# Asymptotic geometric and probabilistic properties of Teichmüller space

Aitor Azemar

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# Abstract

Teichmüller spaces play a pivotal role in the study of dynamics, geometric group theory and conformal geometry. In this thesis we study several geometric and probabilistic aspects of these spaces.

We begin by showing that, while not hyperbolic, Teichmüller space with the Teichmüller distance is statistically hyperbolic with respect to harmonic measures generated by non-elementary measures with finite first moment.

Points in the Teichmüller space of a surface $S$ can be interpreted as conjugacy classes of discrete faithful representations of the fundamental group of $S$ on the group of isometries of the hyperbolic plane, $\mathrm{PSL}(2,\mathbb{R})$. For a given measure on the fundamental group of $S$, this characterization gives us associated measures on uniform lattices on $\mathrm{PSL}(2,\mathbb{R})$. It is a long standing conjecture that the harmonic measures associated to these random walks have dimension strictly smaller than one whenever the measure is admissible and has finite first moment. In this thesis we prove that the conjecture is true outside of a compact subset of the Teichmüller space. Furthermore, we give some sharp bounds for the growth of the drift of the associated random walks in terms of the Teichmüller distance. One key argument is an adaptation of Gouëzel's pivoting techniques to actions of a fixed group on a sequence of hyperbolic metric spaces.

Two commonly studied compactifications of Teichmüller spaces of finite type surfaces are the Gardiner–Masur compactification and the Teichmüller compactification. We finish by showing that these two compactifications are related, proving that the former is finer than the latter. This allows us to prove, among other results, that the Gardiner–Masur compactification is path connected and that its Busemann points are not dense. We also determine for which surfaces the two compactifications are isomorphic, and we show that some horocycles diverge in the Teichmüller compactification based at some point. As an ingredient in one of the proofs we show that the extremal length is not $C^2$ along some paths that are smooth with respect to the piecewise linear structure on measured foliations.

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# Chapter 1

# Introduction

Loosely speaking, the set of all conformal structures on a surface $S$ up to conformal maps is the *moduli space* of $S$, denoted $\mathcal{M}(S)$. By adding a natural topology to moduli space one gets a powerful geometric approach to the related classification problem. However, under such natural topology the resulting space turns out to be rather complicated, and any deeper study requires some additional constructions. If instead of just considering the conformal structures we also keep track of the position of the curves up to isotopy we get the Teichmüller space, a simply connected space which serves as a "universal cover" of moduli space. The topological simplicity of Teichmüller spaces turn them into crucial tools in the study of algebraic geometry. Furthermore, the deck transformations of the cover are the rearrangements of the curves, that is, the mapping class group of the surface, which makes Teichmüller spaces a prime tool for the study of such groups through geometric group theory approaches.

It is possible to give some further structure to Teichmüller spaces by declaring that the distance between two marked surfaces is large if it is conformally hard to send one to the other. Precisely, the Teichmüller metric is defined by setting the distance between two marked conformal structures as one half of the logarithm of the minimal $K$ such that there is a $K$-quasiconformal map between the surfaces. This metric turns out to be complete, uniquely geodesic and, while not hyperbolic, it has many properties typical of negatively curved spaces, as we shall see in Chapter 3. One of the main interests of such metric is that geodesic rays are related to interesting dynamical actions. Namely, it is possible to identify the resulting cotangent vectors with flat structures, and the resulting geodesic flow with the dynamical action resulting from stretching the flat structure horizontally and contracting it vertically. By forgetting the marking we obtain the same action on translation structures, which results in a rather interesting action with applications in the study of the orbits of rational billiards.

Due to the uniformization theorem there is a correspondence between conformal structures and constant curvature geometries, which makes Teichmüller spaces a useful tool in the study of hyperbolic geometry. Since the universal cover of hyperbolic surfaces is the hyperbolic plane, the Teichmüller space of a surface $S$ can also be seen as the space of faithful discrete representations of the fundamental group of $S$ onto the isometry group of the hyperbolic plane. This representation can be used to study dynamical aspects of the discrete actions on the hyperbolic plane, as we shall see in Chapter 4.

The dynamical and geometric properties of Teichmüller spaces are an extremely active and important area of modern research. One frequent object of study in the field of dynamics is random walks, where we fix a probability measure on some group acting on a space and repeatedly concatenate elements of that group. This generates a random sequence of group elements, which when applied to a point in our space results in a random sequence of points. Random walks are used extensively in modelling, but can also be used to obtain geometric properties of our space. The asymptotic properties of random walks indicate the behaviour of applying one random transformation many times to the space, which has innate interest. In this thesis we shall study some properties of random walks on Teichmüller space, driven by the mapping class group, as well as random walks on the hyperbolic plane, driven by images via faithful representations of the fundamental group of a surface.

A tool frequently used in the study of random walks is that of a compactification. In this type of constructions one densely embeds the original space in a compact set. This guarantees that each sequence, such as the one obtained by the random walk, has a converging subsequence to some point. If one then proves that such point is almost surely unique then one has that the random walk converges almost surely in the compactification. This rather standard reasoning can be applied in many cases, as is the case for Teichmüller spaces. The compactification, however, is not unique, and often several are defined. The final contribution of this thesis is an in depth study of the the *horofunction* compactification, a general compactification defined for metric spaces, in the setting of Teichmüller spaces. The construction of said compactification is rather general, and only requires an underlying proper metric space. The study showcases a unique example of a horofunction boundary, different from previously studied examples. While originally motivated by the study of random walks, we find that random walks converge almost surely to a nowhere dense set within the boundary of such compactification.

## 1.1 Statistical hyperbolicity

Given a random walk converging almost surely towards a boundary, it is possible to define a probability measure $\nu$ on said boundary by setting the measure of a set as the probability that the random walk converges to a point in that set. Such a measure is called the *hitting* measure on the boundary. In the case of Teichmüller spaces, this measure can be pushed to a measure on the set of geodesics.

Loosely speaking, a space is *statistically hyperbolic* with respect to some measure in the set of geodesics if the space is on average hyperbolic at large scales. In Chapter 3 we prove that Teichmüller spaces are indeed statistically hyperbolic with respect to measures obtained by random walks. Precisely, we show the following.

**Theorem 1.1.1.** *Let $\mu$ be a measure with finite first moment on* $\mathrm{MCG}(S)$ *and let $\nu$ be the resulting hitting measure on the Thurston boundary. Then, the Teichmüller space of $S$ with respect to the corresponding geodesic sampling obtained by the hitting measure $\nu$ is statistically hyperbolic.*

See Section 3.1 for a definition of statistical hyperbolicity, as well as a precise statement of this result. This contrasts the well known result of Masur–Wolf [MW95], stating that Teichmüller spaces are not hyperbolic, as well as complements the result of Dowdall–Duchin–Masur [DDM14], which states that Teichmüller spaces are statistically hyperbolic with respect to several Lebesgue-class measures. Note that Teichmüller spaces contain many obstructions to hyperbolicity. For example, as shown by Minsky [Min96], it is possible to embed product regions up to additive constants. Our result shows that, according to the harmonic measures, these obstructions are statistically insignificant.

Under some conditions on the measure $\mu$, the resulting hitting measure on the set of geodesics is singular with respect to Lebesgue, so this result is different than the one obtained by Dowdall–Duchin–Masur.

## 1.2 Singularity conjecture

The standard compactification for the hyperbolic plane $\mathbb{H}^2$ is the Gromov compactification, or, equivalently in this case, the visual compactification. Given a uniform lattice $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ and a measure $\mu$ on $\Gamma$ with finite first moment the resulting random walk converges almost surely to the boundary. Furthermore, the distance between the random walk and the starting point grows asymptotically linearly. That is, there some well defined $\mathrm{Drift}(\mu,\Gamma) > 0$ such that for almost every path we have

$$\lim_{n\to\infty} \frac{d(w_n^\mu x_0, x_0)}{n} = \mathrm{Drift}(\mu,\Gamma),$$

where $w_n^\mu$ is the random walk after $n$ steps and $x_0$ is our starting fixed point. Note that $\mathrm{Drift}(\mu,\Gamma)$ does not depend on $x_0$ nor the sample path. The drift is related with the Hausdorff dimension of the measure $\nu$ on the boundary by the formula

$$\dim_{Haus}(\nu(\mu,\Gamma)) = \frac{h(\mu)}{\mathrm{Drift}(\mu,\Gamma)},$$

where $h(\mu)$ is the entropy of the measure, which is a purely combinatorial object. The famous conjecture by Deroin–Kleptsyn–Navas [DKN09a], and more generally by Karlsson–Ledrappier [KL11] states that $\dim_{Haus}(\nu(\mu,\Gamma)) < 1$ whenever the measure has finite first moment. Recent developments by Kosenko–Tiozzo [KT22] show that for a finite amount of cases the conjecture is true. In Chapter 4 we show that, for any suitable measure $\mu$, the conjecture is true for sufficiently degenerated lattices.

For a given compact surface $S$ we can fix a measure $\mu$ on $\pi_1(S,p)$, and then consider faithful representations $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$. We shall denote the drift obtained by the lattice $\rho(\pi_1(S))$ and the measure $\rho^*\mu$ as $\mathrm{Drift}(\mu,\rho)$, and $\nu(\mu,\rho)$ the corresponding hitting measure. As explained briefly before, the Teichmüller space of a surface $S$ can also be seen as the space of faithful discrete representations of the fundamental group $\pi_1(S,p)$ on the space of isometries of the hyperbolic plane, $\mathrm{PSL}(2,\mathbb{R})$, up to conjugation by elements in $\mathrm{PGL}(2,\mathbb{R})$. That is, a point $\rho \in \mathcal{T}(S)$ can be interpreted as a discrete faithful representation $\rho : \pi_1(S,p) \to \mathrm{PSL}(2,\mathbb{R})$ up to conjugation. Therefore, given a measure $\mu$ on $\pi_1(S,p)$ and a point $\rho \in \mathcal{T}(S)$, there is an associated family of random walks on the hyperbolic plane. The associated random walks are not unique, as the representation is not unique. However, the associated drifts, and hence the dimensions of the harmonic measures, are invariant by conjugation, so this is not an issue.

We have then a map $\dim_{Haus}(\nu(\mu,\cdot)) : \mathcal{T}(S) \to \mathbb{R}^+$, and the singularity conjecture can be reestated as claiming that, whenever $\mu$ has finite first moment, this function is strictly smaller than 1 on all $\mathcal{T}(S)$. In Chapter 4 we prove that the conjecture is true outside of some compact subset of Teichmüller space. More precisely, we show the following.

**Theorem 1.2.1.** *Let $\mu$ be an admissible measure on $\pi_1(S)$ with finite first moment. For any $\varepsilon > 0$ there is a compact $K_\varepsilon \subset \mathcal{T}(S)$ such that, for any $\rho$ outside of that compact,*

$$\dim_{Haus}(\nu(\mu,\rho)) < \varepsilon.$$

See Section 4.1 for the definition of *admissible* and *finite first moment*. The precise relation we find is slightly more refined. In Section 4.4 we show that the dimension decreases at least inversely proportionally to the distance to some basepoint, and in Section 4.5 we show it decreases exponentially along almost all geodesics, with respect to both Lebesgue class measures and hitting measures. The main tool of the proof is the novel pivoting technique developed by Gouëzel [Gou22a] and Baik–Choi–Kim [BCK23].

## 1.3 Horofunction compactification

The horofunction compactification is a general compactification that only requires the underlying metric space to be proper. Furthermore, the action of the isometry group of the underlying space can always be extended continuously to a continuous action on the compactification, and geodesic rays always converge to a point. These two last properties make the horofunction compactification a potentially powerful tool for the study of metric spaces. However, in many cases the horofunction construction results in a complicated space, which makes its use difficult.

Focusing on the case of Teichmüller spaces, the convergence of geodesics contrasts with the most commonly studied compactification, the Thurston compactification. While the latter compactification admits a continuous extension of the $\mathrm{MCG}(S)$ action to the whole compactification, some geodesics do not converge to points in the boundary, and in fact have large accumulation sets. There is another well studied compactification of Teichmüller space, the visual, or Kerchoff's, compactification, which has converging geodesics. However, the action of the $\mathrm{MCG}(S)$ can not be extended continuously to the whole compactification. The horofunction compactification avoids these two complications, which makes it particularly interesting. However, while the other two compactifications described result in simple spherical boundaries, this is not the case for Teichmüller spaces, as we shall see in Chapter 6.

One relevant set of points within the horoboundary is that of the *Busemann* points, which are those that can be reached by sequences along geodesics. We prove the following result regarding the sparsity of Busemann points for Teichmüller spaces.

**Theorem 1.3.1.** *Let $S$ be a closed surface of genus $g$ with $p$ marked points. Then the Busemann points are not dense in the horoboundary of $\mathcal{T}(S)$ whenever $3g+p \geq 5$.*

This answers a previously asked question by Liu–Su [LS14]. In subsequent joint work with Fortier Bourque we strengthen the previous result to show that Busemann points are actually nowhere dense within the horoboundary.

We also find that although random walks within this compactification do converge to the boundary, the resulting hitting measure is supported in a nowhere dense set (see Section 6.5.2 for details). Intuitively, this result tells us that the horofunction compactification adds many more points than the ones that are actually needed for the study of random walks. This does not mean that the horofunction compactification can not be used for the study of random walks, as the work of Maher–Tiozzo [MT18] clearly shows. Related to the previous result, the nowhere dense set is a subset of the Busemann points with Lebesgue measure one. No previous example of the horoboundary had been shown to exhibit such behaviour.

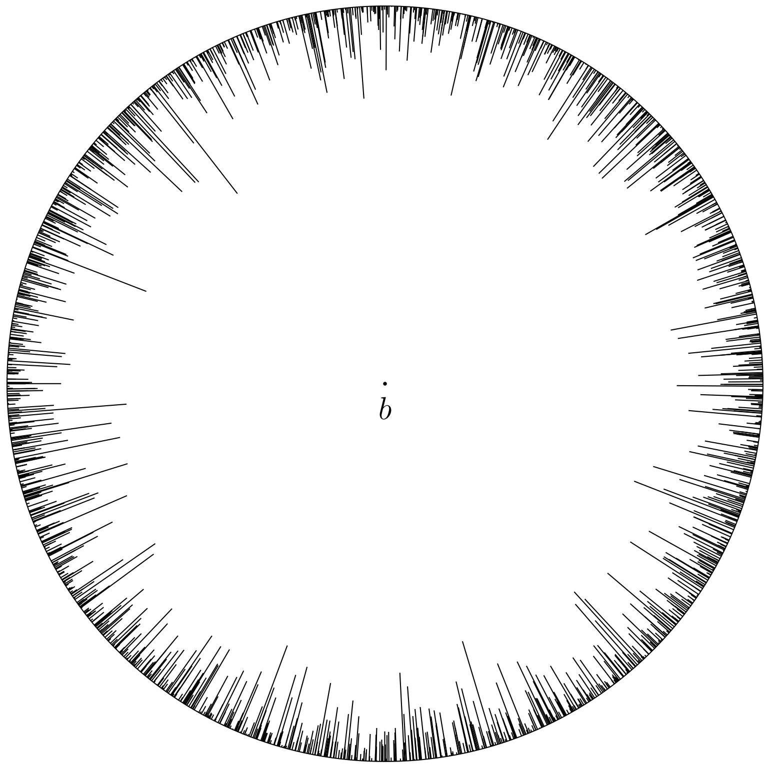



Figure 1.1: Sketch of the shape of the horoboundary of the Teichmüller metric for surfaces without boundary.

Figure 1.1 shows a sketch of what we think the horoboundary looks like based on the results of Chapter 6. The outer circle represents the section given by Theorem 6.6.1. Each line perpendicular to the sphere represents one of the fibers induced by the projection map, so it is associated with a unique Teichmüller ray starting at $b$. Note that while by Proposition 5.2.11 the fibers are path connected, by Theorem 6.4.10 they are bigger than segments in some cases. Furthermore, a priori they might not be contractible.

The nearest point to the basepoint $b$ of each fiber represents the Busemann point associated to the geodesic joining $b$ to the fiber. This point could indeed be considered the nearest point to $b$ from the fiber, as one can access it in a straight way, through a geodesic exiting $b$. On the other hand, the points in the outer circle represents the points associated to the section alluded to earlier. These can be accessed through a sequence of Busemann points whose associated fiber is a point, which can be considered as the most tangentially possible way to reach points in the boundary.

Following a result by Masur [Mas82a], with respect to the measure on the fibers induced by the Lebesgue measure on the set of Teichmüller rays exiting $b$, almost all the fibers are actually points. As we shall see in Theorem 6.5.5 these points are nowhere dense in the boundary.

Note that there exist paths within the horoboundary connecting the fibers without passing through the section, and a priori there may be paths not represented in the sketch along which the fibers vary continuously.

The path continuity of the fibers, as well as the existence of a section, allows us to prove the following result.

**Theorem 1.3.2.** *The horoboundary of any Teichmüller space of real dimension at least 2 is path connected.*

Finally, the connection between the horoboundary and the Gardiner–Masur boundary allows us to translate some of these results to limit theorems regarding the values of certain extremal lengths, extending results of Walsh [Wal19]. See Section 6.7 for details.

## 1.4 Structure of the thesis

In Chapter 2 we give a general introduction to Teichmüller spaces, as well as the tools and concepts needed for the rest of the thesis. Chapter 3 deals with the statistical hyperbolicity of the harmonic measure for Teichmüller space. In Chapter 4 we explain the bounds found for the drift of a random walk as a function on Teichmüller space. Finally, in Chapters 5 and 6 we detail our findings regarding the horofunction compactification, first for metric spaces, and later for Teichmüller spaces. Each chapter starts with a brief introduction, as well as the necessary background material specific to that that chapter.

## 1.5 Disclosures

The work contains both preprints and published works of the author, were some of the works are collaborations. In particular, the material in Chapter 3 comes from a collaboration with Gadre and Jeffreys [AGJ22]; the material in Chapter 4 results from a collaboration with Gadre, Gouëzel, Haettel, Lessa and Uyanik [AGG$^+$22], as well as a preprint from the author [Aze23]. Finally, the material in Chapters 5 and 6 come from a preprint of the author [Aze21].

# Chapter 2

# General background

## 2.1 Metric geometry

### 2.1.1 Basic definitions

Let $(X, d)$ be a metric space. We shall say that a map $\gamma$ from an interval $I \subset \mathbb{R}$ to $X$ is a *geodesic* if it is an isometric embedding, that is, if $d(\gamma(t), \gamma(s)) = |t - s|$. We shall consider two geodesics to be equal if their image is equal and have the same orientation. If for any two points $a, b \in X$ there exists a path $[a, b]$ joining them, with length equal to the distance between $a$ and $b$ we say that $X$ is *geodesic*. That path may not be unique, and by $[a, b]$ we mean any of them. If the path is unique for every $a, b \in X$ then we say $X$ is *uniquely geodesic*. Furthermore, we say that the space is *proper* if the closed balls $D(x, r) = \{p \in X \mid d(p, x) \leq r\}$ are compact. If geodesic segments can be extended uniquely, that is, if for any geodesic segment $\gamma_1$ there is a unique bi-infinite geodesic $\gamma_2$ such that $\gamma_1 \cap \gamma_2 = \gamma_1$, we say that the space is *straight*.

Most of the spaces we work with in this thesis satisfy all the properties described so far. In particular, Teichmüller space, with the Teichmüller metric, is proper, uniquely geodesic and straight.

### 2.1.2 Hyperbolic geometry

Let $(X, d)$ be a geodesic metric space, Given a set $A \subset X$ and $r > 0$, we will denote by $N(A, r)$ the closed $r$-neighbourhood of $A$, that is, $N(A, r) = \{x \in X \mid d(x, A) \leq r\}$. Given $\delta > 0$, we say that $X$ is *$\delta$-hyperbolic* if its triangles are $\delta$-slim, meaning that for any three points $a, b, c \in X$ and any three geodesics $[a, b]$, $[b, c]$ and $[c, a]$ we have $[a, b] \subset N([b, c] \cup [c, a], \delta)$. We say that $X$ is *Gromov hyperbolic* if there exists a $\delta \geq 0$ such that it is $\delta$-hyperbolic. We will use some properties of Gromov hyperbolic spaces, as well as their standard compactification, thorough this thesis, a standard reference for these topics is [BH13, Part III, Chapter H].

Given two points $a, b \in X$ and a basepoint $p \in X$, the *Gromov product* of $a$ and $b$ is defined as

$$(a \cdot b)_p = \frac{1}{2}(d(p, a) + d(p, b) - d(a, b)).$$

It can be seen directly from the definition of Gromov hyperbolicity that the Gromov product of $a$ and $b$ under the basepoint $p$ can be interpreted geometrically as the distance of $p$ to the geodesic $[a, b]$ up to the additive constant $\delta$.

### 2.1.3 Compactifications

A compactification of a space functions, among other things, as a way of characterizing convergence to infinity. Formally, a *compactification* of a topological space $X$ is a pair $(f, \overline{X})$, where $\overline{X}$ is a compact topological space and $f : X \to \overline{X}$ is an embedding with $f(X)$ dense in $\overline{X}$. The boundary of a compactification $\partial \overline{X} = \overline{X} - X$ then describes the different ways of converging to infinity provided by that compactification. We shall usually identify the points in $X$ with the ones in $\overline{X}$ via the map $f$, and say that a sequence $(x_n) \subset X$ converges in $\overline{X}$ if $f(x_n)$ converges.

A compactification $(f_1, X_1)$ of $X$ is *finer* than another one $(f_2, X_2)$ if there exists a continuous extension $\overline{f}_2 : X_1 \to X_2$ of $f_2$ such that $\overline{f}_2 \circ f_1 = f_2$. Since $f_2(X)$ is dense in $X_2$, the continuous extension $\overline{f}_2$ is surjective. Furthermore, we can restrict the map $\overline{f}_2$ to the boundary to get a surjective map $\overline{f}_2\big|_{\partial X_1} : \partial X_1 \to \partial X_2$, which can be seen as a projection. Having a compactification finer than another ones means, from an intuitive point of view, that the finer compactification catalogs more ways of converging to infinity than the other one. Namely, any sequence in $X$ converging in the finer compactification converges also in the coarser one, while the opposite may not be true.

We say that two compactifications are isomorphic if each one is finer than the other one. The following Lemma found in [Wal19, Lemma 17] gives an intuitive understanding of the finer relation.

**Lemma 2.1.1.** *Let $(f_1, X_1)$ and $(f_2, X_1)$ be two compactifications of a Hausdorff space $X$ such that $f_2$ extends continuously to an injective map $\overline{f}_2 : X_1 \to X_2$. Then the two compactifications are isomorphic.*

In other words, the previous Lemma tells us that if the projection $\overline{f}_2$ is actually injectuve, the two compactifications are isometric.

We will usually refer to the space $\overline{X}$ as the compactification when the embedding is either unique or the relevant properties are invariant under the different choices of such. Since the images of $X$ by the embedding are dense, the extensions we get to compare the compactifications are unique. That is, we have the following result

**Lemma 2.1.2.** *Let $(f_1, X_1)$ and $(f_2, X_2)$ be two compactifications of a Hausdorff space $X$ such that $X_1$ is finer than $X_2$. Then the extension $\overline{f_2} : X_1 \to X_2$ is unique.*

*Proof.* For any $x \in X$ we have $\overline{f_2}(f_1(x)) = f_2(x)$. Hence, the image of $\overline{f_2}$ is determined on a dense subset of $X_1$, so by continuity it is determined on $X_1$. □

### Visual compactification

Let $X$ be a proper, uniquely geodesic, straight space. The visual compactification was introduced by Eberlein–O'Neill in [EO73], as a generalization of the Poincaré disk model. Geometrically, the visual compactification describes what an observer at a basepoint $b \in X$ in $X$ would see as particles get further and further away from $b$. That is, a sequence in the visual compactification based at $b$ converges if and only if the sequence gets further away from $b$, and the direction of the ray connecting $b$ and the sequence converges. To give this notion a formal definition let $D_b$ be the set of infinite geodesic rays starting at $b$, with the topology given by uniform convergence on compact sets. Furthermore, denote $S^1_b = \{x \in X \mid d(x, b) = 1\}$ the sphere of radius 1 around $b$.

**Lemma 2.1.3.** *The map from $D_b$ to $S^1_b$ defined by sending $\gamma \in D_b$ to $\gamma(1)$ is a homeomorphism.*

*Proof.* Since the topology on $D_b$ is given by uniform convergence on compact sets, the point $\gamma(1)$ varies continuously with respect to $\gamma$.

