashv [X, -]$. Set $X^\vee := [X, 1]$ and define the morphism $\mathrm{ev}^i_{X,A} \colon X *_i A \to [X^\vee, A]_i$ as the following composite:

$$X *_i A \xrightarrow{u^i_{X^\vee, X *_i A}} [X^\vee, X^\vee *_i (X *_i A)]_i \xrightarrow[\cong]{[X^\vee, \alpha^i_{X^\vee,X,A}]_i} [X^\vee, (X^\vee \otimes X) *_i A]_i$$

$$\xrightarrow[\cong]{[X^\vee, \sigma_{X^\vee,X} *_i A]_i} [X^\vee, (X \otimes X^\vee) *_i A]_i \xrightarrow{[X^\vee, c_{X,1} *_i A]_i} [X^\vee, 1 *_i A]_i \xrightarrow[\cong]{[X^\vee, l^i_A]_i} [X^\vee, A]_i.$$

Definition 3.1.4. A functor $F \colon \mathcal{K}_1 \to \mathcal{K}_2$, between $\mathcal{T}$-modules $\mathcal{K}_1$ and $\mathcal{K}_2$, is called *action-preserving* if there is a natural isomorphism $\phi \colon F(- *_1 -) \to - *_2 F(-)$ between the functors $F(- *_1 -), - *_2 F(-) \colon \mathcal{T} \times \mathcal{K}_1 \to \mathcal{K}_2$ such that, for all $X, Y \in \mathcal{T}$ and for all $A \in \mathcal{K}_1$, the following diagrams commute:

$$\begin{array}{ccccc} F(X *_1 (Y *_1 A)) & \xrightarrow{\phi_{X, Y *_1 A}} & X *_2 F(Y *_1 A) & \xrightarrow{X *_2 \phi_{Y,A}} & X *_2 (Y *_2 FA) \\ \downarrow{\scriptstyle F\alpha^1_{X,Y,A}} & & & & \downarrow{\scriptstyle \alpha^2_{X,Y,FA}} \\ F((X \otimes Y) *_1 A) & & \xrightarrow{\phi_{X \otimes Y, A}} & & (X \otimes Y) *_2 FA, \end{array}$$

$$\begin{array}{ccc} F(1 *_1 A) & \xrightarrow{\phi_{1,A}} & 1 *_2 FA \\ {\scriptstyle Fl^1_A}\downarrow & \swarrow {\scriptstyle l^2_{FA}} & \\ FA. & & \end{array}$$

Definition 3.1.5. A functor $G\colon \mathcal{K}_2 \to \mathcal{K}_1$, between $\mathcal{T}$-modules $\mathcal{K}_2$ and $\mathcal{K}_1$, is called hom-*preserving* if there is a natural isomorphism $\psi\colon [-, G(-)]_1 \to G[-,-]_2$ between the functors $[-, G(-)]_1$, $G[-,-]_2\colon \mathcal{T}^{\mathrm{op}} \times \mathcal{K}_2 \to \mathcal{K}_1$ such that, for all $X, Y \in \mathcal{T}$ and for all $B \in \mathcal{K}_2$, the following diagrams commute:

$$\begin{array}{ccccc} [X, [Y, GB]_1]_1 & \xrightarrow{[X, \psi_{Y,B}]_1} & [X, G[Y,B]_2]_1 & \xrightarrow{\psi_{X,[Y,B]_2}} & G[X, [Y,B]_2]_2 \\ {\scriptstyle \beta^1_{X,Y,GB}}\downarrow & & & & \downarrow{\scriptstyle G\beta^2_{X,Y,B}} \\ [X \otimes Y, GB]_1 & & \xrightarrow{\qquad\psi_{X \otimes Y, B}\qquad} & & G[X \otimes Y, B]_2, \end{array}$$

$$\begin{array}{ccc} GB & & \\ {\scriptstyle m^1_{GB}}\downarrow & \searrow {\scriptstyle Gm^2_B} & \\ [1, GB]_1 & \xrightarrow[\psi_{1,B}]{} & G[1,B]_2. \end{array}$$

Lemma 3.1.6. *Let $\mathcal{T}$ and $\mathcal{K}$ be triangulated categories with $\mathcal{T}$ compactly generated and let $F_1, F_2\colon \mathcal{T} \to \mathcal{K}$ be coproduct-preserving triangulated functors (or contravariant triangulated functors that send coproducts to products). If there is a natural transformation $\theta\colon F_1 \to F_2$ such that θ_x is an isomorphism, for all $x \in \mathcal{T}^{\mathrm{c}}$, then θ is a natural isomorphism.*

Proof. The subcategory $\mathcal{X} = \{X \in \mathcal{T} \mid \theta_X\colon F_1X \to F_2X \text{ is an isomorphism}\}$ is a localizing subcategory of $\mathcal{T}$ that contains $\mathcal{T}^{\mathrm{c}}$. Consequently, $\mathcal{X} = \mathcal{T}$ and this proves the statement. $\square$

Lemma 3.1.7. *Let $F\colon \mathcal{K}_1 \to \mathcal{K}_2$ be a coproduct and action-preserving triangulated functor between $\mathcal{T}$-modules and let $G\colon \mathcal{K}_2 \to \mathcal{K}_1$ be the right adjoint to F. Then G is* hom-*preserving. If G is coproduct-preserving, then G is action-preserving. If F is product-preserving, then F is* hom-*preserving.*

Proof. We denote by $\eta\colon \mathrm{Id}_{\mathcal{K}_1} \to GF$ and $\varepsilon\colon FG \to \mathrm{Id}_{\mathcal{K}_2}$ the unit and the counit, respectively, of the adjunction $F \dashv G$. Let $A \in \mathcal{K}_1$, $B \in \mathcal{K}_2$ and $X \in \mathcal{T}$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}_1}(A, [X, GB]_1) &\cong \mathrm{Hom}_{\mathcal{K}_1}(X * A, GB) \\ &\cong \mathrm{Hom}_{\mathcal{K}_2}(F(X * A), B) \\ &\cong \mathrm{Hom}_{\mathcal{K}_2}(X * FA, B) \\ &\cong \mathrm{Hom}_{\mathcal{K}_2}(FA, [X, B]_2) \\ &\cong \mathrm{Hom}_{\mathcal{K}_1}(A, G[X, B]_2). \end{aligned}$$

For $A = [X, GB]_1$, the image of the identity morphism on $[X, GB]_1$ under the above series of isomorphisms provides a natural (in both variables) isomorphism

$\psi_{X,B}\colon [X,GB]_1 \to G[X,B]_2$ that satisfies the conditions of Definition 3.1.5, showing that G is hom-preserving. More precisely, $\psi_{X,B}$ is the following composite:

$$[X,GB]_1 \xrightarrow{\eta_{[X,GB]_1}} GF[X,GB]_1 \xrightarrow{Gu^2_{X,F[X,GB]_1}} G[X, X *_2 F[X,GB]_1]_2$$

$$\xrightarrow{G[X,\phi^{-1}_{X,[X,GB]_1}]_2} G[X, F(X *_1 [X,GB]_1)]_2 \xrightarrow{G[X,F(c^1_{X,GB})]_2} G[X,FGB]_2$$

$$\xrightarrow{G[X,\varepsilon_B]_2} G[X,B]_2.$$

Now suppose that G preserves coproducts. We define a natural transformation $\xi_{X,B}\colon X *_1 GB \to G(X *_2 B)$ as the composite:

$$X *_1 GB \xrightarrow{\eta_{X *_1 GB}} GF(X *_1 GB) \xrightarrow{G\phi_{X,GB}} G(X *_2 FGB) \xrightarrow{G(X *_2 \varepsilon_B)} G(X *_2 B).$$

We claim that the square

$$\begin{array}{ccc}
X *_1 GB & \xrightarrow{\xi_{X,B}} & G(X *_2 B) \\
{\scriptstyle \mathrm{ev}^1_{X,GB}}\downarrow & & \downarrow{\scriptstyle G\mathrm{ev}^2_{X,B}} \\
[X^\vee, GB]_1 & \xrightarrow[\cong]{\psi_{X^\vee,B}} & G[X^\vee, B]_2
\end{array} \tag{3.1.8}$$

commutes. First, square (3.1.8) can be expanded as follows:

$$\begin{array}{ccccccc}
X *_1 GB & \xrightarrow{\eta_{X *_1 GB}} & GF(X *_1 GB) & \xrightarrow{G\phi_{X,GB}} & G(X *_2 FGB) & \xrightarrow{G(X *_2 \varepsilon_B)} & G(X *_2 B) \\
{\scriptstyle \mathrm{ev}^1_{X,GB}}\downarrow & (1) & \downarrow{\scriptstyle GF\mathrm{ev}^1_{X,GB}} & & \downarrow{\scriptstyle G\mathrm{ev}^2_{X,FGB}} & (3) & \downarrow{\scriptstyle G\mathrm{ev}^2_{X,B}} \\
[X^\vee, GB]_1 & \xrightarrow{\eta_{[X^\vee,GB]_1}} & GF[X^\vee, GB]_1 & (2) & G[X^\vee, FGB]_2 & \xrightarrow{G[X^\vee,\varepsilon_B]_2} & G[X^\vee, B]_2 \\
& & {\scriptstyle Gu^2_{X^\vee,F[X^\vee,GB]_1}}\downarrow & & \uparrow{\scriptstyle G[X^\vee,F(c^1_{X^\vee,GB})]_2} & & \\
& & G[X^\vee, X^\vee *_2 F[X^\vee,GB]_1]_2 & \xrightarrow[G[X^\vee,\phi^{-1}_{X^\vee,[X^\vee,GB]_1}]_2]{} & G[X^\vee, F(X^\vee *_1 [X^\vee,GB]_1)]_2 & &
\end{array}$$

where square (1) commutes by naturality of η and square (3) commutes by naturality of ev^2_X. Therefore, in order to show that (3.1.8) commutes, it suffices to show that diagram (2) commutes. We will prove this slightly more generally. We claim that the following diagram commutes:

$$\begin{array}{ccc}
F(X *_1 A) & \xrightarrow{\phi_{X,A}} & X *_2 FA \\
{\scriptstyle F\mathrm{ev}^1_{X,A}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}^2_{X,FA}} \\
F[X^\vee, A]_1 & & [X^\vee, FA]_2 \\
{\scriptstyle u^2_{X^\vee,F[X^\vee,A]_1}}\downarrow & & \uparrow{\scriptstyle [X^\vee,F(c^1_{X^\vee,A})]_2} \\
[X^\vee, X^\vee *_2 F[X^\vee,A]_1]_2 & \xrightarrow[{[X^\vee,\phi^{-1}_{X^\vee,[X^\vee,A]_1}]_2}]{} & [X^\vee, F(X^\vee *_1 [X^\vee, A]_1)]_2.
\end{array} \tag{3.1.9}$$

Set

$$
\begin{aligned}
f_1 &= u^2_{X^\vee, F[X^\vee, X^\vee *_1 (X *_1 A)]_1}, & g_1 &= [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, X^\vee *_1 (X *_1 A)]_1}]_2,\\
f_2 &= u^2_{X^\vee, F[X^\vee, (X^\vee \otimes X) *_1 A]_1}, & g_2 &= [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, (X^\vee \otimes X) *_1 A]_1}]_2,\\
f_3 &= u^2_{X^\vee, F[X^\vee, (X \otimes X^\vee) *_1 A]_1}, & g_3 &= [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, (X \otimes X^\vee) *_1 A]_1}]_2,\\
f_4 &= u^2_{X^\vee, F[X^\vee, 1 *_1 A]_1}, & g_4 &= [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, 1 *_1 A]_1}]_2,\\
f_5 &= u^2_{X^\vee, F[X^\vee, A]_1}, & g_5 &= [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, A]_1}]_2,
\end{aligned}
$$

$$
\begin{aligned}
h_1 &= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ [X^\vee, \phi_{X^\vee, X *_1 A}]_2 \circ [X^\vee, Fc^1_{X^\vee *_1 (X *_1 A)}]_2,\\
h_2 &= [X^\vee, \phi_{X^\vee \otimes X, A}]_2 \circ [X^\vee, Fc^1_{X^\vee, (X^\vee \otimes X) *_1 A}]_2,\\
h_3 &= [X^\vee, \phi_{X \otimes X^\vee, A}]_2 \circ [X^\vee, Fc^1_{X^\vee, (X \otimes X^\vee) *_1 A}]_2,\\
h_4 &= [X^\vee, \phi_{1,A}]_2 \circ [X^\vee, Fc^1_{X^\vee, 1 *_1 A}]_2,\\
h_5 &= [X^\vee, F(c^1_{X^\vee, A})]_2,
\end{aligned}
$$

and expand diagram (3.1.9) as below:

$$
\begin{array}{ccccccc}
F(X *_1 A) & \multicolumn{5}{c}{\xrightarrow[\cong]{\phi_{X,A}}} & X *_2 FA \\
\Big\downarrow{\scriptstyle Fu^1_{X^\vee, X *_1 A}} & & & & & & \Big\downarrow{\scriptstyle u^2_{X^\vee, X *_2 FA}} \\
F[X^\vee, X^\vee *_1 (X *_1 A)]_1 & \xrightarrow{f_1} & [X^\vee, X^\vee *_2 F[X^\vee, X^\vee *_1 (X *_1 A)]_1]_2 & \xrightarrow{g_1} & [X^\vee, F(X^\vee *_1 [X^\vee, X^\vee *_1 (X *_1 A)]_1)]_2 & \xrightarrow{h_1} & [X^\vee, X^\vee *_2 (X *_2 FA)]_2 \\
\Big\downarrow{\scriptstyle F[X^\vee, \alpha^1_{X^\vee, X, A}]_1} & (1) & \Big\downarrow{\scriptstyle [X^\vee, X^\vee *_2 F[X^\vee, \alpha^1_{X^\vee, X, A}]_1]_2} & (2) & \Big\downarrow{\scriptstyle [X^\vee, F(X^\vee *_1 [X^\vee, \alpha^1_{X^\vee, X, A}]_1)]_2} & (3) & \Big\downarrow{\scriptstyle [X^\vee, \alpha^2_{X^\vee, X, FA}]_2} \\
F[X^\vee, (X^\vee \otimes X) *_1 A]_1 & \xrightarrow{f_2} & [X^\vee, X^\vee *_2 F[X^\vee, (X^\vee \otimes X) *_1 A]_1]_2 & \xrightarrow{g_2} & [X^\vee, F(X^\vee *_1 [X^\vee, (X^\vee \otimes X) *_1 A]_1)]_2 & \xrightarrow{h_2} & [X^\vee, (X^\vee \otimes X) *_2 FA]_2 \\
\Big\downarrow{\scriptstyle F[X^\vee, \sigma_{X^\vee, X} *_1 A]_1} & (4) & \Big\downarrow{\scriptstyle [X^\vee, X^\vee *_2 F[X^\vee, \sigma_{X^\vee, X} *_1 A]_1]_2} & (5) & \Big\downarrow{\scriptstyle [X^\vee, F(X^\vee *_1 [X^\vee, \sigma_{X^\vee, X} *_1 A]_1)]_2} & (6) & \Big\downarrow{\scriptstyle [X^\vee, \sigma_{X^\vee, X} *_2 FA]_2} \\
F[X^\vee, (X \otimes X^\vee) *_1 A]_1 & \xrightarrow{f_3} & [X^\vee, X^\vee *_2 F[X^\vee, (X \otimes X^\vee) *_1 A]_1]_2 & \xrightarrow{g_3} & [X^\vee, F(X^\vee *_1 [X^\vee, (X \otimes X^\vee) *_1 A]_1)]_2 & \xrightarrow{h_3} & [X^\vee, (X \otimes X^\vee) *_2 FA]_2 \\
\Big\downarrow{\scriptstyle F[X^\vee, c_{X,1} *_1 A]_1} & (7) & \Big\downarrow{\scriptstyle [X^\vee, X^\vee *_2 F[X^\vee, c_{X,1} *_1 A]_1]_2} & (8) & \Big\downarrow{\scriptstyle [X^\vee, F(X^\vee *_1 [X^\vee, c_{X,1} *_1 A]_1)]_2} & (9) & \Big\downarrow{\scriptstyle [X^\vee, c_{X,1} *_2 FA]_2} \\
F[X^\vee, 1 *_1 A]_1 & \xrightarrow{f_4} & [X^\vee, X^\vee *_2 F[X^\vee, 1 *_1 A]_1]_2 & \xrightarrow{g_4} & [X^\vee, F(X^\vee *_1 [X^\vee, 1 *_1 A]_1)]_2 & \xrightarrow{h_4} & [X^\vee, 1 *_2 FA]_2 \\
\Big\downarrow{\scriptstyle F[X^\vee, l^1_A]_1} & (10) & \Big\downarrow{\scriptstyle [X^\vee, X^\vee *_2 F[X^\vee, l^1_A]_1]_2} & (11) & \Big\downarrow{\scriptstyle [X^\vee, F(X^\vee *_1 [X^\vee, l^1_A]_1)]_2} & (12) & \Big\downarrow{\scriptstyle [X^\vee, l^2_{FA}]_2} \\
F[X^\vee, A]_1 & \xrightarrow{f_5} & [X^\vee, X^\vee *_2 F[X^\vee, A]_1]_2 & \xrightarrow{g_5} & [X^\vee, F(X^\vee *_1 [X^\vee, A]_1)]_2 & \xrightarrow{h_5} & [X^\vee, FA]_2
\end{array}
$$

It is clear that the squares $(1), (2), \ldots, (12)$ commute. Thus, it remains to show that $u^2_{X^\vee, X*_2 FA} \circ \phi_{X,A} = h_1 \circ g_1 \circ f_1 \circ F u^1_{X^\vee, X*_1 A}$. Indeed,

$$\begin{aligned} h_1 \circ g_1 \circ f_1 \circ F u^1_{X^\vee, X*_1 A} &= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ [X^\vee, \phi_{X^\vee, X*_1 A}]_2 \\ &\circ [X^\vee, F c^1_{X^\vee *_1 (X*_1 A)}]_2 \circ [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, X^\vee *_1 (X*_1 A)]_1}]_2 \\ &\circ u^2_{X^\vee, F[X^\vee, X^\vee *_1 (X*_1 A)]_1} \circ F u^1_{X^\vee, X*_1 A} \\ &= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ [X^\vee, \phi_{X^\vee, X*_1 A}]_2 \\ &\circ [X^\vee, F c^1_{X^\vee *_1 (X*_1 A)}]_2 \circ [X^\vee, \phi^{-1}_{X^\vee, [X^\vee, X^\vee *_1 (X*_1 A)]_1}]_2 \\ &\circ [X^\vee, X^\vee *_2 F u^1_{X^\vee, X*_1 A}]_2 \circ u^2_{X^\vee, F(X*_1 A)} \\ &= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ [X^\vee, \phi_{X^\vee, X*_1 A}]_2 \\ &\circ [X^\vee, F c^1_{X^\vee *_1 (X*_1 A)}]_2 \circ [X^\vee, F(X^\vee *_1 u^1_{X^\vee, X*_1 A})]_2 \\ &\circ [X^\vee, \phi^{-1}_{X^\vee, X*_1 A}]_2 \circ u^2_{X^\vee, F(X*_1 A)} \\ &= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ [X^\vee, \phi_{X^\vee, X*_1 A}]_2 \\ &\circ [X^\vee, \phi^{-1}_{X^\vee, X*_1 A}]_2 \circ u^2_{X^\vee, F(X*_1 A)} \\ &= [X^\vee, X^\vee *_2 \phi_{X,A}]_2 \circ u^2_{X^\vee, F(X*_1 A)} \\ &= u^2_{X^\vee, X*_2 FA} \circ \phi_{X,A}. \end{aligned}$$

We conclude that diagram (3.1.9) commutes and, as a result, square (3.1.8) commutes. If $X \in \mathcal{T}^c$, then by [**Ste13**, Lemma 4.6], $\mathrm{ev}^i_{X,-}$ is an isomorphism. Consequently, the restriction of $\xi_{-,B}\colon -*_1 GB \to G(- *_2 B)$ to the compact objects of $\mathcal{T}$ is a natural isomorphism. Since the triangulated functors $- *_1 GB$ and $G(- *_2 B)$ are coproduct-preserving, it follows by Lemma 3.1.6 that $\xi_{-,B}$ is a natural isomorphism. It is easy to verify that the conditions of Definition 3.1.4 are satisfied. We conclude that G is action-preserving.

The proof that if F preserves products, then F is hom-preserving is similar and left to the interested reader. □

Let $\mathcal{K}$ be a $\mathcal{T}$-module. A subcategory $\mathcal{L} \subseteq \mathcal{K}$ is called a *localizing submodule* if $\mathcal{L}$ is a localizing subcategory such that $X * A \in \mathcal{L}, \forall X \in \mathcal{T}, \forall A \in \mathcal{L}$. The collection of localizing submodules of $\mathcal{K}$ is denoted by $\mathrm{Loc}^*(\mathcal{K})$. A subcategory $\mathcal{C} \subseteq \mathcal{K}$ is called a *colocalizing* hom-*submodule* if $\mathcal{C}$ is a colocalizing subcategory such that $[X, A]_* \in \mathcal{C}, \forall X \in \mathcal{T}, \forall A \in \mathcal{C}$. The collection of colocalizing hom-submodules of $\mathcal{K}$ is denoted by $\mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K})$. Let A be an object of $\mathcal{K}$. The localizing (resp. colocalizing) submodule of $\mathcal{K}$ generated (resp. cogenerated) by A, i.e., the smallest localizing (resp. colocalizing) submodule of $\mathcal{K}$ that contains A, is denoted by $\mathrm{loc}^*(A)$ (resp. $\mathrm{coloc}^{\mathrm{hom}}(A)$). Specializing to the case $\mathcal{K} = \mathcal{T}$ and $* = \otimes$, we obtain the notions of localizing tensor-ideal and colocalizing left hom-ideal.

We define the *annihilator* of $\mathcal{K}$ in $\mathcal{T}$ as the subcategory

$$\mathrm{Ann}_{\mathcal{T}}(\mathcal{K}) \coloneqq \{X \in \mathcal{T} \mid X * A = 0, \forall A \in \mathcal{K}\} \subseteq \mathcal{T},$$

which is equal to $\bigcap_{A \in \mathcal{K}} \mathrm{Ker}(- * A)$ and hence a localizing ideal of $\mathcal{T}$. If $\mathrm{Ann}_{\mathcal{T}}(\mathcal{K}) = 0$ (for instance, when $\mathcal{K} = \mathcal{T}$ and $* = \otimes$) then $\mathcal{K}$ is called a *conservative* $\mathcal{T}$-module.

