Essays on Contemporary Monetary and Fiscal Policies

Vasileios Rafail Karaferis

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Adam Smith Business School
College of Social Sciences
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Abstract

The thesis consists of three chapters. All chapters study monetary and fiscal policy interactions in tractable heterogeneous agent economies. Tractable Heterogeneous Agents New Keynesian (THANK) models use simplifying assumptions regarding how inequality enters the economy and allow us to study the qualitative rather than the quantitative difference between heterogeneous agent New Keynesian models and the nested representative agent framework. These simplifications allow us to observe the inter-temporal trade-offs and study the distributional consequences of monetary and/fiscal policy in general equilibrium settings, without having to sacrifice the analytical tractability of the key results. As such, throughout this thesis we use these frameworks to study monetary and fiscal policy interactions.

In chapter one, we aim to better understand how the Heterogeneous Agent New Keynesian (HANK) economy differs from the nested Representative Agent New Keynesian (RANK) framework and how the policy mix is affected. We employ the analytically tractable HANK framework of Acharya et al. (2023) that combines the overlapping generations structure of Blanchard (1985) with the incomplete market model of Aiyagari (1994). We augment the model along two dimensions. First, we introduce a general debt structure and depart from the assumption that aggregate debt exists in zero net supply. An assumption that limits the models predictions regarding the long-run equilibrium as well as its dynamics. Next, we introduce "declining labour efficiencies", as a proxy for the time spent in retirement, which results in richer inter- generational wealth inequality. We show that the parameter space that ensures a unique rational expectation equilibrium changes as we add more layers of heterogeneity. Finally, we argue that in the absence of aggregate risk, the fiscally- led policy mix (AF/PM) is preferred by a policy maker who values "equity" more than "efficiency" to the alternative monetary-led policy mix (AM/PF) in response to a transitionary (" mit") shock since it causes smaller deviations in inequality as well as in the MPC -out-of- cash on hand.

Next, in chapter two, we extend once more the framework of Acharya et al. (2023) to analyse optimal monetary and fiscal policy in a tractable heterogeneous agent New Keynesian (HANK) economy where overlapping generations of households wish to save for retirement and precautionary reasons. While monetary policy can affect the households' ability to self-insure against

shocks, fiscal policy has a greater impact on such behaviour both in steady-state and in response to aggregate shocks. A policy maker, even one wishing to minimize inequality solely, would, in steady state, provide insufficient government debt to enable households to save for retirement and accumulate precautionary savings. This is because they prefer to suppress interest rates below households' rate of time preference, facilitating borrowing in the face of idiosyncratic shocks. The Ramsey policy maker faces a trade-off between " equity" and " efficiency" and due to the costs of servicing that debt with distortionary taxation will issue even less debt, driving equilibrium interest rates down further. We explore the relative efficacy of monetary and fiscal policy in responding to aggregate shocks in this environment, under different tax instruments.

Furthermore, in chapter three, we study optimal monetary policy in a tractable HANK environment with meaningful amount of government debt. The model admits both idiosyncratic and aggregate risk. The idiosyncratic shocks are uncorrelated between each other as well as with the aggregate shock. We assume that there exists a consolidated monetary- fiscal authority. The monetary authority pursues optimal (Ramsey) monetary policy whilst the fiscal authority follows a simple non- linear tax rule. Our aim is to provide a clear distinction between the notions of discontinuous labour market participation (DLMP) and infrequent asset market participation (IAMP), which are typically intertwined in the literature. In a HANK- DLMP model, constrained households are able to use assets to smooth their inter- temporal consumption. As such, the long run equilibrium as well as the model's dynamics under optimal monetary policy are different from both the nested representative agent model and from the HANK- IAMP framework. We demonstrate that DLMP frictions are an important source of heterogeneity on their own merit and should not be overlooked. Finally, we find that despite the presence of imperfect risk sharing, the model delivers perfect self-insurance and the policy maker in our framework will not deviate from price stability in steady state (Woodford (2003)). This result is unaffected by the amount of outstanding government debt or the presence of direct redistribution.

Finally, throughout this thesis we investigate how household heterogeneity affects the conduct of monetary and fiscal policy. We explore the merits of different modelling assumptions regarding inequality and focus on how policy changes as we add more layers of heterogeneity. All models are calibrated for the US economy for the period 1985- 2021.

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List of Abbreviations

AF Active fiscal

AM Active monetary

BY Blanchard- Yaari

CARA Constant absolute risk aversion

CRRA Constant relative risk aversion

DIP Declining income profiles

DLMP Discontinuous labour market participation

HA Heterogeneous agents

HAIM Heterogeneous agent incomplete markets

HANK Heterogeneous agent New Keynesian

IAMP Infrequent asset market participation

iid Independently and identically distributed

IM Incomplete markets

MPC Marginal propensity to consume

MU Marginal Utility

NK New Keynesian

OLG Overlapping generations

PF Passive fiscal

PM Passive monetary

PRANK Pseudo- Representative agent New Keynesian

PY Perpetual Youth

RA Representative agent

RANK Representative agent New Keynesian

ss Steady State

TANK Two- agent New Keynesian

THANK Tractable Heterogeneous agent New Keynesian

Dedication of Thesis

Στην οικογένειά μου και στην μνήμη του Λουκά μου...

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Declaration

“I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.“

Signature

Vasileios Rafail Karaferis

Motivation

The main research theme of this thesis is the interaction between monetary and fiscal policy in a tractable heterogeneous agent environment. The first and second chapters build on the seminal work of Acharya and Dogra (2020) and Acharya et al. (2023), who created a tractable OLG-HANK framework to study "monetary" puzzles and optimal monetary policy, respectively. This model combines the Perpetual Youth (PY, henceforth) household structure of Blanchard (1985) with the standard incomplete market heterogeneous agent model of the Aiyagari (1994) tradition in a standard New Keynesian environment. Hence, we refer to this class of models as OLG-HANK.

Our contribution lies in developing the fiscal side of the framework to jointly study monetary and fiscal policy. This is done by introducing a meaningful supply of government bonds, assets of different maturities, different tax instruments, and phased retirement.

The motivation behind this extension has been twofold. First, monetary policy affects inequality through the real interest rate, and these effects are transitory. This means that the duration of the effects is linked to price stickiness. More specifically, in this framework, monetary policy affects inequality through the "income-risk" channel and the "self-insurance" channel. That is, the cyclicality of the variance of the partially uninsurable income risk will influence the extent to which monetary policy can mitigate the magnitude of the idiosyncratic risk following an unanticipated aggregate shocks ("income-risk" channel). Whereas the "self-insurance" channel indicates that by lowering the interest rate, the cost of borrowing reduces, and households are able to use assets to smooth their consumption and partially self-insure against adverse realizations of the idiosyncratic income shock.

On the other hand, fiscal policy has a permanent effect on inequality. Our economic environment features an endogenously determined non-trivial wealth distribution without imposing exogenous binding borrowing limits; in fact, it shapes the distribution of consumption in the economy. In this framework, distortionary taxes, as well as the supply of government bonds, fulfill roles that are not present in the standard New Keynesian model. Distortionary taxes still cause efficiency losses as they discourage households' supply of labor. However, they also reduce the variance between the pre-tax and post-tax labor income. Since high taxation diminishes the difference in the labor income of households who draw high and those who draw low realizations of the idiosyncratic shock, causing intra-generational income inequality to drop. Still, higher taxation reduces households' ability to borrow against their expected future income. Similarly, a higher supply of government bonds allows households to at least partially fulfill their desire to save for retirement as well as for precautionary reasons but at the cost of higher servicing costs.

Additionally, contrary to the original paper, we also depart from the assumption that newly-born agents receive a lump-sum transfer that essentially guarantees that all cohorts enjoy the same average consumption and wealth, effectively eliminating inter-generational wealth inequality. Furthermore, the assumption of phased retirement acts as another form of inequality since it increases the magnitude of the OLG channel, increasing households' desire to save and smooth their consumption over their lifetimes.

Thus, the model manages to incorporate intra-generational income inequality as well as inter-generational wealth inequality without losing its analytic tractability. This is made possible by relying on constant absolute risk aversion (CARA, henceforth) household preferences and normally distributed idiosyncratic shocks while abstaining from aggregate risk. As such, when investigating the model's dynamics, we consider a one-time unanticipated shock to the perfect foresight path.

In the first chapter, we look at monetary and fiscal policy interactions in an economy where both policies are conducted by following simple rules. In this environment, the real interest rate deviates from its steady-state value only if inflation deviates from the steady-state target of zero inflation. Similarly, taxes deviate from their equilibrium value if and only if the value of the aggregate government debt deviates from the exogenously given target. We examine how the steady-state allocations, determinacy properties, as well as the system's response to aggregate (unanticipated) shocks depend on each added layer of heterogeneity and on the fiscal instruments available.

In the second chapter, we focus on optimal monetary and fiscal policy. The model features a policymaker who combines the powers and responsibilities of the monetary and fiscal authorities and has access to commitment technology. The policymaker considers the inequality present in the economy when optimally deciding on the level of policy instruments. Specifically, these instruments include the real interest rate and the aggregate supply of government bonds financed by distortionary income taxes. Households still wish to save for retirement and insure against idiosyncratic risk. Thus, we define a "golden rule" of steady-state savings as a benchmark where the government supplies enough bonds to satisfy households' consumption-smoothing desires, which in turn pushes the steady-state real interest rate to align with the rate of time preference. This question of a modified "golden" rule of savings is not new to the OLG literature, but our work creates a benchmark due to the inclusion of the incomplete market component.

The problem that the policymaker faces is not trivial. It is illustrated by the social welfare function that the policymaker wishes to maximize. This function consists of a consumption maximization component that reflects the typical desire for "efficiency" known from the NK literature, as well as an inequality component. The policy maker's problem is further complicated by the fact that inter-generational consumption/wealth inequality and intra-generational income inequality move in opposite directions. An increase in the aggregate supply of bonds allows households to save more but at the cost of higher income taxes and a higher real interest rate. As the interest rate rises, the cost of using assets to self-insure against idiosyncratic risk also rises, increasing intra-generational income inequality.

We examine this issue under different scenarios: where the policymaker is fully optimal, maximizes a welfare criterion that incorporates only an efficiency component, or focuses solely on reducing inequality in the economy.

In Chapter 3, we again study the interactions between optimal monetary policy and fiscal policy. Here, the policymaker pursues optimal monetary policy under commitment in a tractable heterogeneous agent environment with a meaningful supply of government bonds. However, as in Chapter 1, the fiscal authority follows a simple rule where taxes deviate from their steady-state level only if the value of government debt exceeds an exogenous target. We extend the framework of Chien and Wen (2021) by introducing nominal rigidities, transfers to constrained households, and long-term government bonds. The model features (exogenous) stochastic transitions between labor market participation and non-participation, deviating from the assumption of Keynesian-constrained consumers who face infrequent asset market participation (IAMP). Unlike the original paper, constrained households do not face equilibrium binding borrowing constraints but are subject to portfolio re-balancing costs. In the absence of borrowing constraints on constrained households, the policymaker is unable to affect consumption inequality in the steady state or in response to an aggregate shock. Nonetheless, households have unequal exposure to an aggregate shock.

Even in the extreme case where non-participating households can freely adjust their asset positions and thus have the same exposure to a change in the interest rate (as in the RANK case), optimal monetary policy still differs from the RANK benchmark because the policy maker's actions will reallocate wealth following an aggregate shock. Therefore, our tractable HANK model with discontinuous labor market participation (DLMP) features two types of Ricardian consumers.

The policymaker has access to commitment technology and considers the behavior of the fiscal authority before solving their program. In this chapter, for comparability purposes, we assume that all economies display the same annualized debt-to-GDP ratio, the same government consumption, and, where applicable, the same level of exogenous transfers to constrained households. We compare the steady-state allocations and aggregate dynamics of both the nested representative agent model and the tractable HANK model with infrequent asset market participation (IAMP). As noted by Cantore and Freund (2021), this latter HANK specification is an extreme case of our model where constrained households face a binding borrowing constraint that prohibits them from taking any asset position and incurs infinite portfolio re-balancing costs.

Essentially, we start with the standard representative agent specification and show how each additional layer of heterogeneity affects both steady-state allocations and optimal policy dynamics. This is done through a combination of theoretical results and numerical simulations.

Chapter 1

Monetary and fiscal policy interactions in a tractable HANK economy

Based on joint work with T. Kirsanova and C. Leith

Abstract

In this paper, we aim to better understand how the Heterogeneous Agent New Keynesian (HANK) economy differs from the nested Representative Agent New Keynesian (RANK) framework and how the policy mix is affected. We employ the analytically tractable HANK framework of Acharya et al. (2023) that combines the overlapping generations structure of Blanchard (1985) with the incomplete market model of Aiyagari (1994). We augment the model along two dimensions. First, we introduce a general debt structure and depart from the assumption that aggregate debt exists in zero net supply. An assumption that limits the models predictions regarding the long- run equilibrium affects its dynamics. Next, we further introduce “declining labour efficiencies“, as a proxy for the time spent in retirement, which results in richer inter- generational wealth inequality. We show that the parameter space that ensures a unique ration expectation equilibrium changes as we add more layers of heterogeneity. Finally, we argue that in the absence of aggregate risk, the fiscally- led policy mix (AF/PM) is preferred to the alternative monetary-led policy mix (AM/PF) in response to a transitionary ("mit") shock since it causes smaller deviations in inequality. The model is calibrated for the US economy for the period 1985- 2021.

1.1 Introduction

In the aftermath of the financial crisis, the heterogeneous agent New Keynesian (HANK, henceforth) framework emerged as an apparatus to study the distributional consequences of monetary and fiscal policy. Although reduction of inequality is not part of the mandate of any central bank yet, prominent policy makers have devoted entire speeches calling for more research on the implications of monetary policy on inequality and redistribution (See Carney 2016 ; Draghi 2016; Yellen 2017).

In this paper we study the interactions between monetary and fiscal policy in a HANK economy. Our model is an extension of the seminal work of Acharya & Dogra (2020) and Acharya et al. (2023) who added uninsurable income risk and a Perpetual Youth (PY, hereafter) structure to an otherwise standard representative agent New Keynesian (RANK, henceforth) model. This framework falls under the umbrella of the so- called tractable HANK (THANK) models. This class of models rely on simplifying assumption regarding the way that inequality enters the economy. They are used to study the qualitative rather than the quantitative difference between the HANK and the nested RANK specification.

The framework features a CARA utility function that allows for linear aggregation and thus, we are able to solve the model without having to impose either a degenerate wealth distribution or a zero liquidity constraint. We further augment the framework by introducing declining income profiles (See Blanchard (1985)) to allow for richer inter-generational wealth inequality and also expand the fiscal side of the economy. More specifically, we allow assets of different maturities (short and long term bonds) to enter the households' balanced sheets and a realistic equilibrium debt- to - GDP ratio. The model abstracts from marginal propensity to consume (MPC, henceforth) heterogeneity and instead focuses on the role that uninsurable income risk and inter-generational wealth inequality play in order to identify differences in the policy mix compared to the nested representative agent economy. We further assume that the only source of uncertainty in the model comes from the idiosyncratic histories and from the households' stochastic lifespans.

We also investigate the differences in the long- run equilibrium, determinancy areas as well as dynamics between the RANK and a plethora of different (nested) HANK specifications. We find that with each extra layer of inequality that we add, both the long- run equilibrium as well as the areas of determinancy (See, Leeper 1991) are affected. Following Auclert (2019), we only consider the policy response to a transitory ("mit") aggregate shock. Hence, after the one- time aggregate shock is realised, households have perfect foresight. The consolidated monetary- fiscal authority relies on simple rules to conduct monetary and fiscal policy. We keep the policy response coefficients constant across the different specifications and thus, the monetary and fiscal policy have identical responses across all frameworks. However, the joint monetary- fiscal policy response in specifications that feature household heterogeneity always has redistributive effects. In line with Auclert (2019), we find that since our framework abstracts from marginal propensity to consume heterogeneity and binding equilibrium borrowing constraints, following an aggregate shock, the policy response will affect the wealth disparity in the economy but the policy maker will still be unable to redistribute consumption towards younger/poorer households.

As we know from the seminal work of Leeper (1991), there exist two areas of the parameter space that ensure a unique rational expectations equilibrium. The first regime requires a monetary-led policy mix whilst the alternative requires a fiscally- led policy mix. We find that although both policy mixtures are able to stabilise the economy in response an unanticipated aggregate shock, a fiscally- led regime might be preferable for the HANK economy, since it is consistent with a smaller deviation in the dispersion of wealth as well as smaller initial deviations in the sensitivity of individual consumption to changes in individual income, in response to the shock. However, if the policy maker is willing to sacrifice "equity" for " efficiency" the monetary led regime is still preferred.

1.2 Literature Review

The paper contributes to large literature that studies the distributional consequences of monetary and/or fiscal policy using discrete- time tractable HANK or simply THANK models (See for instance, Bilbiie 2008; Bilbiie 2024; Broer et al. 2020; Challe 2020; Cantore & Freund 2021; Debortoli & Galí 2018 ; Komatsu 2023; among others). Papers in this literature rely on simplifying assumption regarding how inequality enters the economy and focus on the qualitative rather than the quantitative difference between HANK and the nested RANK model. These simplifications allow us to observe the inter-temporal trade-offs and study the distributional consequences of monetary and/fiscal policy in a general equilibrium setting, without having to sacrifice the analytical tractability of the key results. Most papers in this literature rely on generalisations of the seminal work of Galí et al. (2007) and Bilbiie (2008). A key advantage of these specifications is that this class of HANK models can be solved and/or estimated using well- known techniques from the representative agent literature[1]. Yet, they are able to trace the dynamics of the large quantitative models quite closely. However, more recent studies have also exploited the concept of "Recursive Contracts" of Marcet & Marimon (2019) or the assumption of "truncated idiosyncratic histories" of Le Grand et al. (2022) to employ larger- more quantitative- models and still obtain some analytical results without moving to a continuous- time set up. However, most of these studies still rely on the assumption a “zero liquidity limit“ or a degenerate wealth distribution to further simplify their analysis.

Papers in this literature use marginal propensity to consume (MPC, hereafter) heterogeneity and/or partially uninsurable income risk to explore the differences between HANK and RANK economies. Following Acharya & Dogra (2020) and Acharya et al. (2023) we abstract from MPC heterogeneity and instead rely only on cross- sectional income inequality and inter-generational wealth inequality to study the differences between the two frameworks. As discussed above, we augment the friction from the OLG channel by including declining labour efficiency. This assumption strengthens the “Overlapping generations“- wealth inequality channel that shapes the long-run equilibrium, areas of determinancy as well as the model dynamics. Our paper complements the work the Monacelli & Colarieti (2022), who augment the seminal model of Acharya & Dogra (2020) by including richer cross- sectional income inequality. They rely on the assumption of heteroskedastic income processes to introduce MPC heterogeneity in a parsimonious way. The papers that build on the seminal work of Acharya & Dogra (2020) make use of the CARA utility and (iid) normally distributed idiosyncratic shocks that considerably simplifies the aggregation process. As such, we can operate under the assumption that there exist an infinite type of agents in the economy without turning the wealth distribution into an infinite dimension object.

Finally, our work also complements the studies who have examined monetary and/or fiscal policy using standard New Keynesian environments embed with Blanchard- Yaari consumers. Leith & Wren-Lewis (2000) first investigated monetary and fiscal policy interactions in an economy with non- Ricardian agents. Chadha & Nolan (2007) provided a systematic characterisation of mone-

[1]See for instance, Judd 1998, Miranda & Fackler 2004, Maußner 2005

tary and fiscal policy rules in a OLG- RANK business cycle economy. Kirsanova et al. (2007) used an open- economy version of the OLG- RANK framework to examine fiscal policy issues in a monetary union. Whilst, Leith & Von Thadden (2008) introduced the Blanchard- Yaari consumer structure into a New Keynesian with capital accumulation to study how government debt affects the determinancy of the system. Finally, Nistico (2016) and Rigon & Zanetti (2018) used a similar OLG- RANK framework to study how optimal monetary policy and/or fiscal policy change in environments with Non- Ricardian consumers.

1.3 Model

The economic environment that we describe in this chapter follows closely the model of Acharya et al. (2023). Our model is a combination of a Bewley (1977), Huggett (1993) and Aiyagari (1994) economy in which households face (partially) uninsurable idiosyncratic income risk coupled with the overlapping generations structure of Blanchard (1985). The model employs Constant Absolute Risk Aversion (CARA) preferences and normally distributed shocks to individual household labour supply to develop a tractable heterogeneous agent model for the analysis of monetary policy. The model is capable of describing both macroeconomic aggregates and measuring social welfare whilst accounting for heterogeneity.

Following Galí (2021), we refer to this class of models as OLG- HANK. This adoption of the Blanchard- Yaari (BY, henceforth) or “perpetual youth" model has significant advantages. The main benefits of Blanchard’s economy over the competing overlapping generations frameworks is that the average lifetime of an agent can be parameterised. That is to say that the “PY" model nests within itself the infinitely lived agent model (as a special case). For us, the adoption of this “perpetual youth" structure a- la Blanchard- Yaari (1985), practically means that in every period each household faces a constant probability of survival (ϑ). For simplicity, the population is fixed and normalised to 1. Hence,the size of a newly born cohort at any date t is $(1-\vartheta)$ and the size of a cohort at period t of agents born at any date $s<t$ is $(1-\vartheta)\,\vartheta^{t-s}$.

However, this structure also harbours additional advantages. Most importantly, it allows us to solve the model without having to impose any borrowing constraints. Since, the existence of transversality conditions permits us to refrain from imposing any restrictions on asset trading (household debt level) whilst ruling out ponzi-schemes. This endogenously determines a steady-state distribution of wealth, affecting the response to shocks and implying an additional externalities that are absent in models without government debt. Overlapping generations of households will decide whether to save by purchasing risk free assets, not internalizing the impact of these decisions on the equilibrium real interest rate . A feature absent from both the standard representative agent model as well as the heterogeneous agent model of Acharya & Dogra (2020) and Acharya et al. (2023) where government debt is in zero net supply (See Acemoglu 2008 chapter 9, for a textbook exposition).

Next, we augment the OLG- channel by assuming that the *dis- utility of supplying labor income decreases with age* in order to mimic economic retirement. This approach generates a desire to save in anticipation of falling incomes, akin to saving for retirement and provides us with richer-inter-generational wealth inequality.

In our framework, agents can hold two types of (risk- free) actuarial bonds which they buy from the stand-in financial intermediary. These intermediaries are just an aggregation device used to convert actuarial bonds into government bonds.

The production side of the economy is kept deliberately simple. As discussed above the model features a consolidated monetary- fiscal authority. The policy maker conducts monetary and fiscal policy following simple rules. Additionally, the fiscal authority is also tasked with issuing government debt, public consumption and provide subsidies to households.

Finally, we also abstract from aggregate risk and only allow for a one-time unanticipated transitionary shock at date $t = 0$, after which all households have perfect foresight. The complete derivation of the model and the proofs of the propositions can be found in appendix A.

1.3.1 Households

The economy is populated by cohorts of Blanchard- Yaari individuals that have constant survival probability in any period, $0 < \vartheta < 1$, see Blanchard (1985). At any time t, an individual i which belongs to generation born at time $s \leq t$ derives utility from age-dependent real private consumption $c_t^s(i)$ and real government consumption G_t. They also derive dis-utility from labour supply, $l_t^s(i)$, and, exogenously, dist-utility rises with age reflecting a desire to retire, $\Theta_t^s = \varkappa(t-s)$. This gradual withdrawal from the labour market will create a desire to save for 'retirement' and will ensure that the government wishes to issue a plausible level of government debt in the Ramsey steady-state. Crucially, households face uninsurable idiosyncratic shocks to dis-utility from labour $\xi_t^s(i) \sim \left(\bar{\xi}, \sigma_t^2\right)$; these shocks are independent across time and individuals. The variance of this shock may vary with economic activity and there is only idiosyncratic risk.

We assume CARA preferences so utility takes form:

$$U_s = \mathbb{E}_i \sum_{t=s}^{\infty} (\beta\vartheta)^{t-s} \left(-\frac{1}{\gamma} e^{-\gamma(c_t^s(i)+\chi G_t)} - \rho e^{\frac{1}{\rho}(l_t^s(i)+\Theta_t^s-\xi_t^s(i))} \right)$$

Individuals invest in long and short term nominal actuarial bonds $\mathscr{A}_t^{L,s}(i)$ and $\mathscr{A}_t^{S,s}(i)$. The short-term bonds are issued at price $\tilde{q}_t$, paying out one unit of current one period later. While, following Woodford (2001), the longer-term bonds, issued at price $\tilde{P}_t^M$, pay an initial coupon of one unit of currency which falls to ς^s, s period's later. Longer maturity debt matters as, following shocks, the revaluation effects on wealth held in the form of longer-term bonds through fluctuations in bond prices will be greater, which, in turn, will affect the impact of that shock on the distribution of wealth (See Leeper & Leith (2016*a*) for a discussion). Households receive after tax-wages, $(1-\tau_t)P_t w_t l_t^s(i)$, where the labor income tax, levied at rate τ_t, is the the sole source of government tax revenues in our benchmark model. We also introduce a lump-sum tax, $P_t T_t$, which will used to replace distortionary taxation as a means of eliminating the effects of tax distortions for demonstrative purposes only. Each household receives dividends, $P_t d_t$.[2] Their budget constraint at time t is

$$\begin{aligned}
&P_t c_t^s(i) + \tilde{P}_t^M \mathscr{A}_{t+1}^{L,s}(i) + \tilde{q}_t \mathscr{A}_{t+1}^{S,s}(i) \\
&= \left(1+\varsigma \tilde{P}_t^M\right) \mathscr{A}_t^{L,s}(i) + \mathscr{A}_t^{S,s}(i) \\
&+ (1-\tau_t) P_t w_t l_t^s(i) + P_t d_t - P_t T_t
\end{aligned} \tag{1.1}$$

Each individual is born with zero bond holdings, $\mathscr{A}_s^{L,s} = \mathscr{A}_s^{S,s} = 0$ and there is no fiscal transfer to newborns and/or wealth tax on existing households to ensure *ex ante* equality between all households as in Acharya et al. (2023).

We define real asset holdings (i.e. the ratio of the number of each type of assets to the price level) as,

$$a_t^{J,s}(i) = \frac{\mathscr{A}_t^{J,s}(i)}{P_{t-1}}, J \in \{L, S\} \tag{1.2}$$

and introduce a measure of real assets

$$A_t^s(i) = \frac{\left(1+\varsigma \tilde{P}_t^M\right) a_t^{L,s}(i) + a_t^{S,s}(i)}{(1+\pi_t)} \tag{1.3}$$

Then, we can re-write the budget constraint in real terms as:

$$\frac{\vartheta}{R_t} A_{t+1}^s(i) = A_t^s(i) + y_t^s(i) - c_t^s(i) \tag{1.4}$$

where net household income is defined as,

$$y_t^s(i) = \eta_t l_t^s(i) + d_t - T_t, \tag{1.5}$$

[2]For simplicity we assume that dividends are shared equally across households. It would be possible to allow dividends to vary with household labor supply or the state of the economy as in Acharya & Dogra (2020). In our economy another possibility might be to allow dividends paid to individual households to vary with age, reflecting re-balancing of portfolios from equities to bonds over the life-cycle.

the post-tax wage is

$$\eta_t = (1-\tau_t)\, w_t, \tag{1.6}$$

and we can define the ex ante real interest rate R_t as follows,

$$\frac{\vartheta}{R_t} = \tilde{q}_t\,(1+\pi_{t+1})\,.$$

Note that the ex post real rate will differ depending on the proportion of short and long-term bonds the household possesses in the presence of aggregate 'shocks' to the perfect foresight equilibrium path since additional capital gains/losses are possible on long-term bonds when the path of interest rates differ from what was expected.

The solution to an individual's optimisation problem can be summarized by the following Proposition derived in Appendix A2.9.1.

Proposition 1 (Individual's Optimisation) *In equilibrium, the optimal date t consumption and labour supply decisions of a household i born at date s are,*

$$c_t^s(i) = \mathscr{C}_t - \chi G_t + \mu_t m_t^s(i) \tag{1.7}$$

$$l_t^s(i) = \rho \ln(\eta_t) - \Theta_t^s - \rho\gamma(c_t^s(i) + \chi G_t) + \xi_t^s(i) \tag{1.8}$$

where

$$m_t^s(i) = A_t^s(i) - \varphi_t \Theta_t^s + \eta_t\left(\xi_t^s(i) - \bar{\xi}\right)$$

is demeaned 'cash-on-hand', $\mathscr{C}_t$ is a measure of common consumption, μ_t is the 'marginal propensity to consume (MPC) out of cash-on-hand and φ_t is the after-tax value of the human wealth of an individual supplying one unit of labor supply. This latter variable is used to value the income lost to retirement within households and for the population as a whole. The evolution of these variables is given according to:

$$\frac{1}{\mu_t} = \frac{\vartheta}{R_t\mu_{t+1}} + (1+\rho\gamma\eta_t) \tag{1.9}$$

$$\varphi_t = \eta_t + \frac{\vartheta}{R_t}\varphi_{t+1} \tag{1.10}$$

$$\begin{aligned}\mathscr{C}_t = &-\frac{\mu_t\vartheta}{R_t\mu_{t+1}\gamma}\ln(\beta R_t) + \frac{\vartheta\mu_t}{R_t\mu_{t+1}}\mathscr{C}_{t+1} - \frac{\vartheta\mu_t}{R_t\mu_{t+1}}\frac{\gamma}{2}\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2 \\ &-\mu_t\frac{\vartheta}{R_t}\varkappa\varphi_{t+1} + \mu_t\left(\eta_t\left(\rho\log(\eta_t) + \bar{\xi}\right) + d_t - T_t + \chi G_t\right)\end{aligned} \tag{1.11}$$

where $R_t = \frac{\vartheta}{\tilde{q}_t(1+\pi_{t+1})}$ is real interest rate.

The solution to the household's optimization problem implies that their consumption equals a measure of consumption, $\mathscr{C}_t$, which only depends on aggregate variables, after adjusting for the substitutability between private and public consumption in utility, χG_t, plus a term that is idiosyncratic, $\mu_t m_t^s(i)$. This final term depends on household i's cash-in-hand, $m_t^s(i)$, which comprises their financial assets, $A_t^s(i)$, minus the age-dependent loss of human wealth due to retirement that period, $\varphi_t \Theta_t^s$, and the extent to which their labor income varies due to their idiosyncratic shock to labor dis-utility, differing from the population average, $\eta_t\left(\xi_t^s(i) - \bar{\xi}\right)$. In turns, the household labor supply depends positively on the post-tax real wage, negatively on consumption, with adjustments made for both age-dependent retirement and idiosyncratic shocks to the disutility of labor supply.

A negative shock to labor supply, $\xi_t^s(i) < \bar{\xi}$, reduces household income and results in a fall in consumption, where $\frac{\partial c_t^s(i)}{\partial \xi_t^s(i)} = \mu_t \eta_t = \mu_t(1-\tau_t) w_t$. This fall will be greater the higher the marginal propensity to consume out of cash-on-hand, μ_t, and the greater the post-tax real wage. Households are therefore more insulated from the direct impact of the shock the higher the tax rates. As a result of the fall in consumption, they will work harder, where $\frac{\partial l_t^s(i)}{\partial \xi_t^s(i)} = 1 - \gamma\rho\mu_t\eta_t = 1 - \gamma\rho\mu_t(1-\tau_t) w_t < 1$. Again, a lower marginal propensity to consume and a higher tax rate will reduce the household's desire to maintain consumption by working harder in the period of the shock. Aside from working harder, the household can also maintain consumption through borrowing. Its ability to do so is implicit in the marginal propensity to consume.

We can iterate the marginal propensity to consume out of cash-on-hand forwards to obtain:

$$\frac{\mu_t}{R_t} = \left[\sum_{s=0}^{\infty} \frac{\vartheta^s (1+\rho\gamma(1-\tau_{t+s}) w_{t+s})}{\prod_{j=1}^{s+1} R_{t+j-1}}\right]^{-1}. \tag{1.12}$$

This formula is the same as in Acharya et al. (2023), except it incorporates dependency on the future *post-tax* real wage rate. It indicates that the propensity to consume increases with interest rates but decreases with future post-tax wages. Therefore, after experiencing a negative idiosyncratic shock to labor supply, which reduces their cash-on-hand, $m_t^s(i)$, households can maintain consumption closer to $\mathscr{C}_t$ when the marginal propensity to consume is low. This occurs when interest rates are low, making borrowing to smooth consumption less costly, or when post-tax wages are expected to be higher in the future, making it less expensive to repay any borrowing. Additionally, the presence of the tax rate implies that a lower tax rate makes it less costly (in utility terms) to increase future labor supply to pay off any debt incurred to smooth consumption. Thus, future distortionary taxation inhibits self-insurance, although high tax rates at the time of the shock mitigate its direct impact, as part of the lost income would have been taxed anyway.

Meanwhile, the component of household consumption driven by aggregate variables, $\mathscr{C}_t$, can be iterated forwards to obtain:

$$\mathscr{C}_t = -\frac{1}{\gamma}\sum_{s=0}^{\infty} Q_{t+s,t}\frac{\mu_t}{\mu_{t+s}}\ln(\beta R_{t+s}) - \frac{\gamma\mu_t}{2}\sum_{s=0}^{\infty} Q_{t+s,t}\mu_{t+s}^2 w_{t+s}^2 (1-\tau_{t+s})^2\sigma_{t+s}^2$$
$$+\mu_t\sum_{s=0}^{\infty} Q_{t+s,t}\bar{y}_{t+s} - \varkappa\mu_t\sum_{s=1}^{\infty} Q_{t+s,t}\varphi_{t+s}. \tag{1.13}$$

The first term has the same interpretation as in Acharya & Dogra (2020), capturing the impact of variations in interest rates relative to the impatience of households. If interest rates are typically higher than the rate of time preference, current consumption will be lower as households increase savings and cut current consumption. The discount factor, $Q_{t+s,t} = \frac{\vartheta^s}{\prod_{j=0}^{s-1} R_{t+j}}$, accounts for both the interest rate on financial assets and the probability of death, $1-\vartheta$. The second term is attributable to precautionary savings. A higher variance of idiosyncratic shocks, σ_{t+s}^2, increases the variance of post-tax income, $w_{t+s}^2(1-\tau_{t+s})^2\sigma_{t+s}^2$, which, after applying the marginal propensity to consume, captures the variance in consumption across households, $\mu_{t+s}^2 w_{t+s}^2(1-\tau_{t+s})^2\sigma_{t+s}^2$. The third term represents the discounted value of per capita post-tax income from labor, dividends, and transfers, after adjusting for the utility generated by public consumption, $\bar{y}_t = \left(\eta_t\left(\rho\log(\eta_t)+\bar{\xi}\right)+d_t-T_t+\chi G_t\right)$. Lastly, the equation includes the discounted value of the income lost due to the gradual retirement of the population throughout their working lives. Taxation affects this measure of aggregate consumption through its impact on the marginal propensity to consume, as discussed above, positively by reducing the variance of post-tax income but negatively by reducing the level of post-tax income and, therefore, the discounted value of that income.

It follows that net income (2.5) can be written as

$$y_t^s(i) = \eta_t\left(\rho\log(\eta_t)+\xi_t^s(i)\right) - \eta_t\Theta_t^s - \rho\gamma\chi\eta_t G_t - \rho\gamma\eta_t c_t^s(i) + d_t - T_t. \tag{1.14}$$

Aggregation of the household budget constraint yields

$$\frac{\vartheta}{R_t}A_{t+1} = \vartheta A_t + y_t - c_t, \tag{1.15}$$

where

$$A_t = \frac{\left(1+\varsigma\tilde{P}_t^M\right)a_t^L + a_t^S}{1+\pi_t},$$

and a_t^J is an aggregation of long term ($J=L$) and short term ($J=S$) bonds.

The straightforward aggregation of income (2.14) yields:

$$y_t = \eta_t\left(\rho\log(\eta_t)+\bar{\xi}\right) - \frac{\varkappa\vartheta}{(1-\vartheta)}\eta_t - \rho\gamma\eta_t\chi G_t - \rho\gamma\eta_t c_t + d_t - T_t, \tag{1.16}$$

and aggregation of (2.7) yields

$$c_t = \mathscr{C}_t - \chi G_t + \mu_t \vartheta \left(A_t - \frac{\varkappa}{1-\vartheta} \varphi_t \right). \tag{1.17}$$

This latter expressions indicates that per capita consumption equals the consumption measure, $\mathscr{C}_t$, driving individual household consumption in (2.7), after adjusting for the substitutability between private and public consumption, χG_t, and the extent to which, in aggregate, households have successfully saved for retirement. $A_t > \frac{\varkappa}{1-\vartheta}\varphi_t$ implies that household financial wealth exceeds the loss of human wealth due to retirement across the population.

Aggregated first order conditions for the individuals' problem yield the following relationships, derived in Appendix 2.9.3.

Proposition 2 (Aggregated Individuals' Optimisation) *In equilibrium, the optimal date t the aggregate total consumption and labour supply decisions are:*

$$x_t = -\frac{1}{\gamma} \log\left(\beta R_t\right) + x_{t+1} + \mu_{t+1}\left(1-\vartheta\right) A_{t+1} - \frac{\gamma}{2} \mu_{t+1}^2 \eta_{t+1}^2 \sigma_{t+1}^2 - \varkappa \mu_{t+1} \varphi_{t+1}, \tag{1.18}$$

$$n_t = \rho \log \eta_t - \frac{\varkappa \vartheta}{1-\vartheta} + \bar{\xi} - \rho \gamma x_t, \tag{1.19}$$

$$x_t = c_t + \chi G_t, \tag{1.20}$$

$$\mathscr{C}_t = x_t - \mu_t \vartheta \left(A_t - \frac{\varkappa}{(1-\vartheta)} \varphi_t \right). \tag{1.21}$$

The dynamics of x_t resemble that of consumption in a representative agent model, but with notable differences. Typically, consumption is expected to grow whenever the interest rate exceed the rate of time preference, $\beta R_t > 1$. In other words, consumption jumps down when interest rates unexpectedly rise, as the discounted value of future post-tax income across the economy falls. Consumption then recovers as interest rates return to normal levels. However, there is an additional term, $\mu_{t+1}\left(1-\vartheta\right) A_{t+1}$, attributable to the aggregation across finitely-lived generations. This term would not exist if households were infinitely lived and $\vartheta = 1$. Instead, finite lives imply that government debt (which is mapped to households assets as $B_t = \vartheta A_t$) are net assets for households. Households currently alive do not expect paying for all the surpluses backing government debt, implying that any increase in those assets increases consumption. As above, the term $\frac{\gamma}{2}\mu_{t+1}^2 \eta_{t+1}^2 \sigma_{t+1}^2$ measures the variance of consumption across households due to idiosyncratic shocks, providing a motive for precautionary saving, which in turn reduces current consumption. Finally, consumption is reduced by the ongoing loss of post-tax income due to retirement.

It is helpful to consider the steady-state of this relationship to see how these additional factors influence interest rates:

$$\frac{1}{\gamma}\log\left(\beta R\right) = \mu\left(1-\vartheta\right)\left(A - \frac{\varkappa}{1-\vartheta}\varphi\right) - \frac{\gamma}{2}\mu^2\eta^2\sigma^2. \tag{1.22}$$

In the absence of idiosyncratic risk or finite lives, the steady-state interest rate in a representative agent economy would be consistent with household preferences, $\beta R = 1$. However, the desire for precautionary savings drive down the steady-state interest rate relative to these preferences, while the accumulation of assets beyond what is needed to fund retirement in an OLG economy, $A > \frac{\varkappa}{1-\vartheta}\varphi$, raises interest rates. If the government could provide sufficient assets for households to satiate their desire for precautionary savings and their need to smooth consumption in retirement, then the steady-state interest rate would equal the households rate of time preference, provided :

$$B - \frac{\varkappa}{1-\vartheta}\varphi = \frac{1}{2}\frac{\vartheta}{1-\vartheta}\gamma\mu\eta^2\sigma^2. \tag{1.23}$$

This is crucial for determining the infimum of the fiscal response coefficient that ensures passive fiscal policy.

1.3.2 Firms

The economy features two production sectors. A perfectly competitive final good producing sector as well as a monopolistically competitive intermediate good sector. The final good producing firms are identical and thus, we model this sector as a single stand-in aggregate firm that is the typical CES aggregator- that combines intermediate varieties into the final good:

$$Y_t = \left[\int_0^1 \left(y_t\left(j\right)\right)^{\frac{\varepsilon_t - 1}{\varepsilon_t}} dj\right]^{\frac{\varepsilon_t}{\varepsilon_t - 1}}$$

where, Y_t denotes the quantity of the final good, $y_t\left(j\right)$ denotes the demand for intermediate input j, and $\varepsilon_t > 1$ governs the elasticity of substitution between any two intermediate varieties.

There is continuum $j \in [0,1]$ of intermediate good producing firms, each producing a differentiated variety. Each firm j produces its differentiated product according to the production function

$$y_t\left(j\right) = z_t h_t\left(j\right)$$

where, $h_t\left(j\right)$ stands for the labour demand of firm j whist z_t is the aggregate technology (TFP) shock. We abstract from aggregate risk and only permit a one- time anticipated aggregate shock to the level of labour productivity z_0, at $t = 0$. Under the assumption that the shock decays geometrically:

$$\log(z_t) = \rho_z^t \log(z_0) + e_t$$

Intermediate firms face a quadratic cost a- la- Rotemberg (1982) when changing their prices. The firm's problem becomes choosing $\{P_t(j)\}_{t=0}^{\infty}$ in order to maximise :

$$\max_{P_t(j)} \sum_{t=0}^{\infty} m_{0,t} \left(\left(\frac{P_t(j)}{P_t} - (1-s)\frac{w_t}{z_t} \right) y_t(j) - \frac{\Phi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \right)$$

subject to monopolistic demand

$$y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon_t} Y_t$$

Solving the firms' profit maximisation problem yields the *NK Price Phillips Curve*

$$\Phi\pi_t(1+\pi_t)(1+r_t) = (1-\varepsilon_t + (1-s)\varepsilon_t \frac{w_t}{z_t})(1+r_t) + \Phi\mathbb{E}_t \left(\pi_{t+1}(1+\pi_{t+1}) \frac{Y_{t+1}}{Y_t} \right) \quad \text{(NKPC)}$$

Their profit is distributed as dividend:

$$d_t = (Y_t - (1-s)w_t H_t) - \frac{\Phi}{2}\pi_t^2 Y_t. \tag{1.24}$$

where, H_t stands for the aggregate labour demand[3].

1.3.3 Government

The model features a consolidated monetary-fiscal authority. In each period, the monetary authority chooses the level of the nominal interest rate $\{I_t\}$. Whereas, the fiscal authority chooses the level of the outstanding government debt$\{b_t^L, b_t^S\}$. Given the aggregate bond supply, the level of subsidies $s \in [0,1]$ and public spending$\{G_t = G_0 \in \mathbb{R}_+, \forall t\}$ then, they adjust the level of taxes (τ_t, T_t^R) to ensure fiscal solvency. Since, changes in government spending typically involve parliamentary procedures, we are going to assume that they are held constant and determined exogenously. We further assume that the maturity of long-term bonds is the same as the maturity of actuarial bonds.

The government budget constraint (GBC, henceforth) is given as

$$P_t^M \mathscr{B}_{t+1}^L + q_t \mathscr{B}_{t+1}^S = \left(1 + \varsigma P_t^M\right) \mathscr{B}_t^L + \mathscr{B}_t^S + P_t G_t - \tau_t P_t w_t n_t - P_t T_t$$

[3]See Appendix 1.8.2 for the aggregation and the derivation of the market clearing conditions. Labour market clearing requires aggregate labour supply to equal aggregate labour demand hence, $H_t := \int_0^1 n_t(j)\,dj = (1-\vartheta)\sum_{s=-\infty}^{t} \vartheta^{t-s} \int_0^1 l_t^s(i)\,di = n_t$.

Where, the Lump Sum taxes/transfers are defined as

$$T_t = sw_t H_t - T_t^R$$

We can re-write the GBC in real terms, using the nominal interest rate as:

$$(1+\pi_{t+1})\left[\frac{P_t^M}{(1+\pi_{t+1})}b_t^L + \frac{b_t^S}{I_t}\right] + \tau_t w_t H_t = b_{t-1}^S + \left(1+\varsigma P_t^M\right) b_{t-1}^L + G_t + T_t$$

where,

$$B_t = \frac{\left(\left(1+\varsigma P_t^M\right) b_t^L + b_t^S\right)}{(1+\pi_t)}$$

and

$$b_t^J = \frac{\mathscr{B}_t^J}{P_{t-1}}, J \in \{L, S\}.$$

In the benchmark case, we assume that the government raises revenue using distortionary income taxes $\{\tau_t\}$. However, we also investigate an alternative scenario where the only Lump Sum taxes are available. We further assume that both taxes $\left(\tau_t, T_t^R\right)$ and the nominal interest rate $(I_t = R_t(1+\pi_{t+1}))$ each follows a simple rule. To simplify our analysis further, we set the short term debt in zero net supply $\left(b_{t+1}^S = b_t^S = 0, \forall t\right)$ thus, we can immediately drop it from the GBC altogether. Hence,

$$(1+\pi_{t+1})\left[\frac{P_t^M}{(1+\pi_{t+1})}b_{t+1}^L\right] + \tau_t w_t H_t = \left(1+\varsigma P_t^M\right) b_t^L + G_t + T_t \tag{1.25}$$

1.3.4 Policy Rules

The policy maker conducts monetary policy by following a simple interest rate rule. The nominal interest rate (I_t) deviates from its long term value in response to deviations in inflation or in the output gap. Throughout this chapter we operate under the assumption that the inflation target is zero $(\pi^* = 0)$. So, the Taylor- like rule takes the form

$$\frac{I_t}{\bar{I}} = \left(\frac{1+\pi_t}{1+\pi^*}\right)^{\phi_\pi}\left(\frac{Y_t}{\bar{Y}}\right)^{\phi_Y}$$

A value of $\phi_\pi > 1$ indicates that the so- called "Taylor Principle" is satisfied and the gross nominal interest rate $(I_t = 1 + i_t)$ adjusts by more compared to the size of the inflation deviation. Although we have included the general form of the Taylor rule, throughout this paper we are going to assume that $\phi_Y = 0$. Since, as shown by Schmitt-Grohé & Uribe (2007) “Interest-rate rules that feature a positive response to output can lead to significant welfare losses.“

Similarly, the fiscal policy follows a simple non- linear rule where taxes deviate from their steady state value if and only if the value of outstanding government debt $\left(P_t^M b_t^L\right)$ deviates from the equilibrium target.

Hence, if the government has access only to distortionary income taxation (τ_t) then the feedback rule is given as:

$$\tau_t = \bar{\tau} \cdot \left(\frac{P_t^M b_t^L}{P^M \bar{b}^L}\right)^{\phi_b}$$

Whereas, if the fiscal authority raises revenue using Lump Sum taxes $\left(T_t^R\right)$ instead, the tax rule takes the form

$$T_t^R = \bar{T} \cdot \left(\frac{P_t^M b_t^L}{P^M \bar{b}^L}\right)^{\phi_b}$$

The rules abstract from any lagged interest rate or tax terms since, as stated in Leith & Wren-Lewis (2000) tax smoothing would indicate " that fiscal policy is conducted in very similar manner to optimal discretionary policy" and this is not the focus of this chapter.

Financial Intermediaries

Financial intermediaries trade actuarial and government bonds.

The real profit of intermediaries is the difference between total bonds and total amount of actuarial bonds in the economy in $t+1$:

$$\Pi = \left(1 + \varsigma P_{t+1}^M\right) b_{t+1}^L + b_{t+1}^S - \left(1 + \varsigma \tilde{P}_{t+1}^M\right) \vartheta a_{t+1}^L - \vartheta a_{t+1}^S \tag{1.26}$$

where b_{t+1}^J are total government bonds and ϑa_{t+1}^J are total actuarial bonds at time $t+1$, i.e. $\vartheta a_{t+1}^J = (1-\vartheta) \sum_{s=-\infty}^{t+1} \vartheta^{t+1-s} \int_0^1 a_{t+1}^{J,s}(i)\, di.$

The intermediaries maximise (2.27) subject to constraint:

$$-\tilde{P}_t^M a_{t+1}^L - \tilde{q}_t a_{t+1}^S + P_t^M b_{t+1}^L + q_t b_{t+1}^S \leqslant 0. \tag{1.27}$$

Optimization yields

$$\frac{1}{\tilde{q}_t} = \frac{\left(1+\varsigma\tilde{P}^M_{t+1}\right)}{\tilde{P}^M_t} \tag{1.28}$$

$$\tilde{q}_t = \vartheta q_t \tag{1.29}$$

$$\frac{1}{q_t} = \frac{\left(1+\varsigma P^M_{t+1}\right)}{P^M_t} \tag{1.30}$$

and so the profit is zero.

Denote short-term nominal interest rate

$$\frac{1}{1+i_t} = q_t \tag{1.31}$$

and the real gross interest rate is

$$R_t = 1+r_t = \frac{\vartheta}{\tilde{q}_t\left(1+\pi_{t+1}\right)} = \frac{1}{q_t\left(1+\pi_{t+1}\right)} = \frac{1+i_t}{1+\pi_{t+1}} \tag{1.32}$$

1.3.5 Market Clearing

We use individual budget constraint (2.15), Government budget constraint (2.26), profit of financial intermediaries (2.27), aggregation of income (2.5), profit of monopolistic firms (2.25) yield the market clearing condition, or the resource constraint:

$$Y_t = c_t + G_t + \frac{\Phi}{2}\pi_t^2 Y_t \tag{1.33}$$

Now, using (2.27) we can rewrite consumption decision (2.21) in terms of aggregate debt:

$$\mathscr{C}_t = c_t + \chi G_t - \mu_t\left(B_t - \frac{\vartheta\varkappa}{1-\vartheta}\varphi_t\right) \tag{1.34}$$

Finally, as shown in Appendix 1.8.2, the clearing of the labour market requires the aggregate labour supply (H_t) to equal aggregate labour demand (n_t) hence,

$$H_t := \int_0^1 n_t(j)\,dj = (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 l_t^s(i)\,di = n_t \tag{1.35}$$

And, in an effort to provide the reader with more consistent notation for the rest of the chapter we will denote the aggregate labour supply or demand by n_t .

1.3.6 Competitive Equilibrium

The private sector equilibrium $\{x_t, Y_t, \pi_t, w_t, b_t^L, P_t^M, R_t, \mu_t, \varphi_t, \sigma_t^2\}$ given policy $\{I_t, G_t, T_t^p, \tau_t\}$ and deterministic disturbances z_t and ε_t is given by the following system:

Aggregate Consumption Euler Equation

$$x_t = \begin{bmatrix} -\frac{1}{\gamma}\log(\beta R_t) + x_{t+1} + \frac{(1-\vartheta)}{\vartheta}\mu_{t+1}\left(\frac{P_t^M}{q_t}b_{t+1}^M + b_{t+1}^S\right) \\ -\frac{\gamma}{2}\mu_{t+1}^2(1-\tau_0)^2\omega_{t+1} - \bar{\varkappa}\mu_{t+1}\phi_{t+1} \end{bmatrix}$$

MPC recursion

$$\frac{1}{\mu_t} = \frac{\vartheta}{R_t\mu_{t+1}} + (1+\rho\gamma\eta_t)$$

Labour decline recursion

$$\varphi_t = (1-\tau_t)w_t + \frac{\vartheta}{R_t}\varphi_{t+1}$$

Output

$$\frac{Y_t}{z_t} = \begin{bmatrix} \rho\log((1-\tau_t)w_t) + \bar{\xi} \\ -\varkappa\frac{\vartheta}{1-\vartheta} - \rho\gamma x_t \end{bmatrix}$$

Phillips curve

$$\pi_t(1+\pi_t) = \left[\frac{1-\varepsilon_t + (1-s)\varepsilon_t\frac{w_t}{z_t}}{\Phi} + \frac{1}{R_t}\pi_{t+1}(1+\pi_{t+1})\frac{Y_{t+1}}{Y_t}\right]$$

Resource Constraint.

$$\left(1-\frac{\Phi}{2}\pi_t^2\right)Y_t = x_t + (1-\chi)G_t - \frac{1}{\gamma}\log(1+\tau_t^c)$$

Government budget constraint

$$P_t^M b_{t+1}^M = \frac{1}{(1+\pi_{t+1})}\left(\left(1+\varsigma P_t^M\right)b_t^M + G_t - \tau_t w_t n_t - T_t + s w_t n_t\right)$$

Bond Pricing equation

$$P_t^M R_t = \frac{\left(1+\varsigma P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)}$$

$$R_t = \frac{I_t}{1+\pi_{t+1}}$$

Idiosyncratic income risk (cyclicality equation)

$$\omega_t = w^2\sigma^2 \exp\left(2\phi_\sigma\left(Y_t - Y\right)\right)$$

where in the last equation, following Acharya et al. (2023), we assumed that risk is pro-cyclical if $\phi_\sigma > 0$ *and counter-cyclical if* $\phi_\sigma < 0$.

Social Welfare Function

We define the aggregate welfare function at time $t = 0$ as:

$$\mathbb{W}_0 = (1-\vartheta)\left(\sum_{s=-\infty}^{0} \vartheta^{-s}\int_0^1 W_0^s(i)\,di + \sum_{s=1}^{\infty}\beta^s \int_0^1 W_s^s(i)\,di\right), \tag{1.36}$$

where the first term represents the utility of generations that are alive at time zero. The currently alive are treated equally after accounting for their relative size. The second term represents the utility of unborn generations, with $s > 0$, and the utility of each such generation is discounted with weight β^s. Appendix 2.9.6 shows that this welfare measure can be written as follows:

$$\mathbb{W}_0 = \sum_{t=0}^{\infty}\beta^t \mathbb{U}_t,$$

where

$$\mathbb{U}_t = -\frac{1}{\gamma}\left(1+\gamma\rho\eta_t\right)e^{-\gamma x_t} S_t, \tag{1.37}$$

and S_t satisfies the recursion:

$$S_t = \left(\vartheta e^{-\frac{\gamma}{\vartheta}W_t} S_{t-1} + 1 - \vartheta\right) e^{\gamma W_t} e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}. \tag{1.38}$$

Here

$$W_t = \mu_t\left(B_t - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi_t\right) \tag{1.39}$$

measures the extent to which society has succeeded in financing its retirement. It extends the form of the welfare function considered in Acharya et al. (2023) by accounting for inter- generational inequality as well as the distribution of consumption driven by idiosyncratic shocks. The first part of the social welfare function captures the utility generated by per capita levels of private and public consumption, less the disutility of labor supply. The second element adjusts that measure for the welfare effects of inequality, driven by both idiosyncratic shocks and the distribution of consumption and labor supply across generations due to the endogenous accumulation of assets and age-related withdrawal from the labor market.

Although, we are not concerned with the conduct of optimal policy in this chapter, the above derivations allows us to obtain a simple but micro- founded inequality statistic. As such, we are going to use this recursion for the wealth inequality in the economy to gauge impact of the policy response to a transitionary shock. From the representative agent literature, we know that a monetary policy led regime is more efficient. However, in heterogeneous agent economies the policy response is going to redistribute wealth amongst households and as such the policy maker is faced with a trade- off between “equity“ and “efficiency“. Hence, we can use this Inequality statistic to better understand these trade- offs.

1.4 Calibration & Simulations

The model is calibrated to a quarterly frequency. The calibration of most parameters is standard and generally follows the one in Acharya & Dogra (2020). We calibrate the household discount rate $\beta = (1.02)^{-1/4}$ to match the real interest rate of 2% per annum, which is the average in the US over the Great Moderation period (1984-2021). The coefficient of relative risk aversion is set to $\gamma = 3$ whilst while the Frisch elasticity of substitution is set at $\rho = 1/3$ to match the empirical evidence (see, e.g., Fagereng et al. (2017); Christelis et al. (2015)).

Fiscal parameters are based on data over the same period. Specifically, the coefficient ς sets the maturity of government debt to be 20 quarters, which is a close match of the 5.4 years observed in the data (Fund 2016). The parameter G generates a spending share $G/Y = 0.15$, see IMF IFS data.[4] For simplicity, the relative weight on government consumption in utility, χ, is set to 0.05, which is a free parameter that generally ensures that government expenditure does not get fully wasted.

The elasticity of substitution between goods, ε, is set to 4.3 based on evidence in Hall (2018) and corresponds to an approximate 31% average mark up.

Our model features nominal rigidities following Rotemberg (1982). The majority of recent papers in the macro literature that calibrate their frameworks for the US economy choose prices to change every 10 months (See Klenow & Kryvtsov (2008); Klenow & Malin (2010); Nakamura & Steinsson (2008) ; Gopinath & Rigobon (2008) and Kehoe & Midrigan (2015).). As the Rotemberg

[4]The relevant data series are NGDP_XDC and NCGG_XDC.

(1982) and Calvo (1983) models generate isomophic linearized New Keynesian Phillips curves, the equivalent Rotemberg model parameter is $\Phi = 37.5$.

The parameter ($\bar{\xi}$) capturing the average endowment of time available for work is set to 2, which normalizes output to be close to one.

We choose the survival rate to be consistent with an average lifespan of 80 years, see SSA data.[5] The declining labor supply efficiency parameter, $\varkappa$, is chosen to be consistent with 20 years of retirement, in line with the US data over the last 50 years.[6]

We follow Guvenen et al. (2014), who document the standard deviation of one-year growth rate of log earnings to be about 0.5. This yields $\sigma = 0.33$ for the baseline calibration.

Furthermore we calibrate the persistence of deterministic disturbances for productivity and elasticity of substitution to be 0.95 and 0.9 respectively. This again follows Acharya et al. (2023) who adopt the empirical estimates of Bayer et al. (2020).

Finally, we consider two distinct values for the interest rate feedback coefficient on inflation and the fiscal response coefficient on aggregate debt. When the monetary policy is active, the Taylor principle needs to be satisfied and hence, $\phi_\pi = 1.5$. Where as, when monetary policy is passive $\phi_\pi = 0.95$. In line with results from table 1.3, when the fiscal policy is passive $\phi_b = 0.2$ and under active fiscal policy the tax feedback on debt is set to $\phi_b = 0.04$.

All computations regarding the model dynamics were implemented in the RISE toolbox (Maih 2015).The summary of coefficients is given in Table 1.1.

1.5 Discussion

In this section we discuss the monetary and fiscal policy interactions in an analytically tractable HANK environment. First, we analyse the model's long- run equilibrium and then we move on to discuss the determinancy properties as well as the model's dynamics in response to transitionary ("mit") shock.

We compare steady state of the standard the representative agent model against a plethora of HANK specifications. More specifically, we begin with the standard HANK model of Acharya & Dogra (2020), where the only source of risk comes from the households' individual history of idiosyncratic shock. Next, following Acharya et al. (2023), we introduce a Blanchard- Yaari structure to the consumer side. Allowing households to have stochastic finite lifespans, introduces inter-generational consumption/ wealth inequality to the model. This additional source of heterogeneity alters both the long- run equilibrium as well as the policy response. Finally, we further augment the OLG channel by including an additive declining income profiles component that fur-

[5] See Period Life Table at www.ssa.gov.

[6] See https://crr.bc.edu/wp-content/uploads/2024/04/Average-retirement-age_2021-CPS.pdf

Table 1.1: Calibration

Description	Parameter	Value	Source
Time discount factor	β	0.995	data
survival rate	ϑ	0.9961	data
Labour efficiency parameter	$\varkappa$	0.00011	data
Demand elasticity	ε	4.2	Hall (2018)
Average price duration	δ_θ	0.75	Kehoe & Midrigan (2015)
Debt Maturity	ς	20	Leeper & Zhou (2021)
Preference for public good	χ	0.0	Free parameter
Average idiosyncratic Productivity	$\bar{\xi}$	2	Acharya & Dogra (2020)
Risk aversion coefficient	γ	3	Acharya & Dogra (2020)
Inverse of Frisch elasticity	ρ	1/3	Fagereng et al. (2017)
Government Spending	$\frac{G}{Y}$	0.15	data
Persistence of TFP shock	ρ_z	0.95	Bayer et al. (2020)
Persistence of Cost- push shock	ρ_ε	0.9	Bayer et al. (2020)
St Dev of idiosyncratic earnings	σ	0.33	data
Policy feedback coefficients			
Monetary policy feedback on Inflation	ϕ_π	1.5; 0.95	
Fiscal policy feedback on Debt	ϕ_b	0.2; 0.04	

ther enriches the inter- generational wealth heterogeneity. As discussed above, the idea stems from the seminal work of Blanchard (1985), where households become less productive over time (i.e. loss of productivity due to health deterioration). This latter component creates a stronger consumption smoothing motive for the households and thus, increases the aggregate asset demand. We refer this final extension HANK- OLG- DIP.

We show that in the presence of meaningful amount of aggregate debt, the OLG structure of the model has a bigger impact than the market incompleteness (IM, henceforth). Since, the sole presence of uninsurable income risk creates very small quantitative differences from the nested RANK model. As shown by Auclert (2019), for the monetary policy to have a redistributive role[7], households in the economy must have unequal exposure to aggregate shocks. The same requirement is also necessary for the fiscal policy to be able to redistribute consumption following a shock. Our model does not feature a binding equilibrium borrowing limit or unequal access to the financial market hence, there is no redistribution channel present in the transmission mechanism of either monetary or fiscal policy. Once again, following the recent HANK literature, the absence of redistribution channels speaks to the policy maker's inability to redistribute consumption across households following a shock. Since, in response to an unexpected “transitionary“ ("mit") shock, the policy response will always redistribute wealth across agents.

[7]In the language of Auclert (2019), the monetary policy has a redistributive role if a change in the interest rate can redistribute consumption (not just wealth) amongst consumers.

Finally, we find that the inclusion of the BY structure has a profound effect on both the model's steady state and in the determinancy regions. Across all specifications, we operate under the assumption that agents enter the market with zero wealth. And, unlike the seminal paper of Acharya & Dogra (2020) and Acharya et al. (2023), we allow the time that a household spends in the market to matter. That is to say that we do not focus our analysis around the "egalitarian" equilibrium where every cohort has the same wealth. Hence, as in the standard perpetual youth model, households' consumption smoothing objective makes them borrow when they are young and repay their debt when older. As a result, in the HANK- OLG framework with declining income profiles, agents exhibit the strongest consumption smoothing desire that delivers the highest aggregate asset demand.

1.5.1 Steady State

In this section we compare the long- run equilibrium of the different (nested) HANK specifications that we consider against the benchmark RANK model. Table 1.2 below shows the models' steady state under different assumptions on heterogeneity. Column 1 and column 2 report the steady state of the standard RANK and HANK environments, respectively. These specifications are the closest to Acharya & Dogra (2020). In these frameworks agents are infinitely lived and the only source of uncertainty in the economy comes from the different idiosyncratic histories of the households. Next, as discussed above, we augment our HANK specification by introducing a Blanchard- Yaari households structure so that the model can also feature heterogeneity due to consumers' stochastic life- spans. As such, column 3 reports the steady state of this HANK- OLG specification. Once again in this version of the economy, households heterogeneity stems not not only due to the different histories of the idiosyncratic shock but also from the fact that household wealth is also proportional to the time they have spend in the market. Finally, in column 4, we present the long-run equilibrium of the HANK- OLG model with declining income profiles (HANK- OLG- DIP). In this last specification we have further augmented the frictions coming from the overlapping generations channel causing an increase in the consumption smoothing motive whilst lowering the aggregate labour supply. As discussed above, this decline labour productivity is captured by a linear term that enter the individual household's budget constraint, capturing the fact that as the agent grows older there is a propositional loss in productivity due health deterioration.

The steady state amount of aggregate asset holdings is set exogenously to reflect an annualised debt to GDP ratio of $+43\%$ for all models.

As expected, we find that across all HANK specifications, the presence of incomplete asset markets and uninsurable income risk causes the steady state (ss,henceforth) real interest rate to be consistently different from the rate of time preference ($R \neq \beta$). And since, we only consider the (efficient) zero- inflation steady state, the notion of the real and nominal interest rate become indistinguishable. As discussed above in the household block, the steady state interest rate is determined by four main components:

$$R = \underbrace{\frac{1}{\beta}}_{RANK} \exp\left(\underleftrightarrow{-\frac{1}{2}\gamma^2\mu^2\left(1-\tau_0\right)^2 w^2\sigma^2}_{IM} + \underleftrightarrow{\gamma\frac{(1-\vartheta)}{\vartheta}\mu\left(1+\varsigma P^M\right)b^M}_{OLG} - \underleftrightarrow{\gamma\bar{\varkappa}\mu\bar{\phi}}_{DLP} \right)$$

In the absence of income inequality and OLG frictions the model reduces to a standards RANK model where $R = \frac{1}{\beta}$ (RANK component). From the Hugget- Bewley- Aiyagari literature (See, Ljungqvist & Sargent 2018 Chapter 17 for a textbook treatment) we know that in the presence of uninsurable income risk and infinite- lived households, the equilibrium interest rate is found to be below the rate of time preference $\left(R = \frac{1}{\beta}\exp\left(-\frac{1}{2}\gamma^2\mu^2\left(1-\tau_0\right)^2 w^2\sigma^2\right) < \frac{1}{\beta}\right)$ (IM component). However, the OLG frictions present in our model also shape the equilibrium interest rate. From table 1.2, we observe that the HANK- OLG framework, delivers a higher equilibrium interest rate than both the plain- HANK model as well as the nested RANK. This is due to the effects of the OLG component $\left(\frac{(1-\vartheta)}{\vartheta}\mu\frac{P^M}{q}b^M\right)$ and our modelling assumption that government debt is positive in steady state. Everything else constant, the higher the steady state amount of outstanding government debt, the higher the equilibrium real interest rate.

In contrast, both the nested HANK model (with infinitely lived agents) and OLG- HANK with declining income profiles, display lower steady state value for the real interest rate. In the HANK- OLG framework with declining income profiles, the additional term associated with the decline in the labour efficiency moves in the same direction as the idiosyncratic (IM) component. Thus, reducing the effect of positive (aggregate) asset holdings on the real interest rate $\left(\frac{(1-\vartheta)}{\vartheta}\mu\frac{P^M}{q}b^M - \varkappa\mu\phi\right)$.

Intuitively, in this specifications, households display the largest precautionary savings motive. As such, they require lower compensation to hold the exogenous amount of outstanding government debt.

Table 1.2: Steady State Comparison between RANK and all HANK specifications.

		RANK		HANK		HANK- OLG		HANK– OLG- DIP	
Key Parameters		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Survival Rate	θ	1	1	1	1	0.9961	0.9961	0.9961	0.9961
Decline in labour supply	$\varkappa$	0	0	0	0	0	0	$1e-4$	$1e-4$
Steady State									
Income tax	τ	0.20758	–	0.20757	–	0.20782	–	0.21047	–
Lump Sum Tax	T^p	–	0.156891	–	0.156885	–	0.15706	–	0.15668
Inequality	S	–	–	1.00013	1.00011	1.00255	1.003198	1.002152	1.002897
Aggregate Consumption	c	0.8419	0.880678	0.8419	0.880678	0.84185	0.880678	0.82724	0.86663
MPC (cash-on-hand)	μ	0.003112	0.002838	0.00312	0.00284	0.0056	0.00509	0.00553	0.00501
Inflation	π	0	0	0	0	0	0	0	0
Real Interest rate	R	1.005025	1.005025	1.005022	1.005022	1.00513	1.00512	1.0050023	1.004975
Aggregate Output	Y	0.9899	1.02868	0.9899	1.02868	0.98985	1.02868	0.97524	1.0146
Wage	w	0.7619	0.7619	0.7619	0.7619	0.7619	0.7619	0.7619	0.7619
Real Gov. Debt	b^M	0.08556	0.08891	0.08556	0.08891	0.08573	0.08908	0.084254	0.08761
Asset Prices	P^M	19.9	19.9	19.9011	19.9014	19.859	19.862	19.91	19.92
Annualised Debt- to- GDP	$\frac{b^M \cdot P^M}{4 \cdot Y}$	0.43	0.43	0.43	0.43	0.43	0.43	0.43	0.43

As a result, this specification delivers the lowest steady state value for the real interest rate. However, the steady state level of the real interest also depends on the tax instrument available. In fact, there is a causal relationship between the two. Under lump sum taxes (T), the size of the IM component increases thus, forcing the real interest rate to take a smaller value.

Furthermore, from the no- arbitrage condition we know that bond prices are inversely related to the nominal interest rate. So, in the zero inflation steady state $(\pi = 0)$, the specification that reported the lowest (highest) steady state interest rate will also display the highest (lowest) steady state bond price. As such, we find that the HANK- OLG framework that features declining labour productivity reports the highest equilibrium asset price whilst the lowest value is found in the HANK- OLG specification.

As expected, the aggregate labour supply takes the highest equilibrium value in the plain RANK model with infinitely- lived agents. Additionally, since the effect of the (partially-) uninsurable income risk is not quantitatively significant, the HANK and nested RANK both report the same steady state output. However, from the expression for the optimal labour supply we see that the inclusion of a BY structure causes households' to have a lower optimal labour supply.

$$n = \rho \log \eta - \frac{\varkappa \vartheta}{1-\vartheta} + \bar{\xi} - \rho \gamma x$$

Consequently, the more layers of OLG frictions are introduces, the model delivers lower steady state aggregate aggregate labour supply and hence, lower aggregate output. After all, as shown by Ascari & Rankin (2007), in the BY model with standard preferences and endogenous labour supply, as households grow older, they want to the number of hours that they provide to the market and instead, rely more on their financial wealth for consumption. In fact households who belong to very old cohorts ideally would want to have a negative labour supply. So naturally, the optimal aggregate labour decreases and these effects from the OLG channel are further amplified with the addition of declining labour efficiency.

Moreover, aggregate consumption follows the same pattern. After all, aggregate consumption is connected to aggregate output through the economy's resource constraint. Additionally, we can also observe that all specifications feature the same equilibrium wage rate. Since, at the zero inflation steady state, the equilibrium wage rate depends solely on the elasticity of substitution between intermediate varieties(ε).