On the other hand, since the space is straight and has unique geodesics, given any point $a \in S^1_b$ there is a unique geodesic ray starting at $b$ and passing through $a$. This is the inverse to the map obtained by evaluating the geodesics. To see that the relation is continuous we consider a sequence $(a_n) \subset S^1_b$ converging to some $a$, and denote $(\gamma_n)$ and $\gamma$ the associated geodesics. Assume $\gamma_n$ does not converge to $\gamma$. Then we have a subsequence without $\gamma$ as an accumulation point. For any $t > 0$, the geodesic segments $\gamma|_{[0,t]}$ are contained in the ball of radius $t$, which is compact, as $X$ is proper. As these are geodesics we have equicontinuity, so by Arzelà–Ascoli we can take a subsequence converging uniformly to some path $\gamma'$. Since the distance function is continuous, $\gamma'$ is a geodesic. Furthermore, $\gamma'(1) = \lim_{n\to\infty} \gamma_n(1) = \lim_{n\to\infty} a_n = a$. By uniqueness of geodesics, $\gamma'$ and $\gamma$ are equal when restricted to $[0, 1]$, which by straightness implies they are equal. Hence, $\gamma_n$ converges to $\gamma$ uniformly on the compact $[0, t]$. □

Following a similar reasoning it is possible to show the following, still under the same hypotheses on $X$.

**Lemma 2.1.4.** *The space $X$ is homeomorphic to $D_b \times [0, \infty)/D_b \times \{0\}$.*

*Proof.* We define the map $C : D_b \times [0, \infty)/D_b \times \{0\} \to X$ given by $C(\theta, r) = \theta(r)$. This is well defined, as $C(\theta, 0) = b$ for any $\theta \in D_b$. Furthermore, this is a bijection, since for every $x \in X - \{b\}$ there is a unique geodesic ray from $b$ to $x$. The map is continuous, as the topology on $D_b$ is given by uniform convergence on compact sets. To see that the inverse is continuous consider a sequence $a_n \in X$ converging to some $a \in X$. If $a = b$, then $d(a_n, b) \to 0$, so we have continuity. Otherwise we denote $r_n = d(a_n, b)$ and $r = d(a, b)$. We have $r_n \to r$, so denoting $(\gamma_n)$ and $\gamma$ the unique geodesic in $D_b$ such that $\gamma_n(r_n) = a_n$ and $\gamma(r) = a$ and applying Arzelà–Ascoli's theorem in the same way as in Lemma 2.1.3, we have that $\gamma_n$ converges to $\gamma$. □

The space $D_b \times [0, \infty)/D_b \times \{0\}$ can be included into the compact space $D_b \times [0, \infty]/D_b \times \{0\}$, which can be written as $(D_b \times [0, \infty)/D_b \times \{0\}) \cup D_b \times \{\infty\}$. Using the homeomorphism from Lemma 2.1.4, we can use this inclusion to give a compact topology on the space $X \cup D_b$. The *visual compactification* is defined as the pair $(i, X \cup D_b)$, where $i$ is the inclusion $i : X \to X \cup D_b$ and the topology on the space $X \cup D_b$ is the one we just defined. We shall denote $X \cup D_b$ as $\overline{X}^v_b$, or $\overline{X}^v$ when the basepoint is not relevant to the discussion.

**Gromov compactificaiton**

Let $(X, d)$ be a proper Gromov hyperbolic space. Given two sequences $(a_n), (b_n) \subset X$ we say that they are equivalent, and write $(a_n) \sim (b_n)$, if for some (and hence any) $p \in X$ we have $(a_n, b_n)_p \to \infty$. The Gromov boundary $\partial\overline{X}^G$ is defined as the equivalence set of sequences $(a_n) \subset X$ such that $(a_n, a_m)_p \to \infty$ for any $p \in X$, with the defined relation. The Gromov product between two elements of the boundary can be defined by

$$(a \cdot b)_p := \sup \liminf_{m,n\to\infty} (a_m \cdot b_n)_p,$$

where the supremum is taken over all sequences $(a_m)$, $(b_n)$ related to $a, b$ respectively. Furthermore, if $a \in X$ and $b \in \partial\overline{X}^G$, we can use the same definition replacing the sequence $(a_m)$ by $a$. We can give a topology to the set $X \cup \partial\overline{X}^G$ by adding the open sets $V(a, r) = \{b \in X \cup \partial\overline{X}^G | (a \cdot b)_p > r\}$. The resulting topology is compact, and the inclusion $i : X \to X \cup \partial\overline{X}^G$ is continuous. Hence, $(i, X \cup \partial\overline{X}^G)$ with the described topology is a compactification, usually called the *Gromov compactification*. Using the triangle inequality it is straightforward to check that this construction does not depend on the basepoint $p$. We shall denote the Gromov compactification as $\overline{X}^G$.

Let $(X, d)$ be straight and uniquely geodesic, as well as proper and Gromov hyperbolic. Then, both the visual and the Gromov compactifications are defined. Let $\gamma : [0, \infty) \to X$ be a geodesic ray starting at $p$. Then, $(\gamma(t) \cdot \gamma(s))_p = \min(t, s)$, so $\gamma(t)$ converges to a unique point in the Gromov compactification. Therefore, we have a map $f : \overline{X}^v_p \to \overline{X}^G$, defined by sending each point $x \in X \cup D_p$ to either $x \in X \cup \partial\overline{X}^G$ if $x \in X$ or to $\lim_{s\to\infty} x(s) \in \partial\overline{X}^G$ if $x \in D_p$. Denote $i_v, i_g$ the inclusion from the visual and Gromov compactification respectively. Then it is clear than $f(i_v) = i_g$, since $f(i_v(x)) = x$ for all $x \in X$. Furthermore, Let $(\gamma_n(t_n)) \subset \partial\overline{X}^v$ be a sequence converging to $\gamma \in D_p$. By the definition of convergence in the visual compactification we have that $\gamma_n$ converges locally uniformly to $\gamma$. Hence, for each $n$ we have some $s_n$ such that $d(\gamma_n(s), \gamma(s)) < \delta$ for all $s < s_n$, and $s_n \to \infty$. Then, since the Gromov product between two points with respect to $p$ is the distance of the geodesic between those two points to $p$, up to the constant $\delta$,

$$(\gamma_n(t_n) \cdot \gamma(t_n))_p \geq d(p, [\gamma_n(t_n), \gamma(t_n)] - \delta \geq s_n - 2\delta.$$

So $(\gamma_n(t_n) \cdot f(\gamma))_p \to \infty$, and hence $\gamma_n(t_n) \to f(\gamma_n(t_n))$ in the Gromov compactification.

Therefore, $f$ is continuous and the visual compactification is finer than the Gromov compactification.

### Horofunction compactification

Let $(X, d)$ be a proper metric space. Given a basepoint $b \in X$, one can embed $X$ into the space of continuous functions from $X$ to $\mathbb{R}$ via the map $h : X \to C(X)$ defined by

$$h(x)(\cdot) := d(x, \cdot) - d(x, b).$$

The topology given to $C(X)$ is that of uniform convergence on compact sets. The map $h$ is indeed continuous, as the distance function is continuous. Furthermore, $h$ is injective, as $h(x)$ has a strict global minimum at $x$. It can also be proven that since $X$ is proper, $h$ is an embedding. For more details about this construction see, for example, [Wal14a, Section 2]. Furthermore, the properness of $X$ implies it is second countable, so the closure of $h(X)$ is compact, Hausdorff and second countable. We shall denote the closure of $h(X)$ on $C(X)$ as $\overline{X}^h$. The *horofunction compactification* is defined as the pair $(h, \overline{X}^h)$. We call the set $\partial\overline{X}^h = \overline{X}^h - X$ the *horofunction boundary* or *horoboundary*, and we call its members *horofunctions*. If we want to specify the chosen basepoint we write $\overline{X}^h_b$. However, it is possible to see that quotienting the compactification by letting $f \sim g$ whenever the difference is constant we get an isomorphic compactification, showing that the horofunction compactification does not depend on the basepoint.

Usually the easier points to identify in the horoboundary are the Busemann points. These are the ones that can be reached as a limit along almost geodesics, which is a slight weakening of the notion of geodesic by allowing an additive constant approaching 0. That is, a path $\gamma : [0, \infty) \to X$ is an *almost geodesic* if for each $\varepsilon > 0$,

$$|d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t| < \varepsilon$$

for all $s$ and $t$ large enough, with $s \leq t$. Rieffel [Rie02] proved that every almost geodesic converges to a limit in $\partial\overline{X}^h$. A horofunction is called a *Busemann point* if there exists an almost geodesic converging to it. We shall denote the Busemann point associated in this way to the almost geodesic $\gamma$ by $B_\gamma$.

We shall prove in Section 5.2 that the horofunction compactification is finer than the visual compactification, whenever both compactifications are defined. Therefore, if the Gromov compactification is also defined, the horofunction compactification is also finer than the Gromov compactification.

## 2.2 Teichmüller spaces

A *surface with marked points* $S$ is a pair $(\Sigma, P)$, where $\Sigma$ is a compact, orientable surface with possibly empty boundary, and $P \subset \Sigma$ is a finite, possibly empty, set of points, where we allow points to be on the boundary. The *Teichmüller space* $\mathcal{T}(S)$ is the set of equivalence classes of pairs $(X, f)$ where $X$ is a Riemann surface and $f : \Sigma \to X$ is an orientation-preserving homeomorphism. Two pairs $(X, f)$ and $(Y, g)$ are equivalent if there is a conformal diffeomorphism $h : X \to Y$ such that $g^{-1} \circ h \circ f$ is isotopic to identity rel $P$.

The *Teichmüller distance* between two points $[(X, f)], [(Y, g)] \in \mathcal{T}(S)$ is defined as the value $\frac{1}{2} \log \inf K$, where the infimum is taken over all $K \geq 1$ such that there exists a $K$-quasiconformal homeomorphism $h : X \to Y$ with $g^{-1} \circ h \circ f$ isotopic to identity rel $P$. There is a quasiconformal homeomoprhism realizing the infimum. Furthermore, it is possible to assign a smooth structure compatible with the metric, provided by the Fenchel–Nielsen coordinates. See [FM12, Part 2] for some background on the Teichmüller metric and the Fenchel–Nielsen coordinates. While it is possible to define several other metrics on Teichmüller space, in this thesis we focus on the Teichmüller distance. With this metric, Teichmüller space is not hyperbolic, but satisfies many hyperbolic-like characteristics. Notably, given any geodesic there is an embedding of the hyperbolic plane $\mathbb{H}^2$ into $\mathcal{T}(S)$ containing said geodesic. Such an embedding is called *Teichmüller disk*. $\mathcal{T}(S)$ satisfies all the properties discussed in Section 2.1.1. Namely, $\mathcal{T}(S)$ is proper, uniquely geodesic and straight.

### 2.2.1 Quadratic differentials and flat metrics

A *quadratic differential* on a Riemann surface $X$ is a map $q : TX \to \mathbb{C}$ such that $q(\lambda v) = \lambda^2 q(v)$ for every $\lambda \in \mathbb{C}$ and $v \in TX$. Considering only holomorphic quadratic differentials with finite area $\int_X |q|$ with simple poles only at the marked points we get a characterization of the cotangent space to the Teichmüller space based at $[(X, f)]$. We shall denote the set of holomorphic quadratic differentials on $\rho \in \mathcal{T}(S)$ as $Q(\rho)$, and the subset of $Q(\rho)$ with area 1 as $Q^1(\rho)$. The union of all $Q(\rho)$ gives a bundle $Q$ over $\mathcal{T}(S)$. By contour integration and a choice of square root, each $q \in Q(x)$ defines a half-translation structure on $S$. That is, it defines charts to $\mathbb{C} = \mathbb{R}^2$ with half-translation transition functions of the form $z \to \pm z + c$. The $SL(2,\mathbb{R})$-action on $\mathbb{R}^2$ preserves the area and also the form of the transition functions. Hence, it descends to an action on $Q$. The compact part $SO(2,\mathbb{R})$ acts by rotations and preserves the conformal structure. The diagonal part of the action given by

$$\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$$

is called the *Teichmüller flow* and we will denote it by $\phi_t$. Given a point $\rho \in \mathcal{T}(S)$ and a quadratic differential $q \in Q^1(\rho)$ the path $R(q; \cdot) : (-\infty, \infty) \to \mathcal{T}(S)$ defined by setting $R(q;t)$ as the projection to $\mathcal{T}(S)$ of $\phi_t q$ is a geodesic with constant speed 1. Furthermore, it is the unique geodesic such that $\gamma(0) = p$ and $\gamma'(0) = |q|/q$. To ease the notation, we shall denote $R(q;\cdot)$ as $R(q)$.

### 2.2.2 Measured foliations

A *multicurve* on $S$ is an embedded 1-dimensional submanifold of $\Sigma \backslash P$ with boundary in $\partial\Sigma \backslash P$ such that

- no circle component bounds a disk with at most 1 marked point;
- no arc component bounds a disk with no interior marked points and at most 1 marked point on $\partial\Sigma$ and
- no two components are isotopic to each other in $\Sigma$ rel $P$.

Each of the components is called *curve.* A *weighted multicurve* is a multicurve together with a positive weight associated to each curve. We shall consider (weighted) multicurves up to isotopy rel $P$. If a simple curve is a circle we shall denote it *closed curve*, and *proper arc* otherwise.

A *measured foliation* on $S$ is a foliation with isolated prong singularities, where we allow 1-prong singularities at marked points, equipped with an invariant transverse measure $\mu_F$. For a standard reference on measured foliations see [FLP12, Exposé 5]. Denoting $\alpha_i$ and $w_i$ the components of the multicurve and the weights of each individual curve, the intersection number $i(\alpha, F)$ is defined as $\inf \sum_i w_i \int_{\alpha_i} |\mu_F| d\alpha_i$, where the infimum is taken over all representatives of $\alpha$. Two measured foliations $F$ and $G$ are *equivalent* if $i(\alpha, F) = i(\alpha, G)$ for every multicurve $\alpha$. We shall always consider measured foliations up to this equivalence relation. The set of measured foliations is usually denoted as $\mathcal{MF}$, and its topology is defined in such a way that a sequence $(F_n) \subset \mathcal{MF}$ converges to $F$ if and only if $i(\alpha, F_n)$ converges to $i(\alpha, F)$ for every multicurve $\alpha$.

Given a quadratic differential one can define the *vertical foliation* as the union of *vertical trajectories*, that is, maximal smooth paths $\gamma$ such that $q(\gamma'(t)) < 0$ for every $t$ in the interior of the domain. This foliation can be equipped with the transverse measure given by $|\operatorname{Re}\sqrt{q}|$. This measured foliation is called the *vertical measured foliation* of $q$, and shall be denoted as $V(q)$. This map is actually an homeomorphism. As such, given a measured foliation $F$ and a complex structure $X$ there is a unique quadratic differential $q_{F,X}$ on $X$ such that $V(q_{F,X}) = F$. This quadratic differential is usually called the *Hubbard–Masur* differential associated to $F$ on $X$ [HM79]. Furthermore, for each $\lambda > 0$ we have $q_{\lambda F,X} = \lambda q_{F,X}$. Similarly, the *horizontal foliation* $H(q)$ can be defined as the union of maximal smooth paths $\gamma$ such that $q(\gamma'(t)) > 0$, with the transverse measure $|\operatorname{Im}\sqrt{q}|$.

It is possible to associate a measured foliation to each weighted multicurve by thickening each proper arc and closed curve to a rectangle or cylinder respectively with width equal to the weight of the curve, and then collapsing the rest of the surface. The intersection numbers are maintained by this construction. This association is injective, and hence we shall consider the set of weighted multicurves as a subset of the measured foliations, and use both expressions of weighted multicurve indistinctly.

By removing the critical graph, a measured foliation is decomposed into a finite number of connected components, each of which is either a thickened curve, or a minimal component which does not intersect the boundary, in which every leaf is dense [Str84, Chapter 24.3]. Each transverse measure within the minimal components can be further decomposed into a sum of finitely many projectively distinct ergodic measures. A foliation $F'$ is an indecomposable component of $F$ if it is either a thickened curve or a minimal component with a transverse measure that cannot be decomposed as a sum of more than one projectively distinct ergodic measure. Every foliation can be decomposed uniquely into a union of indecomposable foliations. For a surface of genus $g$ with no boundaries nor marked points Papadopoulos shows [Pap86] that the maximum number of indecomposable components for any foliation is $3g - 3$. It is possible to get an upper bound for foliations on surfaces with boundary and marked points by swapping the marked points for boundaries and using the doubling trick we will explain in Section 6.2.1.

It was shown by Thurston that for surfaces without boundary it is possible to achieve a dense subset by restricting to simple closed curves, see Fathi–Laudenbach–Poénaru [FLP12] for a reference. When there are boundaries the picture gets slightly more complicated, but it has been shown by Kahn–Pilgrim–Thurston [KPT22, Proposition 2.12] that multicurves can be seen as a dense subset. More precisely, they show the following.

**Proposition 2.2.1** (Kahn–Pilgrim–Thurston)**.** *Let $F$ be a measured foliation in $S$ not containing proper arcs. Then there exists a sequence of multicurves composed solely of closed curves approaching $F$.*

The result can be extended to any foliation by cutting along the proper arcs and approaching the foliation in the resulting surfaces by multicurves. Then, joining the multicurves from the proposition with the proper arcs and the adequate weights we get a sequence of multicurves converging to our original foliation.

### 2.2.3 Extremal length

Given a marked conformal structure on $S$, that is, a point $\rho \in \mathcal{T}(S)$, the *extremal length* of $F$ on $\rho$ is defined as

$$\mathrm{Ext}_\rho(F) := \int_X |q_{F,\rho}|.$$

The map $\mathrm{Ext} : \mathcal{MF}(S) \times \mathcal{T}(S) \to \mathbb{R}$ is continuous and homogeneous of degree 2 in the first variable.

Given two points $x, y \in \mathcal{T}(S)$ we can define the function

$$K_{x,y} := \sup_{F \in P_b} \frac{\mathrm{Ext}_x(F)}{\mathrm{Ext}_y(F)},$$

where $P_b$ are the measured foliations $F$ satisfying $\mathrm{Ext}_b(F) = 1$. As revealed by Kerckhoff's formula [Ker80], the value $\frac{1}{2}\log K_{x,y}$ coincides with the usual definition of the Teichmüller distance $d_{\mathrm{Teich}}(x, y)$.

The characterization above points towards exponential growth of the extremal length of some foliations along geodesics. In fact, Walsh proves that whenever the foliation $F$ is not aligned with the horizontal foliation of the quadratic differential associated with the geodesic, the growth is exponential. Furthermore, we have an explicit formula for the rate of growth.

**Theorem 2.2.2** ([Wal19, Theorem 1 and Lemma 3])**.** *Let $S$ be a surface without boundary, $R(q) : \mathbb{R}_+ \to \mathcal{T}(S)$ be the Teichmüller ray with initial unit-area quadratic differential $q$, and $F$ be a measured foliation. Then,*

$$\lim_{t\to\infty} e^{-2t} \operatorname{Ext}_{R(q;t)}(F) = \sum_j \frac{i(G_j, F)^2}{i(G_j, H(q))},$$

*and*

$$e^{-2t} \operatorname{Ext}_{R(q;t)}(F) \geq \sum_j \frac{i(G_j, F)^2}{i(G_j, H(q))},$$

*where the $\{G_j\}$ are the indecomposable components of the vertical foliation $V(q)$, and $H(q)$ is the horizontal foliation.*

### 2.2.4 Moduli space and the mapping class group

The mapping class group of $S = (\Sigma, P)$, denoted $\mathrm{MCG}(S)$, is defined as the set of equivalence classes of diffeomorphisms $h : \Sigma \to \Sigma$ fixing the marked points. Two diffeomorphisms $h_1, h_2 : \Sigma \to \Sigma$ are equivalent if $h_2 \circ h_1^{-1}$ is isotopic to the identity $\mathrm{rel}(P)$. The mapping class group acts on Teichmüller space by changing the marking. Namely, given $[(X, f)] \in \mathcal{T}(S)$ and $h \in \mathrm{MCG}(S)$ we define $h([(X, f)]) = [(X, f \circ h^{-1})]$. This action is well defined, since for any $h_1 \sim h_2 \in \mathrm{MCG}(S)$ and $(X, f) \sim (Y, g)$ we have some conformal diffeomorphism $h : X \to Y$ such that $g^{-1} \circ h \circ f$ is isopotic to the identity $\mathrm{rel}\, P$, so $(g \circ h_2^{-1})^{-1} \circ h \circ (f \circ h_1^{-1}) = h_2\, (g^{-1} \circ h \circ f) \circ h_1^{-1}$ is also isotopic to the identity $\mathrm{rel}(P)$, by concatenating first the isotopy between $g^{-1} \circ h \circ f$ and the identity, and then the isotopy between $h_2 \circ h_1^{-1}$ and the identity. Since both isotopies leave $P$ fixed, the concatenation also leaves $P$ fixed.

The action of $\mathrm{MCG}(S)$ on $\mathcal{T}(S)$ is an isometric action. Indeed, let $[(X, f)], [(Y, g)] \in T(S)$ be two points, and let $\phi_K : X \to Y$ be the $K-$quasiconformal map realizing the Teichmüller distance, that is, such that $\frac{1}{2} \log K = d([(X, f)], [(Y, g)])$. Then, as in the last paragraph, for any $h \in \mathrm{MCG}(S)$ we have $(g \circ h^{-1})^{-1} \circ \phi_K \circ (g \circ h^{-1})$ is isotopic to the identity $\mathrm{rel}\, P$, so $d(h([(X, f)]), h([(Y, g)])) \leq d([(X, f)], [(Y, g)])$. Since $h$ is invertible we get that $h$ is an isometry.

The action of $\mathrm{MCG}(S)$ on $\mathcal{T}(S)$ is properly discontinuous, so we can define the quotient $\mathcal{M}(S)$ as $\mathcal{T}(S)/\mathrm{MCG}(S)$. This quotient space is the *moduli space* of $S$, and can be seen as the set of conformal structures on $S$, without considering the marking.

By the Nielsen-Thurston classification, an element of the mapping class group is either finite order, reducible or pseudo-Anosov. A finite order element is an automorphism of some Riemann surface. A reducible element has some power that fixes a multi-curve on the surface. A pseudo-Anosov element $f$ has a Teichmüller axis: an $f$-invariant bi-infinite Teichmüller geodesic along which the map translates realising the infimum of $d_{\mathrm{Teich}}(x, f(x))$ over $\mathcal{T}(S)$ by this translation. This description of a pseudo-Anosov map implies that the Teichmüller axis is unique and that its vertical and horizontal measured foliations are uniquely ergodic.

### 2.2.5 Metric length and the thick part of Teichmüller space

By the uniformization theorem, if the Euler characteristic of $S$ is negative then for each $[(X, f)] \in \mathcal{T}(S)$ there is a unique hyperbolic metric in $X - f(P)$. Let $\mathcal{S}$ be the set of isotopy classes of simple closed curves on $S$, and let $\gamma \in S$. We define the metric length of $\gamma$ in $[(X, f)]$ as the infimum length of $f(\gamma)$ under the unique hyperbolic metric, and denote it as $\mathrm{Hyp}_{[(X,f)]}(\gamma)$.

Given $\varepsilon > 0$, a point $[(X, f)] \in \mathcal{T}(S)$ is $\varepsilon-$*thick* if $\mathrm{Hyp}_{[(X,f)]}(\gamma) \geq \varepsilon$ for all $\gamma \in \mathcal{S}$. We let $\mathcal{T}_\varepsilon(S)$ be the subset of $\mathcal{T}(S)$ consisting of all the $\varepsilon-$thick points. The metric on $[(X, f)]$ is invariant under the aciton of $\mathrm{MCG}(S)$, so if $\rho \in \mathcal{T}(S)$ is $\varepsilon-$thick, so is $h(\rho)$ for any $h \in \mathrm{MCG}(S)$. Hence, $\mathrm{MCG}(S)(\mathcal{T}_\varepsilon(S)) = \mathcal{T}_\varepsilon(S)$. The $\varepsilon$-thick part of Teichmüller space is where the most regular and hyperbolic-like behaviour is observed. For example, large segments contained in the thick part of edges of geodesic triangles are close to one of the other two edges, as proven by Rafi [Raf14]. The Mumford compactness theorem says that for any $\varepsilon > 0$, the thick part $\mathcal{M}_\varepsilon(S)$ is compact.

Since the thick part of Teichmüller space exhibits regularity, we shall be interested in geodesics that spend a positive proportion of their time in the thick part. For a geodesic $\gamma$, time $t > 0$ and $\varepsilon > 0$ we define

$$\mathrm{Thick}^{\%}_{\varepsilon}(\gamma, t) := \frac{|\{0 \leq s \leq t \,:\, \gamma(s) \in \mathcal{T}_\varepsilon(S)\}|}{t}.$$

We shall observe that most geodesics spend an arbitrarily large proportion in the $\varepsilon$-thick space, provided $\varepsilon$ is small enough.