Lemma 3.1.10. *Let $\mathcal{K}_1, \mathcal{K}_2$ be two $\mathcal{T}$-modules and let $\mathcal{A} \subseteq \mathrm{Ob}(\mathcal{K}_1)$ and $\mathcal{B} \subseteq \mathrm{Ob}(\mathcal{K}_2)$.*

(a) *If* $F\colon \mathcal{K}_1 \to \mathcal{K}_2$ *is a coproduct and action-preserving triangulated functor, then* $F(\mathrm{loc}^*(\mathcal{A})) \subseteq \mathrm{loc}^*(F\mathcal{A})$.
(b) *If* $F\colon \mathcal{K}_1^{\mathrm{op}} \to \mathcal{K}_2$ *is a triangulated functor that sends coproducts to products and* $F(X * A) \cong [X, FA]_*$, $\forall X \in \mathcal{T}$, $\forall A \in \mathcal{K}_1$, *then* $F(\mathrm{loc}^*(\mathcal{A})) \subseteq \mathrm{coloc}^{\mathrm{hom}}(F\mathcal{A})$.
(c) *If* $G\colon \mathcal{K}_2 \to \mathcal{K}_1$ *is a product and* hom*-preserving triangulated functor, then* $G(\mathrm{coloc}^{\mathrm{hom}}(\mathcal{B})) \subseteq \mathrm{coloc}^{\mathrm{hom}}(G\mathcal{B})$.

PROOF. We will prove (a). The subcategory $\mathcal{X} = \{A \in \mathcal{K}_1 \mid FA \in \mathrm{loc}^*(F\mathcal{A})\}$ is a localizing submodule of $\mathcal{K}_1$ that contains $\mathcal{A}$. Therefore, $\mathcal{X}$ contains $\mathrm{loc}^*(\mathcal{A})$, proving the statement. The proofs of (b) and (c) are similar. □

Lemma 3.1.11. *Let* $\mathcal{K}$ *be a* $\mathcal{T}$*-module and assume that* $\mathcal{T} = \mathrm{loc}(1)$*. Then every colocalizing subcategory of* $\mathcal{K}$ *is a* hom*-submodule.*

PROOF. Let $\mathcal{C}$ be a colocalizing subcategory of $\mathcal{K}$ and let $A \in \mathcal{C}$. The collection $\mathcal{X} = \{X \in \mathcal{T} \mid [X, A]_* \in \mathcal{C}\}$ is a localizing subcategory of $\mathcal{T}$ that contains 1. It follows that $\mathcal{X} = \mathcal{T}$, i.e., $[X, A]_* \in \mathcal{C}$, $\forall X \in \mathcal{T}$. Hence, $\mathcal{C}$ is a hom-submodule of $\mathcal{K}$. □

3.2. Stratification–costratification

Fix a big tt-category $\mathcal{T}$ and a $\mathcal{T}$-module $\mathcal{K}$. We always assume that $\mathcal{K}$ is compactly generated. Let $I_{\mathcal{T}}$ and $I_{\mathcal{K}}$ be the cogenerators of $\mathcal{T}$ and $\mathcal{K}$, respectively, as in Recollection 1.1.13.

Support–cosupport. Fix a topological space S.

Definition 3.2.1 (See also [**BHS23b**, Definition 7.1])**.** A *support data for* $\mathcal{T}$ *with values in* S is a map $\mathrm{s}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ that satisfies the following properties:
(a) $\mathrm{s}(0) = \varnothing \quad \& \quad \mathrm{s}(1) = S$.
(b) $\mathrm{s}(\coprod X_i) = \bigcup \mathrm{s}(X_i)$.
(c) $\mathrm{s}(\Sigma X) = \mathrm{s}(X)$.
(d) $\mathrm{s}(Y) \subseteq \mathrm{s}(X) \cup \mathrm{s}(Z)$, for any triangle $X \to Y \to Z \to \Sigma X$.
(e) $\mathrm{s}(X \otimes Y) \subseteq \mathrm{s}(X) \cap \mathrm{s}(Y)$.

Definition 3.2.2. A *cosupport data for* $\mathcal{T}$ *with values in* S is a map $\mathrm{c}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ that satisfies the following properties:
(a) $\mathrm{c}(0) = \varnothing \quad \& \quad \mathrm{c}(I_{\mathcal{T}}) = S$.
(b) $\mathrm{c}(\prod X_i) = \bigcup \mathrm{c}(X_i)$.
(c) $\mathrm{c}(\Sigma X) = \mathrm{c}(X)$.
(d) $\mathrm{c}(Y) \subseteq \mathrm{c}(X) \cup \mathrm{c}(Z)$, for any triangle $X \to Y \to Z \to \Sigma X$.
(e) $\mathrm{c}([X, Y]) \subseteq \mathrm{c}([X, I_{\mathcal{T}}]) \cap \mathrm{c}(Y)$.

Remark 3.2.3. Let $\mathrm{c}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ be a cosupport data. It is a straightforward verification, using the properties of $[-, I_{\mathcal{T}}]$, that setting $\mathrm{s}(X) = \mathrm{c}([X, I_{\mathcal{T}}])$ gives rise to a support data $\mathrm{s}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$.

Definition 3.2.4. Let $\mathrm{c}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ be a cosupport data. The support data $\mathrm{s}\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S)$ defined by $\mathrm{s}(X) = \mathrm{c}([X, I_{\mathcal{T}}])$ is called *the support data induced by* c. We say that (s, c) is a *support–cosupport pair.*

Lemma 3.2.5. *Let* $\Gamma\colon S \to \mathrm{Ob}(\mathcal{T})$ *be a map such that* $\Gamma_s \coloneqq \Gamma(s) \neq 0$, $\forall s \in S$*. Then the maps*

$$\mathrm{s}_\Gamma\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S),\ \mathrm{s}_\Gamma(X) = \{s \in S \mid \Gamma_s \otimes X \neq 0\}$$
$$\mathrm{c}_\Gamma\colon \mathrm{Ob}(\mathcal{T}) \to \mathcal{P}(S),\ \mathrm{c}_\Gamma(X) = \{s \in S \mid [\Gamma_s, X] \neq 0\}$$

are a support and cosupport data, respectively. Moreover, s_Γ *is induced by* c_Γ*.*

PROOF. That s_Γ and c_Γ are a support and cosupport data, respectively, follows from the fact that $\Gamma_s \otimes -$ is a coproduct-preserving triangulated functor and $[\Gamma_s, -]$ is a product-preserving triangulated functor. Let $X \in \mathcal{T}$. The claim that s_Γ is induced by c_Γ follows from the isomorphism $[\Gamma_s, [X, I_\mathcal{T}]] \cong [\Gamma_s \otimes X, I_\mathcal{T}]$ and the fact that $I_\mathcal{T}$ is a cogenerator of $\mathcal{T}$. □

Definition 3.2.6. A support–cosupport pair $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ is called *good* if it is induced by a map $\Gamma\colon S \to \mathrm{Ob}(\mathcal{T})$ such that $\Gamma_s \otimes \Gamma_r = 0$, $\forall s \neq r$ and $\Gamma_s \otimes \Gamma_s \cong \Gamma_s \neq 0$, $\forall s \in S$.

Remark 3.2.7. An important feature of a good support–cosupport pair $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ is that $\mathrm{s}_\Gamma(\Gamma_s) = \mathrm{c}_\Gamma([\Gamma_s, I_\mathcal{T}]) = \{s\}$, $\forall s \in S$.

Example 3.2.8. The following are good support–cosupport pairs on $\mathcal{T}$:

(a) Assuming that the frame of smashing ideals of $\mathcal{T}$ is a spatial frame, the big smashing support–cosupport pair $(\mathrm{supp}^{\mathrm{s}}, \mathrm{cosupp}^{\mathrm{s}})$:

$$\begin{aligned} \mathrm{supp}^{\mathrm{s}}(X) &= \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid f_P \otimes X \neq 0\}, \\ \mathrm{cosupp}^{\mathrm{s}}(X) &= \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid [f_P, X] \neq 0\}. \end{aligned}$$

Since $\mathrm{Ker}(f_P \otimes -) = P$, it holds that $P \in \mathrm{supp}^{\mathrm{s}}(X)$ if and only if $X \notin P$. Using the equality $P^\perp = \mathrm{Im}(f_P \otimes -)$, one can deduce that $\mathrm{Ker}[f_P, -] = (P^\perp)^\perp$, so $P \in \mathrm{cosupp}^{\mathrm{s}}(X)$ if and only if $X \notin (P^\perp)^\perp$.

(b) Assuming further that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D, the small smashing support–cosupport pair $(\mathrm{Supp}^{\mathrm{s}}, \mathrm{Cosupp}^{\mathrm{s}})$:

$$\begin{aligned} \mathrm{Supp}^{\mathrm{s}}(X) &= \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid \Gamma_P \otimes X \neq 0\}, \\ \mathrm{Cosupp}^{\mathrm{s}}(X) &= \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid [\Gamma_P, X] \neq 0\}. \end{aligned}$$

(c) Assuming that every point of $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ is visible, the Balmer–Favi support–cosupport pair $(\mathrm{Supp}, \mathrm{Cosupp})$:

$$\begin{aligned} \mathrm{Supp}(X) &= \{\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) \mid g_\mathfrak{p} \otimes X \neq 0\}, \\ \mathrm{Cosupp}(X) &= \{\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) \mid [g_\mathfrak{p}, X] \neq 0\}. \end{aligned}$$

(d) If R is a graded commutative noetherian ring and $\mathcal{T}$ is R-linear, the BIK support–cosupport pair $(\mathrm{supp}_R, \mathrm{cosupp}_R)$:

$$\begin{aligned} \mathrm{supp}_R(X) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \Gamma_\mathfrak{p} 1 \otimes X \neq 0\}, \\ \mathrm{cosupp}_R(X) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid [\Gamma_\mathfrak{p} 1, X] \neq 0\}. \end{aligned}$$

See [**BIK11a, BIK12**].

Remark 3.2.9. Suppose that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D and let $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ with associated idempotent $\Gamma_P = e_\mathcal{S} \otimes f_P$. If X is an object of $\mathcal{T}$ such that $P \in \mathrm{Cosupp}^{\mathrm{s}}(X)$, then $0 \neq [\Gamma_P, X] = [e_\mathcal{S} \otimes f_P, X] \cong [e_\mathcal{S}, [f_P, X]]$. Hence, $[f_P, X] \neq 0$. In other words, $P \in \mathrm{cosupp}^{\mathrm{s}}(X)$. This shows that $\mathrm{Cosupp}^{\mathrm{s}}(X) \subseteq \mathrm{cosupp}^{\mathrm{s}}(X)$, $\forall X \in \mathcal{T}$.

Proposition 3.2.10. *Assuming that* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ *is* T_D*, the small and big smashing cosupports coincide, i.e.,* $\mathrm{Cosupp}^{\mathrm{s}}(X) = \mathrm{cosupp}^{\mathrm{s}}(X)$, $\forall X \in \mathcal{T}$*, if and only if every point of* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ *is closed, i.e.,* $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ *is* T_1*.*

PROOF. Since $\mathrm{Supp}^{\mathrm{s}}(-) = \mathrm{Cosupp}^{\mathrm{s}}([-, I_{\mathcal{T}}])$ and $\mathrm{supp}^{\mathrm{s}}(-) = \mathrm{cosupp}^{\mathrm{s}}([-, I_{\mathcal{T}}])$, if the small and big smashing cosupports coincide, then so do the small and big smashing supports. By Proposition 2.2.28, it follows that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_1. Conversely, if $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_1 and $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, then $V_P = \{P\}$. This implies that $\Gamma_P = f_P$. So, $[\Gamma_P, X] = 0$ if and only if $[f_P, X] = 0$, for all $X \in \mathcal{T}$. Hence, $\mathrm{Cosupp}^{\mathrm{s}} = \mathrm{cosupp}^{\mathrm{s}}$. □

Remark 3.2.11. Assume that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D and consider the small smashing support–cosupport. Then $\mathrm{Cosupp}^{\mathrm{s}}(1) = \{P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid [\Gamma_P, 1] \neq 0\}$. There are many cases where $\mathrm{Cosupp}^{\mathrm{s}}(1) \neq \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. For instance, $\mathrm{Cosupp}^{\mathrm{s}}(\mathbb{Z}_p) = \{(p)\} \neq \mathrm{Spc}^{\mathrm{s}}(\mathrm{D}(\mathbb{Z}_p)) \cong \mathrm{Spec}(\mathbb{Z}_p) = \{(0), (p)\}$. For more examples and results concerning the cosupport in derived categories of commutative noetherian rings, see [**Tho18**].

Let $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ be a support–cosupport pair on $\mathcal{T}$ induced by a map $\Gamma \colon S \to \mathrm{Ob}(\mathcal{T})$ and define the maps

$$\mathrm{s}_\Gamma^* \colon \mathrm{Ob}(\mathcal{K}) \to \mathcal{P}(S),\ \mathrm{s}_\Gamma^*(A) = \{s \in S \mid \ \Gamma_s * A \ \neq 0\},$$
$$\mathrm{c}_\Gamma^* \colon \mathrm{Ob}(\mathcal{K}) \to \mathcal{P}(S),\ \mathrm{c}_\Gamma^*(A) = \{s \in S \mid [\Gamma_s, A]_* \neq 0\}.$$

Lemma 3.2.12. *The maps s_Γ^* and c_Γ^* satisfy the following properties:*

(a) $\mathrm{s}_\Gamma^*(0) = \varnothing$.
(b) $\mathrm{s}_\Gamma^*(\coprod A_i) = \bigcup \mathrm{s}_\Gamma^*(A_i)$.
(c) $\mathrm{s}_\Gamma^*(\Sigma A) = \mathrm{s}_\Gamma^*(A)$.
(d) $\mathrm{s}_\Gamma^*(B) \subseteq \mathrm{s}_\Gamma^*(A) \cup \mathrm{s}_\Gamma^*(C)$, *for any triangle $A \to B \to C \to \Sigma A$ of $\mathcal{K}$.*
(e) $\mathrm{s}_\Gamma^*(X * A) \subseteq \mathrm{s}_\Gamma(X) \cap \mathrm{s}_\Gamma^*(A)$.

(f) $\mathrm{c}_\Gamma^*(0) = \varnothing$.
(g) $\mathrm{c}_\Gamma^*(\prod A_i) = \bigcup \mathrm{c}_\Gamma^*(A_i)$.
(h) $\mathrm{c}_\Gamma^*(\Sigma A) = \mathrm{c}_\Gamma^*(A)$.
(i) $\mathrm{c}_\Gamma^*(B) \subseteq \mathrm{c}_\Gamma^*(A) \cup \mathrm{c}_\Gamma^*(C)$, *for any triangle $A \to B \to C \to \Sigma A$ of $\mathcal{K}$.*
(j) $\mathrm{c}_\Gamma^*([X, A]_*) \subseteq \mathrm{c}_\Gamma([X, I_{\mathcal{T}}]) \cap \mathrm{c}_\Gamma^*(A)$.

PROOF. The argument is essentially the same as the one given in Lemma 3.2.5. The property $\mathrm{c}_\Gamma^*([X, A]_*) \subseteq \mathrm{c}_\Gamma([X, I_{\mathcal{T}}]) \cap \mathrm{c}_\Gamma^*(A)$ follows from Lemma 3.1.2. □

The (co)local-to-global principle and (co)minimality. We define two pairs of inclusion-preserving maps

$$\mathcal{P}(S) \underset{\sigma_{\mathrm{s}_\Gamma^*}}{\overset{\tau_{\mathrm{s}_\Gamma^*}}{\rightleftarrows}} \mathrm{Loc}^*(\mathcal{K}) \qquad \& \qquad \mathcal{P}(S) \underset{\sigma_{\mathrm{c}_\Gamma^*}}{\overset{\tau_{\mathrm{c}_\Gamma^*}}{\rightleftarrows}} \mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K})$$

by the formulas

$$\tau_{\mathrm{s}_\Gamma^*}(W) = \{A \in \mathcal{K} \mid \mathrm{s}_\Gamma^*(A) \subseteq W\} \quad \& \quad \sigma_{\mathrm{s}_\Gamma^*}(\mathcal{L}) = \bigcup_{A \in \mathcal{L}} \mathrm{s}_\Gamma^*(A),$$
$$\tau_{\mathrm{c}_\Gamma^*}(W) = \{A \in \mathcal{K} \mid \mathrm{c}_\Gamma^*(A) \subseteq W\} \quad \& \quad \sigma_{\mathrm{c}_\Gamma^*}(\mathcal{C}) = \bigcup_{A \in \mathcal{C}} \mathrm{c}_\Gamma^*(A).$$

It is clear from the properties of s_Γ^* and c_Γ^* that the maps $\tau_{\mathrm{s}_\Gamma^*}, \sigma_{\mathrm{s}_\Gamma^*}, \tau_{\mathrm{c}_\Gamma^*}, \sigma_{\mathrm{c}_\Gamma^*}$ are well-defined. Moreover, $\mathrm{Im}\,\sigma_{\mathrm{s}_\Gamma^*} \subseteq \mathcal{P}(\sigma_{\mathrm{s}_\Gamma^*}(\mathcal{K}))$ and $\mathrm{Im}\,\sigma_{\mathrm{c}_\Gamma^*} \subseteq \mathcal{P}(\sigma_{\mathrm{c}_\Gamma^*}(\mathcal{K}))$. In fact, $\sigma_{\mathrm{s}_\Gamma^*}(\mathcal{K}) = \sigma_{\mathrm{c}_\Gamma^*}(\mathcal{K}) = \mathrm{c}_\Gamma^*(I_{\mathcal{K}})$. The first equality follows from the adjunction $\Gamma_s * - \dashv [\Gamma_s, -]_*$. The second equality is a special case of Lemma 3.2.18 using the fact that $\mathcal{K} = \mathrm{coloc}^{\mathrm{hom}}(I_{\mathcal{K}})$. If $\mathcal{K}$ is a conservative $\mathcal{T}$-module, then $\Gamma_s * - \neq 0, \forall s \in S$. Hence, in this case, $\sigma_{\mathrm{s}_\Gamma^*}(\mathcal{K}) = S$. For $\mathcal{K} = \mathcal{T}$ and $* = \otimes$, we obtain the maps $\tau_{\mathrm{s}_\Gamma}, \sigma_{\mathrm{s}_\Gamma}, \tau_{\mathrm{c}_\Gamma}, \sigma_{\mathrm{c}_\Gamma}$.

Definition 3.2.13.
(a) $\mathcal{K}$ is *stratified* by Γ if $\tau_{\mathrm{s}_\Gamma^*}$ and $\sigma_{\mathrm{s}_\Gamma^*}$, between $\mathcal{P}(\mathrm{c}_\Gamma^*(I_\mathcal{K}))$ and $\mathrm{Loc}^*(\mathcal{K})$, are mutually inverse bijections.
(b) $\mathcal{K}$ is *costratified* by Γ if $\tau_{\mathrm{c}_\Gamma^*}$ and $\sigma_{\mathrm{c}_\Gamma^*}$, between $\mathcal{P}(\mathrm{c}_\Gamma^*(I_\mathcal{K}))$ and $\mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K})$, are mutually inverse bijections.

Remark 3.2.14. Since we will always work with a fixed support–cosupport pair induced by a map $\Gamma\colon S \to \mathrm{Ob}(\mathcal{T})$, we will omit the reference to Γ in Definition 3.2.13 and say "$\mathcal{K}$ is stratified" and "$\mathcal{K}$ is costratified", respectively. We will mention explicit support–cosupport pairs where appropriate.

Definition 3.2.15 (For (a), see [**Ste13**, Definition 6.1])**.**
(a) $\mathcal{K}$ satisfies the *local-to-global principle* if
$$\mathrm{loc}^*(A) = \mathrm{loc}^*(\Gamma_s * A \mid s \in S),\ \forall A \in \mathcal{K}.$$
(b) $\mathcal{K}$ satisfies *minimality* if, for all $s \in S$, $\mathrm{loc}^*(\Gamma_s * A \mid A \in \mathcal{K}\}$ is minimal in $\mathrm{Loc}^*(\mathcal{K})$ in the sense that it does not contain any non-zero proper localizing submodule of $\mathcal{K}$.
(i) $\mathcal{K}$ satisfies the *colocal-to-global principle* if
$$\mathrm{coloc}^{\mathrm{hom}}(A) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid s \in S),\ \forall A \in \mathcal{K}.$$
(ii) $\mathcal{K}$ satisfies *cominimality* if, for all $s \in S$, $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_*)$ is minimal in $\mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K})$ in the sense that it does not contain any non-zero proper colocalizing hom-submodule of $\mathcal{K}$.

Remark 3.2.16. Let $X \in \mathcal{T}$. Since $\mathcal{K} = \mathrm{coloc}(I_\mathcal{K}) = \mathrm{coloc}^{\mathrm{hom}}(I_\mathcal{K})$, Lemma 3.1.10 for the functor $[X, -]_*$ implies that $\mathrm{coloc}^{\mathrm{hom}}([X, I_\mathcal{K}]_*) = \mathrm{coloc}^{\mathrm{hom}}([X, A]_* \mid A \in \mathcal{K})$. In particular, $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_*) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid A \in \mathcal{K}),\ \forall s \in S$.

Remark 3.2.17. It is clear from the definition of s_Γ^* and c_Γ^* that
$$\mathrm{loc}^*(\Gamma_s * A \mid s \in S) = \mathrm{loc}^*(\Gamma_s * A \mid s \in \mathrm{s}_\Gamma^*(A)),$$
$$\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid s \in S) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid s \in \mathrm{c}_\Gamma^*(A)).$$
In addition, if $\mathcal{K}$ satisfies the local-to-global (resp. colocal-to-global) principle, then s_Γ^* (resp. c_Γ^*) detects vanishing, i.e., $\mathrm{s}_\Gamma^*(A) = \varnothing \Rightarrow A = 0$ and similarly for c_Γ^*. For the case $\mathcal{K} = \mathcal{T}$ and $* = \otimes$, it holds that codetection implies detection, since $\varnothing = \mathrm{s}_\Gamma(X) = \mathrm{c}_\Gamma([X, I_\mathcal{T}])$ implies $[X, I_\mathcal{T}] = 0$, so $X = 0$.

For the rest of the section, fix a good support–cosupport pair $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ on $\mathcal{T}$.

Lemma 3.2.18. *Let $\mathcal{A}$ be a collection of objects of $\mathcal{K}$. Then we have the following equalities of subsets of S:*
(a) $\sigma_{\mathrm{s}_\Gamma^*}(\mathrm{loc}^*(\mathcal{A})) = \bigcup_{A \in \mathcal{A}} \mathrm{s}_\Gamma^*(A)$.
(b) $\sigma_{\mathrm{c}_\Gamma^*}(\mathrm{coloc}^{\mathrm{hom}}(\mathcal{A})) = \bigcup_{A \in \mathcal{A}} \mathrm{c}_\Gamma^*(A)$.

PROOF. We will prove the case of $\sigma_{\mathrm{c}_\Gamma^*}$. Let s be an element of S. Then
$$\begin{aligned} s \notin \bigcup_{A \in \mathcal{A}} \mathrm{c}_\Gamma^*(A) &\Leftrightarrow \mathcal{A} \subseteq \mathrm{Ker}[\Gamma_s, -]_* \\ &\Leftrightarrow \mathrm{coloc}^{\mathrm{hom}}(\mathcal{A}) \subseteq \mathrm{Ker}[\Gamma_s, -]_* \\ &\Leftrightarrow s \notin \bigcup_{A \in \mathrm{coloc}^{\mathrm{hom}}(\mathcal{A})} \mathrm{c}_\Gamma^*(A) = \sigma_{\mathrm{c}_\Gamma^*}(\mathrm{coloc}^{\mathrm{hom}}(\mathcal{A})). \end{aligned}$$
$\square$

Remark 3.2.19. Let $A \in \mathcal{K}$ and $s \in S$. It holds that $\mathrm{c}_\Gamma^*([\Gamma_s, A]_*) \subseteq \{s\}$. Hence, if $[\Gamma_s, A]_* \neq 0$ (i.e., $s \in \mathrm{c}_\Gamma^*(A)$) then $\mathrm{c}_\Gamma^*([\Gamma_s, A]_*) = \{s\}$. In particular, if $s \in \mathrm{c}_\Gamma^*(I_\mathcal{K})$, then $\mathrm{c}_\Gamma^*([\Gamma_s, I_\mathcal{K}]_*) = \{s\}$.