Furthermore, across specifications, the steady state tax rate depends on the value of outstanding government debt, the equilibrium level of public spending as well as the size of the tax base. Hence, from the steady state expression for the government budget constraint, we know that steady state taxes (τ, T) are given as:

1. If the policy maker has access to only distortionary income taxes $(T = 0)$:

$$\tau = \left((1-q)\frac{B}{Y} + \frac{G_0}{Y}\right)\frac{z}{w}$$

2. Whereas, if the policy maker has access to only Lump Sum taxes instead, then.:

$$T = \left((1-q)\frac{B}{Y} + \frac{G_0}{Y}\right)Y$$

As such, with $\frac{B}{Y}, G_0, z, w$ held constant across all HANK specifications, the steady state tax level depends inversely on the steady state value of output and the equilibrium real interest rate. Hence, it come as no surprise the highest steady state income tax is reported in the HANK- OLG-DIP specification and the lowest in the plain HANK with infinitely lived agents.

Finally, let us turn our attention to the marginal propensity to consume (MPC) out- of- cash-on hand (μ). This μ recursion captures the sensitivity of individual consumption to changes in individual income (See, Acharya & Dogra 2020; Acharya et al. 2023). The steady state value of μ is determined by the interplay between net labour income (η), the survival rate(ϑ) and the real interest rate $(R = 1 + r)$.

$$\mu = \frac{(R-\vartheta)}{(1+\rho\gamma\eta)R}$$

As a result, we find that our benchmark HANK- OLG model features the highest MPC out-of- cash- on- hand. In any case, any variation of plain HANK model augmented with an OLG-component will always exhibit higher steady state MPC. Intuitively, the inclusion of finite lifetimes introduces an additional source of inequality in the economy (overlapping generations wealth inequality) causing individual consumption to become more sensitive to changes in the individual income.

1.5.2 Log-linear approximation

For the remainder of the chapter, I work with a log-linear approximation of the system around the efficient steady state. Variables without time-subscript represent steady- state values and hatted variables are log deviations from the zero inflation steady state. Whereas, variables with time subscript that are not hatted represent the log deviation of variables with zero steady state level. The (reduced) linearised system is as follows:

1. Government Budget Constraint

$$\frac{\left(1+\varsigma P^M\right)}{I} b^M \left(\hat{b}_{t+1}^M + \pi_{t+1}\right) + \frac{\varsigma P^M b^M}{I} \hat{P}_{t+1}^M = \left(\begin{array}{c} \tau w \frac{Y}{z} \left(\frac{z\rho+Y}{z\rho}\right) \hat{z}_t \\ \varsigma P^M b^M \left(\hat{P}_t^M\right) \\ + \left(1+\varsigma P^M\right) b^M \left(\hat{b}_t^M\right) \\ -\tau w \frac{Y}{z} \left(1 + \left(\frac{1+\gamma z \rho}{z\rho}\right) Y\right) \hat{Y}_t \\ -w \frac{Y}{z} \left(\frac{\tau}{1-\tau}\right) \hat{\tau}_t \\ -T \left(\hat{T}_t\right) \\ + \left(\frac{\left(1+\varsigma P^M\right)}{I} b^M\right) \hat{I}_t \end{array}\right)$$

2. Labour decline recursion

$$\frac{\vartheta \bar{\phi}}{R} \hat{\phi}_{t+1} + \frac{\vartheta \varphi}{R} \pi_{t+1} = \left(\begin{array}{c} (1-\tau) w \left[\frac{Y}{z\rho} \hat{z}_t\right] \\ +\bar{\phi} \hat{\phi}_t \\ -(1-\tau) w \left(\frac{1+\gamma z \rho}{z\rho}\right) Y \hat{Y}_t \\ +\frac{\vartheta \varphi}{R} \hat{I}_t \end{array}\right)$$

3. The μ Recursion

$$\left(\frac{\vartheta}{R}\right) \hat{\mu}_{t+1} - \left(\frac{\vartheta}{R}\right) \pi_{t+1} = \hat{\mu}_t + \gamma \eta \mu \left(\frac{1+\gamma z \rho}{z}\right) Y \hat{Y}_t - \gamma \eta \mu \frac{Y}{z} \hat{z}_t - \left(\frac{\vartheta}{R}\right) \hat{I}_t$$

4. Bond Pricing Equations

$$\hat{P}_t^M + \hat{I}_t = \left(\frac{\varsigma P^M}{1+\varsigma P^M}\right) \hat{P}_{t+1}^M$$

5. Aggregate Consumption Euler Equation

$$Y \hat{Y}_t + \frac{1}{\gamma} \hat{I}_t = \left[\begin{array}{c} +\frac{1}{\gamma} \pi_{t+1} \\ + \left(1 - \gamma \bar{\mu}^2 (1-\tau_0)^2 \omega \phi_Y\right) Y \hat{Y}_{t+1} \\ +\frac{(1-\vartheta)}{\vartheta} \bar{\mu} \varsigma P^M b^M \left(\hat{P}_{t+1}^M\right) \\ + \left(\frac{(1-\vartheta)}{\vartheta} \bar{\mu} \left(1+\varsigma P^M\right) b^M - \gamma \bar{\mu}^2 (1-\tau_0)^2 \omega - \bar{\varkappa} \bar{\mu} \bar{\phi}\right) \left(\hat{\mu}_{t+1}\right) \\ -\bar{\varkappa} \bar{\mu} \bar{\phi} \left(\hat{\phi}_{t+1}\right) \\ +\frac{(1-\vartheta)}{\vartheta} \bar{\mu} \left(1+\varsigma P^M\right) b^M \hat{b}_{t+1}^M \end{array}\right]$$

6. The NK Phillips curve

$$\beta\Phi\pi_{t+1} = \Phi\pi_t + \left(1+\frac{Y}{z\rho}\right)\varepsilon\frac{w}{z}\hat{z}_t + \left(1-\frac{w}{z}\right)\varepsilon\hat{\varepsilon}_t - \varepsilon\frac{w}{z}\left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_t - \varepsilon\frac{w}{z}\frac{\tau}{1-\tau}\hat{\tau}_t$$

7. Taylor Rule

$$\hat{I}_t = \phi_\pi\pi_t + \phi_Y\hat{Y}_t$$

8. Fiscal rule

- If the government raises revenue using Distortionary Income taxes:

$$\hat{\tau}_t = \phi_b\hat{b}_t^M + \phi_b\hat{P}_t^M$$

- If the government raises revenue using Lump Sum taxes:

$$\hat{T}_t^R = \phi_b\hat{b}_t^M + \phi_b\hat{P}_t^M$$

However, when calculating the model's dynamics we also make use of the following auxiliary expressions:

9. The Wage rate

$$\hat{w}_t = \left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_t - \frac{Y}{z\rho}\hat{z}_t + \frac{\tau}{1-\tau}\hat{\tau}_t$$

10. Net labour Income

$$\hat{\eta}_t = \left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_t - \frac{Y}{z\rho}\hat{z}_t$$

11. Inequality Recursion

$$S\hat{S}_t = \left(\begin{array}{c} \vartheta e^{-\gamma\frac{(1-\vartheta)}{\vartheta}W} e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}\left(S\hat{S}_{t-1}\right) + \gamma\left((2-\vartheta)e^{\gamma W} - S\vartheta e^{-\gamma\frac{(1-\vartheta)}{\vartheta}W}\right)e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}\left(W\hat{W}_t\right) \\ +\left(\gamma^2\mu^2\eta^2\sigma^2\left(\hat{\mu}_t + \hat{\eta}_t + \phi Y\hat{Y}_t\right)\right)(2-\vartheta)e^{\gamma W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2} \end{array}\right)$$

12. Proxy for the effect of the OLG component on aggregate welfare

$$W\hat{W}_t = W\hat{\mu}_t + \mu\left(1+\varsigma P^M\right)b^L\left(\frac{\varsigma P^M}{1+\varsigma P^M}P_t^M + \hat{b}_t^L - \pi_t\right) - \mu\frac{\vartheta\varkappa}{(1-\vartheta)}\left(\varphi\hat{\varphi}_t\right)$$

(The complete derivations of the log-linearised system can be found in Appendix C.)

1.5.3 Determinancy in an analytically tractable HANK-OLG economy

Leeper (1991) first showed the existence of two areas for the parameter space that guarantee determinacy of a unique rational expectations equilibrium (REE, henceforth).

The first regime features active monetary policy and passive fiscal policy (AM/PF or monetary-led policy mix). In this regime the monetary authority adjusts the nominal interest rate by more compared to an inflation deviation ($\phi_\pi > 1$) whilst the fiscal authority focuses on debt sustainability. So, in response to an aggregate shock, taxes deviate as to keep the dynamics of the real value of outstanding government debt on a stable path.

The second regime displays passive monetary policy and active fiscal policy (PM/AF or fiscal-led policy mix). In this scenario, the monetary authority does not display a strong reaction to excess inflation. Namely, the inflation reaction coefficient, in the monetary rule, is less unity ($\phi_\pi < 1$) and the fiscal authority no longer focuses on adjusting fiscal surpluses to stabilise the aggregate debt.

In this section we examine how the size of the fiscal response coefficient necessary to ensure a stable equilibrium, is affected by the presence of (partially) uninsurable income risk and/ or the overlapping wealth inequality.

From the linearised government budget constraint, we observe that the size of the debt stabilisation coefficient primarily depends on the type of tax instrument(s) available and size of equilibrium interest rate. In the most general case, the log- linearised government budget constraint takes the form:

$$\frac{\varsigma}{R}\hat{P}_{t+1}^M + \hat{b}_{t+1}^M = \left(\frac{(\tau-s)\,w}{P^M b^M}\left(\frac{Y}{z}\right)(\hat{z}_t) + R\hat{b}_t^M + \varsigma\hat{P}_t^M - \frac{\tau w}{P^M b^M}\frac{Y}{z}(\hat{\tau}_t) - \frac{T}{P^M b^M}\left(\hat{T}_t\right) + \hat{R}_t\right)$$

- If the fiscal authority has access only to distortionary income taxes $\left(\hat{T}_t = 0\right)$ then,

$$\frac{\varsigma}{R}\hat{P}_{t+1}^M + \hat{b}_{t+1}^M = \left(\frac{(\tau-s)\,w}{P^M b^M}\left(\frac{Y}{z}\right)(\hat{z}_t) + R\hat{b}_t^M + \varsigma\hat{P}_t^M - \frac{\tau w}{P^M b^M}\frac{Y}{z}(\hat{\tau}_t) + \hat{R}_t\right)$$

where,

$$\hat{\tau}_t = \phi_b \hat{b}_t^M + \phi_b \hat{P}_t^M$$

Then, if the fiscal authority is concerned with stabilising the aggregate debt, we find ourselves in monetary- led regime and the fiscal feedback coefficient has to be

$$\phi_b^\tau > \frac{z(R-1)}{\tau w Y} P^M b^M$$

to ensure a stable path for the debt dynamics

$$\left(R - \phi_b \frac{\tau w}{P^M b^M} \frac{Y}{z} \right) < 1$$

In this monetary- led regime, the Taylor principle needs to be satisfied ($\phi_\pi > 1$) regardless of the tax instrument available. Now, following convention[8] we have chosen the value of inflation reaction coefficient to be $\phi_\pi = 1.5$.

- On the other hand, when the government can only use Lump Sum taxes to raise tax revenue ($\hat{\tau}_t = 0$), the linearised government budget constraint takes the form

$$\frac{\varsigma}{R} \hat{P}_{t+1}^M + \hat{b}_{t+1}^M = \left(\frac{(\tau - s) w}{P^M b^M} \left(\frac{Y}{z} \right) (\hat{z}_t) + R \hat{b}_t^M + \varsigma \hat{P}_t^M - \frac{T}{P^M b^M} (\hat{T}_t) + \hat{R}_t \right)$$

where,

$$\hat{T}_t = \phi_b \hat{b}_t^M + \phi_b \hat{P}_t^M$$

In this case, if the fiscal authority wishes to ensure stable debt dynamics (i.e. pursue passive fiscal policy), then the fiscal response coefficient (ϕ_b^T) has to be

$$\phi_b^T > (R-1) \frac{P^M b^M}{T}$$

to once again ensure that

$$\left(R - \phi_b \frac{T}{P^M b^M} \right) < 1$$

Whilst once more, active monetary policy is requires that we set the inflation response coefficient to $\phi_\pi = 1.5$.

[8]See the Calibration section for the relevant discussion on parameter values.

It is evident that the size of the fiscal reaction coefficient has three key determinants. Firstly, it depends (inversely) on the steady state amount of tax revenue. As such, when the government raises revenue using only distortionary income taxes, the aggregate labour supply and thus aggregate output are lower. Next, the steady state amount of the real government debt$\left(b^M\right)$. This steady state quantity is set to correspond to an exogenous and fixed debt to GDP ratio. Everything else equal, whether $\phi_b^\tau \lesseqgtr \phi_b^T$ depends on the size of the tax revenue of each specification (See proposition 3 below).

Proposition 3 *Since, aggregate output is always higher when the government has access to Lump Sum taxes* $\left(Y^T > Y^\tau\right)$*, everything else equal, whether* $\phi_b^\tau \lesseqgtr \phi_b^T$ *depends on the size of the tax revenue* $\left(\tau wn \lesseqgtr T\right)$ *as well as on the steady state real interest rate* (R)*, in each specification.*

Moreover, the steady state value of the real interest rate (R), is determined by the assumptions regarding market (in)completeness and the size of the OLG frictions. As such, the HANK and HANK- OLG framework with declining labour efficiency, where $R < \frac{1}{\beta}$, require a smaller fiscal response coefficient to stabilise debt. As discussed above, we know that across the four specification the steady state real interest rate is found to take the smallest value in the HANK- OLG with decline labour efficiencies and the highest in the benchmark HANK- OLG. More specifically,

$$R_{HANK-OLG-DIP} < R_{HANK} < R_{RANK} < R_{HANK-OLG}$$

Hence, the necessary size for the fiscal feedback coefficient (on aggregate debt) to ensure stable debt dynamics must be

$$\phi_b^{HANK-OLG-DIP} < \phi_b^{HANK} < \phi_b^{RANK} < \phi_b^{HANK-OLG}$$

This result is confirmed by table 1.3. The table contains the infimum value of the fiscal response coefficient in the monetary- led regime (AM/PF), under different tax instruments. Given our calibration, we find that across all specifications the infimum value of ϕ_b to ensure a stable path for debt dynamics is consistently higher under Lump Sum taxes ($\phi_b^T > \phi_b^\tau$). Whilst, the inflation reaction coefficient (ϕ_π) governing the intensity of monetary policy's response to a deviation in inflation from the steady state target ($\pi^* = 0$) is held constant at 1.5, across all specifications,

Table 1.3: Infimum of the debt response coefficient in monetary- led regime (AM/PF).

	RANK	HANK	HANK- OLG	HANK- OLG- DIP
Fiscal Reaction coefficient				
Under distortionary income tax $\left(\phi_b^\tau\right)$	0.054649364	0.054622388	0.055725357	0.053657674
Under lump sum tax $\left(\phi_b^T\right)$	0.056668501	0.056640819	0.05767765	0.055414387

1.5.4 Determinancy Areas

Figures 1.1 and 1.2, display the determinancy regions under distortionary income taxes and Lump Sum taxes, respectively. As discussed previously, this framework abstracts from aggregate risk and features only idiosyncratic income risk and uncertainty due to households' stochastic life spans. Hence, the model has only one predetermined state variable, the aggregate debt level (b^M). And, in line with the empirical literature, we always operate under the assumption that the idiosyncratic income risk is counter-cyclical (See Guvenen et al. (2014)).

The figures below display the areas of determinacy for the parameter space that guarantee a unique REE equilibrium for the different model specifications. The bottom left corner of each graph corresponds a fiscally- led policy mix whilst the top right corner, where all values of $\phi_\pi > 1$, correspond to a monetary- led policy mix. We compare the parameter space of the standard RANK model to the HANK model of Acharya & Dogra (2020) and then, against the specification that also allow for inter- generational wealth inequality due to the OLG component and the subsequent assumption of declining income profiles.

We observe that, regardless of the tax instrument available, the benchmark HANK- OLG model (without declining labour productivity) displays a unique REE equilibrium in regime with both active monetary and fiscal policy.

Interestingly enough, it is evident from figures 1.1 and 1.2, that in a fiscally- led regime, the maximum size of the inflation response coefficient is proportional to the steady state value of the real interest rate. As such, in the RANK model where $R = \frac{1}{\beta}$ the upper bound for ϕ_π approaches 1.

Conversely, in the HANK and HANK- OLG with declining income profiles, where the equilibrium interest rate is significantly below the rate of time preference, we see that the lower bound is significantly smaller. Finally, as discussed above, the HANK- OLG model that features $R > \frac{1}{\beta}$ and so, the supremum for ϕ_π is marginally above 1 in the fiscally- led regime.

Overall, we can visually confirm that across all specifications, the minimum fiscal response coefficient to ensure stable debt dynamics is consistently higher under Lump Sum taxes. The inclusion of partially uninsurable income risk has narry an effect on the model's determinacy. The difference in the infimum of the fiscal response coefficient is in the 5th decimal place whilst the difference in the inflation response coefficient is marginally larger. As expected, we find that both the addition of inter- generational wealth inequality as well as the tax instrument available play a more significant role in altering the areas of determinancy. And in fact, with each added layer of household heterogeneity we observe a non- negligible change.

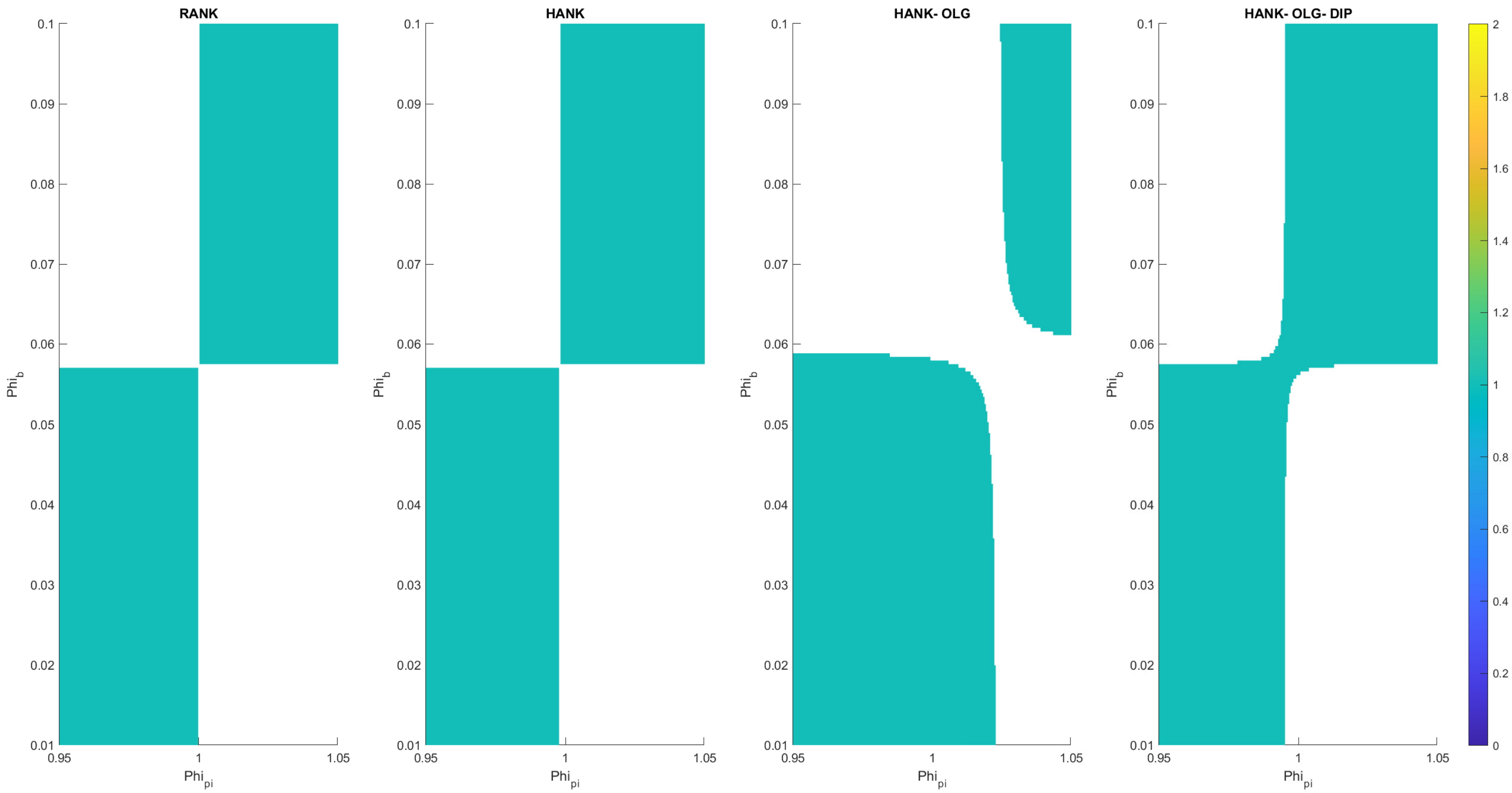

Figure 1.1: Determinancy under Distortionary income tax.

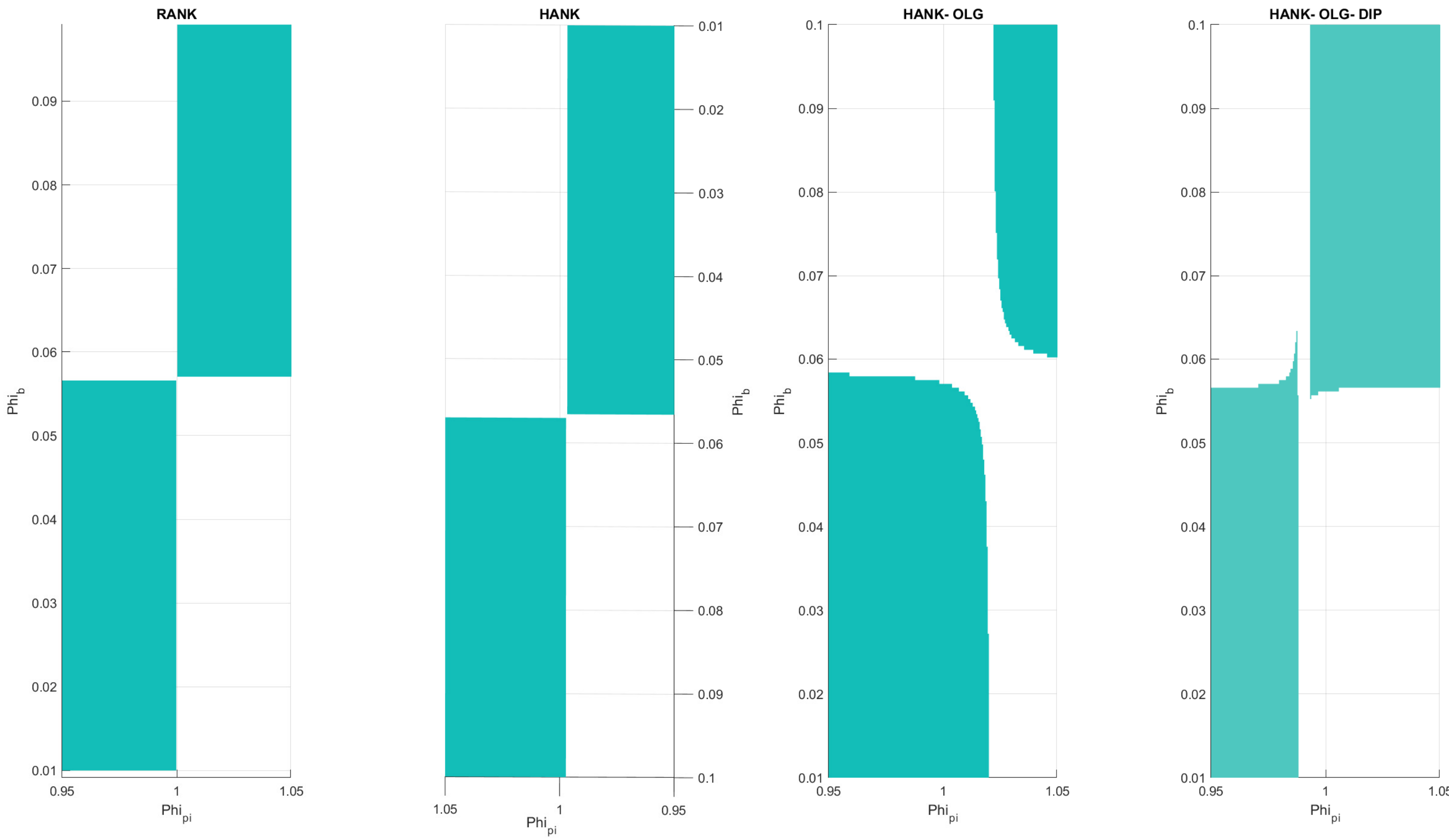

Figure 1.2: Determinancy under Lump Sum tax.

1.5.5 Dynamics under monetary and fiscal policy

In this section, we look at the policy response to an unanticipated transitionary cost push shock. In the appendix B, we also present the policy response to a one- time TFP shock. In either case, after the aggregate shock is realised, households have perfect foresight.

In the benchmark case, illustrated by Figures 1.3 and 1.4, the policy maker has access only to distortions income taxes. Since the model abstracts from marginal propensity to consume heterogeneity and binding equilibrium borrowing limits, the policy response to the shock is unable to redistribute consumption from wealthy/older agents to indebted households[9] (See Auclert (2019)).

Additionally, both monetary and fiscal policy are conducted following simple rules, with the size of the feedback coefficients held constant across all specifications. Hence, the policy response is not affected by the size or the type of inequality that is present in the economy.

In response to a positive (unanticipated) cost push shock, there is an unambiguous effect on inflation regardless of household heterogeneity or the tax instrument available. Following the shock, inflation immediately rises causing the nominal interest rate (NIR) to also rise. The size of this initial jump depends on the policy mix. In the monetary- led regime where the monetary policy reacts more aggressively to excess inflation, the initial jump is considerably smaller (almost half). In either case, the nominal interest rate and tax dynamics are governed by the Taylor rule and the fiscal rule, respectively. So, after the initial response to the shock, both inflation and the interest rate begin to move back towards the steady state. The speed of the convergence is linked to the strength of the inflation response coefficient.

As expected from the no- arbitrage condition, asset prices(P^M) are inversely related to the nominal interest rate. As such, after the shock is realised, asset prices immediately drop and then, the series starts to move back towards the long run equilibrium, tracing the interest rate path.

Furthermore, the initial jump in inflation also reduces the real value of the outstanding government debt. Since, both asset prices and government debt initially falls, so do taxes. After all, regardless of the tax instrument available, the tax response depends solely on the asset price dynamics and the asset demand deviations. In line Schmitt-Grohé & Uribe (2004), following the shock, both taxes and the government debt dynamics resemble almost a random walk. This behaviour of government bonds allows the policy maker to steadily smooth out tax distortions over time.

Similarly, following the realisation of the aggregate shock, aggregate consumption initially drops. This result is driven by the fact that in response to a positive cost push shock, wages fall due to the initial rise in inflation, causing the non- financial income of household to decline. Additionally,the aggregate labour is bounded by the value of $\bar{\xi}$. Hence, households can only

[9]In this framework, indebted households are agents who either recently entered the market or those with a history of drawing "bad" realisation of the idiosyncratic shocks.

partially adjust their labour supply, following the shock. Now, since aggregate consumption and aggregate output are connected through the resource constraint, the two series display near identical dynamics.

Although the policy response is identical across the different frameworks, in the specifications that feature household heterogeneity, the policy response will redistribute wealth across agents. As discussed above, the initial jump in inflation reduces the value of the aggregate asset holdings. So, households who belong to older cohorts as well as those who have a history of “good“ realisations of the idiosyncratic shock experience a decline in their wealth following the shock. Whereas, households who belong to younger cohorts as well as those who consistently drew “low“ realisations of the idiosyncratic productivity shock their welfare improves, following the jump in inflation, since the real value of their outstanding obligations decreases. However, older cohorts make up the majority of the active population. And, at the same time, the initial positive interest rate response to the inflation deviation causes the cost of borrowing to increase.

As such, following the shock, inequality initially rises. In fact, the behaviour of the series is determined by the behaviour of aggregate debt, causing it to behave almost as a unit root, in response to the shock. As expected, the deviation in inequality is considerably larger in the HANK-OLG specifications.

As we move from the HANK to the HANK- OLG specifications, the amount of wealth inequality, in the economy, increases. As discussed above, the HANK- OLG model with declining labour efficiencies displays the largest amount of inequality across the different HANK frameworks. As such, it comes as no surprise that in response to the shock, the sensitivity of individual consumption to changes in individual income (μ) rises the most in this specification. Consequently, this version of the HANK model also experiences the largest initial jump in inequality, following the shock. Interestingly, the initial jump in both μ and S are smaller in the PM/AF regime.

Moreover, the response of the Debt- to- GDP ratio is identical across all specifications. The fact that ratio falls in response to the shock indicates that the initial decrease in aggregate output is larger than the decrease in the value of the outstanding government debt. This result persists in both the AM/PF and AF/PM regimes.

Finally, as shown by figure 1.5, both policy combinations AM/PF or AF/PM will stabilise the economy, in response to the unanticipated aggregate shock. However, for policy maker who is more concerned with “equity“ than “efficiency“, they have an incentive to purse a PM/AF policy mix, following the transitory shock. Since, the fiscally- led regime exhibit the smallest initial increase in both MPC out of cash on hand and in wealth inequality.

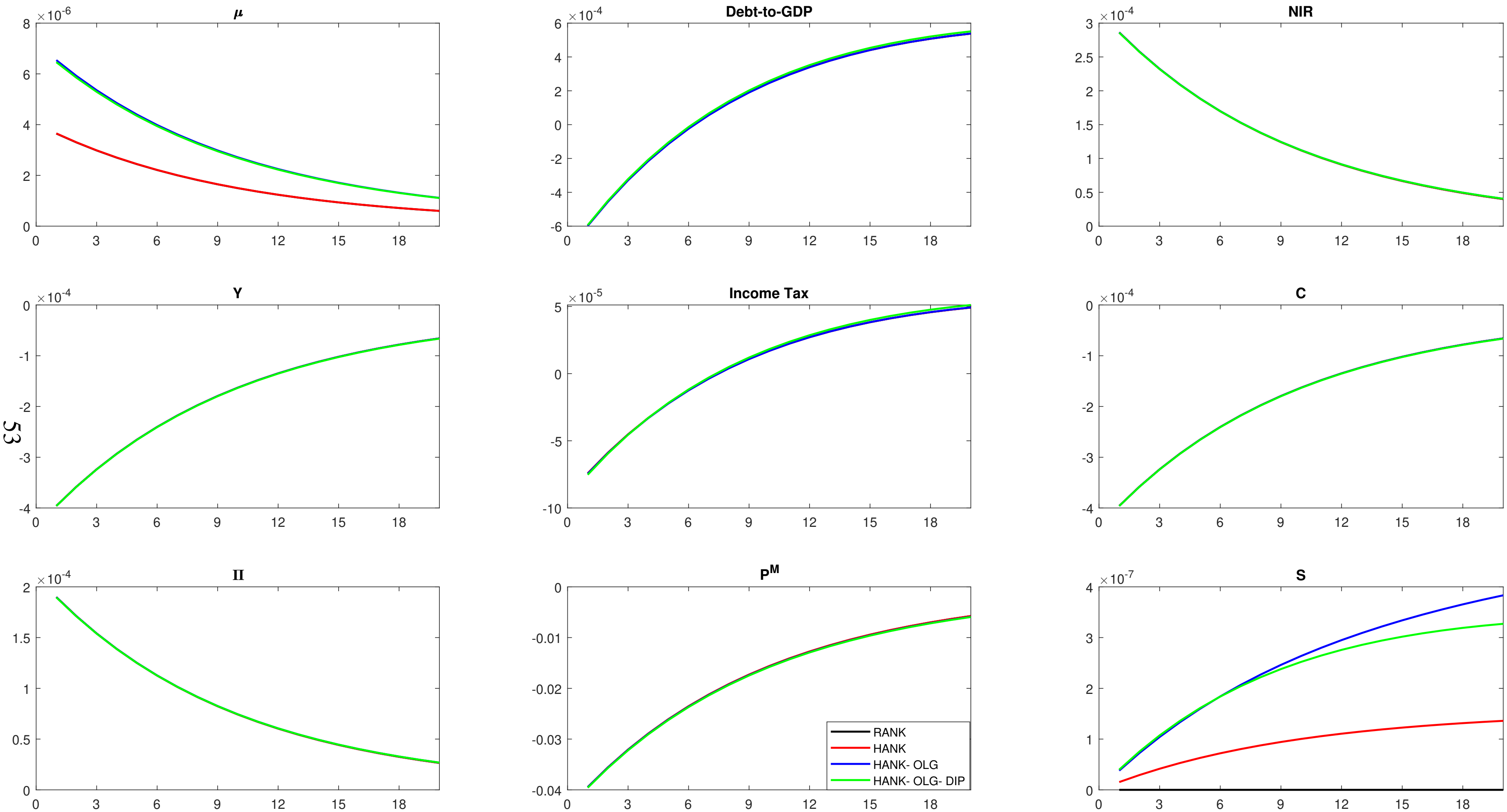

Figure 1.3: Transitionary Cost Push shock:All Cases AM/PF.

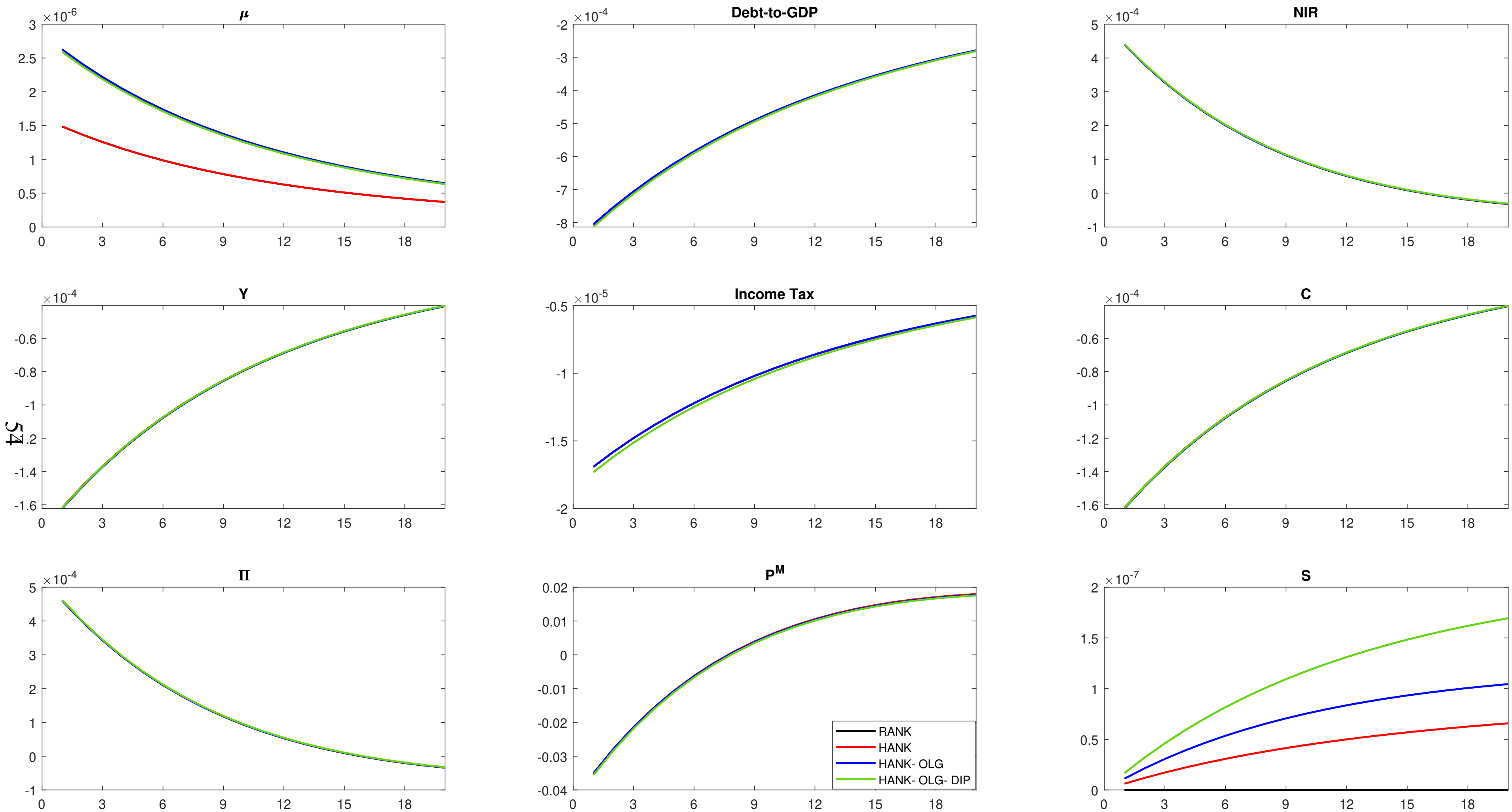

Figure 1.4: Transitionary Cost Push shock:All Cases PM/AF.

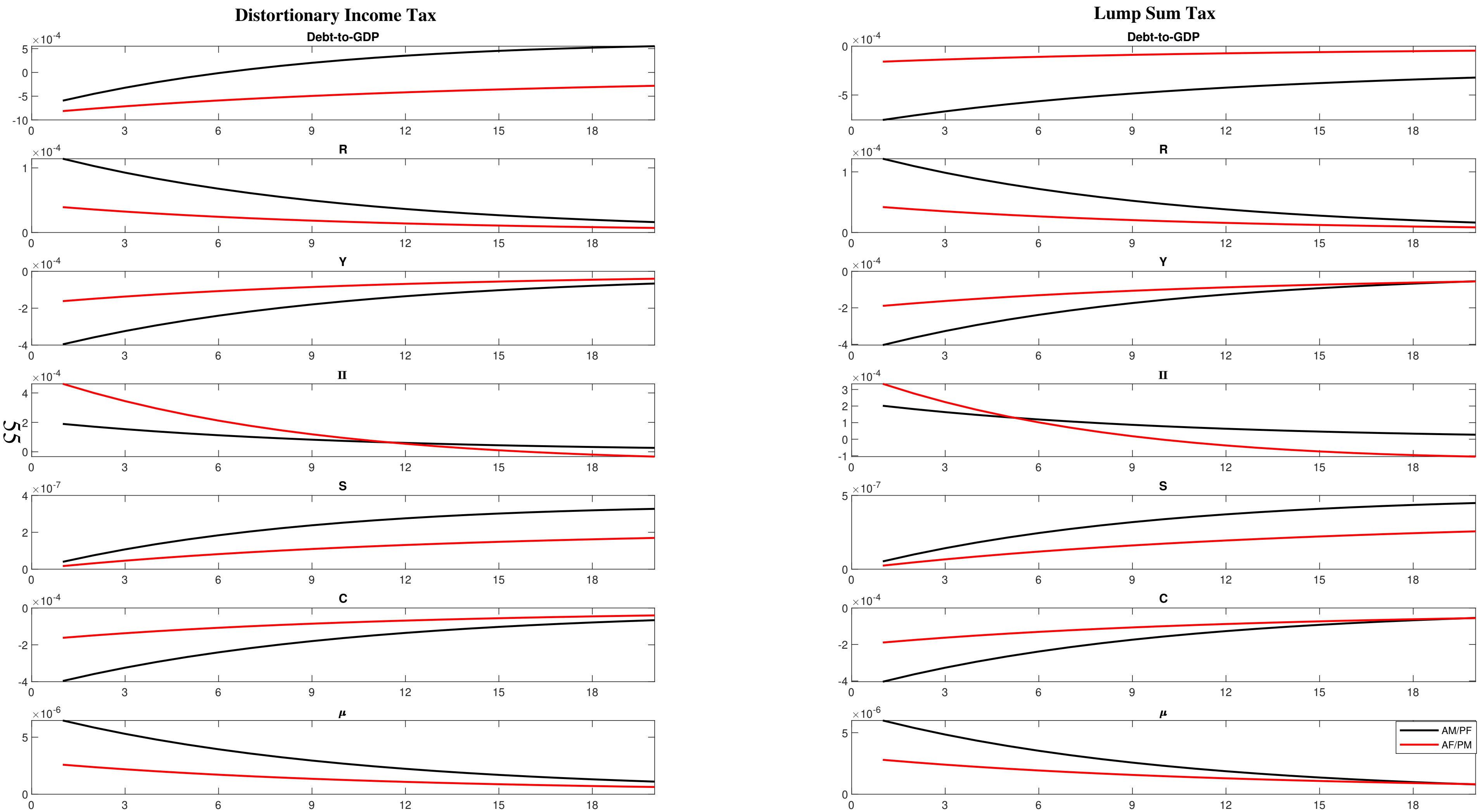

Figure 1.5: Transitionary Cost Push shock: HANK

1.6 Conclusion

We study monetary and fiscal policy interactions in THANK environment that features (partially) insurable income risk and rich inter-generational wealth inequality. The assumption of the CARA utility and normally distributed (iid) idiosyncratic productivity simplify the aggregation process and allows us to compute the distribution of wealth in a discrete- time environment. We find that each layer of heterogeneity that we add to the model, it has a significant impact on both the model's long run equilibrium and determinancy properties. Additionally, deviating from the assumption of "zero- liquidity" implies an additional externalities absent from models without government debt.

Furthermore, in the presence of outstanding of government debt, the OLG frictions play a bigger part in shaping the model's steady state as well as in determining the areas of the parameter space that ensures the existence of unique stable rational expectations equilibrium. Under passive fiscal policy, we show that the infimum coefficient value for the fiscal feedback on debt is determined by the amount/types of inequality and the tax instrument available.

Moreover, the model abstracts from marginal propensity to consume heterogeneity and binding equilibrium borrowing constraints. As such, in response to an aggregate shock the policy response will affect the wealth dispersion among agents but the policy response is still unable to redistribute consumption from wealth/older households to younger/ poorer agents. Although, we abstract of the study of optimal policy, we have derived a utilitarian Social welfare criterion and a corresponding micro- founded wealth inequality statistic.

Finally, despite the fact that the conduct of monetary and fiscal policy is identical across all frameworks, we find that for a policy maker who is more concerned with "equity" rather than "efficiency", pursuing a fiscally- led (AF/PM) mix might be preferable. Since, a fiscally- led regime results in smaller initial jump in inequality as well as in the smallest MPC rise in response to an unanticipated aggregate "transitionary" shock.

1.7 Appendix

1.8 Appendix A

1.8.1 Proof of Proposition 1

Proof. We form the following Lagrangian

$$\begin{aligned} L_s = \mathbb{E}_i \sum_{t=s}^{\infty} (\beta\vartheta)^{t-s} \Big(& -\frac{1}{\gamma} e^{-\gamma(c_t^s(i)+\chi G_t)} - \rho e^{\frac{1}{\rho}(l_t^s(i)+\Theta_t^s-\xi_t^s(i))} \\ & + \lambda_t^s(i) \Big(\Big(c_t^s(i) - \eta_t l_t^s(i) - d_t + T_t + \tilde{P}_t^M a_{t+1}^{L,s}(i) + \tilde{q}_t a_{t+1}^{S,s}(i) \Big) (1+\pi_t) \\ & - \left(1+\varsigma \tilde{P}_t^M\right) a_t^{L,s}(i) - a_t^{S,s}(i) \Big) \Big) \end{aligned}$$

so the FOCs are

$$\begin{aligned} 0 &= e^{-\gamma(c_t^s(i)+\chi G_t)} + \lambda_t^s(i)(1+\pi_t) \\ 0 &= -e^{\frac{1}{\rho}(l_t^s(i)+\Theta_t^s-\xi_t^s(i))} - \lambda_t^s(i)\,\eta_t(1+\pi_t) \\ 0 &= \lambda_t^s(i)\,\tilde{P}_t^M(1+\pi_t) - \mathbb{E}_i\left(1+\varsigma\tilde{P}_{t+1}^M\right)\beta\vartheta\lambda_{t+1}^s(i) \\ 0 &= \lambda_t^s(i)\,\tilde{q}_t(1+\pi_t) - \beta\vartheta\mathbb{E}_i\lambda_{t+1}^s(i) \end{aligned}$$

from where (there is no aggregate risk)

$$\begin{aligned} \lambda_t^s(i) &= -\frac{1}{(1+\pi_t)} e^{-\gamma(c_t^s(i)+\chi G_t)} \\ l_t^s(i) &= \rho\log\eta_t - \gamma\rho\left(c_t^s(i)+\chi G_t\right) - \Theta_t^s + \xi_t^s(i) \\ c_t^s(i) &= -\frac{1}{\gamma}\log\frac{\beta\vartheta}{\tilde{q}_t(1+\pi_{t+1})} + \chi G_{t+1} - \chi G_t - \frac{1}{\gamma}\log\mathbb{E}_i e^{-\gamma c_{t+1}^s(i)} \\ \frac{1}{\tilde{q}_t} &= \frac{\left(1+\varsigma\tilde{P}_{t+1}^M\right)}{\tilde{P}_t^M} \end{aligned}$$

The Euler equation, using normality of consumption distribution, can also be written as

$$c_t^s(i) = -\frac{1}{\gamma}\log\left(\frac{\beta\vartheta}{\tilde{q}_t(1+\pi_{t+1})}\right) + \chi G_{t+1} - \chi G_t + \mathbb{E}_i c_{t+1}^s(i) - \frac{\gamma}{2}\mathbb{V}_i c_{t+1}^s(i). \tag{1.40}$$

To obtain expressions for expectation and variance of consumption, we do the following three steps.

First, substitute labour supply into the budget constraint:

$$A_{t+1}^{s}(i)=\frac{R_t}{\vartheta}\left(A_t^s(i)+X_t-\eta_t\Theta_t^s+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)-(1+\rho\gamma\eta_t)\,c_t^s(i)\right) \tag{1.41}$$

where we denoted

$$X_t=\eta_t\left(\rho\log\eta_t+\bar{\xi}-\rho\gamma\chi G_t\right)+d_t-T_t.$$

Second, assume that individual consumption can be parameterised as

$$c_t^s(i)=\mathscr{X}_t+\mu_t\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)-\varphi_t\Theta_t^s\right) \tag{1.42}$$

Lead it one period:

$$\begin{aligned}c_{t+1}^s(i)&=\mathscr{X}_{t+1}+\mu_{t+1}\left(A_{t+1}^s(i)+\eta_{t+1}\left(\xi_{t+1}^s(i)-\bar{\xi}\right)-\varphi_{t+1}\Theta_{t+1}^s\right)\\&=\mu_{t+1}\left(\frac{R_t}{\vartheta}\left(\begin{array}{c}(1-(1+\rho\gamma\eta_t)\,\mu_t)\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)+X_t\\-(1+\rho\gamma\eta_t)\,\mathscr{X}_t+(-\eta_t+(1+\rho\gamma\eta_t)\,\mu_t\varphi_t)\,\Theta_t^s\end{array}\right)\right)\\&+\mathscr{X}_{t+1}+\mu_{t+1}\eta_{t+1}\left(\xi_{t+1}^s(i)-\bar{\xi}\right)-\mu_{t+1}\varphi_{t+1}\left(\Theta_t^s+\varkappa\right)\end{aligned} \tag{1.43}$$

where in the second line we used the budget constraint, parameterisation (2.60) and the fact that $\Theta_{t+1}^s=\varkappa(t+1-s)=\varkappa(t-s)+\varkappa=\Theta_t^s+\varkappa$.

Finally, we obtain expressions for expectation and variance terms. Because $c_{t+1}^s(i)$ is normally distributed by i, its mean and variance are determined as follows:

$$\begin{aligned}\mathbb{E}_i c_{t+1}^s(i)&=\mathscr{X}_{t+1}+\mu_{t+1}\left(\begin{array}{c}\frac{R_t}{\vartheta}(1-(1+\rho\gamma\eta_t)\,\mu_t)\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)\\+\frac{R_t}{\vartheta}\left(X_t-(1+\rho\gamma\eta_t)\,\mathscr{X}_t\right)\\+\frac{R_t}{\vartheta}\left(-\eta_t+(1+\rho\gamma\eta_t)\,\mu_t\varphi_t\right)\Theta_t^s\end{array}\right)\\&\quad-\mu_{t+1}\varphi_{t+1}\left(\Theta_t^s+\varkappa\right)\\\mathbb{V}_i c_{t+1}^s(i)&=\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2\end{aligned}$$

(note that $\mathbb{E}_i\xi_{t+1}^s(i)=\bar{\xi}$, but $\mathbb{E}_i\xi_t^s(i)=\xi_t^s(i)$, $\mathbb{V}_i\xi_{t+1}^s(i)=\sigma_{t+1}^2$, but $\mathbb{V}_i\xi_t^s(i)=0$).

We now use these expressions and parameterisation (2.60) and substitute them into the consumption Euler equation (2.58) to find coefficients $\mathscr{X}_t,\mu_t$ and φ_t.

Substitution into the Euler equation yields:

$$
\begin{aligned}
&\mathscr{X}_t + \mu_t \left(A_t^s(i) + \eta_t^s \left(\xi_t^s(i) - \bar{\xi}\right) - \varphi_t \Theta_t^s\right) \\
&= -\frac{1}{\gamma}\log(\beta R_t) + \chi G_{t+1} - \chi G_t - \mu_{t+1}\varphi_{t+1}\left(\Theta_t^s + \varkappa\right) - \frac{\gamma}{2}\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2 \\
&+ \mathscr{X}_{t+1} + \mu_{t+1}\frac{R_t}{\vartheta}\left(\begin{array}{c} (1-(1+\rho\gamma\eta_t^s)\mu_t)\left(A_t^s(i) + \eta_t^s\left(\xi_t^s(i) - \bar{\xi}\right)\right) \\ +(X_t - (1+\rho\gamma\eta_t)\mathscr{X}_t) + (-\eta_t + (1+\rho\gamma\eta_t)\mu_t\varphi_t)\Theta_t^s \end{array}\right).
\end{aligned}
$$

Collect coefficients on independent states: $1, A_t^s(i), \xi_t^s(i), \Theta_t^s$. This yields three independent equations on μ_t, κ_t and $\mathscr{X}_t$:

$$
\begin{aligned}
\mathscr{X}_t - \mu_t\eta_t\bar{\xi} = {}& -\frac{1}{\gamma}\log(\beta R_t) + \chi\tilde{G}_{t+1} - \chi\tilde{G}_t + \mathscr{X}_{t+1} - \frac{\gamma}{2}\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2 \\
&+ \mu_{t+1}\left(\frac{R_t}{\vartheta}(X_t - (1+\rho\gamma\eta_t)\mathscr{X}_t) - \frac{R_t}{\vartheta}(1-(1+\rho\gamma\eta_t)\mu_t)\eta_t\bar{\xi}\right) \\
&- \mu_{t+1}\varphi_{t+1}\varkappa
\end{aligned} \tag{1.44}
$$

$$
\mu_t = \mu_{t+1}\frac{R_t}{\vartheta}(1-(1+\rho\gamma\eta_t)\mu_t) \tag{1.45}
$$

$$
-\mu_t\varphi_t = \mu_{t+1}\left(\frac{R_t}{\vartheta}(-\eta_t + (1+\rho\gamma\eta_t)\mu_t\varphi_t)\right) - \mu_{t+1}\varphi_{t+1} \tag{1.46}
$$

Provided that $\mu_t \neq 0$ The dynamic equation on evolution of the marginal propensity to consume ot of cash in hands can be expressed as:

$$
\frac{1}{\mu_t} - (1+\rho\gamma\eta_t) = \frac{\vartheta}{R_t\mu_{t+1}} \tag{1.47}
$$

the equation for φ_t becomes*[name it!]*

$$
\varphi_t = \eta_t + \frac{\vartheta}{R_t}\varphi_{t+1} \tag{1.48}
$$

and the evolution of the measure of aggregate consumption $\mathscr{X}_t$ is:

$$
\begin{aligned}
\mathscr{X}_t = {}& -\frac{\vartheta\mu_t}{\gamma\mu_{t+1}R_t}\log(\beta R_t) + \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\mathscr{X}_{t+1} + \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\chi G_{t+1} \\
&- \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\chi G_t + \mu_t X_t - \frac{\vartheta\mu_t}{R_t}\varkappa\varphi_{t+1} - \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\frac{\gamma}{2}\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2
\end{aligned}
$$

Introduce new variable:

$$
\mathscr{C}_t = \mathscr{X}_t + \chi G_t
$$

then we arrive to

$$\mathscr{C}_t = -\frac{\vartheta\mu_t}{\gamma\mu_{t+1}R_t}\log(\beta R_t) + \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\mathscr{C}_{t+1} - \frac{\vartheta\mu_t}{R_t}\varkappa\varphi_{t+1} - \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\frac{\gamma}{2}\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2$$
$$+\mu_t\left(\eta_t\left(\rho\log(\eta_t)+\bar{\xi}\right)+d_t-T_t+\chi G_t\right)$$

after all terms with G_t are combined. ■

1.8.2 Aggregation

Define aggregate consumption, income, labour supply and labour demand

$$c_t := (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 c_t^s(i)\,di$$
$$y_t := (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 y_t^s(i)\,di$$
$$H_t := \int_0^1 h_t(j)\,dj$$
$$n_t := (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 l_t^s(i)\,di$$

From labour market clearing conditions we know that aggregate labour supply must equal aggregate labour demand hence:

$$n_t := \int_0^1 n_t(j)\,dj = (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 l_t^s(i)\,di = H_t$$

Define aggregate actuarial bonds, $J=\{S,L\}$:

$$\vartheta a_t^J := (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 a_t^{J,s}(i)\,di.$$

To aggregate the household budget constraint, we need to compute $(1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 a_{t+1}^{J,s}(i)\,di$.

Note that

$$\vartheta a_{t+1}^{J}=\left(1-\vartheta\right)\sum_{s=-\infty}^{t+1}\vartheta^{t+1-s}\int_{0}^{1}a_{t+1}^{J,s}\left(i\right)di=\left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t+1-s}\int_{0}^{1}a_{t+1}^{J,s}\left(i\right)di$$
$$+\left(1-\vartheta\right)\int_{0}^{1}a_{t+1}^{J,t+1}\left(i\right)di$$
$$=\vartheta\left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_{0}^{1}a_{t+1}^{J,s}\left(i\right)di$$

then

$$a_{t+1}^{J}=\left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_{0}^{1}a_{t+1}^{J,s}\left(i\right)di$$

It follows

$$\vartheta A_{t}=\left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_{0}^{1}A_{t}^{s}\left(i\right)di,$$
$$A_{t+1}^{s}=\left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_{0}^{1}A_{t+1}^{s}\left(i\right)di,$$

where

Finally, note that

$$\sum_{s=-\infty}^{t}\vartheta^{t-s}\left(t-s\right)=\vartheta^{t-t}\left(t-t\right)+\vartheta^{t-t+1}\left(t-t+1\right)+\vartheta^{t-t+2}\left(t-2\right)=\ldots$$
$$=\sum_{k=1}^{\infty}k\vartheta^{k}=\frac{\vartheta}{\left(1-\vartheta\right)^{2}}$$

so that agggregation of sick days yields

$$\left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_{0}^{1}\Theta_{t}^{s}di=\left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_{0}^{1}\varkappa\left(t-s\right)di$$
$$=\varkappa\left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\left(t-s\right)$$
$$=\frac{\varkappa\vartheta}{1-\vartheta}$$

Aggregation of household budget constrain (2.4) yields:

$$\frac{\vartheta}{R_{t}}\frac{\left(\frac{\tilde{P}_{t}^{M}}{\tilde{q}_{t}}a_{t+1}^{L}+a_{t+1}^{S}\right)}{\left(1+\pi_{t+1}\right)}=\vartheta\frac{\left(\left(1+\varsigma\tilde{P}_{t}^{M}\right)a_{t}^{L}+a_{t}^{S}\right)}{\left(1+\pi_{t}\right)}+y_{t}-c_{t}\tag{1.49}$$

or

$$\frac{\vartheta}{R_t}A_{t+1} = \vartheta A_t + y_t - c_t$$

where

$$A_t = \frac{\left(\left(1+\varsigma \tilde{P}_t^M\right) a_t^L + a_t^S\right)}{(1+\pi_t)} = \frac{\left(\frac{\tilde{P}_{t-1}^M}{\tilde{q}_{t-1}} a_t^L + a_t^S\right)}{(1+\pi_t)}$$

and

$$y_t = \eta_t n_t + d_t - T_t.$$

1.8.3 Proof of Proposition 2

Proof. We start with the derived relationship:

$$\begin{aligned}\mathscr{X}_t = &-\frac{\vartheta\mu_t}{\gamma\mu_{t+1}R_t}\log(\beta R_t) + \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\mathscr{X}_{t+1} + \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\chi G_{t+1} \\ &- \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\chi G_t + \mu_t X_t - \frac{\vartheta\mu_t}{R_t}\varkappa\varphi_{t+1} - \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\frac{\gamma}{2}\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2\end{aligned}$$

Recall that

$$\mathscr{X}_t = \mathscr{C}_t - \chi G_t$$

and

$$c_t = \mathscr{C}_t - \chi G_t + \mu_t\left(\vartheta A_t - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi_t\right)$$

So we can parameterise

$$\begin{aligned}\mathscr{X}_t = \mathscr{C}_t - \chi G_t &= c_t - \mu_t\left(\vartheta A_t - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi_t\right) \\ \mathscr{X}_{t+1} &= c_{t+1} - \mu_{t+1}\left(\vartheta A_{t+1} - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi_{t+1}\right)\end{aligned}$$

and substitute these two relationships

$$\begin{aligned}&c_t - \mu_t\left(\vartheta A_t - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi_t\right) \\ &= -\frac{\vartheta\mu_t}{\gamma\mu_{t+1}R_t}\log(\beta R_t) + \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\left(c_{t+1} - \mu_{t+1}\left(\vartheta A_{t+1} - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi_{t+1}\right)\right) \\ &+ \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\chi G_{t+1} - \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\chi G_t + \mu_t X_t - \frac{\vartheta\mu_t}{R_t}\varkappa\varphi_{t+1} - \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\frac{\gamma}{2}\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2\end{aligned}$$

Substitute

$$X_t = \eta_t \left(\rho \log(\eta_t) + \bar{\xi} - \rho\gamma\chi G_t\right) + d_t - T_t$$

and use budget constraint

$$\vartheta A_t = \frac{\vartheta}{R_t} A_{t+1} - \eta_t \left(\rho \log(\eta_t) + \bar{\xi}\right) + \frac{\varkappa\vartheta}{(1-\vartheta)}\eta_t + \rho\gamma\eta_t\chi G_t + \rho\gamma\eta_t c_t - d_t + T_t + c_t$$

and (2.9)-(2.10) to arrive to the following Euler equation

$$c_t + \chi G_t = -\frac{1}{\gamma}\log(\beta R_t) + c_{t+1} + \chi G_{t+1} + (1-\vartheta)\mu_{t+1}A_{t+1} - \frac{\gamma}{2}\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2 - \mu_{t+1}\varkappa\varphi_{t+1}$$

Labour supply (2.8) is is straightforwardly aggregated to

$$n_t = \rho \log(\eta_t) - \varkappa\frac{\vartheta}{1-\vartheta} - \rho\gamma(c_t + \chi G_t) + \bar{\xi}$$

■

Derivation of Phillips Curve

Firm j solves the following optimization problem

$$\max_{P_t(j)} \sum_{t=0}^{\infty} \beta^t \left(\left(\frac{P_t(j)}{P_t} Y_t(j) - (1-s) w_t n_t(j) \right) - \frac{\Phi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \right)$$

subject to monopolistic demand

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon_t} Y_t$$

and production function

$$Y_t(j) = z_t n_t(j)$$

Substitute

$$\max_{P_t(j)} \sum_{t=0}^{\infty} \beta^t \left(\left(\frac{P_t(j)}{P_t} - (1-s)\frac{w_t}{z_t} \right) \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon_t} Y_t - \frac{\Phi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \right)$$

to yield the following first order condition:

$$0=\beta^t\left((1-\varepsilon_t)\left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_t}\frac{Y_t}{P_t}+\varepsilon_t(1-s)\frac{w_t}{z_t}\left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_t-1}\frac{Y_t}{P_t}-\Phi\left(\frac{P_t(j)}{P_{t-1}(j)}-1\right)\frac{Y_t}{P_{t-1}(j)}\right)$$
$$+\beta^{t+1}\left(\Phi\left(\frac{P_{t+1}(j)}{P_t(j)}-1\right)Y_{t+1}\frac{P_{t+1}(j)}{P_t^2(j)}\right)$$

All firms are identical so $P_t(j)=P_t$ and, therefore:

$$\pi_t(1+\pi_t)=\frac{1-\varepsilon_t+(1-s)\varepsilon_t\frac{w_t}{z_t}}{\Phi}+\beta\frac{Y_{t+1}}{Y_t}\pi_{t+1}(1+\pi_{t+1})$$

The profit of firms, distributed as dividends

$$d_t=(Y_t-(1-s)w_tn_t)-\frac{\Phi}{2}\pi_t^2Y_t$$

Financial Intermediaries

Financial intermediaries trade actuarial and government bonds.

At time t they buy short and long-term actuarial bonds and pay with short and long term government bonds, so the budget constraint of intermediaries is

$$-\tilde{P}_t^M a_{t+1}^L-\tilde{q}_t a_{t+1}^S+P_t^M b_{t+1}^L+q_t b_{t+1}^S\leqslant 0, \tag{1.50}$$

where $a_{t+1}^J=(1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 a_{t+1}^{J,s}(i)\,di$.

Their profit one period later is, therefore

$$\Pi=\left(1+\varsigma P_{t+1}^M\right)b_{t+1}^L+b_{t+1}^S-\left(1+\varsigma\tilde{P}_{t+1}^M\right)\vartheta a_{t+1}^L-\vartheta a_{t+1}^S$$

where b_{t+1}^J are total government bonds at time $t+1$, and ϑa_{t+1}^J are total actuarial bonds at time $t+1$, i.e. $\vartheta a_{t+1}^J=(1-\vartheta)\sum_{s=-\infty}^{t+1}\vartheta^{t+1-s}\int_0^1 a_{t+1}^{J,s}(i)\,di$.

The Lagrangian is

$$\Pi=\left(1+\varsigma P_{t+1}^M\right)b_{t+1}^L+b_{t+1}^S-\left(1+\varsigma\tilde{P}_{t+1}^M\right)\vartheta a_{t+1}^L-\vartheta a_{t+1}^S$$
$$+\lambda_t\left(-\tilde{P}_t^M a_{t+1}^L-\tilde{q}_t a_{t+1}^S+P_t^M b_{t+1}^L+q_t b_{t+1}^S\right)$$

and the first order conditions are:

$$\begin{aligned}
&\frac{\partial}{\partial b_{t+1}^{L}} : \left(1+\varsigma P_{t+1}^{M}\right) + \lambda_t P_t^M \\
&\frac{\partial}{\partial b_{t+1}^{S}} : 1+\lambda_t q_t \\
&\frac{\partial}{\partial a_{t+1}^{L}} : -\left(1+\varsigma \tilde{P}_{t+1}^{M}\right)\vartheta - \lambda_t \tilde{P}_t^M \\
&\frac{\partial}{\partial a_{t+1}^{S}} : -\vartheta - \lambda_t \tilde{q}_t
\end{aligned}$$

From where we have:

$$\frac{1}{\tilde{q}_t} = \frac{\left(1+\varsigma \tilde{P}_{t+1}^{M}\right)}{\tilde{P}_t^M} \tag{1.51}$$

$$\tilde{q}_t = \vartheta q_t \tag{1.52}$$

$$\frac{1}{q_t} = \frac{\left(1+\varsigma P_{t+1}^{M}\right)}{P_t^M} \tag{1.53}$$

and so the profit is zero:

$$\begin{aligned}
\Pi &= \left(1+\varsigma P_{t+1}^{M}\right) b_{t+1}^{L} + b_{t+1}^{S} - \vartheta a_{t+1}^{S} - \vartheta\left(1+\varsigma \tilde{P}_{t+1}^{M}\right) a_{t+1}^{L} \\
&= \frac{1}{q_t}\left(P_t^M b_{t+1}^{L} + q_t b_{t+1}^{S} - \tilde{q}_t a_{t+1}^{S} - \tilde{P}_t^M a_{t+1}^{L}\right) = 0
\end{aligned} \tag{1.54}$$

and

$$R_t = \frac{\vartheta}{\tilde{q}_t\left(1+\pi_{t+1}\right)} = \frac{1}{q_t\left(1+\pi_{t+1}\right)} \tag{1.55}$$

1.8.4 Social Welfare Function

Aggregation of Welfare

Recall that

$$\begin{aligned}
l_t^s(i) &= \rho\log\eta_t - \Theta_t^s - \rho\gamma\left(c_t^s(i) + \chi G_t\right) + \xi_t^s(i) \\
c_t^s(i) &= \mathscr{C}_t - \chi G_t + \mu_t m_t^s(i) \\
m_t^s(i) &= A_t^s(i) + \eta_t\left(\xi_t^s(i) - \bar{\xi}\right) - \varphi_t \Theta_t^s
\end{aligned}$$

so the (remaining at p) life-time utility of an agent born at s at time $p > s$ can be written as (substitute labour supply)

$$W_p^s(i) = \sum_{t=p}^{\infty} (\beta\vartheta)^{t-p} U_t^s(i) \tag{1.56}$$

where

$$\begin{aligned}
U_t^s(i) &= -\frac{1}{\gamma} e^{-\gamma(c_t^s(i)+\chi G_t)} - \rho e^{\frac{1}{\rho}(l_t^s(i)+\Theta_t^s-\xi_t^s(i))} \\
&= -\frac{1}{\gamma} e^{-\gamma(c_t^s(i)+\chi G_t)} - \rho e^{\frac{1}{\rho}(\rho\log(\eta_t)-\rho\gamma(c_t^s(i)+\chi G_t))} \\
&= -\frac{1}{\gamma} e^{-\gamma(c_t^s(i)+\chi G_t)} - \rho\eta_t e^{-\gamma(c_t^s(i)+\chi G_t)} \\
&= -\frac{1}{\gamma}(1+\gamma\rho\eta_t) e^{-\gamma(c_t^s(i)+\chi G_t)} \\
&= -\frac{1}{\gamma}(1+\gamma\rho\eta_t) e^{-\gamma(\mathscr{C}_t+\mu_t m_t^s(i))}
\end{aligned}$$

The social welfare function at time $t = 0$ is defined as

$$\mathbb{W}_0 = (1-\vartheta) \sum_{s=-\infty}^{0} \vartheta^{-s} \int_0^1 W_0^s(i)\, di + \sum_{s=1}^{\infty} (1-\vartheta)\beta^s \int_0^1 W_s^s(i)\, di \tag{1.57}$$

where the first term is utility of generations that are alive at time zero. The second term is utility of unborn generations, with $s > 0$, each such generation is treated with weight β^s.

We can rewrite the welfare function in a more convenient way. Denote

$$\mathscr{U}_t^s = -\frac{1}{\gamma}(1+\gamma\rho\eta_t) \int_0^1 e^{-\gamma(\mathscr{C}_t+\mu_t m_t^s(i))} di$$

is t-period utility of a *cohort* born at time s.

Then

$$\begin{aligned}
\frac{\mathbb{W}_0}{(1-\vartheta)} &= \mathscr{U}_0^0 + \vartheta\mathscr{U}_0^{-1} + \vartheta^2\mathscr{U}_0^{-2} + \ldots \\
&+ \beta\left(\mathscr{U}_0^1 + \vartheta\mathscr{U}_0^0 + \vartheta^2\mathscr{U}_0^{-1} + \ldots\right) + \ldots \\
&+ \beta^t\left(\mathscr{U}_t^t + \vartheta\mathscr{U}_t^{t-1} + \vartheta^2\mathscr{U}_t^{t-2} + \ldots + \vartheta^s\mathscr{U}_t^{t-s}\right) + \ldots \\
&= \sum_{t=0}^{\infty} \beta^t \sum_{s=0}^{\infty} \vartheta^s \mathscr{U}_t^{t-s} = \sum_{t=0}^{\infty} \beta^t \sum_{v=-\infty}^{t} \vartheta^{t-v} \mathscr{U}_t^v
\end{aligned}$$

where in the last line we used new index $v = t - s$.

Recycling notation, we get

$$\mathbb{W}_0 = -\frac{1}{\gamma}\sum_{t=0}^{\infty}\beta^t\left(1+\gamma\rho\eta_t\right)e^{-\gamma\mathscr{C}_t}\left(\left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 e^{-\gamma\mu_t m_t^s(i)}di\right) \tag{1.58}$$

$$\mathbb{W}_0 = -\frac{1}{\gamma}\sum_{t=0}^{\infty}\beta^t\left(1+\gamma\rho\eta_t\right)e^{-\gamma\mathscr{C}_t}\left(\left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 e^{-\gamma\mu_t m_t^s(i)}di\right) \tag{1.59}$$

Denote

$$\Sigma_t = \left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 e^{-\gamma\mu_t m_t^s(i)}di$$

so that

$$\mathbb{W}_0 = \sum_{t=0}^{\infty}\beta^t\mathbb{U}_t$$

where

$$\mathbb{U}_t = -\frac{1}{\gamma}\left(1+\gamma\rho\eta_t\right)e^{-\gamma\mathscr{C}_t}\Sigma_t$$

Here $\left(1+\gamma\rho\eta_t\right)e^{-\gamma\mathscr{C}_t}$ only depends on aggregate variables, so will be the same for a representative agent.

$\Sigma_t = \left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 e^{-\gamma\mu_t m_t^s(i)}di$ is a welfare cost of inequality. It is increasing in the within cohort dispersion of consumption. If there is risk then Σ_t is increasing, and this decreases the overall level of welfare.