### 2.2.6 Compactifications of Teichmüller space

Teichmüller space with the Teichmüller metric is not hyperbolic, so the Gromov boundary can not be defined. However, the visual and the horofunction compactification are both well defined. There are many more ad-hoc compactifications defined for Teichmüller space. In this thesis we will deal with two of them: Gardiner–Masur's and Thurston's compactification.

#### Gardiner–Masur's compactification

For a surface $S$ with marked points and empty boundary we can embed $\mathcal{T}(S)$ into the space of projective continuous functions from the set $\mathcal{S}$ of simple closed curves on $S$ to $\mathbb{R}$ via the map $\phi : \mathcal{T}(S) \to P(\mathbb{R}^{\mathcal{S}})$ defined by

$$\phi(X) = \left[\mathrm{Ext}_X(\alpha)^{1/2}\right]_{\alpha\in\mathcal{S}},$$

where the square brackets indicate a projective vector. Gardiner and Masur show [GM91] that this map is indeed an embedding, and that $\phi(\mathcal{T}(S))$ is precompact. The Gardiner–Masur compactification of a surface without boundary is then defined as the pair $(\phi, \overline{\phi(\mathcal{T}(S))})$.

While this definition works fine for surfaces without boundary, some tweaks have to be done when dealing with surfaces with boundary. We shall detail the extended construction in Chapter 6, as well as extend the required results.

One of the relevant features of the Gardiner–Masur compactification is that it coincides with the horofunction compactification with respect to the Teichmüller distance. Indeed, Liu–Su [LS14] and Walsh [Wal19] prove that for surfaces without boundary these two compactifications are isomorphic. We shall extend the relevant results to surfaces with boundary in Chapter 6.

#### Thurston's compactification

A similar construction to Gardiner–Masur's compactification had been done previously by using the metric length instead of the extremal length. We shall not be dealing with this compactification for surfaces with non-empty boundary, so we shall quickly go over the definition for surfaces with empty boundary. As before, we can embed $\mathcal{T}(S)$ into the space of projective continuous functions from the set $\mathcal{S}$ of simple closed curves on $S$ to $\mathbb{R}$ via the map $\varphi : \mathcal{T}(S) \to P(\mathbb{R}^{\mathcal{S}})$ defined by

$$\varphi(X) = \left[\mathrm{Hyp}_X(\alpha)^{1/2}\right]_{\alpha \in \mathcal{S}},$$

where the only difference with respect to the Gardiner–Masur construction is that we use the hyperbolic length instead of the extremal length. This is indeed an embedding, and the image is precompact. The compactification of the image, toghether with the embedding, is called the *Thurston compactification.*

The set of projective measured foliations $\mathcal{PMF}(S)$ is defined by considering the equivalence classes of measured foliations, where two measured foliations $F, G \in \mathcal{MF}(S)$ are equivalent if they are equal as foliations and the measure of one is a non-zero multiple of the other. The boundary of the Thurston compactification can be naturally identified with $\mathcal{PMF}(S)$. As shown by Lenzhen [Len08], not all geodesics converge to the Thurston boundary. However, geodesics associated to quadratic differentials whose vertical foliation is uniquely ergodic do, as shown by Masur [Mas82b]. In particular, given a quadratic differential $q$ based at $x \in \mathcal{T}(S)$ whose vertical foliation is the uniquely ergodic foliation $F$, then $R(q; \cdot)$ converges to the projectivized version of $F$ in $\mathcal{PMF}(S)$.

It is possible to endow $\mathcal{PMF}(S)$ with a natural family of measures, the *Lebesgue measure class*, by pulling back Lebesgue measures from the charts defined by using train tracks coordinates. For more details we refer, for example, to [Gad14, Section 2.2]. These measures are supported in the set of projective measured foliations, so it is possible to get a measure on the set of geodesic rays exiting some basepoint $x \in \mathcal{T}(S)$, which as we recall can be identified with $Q^1(x)$. It follows from Dowdall–Duchin–Masur [DDM14, Proposition 5.5] that, according to the resulting Lebesgue measures, almost all geodesics spend eventually an arbitrarily large proportion of time in the thick space. Precisely,

**Theorem 2.2.3.** *Dowdall–Duchin–Masur Let $o \in \mathcal{T}(S)$ be a basepoint in Teichmüller space. Furthermore, let $\lambda$ be the Lebesgue measure on $Q^1(o)$. Then, for all $0 < \xi < 1$ there is some $\varepsilon(\xi) > 0$ such that for $\sigma$-almost every $q$ there is $t_q^\xi < \infty$ such that*

$$\mathrm{Thick}^{\%}_{\varepsilon(\xi)}(R(q), t) \geq \xi$$

*for all $t \geq t_q^\xi$.*

## 2.3 Random walks and measures on boundaries

Let $G$ be a discrete group and $\mu$ a probablity measure on $G$. The *step space* $\Omega := G^{\mathbb{N}}$ is the space of infinite sequences of group elements, which we consider as a probability space with the product measure $\mathbb{P} := \mu^{\mathbb{N}}$. We will denote *random walk* on $G$ starting at $g_0$ the stochastic process (indexed by $\mathbb{N} \cup 0$) obtained by associating to each $n$ the $G$-valued random variable $w_n : \Omega \to G$

$$(g_1, g_2, \ldots) \to w_n := g_0 g_1 \ldots g_n.$$

In other words, a random walk on $G$ is a time homogeneous, space homogeneous Markov chain with transition probabilities given by $p(g, h) = \mu(g^{-1}h)$. Our random walks will always start at the neutral element, that is, $g_0 = e$.

On all of our cases the group $G$ will act by isometries on some metric space $(X, d)$, and we will be interested on the process we get by applying the random walk to some starting point $x \in X$, i.e., in the process $(w_n x)_{n \in \mathbb{N}}$. This may make the choice of creating the random walk on the group by right multiplication look weird, since this may result in the new process not being a Markov chain. However, by doing it this way we can interpret the steps as going from $w_n x$ to $(w_n g_{n+1} w_n^{-1}) w_n x$, i.e., every step consists on drafting independently an isometry $g_{n+1} \in G$ with probability $\mu$, translating it to the point $w_n x$ (i.e., considering the isometry $w_n g_{n+1} w_n^{-1}$) and applying this new isometry. Since there might be more than one way of translating our isometry, and the way we chose depends on the path $w_n$, we might end up with something that is no longer a Markov chain. However, by making this choice we get something similar to time and space homogeneity, since the distribution of every step will be a random translation of the distribution of the first step, i.e.,

$$\mathbb{P}[w_n x = y | w_{n-1} x = z] = \mathbb{P}[w_{n-1} g_n x = y | w_{n-1} x = z] =$$

$$= \mathbb{P}[g_n x = w_{n-1}^{-1} y | x = w_{n-1}^{-1} z].$$

We will refer to this new process as random walk on $X$ (generated by $(G, \mu)$). This can also be seen as the projection of the random walk on $G$ to $X$.

Sometimes we will consider both forward and backward random walks. The backward random walk is simply the random walk with respect to the reflected measure $\hat{\mu}$ defined by $\hat{\mu}(g) = \mu(g^{-1})$. We then consider bi-infinite sequences as elements of $G^{\mathbb{Z}}$ with the shift $\sigma$ acting as a step of the random walk. For the push-forward $h$ of the product measure $\mu^{\mathbb{Z}}$ on

$G^{\mathbb{Z}}$ the conditional measure for the shift is given by $\mu$. We can separate the forward and backward directions to write $h$ as the product $\nu \times \hat{\nu}$. We call the measure $\nu$ the harmonic measure.

### 2.3.1 Random walks on Teichmüller space

A subgroup of $\mathrm{MCG}(S)$ is non-elementary if it contains a pair of pseudo-Anosov mapping classes with distinct Teichmüller axes. Let $x \in \mathcal{T}(S)$ be a base-point. Kaimanovich-Masur showed that if the support of a probability distribution $\mu$ on $\mathrm{MCG}(S)$ generates a non-elementary subgroup then for $h$-almost every sample path $\omega = (w_n)$ the sequence $w_n x$ converges to a projective class of a uniquely ergodic measured foliation on $S$. See [KM96, Theorem 2.2.4]. So the Kaimanovich-Masur theorem can be rephrased as convergence to the boundary $\mathcal{PMF}(S)$ for $h$-almost every sample path. In particular, the measure $\nu$ on $\mathrm{MCG}(S)^{\mathbb{N}}$ pushes forward to a measure on $\mathcal{PMF}(S)$. We call this the harmonic measure on $\mathcal{PMF}(S)$.

As shown by Maher–Tiozzo [MT18, Theorem 1.2], finite first moment in the curve complex is sufficient for positive linear drift of typical sample paths when projected to the curve complex using the mapping class group action. Since Teichmüller distance is a coarse upper bound for the curve complex distance, finite first $d_{\mathrm{Teich}}$-moment implies finite first moment in the curve complex. Hence we have positive linear drift of sample paths in the curve complex and consequently in Teichmüller space.

Going back to the work of Kaimanovich–Masur, they showed that if $\mu$ has finite entropy and finite first logarithmic moment with respect to the Teichmüller metric then the push-forward measure is measurably isomorphic to $\nu$. See [KM96, Theorem 2.3.1]. For this reason, and to keep the notation simple, we denote the measure on $\mathcal{PMF}(S)$ by $\nu$ even though we do not need the measurable isomorphism.

# Chapter 3

# Statistical hyperbolicity of Teichmüller space

## 3.1 Introduction

The notion of *statistical hyperbolicity*, introduced by Duchin-Lelièvre-Mooney in [DLM12], encapsulates whether a space is on average hyperbolic at large scales, that is, for any point in the space and spheres centred at that point whether as the radius $r \to \infty$ the average distance between pairs of points on the sphere of radius $r$ is $2r$. This corresponds to the distance between two randomly selected point being similar to the length of the path passing through the center of the sphere, which is the case when the corresponding triangle is thin and the points are not too close between them. To make sense of the average distance, one requires measures on spheres.

For many Lebesgue-class measures on Teichmüller space, Dowdall-Duchin-Masur showed that Teichmüller space with the Teichmüller metric is statistically hyperbolic. See [DDM14, Theorems B, C and D]. Here, we consider the same question for the harmonic measures that arise from random walks on the mapping class group determined by probability distributions with finite first moment with respect to the Teichmüller metric, and whose supports generate non-elementary subgroups. As explained in Section 2.3.1, Kaimanovich-Masur showed that under such natural assumptions random walks converge almost surely to the Thurston boundary of Teichmüller space. This defines a harmonic measure on the Thurston boundary, and Kaimanovich-Masur showed that this measure is supported on the set of uniquely ergodic measured foliations. See [KM96, Theorem 2.2.4] for both statements. Since Teichmüller rays with uniquely ergodic vertical foliations asymptotically converge to this vertical foliation, it is possible to pull back the harmonic measure to the unit cotangent space at a base-point. This allows us to equip spheres in Teichmüller space with a harmonic measure. We can then consider the question of whether Teichmüller space is statistically hyperbolic with respect to these measures.

The main theorem of this chapter, proven as a combination of Corollary 3.3.4 from Section 3.3.4, Corollary 3.3.5 from Section 3.3.5, and [DDM14, Theorem A], is:

**Theorem 3.1.1.** *Let $S$ be a surface of finite type. Let $\mu$ be a probability distribution on the mapping class group* $\mathrm{MCG}(S)$ *with finite first moment with respect to the Teichmüller metric, and such that the support generates a non-elementary subgroup. Then the Teichmüller space* $\mathcal{T}(S)$ *with the Teichmüller metric is statistically hyperbolic with respect to the harmonic measure defined by the $\mu$-random walk on* $\mathrm{MCG}(S)$.

When $S$ is a torus or a torus with one marked point or a sphere with four marked points, $\mathcal{T}(S)$ with the Teichmüller metric is isometric to $\mathbb{H}$. If $\mu$ also has finite first moment in the word metric then by a theorem of Guivarch-LeJan [GLJ90], the harmonic measure from the $\mu$-random walk is singular with respect to the Lebesgue measure class.

With respect to the class of Lebesgue measures on $\mathcal{T}(S)$, a similar singularity of harmonic measures also holds when the complex dimension is greater than one. Gadre [Gad14, Theorem 1.1] proved the singularity of harmonic measures from finitely supported random walks on $\mathrm{MCG}(S)$ and Gadre–Maher–Tiozzo [GMT17, Theorem 1.4] proved the singularity for harmonic measures for random walks with finite first moment with respect to a word metric on $\mathrm{MCG}(S)$. Finite first moment with respect to the word metric implies finite first moment with respect to the Teichmüller metric. For this large class of random walks, Thereom 3.1.1 gives a conclusion that is distinct from the main results of Dowdall-Duchin-Masur [DDM14] and, as we outline below, requires different tools.

On the other hand, finite first moment with respect to the Teichmüller metric does not imply finite first moment in the word metric. This is because the mapping class group is distorted under the orbit map to Teichmüller space. For the exceptional surfaces mentioned above whose Teichmüller space is $\mathbb{H}$, Furstenberg showed in [Fur71] that there is a finite first $d_{\mathbb{H}}$-moment random walk whose harmonic measure on $S^1$ is absolutely continuous. Thus, for the exceptional surfaces Theorem 3.1.1 derives statistical hyperbolicity covering both singular and Lebesgue class measures in one statement.

We will first present the proof of Theorem 3.1.1 when the complex dimension of $\mathcal{T}(S)$ is greater than one. This is the harder case. For the exceptional surfaces, that is when $\mathcal{T}(S) = \mathbb{H}$, the proof of Theorem 3.1.1 is obviously easier because the ambient geometry is already hyperbolic. However, as mentioned above, many harmonic measures are singular. So there is something to prove. The argument required in the exceptional case is straightforward and uses the geodesic separation property for harmonic measure that is already formulated in the proof of the harder case of Theorem 3.1.1.

In fact, we present the exceptional case as a special case of a more general theorem when the ambient geometry is hyperbolic. In Section 3.4.1 we prove:

**Theorem 3.1.2.** *Let $\Gamma$ be a lattice in $\mathrm{Isom}(\mathbb{H}^n)$ for $n \geqslant 2$. Let $\mu$ be a probability distribution on $\Gamma$ with finite first $\mathbb{H}^n$-moment such that the support of $\mu$ generates a subgroup that contains a pair of loxodromic elements with distinct axes. Then, with respect to the harmonic measure defined by the $\mu$-random walk on $\Gamma$, the space $\mathbb{H}^n$ with the hyperbolic metric is statistically hyperbolic.*

We note that when $n > 2$ and $\Gamma$ is a non-uniform lattice, Randecker-Tiozzo [RT21, Theorem 2] proved that a harmonic measure arising from a $\mu$ whose support generates $\Gamma$ and has finite $(n-1)^{\mathrm{th}}$ moment with respect to a word metric is singular. For uniform lattices, many classes of harmonic measures are known or conjectured to be singular. For instance, the singularity conjecture of Guivarc'h-Kaimanovich-Ledrappier mentioned in the introduction asserts that harmonic measures that arise from finitely supported random walks on a uniform lattice in $SL(2, \mathbb{R})$ are singular. Furtermore, results of Kosenko–Tiozzo [KT22] as well as the ones we shall present in Chapter 4 show many cases in which the measures are indeed singular. So Theorem 3.1.2 has new content.

From now on we assume that the complex dimension of $\mathcal{T}(S)$ is greater than one and present the proof of Theorem 3.1.1 with that assumption.

### 3.1.1 Strategy of the proof

To derive statistical hyperbolicity, Dowdall-Duchin-Masur set up two properties to check. The first property is the thickness property introduced in Section 2.2.5. It states that as the radius of a sphere goes to infinity a typical radial geodesic segment spends a definite proportion of its time in the thick part of Teichmüller space. The second property is called the separation property. See [DDM14, Definition 6.1]. It states that as the radius of a sphere goes to infinity a typical pair of radial geodesic segments exhibit good separation. For Lebesgue-class visual measures, the ergodicity of the Teichmüller geodesic flow is the key tool in their proof of the thickness property. For rotationally invariant Lebesgue measures, they verify the separation property by disintegrating the measure along and transverse to Teichmüller discs and then use the hyperbolic geometry of these discs.

For random walks, different tools are needed. The main tool is the ergodicity of the shift map on the space of bi-infinite sample paths. This ergodicity can be leveraged to prove that a typical bi-infinite sample path recurs to a neighbourhood of its tracked geodesic with a positive asymptotic frequency. As sample paths lie in a thick part, recurrence implies that the tracked geodesics spend a positive proportion of their time in a thick part. By tweaking the size of the neighbourhood, and hence the thick part, we show that the time spent in the thick part by the tracked geodesic can achieve any positive proportion. While a positive proportion of thickness is suggested by the main theorem in [GMT17], the precise quantitative version that we need here requires some work.

For the separation property, we project two fellow travelling radial geodesic segments to the curve complex. By a theorem of Maher-Tiozzo, a typical sample path makes linear progress in the curve complex. Combining this theorem with the recurrence, we show that the projections of fellow travelling radial geodesic segments must nest into a shadow. Also by a proposition in Maher-Tiozzo, the harmonic measure of a shadow tends to zero in the distance of the shadow from the base-point. This then enables us to conclude the required separation property.

### 3.1.2 Disclosures

This chapter is based on joint work with Vaibhav Gadre and Luke Jeffreys. The work contained in this chapter has been published [AGJ22].

## 3.2 Section specific background

### 3.2.1 Statistical hyperbolicity

Let $(X, d)$ be a metric space. Let $x \in X$. Let $r > 0$. Recall from the introduction that we denote the set $S_x^r = \{x' \in X \text{ such that } d(x, x') = r\}$ the *sphere* of radius $r$ centred at $x$. Suppose $\nu_r$ is a family of probability measures supported on $S_x^r$. Provided the limit exists, one defines a numerical index $E(X) := E(X, x, d, \{\nu_r\})$ by

$$E(X) = \lim_{r \to \infty} \frac{1}{r} \int\limits_{S_x^r \times S_x^r} d(x', x'')\, d\nu_r(x') d\nu_r(x'')$$

A space is said to be *statistically hyperbolic* if $E(X) = 2$. This is motivated by the fact that $E(\mathbb{H}^n) = 2$ for any $n \geq 2$ equipped with the natural measures on spheres. Moreover, it was demonstrated by Duchin-Lelièvre-Mooney [DLM12, Theorem 4] that $E(G) = 2$ for any non-elementary hyperbolic group $G$ with any choice of generating set.

For uniform lattices in $\mathrm{Isom}(\mathbb{H}^n)$, the Green metric defined by the random walk is quasi-isometric to the induced hyperbolic metric through the orbit map. This suggests a derivation of statistical hyperbolicity by reducing the problem to the Duchin-Lelièvre-Mooney result. As our proof of Theorem 3.1.2 covers both uniform and non-uniform lattices, we omit the details for this alternate approach. In any case, it would work only for uniform lattices.

We direct the reader to [DLM12] for further discussion on the sensitivity of $E$. Indeed, it is not quasi-isometrically invariant, and has dependence on the base-point $x$ and the choice of measures $v_r$. Furthermore, $\delta$-hyperbolicity and exponential volume growth are not sufficient to guarantee statistical hyperbolicity.

### 3.2.2 Statistical hyperbolicity for a harmonic measure

Recall that, as explained in Section 2.2.5, the stationary measure generated by a random walk in $\mathcal{T}(S)$ can be pulled to a measure on the set of unit area quadratic differentials for the marked Riemann surface $x \in \mathcal{T}(S)$, denoted $Q^1(x)$. Since Teichmüller space is straight and uniquely geodesic, this gives us a measure on every sphere $S_x^r$ via the map induced by $R(\cdot; r) : Q^1(x) \to \S_x^r$ sending $q$ to $R(q; r)$. Thus, we consider that $\mathcal{T}(S)$ with the Teichmüller metric is statistically hyperbolic with respect to harmonic measure $\nu$ if

$$\lim_{r\to\infty} \frac{1}{r} \int\limits_{Q^1(x)\times Q^1(x)} d(R(q;r), R(q';r))\, d\nu(q)d\nu(q') = 2.$$

### 3.2.3 Statistical hyperbolicity in Teichmüller space

Dowdall-Duchin-Masur [DLM12] reduce statistical hyperbolicity of Teichmüller space with the Teichmüller metric for a family of measures $\{\nu_r\}$ to the verification of two properties: the thickness property [DLM12, Definition 5.2] and the separation property [DLM12, Definition 6.1]. We will now state these properties and in Section 3.4, we will give a quick sketch of how these properties imply statistical hyperbolicity. For those unfamiliar with Dowdall-Duchin-Masur, we recommend reading through the sketch immediately after Definitions 3.2.1 and 3.2.2.

Recall that for a choice of $\varepsilon > 0$ and a geodesic $R(q) : \mathbb{R} \to \mathcal{T}(S)$, we denote the proportion of time $[R(q;0), R(q;t)]$ spends in $\mathcal{T}_\varepsilon(S)$ by

$$\mathrm{Thick}^{\%}_{\varepsilon}(R(q), t) := \frac{|\{0 \leqslant s \leqslant t \;:\; R(q;s) \in \mathcal{T}_\varepsilon(S)\}|}{t}.$$

The precise definition of the thickness and separation properties given by Dowdall-Duchin-Masur refers to sequences of measures in spheres. We shall adapt such definition to a measure on $Q^1(o)$ for some basepoint $o \in \mathcal{T}(S)$. The original definition can be recovered by considering the sequence of measures on spheres obtained by mapping the measure on $Q^1(o)$ to the spheres of radius $r$, $S^r_o$ via the map induced by $R(\cdot;r) : Q^1(o) \to S^r_o$ sending $q$ to $R(q;r)$.

The adapted definition of the thickness property is the following.

**Definition 3.2.1** (Thickness property)**.** *Given a basepoint $o \in \mathcal{T}(S)$, a probability measure $\nu$ on the set of directions $Q^1(o)$ has the thickness property if for all $0 < \theta, \eta < 1$ there exists an $\varepsilon > 0$ such that*

$$\lim_{r\to\infty} \nu\left(\{q \in Q^1(o) \mid \mathrm{Thick}^{\%}_{\varepsilon}(R(q), t) \geqslant \theta \text{ for all } t \in [\eta r, r]\}\right) = 1,$$

*for all $o \in \mathcal{T}(S)$.*

The other required property relates to avoiding measures whose geodesics are too clustered. This is done by requiring that for any $M > 0$, the distance between most geodesic rays is eventually at least $M$. Precisely, the adapted separation property is the following.

**Definition 3.2.2** (Separation property)**.** *Given a basepoint $o \in \mathcal{T}(S)$, a measure $\nu$ on the set of directions $Q^1(o)$ has the separation property if for all $M > 0$ and $0 < \eta < 1$, we have*

$$\lim_{r\to\infty} \nu \times \nu \left( \{ (q, q') \in Q^1(o) \times Q^1(o) \mid d_{\text{Teich}}(R(q;t), R(q';t)) \geqslant M \text{ for all } t \in [\eta r, r] \} \right) = 1,$$

*for all $o \in \mathcal{T}(S)$.*

In the next section, we derive these properties for a harmonic measure that arises from a random walk on the mapping class group determined by a probability distribution with finite first moment with respect to the Teichmüller metric whose support generates a non-elementary subgroup.

## 3.3 Derivation of the Thickness and Separation Properties

### 3.3.1 Recurrence

Let $x$ be a base-point in Teichmüller space. Let $\omega$ be a bi-infinite sample path. As a convenient notation, we let $x_n = w_n x$ for any $n \in \mathbb{Z}$. In particular, this means that $x_0$ is the same as the base-point $x$. For almost every $\omega$, the sequences $x_n$ and $x_{-n}$ converge projectively as $n \to \infty$ to distinct uniquely ergodic measured foliations $\lambda^+$ and $\lambda^-$ respectively. For such sample paths, let $\gamma_\omega$ be the bi-infinite Teichmüller geodesic between $\lambda^+$ and $\lambda^-$. As convenient notation, let $\gamma = \gamma_\omega$ and let $\gamma_n$ be a point of $\gamma$ that is closest to $x_n$. The diameter of the set of closest points is coarsely bounded above by $d_{\text{Teich}}(x_n, \gamma_n)$. As sample paths deviate sub-linearly from their tracked geodesics [Tio15], the choice of the closest point does not affect our estimates. This will become quantitatively precise subsequently.