Lemma 3.2.20. *It holds that* $\sigma_{\mathrm{s}_\Gamma^*} \circ \tau_{\mathrm{s}_\Gamma^*} = \mathrm{Id}$ *and* $\sigma_{\mathrm{c}_\Gamma^*} \circ \tau_{\mathrm{c}_\Gamma^*} = \mathrm{Id}$, *where both composites are restricted to* $\mathcal{P}(\mathrm{c}_\Gamma^*(I_\mathcal{K}))$. *In particular, the respective restrictions of* $\tau_{\mathrm{s}_\Gamma^*}$ *and* $\tau_{\mathrm{c}_\Gamma^*}$ *are injective, while* $\sigma_{\mathrm{s}_\Gamma^*}$ *and* $\sigma_{\mathrm{c}_\Gamma^*}$ *are surjective.*

Proof. We will prove that $\sigma_{\mathrm{c}_\Gamma^*} \circ \tau_{\mathrm{c}_\Gamma^*} = \mathrm{Id}$ (restricted to $\mathcal{P}(\mathrm{c}_\Gamma^*(I_\mathcal{K}))$). To this end, let W be a subset of $\mathrm{c}_\Gamma^*(I_\mathcal{K})$. Since $(\sigma_{\mathrm{c}_\Gamma^*} \circ \tau_{\mathrm{c}_\Gamma^*})(W) = \bigcup_{\mathrm{c}_\Gamma^*(A) \subseteq W} \mathrm{c}_\Gamma^*(A)$, we have $(\sigma_{\mathrm{c}_\Gamma^*} \circ \tau_{\mathrm{c}_\Gamma^*})(W) \subseteq W$. Let s be an element of W. Then $s \in \mathrm{c}_\Gamma^*([\Gamma_s, I_\mathcal{K}]_*) = \{s\} \subseteq W$. Therefore, $s \in (\sigma_{\mathrm{c}_\Gamma^*} \circ \tau_{\mathrm{c}_\Gamma^*})(W)$, completing the proof. □

Theorem 3.2.21. *Let* $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ *be a good support-cosupport pair on* $\mathcal{T}$.

(a) $\mathcal{K}$ *is stratified with respect to* $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ *if and only if* $\mathcal{K}$ *satisfies the local-to-global principle and minimality.*
(b) $\mathcal{K}$ *is costratified with respect to* $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ *if and only if* $\mathcal{K}$ *satisfies the colocal-to-global principle and cominimality.*

Proof. We will only prove (b), since (a) is proved analogously. Suppose that $\mathcal{K}$ is costratified. Then $\sigma_{\mathrm{c}_\Gamma^*}$ is injective. Let A be an object of $\mathcal{K}$. Then

$$\begin{aligned}
\sigma_{\mathrm{c}_\Gamma^*}(\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid s \in \mathrm{c}_\Gamma^*(A))) &= \bigcup_{s \in \mathrm{c}_\Gamma^*(A)} \mathrm{c}_\Gamma^*([\Gamma_s, A]_*) \\
&= \bigcup_{s \in \mathrm{c}_\Gamma^*(A)} \{s\} \\
&= \mathrm{c}_\Gamma^*(A) \\
&= \sigma_{\mathrm{c}_\Gamma^*}(\mathrm{coloc}^{\mathrm{hom}}(A)),
\end{aligned}$$

where the first and last equalities are due to Lemma 3.2.18. Since $\sigma_{\mathrm{c}_\Gamma^*}$ is injective, it follows that $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid s \in \mathrm{c}_\Gamma^*(A)) = \mathrm{coloc}^{\mathrm{hom}}(A)$. Thus, $\mathcal{K}$ satisfies the colocal-to-global principle. In particular, c_Γ^* detects vanishing.

Let s be an element of $\mathrm{c}_\Gamma^*(I_\mathcal{K})$ and A a non-zero object in $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_*)$. Then $\varnothing \neq \mathrm{c}_\Gamma^*(A) \subseteq \mathrm{c}_\Gamma^*([\Gamma_s, I_\mathcal{K}]_*) = \{s\}$. Therefore, $\mathrm{c}_\Gamma^*(A) = \mathrm{c}_\Gamma^*([\Gamma_s, I_\mathcal{K}]_*)$. Since $\sigma_{\mathrm{c}_\Gamma^*}$ is injective, $\mathrm{coloc}^{\mathrm{hom}}(A) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_*)$. Hence, $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_*)$ is minimal.

Suppose that $\mathcal{K}$ satisfies the colocal-to-global principle and cominimality. Let $\mathcal{C} \in \mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K})$. Clearly, $\mathcal{C} \subseteq (\tau_{\mathrm{c}_\Gamma^*} \circ \sigma_{\mathrm{c}_\Gamma^*})(\mathcal{C})$. Let $A \in (\tau_{\mathrm{c}_\Gamma^*} \circ \sigma_{\mathrm{c}_\Gamma^*})(\mathcal{C})$, i.e., $\mathrm{c}_\Gamma^*(A) \subseteq \sigma_{\mathrm{c}_\Gamma^*}(\mathcal{C})$. Then

$$\begin{aligned}
\mathrm{coloc}^{\mathrm{hom}}(A) &= \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid s \in \mathrm{c}_\Gamma^*(A)) \\
&\subseteq \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_* \mid s \in \mathrm{c}_\Gamma^*(A)) \\
&\subseteq \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_* \mid s \in \sigma_{\mathrm{c}_\Gamma^*}(\mathcal{C})) \\
&\subseteq \mathcal{C}.
\end{aligned}$$

The first equality is due to the colocal-to-global principle. The first containment relation follows from Remark 3.2.16, while the second containment relation is clear. For the third containment, if $s \in \sigma_{\mathrm{c}_\Gamma^*}(\mathcal{C})$, then there exists an object $B \in \mathcal{C}$ such that $[\Gamma_s, B]_* \neq 0$. Since $[\Gamma_s, B]_* \in \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_*)$ and the latter is minimal, it follows that $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_*) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, B]_*) \subseteq \mathcal{C}$. We infer that $A \in \mathcal{C}$, proving that $(\tau_{c^*_\Gamma} \circ \sigma_{c^*_\Gamma})(\mathcal{C}) = \mathcal{C}$. So, $\sigma_{c^*_\Gamma}$ is injective and thus, $\sigma_{c^*_\Gamma}$ is bijective. This shows that $\mathcal{K}$ is costratified. □

Remark 3.2.22. Theorem 3.2.21 (b) could be stated slightly more generally, replacing a good support–cosupport pair $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$, in the sense of Definition 3.2.6, with one that satisfies the property stated in Remark 3.2.19, i.e., if A is an object of $\mathcal{K}$ such that $[\Gamma_s, A]_* \neq 0$, then $\mathrm{c}^*_\Gamma([\Gamma_s, A]_*) = \{s\}$. Similarly, the analogous property for Theorem 3.2.21 (a) is: if A is an object of $\mathcal{K}$ such that $\Gamma_s * A \neq 0$, then $\mathrm{s}^*_\Gamma(\Gamma_s * A) = \{s\}$. This observation will be useful in Section 3.5, where we consider the support–cosupport for objects of the derived category of a commutative noetherian ring defined by the residue fields.

Local-to-global implies colocal-to-global. Let $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ be a (not necessarily good) support–cosupport pair on $\mathcal{T}$.

Lemma 3.2.23. *Suppose that $\mathcal{T} = \mathrm{loc}^\otimes(G)$. Then the following hold:*
(a) $\mathrm{loc}^*(A) = \mathrm{loc}^*(G * A)$, $\forall A \in \mathcal{K}$.
(b) $\mathrm{coloc}^{\mathrm{hom}}(A) = \mathrm{coloc}^{\mathrm{hom}}([G, A]_*)$, $\forall A \in \mathcal{K}$.

Proof. We will prove (b). The inclusion $\mathrm{coloc}^{\mathrm{hom}}([G, A]_*) \subseteq \mathrm{coloc}^{\mathrm{hom}}(A)$ is clear. Since $\mathcal{T} = \mathrm{loc}^\otimes(G)$, it holds that $1 \in \mathrm{loc}^\otimes(G)$. By Lemma 3.1.10 for the functor $[-, A]_*$, it follows that $A \cong [1, A]_* \in \mathrm{coloc}^{\mathrm{hom}}([G, A]_*)$. Therefore, $\mathrm{coloc}^{\mathrm{hom}}(A) \subseteq \mathrm{coloc}^{\mathrm{hom}}([G, A]_*)$. The proof of (a) is analogous. □

Remark 3.2.24. An easy generalization of Lemma 3.2.23 is the following: If $\mathcal{T} = \mathrm{loc}^\otimes(\mathcal{G})$, for a collection of objects $\mathcal{G}$, then $\forall A \in \mathcal{K}$: $\mathrm{loc}^*(A) = \mathrm{loc}^*(G * A \mid G \in \mathcal{G})$ and $\mathrm{coloc}^{\mathrm{hom}}(A) = \mathrm{coloc}^{\mathrm{hom}}([G, A]_* \mid G \in \mathcal{G})$.

Proposition 3.2.25 (See also [**Ste13**, Proposition 6.8]). *If $\mathcal{T}$ satisfies the local-to-global principle, then $\mathcal{K}$ satisfies the local-to-global principle and the colocal-to-global principle.*

Proof. Since $\mathcal{T}$ satisfies the local-to-global principle, we have the equality $\mathcal{T} = \mathrm{loc}^\otimes(\Gamma_s \mid s \in S)$. Hence, by Remark 3.2.24, $\mathrm{loc}^*(A) = \mathrm{loc}^*(\Gamma_s * A \mid s \in S)$ and $\mathrm{coloc}^{\mathrm{hom}}(A) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid s \in S)$, for all $A \in \mathcal{K}$. This proves the statement. □

Corollary 3.2.26. *For the case $\mathcal{K} = \mathcal{T}$ we have: if $\mathcal{T}$ satisfies the local-to-global principle, then $\mathcal{T}$ satisfies the colocal-to-global principle.*

Example 3.2.27. Let R be a graded commutative noetherian ring such that $\mathcal{T}$ is R-linear and consider the BIK support–cosupport $(\mathrm{supp}_R, \mathrm{cosupp}_R)$, which takes values in $\mathrm{supp}_R(1) \subseteq \mathrm{Spec}(R)$ — this may not be an equality. As explained in [**BHS23b**, Corollary 7.11], if $\mathcal{T}$ is stratified in the sense of BIK, then $\mathrm{supp}_R(1)$ is homeomorphic to $\mathrm{Spc}(\mathcal{T}^c)$ and the BIK support is identified with the Balmer–Favi support under this homeomorphism. It then follows that $\mathcal{T}$ is stratified by the Balmer–Favi support. Now since the tensor-idempotents $\Gamma_\mathfrak{p} 1$ (defining the BIK support) and the tensor-idempotents $g_\mathfrak{p}$ (defining the Balmer–Favi support) have the same support (which is $\{\mathfrak{p}\}$) it follows that $\mathrm{loc}^\otimes(\Gamma_\mathfrak{p} 1) = \mathrm{loc}^\otimes(g_\mathfrak{p})$. Applying Lemma 3.1.10 for the functor $[-, I_\mathcal{T}]$, it follows that $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_\mathfrak{p} 1, I_\mathcal{T}]) = \mathrm{coloc}^{\mathrm{hom}}([g_\mathfrak{p}, I_\mathcal{T}])$. By Corollary 3.2.26, $\mathcal{T}$ satisfies the colocal-to-global principle with respect to the Balmer–Favi support. Taking into account Theorem 3.2.21, we conclude that if $\mathcal{T}$ is BIK-stratified, then: $\mathcal{T}$ is Balmer–Favi-costratified if and only if $\mathcal{T}$ is BIK-costratified if and only if $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_{\mathfrak{p}}1, I_{\mathcal{T}}])$ is minimal, for all $\mathfrak{p} \in \mathrm{supp}_R(1)$. If $\mathcal{T} = \underline{\mathrm{Mod}}(kG)$ is the stable module category of the group algebra of a finite group G, then $\mathcal{T}$ is BIK-costratified by the canonical action of $H^*(G,k)$; see [**BIK12**, Theorem 11.13]. We infer that $\underline{\mathrm{Mod}}(kG)$ is Balmer–Favi-costratified.

3.3. Prime submodules

In this section we introduce the classes of prime localizing submodules and hom-prime colocalizing submodules of a given $\mathcal{T}$-module $\mathcal{K}$. The class of prime localizing submodules generalizes the class of objectwise-prime localizing tensor-ideals in the context of relative tensor-triangular geometry, while the class of hom-prime colocalizing submodules specializes to the class of hom-prime colocalizing left hom-ideals if $\mathcal{K} = \mathcal{T}$.

Prime localizing and colocalizing submodules. As before, $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ will be a good support-cosupport pair on $\mathcal{T}$ with values in a space S. Given $\mathcal{L} \in \mathrm{Loc}^*(\mathcal{K})$ and $\mathcal{C} \in \mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K})$, we define two subcategories of $\mathcal{T}$ as follows:

$$\mathcal{L}^{\otimes \mathrm{L}} = \{X \in \mathcal{T} \mid X * \mathcal{K} \subseteq \mathcal{L}\},$$
$$\mathcal{C}^{\otimes \mathrm{C}} = \{X \in \mathcal{T} \mid [X, \mathcal{K}]_* \subseteq \mathcal{C}\},$$

where $X * \mathcal{K} \coloneqq \mathrm{loc}^*(X * A \mid A \in \mathcal{K})$ and $[X, \mathcal{K}]_* \coloneqq \mathrm{coloc}^{\mathrm{hom}}([X, A]_* \mid A \in \mathcal{K})$, with the latter also being equal to $\mathrm{coloc}^{\mathrm{hom}}([X, I_{\mathcal{K}}]_*)$. Evidently, if $\mathcal{L}_1 \subseteq \mathcal{L}_2$, then $\mathcal{L}_1^{\otimes \mathrm{L}} \subseteq \mathcal{L}_2^{\otimes \mathrm{L}}$ and if $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\mathcal{C}_1^{\otimes \mathrm{C}} \subseteq \mathcal{C}_2^{\otimes \mathrm{C}}$.

Remark 3.3.1. Clearly, $\mathcal{L}^{\otimes \mathrm{L}}$ and $\mathcal{C}^{\otimes \mathrm{C}}$ are localizing tensor-ideals of $\mathcal{T}$.

Remark 3.3.2. If $\mathcal{K} = \mathcal{T}$ and $* = \otimes$, then $\mathcal{L}^{\otimes \mathrm{L}} = \mathcal{L}$. The inclusion $\mathcal{L}^{\otimes \mathrm{L}} \subseteq \mathcal{L}$ follows from the equality $X \otimes \mathcal{T} = \mathrm{loc}^{\otimes}(X)$, while the inclusion $\mathcal{L} \subseteq \mathcal{L}^{\otimes \mathrm{L}}$ holds because $\mathcal{L}$ is a tensor-ideal.

Definition 3.3.3.
(a) A proper localizing submodule $\mathcal{L} \subseteq \mathcal{K}$ is called *prime* if $X * A \in \mathcal{L}$ implies $X \in \mathcal{L}^{\otimes \mathrm{L}}$ or $A \in \mathcal{L}$.
(b) A proper colocalizing hom-submodule $\mathcal{C} \subseteq \mathcal{K}$ is called hom-*prime* if $[X, A]_* \in \mathcal{C}$ implies $X \in \mathcal{C}^{\otimes \mathrm{C}}$ or $A \in \mathcal{C}$.

Remark 3.3.4. If $\mathcal{K} = \mathcal{T}$ and $* = \otimes$, then the notion of prime localizing submodule recovers the notion of objectwise-prime localizing tensor-ideal; see Remark 3.3.2. The notion of hom-prime colocalizing hom-submodule provides the notion of hom-prime colocalizing left hom-ideal.

Lemma 3.3.5. *Let $\mathcal{L}$ be a prime localizing submodule of $\mathcal{K}$ and let $\mathcal{C}$ be a* hom-*prime colocalizing submodule of $\mathcal{K}$. Then $\mathcal{L}^{\otimes \mathrm{L}}$ and $\mathcal{C}^{\otimes \mathrm{C}}$ are objectwise-prime localizing ideals.*

PROOF. We will prove that $\mathcal{C}^{\otimes \mathrm{C}}$ is objectwise-prime. The proof for $\mathcal{L}^{\otimes \mathrm{L}}$ is analogous. Let $X, Y \in \mathcal{T}$ such that $X \otimes Y \in \mathcal{C}^{\otimes \mathrm{C}}$. Then $[X \otimes Y, A]_* \in \mathcal{C}$, $\forall A \in \mathcal{K}$. By Lemma 3.1.2, $[X, [Y, A]_*]_* \cong [X \otimes Y, A]_*$. Since $\mathcal{C}$ is hom-prime, $X \in \mathcal{C}^{\otimes \mathrm{C}}$ or $[Y, A]_* \in \mathcal{C}$. If $X \notin \mathcal{C}^{\otimes \mathrm{C}}$, then $[Y, A]_* \in \mathcal{C}$, $\forall A \in \mathcal{K}$, i.e., $Y \in \mathcal{C}^{\otimes \mathrm{C}}$. This proves that $\mathcal{C}^{\otimes \mathrm{C}}$ is objectwise-prime. □

The main result of this section, i.e., Theorem 3.3.10, is a consequence of the following series of lemmas.

Lemma 3.3.6. *The following statements hold:*

(a) $\operatorname{Ker}(\Gamma_s \otimes -) \subseteq \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}} = \operatorname{Ker}([\Gamma_s, -]_*)^{\otimes \mathrm{C}}$, $\forall s \in S$.
(b) *If* $\mathcal{K}$ *is conservative, then* $\operatorname{Ker}(\Gamma_s \otimes -) = \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$, $\forall s \in S$.
(c) *If* $\operatorname{Ker}(\Gamma_s \otimes -) = \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$, $\forall s \in S$ *and* s_Γ *detects vanishing, then* $\mathcal{K}$ *is conservative.*

Proof. Let X be an object of $\mathcal{T}$. Then we have $X \in \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$ if and only if $\Gamma_s * (X * A) \cong (\Gamma_s \otimes X) * A = 0$, $\forall A \in \mathcal{K}$, which is equivalent to $(\Gamma_s \otimes X) * - = 0$. Similarly, using the isomorphism $[\Gamma_s \otimes X, -]_* \cong [\Gamma_s, [X, -]_*]_*$, one deduces that $X \in \operatorname{Ker}([\Gamma_s, -]_*)^{\otimes \mathrm{C}}$ if and only if $[\Gamma_s \otimes X, -]_* = 0$. Since $(\Gamma_s \otimes X) * - \dashv [\Gamma_s \otimes X, -]_*$, these two functors are either both the zero functor on $\mathcal{K}$ or none of them is the zero functor. Therefore, $\operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}} = \operatorname{Ker}([\Gamma_s, -]_*)^{\otimes \mathrm{C}}$. Since $\operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}} = \{X \in \mathcal{T} \mid (\Gamma_s \otimes X) * - = 0\}$, it immediately follows that $\operatorname{Ker}(\Gamma_s \otimes -) \subseteq \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$. This proves (a).

If $\mathcal{K}$ is conservative and $(\Gamma_s \otimes X) * - = 0$, then $\Gamma_s \otimes X = 0$. Hence, $\operatorname{Ker}(\Gamma_s \otimes -) = \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$. This proves (b).

Let $X \in \mathcal{T}$ such that $X * - = 0$. Then $(\Gamma_s \otimes X) * - \cong X * (\Gamma_s * -) = 0$, $\forall s \in S$. Therefore, $X \in \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$. This implies that $X \in \operatorname{Ker}(\Gamma_s \otimes -)$, $\forall s \in S$, i.e., $\Gamma_s \otimes X = 0$, $\forall s \in S$. Equivalently, $\mathrm{s}_\Gamma(X) = \varnothing$. Since s_Γ detects vanishing, it follows that $X = 0$. This proves (c). □

Lemma 3.3.7.

(a) *Let* $\mathcal{L}$ *be a prime localizing submodule of* $\mathcal{K}$. *There is at most one* $s \in \mathrm{c}^*_\Gamma(I_\mathcal{K})$ *such that* $\mathcal{L} \subseteq \operatorname{Ker}(\Gamma_s * -)$.
(b) *Let* $\mathcal{C}$ *be a* hom-*prime colocalizing submodule of* $\mathcal{K}$. *There is at most one* $s \in \mathrm{c}^*_\Gamma(I_\mathcal{K})$ *such that* $\mathcal{C} \subseteq \operatorname{Ker}[\Gamma_s, -]_*$.

Proof.

(a) Similar to (b).
(b) Let $s \in \mathrm{c}^*_\Gamma(I_\mathcal{K})$ and suppose that $\mathcal{C} \subseteq \operatorname{Ker}[\Gamma_s, -]_*$. Let $r \in S$ such that $r \neq s$ and let $A \in \mathcal{K}$. Then $[\Gamma_s, [\Gamma_r, A]_*]_* = 0 \in \mathcal{C}$. Since $\mathcal{C}$ is hom-prime, $\Gamma_s \in \mathcal{C}^{\otimes \mathrm{C}} \subseteq \operatorname{Ker}[\Gamma_s, -]_*^{\otimes \mathrm{C}} = \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$ or $[\Gamma_r, A]_* \in \mathcal{C} \subseteq \operatorname{Ker}[\Gamma_s, -]_*$. For the equality $\operatorname{Ker}[\Gamma_s, -]_*^{\otimes \mathrm{C}} = \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$, see Lemma 3.3.6. The former of the two does not hold since $\Gamma_s \in \operatorname{Ker}(\Gamma_s * -)^{\otimes \mathrm{L}}$ if and only if $\Gamma_s * - = 0$, but $s \in \mathrm{c}^*_\Gamma(I_\mathcal{K})$ which means that $\Gamma_s * - \neq 0$. It follows that $\mathcal{C}$ contains all objects $[\Gamma_r, A]_*$, for $r \neq s$ and $A \in \mathcal{K}$. So, if $\mathcal{C} \subseteq \operatorname{Ker}[\Gamma_r, -]_*$, for $r \neq s$ and $r \in \mathrm{c}^*_\Gamma(I_\mathcal{K})$, then $[\Gamma_r, A]_* \in \operatorname{Ker}[\Gamma_r, -]_*$, $\forall A \in \mathcal{K}$. It follows that $[\Gamma_r, -]_* = 0$; thus, $\Gamma_r * - = 0$, which is false since $r \in \mathrm{c}^*_\Gamma(I_\mathcal{K})$. □

Lemma 3.3.8. *If* $\mathcal{K}$ *satisfies the colocal-to-global principle, then the following holds:* $\operatorname{Ker}[\Gamma_s, -]_* = \operatorname{coloc}^{\mathrm{hom}}([\Gamma_r, I_\mathcal{K}]_* \mid r \neq s)$, $\forall s \in S$. *Analogously, if* $\mathcal{K}$ *satisfies the local-to-global principle, then* $\operatorname{Ker}(\Gamma_s * -) = \operatorname{loc}^*(\Gamma_r * A \mid r \neq s,\ A \in \mathcal{K})$, $\forall s \in S$.