1.8.5 Recursion

Derive Σ_t recursion.

$$\begin{aligned}
\Sigma_t &= \left(1-\vartheta\right)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 e^{-\gamma\mu_t m_t^s(i)}di \\
&= \left(1-\vartheta\right)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}\int_0^1 e^{-\gamma\mu_t m_t^s(i)}di + \left(1-\vartheta\right)\int_0^1 e^{-\gamma\mu_t m_t^t(i)}di \\
&= \left(1-\vartheta\right)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma\mu_t\varkappa(t-s)\varphi_t}\int_0^1 e^{-\gamma\mu_t\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)}di \\
&+ \left(1-\vartheta\right)\int_0^1 e^{-\gamma\mu_t\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)}di \\
&= \left(1-\vartheta\right)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma\mu_t\varkappa(t-s)\varphi_t}I_t + \left(1-\vartheta\right)e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}
\end{aligned}$$

where

$$I_t = \int_0^1 e^{-\gamma\mu_t\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)}di = \int_0^1 e^{-\gamma\mu_t A_t^s(i)} e^{-\gamma\mu_t\left(\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)}di$$

Integral I_t is an expectation of a product of two functions (uniformly distributed), and as $A_t^s(i)$ is not correlated with $\left(\xi_t^s(i)-\bar{\xi}\right) see some formula above which expresses wealth as a function of past shocks only, then expe$ $\int_0^1 e^{-\gamma\mu_t\left(\eta_t\left(\xi_t^s(j)-\bar{\xi}\right)\right)}dj \int_0^1 e^{-\gamma\mu_t A_t^s(i)}di = e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}\int_0^1 e^{-\gamma\mu_t A_t^s(i)}di$

Recall the budget constraint (2.59):

$$A_{t+1}^s(i) = \frac{R_t}{\vartheta}\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)+X_t-\eta_t\Theta_t^s-(1+\rho\gamma\eta_t)c_t^s(i)\right)$$

substitute out consumption using (2.60)

$$A_{t+1}^s(i) = \frac{R_t}{\vartheta}\left(\begin{array}{c}(1-(1+\rho\gamma\eta_t)\mu_t)\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)\\ +X_t-(1+\rho\gamma\eta_t)\mathscr{X}_t-(\eta_t-(1+\rho\gamma\eta_t)\mu_t\varphi_t)\Theta_t^s\end{array}\right)$$

and simplify using (2.65) and (2.64)

$$\mu_{t+1}A_{t+1}^s(i) = \left(\begin{array}{c}\mu_t\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)-(\mu_t\varphi_t-\mu_{t+1}\varphi_{t+1})\Theta_t^s\\ +\mu_{t+1}\frac{R_t}{\vartheta}\left(X_t-(1+\rho\gamma\eta_t)\mathscr{X}_t\right)\end{array}\right).$$

Take a lag and substitute this expression into formula for I_t to obtain a recursion for this integral:

$$\begin{aligned}
I_t &= \int_0^1 e^{-\gamma\mu_t\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)}di = e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}\int_0^1 e^{-\gamma\mu_t A_t^s(i)}di\\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}\int_0^1 e^{-\gamma\left(\mu_{t-1}\left(A_{t-1}^s(i)+\eta_{t-1}\left(\xi_{t-1}^s(i)-\bar{\xi}\right)\right)\right)}\\
&\times e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})-(\mu_{t-1}\varphi_{t-1}-\mu_t\varphi_t)\Theta_{t-1}^s\right)}di\\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})-(\mu_{t-1}\varphi_{t-1}-\mu_t\varphi_t)\Theta_{t-1}^s\right)}\\
&\times\int_0^1 e^{-\gamma\left(\mu_{t-1}\left(A_{t-1}^s(i)+\eta_{t-1}\left(\xi_{t-1}^s(i)-\bar{\xi}\right)\right)\right)}di\\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}\left(X_{t-1}-\left(1+\tau_{t-1}^c+\rho\gamma\eta_{t-1}\right)\mathscr{X}_{t-1}\right)-\left(\mu_{t-1}\varphi_{t-1}-\mu_t\varphi_t\frac{\varkappa_t}{\varkappa_{t-1}}\right)\varkappa(t-1-s)\right)}I_{t-1}\\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})-(\mu_{t-1}\varphi_{t-1}\varkappa-\mu_t\varphi_t\varkappa)(t-1-s)\right)}I_{t-1}
\end{aligned}$$

Note that, by definition,

$$
\begin{aligned}
\Sigma_{t-1} &= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-1-s}\int_0^1 e^{-\gamma\mu_{t-1}m_{t-1}^s(i)}di \\
&= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-1-s}\int_0^1 e^{-\gamma\mu_{t-1}\left(A_{t-1}^s(i)+\eta_{t-1}\left(\xi_{t-1}^s(i)-\bar{\xi}\right)-\varphi_{t-1}\Theta_{t-1}^s\right)}di \\
&= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-1-s}e^{-\gamma\mu_{t-1}\left(-\varphi_{t-1}\Theta_{t-1}^s\right)}\int_0^1 e^{-\gamma\mu_{t-1}\left(A_{t-1}^s(i)+\eta_{t-1}\left(\xi_{t-1}^s(i)-\bar{\xi}\right)\right)}di
\end{aligned}
$$

so that

$$
\Sigma_{t-1} = (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-1-s}e^{\gamma\mu_{t-1}\varphi_{t-1}\varkappa(t-1-s)}I_{t-1}
$$

Now, isolate this term:

$$
\begin{aligned}
\Sigma_t &= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma\mu_t\varkappa(t-s)\varphi_t}I_t+(1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \\
&= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma\mu_t\varkappa(t-s)\varphi_t}e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})-(\mu_{t-1}\varphi_{t-1}\varkappa-\mu_t\varphi_t\varkappa)(t-1-s)\right)}I_{t-1} \\
&+(1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}\,(1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma\mu_t\varkappa(t-s)\varphi_t+\gamma(\mu_{t-1}\varphi_{t-1}\varkappa-\mu_t\varphi_t\varkappa)(t-1-s)}e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})\right)}I_{t-1} \\
&+(1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}\,(1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma(\mu_t\varphi_t\varkappa+\mu_{t-1}\varphi_{t-1}\varkappa(t-1-s))}e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})\right)}I_{t-1} \\
&+(1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \\
&= e^{\gamma\mu_t\varphi_t\varkappa}e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})\right)}\vartheta\,(1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s-1}e^{\gamma\mu_{t-1}\varphi_{t-1}\varkappa(t-1-s)}I_{t-1} \\
&+(1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}
\end{aligned}
$$

to obtain recursive relationship:

$$
\Sigma_t = \vartheta e^{-\gamma\mu_t\left(\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})-\varkappa\varphi_t\right)}e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}\Sigma_{t-1}+(1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}
$$

Introduce new variable Z_t to yield

$$
\Sigma_t = \left(e^{-\frac{\gamma}{\vartheta}\mu_t(R_{t-1}Z_{t-1}-\vartheta\varkappa\varphi_t)}\vartheta\Sigma_{t-1}+1-\vartheta\right)e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \tag{1.60}
$$

where

$$\begin{aligned}
Z_t &= X_t - (1+\rho\gamma\eta_t)\,\mathscr{X}_t \\
&= \eta_t\left(\rho\log(\eta_t) + \bar{\xi} - \rho\gamma\chi G_t\right) + d_t - T_t - (1+\rho\gamma\eta_t)(\mathscr{C}_t - \chi G_t) \\
&= \eta_t\left(\rho\log(\eta_t) + \bar{\xi}\right) - (1+\rho\gamma\eta_t)\,\mathscr{C}_t + \chi G_t + d_t - T_t
\end{aligned} \tag{1.61}$$

We can represent Z_t in a different form:

$$c_t + \chi G_t - \vartheta\mu_t\left(A_t - \frac{\varkappa}{1-\vartheta}\varphi_t\right) = \mathscr{C}_t$$

then

$$Z_t = \eta_t\left(\rho\log(\eta_t) + \bar{\xi}\right) - (1+\rho\gamma\eta_t)\left(c_t + \chi G_t - \vartheta\mu_t\left(A_t - \frac{\varkappa}{1-\vartheta}\varphi_t\right)\right) + \chi G_t + d_t - T_t$$

use

$$y_t = \eta_t\rho\log(\eta_t) + \eta_t\bar{\xi} - \eta_t\varkappa\frac{\vartheta}{1-\vartheta} - \rho\gamma\eta_t c_t - \rho\gamma\chi\eta_t G_t + d_t - T_t.$$

to obtain

$$y_t + \varkappa\frac{\vartheta}{(1-\vartheta)}\eta_t + \rho\gamma\eta_t c_t - d_t + T_t + \eta_t\rho\gamma\chi G_t = \eta_t\left(\rho\log(\eta_t) + \bar{\xi}\right).$$

$$Z_t = y_t - c_t + (1+\rho\gamma\eta_t)\,\mu_t\vartheta A_t + \frac{\vartheta}{(1-\vartheta)}\varkappa\eta_t - (1+\rho\gamma\eta_t)\,\mu_t\frac{\vartheta\varkappa}{1-\vartheta}\varphi_t \tag{1.62}$$

$$(1+\rho\gamma\eta_t)\,\mu_t = 1 - \frac{\vartheta\mu_t}{\mu_{t+1}R_t}$$

$$Z_t = y_t - c_t + (1+\rho\gamma\eta_t)\,\mu_t\vartheta A_t - \frac{\vartheta}{(1-\vartheta)}\frac{\vartheta}{\mu_{t+1}R_t}\left(\varkappa\mu_{t+1}\varphi_{t+1} - \varkappa\mu_t\varphi_t\right)$$

Furthermore, the aggregated budget constraint:

$$\frac{\vartheta}{R_t}A_{t+1} - \vartheta A_t = y_t - c_t$$

using which

$$\begin{aligned}
Z_t &= \frac{\vartheta}{R_t}A_{t+1} - \vartheta A_t + (1+\rho\gamma\eta_t)\,\mu_t\vartheta A_t - \frac{\vartheta}{(1-\vartheta)}\frac{\vartheta}{\mu_{t+1}R_t}\left(\varkappa\mu_{t+1}\varphi_{t+1} - \varkappa\mu_t\varphi_t\right) \\
&= \frac{\vartheta}{R_t}A_{t+1} - \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\vartheta A_t - \frac{\vartheta}{(1-\vartheta)}\frac{\vartheta}{\mu_{t+1}R_t}\left(\varkappa\mu_{t+1}\varphi_{t+1} - \varkappa\mu_t\varphi_t\right) \\
&= \frac{\vartheta}{R_t}\left(A_{t+1} - \frac{\vartheta}{(1-\vartheta)}\varkappa\varphi_{t+1}\right) - \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\left(\vartheta A_t - \frac{\vartheta}{(1-\vartheta)}\varkappa\varphi_t\right)
\end{aligned}$$

so that

$$Z_t = \frac{\vartheta}{\mu_{t+1} R_t} \left(\mu_{t+1} \left(A_{t+1} - \frac{\vartheta}{(1-\vartheta)} \varkappa \varphi_{t+1} \right) - \mu_t \left(\vartheta A_t - \frac{\vartheta}{(1-\vartheta)} \varkappa \varphi_t \right) \right) \tag{1.63}$$

It is apparent that if the aggregate asset holding is zero then $Z_t = 0$ and we obtain the same recursive formula for Σ_t as reported in Acharya et al (2020).

Using intermediation constraint (2.28) we rewrite (2.81)

$$\mu_{t+1} R_t Z_t = \mu_{t+1} \left(B_{t+1} - \vartheta \frac{\vartheta}{(1-\vartheta)} \varkappa \varphi_{t+1} \right) - \vartheta \mu_t \left(B_t - \frac{\vartheta}{(1-\vartheta)} \varkappa \varphi_t \right) \tag{1.64}$$

Introduce new variable

$$W_t = \mu_t \left(B_t - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t \right)$$

then

$$\mu_{t+1} R_t Z_t = W_{t+1} - \vartheta W_t + \vartheta \varkappa \mu_{t+1} \varphi_{t+1} \tag{1.65}$$

Denote

$$S_t = e^{\gamma \mu_t \left(B_t - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t \right)} \Sigma_t$$

Then

$$\begin{aligned} \mathbb{U}_t &= -\frac{1}{\gamma} (1 + \gamma \rho \eta_t) \, e^{-\gamma \left(x_t - \mu_t \left(B_t - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t \right) \right)} \Sigma_t \\ &= -\frac{1}{\gamma} (1 + \gamma \rho \eta_t) \, e^{-\gamma x_t} S_t \end{aligned}$$

Use (2.83) to rewrite (2.78)

$$S_t = \left(e^{-\frac{\gamma}{\vartheta} W_t} \vartheta S_{t-1} + 1 - \vartheta \right) e^{\gamma W_t} e^{\frac{1}{2} \gamma^2 \mu_t^2 \eta_t^2 \sigma_t^2} \tag{1.66}$$

1.8.6 Proof of Proposition 3

Let us denote by $\breve{b}$ the exogenous annualised debt to GDP ratio (in steady state) $\left(\breve{b} = \frac{P^M b^M}{4Y} \right)$. Under Lump Sum taxes, if the fiscal authority wishes to ensure a stable path for the debt dynamics then:

$$\phi_b^T > (R-1) \frac{P^M b^M}{T}$$

However, we can re- write this expression using the (annualised) debt- to- GDP ratio that is held constant across the different specifications. In this case,

$$\phi_b^T > 4\breve{b}\,(R-1)\,\frac{Y}{T}$$

Similarly, under distortionary income taxes, passive fiscal policy requires that

$$\phi_b^\tau > 4\breve{b}\,(R-1)\,\frac{z}{\tau w}$$

where, the steady state productivity is normalised to unity $(z=1)$. From the aggregate production function we know that the steady state output is

$$Y = zn = n$$

Hence, we can re- write the expression for the fiscal policy feedback coefficient as

$$\phi_b^\tau > 4\breve{b}\,(R-1)\,\frac{Y}{\tau w n}$$

Hence, everything else equal, the difference in size of the debt stabilisation coefficient under distortionary income taxes and lump sum taxes is determined by the difference in the output to tax revenue ratio. More specifically, we know that aggregate labour supply and thus, aggregate output is always higher under Lump Sum taxes $\left(Y^T > Y^\tau\right)$. As such whether $\phi_b^\tau \lesseqgtr \phi_b^T$ depends on the size of the tax revenue $\left(\tau w n \lesseqgtr T\right)$ and the interest rate level in each specification.

Finally, the size of the fiscal reaction coefficient also depends on the steady state real interest rate (R). Unlike the standard representative agent framework, in our model, the steady state real interest rate is not necessarily equal to the rate of time preference$\left(R \neq \frac{1}{\beta}\right)$. More specifically, our assumption regarding the type(s) of inequality present in our economy, shapes the steady state value of the real interest rate and with, it the size of the fiscal response coefficient necessary to ensure stable debt dynamics.

$$R = \frac{1}{\beta}\exp\left(\underbrace{-\frac{1}{2}\gamma^2\mu^2\left(1-\tau_0\right)^2 w^2\sigma^2}_{IM} + \underbrace{\gamma\frac{(1-\vartheta)}{\vartheta}\mu\left(1+\varsigma P^M\right)b^M}_{OLG} - \underbrace{\gamma\bar{\varkappa}\mu\bar{\phi}}_{DLP}\right)$$

As in Acharya and Dogra (2020), allowing for partially uninsurable income risk, captured by the IM component, decreases the size of the fiscal response necessary to ensure fiscal solvency. The rationale behind this result is that by allowing income inequality, we admit an extra term in the aggregate consumption Euler equation, which drives the steady state real interest down, below the rate of time preference. A well known result the Bewley- Aiyagari literature.

However, even without IM component, having introduced a perpetual youth (OLG) structure in the household side allows aggregate asset holdings to enter the aggregate consumption Euler equation. As such, even in the absence of incomplete asset markets, the OLG- RANK model delivers a different steady state interest rate than the nested RANK and hence, we require a more aggressive fiscal response. Finally, admitting a richer OLG channel, by introducing declining income profiles creates a similar effect as the IM component. Households' labor income declines with age (i.e. lose in productivity due to illness) increases their consumption smoothing motive, causing the overall (steady state) demand for savings to increase and consequently resulting in a fall for the (steady state) interest rate.Thus, the interest effect on the fiscal response coefficient is an aggregate of the outstanding value of the government debt, the amount of uninsurable income risk and the friction created by the households' stochastic lifespans.

1.9 Appendix B

Policy response to an one- time TFP shock

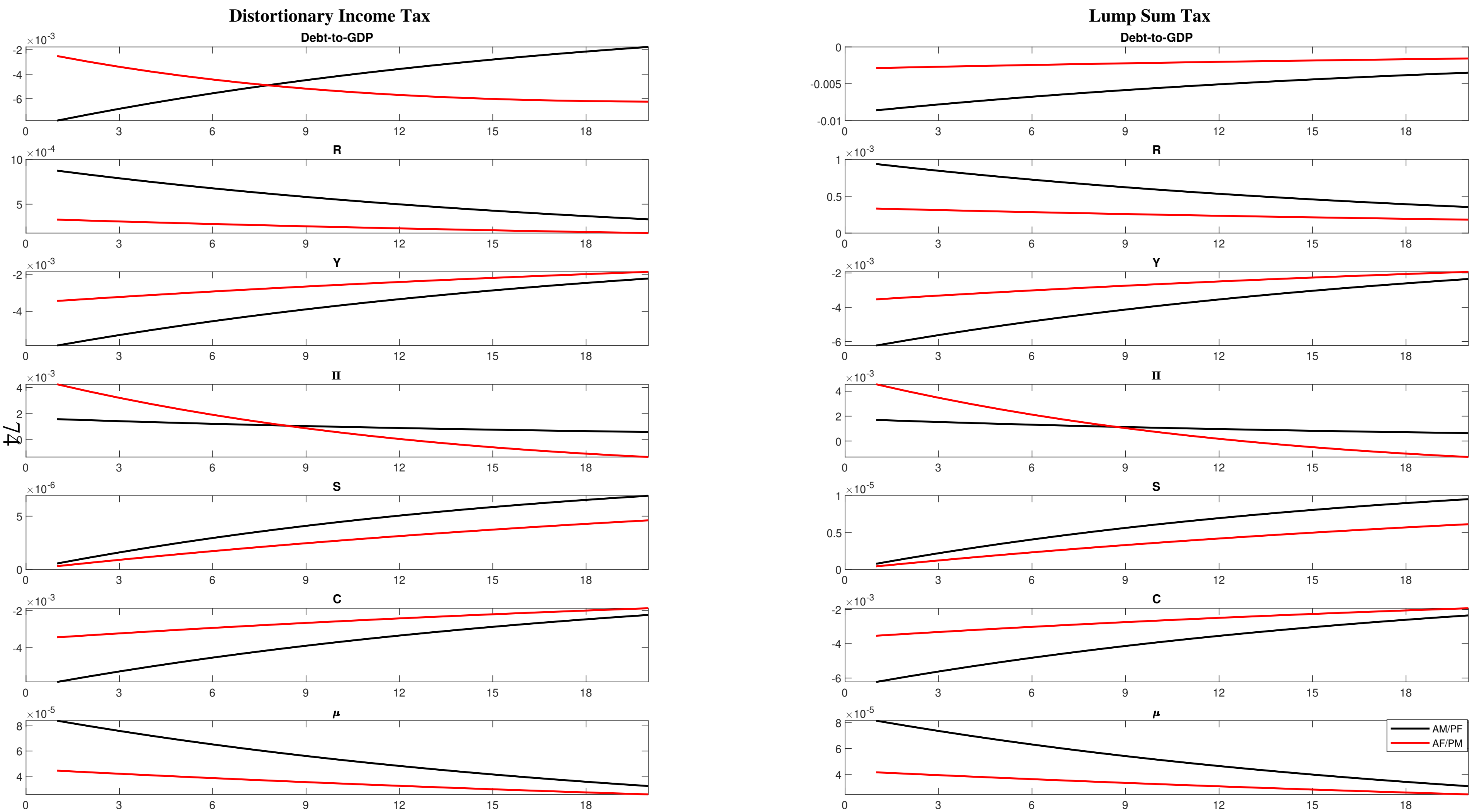

Figure 1.6: Transitionary TFP shock: HANK

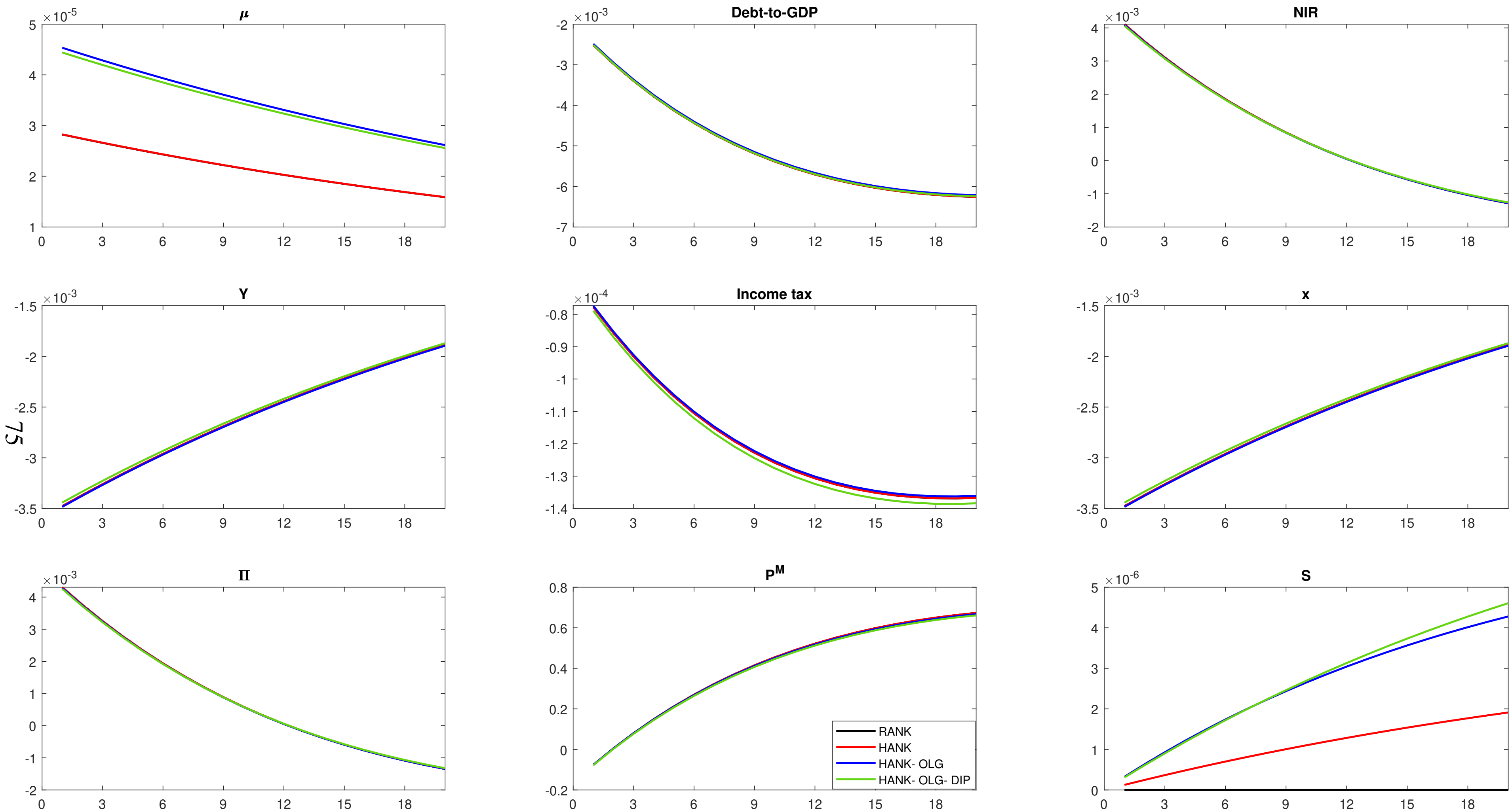

Figure 1.7: Transitionary TFP shock:All Cases PM/AF.

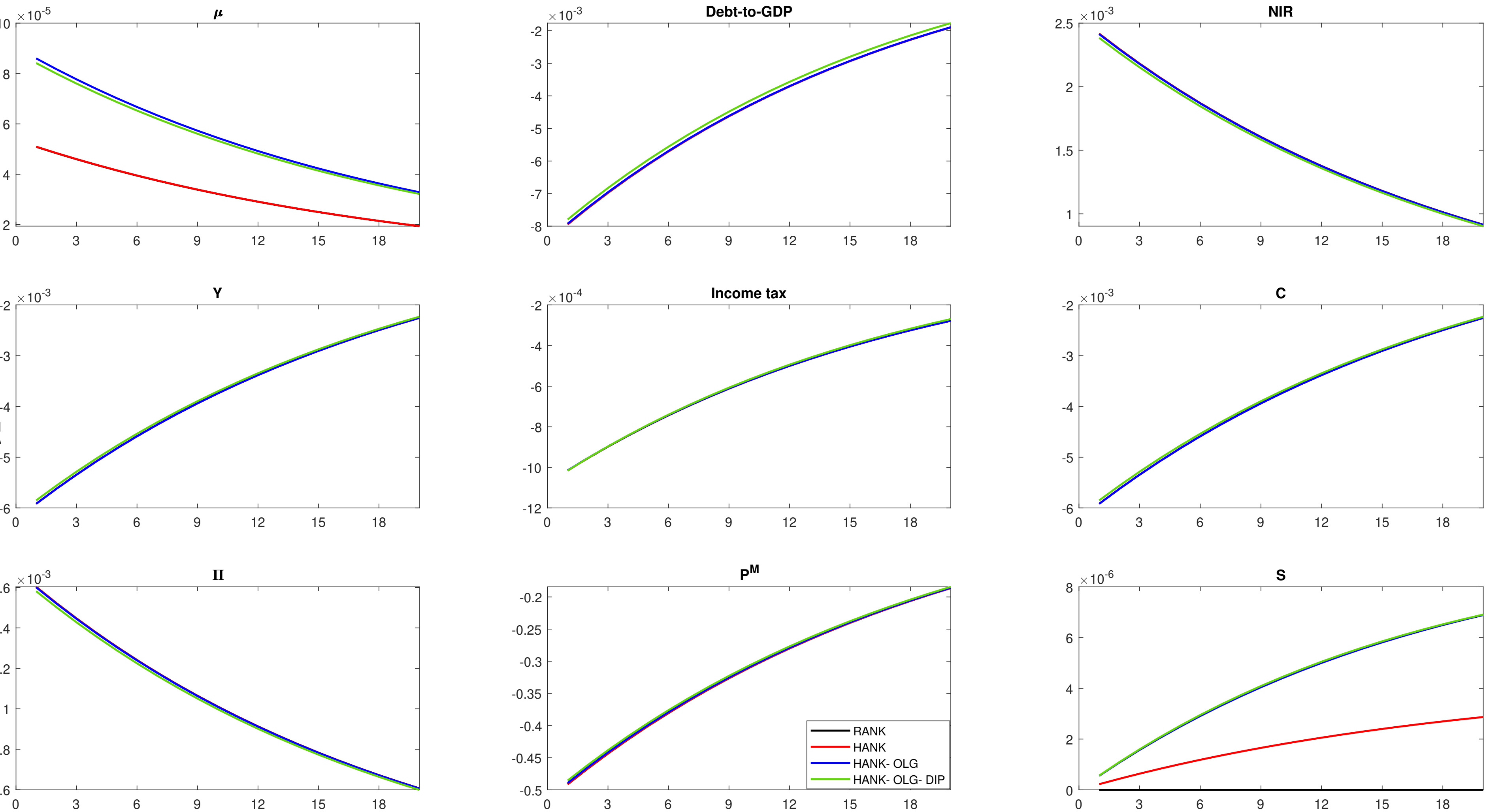

Figure 1.8: Transitionary TFP shock:All Cases AM/PF.

1.10 Appendix C

Derivation of the (Reduced) Log- linearised System

1. Government Budget Constraint

$$(1+\pi_{t+1})\frac{\left(1+\varsigma P_{t+1}^{M}\right)}{I_t}b_{t+1}^{M}=\begin{pmatrix}\left(1+\varsigma P_t^{M}\right)b_t^{M}+G_t\\ -\tau_t w_t\frac{Y_t}{z_t}-T_t+sw_t\frac{Y_t}{z_t}\end{pmatrix}$$

$$(1+\pi_{t+1})\frac{\left(1+\varsigma\left(1+\hat{P}_{t+1}^{M}\right)P^{M}\right)}{I\left(1+\hat{I}_t\right)}b^{M}\left(1+\hat{b}_{t+1}^{M}\right)=\begin{pmatrix}\left(1+\varsigma\left(1+\hat{P}_t^{M}\right)P^{M}\right)b^{M}\left(1+\hat{b}_t^{M}\right)\\ +G_t\\ -\tau\left(1+\hat{\tau}_t\right)w\left(1+\hat{w}_t\right)\frac{Y\left(1+\hat{Y}_t\right)}{z\left(1+\hat{z}_t\right)}\\ -T\left(1+\hat{T}_t\right)\end{pmatrix}$$

$$\frac{\left(1+\varsigma P^{M}\right)}{I}b^{M}\begin{pmatrix}1+\hat{b}_{t+1}^{M}+\pi_{t+1}\\ +\frac{\varsigma P^{M}}{1+\varsigma P^{M}}\hat{P}_{t+1}^{M}-\hat{I}_t\end{pmatrix}=\begin{pmatrix}\left(1+\varsigma P^{M}\right)b^{M}\left(1+\frac{\varsigma P^{M}}{1+\varsigma P^{M}}\hat{P}_t^{M}+\hat{b}_t^{M}\right)+G_t\\ -\tau w\frac{Y}{z}\left(1+\hat{\tau}_t+\hat{w}_t+\hat{Y}_t-\hat{z}_t\right)-T\left(1+\hat{T}_t\right)\end{pmatrix}$$

$$\frac{\left(1+\varsigma P^{M}\right)}{I}b^{M}\left(\hat{b}_{t+1}^{M}+\pi_{t+1}+\frac{\varsigma P^{M}}{1+\varsigma P^{M}}\hat{P}_{t+1}^{M}\right)=\begin{pmatrix}\varsigma P^{M}b^{M}\left(\hat{P}_t^{M}\right)\\ +\left(1+\varsigma P^{M}\right)b^{M}\left(\hat{b}_t^{M}\right)\\ -\tau w\frac{Y}{z}\left(\hat{\tau}_t+\hat{w}_t+\hat{Y}_t-\hat{z}_t\right)-T\left(\hat{T}_t\right)\\ +\left(\frac{\left(1+\varsigma P^{M}\right)}{I}b^{M}\right)\hat{I}_t\end{pmatrix}$$

Recall that

$$\hat{w}_t=\left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_t-\frac{Y}{z\rho}\hat{z}_t+\frac{\tau}{1-\tau}\hat{\tau}_t$$

$$\frac{\left(1+\varsigma P^M\right)}{I} b^M \left(\hat{b}^M_{t+1} + \pi_{t+1} + \frac{\varsigma P^M}{1+\varsigma P^M} \hat{P}^M_{t+1}\right) = \left(\begin{array}{c} \varsigma P^M b^M \left(\hat{P}^M_t\right) \\ +\left(1+\varsigma P^M\right) b^M \left(\hat{b}^M_t\right) \\ -\tau w \frac{Y}{z}\left(\hat{\tau}_t + \left(\frac{1+\gamma z\rho}{z\rho}\right) Y\hat{Y}_t - \frac{Y}{z\rho}\hat{z}_t + \frac{\tau}{1-\tau}\hat{\tau}_t + \hat{Y}_t - \hat{z}_t\right) \\ -T\left(\hat{T}_t\right) \\ +\left(\frac{\left(1+\varsigma P^M\right)}{I} b^M\right)\hat{I}_t \end{array}\right)$$

$$\frac{\left(1+\varsigma P^M\right)}{I} b^M \left(\hat{b}^M_{t+1} + \pi_{t+1}\right) + \frac{\varsigma P^M b^M}{I} \hat{P}^M_{t+1} = \left(\begin{array}{c} +\tau w \frac{Y}{z}\left(\frac{z\rho+Y}{z\rho}\right)\hat{z}_t \\ \varsigma P^M b^M \left(\hat{P}^M_t\right) \\ +\left(1+\varsigma P^M\right) b^M \left(\hat{b}^M_t\right) \\ -\tau w \frac{Y}{z}\left(1+\left(\frac{1+\gamma z\rho}{z\rho}\right) Y\right)\hat{Y}_t \\ -w\frac{Y}{z}\left(\frac{\tau}{1-\tau}\right)\hat{\tau}_t \\ -T\left(\hat{T}_t\right) \\ +\left(\frac{\left(1+\varsigma P^M\right)}{I} b^M\right)\hat{I}_t \end{array}\right)$$

1. Government Budget Constraint

$$\frac{\left(1+\varsigma P^M\right)}{I} b^M \left(\hat{b}^M_{t+1} + \pi_{t+1}\right) + \frac{\varsigma P^M b^M}{I} \hat{P}^M_{t+1} = \left(\begin{array}{c} +\tau w \frac{Y}{z}\left(\frac{z\rho+Y}{z\rho}\right)\hat{z}_t \\ \varsigma P^M b^M \left(\hat{P}^M_t\right) \\ +\left(1+\varsigma P^M\right) b^M \left(\hat{b}^M_t\right) \\ -\tau w \frac{Y}{z}\left(1+\left(\frac{1+\gamma z\rho}{z\rho}\right) Y\right)\hat{Y}_t \\ -w\frac{Y}{z}\left(\frac{\tau}{1-\tau}\right)\hat{\tau}_t \\ -T\left(\hat{T}_t\right) \\ +\left(\frac{\left(1+\varsigma P^M\right)}{I} b^M\right)\hat{I}_t \end{array}\right)$$

2. Labour decline recursion

$$\frac{\vartheta\bar{\phi}}{R}\hat{\phi}_{t+1} = \bar{\phi}\hat{\phi}_t + \tau w\hat{\tau}_t + \frac{\vartheta\varphi}{R}\hat{R}_t - (1-\tau) w\hat{w}_t$$

with,

$$\hat{w}_t = \left(\frac{1+\gamma z\rho}{z\rho}\right) Y\hat{Y}_t - \frac{Y}{z\rho}\hat{z}_t + \frac{\tau}{1-\tau}\hat{\tau}_t$$

$$\hat{\eta}_t = \left(\frac{1+\gamma z\rho}{z\rho}\right) Y\hat{Y}_t - \frac{Y}{z\rho}\hat{z}_t$$

we can re-write it as

$$\frac{\vartheta\bar{\phi}}{R}\hat{\phi}_{t+1} = \begin{pmatrix} \bar{\phi}\hat{\phi}_t + \tau w\hat{\tau}_t + \frac{\vartheta\varphi}{R}\hat{R}_t \\ -(1-\tau)w\left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_t \\ +(1-\tau)w\left[\frac{Y}{z\rho}\hat{z}_t\right] \\ -(\tau w)\hat{\tau}_t \end{pmatrix}$$

$$\frac{\vartheta\bar{\phi}}{R}\hat{\phi}_{t+1} = \begin{pmatrix} (1-\tau)w\left[\frac{Y}{z\rho}\hat{z}_t\right] + \bar{\phi}\hat{\phi}_t \\ -(1-\tau)w\left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_t \\ +\frac{\vartheta\bar{\phi}}{R}\hat{R}_t \end{pmatrix}$$

3. The miu Recursion

$$\frac{1}{\mu}\hat{\mu}_t + \rho\gamma\eta\hat{\eta}_t - \frac{\vartheta}{R\mu}\hat{R}_t = \frac{\vartheta}{R\mu}\hat{\mu}_{t+1}$$

With,

$$\hat{\eta}_t = \left(\frac{1+\gamma z\rho}{z\rho}\right) Y\hat{Y}_t - \frac{Y}{z\rho}\hat{z}_t$$

It can be simplified to

$$\frac{\vartheta}{R\mu}\hat{\mu}_{t+1} = \frac{1}{\mu}\hat{\mu}_t + \rho\gamma\eta\left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_t - \rho\gamma\eta\frac{Y}{z\rho}\hat{z}_t - \frac{\vartheta}{R\mu}\hat{R}_t$$

$$\hat{\mu}_{t+1} = \left(\frac{R}{\vartheta}\right)\hat{\mu}_t + \gamma\eta\mu\left(\frac{R}{\vartheta}\right)\left(\frac{1+\gamma z\rho}{z}\right)Y\hat{Y}_t - \gamma\eta\mu\left(\frac{R}{\vartheta}\right)\frac{Y}{z}\hat{z}_t - \hat{R}_t$$

4.Bond Pricing Equations

$$\hat{P}_t^M + \hat{R}_t + \pi_{t+1} = \left(\frac{\varsigma}{R}\right)\hat{P}_{t+1}^M$$

5.Agg. Consumption Euler Equation

$$\begin{bmatrix} Y\hat{Y}_t + \frac{1}{\gamma}\left(1-\gamma\frac{(1-\vartheta)}{\vartheta}\bar{\mu}RP^M b^M\right)\hat{R}_t \\ -\frac{(1-\vartheta)}{\vartheta}\bar{\mu}RP^M b^M \hat{P}_t^M \end{bmatrix} = \begin{bmatrix} +\left(1-\phi\gamma\bar{\mu}^2(1-\tau_0)^2\omega\right)Y\hat{Y}_{t+1} \\ +\frac{(1-\vartheta)}{\vartheta}\bar{\mu}RP^M b^M\left(\hat{b}_{t+1}^M\right) \\ +\frac{(1-\vartheta)}{\vartheta}\bar{\mu}RP^M b^M \pi_{t+1} \\ +\left(\frac{(1-\vartheta)}{\vartheta}\bar{\mu}RP^M b^M - \gamma\bar{\mu}^2(1-\tau_0)^2\omega - \bar{\varkappa}\bar{\mu}\bar{\phi}\right)(\hat{\mu}_{t+1}) \\ -\bar{\varkappa}\bar{\mu}\bar{\phi}\left(\hat{\phi}_{t+1}\right) \end{bmatrix}$$

6.The NK Phillips curve

$$\Phi\pi_t + \left(1-\varepsilon\frac{w}{z}\right)\varepsilon\hat{\varepsilon}_t - \varepsilon\frac{w}{z}(\hat{w}_t) + \varepsilon\frac{w}{z}(\hat{z}_t) = \beta\Phi\pi_{t+1}$$

with,

$$\hat{w}_t = \left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_t - \frac{Y}{z\rho}\hat{z}_t + \frac{\tau}{1-\tau}\hat{\tau}_t$$

It can be simplified to

$$\beta\Phi\pi_{t+1} = \Phi\pi_t + \left(1-\varepsilon\frac{w}{z}\right)\varepsilon\hat{\varepsilon}_t - \varepsilon\frac{w}{z}\left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_t - \varepsilon\frac{w}{z}\frac{\tau}{1-\tau}\hat{\tau}_t + \left(1+\frac{Y}{z\rho}\right)\varepsilon\frac{w}{z}(\hat{z}_t)$$

The policy block (PB) Now suppose, the log- linearised **Taylor rule**

$$\hat{I}_t = \phi_\pi\pi_t + \phi_Y\hat{Y}_t$$

and, the fiscal rule yield

$$\hat{\tau}_t = \phi_b\hat{b}_t^M + \phi_b\hat{P}_t^M$$

Using the log-linearised Fisher equation $\hat{I}_t = \hat{R}_t + \pi_{t+1}$, we can re- write the

1. Government Budget Constraint

$$\frac{\left(1+\varsigma P^{M}\right)}{I}b^{M}\left(\hat{b}_{t+1}^{M}+\pi_{t+1}\right)+\frac{\varsigma P^{M}b^{M}}{I}\hat{P}_{t+1}^{M}=\left(\begin{array}{c}+\tau w\frac{Y}{z}\left(\frac{z\rho+Y}{z\rho}\right)\hat{z}_{t}\\ \varsigma P^{M}b^{M}\left(\hat{P}_{t}^{M}\right)\\ +\left(1+\varsigma P^{M}\right)b^{M}\left(\hat{b}_{t}^{M}\right)\\ -\tau w\frac{Y}{z}\left(1+\left(\frac{1+\gamma z\rho}{z\rho}\right)Y\right)\hat{Y}_{t}\\ -w\frac{Y}{z}\left(\frac{\tau}{1-\tau}\right)\hat{\tau}_{t}\\ -T\left(\hat{T}_{t}\right)\\ +\left(\frac{\left(1+\varsigma P^{M}\right)}{I}b^{M}\right)\hat{I}_{t}\end{array}\right)$$

2. Labour decline recursion

$$\frac{\vartheta\bar{\phi}}{R}\hat{\phi}_{t+1}+\frac{\vartheta\varphi}{R}\pi_{t+1}=\left(\begin{array}{c}(1-\tau)\,w\left[\frac{Y}{z\rho}\hat{z}_{t}\right]\\ +\bar{\phi}\hat{\phi}_{t}\\ -(1-\tau)\,w\left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_{t}\\ +\frac{\vartheta\varphi}{R}\hat{I}_{t}\end{array}\right)$$

3. The miu Recursion

$$\left(\frac{\vartheta}{R}\right)\hat{\mu}_{t+1}-\left(\frac{\vartheta}{R}\right)\pi_{t+1}=\hat{\mu}_{t}+\gamma\eta\mu\left(\frac{1+\gamma z\rho}{z}\right)Y\hat{Y}_{t}-\gamma\eta\mu\frac{Y}{z}\hat{z}_{t}-\left(\frac{\vartheta}{R}\right)\hat{I}_{t}$$

4.Bond Pricing Equations

$$\hat{P}_{t}^{M}+\hat{I}_{t}=\left(\frac{\varsigma P^{M}}{1+\varsigma P^{M}}\right)\hat{P}_{t+1}^{M}$$

5. Agg. Consumption Euler Equation

$$Y\hat{Y}_{t}+\frac{1}{\gamma}\hat{I}_{t}=\left[\begin{array}{c}+\frac{1}{\gamma}\pi_{t+1}\\ +\left(1-\gamma\bar{\mu}^{2}\left(1-\tau_{0}\right)^{2}\omega\phi_{Y}\right)Y\hat{Y}_{t+1}\\ +\frac{(1-\vartheta)}{\vartheta}\bar{\mu}\varsigma P^{M}b^{M}\left(\hat{P}_{t+1}^{M}\right)\\ +\left(\frac{(1-\vartheta)}{\vartheta}\bar{\mu}\left(1+\varsigma P^{M}\right)b^{M}-\gamma\bar{\mu}^{2}\left(1-\tau_{0}\right)^{2}\omega-\bar{\varkappa}\bar{\mu}\bar{\phi}\right)\left(\hat{\mu}_{t+1}\right)\\ -\bar{\varkappa}\bar{\mu}\bar{\phi}\left(\hat{\phi}_{t+1}\right)\\ +\frac{(1-\vartheta)}{\vartheta}\bar{\mu}\left(1+\varsigma P^{M}\right)b^{M}\hat{b}_{t+1}^{M}\end{array}\right]$$

6.The NK Phillips curve

$$\beta\Phi\pi_{t+1} = \Phi\pi_t + \left(1+\frac{Y}{z\rho}\right)\varepsilon\frac{w}{z}\hat{z}_t + \left(1-\frac{w}{z}\right)\varepsilon\hat{\varepsilon}_t - \varepsilon\frac{w}{z}\left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_t - \varepsilon\frac{w}{z}\frac{\tau}{1-\tau}\hat{\tau}_t$$

Different Discount Factor

$$\frac{\Phi}{R}\pi_{t+1} = \Phi\pi_t + \varepsilon\frac{w}{z}\left(1+\frac{Y}{z\rho}\right)\hat{z}_t + \left(1-\frac{w}{z}\right)\varepsilon\hat{\varepsilon}_t - \varepsilon\frac{w}{z}\left(\frac{1+\gamma z\rho}{z\rho}\right)Y\hat{Y}_t - \varepsilon\frac{w}{z}\left(\frac{\tau}{1-\tau}\right)\hat{\tau}_t$$

The policy block (PB) Now suppose, the log- linearised **Taylor rule**

$$\hat{I}_t = \phi_\pi\pi_t + \phi_Y\hat{Y}_t$$

and, the fiscal rule yields

$$\hat{\tau}_t = \phi_b\hat{b}_t^M + \phi_b\hat{P}_t^M$$

1.11 Matrix Form

- To produce the **Determinacy Graphs** we make use of the following system:

$$
\begin{bmatrix}
\frac{\left(1+\varsigma P^M\right)}{I}b^M & 0 & 0 & \frac{\varsigma P^M b^M}{I} & 0 & \frac{\left(1+\varsigma P^M\right)}{I}b^M \\
0 & \frac{\vartheta\bar{\phi}}{R} & 0 & 0 & 0 & \frac{\vartheta\varphi}{R} \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & \left(\frac{\varsigma P^M}{1+\varsigma P^M}\right) & 0 & 0 \\
\frac{(1-\vartheta)}{\vartheta}\bar{\mu}\left(1+\varsigma P^M\right)b^M & -\bar{\varkappa}\bar{\mu}\bar{\phi} & \alpha_1 & 0 & \left(1-\phi_Y\gamma\bar{\mu}^2\left(1-\tau_0\right)^2\omega\right)Y & \frac{1}{\gamma} \\
0 & 0 & 0 & 0 & 0 & \beta\Phi
\end{bmatrix}
\begin{bmatrix}
\hat{b}^M_{t+1} \\ \hat{\varphi}_{t+1} \\ \hat{\mu}_{t+1} \\ \hat{P}^M_{t+1} \\ \hat{Y}_{t+1} \\ \pi_{t+1}
\end{bmatrix} =
$$

$$
= \begin{bmatrix}
\left(1+\varsigma P^M\right)b^M & 0 & 0 & \varsigma P^M b^M & -\tau w\frac{Y}{z}\left(1+\left(\frac{1+\gamma z\rho}{z\rho}\right)Y\right) & 0 \\
0 & \bar{\phi} & 0 & 0 & -\left(1-\tau\right)w\left(\frac{1+\gamma z\rho}{z\rho}\right)Y & 0 \\
0 & 0 & \left(\frac{R}{\vartheta}\right) & 0 & \gamma\eta\mu\left(\frac{R}{\vartheta}\right)\left(\frac{1+\gamma z\rho}{z}\right)Y & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & Y & 0 \\
0 & 0 & 0 & 0 & -\varepsilon\frac{w}{z}\left(\frac{1+\gamma z\rho}{z\rho}\right)Y & \Phi
\end{bmatrix}
\begin{bmatrix}
\hat{b}^M_t \\ \hat{\varphi}_t \\ \hat{\mu}_t \\ \hat{P}^M_t \\ \hat{Y}_t \\ \pi_t
\end{bmatrix}
+ \begin{bmatrix}
\left(\frac{\left(1+\varsigma P^M\right)}{I}b^M\right) \\
\frac{\vartheta\varphi}{R} \\
-\left(\frac{\vartheta}{R}\right) \\
\frac{1}{\gamma} \\
\frac{1}{\gamma}\left(1-\gamma\frac{(1-\vartheta)}{\vartheta}\bar{\mu}RP^M b^M\right) \\
0
\end{bmatrix}
\left[\hat{I}_t\right]
$$

$$
+ \begin{bmatrix}
-w\frac{Y}{z}\left(\frac{\tau}{1-\tau}\right) \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\varepsilon}{z}\left(\frac{\tau w}{1-\tau}\right)
\end{bmatrix}
\left[\hat{\tau}_t\right] +
\begin{bmatrix}
-T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
\left[\hat{T}_t\right]
$$

$$
\alpha_1 = \left(\frac{(1-\vartheta)}{\vartheta}\bar{\mu}\left(1+\varsigma P^M\right)b^M - \gamma\bar{\mu}^2\left(1-\tau_0\right)^2\omega - \bar{\varkappa}\bar{\mu}\bar{\phi}\right)
$$

Calculating the Policy Block   For Monetary Policy

$$
\left[\hat{I}_t\right]=\begin{bmatrix} \left(\frac{\left(1+\varsigma P^M\right)}{I}b^M\right) \\ \frac{\vartheta\varphi}{R} \\ -\left(\frac{\vartheta}{R}\right) \\ \frac{1}{\gamma} \\ \frac{1}{\gamma}\left(1-\gamma\frac{(1-\vartheta)}{\vartheta}\bar{\mu}RP^Mb^M\right) \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & \phi_Y & \phi_\pi \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 0 & b^M\frac{\phi_Y}{I}\left(P^M\varsigma+1\right) & b^M\frac{\phi_\pi}{I}\left(P^M\varsigma+1\right) \\ 0 & 0 & 0 & 0 & \frac{1}{R}\vartheta\varphi\phi_Y & \frac{1}{R}\vartheta\varphi\phi_\pi \\ 0 & 0 & 0 & 0 & -\frac{1}{R}\vartheta\phi_Y & -\frac{1}{R}\vartheta\phi_\pi \\ 0 & 0 & 0 & 0 & \phi_Y & \phi_\pi \\ 0 & 0 & 0 & 0 & \frac{1}{\gamma}\phi_Y\left(P^MRb^M\gamma\frac{\mu}{\vartheta}(\vartheta-1)+1\right) & \frac{1}{\gamma}\phi_\pi\left(P^MRb^M\gamma\frac{\mu}{\vartheta}(\vartheta-1)+1\right) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

For Fiscal Policy

$$
\left[\hat{\tau}_t\right]=\begin{bmatrix} -\frac{\tau w}{P^Mb^M}\frac{Y}{z} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\varepsilon}{z}\left(\frac{\tau w}{1-\tau}\right) \end{bmatrix}\begin{bmatrix} \phi_b & 0 & 0 & \phi_b & 0 & 0 \end{bmatrix}=\begin{bmatrix} -\frac{1}{P^M}\frac{Y}{b^M}\frac{w}{z}\tau\phi_b & 0 & 0 & -\frac{1}{P^M}\frac{Y}{b^M}\frac{w}{z}\tau\phi_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{w}{z}\tau\varepsilon\frac{\phi_b}{\tau-1} & 0 & 0 & \frac{w}{z}\tau\varepsilon\frac{\phi_b}{\tau-1} & 0 & 0 \end{bmatrix}
$$

$$
\left[\hat{T}_t\right]=\begin{bmatrix} -\frac{T}{P^Mb^M} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \phi_b & 0 & 0 & \phi_b & 0 & 0 \end{bmatrix}=\begin{bmatrix} -\frac{1}{P^M}\frac{T}{b^M}\phi_b & 0 & 0 & -\frac{1}{P^M}\frac{T}{b^M}\phi_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

1.11.1 The dynamics of the log- linearised model: Response to unanticipated (transitionary) aggregate shock (Impulse Responses)

In this section, we present the matrix for the system dynamics in response to an aggregate shock. We are looking at first order perturbations around the zero inflation steady state. As such, the certainty equivalence result holds and we can drop the expectation operator.To calculate system dynamics around the efficient steady state we use the following system. The main difference is that we have included the deterministic shocks.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{\varsigma}{R} & 0 & 1 \\ 0 & 0 & 0 & \frac{\vartheta\bar{\phi}}{R} & 0 & 0 & 0 & \frac{\vartheta\varphi}{R} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{\varsigma}{R}\right) & 0 & 0 \\ 0 & 0 & \frac{(1-\vartheta)}{\vartheta}\bar{\mu}RP^{M}b^{M} & -\bar{\varkappa}\bar{\mu}\bar{\phi} & \alpha_1 & 0 & \left(1-\phi\gamma\bar{\mu}^{2}\left(1-\tau_0\right)^{2}\omega\right) & \frac{1}{\gamma} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\Phi}{R} \end{bmatrix} \begin{bmatrix} \hat{z}_{t+1} \\ \hat{\varepsilon}_{t+1} \\ \hat{b}^{M}_{t+1} \\ \hat{\varphi}_{t+1} \\ \hat{\mu}_{t+1} \\ \hat{P}^{M}_{t+1} \\ \hat{Y}_{t+1} \\ \pi_{t+1} \end{bmatrix} //$$

$$= \begin{bmatrix} \varsigma_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tau w\frac{Y}{z}\left(\frac{z\rho+Y}{z\rho}\right)0 & \varsigma_\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{(\tau-s)w}{P^{M}b^{M}}\left(\frac{Y}{z}\right) & 0 & R & 0 & 0 & \varsigma & \varsigma & 0 \\ (1-\tau)w\frac{Y}{z\rho} & 0 & 0 & \bar{\phi} & 0 & 0 & -(1-\tau)w\left(\frac{1+\gamma z\rho}{z\rho}\right)Y & 0 \\ -\gamma\eta\mu\left(\frac{R}{\vartheta}\right)\frac{Y}{z} & 0 & 0 & 0 & \left(\frac{R}{\vartheta}\right) & 0 & \gamma\eta\mu\left(\frac{R}{\vartheta}\right)\left(\frac{1+\gamma z\rho}{z}\right)Y & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{(1-\vartheta)}{\vartheta}\bar{\mu}RP^{M}b^{M} & Y & 0 \\ \left(1+\frac{Y}{z\rho}\right)\varepsilon\frac{w}{z} & \left(1-\frac{w}{z}\right)\varepsilon & 0 & 0 & 0 & 0 & -\varepsilon\frac{w}{z}\left(\frac{1+\gamma z\rho}{z\rho}\right)Y & \Phi \end{bmatrix} \begin{bmatrix} \hat{z}_t \\ \hat{\varepsilon}_t \\ \hat{b}^{M}_t \\ \hat{\varphi}_t \\ \hat{\mu}_t \\ \hat{P}^{M}_t \\ \hat{Y}_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{\vartheta\varphi}{R} \\ -1 \\ 1 \\ \frac{1}{\gamma}\left(1-\gamma\frac{(1-\vartheta)}{\vartheta}\bar{\mu}RP^{M}b^{M}\right) \\ 0 \end{bmatrix} \left[\hat{I}_t\right]$$

$$
+\begin{bmatrix} 0 \\ 0 \\ -\frac{\tau w}{P^M b^M}\frac{Y}{z} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\varepsilon}{z}\left(\frac{\tau w}{1-\tau}\right) \end{bmatrix} [\hat{\tau}_t] + \begin{bmatrix} 0 \\ 0 \\ -\frac{T}{P^M b^M} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} [\hat{T}_t]
$$

$$
\alpha_1 = \left(\frac{(1-\vartheta)}{\vartheta}\bar{\mu} R P^M b^M - \gamma\bar{\mu}^2 (1-\tau_0)^2\,\omega - \bar{\varkappa}\bar{\mu}\bar{\phi}\right)
$$

Computing the Policy Block For Monetary Policy

$$
[\hat{I}_t] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{\vartheta\varphi}{R} \\ -1 \\ 1 \\ \frac{1}{\gamma}\left(1-\gamma\frac{(1-\vartheta)}{\vartheta}\bar{\mu} R P^M b^M\right) \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \phi_Y & \phi_\pi \end{bmatrix} =
$$

$$
= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \phi_Y & \phi_\pi \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R}\vartheta\varphi\phi_Y & \frac{1}{R}\vartheta\varphi\phi_\pi \\
0 & 0 & 0 & 0 & 0 & 0 & -\phi_Y & -\phi_\pi \\
0 & 0 & 0 & 0 & 0 & 0 & \phi_Y & \phi_\pi \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma}\phi_Y\left(P^M R b^M \gamma \frac{\mu}{\vartheta}(\vartheta-1)+1\right) & \frac{1}{\gamma}\phi_\pi\left(P^M R b^M \gamma \frac{\mu}{\vartheta}(\vartheta-1)+1\right) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

For Fiscal Policy

$$
[\hat{\tau}_t] = \begin{bmatrix}
0 \\
0 \\
-\frac{\tau w}{P^M b^M}\frac{Y}{z} \\
0 \\
0 \\
0 \\
0 \\
-\frac{\varepsilon}{z}\left(\frac{\tau w}{1-\tau}\right)
\end{bmatrix}
\begin{bmatrix} 0 & 0 & \phi_b & 0 & 0 & \phi_b & 0 & 0 \end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{P^M}\frac{Y}{b^M}\frac{w}{z}\tau\phi_b & 0 & 0 & -\frac{1}{P^M}\frac{Y}{b^M}\frac{w}{z}\tau\phi_b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{w}{z}\tau\varepsilon\frac{\phi_b}{\tau-1} & 0 & 0 & \frac{w}{z}\tau\varepsilon\frac{\phi_b}{\tau-1} & 0 & 0
\end{bmatrix}
$$

$$
[\hat{T}_t] = \begin{bmatrix}
0 \\
0 \\
-\frac{T}{P^M b^M} \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix} 0 & 0 & \phi_b & 0 & 0 & \phi_b & 0 & 0 \end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{P^M}\frac{T}{b^M}\phi_b & 0 & 0 & -\frac{1}{P^M}\frac{T}{b^M}\phi_b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Once we have calculated the system dynamics, we make use of the following expressions to retrieve the response of Wealth inequality to the aggregate shock

$$S_t = \left(\vartheta e^{-\frac{\gamma}{\vartheta}W_t} S_{t-1} + 1 - \vartheta\right) e^{\gamma W_t} e^{\frac{1}{2}\gamma^2 \mu_t^2 \eta_t^2 \sigma_t^2}. \tag{1.67}$$

Here

$$\begin{aligned} W_t &= \mu_t \left(B_t - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t\right) & (1.68)\\ &= \mu_t \left(\frac{\left(\left(1+\varsigma P_t^M\right) b_t^L + b_t^S\right)}{(1+\pi_t)} - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t\right) & (1.69)\\ &= \mu_t \left(\frac{\left(\left(1+\varsigma P_t^M\right) b_t^L\right)}{(1+\pi_t)} - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t\right) & (1.70) \end{aligned}$$

$$B_t = \frac{\left(\left(1+\varsigma P_t^M\right) b_t^L + b_t^S\right)}{(1+\pi_t)}$$

$$S_t = \left(\vartheta e^{-\frac{\gamma}{\vartheta}W_t} S_{t-1} + 1 - \vartheta\right) e^{\gamma W_t} e^{\frac{1}{2}\gamma^2 \mu_t^2 \eta_t^2 \sigma_t^2}. \tag{1.71}$$

$$\sigma_t = \sigma^2 \exp\left(2\phi\left(Y_t - Y\right)\right)$$

$$\sigma^2 \left(1 + 2\hat{\sigma}_t\right) = \sigma^2 \exp\left(2\phi\left(Y\left(1+\hat{Y}_t\right) - Y\right)\right)$$

In steady state $\omega = w^2 \sigma^2$

$$\left(1 + 2\hat{\sigma}_t\right) = \underbrace{\left(\exp\left(2\phi Y \hat{Y}_t\right)\right)}_{\approx 1 + 2\phi Y \hat{Y}_t}$$

$$\hat{\sigma}_t = \phi Y \hat{Y}_t$$

Log- linearising inequality around the zero inflation steady state yields:

$$S_t = \left(\vartheta e^{-\frac{\gamma}{\vartheta}W_t} S_{t-1} + 1 - \vartheta\right) e^{\gamma W_t} e^{\frac{1}{2}\gamma^2 \mu_t^2 \eta_t^2 \sigma_t^2}.$$

$$\frac{S_t}{\left(\vartheta e^{-\frac{\gamma}{\vartheta}W_t}S_{t-1}+1-\vartheta\right)}=e^{\gamma W_t}e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}.$$

$$\frac{\left(1+\hat{S}_t\right)S}{\left(\vartheta e^{-\frac{\gamma}{\vartheta}W\left(1+\hat{W}_t\right)}\left(1+\hat{S}_{t-1}\right)S+1-\vartheta\right)}=e^{\gamma W\left(1+\hat{W}_t\right)}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2(1+2\hat{\mu}_t)(1+2\hat{\eta}_t)\left(1+2\phi Y\hat{Y}_t\right)}.$$

$$\frac{\left(1+\hat{S}_t\right)S}{\left(\vartheta\left(1+\hat{S}_{t-1}-\gamma W\hat{W}_t\right)e^{-\frac{\gamma}{\vartheta}W}S+1-\vartheta\right)}=\left(1+\gamma W\hat{W}_t\right)e^{\gamma W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}e^{\gamma^2\mu^2\eta^2\sigma^2\left(\hat{\mu}_t+\hat{\eta}_t+\phi Y\hat{Y}_t\right)}.$$

$$\left(1+\hat{S}_t\right)S=\left(\begin{array}{c}\vartheta\left(1+\hat{S}_{t-1}-\gamma W\hat{W}_t\right)e^{-\frac{\gamma}{\vartheta}W}S\\+1-\vartheta\end{array}\right)\left(\begin{array}{c}1\\+\gamma^2\mu^2\eta^2\sigma^2\left(\hat{\mu}_t+\hat{\eta}_t+\phi Y\hat{Y}_t\right)\\+\gamma W\hat{W}_t\end{array}\right)e^{\gamma W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}$$

$$\left(1+\hat{S}_t\right)S=\left(\begin{array}{c}\left(\begin{array}{c}\vartheta e^{-\frac{\gamma}{\vartheta}W}S+\left(\hat{S}_{t-1}-\gamma W\hat{W}_t\right)\vartheta e^{-\frac{\gamma}{\vartheta}W}S\\+\gamma^2\mu^2\eta^2\sigma^2\left(\hat{\mu}_t+\hat{\eta}_t+\phi Y\hat{Y}_t\right)+\gamma W\hat{W}_t\end{array}\right)e^{\gamma W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}\\+\left(1+\gamma^2\mu^2\eta^2\sigma^2\left(\hat{\mu}_t+\hat{\eta}_t+\phi Y\hat{Y}_t\right)+\gamma W\hat{W}_t\right)(1-\vartheta)e^{\gamma W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}\end{array}\right)$$

$$\left(1+\hat{S}_t\right)S=\left(\begin{array}{c}\left(\begin{array}{c}\vartheta e^{-\frac{\gamma}{\vartheta}W}S+\left(\hat{S}_{t-1}-\gamma W\hat{W}_t\right)\vartheta e^{-\frac{\gamma}{\vartheta}W}S\\+\gamma^2\mu^2\eta^2\sigma^2\left(\hat{\mu}_t+\hat{\eta}_t+\phi Y\hat{Y}_t\right)+\gamma W\hat{W}_t\end{array}\right)e^{\gamma W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}\\+\left(1+\gamma^2\mu^2\eta^2\sigma^2\left(\hat{\mu}_t+\hat{\eta}_t+\phi Y\hat{Y}_t\right)+\gamma W\hat{W}_t\right)(1-\vartheta)e^{\gamma W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}\end{array}\right)$$

We know that in steady state:

$$\begin{aligned}S &= \vartheta e^{-\frac{\gamma}{\vartheta}W}Se^{\gamma W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}+(1-\vartheta)e^{\gamma W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}\\ S\left(1-\vartheta e^{-\gamma\frac{(1-\vartheta)}{\vartheta}W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}\right) &= (1-\vartheta)e^{\gamma W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}\\ S &= \frac{(1-\vartheta)e^{\gamma W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}}{(1-\vartheta e^{-\gamma\frac{(1-\vartheta)}{\vartheta}W}e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2})}.\end{aligned}$$

$$S\hat{S}_t = \left(\begin{array}{c} \vartheta e^{-\gamma \frac{(1-\vartheta)}{\vartheta} W} e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2} \left(S\hat{S}_{t-1} \right) \\ +\gamma \left((2-\vartheta) e^{\gamma W} - S\vartheta e^{-\gamma \frac{(1-\vartheta)}{\vartheta} W} \right) e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2} \left(W\hat{W}_t \right) \\ + \left(\gamma^2\mu^2\eta^2\sigma^2 \left(\hat{\mu}_t + \hat{\eta}_t + \phi Y\hat{Y}_t \right) \right) (2-\vartheta) e^{\gamma W} e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2} \end{array} \right)$$

where,

$$\begin{aligned} W_t &= \mu_t \left(B_t - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t \right) \\ &= \mu_t \left(\frac{\left(\left(1 + \varsigma P_t^M \right) b_t^L + b_t^S \right)}{(1+\pi_t)} - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t \right) \\ &= \mu_t \left(\frac{\left(\left(1 + \varsigma P_t^M \right) b_t^L \right)}{(1+\pi_t)} - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t \right) \end{aligned}$$

$$W \left(1 + \hat{W}_t \right) = \mu \left(1 + \hat{\mu}_t \right) \left(\begin{array}{c} \frac{\left(\left(1 + \varsigma P^M \left(1 + P_t^M \right) \right) \left(1 + \hat{b}_t^L \right) b^L \right)}{(1+\pi_t)} \\ - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi \left(1 + \hat{\varphi}_t \right) \end{array} \right)$$

$$W \left(1 + \hat{W}_t \right) = \left(\begin{array}{c} \mu \left(1 + \varsigma P^M \right) b^L \left(1 + \hat{\mu}_t + \frac{\varsigma P^M}{1+\varsigma P^M} P_t^M + \hat{b}_t^L - \pi_t \right) \\ -\mu \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi \left(1 + \hat{\mu}_t + \hat{\varphi}_t \right) \end{array} \right)$$

$$W\hat{W}_t = \left(\begin{array}{c} \mu \underbrace{\left(\left(1 + \varsigma P^M \right) b^L - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi \right)}_{=W} \hat{\mu}_t \\ +\mu \left(1 + \varsigma P^M \right) b^L \left(\frac{\varsigma P^M}{1+\varsigma P^M} P_t^M + \hat{b}_t^L - \pi_t \right) \\ -\mu \frac{\vartheta \varkappa}{(1-\vartheta)} \left(\varphi \hat{\varphi}_t \right) \end{array} \right)$$

$$\begin{aligned} W\hat{W}_t &= W\hat{\mu}_t + \mu \left(1 + \varsigma P^M \right) b^L \left(\frac{\varsigma P^M}{1+\varsigma P^M} P_t^M + \hat{b}_t^L - \pi_t \right) \\ &\quad -\mu \frac{\vartheta \varkappa}{(1-\vartheta)} \left(\varphi \hat{\varphi}_t \right) \end{aligned}$$

1.12 Steady State

We are looking at zero inflation steady state($\pi = 0$). And, to simplify the analysis, we are going to drop $\chi\tilde{G}_t$ from the household utility function($\chi = 0$), so $x_t = c_t + \chi\tilde{G}_t \Rightarrow x_t = c_t$

$$
\begin{aligned}
1 &: \pi = 0 \\
2 &: b^S = 0 \\
3 &: b^M = 0.43 \cdot \left(\frac{4Y}{P^M}\right) \\
4 &: P^M = \frac{1}{((1+\pi)R - \varsigma)} \\
5 &: LS\_Tax : T = P^M (R-1) b_t^M + G_0 + sw\frac{Y}{z} \\
6 &: DI\_Tax : \tau = \frac{z}{wY}\left(P^M (R-1) b_t^M + G_0 + sw\frac{Y}{z}\right) \\
7 &: \bar{\omega} = w^2\sigma^2 \\
8 &: w = z\left(\frac{\varepsilon - 1}{(1-s)\varepsilon}\right) \\
9 &: \eta = (1-\tau) w \\
10 &: R = \frac{1}{\beta}\exp\left(\gamma\frac{(1-\vartheta)}{\vartheta}\mu\left(P^M R b^M\right) - \frac{\gamma}{2}\mu^2(1-\tau_0)^2 w^2\sigma^2 - \bar{\varkappa}\mu\phi\right) \\
11 &: R = I \\
12 &: \mu = \left(\frac{R-\vartheta}{(1+\rho\gamma\eta)R}\right) \\
13 &: \varphi = (1-\tau) w\left(\frac{R}{R-\vartheta}\right) \\
14 &: x = \left(\begin{array}{c} \frac{z\rho}{(1+\rho\gamma z)}\log((1-\tau)w) + \frac{z\bar{\xi}}{(1+\rho\gamma z)} \\ -\frac{(1-\chi)G_0}{(1+\rho\gamma z)} - \frac{z\bar{\varkappa}}{(1+\rho\gamma z)}\left(\frac{\vartheta}{1-\vartheta}\right) \end{array}\right) \\
15 &: Y = x + (1-\chi) G_0 \\
16 &: W = \mu_t\left(\frac{\left(\left(1+\varsigma P^M\right) b^L\right)}{(1+\pi)} - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi\right) \\
17 &: S = \frac{(1-\vartheta)e^{\gamma W} e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}}{(1-\vartheta e^{-\gamma\frac{(1-\vartheta)}{\vartheta}W} e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2})}
\end{aligned}
$$

Chapter 2

Equity versus Efficiency: Optimal Monetary and Fiscal Policy in a HANK Economy

Based on joint work with T. Kirsanova and C. Leith

Abstract

We extend Acharya et al. (2023) to analyse optimal fiscal policy in a tractable heterogeneous agent New Keynesian (HANK) economy where overlapping generations of households wish to save for retirement and precautionary reasons. While monetary policy can affect the households' ability to self-insure against shocks, fiscal policy has a greater impact on such behavior both in steady-state and following shocks. A policy maker solely focused on minimizing inequality would provide insufficient government debt to enable households to save for retirement and accumulate precautionary savings. Why? The trade off between inter- and intra-generational equity means it wishes to suppress interest rates below households' rate of time preference, facilitating household borrowing in the face of idiosyncratic shocks. The Ramsey policy maker faces a further trade-off - equity versus efficiency - and due to debt service costs will issue even less debt, driving equilibrium interest rates down further. We explore the relative efficacy of monetary and fiscal policy in responding to aggregate shocks in this environment.

2.1 Introduction

There is growing interest in issues of inequality in macroeconomics. Politicians have awoken to the possibility that the policy consensus has not always been felt to benefit all of society, leading populist politicians to highlight dissatisfaction with the status quo (**?**). Central banks are increasingly conscious of the distributional impacts of their policies.[1] Popular treatments debate the relative importance of intra- generational (**?**) versus inter- generational (**?**) inequality. However, technical issues often make it difficult to provide well-founded normative policy recommendations which address these concerns. This paper seeks to make progress in this area by considering (non-trivial) fiscal policies, including the evolution of long-term government debt, alongside monetary policy in an environment where there are meaningful trade-offs between efficiency and equity, both within and across generations.

Specifically, we analyze jointly optimal monetary and fiscal policy in a heterogeneous-agent New Keynesian (HANK) economy. To achieve this, we build upon the insights of Acharya, Challe & Dogra (2023) and Acharya & Dogra (2020), who develop a tractable heterogeneous-agent economy for analyzing optimal monetary policy and monetary policy 'puzzles', respectively. We extend the overlapping-generation model of Acharya et al. (2023) by developing the fiscal side, which was kept deliberately simple given their primary focus on monetary policy. Thus, we introduce long-term government debt, financed by distortionary labor income taxes, and allow it to evolve over

[1] US Fed chairman Jerome **?** used his Jackson Hole speech to stress the desirability of running the economy close to maximum employment in order to spread the benefits of economic growth more widely.

time in accordance with optimal policy. The presence of government debt and the absence of fiscal transfers to new-born generations, implies that intergenerational inequality is driven not only by idiosyncratic labor supply shocks but also by differing levels of accumulated wealth, which optimal policy must address. By assuming that the disutility of labor supply rises with age, we mimic a desire to save for retirement, which augments the motive for precautionary savings and leads, in equilibrium, to the government optimally issuing plausible levels of government debt to facilitate such saving behavior without sub-optimally suppressing interest rates. The differences in wealth across and within generations endogenously generate differences in the exposure to aggregate shocks, which optimal policy will account for.