Let

$$\Lambda_R = \{ \omega \text{ such that } d_{\text{Teich}}(x, \gamma_\omega) < R \} .$$

By Kaimanovich–Masur [KM96, Lemma 1.4.4], the function $\omega \to d_{\text{Teich}}(x, \gamma_\omega)$ is measurable. Recall that $h$ is our notation for the harmonic measure $\nu \times \hat{\nu}$ on bi-infinite sample paths. So

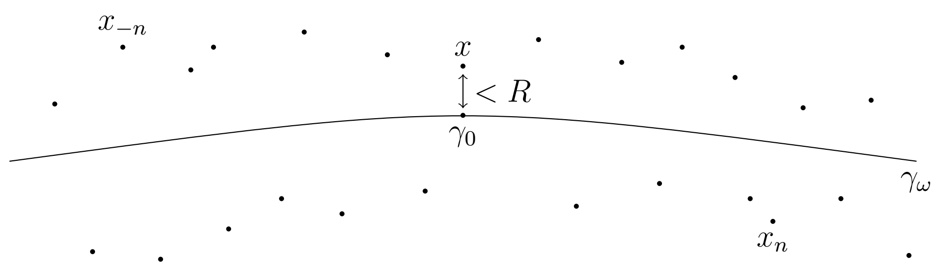



Figure 3.1: A sample path $\omega$ in $\Lambda_R$.

if $R$ is large enough then $h(\Lambda_R) > 0$ and $h(\Lambda_R) \to 1$ as $R \to \infty$.

An integer $k$ will be called an *R-recurrence time* for $\omega$ if $\sigma^k\omega \in \Lambda_R$ where recall that $\sigma$ is the shift map. Suppose $j < k$ are $R$-recurrence times for $\omega$ and denote $d_{\text{Teich}}(x_j, x_k) = 2d$. This distance will be bounded by the sum of the length of each step we do, that is,

$$2d \leqslant \sum_{i=j+1}^{k} d_{\text{Teich}}(x_{i-1}, x_i) = \sum_{i=j+1}^{k} d_{\text{Teich}}(x, g_i x).$$

We let $[\gamma_j, \gamma_k]$ be the segment of $\gamma_\omega$ connecting $\gamma_j$ and $\gamma_k$. We note that

$$\text{Length}[\gamma_j, \gamma_k] \leqslant 2R + 2d.$$

If $2d \leqslant 2R$, then

$$[\gamma_j, \gamma_k] \subset B(x_j, 3R) \cup B(x_k, 3R).$$

So suppose $2d > 2R$. We consider the sub-segments of $[\gamma_j, \gamma_k]$ that might be outside of the union $B(x_j, 3R) \cup B(x_k, 3R)$. We denote the union of these sub-segments by $C_{j,k}$ and let $L(j, k)$ to be the sum of their lengths.

Let $0 < \rho < p < 1$. We choose $R$ large enough such that $h(\Lambda_R) \geqslant p$. Let $n \in \mathbb{N}$ and set

$$E_n^{(1)} = \left\{ \omega \text{ such that } \frac{1}{m} \sum_{0 \leqslant k \leqslant m} \chi_{\Lambda_R}(\sigma^k\omega) < p - \rho \text{ for some } m \geqslant n \right\}$$

where $\chi$ is the indicator function. By ergodicity of the shift map $\sigma$ it follows that $h(E_n^{(1)}) \to 0$ as $n \to \infty$.

Suppose $\omega$ is in the complement of $E_n^{(1)}$. Then the number of times $i \in \{0, \cdots, n\}$ such that $\sigma^i\omega \notin \Lambda_R$ is at most $(1+\rho-p)n$. Let $j_{\min}$ and $j_{\max}$ be the smallest and largest $R$-recurrence times in $\{0,\ldots,n\}$. Then we note that

$$j_{\min} \leqslant (1+\rho-p)n \quad \text{and} \quad j_{\max} \geqslant n-(1+\rho-p)n.$$

By bounding with steps, we get the estimate

$$d_{\mathrm{Teich}}(x, x_{j_{\min}}) \leqslant \sum_{i=1}^{j_{\min}} d_{\mathrm{Teich}}(x, g_i x).$$

We will separate the sum into two sums. The first will contain terms $d_{\mathrm{Teich}}(x, g_i x)$ for which $d_{\mathrm{Teich}}(x, g_i x) \leqslant D$ and the second will contain the rest of the terms. For convenience of notation, we let $B$ be the set of bi-infinite sample paths $\omega$ whose first step $g_1$ satisfies $d_{\mathrm{Teich}}(x, g_1 x) \leqslant D$. The set $B$ depends on the choice of $D$ but we will suppress this from the notation for the moment and point it out when required later. With this notation, the sum above becomes

$$d_{\mathrm{Teich}}(x, x_{j_{\min}}) \leqslant \sum_{i=1}^{j_{\min}} d_{\mathrm{Teich}}(x, g_i x)\chi_B(\sigma^i(\omega)) + \sum_{i=1}^{j_{\min}} d_{\mathrm{Teich}}(x, g_i x)\chi_{\Omega\setminus B}(\sigma^i(\omega)),$$

We can bound the first sum by $\sum_{i=1}^{j_{\min}} D \leqslant (1+\rho-p)Dn$, and the second one by

$$\sum_{i=1}^{n} d_{\mathrm{Teich}}(x, g_i x)\chi_{\Omega\setminus B}(\sigma^i(\omega)).$$

We denote each of the terms of the sum above as $b_i^D$, that is $b_i^D(\omega) = d_{\mathrm{Teich}}(x, g_i x)\chi_{\Omega\setminus B}(\sigma^i(\omega))$. The random variables $b_i^D$ are all independent and identically distributed. Furthermore, $b_i^D(\omega) \leqslant d_{\mathrm{Teich}}(x, g_i x)$, which, since the measure has finite first $d_{\mathrm{Teich}}$-moment, is integrable. By the strong law of large numbers, the sum above when divided by $n$ converges almost surely to $\mathbb{E}[b_i^D]$. Let $C$ denote this expectation.

We conclude that the sets

$$E_n^{(2)} = \left\{\omega \text{ such that } \frac{1}{m}\sum_{i=0}^{m} b_i^D(\omega) > C + c \text{ for some } m \geqslant n\right\}$$

satisfy $h(E_n^{(2)}) \to 0$ as $n \to \infty$ for all $c > 0$.

As $D$ tends to infinity, the random variable $b_i^D$ converges to zero point-wise. By the dominated convergence theorem where we are dominating by $d_{\text{Teich}}(x, gx)$, we get that if $D$ is large enough, then the expectation $C$ is small.

Assume $\omega$ is in the complement of both $E_n^{(1)}$ and $E_n^{(2)}$. Then,

$$d_{\text{Teich}}(x, x_{j_{\min}}) \leqslant ((1 + \rho - p)D + C + c)n,$$

and, by the same reasoning,

$$d_{\text{Teich}}(x_{j_{\max}}, x_n) \leqslant ((1 + \rho - p)D + C + c)n.$$

This implies

$$d_{\text{Teich}}(\gamma_0, \gamma_{j_{\min}}) \leqslant ((1 + \rho - p)D + C + c)n + 2R$$

and

$$d_{\text{Teich}}(\gamma_{j_{\max}}, \gamma_n) \leqslant ((1 + \rho - p)D + C + c)n + 2R.$$

We will now estimate from above the time that the segment $[\gamma_{j_{\min}}, \gamma_{j_{\max}}]$ spends outside the neighbourhood of points along the sample path. A pair $j < k$ of recurrence times is *consecutive* if every $J$ satisfying $j < J < k$ is not a recurrence time.

If $k = j + 1$, the set $C_{j,j+1}$ will be non empty only if $d_{\text{Teich}}(x_j, x_{j+1}) \geqslant 2R$. In that case, if we choose $D$ smaller than $R$, then $L(j, j+1) \leqslant b_{j+1}^D$.

If $k > j + 1$, we have $k - j - 1$ steps taken outside the $R$-neighbourhood of the geodesic. We can split these steps depending on whether they are in $B$ or not. Recall that since we are outside the exceptional set $E_n^{(1)}$, we know that the total number of non-recurrence times is bounded above by $(1 + \rho - p)n$. In the estimate for the sum of $L(j, k)$, we note that each non-recurrence time contributes to at most two terms in the sum. We deduce that

$$\begin{aligned}
\sum_{\substack{j<k \\ \text{consecutive}}} L(j,k) &\leqslant \sum_{\substack{j+1=k \\ \text{consecutive}}} b_{j+1}^D + \sum_{\substack{j+1<k \\ \text{consecutive}}} \sum_{i=j+1}^{k} \left(b_i^D + d_{\text{Teich}}(x, g_i x)\chi_B(\sigma^i(\omega))\right) \\
&\leqslant \sum_{i=1}^{n} b_i^D + 2(1 + \rho - p)nD \\
&\leqslant (2(1 + \rho - p)D + C + c)n.
\end{aligned}$$

### 3.3.2 Linear progress

The Teichmüller metric is sub-additive along sample paths. By Kingman's sub-additive ergodic theorem, there exists a constant $A \geqslant 0$ such that for almost every sample path $\omega$ we have

$$\lim_{n\to\infty} \frac{d_{\mathrm{Teich}}(x_0, x_n)}{n} = A.$$

By Maher–Tiozzo [MT18, Theorem 1.2], $A > 0$.

Let $0 < a < 1$ be a constant smaller than $A$. Let $n \in \mathbb{N}$. Consider the set of sample paths

$$\Omega_n^{(3)} = \{\omega \text{ such that } (A-a)m < d_{\mathrm{Teich}}(x_0, x_m) < (A+a)m \text{ for all } m \geqslant n\}.$$

Let $E_n^{(3)}$ be the complement $\Omega \setminus \Omega_n^{(3)}$. It follows that $h(E_n^{(3)}) \to 0$ as $n \to \infty$.

### 3.3.3 Thickness along tracked geodesics

Let $0 < \theta' < 1$.

We parameterise the tracked geodesic $\gamma = \gamma_\omega$ by unit speed such that at time zero we are at $\gamma_0$, a closest point to $x_0 = x$, and $\gamma(t) \to \lambda^+$ as $t \to \infty$.

Let $\Lambda(r, \theta', \varepsilon')$ be the set of sample paths $\omega$ such that for all $s > r$ we have

$$\mathrm{Thick}^{\%}_{\varepsilon'}(\gamma, s) \geqslant \theta'.$$

**Proposition 3.3.1.** *Given $0 < \theta' < 1$ there exists an $\varepsilon' > 0$ such that*

$$\lim_{r\to\infty} h(\Lambda(r, \theta', \varepsilon')) = 1.$$

*Proof.* Given $R > 0$ there exists $\varepsilon(R) > 0$ such that $B(x_0, 3R) \subset \mathcal{T}_{\varepsilon(R)}(S)$. By equivariance, $B(x_n, 3R) \subset \mathcal{T}_{\varepsilon(R)}(S)$ for all $n \in \mathbb{Z}$.

Suppose that $\omega$ is in the complement of $E_n^{(1)} \cup E_n^{(2)}$. We first prove the proposition along the discrete set of times $\gamma_n$ along $\gamma_\omega$. By the triangle inequality

$$d_{\mathrm{Teich}}(\gamma_0, \gamma_n) \geqslant d_{\mathrm{Teich}}(x_0, x_n) - d_{\mathrm{Teich}}(x_0, \gamma_0) - d_{\mathrm{Teich}}(x_n, \gamma_n).$$

Since $\gamma_0$ is the closest point in $\gamma_\omega$ to $x_0$

$$d_{\mathrm{Teich}}(x_0, \gamma_0) \leqslant d_{\mathrm{Teich}}(x_0, x_{j_{\min}}) + R \leqslant ((1 + \rho - p)D + C + c)n + R.$$

Similarly

$$d_{\mathrm{Teich}}(x_n, \gamma_n) \leqslant d_{\mathrm{Teich}}(x_n, x_{j_{\max}}) + R \leqslant ((1 + \rho - p)D + C + c)n + R.$$

So we get

$$d_{\mathrm{Teich}}(\gamma_0, \gamma_n) \geqslant d_{\mathrm{Teich}}(x_0, x_n) - 2((1 + \rho - p)D + C + c)n - 2R.$$

Further assume that $\omega$ is also in the complement of $E_n^{(3)}$. We deduce from the above estimates that

$$d_{\mathrm{Teich}}(\gamma_0, \gamma_n) \geqslant (A - a)n - 2((1 + \rho - p)D + C + c)n - 2R. \tag{3.3.1}$$

We note that the points in the segment $[\gamma_0, \gamma_n]$ that are not in $\mathcal{T}_{\varepsilon(R)}(S)$ are in the union of the sets $[\gamma_0, \gamma_{j_{\min}}]$, $[\gamma_{j_{\max}}, \gamma_n]$ and the sets $C_{j,k}$ for all consecutive recurrence pairs $j < k$. The individual upper bounds on the lengths of each set in the union gives us the bound on the thick proportion for a choice of $\varepsilon' \leqslant \varepsilon(R)$

$$1 - \mathrm{Thick}^{\%}_{\varepsilon'}(\gamma, d_{\mathrm{Teich}}(\gamma_0, \gamma_n)) \leqslant \frac{4((1 + \rho - p)D + C + c)}{(A - a) - 2((1 + \rho - p)D + C + c) - 2R/n}. \tag{3.3.2}$$

Now we make explicit choices as follows.

- Note that as $R \to \infty$, the proportion $p \to 1$ and hence $\min\{(1 - p)^{-1/2}, R\} \to \infty$.
- Recall that if we tend $D \to \infty$ then $C \to 0$.
- So if we set $D = \min\{(1 - p)^{-1/2}, R\}$ then we can pass to $R$ large enough so that $C$ is small enough.
- Furthermore, note that $D(1 - p) \leqslant (1 - p)^{1/2}$ which can also be made small with $R$ large enough.
- Recall also that $\rho, c$ and $a$ can be chosen after all the other constants have been set, and so we can set them to be small enough.

These choices imply that by choosing $R$ large enough we may arrange that the numerator and the second term in the denominator in (3.3.2) are small. Once $R$ has been fixed the third term in the denominator goes to zero as $n \to \infty$. In conclusion, by choosing $R$ large enough we may arrange the right hand side of (3.3.2) to be smaller than $1 - \theta'$.

Now we conclude the proposition as the time $s \to \infty$ along $\gamma_\omega$. By an argument identical to the derivation of (3.3.1), we get the upper bound

$$d_{\mathrm{Teich}}(\gamma_0, \gamma_n) \leqslant (A + a)n + 2((1 + \rho - p)D + C + c)n + 2R.$$

Given a time $s > 0$, we may choose $n$ to satisfy

$$(A - a)n - 2((1 + \rho - p)D + C + c)n - 2R < s < (A + a)n + 2((1 + \rho - p)D + C + c)n + 2R.$$

When $s$ is large enough, such a choice always exists. Since we are only interested in the limit as $s \to \infty$, we may make this choice. Further tweaking $R$ and hence $p$ and also tweaking $\rho, c$ and $a$ we may arrange that the ratio of the upper bound to the lower bound in the above inequality is as close to one as we want. This implies that as $s \to \infty$ the thick proportion of $[\gamma_0, \gamma(s)]$ is the same as the thick proportion of $[\gamma_0, \gamma_n]$.

Finally, we note that the set of exceptions is the union $E_n^{(1)} \cup E_n^{(2)} \cup E_n^{(3)}$ whose measure tends to zero as $n \to \infty$. In particular, this implies $h(\Lambda(r, \theta', \varepsilon')) \to 1$ as $r \to \infty$, and we are done. $\square$

As a direct consequence of Proposition 3.3.1, we get the following conclusion.

**Proposition 3.3.2.** *Let $0 < \theta' < 1$. Then there exists an $\varepsilon' > 0$ such that for almost every bi-infinite sample path $\omega$ there exists $t_\omega$ such that for all $t > t_\omega$*

$$\mathrm{Thick}^{\%}_{\varepsilon'}(\gamma, t) \geqslant \theta'.$$

### 3.3.4 Thickness along rays

Now let $y$ be some other base-point in $\mathcal{T}(S)$ possibly distinct from the base-point $x_0$ for the random walk. Masur [Mas80, Theorem 2] proved that Teichmüller rays with the same vertical foliation are asymptotic if the foliation is uniquely ergodic. We now use this result to transfer the thickness estimates from tracked geodesics to corresponding rays from $y$. Suppose that

$\omega$ is a typical bi-infinite sample path with the tracked geodesic $\gamma_\omega$. Let $\lambda_\omega^+$ be the projective measured foliation that $\gamma_\omega$ converges to in the forward direction. Let $q_\omega \in Q^1(y)$ be the unit area quadratic differential based at $y$ with vertical foliation $\lambda_\omega^+$.

**Proposition 3.3.3.** *Let $0 < \theta < 1$. Then there exists $\varepsilon > 0$ such that for almost every bi-infinite sample path $\omega$ there is a time $T_\omega > 0$ such that*

$$\mathrm{Thick}_\varepsilon^\%(R(q_\omega), t) \geqslant \theta$$

*for all $t \geqslant T_\omega$.*

*Proof.* By [KM96, Theorem 2.2.4], for almost every $\omega$ the foliation $\lambda_\omega^+$ is uniquely ergodic. By Masur's theorem, there is a time $s > 0$ that depends only on $d_{\mathrm{Teich}}(\gamma_\omega, y)$ such that for all $t \geqslant s$ we have $d_{\mathrm{Teich}}(R(q_\omega; t), \gamma_\omega) < 1/2$. We may choose $\varepsilon \leqslant \varepsilon'$ such that the 1-neighbourhood of $\mathcal{T}_{\varepsilon'}(S)$ is contained in $\mathcal{T}_\varepsilon(S)$. This means after the time $s$ along $R(q_\omega)$ any $\varepsilon'$-thick segment of $\gamma_\omega$ gives an $\varepsilon$-thick segment of $R(q_\omega)$ of at least the same length. We now set $\theta' > \theta$ and use Proposition 3.3.2. Let $t > s$. The total length of $\varepsilon'$-thick segments of $\gamma_\omega$ that are inside a 1-neighbourhood of $[R(q_\omega; s), R(q_\omega; t)]$ is at least $\theta' t - s - d_{\mathrm{Teich}}(\gamma(0), y)$. If $t$ is large enough then $\theta' t - s - d_{\mathrm{Teich}}(\gamma(0), y) > \theta t$, and we are done. $\square$

As an immediate corollary, we get

**Corollary 3.3.4.** *The harmonic measure $\nu$ on $Q^1(y)$ satisfies the thickness property 3.2.1 for any $y \in \mathcal{T}(S)$.*

### 3.3.5 Separation properties

To prove the separation properties, we project to the complex of curves. As proven by Masur–Minsky [MM99], under this projection Teichmüller geodesics give unparameterised quasi-geodesics and thick segments make coarse linear progress. If a pair of Teichmüller rays with good thickness properties up to a distance $r$ are not well-separated then their projections to the curve complex fellow travel. Hence, the endpoints of their projections lie in some shadow and this shadow is pushed further and further out as $r \to \infty$. So roughly speaking, we may conclude the separation properties from knowing that the harmonic measure of these shadows goes to zero. We now give the details of the argument.

The complex of curves $\mathcal{C}(S)$ is a graph whose vertices are isotopy classes of essential simple closed curves on $S$. Two vertices $\alpha$ and $\beta$ have an edge between them if $\alpha$ and $\beta$ have representatives that are disjoint. The curve complex $\mathcal{C}(S)$ is a locally infinite graph with infinite diameter. Masur-Minsky [MM99, Theorem 1.1] showed that $\mathcal{C}(S)$ is in fact $\delta$-hyperbolic. Recall that the Gromov product on $\mathcal{C}(S)$ is defined as follows: given a base-point $x$ and two points $y, z$ in the space, the Gromov product of $y$ and $z$ based at $x$ is defined as

$$(y,z)_x = \frac{1}{2}\left(d_{\mathcal{C}}(x,y) + d_{\mathcal{C}}(x,z) - d_{\mathcal{C}}(y,z)\right)$$

where $d_{\mathcal{C}}$ denotes the distance in the curve complex. Let $\partial\mathcal{C}(S)$ be the Gromov boundary of $\mathcal{C}(S)$. Given a number $\tau > 0$, the shadow of $y$ from $x$ with distance parameter $d_{\mathcal{C}}(x,y) - \tau$ is defined as

$$\mathrm{Shad}_x(y,\tau) = \{z \in \mathcal{C}(S) \cup \partial\mathcal{C}(S) \text{ such that } (y,z)_x \geqslant d_{\mathcal{C}}(x,y) - \tau\}.$$

There are different definitions of shadows in the literature and the one we use here is from Maher–Tiozzo [MT18, Page 197].

To every marked hyperbolic surface $x \in \mathcal{T}(S)$, one can consider a systole on $x$, that is, a shortest closed hyperbolic geodesic on $x$. A systole is always a simple closed curve and hence can be thought of as a vertex in the curve complex $\mathcal{C}(S)$. This defines a coarse projection $\mathrm{sys} : \mathcal{T}(S) \to \mathcal{C}(S)$ from Teichmüller space to the curve complex. By Masur–Minsky [MM99, Theorems 2.3 and 2.6], the projection $\mathrm{sys}(\gamma)$ of a Teichmüller geodesic $\gamma$ is an unparameterised quasi-geodesic in $\mathcal{C}(S)$ with uniform constants that depend only on the surface.

Let $M > 0$. Suppose $q_\omega, q_\eta \in Q^1(y)$ are chosen with respect to the harmonic measure, where $\omega$ and $\eta$ are the associated bi-infinite sample paths. Let $T_\omega$ and $T_\eta$ be the thresholds given by Proposition 3.3.3 for $\omega$ and $\eta$, respectively. Pick $T$ larger than $T_\omega$ and $T_\eta$. Suppose that $d_{\mathrm{Teich}}(R(q_\omega;T), R(q_\eta;T)) < M$. Since the projection by systoles is coarsely Lipschitz, we may continue to assume $d_{\mathcal{C}}(\mathrm{sys}(R(q_\omega;T)), \mathrm{sys}(R(q_\eta;T))) < M$.

We consider the projections $\mathrm{sys}([y, R(q_\omega;T)])$ and $\mathrm{sys}([y, R(q_\eta;T)])$. Since, $R(q_\omega)$ and $R(q_\eta)$ spend at least $\theta$ proportion of their time in the thick part their projections to $\mathcal{C}(S)$ make linear progress, that is there exists a constant $\kappa > 0$ such that $d_{\mathcal{C}}(\mathrm{sys}(y), \mathrm{sys}(R(q_\omega;T)) \geqslant \kappa T$ and $d_{\mathcal{C}}(\mathrm{sys}(y), \mathrm{sys}(R(q_\eta;T)) \geqslant \kappa T$.

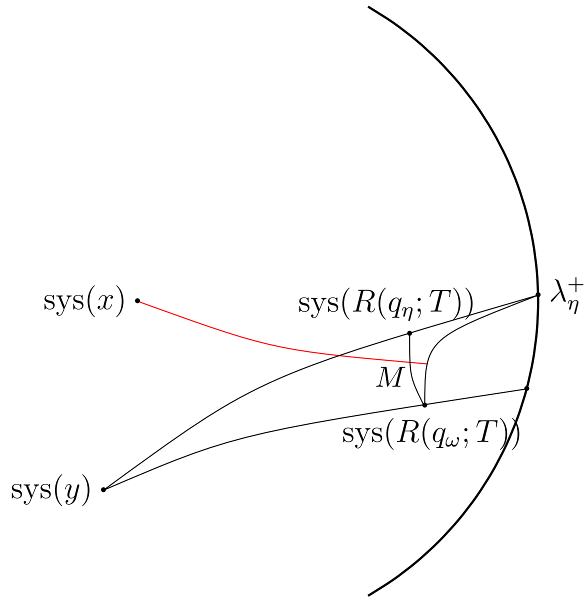



Figure 3.2: The Gromov product $(\mathrm{sys}(R(q_\omega;T)), \lambda^+_\eta)_{\mathrm{sys}(x)}$ in $\mathcal{C}(S)$, up to an additive constant.