Proof. Let $r, s \in S$ such that $r \neq s$. Then $[\Gamma_r, I_\mathcal{K}]_* \in \operatorname{Ker}[\Gamma_s, -]_*$. Therefore, $\operatorname{coloc}^{\mathrm{hom}}([\Gamma_r, I_\mathcal{K}]_* \mid r \neq s) \subseteq \operatorname{Ker}[\Gamma_s, -]_*$. Let $A \in \operatorname{Ker}[\Gamma_s, -]_*$. Then $s \notin \mathrm{c}^*_\Gamma(A)$. Since $\mathcal{K}$ satisfies the colocal-to-global principle,

$$
\begin{aligned}
\operatorname{coloc}^{\mathrm{hom}}(A) &= \operatorname{coloc}^{\mathrm{hom}}([\Gamma_r, A]_* \mid r \in \mathrm{c}^*_\Gamma(A)) \\
&= \operatorname{coloc}^{\mathrm{hom}}([\Gamma_r, A]_* \mid r \neq s) \\
&\subseteq \operatorname{coloc}^{\mathrm{hom}}([\Gamma_r, I_\mathcal{K}]_* \mid r \neq s).
\end{aligned}
$$

See Remark 3.2.17 for the second equality and Remark 3.2.16 for the containment relation. Hence, $A \in \mathrm{coloc}^{\mathrm{hom}}([\Gamma_r, I_{\mathcal{K}}]_* \mid r \neq s)$, completing the proof. The case of $\mathrm{Ker}(\Gamma_s * -)$ is similar and left to the reader. $\square$

Lemma 3.3.9. *Let* $\mathcal{C} \in \mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K})$. *Then*

$$\tau_{\mathrm{c}_\Gamma^*}(\sigma_{\mathrm{c}_\Gamma^*}(\mathcal{C})) = \bigcap_{\substack{\mathcal{C} \subseteq \mathrm{Ker}[\Gamma_s, -]_* \\ s \in \mathrm{c}_\Gamma^*(I_{\mathcal{K}})}} \mathrm{Ker}[\Gamma_s, -]_*.$$

If $\mathcal{K}$ *is costratified, then* $\mathcal{C} = \bigcap_{\substack{\mathcal{C} \subseteq \mathrm{Ker}[\Gamma_s, -]_* \\ s \in \mathrm{c}_\Gamma^*(I_{\mathcal{K}})}} \mathrm{Ker}[\Gamma_s, -]_*$.

Analogously, if $\mathcal{L} \in \mathrm{Loc}^*(\mathcal{K})$, *then*

$$\tau_{\mathrm{s}_\Gamma^*}(\sigma_{\mathrm{s}_\Gamma^*}(\mathcal{L})) = \bigcap_{\substack{\mathcal{L} \subseteq \mathrm{Ker}(\Gamma_s * -) \\ s \in \mathrm{c}_\Gamma^*(I_{\mathcal{K}})}} \mathrm{Ker}(\Gamma_s * -).$$

If $\mathcal{K}$ *is stratified, then* $\mathcal{L} = \bigcap_{\substack{\mathcal{L} \subseteq \mathrm{Ker}(\Gamma_s * -) \\ s \in \mathrm{c}_\Gamma^*(I_{\mathcal{K}})}} \mathrm{Ker}(\Gamma_s * -)$.

PROOF. Let $A \in \mathcal{K}$. Then $A \notin \bigcap_{\mathcal{C} \subseteq \mathrm{Ker}[\Gamma_s, -]_*} \mathrm{Ker}[\Gamma_s, -]_*$ if and only if there exists $s \in S$ such that $\mathcal{C} \subseteq \mathrm{Ker}[\Gamma_s, -]_*$ and $[\Gamma_s, A]_* \neq 0$. Equivalently, $s \notin \sigma_{\mathrm{c}_\Gamma^*}(\mathcal{C})$ and $s \in \mathrm{c}_\Gamma^*(A)$. In other words, $\mathrm{c}_\Gamma^*(A) \nsubseteq \sigma_{\mathrm{c}_\Gamma^*}(\mathcal{C})$. Since $\tau_{\mathrm{c}_\Gamma^*}(\sigma_{\mathrm{c}_\Gamma^*}(\mathcal{C}))$ consists precisely of those $A \in \mathcal{K}$ such that $\mathrm{c}_\Gamma^*(A) \subseteq \sigma_{\mathrm{c}_\Gamma^*}(\mathcal{C})$, it follows that $\tau_{\mathrm{c}_\Gamma^*}(\sigma_{\mathrm{c}_\Gamma^*}(\mathcal{C})) = \bigcap_{\mathcal{C} \subseteq \mathrm{Ker}[\Gamma_s, -]_*} \mathrm{Ker}[\Gamma_s, -]_*$. Finally, if $\mathcal{K}$ is costratified, then $\mathcal{C} = \tau_{\mathrm{c}_\Gamma^*}(\sigma_{\mathrm{c}_\Gamma^*}(\mathcal{C}))$, which proves the statement (the indexing set of the intersection involved in the claimed equalities can be considered to consist of points $s \in \mathrm{c}_\Gamma^*(I_{\mathcal{K}})$ since if $s \notin \mathrm{c}_\Gamma^*(I_{\mathcal{K}})$, then $[\Gamma_s, -]_* = 0$ and so $\mathrm{Ker}[\Gamma_s, -]_* = \mathcal{K}$ so the intersection is not affected). The rest is similar and left to the reader. $\square$

Theorem 3.3.10. *Let* $\mathcal{K}$ *be a costratified* $\mathcal{T}$*-module. Then there is a bijective correspondence between* hom*-prime colocalizing submodules of* $\mathcal{K}$ *and points of* $\mathrm{c}_\Gamma^*(I_{\mathcal{K}})$. *A point* $s \in \mathrm{c}_\Gamma^*(I_{\mathcal{K}})$ *is associated with* $\mathrm{Ker}[\Gamma_s, -]_* = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_r, I_{\mathcal{K}}]_* \mid r \neq s)$.

PROOF. Let $\mathcal{C} \in \mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K})$ be hom-prime. Then, by Lemma 3.3.9, we have $\mathcal{C} = \bigcap_{\substack{\mathcal{C} \subseteq \mathrm{Ker}[\Gamma_s, -]_* \\ s \in \mathrm{c}_\Gamma^*(I_{\mathcal{K}})}} \mathrm{Ker}[\Gamma_s, -]_*$. It follows by Lemma 3.3.7 that $\mathcal{C}$ must be contained in $\mathrm{Ker}[\Gamma_s, -]_*$, for a unique $s \in \mathrm{c}_\Gamma^*(I_{\mathcal{K}})$. Conclusion: $\mathcal{C} = \mathrm{Ker}[\Gamma_s, -]_*$, for a unique $s \in \mathrm{c}_\Gamma^*(I_{\mathcal{K}})$. The equality $\mathrm{Ker}[\Gamma_s, -]_* = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_r, I_{\mathcal{K}}]_* \mid r \neq s)$ was proved in Lemma 3.3.8. $\square$

Using Lemma 3.3.7, Lemma 3.3.8 and Lemma 3.3.9, one can prove with analogous arguments the following:

Theorem 3.3.11. *Let* $\mathcal{K}$ *be a stratified* $\mathcal{T}$*-module. Then there is a bijective correspondence between prime localizing submodules of* $\mathcal{K}$ *and points of* $\mathrm{c}_\Gamma^*(I_{\mathcal{K}})$. *A point* $s \in \mathrm{c}_\Gamma^*(I_{\mathcal{K}})$ *is associated with* $\mathrm{Ker}(\Gamma_s * -) = \mathrm{loc}^*(\Gamma_r * A \mid r \neq s,\ A \in \mathcal{K})$.

The following observation, which is of independent interest and will not play a role in the sequel, showcases a conceptual similarity between the theory of actions of tensor-triangulated categories and the theory of associated primes of modules over rings. To see this, recall the following result: If R is a ring and M is a non-zero R-module such that for every non-zero submodule $N \subseteq M$, it holds that $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(N)$, then $\mathrm{Ann}_R(M)$ is a prime ideal of R.

Proposition 3.3.12. *Let $\mathcal{L}$ be a non-zero localizing submodule of $\mathcal{K}$ such that for every non-zero localizing submodule $\mathcal{L}'$ of $\mathcal{K}$ with $\mathcal{L}' \subseteq \mathcal{L}$, it holds that* $\mathrm{Ann}_{\mathcal{T}}(\mathcal{L}) = \mathrm{Ann}_{\mathcal{T}}(\mathcal{L}')$. *Then* $\mathrm{Ann}_{\mathcal{T}}(\mathcal{L})$ *is an objectwise-prime localizing ideal of* $\mathcal{T}$.

Proof. Let $X, Y \in \mathcal{T}$ such that $X \otimes Y \in \mathrm{Ann}_{\mathcal{T}}(\mathcal{L})$. Then $(X \otimes Y) * \mathcal{L} = 0$. Suppose that $X \notin \mathrm{Ann}_{\mathcal{T}}(\mathcal{L})$, i.e., $X * \mathcal{L} \neq 0$. Then $\mathrm{Ann}_{\mathcal{T}}(X * \mathcal{L}) = \mathrm{Ann}_{\mathcal{T}}(\mathcal{L})$. Since $Y * (X * \mathcal{L}) = (X \otimes Y) * \mathcal{L} = 0$, it follows that $Y \in \mathrm{Ann}_{\mathcal{T}}(X * \mathcal{L})$, so $Y \in \mathrm{Ann}_{\mathcal{T}}(\mathcal{L})$. □

The action and internal-hom formulas.

Definition 3.3.13.

(a) $\mathcal{K}$ satisfies the *Action Formula* (AF) if

$$\mathrm{s}^*_\Gamma(X * A) = \mathrm{s}_\Gamma(X) \cap \mathrm{s}^*_\Gamma(A), \ \forall X \in \mathcal{T}, \ \forall A \in \mathcal{K}.$$

(b) $\mathcal{K}$ satisfies the *Internal-Hom Formula* (IHF) if

$$\mathrm{c}^*_\Gamma([X, A]_*) = \mathrm{c}_\Gamma([X, I_{\mathcal{T}}]) \cap \mathrm{c}^*_\Gamma(A), \ \forall X \in \mathcal{T}, \ \forall A \in \mathcal{K}.$$

(Recall that $\mathrm{c}_\Gamma([X, I_{\mathcal{T}}]) = \mathrm{s}_\Gamma(X)$.)

Proposition 3.3.14.

(a) *If $\mathcal{K}$ satisfies the Action Formula, then* $\mathrm{Ker}(\Gamma_s * -)$ *is a prime localizing submodule, $\forall s \in S$. If $\mathcal{K}$ is a conservative $\mathcal{T}$-module, then the converse holds.*

(b) *If $\mathcal{K}$ satisfies the Internal-Hom Formula, then* $\mathrm{Ker}[\Gamma_s, -]_*$ *is a* hom-*prime colocalizing* hom-*submodule, $\forall s \in S$. If $\mathcal{K}$ is a conservative $\mathcal{T}$-module, then the converse holds.*

Proof.

(a) Similar to (b).

(b) The Internal-Hom Formula can be restated as follows: if $[\Gamma_s \otimes X, A]_* = 0$ then $\Gamma_s \otimes X = 0$ or $[\Gamma_s, A]_* = 0$ — the converse holds by the definition of cosupport. So, if $[X, A]_* \in \mathrm{Ker}[\Gamma_s, -]_*$, then $X \in \mathrm{Ker}(\Gamma_s \otimes -) \subseteq \mathrm{Ker}[\Gamma_s, -]_*^{\otimes \mathrm{C}}$ or $A \in \mathrm{Ker}[\Gamma_s, -]_*$; for the first alternative, see Lemma 3.3.6. This means that $\mathrm{Ker}[\Gamma_s, -]_*$ is hom-prime. Now if $\mathrm{Ker}[\Gamma_s, -]_*$ is hom-prime and $\mathcal{K}$ is a conservative $\mathcal{T}$-module, then $\mathrm{Ker}(\Gamma_s \otimes -) = \mathrm{Ker}[\Gamma_s, -]_*^{\otimes \mathrm{C}}$. Therefore, if $[\Gamma_s \otimes X, A]_* = 0$, then $\Gamma_s \otimes X = 0$ or $[\Gamma_s, A]_* = 0$, which is precisely the statement of the Internal-Hom Formula. □

Proposition 3.3.15.

(a) *If $\mathcal{T}$ satisfies minimality, then $\mathcal{K}$ satisfies the Action Formula and the Internal-Hom Formula.*

(b) *If $\mathcal{K}$ is a conservative $\mathcal{T}$-module and $\mathcal{K}$ satisfies cominimality, then $\mathcal{K}$ satisfies the Internal-Hom Formula.*

(c) *If $\mathcal{T}$ satisfies the Internal-Hom Formula, then $\mathcal{T}$ satisfies the Action Formula.*

Proof. Let $s \in S$, $X \in \mathcal{T}$, $A \in \mathcal{K}$.

(a) If $s \in \mathrm{s}_\Gamma(X) \cap \mathrm{s}^*_\Gamma(A)$, then $\Gamma_s \otimes X \neq 0$ and $\Gamma_s * A \neq 0$. Since $\mathrm{loc}^\otimes(\Gamma_s)$ is minimal, it follows that $\Gamma_s \in \mathrm{loc}^\otimes(\Gamma_s \otimes X)$. Hence, $\Gamma_s * A \in \mathrm{loc}^*((\Gamma_s \otimes X) * A)$. Since $\Gamma_s * A \neq 0$, it holds that $\Gamma_s * (X * A) \cong (\Gamma_s \otimes X) * A \neq 0$. In other words, $s \in \mathrm{s}^*_\Gamma(X * A)$. Conclusion: $\mathcal{K}$ satisfies AF.

Now suppose that $s \in \mathrm{c}_\Gamma([X, I_{\mathcal{T}}]) \cap \mathrm{c}^*_\Gamma(A)$. Then $\Gamma_s \otimes X \neq 0$ and $[\Gamma_s, A]_* \neq 0$ (recall that $\mathrm{c}_\Gamma([X, I_{\mathcal{T}}]) = \mathrm{s}_\Gamma(X)$). Since $\mathrm{loc}^\otimes(\Gamma_s)$ is minimal, it follows that

$\Gamma_s \in \mathrm{loc}^{\otimes}(\Gamma_s \otimes X)$. Hence, $[\Gamma_s, A]_* \in \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s \otimes X, A]_*)$. It follows that $[\Gamma_s \otimes X, A]_* \neq 0$, i.e., $s \in \mathrm{c}^*_\Gamma([X, A]_*)$. Conclusion: $\mathcal{K}$ satisfies IHF.

(b) Suppose that $s \in \mathrm{c}_\Gamma([X, I_\mathcal{T}]) \cap \mathrm{c}^*_\Gamma(A)$. Then $\Gamma_s \otimes X \neq 0$ and $[\Gamma_s, A]_* \neq 0$. Aiming for contradiction, assume that $[\Gamma_s \otimes X, A]_* = 0$. Then $A \in \mathrm{Ker}[\Gamma_s \otimes X, -]_*$. Since $0 \neq [\Gamma_s, A]_* \in \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_*)$, it follows by cominimality of $\mathcal{K}$ that $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_\mathcal{K}]_*) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_*) \subseteq \mathrm{coloc}^{\mathrm{hom}}(A) \subseteq \mathrm{Ker}[\Gamma_s \otimes X, -]_*$. Thus, $[\Gamma_s \otimes X, I_\mathcal{K}]_* = [\Gamma_s \otimes X, [\Gamma_s, I_\mathcal{K}]_*]_* = 0$. So, $[\Gamma_s \otimes X, -]_* = 0$. By the $(\Gamma_s \otimes X) * - \dashv [\Gamma_s \otimes X, -]_*$ adjunction, it follows that $(\Gamma_s \otimes X) * - = 0$, i.e., $\Gamma_s \otimes X \in \mathrm{Ann}_\mathcal{T}(\mathcal{K})$. Since $\mathcal{K}$ is a conservative $\mathcal{T}$-module, $\Gamma_s \otimes X = 0$, which is a contradiction. Conclusion: $\mathcal{K}$ satisfies IHF.

(c) Let $X, Y \in \mathcal{T}$. Then $\mathrm{s}_\Gamma(X \otimes Y) = \mathrm{c}_\Gamma([X \otimes Y, I_\mathcal{T}]) = \mathrm{c}_\Gamma([X, [Y, I_\mathcal{T}]]) = \mathrm{c}_\Gamma([X, I_\mathcal{T}]) \cap \mathrm{c}_\Gamma([Y, I_\mathcal{T}]) = \mathrm{s}_\Gamma(X) \cap \mathrm{s}_\Gamma(Y)$. Conclusion: IHF implies AF. □

Remark 3.3.16. In the case $\mathcal{K} = \mathcal{T}$, the statement of the Action Formula is: $\mathrm{s}_\Gamma(X \otimes Y) = \mathrm{s}_\Gamma(X) \cap \mathrm{s}_\Gamma(Y)$, $\forall X, Y \in \mathcal{T}$. This is known as the Tensor Product Formula (which does not hold in general); see [**BF11, BHS23b**]. See also [**Bal20a**] for a support theory that does satisfy the Tensor Product Formula. On the other hand, the Internal-Hom Formula states: $\mathrm{c}_\Gamma([X, Y]) = \mathrm{s}_\Gamma(X) \cap \mathrm{c}_\Gamma(Y)$, $\forall X, Y \in \mathcal{T}$. For the BIK support, this is equivalent to stratification of $\mathcal{T}$ [**BIK12**, Theorem 9.5].

3.4. Smashing submodules

Let $\mathcal{K}$ be a $\mathcal{T}$-module. Recall our assumption that $\mathcal{K}$ is compactly generated. A smashing submodule of $\mathcal{K}$ is a smashing subcategory $\mathcal{M} \subseteq \mathcal{K}$ that is also a submodule. Specifically, the quotient functor $j_\mathcal{M} \colon \mathcal{K} \to \mathcal{K}/\mathcal{M}$ is a coproduct-preserving and essentially surjective triangulated functor that has a right adjoint $k_\mathcal{M} \colon \mathcal{K}/\mathcal{M} \to \mathcal{K}$ (which is necessarily fully faithful) that preserves coproducts — and products since it is a right adjoint. By Brown representability, $k_\mathcal{M}$ has a right adjoint $\ell_\mathcal{M} \colon \mathcal{K} \to \mathcal{K}/\mathcal{M}$ (which is necessarily essentially surjective) that preserves products. By the relations $j_\mathcal{M} k_\mathcal{M} \cong \mathrm{Id} \cong \ell_\mathcal{M} k_\mathcal{M}$, it follows that $j_\mathcal{M}$ and $\ell_\mathcal{M}$ take the same values on the image of $k_\mathcal{M}$, which is $\mathcal{M}^\perp$. The set of smashing submodules of $\mathcal{K}$ is denoted by $\mathrm{S}^*(\mathcal{K})$.

Next we describe the action of $\mathcal{T}$ on $\mathcal{K}/\mathcal{M}$ induced by the action of $\mathcal{T}$ on $\mathcal{K}$. The category $\mathcal{T} \times \mathcal{K}/\mathcal{M}$ is a triangulated category that is the quotient of $\mathcal{T} \times \mathcal{K}$ over $0 \times \mathcal{M}$, with the quotient functor $\mathcal{T} \times \mathcal{K} \to \mathcal{T} \times \mathcal{K}/\mathcal{M}$ being $\mathrm{Id}_\mathcal{T} \times j_\mathcal{M}$. Since $0 \times \mathcal{M}$ is contained in the kernel of $j_\mathcal{M} \circ *$, it follows that $j_\mathcal{M} \circ *$ factors through $\mathcal{T} \times \mathcal{K}/\mathcal{M}$ via a functor $* \colon \mathcal{T} \times \mathcal{K}/\mathcal{M} \to \mathcal{K}/\mathcal{M}$. It is straightforward to check that this functor is an action of $\mathcal{T}$ on $\mathcal{K}/\mathcal{M}$. If $X \in \mathcal{T}$ and $A = j_\mathcal{M}(B) \in \mathcal{K}/\mathcal{M}$, then $X * A = j_\mathcal{M}(X * B)$. The functor $j_\mathcal{M} \colon \mathcal{K} \to \mathcal{K}/\mathcal{M}$ is action-preserving. We denote by $[-,-]_* \colon \mathcal{T}^{\mathrm{op}} \times \mathcal{K}/\mathcal{M} \to \mathcal{K}/\mathcal{M}$ the relative internal-hom of $\mathcal{K}/\mathcal{M}$. By Lemma 3.1.7, $k_\mathcal{M}$ is action and hom-preserving and $\ell_\mathcal{M}$ is hom-preserving. Moreover, since $I_\mathcal{K}$ (the product of the Brown–Comenetz duals of the compact objects of $\mathcal{K}$) is a pure-injective cogenerator of $\mathcal{K}$ and $\ell_\mathcal{M}$ is an essentially surjective right adjoint, it follows that $\ell_\mathcal{M}(I_\mathcal{K})$ is a pure-injective cogenerator of $\mathcal{K}/\mathcal{M}$. In particular, $\mathcal{K}/\mathcal{M} = \mathrm{coloc}(\ell_\mathcal{M}(I_\mathcal{K}))$.

Now we describe the colocalizing hom-submodules of $\mathcal{K}/\mathcal{M}$. The functor $k_\mathcal{M}$ gives a bijective correspondence between the colocalizing subcategories of $\mathcal{K}/\mathcal{M}$ and the colocalizing subcategories of $\mathcal{K}$ contained in $\mathcal{M}^\perp$. Since $k_\mathcal{M}$ is hom-preserving, this bijection restricts to colocalizing hom-submodules, i.e., the maps

$$\mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K}/\mathcal{M}) \underset{k_{\mathcal{M}}^{-1}}{\overset{k_{\mathcal{M}}}{\rightleftarrows}} \{\mathcal{C} \in \mathrm{Coloc}^{\mathrm{hom}}(\mathcal{K}) \mid \mathcal{C} \subseteq \mathcal{M}^{\perp}\} \tag{3.4.1}$$

are mutually inverse inclusion-preserving bijections. An observation that will be useful in the sequel is that $k_{\mathcal{M}}\,\mathrm{coloc}^{\mathrm{hom}}(j_{\mathcal{M}}(A)) = \mathrm{coloc}^{\mathrm{hom}}(k_{\mathcal{M}}j_{\mathcal{M}}(A))$, $\forall A \in \mathcal{K}$.