In Acharya et al. (2023), there are two channels through which monetary policy impacts inequality. First, the extent to which the variance of idiosyncratic risk is pro- or counter-cyclical allows monetary policy to influence the magnitude of that risk following aggregate shocks—the 'income-risk' channel. Second, by lowering interest rates, monetary policy facilitates households' self-insurance; it becomes cheaper to buy bonds to smooth consumption and easier to borrow against future income when a relatively loose monetary policy expands the economy, raising future income against which one can borrow—the 'self-insurance' channel. They examine the cyclicality of consumption risk to explain how monetary policy should respond to shocks. In our model, distortionary tax rates and the level of debt also impact inequality, and can be more significant.

Distortionary labor taxation mitigates the initial impact of an idiosyncratic income shock since part of the lost income would have been taxed anyway. However, anticipated future tax rates also affect the household's ability to borrow against future post-tax income in order to smooth consumption — expectations of higher future taxes increase the costs of earning income to repay any borrowing undertaken to offset a negative idiosyncratic shock. Additionally, distortionary labor income taxation leads to the usual loss of efficiency by discouraging worker effort.

In our overlapping-generation (OLG) economy, the extent to which the government issues debt to facilitate household saving, both for retirement and as protection against idiosyncratic shocks, affects equilibrium real interest rates, both in response to shocks and in the steady state. While monetary policy only has a transitory (for as long as prices are sticky) impact on real interest rates – and, through that, inequality – debt policy can have a permanent influence on inequality. As a benchmark, we define a 'golden rule' level of steady-state debt that would align the equilibrium real interest rate with households' rate of time preference, enabling savings for retirement and precautionary savings. We explore how optimal policy diverges from this benchmark. A policy maker aiming to minimize inequality only would not issue sufficient debt to ensure that equilibrium interest rates reached this benchmark. Instead, they would allow interest rates to lie below households' rate of time preference, implying there are insufficient assets to enable households to save for retirement, let alone accumulate precautionary savings. This will drive a degree of inter-generational inequality as consumption falls throughout households' lifetimes. The policy maker is prepared to allow this inter- generational inequality since, by suppressing interest rates, the policy maker facilitates household borrowing in the face of negative idiosyncratic shocks, thereby mitigating intra- generational equity. However, the micro-founded social welfare function not only exhibits a concern for equity, but also for efficiency. Taking account of efficiency leads the policy maker to issue even less debt, suppressing interest rates further, as issuing debt crowds out economic activity, especially when taxes are distortionary. Thus, the Ramsey policy maker faces a trade-off between both inter- and intra-generational equity and efficiency, which leads to them issuing debt but to a lesser extent than needed to facilitate saving for retirement and achieving the 'golden rule' interest rate.

We quantify where the balance is struck in these trade-offs and find that Ramsey policy comes close to achieving the minimum level of inequality - inequality is only 0.001% higher than its minimal value under the fully-optimal Ramsey policy, and 0.007% lower than if the policy maker only cared for efficiency. This can also be seen in the steady-state levels of debt issued by the Ramsey policy maker - 54% of GDP - which is only slightly below the level of 58% that would be chosen by a policy maker seeking in minimize inequality alone. In contrast caring only for efficiency would lead to far lower debt levels of 31% of GDP. Therefore, optimal policy is dominated by a concern for equity.

We then turn to consider the response to aggregate shocks. Again we examine how fiscal policy contributes to stabilizing the economy in the face of such shocks, including allowing for the variance of idiosyncratic shocks to be pro-, counter- or a-cyclical. The benchmark 'divine coincidence' result where interest rates would be cut in response to a positive technology shock without generating deflation, only emerges under special circumstances. In our heterogeneous agent OLG economy featuring phased retirement and idiosyncratic income shocks these conditions are: (i) no fiscal policy, other than (ii) a lump-sum tax financed production subsidy which ensures the steady-state is efficient, and (iii) the policy maker only cares about efficiency, not equity. Relaxing these conditions creates a meaningful policy problem. In the absence of fiscal policy, monetary policy engineers a degree of price level control by following an initial period of deflation, with a period of positive inflation. Seeking to manipulate expectations in this way is a common feature of Ramsey policy in the New Keynesian model. However, when reducing inequality becomes part of the policy objective, the monetary policy maker relaxes policy further, which facilitates households' ability to borrow to offset negative idiosyncratic income shocks. Introducing fiscal policy and government debt implies that there is a non-trivial distribution of wealth which impacts the evolution of consumption inequality. Now the policy mix in response to shocks relies on the use of distortionary taxation alongside monetary policy to reduce movements in inflation while simultaneously facilitating households' ability to smooth consumption in the face of idiosyncratic shocks.

An interesting element of our modelled economy is that households can hold longer-term government debt, and not the single period debt often adopted in the literature. Moreover, optimal policy implies significant variance in the holdings of this debt within and across generations. This affects the re-distributional impacts of shocks, especially when they are autocorrelated, since the changes in the entire path of short-term interest rates create capital gains/losses for holders of these longer-term bonds, a dynamic not present with single-period debt. Under the timelessly optimal policies we consider, the policy maker commits to not attempt to unexpectedly induce such redistributions, but when they occur as the result of shocks, the policy maker's policies will be affected by the extent to which the redistributions affect the evolution of inequality.

2.2 Literature Review:

Our work is related to the large literature on Bewley (1977), Huggett (1993) and Aiyagari (1994) economies, where households face uninsurable idiosyncratic risk. As already noted, our approach closely follows that of Acharya & Dogra (2020) and Acharya et al. (2023), utilizing the assumptions of CARA utility and normally distributed idiosyncratic shocks to enable us to derive tractable aggregate relationships and a micro-founded measure of social welfare. Specifically, we extend the framework of Acharya et al. (2023) to allow a meaningful role for fiscal policy, which turns out to have significant implications for both the steady-state trade-off between equity and efficiency and the response to shocks.

The broader Heterogeneous Agent New Keynesian (HANK) literature, which combines household heterogeneity with sticky prices, typically focuses on a positive description of the impact on monetary policy in such economies (see Violante n.d., Sargent 2023) for overviews of the key insights gained from the literature, and Kaplan & Violante (2018) for a survey of the HANK literature, more generally). This focus is largely due to the computational complexity of modelling optimal policy in an environment where the state space is infinite. Recently, there has been progress in addressing these computational issues, beginning to explore normative policy issues, particularly relating to the conduct of monetary policy. See, for example, Bhandari et al. (2021) and Le Grand et al. (2022) for an analysis of optimal policy, and McKay & Wolf (2022) for a characterization of optimal policy rules. Additionally, Nuño & Thomas (2022) utilize results from continuous-time mathematics to track the wealth distribution over time.

A more common approach to addressing normative issues involves making simplifying assumptions to ensure sufficient tractability to analyse optimal policy. For example, it is common to assume that households cannot borrow and government debt is in zero net supply, which implies that, in equilibrium, households do not hold any assets – the so-called *zero liquidity limit*. This assumption eliminates the ability of households to self-insure through saving and/or borrowing and results in a degenerate wealth distribution, allowing for a tractable analysis of optimal policy. Examples of this approach applied to conventional monetary policy include Bilbiie (2008), Bilbiie (2021), **?** and **?**. Auclert (2019) and Bilbiie (2024) *extend this consideration to fiscal policy*.

Our paper uses a different set of simplifying assumptions, specifically CARA utility and normally distributed idiosyncratic shocks. This approach implies that the welfare costs of inequality are captured by a single variable, which evolves recursively, making the Ramsey policy problem tractable despite the heterogeneity. The cost in doing so is that the marginal propensity to consume is common across households, rather than varying with income/wealth. Nevertheless, our approach allows for precautionary savings and borrowing in response to idiosyncratic shocks. Furthermore, we extend Acharya et al. (2023) and the literature imposing a zero liquidity limit by allowing for government debt to be in non-zero supply and determined endogenously. This implies a non-degenerate wealth distribution and that households within and across generations will face different wealth revaluation effects in the face of aggregate shocks.[2]

Our model relies on an OLG structure as in Blanchard (1985) and Yaari (1965), to prevent the distribution of wealth from becoming non-stationary. Optimal policy in such a framework often focuses on the modified Golden Rule of capital accumulation, stating that the marginal product of capital should equal the rate of growth of the population plus the households' rate of time preference – see the textbook treatment in Blanchard & Fischer (1989). Escolano (1992) obtains the same result by considering optimal policy in an OLG economy with endogenous labor supply and various distortionary taxes. This serves as a useful benchmark in interpreting the results in our OLG heterogeneous agent economy subject to idiosyncratic risk.

2.3 The Model

Our model follows that of Acharya et al. (2023), which employs Constant Absolute Risk Aversion (CARA) preferences and normally distributed shocks to individual household labour supply to develop a tractable heterogeneous agent model for the analysis of monetary policy. The model is capable of describing both macroeconomic aggregates and measuring social welfare, accounting for heterogeneity. We undertake the following extensions, which make the modelled economy a tractable framework for examining jointly optimal monetary *and* fiscal policies in the presence of household heterogeneity.

First, we allow for the existence of *government debt*. This endogenously determines a steady-

[2] Acharya et al. (2023) use a fiscal transfer at the point of birth, and apply a wealth tax to existing households, to ensure all households are *ex ante* identical at time t=0. In a previous version of their paper these assumptions were relaxed to include revaluation effects.

state distribution of wealth, affecting the optimal response to shocks and implying an additional externality absent in models without government debt. Overlapping generations of households will decide whether to save by purchasing government debt, not internalizing the impact of these decisions on the equilibrium real interest rate – a feature not present in representative agent models and absent in the heterogeneous agent model of Acharya et al. (2023) when government debt is in zero net supply (see, chapter 9 of Acemoglu 2008, for a discussion).

Second, in exploring the impact of variation in distortionary tax rates, we do not allow the policy maker access to lump-sum transfers as a policy instrument to finance the government's activities. As a result, raising tax revenues to finance government consumption and service government debt will add to the distortions associated with monopolistic competition. *Distortionary taxation* will also impact post-tax inequality generated by idiosyncratic income shocks. Moreover, we do not employ a subsidy with which to offset the inefficiencies due to monopolistic competition.

Third, we assume that the *disutility of supplying labor income decreases with age* in order to mimic economic retirement. This approach generates a desire to save in anticipation of falling incomes, akin to saving for retirement, and allows our model to feature a plausible level of government debt under the Ramsey policy.

Fourth, as we wish to consider the Ramsey policy problem for such an economy, we develop *a measure of social welfare* that accounts for both idiosyncratic shocks within generations and intergenerational inequality driven by the evolution of the wealth distribution over the life cycle. It also captures how these factors interact and affect the welfare implications of aggregate shocks.

2.3.1 Households

The economy is populated by cohorts of Blanchard-Yaari individuals that have constant survival probability in any period, $0 < \vartheta < 1$, see Blanchard (1985). At any time t, an individual i which belongs to generation born at time $s \leq t$ derives utility from age-dependent real private consumption $c_t^s(i)$ and real government consumption G_t. They also derive disutility from labour supply, $l_t^s(i)$, and, exogenously, dist-utility rises with age reflecting a desire to retire, $\Theta_t^s = \varkappa(t-s)$. This gradual withdrawal from the labour market will create a desire to save for 'retirement' and will ensure that the government wishes to issue a plausible level of government debt in the Ramsey steady-state. Crucially, households face uninsurable idiosyncratic shocks to disutility from labour $\xi_t^s(i) \tilde{} N\left(\bar{\xi}, \sigma_t^2\right)$; these shocks are independent across time and individuals. The variance of this shock may vary with economic activity. There is no aggregate risk.

We assume CARA preferences so utility takes form:

$$U_s = \mathbb{E}_i \sum_{t=s}^{\infty} (\beta\vartheta)^{t-s} \left(-\frac{1}{\gamma} e^{-\gamma(c_t^s(i)+\chi G_t)} - \rho e^{\frac{1}{\rho}(l_t^s(i)+\Theta_t^s-\xi_t^s(i))} \right)$$

Individuals invest in long and short term nominal actuarial bonds $\mathscr{A}_t^{L,s}(i)$ and $\mathscr{A}_t^{S,s}(i)$. The short-term bonds are issued at price $\tilde{q}_t$, paying out one unit of current one period later. While, following Curdia & Woodford (2010), the longer-term bonds, issued at price $\tilde{P}_t^M$, pay an initial coupon of one unit of currency which falls to ρ^s, s period's later. Longer maturity debt matters as, following shocks, the revaluation effects on wealth held in the form of longer-term bonds through fluctuations in bond prices will be greater, which, in turn, will affect the impact of that shock on the distribution of wealth – see Leeper & Leith 2016*b* for a discussion. Households receive after tax-wages, $(1-\tau_t)P_t w_t l_t^s(i)$, where the labor income tax, levied at rate τ_t, is the the sole source of government tax revenues in our benchmark model. We also introduce a lump-sum tax, $P_t T_t$, which will used to replace distortionary taxation as a means of eliminating the effects of tax distortions for comparison purposes only. Each household receives dividends, $P_t d_t$.[3] Their budget constraint at time t is

$$\begin{aligned}
&P_t c_t^s(i) + \tilde{P}_t^M \mathscr{A}_{t+1}^{L,s}(i) + \tilde{q}_t \mathscr{A}_{t+1}^{S,s}(i) \\
&= \left(1+\rho\tilde{P}_t^M\right)\mathscr{A}_t^{L,s}(i) + \mathscr{A}_t^{S,s}(i) \\
&+ (1-\tau_t)P_t w_t l_t^s(i) + P_t d_t - P_t T_t
\end{aligned} \tag{2.1}$$

Each individual is born with zero bond holdings, $\mathscr{A}_s^{L,s} = \mathscr{A}_s^{S,s} = 0$ and there is no fiscal transfer to newborns and/or wealth tax on existing households to ensure *ex ante* equality between all households as in Acharya et al. (2023).

Define the ratio of the number of each type of assets to the price level as,

$$a_t^{J,s}(i) = \frac{\mathscr{A}_t^{J,s}(i)}{P_{t-1}}, J \in \{L,S\} \tag{2.2}$$

and introduce a measure of real assets

$$A_t^s(i) = \frac{\left(1+\rho\tilde{P}_t^M\right)a_t^{L,s}(i) + a_t^{S,s}(i)}{(1+\pi_t)} \tag{2.3}$$

Then, we rewrite the budget constraint in real terms:

$$\frac{\vartheta}{R_t}A_{t+1}^s(i) = A_t^s(i) + y_t^s(i) - c_t^s(i) \tag{2.4}$$

[3] For simplicity we assume that dividends are shared equally across households. It would be possible to allow dividends to vary with household labor supply or the state of the economy as in Acharya & Dogra (2020). In our economy another possibility might be to allow dividends paid to individual households to vary with age, reflecting re-balancing of portfolios from equities to bonds over the life-cycle.

where net household income is defined as,

$$y_t^s(i) = \eta_t l_t^s(i) + d_t - T_t, \tag{2.5}$$

the post-tax wage is

$$\eta_t = (1-\tau_t) w_t, \tag{2.6}$$

and we can define the ex ante real interest rate R_t as follows,

$$\frac{\vartheta}{R_t} = \tilde{q}_t (1+\pi_{t+1}).$$

Note that the ex post real rate will differ depending on the proportion of short and long-term bonds the household possesses in the presence of aggregate 'shocks' to the perfect foresight equilibrium path since additional capital gains/losses are possible on long-term bonds when the path of interest rates differ from what was expected.

The solution to an individual's optimisation problem can be summarized by the following Proposition derived in Appendix 2.9.1.

Proposition 4 (Individual's Optimisation) *In equilibrium, the optimal date t consumption and labour supply decisions of a household i born at date s are,*

$$c_t^s(i) = \mathscr{C}_t - \chi G_t + \mu_t m_t^s(i) \tag{2.7}$$

$$l_t^s(i) = \rho \ln(\eta_t) - \Theta_t^s - \rho\gamma(c_t^s(i) + \chi G_t) + \xi_t^s(i) \tag{2.8}$$

where

$$m_t^s(i) = A_t^s(i) - \varphi_t \Theta_t^s + \eta_t \left(\xi_t^s(i) - \bar{\xi}\right)$$

is demeaned 'cash-on-hand', $\mathscr{C}_t$ is a measure of common consumption, μ_t is the 'marginal propensity to consume (MPC) out of cash-on-hand and φ_t is the after-tax value of the human wealth of an individual supplying one unit of labor supply. This latter variable is used to value the income lost to retirement within households and for the population as a whole. These evolve according to

$$\frac{1}{\mu_t} = \frac{\vartheta}{R_t \mu_{t+1}} + (1+\rho\gamma\eta_t) \tag{2.9}$$

$$\varphi_t = \eta_t + \frac{\vartheta}{R_t}\varphi_{t+1} \tag{2.10}$$

$$\begin{aligned} \mathscr{C}_t = & -\frac{\mu_t \vartheta}{R_t \mu_{t+1}\gamma}\ln(\beta R_t) + \frac{\vartheta\mu_t}{R_t\mu_{t+1}}\mathscr{C}_{t+1} - \frac{\vartheta\mu_t}{R_t\mu_{t+1}}\frac{\gamma}{2}\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2 \\ & -\mu_t\frac{\vartheta}{R_t}\varkappa\varphi_{t+1} + \mu_t\left(\eta_t\left(\rho\log(\eta_t) + \bar{\xi}\right) + d_t - T_t + \chi G_t\right) \end{aligned} \tag{2.11}$$

where $R_t = \frac{\vartheta}{\tilde{q}_t(1+\pi_{t+1})}$ *is real interest rate.*

The household's optimization implies that their consumption equals a measure of consumption, $\mathscr{C}_t$, which only depends on aggregate variables, after adjusting for the substitutability between private and public consumption in utility, χG_t, plus a term that is idiosyncratic, $\mu_t m_t^s(i)$. This final term depends on household i's cash-in-hand, $m_t^s(i)$, which comprises their financial assets, $A_t^s(i)$, minus the age-dependent loss of human wealth due to retirement that period, $\varphi_t \Theta_t^s$, and the extent to which their labor income varies due to their idiosyncratic shock to labor disutility, differing from the population average, $\eta_t\left(\xi_t^s(i) - \bar{\xi}\right)$. Household labor supply then depends positively on the post-tax real wage, negatively on consumption, with adjustments made for both age-dependent retirement and idiosyncratic shocks to the disutility of labor supply.

A negative shock to labor supply, $\xi_t^s(i) < \bar{\xi}$, reduces household income and results in a fall in consumption, where $\frac{\partial c_t^s(i)}{\partial \xi_t^s(i)} = \mu_t \eta_t = \mu_t(1-\tau_t)w_t$. This fall will be greater the higher the marginal propensity to consume out of cash-on-hand, μ_t, and the greater the post-tax real wage. Households are therefore more insulated from the direct impact of the shock the higher the tax rates. As a result of the fall in consumption, they will work harder, where $\frac{\partial l_t^s(i)}{\partial \xi_t^s(i)} = 1 - \gamma\rho\mu_t\eta_t = 1 - \gamma\rho\mu_t(1-\tau_t)w_t < 1$. Again, a lower marginal propensity to consume and a higher tax rate will reduce the household's desire to maintain consumption by working harder in the period of the shock. Aside from working harder, the household can also maintain consumption through borrowing. Its ability to do so is implicit in the marginal propensity to consume.

We can iterate the marginal propensity to consume out of cash-on-hand forwards to obtain:

$$\frac{\mu_t}{R_t} = \left[\sum_{s=0}^{\infty} \frac{\vartheta^s\left(1+\rho\gamma(1-\tau_{t+s})w_{t+s}\right)}{\prod_{j=1}^{s+1} R_{t+j-1}}\right]^{-1}. \tag{2.12}$$

This formula is the same as Acharya et al. (2023), except it incorporates dependency on the future *post-tax* real wage rate. It indicates that the propensity to consume increases with interest rates but decreases with future post-tax wages. Therefore, after experiencing a negative idiosyncratic shock to labor supply, which reduces their cash-on-hand, $m_t^s(i)$, households can maintain consumption closer to $\mathscr{C}_t$ when the marginal propensity to consume is low. This occurs when interest rates are low, making borrowing to smooth consumption less costly, or when post-tax wages are expected to be higher in the future, making it less expensive to repay any borrowing. Additionally, the presence of the tax rate implies that a lower tax rate makes it less costly (in utility terms) to increase future labor supply to pay off any debt incurred to smooth consumption. Thus, future distortionary taxation inhibits self-insurance, although high tax rates at the time of the shock mitigate its direct impact, as part of the lost income would have been taxed anyway.

Meanwhile, the component of household consumption driven by aggregate variables, $\mathscr{C}_t$, can be iterated forwards to obtain:

$$\mathscr{C}_t = -\frac{1}{\gamma}\sum_{s=0}^{\infty} Q_{t+s,t}\frac{\mu_t}{\mu_{t+s}}\ln(\beta R_{t+s}) - \frac{\gamma\mu_t}{2}\sum_{s=0}^{\infty} Q_{t+s,t}\mu_{t+s}^2 w_{t+s}^2 (1-\tau_{t+s})^2\sigma_{t+s}^2$$
$$+\mu_t\sum_{s=0}^{\infty} Q_{t+s,t}\bar{y}_{t+s} - \varkappa\mu_t\sum_{s=1}^{\infty} Q_{t+s,t}\varphi_{t+s}. \tag{2.13}$$

The first term has the same interpretation as in Acharya & Dogra (2020), capturing the impact of variations in interest rates relative to the impatience of households. If interest rates are typically higher than the rate of time preference, current consumption will be lower as households increase savings and cut current consumption. The discount factor, $Q_{t+s,t} = \frac{\vartheta^s}{\prod_{j=0}^{s-1} R_{t+j}}$, accounts for both the interest rate on financial assets and the probability of death, $1-\vartheta$. The second term is attributable to precautionary savings. A higher variance of idiosyncratic shocks, σ_{t+s}^2, increases the variance of post-tax income, $w_{t+s}^2(1-\tau_{t+s})^2\sigma_{t+s}^2$, which, after applying the marginal propensity to consume, captures the variance in consumption across households, $\mu_{t+s}^2 w_{t+s}^2(1-\tau_{t+s})^2\sigma_{t+s}^2$. The third term represents the discounted value of per capita post-tax income from labor, dividends, and transfers, after adjusting for the utility generated by public consumption, $\bar{y}_t = \left(\eta_t\left(\rho\log(\eta_t)+\bar{\xi}\right)+d_t-T_t+\chi G_t\right)$. Lastly, the equation includes the discounted value of the income lost due to the gradual retirement of the population throughout their working lives. Taxation affects this measure of aggregate consumption through its impact on the marginal propensity to consume, as discussed above, positively by reducing the variance of post-tax income but negatively by reducing the level of post-tax income and, therefore, the discounted value of that income.

It follows that net income (2.5) can be written as

$$y_t^s(i) = \eta_t\left(\rho\log(\eta_t)+\xi_t^s(i)\right) - \eta_t\Theta_t^s - \rho\gamma\chi\eta_t G_t - \rho\gamma\eta_t c_t^s(i) + d_t - T_t. \tag{2.14}$$

Aggregation of the household budget constraint yields

$$\frac{\vartheta}{R_t}A_{t+1} = \vartheta A_t + y_t - c_t, \tag{2.15}$$

where

$$A_t = \frac{\left(1+\rho\tilde{P}_t^M\right)a_t^L + a_t^S}{1+\pi_t},$$

and a_t^J is an aggregation of long term ($J=L$) and short term ($J=S$) bonds.

The straightforward aggregation of income (2.14) yields:

$$y_t = \eta_t\left(\rho\log(\eta_t)+\bar{\xi}\right) - \frac{\varkappa\vartheta}{(1-\vartheta)}\eta_t - \rho\gamma\eta_t\chi G_t - \rho\gamma\eta_t c_t + d_t - T_t, \tag{2.16}$$

and aggregation of (2.7) yields

$$c_t = \mathscr{C}_t - \chi G_t + \mu_t \vartheta \left(A_t - \frac{\varkappa}{1-\vartheta} \varphi_t \right). \tag{2.17}$$

This latter expressions indicates that per capita consumption equals the consumption measure, $\mathscr{C}_t$, driving individual household consumption in (2.7), after adjusting for the substitutability between private and public consumption, χG_t, and the extent to which, in aggregate, households have successfully saved for retirement. $A_t > \frac{\varkappa}{1-\vartheta} \varphi_t$ implies that household financial wealth exceeds the loss of human wealth due to retirement across the population.

Aggregated first order conditions for the individuals' problem yield the following relationships, derived in Appendix 2.9.3.

Proposition 5 (Aggregated Individuals' Optimisation) *In equilibrium, the optimal date t the aggregate total consumption and labour supply decisions are:*

$$x_t = -\frac{1}{\gamma} \log(\beta R_t) + x_{t+1} + \mu_{t+1}(1-\vartheta) A_{t+1} - \frac{\gamma}{2} \mu_{t+1}^2 \eta_{t+1}^2 \sigma_{t+1}^2 - \varkappa \mu_{t+1} \varphi_{t+1}, \tag{2.18}$$

$$n_t = \rho \log \eta_t - \frac{\varkappa \vartheta}{1-\vartheta} + \bar{\xi} - \rho \gamma x_t, \tag{2.19}$$

$$x_t = c_t + \chi G_t, \tag{2.20}$$

$$\mathscr{C}_t = x_t - \mu_t \vartheta \left(A_t - \frac{\varkappa}{(1-\vartheta)} \varphi_t \right). \tag{2.21}$$

The dynamics of x_t resemble that of consumption in a representative agent model, but with notable differences. Typically, consumption is expected to grow whenever the interest rate exceed the rate of time preference, $\beta R_t > 1$. In other words, consumption jumps down when interest rates unexpectedly rise, as the discounted value of future post-tax income across the economy falls. Consumption then recovers as interest rates return to normal levels. However, there is an additional term, $\mu_{t+1}(1-\vartheta) A_{t+1}$, attributable to the aggregation across finitely-lived generations. This term would not exist if households were infinitely lived and $\vartheta = 1$. Instead, finite lives imply that government debt (which is mapped to households assets as $B_t = \vartheta A_t$) are net assets for households. Households currently alive do not expect paying for all the surpluses backing government debt, implying that any increase in those assets increases consumption. As above, the term $\frac{\gamma}{2} \mu_{t+1}^2 \eta_{t+1}^2 \sigma_{t+1}^2$ measures the variance of consumption across households due to idiosyncratic shocks, providing a motive for precautionary saving, which in turn reduces current consumption. Finally, consumption is reduced by the ongoing loss of post-tax income due to retirement.

It is helpful to consider the steady-state of this relationship to see how these additional factors influence interest rates:

$$\frac{1}{\gamma}\log\left(\beta R\right) = \mu\left(1-\vartheta\right)\left(A - \frac{\varkappa}{1-\vartheta}\varphi\right) - \frac{\gamma}{2}\mu^2\eta^2\sigma^2. \tag{2.22}$$

In the absence of idiosyncratic risk or finite lives, the steady-state interest rate in a representative agent economy would be consistent with household preferences, $\beta R = 1$. However, the desire for precautionary savings drive down the steady-state interest rate relative to these preferences, while the accumulation of assets beyond what is needed to fund retirement in an OLG economy, $A > \frac{\varkappa}{1-\vartheta}\varphi$, raises interest rates. If the government could provide sufficient assets for households to satiate their desire for precautionary savings and their need to smooth consumption in retirement, then the steady-state interest rate would equal the households rate of time preference, provided :

$$B - \frac{\varkappa}{1-\vartheta}\varphi = \frac{1}{2}\frac{\vartheta}{1-\vartheta}\gamma\mu\eta^2\sigma^2. \tag{2.23}$$

This becomes a relevant benchmark when considering Ramsey policy below.

2.3.2 Firms

There is a continuum of monopolistically competitive firms. Each firm produces a differentiated product according to the production technology:

$$Y_t\left(j\right) = z_t n_t\left(j\right), \tag{2.24}$$

where z_t is the level of aggregate productivity.

They face cost of price adjustment *a la* Rotemberg (1982). In the absence of aggregate risk, firm j solves the following optimization problem

$$\max_{P_t(j)} \sum_{t=0}^{\infty} \beta^t \left(\left(\frac{P_t\left(j\right)}{P_t} Y_t\left(j\right) - w_t n_t\left(j\right) \right) - \frac{\Phi}{2}\left(\frac{P_t\left(j\right)}{P_{t-1}\left(j\right)} - 1 \right)^2 Y_t \right),$$

subject to monopolistic demand for its product,

$$Y_t\left(j\right) = \left(\frac{P_t\left(j\right)}{P_t}\right)^{-\varepsilon_t} Y_t,$$

and production function (2.24).

The profit optimization yields (see Appendix 2.9.4) the following nonlinear Phillips curve,

$$\pi_t\left(1+\pi_t\right)=\frac{1-\varepsilon_t+\varepsilon_t\frac{w_t}{z_t}}{\Phi}+\beta\pi_{t+1}\left(1+\pi_{t+1}\right)\frac{Y_{t+1}}{Y_t},,$$

and any profit is distributed as a dividend,

$$d_t=\left(Y_t-w_tn_t\right)-\frac{\Phi}{2}\pi_t^2Y_t. \tag{2.25}$$

2.3.3 Government

The government issues nominal long term and short term bonds, for which the maturity matches that of the actuarial bonds used by households. The government budget constraint in nominal terms is

$$P_t^M\mathscr{B}_{t+1}^L+q_t\mathscr{B}_{t+1}^S=\left(1+\rho P_t^M\right)\mathscr{B}_t^L+\mathscr{B}_t^S+P_tG_t-\tau_tP_tw_tn_t-P_tT_t$$

where P_t^M is price of long-term bonds, and q_t is price of short term bonds. As noted above, the lump sum taxes, P_tT_t, are generally set to zero and only used as a replacement for distortionary tax revenues, $\tau_tP_tw_tn_t$, when we wish to remove the impact of distortionary taxation on optimal policy.

This can be re-written in real terms,

$$\left(1+\pi_{t+1}\right)q_tB_{t+1}=B_t+G_t-\tau_tw_tn_t-T_t \tag{2.26}$$

where

$$B_t=\frac{\left(\left(1+\rho P_t^M\right)b_t^L+b_t^S\right)}{\left(1+\pi_t\right)}$$

and

$$b_t^J=\frac{\mathscr{B}_t^J}{P_{t-1}},J\in\{L,S\}.$$

2.3.4 Financial Intermediaries

Financial intermediaries trade actuarial and government bonds. The real profit of intermediaries is the difference between total bonds and total amount of actuarial bonds in the economy in $t+1$:

$$\Pi=\left(1+\rho P_{t+1}^M\right)b_{t+1}^L+b_{t+1}^S-\left(1+\rho\tilde{P}_{t+1}^M\right)\vartheta a_{t+1}^L-\vartheta a_{t+1}^S, \tag{2.27}$$

where b_{t+1}^J are total government bonds and ϑa_{t+1}^J are total actuarial bonds at time $t+1$, i.e. $\vartheta a_{t+1}^J=(1-\vartheta)\sum_{s=-\infty}^{t+1}\vartheta^{t+1-s}\int_0^1a_{t+1}^{J,s}(i)\,di$.

The intermediaries maximize (2.27) subject to the constraint,

$$-\tilde{P}_t^M a_{t+1}^L - \tilde{q}_t a_{t+1}^S + P_t^M b_{t+1}^L + q_t b_{t+1}^S \leqslant 0. \tag{2.28}$$

and the optimization yields

$$\frac{1}{\tilde{q}_t} = \frac{\left(1+\rho \tilde{P}_{t+1}^M\right)}{\tilde{P}_t^M}, \tag{2.29}$$

$$\tilde{q}_t = \vartheta q_t, \tag{2.30}$$

$$\frac{1}{q_t} = \frac{\left(1+\rho P_{t+1}^M\right)}{P_t^M}, \tag{2.31}$$

such that the intermediaries' profits are zero and the *ex ante* returns on short and long-bonds are equalized. It is important to note, however, that this does not imply that the *ex post* real interest rates will be equalized in the presence of one-off shocks to the perfect foresight equilibrium path.

We denote the short-term nominal interest rate as,

$$\frac{1}{1+i_t} = q_t, \tag{2.32}$$

and the real interest rate is,

$$R_t = \frac{\vartheta}{\tilde{q}_t\left(1+\pi_{t+1}\right)} = \frac{1}{q_t\left(1+\pi_{t+1}\right)} = \frac{1+i_t}{1+\pi_{t+1}}. \tag{2.33}$$

2.3.5 Market Clearing

We use individuals' budget constraints (2.15), the government budget constraint (2.26), profit of financial intermediaries (2.27), aggregation of income (2.5) and the profit of monopolistic firms (2.25) to obtain the resource constraint,

$$Y_t = c_t + G_t + \frac{\Phi}{2}\pi_t^2 Y_t. \tag{2.34}$$

Finally, using (2.27) we can rewrite consumption decision (2.21) in terms of aggregate debt,

$$\mathscr{C}_t = c_t + \chi G_t - \mu_t\left(B_t - \frac{\vartheta \varkappa}{1-\vartheta}\varphi_t\right). \tag{2.35}$$

2.3.6 Private Sector Equilibrium

The dynamic system which determines private sector equilibrium $\{x_t, Y_t, \pi_t, \eta_t, w_t, B_t, P_t^M, R_t, \mu_t, \varphi_t, \sigma_t^2\}$ given policy $\{i_t, G_t, T_t, \tau_t\}$ and deterministic disturbances z_t and ε_t can be written as follows

$$x_t = -\frac{1}{\gamma}\log(\beta R_t) + x_{t+1} + \frac{(1-\vartheta)}{\vartheta}\mu_{t+1}B_{t+1} - \frac{\gamma}{2}\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2 - \varkappa\mu_{t+1}\varphi_{t+1}, \tag{2.36}$$

$$\pi_t(1+\pi_t) = \frac{1-\varepsilon_t+\varepsilon_t\frac{w_t}{z_t}}{\Phi} + \beta\pi_{t+1}(1+\pi_{t+1})\frac{Y_{t+1}}{Y_t}, \tag{2.37}$$

$$\frac{1}{\mu_t} = \frac{\vartheta}{R_t\mu_{t+1}} + (1+\rho\gamma\eta_t), \tag{2.38}$$

$$(1+\pi_{t+1})q_tB_{t+1} = B_t + G_t - \tau_t w_t n_t - T_t, \tag{2.39}$$

$$\frac{Y_t}{z_t} = \rho\log\eta_t + \bar{\xi} - \varkappa\frac{\vartheta}{1-\vartheta} - \rho\gamma x_t, \tag{2.40}$$

$$\eta_t = (1-\tau_t)w_t, \tag{2.41}$$

$$Y_t = x_t + (1-\chi)G_t + \frac{\Phi}{2}\pi_t^2Y_t, \tag{2.42}$$

$$P_t^MR_t = \frac{\left(1+\rho P_{t+1}^M\right)}{(1+\pi_{t+1})}, \tag{2.43}$$

$$R_t = \frac{1+i_t}{1+\pi_{t+1}}, \tag{2.44}$$

$$\varphi_t = \eta_t + \frac{\vartheta}{R_t}\varphi_{t+1}, \tag{2.45}$$

$$\sigma_t^2 = \sigma^2\exp(2\phi(Y_t - Y)), \tag{2.46}$$

where in the last equation, following Acharya & Dogra (2020), we assumed that risk is procyclical if $\phi > 0$ and countercyclical if $\phi < 0$. We shall explore this in Section **??** below, when examining the response to aggregate shocks.

2.3.7 Social Welfare Function

We define the social welfare function at time $t = 0$ as:

$$\mathbb{W}_0 = (1-\vartheta)\left(\sum_{s=-\infty}^{0}\vartheta^{-s}\int_0^1 W_0^s(i)\,di + \sum_{s=1}^{\infty}\beta^s\int_0^1 W_s^s(i)\,di\right), \tag{2.47}$$

where the first term represents the utility of generations that are alive at time zero. The currently alive are treated equally after accounting for their relative size. The second term represents the utility of unborn generations, with $s > 0$, and the utility of each such generation is discounted with weight β^s. Appendix 2.9.6 shows that this welfare measure can be written as follows:

$$\mathbb{W}_0 = \sum_{t=0}^{\infty} \beta^t \mathbb{U}_t,$$

where

$$\mathbb{U}_t = -\frac{1}{\gamma}(1+\gamma\rho\eta_t)\, e^{-\gamma x_t} S_t, \tag{2.48}$$

and S_t satisfies the recursion:

$$S_t = \left(\vartheta e^{-\frac{\gamma}{\vartheta}W_t} S_{t-1} + 1 - \vartheta\right) e^{\gamma W_t} e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}. \tag{2.49}$$

Here

$$W_t = \mu_t \left(B_t - \frac{\vartheta \varkappa}{(1-\vartheta)}\varphi_t\right) \tag{2.50}$$

measures the extent to which society has succeeded in financing its retirement. It extends the form of the welfare function considered in Acharya et al. (2023) by accounting for inter-generational inequality as well as the distribution of consumption driven by idiosyncratic shocks. The first part of the social welfare function captures the utility generated by per capita levels of private and public consumption, less the disutility of labor supply. The second element adjusts that measure for the welfare effects of inequality, driven by both idiosyncratic shocks and the distribution of consumption and labor supply across generations due to the endogenous accumulation of assets and age-related withdrawal from the labor market.

To gain intuition for these effects, it is helpful to consider the steady-state of the measure of the social costs of inequality, S_t. In the steady-state, the expression becomes:

$$S = \frac{(1-\vartheta)\, e^{\gamma W} e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}}{(1-\vartheta e^{-\gamma\frac{(1-\vartheta)}{\vartheta}W} e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2})}. \tag{2.51}$$

Taking the partial derivative of this measure of inequality with respect to W yields:

$$\frac{\partial S}{\partial W} = \gamma S(1 - \frac{(1-\vartheta)}{\vartheta}((1-\vartheta e^{-\gamma\frac{(1-\vartheta)}{\vartheta}W} e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2})^{-1} - 1). \tag{2.52}$$

The choice of W that would minimize steady-state inequality, treating $\mu^2\eta^2$ as given, would be:

$$W = \frac{1}{2}\frac{\vartheta}{1-\vartheta}\gamma\mu^2\eta^2\sigma^2. \tag{2.53}$$

This would be insufficient to eliminate inequality but would facilitate a degree of self-insurance by providing assets for households to undertake both precautionary savings and to save for retirement. Recall that interest rates are consistent with the household's rate of time preference when this exact condition holds – equation (2.23). Therefore, this level of debt is also the one which ensures that the steady-state equilibrium real interest rates equals the households' rate of time preference, $R = \beta^{-1}$. However, it is important to note that this will not be consistent with the Ramsey optimum since the Ramsey policy maker will also take account the endogeneity of $\mu^2\eta^2$ and be concerned with efficiency as well as equity. Increasing W implies higher taxes to sustain the higher debt level and, by raising interest rates, affects the marginal propensity to consume which reduces the ability of households to borrow against future income in the face of negative idiosyncratic shocks. Therefore, in steady-state, even a policy maker with only a concern for inequality would not necessarily support this level of debt. We shall see below that the Ramsey policy maker delivers a level of debt which falls short of ensuring $R = \beta^{-1}$, even if their objective were to minimize inequality. They would then wish to reduce debt further if they also have a concern for efficiency. We explore these trade-off in the next section.

We can also consider special cases to gain further insight. If there were no idiosyncratic shocks, but there was potential inter-generational inequality due to age-related retirment, then this would imply steady-state inequality of:

$$S = \frac{(1-\vartheta)\, e^{\gamma W}}{\left(1-\vartheta e^{-\gamma \frac{(1-\vartheta)}{\vartheta} W}\right)}, \tag{2.54}$$

and we could eliminate inequality if $W = 0$ so that $B = \frac{\vartheta \varkappa}{(1-\vartheta)}\varphi$. In other words, by issuing sufficient debt to absorb the desire to save for retirement and ensuring interest rates are the same as the households' rate of time preference, the policy maker can eliminate steady-state inter-generational inequality. Households would save by buying government bonds to ensure they had sufficient assets to maintain consumption even as their income falls due to retirement. This would achieve consumption equality across generations.

If, however, we reintroduce idiosyncratic shocks while the policy maker balanced the steady-state level of debt with the discounted value of the income lost through phased retirement, such that $W = 0$, then the inequality measure would reduce to:

$$S = \frac{(1-\vartheta)\, e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}}{\left(1-\vartheta e^{\frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2}\right)}, \tag{2.55}$$

so that when $\sigma^2 > 0$, then $S > 1$ due to the costs of idiosyncratic shocks considered by Acharya & Dogra (2020). This situation would imply that $R < \beta^{-1}$ as households still have an additional motive to undertake precautionary savings, beyond saving for retirement. In the absence of sufficient assets to fulfill that desire, interest rates will lie below the households' rate of time preference. We shall explore where the Ramsey policy maker chooses to set the steady-state level of debt in the light of these trade-offs in the next section.

2.4 Optimal Policy

The policymaker seeks to maximize

$$\sum_{t=0}^{\infty} \beta^t \left(-\frac{1}{\gamma} (1 + \gamma \rho \eta_t) S_t^{\xi} \exp(-\gamma x_t) \right), \tag{2.56}$$

subject to the system describing the private sector equilibrium (2.36)-(2.46), the recursion of inequality measure (2.49), and the definition (2.50).

In the policy objective (2.56), we can set parameter ξ to either zero or one. When it is one then the policy maker cares about both equity and efficiency consistent with the micro-founded social welfare function derived above. When ξ is zero, then the policymaker is concerned only with efficiency, not equity. We shall also consider another scenario in which the policy maker cares only about equity and aims to minimize

$$\sum_{t=0}^{\infty} \beta^t S_t. \tag{2.57}$$

We assume that the policymakers have access to a commitment technology and present all first order conditions in Appendix 2.9.8.

2.5 Calibration & Simulations

The model is calibrated to a quarterly frequency. The calibration of most parameters is standard and generally follows the one in Acharya et al. (2023). We calibrate the household discount rate $\beta = (1.02)^{-1/4}$ to match the real interest rate of 2% per annum, which is the average in the US over the Great Moderation period (1984-2021). The coefficient of relative risk aversion is set to $\gamma = 2$, based on evidence in Hall (1988), Campbell & Mankiw (1989) and Attanasio & Weber (1993, 1995), while the Frisch elasticity of substitution is set at $\rho = 1/2$ following empirical evidence in e.g., Fagereng et al. (2017), Christelis et al. (2015).

Fiscal parameters are based on data over the same period. Specifically, the coefficient ρ sets the maturity of government debt to be 20 quarters, which is a close match of the 5.4 years observed in the data (Fund 2016). The parameter G generates a spending share $G/Y = 0.15$, see IMF IFS data.[4] The relative weight on government consumption in utility, χ, is set to 0.05, which is a free parameter that ensures that government expenditure does not get fully wasted.

The elasticity of substitution between goods, ε, is set to 11 based on evidence in Chari et al. (2000) and corresponds to an approximate 10% mark up.

Our model features nominal rigidities following Rotemberg (1982). The majority of recent papers in the macro literature that calibrate their frameworks for the US economy choose prices to change every 10 months (see Klenow & Kryvtsov 2008 and Klenow & Malin 2010). As the Rotemberg (1982) and Calvo (1983) models generate isomophic linearized New Keynesian Phillips curves, the equivalent Rotemberg model parameter is $\Phi = 106.4$. Parameter $\bar{\xi}$ is set to 2, which normalizes output to be close to one.

We choose the survival rate to be consistent with an average lifespan of 80 years, see SSA data.[5] The declining labor supply efficiency parameter, $\varkappa$, is chosen to be 0.0011 which is consistent with 20 years of retirement, in line with the US data over the last 50 years (Center for Retirement Research (2024)).

We follow Guvenen et al. (2014), who document the standard deviation of one-year growth rate of log earnings to be about 0.5. This yields $\sigma = 0.33$ for the baseline calibration.

Finally, we calibrate the persistence of deterministic disturbances for productivity and elasticity of substitution to be 0.95 and 0.9 respectively. This again follows Acharya et al. (2023) who adopt the empirical estimates of Bayer et al. (2020). All computations were implemented in the RISE toolbox (Maih 2015).

2.6 Steady State

In general, we need to solve the steady-state of the fully optimal policy maker's problem numerically. However, there are some interesting special cases which can be solved analytically. The first is in the absence of idiosyncratic shocks, $\sigma^2 = 0$ and where there fiscal policy instrument is a lump-sum rather than distortionary tax, then the steady-state of the economy under the policy maker's plan under commitment is described by the following proposition.

Proposition 6 *With access to lump sum taxes as a policy instrument, then in the absence of idiosyncratic shocks, the steady-state is given by,*

$$R = \frac{1}{\beta},\ W = 0,\ S = 1,\ \pi = 0,\ w = \eta = \frac{\varepsilon - 1}{\varepsilon},\ \varphi = \frac{R\eta}{(R - \vartheta)},\ B = \frac{\vartheta \varkappa}{(1 - \vartheta)}\varphi,\ \textit{and}\ P^M = \frac{1}{R - \rho}$$

[4] The relative data series are NGDP_XDC and NCGG_XDC.

[5] See Period Life Table at www.ssa.gov.

Here the policy maker would choose to eliminate inequality by issuing sufficient debt to facilitate households saving for retirement by the right amount to ensure consumption is constant in steady-state, $B = \frac{\vartheta \varkappa}{(1-\vartheta)}\varphi$. As discussed above this will ensure that the steady-state real interest rate is consistent with the households' rate of time preference, $R = \beta^{-1}$. Issuing more (less) debt than that would drive interest rates above (below) households' rate of time preference and result in consumption rising (falling) over an individual household's life, creating undesirable inter-generational inequality.

If we then maintain the assumption that there are no idiosyncratic shocks, $\sigma^2 = 0$, but the available fiscal instrument is a distortionary tax on labor income, then the Ramsey policy maker would only wish to eliminate inter-generational inequality when there is no age-related increase in the disutility of supplying labor, $\varkappa = 0$.

Proposition 7 *When the available fiscal policy instrument is a distortionary tax on labor income, then $R = \frac{1}{\beta}$ only holds in the Ramsey steady-state in the absence of both idiosyncratic shocks, $\sigma^2 = 0$, and retirement, $\varkappa = 0$.*

This special case implies that the policy maker does not issue debt in steady-state ($B = 0$). With phased retirement ($\varkappa > 0$), in order to ensure interest rates are consistent with the households' rate of time preference, the policy maker would need to issue debt which is costly when debt service costs must be financed through distortionary taxation.

2.7 Discussion

We begin our numerical analysis with an exploration of the steady-state of the model and the trade-offs between equity and efficiency faced by the Ramsey policy-maker. In the first column of Table 2.1 we begin with the Ramsey steady-state of our benchmark economy, which features jointly optimal monetary and fiscal policy. The fiscal policy instrument is a distortionary labor income tax. Households are subject to idiosyncratic income shocks and must plan for a gradual withdrawal from the labor market over their lifetimes. The second column repeats this exercise but removes fiscal policy, leaving the Ramsey planner with only monetary policy as a tool to affect the equilibrium. The third column returns to the benchmark economy but investigates the steady-state that would occur if the Ramsey policy did not prioritize addressing inequality. The final column considers the opposite extreme, where the policy maker focuses solely on equity, seeking to minimize $\sum_{t=0}^{\infty} \beta^t S_t$.

The first point to note, when comparing columns (1) and (2), is the ability of fiscal policy to mitigate inequality. Without fiscal policy, individual households' efforts to save for both precautionary reasons and retirement drive down the equilibrium real interest rate below the households' rate of

Table 2.1: Steady State values in HANK Economy under Commitment (time- inconsistent policy).

		Fully Optimal policy ($\xi = 1$)	Monetary model Optimal Policy	Commitment under Efficiency ($\xi = 0$)	Commitment under Equity
		(1)	(2)	(3)	(4)
Net real interest rate, % per annum	R	1.994%	1.956%	1.980%	1.997%
Propensity to consume	μ	0.00507	0.00459	0.00503	0.00507
Per Capita Consumption	c	0.832	0.960	0.834	0.832
Output	Y	0.980	0.960	0.982	0.980
Inflation rate, % pa	π	0%	0%	0%	0%
Tax rate	τ	0.179	–	0.174	0.180
Debt minus Lost retirement Income	W	-0.0013	-0.0133	-0.0058	-0.0004
Debt to output ratio	$\frac{P^M B}{4Y}$	53%	–	31%	57%
Inequality	S	1.00089	1.00147	1.00096	1.00088
Social Welfare	U	-0.16219	-0.13959	-0.16213	-0.16221

time preference, $R < \beta^{-1}$. Since there are no assets available in aggregate for households to hold, the desire to save for these two motives forces equilibrium returns on saving to fall, discouraging saving behavior. With the introduction of fiscal policy, the government issues a significant amount of debt, facilitating household saving and causing interest rates to rise. However, the amount of debt issued is insufficient to fulfill households' desires to save for retirement, even before considering their additional need for precautionary saving. As a result, interest rates do not reach the households' rate of time preference but fall slightly short. This shortfall is partly because the taxes needed to service the debt create an efficiency-reducing distortion in the economy. This implies a trade-off between equity and efficiency for the Ramsey planner. This trade-off is more clearly seen in the final two columns, which contrast outcomes where the Ramsey policy maker focuses solely on efficiency and equity, respectively. In column (3), where the policy maker focuses solely on per capita averages (setting $\xi = 0$) and is indifferent to inequality, debt levels at 31% of GDP are well below the 53% level adopted by the Ramsey policy maker maximizing social welfare. However, even when the policy maker's sole objective is minimizing inequality, as in column (4), they still fall slightly short of issuing enough debt to ensure $R = \beta^{-1}$. In fact, when inequality is the sole policy objective, the government still does not generate enough debt to allow households to fully save for retirement, as $W < 0$ and debt is only 4% of GDP higher than in the Ramsey policy case, at 57% of GDP.

These results are explored further in Figure 2.1. These plot the steady-state value of variables

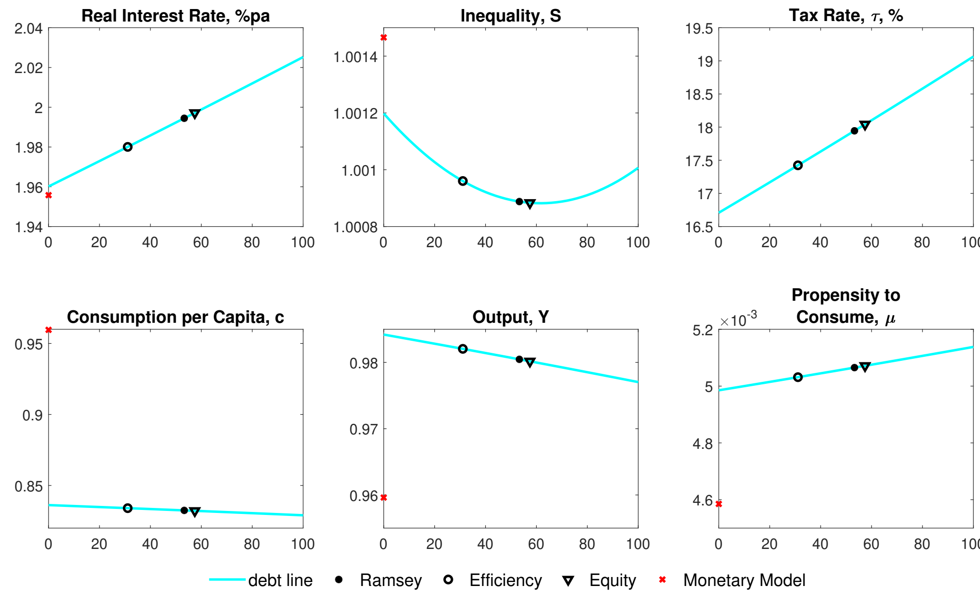

Figure 2.1: Debt-to-GDP Ratio and Steady-State Outcomes

given an equilibrium without inflation, $\pi = 0$, but conditional on a given steady-state debt-to-GDP ratio. More specifically, to construct this figure, we exogenously set the steady-state inflation to zero and create a grid for the annualized debt-to-GDP ratio from 0 to 100. We then solve the model to find its steady state as it now has as many equations as unknowns. This gives us the blue line in the picture. However, we need to draw the reader's attention to the fact that in the three specifications we consider (fully optimal, Efficiency, Equity), the policy maker pursues both optimal monetary and optimal fiscal policy under commitment. The optimal policy specifications will all lie on this line as they deliver zero inflation. In contrast, in the monetary model, there is no fiscal side, and as such, the problem of the fully optimizing policy maker under commitment is fundamentally different. Therefore, it comes as no surprise that, for the monetary model, when the annualized debt-to-GDP ratio is set at zero, the marker for the variables of interest does not lie on the blue line but rather on the Y-axis. Markers are placed on this line, which denote equilibrium outcomes under the Fully Optimal policy under commitment (Ramsey)(star), optimal policy under commitment which is only concerned with per capita averages (hollow circle), optimal policy under commitment which is only concerned with inequality (inverted triangle), and the steady-state of the model without any fiscal policy (cross). The relative position of the markers implies that the trade-off between efficiency and equity is resolved firmly towards equity. Given the decline in labor income as households age, and the subsequent desire to save in anticipation of this 'retirement', the policy maker who is concerned with inequality wishes to issue debt to facilitate saving for retirement and prevent the significant inter-generational inequality that would emerge if there were insufficient assets to smooth consumption over the life-cycle. This leads to steady-state debt levels of 53% of GDP under Ramsey policy, which falls only slightly short of the 57% debt ratio which would occur if the policy maker was solely concerned with inequality. It is interesting to note that even the debt level associated with a desire to minimize inequality is less than needed to drive interest rates to 2% and supply enough assets to support households' desire to save for both retirement and precautionary reasons. The policy maker issues slightly lower debt than this benchmark level even when they only care about inequality, since lower interest rates help households smooth consumption in the face of idiosyncratic shocks. In contrast to the case where minimizing inequality is the primary policy objective, a policy maker concerned with efficiency alone would wish to limit debt issuance to 31% of GDP in order to lower the output losses due to distortionary taxation. The policy maker does not go beyond this by lowering debt further, as the fiscal consolidation that would be implied by further debt reduction is more costly than the steady-state gain from reduced debt-service costs.

As discussed in Propositions 1-3, the desirability of achieving the golden rule interest rate of $R = \beta^{-1}$ depends on the existence of idiosyncratic shocks, the availability of lump-sum taxation, and the need to save for retirement. Recall that the inclusion of phased retirement augments the household desire to save. Now, in addition to the precautionary savings motive where households save to (partially) mitigate the effects of an adverse idiosyncratic shock, they also save to smooth consumption over their life- cycle. Given our parametrisation ($\vartheta = 0.9961$, $\varkappa = 0.0011$), an average household lives for about 80 years and spends approximately 20 of them in retirement. As such, households need to rely on their financial income for one fourth of their lives. As such, the assumption gradual withdrawal from the labour market, creates a sufficiently strong savings motive, enough to generate plausible levels of the aggregate supply of government bonds. Table 2.2 explores these factors further. The first two columns consider the case where the policy maker has access to lump-sum taxation, with and without labor force participation declining with age. The final two columns do the same, but in these, the fiscal instrument is a distortionary tax rate. The figures in brackets are for the same economy, but without idiosyncratic shocks, $\sigma^2 = 0$. We can see that without idiosyncratic risk, the policy maker issues sufficient debt to ensure $R = \beta^{-1}$ and eliminate inequality across the first three columns, but does not do so when saving for retirement becomes relevant. The reason is that retirement requires a sizeable issue of debt to avoid household saving driving down the equilibrium interest rate, but that debt must be serviced through increases in distortionary taxation which are costly in terms of efficiency.

When we consider the same variants, but with idiosyncratic risk, the policy maker always fails to drive interest rates to $R = \beta^{-1}$. In fact, they never issue sufficient debt to allow households to maintain consumption in retirement, even when taxes are lump-sum. The reason is there is a need to suppress interest rates below the households' rate of time preference to facilitate individual households' response to idiosyncratic shocks. When taxes are distortionary, the costs of issuing debt are higher and this inhibits the policy maker further in the sense that interest rates fall further below households' rate of time preference.

Table 2.2: Steady State Values in HANK Economy. Corresponding RANK values are in parentheses.

		Lump Sum Tax		Income Tax	
		$\varkappa = 0$	$\varkappa = 0.0011$	$\varkappa = 0$	$\varkappa = 0.0011$
		(1)	(2)	(3)	(4)
Net real interest rate, % p.a.	R	1.997% (2%)	1.997% (2%)	1.997% (2%)	1.994% (1.997%)
Propensity to consume	μ	0.00464 (0.00464)	0.00464 (0.00464)	0.00503 (0.00503)	0.00506 (0.00507)
Consumption per capita	c	0.8960 (0.8960)	0.8819 (0.8819)	0.848580 (0.8510)	0.8325 (0.8323)
Output	Y	1.0440 (1.0440)	1.0299 (1.0299)	0.9991 (0.9990)	0.9505 (0.9803)
Inflation rate, % p.a.	π	0% (0%)	0% (0%)	0% (0%)	0% (0%)
Tax rate	τ	– (–)	– (–)	0.164 (0.165)	0.179 (0.180)
Lump Sum Taxes	T	0.147 (0.148)	0.162 (0.162)	– (–)	– (–)
Debt minus Lost retirement Income	W	−0.0005 (0.0)	−0.0005 (0.0)	−0.0003 (0.0)	−0.0013 (−0.0010)
Debt to output ratio,	$\frac{P^M B}{4Y}$	−2.8% (0%)	66.5% (69.3%)	−1.7% (0%)	53.3% (54.9%)
Inequality	S	1.0011 (1.0000)	1.0011 (1.0000)	1.0009 (1.0000)	1.0009 (1.000002)
Social Welfare	U	−0.1562 (−0.1560)	−0.1606 (−0.1604)	−0.1575 (−0.1574)	−0.1622 (−0.1621)

2.8 Conclusions

We extended the overlapping-generation heterogeneous-agent model from Acharya et al. (2023), by developing the fiscal side. Specifically, we introduced long-term government debt, financed by distortionary labor income taxes, which evolves over time in accordance with optimal policy. By further assuming that the disutility of labor supply increases with age, we mimic a desire to save for retirement, which augments the motive for precautionary savings. This leads, in equilibrium, to the government optimally issuing plausible levels of government debt to facilitate such saving behaviour. However, this debt issuance is insufficient to reach the 'golden rule' level of debt, which would equate interest rates with households' rate of time preference. Instead, the Ramsey policy maker issues less debt than this level, aiming to suppress interest rates to facilitate households' ability to smooth consumption in the face of idiosyncratic shocks. The extent to which debt falls short of the golden rule benchmark depends on whether taxation is lump-sum or distortionary, the magnitude of the desire to save for retirement, and variance of idiosyncratic shocks. Without idiosyncratic shocks, we would achieve the golden rule level of debt if taxes were lump sum or if there was no desire to save for for retirement when taxation is distortionary. With idiosyncratic shocks, we would never issue this much debt, and households would not be able to fully maintain consumption in retirement. In this sense, the Ramsey policy maker faces a trade-off between inter-generational inequality due to retirement and within-generation inequality due to idiosyncratic shocks. Exploring the trade-off between both forms of equity and efficiency in maximizing social welfare, our numerical results suggest that this is resolved substantially in favor of equity, with debt levels significantly above those a policy maker concerned only with efficiency would generate.

Introducing fiscal policy to the heterogeneous agent model also highlights the role fiscal policy plays in mitigating inequality. As stressed above, debt plays an important role in facilitating precautionary and retirement saving. The distortionary labor taxes used to finance that debt also serve to mitigate inequality by reducing the impact of idiosyncratic shocks on net income, although high future tax rates will reduce the future post-tax income that households can borrow against to smooth consumption in the face of negative idiosyncratic shocks. Examining the response to technology shocks, we find that a policy maker who only cares about efficiency would relax both monetary and fiscal policies in response to a positive aggregate technology shock – a version of the divine coincidence. However, as soon as the policy maker cares about inequality, the divine coincidence is broken. In a monetary-only model, monetary policy shifts away from inflation stabilization towards output stabilization and policy is more relaxed than it would be in the RANK economy in the face of an identical positive technology shock, to the extent that inflation rises. This result is discussed in Acharya et al. (2023). In contrast, when we introduce fiscal policy, the break from the divine coincidence operates in the opposite direction. The movement in tax rates designed to facilitate consumption smoothing are applied alongside a monetary policy that implies inflation falls. These fiscal movements are more pronounced the greater the desire to mitigate inequality and the longer maturity of the issued debt stock.

2.9 Appendix

2.9.1 Proof of Proposition 1

Proof. We form the following Lagrangian

$$
\begin{aligned}
L_s = \mathbb{E}_i \sum_{t=s}^{\infty} (\beta\vartheta)^{t-s} \Bigg(& -\frac{1}{\gamma} e^{-\gamma(c_t^s(i)+\chi G_t)} - \rho e^{\frac{1}{\rho}(l_t^s(i)+\Theta_t^s-\xi_t^s(i))} \\
& + \lambda_t^s(i) \Big(\Big(c_t^s(i) - \eta_t l_t^s(i) - d_t + T_t + \tilde{P}_t^M a_{t+1}^{L,s}(i) + \tilde{q}_t a_{t+1}^{S,s}(i) \Big)(1+\pi_t) \\
& - \left(1+\rho\tilde{P}_t^M\right) a_t^{L,s}(i) - a_t^{S,s}(i) \Big) \Bigg)
\end{aligned}
$$

so the FOCs are

$$
\begin{aligned}
0 &= e^{-\gamma(c_t^s(i)+\chi G_t)} + \lambda_t^s(i)(1+\pi_t) \\
0 &= -e^{\frac{1}{\rho}(l_t^s(i)+\Theta_t^s-\xi_t^s(i))} - \lambda_t^s(i)\eta_t(1+\pi_t) \\
0 &= \lambda_t^s(i)\tilde{P}_t^M(1+\pi_t) - \mathbb{E}_i\left(1+\rho\tilde{P}_{t+1}^M\right)\beta\vartheta\lambda_{t+1}^s(i) \\
0 &= \lambda_t^s(i)\tilde{q}_t(1+\pi_t) - \beta\vartheta\mathbb{E}_i\lambda_{t+1}^s(i)
\end{aligned}
$$

from where (there is no aggregate risk)

$$
\begin{aligned}
\lambda_t^s(i) &= -\frac{1}{(1+\pi_t)} e^{-\gamma(c_t^s(i)+\chi G_t)} \\
l_t^s(i) &= \rho\log\eta_t - \gamma\rho\left(c_t^s(i)+\chi G_t\right) - \Theta_t^s + \xi_t^s(i) \\
c_t^s(i) &= -\frac{1}{\gamma}\log\frac{\beta\vartheta}{\tilde{q}_t(1+\pi_{t+1})} + \chi G_{t+1} - \chi G_t - \frac{1}{\gamma}\log\mathbb{E}_i e^{-\gamma c_{t+1}^s(i)} \\
\frac{1}{\tilde{q}_t} &= \frac{\left(1+\rho\tilde{P}_{t+1}^M\right)}{\tilde{P}_t^M}
\end{aligned}
$$

The Euler equation, using normality of consumption distribution, can also be written as

$$
c_t^s(i) = -\frac{1}{\gamma}\log\left(\frac{\beta\vartheta}{\tilde{q}_t(1+\pi_{t+1})}\right) + \chi G_{t+1} - \chi G_t + \mathbb{E}_i c_{t+1}^s(i) - \frac{\gamma}{2}\mathbb{V}_i c_{t+1}^s(i). \tag{2.58}
$$

To obtain expressions for expectation and variance of consumption, we do the following three steps.

First, substitute labour supply into the budget constraint:

$$
A_{t+1}^s(i) = \frac{R_t}{\vartheta}\left(A_t^s(i) + X_t - \eta_t\Theta_t^s + \eta_t\left(\xi_t^s(i) - \bar{\xi}\right) - (1+\rho\gamma\eta_t)c_t^s(i)\right) \tag{2.59}
$$

where we denoted

$$X_t = \eta_t \left(\rho \log \eta_t + \bar{\xi} - \rho \gamma \chi G_t\right) + d_t - T_t.$$

Second, assume that individual consumption can be parameterized as

$$c_t^s(i) = \mathscr{X}_t + \mu_t \left(A_t^s(i) + \eta_t \left(\xi_t^s(i) - \bar{\xi}\right) - \varphi_t \Theta_t^s\right) \tag{2.60}$$

Lead it one period:

$$\begin{aligned} c_{t+1}^s(i) &= \mathscr{X}_{t+1} + \mu_{t+1} \left(A_{t+1}^s(i) + \eta_{t+1} \left(\xi_{t+1}^s(i) - \bar{\xi}\right) - \varphi_{t+1} \Theta_{t+1}^s\right) \\ &= \mu_{t+1} \left(\frac{R_t}{\vartheta} \left(\begin{array}{c} (1 - (1 + \rho\gamma\eta_t)\mu_t)\left(A_t^s(i) + \eta_t\left(\xi_t^s(i) - \bar{\xi}\right)\right) + X_t \\ -(1 + \rho\gamma\eta_t)\mathscr{X}_t + (-\eta_t + (1 + \rho\gamma\eta_t)\mu_t\varphi_t)\Theta_t^s \end{array} \right) \right) \\ &\quad + \mathscr{X}_{t+1} + \mu_{t+1}\eta_{t+1}\left(\xi_{t+1}^s(i) - \bar{\xi}\right) - \mu_{t+1}\varphi_{t+1}\left(\Theta_t^s + \varkappa\right) \end{aligned} \tag{2.61}$$

where in the second line we used the budget constraint, parameterisation (2.60) and the fact that $\Theta_{t+1}^s = \varkappa(t+1-s) = \varkappa(t-s) + \varkappa = \Theta_t^s + \varkappa$.

Finally, we obtain expressions for expectation and variance terms. Because $c_{t+1}^s(i)$ is normally distributed by i, its mean and variance are determined as follows:

$$\begin{aligned} \mathbb{E}_i c_{t+1}^s(i) &= \mathscr{X}_{t+1} + \mu_{t+1} \left(\begin{array}{c} \frac{R_t}{\vartheta}(1 - (1 + \rho\gamma\eta_t)\mu_t)\left(A_t^s(i) + \eta_t\left(\xi_t^s(i) - \bar{\xi}\right)\right) \\ +\frac{R_t}{\vartheta}(X_t - (1 + \rho\gamma\eta_t)\mathscr{X}_t) \\ +\frac{R_t}{\vartheta}(-\eta_t + (1 + \rho\gamma\eta_t)\mu_t\varphi_t)\Theta_t^s \end{array} \right) \\ &\quad - \mu_{t+1}\varphi_{t+1}\left(\Theta_t^s + \varkappa\right) \\ \mathbb{V}_i c_{t+1}^s(i) &= \mu_{t+1}^2 \eta_{t+1}^2 \sigma_{t+1}^2 \end{aligned}$$

(note that $\mathbb{E}_i \xi_{t+1}^s(i) = \bar{\xi}$, but $\mathbb{E}_i \xi_t^s(i) = \xi_t^s(i)$, $\mathbb{V}_i \xi_{t+1}^s(i) = \sigma_{t+1}^2$, but $\mathbb{V}_i \xi_t^s(i) = 0$).

We now use these expressions and parameterisation (2.60) and substitute them into the consumption Euler equation (2.58) to find coefficients $\mathscr{X}_t, \mu_t$ and φ_t.