Denote by $\lambda^+_\eta$ the limiting point of $\mathrm{sys}(R(q_\eta))$. By hyperbolicity of $\mathcal{C}(S)$ there is a constant $r > 0$ such that the Gromov product between $\lambda^+_\eta$ and $\mathrm{sys}(R(q_\omega;T))$ satisfies

$$(\mathrm{sys}(R(q_\omega;T)), \lambda^+_\eta)_{\mathrm{sys}(y)} \geqslant d_{\mathcal{C}}(\mathrm{sys}(y), \mathrm{sys}(R(q_\omega;T))) - M - r.$$

In order to estimate harmonic measures, we will now pass to $\mathrm{sys}(x)$ as the base-point for Gromov products. By the triangle inequality

$$d_{\mathcal{C}}(\mathrm{sys}(x), \mathrm{sys}(R(q_\omega;T))) \geqslant d_{\mathcal{C}}(\mathrm{sys}(y), \mathrm{sys}(R(q_\omega;T))) - d_{\mathcal{C}}(\mathrm{sys}(x), \mathrm{sys}(y)).$$

Note that $d_{\mathcal{C}}(\mathrm{sys}(x), \mathrm{sys}(R(q_\omega;T)))$ goes to infinity as $d_{\mathcal{C}}(\mathrm{sys}(y), \mathrm{sys}(R(q_\omega;T)))$ does. We have $(\mathrm{sys}(R(q_\omega;T), \lambda^+_\eta)_{\mathrm{sys}(x)} \geqslant d_{\mathcal{C}}(\mathrm{sys}(x), \mathrm{sys}(R(q_\omega;T))) - r'$, where $r' = M + r + 2d_{\mathcal{C}}(\mathrm{sys}(x), \mathrm{sys}(y))$. Then $\lambda^+_\eta$ is contained in $\mathrm{Shad}_{\mathrm{sys}(x)}(\mathrm{sys}(R(q_\omega;T)), r')$. By [MT18, Proposition 5.1], the supremum of the harmonic measure of $\mathrm{Shad}_{\mathrm{sys}(x)}(\mathrm{sys}(R(q_\omega;T)), r')$ tends to zero as $T \to \infty$.

As an immediate corollary we get

**Corollary 3.3.5.** *The harmonic measure $\nu$ on $Q^1(y)$ satisfies the separation property 3.2.2 for any $y \in \mathcal{T}(S)$.*

For exceptional moduli or more generally for lattices in $\mathrm{Isom}(\mathbb{H}^n)$, the ambient geometry is hyperbolic. So we may directly use shadows/ half-spaces in $\mathbb{H}^n$. As above, the harmonic measure of shadows decays to zero as the distance from the base-point goes to infinity. So the proof can be carried out exactly as above to conclude the separation property.

## 3.4 Statistical Hyperbolicity

For the sake of completeness, we now sketch the spherical version of the argument given by Dowdall-Duchin-Masur [DDM14, Theorem 7.1] of how the thickness and separation properties imply statistical hyperbolicity. The idea is to mimic a proof of the fact that $E(\mathbb{H}^n) = 2$ for the natural measures on spheres which makes use of the $\delta$-hyperbolicity of $\mathbb{H}^n$, and of the speed of separation of geodesics.

However, as discussed above, Teichmüller space is neither $\delta$-hyperbolic, nor negatively-curved in the sense of Busemann. The motivation for the separation property 3.2.2 is to show instead that most pairs of geodesics after some threshold time become separated by a definite amount. This replaces the use of the negative curvature of $\mathbb{H}^n$. The combination of the following theorem of Dowdall-Duchin-Masur [DDM14, Theorem A] and the thickness property 3.2.1 then replaces the use of the $\delta$-hyperbolicity of $\mathbb{H}^n$.

**Theorem 3.4.1** ( [DDM14], Theorem A)**.** *For any $\varepsilon > 0$ and any $0 < \theta' \leqslant 1$, there exist constants $C$ and $L$ such that for any geodesic sub-interval $I \subset [x, x'] \subset \mathcal{T}(S)$ of length at least $L$ and spending at least $\theta$ proportion of its time in $\mathcal{T}_\varepsilon(S)$, we have*

$$I \cap \mathrm{Nbhd}_C([x, x''] \cup [x', x'']) \neq \varnothing,$$

*for all $x'' \in \mathcal{T}(S)$.*

We now sketch the proof of statistical hyperbolicity using Theorem 3.4.1 with the thickness and separation properties adapted to measures on $Q^1(x)$, namely Definition 3.2.1 and Definition 3.2.2.

Let $x \in \mathcal{T}(S)$. We choose $\theta$ large enough, say $\theta = 3/4$, and then for any $0 < p < \eta < 1/3$ we let $\varepsilon = \varepsilon(\theta, \eta) > 0$ be that guaranteed by the thickness property 3.2.1. Now choose $\theta' < \theta$, say $\theta' = 1/2$, and let $C$ and $L$ be the constants given by Theorem 3.4.1 for our choice of $\theta'$ and $\varepsilon$. The thickness and separation properties then imply that for all $r$ large enough, there exists a subset $P_r \subset Q^1(x) \times Q^1(x)$ whose complement has $\nu \times \nu$-measure at most $p$ and is such that for all $(q, q') \in P_r$ we have $d_{\mathrm{Teich}}(R(q;t), R(q';t)) \geqslant 3C$, and

$$\mathrm{Thick}^{\%}_{\varepsilon}(R(q), t), \mathrm{Thick}^{\%}_{\varepsilon}(R(q'), t) \geqslant \theta,$$

for all $\eta r \leqslant t \leqslant r$. It can be checked that $B(R(q;t), C) \cap R(q') = \varnothing$ for all such $q, t$.

Choosing $r$ large enough, we can arrange that the interval $I_r = [R(q;\eta r), R(q; 2\eta r)]$ spends at least $\theta'$ proportion of its time in $\mathcal{T}_{\varepsilon}(S)$, has length at least $L$ and, since $\eta < 1/3$, is contained in $[x, R(q;r)]$. By applying Theorem 3.4.1, we must have that $I_r \cap \mathrm{Nbhd}_C([x, R(q';r)] \cup [R(q;r), R(q';r)]) \neq \varnothing$ and, since we have already noted that $I_r \cap \mathrm{Nbhd}_C([x, R(q';r)]) = \varnothing$, it then follows that there exists a point in $[R(q;r), R(q';r)]$ at distance at most $2\eta r + C$ from $x$. Hence we have that

$$d_{\mathrm{Teich}}(R(q;r), R(q';r)) \geqslant (2 - 4\eta)r - 2C,$$

for all $q, q' \in P_r$. From which it follows that

$$\begin{aligned}
E(X) &:= \lim_{r\to\infty} \frac{1}{r} \int_{Q^1(x)\times Q^1(x)} d_{\mathrm{Teich}}(R(q;r), R(q';r)) \; d\nu(q)\, d\nu(q') \\
&\geqslant \liminf_{r\to\infty} \frac{1}{r} \int_{Q^1(x)\times Q^1(x)} d_{\mathrm{Teich}}(R(q;r), R(q';r)) \; d\nu(q)\, d\nu(q') \\
&\geqslant \liminf_{r\to\infty} \frac{1}{r}(1-p)((2-4\eta)r - 2C) \\
&= (1-p)(2-4\eta).
\end{aligned}$$

Hence, the result follows as $p$ and $\eta$ can be taken to be arbitrarily small.

### 3.4.1 Proof of Theorem 3.1.2

We now give a quick proof of Theorem 3.1.2. The ambient geometry in $\mathbb{H}^n$ is already hyperbolic. So we can simply bypass the thickness discussion and directly invoke the separation property for the harmonic measure discussed at the end of Section 3.3.5. The proof of statistical hyperbolicity then follows the one in Section 3.4 and is simpler.

# Chapter 4

# Growth of the drift as a function of Teichmuller space

## 4.1 Introduction

In this section $S$ shall denote a compact oriented surface with empty boundary, no marked points and negative Euler characteristic with a basepoint $p$. Let $\Gamma = \pi_1(S, p)$ and let $\mu$ be a probability measure on $\Gamma$ that is admissible, i.e., the semigroup generated by the support of $\mu$ is equal to $\Gamma$. Further assume that $\mu$ has finite first moment. Consider a random walk $Z_n = g_1 \cdots g_n$ where $g_i$ are i.i.d. elements of $\Gamma$ with distribution $\mu$. Fixing a complete hyperbolic metric $\rho$ on $S$, define

$$\mathrm{Drift}(\rho) \coloneqq \lim_{n \to \infty} \frac{|Z_n|_\rho}{n}$$

where $|Z_n|_\rho$ denotes the $\rho$-length of the unique hyperbolic geodesic representing the free homotopy class of the element $Z_n$. The limit above exists almost surely, and is well defined. The quantity $\mathrm{Drift}(\rho)$ is called the *speed (or drift)* of the random walk on $(S, \rho)$.

In this section we shall study how $\mathrm{Drift}(\rho)$ changes as one varies the hyperbolic metric $\rho$ on $S$, or in other words as $[\rho]$ varies over points in the Teichmüller space $\mathcal{T}(S)$. When one moves in Teichmüller space, some curves get longer but others get shorter, so the behavior is not obvious. However, one expects that most curves get longer, so one should expect $\mathrm{Drift}(\rho)$ to tend to infinity as $\rho$ diverges to infinity in Teichmüller space. There is a difficulty, though, that most curves become more and more parallel to each other (up to orientation) when $\rho$ converges to a point at infinity, as they align asymptotically with the measured foliation at infinity. Since the steps of the random walks are equivalent to travelling along the lift of the curves in the universal cover, this means that consecutive steps of the random walk are likely to be both large, but in opposite directions, thereby cancelling each other effectively and possibly not contributing to the drift. Our main theorem shows that the former effect dominates the latter: The drift indeed tends to infinity at infinity. However, this discussion hints at the fact that this is not straightforward, and indeed our proof is rather indirect. Precisely, as a corollary of Proposition 4.4.1, proven in Section 4.4, we obtain the following theorem.

**Theorem 4.1.1** (Drift is a proper function on Teichmüller space)**.** *Assume that the probability measure $\mu$ on $\Gamma$ is admissible and has a finite first moment. Then the function* $\mathrm{Drift} : \mathcal{T}(S) \to [0, +\infty)$ *is proper.*

Recall that a function is proper if the preimage of any compact set is compact. In our particular case, since Drift is non-negative, and, by Furstenberg's formula [KL11, Theorem 18], continuous, properness is equivalent to the statement that for each $C > 0$ there exists a compact set $K \subset \mathcal{T}(S)$ such that $\mathrm{Drift}(\rho) > C$ for all $[\rho] \notin K$.

More quantitatively, given a finite filling set of curves $F$, we find that the drift is equal up to a constant multiple to the maximum hyperbolic length within the set $F$.

**Theorem 4.1.2.** *Let $F$ be a finite filling set of curves, and let $M_\rho^F$ be the maximum hyperbolic length among the curves under the hyperbolic geometry defined by $\rho$. Then there is some $K < \infty$ such that*

$$\frac{1}{K} M_\rho^F \leq \mathrm{Drift}_\mu(\rho) \leq K M_\rho^F$$

*for all $\rho \in \mathcal{T}(S)$.*

The theorem above is proven in Section 4.3, as a combination of the upper bound of Lemma 4.3.2 and the lower bound of Lemma 4.3.4.

Studying the relation between $M_\rho^F$ and the Teichmüller distance it is possible to find characterisations of the asymptotic behaviour of the drift. In Section 4.4 we study the linear behaviour, and show that the drift grows at least linearly with respect to the Teichmüller distance, and that the linear bound is sharp if we move along a Teichmüller ray generated by a Jenkins–Strebel differential.

**Theorem 4.1.3.** *Let $o \in \mathcal{T}(S)$ be a basepoint. There is some constant $c(o) > 0$ such that, for any $\rho \in \mathcal{T}(S)$ we have*

$$\mathrm{Drift}_\mu(\rho) \geq c(o) d_{\mathrm{Teich}}(\rho, o).$$

*Furthermore, let $q \in Q^1(o)$ be Jenkins–Strebel. Then, there is some $C(q, o) < \infty$ such that*

$$\mathrm{Drift}_\mu(R(q; t)) \leq C(q, o)t$$

*for all $t \geq 1$.*

In Section 4.5 we continue by showing that the typical behaviour is not linear growth, but exponential.

**Corollary 4.1.4.** *Let $o \in \mathcal{T}(S)$ be a basepoint, $0 < \theta < 1$ and $\lambda$ be the Lebesgue measure on $Q^1(o)$. Then, for $\lambda-$almost all directions $q \in Q^1(o)$ we have*

$$e^{\theta t} < \mathrm{Drift}_\mu(R(q; t)) < e^{\frac{1}{\theta} t},$$

*for any $t > t(q)$, where $t(q)$ is some finite time depending on $q$.*

The previous results applies also to harmonic measures. See Theorem 4.5.7 for the precise statement.

We find that the behaviour does not induce a duality. By combining the two previous results we find in Section 4.6 geodesics along which the growth oscillates between being exponential and almost linear.

**Corollary 4.1.5.** *Let $o \in \mathcal{T}(S)$ be a fixed basepoint. There is a constant $c(o) > 0$ such that, for any increasing diverging function $f : \mathbb{R}_+ \to \mathbb{R}_+$ there is some $q \in Q^1(o)$ and diverging increasing sequences $(t_n)$, $(s_n) \subset \mathbb{R}_+$ such that*

$$\mathrm{Drift}_\mu(R(q; t_n)) > c(o)e^{t_n}$$

*and*

$$\mathrm{Drift}_\mu(R(q; s_n)) < f(s_n)s_n.$$

As a part of the proof we also show that the quasiconvexity of hyperbolic lengths along Teichmüller geodesics proven by Lenzhen–Rafi [LR11] can be adapted to the drift.

**Corollary 4.1.6.** *Let $\mu$ be an admissible measure on $\pi_1(S,p)$ with finite first moment. There exists a constant $K' > 0$ such that for any Teichmüller geodesic $G$ and points $x, y, z \in \mathcal{T}(S)$ appearing in that order along $G$, we have*

$$\mathrm{Drift}_\mu(y) \leq K' \max(\mathrm{Drift}_\mu(x), \mathrm{Drift}_\mu(z)).$$

Our strategy to prove the lower bound of the drift function in Theorem 4.1.2 is through a compactification argument. By rescaling the metrics by $1/M^F_\rho$ we get a certain compactification of our space, where the action extends continuously. Then we find that the action on such boundary space satisfies the necessary properties for Gouëzel [Gou22a] and Baik–Choi–Kim [BCK23] pivotal arguments, which gives a positive linear drift. Then, by continuity of the action, we deduce that the properties are satisfied *inside* the rescaled Teichmüller space, obtaining a positive drift inside. Undoing the rescaling we get the drift is proper. Note that in [AGG+22] we perform a more complicated argument, proving that the drift itself is positive at the boundary, and then showing that the drift is lower semicontinuous on the compactification. However, such refinement is not needed to obtain the main results, and this simplification allows us to present a more complete proof.

Note that the drift can also be considered as a Lyapunov exponent. More precisely, if $[\rho] \in \mathcal{T}(S)$ is a point in Teichmüller space, we can consider $[\rho]$ as a conjugacy class of a discrete, faithful representation $\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{R})$. Indeed, if we fix a matrix norm $\|\cdot\|$ on $\mathrm{PSL}(2, \mathbb{R})$, we have

$$\mathrm{Drift}(\rho) = \lim_{n \to +\infty} \frac{1}{n} \int \log \|\rho(g_1) \cdots \rho(g_n)\| \, \mathrm{d}\mu(g_1) \cdots \mathrm{d}\mu(g_n),$$

which is the Lyapunov exponent of the random walk on $\mathrm{PSL}(2, \mathbb{R})$.

A related argument was used in [DF19] to study continuity of Lyapunov exponents for certain meromorphic families of representations in $\mathrm{SL}(2, \mathbb{C})$, with the same idea which consists in looking at the scaled limiting action on an $\mathbb{R}$-tree. One simple situation where both approaches can be used is the case where the hyperbolic structure degenerates by only pinching a simple closed curve: this degeneracy can also be described by a meromorphic family of representations. However, in the context of representations of surface groups into $\mathrm{PSL}(2, \mathbb{R})$, the setting of our continuity result below is more general.

The main feature of this argument is that the representations at boundary points do not live on the same space as the original representations: the group acts on an $\mathbb{R}$-tree instead of the hyperbolic disk. These representations are constructed in [Bes88] and [Pau88] (following previous work [CM87] and [MS84]). In particular, the topological type of Gromov boundaries changes in the limit. This means that the usual continuity argument for the drift, relying on the convergence of stationary measures on the boundary (see [EK13]), does not work. However, we are able to obtain uniform lower bounds for the drift in this context thanks to the pivotal times argument of [Gou22a].

Returning to the setting of Theorem 4.1.1, it is well known that $\mathrm{Drift}(\rho) > 0$ for all $\rho$ and the random walk driven by $\mu$ converges to the boundary almost surely. In order words, the limit

$$X_\infty = \lim_{n\to\infty} \rho(Z_n)o,$$

exists almost surely and $X_\infty$ is in the visual boundary $\partial\mathbb{H}$ of $\mathbb{H}$, (cf. [Kai00]). Recall that this limit defines a *hitting measure* on $\partial\mathbb{H}$ as follows: for any Borel set $U \subset \partial\mathbb{H}$,

$$\nu_\rho(U) \coloneqq \mathbb{P}(\lim_{n\to\infty} \rho(Z_n)o \in U).$$

The measure $\nu_\rho$ is the unique $\mu$-stationary measure on the visual boundary for the $\rho$-action. Denoting by $\dim(\nu)$ the Hausdorff dimension of $\nu$, we recall the following conjecture mentioned in the introduction.

**Conjecture** (Singularity Conjecture). *If $\mu$ is admissible and has finite support, then there exists $\kappa < 1$ such that*

$$\dim(\nu_\rho) \le \kappa$$

*for all $[\rho] \in T(M)$.*

The conjecture above, stated in [DKN09b, Conjecture 1.21] and more generally in [KL11], remains open in spite of some recent progress made in [KT22]. We remark that for all $\rho$ there exists $\mu$ *with infinite support* on $\Gamma$ such that $\dim(\nu_\rho) = 1$. This follows from the Furstenberg-Lyons-Sullivan discretization of Brownian motion [LS84], and also from more general results of Connell and Muchnik [CM07].

The drift of the random walk is closely related to the Hausdorff dimension of the stationary measure. Work in [Tan19] shows that:

$$\dim(\nu_\rho) = \frac{h(\mu)}{\mathrm{Drift}(\rho)},$$

where $h(\mu)$ denotes the entropy of the random walk.

Therefore results 4.1.1 to 4.1.5 immediately translate into statements about the behaviour of the Hausdorff dimension of the stationary measure. For example, from Theorem 4.1.1:

**Corollary 4.1.7** (Dimension drop of stationary measures)**.** *For each $\varepsilon > 0$ there exists a compact $K \subset \mathcal{T}(S)$ such that $\dim(\nu_\rho) < \varepsilon$ for all $[\rho] \notin K$.*

*Proof.* This follows immediately from Theorem 4.1.1 and the above formula for $\dim(\nu_\rho)$. □

### 4.1.1 Disclosures

This chapter is based on joint work with Vaibhav Gadre, Sébastien Gouëzel, Thomas Haettel, Pablo Lessa and Caglar Uyanik [AGG$^+$22], as well as a follow up paper by the author [Aze23].

## 4.2 Section specific background

### 4.2.1 Converging actions

A key notion we shall use in our proof is that of *Converging actions*, which we introduce in this section.

**Definition 4.2.1.** *Consider a group $\Gamma$, and a sequence of pointed metric spaces $(X_k, o_k)_{k\in\mathbb{N}\cup\{\infty\}}$, each of them endowed with an isometric action $\rho_k$ of $\Gamma$. We say that this sequence of actions converges if, for each $g \in \Gamma$, the distance $\mathrm{dist}(o_k, \rho_k(g)o_k)$ converges to $\mathrm{dist}(o_\infty, \rho_\infty(g)o_\infty)$ as $k \to \infty$.*

Following work of Bestvina [Bes88] and Paulin [Pau88], for any diverging sequence $(\rho_k) \subset \mathcal{T}(S)$ it is possible to pick a subsequence and sequence of numbers $(\tilde{M}_k)$ such that, by rescaling the hyperbolic space by $\frac{1}{\tilde{M}_k}$ we have convergence of the action $\rho_k$ to some non-empty $\mathbb{R}$-tree. Precisely, fix a finite symmetric generating set $F \subset \Gamma$ containing the identity.

By [Bes88, Proposition 2.1] there exists for each $\rho$ a basepoint $o_\rho \in \mathbb{H}$ such that

$$\max_{\gamma \in F} \operatorname{dist}(o_\rho, \rho(\gamma) o_\rho) = \min_{x \in \mathbb{H}} \max_{\gamma \in F} \operatorname{dist}(x, \rho(\gamma) x).$$

We define the rescaling factor $\tilde{M}_\rho = \tilde{M}_\rho^F$ as the common value of both sides of the equation above. From the definition, it follows that $\tilde{M}_\rho$ is continuous and proper on $\mathcal{T}(S)$. Consider the rescaled distance $\operatorname{dist}_\rho = \tilde{M}_\rho^{-1} \operatorname{dist}$ on $\mathbb{H}$.

Recall that an $\mathbb{R}$-tree is a non-empty metric space which is 0-hyperbolic and such that every pair of points is joined by a unique geodesic. The following is the main result of [Bes88] and [Pau88] (following previous work [CM87] and [MS84]).

**Lemma 4.2.2.** *Each sequence $\rho_n$ such that $[\rho_n]$ leaves every compact subset in $\mathcal{T}(M)$, has a subsequence $\rho_{n_k}$ with $n_k \to +\infty$ such that $\rho_{n_k}$ when viewed as an action on $\mathbb{H}$ endowed with the distance $\operatorname{dist}_{\rho_{n_k}}$ and the basepoint $o_{\rho_{n_k}}$, converges to an action $\rho_T$ on an $\mathbb{R}$-tree $(T, \operatorname{dist}_T, o_T)$.*

*Furthermore, the group $\rho_T(\Gamma)$ acts minimally on $T$ (i.e., there is no proper closed invariant subtree), and for any arc $I$ in $T$ the set of $\gamma \in \Gamma$ such that $\rho_T$ stabilizes $I$ is a virtually abelian subgroup of $\Gamma$.*

We now verify that the action $\rho_T$ is non-elementary.

**Lemma 4.2.3** (Non-elementary action on the $\mathbb{R}$-tree)**.** *Let $\rho_T$ be a representation of $\Gamma$ into the isometry group of an $\mathbb{R}$-tree $(T, \operatorname{dist}_T)$ with the property that stabilizers of arcs are virtually abelian.*

*Then there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $\rho_T(\gamma_1)$ and $\rho_T(\gamma_2)$ are loxodromic isometries of $T$ along geodesics whose intersubsection is either empty or a compact arc.*

*Proof.* The action of $\rho_T(\Gamma)$ is irreducible in the sense that there is no global fixed point on the boundary at infinity (see [Pau89, Proposition 2.6]). The existence of the two required loxodromic elements now follows from [Chi01, Proposition 3.7]. □

### 4.2.2 Pivot technique

In this section we go over the pivoting technique developed by Gouëzel [Gou22a] and Baik–Choi–Kim [BCK23], adapting them to our setting. The first notion we need to introduce the pivot technique is that of *Schottky* sets. These sets of isometries encapsulate the low likeness of random walks to backtrack within negatively curved spaces. Precisely, a finite set of isometries $S$ acting on a Gromov hyperbolic space $X$, is said to be $(\eta, C, D)$-Schottky if the following three conditions are satisfied:

1. For all $x, y \in X$ the proportion of $a \in S$ such that $(x, ay)_o \leq C$ is at least $1 - \eta$.

2. For all $x, y \in X$ the proportion of $a \in S$ such that $(x, a^{-1}y)_o \leq C$ is at least $1 - \eta$.

3. For all $a \in S$ one has $\mathrm{dist}(o, ao) \geq D$.

The pivoting strategy has two main steps. The first one consists in showing that a sequence with a linear amount of isometries sampled from Schottky pairs has linear drift. That is, let $(r_i) \subset \Gamma$ be a fixed sequence of isometries, and let $(s_k) = (a_k b_k)$ be a sequence of random variables sampled such that each $a_k$ and $b_k$ is sampled uniformly and independently from the Schottky set $S$. Then, one shows that the sequence $(y_n) = (r_0 s_0 r_1 s_1 \dots r_n s_n r_{n+1})$ exhibits linear displacement with high probability, independently of the sequence $(r_i)$.

**Proposition 4.2.4** ([Gou22a, Proposition 4.1])**.** *There exists a uniform $\kappa > 0$ such that, for any $C, \delta > 0$, $\eta < \frac{1}{100}$, $D > 20C + 100\delta + 1$, $\delta$-Gromov hyperbolic space $X$ with isometry group $\Gamma$, $(\eta, C, D)$-Schottky set $S \subset \Gamma$, sequence $(r_n) \subset \Gamma$ and basepoint $o \in X$ we have*

$$\mathbb{P}(d(o, y_n o) < \kappa n) \leq e^{-\kappa n},$$

*for all $n \geq 0$, where $y_n = r_0 s_0 r_1 \dots r_n s_n r_{n+1}$ is the sequence defined above.*

The main idea for the proof is that, since the elements $r_i$ are fixed, the probability that each $s_n$ backtracks is small. Extra care has to be given to the fact that it is possible to backtrack several pivoting positions at once, and this is solved by showing that the probability of doing so is exponentially small on the backtracking.