Let $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma)$ be a good support–cosupport pair on $\mathcal{T}$. We denote the induced support–cosupport on $\mathcal{K}/\mathcal{M}$ by $(\mathrm{s}_\Gamma^{\mathcal{M}}, \mathrm{c}_\Gamma^{\mathcal{M}})$. Specifically,

$$\mathrm{s}_\Gamma^{\mathcal{M}}(j_{\mathcal{M}}(A)) = \{s \in S \mid \ j_{\mathcal{M}}(\Gamma_s * A) \neq 0\ \},$$
$$\mathrm{c}_\Gamma^{\mathcal{M}}(j_{\mathcal{M}}(A)) = \{s \in S \mid [\Gamma_s, j_{\mathcal{M}}(A)]_* \neq 0\}.$$

Then $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle if

$$\mathrm{coloc}^{\mathrm{hom}}(j_{\mathcal{M}}(A)) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, j_{\mathcal{M}}(A)]_* \mid s \in S),\ \forall A \in \mathcal{K}$$

and $\mathcal{K}/\mathcal{M}$ satisfies cominimality if $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})]_*)$ is a minimal colocalizing hom-submodule of $\mathcal{K}/\mathcal{M}$, for all $s \in S$. Finally, let $S_{\mathcal{M}} = \{s \in S \mid [\Gamma_s, I_{\mathcal{K}}]_* \in \mathcal{M}^{\perp}\}$.

Proposition 3.4.2. *Let $\mathcal{M} \in \mathrm{S}^*(\mathcal{K})$. The following are equivalent:*

(a) $\mathcal{K}/\mathcal{M}$ *satisfies the colocal-to-global principle.*

(b) $\mathrm{coloc}^{\mathrm{hom}}(B) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, B]_* \mid s \in S)$, $\forall B \in \mathcal{M}^{\perp}$.

As a result, if $\mathcal{K}$ satisfies the colocal-to-global principle, then $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle.

Proof. Let A be an object of $\mathcal{K}$ and set

$$\mathcal{C}_1 = \mathrm{coloc}^{\mathrm{hom}}(j_{\mathcal{M}}(A)),$$
$$\mathcal{C}_2 = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, j_{\mathcal{M}}(A)]_* \mid s \in S),$$
$$\mathcal{D}_1 = \mathrm{coloc}^{\mathrm{hom}}(k_{\mathcal{M}}j_{\mathcal{M}}(A)),$$
$$\mathcal{D}_2 = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, k_{\mathcal{M}}j_{\mathcal{M}}(A)]_* \mid s \in S).$$

Under the bijection (3.4.1), $\mathcal{C}_1$ corresponds to $\mathcal{D}_1$, while $\mathcal{C}_2$ corresponds to $\mathcal{D}_2$ (recall that $k_{\mathcal{M}}$ is hom-preserving). So, if $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle, then $\mathcal{C}_1 = \mathcal{C}_2$. Hence, $\mathcal{D}_1 = \mathcal{D}_2$. Since $\mathrm{Im}\, k_{\mathcal{M}}j_{\mathcal{M}} = \mathcal{M}^{\perp}$, (b) follows. On the other hand, if (b) holds, then $\mathcal{D}_1 = \mathcal{D}_2$. As a result, $\mathcal{C}_1 = \mathcal{C}_2$, i.e., $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle. This proves (a).

If $\mathcal{K}$ satisfies the colocal-to-global principle, then

$$\mathrm{coloc}^{\mathrm{hom}}(A) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, A]_* \mid s \in S),\ \forall A \in \mathcal{K},$$

so the equality certainly holds for $A \in \mathcal{M}^{\perp}$. Therefore, $\mathcal{K}/\mathcal{M}$ satisfies the colocal-to-global principle by the equivalence (a) $\Leftrightarrow$ (b). ☐

Proposition 3.4.3. *Suppose that $s \in S_{\mathcal{M}}$. Then* $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$ *is a minimal colocalizing* hom-*submodule of $\mathcal{K}$ if and only if* $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})]_*)$ *is a minimal colocalizing* hom-*submodule of $\mathcal{K}/\mathcal{M}$.*

Proof. Since $s \in S_{\mathcal{M}}$, it holds that $[\Gamma_s, I_{\mathcal{K}}]_* \in \mathcal{M}^{\perp}$. Hence, $[\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})]_* \cong \ell_{\mathcal{M}}[\Gamma_s, I_{\mathcal{K}}]_* \cong j_{\mathcal{M}}[\Gamma_s, I_{\mathcal{K}}]_*$. So, under the bijection (3.4.1), $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})]_*)$ corresponds to $\mathrm{coloc}^{\mathrm{hom}}(k_{\mathcal{M}}j_{\mathcal{M}}[\Gamma_s, I_{\mathcal{K}}]_*) = \mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$, with the last equality again because $[\Gamma_s, I_{\mathcal{K}}]_* \in \mathcal{M}^{\perp}$. Consequently, $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$ is minimal if and only if $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, \ell_{\mathcal{M}}(I_{\mathcal{K}})]_*)$ is minimal. ☐

Combining Proposition 3.4.2, Proposition 3.4.3 and Theorem 3.2.21, we obtain the following result.

Theorem 3.4.4. *Let $\{\mathcal{M}_s\}_{s\in S}$ be a collection of smashing submodules of $\mathcal{K}$ such that $s \in S_{\mathcal{M}_s}$, for all $s \in S$. Then:*

(a) *$\mathcal{K}$ satisfies cominimality if and only if $\mathcal{K}/\mathcal{M}_s$ satisfies cominimality, for all $s \in S$.*

(b) *Suppose that $\mathcal{K}$ satisfies the colocal-to-global principle. Then $\mathcal{K}$ is costratified if and only if $\mathcal{K}/\mathcal{M}_s$ is costratified, for all $s \in S$.*

Proof. Since $s \in S_{\mathcal{M}_s}$, for all $s \in S$, by Proposition 3.4.3 $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, I_{\mathcal{K}}]_*)$ is a minimal colocalizing hom-submodule of $\mathcal{K}$ if and only if $\mathrm{coloc}^{\mathrm{hom}}([\Gamma_s, \ell_{\mathcal{M}_s}(I_{\mathcal{K}})]_*)$ is a minimal colocalizing hom-submodule of $\mathcal{K}/\mathcal{M}_s$, for all $s \in S$. In other words, $\mathcal{K}$ satisfies cominimality if and only if $\mathcal{K}/\mathcal{M}_s$ satisfies cominimality, for all $s \in S$. This proves (a). If $\mathcal{K}$ satisfies the colocal-to-global principle, then by Proposition 3.4.2, it follows that $\mathcal{K}/\mathcal{M}_s$ satisfies the colocal-to-global principle, for all $s \in S$. Statement (b) now follows from (a) and Theorem 3.2.21. □

We will apply Theorem 3.4.4 to the case $\mathcal{K} = \mathcal{T}$, $* = \otimes$, $S = \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ and $(\mathrm{s}_\Gamma, \mathrm{c}_\Gamma) = (\mathrm{Supp}^{\mathrm{s}}, \mathrm{Cosupp}^{\mathrm{s}})$ (under Hypothesis 1.4.3 and assuming $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D). In this case, if $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, then $S_P = \{Q \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) \mid [\Gamma_Q, I] \in P^\perp\}$. Since $\Gamma_P = e_{\mathcal{S}} \otimes f_P$, for some $\mathcal{S} \in \mathrm{S}^{\otimes}(\mathcal{T})$, and $P^\perp = \mathrm{Im}[f_P, -]$, it follows that $[\Gamma_P, I] = [e_{\mathcal{S}} \otimes f_P, I] \cong [f_P, [e_{\mathcal{S}}, I]] \in P^\perp$. In other words, $P \in S_P$. This leads to the following result:

Corollary 3.4.5. *Suppose that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D. Then:*

(a) *$\mathcal{T}$ satisfies cominimality if and only if $\mathcal{T}/P$ satisfies cominimality, for all $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.*

(b) *Suppose that $\mathcal{T}$ satisfies the colocal-to-global principle. Then $\mathcal{T}$ is costratified if and only if $\mathcal{T}/P$ is costratified, for all $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$.*

Proof. The result is a direct consequence of Theorem 3.4.4, taking into account the preceding discussion. □

Corollary 3.4.6. *Suppose that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ is T_D and that $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T}) = \bigcup_{j\in J} V_{\mathcal{S}_j}$ is a cover of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ by closed subsets. If $\mathcal{T}/\mathcal{S}_j$ satisfies cominimality, for all $j \in J$, then $\mathcal{T}$ satisfies cominimality. If, moreover, $\mathcal{T}$ satisfies the colocal-to-global principle, then $\mathcal{T}$ is costratified.*

Proof. Let $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$. Then $P \in V_{\mathcal{S}_j}$, for some $j \in J$. This means that $\mathcal{S}_j \subseteq P$. Let $j_{\mathcal{S}_j} : \mathcal{T} \to \mathcal{T}/\mathcal{S}_j$ be the quotient functor. Then $j_{\mathcal{S}_j}(P)$ is a smashing ideal of $\mathcal{T}/\mathcal{S}_j$ such that $(\mathcal{T}/\mathcal{S}_j)/j_{\mathcal{S}_j}(P) \simeq \mathcal{T}/P$. Since $\mathcal{T}/\mathcal{S}_j$ satisfies cominimality, it follows by Corollary 3.4.5 that $\mathcal{T}/P$ satisfies cominimality. Since this is true for all $P \in \mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$, again by Corollary 3.4.5, we conclude that $\mathcal{T}$ satisfies cominimality. The "moreover" part follows by Theorem 3.2.21. □

Essentially via the same arguments (left to the reader) one obtains the analogous results for the Balmer spectrum and the Balmer–Favi support. What one needs to note for Corollary 3.4.8 is that, compared to $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ where the smashing ideals stand in bijection with open subsets of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ (thus closed covers of $\mathrm{Spc}^{\mathrm{s}}(\mathcal{T})$ are necessary) the thick ideals of $\mathcal{T}^{\mathrm{c}}$ — and by extension the compactly generated smashing ideals of $\mathcal{T}$ — stand in bijection with Thomason subsets of $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$; hence, a cover by complements of Thomason subsets is needed.

Corollary 3.4.7. *Suppose that every point of* $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ *is visible. Then:*

(a) $\mathcal{T}$ *satisfies cominimality if and only if* $\mathcal{T}/\operatorname{loc}^{\otimes}(\mathfrak{p})$ *satisfies cominimality, for all* $\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$.

(b) *Suppose that* $\mathcal{T}$ *satisfies the colocal-to-global principle. Then* $\mathcal{T}$ *is costratified if and only if* $\mathcal{T}/\operatorname{loc}^{\otimes}(\mathfrak{p})$ *is costratified, for all* $\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$.

Corollary 3.4.8. *Suppose that every point of* $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ *is visible and that* $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) = \bigcup_{j \in J} U_j$ *is a cover of* $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ *by complements of Thomason subsets. If* $\mathcal{T}(U_j)$ *satisfies cominimality, for all* $j \in J$, *then* $\mathcal{T}$ *satisfies cominimality. If, moreover,* $\mathcal{T}$ *satisfies the colocal-to-global principle, then* $\mathcal{T}$ *is costratified.*

In view of applications involving singularity categories of schemes, which is dealt with in Chapter 4, we need a version of Corollary 3.4.8 for the more general case of a $\mathcal{T}$-module $\mathcal{K}$. Let $\mathcal{S}$ be a compactly generated localizing ideal of $\mathcal{T}$ and set $\mathcal{M} = \mathcal{S} * \mathcal{K}$. Then $\mathcal{M}$ is a compactly generated localizing submodule of $\mathcal{K}$; see [**Ste13**, Section 4]. The action of $\mathcal{T}$ on $\mathcal{K}$ induces, as already discussed previously, an action of $\mathcal{T}$ on $\mathcal{K}/\mathcal{M}$. Because of the way $\mathcal{M}$ is defined, it follows that there is an induced action of $\mathcal{T}/\mathcal{S}$ on $\mathcal{K}/\mathcal{M}$ and a colocalizing subcategory of $\mathcal{K}/\mathcal{M}$ is a hom $\mathcal{T}$-submodule if and only if it is a hom $\mathcal{T}/\mathcal{S}$-submodule.

Assuming that every point of $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ is visible, let V be a Thomason subset of $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ and let $U = \mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) \setminus V$ and consider the localizing ideal $\mathcal{T}_V$ generated by those compact objects of $\mathcal{T}$ whose support is contained in V. By definition, $\mathcal{T}_V$ is compactly generated and hence smashing, so there are associated left and right (respectively) idempotents e_V and f_V such that $\mathcal{T}_V = \operatorname{loc}^{\otimes}(e_V) = \operatorname{Ker}(f_V \otimes -) = \operatorname{Im}(e_V \otimes -)$. We denote by $\mathcal{T}(U)$ the category $\mathcal{T}/\mathcal{T}_V$. It holds that $\mathrm{Spc}(\mathcal{T}(U)^{\mathrm{c}}) \cong U$ and we will treat this homeomorphism as an identification. Let $\mathcal{K}_V = \mathcal{T}_V * \mathcal{K}$ and let $\mathcal{K}(U) = \mathcal{K}/\mathcal{K}_V$. By the previous paragraph, $\mathcal{K}_V$ is a compactly generated localizing submodule of $\mathcal{K}$ and there is an induced action of $\mathcal{T}(U)$ on $\mathcal{K}(U)$ such that a colocalizing subcategory of $\mathcal{K}(U)$ is a hom $\mathcal{T}$-submodule if and only if it is a hom $\mathcal{T}(U)$-submodule. Further, $\mathcal{K}_V = \operatorname{Im}(e_V * -) = \operatorname{Ker}(f_V * -)$ and $\mathcal{K}_V^{\perp} = \operatorname{Im}[e_V, -]_* = \operatorname{Ker}[f_V, -]_*$. By this last observation, it follows that $S_{\mathcal{K}_V} \coloneqq \{\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) \mid [g_{\mathfrak{p}}, I_{\mathcal{K}}]_* \in \mathcal{K}_V^{\perp}\} = U$.

The following result is the analogue of [**Ste13**, Theorem 8.11] for colocalizing hom-submodules.

Theorem 3.4.9. *Suppose that every point of* $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ *is visible and that* $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}}) = \bigcup_{j \in J} U_j$ *is a cover of* $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$ *by complements of Thomason subsets. If* $\mathcal{K}(U_j)$ *(as a* $\mathcal{T}(U_j)$*-module) satisfies cominimality, for all* $j \in J$, *then* $\mathcal{K}$ *satisfies cominimality. If, moreover,* $\mathcal{K}$ *satisfies the colocal-to-global principle, then* $\mathcal{K}$ *is costratified.*

Proof. If $\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$, then there exists $j_{\mathfrak{p}} \in J$ such that $\mathfrak{p} \in U_{j_{\mathfrak{p}}}$. Fix such a $j_{\mathfrak{p}} \in J$, for each $\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$. Let $V_{j_{\mathfrak{p}}}$ be the complement of $U_{j_{\mathfrak{p}}}$. Then we have a collection $\{\mathcal{K}_{V_{j_{\mathfrak{p}}}}\}_{\mathfrak{p} \in \mathrm{Spc}(\mathcal{T}^{\mathrm{c}})}$ of smashing submodules of $\mathcal{K}$ such that $\mathfrak{p} \in S_{\mathcal{K}_{V_{j_{\mathfrak{p}}}}}$ since the latter is equal to $U_{j_{\mathfrak{p}}}$. The result now follows by an immediate application of Theorem 3.4.4. □

3.5. Derived categories of noetherian rings and schemes

Throughout, R will denote a commutative noetherian ring. In [**Nee11**], Neeman proved that there is a bijective correspondence between colocalizing subcategories of $\mathrm{D}(R)$ and subsets of $\mathrm{Spec}(R)$. In this section, we give a more streamlined proof of Neeman's theorem by using the general machinery we developed; specifically Theorem 3.2.21 and Corollary 3.2.26. As a direct consequence, we obtain a complete description of the RHom-prime colocalizing subcategories of $\mathrm{D}(R)$ in terms of the residue fields of R. Further, using Corollary 3.4.8, we prove that the derived category of quasi-coherent sheaves over a noetherian separated scheme is costratified.

We will use the cosupport taking values in $\mathrm{Spec}(R)$ defined by the residue fields $k(\mathfrak{p})$. Specifically, if $X \in \mathrm{D}(R)$, then $\mathrm{Cosupph}(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{RHom}_R(k(\mathfrak{p}), X) \neq 0\}$. We use the notation Cosupph to avoid conflict with the Balmer–Favi cosupport. Note that since $\mathrm{D}(R)$ is generated by its tensor-unit, every colocalizing subcategory of $\mathrm{D}(R)$ is a left RHom-ideal; apply Lemma 3.1.11 with $\mathcal{T} = \mathcal{K} = \mathrm{D}(R)$ and $* = \otimes^{\mathrm{L}}_R$ and $[-,-]_* = \mathrm{RHom}_R(-,-)$. We denote by I_R the cogenerator of $\mathrm{D}(R)$ that is the product of the Brown-Comenetz duals of the compact objects; see Recollection 1.1.13.

Lemma 3.5.1. *Let* $\mathfrak{p} \in \mathrm{Spec}(R)$. *Then* $\mathrm{RHom}_R(k(\mathfrak{p}), X) \cong \bigoplus_{i\in\mathbb{Z}} \Sigma^i k(\mathfrak{p})^{(J_i)} \cong \prod_{i\in\mathbb{Z}} \Sigma^i k(\mathfrak{p})^{(J_i)}$*, for some sets* J_i*, for all* $X \in \mathrm{D}(R)$. *The same holds for the complex* $\mathrm{RHom}_R(X, k(\mathfrak{p}))$.

Proof. Let E be a K-injective resolution of X. Then $\mathrm{RHom}_R(k(\mathfrak{p}), X)$ is the Hom-complex $\mathrm{Hom}_R(k(\mathfrak{p}), E)$. This is a complex of $k(\mathfrak{p})$-vector spaces, therefore it must be quasi-isomorphic to its cohomology complex with zero differential (which also has $k(\mathfrak{p})$-vector spaces as terms; thus coproducts of copies of $k(\mathfrak{p})$). For $\mathrm{RHom}_R(X, k(\mathfrak{p}))$, pick a K-projective resolution of X instead of a K-injective resolution and argue in an identical manner. □

Lemma 3.5.2. *Let X be an object of* $\mathrm{D}(R)$ *such that* $\mathrm{RHom}_R(k(\mathfrak{p}), X) \neq 0$. *Then* $\mathrm{coloc}(k(\mathfrak{p})) = \mathrm{coloc}(\mathrm{RHom}_R(k(\mathfrak{p}), X)) \subseteq \mathrm{coloc}(X)$.

Proof. It holds that $\mathrm{RHom}_R(k(\mathfrak{p}), X) \cong \prod_{i\in\mathbb{Z}} \Sigma^i k(\mathfrak{p})^{(J_i)}$. Since the inclusion $k(\mathfrak{p})^{(J_i)} \hookrightarrow k(\mathfrak{p})^{J_i}$ is a map of $k(\mathfrak{p})$-vector spaces, it must split. So, $k(\mathfrak{p})^{(J_i)}$ is a summand of $k(\mathfrak{p})^{J_i}$. This implies that $k(\mathfrak{p})^{(J_i)} \in \mathrm{coloc}(k(\mathfrak{p}))$ and consequently, $\prod_{i\in\mathbb{Z}} \Sigma^i k(\mathfrak{p})^{(J_i)} \in \mathrm{coloc}(k(\mathfrak{p}))$. Thus, $\mathrm{coloc}(\mathrm{RHom}_R(k(\mathfrak{p}), X)) \subseteq \mathrm{coloc}(k(\mathfrak{p}))$. By the fact that $k(\mathfrak{p})$ is a summand of $\mathrm{RHom}_R(k(\mathfrak{p}), X)$, it follows that $\mathrm{coloc}(k(\mathfrak{p})) \subseteq \mathrm{coloc}(\mathrm{RHom}_R(k(\mathfrak{p}), X))$. Since $\mathrm{D}(R)$ is generated by its tensor-unit, every colocalizing subcategory of $\mathrm{D}(R)$ is a left RHom-ideal. Hence, $\mathrm{RHom}_R(k(\mathfrak{p}), X) \in \mathrm{coloc}(X)$. This completes the proof. □

Proposition 3.5.3. *The category* $\mathrm{D}(R)$ *satisfies the colocal-to-global principle (in particular,* Cosupph *detects vanishing) and, for each* $\mathfrak{p} \in \mathrm{Spec}(R)$*, it holds that* $\mathrm{Cosupph}(k(\mathfrak{p})) = \{\mathfrak{p}\}$.

Proof. Since $\mathrm{D}(R)$ satisfies the local-to-global principle [**Nee92**], by Corollary 3.2.26, $\mathrm{D}(R)$ satisfies the colocal-to-global principle and, by Remark 3.2.17, Cosupph detects vanishing. Hence, $\mathrm{Cosupph}(k(\mathfrak{p})) \neq \varnothing$. Let $\mathfrak{q} \in \mathrm{Spec}(R)$ such that $\mathfrak{p} \neq \mathfrak{q}$. By Lemma 3.5.1, $\mathrm{RHom}_R(k(\mathfrak{p}), k(\mathfrak{q}))$ is quasi-isomorphic to a complex whose terms are of the form $k(\mathfrak{p})^{(I)} \cong k(\mathfrak{q})^{(J)}$ and these are both $k(\mathfrak{p})$ and $k(\mathfrak{q})$-vector spaces. Since $\mathfrak{p} \neq \mathfrak{q}$, this can only happen if the indexing sets I and J are empty. Hence, $\mathrm{RHom}_R(k(\mathfrak{p}), k(\mathfrak{q})) = 0$. Consequently, $\mathrm{Cosupph}(k(\mathfrak{p})) = \{\mathfrak{p}\}$. □

Theorem 3.5.4 ([**Nee11**])**.** *Let R be a commutative noetherian ring. Then* $\mathrm{D}(R)$ *is costratified.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and let X be a non-zero object in $\operatorname{coloc}(k(\mathfrak{p}))$. Then $\operatorname{coloc}(X) \subseteq \operatorname{coloc}(k(\mathfrak{p}))$. By Proposition 3.5.3, $\mathrm{D}(R)$ satisfies the colocal-to-global principle, Cosupph detects vanishing and $\operatorname{Cosupph}(k(\mathfrak{p})) = \{\mathfrak{p}\}$. By Lemma 3.2.18, it follows that $\operatorname{Cosupph}(X) = \{\mathfrak{p}\}$, i.e., $\operatorname{RHom}_R(k(\mathfrak{p}), X) \neq 0$. Lemma 3.5.2 implies that $\operatorname{coloc}(X) = \operatorname{coloc}(k(\mathfrak{p}))$. So, $\operatorname{coloc}(k(\mathfrak{p}))$ is a minimal colocalizing subcategory. Moreover, Lemma 3.5.2 implies that $\operatorname{coloc}(k(\mathfrak{p})) = \operatorname{coloc}(\operatorname{RHom}_R(k(\mathfrak{p}), I_R))$ and so, $\operatorname{coloc}(\operatorname{RHom}_R(k(\mathfrak{p}), I_R))$ is minimal. In conclusion, $\mathrm{D}(R)$ satisfies both the colocal-to-global principle and cominimality; so, Theorem 3.2.21 implies that $\mathrm{D}(R)$ is costratified; see also Remark 3.2.22. □

Theorem 3.5.5. *The* RHom*-prime colocalizing subcategories of* $\mathrm{D}(R)$ *correspond to points of* $\operatorname{Spec}(R)$*. The correspondence is given by associating* $\mathfrak{p} \in \operatorname{Spec}(R)$ *with* $\operatorname{Ker}\operatorname{RHom}_R(k(\mathfrak{p}), -) = \operatorname{coloc}(k(\mathfrak{q}) \mid \mathfrak{q} \neq \mathfrak{p})$.