Substitution into the Euler equation yields:

$$\begin{aligned} &\mathscr{X}_t + \mu_t \left(A_t^s(i) + \eta_t^s \left(\xi_t^s(i) - \bar{\xi}\right) - \varphi_t \Theta_t^s\right) \\ &= -\frac{1}{\gamma} \log(\beta R_t) + \chi G_{t+1} - \chi G_t - \mu_{t+1}\varphi_{t+1}\left(\Theta_t^s + \varkappa\right) - \frac{\gamma}{2}\mu_{t+1}^2 \eta_{t+1}^2 \sigma_{t+1}^2 \\ &+ \mathscr{X}_{t+1} + \mu_{t+1} \frac{R_t}{\vartheta} \left(\begin{array}{c} (1 - (1 + \rho\gamma\eta_t^s)\mu_t)\left(A_t^s(i) + \eta_t^s\left(\xi_t^s(i) - \bar{\xi}\right)\right) \\ +(X_t - (1 + \rho\gamma\eta_t)\mathscr{X}_t) + (-\eta_t + (1 + \rho\gamma\eta_t)\mu_t\varphi_t)\Theta_t^s \end{array} \right). \end{aligned}$$

Collect coefficients on independent states: $1, A_t^s(i), \xi_t^s(i), \Theta_t^s$. This yields three independent equations on μ_t, κ_t and $\mathscr{X}_t$:

$$\mathscr{X}_t - \mu_t \eta_t \bar{\xi} = -\frac{1}{\gamma} \log(\beta R_t) + \chi \tilde{G}_{t+1} - \chi \tilde{G}_t + \mathscr{X}_{t+1} - \frac{\gamma}{2} \mu_{t+1}^2 \eta_{t+1}^2 \sigma_{t+1}^2 \\ + \mu_{t+1} \left(\frac{R_t}{\vartheta} \left(X_t - (1 + \rho \gamma \eta_t) \mathscr{X}_t \right) - \frac{R_t}{\vartheta} \left(1 - (1 + \rho \gamma \eta_t) \mu_t \right) \eta_t \bar{\xi} \right) \\ - \mu_{t+1} \varphi_{t+1} \varkappa \tag{2.62}$$

$$\mu_t = \mu_{t+1} \frac{R_t}{\vartheta} \left(1 - (1 + \rho \gamma \eta_t) \mu_t \right) \tag{2.63}$$

$$-\mu_t \varphi_t = \mu_{t+1} \left(\frac{R_t}{\vartheta} \left(-\eta_t + (1 + \rho \gamma \eta_t) \mu_t \varphi_t \right) \right) - \mu_{t+1} \varphi_{t+1} \tag{2.64}$$

Provided that $\mu_t \neq 0$ The dynamic equation on evolution of the marginal propensity to consume ot of cash in hands can be expressed as:

$$\frac{1}{\mu_t} - (1 + \rho \gamma \eta_t) = \frac{\vartheta}{R_t \mu_{t+1}} \tag{2.65}$$

the equation for φ_t becomes*[name it!]*

$$\varphi_t = \eta_t + \frac{\vartheta}{R_t} \varphi_{t+1} \tag{2.66}$$

and the evolution of the measure of aggregate consumption $\mathscr{X}_t$ is:

$$\mathscr{X}_t = -\frac{\vartheta \mu_t}{\gamma \mu_{t+1} R_t} \log(\beta R_t) + \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \mathscr{X}_{t+1} + \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \chi G_{t+1} \\ - \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \chi G_t + \mu_t X_t - \frac{\vartheta \mu_t}{R_t} \varkappa \varphi_{t+1} - \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \frac{\gamma}{2} \mu_{t+1}^2 \eta_{t+1}^2 \sigma_{t+1}^2$$

Introduce new variable:

$$\mathscr{C}_t = \mathscr{X}_t + \chi G_t$$

then we arrive to

$$\mathscr{C}_t = -\frac{\vartheta \mu_t}{\gamma \mu_{t+1} R_t} \log(\beta R_t) + \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \mathscr{C}_{t+1} - \frac{\vartheta \mu_t}{R_t} \varkappa \varphi_{t+1} - \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \frac{\gamma}{2} \mu_{t+1}^2 \eta_{t+1}^2 \sigma_{t+1}^2 \\ + \mu_t \left(\eta_t \left(\rho \log(\eta_t) + \bar{\xi} \right) + d_t - T_t + \chi G_t \right)$$

after all terms with G_t are combined. ■

2.9.2 Aggregation

Define aggregate consumption, income and labour

$$c_t := (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 c_t^s(i)\,di$$
$$y_t := (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 y_t^s(i)\,di$$
$$n_t := \int_0^1 n_t(j)\,dj = (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 l_t^s(i)\,di$$

Define aggregate actuarial bonds, $J=\{S,L\}$:

$$\vartheta a_t^J := (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 a_t^{J,s}(i)\,di.$$

To aggregate the household budget constraint, we need to compute $(1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 a_{t+1}^{J,s}(i)\,di$. Note that

$$\begin{aligned}\vartheta a_{t+1}^J &= (1-\vartheta)\sum_{s=-\infty}^{t+1}\vartheta^{t+1-s}\int_0^1 a_{t+1}^{J,s}(i)\,di = (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t+1-s}\int_0^1 a_{t+1}^{J,s}(i)\,di\\ &+(1-\vartheta)\int_0^1 a_{t+1}^{J,t+1}(i)\,di\\ &= \vartheta(1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 a_{t+1}^{J,s}(i)\,di\end{aligned}$$

then

$$a_{t+1}^J = (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 a_{t+1}^{J,s}(i)\,di$$

It follows

$$\vartheta A_t = (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 A_t^s(i)\,di,$$
$$A_{t+1}^s = (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 A_{t+1}^s(i)\,di,$$

where

Finally, note that

$$\sum_{s=-\infty}^{t} \vartheta^{t-s}(t-s) = \vartheta^{t-t}(t-t) + \vartheta^{t-t+1}(t-t+1) + \vartheta^{t-t+2}(t-2) = \dots$$
$$= \sum_{k=1}^{\infty} k\vartheta^{k} = \frac{\vartheta}{(1-\vartheta)^2}$$

so that agggregation of sick days yields

$$\begin{aligned}(1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 \Theta_t^s di &= (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 \varkappa(t-s)\,di \\ &= \varkappa(1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}(t-s) \\ &= \frac{\varkappa\vartheta}{1-\vartheta}\end{aligned}$$

Aggregation of household budget constrain (2.4) yields:

$$\frac{\vartheta}{R_t}\frac{\left(\frac{\tilde{P}_t^M}{\tilde{q}_t}a_{t+1}^L + a_{t+1}^S\right)}{(1+\pi_{t+1})} = \vartheta\frac{\left(\left(1+\rho\tilde{P}_t^M\right)a_t^L + a_t^S\right)}{(1+\pi_t)} + y_t - c_t \tag{2.67}$$

or

$$\frac{\vartheta}{R_t}A_{t+1} = \vartheta A_t + y_t - c_t$$

where

$$A_t = \frac{\left(\left(1+\rho\tilde{P}_t^M\right)a_t^L + a_t^S\right)}{(1+\pi_t)} = \frac{\left(\frac{\tilde{P}_{t-1}^M}{\tilde{q}_{t-1}}a_t^L + a_t^S\right)}{(1+\pi_t)}$$

and

$$y_t = \eta_t n_t + d_t - T_t.$$

2.9.3 Proof of Proposition 2.

Proof. We start with the derived relationship:

$$\begin{aligned}\mathscr{X}_t = &-\frac{\vartheta\mu_t}{\gamma\mu_{t+1}R_t}\log(\beta R_t) + \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\mathscr{X}_{t+1} + \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\chi G_{t+1} \\ &- \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\chi G_t + \mu_t X_t - \frac{\vartheta\mu_t}{R_t}\varkappa\varphi_{t+1} - \frac{\vartheta\mu_t}{\mu_{t+1}R_t}\frac{\gamma}{2}\mu_{t+1}^2\eta_{t+1}^2\sigma_{t+1}^2\end{aligned}$$

Recall that

$$\mathscr{X}_t = \mathscr{C}_t - \chi G_t$$

and

$$c_t = \mathscr{C}_t - \chi G_t + \mu_t \left(\vartheta A_t - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t \right)$$

So we can parameterize

$$\mathscr{X}_t = \mathscr{C}_t - \chi G_t = c_t - \mu_t \left(\vartheta A_t - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t \right)$$
$$\mathscr{X}_{t+1} = c_{t+1} - \mu_{t+1} \left(\vartheta A_{t+1} - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_{t+1} \right)$$

and substitute these two relationships

$$\begin{aligned}
& c_t - \mu_t \left(\vartheta A_t - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_t \right) \\
& = -\frac{\vartheta \mu_t}{\gamma \mu_{t+1} R_t} \log(\beta R_t) + \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \left(c_{t+1} - \mu_{t+1} \left(\vartheta A_{t+1} - \frac{\vartheta \varkappa}{(1-\vartheta)} \varphi_{t+1} \right) \right) \\
& + \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \chi G_{t+1} - \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \chi G_t + \mu_t X_t - \frac{\vartheta \mu_t}{R_t} \varkappa \varphi_{t+1} - \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \frac{\gamma}{2} \mu_{t+1}^2 \eta_{t+1}^2 \sigma_{t+1}^2
\end{aligned}$$

Substitute

$$X_t = \eta_t \left(\rho \log(\eta_t) + \bar{\xi} - \rho \gamma \chi G_t \right) + d_t - T_t$$

and use budget constraint

$$\vartheta A_t = \frac{\vartheta}{R_t} A_{t+1} - \eta_t \left(\rho \log(\eta_t) + \bar{\xi} \right) + \frac{\varkappa \vartheta}{(1-\vartheta)} \eta_t + \rho \gamma \eta_t \chi G_t + \rho \gamma \eta_t c_t - d_t + T_t + c_t$$

and (2.9)-(2.10) to arrive to the following Euler equation

$$c_t + \chi G_t = -\frac{1}{\gamma} \log(\beta R_t) + c_{t+1} + \chi G_{t+1} + (1-\vartheta) \mu_{t+1} A_{t+1} - \frac{\gamma}{2} \mu_{t+1}^2 \eta_{t+1}^2 \sigma_{t+1}^2 - \mu_{t+1} \varkappa \varphi_{t+1}$$

Labour supply (2.8) is is straightforwardly aggregated to

$$n_t = \rho \log(\eta_t) - \varkappa \frac{\vartheta}{1-\vartheta} - \rho \gamma (c_t + \chi G_t) + \bar{\xi}$$

■

2.9.4 Derivation of Phillips Curve

Firm j solves the following optimization problem

$$\max_{P_t(j)} \sum_{t=0}^{\infty} \beta^t \left(\left(\frac{P_t(j)}{P_t} Y_t(j) - (1-s) w_t n_t(j) \right) - \frac{\Phi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \right)$$

subject to monopolistic demand

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon_t} Y_t$$

and production function

$$Y_t(j) = z_t n_t(j)$$

Substitute

$$\max_{P_t(j)} \sum_{t=0}^{\infty} \beta^t \left(\left(\frac{P_t(j)}{P_t} - (1-s) \frac{w_t}{z_t} \right) \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon_t} Y_t - \frac{\Phi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \right)$$

to yield the following first order condition:

$$0 = \beta^t \left((1-\varepsilon_t) \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon_t} \frac{Y_t}{P_t} + \varepsilon_t (1-s) \frac{w_t}{z_t} \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon_t - 1} \frac{Y_t}{P_t} - \Phi \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right) \frac{Y_t}{P_{t-1}(j)} \right)$$
$$+ \beta^{t+1} \left(\Phi \left(\frac{P_{t+1}(j)}{P_t(j)} - 1 \right) Y_{t+1} \frac{P_{t+1}(j)}{P_t^2(j)} \right)$$

All firms are identical so $P_t(j) = P_t$ and, therefore:

$$\pi_t (1+\pi_t) = \frac{1 - \varepsilon_t + (1-s) \varepsilon_t \frac{w_t}{z_t}}{\Phi} + \beta \frac{Y_{t+1}}{Y_t} \pi_{t+1} (1+\pi_{t+1})$$

The profit of firms, distributed as dividends

$$d_t = (Y_t - (1-s) w_t n_t) - \frac{\Phi}{2} \pi_t^2 Y_t$$

2.9.5 Financial Intermediaries

Financial intermediaries trade actuarial and government bonds.

At time t they buy short and long-term actuarial bonds and pay with short and long term government bonds, so the budget constraint of intermediaries is

$$-\tilde{P}_t^M a_{t+1}^L - \tilde{q}_t a_{t+1}^S + P_t^M b_{t+1}^L + q_t b_{t+1}^S \leqslant 0, \tag{2.68}$$

where $a^J_{t+1} = (1-\vartheta)\sum_{s=-\infty}^{t} \vartheta^{t-s} \int_0^1 a^{J,s}_{t+1}(i)\,di$.

Their profit one period later is, therefore

$$\Pi = \left(1+\rho P^M_{t+1}\right) b^L_{t+1} + b^S_{t+1} - \left(1+\rho \tilde{P}^M_{t+1}\right) \vartheta a^L_{t+1} - \vartheta a^S_{t+1}$$

where b^J_{t+1} are total government bonds at time $t+1$, and ϑa^J_{t+1} are total actuarial bonds at time $t+1$, i.e. $\vartheta a^J_{t+1} = (1-\vartheta)\sum_{s=-\infty}^{t+1} \vartheta^{t+1-s} \int_0^1 a^{J,s}_{t+1}(i)\,di$.

The Lagrangian is

$$\begin{aligned}\Pi &= \left(1+\rho P^M_{t+1}\right) b^L_{t+1} + b^S_{t+1} - \left(1+\rho \tilde{P}^M_{t+1}\right) \vartheta a^L_{t+1} - \vartheta a^S_{t+1} \\ &\quad + \lambda_t \left(-\tilde{P}^M_t a^L_{t+1} - \tilde{q}_t a^S_{t+1} + P^M_t b^L_{t+1} + q_t b^S_{t+1}\right)\end{aligned}$$

and the first order conditions are:

$$\begin{aligned}&\frac{\partial}{\partial b^L_{t+1}} : \left(1+\rho P^M_{t+1}\right) + \lambda_t P^M_t \\ &\frac{\partial}{\partial b^S_{t+1}} : 1 + \lambda_t q_t \\ &\frac{\partial}{\partial a^L_{t+1}} : -\left(1+\rho \tilde{P}^M_{t+1}\right)\vartheta - \lambda_t \tilde{P}^M_t \\ &\frac{\partial}{\partial a^S_{t+1}} : -\vartheta - \lambda_t \tilde{q}_t\end{aligned}$$

From where we have:

$$\frac{1}{\tilde{q}_t} = \frac{\left(1+\rho \tilde{P}^M_{t+1}\right)}{\tilde{P}^M_t} \tag{2.69}$$

$$\tilde{q}_t = \vartheta q_t \tag{2.70}$$

$$\frac{1}{q_t} = \frac{\left(1+\rho P^M_{t+1}\right)}{P^M_t} \tag{2.71}$$

and so the profit is zero:

$$\begin{aligned}\Pi &= \left(1+\rho P^M_{t+1}\right) b^L_{t+1} + b^S_{t+1} - \vartheta a^S_{t+1} - \vartheta\left(1+\rho \tilde{P}^M_{t+1}\right) a^L_{t+1} \\ &= \frac{1}{q_t}\left(P^M_t b^L_{t+1} + q_t b^S_{t+1} - \tilde{q}_t a^S_{t+1} - \tilde{P}^M_t a^L_{t+1}\right) = 0\end{aligned} \tag{2.72}$$

and

$$R_t = \frac{\vartheta}{\tilde{q}_t \left(1+\pi_{t+1}\right)} = \frac{1}{q_t \left(1+\pi_{t+1}\right)} \tag{2.73}$$

2.9.6 Social Welfare Function

Aggregation of Welfare

Recall that

$$\begin{aligned}
l_t^s(i) &= \rho \log \eta_t - \Theta_t^s - \rho\gamma\left(c_t^s(i) + \chi G_t\right) + \xi_t^s(i) \\
c_t^s(i) &= \mathscr{C}_t - \chi G_t + \mu_t m_t^s(i) \\
m_t^s(i) &= A_t^s(i) + \eta_t \left(\xi_t^s(i) - \bar{\xi}\right) - \varphi_t \Theta_t^s
\end{aligned}$$

so the (remaining at p) life-time utility of an agent born at s at time $p > s$ can be written as (substitute labour supply)

$$W_p^s(i) = \sum_{t=p}^{\infty} (\beta\vartheta)^{t-p} U_t^s(i) \tag{2.74}$$

where

$$\begin{aligned}
U_t^s(i) &= -\frac{1}{\gamma} e^{-\gamma(c_t^s(i)+\chi G_t)} - \rho e^{\frac{1}{\rho}\left(l_t^s(i)+\Theta_t^s-\xi_t^s(i)\right)} \\
&= -\frac{1}{\gamma} e^{-\gamma(c_t^s(i)+\chi G_t)} - \rho e^{\frac{1}{\rho}\left(\rho\log(\eta_t)-\rho\gamma(c_t^s(i)+\chi G_t)\right)} \\
&= -\frac{1}{\gamma} e^{-\gamma(c_t^s(i)+\chi G_t)} - \rho\eta_t e^{-\gamma(c_t^s(i)+\chi G_t)} \\
&= -\frac{1}{\gamma}\left(1+\gamma\rho\eta_t\right) e^{-\gamma(c_t^s(i)+\chi G_t)} \\
&= -\frac{1}{\gamma}\left(1+\gamma\rho\eta_t\right) e^{-\gamma(\mathscr{C}_t+\mu_t m_t^s(i))}
\end{aligned}$$

The social welfare function at time $t = 0$ is defined as

$$\mathbb{W}_0 = (1-\vartheta) \sum_{s=-\infty}^{0} \vartheta^{-s} \int_0^1 W_0^s(i)\, di + \sum_{s=1}^{\infty} (1-\vartheta)\, \beta^s \int_0^1 W_s^s(i)\, di \tag{2.75}$$

where the first term is utility of generations that are alive at time zero. The second term is utility of unborn generations, with $s > 0$, each such generation is treated with weight β^s.

We can rewrite the welfare function in a more convenient way. Denote

$$\mathscr{U}_t^s = -\frac{1}{\gamma}\left(1+\gamma\rho\eta_t\right) \int_0^1 e^{-\gamma(\mathscr{C}_t+\mu_t m_t^s(i))} di$$

is t-period utility of a *cohort* born at time s.

Then

$$\begin{aligned}\frac{\mathbb{W}_0}{(1-\vartheta)} &= \mathscr{U}_0^0 + \vartheta \mathscr{U}_0^{-1} + \vartheta^2 \mathscr{U}_0^{-2} + \dots \\ &+ \beta \left(\mathscr{U}_0^1 + \vartheta \mathscr{U}_0^0 + \vartheta^2 \mathscr{U}_0^{-1} + \dots \right) + \dots \\ &+ \beta^t \left(\mathscr{U}_t^t + \vartheta \mathscr{U}_t^{t-1} + \vartheta^2 \mathscr{U}_t^{t-2} + \dots + \vartheta^s \mathscr{U}_t^{t-s} \right) + \dots \\ &= \sum_{t=0}^{\infty} \beta^t \sum_{s=0}^{\infty} \vartheta^s \mathscr{U}_t^{t-s} = \sum_{t=0}^{\infty} \beta^t \sum_{v=-\infty}^{t} \vartheta^{t-v} \mathscr{U}_t^v\end{aligned}$$

where in the last line we used new index $v = t - s$.

Recycling notation, we get

$$\mathbb{W}_0 = -\frac{1}{\gamma} \sum_{t=0}^{\infty} \beta^t (1 + \gamma \rho \eta_t) e^{-\gamma \mathscr{C}_t} \left((1-\vartheta) \sum_{s=-\infty}^{t} \vartheta^{t-s} \int_0^1 e^{-\gamma \mu_t m_t^s(i)} di \right) \tag{2.76}$$

$$\mathbb{W}_0 = -\frac{1}{\gamma} \sum_{t=0}^{\infty} \beta^t (1 + \gamma \rho \eta_t) e^{-\gamma \mathscr{C}_t} \left((1-\vartheta) \sum_{s=-\infty}^{t} \vartheta^{t-s} \int_0^1 e^{-\gamma \mu_t m_t^s(i)} di \right) \tag{2.77}$$

Denote

$$\Sigma_t = (1-\vartheta) \sum_{s=-\infty}^{t} \vartheta^{t-s} \int_0^1 e^{-\gamma \mu_t m_t^s(i)} di$$

so that

$$\mathbb{W}_0 = \sum_{t=0}^{\infty} \beta^t \mathbb{U}_t$$

where

$$\mathbb{U}_t = -\frac{1}{\gamma} (1 + \gamma \rho \eta_t) e^{-\gamma \mathscr{C}_t} \Sigma_t$$

Here $(1 + \gamma \rho \eta_t) e^{-\gamma \mathscr{C}_t}$ only depends on aggregate variables, so will be the same for a representative agent.

$\Sigma_t = (1-\vartheta) \sum_{s=-\infty}^{t} \vartheta^{t-s} \int_0^1 e^{-\gamma \mu_t m_t^s(i)} di$ is a welfare cost of inequality. It is increasing in the within cohort disperstion of consumption. If there is risk then Σ_t is increasing, and this decreases the overall level of welfare.

2.9.7 Recursion

Derive Σ_t recursion.

$$
\begin{aligned}
\Sigma_t &= (1-\vartheta)\sum_{s=-\infty}^{t}\vartheta^{t-s}\int_0^1 e^{-\gamma\mu_t m_t^s(i)}di \\
&= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}\int_0^1 e^{-\gamma\mu_t m_t^s(i)}di + (1-\vartheta)\int_0^1 e^{-\gamma\mu_t m_t^t(i)}di \\
&= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma\mu_t\varkappa(t-s)\varphi_t}\int_0^1 e^{-\gamma\mu_t\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)}di \\
&+ (1-\vartheta)\int_0^1 e^{-\gamma\mu_t\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)}di \\
&= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma\mu_t\varkappa(t-s)\varphi_t}I_t + (1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}
\end{aligned}
$$

where

$$
I_t = \int_0^1 e^{-\gamma\mu_t\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)}di = \int_0^1 e^{-\gamma\mu_t A_t^s(i)}e^{-\gamma\mu_t\left(\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)}di
$$

Integral I_t is an expectation of a product of two functions (uniformly distributed), and as $A_t^s(i)$ is not correllated with $\left(\xi_t^s(i)-\bar{\xi}\right)$ *[see some formula above which expresses wealth as a function of past shocks only - TK ref to an equation]*, then expectation of a product is equal to a product of expectations, we can write

$$
I_t = \int_0^1 e^{-\gamma\mu_t\left(\eta_t\left(\xi_t^s(j)-\bar{\xi}\right)\right)}dj\int_0^1 e^{-\gamma\mu_t A_t^s(i)}di = e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}\int_0^1 e^{-\gamma\mu_t A_t^s(i)}di
$$

Recall the budget constraint (2.59):

$$
A_{t+1}^s(i) = \frac{R_t}{\vartheta}\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)+X_t-\eta_t\Theta_t^s-(1+\rho\gamma\eta_t)\,c_t^s(i)\right)
$$

substitute out consumption using (2.60)

$$
A_{t+1}^s(i) = \frac{R_t}{\vartheta}\begin{pmatrix} (1-(1+\rho\gamma\eta_t)\,\mu_t)\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right) \\ +X_t-(1+\rho\gamma\eta_t)\,\mathscr{X}_t-(\eta_t-(1+\rho\gamma\eta_t)\,\mu_t\varphi_t)\,\Theta_t^s \end{pmatrix}
$$

and simplify using (2.65) and (2.64)

$$
\mu_{t+1}A_{t+1}^s(i) = \begin{pmatrix} \mu_t\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)-(\mu_t\varphi_t-\mu_{t+1}\varphi_{t+1})\,\Theta_t^s \\ +\mu_{t+1}\frac{R_t}{\vartheta}\left(X_t-(1+\rho\gamma\eta_t)\,\mathscr{X}_t\right) \end{pmatrix}.
$$

Take a lag and substitute this expression into formula for I_t to obtain a recursion for this integral:

$$
\begin{aligned}
I_t &= \int_0^1 e^{-\gamma\mu_t\left(A_t^s(i)+\eta_t\left(\xi_t^s(i)-\bar{\xi}\right)\right)}di = e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}\int_0^1 e^{-\gamma\mu_t A_t^s(i)}di \\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}\int_0^1 e^{-\gamma\left(\mu_{t-1}\left(A_{t-1}^s(i)+\eta_{t-1}\left(\xi_{t-1}^s(i)-\bar{\xi}\right)\right)\right)} \\
&\times e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})-(\mu_{t-1}\varphi_{t-1}-\mu_t\varphi_t)\Theta_{t-1}^s\right)}di \\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})-(\mu_{t-1}\varphi_{t-1}-\mu_t\varphi_t)\Theta_{t-1}^s\right)} \\
&\times \int_0^1 e^{-\gamma\left(\mu_{t-1}\left(A_{t-1}^s(i)+\eta_{t-1}\left(\xi_{t-1}^s(i)-\bar{\xi}\right)\right)\right)}di \\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}\left(X_{t-1}-\left(1+\tau_{t-1}^c+\rho\gamma\eta_{t-1}\right)\mathscr{X}_{t-1}\right)-\left(\mu_{t-1}\varphi_{t-1}-\mu_t\varphi_t\frac{\varkappa_t}{\varkappa_{t-1}}\right)\varkappa(t-1-s)\right)}I_{t-1} \\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})-(\mu_{t-1}\varphi_{t-1}\varkappa-\mu_t\varphi_t\varkappa)(t-1-s)\right)}I_{t-1}
\end{aligned}
$$

Note that, by definition,

$$
\begin{aligned}
\Sigma_{t-1} &= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-1-s}\int_0^1 e^{-\gamma\mu_{t-1}m_{t-1}^s(i)}di \\
&= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-1-s}\int_0^1 e^{-\gamma\mu_{t-1}\left(A_{t-1}^s(i)+\eta_{t-1}\left(\xi_{t-1}^s(i)-\bar{\xi}\right)-\varphi_{t-1}\Theta_{t-1}^s\right)}di \\
&= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-1-s}e^{-\gamma\mu_{t-1}\left(-\varphi_{t-1}\Theta_{t-1}^s\right)}\int_0^1 e^{-\gamma\mu_{t-1}\left(A_{t-1}^s(i)+\eta_{t-1}\left(\xi_{t-1}^s(i)-\bar{\xi}\right)\right)}di
\end{aligned}
$$

so that

$$
\Sigma_{t-1} = (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-1-s}e^{\gamma\mu_{t-1}\varphi_{t-1}\varkappa(t-1-s)}I_{t-1}
$$

Now, isolate this term:

$$
\begin{aligned}
\Sigma_t &= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma\mu_t\varkappa(t-s)\varphi_t}I_t + (1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \\
&= (1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma\mu_t\varkappa(t-s)\varphi_t}e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \\
&\times e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})-(\mu_{t-1}\varphi_{t-1}\varkappa-\mu_t\varphi_t\varkappa)(t-1-s)\right)}I_{t-1} \\
&+ (1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}(1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma\mu_t\varkappa(t-s)\varphi_t+\gamma(\mu_{t-1}\varphi_{t-1}\varkappa-\mu_t\varphi_t\varkappa)(t-1-s)} \\
&\times e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})\right)}I_{t-1} \\
&+ (1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \\
&= e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}(1-\vartheta)\sum_{s=-\infty}^{t-1}\vartheta^{t-s}e^{\gamma(\mu_t\varphi_t\varkappa+\mu_{t-1}\varphi_{t-1}\varkappa(t-1-s))} \\
&\times e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})\right)}I_{t-1} \\
&+ (1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \\
&= e^{\gamma\mu_t\varphi_t\varkappa}e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}e^{-\gamma\left(\mu_t\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})\right)}\vartheta(1-\vartheta) \\
&\times\sum_{s=-\infty}^{t-1}\vartheta^{t-s-1}e^{\gamma\mu_{t-1}\varphi_{t-1}\varkappa(t-1-s)}I_{t-1} + (1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}
\end{aligned}
$$

to obtain recursive relationship:

$$
\Sigma_t = \vartheta e^{-\gamma\mu_t\left(\frac{R_{t-1}}{\vartheta}(X_{t-1}-(1+\rho\gamma\eta_{t-1})\mathscr{X}_{t-1})-\varkappa\varphi_t\right)}e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}\Sigma_{t-1} + (1-\vartheta)\,e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2}
$$

Introduce new variable Z_t to yield

$$
\Sigma_t = \left(e^{-\frac{\gamma}{\vartheta}\mu_t(R_{t-1}Z_{t-1}-\vartheta\varkappa\varphi_t)}\vartheta\Sigma_{t-1}+1-\vartheta\right)e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \tag{2.78}
$$

where

$$
\begin{aligned}
Z_t &= X_t-(1+\rho\gamma\eta_t)\,\mathscr{X}_t \\
&= \eta_t\left(\rho\log(\eta_t)+\bar{\xi}-\rho\gamma\chi G_t\right)+d_t-T_t-(1+\rho\gamma\eta_t)(\mathscr{C}_t-\chi G_t) \\
&= \eta_t\left(\rho\log(\eta_t)+\bar{\xi}\right)-(1+\rho\gamma\eta_t)\mathscr{C}_t+\chi G_t+d_t-T_t
\end{aligned} \tag{2.79}
$$

We can represent Z_t in a different form:

$$c_t + \chi G_t - \vartheta \mu_t \left(A_t - \frac{\varkappa}{1-\vartheta} \varphi_t \right) = \mathscr{C}_t$$

then

$$Z_t = \eta_t \left(\rho \log(\eta_t) + \bar{\xi} \right) - (1 + \rho\gamma\eta_t) \left(c_t + \chi G_t - \vartheta \mu_t \left(A_t - \frac{\varkappa}{1-\vartheta} \varphi_t \right) \right) + \chi G_t + d_t - T_t$$

use

$$y_t = \eta_t \rho \log(\eta_t) + \eta_t \bar{\xi} - \eta_t \varkappa \frac{\vartheta}{1-\vartheta} - \rho\gamma\eta_t c_t - \rho\gamma\chi\eta_t G_t + d_t - T_t.$$

to obtain

$$y_t + \varkappa \frac{\vartheta}{(1-\vartheta)} \eta_t + \rho\gamma\eta_t c_t - d_t + T_t + \eta_t \rho\gamma\chi G_t = \eta_t \left(\rho \log(\eta_t) + \bar{\xi} \right).$$

$$Z_t = y_t - c_t + (1 + \rho\gamma\eta_t) \mu_t \vartheta A_t + \frac{\vartheta}{(1-\vartheta)} \varkappa \eta_t - (1 + \rho\gamma\eta_t) \mu_t \frac{\vartheta\varkappa}{1-\vartheta} \varphi_t \tag{2.80}$$

$$(1 + \rho\gamma\eta_t) \mu_t = 1 - \frac{\vartheta \mu_t}{\mu_{t+1} R_t}$$

$$Z_t = y_t - c_t + (1 + \rho\gamma\eta_t) \mu_t \vartheta A_t - \frac{\vartheta}{(1-\vartheta)} \frac{\vartheta}{\mu_{t+1} R_t} \left(\varkappa \mu_{t+1} \varphi_{t+1} - \varkappa \mu_t \varphi_t \right)$$

Furthermore, the aggregated budget constraint:

$$\frac{\vartheta}{R_t} A_{t+1} - \vartheta A_t = y_t - c_t$$

using which

$$\begin{aligned}
Z_t &= \frac{\vartheta}{R_t} A_{t+1} - \vartheta A_t + (1 + \rho\gamma\eta_t) \mu_t \vartheta A_t - \frac{\vartheta}{(1-\vartheta)} \frac{\vartheta}{\mu_{t+1} R_t} \left(\varkappa \mu_{t+1} \varphi_{t+1} - \varkappa \mu_t \varphi_t \right) \\
&= \frac{\vartheta}{R_t} A_{t+1} - \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \vartheta A_t - \frac{\vartheta}{(1-\vartheta)} \frac{\vartheta}{\mu_{t+1} R_t} \left(\varkappa \mu_{t+1} \varphi_{t+1} - \varkappa \mu_t \varphi_t \right) \\
&= \frac{\vartheta}{R_t} \left(A_{t+1} - \frac{\vartheta}{(1-\vartheta)} \varkappa \varphi_{t+1} \right) - \frac{\vartheta \mu_t}{\mu_{t+1} R_t} \left(\vartheta A_t - \frac{\vartheta}{(1-\vartheta)} \varkappa \varphi_t \right)
\end{aligned}$$

so that

$$Z_t = \frac{\vartheta}{\mu_{t+1} R_t} \left(\mu_{t+1} \left(A_{t+1} - \frac{\vartheta}{(1-\vartheta)} \varkappa \varphi_{t+1} \right) - \mu_t \left(\vartheta A_t - \frac{\vartheta}{(1-\vartheta)} \varkappa \varphi_t \right) \right) \tag{2.81}$$

It is apparent that if the aggregate asset holding is zero then $Z_t = 0$ and we obtain the same recursive formula for Σ_t as reported in Acharya et al (2020).

Using inter-mediation constraint (2.28) we rewrite (2.81)

$$\mu_{t+1} R_t Z_t = \mu_{t+1}\left(B_{t+1} - \vartheta\frac{\vartheta}{(1-\vartheta)}\varkappa\varphi_{t+1}\right) - \vartheta\mu_t\left(B_t - \frac{\vartheta}{(1-\vartheta)}\varkappa\varphi_t\right) \tag{2.82}$$

Introduce new variable

$$W_t = \mu_t\left(B_t - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi_t\right)$$

then

$$\mu_{t+1} R_t Z_t = W_{t+1} - \vartheta W_t + \vartheta\varkappa\mu_{t+1}\varphi_{t+1} \tag{2.83}$$

Denote

$$S_t = e^{\gamma\mu_t\left(B_t - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi_t\right)}\Sigma_t$$

Then

$$\begin{aligned}\mathbb{U}_t &= -\frac{1}{\gamma}(1+\gamma\rho\eta_t)\, e^{-\gamma\left(x_t - \mu_t\left(B_t - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi_t\right)\right)}\Sigma_t \\ &= -\frac{1}{\gamma}(1+\gamma\rho\eta_t)\, e^{-\gamma x_t} S_t\end{aligned}$$

Use (2.83) to rewrite (2.78)

$$S_t = \left(e^{-\frac{\gamma}{\vartheta}W_t}\vartheta S_{t-1} + 1 - \vartheta\right) e^{\gamma W_t} e^{\frac{1}{2}\gamma^2\mu_t^2\eta_t^2\sigma_t^2} \tag{2.84}$$

2.9.8 Optimal Policy Under Commitment

The Lagrangian is

$$
\begin{aligned}
L = &\sum_{t=0}^{\infty} \tilde{\beta}^t \left(-\frac{1}{\gamma} \left(1+\gamma\rho\eta_t\right)^{\psi} \exp\left(-\psi\gamma x_t\right) S_t^{\xi} \right) \\
&+ \sum_{t=0}^{\infty} \tilde{\beta}^t M_{1,t} \left(\begin{array}{c} -\frac{1}{\gamma}\log\left(\beta R_t\right) + x_{t+1} + \frac{(1-\vartheta)}{\vartheta}\mu_{t+1}B_{t+1} - \varkappa\mu_{t+1}\varphi_{t+1} \\ -\frac{\gamma}{2}\mu_{t+1}^2\left(1-\tau_{t+1}\right)^2\omega_{t+1} - x_t \end{array} \right) \\
&+ \sum_{t=0}^{\infty} \tilde{\beta}^t M_{2,t} \left(\frac{1-\varepsilon_t + (1-s)\,\varepsilon_t \frac{w_t}{z_t}}{\Phi} Y_t\beta^{-1} - \pi_t\left(1+\pi_t\right)Y_t\beta^{-1} + \pi_{t+1}\left(1+\pi_{t+1}\right)Y_{t+1} \right) \\
&+ \sum_{t=0}^{\infty} \tilde{\beta}^t M_{3,t} \left(\frac{\vartheta}{\mu_{t+1}} + \left(1+\rho\gamma\eta_t\right)R_t - \frac{R_t}{\mu_t} \right) \\
&+ \sum_{t=0}^{\infty} \tilde{\beta}^t M_{4,t} \left(-\log S_t + \frac{1}{2}\gamma^2\mu_t^2\left(1-\tau_t\right)^2\omega_t + \gamma W_t + \log\left(e^{-\frac{\gamma}{\vartheta}W_t}\vartheta S_{t-1} + 1 - \vartheta\right) \right) \\
&+ \sum_{t=0}^{\infty} \tilde{\beta}^t M_{5,t} \left(\left(\frac{\left(1+\rho P_t^M\right) b_t^L}{\left(1+\pi_t\right)} + G_t - \tau_t \frac{w_t}{z_t} Y_t - T_t^p \right) R_t - \frac{\left(1+\rho P_{t+1}^M\right) b_{t+1}^L}{\left(1+\pi_{t+1}\right)} \right) \\
&+ \sum_{t=0}^{\infty} \tilde{\beta}^t M_{6,t} \left(\rho\log\left(\eta_t\right) + \bar{\xi} - \frac{\varkappa\vartheta}{1-\vartheta} - \rho\gamma x_t - \frac{Y_t}{z_t} \right) + \sum_{t=0}^{\infty} \tilde{\beta}^t M_{7,t}\left(\left(1-\tau_t\right)w_t - \eta_t\right) \\
&+ \sum_{t=0}^{\infty} \tilde{\beta}^t M_{8,t} \left(\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)} - P_t^M R_t \right) + \sum_{t=0}^{\infty} \tilde{\beta}^t M_{9,t} \left(x_t + (1-\chi)\,G_t - \left(1 - \frac{\Phi}{2}\pi_t^2\right) Y_t \right) \\
&+ \sum_{t=0}^{\infty} \tilde{\beta}^t M_{10,t} \left(W_t - \mu_t \left(\frac{\left(1+\rho P_t^M\right) b_t^L}{\left(1+\pi_t\right)} - \frac{\vartheta}{(1-\vartheta)}\varkappa\varphi_t \right) \right) \\
&+ \sum_{t=0}^{\infty} \tilde{\beta}^t M_{11,t}\left(R_t\eta_t + \vartheta\varphi_{t+1} - R_t\varphi_t\right) + \sum_{t=0}^{\infty} \tilde{\beta}^t M_{12,t}\left(w^2\sigma^2\exp\left(2\phi\left(Y_t - Y\right)\right) - \omega_t\right)
\end{aligned}
$$

and the FOCs are

$$
1 : \frac{\partial L}{\partial \varphi_t} = -\tilde{\beta}^{-1} M_{1,t-1}\varkappa\mu_t + M_{10,t}\mu_t \frac{\vartheta\varkappa}{(1-\vartheta)} - M_{11,t}R_t + \tilde{\beta}^{-1}M_{11,t-1}\vartheta
$$

$$
\begin{aligned}
2 : \frac{\partial L}{\partial \mu_t} = &\tilde{\beta}^{-1} M_{1,t-1} \left(\frac{(1-\vartheta)}{\vartheta} \frac{\left(1+\rho P_t^M\right) b_t^L}{\left(1+\pi_t\right)} - \varkappa\varphi_t - \gamma\mu_t\left(1-\tau_t\right)^2\omega_t \right) \\
&+ M_{3,t}\frac{R_t}{\mu_t^2} - \tilde{\beta}^{-1}M_{3,t-1}\frac{\vartheta}{\mu_t^2} + M_{4,t}\gamma^2\mu_t\left(1-\tau_t\right)^2\omega_t \\
&- M_{10,t}\left(\frac{\left(1+\rho P_t^M\right) b_t^L}{\left(1+\pi_t\right)} - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi_t \right)
\end{aligned}
$$

$$
3 : \frac{\partial L}{\partial w_t} = M_{2,t}\frac{(1-s)}{\Phi}\frac{\varepsilon_t}{z_t}Y_t\beta^{-1} - M_{5,t}\frac{\tau_t}{z_t}Y_tR_t + M_{7,t}\left(1-\tau_t\right)
$$

$$
\begin{aligned}
4: \frac{\partial L}{\partial \eta_t} &= -\rho \psi S_t^{\xi} \left(1+\gamma\rho\eta_t\right)^{\psi-1} \exp\left(-\psi\gamma x_t\right) \\
&+ M_{3,t}\rho\gamma R_t + M_{6,t}\frac{\rho}{\eta_t} - M_{7,t} + M_{11,t} R_t
\end{aligned}
$$

$$
\begin{aligned}
5: \frac{\partial L}{\partial R_t} &= -M_{1,t}\frac{1}{\gamma R_t} + M_{3,t}\left(1+\rho\gamma\eta_t - \frac{1}{\mu_t}\right) \\
&+ M_{5,t}\left(\frac{\left(1+\rho P_t^M\right) b_t^L}{\left(1+\pi_t\right)} + G_t - \tau_t \frac{w_t}{z_t} Y_t - T_t^p\right) \\
&- M_{8,t} P_t^M + M_{11,t}\left(\eta_t - \varphi_t\right)
\end{aligned}
$$

$$
\begin{aligned}
6: \frac{\partial L}{\partial B_{t+1}} &= M_{1,t}\frac{(1-\vartheta)}{\vartheta}\mu_{t+1}\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)} - M_{5,t}\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)} \\
&+ \tilde{\beta} M_{5,t+1} R_{t+1}\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)} - \tilde{\beta} M_{10,t+1}\mu_{t+1}\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)}
\end{aligned}
$$

$$
\begin{aligned}
7: \frac{\partial L}{\partial P_t^M} &= \left(\tilde{\beta}^{-1} M_{1,t-1}\frac{(1-\vartheta)}{\vartheta}\mu_t + M_{5,t} R_t - \tilde{\beta}^{-1} M_{5,t-1} - M_{10,t}\mu_t\right)\frac{\rho b_t^L}{\left(1+\pi_t\right)} \\
&- M_{8,t} R_t + \tilde{\beta}^{-1} M_{8,t-1}\frac{\rho}{\left(1+\pi_t\right)}
\end{aligned}
$$

$$
\begin{aligned}
8: \frac{\partial L}{\partial S_t} &= -\frac{1}{\gamma}\left(1+\gamma\rho\eta_t\right)^{\psi} \exp\left(-\psi\gamma x_t\right) \xi S_t^{\xi-1} \\
&- M_{4,t}\frac{1}{S_t} + \tilde{\beta} M_{4,t+1}\frac{e^{-\frac{\gamma}{\vartheta} W_{t+1}}\vartheta}{\left(e^{-\frac{\gamma}{\vartheta} W_{t+1}}\vartheta S_t + 1 - \vartheta\right)}
\end{aligned}
$$

$$
9: \frac{\partial L}{\partial W_t} = M_{4,t}\gamma\left(1 - \frac{e^{-\frac{\gamma}{\vartheta} W_t} S_{t-1}}{\left(e^{-\frac{\gamma}{\vartheta} W_t}\vartheta S_{t-1} + 1 - \vartheta\right)}\right) + M_{10,t}
$$

$$10:\frac{\partial L}{\partial \pi_t}=-\tilde{\beta}^{-1}M_{1,t-1}\frac{(1-\vartheta)}{\vartheta}\mu_t\frac{\left(1+\rho P_t^M\right)b_t^L}{(1+\pi_t)^2}-M_{2,t}(1+2\pi_t)Y_t\beta^{-1}+\tilde{\beta}^{-1}M_{2,t-1}(1+2\pi_t)Y_t$$
$$-M_{5,t}R_t\frac{\left(1+\rho P_t^M\right)b_t^L}{(1+\pi_t)^2}+\tilde{\beta}^{-1}M_{5,t-1}\frac{\left(1+\rho P_t^M\right)b_t^L}{(1+\pi_t)^2}$$
$$-\tilde{\beta}^{-1}M_{8,t-1}\frac{\left(1+\rho P_t^M\right)}{(1+\pi_t)^2}+M_{9,t}\Phi\pi_tY_t+M_{10,t}\mu_t\frac{\left(1+\rho P_t^M\right)b_t^L}{(1+\pi_t)^2}$$

$$11:\frac{\partial L}{\partial x_t}=\psi(1+\gamma\rho\eta_t)^{\psi}\exp(-\psi\gamma x_t)S_t^{\xi}-M_{1,t}+\tilde{\beta}^{-1}M_{1,t-1}-M_{6,t}\rho\gamma+M_{9,t}$$

$$12:\frac{\partial L}{\partial Y_t}=+M_{2,t}\left(\frac{1-\varepsilon_t+(1-s)\varepsilon_t\frac{w_t}{z_t}}{\Phi}-\pi_t(1+\pi_t)\right)\beta^{-1}$$
$$+\tilde{\beta}^{-1}M_{2,t-1}\pi_t(1+\pi_t)-M_{5,t}\tau_t\frac{w_t}{z_t}R_t-M_{6,t}\frac{1}{z_t}$$
$$-M_{9,t}\left(1-\frac{\Phi}{2}\pi_t^2\right)+2\phi M_{12,t}w^2\sigma^2\exp(2\phi(Y_t-Y))$$

$$13a:\frac{\partial L}{\partial \tau_t}=\tilde{\beta}^{-1}M_{1,t-1}\gamma\mu_t^2(1-\tau_t)\omega_t-M_{4,t}\gamma^2\mu_t^2(1-\tau_t)\omega_t-M_{5,t}\frac{w_t}{z_t}Y_tR_t-M_{7,t}w_t$$
$$13b:\frac{\partial L}{\partial T_t^p}=-M_{5,t}R_t$$

here equation 13a is in case of labour income taxes, and equation 13b is there are lump sum taxes.

2.9.9 Steady State

The FOCS in the steady state can be written as:

$$1:\frac{\partial L}{\partial \varphi_t}=-\tilde{\beta}^{-1}M_1\varkappa\mu_t+M_{10}\mu_t\frac{\vartheta\varkappa}{(1-\vartheta)}+M_{11}\left(\vartheta\tilde{\beta}^{-1}-R_t\right)$$

$$2:\frac{\partial L}{\partial \mu_t}=\tilde{\beta}^{-1}M_1\left(\frac{(1-\vartheta)}{\vartheta}\frac{\left(1+\rho P^M\right)b^L}{(1+\pi_t)}-\varkappa\varphi-\gamma\mu(1-\tau)^2\omega\right)$$
$$+M_3\frac{R}{\mu^2}-\tilde{\beta}^{-1}M_3\frac{\vartheta}{\mu^2}+M_4\gamma^2\mu(1-\tau)^2\omega$$
$$-M_{10}\left(\frac{\left(1+\rho P^M\right)b^L}{(1+\pi)}-\frac{\vartheta\varkappa}{(1-\vartheta)}\varphi\right)$$

$$3:\frac{\partial L}{\partial w_t}=M_2\frac{(1-s)}{\Phi}\varepsilon Y\beta^{-1}-M_5\tau YR+M_7(1-\tau)$$

$$4: \frac{\partial L}{\partial \eta_t} = -\rho \psi S^{\xi} (1+\gamma\rho\eta)^{\psi-1} \exp(-\psi\gamma x)$$
$$+M_3\rho\gamma R + M_6\frac{\rho}{\eta} - M_7 + M_{11}R$$

$$5: \frac{\partial L}{\partial R_t} = -M_1\frac{1}{\gamma R} + M_3\left(1+\rho\gamma\eta - \frac{1}{\mu}\right)$$
$$+M_5\left(\frac{\left(1+\rho P^M\right)b^L}{(1+\pi)} + G - \tau w Y - T^p\right)$$
$$-M_8P^M + M_{11}(\eta - \varphi)$$

$$6: \frac{\partial L}{\partial B_{t+1}} = M_1\frac{(1-\vartheta)}{\vartheta}\mu\frac{\left(1+\rho P^M\right)}{(1+\pi)} - M_5\frac{\left(1+\rho P^M\right)}{(1+\pi)}$$
$$+\tilde{\beta}M_5R\frac{\left(1+\rho P^M\right)}{(1+\pi)} - \tilde{\beta}M_{10}\mu\frac{\left(1+\rho P^M\right)}{(1+\pi)}$$

$$7: \frac{\partial L}{\partial P_t^M} = \left(\tilde{\beta}^{-1}M_1\frac{(1-\vartheta)}{\vartheta}\mu + M_5R - \tilde{\beta}^{-1}M_5 - M_{10}\mu\right)\frac{\rho b^L}{(1+\pi)}$$
$$-M_8R + \tilde{\beta}^{-1}M_8\frac{\rho}{(1+\pi)}$$

$$8: \frac{\partial L}{\partial S_t} = -\frac{1}{\gamma}(1+\gamma\rho\eta)^{\psi}\exp(-\psi\gamma x)\,\xi S^{\xi-1}$$
$$-M_4\frac{1}{S} + \tilde{\beta}M_4\frac{e^{-\frac{\gamma}{\vartheta}W}\vartheta}{\left(e^{-\frac{\gamma}{\vartheta}W}\vartheta S + 1 - \vartheta\right)}$$

$$9: \frac{\partial L}{\partial W_t} = M_4\gamma\left(1 - \frac{e^{-\frac{\gamma}{\vartheta}W}S}{\left(e^{-\frac{\gamma}{\vartheta}W}\vartheta S + 1 - \vartheta\right)}\right) + M_{10}$$

$$
\begin{aligned}
10: \frac{\partial L}{\partial \pi_t} = & -\tilde{\beta}^{-1} M_1 \frac{(1-\vartheta)}{\vartheta} \mu \frac{\left(1+\rho P^M\right) b^L}{(1+\pi)^2} - M_2 (1+2\pi) Y \beta^{-1} + \tilde{\beta}^{-1} M_2 (1+2\pi) Y \\
& - M_5 R \frac{\left(1+\rho P^M\right) b^L}{(1+\pi)^2} + \tilde{\beta}^{-1} M_5 \frac{\left(1+\rho P^M\right) b^L}{(1+\pi)^2} \\
& - \tilde{\beta}^{-1} M_8 \frac{\left(1+\rho P^M\right)}{(1+\pi)^2} + M_9 \Phi \pi Y + M_{10} \mu \frac{\left(1+\rho P^M\right) b^L}{(1+\pi)^2}
\end{aligned}
$$

$$
11: \frac{\partial L}{\partial x_t} = \psi (1+\gamma\rho\eta)^{\psi} \exp(-\psi\gamma x) S^{\xi} - M_1 + \tilde{\beta}^{-1} M_1 - M_6 \rho\gamma + M_9
$$

$$
\begin{aligned}
12: \frac{\partial L}{\partial Y_t} = & +M_2 \left(\frac{1-\varepsilon+(1-s)\varepsilon w}{\Phi} - \pi(1+\pi) \right) \beta^{-1} \\
& + \tilde{\beta}^{-1} M_2 \pi (1+\pi) - M_5 \tau w R - M_6 \\
& - M_9 \left(1 - \frac{\Phi}{2} \pi^2 \right) + 2\phi M_{12} w^2 \sigma^2
\end{aligned}
$$

$$
13a: \frac{\partial L}{\partial \tau_t} = \tilde{\beta}^{-1} M_1 \gamma \mu^2 (1-\tau) \omega - M_4 \gamma^2 \mu^2 (1-\tau) \omega - M_5 w Y R - M_7 w
$$

$$
13b: \frac{\partial L}{\partial T_t^p} = -M_5 R
$$

Additionally, the system of constraints can be written as follows.

$$
\begin{aligned}
14 &: 0 = -\frac{1}{\gamma}\log(\beta R) + \frac{(1-\vartheta)}{\vartheta}\mu B - \varkappa\mu\varphi - \frac{\gamma}{2}\mu^2\eta^2\sigma^2 \\
15 &: 0 = \frac{1-\varepsilon_t + (1-s)\,\varepsilon_t \frac{w_t}{z_t}}{\Phi} Y_t \beta^{-1} - \pi(1+\pi) Y \beta^{-1} + \pi(1+\pi) Y \\
16 &: 0 = \frac{\vartheta\mu}{(\mu(1+\rho\gamma\eta)-1)R} + \mu \\
17 &: 0 = -\log S + \frac{1}{2}\gamma^2\mu^2\eta^2\sigma^2 + \gamma W + \log\left(e^{-\frac{\gamma}{\vartheta}W}\vartheta S + 1 - \vartheta\right) \\
18 &: 0 = (B + G - \tau w Y - T^p)R - B \\
19 &: 0 = \rho\log(\eta) + \bar{\xi} - \frac{\varkappa\vartheta}{1-\vartheta} - \rho\gamma x - Y \\
20 &: 0 = (1-\tau)w - \eta \\
21 &: 0 = \frac{(1+\rho P^M)}{(1+\pi)} - P^M R \\
22 &: 0 = x + (1-\chi)G - \left(1 - \frac{\Phi}{2}\pi^2\right)Y \\
23 &: 0 = W - \mu\left(B - \frac{\vartheta\varkappa}{(1-\vartheta)}\varphi\right) \\
24 &: 0 = R\eta + \vartheta\varphi - R\varphi
\end{aligned}
$$

2.9.10 Proof of Proposition 3

Because T_t^p is a policy instrument, the budget constraint (2.39) does not bind in the steady state, and $M_5 = 0$ as follows from the FOC wrt T_t^p.

From the FOC wrt b_{t+1} it follows that

$$M_{10} = M_1 \frac{(1-\vartheta)}{\vartheta\tilde{\beta}} \tag{2.85}$$

then using this in the FOC wrt P_t^M we get $M_8 = 0$ meaning that equation (2.43) does not bind in the steady state.

Use (2.85) to substitute M_{10} into the FOC wrt to φ, to yield $M_{11} = 0$.

Then, in case of RANK $\sigma = 0$ and so that $\omega = 0$, use (2.85) in the FOC wrt μ to yield $M_3 = 0$.

The FOC wrt R yields $M_1 = M_{10} = 0$.

The FOC wrt S suggests that $M_4 \neq 0$ as the derivative of the utility is never zero.

Finally, the FOC wrt W can be written as

$$\frac{\partial L}{\partial W_t} = M_4 \gamma \frac{(1-\vartheta)\left(1-e^{-\frac{\gamma}{\vartheta}W}S\right)}{\left(e^{-\frac{\gamma}{\vartheta}W}\vartheta S + 1 - \vartheta\right)} = 0$$

from where

$$S = e^{\frac{\gamma}{\vartheta}W}$$

Substituting this into the evolution of S equation yields

$$0 = -\frac{\gamma}{\vartheta}W + \gamma W$$

so that $W = 0$, and $S = 1$.

The Euler equation then implies

$$R = \frac{1}{\beta}.$$

All other results in Proposition 3 follow trivially.

It is essential that $\sigma = 0$, the result will break down otherwise.

We see that the level of debt in the steady state will be determined by the rate that labor force participation declines with age.

2.9.11 Proof of Proposition 4

[I don't think this works. You can have $R = \frac{1}{\beta}$ without $B = 0$ and $\varkappa = 0$. Is it just that we can show a steady state given $\varkappa = 0$ even when taxes are distortionary, which has the properties $R = \frac{1}{\beta}$ and $B = 0$?]

We need $\tilde{\beta} = \beta = \frac{1}{R}$.

In a RANK economy, we show that if $R = \frac{1}{\beta}$ then it must be $B = 0$ and $\varkappa = 0$, so this is only possible if there is no declining income.

Suppose $R = \frac{1}{\beta}$ then from the Euler equation (2.36) it follows that $W = 0$ and from inequality recursion (**??**) it follows that $S = 1$.

The FOC wrt b_{t+1} combined with the FOC wrt P_t^M yields $M_8 = 0$.

From the FOC wrt debt, it follows that $M_{10} = M_1 \frac{(1-\vartheta)}{\vartheta \tilde{\beta}}$. Substitute M_{10} into the FOC wrt to φ, to yield $M_{11} = 0$.

Then, in case of RANK $\sigma = 0$ and the FOC wrt μ yields $M_3 = 0$, once we take into account the relationship between M_{10} and M_1.

The FOC wrt S suggests that $M_4 \neq 0$ as the derivative of utility is never zero.

The FOC wrt W and $S=1$ yields $M_{10}=0$, so that $M_1=0$.
The FOC wrt η yields $M_7=0$
The next point to make is that $M_5 \neq 0$.
Indeed,
(i) From FOC wrt Y we have $M_6=-M_5\tau wR-M_9$
(ii) FOCs wrt τ and x yield the equation for M_5 :

$$(1-\eta)\,S^{\xi}\exp\left(-\gamma x\right)+M_5\left(\tau w\rho\gamma+\left(\frac{\eta}{\rho}Y-\tau w\right)(\rho\gamma+1)\right)R=0$$

this equation has non-trivial solution.

Chapter 3

Optimal monetary policy in a HANK economy with meaningful government debt: A tale of two Ricardian consumers

Abstract

We study optimal monetary policy in a tractable HANK environment. The model admits both idiosyncratic and aggregate risk. We assume that there exists a consolidated monetary-fiscal authority. The monetary authority pursues optimal (Ramsey) monetary policy whilst the fiscal authority follows a simple tax rule. Our aim is to provide a clear distinction between the notions of discontinuous labour market participation (DLMP) and infrequent asset market participation (IAMP), which are typically intertwined in the literature. In a HANK- DLMP model, constrained households are able to use assets to smooth their inter- temporal consumption. As such, the long run equilibrium as well as the model's dynamics under optimal monetary policy are different from both the nested representative agent model and from the HANK- IAMP framework. We demonstrate that DLMP frictions are an important source of heterogeneity on their own merit and should not be overlooked. Finally, we find that as was the case with the representative agent model, the policy maker in our framework will not deviate from price stability in steady state (Woodford 2003). This result is unaffected by the amount of outstanding government debt or the presence of direct redistribution. The model is calibrated for the US economy for the period 1985- 2021.

3.1 Introduction

We study optimal Ramsey monetary policy in heterogeneous agent New Keynesian (HANK, henceforth) environment. Our framework falls under the umbrella of the the so - called analytically tractable HANK (THANK) models. Over the last decade, researchers have employed these models to study the distributional consequences of monetary and fiscal policy (See for instance, Bilbiie 2008;Broer et al. 2020; Challe 2020; Cantore & Freund 2021; Chien & Wen 2021; among others). To retain maximal tractability, these models rely on simplifying assumptions regarding how inequality enters the economy and focus on the qualitative rather than the quantitative differences from the nested representative agent environment.

In this paper, we wish to disentangle the notion of discontinuous labour market participation (DLMP, henceforth) from the infrequent asset market participation (IAMP, henceforth) assumption. We demonstrate that exogenous unemployment or rather discontinuous labour market participation is an important source of frictions on their own merit- separate from the IAMP frictions-and should not be overlooked. We extend the heterogeneous agent framework of Chien & Wen (2021) by adding nominal rigidities and direct wealth transfers to study optimal (Ramsey) monetary policy in a discontinuous labour market participation (DLMP) environment.

This framework is a simplified Aiyagari (1994) type model that admits two idiosyncratic employment status shocks as well as aggregate uncertainty. In each period, there is a constant share of "constrained" and "unconstrained" households in the economy. By unconstrained households we refer to the mass of agents who receive a positive realisation of the idiosyncratic state and are thus able to either enter or to continue having access to the labour market. As such, we use the term "employed" throughout this paper to refer to participating households and the term "unemployed" to refer to non- participating agents. As in Bilbiie & Ragot (2021) and Chien & Wen (2021) this simplification is made possible by adopting an extension of the "Big representative family" metaphor of Lucas Jr (1990). Following Bilbiie & Ragot (2021), we assume that "all households belong to a representative family and that the family head maximizes the inter-temporal welfare of all members using a utilitarian welfare criterion". However, the planner has access to limited risk sharing technology and unlike the standard indivisible labour model of Hansen (1985), employed consumers enjoy higher consumption and thus, higher utility. We can think of households as being in two "states" or rather, two "islands". In each period, once the aggregate state is realised, the family head pools resources/assets between households who are on the same island but they are unable to pool assets between islands or rather across consumer types. The key assumption is that all allocations made by the family head, take place before agents realise their idiosyncratic employment status for the current period. After the idiosyncratic status is drawn, the family head is unable to redistribute assets and thus, the model exhibits imperfect risk sharing[1].

In our set up, all households can either save or borrow in the form of both short term and long term government bonds. In line with the existing literature, we refer to a household's ability to hold assets as "having access to the financial market". Unlike Chien & Wen (2021), constrained households do not face a binding borrowing constraint in equilibrium and hence, the model delivers "perfect self- insurance" (See proposition 8, below). We show that in order to exist redistribution channels[2] in the transmission mechanism of monetary policy, households need to have unequal exposure to aggregate shocks (See propositions 14- 15). For our benchmark calibration, households can freely adjust their asset position at any moment and as such, optimal monetary policy can affect the dispersion of wealth in the economy but is unable to redistribute consumption from unconstrained to constrained households. To explore the redistributive consequences of monetary policy, we follow Cantore & Freund (2021) and introduce portfolio adjustment costs only for con-

[1]Proposition 8 below shows that for a particular parametrisation of the (un)employment transition matrix, the model reduces to an indivisible labour model in which case there is perfect risk sharing technology.

[2]In the language of Auclert (2019), the monetary policy has a redistributive role if a change in the interest rate can redistribute consumption (not just wealth) amongst consumers.

strained consumers. These frictions have no effect on the model's steady state but they do influence the optimal policy dynamics. So, in response to a shock, all consumers are able to adjust their net asset position to smooth their respective inter- temporal consumption. Yet, constrained consumers incur a cost proportional to the size of the adjustment. Our framework also nests the HANK-IAMP model as a special case. This scenario requires a "zero- liquidity" constraint imposed on constrained households and also, very large (near infinite) portfolio adjustment costs ($\Omega \to \infty$).
To simplify the tracking of the wealth distribution, we adopt the assumption of Bilbiie & Ragot (2021) where the asset holdings of each household type take a single value in each period and, this value depends on the most recent realisation of the idiosyncratic shock. Hence, all households with the same idiosyncratic realisation have the same consumption, labour supply and asset holdings[3] (perfect insurance within type).

In our framework, constrained or "unemployed" households can consume out of their financial wealth as well as through government subsidies. These "unemployment benefits" are paid as direct wealth transfers from employed to unemployed consumers, financed via taxes levied on unconstrained households[4]. Although these transfers guarantee that constrained households will always have positive consumption, their size is not able to affect the equilibrium consumption inequality in this economy (See proposition 10, below). This result sound be paradoxical at first, but it is quite intuitive in the context of this model. Since, no consumer type is faced with a binding equilibrium borrowing constraint, neither the level of transfers nor the level of household wealth can affect the steady state consumption inequality. This is because households are able to use the asset market to ensure that they consume the bundle associated with highest welfare for their type. Still the fact that there exists a non- trivial amount of consumption inequality in this economy alters the optimal policy's response compared to the nested RANK model. Unlike the standard Aiyagari (1994) model, our environment features "perfect self-insurance". Meaning that despite the presence of imperfect risk sharing (i.e. unconstrained agents enjoy higher consumption), in steady state, the interest rate is going to be equal to the rate of time preference (See proposition 8, below) and households enjoy perfect insurance within type.

Overall, we find that in our benchmark HANK- DLMP framework the policy maker is not faced with a trade- off between "equity" and "efficiency" and thus, the model delivers the equilibrium price stability result of Woodford (2003). In our HANK- DLMP environment monetary policy is concerned with preserving the price stability objective whilst the fiscal policy follows a simple rule to ensure fiscal solvency. The fully optimal policy maker under commitment, takes the tax rule

[3]If we were to allow households to keep their wealth when they receive a shock that forces them to switch type then the assumption of perfect insurance within type would no longer hold. As different households would have different idiosyncratic histories.

[4]Following Bilbiie et al. (2020) we refer to the fact that direct wealth transfers are made only to constrained consumers as having a progressive tax system.

under consideration when solving their program. Meaning that the supply of government bonds is optimally chosen in each period whereas taxes adjust to return government debt to the target. Since, in the absence of unequal exposure to aggregate shocks, the policy response will affect the wealth dispersion but will not be able to redistribute consumption. Although, our paper does not contribute to the debate whether targeting inequality should be a distinct policy objective[5]) for a discussion on optimal policy and the dilemma between "equity" and "efficiency"., the presence of consumption inequality alters both the long- run equilibrium of the model as well as the dynamics under optimal monetary policy. As a result, our model's predictions differ from both the nested RANK and also, from the standard HANK- IAMP environment.

3.2 Related Literature

In the last decade, a burgeoning literature has emerged exploring how monetary and fiscal policy differ in HANK environments compared to the standard representative agent model. The literature is split between studies who use on large quantitative HANK models and those who rely on tractable HANK or THANK models. This first class of HANK models focuses on exploring the quantitative difference between HANK and RANK environments. This strand of the literature either relies on continuous time models (See Ahn et al. 2017; Kaplan & Violante 2018; Achdou et al. 2022; Nuño & Thomas 2022; Fernández-Villaverde et al. 2023) or on the assumption of very large but finite number of different household types in order to compute the wealth distribution. Representative examples of discrete- time models who assume large but finite number of different household types are Bhandari et al. (2021) and Le Grand et al. (2022) for an analysis of optimal policy, Cui & Sterk (2021) on unconventional monetary policy and McKay & Wolf (2022) for study of optimal policy rules. These models do an excellent job of tracking the data, however, their complexity turns them into a kind of black box. However, we have to admit that by adopting the assumption of "Recursive contracts"- made by Marcet & Marimon (2019) or the assumption of "truncated" idiosyncratic histories of Le Grand et al. (2022), newer studies afford more tractability compared to past generations of quantitative heterogeneous- agent- incomplete- market (HAIM, hereafter) models.

On the other hand, THANK models are used to study the qualitative difference between HANK and the nested RANK economies. Still, they are able to trace the dynamics of the large quantitative models. As noted above, our model falls under the umbrella of the THANK models. We introduce nominal price rigidities to the THANK framework of Chien & Wen (2021) to study optimal monetary policy in an environment that allows for both idiosyncratic and aggregate risk but features only Ricardian consumers. Our work complements the studies of Auclert (2019), Bilbiie & Ragot

[5]See Chang (2022), Acharya et al. (2023), Hansen et al. (2023) and Karaferis et al. (2024)

(2021), Bilbiie (2024), Challe (2020), Chang (2022) and Hansen et al. (2023), amongst others who rely either on tractable TANK or simplified HANK models to study how optimal monetary policy differs in the presence of agent heterogeneity. Typically, the literature relies on generalisation of the seminal TANK model of Galí et al. (2007) and Bilbiie (2008) which features a mass of unconstrained Ricardian consumers and a mass of constrained “Keynesian households. Constrained consumers are typically assumed to be hand- to- mouth agents and are forced to consume their entire income in each period. As such, agents in these economies have unequal exposure to aggregate shocks thus, creating a redistributive role for monetary policy.

Auclert (2019) builds a discrete- time HANK model where the only source of uncertainty comes from the households' individual history of idiosyncratic shock. Auclert uses this framework to study optimal monetary policy. They identify three key channels through which monetary policy can affect redistribution. These are the earnings heterogeneity channel, the Fisher channel and the interest rate exposure channel. These channels shape the response of individual consumption and labour supply to the unanticipated transitory aggregate shock. They also find that the maturity structure of the households' balanced sheet is crucial for determining the welfare as well as the wealth effects of the shock. This result is also found in **?** but in the context of the RANK model. Including assets with longer maturities in the household's balance sheet creates a re- evaluation wealth in response to the shock. As such, even thought the maturity structure of the households' balanced sheet is not going to affect the model's steady state (under Commitment), it still alters the transmission mechanism of optimal Ramsey monetary policy. Furthermore, Chang (2022) and Hansen et al. (2023) also study the transmission of monetary policy in a THANK environment. They investigate whether the central bank should pursue reduction of inequality as an added objective separate to price stability and the reduction of the output gap. Similarly, Karaferis et al. (2024) investigate a similar research question, in a tractable HANK environment but unlike most studies, this paper looks at optimal monetary and optimal fiscal policy interactions without imposing “zero- liquidity“ in steady state. The paper augments the HANK framework of Acharya et al. (2023) that allows for a continuum of different household types but relies on a CARA utility function to considerably simplify the aggregation process and allow for analytic tractability.

As discussed above, since our HANK- DLMP framework features only Ricardian consumers there is no re- distributive role for monetary policy following a shock, in the benchmark case (See proposition 13, below). In the language of Auclert (2019) since all households in the economy share the same inter- temporal substitution there is no redistributive effects[6] of monetary policy. Intuitively, with both consumer types being able to use assets to smooth their inter- temporal

[6]When discussing the presence of "redistribution channels" or “redistribution effects“ in the transmission mechanism for the optimal monetary policy, we refer to the monetary policy's ability to redistribute consumption between household types in response to an aggregate shock.

consumption, a change in the real interest rate will have the same effect on the consumption of each type. However, by imposing a quadratic portfolio adjustment cost on constrained consumers, households have unequal exposure to an aggregate shock which gives rise to a redistributive channel for optimal policy (See proposition 14). As discussed above, due to the absence of a binding borrowing limit for unemployed consumers, in equilibrium, the model features complete asset markets. Hence, changes in direct redistribution cannot affect the equilibrium consumption inequality either. Hence, in equilibrium the policy maker is not facing any trade-off between “equity“ and “efficiency“.

Finally, our paper also complement the literature that investigates the joint effects of household heterogeneity and unemployment. This is typically done by incorporating search and matching frictions (SAM, henceforth) into an otherwise standard TANK[7] or THANK model (See for instance, Ravn & Sterk 2017 ;Challe 2020; Broer et al. (2023); **?**). Unlike, Challe (2020) or Ravn & Sterk (2017) who employ an analytical THANK models with endogenous unemployment risk (via SAM frictions), Chien & Wen (2021) assume that unemployment is exogenous. As such, their framework nests both the indivisible labour model of Hansen (1985) and the HANK- IAMP framework as special cases. Broer et al. (2023) introduced SAM frictions and non- zero amount of aggregate short term government debt to the THANK model of Bilbiie (2024). Their framework considers the joint effects of IAMP frictions and unemployment. However, they use it to study the fiscal multipliers. In their framework monetary policy can affect both the size of unemployment as well as the length of the unemployment spell. Their model is a generalisation of Debortoli & Galí (2018) and Komatsu (2023), who use a TANK model with SAM frictions to study the transmission of monetary policy in a tractable environment with household heterogeneity.

3.3 The Model

The economy is populated by a continuum of households. Consumers can be either unconstrained (i.e. employed) or constrained. And agents move randomly across the two idiosyncratic employment states. Employed households are the typical optimizing consumers. Constrained or "unemployed" agents consume out of their financial wealth and out of government subsidies. Next, there is a producer side. This component is kept deliberately simple, consisting of perfectly competitive final goods producers and monopolistically competitive intermediate variety producing firms. The model features only price rigidities in the style of Rotemberg (1982). Finally, there is a policy maker who combines the role of the monetary and fiscal authority. The monetary authority pursues optimal (Ramsey) policy whilst the fiscal authority follows a simple tax rule. The fiscal authority also provides subsidies to constrained consumers in Lump Sum fashion and issues government

[7]TANK is an abbreviation that refers to the Two Agent New Keynesian model.

debt. The complete derivations for each policy block can be found in appendices B- D.

3.3.1 Households

There is a mass 1 of households, indexed by $i \in [0,1]$, who discount the future at rate β. They derive utility from consumption c_t^i and dis-utility from labor supply l_t^i. The felicity function is:

$$\ln\left(C_t^i\right) - \frac{\left(H_t^i\right)^{1+\varphi}}{1+\varphi}$$

As in Bilbiie & Ragot (2021) and Chien & Wen (2021) to reduce the complexity of the problem, we rely on simplifying assumption regarding the way heterogeneity enters the model. As discussed above, this is done using a generalization of the Lucas Jr (1990) "big representative family" metaphor[8].

The model allows for an idiosyncratic employment status shock in each period as well as aggregate uncertainty. The idiosyncratic state is denoted by $\nu_t \in \{e,u\}$. This shock is assumed to be identically and independently distributed (iid) across consumers and follows a two- state Markov process. These shocks are not uncorrelated with each other as well as with the aggregate shock. Namely, if $\nu_t = \{e\}$, then households are able to participate in the labour market in which case, they are earn labour and dividend income. Whereas, if $\nu_t = \{u\}$, individuals are not able to access the labour market and consume only via an unemployment subsidy (T_t^u) and/or through their financial wealth. In this framework, all consumers can have any position in both assets. As such, depending on the average mark up (λ) and the replacement rate (ϑ) which determines the level of direct wealth transfers, constrained households can be either "savers" or "borrowers". In the benchmark case, there are no entry or trading barriers in the asset market, so households regardless of their idiosyncratic status are free to adjust their asset position in each period. In order to create a redistributive role for monetary policy, we follow Cantore & Freund (2021) and introduce portfolio adjustment costs only for the constrained consumers.