In the second step one shows that if the support of the measure $\mu$ driving the random walk $(w_n)$ generates a Schottky set $S$, then the random walk $(w_n)$ contains in some sense the process $y_{\tau(n)}$, where $\tau(n)$ grows linearly with high probability. Crucially for our application, the growth of $\tau(n)$ depends solely on the measure $\mu$, and its combinatorial relation with $S$. While this latter property can be infered from Gouëzel's proof, it is not stated explicitly. Here we provide an explicit re-statement.

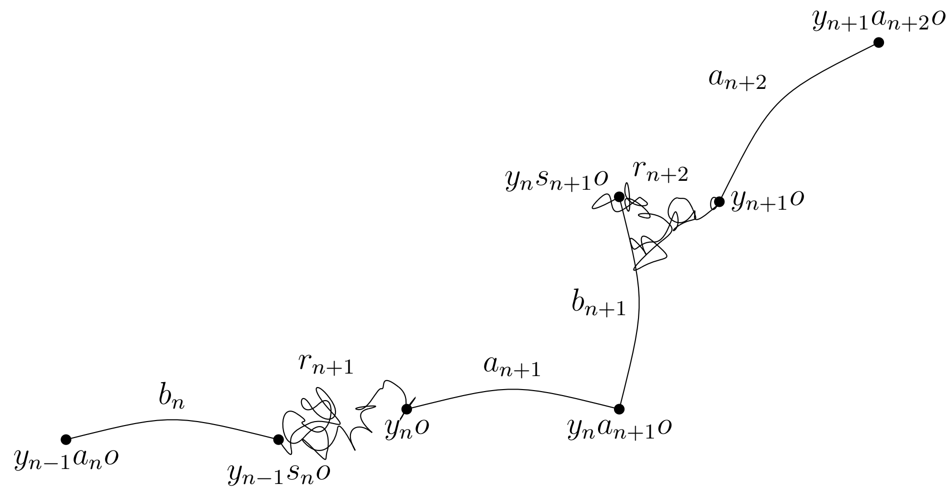



Figure 4.1: Sketch of the sequence $(y_n)$ in Proposition 4.2.4. The elements $r_k$ are fixed, while the elements $a_k$ and $b_k$ are sampled uniformly from the Schottky set.

**Theorem 4.2.5** (Explicit version of [Gou22a, Theorem 1.1])**.** *Let $\Gamma$ be a group, $S \subset \Gamma$ be a subset and $\mu$ be a measure on $\Gamma$ such that the semigroup generated by its support contains $S$. Then, there is $c, n_0 > 0$ such that, for any $C, \delta, \eta, D, X, o$ as in Proposition 4.2.4 for which $\Gamma$ acts on $X$ by isometries and $S$ is $(\eta, C, D)$-Schottky, the random walk $(w_n)$ driven by $\mu$ satisfies*

$$\mathbb{P}(d(o, w_n o) < cn) \leq e^{-cn}$$

*for all $n \geq n_0$.*

*Proof.* Let $M$ be big enough so the support of $\mu^M$ contains $S$. Since the set $S$ is finite, there is some $\alpha > 0$ such that $\mu^M(g) \geq \frac{\alpha}{|S|}$ for all $g \in S$. The key idea for this step is to set $N = 2M$ and write the measure $\mu^N$ as the sum $\alpha\mu_S^2 + (1-\alpha)\nu$, where $\mu^S$ is the uniform measure on $S$ and $\nu$ is the remaining measure. Then, the random walk $(\tilde{w}_n) = (w_{Nn})$ can be reconstructed by considering Bernoulli variables $\varepsilon_i$ such that $\mathbb{P}(\varepsilon_i = 1) = \alpha$, random variables $s_i$ sampled according to $\mu_S^2$ and random variables $g_i$ sampled according to $\nu$. Precisely, we let $(\tilde{w}_n) = \gamma_0\gamma_1 \dots \gamma_n$, where $\gamma_i = s_i$ if $\varepsilon_i = 1$ and $\gamma_i = g_i$ otherwise. Let $t_0 < t_1 < \dots t_k < \dots$ be the times for which $\varepsilon_{t_j} = 1$. Furthermore, let $\tau(n) < n$ be the largest index $j$ such that $t_{\tau(n)} < n$. Then, denoting $s'_i = s_{t_i}$ and $r_i = g_{t_{i-1}+1}g_{t_{i-1}+2}\dots g_{t_i-1}$, where for convenience we set $t_{-1} = -1$, and $\tilde{r}_{\tau(n)+1} = g_{\tau(n)+1}\dots g_n$ we have $(\tilde{w}_n) = (r_0 s_0 r_1 \dots r_{\tau(n)} s_{\tau(n)} \tilde{r}_{\tau(n)+1})$. The random variables $(s_i)$ are independent of $(r_i)$ and $\tau(n)$. Therefore, by Proposition 4.2.4 there is some uniform $\kappa > 0$ such that

$$\mathbb{P}(d(o, \tilde{w}_n o) < \kappa\tau(n)|(r_i), \tau(n)) \leq e^{-\kappa\tau(n)}.$$

That is, the above probability is smaller than $e^{-\kappa\tau(n)}$ for any conditioning on the sequence $(r_i)$. Hence, we can remove the conditioning to the sequence $r_i$, getting

$$\mathbb{P}\left(d(o, \tilde{w}_n o) < \kappa\tau(n)|\tau(n)\right) \leq e^{-\kappa\tau(n)}.$$

Finally, $\tau(n)$ counts the amount of successful Bernoulli trials with parameter $\alpha$ after $n$ attempts. That is, $\tau(n)$ follows a binomial distribution $B(n, \alpha)$. In particular,

$$\mathbb{P}\left(\tau(n) \leq \frac{\alpha}{2}n\right) \leq \exp\left(-\frac{\alpha^2}{2}n\right).$$

Therefore, splitting the event $\{d(o, w_n o) < \frac{2\kappa}{\alpha}n\}$ into the cases where $\tau(n) > \frac{\alpha}{2}n$ and $\tau(n) \leq \frac{\alpha}{2}n$ we get

$$\begin{aligned}\mathbb{P}\left(d(o, w_n o) \leq \frac{\kappa\alpha}{2}n\right) \leq \mathbb{P}\left(d(o, w_n o) \leq \kappa\tau(n) \cap \tau(n) > \frac{\alpha}{2}n\right) + \mathbb{P}\left(\tau(n) \leq \frac{\alpha}{2}n\right)\\ \leq e^{-\kappa\frac{\alpha^2}{2}n} + e^{-\frac{\alpha^2}{2}n}.\end{aligned}$$

There is some $c'$ and $n_0$ such that $e^{-\kappa\frac{\alpha^2}{2}n} + e^{-\frac{\alpha^2}{2}n} < e^{-c'n}$ for all $n > n_0$. Therefore, the theorem follows from setting $c = \max\left(\frac{\kappa\alpha}{2}, c'\right)$. □

The following is a sufficient condition for a set $S$ to be Schottky with certain parameters, which depends on checking conditions involving only a finite number of points. Since the notion of convergence we use in Definition 4.2.1 only gives controls for finitely many points at a time, this criterion will enable us to construct finite sets which are Schottky sets uniformly along a converging family of representations.

**Lemma 4.2.6** (Schottky set criterion)**.** *Let $(X, \mathrm{dist})$ be a $\delta$-hyperbolic metric space with a basepoint $o \in X$. Suppose $S$ is a finite symmetric set of isometries of $X$ such that $c_1 + 2\delta < c_2/2$ where $c_1 = \max_{g \neq h, g,h \in S}(go, ho)_o$, and $c_2 = \min_{g \in S} \mathrm{dist}(o, go)$.*

*Then $S$ is an $(\eta, C, D)$-Schottky set with $\eta = \frac{2}{\#S}$ and $C = c_1 + 3\delta$ and $D = c_2$.*

*Proof.* Let $\varepsilon \in (2\delta, c_2/2 - c_1)$ and for each $g \in S$ set

$$V(g) = \{x \in X : (x, go)_o \geq c_1 + \varepsilon\}.$$

*Claim 1: If $g \neq h$ then $V(g) \cap V(h) = \emptyset$.*

Indeed if $x \in V(g) \cap V(h)$ then one would have

$$c_1 + \varepsilon \leq \min\{(x, go)_o, (x, ho)_o\} \leq (go, ho)_o + \delta \leq c_1 + \delta,$$

contradicting the fact that $\delta < \varepsilon$.

*Claim 2: If $x \notin V(g^{-1})$ then $gx \in V(g)$.*

To see this observe that from the first condition one has

$$\frac{\operatorname{dist}(o, x) + \operatorname{dist}(o, g^{-1}o) - \operatorname{dist}(x, g^{-1}o)}{2} < c_1 + \varepsilon,$$

while if $gx \notin V(g)$ we would have

$$\frac{\operatorname{dist}(o, gx) + \operatorname{dist}(o, go) - \operatorname{dist}(gx, go)}{2} < c_1 + \varepsilon.$$

Taking the sum this would imply

$$c_2 \leq \operatorname{dist}(o, go) < 2c_1 + 2\varepsilon,$$

contradicting the fact that $\varepsilon < \frac{1}{2}c_2 - c_1$.

*Claim 3: If $x \in V(g)$ and $y \in V(h)$ for $g \neq h$ then $(x, y)_o \leq c_1 + 2\delta$.*

By hyperbolicity one has

$$\min((x, go)_o, (x, ho)_o) \leq (go, ho)_o + \delta \leq c_1 + \delta.$$

Since $(x, go)_o \geq c_1 + \varepsilon > c_1 + \delta$ this implies that $(x, ho)_o \leq c_1 + \delta$. From this we obtain

$$\min((y, ho)_o, (x, y)_o) \leq (x, ho)_o + \delta \leq c_1 + 2\delta,$$

but since $(y, ho)_o \geq c_1 + \varepsilon > c_1 + 2\delta$ this implies $(x, y)_o \leq c_1 + 2\delta$ as claimed.

*Claim 4: $S$ is $(\eta, C, D)$-Schottky for the constants in the statement.*

Let us check the first property in the definition of Schottky sets, as the second one follows by symmetry of $S$ and the third one comes from the definition of $c_2$. Given $x, y \in X$ let $a_1, a_2 \in S$ be distinct and such that $x \notin V(a_i^{-1})$ for $i = 1, 2$. By Claim 1 the $a_i$ are chosen among at least $\#S - 1$ elements of $S$. By Claim 2 one has $a_i x \in V(a_i)$ for $i = 1, 2$. By hyperbolicity and Claim 3 one has

$$\min((a_1 x, y)_o, (a_2 x, y)_o) \leq (a_1 x, a_2 x)_o + \delta \leq c_1 + 3\delta = C.$$

This implies that either $(a_1 x, y)_o \leq C$ or $(a_2 x, y)_o \leq C$. Hence the subset of $S$ consisting of elements with $(ax, y)_o > C$ can have at most two elements. $\square$

## 4.3 Bounds in terms of maximum curve length

Let $F$ be a fixed finite filling set of closed curves on $S$. Let $\varepsilon > 0$ and let $\mu_\varepsilon = (1-\varepsilon)\mu + \varepsilon\delta_e$ be a relaxation of $\mu$, where we introduce a slight probability of the random walk not moving at each step. For any $\rho \in \mathcal{T}(S)$ we have $\mathrm{Drift}_{\mu_\varepsilon}(\rho) = (1-\varepsilon)\,\mathrm{Drift}_\mu(\rho)$. Furthermore, since $\mu$ is admissible, there is some $k \in \mathbb{N}$ such that the curves associated to the group elements of $\mathrm{supp}(\mu_\varepsilon^k)$ contain $F$. Then, $\mathrm{Drift}_{\mu_\varepsilon^k}(\rho) = k(1-\varepsilon)\,\mathrm{Drift}_\mu(\rho)$. Since all the results in this chapter regarding the drift are true up to multiplicative constants we will assume that $\mathrm{supp}(\mu)$ already contains $F$. Denote $M_\rho^F = \max_{\gamma\in F}(\mathrm{Hyp}_\rho(\gamma))$. Note that

$$\tilde{M}_\rho^F = \min_{x\in\mathbb{H}} \max_{\gamma\in F} \mathrm{dist}(x, \rho(\gamma)x) \geq \max_{\gamma\in F} \mathrm{Hyp}\,\rho(\gamma) = M_\rho^F \tag{4.3.1}$$

We shall first prove the following well known result. The proof is similar to the one done by Minsky [Min93, Lemma 4.7]

**Lemma 4.3.1.** *Let $F$ be a finite filling set of curves in $S$ and let $o \in \mathcal{T}(S)$ be a fixed basepoint. Then, there is a constant $K$ such that, for any $\rho \in \mathcal{T}(S)$ and free curve $\gamma$ in $S$ we have,*

$$\mathrm{Hyp}_\rho(\gamma) \leq K M_\rho^F\, \mathrm{Hyp}_o(\gamma).$$

*Proof.* Let $D_\rho$ be the maximum diameter of the polygons formed by the distance minimizing configuration of the curves $F$, and $\partial D_\rho$ its perimeter. Since the curves of $F$ cut $S$ into polygons, the value of $\mathrm{Hyp}_\rho(\gamma)$ is bounded by the number of intersections of the curve $\gamma$

with $F$ multiplied by the maximum diameter of the polygons. That is,

$$\mathrm{Hyp}_\rho(\gamma) \leq D_\rho \sum_{\alpha \in F} i(\gamma, \alpha) \leq \partial D_\rho \sum_{\alpha \in F} i(\gamma, \alpha).$$

Under the metric defined by $o$, for any closed curve $\alpha$ we can isometrically embed an annulus around $\alpha$ of thickness $\delta_o(\alpha) > 0$. Then, for any other curve $\gamma$ we have $i(\alpha, \gamma)\delta_o(\alpha) \leq \mathrm{Hyp}_o(\gamma)$. Hence,

$$\mathrm{Hyp}_\rho(\gamma) \leq \partial D_\rho \sum_{\alpha \in F} \frac{1}{\delta(\alpha)} \mathrm{Hyp}_o(\gamma).$$

Finally, the maximum perimeter of the polygons is smaller than twice the sum of the lengths of all the curves in $F$, so, denoting by $|F|$ the cardinality of $F$,

$$\partial D_p \leq 2 \sum_{\alpha \in F} \mathrm{Hyp}_p(\alpha) \leq M_p^F 2|F|.$$

We get the result by setting $K = 2|F| \sum_{\alpha \in F} \frac{1}{\delta(\alpha)}$. ◻

Since the number of curves in $F$ is finite, $M_\rho^F$ is finite for all $\rho \in \mathcal{T}(S)$. Hence, for any other $\rho \in \mathcal{T}(S)$ we have

$$\sum_{g \in \Gamma} |g|_\rho \mu(g) \leq K M_\rho^F \sum_{g \in \Gamma} |g|_o \mu(g) < \infty.$$

Therefore, if $\mu$ has finite first moment with respect so some basepoint $o \in \mathcal{T}(S)$, it has finite first moment for all $\rho \in \mathcal{T}(S)$. That is, it makes sense to say that the measure $\mu$ has finite first moment if it has finite first moment with respect to at least one (and hence any) point in $\mathcal{T}(S)$.

Using Lemma 4.3.1 it is relatively straightforward to check that the value $M_\rho^F$ serves to give an upper bound.

**Lemma 4.3.2.** *There exists a constant $C < \infty$ such that* $\mathrm{Drift}([\rho]) \leq C M_\rho^F$ *for all* $[\rho] \in \mathcal{T}(S)$.

*Proof.* Fix some basepoint $o \in \mathcal{T}(S)$. By Lemma 4.3.1 there is some $K < \infty$ such that

$|Z_n|_\rho \leq KM_p|Z_n|_o$. Therefore,

$$\mathrm{Drift}_\mu(\rho) = \lim_{n\to\infty} \frac{|Z_n|_\rho}{n} \leq KM_p^F \lim_{n\to\infty} \frac{|Z_n|_o}{n} = KM_p^F\,\mathrm{Drift}_\mu(o).$$

□

The lower bound is significantly more involved. To have lighter notation, we will keep the action implicit and write $go_k$ instead of $\rho_k(g)o_k$. Since there is only one possible action for each basepoint $o_k$, this should not create confusion.

Let us fix a sequence of pointed $\delta$-hyperbolic spaces $(X_k, o_k)$ endowed with actions of a group $\Gamma$, and assume that $\rho_k$ converges to $\rho_\infty$ in the sense of Definition 4.2.1. Let also $\mu$ be probability measures on $\Gamma$ such that $\rho_{\infty *}\mu$ is non-elementary on $X_\infty$.

The following lemma is a classical application of a ping-pong argument.

**Lemma 4.3.3.** *Let $\eta > 0$. Then there exists $C > 0$ such that, for any $D > 0$, there exist $N$ and a finite symmetric set $S$ in $\Gamma$ in the support of $\mu_\infty^N$ such that $\#S \geq 2/\eta$ and*

$$\max_{g\neq h, g,h\in S} (go_\infty, ho_\infty)_{o_\infty} < C - 3\delta, \quad \min_{g\in S} \mathrm{dist}(o_\infty, go_\infty) > D. \tag{4.3.2}$$

*Proof.* This follows readily from the proof techniques of [BMSS23, Proposition A.2] or [Gou22b, Proposition 3.12]. □

Let $\eta > 0$. For suitable $C$ and $D$, we can consider a set $S$ as in Lemma 4.3.3. By definition of converging actions and since the number of relations is finite, for large $n$ the inequalities in (4.3.2) also hold for $\rho_k$. By Lemma 4.2.6, it follows that $\rho_k(S)$ is an $(\eta, C, D)$-Schottky set, uniformly for all large enough $k$. We can then use this Schottky set in cojuction with Proposition 4.2.4, to obtain quantitative estimates that are uniform in $k$. Going back on the refactorization we obtain a lower bound in terms of the rescaling factor $M_\rho$.

We can then prove the lower bound.

**Lemma 4.3.4.** *There exists a constant $c > 0$ such that* $\mathrm{Drift}([\rho]) \geq cM_\rho$ *for all* $[\rho] \in \mathcal{T}(M)$.

*Proof.* Suppose by contradiction that we may find a diverging sequence of representations $\rho_k$ for which $\mathrm{Drift}([\rho_k])/M_{\rho_k}$ tends to zero. By Lemma 4.2.2 we can take a subsequence such that the rescaled action converges to some $\rho_T$ on an $\mathbb{R}$-tree. Relabel that subsequence as $\rho_k$. Let $\eta < 1/100$ and let $C$ given by Lemma 4.3.3 for this value of $\eta$. Take $D = 20C + 100\delta + 1$.

By Lemma 4.2.3 the action of $\rho_T$ is non elementary, so by Lemma 4.3.3 and the discussion that follows it there is some $k_0$ and a symmetric set $S \subseteq \Gamma$ such that $\rho_k(S)$ is $(\eta, C, D)$ Schottky for all $k \geq k_0$. Then, by Theorem 4.2.5 there is some $c, n_0$ such that

$$\mathbb{P}\left(\frac{\operatorname{dist}_{\rho_k}(o_k, \rho_k(w_n)o_k)}{\tilde{M}_{\rho_k}} < cn\right) \leq e^{-cn}$$

for all $n \geq n_0$ and $k \geq k_0$. Therefore, for all $k \geq k_0$,

$$\lim_{n\to\infty} \frac{\operatorname{dist}_{\rho_k}(o_k, \rho_k(w_n)o_k)}{\tilde{M}_{\rho_k} n} \geq c$$

almost surely. Hence,

$$\operatorname{Drift}(\rho_k) = \tilde{M}_{\rho_k} \lim_{n\to\infty} \frac{\operatorname{dist}_{\rho_k}(o_k, \rho_k(w_n)o_k)}{\tilde{M}_{\rho_k} n} \geq \tilde{M}_{\rho_k} c \geq M_{\rho_k} c,$$

giving us a contradiction. $\square$

The proof of Theorem 4.1.2 is a result of combining Lemmas 4.3.2 and 4.3.4.

## 4.4 Linear bounds

We shall first prove the following lower bound:

**Proposition 4.4.1.** *For each $[\rho_0] \in \mathcal{T}(S)$ there exists a constant $c > 0$ such that*

$$\operatorname{Drift}([\rho]) \geq c \operatorname{dist}_{Teich}([\rho], [\rho_0])$$

*for all $[\rho] \in \mathcal{T}(M)$.*

In view of Lemma 4.3.4, to prove the lower linear bound it suffices to find a set of curves $F$ such that $M_\rho^F$ can be bound from below by a multiple of $\operatorname{dist}_{Teich}([\rho_0], [\rho])$. We fix $F \subset \Gamma$ to be a subset that is finite, symmetric and filling.

Let $\operatorname{Ext}_\rho(\gamma)$ denote the extremal length of the curve $\gamma$ under the conformal structure provided by $\rho$. As proven by Maskit [Mas85, Corollary 3], we have

$$\frac{1}{2} \operatorname{Hyp}_\rho(\gamma) e^{\operatorname{Hyp}_\rho(\gamma)/2} \geq \operatorname{Ext}_\rho(\gamma), \tag{4.4.1}$$

so it suffices to obtain a lower bound on extremal length. This will be obtained from the result of Walsh [Wal19, Lemma 3] explained in Section 2.2.3.

Recalling Walsh's result, for a unit area quadratic differential $q$ based at some basepoint $[\rho_0]$, denote $R(q;t)$ the point in Teichmüller space obtained after following a Teichmüller ray for time $t > 0$ in the direction provided by $q$. Let $V(q)$ (respectively $H(q)$) be the vertical (respectively horizontal) foliation of $q$, and let $G_j$ be the vertical components of $V(q)$. Then, Walsh proved the following inequality

$$e^{-2t}\operatorname{Ext}_{R(q;t)}(\gamma) \geq \sum_{G_j} \frac{i(G_j,\gamma)^2}{i(G_j,H(q))}, \tag{4.4.2}$$

Denoting $E_q(\gamma) = \sum_{G_j} \frac{i(G_j,\gamma)^2}{i(G_j,H(q))}$, we will use the fact that $F$ is filling to derive a uniform (over $q$) lower bound on $\max_{\gamma\in F} E_q(\gamma)$.

**Lemma 4.4.2.** *Given a basepoint $[\rho_0] \in \mathcal{T}(M)$ there is some $c > 0$ such that*

$$\inf_{q\in T^1([\rho_0])} \max_{\gamma\in F} E_q(\gamma) > c,$$

*where the infimum is taken over all unit area quadratic differentials at $[\rho_0]$.*

*Proof.* For any $q \in T^1([\rho_0])$, we have $i(G_j, H(q)) \leq i(G(q), H(q)) = \operatorname{Area}(q) = 1$. This implies $E_q(\gamma) \geq \sum_{G_j} i(V_j,\gamma)^2$.

Assume we have a sequence $q_n$ of unit area quadratic differentials at $[\rho_0]$ such that $\max_{\gamma\in F} E_{q_n}(\gamma)$ converges to 0 for all $j$. Since the space of unit area quadratic differentials at a basepoint is compact we can pass to a subsequence that converges to some $q$. Furthermore, since geometric intersection number is continuous we have $E_q(\gamma) = 0$ for all $\gamma \in F$. In particular, this implies $i(V(q),\gamma) = 0$ for all $\gamma \in F$. This is impossible because $F$ is a filling set. $\square$

We use the above lemma to get the following global lower bound on the maximal lengths over $F$.

**Lemma 4.4.3.** *Given a basepoint $[\rho_0] \in \mathcal{T}(M)$ there are some $c_1, c_2 > 0$ such that, for all $[\rho] \in T(M)$,*

$$\max_{\gamma\in F} \operatorname{Ext}_\rho(\gamma) \geq c_1 e^{2\operatorname{dist}_{Teich}([\rho_0],[\rho])}$$

*and hence*

$$M_\rho^F = \max_{\gamma\in F} \mathrm{Hyp}_\rho(\gamma) \geq c_2 \,\mathrm{dist}_{Teich}([\rho_0],[\rho])$$

*where* $\mathrm{dist}_{Teich}$ *denotes the Teichmüller distance.*

*Proof.* Let $q$ be such that $[\rho] = R(q; \mathrm{dist}_{Teich}([\rho_0],[\rho]))$. By Equation (4.4.2) we have

$$\max_{\gamma\in F} \mathrm{Ext}_\rho(\gamma) \geq e^{2\,\mathrm{dist}([\rho_0],[\rho])} \max_{\gamma\in F} E_q(\gamma),$$

and so by Lemma 4.4.2 we get the first inequality. By Equation (4.4.1) we have

$$\max_{\gamma\in F} \frac{1}{2}\mathrm{Hyp}_\rho(\gamma) + \log \max_{\gamma\in F} \frac{1}{2}\mathrm{Hyp}_\rho(\gamma) \geq 2\,\mathrm{dist}([\rho_0],[\rho]) + \log(c_1),$$

so the second inequality in the lemma is asymptotically satisfied for any $c_2$ slightly smaller than 2. Furthermore, given any bounded domain we can choose $c_2$ small enough so the inequality is satisfied. □

The proof of Proposition 4.4.1 is a corollary of Lemma 4.4.3 and Lemma 4.3.4. The proof of Theorem 4.1.1 is then a corollary of Proposition 4.4.1.