Proof. Since $\mathrm{D}(R)$ is costratified, and a conservative $\mathrm{D}(R)$-module, Theorem 3.3.10 implies that the RHom-prime colocalizing subcategories of $\mathrm{D}(R)$ are precisely of the form $\operatorname{Ker}\operatorname{RHom}_R(k(\mathfrak{p}), -) = \operatorname{coloc}(\operatorname{RHom}_R(k(\mathfrak{q}), I_R) \mid \mathfrak{q} \neq \mathfrak{p})$ and the claimed equality is due to Lemma 3.5.2. □

Remark 3.5.6. One could also work with the Balmer–Favi support–cosupport (or the smashing support–cosupport since $\mathrm{D}(R)$ satisfies the Telescope Conjecture [**Nee92**]; see also Lemma 2.5.2 and [**BS23**, Section 6]). There is a homeomorphism between $\operatorname{Spc}(\mathrm{D}^{\mathrm{perf}}(R))$ and $\operatorname{Spec}(R)$ [**Nee92**]. Using this homeomorphism, we can express the Balmer–Favi support–cosupport via $\operatorname{Spec}(R)$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the associated Balmer–Favi idempotent is $g_{\mathfrak{p}} = K_\infty(\mathfrak{p}) \otimes R_{\mathfrak{p}}$ (where $K_\infty(\mathfrak{p})$ is the stable Koszul complex; see Section 4.1 for the definition) and $R_{\mathfrak{p}}$ is the localization of R at $\mathfrak{p}$. Let $X \in \mathrm{D}(R)$. Then

$$\operatorname{Supp}(X) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid g_{\mathfrak{p}} \otimes X \neq 0\},$$
$$\operatorname{Cosupp}(X) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{RHom}_R(g_{\mathfrak{p}}, X) \neq 0\}.$$

It holds that $\operatorname{loc}(g_{\mathfrak{p}}) = \operatorname{loc}(k(\mathfrak{p}))$ [**Ste18**, Lemma 3.22]. Therefore,

$$\operatorname{coloc}(\operatorname{RHom}_R(g_{\mathfrak{p}}, X)) = \operatorname{coloc}(\operatorname{RHom}_R(k(\mathfrak{p}), X)) = \operatorname{coloc}(k(\mathfrak{p})),$$

with the last equality by Lemma 3.5.2 (provided that $\operatorname{RHom}_R(k(\mathfrak{p}), X) \neq 0$). Since $\mathrm{D}(R)$ is stratified by the Balmer–Favi support [**BHS23b**, Theorem 5.8], in particular it satisfies the local-to-global principle, $\mathrm{D}(R)$ must also satisfy the colocal-to-global principle; see Corollary 3.2.26. The equality $\operatorname{coloc}(\operatorname{RHom}_R(g_{\mathfrak{p}}, X)) = \operatorname{coloc}(k(\mathfrak{p}))$ shows that $\mathrm{D}(R)$ satisfies cominimality with respect to the Balmer–Favi support–cosupport. Therefore, by Theorem 3.2.21, $\mathrm{D}(R)$ is costratified with respect to the Balmer–Favi support–cosupport.

Next, we include an example of a category that is not costratified. Recall that a commutative ring R is called *absolutely flat* if every R-module is flat and R is called *semi-artinian* if every non-zero homomorphic image of R, in the category of R-modules, contains a simple submodule. An R-module E is called *superdecomposable* if E does not admit any non-zero indecomposable summands.

Example 3.5.7 ([**Ste14a**])**.** Let R be an absolutely flat ring that is not semi-artinian. For example, $R = \prod_{\mathbb{N}} k$, where k is a field. Then there exists a superdecomposable injective R-module E. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $\operatorname{RHom}_R(k(\mathfrak{p}), E) = \operatorname{Hom}_R(k(\mathfrak{p}), E)$. Suppose that there exists a non-zero map $k(\mathfrak{p}) \to E$. Since R is absolutely flat, $k(\mathfrak{p})$ is simple and injective. Therefore, the map $k(\mathfrak{p}) \to E$ is a split monic and so, $k(\mathfrak{p})$ is a summand of E, which is a contradiction. This shows that $\mathrm{RHom}_R(k(\mathfrak{p}), E) = 0$, for all $\mathfrak{p} \in \mathrm{Spec}(R)$, i.e., $\mathrm{Cosupph}(E) = \varnothing$; showcasing the failure of the cosupport to detect vanishing and consequently, the failure of the colocal-to-global principle. Hence, $\mathrm{D}(R)$ is not costratified.

Theorem 3.5.8. *The derived category* $\mathrm{D}(X)$ *of quasi-coherent sheaves over a noetherian separated scheme* X *is costratified.*

Proof. By [**Ste13**, Corollary 8.13], $\mathrm{D}(X)$ is stratified. In particular, $\mathrm{D}(X)$ satisfies the local-to-global principle. Hence, by Corollary 3.2.26, $\mathrm{D}(X)$ satisfies the colocal-to-global principle. So, it suffices to prove cominimality. Let $\{U_i\}_{i\in I}$ be an open affine cover of X. As X is noetherian, any open subset of X is quasi-compact, so its complement is Thomason. The corresponding smashing localization $\mathrm{D}(X)(U_i)$ is equivalent to $\mathrm{D}(U_i)$. The latter is costratified (in particular it satisfies cominimality) by Theorem 3.5.4. The result follows by Corollary 3.4.8. ☐

CHAPTER 4

Colocalizing subcategories of singularity categories

In this chapter, utilizing the results established in Chapter 3 concerning costratification in relative tensor-triangular geometry, we classify the colocalizing subcategories of the singularity category of a locally hypersurface ring and then we generalize this classification to singularity categories of schemes with hypersurface singularities. The results of this chapter first appeared in [**Ver23b**].

The chapter is organized as follows: In Section 4.1, we recall some basic facts about the tensor-triangular geometry of the derived category of a ring and its action on the singularity category. Then we study the relative internal-hom functor on the singularity category. In Section 4.2, we present a few notions concerning Gorenstein rings and Gorenstein-injective modules, with the key point being that, over a Gorenstein ring, the stable category of Gorenstein-injective modules is equivalent to the singularity category. In Section 4.3, using the concept of endofiniteness, we prove that the images of the residue fields under the stabilization functor are pure-injective and in Section 4.4 we obtain cogenerators for certain subcategories of the singularity category. Finally, in Section 4.5 and Section 4.6, we prove the announced classification theorems.

4.1. Singularity categories

Convention. Throughout, R will denote a commutative noetherian ring.

Let f be an element of R. The *stable Koszul complex* associated with f is $K_\infty(f) = R \to R_f$, where R sits in degree 0 and R_f, the localization of R at f, sits in degree 1 and the map $R \to R_f$ is the localization homomorphism. Let $I = (f_1, \ldots, f_n)$ be an ideal of R and set $K_\infty(I) \coloneqq K_\infty(f_1) \otimes_R \cdots \otimes_R K_\infty(f_n)$. Up to quasi-isomorphism, $K_\infty(I)$ does not depend on the choice of generators for the ideal I. Since $K_\infty(I)$ is a bounded complex of flat R-modules, $K_\infty(I)$ is K-*flat*, which means that tensoring with $K_\infty(I)$ preserves quasi-isomorphisms.

Lemma 4.1.1. *Let I, J be ideals of R with $J \subseteq I$. Then we have the isomorphism $K_\infty(I/J) \cong R/J \otimes_R K_\infty(I)$.*

Proof. Let $I = (f_1, \ldots, f_n)$. Then $I/J = (\overline{f_1}, \ldots, \overline{f_n})$. Since

$$K_\infty(\overline{f_i}) = (R/J \to (R/J)_{\overline{f_i}}) \cong (R/J \otimes_R (R \to R_{f_i})) = (R/J \otimes_R K_\infty(f_i)),$$

we have $K_\infty(I/J) = \bigotimes_{i=1}^n K_\infty(\overline{f_i}) \cong (\bigotimes_{i=1}^n R/J) \otimes_R K_\infty(I)$. Since $R/J \otimes_R R/J \cong (R/J)/(JR/J) \cong R/J$, we conclude that $K_\infty(I/J) \cong R/J \otimes_R K_\infty(I)$. □

Recollection 4.1.2. Recall that the derived category $\mathrm{D}(R)$ is a big tt-category whose subcategory of compact objects is $\mathrm{D}^{\mathrm{perf}}(R)$ the subcategory of perfect complexes, i.e., bounded complexes of finitely generated projective R-modules up to quasi-isomorphism. Treating the homeomorphism $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(R)) \cong \operatorname{Spec} R$ as an identification, if $\mathfrak{p} \in \operatorname{Spec} R$, then the Balmer–Favi idempotent associated with $\mathfrak{p}$ is $g_{\mathfrak{p}} = K_\infty(\mathfrak{p}) \otimes_R R_{\mathfrak{p}} \in \mathrm{D}(R)$, where $R_{\mathfrak{p}}$ is the localization of R at $\mathfrak{p}$. Note that $g_{\mathfrak{p}}$ is K-flat. The Balmer–Favi support (resp. cosupport) of an object $X \in \mathrm{D}(R)$ is $\operatorname{Supp}(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid g_{\mathfrak{p}} \otimes_R X \neq 0\}$ (resp. $\operatorname{Cosupp}(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{RHom}_R(g_{\mathfrak{p}}, X) \neq 0\}$).

Recollection 4.1.3. The *singularity category* of R is $\mathrm{S}(R) \coloneqq \mathrm{K}_{\mathrm{ac}}(\operatorname{Inj} R)$ the homotopy category of acyclic complexes of injective R-modules. By [**Kra05b**], $\mathrm{S}(R)$ is a compactly generated triangulated category and there is a recollement

$$\mathrm{S}(R) \;\underset{\underset{I_\rho}{\longleftarrow}}{\overset{\overset{I_\lambda}{\longleftarrow}}{\xrightarrow{\;I\;}}}\; \mathrm{K}(\operatorname{Inj} R) \;\underset{\underset{Q_\rho}{\longleftarrow}}{\overset{\overset{Q_\lambda}{\longleftarrow}}{\xrightarrow{\;Q\;}}}\; \mathrm{D}(R),$$

where I is the inclusion and Q is the composite $\mathrm{K}(\operatorname{Inj} R) \hookrightarrow \mathrm{K}(R) \twoheadrightarrow \mathrm{D}(R)$. The functor $I_\lambda Q_\rho \colon \mathrm{D}(R) \to \mathrm{S}(R)$, called the *stabilization functor*, induces an equivalence of triangulated categories between the idempotent completion of $\mathrm{D}_{\mathrm{Sg}}(R) = \mathrm{D}^{\mathrm{b}}(\operatorname{mod} R)/\mathrm{D}^{\mathrm{perf}}(R)$ and the subcategory of compact objects of $\mathrm{S}(R)$. When there are multiple rings involved, we will use the notation $I_\lambda Q_\rho^R$.

Recollection 4.1.4. The ring R is called a *regular* ring if $R_{\mathfrak{p}}$ is a *regular local* ring, i.e., $\operatorname{gldim} R_{\mathfrak{p}} < \infty$, for all $\mathfrak{p} \in \operatorname{Spec} R$ (this is just one of the many equivalent definitions of a regular local ring). It is a fact that R is regular if and only if $\mathrm{S}(R) = 0$. The *singular locus* of R is $\operatorname{Sing} R = \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{gldim} R_{\mathfrak{p}} = \infty\}$, i.e., $\operatorname{Sing} R$ consists of those prime ideals $\mathfrak{p}$ such that $R_{\mathfrak{p}}$ is not a regular local ring. Clearly, R is a regular ring if and only if $\operatorname{Sing} R = \varnothing$.

A commutative noetherian local ring S is called a *hypersurface* ring if the completion of S at its unique maximal ideal is isomorphic to the quotient of a regular ring by a regular element. It holds that a hypersurface ring is Gorenstein. We say that R is a *locally hypersurface* ring if $R_{\mathfrak{p}}$ is a hypersurface ring, for all $\mathfrak{p} \in \operatorname{Spec} R$. It holds that a locally hypersurface ring is Gorenstein; see Definition 4.2.1.

Recollection 4.1.5. By [**Ste14b**], there is an *action* $* \colon \mathrm{D}(R) \times \mathrm{S}(R) \to \mathrm{S}(R)$, i.e., a coproduct-preserving triangulated bifunctor that satisfies the following properties: $X * (Y * A) \cong (X \otimes^{\mathrm{L}}_R Y) * A$ and $R * A \cong A$, $\forall X, Y \in \mathrm{D}(R)$, $\forall A \in \mathrm{S}(R)$. This action is defined as follows: If $X \in \mathrm{D}(R)$ and $A \in \mathrm{S}(R)$, then $X * A = \widetilde{X} \otimes_R A$, where $\widetilde{X}$ is a K-*flat resolution* of X, i.e., $\widetilde{X}$ is a K-flat complex quasi-isomorphic to X.

Remark 4.1.6. The action of $\mathrm{D}(R)$ on $\mathrm{S}(R)$ does not depend on the choice of K-flat resolutions, in the sense that different K-flat resolutions of the same object yield naturally isomorphic functors. Moreover, any complex of R-modules admits a K-flat resolution that consists of flat R-modules; see [**Mur07**, Corollary 3.22] for a more general version of this result concerning schemes. Hence, when we consider the action of $\mathrm{D}(R)$ on $\mathrm{S}(R)$, we can assume that all K-flat resolutions involved consist of flat R-modules.

The discussion in the beginning of Section 3.1 applies to the case of the singularity category. To be specific, if $X \in \mathrm{D}(R)$, then the functor $X * - \colon \mathrm{S}(R) \to \mathrm{S}(R)$ is a coproduct-preserving triangulated functor. Since $\mathrm{S}(R)$ is compactly generated $X * -$ has a right adjoint $[X, -]$ called $[X, -]$ the *relative internal-hom*. Note that $[R, -] \cong \mathrm{Id}_{\mathrm{S}(R)}$, since $[R, -]$ is the right adjoint of $R * - \cong \mathrm{Id}_{\mathrm{S}(R)}$.

Let $\widetilde{X}$ be a K-flat resolution of X. We have an adjunction

$$\mathrm{K}(R) \underset{\mathrm{Hom}_R(\widetilde{X},-)}{\overset{\widetilde{X}\otimes_R -}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{K}(R) \tag{4.1.7}$$

that restricts to an adjunction on $\mathrm{S}(R)$. We will show this next, using a result of Emmanouil [**Emm23**] which was based on work of Šťovíček [**Sto14**].

Proposition 4.1.8. *Let F be a* K*-flat complex of flat R-modules and let A be an acyclic complex of injective R-modules. Then* $\mathrm{Hom}_R(F, A)$ *is an acyclic complex of injective R-modules. In particular, the adjunction* (4.1.7) *restricts to an adjunction on* $\mathrm{S}(R)$.

Proof. Since A is an acyclic complex that consists of injective R-modules, hence of pure-injective R-modules, it holds that $\Sigma^n A$ is an acyclic complex of pure-injective R-modules, for all $n \in \mathbb{Z}$. By [**Emm23**, Proposition 3.1], it follows that $\mathrm{Hom}_{\mathrm{K}(R)}(F, \Sigma^n A) = 0$. Thus, $\mathrm{H}^n(\mathrm{Hom}_R(F, A)) = \mathrm{Hom}_{\mathrm{K}(R)}(F, \Sigma^n A) = 0,\ \forall n \in \mathbb{Z}$. In other words, $\mathrm{Hom}_R(F, A)$ is acyclic. Each term of the complex $\mathrm{Hom}_R(F, A)$ is a product of R-modules of the form $\mathrm{Hom}_R(M, N)$, where M is a flat R-module and N is an injective R-module. So, the functor $\mathrm{Hom}_R(-, \mathrm{Hom}_R(M, N))$, which is naturally isomorphic to $\mathrm{Hom}_R(M \otimes_R -, N)$, is exact. Equivalently, $\mathrm{Hom}_R(M, N)$ is an injective R-module. Since injective R-modules are closed under products, $\mathrm{Hom}_R(F, A)$ consists of injective R-modules.

By what we just proved, we conclude that the restrictions of the functors involved in (4.1.7) on $\mathrm{S}(R)$ take values in $\mathrm{S}(R)$. Hence, we obtain the adjunction

$$\mathrm{S}(R) \underset{\mathrm{Hom}_R(\widetilde{X},-)}{\overset{\widetilde{X}\otimes_R -}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{S}(R). \qquad \square$$

Corollary 4.1.9. *Let $X \in \mathrm{D}(R)$ and let $\widetilde{X}$ be a* K*-flat resolution of X that consists of flat R-modules. Then* $[X, -] = \mathrm{Hom}_R(\widetilde{X}, -)\colon \mathrm{S}(R) \to \mathrm{S}(R)$.

Proof. The claim follows immediately from Proposition 4.1.8 due to the fact that $\mathrm{Hom}_R(\widetilde{X}, -)$ is right adjoint to $\widetilde{X} \otimes_R - = X * -$. $\square$

Lemma 4.1.10. *Every colocalizing subcategory $\mathcal{C}$ of* $\mathrm{S}(R)$ *is a* hom*-submodule over* $\mathrm{D}(R)$ *(meaning that* $[X, A] \in \mathcal{C}$, $\forall X \in \mathrm{D}(R)$, $\forall A \in \mathcal{C}$*).*

Proof. Since $\mathrm{D}(R) = \mathrm{loc}(R)$, the claim follows by Lemma 3.1.11. $\square$

4.2. Gorenstein rings

In this section, we recall some facts about Gorenstein rings, Gorenstein-injective and Gorenstein-projective modules and the stable category of Gorenstein-injective modules.

Definition 4.2.1. Let R be a commutative noetherian ring and M an R-module.

(a) The ring R is called *Gorenstein* if R has finite injective dimension as an R-module.
(b) The R-module M is called *Gorenstein-injective* if there exists an acyclic complex C that consists of injective R-modules such that $\mathrm{Hom}_R(I, C)$ is acyclic, for all injective R-modules I and $M = \mathrm{Z}^0 C$ the kernel of the zeroth differential of C. Such a complex C is called a *complete injective resolution* of M.

(c) The R-module M is called *Gorenstein-projective* if there exists an acyclic complex C that consists of projective R-modules such that $\mathrm{Hom}_R(C, P)$ is acyclic, for all projective R-modules P and $M = \mathrm{Z}^0 C$ the kernel of the zeroth differential of C. Such a complex is called a *complete projective resolution* of M.
(d) The *Gorenstein-injective envelope* of M is (if it exists) a Gorenstein-injective R-module $G_R(M)$ together with a morphism $f\colon M \to G_R(M)$ such that the following two conditions hold: First, for all Gorenstein-injective R-modules G and morphisms $g\colon M \to G$, there exists a morphism $h\colon G_R(M) \to G$ such that $g = h \circ f$. Second, if $h\colon G_R(M) \to G_R(M)$ is a morphism such that $h \circ f = f$, then h is an isomorphism.
(e) The *Gorenstein-projective cover* of M is (if it exists) a Gorenstein-projective R-module $G^R(M)$ together with a morphism $f\colon G^R(M) \to M$ such that the following two conditions hold: First, for all Gorenstein-projective R-modules G and morphisms $g\colon G \to M$, there exists a morphism $h\colon G \to G^R(M)$ such that $g = f \circ h$. Second, if $h\colon G^R(M) \to G^R(M)$ is a morphism such that $f = f \circ h$, then h is an isomorphism.

Proposition 4.2.2 ([**EJ00**, Theorem 11.3.2, Theorem 11.6.9])**.** *If R is a Gorenstein ring, then any R-module admits a Gorenstein-injective envelope. If, moreover, R is local, then any finitely generated R-module admits a finitely generated Gorenstein-projective cover.*

Recollection 4.2.3. The category $\mathrm{GInj}\, R$ of Gorenstein-injective R-modules is an exact subcategory of $\mathrm{Mod}\, R$ with exact sequences those short exact sequences of Gorenstein-injective R-modules. In fact, $\mathrm{GInj}\, R$ is a Frobenius exact category, i.e., $\mathrm{GInj}\, R$ has enough projectives and enough injectives and its projective and injective objects coincide: they are precisely the injective R-modules. So, $\underline{\mathrm{GInj}}\, R$ the stable category of Gorenstein-injective R-modules is a triangulated category. According to [**Kra05b**, Proposition 7.13], if R is a Gorenstein ring, there is an equivalence of triangulated categories $\mathrm{S}(R) \xrightarrow{\simeq} \underline{\mathrm{GInj}}\, R$ given by mapping $A \in \mathrm{S}(R)$ to $\mathrm{Z}^0 A$ with inverse given by sending a Gorenstein-injective R-module M to a complete injective resolution $C(M)$. Furthermore, by [**Ste14b**, Corollary 4.8], the functors $G_R, \mathrm{Z}^0 I_\lambda Q_\rho \colon \mathrm{Mod}\, R \to \underline{\mathrm{GInj}}\, R$ are naturally isomorphic.

4.3. Endofiniteness, pure-injectivity and residue fields

The goal of this section is to prove that, for a Gorenstein ring, the images of the residue fields under the stabilization functor are pure-injective objects of the singularity category, which is a key result that we will need in the sequel.

Remark 4.3.1. Let $\mathfrak{p}$ be a prime ideal of R. Since $R_\mathfrak{p} \otimes_R - \colon \mathrm{Mod}\, R \to \mathrm{Mod}\, R_\mathfrak{p}$ is an exact functor with right adjoint $\mathrm{res}\colon \mathrm{Mod}\, R_\mathfrak{p} \to \mathrm{Mod}\, R$, it follows that res preserves injective modules. Moreover, since R is noetherian, localization preserves injective modules, so $R_\mathfrak{p} \otimes_R -$ preserves injective modules. Consequently, we have an adjunction

$$\mathrm{S}(R) \underset{\mathrm{res}}{\overset{R_\mathfrak{p} \otimes_R -}{\rightleftarrows}} \mathrm{S}(R_\mathfrak{p}).$$

Since the categories $\mathrm{S}(R)$ and $\mathrm{S}(R_\mathfrak{p})$ are compactly generated and res preserves coproducts, by Recollection 1.1.12, it follows that $\mathrm{res}\colon \mathrm{S}(R_\mathfrak{p}) \to \mathrm{S}(R)$ preserves pure-injective objects.

An object X of a triangulated category $\mathcal{T}$ is called *endofinite* if, for all compact objects C of $\mathcal{T}$, it holds that $\mathrm{Hom}_{\mathcal{T}}(C,X)$ is a finite length module over $\mathrm{End}_{\mathcal{T}}(X)$. By [**Kra99**, Theorem 1.2] (see also [**KR00**, Proposition 3.3] and [**Kra23**, Proposition 5.6]) endofinite objects are pure-injective.

Proposition 4.3.2. *Let $R=(R,\mathfrak{m},k)$ be a local Gorenstein ring. Then $I_\lambda Q_\rho(k)$ is an endofinite (hence a pure-injective) object of* $\mathrm{S}(R)$.