We denote by $p^{e|e}$ the probability of a household having access to the labour market in period $t+1$, provided that they had access in period t (hence, the probability of moving "islands" - switch to not participating- is $p^{e|u}$). Similarly, we use $p^{u|u}$ to indicate the probability of keep being unemployed (not participating) in period $t+1$, conditional upon not participating in the labour market in period t (thus, the probability to switch to participating is denoted by $p^{u|e}$). The unconditional probabilities of the idiosyncratic employment and unemployment shocks are $p^e = \frac{p^{e|u}}{p^{e|u}+p^{u|e}}$ and

[8]The use of this generalisation of the "family head" and "island" metaphors has been widely used in the THANK literature. This simplifying tool has been generalized further, in different contexts, by Le Grand et al. (2022), Chien & Wen (2021), Bilbiie & Ragot (2021) and Bilbiie (2024); among other. Similarly, Heathcote & Perri (2018) have also relied on a generalization of Lucas Jr (1990) metaphor but they use a different simplifying family structure.

$p^u = \frac{p^{u|e}}{p^{e|u}+p^{u|e}}$ and must also satisfy $p^e + p^u = 1$. Relying on the law of large numbers, these probabilities reflect the share of employed (participating) and unemployed (not participating) households in aggregate population.

Furthermore, households belong to a big representative family whose head maximizes the intertemporal welfare of family members using a utilitarian welfare criterion. Still, the family head has access only to limited risk sharing technology due to the timing of the decisions.

The timing of the decisions is as follows. In each period, the family head first pools resources within the "island". Then, the aggregate shocks are realised and the family head decides the consumption plan and labour supply plan for each household, on each island. Following that, the households realise their idiosyncratic employment status for the period and move to the corresponding island. Since the family head cannot make transfers to households after the idiosyncratic shock is realised, they will have to take this as a constraint when solving their program. Once again, the timing of the actions is responsible for the imperfect risk sharing. To simplify our analysis, as in Bilbiie & Ragot (2021), in each period instead of keeping track of the entire distribution of wealth for each household type as in the fully-fledged Bewley-Hugget-Aiyagri model, we are going to assume that households' wealth can only two values. And, these values depend only on the households' most recent realisation of the idiosyncratic state.

The head of the representative family wishes to maximise the following Welfare criterion

$$U_o = \sum_{t=0}^{\infty} \left(\beta^t\right) \left[p^e \left(\ln\left(C_t^e\right) - \frac{\left(H_t^e\right)^{1+\varphi}}{1+\varphi} \right) + p^u \ln\left(C_t^u\right) \right]$$

subject to the budget constraint of each type.

- If the household is allowed to participate in the labour market then, the budget constraint of an employed consumer takes the form

$$\left(\begin{array}{c} C_t^e + (1+\pi_{t+1}) P_t^M \left(\frac{\alpha_{t+1}^{M(e)}}{p^e} \right) \\ + \left[\frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \right] \left(\frac{\alpha_{t+1}^e}{p^e} \right) \end{array} \right) = \left(\begin{array}{c} (1-\tau_t)\, w_t H_t^e + D_t^e \\ + \left(1+\rho P_t^M\right) \left(\frac{\hat{\alpha}_t^{M(e)}}{p^e} \right) + \left(\frac{\hat{\alpha}_t^e}{p^e} \right) - \frac{T_t^e}{p^e} \end{array} \right)$$

where,

$$\hat{\alpha}_t^{M(e)} = p^{e|e} a_t^{M(e)} + p^{e|u} a_t^{M(u)}$$

$$\hat{\alpha}_t^e = p^{e|e} a_t^e + p^{e|u} a_t^u$$

C_t^e is the consumption level of an employed consumer in period t. P_t^M and $\hat{\alpha}_t^{M(e)}$ are the price and the quantity of long- term government bonds, respectively. Similarly, $\hat{\alpha}_t^e$ stands for the quantity of short- term government bonds held by an employed household. The dividends distributed to an employed household are given as D_t^e but consumers do not internalise them. Furthermore, H_t^e and w_t represent the employed consumer's labour supply and the prevailing wage rate, respectively. Finally, τ_t stands for the current level of the distortionary income tax whilst T_t^e is current period's Lump Sum tax. In the baseline scenario, we assume that the policy maker has access to only one tax instrument and also, that taxes in general are levied only against unconstrained consumers.

- Similarly, if household is unable to participate in the labour market in the current period then, this household type's budget constraint takes the form

$$\left(\begin{array}{c} p^u C_t^u \\ +(1+\pi_{t+1}) P_t^M \alpha_{t+1}^{M(u)} \\ +\frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \alpha_{t+1}^u \end{array}\right) = \left(\begin{array}{c} p^u D_t^u + \left(1+\rho P_t^M\right) \hat{\alpha}_t^{M(u)} + \hat{\alpha}_t^u + T_t^u \\ -\frac{\Omega}{2}\left(NAP_t^u - NAP^u\right)^2 \end{array}\right)$$

where,

$$\hat{\alpha}_t^{M(u)} = a_t^{M(e)} p^{u|e} + a_t^{M(u)} p^{u|u}$$

$$\hat{\alpha}_t^u = a_t^e p^{u|e} + a_t^u p^{u|u}$$

$$\begin{aligned} NAP_t^u &= \left(\begin{array}{c} (1+\pi_{t+1}) P_t^M \alpha_{t+1}^{M(u)} + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \alpha_{t+1}^u \\ -\left(\begin{array}{c} \left(1+\rho P_t^M\right)\left(a_t^{M(e)} p^{u|e} + a_t^{M(u)} p^{u|u}\right) \\ +\left(a_t^e p^{u|e} + a_t^u p^{u|u}\right) \end{array}\right) \end{array}\right) \\ &\equiv T_t^u - p^u C_t^u \end{aligned}$$

With C_t^u being the consumption level of an unemployed household in period t. P_t^M and $\hat{\alpha}_t^{M(u)}$ are again the price and the quantity long- term government bonds, respectively. Now, $\hat{\alpha}_t^u$ stands for the quantity of short- term government bonds held by an unemployed consumer. D_t^u stands for the share of dividends paid to an unemployed household. Next, NAP_t^u refers to the net asset position of "unemployed" whilst Ω is a parameter controlling the size of the portfolio adjustment cost. Finally,

T_t^u represents the lump sum transfer (i.e.unemployment benefits) paid to the constrained household type, in each period.

Now, solving the family head's program yields:

1. The Bond Pricing equation

$$P_t^M = \mathbb{E}_t \left(\frac{\left(1+\rho P_{t+1}^M\right)}{(1+r_t)(1+\pi_{t+1})} \right)$$

where, the stochastic discount factor (SDF) takes the form

$$\begin{aligned} SDF &\equiv \mathbb{E}_t \left(\beta \left[p^{e|e} \frac{\psi_{t+1}^e}{\psi_t^e} + p^{u|e} \left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right) \frac{\psi_{t+1}^u}{\psi_t^e} \right] \right) \\ &= \mathbb{E}_t \left(\beta \left[\frac{p^{e|u}}{(1+\Omega(NAP_t^u - NAP^u))} \frac{\psi_{t+1}^e}{\psi_t^u} + p^{u|u} \frac{\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)}{(1+\Omega(NAP_t^u - NAP^u))} \frac{\psi_{t+1}^u}{\psi_t^u} \right] \right) \\ &= \frac{1}{(1+r_t)} \end{aligned}$$

2. The Consumption Euler Equation for each household type:

   - The Consumption Euler Equation for the unconstrained household type takes the form:

$$\frac{\psi_t^e}{(1+r_t)} = \beta \left(p^{e|e} \mathbb{E}_t \left(\psi_{t+1}^e\right) + p^{u|e} \left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right) \mathbb{E}_t \left(\psi_{t+1}^u\right) \right)$$

   - Whilst, the Consumption Euler Equation for the constrained household type is found to be:

$$(1+\Omega(NAP_t^u - NAP^u)) \frac{\psi_t^u}{(1+r_t)} = \beta \left(p^{e|u} \mathbb{E}_t \left(\psi_{t+1}^e\right) + p^{u|u} \left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right) \mathbb{E}_t \left(\psi_{t+1}^u\right) \right)$$

where,

$$\psi_t^e = (C_t^e)^{-1}$$

$$\psi_t^u = (C_t^u)^{-1}$$

are the marginal utility of consumption for each type.

3. The optimal labour supply of the unconstrained consumer type is given by

$$(1-\tau_t)\, w_t - C_t^e \left(H_t^e\right)^{\varphi} = 0$$

In general, the entries of the stochastic transitional matrix for the idiosyncratic employment shock are going to determine whether $C_t^e \lesseqgtr C_t^u$. This property is absent from the standard Aiyagari-type models. Choosing a symmetric probability matrix for the (un)employment shock $(p^{u|u} = p^{e|e})$ guarantees that both consumer types enjoy the same steady state consumption level. However, we choose to focus only in the case with imperfect risk sharing where, $(C_t^e > C_t^u)$. And in the absence of binding borrowing limits on constrained households the model also delivers "perfect self- insurance" (See proposition 8).

Proposition 8 *Under the assumption that both household types are a mass of Ricardian consumers, the model delivers perfect self- insurance* $\left(1+r=\frac{1}{\beta}\right)$. *Now, since we choose to focus only on the case of imperfect risk sharing* $(C_t^e > C_t^u)$, *we further require the idiosyncratic state to be persistent* $(p^{u|u} \geqslant p^{e|e})$ *which in turns implies both* $p^{u|e} + p^{e|e} \leq 1$ *and* $p^{u|u} + p^{e|u} \geqslant 1$.

(See Appendix A.1 for the proof of Proposition)

3.3.2 Firms

The economy features two production sectors. A perfectly competitive final good producing sector as well as a monopolistically competitive intermediate good sector. The final good producing firms are identical and thus, we model this sector as a single stand-in aggregate firm that is the typical CES aggregator- that combines intermediate varieties into the final good:

$$Y_t = \left[\int_0^1 \left(y_t\left(j\right)\right)^{\frac{\varepsilon_t - 1}{\varepsilon_t}} dj\right]^{\frac{\varepsilon_t}{\varepsilon_t - 1}}$$

where, Y_t denotes the quantity of the final good, $y_t(j)$ denotes the demand for intermediate input j, and $\varepsilon_t > 1$ governs the elasticity of substitution between any two intermediate varieties.

There is continuum $j \in [0,1]$ of intermediate good producing firms, each producing a differentiated variety. Each firm j produces its differentiated product according to the production function

$$y_t(j) = z_t h_t(j)$$

where, z_t is the aggregate technology (TFP) shock. We assume that technology is determined exogenously and follows an AR(1) process

$$z_{t+1} = \rho_z z_t + \varepsilon_{t+1}^z$$

where, $\varepsilon_{t+1}^z \overset{iid}{\sim} N\left(0, \sigma_z^2\right)$. Whereas, $h_t\left(j\right)$ stands for the labour demand of firm j.

Intermediate firms face a quadratic cost a- la- Rotemberg (1982) when changing their prices. The firm's problem becomes choosing $\left\{P_t\left(j\right)\right\}_{t=0}^{\infty}$ in order to maximise :

$$\max_{P_t(j)} \sum_{t=0}^{\infty} m_{0,t} \left(\left(\frac{P_t\left(j\right)}{P_t} - \left(1-s\right)\frac{w_t}{z_t} \right) y_t\left(j\right) - \frac{\Phi}{2}\left(\frac{P_t\left(j\right)}{P_{t-1}\left(j\right)} - 1 \right)^2 Y_t \right)$$

subject to monopolistic demand

$$y_t\left(j\right) = \left(\frac{P_t\left(j\right)}{P_t} \right)^{-\varepsilon_t} Y_t$$

Solving the firms' profit maximisation problem yields the *NK Price Phillips Curve*

$$\Phi\pi_t\left(1+\pi_t\right)\left(1+r_t\right) = \left(1-\varepsilon_t+\left(1-s\right)\varepsilon_t\frac{w_t}{z_t}\right)\left(1+r_t\right) + \Phi\mathbb{E}_t\left(\pi_{t+1}\left(1+\pi_{t+1}\right)\frac{Y_{t+1}}{Y_t}\right) \quad \text{(NKPC)}$$

3.3.3 Government

The model features a consolidated monetary-fiscal authority. The monetary authority has access to "commitment technology" and in every period, they optimally choose the level of the real interest rate $\{r_t\}$. Whereas, the fiscal authority optimally chooses the level of the outstanding government debt $\{b_t, b_t^M\}$. Then, given the outstanding government debt, the size of the direct wealth transfers made to the unemployed households (T_t^u) as well as the level of public spending$\{G_t = G_0 \in \mathbb{R}_+, \forall t\}$ they adjust the level of taxes to ensure fiscal solvency, following a simple rule. However, since the aim of the paper is the study of optimal monetary policy, we set the income taxes to zero ($\tau_t = \tau = 0, \forall t$). Thus removing the distortionary effects of income taxation from the optimal policy. Furthermore, since, changes in government spending and/or unemployment benefits typically involve parliamentary procedures, we are going to assume that they are held constant and are determined exogenously.

As such, the government budget constraint (GBC, henceforth), in real terms, takes the form

$$\left(1+\pi_{t+1}\right)P_t^M b_t^M + \frac{b_t}{\left(1+r_t\right)} + \tau_t w_t H_t = b_{t-1} + \left(1+\rho P_t^M\right) b_{t-1}^M + G_t + T_t^u - T_t^e$$

Where, the (aggregate) Lump Sum taxes/transfers in the economy are defined as

$$T_t = T_t^u - T_t^e$$

the difference between the direct wealth transfers (i.e. proxy for unemployment benefits) and the Lump Sum taxes imposed on unconstrained households. In the benchmark model, we have have set the wealth transfers as

$$T_t^u = \vartheta(1-\tau_t)w_t H_t^e p^u$$

With $\vartheta \in (0,1]$ being the parameter controlling the size of the (Lump Sum) unemployment benefits relative the the employed household's net labour income. Hence, we refer to ϑ as the replacement rate. If instead the government raises tax revenue using only Lump Sum taxes, then unemployment benefits take the form:

$$T_t^u = \vartheta(w_t H_t^e - \frac{T_t^e}{p^e})p^u$$

As discussed above, the tax instrument available always follows a simple rule. Since, the fiscal authority has access only to Lump Sum taxes :

$$T_t^e = \bar{T}^e \cdot \left(\frac{P_t^M b_t^M}{P^M b^{M(*)}}\right)^{\psi_b} \cdot \exp\left(\ln\left(\psi_t^b\right)\right)$$

Where, $b^{M(*)}$ stands for the exogenous steady state target for the real debt and,

$$\begin{aligned} \ln\left(\psi_t^b\right) &= \rho_\psi \ln\left(\psi_{t-1}^b\right) + \varepsilon_t^\psi \\ & \varepsilon_t^\psi \overset{iid}{\sim} N\left(0, \sigma_\psi^2\right) \end{aligned}$$

To simplify our analysis, we further assume that the short- term government bond is in zero net supply ($b_{t-1} = b_t = 0, \forall t$). Thus, we can simply drop both short- term bonds and distortionary income taxes from the government budget constraint. Finally, as discussed above, we are going to assume that the Lump Sum transfers made to constrained households are held constant ($T_t^u = \bar{T}^u, \forall t$) in the benchmark HANK- DLMP specification.

3.3.4 Market Clearing

In equilibrium, we require all markets to clear. Thus, by combining the households' budget constraint with the expression for the aggregate dividends and the government budget constraint, we obtain **the (aggregate) resource constraint**:

$$p^e C_t^e + p^u C_t^u = \left(1 - \frac{\Phi}{2}\pi_t^2\right) Y_t - G_t - \frac{\Omega}{2}\left(NAP_t^u - NAP^u\right)^2$$

Consumption Inequality(S_t)

Following Debortoli & Galí (2018) and Komatsu (2023), we employ a simple measure of consumption inequality (S_t):

$$S_t = 1 - \frac{C_t^u}{C_t^e}$$

This is an index of the consumption gap between the "constrained" and "unconstrained" households. If both consumer types have the same consumption level $(\vartheta = 1)$, then $S_t = 0$. However, in our model constrained consumers always receive transfers that are lower compared to the unconstrained agents' net labour income, $S_t \in (0, 1)$.

3.4 Competitive Equilibrium

The competitive equilibrium consists of a price vector $\left(p_t, w_t, P_t^M\right)$, a vector of policy instruments $\left(b_t, b_t^M, \tau_t, T_t^e, T_t^u, r_t\right)$ and an allocation $\left(C_t^e, C_t^u, H_t^e, b_{t+1}^M, b_{t+1}, NAP_t^u\right)$ that induces both the asset market and the goods market to clear, whilst each household type gets to maximise their individual utility. The private sector equilibrium is described by the following set of equations.

The marginal utility of consumption for employed

$$\psi_t^e = \left(C_t^e\right)^{-1} \qquad \text{(eq.(1))}$$

The marginal utility of consumption for unemployed

$$\psi_t^u = \left(C_t^u\right)^{-1} \qquad \text{(eq.(2))}$$

The Euler equation for an employed household

$$\frac{\psi_t^e}{(1+r_t)} = \beta\left(p^{e|e}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|e}\left(1 + \Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right) \qquad \text{(eq.(3))}$$

The Euler equation for an unemployed household

$$(1+\Omega(NAP_t^u-NAP^u))\frac{\psi_t^u}{(1+r_t)}=\beta\left(p^{e|u}\mathbb{E}_t\left(\psi_{t+1}^e\right)+p^{u|u}\left(1+\Omega\left(NAP_{t+1}^u-NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right) \tag{eq.(4)}$$

The labour supply equation for the employed household

$$(1-\tau_t)\,w_t-C_t^e(H_t^e)^{\varphi}=0 \tag{eq.(5)}$$

The Aggregate resource constraint

$$p^eC_t^e+p^uC_t^u=\left(1-\frac{\Phi}{2}\pi_t^2\right)Y_t-G_t-\frac{\Omega}{2}\left(NAP_t^u-NAP^u\right)^2 \tag{eq.(6)}$$

Bond Pricing Equation

$$P_t^M=\mathbb{E}_t\left(\frac{\left(1+\rho P_{t+1}^M\right)}{(1+r_t)\,(1+\pi_{t+1})}\right) \tag{eq.(7)}$$

New Keynesian Price Phillips Curve (NKPC)

$$\Phi\pi_t\,(1+\pi_t)\,(1+r_t)=(1-\varepsilon_t+(1-s)\,\varepsilon_t\frac{w_t}{z_t})(1+r_t)+\Phi\mathbb{E}_t\left(\pi_{t+1}\,(1+\pi_{t+1})\frac{Y_{t+1}}{Y_t}\right) \tag{eq.(8)}$$

Government Budget Constraint (GBC)

$$(1+\pi_{t+1})\,P_t^Mb_t^M+\tau_tw_tH_t^ep^e=\left(1+\rho P_t^M\right)b_{t-1}^M+G_t+T_t^u-T_t^e \tag{eq.(9)}$$

Definition of Consumption inequality

$$S_t=1-\frac{C_t^u}{C_t^e} \tag{eq.(10)}$$

Aggregate Labour Supply[9]

$$H_t=p^eH_t^e \tag{eq.(11)}$$

Aggregate Production

$$Y_t=z_t\,(p^eH_t^e) \tag{eq.(12)}$$

[9] Appendix C contains all the aggregation calculations.

Definition of Inflation

$$1+\pi_{t+1}=\frac{p_{t+1}}{p_t} \qquad \text{(eq.(14))}$$

Net asset position of Constrained households (NAP_t^u)

$$\begin{aligned} NAP_t^u &= \left(\begin{array}{c} (1+\pi_{t+1})\,P_t^M\alpha_{t+1}^{M(u)}+\frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)}\alpha_{t+1}^u \\ -\left(\begin{array}{c} \left(1+\rho P_t^M\right)\left(a_t^{M(e)}p^{u|e}+a_t^{M(u)}p^{u|u}\right) \\ +\left(a_t^e p^{u|e}+a_t^u p^{u|u}\right) \end{array} \right) \end{array} \right) (eq.(15)) \\ &\equiv T_t^u-p^uC_t^u \end{aligned}$$

3.5 Social Welfare Function

The policy maker maximizes an inter-temporal utilitarian welfare criterion. This social welfare function represents the aggregate welfare in the economy and as such, it is calculated as the weighted sum of the utility of each household type weighted by each type's share in the total population. Given these Pareto weights, the policy maker's objective can be written as maximizing

$$\sum_{t=0}^{\infty}(\beta)^t W_t$$

where W_t is the period t felicity function of the planner. The Social Welfare criterion takes the form:

$$W_t=p^u\ln(C_t^u)+p^e\ln(C_t^e)-p^e\frac{(1-\tau_t)\,w_t}{(1+\varphi)\,C_t^e}H_t^e$$

Alternatively, using the definition of consumption inequality, we can re- write the Social Welfare felicity as

$$W_t=p^u\ln(1-S_t)+p^u\ln(C_t^e)-p^e\frac{(1-\tau_t)\,w_t}{(1+\varphi)\,C_t^e}H_t^e$$

Meaning that the aggregate welfare function is increasing in the consumption of the each households type and decreasing in consumption inequality. Additionally, the policy maker takes into account the dis-utility caused by the aggregate labour supply as they seek to maximise the aggregate welfare. Since, we consider only Lump Sum taxes, we can drop the distortionary income tax term from the social welfare function.

Optimal Policy under Commitment

The Utilitarian policy maker wishes to maximise the Social Welfare criterion

$$W_t = p^u \ln(1 - S_t) + p^u \ln(C_t^e) - p^e \frac{w_t H_t^e}{(1+\varphi) C_t^e}$$

Subject to: the Consumption Euler equation of both types (eq.(3)-eq.(4)), the optimal labour supply (eq.(5)),the aggregate resource constraint(eq.(6)), the bond pricing equation (eq.(7)), the Phillips Curve (eq.(8)), the government budget constraint (eq.(9)), the definition of consumption inequality(eq.(10)) and the expression for the aggregate production (eq.(12)). The problem may also include auxiliary equations describing the marginal utility of consumption (eq.(1)- eq.(2) or the expression for the net asset position of constrained households (eq.(15)).
(See the policy maker's problem in detail the Appendix.)

3.6 Calibration and Simulations

We calibrate our model for the US economy for the period 1985-2021. The model is calibrated to quarterly frequency. In our benchmark HANK framework,we set the coefficient of relative risk aversion to $\gamma = 1$ whilst the Frisch elasticity of substitution is set at $\rho = 1/4$, to match the empirical evidence of Chetty (2012). Next, we follow the empirical estimates of Bayer et al. (2020) and set the persistence of the cost- push shock to $\rho_\varepsilon = 0.9$. The policy maker's discount rate, β is 0.9975 to replicate the annual real interest rate of 1 % percent, close to the US average. The annual real interest rate is calculated, using annualised data on the short term nominal interest rate from the FRED [10]. For simplicity, we further assume that all blocks in the economy (Consumers,firms and Policy maker) discount the future using the same discount factor.

As discussed above, we have adopted the Lucas Jr (1990) metaphor of a "big representative" family which results reduces the complexity of the analysis dramatically. This assumption, reduces the HANK model to a generalised TANK, where households switch type in each period with a given probability. We have chosen the entries of the transition matrix to reflect a "labour market participation rate" of $p^e = 0.65$. Consequently, the rate of non- participation in the labour market is $p^u = 1 - p^e = 0.35$, in line the US Bureau of Labor Statistics (BLS) data[11] for the 1985 -2021. Hence, the corresponding transition matrix is

[10] The data series used is REAINTRATREARAT1MO and was taken from the FRED database (https://fred.stlouisfed.org/).

[11] The data on the participation rate was the LNS11300000 and it taken from the BLS database(https://www.bls.gov/).

$$pp = \begin{bmatrix} p^{e|e} & p^{e|u} \\ p^{u|e} & p^{u|u} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.054 & 0.946 \end{bmatrix}$$

Where, $p^e = \frac{p^{e|u}}{p^{e|u}+p^{u|e}} \approx 0.65$ and $p^u = \frac{p^{u|e}}{p^{e|u}+p^{u|e}} \approx 0.35$ and also, satisfy $p^e + p^u = 1$. The law of large numbers informs us that these probabilities can be used as a proxy for the share of participating and non- participating households in the population. However, for the model to display the standard property of imperfect risk sharing so that constrained consumers have larger marginal utility of consumption, the entries of the transition matrix must also satisfy $p^{u|u} > p^{e|e}$ and $p^{u|e} + p^{e|e} < 1$ and $p^{u|u} + p^{e|u} > 1$ (See proposition 8, above).

For simplicity, the steady state level of the government expenditure to GDP ratio (G/Y) is chosen to be 0.0. Following the empirical estimates of Nekarda & Ramey (2011) we set the replacement rate $\vartheta = 0.4$. However, in the HANK- IAMP literature, the ϑ parameter is treated as free variable. More specifically, authors choose the value of ϑ so that the model delivers a specific amount of steady state consumption inequality. As such, when considering the HANK- IAMP specification, we set $\vartheta = 0.464$ so that the model to deliver (approximately) the same amount of steady state consumption inequality as our HANK- DLMP framework. We also consider an alternative parametrization where following the estimates of Karabarbounis & Chodorow-Reich (2014), we have set $\vartheta = 0.8$.

Following Leeper & Leith (2021), who calibrate their model for the same period, we have set the maturity of the outstanding debt $m = 5$ years (20 Quarters) since the average maturity of the US government debt is found to be about 5.4 years whilst the annualised debt to GDP ratio is found to be $+43\%$ [12].

As discussed above we further simplify our analysis by setting the aggregate supply of short term debt to zero. Now as for the portfolio adjustment costs, we consider two values for Ω. In the benchmark case, $\Omega = 0$, meaning that there are no trading barriers in the asset market and all agents are free to adjust their position at any point in time. We also consider an alternative specification where, as in Cantore & Freund (2021), we set $\Omega = 0.07$. This latter specification with $\Omega > 0$ allows us to explore the redistributive role of monetary.

Our model features nominal rigidities a- la Rotemberg (1982). The Rotemberg (1982) and Calvo (1983) models deliver the same results for the linearized version of the New Keynesian Phillips curve. We allow prices to adjust every 10 months (approximately). This price spell has been suggested by the majority of the empirical work[13].

The rigidity of the aggregate price level matches that of a typical Calvo model where micro-

[12]This calculation is done using data from the Federal Reserve Bank of Atlanta (https://www.atlantafed.org/) and is available in the the appendix of Leeper & Leith (2021); Leeper et al., 2021.

[13]See Klenow & Kryvtsov 2008,Gopinath & Rigobon 2008, Klenow & Malin 2010 and Kehoe & Midrigan 2015.

Table 3.1: Baseline Parameterisation

Key Parameters		Value	Source
Time discount factor	β	0.9975	FRED database
Coefficient of Risk aversion	γ	1	Bilbiie & Ragot (2021)
Inverse of Frisch Elasticity	ρ	$\frac{1}{4}$	Chetty (2012)
Average Mark- up	λ	31%	Hall (2018)
Average Price Stickiness	θ	0.75	Kehoe & Midrigan (2015)
Replacement rate	ϑ	0.4	Nekarda & Ramey (2011)
Annualised Debt-to-GDP	$\frac{P^M b^M}{4Y}$	43%	Leeper & Leith (2021)
Participation rate	p^e	0.65	BLS database
Fiscal response coefficient	ψ_b	0.0025	Kirsanova & Wren-Lewis (2012)
Persistence of the cost- push shock	ρ_ε	0.9	Bayer et al. (2020))
Portfolio Adjustment cost	Ω	0|0.07	Cantore & Freund (2021)

level prices change about once per year. Thus, we have chosen the average price stickiness to be $\delta = 0.75$, implying prices change approximately once every 10 months. Moreover, in line with the empirical estimates of Hall (2018), we have set the elasticity of substitution between intermediate varieties to be at $\varepsilon = 4.2$. Implying an average mark- up of about $\lambda = 31\%$. Furthermore, following Kirsanova & Wren-Lewis (2012) we have chosen the value of the fiscal response coefficient (ψ_b) to 0.0025. Since, this is the lowest value that ensured the system's stability. Thus, guaranteeing an active monetary policy. All computations presented in this paper were implemented in the RISE toolbox (See Maih, 2015).

3.7 Discussion

In this section we discuss the model's long- run equilibrium and dynamics under optimal monetary policy. First, we focus on the non- stochastic steady state and compare the long- run equilibrium of our benchmark HANK- DLMP economy to the steady state of the nested RANK model and also, against the steady state of the HANK- IAMP framework. For simplicity, we abstract from wage rigidity and assume that only prices are sticky across all specifications. We further assume that the size of the constrained population as well as the transitional probabilities are the same across the different THANK specifications. Overall, we find that in steady state the HANK- DLMP model exhibits higher consumption inequality, lower consumption for each type and thus, lower aggregate welfare. In our HANK framework, constrained consumers are not facing any binding borrowing constraints and they can be either "borrowers" or "savers", in equilibrium.

In fact, as shown by table 3.3 for moderate values of the replacement rate (ϑ) we observe that constrained consumers tend to be borrowers in equilibrium. As discussed above, we have set the income of unemployed households to equal to a fraction of the after- tax labour income of their unconstrained counterparts. As such, a lower average mark up increases the optimal labour supply of unconstrained consumers and with it size of the direct wealth transfers. A higher value for the elasticity of substitution (ε) increases the prevailing wage and hence the optimal labour supply whilst causing the dividend income to fall. Hence, as ε increases so does the non- financial income of constrained consumers. Once again, in the language of Bilbiie et al. (2020), we consider only a progressive tax system, as wealth transfers only redistribute wealth from unconstrained to constrained consumers.

3.7.1 The Non- Stochastic Steady State

In this subsection, we first discuss the differences in the long-run equilibrium between the HANK model with discontinuous labour market participation (HANK-DLMP) and the nested representative agent model. As discussed earlier, the main focus of our chapter is the conduct of optimal (Ramsey) monetary policy. To avoid the impact of distortionary taxation on optimal policy, we assume that the policymaker raises revenue using only lump-sum taxes. The fiscal authority follows a simple rule, where taxes deviate from their steady state value only when outstanding government debt deviates from the steady-state target. As shown by the definition of the tax rule, this target is exogenous. For comparability, the steady state level of outstanding government debt is set exogenously to reflect an annualized debt-to-GDP ratio of 43%, across all specifications.

Intuitively, since the focus of this chapter is to demonstrate how optimal monetary policy differs from the nested RANK model due to the presence of different Ricardian consumer types in an environment with a meaningful supply of government bonds, it follows that the exogenous aspect of the fiscal environment should be kept constant.

The presence of non- trivial amount of steady state consumption inequality somewhat alters the long- run equilibrium results of the nested RANK model. Although we include a direct wealth transfer mechanism to guarantee that constrained households are always able to have positive consumption, the size of transfer is not able to affect the steady state consumption inequality. After all, in an economy populated by only Ricardian consumers neither monetary nor fiscal policy can affect the equilibrium level of consumption inequality (See propositions 9 and 10).

Proposition 9 below explains the observed differences in the long- run equilibrium level of taxes, output and aggregate consumption between our benchmark HANK-DLMP framework and the nested RANK. Whereas, **proposition 10** explains the steady state consumption inequality found in the HANK- DLMP framework.

Proposition 9 *Regardless of the tax instrument available, in the zero inflation steady state ($\pi = 0$), the presence of non- participating households guarantees that the HANK- DLMP model will* *deliver higher steady state taxes than the nested RANK ($\bar{\tau}^{HANK} > \bar{\tau}^{RANK}$ or $\bar{T}^{HANK} > \bar{T}^{RANK}$) and lower aggregate consumption ($\bar{C}^{HANK} < \bar{C}^{RANK}$).*

(See Proof in Appendix A.2)

Intuitively, since a share of the population is unable to supply labour to the market and rely on government subsidies, both the aggregate labour supply and the tax base are smaller compared to the nested RANK.

Proposition 10 *In our HANK- DLMP framework, constrained households do not face a binding budget constraint in equilibrium. As such, the steady state level of consumption inequality (S) is independent of each household type's wealth or the level of the progressive wealth transfers (T^u). In fact, (S) depends solely on the transitional probabilities of the idiosyncratic (un)employment shock:*

$$
\begin{aligned}
S &= 1 - \frac{p^{u|e}}{\left(1 - p^{e|e}\right)} \\
&= 1 - \frac{p^{u|u}}{\left(1 - p^{e|u}\right)}
\end{aligned}
$$

(See Proof in Appendix A.3)

Moreover, we also compare the steady state of our benchmark HANK- DLMP model against that of the tractable HANK- IAMP model that we typically encounter in the literature. We consider two alternative values for the replacement rate. One value follows Bilbiie & Ragot (2021) approach and the alternative, delivers approximately the same steady state amount of consumption inequality as our HANK- DLMP framework. Despite the different assumptions regarding the non- financial income of constrained consumers, we can still think of the HANK- IAMP model as a special/nested case of our HANK- DLMP framework. This latter HANK specification corresponds to a scenario where constrained consumers face a binding equilibrium borrowing constraint and infinite portfolio adjustment costs.

Proposition 11 below explains the observed differences in the long- run equilibrium level of taxes between the HANK- IAMP framework and the nested RANK. Whereas, **proposition 12** explains the differences in the steady state consumption inequality observed across the different THANK frameworks.

Proposition 11 *The HANK- IMAP model with imperfect self- insurance ($\tilde{r} \neq r$) will always deliver lower equilibrium aggregate production than the nested RANK ($Y^{HANK} < Y^{RANK}$). Given our modelling assumptions that government spending and the debt- to- GDP ratio are exogenous and held constant across all specifications then, lump sum taxes are going to be higher in the HANK-*

IAMP framework ($T^{HANK} > T^{RANK}$) if and only if (iff, hereafter) $\tilde{r} > r\frac{Y^{HANK}}{Y^{RANK}}$. Whereas, under distortionary income taxes, $\tau^{HANK} > \tau^{RANK}$ iff $\tilde{r} > r\frac{w^{RANK}}{w^{HANK}}$.

(See Proof in Appendix A.2)

Interestingly, in the standard HANK- IAMP model with imperfect self- insurance, how steady state taxes relate the equilibrium tax level of the RANK model, depends on the tax instrument available as well as on the equilibrium level of the real interest rate and inflation. Furthermore, in this HANK specification, constrained consumers are unable to take any position in the asset market. Hence, changes in the amount of direct redistribution have a direct impact on the overall steady state amount of consumption inequality (S). This result is presented formally in proposition 12, below.

Proposition 12 *In the HANK- IAMP framework, the steady state consumption inequality depends not only on the transitional probabilities of the idiosyncratic state but also on the labour supply of the Ricardian household as well as on the replacement rate and the ratio of the value of the outstanding government debt to the after tax labour income. In steady state, consumption inequality takes the form:*

$$S = 1 - \vartheta \left(H^e\right)^{(1+\varphi)} - \left(\frac{p^{e|u}}{p^u}\right) \frac{\left(1+\rho P^M\right) b}{\left(1-\tau\right) w H^e} \left(H^e\right)^{1+\varphi}$$

(See Proof in Appendix A.3)

Hence, in this latter HANK specification, everything else equal, a higher value for the exogenous replacement rate (ϑ), translates to lower steady state consumption inequality. Since, a higher value for ϑ closes the gap between the labour productivity of optimising and hand- to- mouth agents without creating the need for a higher tax level to ensure fiscal solvency. Thus resulting in an overall higher steady state aggregate welfare. As discussed above, this exclusion of constrained consumers from financial markets, means that this household type is always facing a binding borrowing constraint ($\alpha_t^u = \alpha_t^{M(u)} = 0, \forall t$) coupled with an infinite portfolio adjustment cost ($\Omega \to \infty$). In this case, both monetary and fiscal policy can affect both the equilibrium level consumption inequality and are able to redistribute both consumption and wealth following an aggregate shock.

RANK vs HANK

Table 3.2 below displays a comparison between the steady state of the different HANK specifications and the nested RANK model. First, we look at the long- run equilibrium of our benchmark HANK framework. In the baseline HANK- DLMP model, the policy maker levies taxes only on unconstrained households. The policy maker uses the revenue raised through taxation to ensure fiscal solvency and redistributes part of it to constrained agents (i.e. unemployed workers) to ensure that constrained agents always have positive consumption.

Table 3.2: Steady State comparison of RANK vs HANK.

Steady State		**RANK**	**HANK- DLMP**	**HANK- IAMP**	
		(1)	(2)	(3)	(4)
Parameters					
Non- participation. rate	p^u	0.0%	35%	35%	35%
Replacement rate	ϑ	–	40%	80%	46.4%
Steady State					
Consumption of Unconstrained	C^R	0.7966	0.755	0.8004	0.79
Consumption of Constrained	C^u	–	0.4077	0.5905	0.4262
Lump Sum Tax	T^e	0.0436	0.1264	0.0797	0.0429
Real Interest rate	R	1.0025	1.0025	1.0301	1.0024
Net Inflation	$\bar{\pi}$	0	0	0.0033	-0.0002
Aggregate. Output	$\bar{Y}$	0.8366	0.6733	0.7669	0.7025
Real Wages	$\bar{w}$	0.7619	0.7619	0.7628	0.7619
Real Gov. Debt	b^M	0.0721	0.058	0.107	0.0602
Asset Prices	P^M	19.95	19.95	12.333	20.0654
Annualised Debt-to-GDP	$\frac{P^M b^M}{4Y}$	0.43	0.43	0.43	0.43
Social Welfare	W	-0.8675	-1.0405	-0.9499	-0.9929
Cons. Inequality	S	1	0.46	0.2623	0.4605

Conversely, in the nested RANK as well as in the HANK- IAMP model taxes are levied on the entire populations. In this latter HANK framework constrained agents consume only out of the exogenous transfer that they receive. As discussed above, in spite of the different assumptions regarding the constrained households' lack of access tot he financial market, this alternative HANK specification can still be thought a special case of HANK- DLMP model with a zero- liquidity constraint imposed on constrained households and infinite portfolio adjustment costs ($\Omega \to \infty$). Comparing the long run equilibrium of RANK model against the different HANK specifications yields several interesting results.

In line with proposition 8, we observe that the HANK- DLMP model delivers the "perfect self-insurance" result in equilibrium. As such, the steady state real interest rate is always found to be equal to the rate of time preference $\left(1+r=\frac{1}{\beta}\right)$. However, this result is absent from the HANK-IAMP framework. In this economy, the presence of binding equilibrium borrowing constraints cause proposition 8 to fail. From Bilbiie (2024) we know that this framework features "perfect insurance only within type" and in line with the Aiyagari (1994) literature, the equilibrium interest rate is different from the rate of time preference. As discussed above, we consider two distinct values for δ. In column 4, the value of delta is chosen so that the model delivers approximately the same steady state consumption inequality as the benchmark HANK- DLMP model. Whereas, in column 3 the value of δ delivers a steady state amount of consumption inequality that roughly follows the estimates of Karabarbounis & Chodorow-Reich (2014). Interestingly, when the two HANK frameworks deliver (approximately) the same amount of equilibrium consumption inequality, the real interest rate in the HANK- IAMP model is just below the rate of time preference and the steady state inflation is mildly negative. However, with higher replacement rate, the model delivers an equilibrium real interest rate above the value implied by complete markets and positive steady state inflation. In line with proposition 11, we find that in the HANK- IAMP specification reported in column 3, where $(\tilde{r} > \frac{1}{\beta} - 1)$ the steady state tax level is higher than the equilibrium tax rate found in the nested RANK. Whilst, column 4 displays the opposite result.

This is hardly surprising since the policy maker is maximising a Utilitarian Welfare criterion and they are not less "egalitarian" than households. As shown by Chang (2022), when monetary policy has an additional redistributive role due to the presence of binding borrowing limits on constrained households, the central bank is unable to simultaneously close the output gap, preserve the price stability and pursue reduction in inequality as an added objective. According to Chang 2022, for the HANK- IAMP model to deliver zero inflation in steady state, the policy maker is required to place higher weight on price stability compared to the other policy objectives.

On the other hand, our HANK- DLMP model does not feature binding borrowing limits for constrained households (in equilibrium) and thus, neither monetary policy nor fiscal policy can affect the steady state level of consumption inequality (See proposition 10). Hence, the result of Woodford 2003 that states “when the monetary authority has access to commitment technology the New Keynesian model will deliver zero inflation in steady state“, still holds. Even in the presence of unemployment, the economy is populated only by Ricardian consumers. As such, the policy maker is not facing any trade- off between “efficiency“ and “equity“ in equilibrium. Thus, the policy maker has no incentive to deviate from the steady state price stability objective. As a result, both the HANK- DLMP model and the nested representative agent framework always delivering the same steady state interest rate and inflation.

Once again, in line with proposition 10, the steady state consumption inequality in the HANK- DLMP framework depends only on the transitional probabilities of the idiosyncratic shock. So, given our calibration, we find that $S = 0.46$. Conversely, in the HANK- IAMP framework, the different values for δ result in the distinct levels of equilibrium consumption inequality (See proposition 12). In fact, as shown by columns 3 and 4, an increase in δ translates in higher consumption for constrained consumers and lower consumption inequality. Additionally, since an increase in δ does not require an increase in taxation to ensure fiscal solvency, in fact the opposite, it also results in higher aggregate welfare.

Moreover, as shown by proposition 9 above, regardless of the tax instrument available, the RANK model always delivers higher steady state aggregate consumption and lower taxes than the HANK- DLMP framework and the HANK- IAMP framework. In this latter specification,

Table 3.3: Steady state behaviour of Constrained consumers

HANK- DLMP							
Key Parameters							
Participation rate	p^e	65%	65%	65%	65%	65%	65%
Replacement rate	ϑ	40%	70%	40%	70%	40%	70%
Average Mark up	λ	31%	31%	15%	15%	20%	20%
Steady State							
Consumption of Unemployed	C^u	0.4077	0.4077	0.4384	0.4384	0.4537	0.4537
Direct Wealth transfers (Individual)	$\frac{T^u}{p^u}$	0.23819	0.21025	0.28254	0.24937	0.30624	0.270242
Net Asset Position of Unemployed	NAP^u	-0.16951	-0.19745	-0.15586	-0.18903	-0.14746	-0.18346

constrained households are forced to consume their entire non- financial income in each period which is an exogenous fraction of the Ricardian household type's net labour income. As a result, this frameworks assumes a higher aggregate labour supply than the HANK- DLMP model which also translates to higher output and a lower tax rate. Hence, the HANK- IAMP specification exhibits lower steady state taxes, higher aggregate production and higher aggregate consumption than the HANK- DLMP model.

The rationale behind these differences is that the HANK- IAMP specification does not feature direct wealth transfers. Yet, the non- financial income of constrained households is linked to the net income of their unconstrained counterparts. Now, since taxes are levied on the entire population, even if both models feature the same steady state consumption inequality, the aggregate consumption as well as the consumption of each type will be lower in the HANK- DLMP model. Nevertheless since constrained consumers in the HANK- IAMP model are always assumed to have lower labour supply than their unconstrained counterparts, the overall labour supply is still lower compared to the nested RANK. And, with aggregate consumption and aggregate output being linked thought the aggregate resource constraint we can conjecture that aggregate consumption is higher in the representative agent model.

Finally, from table 3.3 we can observe the steady state behaviour of unemployed consumers in the HANK- DLMP model. We find that the positive effect of a higher replacement rate is dominated by the adverse effect of higher taxation. Since both household types are Ricardian, a change in the replacement rate does not affect the optimal consumption plan. Moreover, given the average mark up in the economy, a replacement rate of $\vartheta = 40\%$ does not provide unemployed consumers with enough income to obtain the steady state consumption plan that is consistent with the maximisation of the policy maker's program. As such, constrained consumers are borrowers in equilibrium, causing the aggregate asset holdings in the HANK economy to be lower than in the nested RANK specification. Finally, a change the average mark up in the economy results in a change in the optimal consumption plan. As the elasticity of substitution between intermediate varieties increases, so does the equilibrium wage and thus the non- financial income of constrained consumers. Hence, as the average mark- up in the economy decreases, the net asset position of unemployed households improves.

3.8 Dynamics under optimal monetary policy

In this section, we discuss the aggregate dynamics of our THANK framework under optimal monetary policy, in response to an unanticipated positive cost push shock. First, we provide a comparison of the optimal monetary policy response between the RANK, HANK- DLMP and HANK-IAMP frameworks, under Lump Sum taxes.

This comparison allows us to investigate how the policy maker's response changes from the inclusion of household heterogeneity as well as due to the presence of redistributive channels in the transmission of optimal monetary policy. Propositions 13 below, explains why the sheer presence of household heterogeneity does not imply a redistributive role for monetary policy. Whilst propositions 14 and 15 clarify the conditions necessary for the presence of redistribution channels in the transmission mechanism of monetary policy in the two HANK economies.

Proposition 13 *In our benchmark HANK- DLMP model ($\Omega = 0$) all agents have the same elasticity of inter-temporal substitution. Hence, a change in the real interest rate has the same effect on the marginal utility of consumption of both household types:*

$$\frac{dR_t}{R_t} = \frac{d\psi_t^e}{\psi_t^e} = \frac{d\psi^u}{\psi_t^u}$$

(See Proof in Appendix A.4)
Thus, the admitting including household heterogeneity does not imply a role for redistributive channels in the monetary transmission mechanism. In the language of Auclert (2019), despite the presence of imperfect risk sharing, we are still able to obtain the representative- agent response ($\frac{dR_t}{R_t} = \frac{dC_t}{C_t}$). In our benchmark scenario where $\Omega = 0$, despite the fact that optimal monetary policy cannot redistribute consumption across households, it can still affect the wealth dispersion in the economy. As such, despite the lack of a redistribution motive for consumption, the optimal monetary policy is still fundamentally different compared to the nested RANK.

Next, proposition 14 below shows that by introducing portfolio adjustment costs to constrained consumers (only) ($\Omega > 0$), households in our HANK- DLMP economy no longer have equal exposure to aggregate shocks. And naturally, this creates a redistributive role for monetary policy in response to the shock.

Proposition 14 *In the HANK- DLMP model with portfolio adjustment costs, constrained consumers face trading barrier. In this case, a change in the real interest does not have the same effect on the marginal utility of consumption of both agent types ($\frac{d\psi_t^u}{dR_t} \neq \frac{d\psi_t^e}{dR_t}$). For an unconstrained consumer we find that :*

$$\frac{dR_t}{R_t} = \frac{d\psi_t^e}{\psi_t^e} - p^{u|e}\frac{R_t}{\psi_t^e}\left[\frac{d}{dR_t}\left[\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right]\right]dR_t$$

Whereas for a constrained consumer we observe that

$$\frac{dR_t}{R_t} = \frac{d\psi_t^u}{\psi_t^u} - \frac{\beta}{\psi_t^u}\left[\frac{d}{dR_t}\left(\mathbb{E}_t\left(\begin{array}{c} p^{e|u}\left(\frac{\psi_{t+1}^e}{(1+\Omega(NAP_t^u - NAP^u))}\right) \\ +p^{u|u}\frac{\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)}{(1+\Omega(NAP_t^u - NAP^u))}\psi_{t+1}^u \end{array}\right)\right)\right]dR_t$$

(See Proof in Appendix A.4)

In response to an aggregate shock, consumers want to adjust their asset position to smooth their inter-temporal consumption. Since constrained consumers now face a quadratic portfolio adjustment cost, they have unequal exposure to the aggregate shock compared to their unconstrained counterparts. Thus, a change in the real interest rate will redistribute consumption across household types. Nevertheless, the policy maker maximises a utilitarian welfare criterion where the utility of consumption for each type is weighted by their respective share in the total population. So, even though monetary policy can redistribute consumption, the policy maker places a higher weight on the consumption of unconstrained households. Similarly, proposition 15 below explains why we find a redistributive role for monetary policy in the HANK- IAMP framework.

Proposition 15 *In the HANK- IAMP model, a change in the real interest does not have the same effect on the marginal utility of consumption on both agent types. For the Ricardian consumer type we can still obtain the same response as in the benchmark HANK- DLMP model:*

$$\frac{dR_t}{R_t} = \frac{d\psi_t^e}{\psi_t^e}$$

Whereas, for the Keynesian household type, a change in the real interest has a different effect on their marginal utility of consumption (ψ_t^u):

$$\frac{d\psi_t^u}{\psi_t^u} = \frac{dR_t}{R_t}\left[\frac{d}{dR_t}\left(\frac{p^u}{T_t^u} + \frac{p^u}{p^{u|e}}\frac{1}{\left(\left(1+\rho P_t^M\right)b_t^M + b_t\right)}\right)\right]\frac{R_t}{\psi_t^u}$$

(See Proof in Appendix A.4)

In the HANK- IAMP specification, even in the event that aggregate debt exists in zero net supply, a change interest rate will have a distinct effect on the consumption of each type ($\frac{dR_t}{R_t} = \frac{d\psi_t^e}{\psi_t^e}$

& $\frac{d\psi_t^u}{dR_t} = 0$). As such, distinct household types in the economy described by either the HANK-DLMP model with financial frictions ($\Omega > 0$) or the HANK- IAMP model, have unequal exposure to the aggregate shock. Once more, this implies a role for redistributive channels in the monetary transmission mechanism.

Furthermore, propositions 14 and 15 above, also demonstrate the importance of the assumptions pertaining to the maturity structure of the households' balanced sheet. In an economy that features meaningful amount of aggregate debt, the effect of a change in the interest rate on household consumption depends on the re- evaluation effect of the agent's net asset position. As discussed above, this re- evaluation effect is linked to the asset price of bonds with longer maturity and shape the individual consumption and labour supply response.

Finally, we also explore how the different assumptions regarding the wealth transfers in the HANK- DLMP framework shape the optimal policy response. Allowing for time- varying transfers alters the policy maker's information set. And although the differences in the long- run equilibrium level for the key variables are quantitatively negligible, these assumptions can have a significant impact on the model's dynamics.

3.8.1 Response to an Unanticipated Cost- push shock

We now look at the dynamics, under optimal monetary policy, in response to an unanticipated unit (positive) cost push Shock. Taxes follow a simple rule that feed backs on debt. The first column reports the dynamics of the RANK model. The second column shows the response to the unanticipated shock of our benchmark HANK- DLMP economy with constant wealth transfers (T^u) and no entry or trading barriers in the asset market ($\Omega = 0$). Both specifications feature only Ricardian consumers who can freely adjust their asset position to smooth their inter-temporal consumption, in response to the shock. As such, despite the presence of household heterogeneity, the policy maker is unable to affect consumption inequality. Yet, their response will still impact the wealth dispersion in the economy (See proposition 13). Columns 3 and 4 show the two specifications where optimal monetary policy has a "redistributive role"[14] in response to a shock. In these two economies, households have unequal exposure to the aggregate shock (See propositions 14 and 15). Column 3 depicts the response of our HANK- DLMP model with constant wealth transfers (T^u) and portfolio adjustment costs imposed only on constrained consumers ($\Omega = 0.07$). Whilst column 4 reports the dynamics of the standard HANK- IAMP framework with $\delta = 0.8$. This value for the replacement rate is chosen so that the equilibrium consumption inequality is in line the empirical estimates.

[14]Once again, when referring to the "redistributive role" of monetary policy, we are making reference to its capacity to redistribute consumption among households.

Following Kirsanova & Wren-Lewis (2012), we set the fiscal feedback coefficient to be as small as possible to discourage the policy maker from pursuing passive monetary policy. Both the RANK and the HANK- IAMP frameworks are nested in our HANK- DLMP model. As discussed above, the HANK- IAMP can be thought as a special case where constrained households are bounded by a zero liquidity constraint and infinite portfolio adjustment costs. Whilst, the RANK specification corresponds to a version of the HANK- DLMP model with perfect risk sharing and no financial friction or binding equilibrium borrowing constraint for the unemployed.

As expected, the unanticipated positive cost push shock causes inflation to overshoot and reduce the value of the outstanding government debt across all specifications. Since, the fiscal policy is actively trying to ensure stable debt dynamics, the monetary authority pursues contractionary monetary policy following the shock, to suppress inflation. This initial inflation jump reduces the real value of the outstanding government debt causing taxes to drop following the shock. More specifically, the tax dynamics are determined jointly by the response of asset prices and the debt response. From the no- arbitrage condition, we know that asset prices are inversely related to the nominal interest rate. As such, since both the real interest rate as well as inflation rise immediately after the positive cost- push shock is realised, both asset prices and real aggregate debt initially fall. From figure 1, we can observe that since the dynamics of government debt do not display large deviations in response to the aggregate shock, the tax dynamics are determined by the dynamics of bond price. However, in line with well- known results of the optimal policy literature, the government debt "follows a path very close to a unit-root process, mirroring the path of debt under joint monetary–fiscal optimisation." (See Kirsanova & Wren-Lewis 2012).

As in Dávila & Schaab (2022), in response to the unanticipated positive demand shock, the initial inflation response is found to be stronger in the HANK- DLMP environments. Intuitively, the monetary authority has an incentive to use inflation to redistribute wealth and/or consumption towards constrained households. Even though, in the benchmark HANK- DLMP model, all households have the same exposure to the aggregate shock, allowing for positive inflation still redistributes wealth to unemployed consumers. Since, constrained consumers are typically found to be borrowers in equilibrium.

Furthermore, even in the presence of portfolio adjustment costs on constraint workers, the same argument still holds. In this case, in response to the shock, the monetary transmission mechanism can redistributive both consumption and wealth across households (See propositions 14). Following the shock, consumers want to adjust their net asset position and smooth their inter- temporal consumption. Constrained consumers face a quadratic adjustment cost and hence, these trading barriers penalise them proportionally to the size of the adjustment. Yet, in equilibrium this mass of households are still borrowers. So even if they are not able to freely adjust their portfolios following the shock, the initial rise in inflation still decreases the real value of their debt obligations.

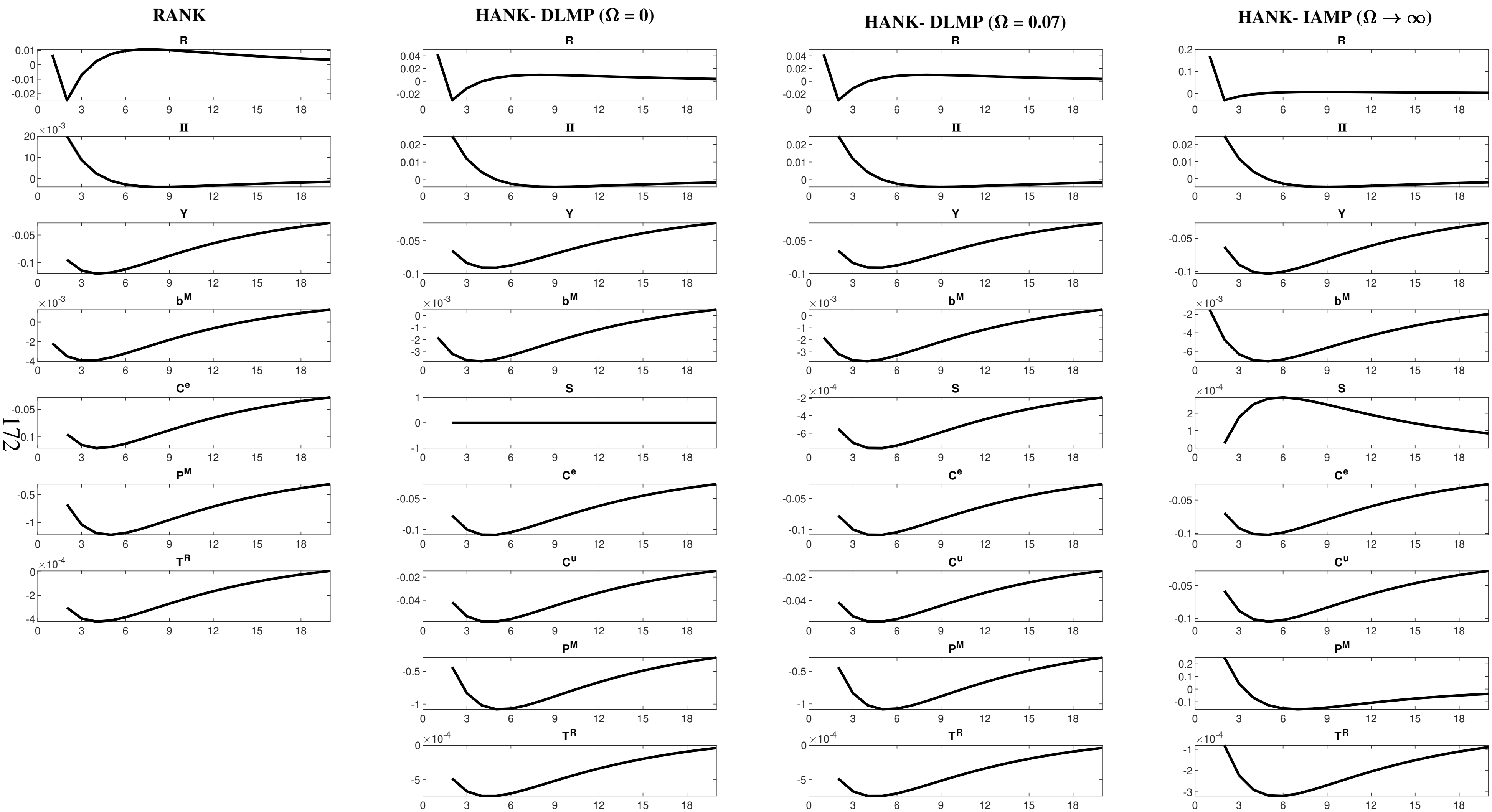

Figure 3.1: Policy response to a cost- push shock across all specifications.

On the other hand, this result is absent from the standard HANK- IAMP framework. In this HANK specification constrained consumers face a binding “zero- liquidity“ equilibrium constraint and very large (near infinite) portfolio adjustment costs. In this economy, the positive inflation deviation has an adverse effect on both household types. Nevertheless, unconstrained households are still free to adjust their portfolios’ and smooth their inter- temporal consumption. As such, in this HANK specification, consumption initially inequality rises in response to the positive cost push shock.

Moreover, after the initial jump, inflation falls below its steady state value and remains below for a number of periods. This behaviour is consistent with the evidence from the representative agent framework literature (See Woodford 2003 for a textbook treatment). However, the magnitude of the deviation differs across frameworks.

As expected, we find that the dynamics of aggregate output mirror the response of aggregate consumption. Since, unconstrained consumers make up a much higher percentage of the population, we find that the output dynamics closely traces the consumption dynamics of the unconstrained consumers, in specifications with household heterogeneity.

All in all, we find that the dynamics of the benchmark HANK- DLMP model are closest to the nested RANK model. This result comes as no surprise since, in line with proposition 13, we know that policy maker is not faced with a trade- off between “equity“ and “efficiency“ in these frameworks. Still, due to the very high persistence of the shock after twenty periods the economy has returned to steady state.

3.8.2 HANK- DLMP: Constant vs Time- varying Transfers

Figures 3.2 depicts a comparison between the two specifications of the HANK- DLMP model. In the benchmark case, we operate under the assumption that non- participating households receive a constant transfer in each period. This Lump Sum wealth transfer is equal to a constant fraction of the steady state after- tax labour income of their unconstrained counterparts ($T^u = \vartheta(wH^e - \frac{T^e}{p^e})p^u$). In the alternative specification, we relax this assumption and although the replacement rate is held constant, we allow the overall transfer levels made to non- participating households to be time- varying ($T_t^u = \vartheta(w_t H_t^e - \frac{T_t^e}{p^e})p^u$). In either case, we look at the policy response with and without redistributive channels in the transmission mechanism of optimal monetary policy ($\Omega = 0$ or $\Omega = 0.07$). This allows us to ascertain how the transmission mechanism of optimal monetary is affected by the assumptions regarding the direct wealth transfers, in response to an unanticipated aggregate shock.

As discussed above, in the latter scenario with time- varying wealth transfers, the policy maker has an augmented information set. More specifically, when the policy maker solves their program,

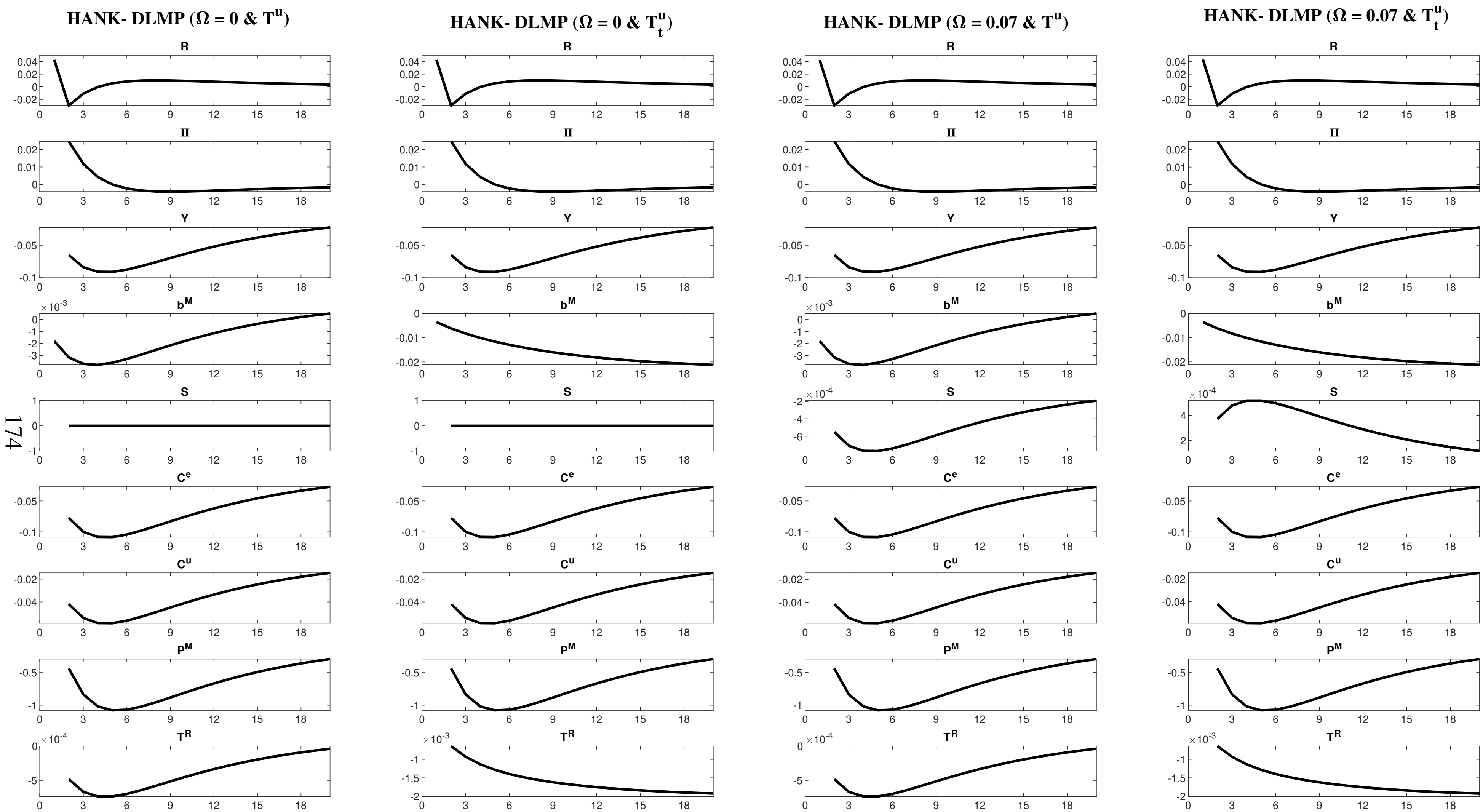

Figure 3.2: Policy response to a cost- push shock across all specifications.

they take into account how the optimal choice of the aggregate labour supply and the wage rate, will affect both participating and non- participating households. Still, under our assumption that the policy maker wishes to maximise a Utilitarian Welfare criterion, they still place higher weight on the consumption (utility) of unconstrained households.

Regardless of household heterogeneity, there is a unambiguous effect on inflation in response to a positive cost push shock. As the shock hits the economy, inflation initially overshoots and then drops below the steady state value, where it remains until the series converges back to the long- run equilibrium. As discussed above this initial jump in inflation reduces the value of the outstanding government debt.

As discussed above, in response to the shock, the debt dynamics resemble almost a unit- root. After the initially drop in the real value of outstanding government debt, the series takes very long time to return to equilibrium. However, relaxing the assumption of constant wealth transfers alters the dynamics of the aggregate debt, consumption inequality and taxes.

From the asset pricing equation we know that bond prices are inversely related to the nominal interest rate. Following the shock, since both the real interest rate and inflation initially rise, asset prices fall following the shock. However, under time- varying wealth transfers, the tax dynamics no longer mirror the dynamics of asset prices. In this case, the debt dynamics are much larger and thus the tax response seems to be a weighted average of the response of both series.

Furthermore, the presence of a redistributive role for monetary policy does not affect the consumption dynamics. Absence the effects of distortionary taxation on optimal policy, consumption (of both types) displays near identical dynamics across all the different frameworks. And, since aggregate consumption is linked with aggregate output through the resource constrained, we observe that output and consumption display near identical responses.

Finally, from the dynamics of aggregate output and inflation, respectively, we know that in response to the shock, both aggregate the labour supply and wages initially fall. And, their initial decrease is much larger than the initial decline in taxes. As such, in the HANK- DLMP model with financial frictions and time- varying transfers, the initial pursuit of contractionary (“active“) monetary policy in response to a rise in inflation causes consumption inequality to initially jump. Intuitively, the rise in inflation may have have reduced the real value of the outstanding debt obligations of constrained consumers however, the decrease in their non- financial income- due to both positive inflation and lower level of wealth transfers coupled the increased cost of borrowing worsens the financial condition of all households. Since, constrained consumers face trading barriers in the asset market, the overall consumption inequality initially rises following the shock. As output and wages start to move back towards their steady state value, the financial condition of constrained households begin to improve and inequality starts to fall. Yet, due to the very high persistence of the shock after twenty periods the economy has returned to steady state.

3.9 Conclusion

The paper shows that unemployment or rather discontinuous labour market participation is an important source of household heterogeneity separate to infrequent asset market participation frictions and it should not be overlooked.

Although the HANK- DLMP environment nests both the representative agent model as well as the HANK- IAMP as special cases, it is still fundamentally different from both. This difference is highlighted in the presence of financial frictions. In the baseline specification, there are no trading barriers in the asset market and thus, all households have equal exposure to an aggregate shock. In this case, optimal monetary policy can redistribute wealth but no consumption across agents, following a shock. Hence, even without redistributive channels in the transmission mechanism of monetary policy, the optimal policy response is still different to the nested RANK model. To ensure that constrained households always have positive consumption, we have departed from the assumption of Chien & Wen (2021) that non- participating households can consume only out of their financial wealth, and instead, we have introduced direct wealth transfers in each period. In the language of Bilbiie et al. (2020), we consider only a progressive tax system since wealth transfers are only made from unconstrained to constrained consumers. These transfers are set to be equal to constant fraction of the after tax labour income of an unconstrained (i.e. employed) agent. Whether these transfers are constant or time- varying does not affect significantly the long- run equilibrium of the model. Still, the assumption has a non- negligible effect on the optimal monetary policy response to the unexpected shock. Since, with time- varying transfers the policy maker has an augmented information set.

In the benchmark HANK model, these transfers are financed by levying taxes on unconstrained households. As such, their level directly affects the level of taxes necessary to ensure fiscal solvency. However, the steady state level of consumption inequality depends solely on the transitional probabilities of the idiosyncratic (un)employment shock and is unaffected by the level of the direct wealth transfers (See proposition 10). Next, we show that following a shock, monetary policy can affect redistribution only in models that feature unequal access to the financial markets and/or Keynesian consumer types (See proposition 14 and 15). Furthermore, contrary to the predictions of the typical HANK model, our HANK- DLMP specification also delivers perfect self-insurance despite the presence of imperfect risk sharing and/or portfolio adjustment costs for constrained consumers (See proposition 8). We further we find that the Woodford (2003) result of zero steady state inflation always holds for our benchmark HANK specification since the policy maker is not faced with a trade- off between “equity“ and “efficiency“ in equilibrium.

The same result is not present in the HANK- IAMP model. In line with by Chang (2022), we find that when there is a binding equilibrium borrowing constraint on constrained households, a policy maker who maximises a Utilitarian Welfare criterion cannot simultaneously pursue equilibrium price stability objective, close the output gap and target reduction in inequality.

All in all, optimal monetary policy is fundamentally different in the HANK- DLMP environment compared to both the typical HANK- IAMP framework and the nested RANK model. Discontinuous labour market participation frictions are an important source of household heterogeneity that should not be overlooked.

3.10 Appendix

The appendices are available upon request.