The main ingredient for the upper bound is the following result, established by Masur [Mas82b, End of the proof of Theorem 1.1], which finds limiting values for hyperbolic lengths along Jenkins–Strebel quadratic differentials.

**Theorem 4.4.4** (Masur)**.** *Let $q$ be a unit area Jenkins–Strebel quadratic differential and let $\alpha_1, \ldots, \alpha_k$ be its core curves. Then, for any sequence $(\rho_n) \subset \mathcal{T}(S)$ converging to $q$ in the visual compactification and any curve $\gamma$ in $S$ we have*

$$\lim_{n\to\infty} \frac{\mathrm{Hyp}_{\rho_n}(\gamma)}{4 d_{\mathrm{Teich}}(o,\rho_n)} = \sum_{i=1}^{k} i(\alpha_i,\gamma).$$

*Proof of Theorem 4.1.3.* By Proposition 4.4.1 we only have to prove the upper bound. Let $\alpha_1, \ldots, \alpha_k$ be the core curves of the vertical foliation of $q$ and let $\gamma_1, \ldots, \gamma_l$ be the curves within the filling set $F$. Let $\delta > 0$. Then, by Theorem 4.4.4 for each $\gamma_j$ we have a $t_j$ such that, for all $t > t_j$, we have

$$\mathrm{Hyp}_{R(q;t)}(\gamma_j) < 4\left(\sum_{i=1}^{k} i(\alpha_i,\gamma_j) + \delta\right) t.$$

Hence we can take $C(q,j)$ big enough so $\mathrm{Hyp}_{R(q;t)}(\gamma_j) < C(q,j)t$ for all $t \geq 1$. The theorem follows by setting $C(q) = K \max_{j\leq k} C(q,j)$, where $K$ is the constant given by Theorem 4.1.2.

□

## 4.5 Exponential bounds

The goal of this section is to prove that the standard behaviour of the drift is exponential growth with respect to the Teichmüller distance. We begin by observing that as a direct result of Wolpert's Lemma, the growth can not be higher than exponential.

**Proposition 4.5.1.** *Given a basepoint $o \in \mathcal{T}(S)$, there exists some constant $C(o)$ such that* $\mathrm{Drift}_\mu(\rho) \leq \mathrm{Drift}_\mu(o)e^{2d_{\mathrm{Teich}}(\rho,o)}$.

*Proof.* As proven by Wolpert [Wol79, Lema 3.1] for any two points $o, \rho \in \mathcal{T}(S)$ and loop $\gamma$ we have

$$\mathrm{Hyp}_\rho(\gamma) \leq e^{2d_{\mathrm{Teich}}(o,\rho)}\,\mathrm{Hyp}_o(\gamma).$$

Therefore,

$$\mathrm{Drift}_\mu(\rho) = \lim_{n\to\infty} \frac{\mathrm{Hyp}_\rho(Z_n)}{n} \leq e^{2d_{\mathrm{Teich}}(o,\rho)} \lim_{n\to\infty} \frac{\mathrm{Hyp}_o(Z_n)}{n} = e^{2d_{\mathrm{Teich}}(o,\rho)}\,\mathrm{Drift}_\mu(o).$$

□

The following result by Choi–Rafi [CR07, Theorem B] allows us to improve the previous upper bound, as well as get a lower bound for the growth of the drift for points in the thick part of Teichmüller space. Recall that $\mathcal{T}_\varepsilon(S)$ denotes the $\varepsilon$-thick part of Teichmüller space.

**Theorem 4.5.2** (Choi–Rafi)**.** *Fix $\varepsilon > 0$ and $o \in \mathcal{T}_\varepsilon(S)$. There is a finite filling set of closed curves $G$ and a constant $D > 0$ such that, for any $\rho \in T_\varepsilon(S)$ we have*

$$\left| d_{\mathrm{Teich}}(\rho,o) - \log\left(\max_{\alpha\in G} \frac{\mathrm{Hyp}_\rho(\alpha)}{\mathrm{Hyp}_o(\alpha)}\right)\right| \leq D.$$

**Proposition 4.5.3.** *Let $\varepsilon > 0$ and let $o \in \mathcal{T}_\varepsilon(S)$. Then, there exists constants $c, C > 0$ such that for any $\rho \in \mathcal{T}_\varepsilon(S)$ we have*

$$ce^{d_{\mathrm{Teich}}(\rho,o)} \leq \mathrm{Drift}_\mu(\rho) \leq Ce^{d_{\mathrm{Teich}}(\rho,o)}.$$

*Proof.* Let $G$ be the filling set of curves from Theorem 4.5.2. For any curve $\gamma \in G$ we have

$$\mathrm{Hyp}_\rho(\gamma) = \mathrm{Hyp}_o(\gamma)\frac{\mathrm{Hyp}_\rho(\gamma)}{\mathrm{Hyp}_o(\gamma)} \leq \mathrm{Hyp}_o(\gamma)\max_{\alpha\in G}\frac{\mathrm{Hyp}_\rho(\alpha)}{\mathrm{Hyp}_o(\alpha)}.$$

Then, taking the maximum among all $\gamma \in G$ in the previous inequality we have, by Theorem 4.1.2, some $K > 0$ such that

$$\mathrm{Drift}_\mu(\rho) \leq K\max_{\alpha\in G}\mathrm{Hyp}_\rho(\alpha) \leq K\max_{\alpha\in G}\mathrm{Hyp}_o(\alpha)e^D e^{d_{\mathrm{Teich}}(\rho,o)},$$

where the last inequality follows from applying Theorem 4.5.2. On the other hand,

$$\frac{\max_{\alpha\in G}\mathrm{Hyp}_\rho(\alpha)}{\min_{\alpha\in G}\mathrm{Hyp}_o(\alpha)} \geq \max_{\alpha\in G}\frac{\mathrm{Hyp}_\rho(\alpha)}{\mathrm{Hyp}_o(\alpha)},$$

so similarly we have

$$\mathrm{Drift}_\mu(\rho) \geq \frac{1}{K}\max_{\alpha\in G}\mathrm{Hyp}_\rho(\alpha) \geq \frac{1}{K}\min_{\alpha\in G}\mathrm{Hyp}_o(\alpha)e^{-D}e^{d_{\mathrm{Teich}}(\rho,o)}.$$

Hence, the result follows from setting the values $c = \frac{1}{K}\min_{\alpha\in G}(\mathrm{Hyp}_o(\alpha))e^{-D}$ and $C = K\max_{\alpha\in G}(\mathrm{Hyp}_o(\alpha))e^D$. □

By Mumford's compactness the preimage of every bounded subset of moduli space is contained in some $\varepsilon$-thick part of Teichmüller space for $\varepsilon$ small enough. Hence we have the following result.

**Corollary 4.5.4.** *Let $\gamma : [0,\infty) \to \mathcal{T}(S)$ be a Teichmüller ray such that its image is bounded in moduli space. Then, there is $c, C > 0$ such that*

$$ce^t \leq \mathrm{Drift}_\mu(\gamma(t)) \leq Ce^t.$$

To prove that for almost all directions the growth is exponential we first observe that the growth of the drift is, up to a constant, not lost for big enough times. The main ingredient we shall use in the proof is the following.

**Theorem 4.5.5** (Lenzhen–Rafi [LR11, Theorem A])**.** *There exists a constant $K > 0$ such that for every closed curve $\gamma$, any Teichmüller geodesic $G$ and points $x, y, z \in \mathcal{T}(S)$ appearing* *in that order along $G$, we have*

$$\mathrm{Hyp}_y(\gamma) \leq K \max(\mathrm{Hyp}_x(\gamma), \mathrm{Hyp}_z(\gamma)).$$

By Theorem 4.1.2 Lenzhen–Rafi's result translates directly to the drift. That is, we have Corollary 4.1.6.

*Proof of Corollary 4.1.6.* By Theorems 4.1.2 and 4.5.5 there are constants $C, K > 0$ such that

$$\mathrm{Drift}_\mu(y) \leq C \max_{\gamma \in F} \mathrm{Hyp}_y(\gamma) \leq KC \max_{\gamma \in F} (\max(\mathrm{Hyp}_x(\gamma), \mathrm{Hyp}_z(\gamma))$$

Switching the order of the maximums we have

$$\mathrm{Drift}_\mu(y) \leq KC \max\left(\max_{\gamma \in F} \mathrm{Hyp}_x(\gamma), \max_{\gamma \in F} \mathrm{Hyp}_z(\gamma)\right) \leq KC^2 \max(\mathrm{Drift}_\mu(x), \mathrm{Drift}_\mu(z)).$$

□

Furthermore, the drift is a proper function, so along any Teichmüller ray $R(q)$, the value of $\mathrm{Drift}_\mu(R(q;t))$ is eventually larger than $\mathrm{Drift}_\mu(R(q;0)$. Hence, we have the following.

**Lemma 4.5.6** (Drift is quasi increasing)**.** *There is some constant $K > 0$ such that, for any basepoint $o \in \mathcal{T}(S)$ there is some time $t_o$ for which*

$$\mathrm{Drift}_\mu(R(q;s)) \leq K \,\mathrm{Drift}_\mu(R(q;t))$$

*for any $t > t_o$, $t > s > 0$ and $q \in Q^1(o)$.*

*Proof.* Let $t_o$ be big enough so $\mathrm{Drift}_\mu(\rho) > \mathrm{Drift}_\mu(o)$ for any $\rho$ such that $d_{\mathrm{Teich}}(\rho, o) > t_o$. By Proposition 4.4.1 such a $t_o$ exists. Then, for any quadratic differential $q$ based at $o$, $t > t_o$ and $t > s > 0$ we have, by Corollary 4.1.6,

$$\mathrm{Drift}_\mu(R(q;s)) \leq K \max\left(\mathrm{Drift}_\mu(R(q;0), \mathrm{Drift}_\mu(R(q;t))\right)) = K\,\mathrm{Drift}_\mu(R(q;t))$$

□

Lemma 4.5.6, combined with Proposition 4.5.3, gives upper and lower bound on the growth of the drift along a geodesic provided said geodesic does not spend too much continued time outside the thick part. For a given geodesic $R(q)$, $\varepsilon > 0$ and $t > 0$, we aim to find some control on the largest $s_q^{bot}(t) \leq t$ and lowest $s_q^{top}(t) \geq t$ such that $R(q; s_q^{bot}(t)), R(q; s_q^{top}(t)) \in \mathcal{T}_\varepsilon(S)$. Given a geodesic ray $\gamma : \mathbb{R}_+ \to \mathcal{T}(S)$, we defined the proportion of the amount of time spent in the thick part up to time $t$ as

$$\mathrm{Thick}_\varepsilon^{\%}(\gamma, t) := \frac{|\{0 \leq s \leq t \,:\, \gamma(s) \in \mathcal{T}_\varepsilon(S)\}|}{t}.$$

**Theorem 4.5.7.** *Let $o \in \mathcal{T}(S)$ be a basepoint in Teichmüller space. Furthermore, let $\sigma$ be a measure on $Q^1(o)$ such that for all $0 < \xi < 1$ there is some $\varepsilon(\xi) > 0$ such that for $\sigma$-almost every $q$ there is $t_q^\xi < \infty$ such that*

$$\mathrm{Thick}_{\varepsilon(\xi)}^{\%}(R(q), t) \geq \xi$$

*for all $t \geq t_q^\xi$.*

*Fix then $0 < \theta < 1$. For $\sigma$-almost all directions $q \in Q^1(o)$ there is $t(q, \theta) < \infty$ such that*

$$e^{\theta t} < \mathrm{Drift}_\mu(R(q; t)) < e^{\frac{1}{\theta} t}$$

*for all $t > t(q, \theta)$.*

*Proof.* Given $0 < \theta < 1$, let $\xi = \frac{1+\theta}{2}$. For a given $q \in Q^1(o)$ and $t > 0$ let $s_q^{top}(t) \geq t$ be the smallest time larger than $t$ such that $R(q; s_q^{top}(t)) \in \mathcal{T}_{\varepsilon(\xi)}(S)$. The time spent outside $\mathcal{T}_{\varepsilon(\xi)}(S)$ directly after $t$ is $s_q^{top}(t) - t$. Hence, $\frac{s_q^{top}(t) - (s_q^{top}(t) - t)}{s_q^{top}(t)} > \xi$. Therefore, $s_q^{top}(t) < \frac{1}{\xi} t$. Hence, By Lemma 4.5.6 we have $K, t_o > 0$ such that, for all $t > t_o$,

$$\mathrm{Drift}_\mu(R(q; t)) \leq K \,\mathrm{Drift}_\mu(R(q; s_q^{top}(t))).$$

Since $R(q; s_q^{top}(t))$ is in the $\varepsilon$-thick part of Teichmüller space we have, by Proposition 4.5.3,

$$\mathrm{Drift}_\mu(R(q; t)) \leq K \,\mathrm{Drift}_\mu(R(q; s_q^{top}(t)) \leq CKe^{s_q^{top}(t)} \leq CKe^{\frac{1}{\xi} t}.$$

Similarly, denoting $s_q^{bot}(t)$ the largest time smaller than $t$ such that $R(q; s_q^{top}(t)) \in \mathcal{T}_{\varepsilon(\xi)}(S)$ we get $s_q^{bot}(t) > \xi t$. Following the same reasoning we get

$$\mathrm{Drift}_\mu(R(q; t)) \geq \frac{1}{K} \,\mathrm{Drift}_\mu(R(q; s_q^{bot}(t)) \geq \frac{c}{K} e^{s_q^{bot}(t)} \geq \frac{c}{K} e^{\xi t}.$$

Since $\xi < \theta < 1$ there is some $t_\theta$ such that $e^{\theta t} > \frac{c}{K} e^{\xi t}$ and $e^{\frac{1}{\theta} t} < CK e^{\frac{1}{\xi} t}$ for all $t \geq t_\theta$. Hence, the theorem follows from setting $t(q, \theta) = \max(t_q, t_\theta, t_o)$. □

Recall that it follows from Downdall–Duchin–Masur [DDM14, Proposition 5.5] that the hypothesis of Theorem 4.5.7 is satisfied by a wide variety of Lebesgue-class measures, giving us a proof of Corollary 4.1.4. Furthermore, as we have shown in Proposition 3.3.3, the property is also satisfied for harmonic measures generated by non-elementary measures on the mapping class group with finite first moment.

The previous Theorem can be used to get the following limiting results.

**Corollary 4.5.8.** *Let $\sigma$ be a measure on $Q^1(o)$ satisfying the hypothesis of Theorem 4.5.7. Then, for any $\varepsilon > 0$ we have, for $\sigma$-almost every $q \in Q^1(o)$,*

$$\liminf_{t\to\infty} \frac{\mathrm{Drift}_\mu(R(q;t))}{e^{(1-\varepsilon)t}} = \infty$$

*and*

$$\limsup_{t\to\infty} \frac{\mathrm{Drift}_\mu(R(q;t))}{e^{(1+\varepsilon)t}} = 0.$$

*Furthermore, there is $\sigma$-almost surely some $0 < c(q) < \infty$ such that*

$$\liminf_{t\to\infty} \frac{\mathrm{Drift}_\mu(R(q;t))}{e^{t}} \leq c(q) \leq \limsup_{t\to\infty} \frac{\mathrm{Drift}_\mu(R(q;t))}{e^{t}}.$$

*Proof.* Let $\theta = 1 - \varepsilon$. Then, by Corollary 4.1.4 there is some $c > 0$ such that $\sigma$-almost every $q \in Q^1(o)$ we have

$$\mathrm{Drift}_\mu(R(q;t)) \geq e^{(1-\varepsilon)t}$$

for all $t$ big enough. Hence, the first relation follows. Similarly for $\theta = \frac{1}{1+\varepsilon}$ we get the second relation. The last relation follows from taking a diverging sequence of times $(t_n)$ such that $R(q;t_n) \in \mathcal{T}_\varepsilon(S)$ and applying Proposition 4.5.3. □

## 4.6 Variable growth

In this section we prove that there is some geodesic along which we have variable growth. The basic idea of the proof is alternating Proposition 4.5.3 and Theorem 4.1.3, using the fact that both results apply to dense sets of directions.

*Proof of Corollary 4.1.5.* Fix $\delta, \varepsilon > 0$ such that the basepoint $o \in \mathcal{T}(S)$ is in the $\varepsilon$-thick part of Teichmüller space. We shall build inductively a sequence of quadratic differentials $(j_k)$, as well as sequences of times $(s_k)$, $(t_k)$ such that, $\mathrm{Drift}_\mu(R(j_k; s_i)) < \left(f(s_i) - \delta 2^{-2(k-i)}\right) s_i$ for each $i \leq k$ and each $R(j_k; t_i)$ is at most at distance $\delta\left(2 - 2^{-2(k-i)}\right)$ from $\mathcal{T}_\varepsilon$ for each $i \leq k$. The theorem will follow by taking an accumulation point of such sequence.

For $k = 0$ let $j_0 \in Q^1(o)$ be a Jenkins-Strebel quadratic differential. By Theorem 4.1.3 there are constants $C(j_0), t(j_0)$ such that $\mathrm{Drift}_\mu(R(j_0; t)) < C(j_0)t$ for all $t > t(j_0)$. Let $s_0$ be big enough such that $f(s_0) > C(j_0) + \delta$ and $s_0 > t(j_0)$. Finally, let $t_0 = 0$.

Assume then we have the sequence up to $k$. The set of recurrent directions to $\mathcal{T}_\varepsilon$ is dense, so we can take a sequence of quadratic differentials $(q^n) \subset Q^1(o)$ spawning recurrent geodesics and converging to $j_k$. Since $q^n \to j_k$, the geodesics $R(q^n; \cdot)$ converge to the geodesic $R(j_k; \cdot)$ pointwise. The drift is a continuous function with respect to Teichmüller space, so $\mathrm{Drift}_\mu(R(q^n; s_i)) \to \mathrm{Drift}_\mu(R(j_k; s_i)) \leq (f(s_i) - \delta 2^{-2(k-i)}) s_i$ for each $i \leq k$. Let $q$ be the first element of the sequence $(q^n)$ such that

$$\mathrm{Drift}_\mu(R(q; s_i)) < (f(s_i) - \delta 2^{-(2(k-i)+1)}) s_i$$

and

$$d_{\mathrm{Teich}}(R(q; t_i), R(j_k; t_i)) < \delta 2^{-(2(k-i)+1)}$$

for all $i \leq k$. The geodesic $R(q)$ is recurrent, so we can fix $t_{k+1} > s_k + 1$ such that $R(q; t_{k+1}) \in \mathcal{T}_\varepsilon$.

The set of Jenkins–Strebel directions is dense, so we can take a sequence $(j^n)$ converging to $q$. As before, the convergence within the sequences is pointwise, so $\mathrm{Drift}_\mu(R(j^n; s_i)) \to \mathrm{Drift}_\mu(R(q; s_i)) \leq (f(s_i) - \delta 2^{-(2(k-i)+1)}) s_i$ for each $i \leq k$. Let $j_{k+1}$ be the first element of the sequence $(j^n)$ such that

$$\mathrm{Drift}_\mu(R(j_{k+1}; s_i)) \leq (f(s_i) - \delta 2^{-2(k+1-i)}) s_i$$

and

$$d_{\mathrm{Teich}}(R(j_{k+1}; t_i), R(q; t_i)) < \delta 2^{-2(k+1-i)}$$

for all $i \leq k + 1$. As before, there is some $C(j_{k+1})$ such that $\mathrm{Drift}_\mu(R(j_{k+1}; t)) < C(j_{k+1})t$, so let $s_{k+1}$ be the first time larger than $t_{k+1}$ such that $f(s_{k+1}) > C(s_{k+1}) + \delta$. Furthermore, for $i \leq k$

$$d_{\text{Teich}}(R(j_{k+1};t_i),\mathcal{T}_\varepsilon) \leq d_{\text{Teich}}(R(j_{k+1};t_i),R(q;t_i))+d_{\text{Teich}}(R(q;t_i),R(j_k;t_i))+d_{\text{Teich}}(R(j_k;t_i),\mathcal{T}_\varepsilon)$$
$$< \delta\left(2^{-(2(k-i)+1)}+2^{-2(k+1-i)}+2-2^{-(2(k-i))}\right) = \delta\left(2-2^{-2(k+1-i)}\right)$$

and for $i = k+1$ we have, since $R(q_{k+1};t_{k+1}) \in \mathcal{T}_\varepsilon$,

$$d_{\text{Teich}}(R(j_{k+1};t_{k+1}),\mathcal{T}_\varepsilon) \leq d_{\text{Teich}}(R(j_{k+1};t_{k+1}),R(q_{k+1};t_{k+1})) < \delta.$$

Hence, we have completed the induction step.

Let $q_f$ be an accumulation point of the sequence $(j_k)$. There is then a subsequence, relabeled $(j_k)$ converging to $q_f$. By pointwise convergence, $\text{Drift}_\mu(R(j_k;s_i)) \to \text{Drift}_\mu(R(q_f;s_i)$, so $\text{Drift}_\mu(R(q_f;s_i) < f(s_i)s_i$ for each $i$. Furthermore, $R(j_k;t_i) \to R(q_f;t_i)$, so for each $i$ the points $R(q_f;t_i)$ are at most at distance $2\delta$ from the $\mathcal{T}_\varepsilon$. Therefore, there is some $\varepsilon'$ such that $R(q_f;t_i) \in T_{\varepsilon'}$ for all $i$. Hence, by Proposition 4.5.3 there is some $c > 0$ such that $\text{Drift}_\mu(R(q_f;t_i)) > ce^{t_i}$. Furthermore, $t_{k+1} > s_k+1 > t_k+1$, so the sequence $(t_k)$ diverges to infinity, and so does $(s_k)$. Finally, the geodesic $R(q_f;\cdot)$ is recurrent, so by Masur's criterion [Mas92, Theorem 1.1] the vertical foliation of $q_f$ is uniquely ergodic. □

## 4.7 Singularity conjecture and open questions

In this last subsection we return to the singularity conjecture and dimension drop of stationary measures. Recall from the introduction that by the results of [Tan19], the stationary measure $\nu_\rho$ is exact dimensional and its dimension is given by

$$\dim(\nu_\rho) = \frac{h}{\text{Drift}(\rho)}, \tag{4.7.1}$$

where $h = h(\mu)$ is the asymptotic (or Avez) entropy defined by

$$h = \lim_{n\to+\infty} \frac{1}{n} H(Z_n),$$

and $H(Z) = -\sum_{g\in\text{supp}(Z)} \mathbb{P}(Z=g)\log(\mathbb{P}(Z=g))$ denotes the Shannon entropy of the random variable $Z$. Note that $h$ does not depend on the representation $\rho$.

Recall the singularity conjecture from the introduction:

**Conjecture** (Singularity conjecture)**.** *If $\mu$ is admissible and has finite support then there exists $\delta < 1$ such that*

$$\dim(\nu_\rho) \leq \delta$$

*for all $[\rho] \in \mathcal{T}(M)$.*

Since the visual boundary is one-dimensional, equation (4.7.1) implies that $h \leq \mathrm{Drift}(\rho)$ for all $\rho$. The singularity conjecture then amounts to this inequality being strict on all of $\mathcal{T}(M)$.

Let us record some basic properties of Drift.

**Proposition 4.7.1.** *The function* $\mathrm{Drift} : \mathcal{T}(S) \to (0, +\infty)$ *is continuous.*

*Proof.* This follows from Furstenberg formula [KL11, Theorem 18] for speed and convergence of the stationary measures. □

One basic result from $\mathrm{Drift}(p)$ being continuous and proper is as follows

**Corollary 4.7.2.** *The functions* $\mathrm{Drift} : \mathcal{T}(S) \to [h, +\infty)$ *and* $\dim(\nu) : \mathcal{T}(S) \to (0, 1]$ *attain their minimum and maximum respectively.*

It is natural to ask then the following question

**Question 4.7.3.** *Does* $\dim(\nu_\rho)$ *attain its maximum at a unique point in $\mathcal{T}(S)$? Equivalently is* Drift *minimized at a unique point?*

When the maximal dimension is 1 and $\mu$ is symmetric (i.e., $\mu(g) = \mu(g^{-1})$ for all $g$) the answer to the previous question is affirmative. That is, we have the following.