Proof. Since $k=R/\mathfrak{m}$ is a finitely generated R-module, it follows that k is an object of $\mathrm{D}_{\mathrm{Sg}}(R)$. As we have already discussed in Recollection 4.1.3, the subcategory of compact objects of $\mathrm{S}(R)$ is equivalent to the closure under summands of the image of $\mathrm{D}_{\mathrm{Sg}}(R)$ under $I_\lambda Q_\rho$. Further, by [**Orl06**, Lemma 1.11], every object of $\mathrm{D}_{\mathrm{Sg}}(R)$ is of the form $\Sigma^i M$, where M is a finitely generated R-module. Consequently, it suffices to show that $\mathrm{Hom}_{\mathrm{D}_{\mathrm{Sg}}(R)}(\Sigma^i M,k)$ is a module of finite length over $\Lambda\coloneqq\mathrm{End}_{\mathrm{D}_{\mathrm{Sg}}(R)}(k)$. Since R is a local Gorenstein ring and M is a finitely generated R-module, it follows that M has a finitely generated Gorenstein-projective cover $G^R(M)$; see Proposition 4.2.2. Let $c(M)$ be a complete projective resolution of $G^R(M)$ that consists of finitely generated projective R-modules. Now we compute:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}_{\mathrm{Sg}}(R)}(\Sigma^i M,k) &\cong \underline{\mathrm{Hom}}_R(M,\Sigma^{-i}G_R(k))\\ &\cong \widehat{\mathrm{Ext}}_R^{-i}(M,k)\\ &= \mathrm{H}^{-i}(\mathrm{Hom}_R(c(M),k))\end{aligned}$$

with the second isomorphism by [**Kra05b**, Proposition 7.7] and the subsequent equality by definition. Since $c(M)$ is a complex of finitely generated R-modules, it follows that $\mathrm{Hom}_R(c(M),k)$ is a complex of finite dimensional k-vector spaces. Hence, $\mathrm{H}^{-i}(\mathrm{Hom}_R(c(M),k))$ is a finite dimensional k-vector space. Consequently, $\mathrm{Hom}_{\mathrm{D}_{\mathrm{Sg}}(R)}(\Sigma^i M,k)$ is a finite dimensional k-vector space. In particular, for $M=k$ and $i=0$, we have that Λ is a finite dimensional k-algebra. So, $\mathrm{Hom}_{\mathrm{D}_{\mathrm{Sg}}(R)}(\Sigma^i M,k)$ is a Λ-module that is finite dimensional over k, thus it has finite length over Λ. In conclusion, k is an endofinite object of $\mathrm{D}_{\mathrm{Sg}}(R)$ and so $I_\lambda Q_\rho(k)$ is an endofinite (hence a pure-injective) object of $\mathrm{S}(R)$. □

Proposition 4.3.3. *Let R be a Gorenstein ring. Then $I_\lambda Q_\rho(k(\mathfrak{p}))$ is a pure-injective object of* $\mathrm{S}(R)$.

Proof. Since R is a Gorenstein ring, we have that $R_\mathfrak{p}$ is a local Gorenstein ring. Hence, by Proposition 4.3.2, $I_\lambda Q_\rho^{R_\mathfrak{p}}(k(\mathfrak{p}))$ is a pure-injective object of $\mathrm{S}(R_\mathfrak{p})$. We have $I_\lambda Q_\rho^R(k(\mathfrak{p}))=I_\lambda Q_\rho^R(\mathrm{res}\,k(\mathfrak{p}))=\mathrm{res}\,I_\lambda Q_\rho^{R_\mathfrak{p}}(k(\mathfrak{p}))$. According to Remark 4.3.1, $\mathrm{res}\colon \mathrm{S}(R_\mathfrak{p})\to\mathrm{S}(R)$ preserves pure-injective objects. Hence, $I_\lambda Q_\rho^R(k(\mathfrak{p}))$ is pure-injective. □

4.4. Cogeneration

Our goal here is to prove that if $R=(R,\mathfrak{m},k)$ is a hypersurface ring, then the image of the functor $[g_\mathfrak{m},-]\colon \mathrm{S}(R)\to\mathrm{S}(R)$ is cogenerated by $I_\lambda Q_\rho(k)$. The key results we will need are Proposition 4.3.3 and the following:

Lemma 4.4.1. *Let $R = (R, \mathfrak{m}, k)$ be a local Gorenstein ring. Then, in* $\mathrm{S}(R)$*,*

$$\operatorname{Hom}_R(K_\infty(\mathfrak{m}), I_\lambda Q_\rho(k(\mathfrak{p}))) \cong \begin{cases} I_\lambda Q_\rho(k), & \mathfrak{p} = \mathfrak{m}, \\ 0, & \mathfrak{p} \neq \mathfrak{m}. \end{cases} \tag{4.4.2}$$

Proof. We will first prove (4.4.2) for the case $\mathfrak{p} = \mathfrak{m}$, by induction on the Krull dimension of R. In the case $\dim R = 0$, we have $\operatorname{Spec} R = \{\mathfrak{m}\}$. Thus, $\mathfrak{m}$ consists of nilpotent elements. Let $\mathfrak{m} = (f_1, \ldots, f_n)$. Then each f_i is nilpotent. Therefore, $R_{f_i} = 0$ and so $K_\infty(\mathfrak{m}) = \bigotimes_{i=1}^n K_\infty(f_i) = \bigotimes_{i=1}^n R \cong R$. We have $\operatorname{Hom}_R(K_\infty(\mathfrak{m}), I_\lambda Q_\rho(k)) \cong \operatorname{Hom}_R(R, I_\lambda Q_\rho(k)) \cong I_\lambda Q_\rho(k)$.

Now let $d > 0$ and assume that (4.4.2) (for $\mathfrak{p} = \mathfrak{m}$) holds for all local Gorenstein rings of dimension strictly less than d and suppose that $\dim R = d$. Let $x \in \mathfrak{m}$ be a regular element (such an element exists by the prime avoidance lemma and our assumption that $\dim R = d > 0$). Then $R/(x)$ is a local Gorenstein ring with residue field $(R/(x))/(\mathfrak{m}/(x)) \cong R/\mathfrak{m} = k$ and $\dim R/(x) = d-1$. By Lemma 4.1.1, $K_\infty(\mathfrak{m}/(x)) \cong R/(x) \otimes_R K_\infty(\mathfrak{m})$. We have

$$\begin{aligned} \operatorname{Hom}_R(K_\infty(\mathfrak{m}), I_\lambda Q_\rho^R(k)) &\cong \operatorname{Hom}_R(K_\infty(\mathfrak{m}), \operatorname{res} I_\lambda Q_\rho^{R/(x)}(k)) \\ &\cong \operatorname{res} \operatorname{Hom}_{R/(x)}(R/(x) \otimes_R K_\infty(\mathfrak{m}), I_\lambda Q_\rho^{R/(x)}(k)) \\ &\cong \operatorname{res} \operatorname{Hom}_{R/(x)}(K_\infty(\mathfrak{m}/(x)), I_\lambda Q_\rho^{R/(x)}(k)) \\ &\cong \operatorname{res} I_\lambda Q_\rho^{R/(x)}(k) \\ &\cong I_\lambda Q_\rho^R(k). \end{aligned}$$

The first and last isomorphisms hold because the singularity category of a Gorenstein ring is equivalent to the stable category of Gorenstein-injective modules and $\mathrm{Z}^0 I_\lambda Q_\rho = G(-)$ on modules (see Recollection 4.2.3) and by [**Ste14b**, Remark 6.11], taking Gorenstein-injective envelopes commutes with restriction. The second isomorphism follows from the internal-hom version of the adjunction

$$\mathrm{K}(R) \underset{\text{res}}{\overset{R/(x)\otimes_R -}{\rightleftarrows}} \mathrm{K}(R/(x)),$$

which asserts that the functors

$$\operatorname{res} \operatorname{Hom}_{R/(x)}(R/(x) \otimes_R -, -),\ \operatorname{Hom}_R(-, \operatorname{res}(-)) \colon \mathrm{K}(R)^{\mathrm{op}} \times \mathrm{K}(R/(x)) \to \mathrm{K}(R)$$

are naturally isomorphic. The fourth isomorphism holds by the inductive hypothesis. This completes the proof of (4.4.2) for the case $\mathfrak{p} = \mathfrak{m}$.

Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal of R. Then necessarily $\mathfrak{p} \subsetneq \mathfrak{m}$ since $\mathfrak{m}$ contains all prime ideals. We have

$$\begin{aligned} \operatorname{Hom}_R(K_\infty(\mathfrak{m}), I_\lambda Q_\rho^R(k(\mathfrak{p}))) &\cong \operatorname{Hom}_R(K_\infty(\mathfrak{m}), \operatorname{res} I_\lambda Q_\rho^{R_\mathfrak{p}}(k(\mathfrak{p}))) \\ &\cong \operatorname{res} \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p} \otimes_R K_\infty(\mathfrak{m}), I_\lambda Q_\rho^{R_\mathfrak{p}}(k(\mathfrak{p}))) \\ &= 0. \end{aligned}$$

The first two isomorphisms are justified in the same way as in the calculation in the previous paragraph, replacing $R/(x)$ with $R_\mathfrak{p}$. Since $\mathfrak{p} \subsetneq \mathfrak{m}$, it follows that $\operatorname{loc}(K_\infty(\mathfrak{m})) = \mathcal{M} \subsetneq \mathcal{P} = \operatorname{Ker}(R_\mathfrak{p} \otimes_R -)$, where $\mathcal{P}$ and $\mathcal{M}$ are the smashing subcategories of $\mathrm{D}(R)$ corresponding to $\mathfrak{p}$ and $\mathfrak{m}$, respectively. Hence, $K_\infty(\mathfrak{m}) \otimes_R R_\mathfrak{p} = 0$. This explains the last equality, completing the proof. □

The following lemma is well-known and easy to prove. We present it for the convenience of the reader.

Lemma 4.4.3. *Let $G\colon \mathcal{T} \to \mathcal{U}$ be a product-preserving triangulated functor between triangulated categories with products and let $\mathcal{X}$ be a collection of objects of $\mathcal{T}$. Then* $G\,\mathrm{coloc}(\mathcal{X}) \subseteq \mathrm{coloc}(G\mathcal{X})$.

Proof. It is straightforward to verify that $\mathcal{L} = \{X \in \mathcal{T} \mid GX \in \mathrm{coloc}(G\mathcal{X})\}$ is a colocalizing subcategory of $\mathcal{T}$ that contains $\mathcal{X}$. It follows that $\mathrm{coloc}(\mathcal{X}) \subseteq \mathcal{L}$, which proves the statement. □

Recollection 4.4.4. Let $\mathcal{T}$ be a compactly generated triangulated category. If $\mathcal{X}$ is a cogenerating set of objects of $\mathcal{T}$ (in the sense that ${}^{\perp}\mathcal{X} = 0$) that consists of pure-injective objects, then $\mathcal{T} = \mathrm{coloc}(\mathcal{X})$, which means that the smallest colocalizing subcategory of $\mathcal{T}$ that contains $\mathcal{X}$ is $\mathcal{T}$. This holds because the pure-injective objects of $\mathcal{T}$ form a perfect cogenerating set in the sense of [**Kra02**]. Also, [**BCHS23**, Section 9] provides a detailed account on the concept of perfect (co)generation.

Proposition 4.4.5. *Let R be a locally hypersurface ring. Then*

$$\mathrm{S}(R) = \mathrm{coloc}(I_\lambda Q_\rho(k(\mathfrak{p})) \mid \mathfrak{p} \in \mathrm{Sing}\, R).$$

Proof. Let $\mathcal{L} = {}^{\perp}\{\Sigma^n I_\lambda Q_\rho(k(\mathfrak{p})) \mid n \in \mathbb{Z},\, \mathfrak{p} \in \mathrm{Sing}\, R\}$. Then $\mathcal{L}$ is a localizing subcategory of $\mathrm{S}(R)$ that does not contain any $I_\lambda Q_\rho(k(\mathfrak{p}))$, for $\mathfrak{p} \in \mathrm{Sing}\, R$. Indeed, if $I_\lambda Q_\rho(k(\mathfrak{p})) \in \mathcal{L}$, then $\mathrm{Hom}_{\mathrm{S}(R)}(I_\lambda Q_\rho k(\mathfrak{p}), I_\lambda Q_\rho k(\mathfrak{p})) = 0$ and this implies that $I_\lambda Q_\rho(k(\mathfrak{p})) = 0$, which is false when $\mathfrak{p} \in \mathrm{Sing}\, R$. Consequently, by [**Ste14b**, Theorem 6.13], we have $\mathcal{L} = 0$, i.e., $\{\Sigma^n I_\lambda Q_\rho(k(\mathfrak{p})) \mid n \in \mathbb{Z},\, \mathfrak{p} \in \mathrm{Sing}\, R\}$ is a cogenerating set for $\mathrm{S}(R)$. Since, by Proposition 4.3.3, the objects $I_\lambda Q_\rho(k(\mathfrak{p}))$ are pure-injective, it follows that $\mathrm{S}(R) = \mathrm{coloc}(I_\lambda Q_\rho(k(\mathfrak{p})) \mid \mathfrak{p} \in \mathrm{Sing}\, R)$; see Recollection 4.4.4. □

Proposition 4.4.6. *Let $R = (R, \mathfrak{m}, k)$ be a hypersurface ring. Then* $[g_{\mathfrak{m}}, \mathrm{S}(R)] = \mathrm{coloc}(I_\lambda Q_\rho(k))$.

Proof. Since $[g_{\mathfrak{m}}, -] = \mathrm{Hom}_R(g_{\mathfrak{m}}, -)\colon \mathrm{S}(R) \to \mathrm{S}(R)$ is a product-preserving triangulated functor, by Lemma 4.4.3 and Proposition 4.4.5 we have $[g_{\mathfrak{m}}, \mathrm{S}(R)] = \mathrm{coloc}([g_{\mathfrak{m}}, I_\lambda Q_\rho(k(\mathfrak{p}))] \mid \mathfrak{p} \in \mathrm{Sing}\, R) = \mathrm{coloc}(I_\lambda Q_\rho(k))$, with the last equality due to Lemma 4.4.1. □

4.5. Locally hypersurface rings

The main result of this section is Theorem 4.5.7, which classifies the colocalizing subcategories of the singularity category $\mathrm{S}(R)$ of a locally hypersurface ring R in terms of the singular locus $\mathrm{Sing}\, R$.

Cosupport and costratification. Let A be an object of $\mathrm{S}(R)$. The *cosupport* of A is $\mathrm{Cosupp}(A) = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{Hom}_R(g_{\mathfrak{p}}, A) \neq 0\}$.

Lemma 4.5.1. *The assignment* $\mathrm{Cosupp}\colon \mathrm{Ob}\,\mathrm{S}(R) \to \mathcal{P}(\mathrm{Spec}\, R)$ *satisfies the following properties:*

(a) $\mathrm{Cosupp}(0) = \varnothing$.
(b) $\mathrm{Cosupp}(\prod A_i) = \bigcup \mathrm{Cosupp}(A_i)$.
(c) $\mathrm{Cosupp}(\Sigma A) = \mathrm{Cosupp}(A)$.
(d) $\mathrm{Cosupp}(A) \subseteq \mathrm{Cosupp}(B) \cup \mathrm{Cosupp}(C)$, *for all triangles $A \to B \to C$ of* $\mathrm{S}(R)$.
(e) $\mathrm{Cosupp}([X, A]) \subseteq \mathrm{Supp}(X) \cap \mathrm{Cosupp}(A)$.

Proof. See Lemma 3.2.12. □

Remark 4.5.2. Recall that by Lemma 4.1.10, all colocalizing subcategories of $\mathrm{S}(R)$ are hom-submodules and so all results concerning colocalizing hom-submodules apply to all colocalizing subcategories of $\mathrm{S}(R)$.

We denote by $I_{\mathrm{S}(R)}$ the product of the Brown–Comenetz duals of the compact objects of $\mathrm{S}(R)$. Then $I_{\mathrm{S}(R)}$ is a pure-injective cogenerator of $\mathrm{S}(R)$; see Recollection 4.4.4.

We define the maps

$$\mathrm{Coloc}(\mathrm{S}(R)) \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \mathcal{P}(\mathrm{Spec}\, R),$$

where $\sigma(\mathcal{C}) = \bigcup_{A \in \mathcal{C}} \mathrm{Cosupp}(A)$ and $\tau(W) = \{A \in \mathrm{S}(R) \mid \mathrm{Cosupp}(A) \subseteq W\}$. The maps σ and τ are inclusion-preserving. By Section 3.2 and [**Ste14b**, Proposition 5.7], it follows that $\sigma(\mathrm{S}(R)) = \mathrm{Cosupp}(I_{\mathrm{S}(R)}) = \mathrm{Sing}\, R$. Hence, $\sigma(\mathcal{C}) \subseteq \sigma(\mathrm{S}(R)) = \mathrm{Sing}\, R$. This shows that $\sigma\colon \mathrm{Coloc}(\mathrm{S}(R)) \to \mathcal{P}(\mathrm{Sing}\, R)$ is well-defined. From now on, we will consider the codomain of σ and the domain of τ to be $\mathcal{P}(\mathrm{Sing}\, R)$. It holds that $\sigma \circ \tau = \mathrm{Id}$; see Lemma 3.2.20.

Our goal is to prove that if R is a locally hypersurface ring, then $\mathrm{S}(R)$ is costratified in the sense of Definition 3.2.13, i.e., that $\tau \circ \sigma = \mathrm{Id}$. To achieve this, we will first treat the case of R being a hypersurfce ring by using the equivalent characterization of costratification given by Theorem 3.2.21 and then Theorem 3.4.4 will allow us to prove the more general case of locally hypersurface rings. Let us begin by spelling out in detail what it means for $\mathrm{S}(R)$ to satisfy the colocal-to-global principle and cominimality. According to Definition 3.2.15 (taking into account Lemma 4.1.10):

(a) $\mathrm{S}(R)$ satisfies the *colocal-to-global principle* if

$$\mathrm{coloc}(A) = \mathrm{coloc}(\mathrm{Hom}_R(g_{\mathfrak{p}}, A) \mid \mathfrak{p} \in \mathrm{Sing}\, R),\ \forall A \in \mathrm{S}(R).$$

(b) $\mathrm{S}(R)$ satisfies *cominimality* if $\mathrm{coloc}(\mathrm{Hom}_R(g_{\mathfrak{p}}, I_{\mathrm{S}(R)}))$ is a minimal colocalizing subcategory of $\mathrm{S}(R)$, $\forall \mathfrak{p} \in \mathrm{Sing}\, R$.

Theorem 3.2.21 states that $\mathrm{S}(R)$ is costratified if and only if $\mathrm{S}(R)$ satisfies the colocal-to-global principle and cominimality.

Colocalizing subcategories of $\mathrm{S}(R)$, for a locally hypersurface ring R.

Lemma 4.5.3. *Let $R = (R, \mathfrak{m}, k)$ be a local Gorenstein ring and let $x \in R$ be a regular element. Let G be a non-zero object of $[g_{\mathfrak{m}}, \mathrm{S}(R)]$ and set $\widetilde{M} = \mathrm{Hom}_R(R/(x), G)$ viewed as a complex of $R/(x)$-modules. Then $\widetilde{M} \in [g_{\mathfrak{m}/(x)}, \mathrm{S}(R/(x))]$ and $\widetilde{M} \neq 0$.*

PROOF. Since $g_{\mathfrak{m}} = K_\infty(\mathfrak{m})$ is a left idempotent, it holds that $[g_{\mathfrak{m}}, \mathrm{S}(R)] = \mathrm{Im}[g_{\mathfrak{m}}, -]$. Therefore, $\mathrm{Hom}_R(g_{\mathfrak{m}}, G) = [g_{\mathfrak{m}}, G] \cong G$. We have

$$\begin{aligned}
\mathrm{res}[g_{\mathfrak{m}/(x)}, \widetilde{M}] &= \mathrm{res}\,\mathrm{Hom}_{R/(x)}(g_{\mathfrak{m}} \otimes_R R/(x), \widetilde{M}) \\
&\cong \mathrm{Hom}_R(g_{\mathfrak{m}}, \mathrm{res}\,\widetilde{M}) \\
&= \mathrm{Hom}_R(g_{\mathfrak{m}}, \mathrm{Hom}_R(R/(x), G)) \\
&\cong \mathrm{Hom}_R(g_{\mathfrak{m}} \otimes_R R/(x), G) \\
&\cong \mathrm{Hom}_R(R/(x), \mathrm{Hom}_R(g_{\mathfrak{m}}, G)) \\
&\cong \mathrm{Hom}_R(R/(x), G) \\
&= \mathrm{res}\,\widetilde{M}.
\end{aligned}$$

Hence, $[g_{\mathfrak{m}/(x)}, \widetilde{M}] \cong \widetilde{M}$. This shows that $\widetilde{M} \in \mathrm{Im}[g_{\mathfrak{m}/(x)}, -] = [g_{\mathfrak{m}/(x)}, \mathrm{S}(R/(x))]$. By the equivalence between $\mathrm{S}(R)$ and $\underline{\mathrm{GInj}}\,R$, as described in Recollection 4.2.3, if G' is the object of $\underline{\mathrm{GInj}}\,R$ corresponding to G, then $\mathrm{pd}_R \mathrm{Hom}_R(R/(x), G') = \mathrm{id}_R \mathrm{Hom}_R(R/(x), G') = \infty$; see [**Ste14b**, Lemma 6.6]. Thus, $\mathrm{Hom}_R(R/(x), G') \neq 0$ and so $\mathrm{Hom}_R(R/(x), G) \neq 0$. Since $\mathrm{Hom}_R(R/(x), G)$ is the restriction of $\widetilde{M}$, we conclude that $\widetilde{M} \neq 0$. □

A compactly generated triangulated category $\mathcal{T}$ is called *pure-semisimple* if every object of $\mathcal{T}$ is pure-injective.