3.11 Appendix A

Proof of Proposition 8

Full Self- Insurance in the HANK- DLMP framework

Meaning that the **Consumption Euler Equation for an unconstrained household** takes the form

$$\frac{\psi_t^e}{(1+r_t)} = \beta\left(p^{e|e}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|e}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right)$$

Meaning that the **Consumption Euler Equation for an constrained household** takes the form

$$\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\frac{\psi_t^u}{(1+r_t)} = \beta\left(p^{e|u}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|u}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right)$$

Where, the non- participating households' Net asset position (NAP_t^u)

$$\begin{aligned} NAP_t^u &= \left(\begin{array}{c} (1+\pi_{t+1})P_t^M\alpha_{t+1}^{M(u)} + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)}\alpha_{t+1}^u \\ -\left(\left(1+\rho P_t^M\right)\left(a_t^{M(e)}p^{u|e} + a_t^{M(u)}p^{u|u}\right) + \left(a_t^e p^{u|e} + a_t^u p^{u|u}\right)\right) \end{array}\right) \\ &\equiv T_t^u - p^u C_t^u \end{aligned}$$

Hence, the stochastic discount factor is given (SDF)

$$
\begin{aligned}
SDF &\equiv \mathbb{E}_t\left(\beta\left[p^{e|e}\frac{\psi^e_{t+1}}{\psi^e_t}+p^{u|e}\left(1+\Omega\left(NAP^u_{t+1}-NAP^u\right)\right)\frac{\psi^u_{t+1}}{\psi^e_t}\right]\right)\\
&= \mathbb{E}_t\left(\beta\left[\frac{p^{e|u}}{\left(1+\Omega\left(NAP^u_t-NAP^u\right)\right)}\frac{\psi^e_{t+1}}{\psi^u_t}+p^{u|u}\frac{\left(1+\Omega\left(NAP^u_{t+1}-NAP^u\right)\right)}{\left(1+\Omega\left(NAP^u_t-NAP^u\right)\right)}\frac{\psi^u_{t+1}}{\psi^u_t}\right]\right)\\
&= \frac{1}{(1+r_t)}=\frac{1}{R_t}
\end{aligned}
$$

In the Non- Stochastic steady state we have

$$
\mathbb{E}_t\left(\beta\left[p^{e|e}+p^{u|e}\frac{\psi^u}{\psi^e}\right]\right)=\mathbb{E}_t\left(\beta\left[p^{e|u}\frac{\psi^e}{\psi^u}+p^{u|u}\right]\right)=\frac{1}{(1+r)}=\frac{1}{R}
$$

- So, if we require both consumer types enjoy the same consumption

$$
\psi^u=\psi^e\Leftrightarrow C^u=C^e
$$

and

$$
p^{e|e}+p^{u|e}=p^{e|u}+p^{u|u}
$$

Meaning that this case requires a symmetric transition matrix and consequently, the share of constrained and unconstrained consumers are equal. Since the $p^{e|e}, p^{u|e}, p^{e|u}, p^{u|u}$ are the entries of a stochastic transition matrix the must also satisfy that

$$
p^{e|e}+p^{e|u}=1
$$

$$
p^{u|e}+p^{u|u}=1
$$

$$
p^{e|e}+1-p^{u|u}=1-p^{e|e}+p^{u|u}
$$

Hence,

$$
p^{e|e}=p^{u|u}
$$

$$
p^{e|u}=p^{u|e}
$$

So, the share of constrained households is

$$p^e = \frac{p^{e|u}}{p^{e|u} + p^{u|e}} = \frac{p^{u|e}}{p^{e|u} + p^{u|e}} = p^u$$

And thus,

$$\beta = \frac{1}{(1+r)} = \frac{1}{R}$$

- Alternatively, if we want to focus on the more realistic case where

$$\psi^u > \psi^e \Leftrightarrow C^e > C^u$$

then, for the SDF to hold we require that in equilibrium

$$p^{e|e} + p^{u|e}\frac{\psi^u}{\psi^e} = p^{e|u}\frac{\psi^e}{\psi^u} + p^{u|u}$$

$$p^{u|e}\frac{\psi^u}{\psi^e} - p^{e|u}\frac{\psi^e}{\psi^u} = p^{u|u} - p^{e|e}$$

$$p^{u|e}\frac{\psi^u}{\psi^e} - p^{e|u}\frac{\psi^e}{\psi^u} = 1 - p^{u|e} - 1 + p^{e|u}$$

$$p^{u|e}\frac{\psi^u}{\psi^e} + p^{u|e} = p^{e|u}\frac{\psi^e}{\psi^u} + p^{e|u}$$

$$p^{u|e}\left(\frac{\psi^u}{\psi^e} + 1\right) = p^{e|u}\left(\frac{\psi^e}{\psi^u} + 1\right)$$

With, $\psi^u > \psi^e \Rightarrow \left(\frac{\psi^u}{\psi^e} + 1\right) > \left(\frac{\psi^e}{\psi^u} + 1\right)$
So for the equality to hold we require

$$p^{u|e} < p^{e|u}$$

Consequently, using the property of the stochastic transition matrix that $p^{u|e} + p^{u|u} = 1$ and $p^{e|e} + p^{e|u} = 1$ we find that the above result also translates to

$$p^{u|u} > p^{e|e}$$

In which case, the model implies that $p^{u|e} + p^{e|e} < 1$ and $p^{u|u} + p^{e|u} > 1$.

Now, let us turn our focus on the on the "Prefect Self- Insurance" result. Suppose, the model delivers the typical Aiyagari (1994) result that in steady state $C^e > C^u$ and $R < \frac{1}{\beta}$.

Once again, from the expression for the SDF, we have

$$\mathbb{E}_t\left(\beta\left[p^{e|e} + p^{u|e}\frac{\psi^u}{\psi^e}\right]\right) = \mathbb{E}_t\left(\beta\left[p^{e|u}\frac{\psi^e}{\psi^u} + p^{u|u}\right]\right) = \frac{1}{(1+r)} = \frac{1}{R}$$

And in the Non- Stochastic steady state

$$\beta\left[p^{e|e} + p^{u|e}\frac{\psi^u}{\psi^e}\right] = \beta\left[p^{e|u}\frac{\psi^e}{\psi^u} + p^{u|u}\right] = \frac{1}{(1+r)} = \frac{1}{R}$$

Now, if $\frac{1}{R\beta} > 1$ then

$$\left[p^{e|e} + p^{u|e}\frac{\psi^u}{\psi^e}\right] = \left[p^{e|u}\frac{\psi^e}{\psi^u} + p^{u|u}\right] = \frac{1}{\beta(1+r)} = \frac{1}{\beta R} > 1$$

Which requires both

$$p^{e|e} + p^{u|e}\frac{\psi^u}{\psi^e} > 1$$

$$1 - p^{e|u} + p^{u|e}\frac{\psi^u}{\psi^e} > 1$$

$$p^{u|e} > \frac{\psi^e}{\psi^u}p^{e|u}$$

and,

$$p^{e|u}\frac{\psi^e}{\psi^u} + p^{u|u} > 1$$

Since $\psi^u > \psi^e$ whilst $p^{i|j} \in (0,1)$ and at the same time $p^{u|e} + p^{e|e} < 1$ and $p^{u|u} + p^{e|u} > 1$

This means for

$$p^{u|u} > 1 - p^{e|u}\frac{\psi^e}{\psi^u} > 1 - p^{u|e}$$

From the properties of the stochastic matrix we know that

$$p^{u|u} = 1 - p^{u|e}$$

which leads to a contradiction.
On the other hand, if $\frac{1}{R\beta} > 1$ then

$$\left[p^{e|e} + p^{u|e}\frac{\psi^u}{\psi^e}\right] = \left[p^{e|u}\frac{\psi^e}{\psi^u} + p^{u|u}\right] = \frac{1}{\beta(1+r)} = \frac{1}{\beta R} < 1$$

Which requires both

$$p^{e|e} + p^{u|e}\frac{\psi^u}{\psi^e} < 1$$

and,

$$p^{e|u}\frac{\psi^e}{\psi^u} + p^{u|u} < 1$$

Since $p^{i|j} \in (0,1)$ and at the same time $p^{u|e} + p^{e|e} < 1$ and $p^{u|u} + p^{e|u} > 1$

Let's re- arrange the inequalities

$$p^{e|e} < 1 - p^{u|e}\frac{\psi^u}{\psi^e}$$

$$p^{u|u} < 1 - p^{e|u}\frac{\psi^e}{\psi^u}$$

Using the properties of the stochastic matrix that $p^{u|u} = 1 - p^{u|e}$ we have

$$1 - p^{u|e} < 1 - p^{e|u}\frac{\psi^e}{\psi^u}$$

$$p^{e|u} < p^{u|e}\frac{\psi^u}{\psi^e}$$

But since,

$$p^{e|e} < 1 - p^{u|e}\frac{\psi^u}{\psi^e}$$

it implies

$$p^{e|e} < 1 - p^{u|e}\frac{\psi^u}{\psi^e} < 1 - p^{e|u}$$

$$p^{e|e} < 1 - p^{e|u}$$

Which again leads to a contradiction. Hence the model delivers perfect insurance in steady state.

Proof of Proposition 9 & 11

Steady State taxes and Agg. Consumption:

RANK vs HANK- DLMP

Comparison of Steady State Income Tax between the RANK and the HANK -DLMP model

Looking again at the government budget constraint, we can re- write it in terms of tax revenue

$$\tau_t w_t p^e H_t^e + p^e T_t^e = \frac{\left(1+\rho P_t^M\right)}{(1+\pi_t)} b_{t-1}^M - P_t^M b_t^M + G_t + p^u T_t^u$$

Using the definition of the asset pricing equation, we can substitute $\frac{\left(1+\rho P_t^M\right)}{(1+\pi_t)}$ with $P_{t-1}^M\left(1+r_{t-1}\right)$

$$\tau_t w_t p^e H_t^e + p^e T_t^e = P_{t-1}^M\left(1+r_{t-1}\right) b_{t-1}^M - P_t^M b_t^M + G_t + p^u T_t^u$$

In the baseline model, the policy maker uses only distortionary income taxes to raise revenue $(T_t^e = 0)$ and pays Lump Sum Transfers to unemployed households,

$$\tau_t w_t p^e H_t^e = P_{t-1}^M\left(1+r_{t-1}\right) b_{t-1}^M - P_t^M b_t^M + G_t + p^u T_t^u$$

So, the level of the distortionary income tax is found to be

$$\tau_t = \frac{P_{t-1}^M\left(1+r_{t-1}\right) b_{t-1}^M - P_t^M b_t^M}{p^e H_t^e w_t} + \frac{G_t + p^u T_t^u}{p^e H_t^e w_t}$$

The rationale behind this assumption($T_t^e = 0$ and $\tau_t \geqslant 0$) is very straight forward. In reality we do notobserve Lump Sum taxes but unemployment benefits are paid in Lump Sum fashion. Thus, even in the absence of direct redistribution $(T_t^u = 0)$, since a smaller fraction of the economy's population is getting taxed $\left(p^e H_t^e < H_t^{RANK}\right)$, the marginal tax rate needs to be higher in order to insure fiscal solvency.

By construction, both in the HANK- DLMP framework as well as in the nested RANK economy, output(Y_t) is produced using only labour(H_t) and technology(z_t).

In the HANK economy, aggregate output is given as

$$Y_t^H = p^e H_t^e z_t$$

whereas, in the nested representative agent model

$$Y_t^R = H_t z_t$$

So, in steady state

$$Y^H = p^e H^e \bar{z}$$

and

$$Y^R = H\bar{z}$$

where, $\bar{z} = 1$

Meaning that the steady state tax in each economy is given as

$$(\tau^{HANK} \equiv)\tau^H = \frac{r}{w}\frac{P^M b^M}{Y^H} + \frac{1}{w}\frac{G_0 + p^u T^u}{Y^H}$$

and

$$(\tau^{RANK} \equiv)\tau^R = \frac{r}{w}\frac{P^M b^M}{Y^R} + \frac{1}{w}\frac{G_0}{Y^R}$$

As shown by proposition 8, in steady state the HANK- DLMP framework features perfect self insurance hence, $r^{HANK} = r^{RANK} = r = \frac{1}{\beta} - 1$. Furthermore, in the zero inflation steady state$\left(\pi^{HANK} = \pi^{RANK} = 0\right)$, wages depend only on the elasticity of substitution between intermediate varieties and since the two models **share the same calibration** $\left(\varepsilon^{HANK} = \varepsilon^{RANK}, G_0 = 0\right)$, $w^{RANK} = w^{HANK} = w = \bar{z}\frac{\varepsilon - 1}{(1-s)\varepsilon}$. Finally, although the aggregate amount of outstanding government debt may vary between the two economies, the steady state debt to GDP ratio is exogenously fixed $\bar{b} = \frac{P^M b^M}{4Y^H} = \frac{P^M b^M}{4Y^R} > 0$.

$$\begin{aligned} w\tau^{H} &= 4\bar{b}r + \frac{G_0 + T^u}{Y^H} \\ &= 4\bar{b}r + \frac{T^u}{Y^H} \end{aligned}$$

$$\begin{aligned} w\tau^{RANK} &= 4\bar{b}r + \frac{G_0}{Y^R} \\ &= 4\bar{b}r \end{aligned}$$

$$\frac{\tau^H}{\tau^R} = 1 + \frac{T^u}{4\bar{b}rY^H}$$

Meaning that under a progressive tax system ($T^u > 0$)

$$\tau^{HANK} > \tau^{RANK}$$

Now, if instead the policy maker raises tax revenue using Lump sum taxes($T \geqslant 0$ and $\tau_t = 0$), the tax revenue in each economy is given as

$$T_t^e = P_{t-1}^M (1 + r_{t-1}) b_{t-1}^M - P_t^M b_t^M + G_t + T_t^u$$

and

$$T_t^R = P_{t-1}^M (1 + r_{t-1}) b_{t-1}^M - P_t^M b_t^M + G_t$$

In steady state, under the same calibration (as discussed above),

$$\begin{aligned} T^e &= rP^M b^M + G_0 + T^u \\ &= 4\bar{b}Y^H r + T^u \end{aligned}$$

With $T^u = \vartheta \left(w^H H^e - \frac{T^e}{p^e} \right) p^u$

$$T^e = 4\bar{b}Y^H r + \vartheta \left(w^H H^e - \frac{T^e}{p^e} \right) p^u$$

$$\left(1 + \frac{p^u}{p^e}\right) T^e = 4\bar{b}Y^H r + \vartheta \left(w^H H^e\right) p^u$$

Now, since $Y^H = \bar{z}p^e H^e = p^e H^e$ and $p^u + p^e = 1$

$$\left(\frac{1}{p^e}\right) T^e = 4\bar{b}Y^H r + \vartheta\left(w^H H^e\right) p^u$$

$$T^e = 4p^e\bar{b}Y^H r + \vartheta\left(w^H H^e p^e\right) p^u$$

$$T^e = 4p^e\bar{b}Y^H r + \vartheta p^u w^H Y^H$$

Once again, we have assumed that $G_0 = 0$ (across all specifications). Similarly, in the steady state of the RANK economy

$$\begin{aligned} T^R &= rP^M b^M + G_0 \\ &= 4\bar{b}Y^R r \end{aligned}$$

Hence,

$$\frac{T^e}{T^R} = \frac{4p^e\bar{b}Y^H r}{4\bar{b}Y^R r} + \frac{\vartheta p^u w^H Y^H}{4\bar{b}Y^R r}$$

$$\frac{T^e}{T^R} = p^e\frac{Y^H}{Y^R} + p^u\frac{\vartheta w^H}{4\bar{b}r}\frac{Y^H}{Y^R}$$

$$\frac{T^e}{T^R} = \left(1 - p^u + p^u\frac{\vartheta w^H}{4\bar{b}r}\right)\frac{Y^H}{Y^R}$$

$$\frac{T^e}{T^R} = \left(1 + p^u\frac{\vartheta w^H - 4\bar{b}r}{4\bar{b}r}\right)\frac{Y^H}{Y^R}$$

Since, $4\bar{b}r < \vartheta w^H \Rightarrow p^u\frac{\vartheta w^H - 4\bar{b}r}{4\bar{b}r} > 1$

Namely, for our benchmark calibration, $\left(1 + p^u\frac{\vartheta w^H - 4\bar{b}r}{4\bar{b}r}\right) \approx 25.456$.
Hence,

$$T^e > T^R$$

Now, let's look at steady state aggregate consumption. Using the steady state expression for the aggregate resource in each economy, we can re- write the steady state aggregate consumption as

$$C^{RANK} = \left(1 - \frac{\Phi}{2}\pi^2\right) Y^R - G_0$$

and

$$p^e C^e + p^u C^u = \left(\left(1 - \frac{\Phi}{2}\pi^2\right) Y^H - \frac{\Omega}{2}\left(NAP^u - NAP^u\right)^2 - G_0\right)$$

$$C^{HANK} = \left(1 - \frac{\Phi}{2}\pi^2\right) Y^H - G_0$$

In the benchmark calibration, $G_0 = 0$ meaning that in the zero inflation steady state

$$\frac{C^{HANK}}{C^{RANK}} = \frac{Y^H}{Y^R} < 1$$

$$C^{HANK} < C^{RANK}$$

RANK vs HANK- IAMP

In the HANK- IAMP model, the aggregate labour supply is always lower compared to nested representative agent model

$$H_t^R > p^e H_t^e$$

So, in steady state,

$$H^R > p^e H^e$$

Once again, let us look at the government budget constraint. We can re- write the government budget constraint in terms of tax revenue

$$\tau_t w_t \left(p^e H_t^e\right) + T_t^e = \frac{\left(1 + \rho P_t^M\right)}{\left(1 + \pi_t\right)} b_{t-1}^M - P_t^M b_t^M + G_t + T_t^u$$

Using the definition of the asset pricing equation, we can substitute $\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)}$ with $P_{t-1}^M \left(1 + \tilde{r}_{t-1}\right)$

$$\tau_t w_t \left(p^e H_t^e\right) + T_t^e = P_{t-1}^M \left(1+\tilde{r}_{t-1}\right) b_{t-1}^M - P_t^M b_t^M + G_t + T_t^u$$

In the baseline model, the central planner uses only distortionary income taxes raise tax revenue $(T_t^e = 0)$ and pays Lump Sum Transfers to unemployed households,

$$\tau_t w_t \left(p^e H_t^e\right) = P_{t-1}^M \left(1+\tilde{r}_{t-1}\right) b_{t-1}^M - P_t^M b_t^M + G_t + T_t^u$$

So, the level of the distortionary income tax is found to be

$$\tau_t^H = \frac{P_{t-1}^M \left(1+\tilde{r}_{t-1}\right) b_{t-1}^M - P_t^M b_t^M}{\left(p^e H_t^e\right) w_t} + \frac{G_t + T_t^u}{\left(p^e H_t^e\right) w_t}$$

The rationale behind this assumption$(T_t^e = 0$ and $\tau_t \geqslant 0)$ is very straight forward. In reality we do not observe Lump Sum taxes but unemployment benefits are paid in Lump Sum fashion. Thus, even in the absence of direct redistribution $(T_t^u = 0)$, since a smaller fraction of the economy's population is getting taxed $\left(p^e H_t^e < H_t^{RANK}\right)$, the marginal tax rate needs to be higher in order to insure fiscal solvency.

By construction, both in the HANK- IAMP framework as well as in the nested RANK economy, output(Y_t) is produced using only labour(H_t) and technology(z_t).

In the HANK economy, aggregate output is given as

$$Y_t^H = \left(p^e H_t^e + p^u \delta\right) z_t$$

whereas, in the nested representative agent model

$$Y_t^R = H_t z_t$$

So, in steady state

$$Y^H = \left(p^e H^e\right) \bar{z}$$

and

$$Y^R = H \bar{z}$$

Since, the HANK frame work features lower aggregate labour supply,

$$Y^R > Y^H$$

With $\bar{z} = 1$ and $T_t^u = T^u = 0$, we can re-write the expression for the steady state tax in each economy as

$$\tau^H = \frac{\tilde{r}}{w} \frac{P^M b^M}{Y^H} + \frac{1}{w} \frac{G_0}{Y^H}$$

and

$$\tau^{RANK} = \frac{r}{w} \frac{P^M b^M}{Y^R} + \frac{1}{w} \frac{G_0}{Y^R}$$

For our calibration, both the steady state debt- to- GDP ratio and government spending are held constant: $\frac{P^M b^M}{Y} = \bar{\zeta}$ and $G_0 = 0$.

$$\tau^H = \frac{\tilde{r}}{w_H} \bar{\zeta}$$

and

$$\tau^{RANK} = \frac{r}{w_R} \bar{\zeta}$$

So,

$$\begin{aligned} \frac{\tau^{HANK}}{\tau^{RANK}} &= \frac{\frac{\tilde{r}}{w_H}}{\frac{r}{w_R}} \\ &= \left(\frac{\tilde{r}}{r} \frac{w_R}{w_H} \right) \end{aligned}$$

So, whether $\tau^{HANK} \gtreqless \tau^{RANK}$ depends on the $\tilde{r} \gtreqless r \frac{w_R}{w_H}$.

Under Lump Sum taxes :

$$T_t^e = P_{t-1}^M (1 + r_{t-1}) b_{t-1}^M - P_t^M b_t^M + G_t$$

So, the steady state level of the lump sum tax is found to be

$$T_t^e = \tilde{r} P^M b^M + G_o$$

or equivalently,

$$T^e = \tilde{r}Y^H\bar{\zeta} + G_o$$

And for the RANK economy

$$T = rY^R\bar{\zeta} + G_o$$

Given our modelling assumption that $G_o = 0$. The ratio of taxes is

$$\frac{T^e}{T} = \frac{\tilde{r}}{r}\underbrace{\frac{Y^H}{Y^R}}_{<1}$$

So, whether $T^e \gtreqless T$ depends on the $\tilde{r} \gtreqless r\frac{Y^H}{Y^R}$.

Proof of Proposition 10 & 12

Steady State Consumption Inequality: HANK- DLMP vs HANK- IAMP

Recall that in the HANK- DLMP model both consumer types are optimising. As such, we can find the optimal consumption bundle for each type using their respective consumption Euler equation:

The marginal utility of consumption for employed

$$\psi_t^e = (C_t^e)^{-1} \tag{eq.(1)}$$

The marginal utility of consumption for unemployed

$$\psi_t^u = (C_t^u)^{-1} \tag{eq.(2)}$$

The Euler equation of the employed households

$$\frac{\psi_t^e}{(1+r_t)} = \beta\left(p^{e|e}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|e}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right) \tag{eq.(3)}$$

The Euler equation of the unemployed households

$$\left(1-\Omega\left(NAP_t^u - NAP^u\right)\right)\frac{\psi_t^u}{(1+r_t)} = \beta\left(p^{e|u}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|u}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right) \tag{eq.(4)}$$

Where,

The non- participating households' Net asset position (NAP_t^u)

$$
\begin{aligned}
NAP_t^u &= \left(\begin{array}{c} (1+\pi_{t+1})\, P_t^M \alpha_{t+1}^{M(u)} + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \alpha_{t+1}^u \\ -\left(\left(1+\rho P_t^M\right) \left(a_t^{M(e)} p^{u|e} + a_t^{M(u)} p^{u|u} \right) + \left(a_t^e p^{u|e} + a_t^u p^{u|u} \right) \right) \end{array} \right) \\
&\equiv T_t^u - p^u C_t^u
\end{aligned}
$$

The portfolio adjustment costs found in the constrained consumers budget constraint do not affect the long- run equilibrium. As such, in steady state the model delivers perfect insurance hence, $\beta(1+r)=1$. Hence, using the definition of consumption inequality

$$
\begin{aligned}
S_t &= 1 - \frac{\psi_t^e}{\psi_t^u} \\
&= 1 - \frac{C_t^u}{C_t^e}
\end{aligned}
$$

it is evident that in steady state

$$
\begin{aligned}
1 - S &\equiv \frac{\psi^e}{\psi^u} \\
&= \frac{p^{u|e}}{\left(1 - p^{e|e}\right)} \\
&= \frac{p^{u|u}}{\left(1 - p^{e|u}\right)}
\end{aligned}
$$

Thus, the steady state consumption inequality in the HANK- DLMP model is found to be

$$
\begin{aligned}
S &= 1 - \frac{p^{u|e}}{\left(1 - p^{e|e}\right)} \\
&= 1 - \frac{p^{u|u}}{\left(1 - p^{e|u}\right)}
\end{aligned}
$$

and depends only on the probabilities of the idiosyncratic shock.

In sharp contract, in the HANK- IAMP model changes in direct redistribution have a direct impact on the steady state amount of consumption inequality. First, let us consider the HANK-IAMP framework of Bilbiie & Ragot (2021). In HANK specification, constrained households consume only out of the exogenous wealth transfer.

The budget constraint of a constrained household takes the form

$$p^u C_t^u = T_t^u + p^{e|u}\left(1+\rho P_t^M\right) b_{t-1}$$

Where

$$T_t^u = p^u\left(1-\tau_t\right) w_t H_t^e \delta$$

with δ controlling the fraction of net labour income that constrained households receive. In line HANK- IAMP literature, we retain the assumption that $\delta < 1$.

Moreover, from the optimal labour supply of the unconstrained type we know that

$$C_t^e = \left(1-\tau_t\right) w_t \left(H_t^e\right)^{-\varphi}$$

Using the definition of consumption inequality (S_t) we get that

$$S_t = 1 - \frac{C_t^u}{C_t^e}$$

$$\begin{aligned} S_t &= 1 - \frac{\left(1-\tau_t\right) w_t \delta H_t^e + \frac{p^{e|u}}{p^u}\left(1+\rho P_t^M\right) b_{t-1}}{\left(1-\tau_t\right) w_t \left(H_t^e\right)^{-\varphi}} \\ &= 1 - \frac{\delta}{\left(H_t^e\right)^{-\varphi-1}} - \left(\frac{p^{e|u}}{p^u}\right)\frac{\left(1+\rho P_t^M\right) b_{t-1}}{\left(1-\tau_t\right) w_t \left(H_t^e\right)^{-\varphi}} \\ &= 1 - \frac{\delta}{\left(H_t^e\right)^{-(1+\varphi)}} - \left(\frac{p^{e|u}}{p^u}\right)\frac{\left(1+\rho P_t^M\right) b_{t-1}}{\left(1-\tau_t\right) w_t H_t^e}\left(H_t^e\right)^{1+\varphi} \end{aligned}$$

As such, everything else equal, a higher value of δ translates to lower consumption inequality. So, in steady state

$$S = 1 - \delta\left(H^e\right)^{1+\varphi} - \left(\frac{p^{e|u}}{p^u}\right)\frac{\left(1+\rho P^M\right) b}{\left(1-\tau\right) w H^e}\left(H^e\right)^{1+\varphi}$$

Proof of Proposition 13, 14 & 15

Redistribution Channels for the transmission mechanism of the Monetary Policy

In the absence of financial market frictions ($\Omega = 0$), the consumption Euler equation for each type is given as

Consumption Euler Equation for an employed household:

$$\psi_t^e = \beta R_t \left[p^{e|e} \mathbb{E}_t \left(\psi_{t+1}^e \right) + p^{u|e} \mathbb{E}_t \left(\psi_{t+1}^u \right) \right]$$

Consumption Euler Equation for an unemployed household:

$$\psi_t^u = \beta R_t \left[p^{e|u} \mathbb{E}_t \left(\psi_{t+1}^e \right) + p^{u|u} \mathbb{E}_t \left(\psi_{t+1}^u \right) \right]$$

Let us look at how a change in the interest rate affect the consumption of each type.

For the unconstrained consumer type:

$$\frac{d\psi_t^e}{dR_t} = \underbrace{\beta \left(p^{e|e} \mathbb{E}_t \left(\psi_{t+1}^e \right) + p^{u|e} \mathbb{E}_t \left(\psi_{t+1}^u \right) \right)}_{=\frac{\psi_t^e}{R_t}}$$

$$d\psi_t^e = \frac{\psi_t^e}{R_t} dR_t$$

$$\frac{d\psi_t^e}{\psi_t^e} = \frac{dR_t}{R_t}$$

Similarly, from the **Consumption Euler Equation for an unemployed household** we have that

$$\begin{aligned} \frac{d\psi_t^u}{dR_t} &= \underbrace{\beta \left(p^{e|u} \mathbb{E}_t \left(\psi_{t+1}^e \right) + p^{u|u} \mathbb{E}_t \left(\psi_{t+1}^u \right) \right)}_{=\frac{\psi_t^u}{R_t}} \\ &= \frac{\psi_t^u}{R_t} \end{aligned}$$

$$\frac{d\psi^u}{\psi_t^u} = \frac{dR_t}{R_t}$$

Hence,

$$\frac{dR_t}{R_t} = \frac{d\psi_t^e}{\psi_t^e} = \frac{d\psi^u}{\psi_t^u}$$

In the absence of financial market frictions, a change in the real interest rate has the same effect on the consumption of each type. In which, monetary policy may affect wealth disparity in the Economy but it is not able to redistribute consumption from unconstrained to constrained consumer. In this case, we state that there are redistribution channels in the transmission mechanism of monetary policy.

However, by introducing a portfolio adjustment cost for constrained households, we find that consumers in the economy no longer have equal exposure to aggregate shocks.

The Euler equation of the employed households

$$\frac{\psi_t^e}{R_t} = \beta\left(p^{e|e}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|e}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right) \tag{eq.(3)}$$

The Euler equation of the unemployed households

$$\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\frac{\psi_t^u}{R_t} = \beta\left(p^{e|u}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|u}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right) \tag{eq.(4)}$$

Where,

The non- participating households' Net asset position (NAP_t^u) is given as

$$\begin{aligned} NAP_t^u &= \left(\begin{array}{c} (1+\pi_{t+1})P_t^M\alpha_{t+1}^{M(u)} + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})R_t}\alpha_{t+1}^u \\ -\left(\left(1+\rho P_t^M\right)\left(a_t^{M(e)}p^{u|e} + a_t^{M(u)}p^{u|u}\right) + \left(a_t^e p^{u|e} + a_t^u p^{u|u}\right)\right) \end{array}\right) \\ &\equiv T_t^u - p^u C_t^u \end{aligned}$$

For the unconstrained consumer type, a change in the real interest rate yields

$$\frac{d\psi_t^e}{dR_t} = \left(\begin{array}{c} \underbrace{\beta \left(\begin{array}{c} p^{e|e}\mathbb{E}_t\left(\psi_{t+1}^e\right) \\ +p^{u|e}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right) \end{array} \right)}_{=\frac{\psi_t^e}{R_t}} \\ +p^{u|e}R_t\frac{d}{dR_t}\left[\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right] \end{array} \right)$$

$$\frac{d\psi_t^e}{dR_t} = \frac{\psi_t^e}{R_t} + p^{u|e}R_t\frac{d}{dR_t}\left[\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right]$$

..

$$\frac{dR_t}{R_t} = \frac{d\psi_t^e}{\psi_t^e} - p^{u|e}\frac{R_t}{\psi_t^e}\left[\frac{d}{dR_t}\left[\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right]\right]dR_t$$

Similarly, for the constrained household type

$$\frac{\psi_t^u}{R_t} = \beta\mathbb{E}_t\left(p^{e|u}\left(\frac{\psi_{t+1}^e}{\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)}\right) + p^{u|u}\frac{\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)}{\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)}\psi_{t+1}^u\right)$$

A change in the real interest rate yields

$$\frac{d\psi_t^u}{dR_t} = \frac{\psi_t^u}{R_t} + \beta\frac{d}{dR_t}\left(\mathbb{E}_t\left(\begin{array}{c} p^{e|u}\left(\frac{\psi_{t+1}^e}{(1+\Omega(NAP_t^u - NAP^u))}\right) \\ +p^{u|u}\frac{\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)}{(1+\Omega(NAP_t^u - NAP^u))}\psi_{t+1}^u \end{array}\right)\right)$$

$$\frac{d\psi_t^u}{\psi_t^u} = \frac{dR_t}{R_t} + \frac{\beta}{\psi_t^u}\left[\frac{d}{dR_t}\left(\mathbb{E}_t\left(\begin{array}{c} p^{e|u}\left(\frac{\psi_{t+1}^e}{(1-\Omega(NAP_t^u - NAP^u))}\right) \\ +p^{u|u}\frac{\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)}{(1-\Omega(NAP_t^u - NAP^u))}\psi_{t+1}^u \end{array}\right)\right)\right]dR_t$$

$$\frac{dR_t}{R_t} = \frac{d\psi_t^u}{\psi_t^u} - \frac{\beta}{\psi_t^u}\left[\frac{d}{dR_t}\left(\mathbb{E}_t\left(\begin{array}{c} p^{e|u}\left(\frac{\psi_{t+1}^e}{(1+\Omega(NAP_t^u - NAP^u))}\right) \\ +p^{u|u}\frac{\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)}{(1+\Omega(NAP_t^u - NAP^u))}\psi_{t+1}^u \end{array}\right)\right)\right]dR_t$$

Since, only constrained consumers face a portfolio adjustment cost, a change in the interest rate will not have the same effect on the consumption of each type. Monetary policy has now a retribution channel and a change in the real interest rate will not only affect wealth disparity but also the consumption disparity.

Now, looking in the HANK- IAMP framework , we observe that unconstrained consumers are identical to those in the benchmark HANK- DLMP framework $(\Omega = 0)$ and thus,

$$\frac{d\psi_t^e}{\psi_t^e} = \frac{dR_t}{R_t}$$

On the hand, constrained consumers are mass of Keynesian non- optimising agents whose consumption in every period is

$$
\begin{aligned}
C_t^u &= \frac{T_t^u}{p^u} + \frac{p^{u|e}}{p^u}\left(\left(1+\rho P_t^M\right) a_t^{M(e)} + a_t^e\right) \\
&= \frac{T_t^u}{p^u} + \frac{p^{u|e}}{p^u}\left(\left(1+\rho P_t^M\right) b_t^M + b_t\right)
\end{aligned}
$$

Meaning that

$$
\psi_t^u = \frac{p^u}{T_t^u} + \frac{p^u}{p^{u|e}} \frac{1}{\left(\left(1+\rho P_t^M\right) b_t^M + b_t\right)}
$$

$$
\frac{d\psi_t^u}{dR_t} = \frac{d}{dR_t}\left(\frac{p^u}{T_t^u} + \frac{p^u}{p^{u|e}} \frac{1}{\left(\left(1+\rho P_t^M\right) b_t^M + b_t\right)}\right)
$$

Hence,

$$
\frac{d\psi_t^u}{\psi_t^u} = \frac{dR_t}{R_t}\left[\frac{d}{dR_t}\left(\frac{p^u}{T_t^u} + \frac{p^u}{p^{u|e}} \frac{1}{\left(\left(1+\rho P_t^M\right) b_t^M + b_t\right)}\right)\right]\frac{R_t}{\psi_t^u}
$$

We can conclude that in order to have a redistributive role for monetary policy households in the economy need to display unequal exposure to an aggregate shock.

3.12 Appendix B

The Household Optimisation Problem

Optimisation Problem of the head of each (representative) family The head of each (representative) family wishes to maximise the following Welfare criterion

$$
\begin{aligned}
U_o &= \sum_{t=0}^{\infty}\left(\beta^t\right)\left[p^e\left(\ln\left(C_t^e\right) - \frac{\left(H_t^e\right)^{1+\varphi}}{1+\varphi}\right) + p^u \ln\left(C_t^u\right)\right] \\
&= \sum_{t=0}^{\infty}\left(\beta^t\right)\left[p^e \ln\left(C_t^e\right) + p^u \ln\left(C_t^u\right) - p^e \frac{\left(H_t^e\right)^{1+\varphi}}{1+\varphi}\right]
\end{aligned}
$$

subject to the budget constraint given that the household might be either

1. employed

$$p^e C_t^e + (1+\pi_{t+1}) P_t^M \alpha_{t+1}^{M(e)} + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \left(\alpha_{t+1}^e\right) = (1-\tau_t) w_t \underbrace{H_t^e p^e}_{=H_t} + p^e D_t^e + \left(1+\rho P_t^M\right) \hat{\alpha}_t^{M(e)} + \hat{\alpha}_t^e - T_t^e$$

or,

2. unemployed

$$\begin{pmatrix} p^u C_t^u \\ +(1+\pi_{t+1}) P_t^M \alpha_{t+1}^{M(u)} \\ +\frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \alpha_{t+1}^u \end{pmatrix} = \begin{pmatrix} p^u D_t^u + \left(1+\rho P_t^M\right) \hat{\alpha}_t^{M(u)} + \hat{\alpha}_t^u + T_t^u \\ -\frac{\Omega}{2}\left(NAP_t^u - NAP^u\right)^2 \end{pmatrix}$$

$$p^u C_t^u + (1+\pi_{t+1}) P_t^M \alpha_{t+1}^{M(u)} + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \alpha_{t+1}^u = \begin{pmatrix} \underbrace{p^u D_t^u}_{=0} + T_t^u + \left(1+\rho P_t^M\right)\left(a_t^{M(e)} p^{u|e} + a_t^{M(u)} p^{u|u}\right) \\ + \left(a_t^e p^{u|e} + a_t^u p^{u|u}\right) - \frac{\Omega}{2}\left(NAP_t^u - NAP^u\right)^2 \end{pmatrix}$$

where, the household's Net asset position (NAP_t^u)

$$\begin{aligned} NAP_t^u &= \begin{pmatrix} (1+\pi_{t+1}) P_t^M \alpha_{t+1}^{M(u)} + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \alpha_{t+1}^u \\ -\left(\left(1+\rho P_t^M\right)\left(a_t^{M(e)} p^{u|e} + a_t^{M(u)} p^{u|u}\right) + \left(a_t^e p^{u|e} + a_t^u p^{u|u}\right)\right) \end{pmatrix} \\ &\equiv T_t^u - p^u C_t^u \end{aligned}$$

Setting up the Lagrangian:

$$\mathscr{L}_t = \max_{\{C_t^e, H_t, b_{t+1}\}_{t=s}^{\infty}} E_t \sum_{t=s}^{\infty} (\beta)^{t-s} \begin{pmatrix} p^e \ln(C_t^e) + p^u \ln(C_t^u) - p^e \frac{(H_t^e)^{1+\varphi}}{1+\varphi} \\ +\psi_t^e \begin{pmatrix} (1-\tau_t) w_t H_t^e p^e - T_t^e + D_t^e p^e - p^e C_t^e \\ +\left(1+\rho P_t^M\right)\left(a_t^{M(e)} p^{e|e} + a_t^{M(u)} p^{e|u}\right) + \left(a_t^e p^{e|e} + a_t^u p^{e|u}\right) \\ -(1+\pi_{t+1}) P_t^M \left(a_{t+1}^{M(e)}\right) - \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \left(a_{t+1}^e\right) \end{pmatrix} \\ +\psi_t^u \begin{pmatrix} p^u D_t^u + T_t^u - p^u C_t^u \\ +\left(1+\rho P_t^M\right)\left(a_t^{M(e)} p^{u|e} + a_t^{M(u)} p^{u|u}\right) - (1+\pi_{t+1}) P_t^M \alpha_{t+1}^{M(u)} \\ +\left(a_t^e p^{u|e} + a_t^u p^{u|u}\right) - \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \alpha_{t+1}^u \\ -\frac{\Omega}{2} \begin{pmatrix} (1+\pi_{t+1}) P_t^M \alpha_{t+1}^{M(u)} - \left(1+\rho P_t^M\right)\left(a_t^{M(e)} p^{u|e} + a_t^{M(u)} p^{u|u}\right) \\ +\frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \alpha_{t+1}^u - \left(a_t^e p^{u|e} + a_t^u p^{u|u}\right) \\ -NAPU \end{pmatrix}^2 \end{pmatrix} \end{pmatrix}$$

FOCs

$$\frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0 : \left(\frac{p^u}{C_t^u}\right) - p^u \psi_t^u = 0$$

$$1 \qquad \frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0 : \psi_t^u = \frac{1}{C_t^u}$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0 : \frac{p^e}{C_t^e} - p^e \psi_t^e = 0$$

$$2 \qquad \frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0 : \psi_t^e = \frac{1}{C_t^e}$$

$$\begin{aligned}
\frac{\partial \mathscr{L}_t}{\partial H_t^e} &= 0 : -p^e \left(H_t^e\right)^{\varphi} + \psi_t^e \left(1-\tau_t\right) w_t p^e = 0 \\
\frac{\partial \mathscr{L}_t}{\partial H_t^e} &= 0 : \left(H_t^e\right)^{\varphi} = \psi_t^e \left(1-\tau_t\right) w_t \\
\frac{\partial \mathscr{L}_t}{\partial H_t^e} &= 0 : C_t^e \left(H_t^e\right)^{\varphi} = \left(1-\tau_t\right) w_t
\end{aligned}$$

Alternatively, given that only employed households supply labour, we can re-write the expression in terms of the aggregate labour supply

$$3 : C_t^e \left(\frac{H_t}{p^e}\right)^{\varphi} = \left(1-\tau_t\right) w_t$$

$$\frac{\partial \mathscr{L}_t}{\partial \alpha_{t+1}^{M(e)}} = 0 : \begin{pmatrix} -(\beta)^{t-s} \psi_t^e \mathbb{E}_t \left(\left(1+\pi_{t+1}\right) P_t^M\right) \\ +(\beta)^{t+1-s} \mathbb{E}_t \left(\psi_{t+1}^e \left(1+\rho P_{t+1}^M\right) p^{e|e}\right) \\ +(\beta)^{t+1-s} \mathbb{E}_t \psi_{t+1}^u \left(\left(1+\rho P_{t+1}^M\right) p^{u|e}\right) \\ +(\beta)^{t+1-s} \Omega \left(NAP_{t+1}^u - NAP^u\right) \mathbb{E}_t \psi_{t+1}^u \left(\left(1+\rho P_{t+1}^M\right) p^{u|e}\right) \end{pmatrix} = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial \alpha_{t+1}^{M(e)}} = 0 : P_t^M = \mathbb{E}_t \left(\beta \left(\frac{\psi_{t+1}^e}{\psi_t^e} p^{e|e} + \left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right) \frac{\psi_{t+1}^u}{\psi_t^e} p^{u|e}\right) \frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)}\right)$$

$$4: P_t^M = \mathbb{E}_t\left(\beta\left(\frac{\psi_{t+1}^e}{\psi_t^e}p^{e|e} + \left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\frac{\psi_{t+1}^u}{\psi_t^e}p^{u|e}\right)\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)}\right)$$

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial \alpha_{t+1}^{M(u)}} &= 0: \begin{pmatrix} +(\beta)^{t+1-s}p^{e|u}\mathbb{E}_t\left(\psi_{t+1}^e\left(1+\rho P_{t+1}^M\right)\right) \\ -(\beta)^{t-s}\psi_t^u\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)P_t^M\mathbb{E}_t\left(1+\pi_{t+1}\right) \\ +(\beta)^{t+1-s}p^{u|u}\mathbb{E}_t\left(\psi_{t+1}^u\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\left(1+\rho P_{t+1}^M\right)\right) \end{pmatrix} = 0 \\
\frac{\partial \mathcal{L}_t}{\partial \alpha_{t+1}^{M(u)}} &= 0: \begin{pmatrix} \beta\mathbb{E}_t\left(\left(1+\rho P_{t+1}^M\right)\left(p^{e|u}\psi_{t+1}^e + p^{u|u}\psi_{t+1}^u\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\right)\right) \\ = \psi_t^u\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)P_t^M\mathbb{E}_t\left(1+\pi_{t+1}\right) \end{pmatrix} \\
P_t^M &= \mathbb{E}_t\left(\beta\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)}\left(p^{e|u}\frac{\psi_{t+1}^e}{\psi_t^u} + p^{u|u}\frac{\psi_{t+1}^u}{\psi_t^u}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\right)\right)
\end{aligned}$$

$$5: P_t^M = \frac{1}{\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)}\mathbb{E}_t\left(\beta\left(p^{e|u}\frac{\psi_{t+1}^e}{\psi_t^u} + p^{u|u}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\frac{\psi_{t+1}^u}{\psi_t^u}\right)\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)}\right)$$

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial \alpha_{t+1}^e} &= 0: \begin{pmatrix} -(\beta)^{t-s}\frac{\psi_t^e}{(1+r_t)} \\ +(\beta)^{t+1-s}p^{e|e}\mathbb{E}_t\psi_{t+1}^e \\ +(\beta)^{t+1-s}p^{u|e}\mathbb{E}_t\left(\psi_{t+1}^u\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\right) \end{pmatrix} = 0 \\
\frac{\partial \mathcal{L}_t}{\partial \alpha_{t+1}^e} &= 0: \beta\mathbb{E}_t\left(p^{e|e}\psi_{t+1}^e + p^{u|e}\psi_{t+1}^u\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\right) = \frac{\psi_t^e}{(1+r_t)}
\end{aligned}$$

The consumption Euler equation of an employed household is

$$6: \psi_t^e = \beta\left(1+r_t\right)\left(p^{e|e}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|e}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\mathbb{E}_t\left(\psi_{t+1}^u\right)\right)$$

$$\frac{\partial \mathcal{L}_t}{\partial \alpha_{t+1}^u} = 0: \begin{pmatrix} (\beta)^{t+1-s}p^{e|u}\mathbb{E}_t\left(\psi_{t+1}^e\right) - (\beta)^{t-s}\psi_t^u\left(\frac{1}{(1+r_t)}\right) \\ +(\beta)^{t+1-s}\mathbb{E}_t\left(\psi_{t+1}^u\right)p^{u|u} - (\beta)^{t-s}\psi_t^u\frac{\Omega\left(NAP_t^u - NAP^u\right)}{(1+r_t)} \\ +p^{u|u}(\beta)^{t+1-s}\mathbb{E}_t\left(\psi_{t+1}^u\right)\mathbb{E}_t\left(\Omega\left(NAP_{t+1}^u - NAP^u\right)\right) \end{pmatrix} = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial \alpha^u_{t+1}} = 0 : \psi^u_t = \frac{(1+r_t)\beta}{(1+\Omega(NAP^u_t - NAP^u))}\left(p^{e|u}\mathbb{E}_t\left(\psi^e_{t+1}\right) + \mathbb{E}_t\left(\psi^u_{t+1}\left(1+\Omega\left(NAP^u_{t+1} - NAP^u\right)\right)\right)p^{u|u}\right)$$

The consumption Euler equation of an unemployed household is

$$7 : \psi^u_t = \frac{\beta(1+r_t)}{(1+\Omega(NAP^u_t - NAP^u))}\left(p^{e|u}\mathbb{E}_t\left(\psi^e_{t+1}\right) + p^{u|u}\left(1+\Omega\left(NAP^u_{t+1} - NAP^u\right)\right)\mathbb{E}_t\left(\psi^u_{t+1}\right)\right)$$

Now, combining either eq.(4) with eq. (6) or eq.(5) with eq.(7) yields the expression for the Bond Pricing equation

$$P^M_t = \left(\mathbb{E}_t\left(\frac{\left(1+\rho P^M_{t+1}\right)}{(1+r_t)(1+\pi_{t+1})}\right)\right)$$

meaning that the stochastic discount factor (SDF) is given as

$$\begin{aligned} SDF &\equiv \mathbb{E}_t\left(\beta\left[p^{e|e}\frac{\psi^e_{t+1}}{\psi^e_t} + p^{u|e}\left(1+\Omega\left(NAP^u_{t+1} - NAP^u\right)\right)\frac{\psi^u_{t+1}}{\psi^e_t}\right]\right) \\ &= \mathbb{E}_t\left(\beta\left[\frac{p^{e|u}}{(1+\Omega(NAP^u_t - NAP^u))}\frac{\psi^e_{t+1}}{\psi^u_t} + p^{u|u}\frac{\left(1+\Omega\left(NAP^u_{t+1} - NAP^u\right)\right)}{(1+\Omega(NAP^u_t - NAP^u))}\frac{\psi^u_{t+1}}{\psi^u_t}\right]\right) \\ &= \frac{1}{(1+r_t)} = \frac{1}{R_t} \end{aligned}$$

Meaning that the **Consumption Euler Equation for an employed household** takes the form

$$\frac{\psi^e_t}{(1+r_t)} = \beta\left(p^{e|e}\mathbb{E}_t\left(\psi^e_{t+1}\right) + p^{u|e}\left(1+\Omega\left(NAP^u_{t+1} - NAP^u\right)\right)\mathbb{E}_t\left(\psi^u_{t+1}\right)\right)$$

Meaning that the **Consumption Euler Equation for an unemployed household** takes the form

$$(1+\Omega(NAP^u_t - NAP^u))\frac{\psi^u_t}{(1+r_t)} = \beta\left(p^{e|u}\mathbb{E}_t\left(\psi^e_{t+1}\right) + p^{u|u}\left(1+\Omega\left(NAP^u_{t+1} - NAP^u\right)\right)\mathbb{E}_t\left(\psi^u_{t+1}\right)\right)$$

Where,

The household's Net asset position (NAP_t^u)

$$\begin{aligned} NAP_t^u &= \left(\begin{array}{c} (1+\pi_{t+1}) P_t^M \alpha_{t+1}^{M(u)} + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \alpha_{t+1}^u \\ - \left(\begin{array}{c} \left(1+\rho P_t^M\right) \left(a_t^{M(e)} p^{u|e} + a_t^{M(u)} p^{u|u}\right) \\ + \left(a_t^e p^{u|e} + a_t^u p^{u|u}\right) \end{array} \right) \end{array} \right) \\ &= T_t^u - p^u C_t^u \end{aligned}$$

3.13 Appendix C

Firms' Problem

Phillips Curve

Intermediate firms face a quadratic cost when changing their Price. The firm's problem becomes choosing

$\{P_t(j)\}_{t=0}^{\infty}$ in order to maximise :

$$\max_{P_t(j)} \sum_{t=0}^{\infty} m_{0,t} \left(\left(\frac{P_t(j)}{P_t} - (1-s) \frac{w_t}{z_t} \right) y_t(j) - \frac{\Phi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \right)$$

subject to monopolistic demand

$$y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon_t} Y_t$$

Substitute

$$\max_{P_t(j)} \sum_{t=0}^{\infty} m_{0,t} \left(\left(\frac{P_t(j)}{P_t} - (1-s) \frac{w_t}{z_t} \right) \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon_t} Y_t - \frac{\Phi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \right)$$

$$0 = \left[\begin{array}{c} m_{0,t} \left(\begin{array}{c} (1-\varepsilon_t) \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_t} \frac{Y_t}{P_t} \\ +\varepsilon_t (1-s) \frac{w_t}{z_t} \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_t - 1} \frac{Y_t}{P_t} - \Phi \left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right) \frac{Y_t}{P_{t-1}(j)} \end{array} \right) \\ +\mathbb{E}_t \left(m_{0,t+1} \left(\Phi \left(\frac{P_{t+1}(j)}{P_t(j)} - 1 \right) Y_{t+1} \frac{P_{t+1}(j)}{P_t^2(j)} \right) \right) \end{array} \right]$$

All firms are identical so $P_t(j) = P_t$ and

$$\pi_t (1+\pi_t) = \frac{1 - \varepsilon_t + (1-s) \varepsilon_t \frac{w_t}{z_t}}{\Phi} + \mathbb{E}_t \left(\frac{m_{0,t+1}}{m_{0,t}} \frac{Y_{t+1}}{Y_t} \pi_{t+1} (1+\pi_{t+1}) \right)$$

The stochastic discount factor isT_t^u

$$m_{0,t} = \Pi_{s=0}^{t} \left(1 + r_s\right)^{-1}$$

So, we can re- write the *NK Price Phillips Curve*

$$\Phi\pi_t \left(1+\pi_t\right)\left(1+r_t\right) = \left(1 - \varepsilon_t + (1-s)\,\varepsilon_t \frac{w_t}{z_t}\right)\left(1+r_t\right) + \Phi\mathbb{E}_t \left(\pi_{t+1}\left(1+\pi_{t+1}\right)\frac{Y_{t+1}}{Y_t}\right) \quad \text{(NKPC)}$$

3.14 Appendix D

Aggregation and Market Clearing

The obtain the aggregate system, we need to first to aggregate the individual relationships found from the individual household's.

Namely, the aggregate labour supply is

$$\begin{aligned} H_t^s &= \int_0^1 H_t^i di \\ &= \int_0^{p^u} H_t^u du + \int_{p^u}^1 H_t^e de \\ &= \int_0^{p^u} H_t^u du + \int_{(1-p^e)}^1 H_t^e de \\ &= p^e H_t^e + p^u H_t^u \end{aligned}$$

However, given that only employed households are able to supply labour to the market $\left(H_t^u = 0\right)$, the aggregate labour supply is

$$H_t^s = p^e H_t^e$$

The aggregate labour demand

$$H_t^d = \int_0^1 h_t\left(j\right) dj$$

For labour market to clear, we require

$$H_t^s = H_t^d = H_t$$

Meaning that aggregate labor demand is equal to the labour supply of employed households

$$H_t = p^e H_t^e$$

Similarly, the aggregate consumption is

$$\begin{aligned} C_t &= \int_0^1 C_t^i di \\ &= p^e C_t^e + p^u C_t^u \end{aligned}$$

Whilst the aggregate dividends take the form

$$\begin{aligned} D_t &= \int_0^1 D_t^i di \\ &= p^e D_t^e + p^u D_t^u \end{aligned}$$

Unless otherwise specified, we are going to assume that dividends are equally distributed across employed households only $(D_t^u = 0)$.

$$D_t^e = \frac{D_t}{p^e}$$

Unemployed households receive a Lump Sum (direct wealth) transfer of $\frac{T_t^u}{p^u}$. The aggregate amount paid to constrained households is

$$\begin{aligned} T_t^u &= \int_0^{p^u} \frac{T_t^u}{p^u} di \\ &= p^u \frac{T_t^u}{p^u} \end{aligned}$$

Similarly, assuming that the transfer is financed via Lump Sum taxes imposed on the employed consumers then,

$$\begin{aligned} T_t^e &= \int_0^{p^e} \frac{T_t^e}{p^e} di \\ &= p^e \frac{T_t^e}{p^e} \end{aligned}$$

The aggregate level of transfers is given as

$$T_t = T_t^u - T_t^e$$

Additionally, for markets to clear we require the aggregate stock of private savings (of either duration) should equal the aggregate supply of government debt.

For the long term assets, we have

$$b_{t+1}^M = \alpha_{t+1}^{M(e)} + \alpha_{t+1}^{M(u)}$$

whilst for the short- term (1-period) assets

$$b_{t+1} = \alpha_{t+1}^e + \alpha_{t+1}^u$$

As explained in the government block, the supply of short term assets is set to zero.

Finally, since we require all markets to clear, combining the household budget constraint with the expression for aggregate dividends and the government budget constraint, we obtain **the (aggregate) resource constraint**

Aggregate Resource Constraint The Budget Constraint of

1. employed

$$(1+\pi_{t+1}) P_t^M \left(a_{t+1}^{M(e)}\right) + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \left(a_{t+1}^e\right) + p^e C_t^e = \begin{pmatrix} (1-\tau_t) w_t H_t^e p^e - T_t^e + D_t^e p^e \\ + \left(1+\rho P_t^M\right) \left(a_t^{M(e)} p^{e|e} + a_t^{M(u)} p^{e|u}\right) \\ + \left(a_t^e p^{e|e} + a_t^u p^{e|u}\right) \end{pmatrix}$$

2. unemployed

$$p^u C_t^u + (1+\pi_{t+1}) P_t^M \alpha_{t+1}^{M(u)} + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \alpha_{t+1}^u = \begin{pmatrix} \underbrace{p^u D_t^u}_{=0} + T_t^u + \left(1+\rho P_t^M\right) \left(a_t^{M(e)} p^{u|e} + a_t^{M(u)} p^{u|u}\right) \\ + \left(a_t^e p^{u|e} + a_t^u p^{u|u}\right) - \frac{\Omega}{2} \left(NAP_t^u - NAP^u\right)^2 \end{pmatrix}$$

Combining the two yields

$$\begin{pmatrix} p^e C_t^e + p^u C_t^u \\ +(1+\pi_{t+1}) P_t^M \left(\alpha_{t+1}^{M(e)} + \alpha_{t+1}^{M(u)}\right) \\ +\frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \left(\alpha_{t+1}^e + \alpha_{t+1}^u\right) \end{pmatrix} = \begin{pmatrix} (1-\tau_t) w_t H_t + D_t \\ +\left(1+\rho P_t^M\right)\left(a_t^{M(e)}\left(p^{e|e} + p^{u|e}\right) + a_t^{M(u)}\left(p^{e|u} + p^{u|u}\right)\right) \\ +\left(a_t^e\left(p^{e|e} + p^{u|e}\right) + a_t^u\left(p^{e|u} + p^{u|u}\right)\right) \\ -\frac{\Omega}{2}\left(NAP_t^u - NAP^u\right)^2 \\ +T_t^u - T_t^e \end{pmatrix}$$

Recall that aggregate government debt

$$\begin{aligned} b_t &= \alpha_{t+1}^e + \alpha_{t+1}^u \\ b_t^M &= \alpha_{t+1}^{M(e)} + \alpha_{t+1}^{M(u)} \end{aligned}$$

$$\begin{pmatrix} p^e C_t^e + p^u C_t^u \\ +(1+\pi_{t+1}) P_t^M b_t^M \\ +\frac{1}{(1+r_t)} b_t \end{pmatrix} = \begin{pmatrix} (1-\tau_t) w_t H_t + D_t + \left(1+\rho P_t^M\right) b_{t-1}^M + b_{t-1} \\ -\frac{\Omega}{2}\left(NAP_t^u - NAP^u\right)^2 \\ +T_t^u - T_t^e \end{pmatrix}$$

The government budget constraint

$$(1+\pi_{t+1}) P_t^M b_t^M + \frac{b_t}{(1+r_t)} + \tau_t w_t H_t = b_{t-1} + \left(1+\rho P_t^M\right) b_{t-1}^M + G_t + T_t^u - T_t^e$$

Combining the two yields

$$\left(p^e C_t^e + p^u C_t^u + G_t\right) = w_t H_t + D_t - \frac{\Omega}{2}\left(NAP_t^u - NAP^u\right)^2$$

$$p^e C_t^e + p^u C_t^u = \left(w_t H_t^e p^e + D_t - G_t - \frac{\Omega}{2}\left(NAP_t^u - NAP^u\right)^2\right)$$

With aggregate dividends are

$$D_t = \left(1 - \frac{\Phi}{2}\pi_t^2\right) Y_t - w_t H_t^e p^e$$

So, the **Aggregate Budget Constraint**

$$p^e C_t^e + p^u C_t^u = \left(1 - \frac{\Phi}{2}\pi_t^2\right) Y_t - G_t - \frac{\Omega}{2}\left(NAP_t^u - NAP^u\right)^2$$

where, the household's Net asset position $(NAPU_t)$

$$\begin{aligned} NAP_t^u &= \left(\begin{array}{c} (1+\pi_{t+1}) P_t^M \alpha_{t+1}^{M(u)} + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \alpha_{t+1}^u \\ -\left(\left(1+\rho P_t^M\right) \left(a_t^{M(e)} p^{u|e} + a_t^{M(u)} p^{u|u} \right) + \left(a_t^e p^{u|e} + a_t^u p^{u|u} \right) \right) \end{array} \right) \\ &\equiv T_t^u - p^u C_t^u \end{aligned}$$

3.15 Appendix E

Social Welfare Function

The household preferences for consumption and labour, are captured by the standard CRRA felicity

$$U_t^i = \ln\left(C_t^i\right) - \frac{\left(H_t^i\right)^{1+\varphi}}{1+\varphi}$$

Where, the type of household is indexed by $i = \{R, u\}$.

The aggregate welfare function that the policy maker seeks to maximize is the aggregate utility function of the economy's population. As in Chien & Wen (2021), social welfare function takes the same form as the function the head of each family wishes to maximises under a different set of constraints.

$$\begin{aligned} W_t &= \int_0^1 U_t^i di \\ &= \int_0^{p^u} U_t^u du + \int_{p^u}^1 U_t^e de \\ &= p^e \ln(C_t^e) + p^u \ln(C_t^u) - p^e \frac{(H_t^e)^{1+\varphi}}{1+\varphi} \end{aligned}$$

To simplify our analysis, we can substitute in the expression for the optimal labour supply

$$H_t = H_t^e p^e$$

We can re-write the aggregate welfare as:

$$
\begin{aligned}
W_t &= p^e \ln(C_t^e) + p^u \ln(C_t^u) - p^e \frac{\left(\frac{H_t}{p^e}\right)^{1+\varphi}}{1+\varphi} \\
&= (1-p^u)\ln(C_t^e) + p^u \ln(C_t^u) - p^e \frac{\left(\frac{H_t}{p^e}\right)^{1+\varphi}}{1+\varphi} \\
&= p^u \underbrace{\ln\left(\frac{C_t^u}{C_t^e}\right)}_{=\ln(1-S_t)} + \ln(C_t^e) - p^e \frac{\left(\frac{H_t}{p^e}\right)^{1+\varphi}}{1+\varphi}
\end{aligned}
$$

From the household's optimality conditions we have that the labour supply equation for the employed household is given as

$$
\frac{(1-\tau_t)\,w_t}{C_t^e} = (H_t)^{\varphi}
$$

As such, we can re-write the **Social Welfare function** as

$$
\begin{aligned}
W_t &= p^e \ln(C_t^e) + p^u \ln(C_t^u) - p^e \frac{(1-\tau_t)\,w_t}{C_t^e\,(1+\varphi)} H_t^e \\
&= p^u \ln(1-S_t) + \ln(C_t^e) - p^e \frac{(1-\tau_t)\,w_t}{C_t^e\,(1+\varphi)} H_t^e
\end{aligned}
$$

We can easily observe from this expression that the negative effect inequality(S_t) has on aggregate welfare(W_t) is proportional to the size of the "unemployed" population (p^u). Also, although the central planner might wish to reduce $(1-S_t)$,the will never try to eliminate it completely. Finally, when the policy maker has access to distortionary income taxes, we need to also substitute in the tax rule. Hence, The Social Welfare function takes the form

$$
W_t = p^u \ln(1-S_t) + \ln(C_t^e) - p^e \left(1 - \bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) \frac{w_t}{C_t^e\,(1+\varphi)} H_t^e
$$

3.16 Appendix F

The policy maker's problem (HANK- DLMP model)

- In the benchmark case, we assume that wealth transfers are exogenous and constant $(T_t^u = T^u)$

Under distortionary income taxes

$$\mathcal{L}_t = \sum_{t=0}^{\infty} (\beta)^t \left[\begin{array}{c} \left(p^u \ln(C_t^u) + p^e \ln(C_t^e) - p^e \left(1 - \bar{\tau} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) \frac{w_t}{C_t^e (1+\varphi)} H_t^e \right) \\ +\lambda_t^1 \left(p^e C_t^e + p^u C_t^u + G_t + \frac{\Omega}{2} \left(NAP_t^u - NAP^u\right)^2 - \left(1 - \frac{\Phi}{2}\pi_t^2\right) Y_t \right) \\ +\lambda_t^2 \left(\beta \mathbb{E}_t \left(p^{e|e} \left(C_{t+1}^e\right)^{-1} + p^{u|e} \left(1 + \Omega \left(NAP_{t+1}^u - NAP^u\right)\right) \left(C_{t+1}^u\right)^{-1} \right) - \frac{(C_t^e)^{-1}}{R_t} \right) \\ +\lambda_t^8 \left(\begin{array}{c} \beta \mathbb{E}_t \left(p^{e|u} \left(C_{t+1}^e\right)^{-1} + p^{u|u} \left(1 + \Omega \left(NAP_{t+1}^u - NAP^u\right)\right) \left(C_{t+1}^u\right)^{-1} \right) \\ -\left(1 + \Omega \left(NAP_t^u - NAP^u\right)\right) \frac{(C_t^u)^{-1}}{R_t} \end{array} \right) \\ +\lambda_t^3 \left(\mathbb{E}_t \left(\frac{(1+\rho P_{t+1}^M)}{(1+\pi_{t+1})} \right) - R_t P_t^M \right) \\ +\lambda_t^4 \left(\left[\left(1 - \varepsilon_t + (1-s)\varepsilon_t \frac{w_t}{z_t}\right) - \Phi \pi_t (1+\pi_t) \right] Y_t R_t + \Phi \mathbb{E}_t \left(\pi_{t+1}(1+\pi_{t+1}) Y_{t+1}\right) \right) \\ +\lambda_t^5 \left(\left(1 - \bar{\tau} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) w_t (C_t^e)^{-1} - (H_t^e)^{\varphi} \right) \\ +\lambda_t^6 \left(\left(\frac{(1+\rho P_t^M)}{(1+\pi_t)} b_t^L + G_t + T_t^u - \bar{\tau} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} w_t p^e H_t^e - T_t^p \right) R_t - \mathbb{E}_t \left(\frac{(1+\rho P_{t+1}^M)}{(1+\pi_{t+1})} b_{t+1}^L \right) \right) \\ +\lambda_t^7 \left(p^e H_t^e z_t - Y_t \right) \\ +\lambda_t^9 \left(T_t^u - p^u C_t^u - NAP_t^u \right) \end{array} \right]$$

$$FOCs$$

$$\frac{\partial \mathcal{L}_t}{\partial C_t^e} = 0: \begin{bmatrix} \frac{p^e}{C_t^e}\left(1+\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{w_t}{C_t^e(1+\varphi)}H_t^e\right) \\ +p^e\lambda_t^1 \\ -p^{e|e}\left(C_t^e\right)^{-2}\lambda_{t-1}^2+\lambda_t^2\frac{\left(C_t^e\right)^{-2}}{R_t} \\ -\lambda_{t-1}^8 p^{e|u}\left(C_t^e\right)^{-2} \\ -\lambda_t^5\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t\left(C_t^e\right)^{-2} \end{bmatrix} = 0$$

Multiply across by $\left(C_t^e\right)^2$

$$\frac{\partial \mathcal{L}_t}{\partial C_t^e} = 0: \begin{bmatrix} p^e C_t^e + p^e\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{w_t}{(1+\varphi)}H_t^e \\ +p^e\lambda_t^1\left(C_t^e\right)^2 - p^{e|u}\lambda_{t-1}^8 \\ -p^{e|e}\lambda_{t-1}^2 \\ +\frac{\lambda_t^2}{R_t} \\ -\lambda_t^5\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t \end{bmatrix} = 0$$

$$\frac{\partial \mathcal{L}_t}{\partial NAP_t^u} = 0: \lambda_t^1\Omega\left(NAP_t^u - NAP^u\right) + \Omega p^{u|e}\lambda_{t-1}^2\left(C_t^u\right)^{-1} + \Omega p^{u|u}\lambda_{t-1}^8\left(C_t^u\right)^{-1} - \Omega\frac{\left(C_t^u\right)^{-1}}{R_t}\lambda_t^8 - \lambda_t^9 = 0$$

$$\frac{\partial \mathcal{L}_t}{\partial C_t^u} = 0: \begin{bmatrix} \frac{p^u}{C_t^u} + p^u\lambda_t^1 - p^{u|e}\lambda_{t-1}^2\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(C_t^u\right)^{-2} \\ -\lambda_{t-1}^8 p^{u|u}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(C_t^u\right)^{-2} \\ +\lambda_t^8\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(\frac{\left(C_t^u\right)^{-2}}{R_t}\right) - \lambda_t^9 p^u \end{bmatrix} = 0$$

$$\frac{\partial \mathcal{L}_t}{\partial C_t^u} = 0: \begin{bmatrix} p^u C_t^u + \lambda_t^1 p^u\left(C_t^u\right)^2 - \lambda_{t-1}^2 p^{u|e}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) \\ -\lambda_{t-1}^8 p^{u|u}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) \\ +\frac{\lambda_t^8}{R_t}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) - \lambda_t^9 p^u\left(C_t^u\right)^2 \end{bmatrix} = 0$$

:

$$\frac{\partial \mathcal{L}_t}{\partial Y_t} = 0: \begin{bmatrix} -\lambda_t^1\left(1-\frac{\Phi}{2}\pi_t^2\right)Y_t \\ +\lambda_t^4\left(\left(1-\varepsilon_t+(1-s)\varepsilon_t\frac{w_t}{z_t}\right)-\Phi\pi_t\left(1+\pi_t\right)\right)R_t Y_t \\ +\frac{\lambda_{t-1}^4}{\beta}\Phi\left(1+\pi_t\right)\pi_t Y_t \\ -\lambda_t^7 Y_t \end{bmatrix} = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial \pi_t} = 0 : \left[\begin{array}{c} \lambda_t^1 \left(\Phi \pi_t Y_t \right) - \frac{\lambda_{t-1}^3}{\beta} \left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2} \right) \\ -\lambda_t^4 \Phi \left(1+2\pi_t\right) Y_t R_t \\ +\frac{\lambda_{t-1}^4}{\beta} \Phi \left(1+2\pi_t\right) Y_t \\ -\lambda_t^6 \left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2} b_t^M \right) R_t + \frac{\lambda_{t-1}^6}{\beta} \left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2} \right) b_t^M \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial R_t} = 0 : \left[\begin{array}{c} +\lambda_t^2 \frac{\left(C_t^e\right)^{-1}}{\left(R_t\right)^2} + \lambda_t^8 \left(1 + \Omega \left(NAP_t^u - NAP^u\right)\right) \frac{\left(C_t^u\right)^{-1}}{\left(R_t\right)^2} - \lambda_t^3 P_t^M \\ +\lambda_t^4 \left(\left(1 - \varepsilon_t + (1-s)\, \varepsilon_t \frac{w_t}{z_t}\right) - \Phi \pi_t \left(1+\pi_t\right) \right) Y_t \\ +\lambda_t^6 \left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)} b_t^L + G_t + T_t^u - \bar{\tau} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} w_t p^e H_t^e - T_t^p \right) \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial w_t} = 0 : \left[\begin{array}{c} -p^e \left(1 - \bar{\tau} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} \right) \frac{H_t^e}{C_t^e (1+\varphi)} \\ +\lambda_t^4 (1-s)\, \varepsilon_t \frac{Y_t}{z_t} R_t \\ +\lambda_t^5 \left(1 - \bar{\tau} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} \right) \left(C_t^e\right)^{-1} \\ -\lambda_t^6 \left(p^e \bar{\tau} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} H_t^e \right) R_t \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial H_t^e} = 0 : \left[\begin{array}{c} -p^e \left(1 - \bar{\tau} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} \right) \frac{w_t}{C_t^e (1+\varphi)} \\ -\varphi \lambda_t^5 \left(H_t^e\right)^{\varphi - 1} \\ -\lambda_t^6 \left(\bar{\tau} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} w_t p^e \right) R_t \\ +p^e z_t \lambda_t^7 \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial H_t^e} = 0 : \left[\begin{array}{c} -p^e \left(1 - \bar{\tau} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} \right) \frac{w_t}{C_t^e (1+\varphi)} H_t^e \\ -\varphi \lambda_t^5 \left(H_t^e\right)^{\varphi} \\ -\lambda_t^6 \left(\bar{\tau} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} w_t p^e H_t^e \right) R_t \\ +\lambda_t^7 Y_t \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial P_t^M}=0: \begin{bmatrix} \left(p^e \bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} \frac{w_t}{C_t^e(1+\varphi)} H_t^e\right) \\ -\lambda_t^3 R_t \\ +\left(\frac{\lambda_{t-1}^3}{\beta}\right)\left(\frac{\rho}{(1+\pi_t)}\right) \\ -\lambda_t^5\left(\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) w_t\left(C_t^e\right)^{-1} \\ +\lambda_t^6\left[\left(\frac{\rho}{(1+\pi_t)}\right) b_t^M-\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} w_t p^e H_t^e\right] R_t \\ -\frac{\lambda_{t-1}^6}{\beta}\left(\frac{\rho}{(1+\pi_t)} b_t^M\right) \end{bmatrix}=0$$

$$\frac{\partial \mathscr{L}_t}{\partial b_{t+1}^M}=0: \begin{bmatrix} \left(\bar{\tau}\left(\frac{\phi_b}{b^M}\right)\left(\frac{P_t^M}{P^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_{b-1}} \frac{w_t}{C_t^e(1+\varphi)} p^e H_t^e\right) \\ -\lambda_t^5\left(\bar{\tau}\left(\frac{\phi_b}{b^M}\right)\left(\frac{P_t^M}{P^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b-1}\right) w_t\left(C_t^e\right)^{-1} \\ -\lambda_t^6\left(\bar{\tau}\left(\frac{\phi_b}{b^M}\right)\left(\frac{P_t^M}{P^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b-1} w_t p^e H_t^e R_t+\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)}\right) \\ +\beta \lambda_{t+1}^6\left(\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)}\right) R_{t+1} \end{bmatrix}=0$$

Under Lump Sum taxes

$$\mathscr{L}_t =_{t=0}^{\infty}(\beta)^t \begin{bmatrix} \left(p^u \ln\left(C_t^u\right)+p^e \ln\left(C_t^e\right)-p^e \frac{w_t}{C_t^e(1+\varphi)} H_t^e\right) \\ +\lambda_t^1\left(p^e C_t^e+p^u C_t^u+G_t+\frac{\Omega}{2}\left(NAP_t^u-NAP^u\right)^2-\left(1-\frac{\Phi}{2} \pi_t^2\right) Y_t\right) \\ +\lambda_t^2\left(\beta \mathbb{E}_t\left(p^{e|e}\left(C_{t+1}^e\right)^{-1}+p^{u|e}\left(1+\Omega\left(NAP_{t+1}^u-NAP^u\right)\right)\left(C_{t+1}^u\right)^{-1}\right)-\frac{\left(C_t^e\right)^{-1}}{R_t}\right) \\ +\lambda_t^8\begin{pmatrix} \beta \mathbb{E}_t\left(p^{e|u}\left(C_{t+1}^e\right)^{-1}+p^{u|u}\left(1+\Omega\left(NAP_{t+1}^u-NAP^u\right)\right)\left(C_{t+1}^u\right)^{-1}\right) \\ -\left(1+\Omega\left(NAP_t^u-NAP^u\right)\right) \frac{\left(C_t^u\right)^{-1}}{R_t} \end{pmatrix} \\ +\lambda_t^3\left(\mathbb{E}_t\left(\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)}\right)-R_t P_t^M\right) \\ +\lambda_t^4\left(\left[\left(1-\varepsilon_t+(1-s) \varepsilon_t \frac{w_t}{z_t}\right)-\Phi \pi_t\left(1+\pi_t\right)\right] Y_t R_t+\Phi \mathbb{E}_t\left(\pi_{t+1}\left(1+\pi_{t+1}\right) Y_{t+1}\right)\right) \\ +\lambda_t^5\left(w_t\left(C_t^e\right)^{-1}-\left(H_t^e\right)^{\varphi}\right) \\ +\lambda_t^6\left(\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)} b_t^L+G_t+T_t^u-T^p\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) R_t-\mathbb{E}_t\left(\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)} b_{t+1}^L\right)\right) \\ +\lambda_t^7\left(p^e H_t^e z_t-Y_t\right) \\ +\lambda_t^9\left(T_t^u-p^u C_t^u-NAP_t^u\right) \end{bmatrix}$$

FOCs

$$\frac{\partial \mathcal{L}_t}{\partial C_t^e} = 0: \left[\begin{array}{c} \frac{p^e}{C_t^e}\left(1+\frac{w_t}{C_t^e(1+\varphi)}H_t^e\right) \\ +p^e\lambda_t^1 \\ -p^{e|e}\left(C_t^e\right)^{-2}\lambda_{t-1}^2+\lambda_t^2\frac{\left(C_t^e\right)^{-2}}{R_t} \\ -\lambda_{t-1}^8 p^{e|u}\left(C_t^e\right)^{-2} \\ -\lambda_t^5 w_t\left(C_t^e\right)^{-2} \end{array}\right] = 0$$

Multiply across by $\left(C_t^e\right)^2$

$$\frac{\partial \mathcal{L}_t}{\partial C_t^e} = 0: \left[\begin{array}{c} p^e C_t^e + p^e\frac{w_t}{(1+\varphi)}H_t^e \\ +p^e\lambda_t^1\left(C_t^e\right)^2 - p^{e|u}\lambda_{t-1}^8 \\ -p^{e|e}\lambda_{t-1}^2 \\ +\frac{\lambda_t^2}{R_t} \\ -\lambda_t^5 w_t \end{array}\right] = 0$$

$$\frac{\partial \mathcal{L}_t}{\partial NAP_t^u} = 0: \lambda_t^1\Omega\left(NAP_t^u - NAP^u\right) + \Omega p^{u|e}\lambda_{t-1}^2\left(C_t^u\right)^{-1} + \Omega p^{u|u}\lambda_{t-1}^8\left(C_t^u\right)^{-1} - \Omega\frac{\left(C_t^u\right)^{-1}}{R_t}\lambda_t^8 - \lambda_t^9 = 0$$

$$\frac{\partial \mathcal{L}_t}{\partial C_t^u} = 0: \left[\begin{array}{c} \frac{p^u}{C_t^u} + p^u\lambda_t^1 - p^{u|e}\lambda_{t-1}^2\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(C_t^u\right)^{-2} \\ -\lambda_{t-1}^8 p^{u|u}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(C_t^u\right)^{-2} \\ +\lambda_t^8\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(\frac{\left(C_t^u\right)^{-2}}{R_t}\right) - \lambda_t^9 p^u \end{array}\right] = 0$$

$$\frac{\partial \mathcal{L}_t}{\partial C_t^u} = 0: \left[\begin{array}{c} p^u C_t^u + \lambda_t^1 p^u\left(C_t^u\right)^2 - \lambda_{t-1}^2 p^{u|e}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) \\ -\lambda_{t-1}^8 p^{u|u}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) \\ +\frac{\lambda_t^8}{R_t}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) - \lambda_t^9 p^u\left(C_t^u\right)^2 \end{array}\right] = 0$$

:

$$\frac{\partial \mathcal{L}_t}{\partial Y_t} = 0: \left[\begin{array}{c} -\lambda_t^1\left(1-\frac{\Phi}{2}\pi_t^2\right)Y_t \\ +\lambda_t^4\left(\left(1-\varepsilon_t+(1-s)\varepsilon_t\frac{w_t}{z_t}\right) - \Phi\pi_t\left(1+\pi_t\right)\right)R_t Y_t \\ +\frac{\lambda_{t-1}^4}{\beta}\Phi\left(1+\pi_t\right)\pi_t Y_t \\ -\lambda_t^7 Y_t \end{array}\right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial \pi_t}=0:\left[\begin{array}{c}\lambda_t^1\left(\Phi \pi_t Y_t\right)-\frac{\lambda_{t-1}^3}{\beta}\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2}\right) \\ -\lambda_t^4 \Phi\left(1+2 \pi_t\right) Y_t R_t \\ +\frac{\lambda_{t-1}^4}{\beta} \Phi\left(1+2 \pi_t\right) Y_t \\ -\lambda_t^6\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2} b_t^M\right) R_t+\frac{\lambda_{t-1}^6}{\beta}\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2}\right) b_t^M\end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial R_t}=0:\left[\begin{array}{c}+\lambda_t^2 \frac{\left(C_t^e\right)^{-1}}{\left(R_t\right)^2}+\lambda_t^8\left(1+\Omega\left(N A P_t^u-N A P^u\right)\right) \frac{\left(C_t^u\right)^{-1}}{\left(R_t\right)^2}-\lambda_t^3 P_t^M \\ +\lambda_t^4\left(\left(1-\varepsilon_t+(1-s) \varepsilon_t \frac{w_t}{z_t}\right)-\Phi \pi_t\left(1+\pi_t\right)\right) Y_t \\ +\lambda_t^6\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)} b_t^L+G_t+T_t^u-T^p\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial w_t}=0:\left[\begin{array}{c}-p^e \frac{H_t^e}{C_t^e(1+\varphi)} \\ +\lambda_t^4(1-s) \varepsilon_t \frac{Y_t}{z_t} R_t \\ +\lambda_t^5\left(C_t^e\right)^{-1}\end{array}\right]=0$$

$$\begin{aligned}\frac{\partial \mathscr{L}_t}{\partial H_t^e} &= 0:\left[\begin{array}{c}-p^e \frac{w_t}{C_t^e(1+\varphi)} \\ -\varphi \lambda_t^5\left(H_t^e\right)^{\varphi-1} \\ +p^e z_t \lambda_t^7\end{array}\right]=0 \\ \frac{\partial \mathscr{L}_t}{\partial H_t^e} &= 0:\left[\begin{array}{c}-p^e \frac{w_t}{C_t^e(1+\varphi)} H_t^e \\ -\varphi \lambda_t^5\left(H_t^e\right)^{\varphi} \\ +\lambda_t^7 Y_t\end{array}\right]=0\end{aligned}$$

$$\frac{\partial \mathscr{L}_t}{\partial P_t^M}=0:\left[\begin{array}{c}-\lambda_t^3 R_t \\ +\left(\frac{\lambda_{t-1}^3}{\beta}\right)\left(\frac{\rho}{\left(1+\pi_t\right)}\right) \\ -\lambda_t^5\left(\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) w_t\left(C_t^e\right)^{-1} \\ +\lambda_t^6\left[\left(\frac{\rho}{\left(1+\pi_t\right)}\right) b_t^M-\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} T^p\right] R_t \\ -\frac{\lambda_{t-1}^6}{\beta}\left(\frac{\rho}{\left(1+\pi_t\right)} b_t^M\right)\end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial b_{t+1}^M}=0:\left[\begin{array}{c}-\lambda_t^6\left(\left(\frac{\phi_b}{b^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} T^p R_t+\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)}\right) \\ +\beta \lambda_{t+1}^6\left(\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)}\right) R_{t+1}\end{array}\right]=0$$

- Now allowing transfers to be time-varying alters the policy maker's information set regardless of the tax instrument available.