Lemma 4.5.4. *Let $\mathcal{T}$ be a pure-semisimple triangulated category such that the only localizing subcategories of $\mathcal{T}$ are 0 and $\mathcal{T}$. Then the only colocalizing subcategories of $\mathcal{T}$ are 0 and $\mathcal{T}$.*

PROOF. Let X be a non-zero object of $\mathcal{T}$. Then ${}^\perp\mathrm{coloc}(X)$ is either 0 or $\mathcal{T}$. The latter is false, since in that case X would have to be 0. So, ${}^\perp\{\Sigma^n X \mid n \in \mathbb{Z}\} = {}^\perp\mathrm{coloc}(X) = 0$. This means that the set of suspensions of X is a cogenerating set of $\mathcal{T}$. Since $\mathcal{T}$ is pure-semisimple, X is pure-injective. Consequently, by Recollection 4.4.4, $\mathcal{T} = \mathrm{coloc}(X)$. This shows that $\mathcal{T}$ is cogenerated by any of its non-zero objects. As a result, the only colocalizing subcategories of $\mathcal{T}$ are 0 and $\mathcal{T}$. □

Proposition 4.5.5. *Let R be an artinian hypersurface ring with unique maximal ideal $\mathfrak{m}$. Then $[g_{\mathfrak{m}}, \underline{\mathrm{GInj}}\,R]$ (resp. $[g_{\mathfrak{m}}, \mathrm{S}(R)]$) is a minimal colocalizing subcategory of $\underline{\mathrm{GInj}}\,R$ (resp. $\mathrm{S}(R)$).*

PROOF. Every R-module is Gorenstein-injective, i.e., $\underline{\mathrm{Mod}}\,R = \underline{\mathrm{GInj}}\,R$ and further, $\underline{\mathrm{Mod}}\,R$ is a pure-semisimple compactly generated triangulated category; see the explanations in the proof of [**Ste14b**, Lemma 6.8]. It follows by Lemma 4.5.4 that the only colocalizing subcategories of $\underline{\mathrm{Mod}}\,R$ are 0 and $\underline{\mathrm{Mod}}\,R$. Consequently, $[g_{\mathfrak{m}}, \underline{\mathrm{GInj}}\,R] = \underline{\mathrm{Mod}}\,R = \underline{\mathrm{GInj}}\,R$ is a minimal colocalizing subcategory of $\underline{\mathrm{GInj}}\,R$. By the equivalence $\mathrm{S}(R) \simeq \underline{\mathrm{GInj}}\,R$, it also holds that $[g_{\mathfrak{m}}, \mathrm{S}(R)] = \mathrm{S}(R)$ is a minimal colocalizing subcategory of $\mathrm{S}(R)$. □

Proposition 4.5.6. *Let $R = (R, \mathfrak{m}, k)$ be a hypersurface ring. Then $[g_{\mathfrak{m}}, \mathrm{S}(R)]$ is a minimal colocalizing subcategory of $\mathrm{S}(R)$.*

PROOF. If $\dim R = 0$, then R is an artinian hypersurface and the claim holds by Proposition 4.5.5. Now suppose that $\dim R = n > 0$ and that the claim holds for all hypersurface rings of dimension strictly less than n. There exists a regular element $x \in R$ such that $R/(x)$ is a hypersurface and $\dim R/(x) = n-1$; see the explanations in the proof of [**Ste14b**, Theorem 6.12]. Let G be a non-zero object of $[g_{\mathfrak{m}}, \mathrm{S}(R)]$ and set $M = \mathrm{Hom}_R(R/(x), G)$ and denote by $\widetilde{M}$ the complex M viewed as a complex of $R/(x)$-modules. By Lemma 4.5.3, it holds that $\widetilde{M} \in [g_{\mathfrak{m}/(x)}, \mathrm{S}(R/(x))]$. Further, $\mathrm{res}\,\widetilde{M} = M \in \mathrm{coloc}(G)$. The last assertion holds because $\mathrm{coloc}(G)$ is a hom-submodule; see Lemma 4.1.10 and Recollection 4.2.3. By the inductive hypothesis, $[g_{\mathfrak{m}/(x)}, \mathrm{S}(R/(x))]$ is minimal and by Lemma 4.5.3, $\widetilde{M} \neq 0$. Hence, $\mathrm{coloc}(\widetilde{M}) = [g_{\mathfrak{m}/(x)}, \mathrm{S}(R/(x))] = \mathrm{coloc}(I_\lambda Q_\rho^{R/(x)}(k))$, with the last equality by Proposition 4.4.6. Since res is a product-preserving triangulated functor, it follows by Lemma 4.4.3 that $\mathrm{res}\,\mathrm{coloc}(\widetilde{M}) \subseteq \mathrm{coloc}(\mathrm{res}\,\widetilde{M}) = \mathrm{coloc}(M)$. The latter is contained in $\mathrm{coloc}(G)$. Consequently, $\mathrm{res}\,\mathrm{coloc}(I_\lambda Q_\rho^{R/(x)}(k)) \subseteq \mathrm{coloc}(G)$ and so $I_\lambda Q_\rho^R(k) = \mathrm{res}\, I_\lambda Q_\rho^{R/(x)}(k) \in \mathrm{coloc}(G)$. So, $\mathrm{coloc}(I_\lambda Q_\rho^R(k)) \subseteq \mathrm{coloc}(G)$. We infer that $\mathrm{coloc}(G) = \mathrm{coloc}(I_\lambda Q_\rho^R(k)) = [g_{\mathfrak{m}}, \mathrm{S}(R)]$ (with the last equality by Proposition 4.4.6) and so $[g_{\mathfrak{m}}, \mathrm{S}(R)]$ is a minimal colocalizing subcategory of $\mathrm{S}(R)$. □

Theorem 4.5.7. *Let R be a locally hypersurface ring. Then $\mathrm{S}(R)$ is costratified.*

Proof. Since $\mathrm{D}(R)$ satisfies the local-to-global principle, $\mathrm{S}(R)$ satisfies the colocal-to-global principle; see Proposition 3.2.25. Let $\mathfrak{p} \in \mathrm{Sing}(R)$ and set $\mathcal{M}_{\mathfrak{p}} = \mathrm{Ker}(R_{\mathfrak{p}} \otimes_R -\colon \mathrm{S}(R) \to \mathrm{S}(R))$. Then $\mathcal{M}_{\mathfrak{p}}$ is a smashing subcategory of $\mathrm{S}(R)$ and $\mathrm{S}(R_{\mathfrak{p}}) \simeq \mathrm{S}(R)/\mathcal{M}_{\mathfrak{p}}$ and ${\mathcal{M}_{\mathfrak{p}}}^{\perp} = \mathrm{Im}(R_{\mathfrak{p}} \otimes_R -) = \mathrm{Im}[R_{\mathfrak{p}}, -]$. So, $[g_{\mathfrak{p}}, I_{\mathrm{S}(R)}] = [K_\infty(\mathfrak{p}) \otimes_R R_{\mathfrak{p}}, I_{\mathrm{S}(R)}] \cong [R_{\mathfrak{p}}, [K_\infty(\mathfrak{p}), I_{\mathrm{S}(R)}]] \in {\mathcal{M}_{\mathfrak{p}}}^{\perp}$. Since $R_{\mathfrak{p}}$ is a hypersurface, by Proposition 4.5.6, $\mathrm{S}(R_{\mathfrak{p}})$ satisfies cominimality at the unique closed point of $\mathrm{Spec}\, R_{\mathfrak{p}}$. By Proposition 3.4.3, $\mathrm{S}(R)$ satisfies cominimality at $\mathfrak{p}$. Hence, $\mathrm{S}(R)$ is costratified; see Theorem 3.2.21. □

Recall from Definition 3.3.3 that a proper colocalizing subcategory $\mathcal{C}$ of $\mathrm{S}(R)$ is called hom-*prime* if, for all $X \in \mathrm{D}(R)$ and $A \in \mathrm{S}(R)$, if $[X, A] \in \mathcal{C}$, then $[X, I_{\mathrm{S}(R)}] \in \mathcal{C}$ or $A \in \mathcal{C}$.

Theorem 4.5.8. *Let R be a locally hypersurface ring. Then there is a bijective correspondence between points of* $\mathrm{Sing}\, R$ *and* hom-*prime colocalizing subcategories of* $\mathrm{S}(R)$. *A point* $\mathfrak{p} \in \mathrm{Sing}\, R$ *is associated with* $\mathrm{Ker}(\mathrm{Hom}_R(g_{\mathfrak{p}}, -)\colon \mathrm{S}(R) \to \mathrm{S}(R))$, *with the latter being equal to* $\mathrm{coloc}(\mathrm{Hom}_R(g_{\mathfrak{q}}, I_{\mathrm{S}(R)}) \mid \mathfrak{q} \neq \mathfrak{p})$.

Proof. According to Theorem 4.5.7, $\mathrm{S}(R)$ is costratified. By Lemma 4.1.10, every colocalizing subcategory of $\mathrm{S}(R)$ is a hom-submodule. Further, $\mathrm{Sing}\, R = \mathrm{Cosupp}(I_{\mathrm{S}(R)})$. The claim now follows by applying Theorem 3.3.10. □

Example 4.5.9. Let k be a field. Then $R = k[x]/(x^2)$ is a commutative noetherian local ring with unique maximal ideal $\mathfrak{m} = (x)/(x^2)$ and residue field k. In fact, $\mathrm{Spec}\, R = \{\mathfrak{m}\}$, so $\dim R = 0$. The sequence $\cdots \to R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \to k \to 0$ is a minimal projective resolution of k and so $\mathrm{pd}_R\, k = \infty$, implying that $\mathrm{gldim}\, R = \infty$. Hence, R is not a regular ring. It follows that $\mathrm{Sing}\, R = \mathrm{Spec}\, R$. Applying the functor $\mathrm{Hom}_R(-, R)$ to the above projective resolution (after deleting k) results in the sequence $0 \to R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \to \cdots$, which is exact. Therefore, $\mathrm{Ext}_R^n(k, R) = 0$, $\forall n \geq 1$. Since R is a commutative noetherian local ring of finite Krull dimension (equal to 0), by [**Mat89**, Theorem 18.1], it follows that $\mathrm{id}_R R = 0$. Hence, R is a Gorenstein ring. The ideal $\mathfrak{m}$ is nilpotent ($\mathfrak{m}^2 = 0$) and so the completion of R with respect to $\mathfrak{m}$ is isomorphic to R, i.e., R is a complete ring. Since $k[x]$ is regular and x^2 is a regular element of $k[x]$, it follows that R is a hypersurface ring. By Theorem 4.5.7, it follows that $\mathrm{S}(R)$ is costratified and the colocalizing subcategories of $\mathrm{S}(R)$ stand in bijection with $\mathcal{P}(\operatorname{Sing} R) = \{\varnothing, \operatorname{Sing} R\}$ and so $\operatorname{Coloc}(\mathrm{S}(R)) = \{0, \mathrm{S}(R)\}$. One could of course reach this conclusion by invoking Lemma 4.5.4, since R is a hypersurface ring of Krull dimension zero, so $\mathrm{S}(R)$ is pure-semisimple and the only localizing subcategories of $\mathrm{S}(R)$ are 0 and $\mathrm{S}(R)$. The unique hom-prime colocalizing subcategory of $\mathrm{S}(R)$ is 0 and corresponds to the unique point $\mathfrak{m}$ of $\operatorname{Spec} R$.

4.6. Schemes with hypersurface singularities

In this section, we generalize Theorem 4.5.7 to schemes with hypersurface singularities. First let us recall some facts about derived categories of schemes.

Let X be a noetherian separated scheme with structure sheaf $\mathcal{O}_X$. We denote by $\operatorname{QCoh} X$ the abelian category of quasi-coherent $\mathcal{O}_X$-modules and by $\mathrm{D}(X)$ the derived category of $\operatorname{QCoh} X$. The derived category $\mathrm{D}(X)$ is a rigidly-compactly generated tensor-triangulated category with tensor product the left derived tensor product of complexes of $\mathcal{O}_X$-modules and unit $\mathcal{O}_X$ concentrated in degree zero. The subcategory of compact objects of $\mathrm{D}(X)$ is $\mathrm{D}^{\mathrm{perf}}(X)$ the subcategory of complexes that are locally quasi-isomorphic to bounded complexes of locally free sheaves of $\mathcal{O}_X$-modules. There is a homeomorphism $\operatorname{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \cong X$ and we will treat this as an equality. The *singularity category* of X is $\mathrm{S}(X) = \mathrm{K}_{\mathrm{ac}}(\operatorname{Inj} X)$ the homotopy category of acyclic complexes of injective quasi-coherent $\mathcal{O}_X$-modules, which is a compactly generated triangulated category by [**Kra05b**].

The results in the following discussion can be found in [**Ste14b**, Section 7]. Let U be an open subset of X and let $Z = X \setminus U$. Let $\mathrm{D}(X)_Z$ be the localizing subcategory of $\mathrm{D}(X)$ generated by those compact objects supported on Z. We denote by $\mathrm{D}(X)(U)$ the category $\mathrm{D}(X)/\mathrm{D}(X)_Z$. Then there is an equivalence $\mathrm{D}(X)(U) \simeq \mathrm{D}(U)$. There is an action of $\mathrm{D}(X)$ on $\mathrm{S}(X)$ that induces a support theory for objects of $\mathrm{S}(X)$ (and a cosupport theory in the sense of Chapter 3). We denote by $\mathrm{S}(X)(U)$ the localizing subcategory of $\mathrm{S}(X)$ generated by those compact objects supported on Z. The category $\mathrm{S}(X)(U)$ is equivalent to $\mathrm{S}(U)$ and the action of $\mathrm{D}(X)$ on $\mathrm{S}(X)$ gives rise to an action of $\mathrm{D}(U)$ on $\mathrm{S}(U)$. If $\{U_i \cong \operatorname{Spec} R_i\}$ is an open affine cover of X, then the singular locus of X is $\operatorname{Sing} X = \bigcup \operatorname{Sing} R_i$.

Theorem 4.6.1. *Let X be a noetherian separated scheme with hypersurface singularities. Then $\mathrm{S}(X)$ is costratified, i.e., there is a bijective correspondence between $\operatorname{Sing} X$ and the collection of colocalizing* hom-*submodules of $\mathrm{S}(X)$ given by mapping a colocalizing* hom-*submodule of $\mathrm{S}(X)$ to its cosupport.*

Proof. Let $X = \bigcup U_i$ be an open affine cover of X. Then each U_i is isomorphic to $\operatorname{Spec} R_i$ for a commutative noetherian ring R_i that is locally a hypersurface. By the above discussion, we have an action of $\mathrm{D}(U_i) = \mathrm{D}(R_i)$ on $\mathrm{S}(U_i) = \mathrm{S}(R_i)$ and since R_i is locally a hypersurface, $\mathrm{S}(R_i)$ is costratified by Theorem 4.5.7. A direct application of Theorem 3.4.9 implies that $\mathrm{S}(X)$ is costratified. □

Theorem 4.6.2. *Let X be a noetherian separated scheme with hypersurface singularities. Then there is a bijective correspondence between points of $\operatorname{Sing} X$ and* hom-*prime colocalizing submodules of $\mathrm{S}(X)$. A point $x \in \operatorname{Sing} X$ is associated with $\operatorname{Ker}([g_x, -]\colon \mathrm{S}(X) \to \mathrm{S}(X))$, with the latter being equal to $\operatorname{coloc}([g_y, I_{\mathrm{S}(X)}] \mid y \neq x)$.*

PROOF. By Theorem 4.6.1, $\mathrm{S}(X)$ is costratified. The result now follows immediately from Theorem 3.3.10. $\square$

Bibliography

[AJS04] L. Alonso Tarrío, A. Jeremías López, and M. J. Souto Salorio. Bousfield localization on formal schemes. *J. Algebra*, 278(2):585–610, 2004.

[AS16] Benjamin Antieau and Greg Stevenson. Derived categories of representations of small categories over commutative noetherian rings. *Pacific. J. Math.*, 283(1):21–42, 2016.

[AT62] C. E. Aull and W. J. Thron. Separation axioms between T_0 and T_1. *Indag. Math.*, 24:26–37, 1962.

[Bal05] Paul Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[Bal20a] Paul Balmer. Homological support of big objects in tensor-triangulated categories. *J. Éc. polytech. Math.*, 7:1069–1088, 2020.

[Bal20b] Paul Balmer. Nilpotence theorems via homological residue fields. *Tunis. J. Math.*, 2(2):359–378, 2020.

[BC21] Paul Balmer and James Cameron. Computing homological residue fields in algebra and topology. *Proc. Amer. Math. Soc.*, 149(8):3177–3185, 2021.

[BCHS23] Tobias Barthel, Natalia Castellana, Drew Heard, and Beren Sanders. Cosupport in tensor-triangular geometry. Preprint arXiv:2303.13480, 2023.

[BCR97] David J. Benson, Jon F. Carlson, and Jeremy Rickard. Thick subcategories of the stable module category. *Fund. Math.*, 153:59–80, 1997.

[BDS16] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders. Grothendieck-Neeman duality and the Wirthmüller isomorphism. *Compos. Math.*, 152(8):1740–1776, 2016.

[Bel00] Apostolos Beligiannis. Relative homological algebra and purity in triangulated categories. *J. Algebra*, 227:268–361, 2000.

[BF07] Paul Balmer and Giordano Favi. Gluing techniques in triangular geometry. *Q. J. Math.*, 58(4):415–441, 2007.

[BF11] Paul Balmer and Giordano Favi. Generalized tensor idempotents and the telescope conjecture. *Proc. Lond. Math. Soc. (3)*, 102(6):1161–1185, 2011.

[BHS23a] Tobias Barthel, Drew Heard, and Beren Sanders. Stratification and the comparison between homological and tensor triangular support. *Q. J. Math.*, 74(2):747–766, 2023.

[BHS23b] Tobias Barthel, Drew Heard, and Beren Sanders. Stratification in tensor triangular geometry with applications to spectral Mackey functors. *Camb. J. Math.*, 11(4):829–915, 2023.

[BIK11a] David J. Benson, Srikanth B. Iyengar, and Henning Krause. Stratifying triangulated categories. *J. Topol.*, 4(3):641–666, 2011.

[BIK11b] David J. Benson, Srikanth B. Iyengar, and Henning Krause. Stratifying modular representations of finite groups. *Ann. of Math. (2)*, 174(3):1643–1684, 2011.

[BIK12] David J. Benson, Srikanth B. Iyengar, and Henning Krause. Colocalizing subcategories and cosupport. *J. Reine Angew. Math.*, 673:161–207, 2012.

[BKS20] Paul Balmer, Henning Krause, and Greg Stevenson. The frame of smashing tensor-ideals. *Math. Proc. Cambridge Philos. Soc.*, 168(2):323–343, 2020.

[BŠ17] Silvana Bazzoni and Jan Šťovíček. Smashing localizations of rings of weak global dimension at most one. *Adv. Math.*, 305:351–401, 2017.

[BS23] Scott Balchin and Greg Stevenson. Big categories, big spectra. Preprint arXiv:2109.11934, 2023.

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. Nilpotence and stable homotopy theory I. *Ann. of Math.*, 128(2):207–241, 1988.

[EJ00] Edgar E. Enochs and Overtoun M. G. Jenda. Relative homological algebra, volume 30 of de Gruyter Expositions in Mathematics. Walter de Gruyter & Co., Berlin, 2000.

[Emm23] Ioannis Emmanouil. K-flatness and orthogonality in homotopy categories. *Isr J. Math.*, 255:201–230, 2023.

[Fox79] Hans-Bjørn Foxby. Bounded complexes of flat modules. *J. Pure Appl. Algebra (2)*, 15:149–172, 1979.

[Jon82] Peter T. Johnstone. Stone spaces, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1982.

[Kel94] Bernhard Keller. A remark on the generalized smashing conjecture. *Manuscripta Math.*, 84(2):192–198, 1994

[Kra00] Henning Krause. Smashing subcategories and the telescope conjecture – an algebraic approach. *Invent. Math.*, 139:99–133, 2000.

[Kra02] Henning Krause. A Brown representability theorem via coherent functors. *Topology*, 41(4):853–861, 2002.

[Kra05a] Henning Krause. Cohomological quotients and smashing localizations. *Amer. J. Math.*, 127(6):1191–1246, 2005.

[Kra05b] Henning Krause. The stable derived category of a Noetherian scheme. *Compos. Math.*, 141(5):1128–1162, 2005.

[Kra23] Henning Krause. Completions of triangulated categories. Preprint arXiv:2309.01260, 2023.

[Kra99] Henning Krause. Decomposing thick subcategories of the stable module category. *Math. Ann.*, 313:95–108, 1999.

[KR00] Henning Krause and Ulrike Reichenbach. Endofiniteness in stable homotopy theory. *Transactions of the AMS*, 353(1):157–173, 2000.

[KS17] Henning Krause and Greg Stevenson. The derived category of the projective line. In *Spectral Structures and Topological Methods in Mathematics*, 275–297. EMS Publishing House, 2019.

[Mac98] Saunders Mac Lane. Categories for the working mathematician, volume 5 of *Graduate Texts in Mathematics*, Springer–Verlag, New York, second edition, 1998.

[Mat89] Hideyuki Matsumura. Commutative ring theory. *Cambridge Studies in Advanced Mathematics, Cambridge University Press*, 1989.

[May01] John P. May. The additivity of traces in triangulated categories. *Adv. Math.*, 163(1):34–73, 2001.

[Mil92] Haynes Miller. Finite localizations. *Bol. Soc. Mat. Mexicana (2)*, 37(1-2):383–389, 1992. Papers in honor of José Adem (Spanish).

[Mur07] Daniel Murfet. The mock homotopy category of projectives and Grothendieck duality. Ph.D. thesis, 2007.

[Nee01] Amnon Neeman. Triangulated categories, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton NJ, 2001.

[Nee11] Amnon Neeman. Colocalizing subcategories of D(R). *J. Reine Angew. Math.*, 653:221–243, 2011.

[Nee92] Amnon Neeman. The chromatic tower for $D(R)$. *Topology*, 31:519–532, 1992.

[Nee96] Amnon Neeman. The Grothendieck duality theorem via Bousfield's techniques and Brown representability. *J. Amer. Math. Soc.*, 9(1):205–236, 1996.

[Nee98] Amnon Neeman. Brown representability for the dual. *Invent. Math.*, 133:97–105, 1998.

[Orl06] Dmitri Orlov. Triangulated categories of singularities and D-branes in Landau–Ginzburg models. *Tr. Mat. Inst. Steklova*, 246(3):227–248, 2006

[Ric96] Jeremy Rickard. Idempotent modules in the stable category. *J. London Math. Soc. (2)*, 56(1):149–170, 1997.

[Rie16] Emily Riehl. Category Theory in Context. Aurora Modern Math Originals. Dover Publications, 2016.

[Ste13] Greg Stevenson. Support theory via actions of tensor triangulated categories. *J. Reine Angew. Math.*, 681:219–254, 2013.

[Ste14a] Greg Stevenson. Derived categories of absolutely flat rings. *Homology, Homotopy and Applications*, 16(2):45–64, 2014.

[Ste14b] Greg Stevenson. Subcategories of singularity categories via tensor actions. *Compos. Math.*, 150:229–272, 2014.

[Ste17] Greg Stevenson. The local-to-global principle for triangulated categories via dimension functions. *J. Algebra*, 473:406–429, 2017.

[Ste18] Greg Stevenson. A tour of support theory for triangulated categories through tensor triangular geometry. In *Building Bridges Between Algebra and Topology*, 63–101. Birkhäuser, 2018.

[Sto14] Jan Št'ovíček. On purity and applications to coderived and singularity categories. Preprint arXiv:1412.1615, 2014.

[SW21] Liran Shaul and Jordan Williamson. Lifting (co)stratifications between tensor triangulated categories. Isr. J. Math., 2023.

[Tho18] Peder Thompson. Cosupport computations for finitely generated modules over commutative noetherian rings. *J. Algebra*, 511:249–269, 2018.

[Tho97] Robert. W. Thomason. The classification of triangulated categories. *Compos. Math.*, 105:1–27, 1997.

[Ver23a] Charalampos Verasdanis. Costratification and actions of tensor-triangulated categories. Preprint arXiv:2211.04139, 2023.

[Ver23b] Charalampos Verasdanis. Colocalizing subcategories of singularity categories. *J. Algebra*, 662:608–624, 2025.

[Ver23c] Charalampos Verasdanis. Stratification and the smashing spectrum. *Math. Zeit.*, 305:54, 2023.