Under distortionary income taxes, $T_t^u = \vartheta\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) w_t H_t^e p^u$

$$\mathscr{L}_t =_{t=0}^{\infty} (\beta)^t \left[\begin{array}{c} \left(p^u \ln(C_t^u) + p^e \ln(C_t^e) - p^e\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{w_t}{C_t^e(1+\varphi)} H_t^e\right) \\ +\lambda_t^1\left(p^e C_t^e + p^u C_t^u + G_t + \frac{\Omega}{2}(NAP_t^u - NAP^u)^2 - \left(1-\frac{\Phi}{2}\pi_t^2\right)Y_t\right) \\ +\lambda_t^2\left(\beta\mathbb{E}_t\left(p^{e|e}\left(C_{t+1}^e\right)^{-1} + p^{u|e}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\left(C_{t+1}^u\right)^{-1}\right) - \frac{(C_t^e)^{-1}}{R_t}\right) \\ +\lambda_t^8\left(\begin{array}{c} \beta\mathbb{E}_t\left(p^{e|u}\left(C_{t+1}^e\right)^{-1} + p^{u|u}\left(1+\Omega\left(NAP_{t+1}^u - NAP^u\right)\right)\left(C_{t+1}^u\right)^{-1}\right) \\ -\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\frac{(C_t^u)^{-1}}{R_t} \end{array}\right) \\ +\lambda_t^3\left(\mathbb{E}_t\left(\frac{(1+\rho P_{t+1}^M)}{(1+\pi_{t+1})}\right) - R_t P_t^M\right) \\ +\lambda_t^4\left(\left[\left(1-\varepsilon_t + (1-s)\varepsilon_t\frac{w_t}{z_t}\right) - \Phi\pi_t(1+\pi_t)\right]Y_t R_t + \Phi\mathbb{E}_t\left(\pi_{t+1}(1+\pi_{t+1})Y_{t+1}\right)\right) \\ +\lambda_t^5\left(\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) w_t (C_t^e)^{-1} - (H_t^e)^{\varphi}\right) \\ +\lambda_t^6\left(\left(\begin{array}{c} \frac{(1+\rho P_t^M)}{(1+\pi_t)} b_t^L + G_t + \vartheta\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) w_t H_t^e p^u \\ -\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} w_t p^e H_t^e - T_t^p \end{array}\right) R_t - \mathbb{E}_t\left(\frac{(1+\rho P_{t+1}^M)}{(1+\pi_{t+1})} b_{t+1}^L\right)\right) \\ +\lambda_t^7\left(p^e H_t^e z_t - Y_t\right) \\ +\lambda_t^9\left(\vartheta\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) w_t H_t^e p^u - p^u C_t^u - NAP_t^u\right) \end{array}\right]$$

$$FOCs$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0: \left[\begin{array}{c} \frac{p^e}{C_t^e}\left(1+\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{w_t}{C_t^e(1+\varphi)}H_t^e\right) \\ +p^e\lambda_t^1 \\ -p^{e|e}\left(C_t^e\right)^{-2}\lambda_{t-1}^2+\lambda_t^2\frac{\left(C_t^e\right)^{-2}}{R_t} \\ -\lambda_{t-1}^8 p^{e|u}\left(C_t^e\right)^{-2} \\ -\lambda_t^5\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t\left(C_t^e\right)^{-2} \end{array}\right] = 0$$

Multiply across by $\left(C_t^e\right)^2$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0: \left[\begin{array}{c} p^e C_t^e + p^e\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{w_t}{(1+\varphi)}H_t^e \\ +p^e\lambda_t^1\left(C_t^e\right)^2 - p^{e|u}\lambda_{t-1}^8 \\ -p^{e|e}\lambda_{t-1}^2 \\ +\frac{\lambda_t^2}{R_t} \\ -\lambda_t^5\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t \end{array}\right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial NAP_t^u} = 0: \lambda_t^1\Omega\left(NAP_t^u - NAP^u\right) + \Omega p^{u|e}\lambda_{t-1}^2\left(C_t^u\right)^{-1} + \Omega p^{u|u}\lambda_{t-1}^8\left(C_t^u\right)^{-1} - \Omega\frac{\left(C_t^u\right)^{-1}}{R_t}\lambda_t^8 - \lambda_t^9 = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0: \left[\begin{array}{c} \frac{p^u}{C_t^u} + p^u\lambda_t^1 - p^{u|e}\lambda_{t-1}^2\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(C_t^u\right)^{-2} \\ -\lambda_{t-1}^8 p^{u|u}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(C_t^u\right)^{-2} \\ +\lambda_t^8\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(\frac{\left(C_t^u\right)^{-2}}{R_t}\right) - \lambda_t^9 p^u \end{array}\right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0: \left[\begin{array}{c} p^u C_t^u + \lambda_t^1 p^u\left(C_t^u\right)^2 - \lambda_{t-1}^2 p^{u|e}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) \\ -\lambda_{t-1}^8 p^{u|u}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) \\ +\frac{\lambda_t^8}{R_t}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) - \lambda_t^9 p^u\left(C_t^u\right)^2 \end{array}\right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial Y_t} = 0: \left[\begin{array}{c} -\lambda_t^1\left(1-\frac{\Phi}{2}\pi_t^2\right)Y_t \\ +\lambda_t^4\left(\left(1-\varepsilon_t+(1-s)\varepsilon_t\frac{w_t}{z_t}\right)-\Phi\pi_t\left(1+\pi_t\right)\right)R_t Y_t \\ +\frac{\lambda_{t-1}^4}{\beta}\Phi\left(1+\pi_t\right)\pi_t Y_t \\ -\lambda_t^7 Y_t \end{array}\right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial \pi_t}=0:\left[\begin{array}{c}\lambda_t^1\left(\Phi\pi_t Y_t\right)-\frac{\lambda_{t-1}^3}{\beta}\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2}\right)\\ -\lambda_t^4\Phi\left(1+2\pi_t\right)Y_t R_t\\ +\frac{\lambda_{t-1}^4}{\beta}\Phi\left(1+2\pi_t\right)Y_t\\ -\lambda_t^6\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2}b_t^M\right)R_t+\frac{\lambda_{t-1}^6}{\beta}\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2}\right)b_t^M\end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial R_t}=0:\left[\begin{array}{c}+\lambda_t^2\frac{\left(C_t^e\right)^{-1}}{\left(R_t\right)^2}+\lambda_t^8\left(1+\Omega\left(NAP_t^u-NAP^u\right)\right)\frac{\left(C_t^u\right)^{-1}}{\left(R_t\right)^2}-\lambda_t^3 P_t^M\\ +\lambda_t^4\left(\left(1-\varepsilon_t+(1-s)\,\varepsilon_t\frac{w_t}{z_t}\right)-\Phi\pi_t\left(1+\pi_t\right)\right)Y_t\\ +\lambda_t^6\left(\begin{array}{c}\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)}b_t^L+G_t+\vartheta\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t H_t^e p^u\\ -\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}w_t p^e H_t^e-T_t^p\end{array}\right)\end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial w_t}=0:\left[\begin{array}{c}-p^e\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{H_t^e}{C_t^e(1+\varphi)}\\ +\lambda_t^4\left(1-s\right)\varepsilon_t\frac{Y_t}{z_t}R_t\\ +\lambda_t^5\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\left(C_t^e\right)^{-1}\\ +\lambda_t^6\left(\vartheta\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)H_t^e p^u-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}p^e H_t^e\right)R_t\\ +\lambda_t^9\left(\vartheta\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)H_t^e p^u\right)\end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial H_t^e} \;=\; 0: \left[\begin{array}{c} -p^e\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{w_t}{C_t^e(1+\varphi)} \\ -\varphi\lambda_t^5\left(H_t^e\right)^{\varphi-1} \\ +\lambda_t^6\left(\begin{array}{c} \vartheta\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t p^u \\ -\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} w_t p^e \end{array}\right)R_t \\ +p^e z_t\lambda_t^7+\lambda_t^9\left(\vartheta\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t p^u\right) \end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial H_t^e} \;=\; 0: \left[\begin{array}{c} -p^e\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{w_t}{C_t^e(1+\varphi)}H_t^e \\ -\varphi\lambda_t^5\left(H_t^e\right)^{\varphi} \\ +\lambda_t^6\left(\begin{array}{c} \vartheta\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t H_t^e p^u \\ -\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} w_t p^e H_t^e \end{array}\right)R_t \\ +\lambda_t^7 Y_t+\lambda_t^9\left(\vartheta\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t p^u H_t^e\right) \end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial P_t^M}=0: \left[\begin{array}{c} \left(p^e\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\frac{w_t}{C_t^e(1+\varphi)}H_t^e\right) \\ -\lambda_t^3 R_t \\ +\left(\frac{\lambda_{t-1}^3}{\beta}\right)\left(\frac{\rho}{(1+\pi_t)}\right) \\ -\lambda_t^5\left(\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t\left(C_t^e\right)^{-1} \\ +\lambda_t^6\left[\begin{array}{c} \left(\frac{\rho}{(1+\pi_t)}\right)b_t^M-\vartheta\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}w_t H_t^e p^u \\ -\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}w_t p^e H_t^e \end{array}\right]R_t \\ -\frac{\lambda_{t-1}^6}{\beta}\left(\frac{\rho}{(1+\pi_t)}b_t^M\right)-\lambda_t^9\vartheta\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}w_t H_t^e p^u \end{array}\right]=0$$

$$\frac{\partial \mathcal{L}_t}{\partial b^M_{t+1}} = 0 : \left[\begin{array}{c} \left(\bar{\tau} \left(\frac{\phi_b}{b^M_{t+1}} \right) \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} \frac{w_t}{C^e_t (1+\varphi)} p^e H^e_t \right) \\ -\lambda^5_t \left(\bar{\tau} \left(\frac{\phi_b}{b^M_{t+1}} \right) \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} \right) w_t \left(C^e_t \right)^{-1} \\ -\lambda^6_t \left(\begin{array}{c} \vartheta \bar{\tau} \left(\frac{\phi_b}{b^M_{t+1}} \right) \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} w_t p^u H^e_t R_t \\ + \bar{\tau} \left(\frac{\phi_b}{b^M_{t+1}} \right) \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} w_t p^e H^e_t R_t + \frac{\left(1+\rho P^M_{t+1}\right)}{(1+\pi_{t+1})} \end{array} \right) \\ +\beta \lambda^6_{t+1} \left(\frac{\left(1+\rho P^M_{t+1}\right)}{(1+\pi_{t+1})} \right) R_{t+1} - \lambda^9_t \vartheta \bar{\tau} \left(\frac{\phi_b}{b^M_{t+1}} \right) \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} w_t H^e_t p^u \end{array} \right] = 0$$

Under Lump Sum taxes $T^u_t = \vartheta \left(w_t H^e_t - \frac{T^p}{p^e} \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} \right) p^u$

$$\mathcal{L}_t =_{t=0}^{\infty} (\beta)^t \left[\begin{array}{c} \left(p^u \ln \left(C^u_t \right) + p^e \ln \left(C^e_t \right) - p^e \frac{w_t}{C^e_t (1+\varphi)} H^e_t \right) \\ +\lambda^1_t \left(p^e C^e_t + p^u C^u_t + G_t + \frac{\Omega}{2} \left(NAP^u_t - NAP^u \right)^2 - \left(1 - \frac{\Phi}{2} \pi^2_t \right) Y_t \right) \\ +\lambda^2_t \left(\beta \mathbb{E}_t \left(p^{e|e} \left(C^e_{t+1} \right)^{-1} + p^{u|e} \left(1 + \Omega \left(NAP^u_{t+1} - NAP^u \right) \right) \left(C^u_{t+1} \right)^{-1} \right) - \frac{\left(C^e_t\right)^{-1}}{R_t} \right) \\ +\lambda^8_t \left(\begin{array}{c} \beta \mathbb{E}_t \left(p^{e|u} \left(C^e_{t+1} \right)^{-1} + p^{u|u} \left(1 + \Omega \left(NAP^u_{t+1} - NAP^u \right) \right) \left(C^u_{t+1} \right)^{-1} \right) \\ - \left(1 + \Omega \left(NAP^u_t - NAP^u \right) \right) \frac{\left(C^u_t\right)^{-1}}{R_t} \end{array} \right) \\ +\lambda^3_t \left(\mathbb{E}_t \left(\frac{\left(1+\rho P^M_{t+1}\right)}{(1+\pi_{t+1})} \right) - R_t P^M_t \right) \\ +\lambda^4_t \left(\left[\left(1 - \varepsilon_t + (1-s) \varepsilon_t \frac{w_t}{z_t} \right) - \Phi \pi_t \left(1 + \pi_t \right) \right] Y_t R_t + \Phi \mathbb{E}_t \left(\pi_{t+1} \left(1 + \pi_{t+1} \right) Y_{t+1} \right) \right) \\ +\lambda^5_t \left(w_t \left(C^e_t \right)^{-1} - \left(H^e_t \right)^{\varphi} \right) \\ +\lambda^6_t \left(\begin{array}{c} \left(\frac{\left(1+\rho P^M_t\right)}{(1+\pi_t)} b^L_t + G_t + \vartheta \left(w_t H^e_t - \frac{T^p}{p^e} \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} \right) p^u - T^p \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} \right) R_t \\ - \mathbb{E}_t \left(\frac{\left(1+\rho P^M_{t+1}\right)}{(1+\pi_{t+1})} b^L_{t+1} \right) \end{array} \right) \\ +\lambda^7_t \left(p^e H^e_t z_t - Y_t \right) \\ +\lambda^9_t \left(\vartheta \left(w_t H^e_t - \frac{T^p}{p^e} \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} \right) p^u - p^u C^u_t - NAP^u_t \right) \end{array} \right]$$

$$FOCs$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0: \left[\begin{array}{c} \frac{p^e}{C_t^e}\left(1+\frac{w_t}{C_t^e(1+\varphi)}H_t^e\right) \\ +p^e\lambda_t^1 \\ -p^{e|e}\left(C_t^e\right)^{-2}\lambda_{t-1}^2+\lambda_t^2\frac{\left(C_t^e\right)^{-2}}{R_t} \\ -\lambda_{t-1}^8 p^{e|u}\left(C_t^e\right)^{-2} \\ -\lambda_t^5 w_t\left(C_t^e\right)^{-2} \end{array} \right] = 0$$

Multiply across by $\left(C_t^e\right)^2$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0: \left[\begin{array}{c} p^e C_t^e + p^e\frac{w_t}{(1+\varphi)}H_t^e \\ +p^e\lambda_t^1\left(C_t^e\right)^2 - p^{e|u}\lambda_{t-1}^8 \\ -p^{e|e}\lambda_{t-1}^2 \\ +\frac{\lambda_t^2}{R_t} \\ -\lambda_t^5 w_t \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial NAP_t^u} = 0: \lambda_t^1\Omega\left(NAP_t^u - NAP^u\right) + \Omega p^{u|e}\lambda_{t-1}^2\left(C_t^u\right)^{-1} + \Omega p^{u|u}\lambda_{t-1}^8\left(C_t^u\right)^{-1} - \Omega\frac{\left(C_t^u\right)^{-1}}{R_t}\lambda_t^8 - \lambda_t^9 = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0: \left[\begin{array}{c} \frac{p^u}{C_t^u} + p^u\lambda_t^1 - p^{u|e}\lambda_{t-1}^2\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(C_t^u\right)^{-2} \\ -\lambda_{t-1}^8 p^{u|u}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(C_t^u\right)^{-2} \\ +\lambda_t^8\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right)\left(\frac{\left(C_t^u\right)^{-2}}{R_t}\right) - \lambda_t^9 p^u \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0: \left[\begin{array}{c} p^u C_t^u + \lambda_t^1 p^u\left(C_t^u\right)^2 - \lambda_{t-1}^2 p^{u|e}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) \\ -\lambda_{t-1}^8 p^{u|u}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) \\ +\frac{\lambda_t^8}{R_t}\left(1+\Omega\left(NAP_t^u - NAP^u\right)\right) - \lambda_t^9 p^u\left(C_t^u\right)^2 \end{array} \right] = 0$$

:

$$\frac{\partial \mathscr{L}_t}{\partial Y_t} = 0: \left[\begin{array}{c} -\lambda_t^1\left(1-\frac{\Phi}{2}\pi_t^2\right)Y_t \\ +\lambda_t^4\left(\left(1-\varepsilon_t+(1-s)\,\varepsilon_t\frac{w_t}{z_t}\right) - \Phi\pi_t\left(1+\pi_t\right)\right)R_t Y_t \\ +\frac{\lambda_{t-1}^4}{\beta}\Phi\left(1+\pi_t\right)\pi_t Y_t \\ -\lambda_t^7 Y_t \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial \pi_t} = 0 : \left[\begin{array}{c} \lambda_t^1 \left(\Phi \pi_t Y_t\right) - \frac{\lambda_{t-1}^3}{\beta} \left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2} \right) \\ -\lambda_t^4 \Phi \left(1+2\pi_t\right) Y_t R_t \\ +\frac{\lambda_{t-1}^4}{\beta} \Phi \left(1+2\pi_t\right) Y_t \\ -\lambda_t^6 \left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2} b_t^M \right) R_t + \frac{\lambda_{t-1}^6}{\beta} \left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2} \right) b_t^M \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial R_t} = 0 : \left[\begin{array}{c} +\lambda_t^2 \frac{\left(C_t^e\right)^{-1}}{\left(R_t\right)^2} + \lambda_t^8 \left(1 + \Omega \left(NAP_t^u - NAP^u\right)\right) \frac{\left(C_t^u\right)^{-1}}{\left(R_t\right)^2} \\ -\lambda_t^3 P_t^M \\ +\lambda_t^4 \left(\left(1 - \varepsilon_t + (1-s) \varepsilon_t \frac{w_t}{z_t}\right) - \Phi \pi_t \left(1+\pi_t\right) \right) Y_t \\ +\lambda_t^6 \left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)} b_t^L + G_t + \vartheta \left(w_t H_t^e - \frac{T^p}{p^e} \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} \right) p^u - T^p \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} \right) \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial w_t} = 0 : \left[\begin{array}{c} -p^e \frac{H_t^e}{C_t^e (1+\varphi)} \\ +\lambda_t^4 (1-s) \varepsilon_t \frac{Y_t}{z_t} R_t \\ +\lambda_t^6 \vartheta H_t^e p^u \\ +\lambda_t^5 \left(C_t^e\right)^{-1} \\ +\lambda_t^9 \vartheta H_t^e p^u \end{array} \right] = 0$$

$$\begin{aligned} \frac{\partial \mathscr{L}_t}{\partial H_t^e} &= 0 : \left[\begin{array}{c} -p^e \frac{w_t}{C_t^e (1+\varphi)} \\ -\varphi \lambda_t^5 \left(H_t^e\right)^{\varphi - 1} \\ +\lambda_t^6 \vartheta w_t p^u \\ +p^e z_t \lambda_t^7 + \lambda_t^9 \vartheta w_t p^u \end{array} \right] = 0 \\ \frac{\partial \mathscr{L}_t}{\partial H_t^e} &= 0 : \left[\begin{array}{c} -p^e \frac{w_t}{C_t^e (1+\varphi)} H_t^e \\ -\varphi \lambda_t^5 \left(H_t^e\right)^{\varphi} \\ +\lambda_t^6 \vartheta w_t p^u H_t^e \\ +\lambda_t^7 Y_t + \lambda_t^9 \vartheta w_t p^u H_t^e \end{array} \right] = 0 \end{aligned}$$

$$\frac{\partial \mathscr{L}_t}{\partial P_t^M}=0:\left[\begin{array}{c} -\lambda_t^3 R_t \\ +\left(\frac{\lambda_{t-1}^3}{\beta}\right)\left(\frac{\rho}{(1+\pi_t)}\right) \\ -\lambda_t^5\left(\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) w_t\left(C_t^e\right)^{-1} \\ +\lambda_t^6\left[\left(\frac{\rho}{(1+\pi_t)}\right) b_t^M-\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} T^p\right] R_t \\ -\frac{\lambda_{t-1}^6}{\beta}\left(\frac{\rho}{(1+\pi_t)} b_t^M\right) \end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial b_{t+1}^M}=0:\left[\begin{array}{c} -\lambda_t^6\left(\begin{array}{c} \vartheta\left(\frac{\phi_b}{b^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} \frac{T^p}{p^e} p^u R_t \\ +\left(\frac{\phi_b}{b^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} T^p R_t+\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)} \end{array}\right) \\ +\beta \lambda_{t+1}^6\left(\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)}\right) R_{t+1}-\lambda_t^9 \vartheta\left(\frac{\phi_b}{b^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} \frac{T^p}{p^e} p^u \end{array}\right]=0$$

3.17 Appendix G

In this section we derive a tractable HANK model in the tradition of Bilbiie & Ragot (2021) and Bilbiie (2024). Once again, we rely on the family metaphor of **Lucas Jr (1990)** to simplify our analysis. Each individual household belongs to a big representative family. Thee so-called "family head" maximizes the inter temporal welfare of all family members using a utilitarian welfare criterion (all family members are equally weighted). This model also features imperfect risk sharing, in the sense that the family head can reshuffle assets between members- individual with the same realisation of the idiosyncratic shock- but not across different household types. Essentially, the model offers perfect insurance but only within type.

In every period there exist two groups of household types: an optimising Ricardian group and a Hand -to -Mouth (M-to-M, henceforth) group. As in our benchmark HANK model, each household's initial wealth in period t takes only two possible values that depend on the current state of idiosyncratic shock but that are independent of the household's past history. The **idiosyncratic shock determines whether the household will have access to the financial markets (or rather access to a savings vehicle) or not.** To keep the notation as close as possible to our main model, we will assume again that the idiosyncratic stock takes again two values $v_t = \{e, u\}$ where this "e" this time will denote a typical optimising consumer type whilst "u" will denote a constrained or "H-to-M" type.

Both THANK frameworks feature marginal propensity to consume (MPC, henceforth) heterogeneity and full insurance within type.

As in **Bilbiie (2019, 2024)**, the consumption and saving choices of a household households are identical within group. We denote the level of consumption, labor supply, and asset holdings (savings) for the Ricardian households in period t as $c_t^e, H_t^e, \frac{\alpha_{t+1}^{M(e)}}{p^e}$, and $\frac{a_{t+1}^e}{p^e}$. Similarly, for the H-to-M consumer, we use H_t^u, c_t^u, $\frac{\alpha_{t+1}^{M(u)}}{p^u}$ and $\frac{a_{t+1}^u}{p^u}$ to denote the labour supply, consumption and savings. For simplicity, we assume again $2x2$ Markov- transition matrix,

$$pp = \begin{bmatrix} p(e|e) & p(e|u) \\ p(u|e) & p(u|u) \end{bmatrix}$$

Where, the entries denote the probability of one retaining their idiosyncratic status (Optimising or H-to-M household) or the probability of switching type. This Representative family assumption simplifies the model considering thus, allowing us to derive an number of analytical results.

The budget constraints for the Ricardian$\{e\}$ and Non- Ricardian$\{u\}$ households are given, respectively, by

$$C_t^e + (1+\pi_{t+1}) P_t^M \left(\frac{\alpha_{t+1}^{M(e)}}{p^e}\right) + \left[\frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)}\right]\left(\frac{\alpha_{t+1}^e}{p^e}\right) = \left(\begin{array}{c} (1-\tau_t) w_t H_t^e + D_t^e \\ +\left(1+\rho P_t^M\right)\left(\frac{\hat{\alpha}_t^{M(e)}}{p^e}\right) + \left(\frac{\hat{\alpha}_t^e}{p^e}\right) - \frac{T_t^e}{p^e} \end{array}\right)$$

where,

$$\begin{aligned} \hat{\alpha}_t^{M(e)} &= a_t^{M(e)} p^{e|e} + a_t^{M(u)} p^{e|u} = a_t^{M(e)} p^{e|e} \\ \hat{\alpha}_t^e &= a_t^e p^{e|e} + a_t^u p^{e|u} = a_t^e p^{e|e} \end{aligned}$$

Or rather,

$$C_t^e + (1+\pi_{t+1}) P_t^M \left(\frac{\alpha_{t+1}^{M(e)}}{p^e}\right) + \left[\frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)}\right]\left(\frac{\alpha_{t+1}^e}{p^e}\right) = \left(\begin{array}{c} (1-\tau_t) w_t H_t^e + D_t^e \\ +\left(1+\rho P_t^M\right)\left(\frac{p^{e|e}}{p^e}\right) a_t^{M(e)} + \left(\frac{p^{e|e}}{p^e}\right) a_t^e - \frac{T_t^e}{p^e} \end{array}\right)$$

Where, C_t^e is the consumption level in period t, P_t^M and $\hat{\alpha}_t^{M(e)}$ are the price and quantity long term government bonds. $\hat{\alpha}_t^e$ stands for the short government bonds. For simplicity we assume that $\hat{\alpha}_t^e$ exists in zero net supply. The aggregate dividends are given as D_t^e but the households do not internalise them. Furthermore, H_t^e and w_t represent the labour supply and the aggregate wage rate, for the Ricardian type. Finally, τ_t stand for distortionary income tax and T_t for a Lump Sum tax imposed on both consumer types consumers.

We can simplify further the budget constraint for each household type. For the Ricardian type, the budget constraint takes the form:

$$C_t^u + (1+\pi_{t+1}) P_t^M \left(\frac{\alpha_{t+1}^{M(u)}}{p^u}\right) + \frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)} \left(\frac{\alpha_{t+1}^u}{p^u}\right) = \begin{pmatrix} (1-\tau_t) w_t H_t^u + D_t^u \\ +\left(1+\rho P_t^M\right)\left(\frac{\hat{\alpha}_t^{M(u)}}{p^u}\right) + \frac{\hat{\alpha}_t^u}{p^u} + \frac{T_t^u}{p^u} \end{pmatrix}$$

where,

$$\hat{\alpha}_t^{M(u)} = a_t^{M(e)} p^{u|e} + a_t^{M(u)} p^{u|u} = a_t^{M(e)} p^{u|e}$$

$$\hat{\alpha}_t^u = a_t^e p^{u|e} + a_t^u p^{u|u} = a_t^e p^{u|e}$$

Since,

$$a_t^{M(u)} = a_{t+1}^{M(u)} = 0, \forall t$$

$$a_t^u = a_{t+1}^u = 0, \forall t$$

and for the Keynesian type:

$$p^u C_t^u = \left((1-\tau_t) w_t H_t^u p^u + p^{u|e}\left(1+\rho P_t^M\right) a_t^{M(e)} + p^{u|e} a_t^e + T_t^u\right)$$

Since,

$$D_t^u = 0;$$
$$H_t^u = \delta$$

Where, C_t^u is the consumption level in period t, P_t^M and $\hat{\alpha}_t^{M(u)}$ are the price and quantity long term government bonds. $\hat{\alpha}_t^u$ stands for the short government bonds. Since, H- to -M households do not have access to the financial market a_t^u and $a_t^{M(u)}$ are always zero. Finally, H_t^u represents the labour supply of the constrained household type and T_t^u stands for a lump sum transfer to them.

Household (optimisation) Problem

The head of each representative family wishes to maximise the following (utilitarian) welfare criterion

$$
\begin{aligned}
U_o &= \sum_{t=0}^{\infty}\left(\beta^t\right)\left[p^e\left(\ln\left(C_t^e\right)-\frac{\left(H_t^e\right)^{1+\varphi}}{1+\varphi}\right)+p^u\left(\ln\left(C_t^u\right)-\frac{(\delta)^{1+\varphi}}{1+\varphi}\right)\right] \\
&= \sum_{t=0}^{\infty}\left(\beta^t\right)\left[p^e\ln\left(C_t^e\right)+p^u\ln\left(C_t^e\right)-p^e\frac{\left(H_t^e\right)^{1+\varphi}}{1+\varphi}-p^u\frac{(\delta)^{1+\varphi}}{1+\varphi}\right]
\end{aligned}
$$

subject to the budget constraint given that the household is going to be (with a given probability) either

1. Ricardian

$$
p^eC_t^e+(1+\pi_{t+1})P_t^M\alpha_{t+1}^{M(e)}+\left(\frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)}\right)\alpha_{t+1}^e=\left(\begin{array}{c}(1-\tau_t)w_tH_t^ep^e+D_t^ep^e+\left(1+\rho P_t^M\right)p^{e|e}a_t^{M(e)}\\ +p^{e|e}a_t^e-T_t^e\end{array}\right)
$$

or,

2. Non- Ricardian

$$
p^uC_t^u=\left(p^{u|e}\left(1+\rho P_t^M\right)a_t^{M(e)}+p^{u|e}a_t^e+(1-\tau_t)w_t\delta p^u\right)
$$

Solving the hhs optimisation problem

$$
\mathscr{L}_t=\max_{\{C_t^e,H_t,b_{t+1}\}_{t=s}^{\infty}}\mathbb{E}_t\sum_{t=s}^{\infty}(\beta)^{t-s}\left(\begin{array}{c}p^e\ln\left(C_t^e\right)+p^u\ln\left(C_t^u\right)-p^e\frac{\left(H_t^e\right)^{1+\varphi}}{1+\varphi}-p^u\frac{(\delta)^{1+\varphi}}{1+\varphi}\\ +\psi_t^e\left(\begin{array}{c}(1-\tau_t)w_tH_t^ep^e+D_t^ep^e-T_t^e\\ +\left(1+\rho P_t^M\right)p^{e|e}a_t^{M(e)}-(1+\pi_{t+1})P_t^M\alpha_{t+1}^{M(e)}\\ +p^{e|e}a_t^e-\left(\frac{(1+\pi_{t+1})}{(1+\pi_{t+1})(1+r_t)}\right)\alpha_{t+1}^e\\ -p^eC_t^e\end{array}\right)\\ +\psi_t^u\left((1-\tau_t)w_tH_t^up^u+p^{u|e}\left(1+\rho P_t^M\right)a_t^{M(e)}+p^{u|e}a_t^e+T_t^u-p^uC_t^u\right)\end{array}\right)
$$

FOCs

$$\frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0 : \mathbb{E}_t \left(\beta\right)^{t-s} \left(\frac{p^e}{C_t^e} - \psi_t^e p^e\right) = 0$$
$$\frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0 : \psi_t^e = \left(C_t^e\right)^{-1}$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0 : \mathbb{E}_t \left(\beta\right)^{t-s} \left(\frac{p^u}{C_t^u} - p^u \psi_t^u\right) = 0$$
$$\frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0 : \psi_t^u = \left(C_t^u\right)^{-1}$$

$$\frac{\partial \mathscr{L}_t}{\partial H_t^e} = 0 : \mathbb{E}_t \left(\beta\right)^{t-s} \left(-p^e \left(H_t^e\right)^{\varphi} + \psi_t^e \left(\left(1-\tau_t\right) w_t p^e\right)\right) = 0$$
$$\frac{\partial \mathscr{L}_t}{\partial H_t^e} = 0 : C_t^e \left(H_t^e\right)^{\varphi} = \left(1-\tau_t\right) w_t$$

$$\frac{\partial \mathscr{L}_t}{\partial \alpha_{t+1}^e} = 0 : \left(-\left(\beta\right)^{t-s}\left(\frac{\psi_t^e}{\left(1+r_t\right)}\right) + p^{e|e}\left(\beta\right)^{t+1-s}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|e}\left(\beta\right)^{t+1-s}\mathbb{E}_t\left(\psi_{t+1}^u\right)\right) = 0$$
$$\frac{\partial \mathscr{L}_t}{\partial \alpha_{t+1}^e} = 0 : \psi_t^e = \beta\left(1+r_t\right)\left[p^{e|e}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|e}\mathbb{E}_t\left(\psi_{t+1}^u\right)\right]$$

$$\frac{\partial \mathscr{L}_t}{\partial \alpha_{t+1}^{M(e)}} = 0 : \begin{pmatrix} -\left(\beta\right)^{t-s}\psi_t^e\left(\left(1+\pi_{t+1}\right)P_t^M\right) \\ +\left(\beta\right)^{t+1-s}p^{e|e}\mathbb{E}_t\left(\psi_{t+1}^e\right)\mathbb{E}_t\left(1+\rho P_{t+1}^M\right) \\ +\left(\beta\right)^{t+1-s}p^{u|e}\mathbb{E}_t\left(\psi_{t+1}^u\right)\mathbb{E}_t\left(1+\rho P_{t+1}^M\right) \end{pmatrix} = 0$$
$$\frac{\partial \mathscr{L}_t}{\partial \alpha_{t+1}^{M(e)}} = 0 : \psi_t^e\left(\left(1+\pi_{t+1}\right)P_t^M\right) = \beta\left[p^{e|e}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|e}\mathbb{E}_t\left(\psi_{t+1}^u\right)\right]\mathbb{E}_t\left(1+\rho P_{t+1}^M\right)$$
$$\frac{\partial \mathscr{L}_t}{\partial \alpha_{t+1}^{M(e)}} = 0 : P_t^M = \mathbb{E}_t\left(\frac{1+\rho P_{t+1}^M}{\left(1+r_t\right)\left(1+\pi_{t+1}\right)}\right)$$

The Consumption Euler equation takes the form

$$\psi_t^e = \beta\left(1+r_t\right)\left[p^{e|e}\mathbb{E}_t\left(\psi_{t+1}^e\right) + p^{u|e}\mathbb{E}_t\left(\psi_{t+1}^u\right)\right]$$

where, $\psi_t^u = \left(C_t^u\right)^{-1}$ & $\psi_t^e = \left(C_t^e\right)^{-1}$

as such, the stochastic discount factor (SDF) is given as

$$SDF : \frac{1}{\left(1+r_t\right)} = \beta\left[p^{e|e}\frac{\mathbb{E}_t\left(\psi_{t+1}^e\right)}{\psi_t^e} + p^{u|e}\frac{\mathbb{E}_t\left(\psi_{t+1}^u\right)}{\psi_t^e}\right]$$

The bond pricing equation takes the form

$$P_t^M = \mathbb{E}_t\left(\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+r_t\right)\left(1+\pi_{t+1}\right)}\right)$$

Social Welfare Function

The household preferences for consumption and labour, are captured by the standard CRRA felicity

$$U_t^i = \ln\left(C_t^i\right) - \frac{\left(H_t^i\right)^{1+\varphi}}{1+\varphi}$$

Where, the type of household is indexed by $i = \{R, u\}$.

The aggregate welfare function that the policy maker seeks to maximize is the aggregate utility function of the economy's population. As in Chien & Wen (2021), it the same function that the head of each family is wishes to maximises under different set of constraints.

$$\begin{aligned} W_t &= \int_0^1 U_t^i di \\ &= \int_0^{p^u} U_t^u du + \int_{p^u}^1 U_t^e de \\ &= p^e \ln\left(C_t^e\right) + p^u \ln\left(C_t^u\right) - p^u\frac{\left(\delta\right)^{1+\varphi}}{1+\varphi} - p^e\frac{\left(H_t^e\right)^{1+\varphi}}{1+\varphi} \end{aligned}$$

To simplify the problem, we will substitute in the expression for the the optimal labour supply as well as the expression for consumption inequality.

$$
\begin{aligned}
W_t &= p^e \ln\left(C_t^e\right) + p^u \ln\left(C_t^u\right) - p^u \frac{(\delta)^{1+\varphi}}{1+\varphi} - p^e \frac{(1-\tau_t)\, w_t}{C_t^e\,(1+\varphi)} H_t^e \\
&= p^u \ln\left(1-S_t\right) + \ln\left(C_t^e\right) - p^u \frac{(\delta)^{1+\varphi}}{1+\varphi} - p^e \frac{(1-\tau_t)\, w_t}{C_t^e\,(1+\varphi)} H_t^e
\end{aligned}
$$

The Policy maker's optimisation problem

The Ramsey policy maker wishes to maximise the following program

Under distortionary Income taxes

$$
\mathscr{L}_t =_{t=0}^{\infty} (\beta)^t \left[
\begin{array}{c}
p^u \ln\left(C_t^u\right) + p^e \ln\left(C_t^e\right) - p^u \frac{(\delta)^{1+\varphi}}{1+\varphi} - p^e \left(1 - \bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) w_t \frac{H_t^e}{C_t^e(1+\varphi)} \\
+\lambda_t^1 \left(\left(p^e C_t^e + p^u C_t^u\right) + G_t - \left(1 - \frac{\Phi}{2}\pi_t^2\right) Y_t\right) \\
+\lambda_t^2 \left(\beta \mathbb{E}_t \left(p^{e|e}\left(C_{t+1}^e\right)^{-1} + p^{u|e}\left(C_{t+1}^u\right)^{-1}\right) - \frac{\left(C_t^e\right)^{-1}}{R_t}\right) \\
+\lambda_t^8 \left(\left(1 - \bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) w_t \delta p^u + p^{u|e}\left(1+\rho P_t^M\right) b_t^M + T_t^u - p^u C_t^u\right) \\
+\lambda_t^3 \left(\mathbb{E}_t\left(\frac{\left(1+\rho P_{t+1}^M\right)}{R_t\left(1+\pi_{t+1}\right)}\right) - P_t^M\right) \\
+\lambda_t^4 \left(\left(1 - \varepsilon_t + (1-s)\,\varepsilon_t \frac{w_t}{z_t}\right) Y_t - \Phi \pi_t \left(1+\pi_t\right) Y_t + \Phi \beta_f \mathbb{E}_t\left(\pi_{t+1}\left(1+\pi_{t+1}\right) Y_{t+1}\right)\right) \\
+\lambda_t^5 \left(\left(1 - \bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right) w_t - C_t^e \left(H_t^e\right)^{\varphi}\right) \\
+\lambda_t^6 \left(\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)} b_t^L + G_t + T_t^u - \bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b} w_t \left(p^e H_t^e + p^u \delta\right) - T_t^p\right) R_t - \mathbb{E}_t\left(\frac{\left(1+\rho P_{t+1}^M\right)}{\left(1+\pi_{t+1}\right)} b_{t+1}^L\right)\right) \\
+\lambda_t^7 \left(z_t \left(p^e H_t^e + p^u \delta\right) - Y_t\right)
\end{array}
\right.
$$

$$
FOCs
$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0: \begin{bmatrix} \frac{p^e}{C_t^e}\left(1+\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{w_t}{C_t^e(1+\varphi)}H_t^e\right) \\ +p^e\lambda_t^1 \\ -p^{e|e}\left(C_t^e\right)^{-2}\lambda_{t-1}^2+\lambda_t^2\frac{\left(C_t^e\right)^{-2}}{R_t} \\ -\lambda_t^5\left(1-\bar{\tau}\left(\frac{P_t^M b_t^M}{P^M b^M}\right)^{\phi_b}\right)w_t\left(C_t^e\right)^{-2} \end{bmatrix} = 0$$

Multiply across by $\left(C_t^e\right)^2$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0: \begin{bmatrix} p^e C_t^e + p^e\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{w_t}{(1+\varphi)}H_t^e \\ +p^e\lambda_t^1\left(C_t^e\right)^2 \\ -p^{e|e}\lambda_{t-1}^2 \\ +\frac{\lambda_t^2}{R_t} \\ -\lambda_t^5\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t \end{bmatrix} = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0: \left[\frac{p^u}{C_t^u}+p^u\lambda_t^1-p^{u|e}\lambda_{t-1}^2\left(C_t^u\right)^{-2}-\lambda_t^8 p^u\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0: \left[p^u C_t^u+\lambda_t^1 p^u\left(C_t^u\right)^2-\lambda_{t-1}^2 p^{u|e}-\lambda_t^8 p^u\left(C_t^u\right)^2\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial Y_t} = 0: \begin{bmatrix} -\lambda_t^1\left(1-\frac{\Phi}{2}\pi_t^2\right)Y_t \\ +\lambda_t^4\left(\left(1-\varepsilon_t+(1-s)\varepsilon_t\frac{w_t}{z_t}\right)-\Phi\pi_t\left(1+\pi_t\right)\right)R_tY_t \\ +\frac{\lambda_{t-1}^4}{\beta}\Phi\left(1+\pi_t\right)\pi_tY_t \\ -\lambda_t^7Y_t \end{bmatrix} = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial \pi_t} = 0: \begin{bmatrix} \lambda_t^1\left(\Phi\pi_tY_t\right) \\ -\frac{\lambda_{t-1}^3}{\beta}\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2}\right) \\ -\lambda_t^4\Phi\left(1+2\pi_t\right)Y_tR_t \\ +\frac{\lambda_{t-1}^4}{\beta}\Phi\left(1+2\pi_t\right)Y_t \\ -\lambda_t^6\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2}b_t^M\right)R_t+\frac{\lambda_{t-1}^6}{\beta}\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2}\right)b_t^M \end{bmatrix} = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial R_t}=0:\left[\begin{array}{c}+\lambda_t^2\frac{(C_t^e)^{-1}}{(R_t)^2}\\ -\lambda_t^3 P_t^M\\ +\lambda_t^4\left(\left(1-\varepsilon_t+(1-s)\,\varepsilon_t\frac{w_t}{z_t}\right)-\Phi\pi_t\,(1+\pi_t)\right)Y_t\\ +\lambda_t^6\left(\frac{\left(1+\rho P_t^M\right)}{(1+\pi_t)}b_t^L+G_t+T_t^u-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}w_t\left(p^e H_t^e+p^u\delta\right)-T_t^p\right)\end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial w_t}=0:\left[\begin{array}{c}-p^e\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{H_t^e}{C_t^e(1+\varphi)}\\ +\lambda_t^4\,(1-s)\,\varepsilon_t\frac{Y_t}{z_t}R_t\\ +\lambda_t^5\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)(C_t^e)^{-1}\\ -\lambda_t^6\left(p^e\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}H_t^e\right)R_t\\ +\lambda_t^8\left(\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\delta p^u\right)\end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial H_t^e}=0:\left[\begin{array}{c}-p^e\left(1-\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)\frac{w_t}{C_t^e(1+\varphi)}\\ -\varphi\lambda_t^5\,(H_t^e)^{\varphi-1}\\ -\lambda_t^6\left(\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}w_t p^e\right)R_t\\ +p^e z_t\lambda_t^7\end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial P_t^M}=0:\left[\begin{array}{c}\left(p^e\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\frac{w_t}{C_t^e(1+\varphi)}H_t^e\right)\\ -\lambda_t^3 R_t\\ +\left(\frac{\lambda_{t-1}^3}{\beta}\right)\left(\frac{\rho}{(1+\pi_t)}\right)\\ -\lambda_t^5\left(\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t\,(C_t^e)^{-1}\\ +\lambda_t^6\left[\left(\frac{\rho}{(1+\pi_t)}\right)b_t^M-\bar{\tau}\left(\frac{\phi_b}{P_t^M}\right)\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}w_t p^e H_t^e\right]R_t\\ -\frac{\lambda_{t-1}^6}{\beta}\left(\frac{\rho}{(1+\pi_t)}b_t^M\right)\\ +\lambda_t^8\left(-\left(\left(\frac{\phi_b}{P_t^M}\right)\bar{\tau}\left(\frac{P_t^M b_{t+1}^M}{P^M b^M}\right)^{\phi_b}\right)w_t\delta p^u+p^{u|e}\rho b_t^M\right)\end{array}\right]=0$$

$$\frac{\partial \mathcal{L}_t}{\partial b^M_{t+1}} = 0 : \left[\begin{array}{c} \left(\bar{\tau} \left(\frac{\phi_b}{b^M_{t+1}} \right) \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} \frac{w_t}{C^e_t (1+\varphi)} p^e H^e_t \right) \\ -\lambda^5_t \left(\bar{\tau} \left(\frac{\phi_b}{b^M_{t+1}} \right) \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} \right) w_t \left(C^e_t \right)^{-1} \\ -\lambda^6_t \left(\bar{\tau} \left(\frac{\phi_b}{b^M_{t+1}} \right) \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} w_t p^e H^e_t R_t + \mathbb{E}_t \frac{\left(1+\rho P^M_{t+1}\right)}{(1+\pi_{t+1})} \right) \\ +\beta \mathbb{E}_t \lambda^6_{t+1} \left(\frac{\left(1+\rho P^M_{t+1}\right)}{(1+\pi_{t+1})} \right) R_{t+1} \\ +\lambda^8_t \left(\left(-\bar{\tau} \left(\frac{\phi_b}{b^M_{t+1}} \right) \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} \right) w_t \delta p^u \right) \\ +\frac{\lambda^8_{t+1}}{\beta} \left(p^{u|e} \mathbb{E}_t \left(1+\rho P^M_{t+1} \right) \right) \end{array} \right] = 0$$

Under Lump Sum taxes

$$\mathcal{L}_t =_{t=0}^{\infty} (\beta)^t \left[\begin{array}{c} p^u \ln\left(C^u_t\right) + p^e \ln\left(C^e_t\right) - p^u \frac{(\delta)^{1+\varphi}}{1+\varphi} - p^e w_t \frac{H^e_t}{C^e_t (1+\varphi)} \\ +\lambda^1_t \left(\left(p^e C^e_t + p^u C^u_t \right) + G_t - \left(1 - \frac{\Phi}{2} \pi^2_t \right) Y_t \right) \\ +\lambda^2_t \left(\beta \mathbb{E}_t \left(p^{e|e} \left(C^e_{t+1} \right)^{-1} + p^{u|e} \left(C^u_{t+1} \right)^{-1} \right) - \frac{\left(C^e_t\right)^{-1}}{R_t} \right) \\ +\lambda^8_t \left(\left(w_t \delta - \bar{T}^p \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} \right) p^u + p^{u|e} \left(1+\rho P^M_t \right) b^M_t + T^u_t - p^u C^u_t \right) \\ +\lambda^3_t \left(\mathbb{E}_t \left(\frac{\left(1+\rho P^M_{t+1}\right)}{R_t (1+\pi_{t+1})} \right) - P^M_t \right) \\ +\lambda^4_t \left(\left(1 - \varepsilon_t + (1-s) \varepsilon_t \frac{w_t}{z_t} \right) Y_t - \Phi \pi_t (1+\pi_t) Y_t + \Phi \beta_f \mathbb{E}_t \left(\pi_{t+1} (1+\pi_{t+1}) Y_{t+1} \right) \right) \\ +\lambda^5_t \left((1-\bar{\tau}) w_t - C^e_t \left(H^e_t \right)^{\varphi} \right) \\ +\lambda^6_t \left(\left(\frac{\left(1+\rho P^M_t\right)}{(1+\pi_t)} b^L_t + G_t + T^u_t - \bar{T}^p \left(\frac{P^M_t b^M_{t+1}}{P^M b^M} \right)^{\phi_b} \right) R_t - \mathbb{E}_t \left(\frac{\left(1+\rho P^M_{t+1}\right)}{(1+\pi_{t+1})} b^L_{t+1} \right) \right) \\ +\lambda^7_t \left(z_t \left(p^e H^e_t + p^u \delta \right) - Y_t \right) \end{array} \right]$$

$$FOCs$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0: \left[\begin{array}{c} \frac{p^e}{C_t^e}\left(1+\frac{w_t}{C_t^e(1+\varphi)}H_t^e\right) \\ +p^e\lambda_t^1 \\ -p^{e|e}\left(C_t^e\right)^{-2}\lambda_{t-1}^2+\lambda_t^2\frac{\left(C_t^e\right)^{-2}}{R_t} \\ -\lambda_t^5 w_t\left(C_t^e\right)^{-2} \end{array}\right]=0$$

Multiply across by $\left(C_t^e\right)^2$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^e} = 0: \left[\begin{array}{c} p^e C_t^e+p^e\frac{w_t}{(1+\varphi)}H_t^e \\ +p^e\lambda_t^1\left(C_t^e\right)^2 \\ -p^{e|e}\lambda_{t-1}^2 \\ +\frac{\lambda_t^2}{R_t} \\ -\lambda_t^5 w_t \end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0: \left[\frac{p^u}{C_t^u}+p^u\lambda_t^1-p^{u|e}\lambda_{t-1}^2\left(C_t^u\right)^{-2}-\lambda_t^8 p^u\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial C_t^u} = 0: \left[p^u C_t^u+\lambda_t^1 p^u\left(C_t^u\right)^2-\lambda_{t-1}^2 p^{u|e}-\lambda_t^8 p^u\left(C_t^u\right)^2\right]=0$$

:

$$\frac{\partial \mathscr{L}_t}{\partial Y_t}=0: \left[\begin{array}{c} -\lambda_t^1\left(1-\frac{\Phi}{2}\pi_t^2\right)Y_t \\ +\lambda_t^4\left(\left(1-\varepsilon_t+(1-s)\varepsilon_t\frac{w_t}{z_t}\right)-\Phi\pi_t\left(1+\pi_t\right)\right)R_tY_t \\ +\frac{\lambda_{t-1}^4}{\beta}\Phi\left(1+\pi_t\right)\pi_tY_t \\ -\lambda_t^7Y_t \end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial \pi_t}=0: \left[\begin{array}{c} \lambda_t^1\left(\Phi\pi_tY_t\right) \\ -\frac{\lambda_{t-1}^3}{\beta}\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2}\right) \\ -\lambda_t^4\Phi\left(1+2\pi_t\right)Y_tR_t \\ +\frac{\lambda_{t-1}^4}{\beta}\Phi\left(1+2\pi_t\right)Y_t \\ -\lambda_t^6\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2}b_t^M\right)R_t+\frac{\lambda_{t-1}^6}{\beta}\left(\frac{\left(1+\rho P_t^M\right)}{\left(1+\pi_t\right)^2}\right)b_t^M \end{array}\right]=0$$

$$\frac{\partial \mathscr{L}_t}{\partial R_t} = 0 : \left[\begin{array}{c} +\lambda_t{}^2 \frac{(C_t^e)^{-1}}{(R_t)^2} \\ -\lambda_t{}^3 P_t^M \\ +\lambda_t{}^4 \left(\left(1 - \varepsilon_t + (1-s)\,\varepsilon_t \frac{w_t}{z_t} \right) - \Phi \pi_t \left(1 + \pi_t\right) \right) Y_t \\ +\lambda_t{}^6 \left(\frac{\left(1+\rho P_t^M\right)}{(1+\pi_t)} b_t^L + G_t + T_t^u - \bar{T}^p \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} \right) \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial w_t} = 0 : \left[\begin{array}{c} -p^e \frac{H_t^e}{C_t^e(1+\varphi)} \\ +\lambda_t^4 \left(1-s\right) \varepsilon_t \frac{Y_t}{z_t} R_t \\ +\lambda_t^5 \left(C_t^e\right)^{-1} \\ +\lambda_t^8 \left(\delta p^u\right) \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial H_t^e} = 0 : \left[\begin{array}{c} -p^e \frac{w_t}{C_t^e(1+\varphi)} \\ -\varphi \lambda_t^5 \left(H_t^e\right)^{\varphi-1} \\ +p^e z_t \lambda_t^7 \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial P_t^M} = 0 : \left[\begin{array}{c} -\lambda_t^3 R_t + \left(\frac{\lambda_{t-1}^3}{\beta} \right) \left(\frac{\rho}{(1+\pi_t)} \right) \\ +\lambda_t^6 \left[\left(\frac{\rho}{(1+\pi_t)} \right) b_t^M - \bar{T}^p \left(\frac{\phi_b}{P_t^M} \right) \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} \right] R_t \\ -\frac{\lambda_{t-1}^6}{\beta} \left(\frac{\rho}{(1+\pi_t)} b_t^M \right) \\ +\lambda_t^8 \left(- \left(\left(\frac{\phi_b}{P_t^M} \right) \bar{T}^p \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} \right) p^u + p^{u|e} \rho b_t^M \right) \end{array} \right] = 0$$

$$\frac{\partial \mathscr{L}_t}{\partial b_{t+1}^M} = 0 : \left[\begin{array}{c} -\lambda_t^6 \left(\bar{\tau} \left(\frac{\phi_b}{b_{t+1}^M} \right) \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} w_t p^e H_t^e R_t + \mathbb{E}_t \frac{\left(1+\rho P_{t+1}^M\right)}{(1+\pi_{t+1})} \right) \\ +\beta \mathbb{E}_t \lambda_{t+1}^6 \left(\frac{\left(1+\rho P_{t+1}^M\right)}{(1+\pi_{t+1})} \right) R_{t+1} \\ +\lambda_t^8 \left(\left(-\bar{T}^p \left(\frac{\phi_b}{b_{t+1}^M} \right) \left(\frac{P_t^M b_{t+1}^M}{P^M b^M} \right)^{\phi_b} \right) p^u \right) \\ +\frac{\lambda_{t+1}^8}{\beta} \left(p^{u|e} \mathbb{E}_t \left(1 + \rho P_{t+1}^M\right) \right) \end{array} \right] = 0$$

Concluding Remarks

The main contribution of this thesis is the study of monetary and/or fiscal policy in environments with heterogeneous agents.

In the first two chapters, we develop the fiscal side of the OLG-HANK model of Acharya and Dogra (2020) and Acharya et al. (2023) to study monetary and fiscal policy interactions in an environment that features both intra-generational income inequality and inter-generational consumption/wealth inequality. This is primarily achieved by allowing for a non-trivial, endogenously determined, wealth distribution that shapes consumption disparity in the economy and also by deviating from the assumption that households are ex-ante identical.

In the first chapter, we demonstrate how the steady-state allocations, determinacy properties, and the system's response to unanticipated aggregate shocks depend on each layer of heterogeneity introduced and on the fiscal instruments available. We find that the OLG channel, which is the product of Blanchard-Yaari frictions and the households' phased retirement assumption, dominates the incomplete markets channel and shapes our results.

The key variable responsible for both the steady-state allocations and the determinacy properties of the environment is the real interest rate. According to the long-standing representative agent literature, in the case of the infinite-horizon representative agent model (RANK, henceforth), the steady-state real interest rate is equal to the rate of time preference ($R = \frac{1}{\beta}$). In this scenario, households are able to use the asset market to fully insure themselves against idiosyncratic shocks.

However, if the model features (partially) uninsurable idiosyncratic risk, in the Bewley-Hugget-Aiyagari tradition, then in steady state, the framework delivers a value for the real interest rate below the rate of time preference (see Ljungqvist and Sargent, 2020, Ch. 17). Meanwhile, the inclusion of Blanchard-Yaari (1985) consumers results in households' assets entering the aggregate consumption Euler equation. Thus, the value of the equilibrium real interest rate also depends on the aggregate supply of government bonds. This means that a higher supply of government bonds results in a higher equilibrium real interest rate.

Moreover, we also explore the implications of phased retirement, where labor income declines over the life cycle. This feature strengthens the households' savings motive by creating a desire to save in order to smooth consumption over the life cycle. Although phased retirement strengthens the OLG channel, it still suppresses the steady-state real interest rate. We show that the higher the steady-state interest rate, the higher the required value of the fiscal response coefficient necessary to ensure stable debt dynamics. Consequently, the baseline OLG-HANK model of Acharya et al. (2023) requires the strongest fiscal response, while our OLG-HANK model with phased retirement requires the smallest.

In response to a one-time unanticipated aggregate shock, a fiscally-led policy mix (AF/PM) might be preferred by a policymaker who values "equity" more than "efficiency." Although both a fiscally-led policy regime (AF/PM) and a monetary-led mix (AM/PF) are capable of stabilizing the economy (see Grohe-Schmitt and Uribe, 2004), we find that in a scenario where fiscal policy is active, regardless of the tax instrument available, the deviations in inequality and in the sensitivity of individual consumption to changes in adjusted wealth are smaller. Hence, we argue that this policy mix might be preferred by a policymaker who cares more about "equity."

In Chapter 2, we use the framework outlined above to investigate jointly optimal monetary and fiscal policy. The model features a policymaker who combines the powers and responsibilities of both monetary and fiscal authorities and has access to commitment technology. The policymaker takes into account the inequality present in the economy when solving their program. The presence of commitment technology refers to the "time inconsistent" policy that the policymaker pursues.

More specifically, they commit not to try to redistribute wealth or consumption through a policy surprise and instead adjust the level of policy instruments only once in response to an unanticipated aggregate shock. The model allows for deriving a micro-founded social welfare function that consists of an aggregate consumption maximization or "efficiency" component as well as on an "inequality" component. Both inter-generational and intra-generational inequality are captured by a simple recursive variable, which is the same as the "inequality" component of the social welfare function. Thus, contrary to the representative agent case, the fully optimal policymaker is tasked not only with maximizing "efficiency" but also with maximizing "equity."

Households still wish to save to satisfy their consumption-smoothing motive and to partially insure against idiosyncratic risk. As such, we define a "golden rule" of steady-state savings as a benchmark where the government supplies enough bonds to meet households' consumption-smoothing desires. If the golden rule of steady-state government debt is achieved, the real steady-state interest rate is pushed to the rate of time preference. However, a higher real interest rate reduces households' ability to insure against idiosyncratic risk.

Additionally, a higher supply of government bonds requires a higher level of distortionary income taxes to ensure fiscal solvency. High taxes reduce the variance between agents' pre- and post-tax labor income but also reduce households' ability to borrow against their future income if they experience a low realization of an idiosyncratic shock.

Furthermore, high distortionary income taxes cause the standard efficiency losses associated with a decrease in households' willingness to supply labor to the market. Consequently, the policymaker faces a non-trivial dilemma between mitigating inter-generational consumption/wealth inequality and intra-generational income inequality. In other words, they face a trade-off between "equity" and "efficiency."

We consider three distinct scenarios. In the first case, the policy maker maximizes social welfare (the fully optimal case), which includes both an "efficiency" and an "equity" component. In the other two scenarios, we silence one of the two components depending on the policy maker's objectives. In the second scenario, the policymaker focuses solely on "efficiency" maximization, while in the third case, they are concerned only with reducing inequality- "equity" maximisation case.

Although there is no instance where the "golden rule" of steady-state savings is fully achieved, the policy maker with only the "equity" objective comes the closest to achieving the target. Still, the outcome of the fully optimal case is remarkably close, indicating that the policymaker under commitment places more emphasis on "equity" than on "efficiency."

Finally, in Chapter 3, we study optimal monetary policy under commitment in a tractable heterogeneous agent economy with a meaningful supply of government bonds. Similar to Chapter 1, the fiscal policy follows a simple tax rule where taxes deviate from their steady-state level if and only if the value of the outstanding government debt exceeds the exogenous target. More specifically, we extend the framework of Chien and Wen (2021) by introducing nominal rigidities, exogenous transfers to constrained households, and long-term government bonds. The model features (exogenous) stochastic transitions between labor market participation and non-participation, deviating from the assumption of Keynesian-constrained consumers due to households' infrequent asset market participation (IAMP). Unlike the original paper, constrained households do not face equilibrium binding borrowing constraints but are subject to portfolio rebalancing costs in our benchmark case. Hence, our tractable HANK model with discontinuous labor market participation (DLMP) features two types of Ricardian consumers.

We show through propositions and numerical results that even without the presence of Keynesian consumers, binding equilibrium borrowing constraints, or portfolio rebalancing costs on constrained households, optimal policy in THANK environment remains sufficiently different from the representative agent case. Without Keynesian households or binding equilibrium borrowing constraints on non-participating consumers, the policymaker cannot affect steady-state consumption inequality. In fact, we show that in our framework only the choice of the transitional probabilities of the idiosyncratic shock determines the steady-state level of consumption inequality. If the model also abstains from portfolio rebalancing costs on non-participating agents, then the optimal monetary policy is unable to affect consumption inequality even in response to an aggregate shock. Still, the fact that constrained consumers can adjust their portfolios to smooth the effects of the shock implies that the fully optimal policymaker's actions (under commitment) will redistribute wealth across the different household types.

As such, the model features perfect self-insurance, while the entries of the stochastic transition matrix for the idiosyncratic shock are chosen so that we focus on a more realistic case where unconstrained consumers enjoy higher consumption than their constrained counterparts. Since, if the transition matrix is symmetric then, the model reduces to the standard indivisible labor model of Hansen (1985). Our framework delivers the zero steady-state inflation result of Woodford (2003) despite the presence of constrained consumers. Additionally, the fact that only a fraction of the population supplies labor to the market ensures that both the aggregate steady-state consumption and output will always be lower than those found in the steady state of the nested RANK model. Also, the lower tax base guarantees the presence of taxes in the HANK specification. Still, the optimal policy response to the unanticipated aggregate shock is closer to the RANK model than to the tractable HANK model with infrequent asset market participation (IAMP).

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