**Proposition 4.7.4.** *If $\mu$ is admissible symmetric and has finite first moment, then there exists at most one point $[\rho] \in \mathcal{T}(S)$ such that* $\dim(\nu_\rho) = 1$.

*Proof.* As shown by Blachère–Haïssinsky–Mathieu [BHM11, Theorem 1.5], if $\mu$ is symmetric then whenever $\dim(\nu_\rho) = 1$ the measure $\nu_\rho$ is absolutely continuous with respect to the Lebesgue measure on the boundary.

Suppose $\dim(\nu_{\rho_1}) = \dim(\nu_{\rho_2}) = 1$. There exists a quasi-conformal homeomorphism $\varphi : \mathbb{H} \to \mathbb{H}$ such that $\varphi(o) = o$ and $\varphi \circ \rho_1(\gamma) = \rho_2(\gamma) \circ \varphi$ for all $\gamma \in \Gamma$. The quasi-conformal map $\varphi$ extends continuously to the visual boundary in a unique way. Denoting this extension by $\varphi$ as well we have $\varphi_* \nu_{\rho_1} = \nu_{\rho_2}$.

This implies that the restriction of $\varphi$ to the visual boundary is absolutely continuous. However this can only happen if $[\rho_1] = [\rho_2]$, as shown by Agard [Aga85]. □

On the other hand, there are some interesting questions pending regarding the finer variation of the growth. Theorem 4.5.7 shows that along a typical geodesic the drift grows exponentially. However, it does not determine precisely the fluctuations within such exponential growth. Furthermore, Corollary 4.1.5 shows that the behaviour of the drift along a geodesic can vary wildly, so it is natural to ask whether there is some variation within a typical geodesic.

**Question 4.7.5.** *Let $\nu$ be a measure on $Q^1(o)$ satisfying the hypothesis of Theorem 4.5.7. Do we have*

$$0 < \liminf_{t\to\infty} \frac{\mathrm{Drift}_\mu(R(q;t))}{e^t} = \limsup_{t\to\infty} \frac{\mathrm{Drift}_\mu(R(q;t))}{e^t} < \infty$$

*$\nu$-almost surely?*

Note that by Corollary 4.5.8 we have $\limsup_{t\to\infty} \frac{\mathrm{Drift}_\mu(R(q;t))}{e^t} > 0$ and $\liminf_{t\to\infty} \frac{\mathrm{Drift}_\mu(R(q;t))}{e^t} < \infty$. In the proof of Theorem 4.5.7 we have bounded the quotient between the drift and the exponential $e^t$ by a function depending on the maximal continuous time spent in the thin part of Teichmüller space up to time $t$. The growth of these maximal departures may vary differently depending on the measure. On the one hand, following work of Gadre [Gad17, Lemma 5.5] it is reasonable to conjecture that the maximal departure grows slightly faster than $\log(t)$ for the Lebesgue measure. On the other hand, because of exponential decay of subsurface projections for harmonic measures, it may be expected (though this is still unproved) that the largest continuous time spent in the thin part for a harmonic typical Teichmüller geodesic is of the order of $\log(\log(t))$. Therefore, the answer to the previous question might be different for the Lebesgue and harmonic measures.

# Chapter 5

# The horofunction compactification of proper, uniquely geodesic, straight metric spaces

## 5.1 Introduction

The horofunction compactification of a metric space is defined in terms of the metric, so its properties are well aligned for studying the metric properties of the space. For example, all geodesic rays converge to points and isometries of the space can be extended to homeomorphisms of the compactification. This compactification was first introduced by Gromov [Gro81] as a natural, general compactification, based on previous ideas of Busemann. The horofunction compactification has since found several applications, such as obtaining asymptotic properties of random walks on weakly hyperbolic spaces by Maher–Tiozzo [MT18], determining the isometry group of some Hilbert geometries by Lemmens–Walsh [LW11] and obtaining properties of quantum metric spaces by Rieffel [Rie02]. The compactification is obtained by embedding the metric space $X$ into the space $C(X)$ of continuous functions on $X$ via the map $h : X \hookrightarrow C(X)$ defined by

$$h(p)(\cdot) = d(p, \cdot) - d(p, b),$$

where $b \in X$ is an arbitrarily chosen basepoint. As explained, for example, by Walsh [Wal14a, Section 2], if the space $X$ is proper then $h$ is and embedding, the closure of $h(X)$

is compact and the *horofunction compactification* of $X$ is defined as the pair $(h, \overline{h(X)})$. By considering two functions equivalent if they differ by a constant one can show that the compactification does not depend on the basepoint $b$. While this compactification has been rather useful, it is sometimes hard to visualize, and there are not that many examples where the horofunction boundary is explicitly known. Some cases where the horofunction compactification is understood include Hadamard manifolds and some of their quotients, by Dal'bo–Peigné–Sambusetti [DPS12], as well as the Heisenberg group with the Carnot–Carathéodory metric, by Klein–Nicas [KN10], and Hilbert geometries, by Walsh [Wal14b].

On the other hand, for a proper, uniquely geodesic, straight metric space $X$ (see Section 2.1.1 for definitions) the visual compactification based at some point $b \in X$ is defined by pasting the set of geodesic rays exiting $b$, denoted $D_b$, to the space $X$ in such a way that a sequence $(x_n) \subset X$ converges to some ray $\gamma \in D_b$ if the distance $d(b, x_n)$ goes to infinity as $n \to \infty$, and the geodesic ray between $b$ and $x_n$ converges uniformly on compacts to $\gamma$. See Section 2.1.3 for details on the topology of $X \cup D_b$. This compactification may depend on the basepoint $b$, which restricts its usefulness. It can even happen that isometries of $X$ that move the basepoint can not be extended continuously to the compactification, as Kerckhoff showed for Teichmüller spaces [Ker80]. However, the visual compactification usually has a simple geometric interpretation. For example, for a Hadamard manifold, as well as for a Teichmüller space with the Teichmüller metric, this compactification is homeomorphic to a closed ball of the same dimension as the space, where the boundary of that ball is the space of geodesic rays based at $b$. In the context of Teichmüller spaces with the Teichmüller metric, the visual compactification is often called the Teichmüller compactification.

The work in this chapter is aimed towards an application to Teichmüller spaces with the Teichmüller metric. However, to make the work as general as possible we begin our analysis by using some metric properties of the Teichmüller metric. The relationship between the horofunction compactification and the visual compactification is established by observing that, for such a metric space, a sequence converging to a point in the horofunction compactification also converges in the visual compactification. This allows us to build a continuous map $\Pi$ from the horofunction compactification $\overline{h(X)}$ to the visual compactification $X \cup D_b$, showing that the former is finer than the latter.

Given a geodesic ray $\gamma$, the path $\gamma(t)$ converges, as $t \to \infty$, to the *Busemann point* associated to $\gamma$ in the horofunction compactification, which we denote $B_\gamma$. As the map $\Pi$ is defined in terms of sequences it follows that, if $\gamma$ starts at the basepoint $b$, then $\Pi(B_\gamma) = \gamma$.

The existence of the map $\Pi$ shows a strong relation between the horofunction and the visual compactification, which we state in the following result.

**Theorem 5.1.1.** *Let $(X, d)$ be a proper, uniquely geodesic, straight metric space. For any basepoint $b \in X$, there is a continuous surjection $\Pi$ from the horofunction compactification to the visual compactification based at $b$ such that $\Pi(B_\gamma) = \gamma$ for every ray $\gamma$ starting at $b$ and $\Pi(h(p)) = p$ for every $p \in X$.*

*In particular, the horofunction compactification of $X$ is finer than the visual compactification of $X$ based at any point.*

The construction of the map $\Pi$ is done in Section 5.2.1, and the previous thereom can be summed up as a combination of Propositions 5.2.5 and 5.2.6 in said section. Most of the subsequent results in the chapter, as well as the next one, follow as applications of this theorem.

It is not the first time that a map such as $\Pi$ appears in the literature. Similar maps have been found for $\delta$-hyperbolic spaces by Webster–Winchester [WW05]. Walsh defined such a map for Hilbert geometries [Wal14b], which satisfy the hypothesis of the theorem whenever there are no coplanar noncollinear segments in the boundary of the convex set, as shown by de la Harpe [dlH93, Proposition 2]. Furthermore, in the context of Teichmüller spaces without boundary, the map $\Pi$ coincides with the one defined by Liu–Shi [LS22, Definition 3.3].

The map $\Pi$ does not induce a fiber bundle, as its fibers $\Pi^{-1}(\gamma)$ vary from points to higher dimensional sets in some examples (see Theorem 6.4.10). Still, Theorem 5.1.1 characterizes the horoboundary as the disjoint union of all the fibers $\Pi^{-1}(\gamma)$. This gives a starting point to study the structure of the horoboundary, by looking at the structure of these fibers. For this reason, in Section 5.2.2 we study the fibers, and as a corollary of Proposition 5.2.11 we get the following result.

**Proposition 5.1.2.** *Let $\Pi$ be the map between the horofunction compactification and the visual compactification defined above, and let $\gamma$ be a geodesic. Then, $\Pi^{-1}(\gamma)$ is path connected.*

Using this feature of the fibers we get in the same section a characterization of the connectivity of the horoboundary.

**Proposition 5.1.3.** *The horoboundary of a proper, uniquely geodesic straight metric space is connected if and only if its visual boundary based at some point (and hence, any) is connected.*

The analysis of the fibers starts by looking at the *Busemann map* $B$ from the visual compactification $X \cup D_b$ to the horofunction compactification, defined by setting $B(\gamma) = B_\gamma$ for each geodesic ray $\gamma \in D_b$ and $B(p) = h(p)$ for each $p \in X$. With this definition, the map satisfies $\Pi \circ B = \mathrm{id}$. In Section 5.2.2 we show that this map is continuous if and only if the two relevant compactifications are, as compactifications, the same.

**Proposition 5.1.4.** *The visual compactification of a proper, uniquely geodesic, straight metric space based at some point is isomorphic to its horofunction compactification if and only if the Busemann map is continuous.*

The Busemann map is essentially the identity inside $X$, so the only possible points of discontinuity are at the boundary. It is therefore of interest to find a criterion for the continuity of $B$ at the boundary, which as we show in Section 5.2.2 gives a criterion for when the fibers $\Pi^{-1}(\gamma)$ are singletons.

**Proposition 5.1.5.** *Let $X$ be a proper, uniquely geodesic, straight metric space, $b \in X$ a basepoint, $B$ the corresponding Busemann map and furthermore, let $\gamma$ be a geodesic ray based at $b$. Then the following three statements are equivalent:*

1. *The Busemann map $B$ restricted to the boundary is continuous at $\gamma$.*
2. *The fiber $\Pi^{-1}(\gamma)$ is a singleton.*
3. *The Busemann map $B$ is continuous at $\gamma$.*

In other words, we have reduced the continuity of $B$ to the continuity restricted to the boundary. This result can then be applied to different settings to obtain a more precise characterization. In the case of Teichmüller spaces Proposition 5.1.5 can be used to get an explicit criterion for the continuity of the Busemann map in terms of the quadratic differentials associated to the geodesic rays, giving us a characterization of the fibers that are singletons.

The map $\Pi_b$ can be defined for any basepoint $b \in X$. Hence, as an immediate corollary, we get that if a sequence converges in the horofunction compactification, the sequence also converges in all the visual compactifications. Under some extra smoothness assumptions, which shall be explicited in Section 5.2.3, we are able to get the inverse relation. That is, if the sequence converges in all the visual compactificaitons, then the sequence also converges in the horofunction compactification. This is stated as Corollary 5.2.14, in Section 5.2.3. To reach such result we give an alternative definition of the horofunction compactification, under the extra smoothness hypotheses, as the reachable subset of the infinite product of all the visual compactifications. See Theorem 5.2.12 in that same section for details.

### 5.1.1 Disclosures

This chapter is based on the first part of work by the author [Aze21].

## 5.2 Horofunction compactification of metric spaces.

### 5.2.1 The relation between the horofunction compactification and the visual compactification

Fix a uniquely geodesic, proper and straight metric space $(X, d)$ and a basepoint $b \in X$. We will assume $X$ satisfies these hypotheses through this section. For each geodesic ray $\gamma \in \partial\overline{X}^v$ starting at $b$ there is an associated Busemann point $B_\gamma \in \partial\overline{X}^h$. We can extend this map to all the visual compactification by setting it as the identification with the map $h$ on $X$ given by the horofunction compactification. That is, we define the *Busemann map* $B : \overline{X}^v \to \overline{X}^h$ by setting $B(\gamma) = B_\gamma$ for $\gamma \in \partial\overline{X}^v$ and $B(x) = h(x)$ for $x \in X$. The relevance of this map can be seen with the following result.

**Lemma 5.2.1.** *The visual compactification $(i, \overline{X}^v)$ is finer than the horofunction compactification $(h, \overline{X}^h)$ if and only if the map $B$ is continuous.*

*Proof.* We have that $B(i(x)) = h(x)$, so $B$ is an extension of $h$ to $\overline{X}^v$. Hence, if $B$ is continuous, then the visual compactification is finer than the horofunction compactification.

On the other hand, if the visual compactification is finer than the horofunction compactification, then we have a continuous map $f : \overline{X}^v \to \overline{X}^h$. For every $x \in X$, we have $f(i(x)) = h(x) = B(i(x))$. Furthermore, for any ray $\gamma$ starting at the basepoint we have $f(\gamma) = \lim_{t\to\infty} f(i(\gamma(t)) = \lim h(\gamma(t)) = B(\gamma)$. Hence, $B = f$, and $B$ is continuous. □

In general, the Busemann map may not be surjective nor continuous. However, we have the following.

**Proposition 5.2.2.** *For a proper, uniquely geodesic, straight metric space $(X, d)$ the Busemann map is injective.*

*Proof.* For each $x \in X$, the associated function $h(x)$ has a global minimum at $x$, while $B_\gamma$ is unbounded below for every $\gamma \in \partial\overline{X}^v$. Hence, in the interior of $\overline{X}^v$ the map is injective and $B(X) \cap B(\partial\overline{X}^v) = \emptyset$. Assume we have $\gamma, \gamma' \in \partial\overline{X}^v$ such that $\gamma \neq \gamma'$ and $B(\gamma) = B(\gamma') = \xi$. Then, for a given sequence $t_n \to \infty$ we have $\lim_{n\to\infty} h(\gamma(t_n)) = \lim_{n\to\infty} h(\gamma'(t_n)) = \xi$. For any $t \in \mathbb{R}$ and any $n$ such that $t_n > t$ we have

$$h(\gamma(t_n))(\gamma(t)) = d(\gamma(t_n), \gamma(t)) - d(\gamma(t_n), \gamma(0)) = t_n - t - t_n = -t,$$

and similarly for $\gamma'$. Hence $\xi(\gamma(t)) = \xi(\gamma'(t)) = -t$ for all $t$.

Fix now a $t > 0$. We have

$$\begin{aligned} -t = \xi(\gamma'(t)) &= \lim_{n\to\infty} (d(\gamma'(t), \gamma(t_n)) - d(b, \gamma(t_n))) \\ &= \lim_{n\to\infty} (d(\gamma'(t), \gamma(t_n)) - t_n). \end{aligned}$$

That is, there is a sequence $\varepsilon_n$ with $\varepsilon_n \to 0$ such that

$$t_n - t + \varepsilon_n \geq d(\gamma'(t), \gamma(t_n)) \geq t_n - t - \varepsilon_n.$$

for every $n$.

By straightness we can extend $\gamma$ in the negative direction towards $\gamma(-s)$ for some $s > 0$. We shall now show that the geodesic $\gamma$ does not minimize the distance between $\gamma(-s)$ and $\gamma(t_n)$ for $n$ big enough. Since the space is straight, the geodesic segment between $\gamma(-s)$ and $b$ can be extended uniquely, so concatenating it with the segment between $b$ and $\gamma'(t)$ does not result in a geodesic. Hence, the distance between $\gamma'(t)$ and $\gamma(-s)$ is strictly smaller than $s + t$. That is, there is some $\delta > 0$ such that $d(\gamma(-s), \gamma'(t)) < t + s - \delta$. As shown in Figure 5.1 we get a path going from $\gamma(-s)$, to $\gamma(t_n)$, passing through $\gamma'(t)$ that has length less than $t + s - \delta + t_n - t + \varepsilon_n = t_n + s - \delta + \varepsilon_n$. Hence, taking $n$ big enough so that $\varepsilon_n < \delta$ we get that the geodesic segment between $\gamma(-s)$ and $\gamma(t_n)$ is not minimizing. This is a contradiction, from which we conclude that $\gamma = \gamma'$. Therefore, $B$ is injective.

$\square$

Hence, given a Busemman point $\xi$ in $B(\partial\overline{X}^v)$ we have a unique associated geodesic ray

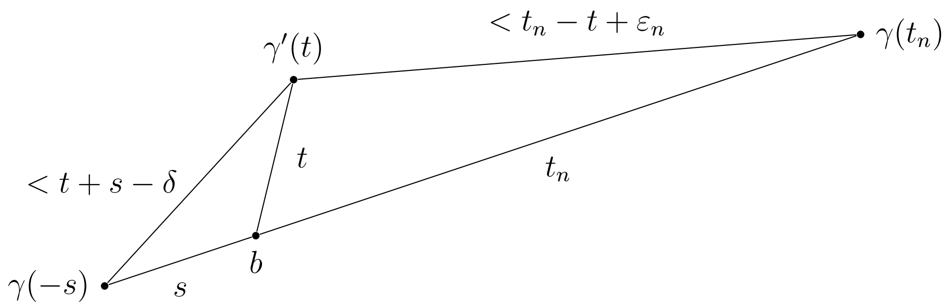



Figure 5.1: The triangles involved in the proof of Proposition 5.2.2.

$\gamma \in \partial\overline{X}^v$ such that $\xi(\gamma(t)) = -t$ for all $t$. Our next aim is to build a similar relation for all other horofunctions. Our approach is similar to the one used by Walsh in [Wal19, Section 7].

We say that a geodesic $\gamma$ is an *optimal geodesic* for a certain horofunction $\xi \in \overline{X}^h$ if $\xi(\gamma(t)) - \xi(\gamma(0)) = -t$ for all $t \in \mathbb{R}$. We shall now see that each function in the horoboundary has at least one optimal geodesic.

**Lemma 5.2.3.** *Let $X$ be a proper, uniquely geodesic, straight metric space and let $\xi \in \partial\overline{X}^h$ be a horofunction. Suppose that $(x_n) \subset X$ converges to $\xi$, with $x_n = \gamma_n(t_n)$, $\gamma_n \in \partial\overline{X}^v$ and $(\gamma_n)$ converging to $\gamma$ as $n \to \infty$. Then $\xi(\gamma(t)) = -t$ for every $t \in \mathbb{R}$. That is, $\gamma(t)$ is an optimal geodesic for $\xi$.*

*Proof.* Fix $t$. We have that

$$\xi(\gamma(t)) = \lim_{n\to\infty}(d(\gamma(t), \gamma_n(t_n)) - d(b, \gamma_n(t_n))) = \lim_{n\to\infty}(d(\gamma(t), \gamma_n(t_n)) - t_n).$$

As $n$ goes to infinity, $\gamma_n$ converges to $\gamma$. Hence by the given topology on the visual boundary, the maps $\gamma_n(\cdot)$ converge uniformly on compact sets to the geodesic $\gamma(\cdot)$. In particular, denoting $d(\gamma(t), \gamma_n(t)) = \varepsilon_n$ we have $\varepsilon_n \to 0$. We get then Figure 5.2, so by the triangle inequality,

$$|d(\gamma(t), \gamma_n(t_n)) - (t_n - t)| = |d(\gamma(t), \gamma_n(t_n)) - d(\gamma_n(t), \gamma_n(t_n))| \leq \varepsilon_n,$$

and so $\xi(\gamma(t)) = -t$. $\square$

Since $\partial\overline{X}^v$ is compact, for any horofunction $\xi \in \partial\overline{X}^h$ and sequence $(x_n) \subset X$ converging

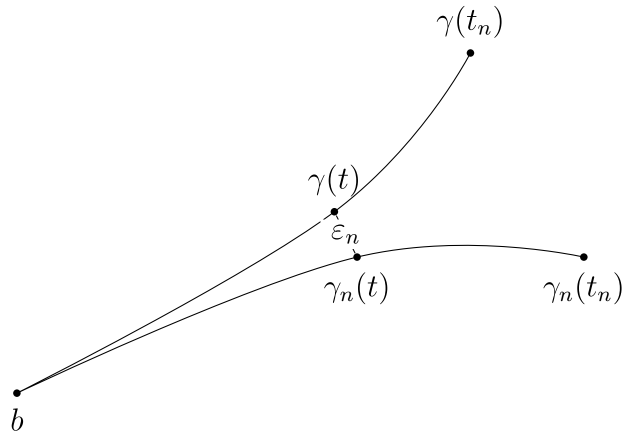



Figure 5.2: In the proof of Lemma 5.2.3, $\gamma_n$ converges to $\gamma$, so $\gamma_n(t)$ converges to $\gamma(t)$, and hence the distance between $\gamma_n(t_n)$ and $\gamma_n(t)$ gets arbitrarily close to the distance between $\gamma_n(t_n)$ and $\gamma(t)$.

to $\xi$ we can take a subsequence such that the hypotheses of Lemma 5.2.3 are satisfied, so each $\xi \in \partial\overline{X}^v$ does have at least one optimal geodesic.

If $\xi$ has another optimal geodesic $\gamma'$ with $\gamma'(0) = \gamma(0)$ we have at least two geodesics along which $\xi(\gamma(t)) = \xi(\gamma'(t)) = -t$ for all $t$. Following a reasoning similar to the one in the proof of Proposition 5.2.2, we get a contradiction. This time, however, we have to be a bit more careful about the distances, as instead of two fixed rays we have a fixed ray and a sequence converging to a distinct fixed ray.

**Proposition 5.2.4.** *Let $\xi \in \partial\overline{X}^h$ and $b \in X$. Then there is a unique optimal geodesic for $\xi$ passing through $b$.*

*Proof.* Let $(x_n) = (\gamma_n(t_n))$ be a sequence converging to $\xi$, with $(\gamma_n) \subset \partial\overline{X}^v$, and take a subsequence such that $\gamma_n$ converges to some geodesic $\gamma$. By Lemma 5.2.3, $\gamma$ is an optimal geodesic. Assume that we have a different optimal geodesic $\gamma'$ passing through $b$.

Using that $h(\gamma_n(t_n))$ converges pointwise to $\xi$ we have

$$\begin{aligned} -t = \xi(\gamma'(t)) &= \lim_{n\to\infty}(d(\gamma'(t), \gamma_n(t_n)) - d(b, \gamma_n(t_n))) \\ &= \lim_{n\to\infty}(d(\gamma'(t), \gamma_n(t_n)) - t_n). \end{aligned}$$

Hence, there is a sequence $\varepsilon_n$ with $\varepsilon_n \to 0$ such that

$$t_n - t + \varepsilon_n \geq d(\gamma'(t), \gamma_n(t_n)) \geq t_n - t - \varepsilon_n.$$

We proceed by showing that for $n$ big enough there is some $s > 0$ such that the geodesic $\gamma_n$ does not minimize the distance between $\gamma_n(-s)$ and $\gamma_n(t_n)$. As in the proof of Proposition 5.2.2, by applying the triangle inequality between $\gamma'(t), \gamma(-s)$ and $b$ we have $d(\gamma'(t), \gamma(-s)) < s+t$. Fix $s > 0$ and pick $\delta > 0$ such that $d(\gamma'(t), \gamma(-s)) < t+s-\delta$. Since $\gamma_n$ converges to $\gamma$ uniformly on compact sets, $\gamma_n(-s)$ converges to $\gamma(-s)$. Hence, $d(\gamma'(t), \gamma_n(-s))$ converges to $d(\gamma'(t), \gamma(-s))$. Then for $n$ big enough we have $d(\gamma'(t), \gamma_n(-s)) < t + s - \delta$. Consider then $n$ big enough so that $\varepsilon_n \leq \delta/2$ as well. The triangle between $\gamma'(t), \gamma_n(-s)$ and $\gamma_n(t_n)$ gives

$$d(\gamma_n(-s), \gamma_n(t_n)) \leq d(\gamma_n(-s), \gamma'(t)) + d(\gamma'(t), \gamma_n(t_n)) < (t + s - \delta) + (t_n - t + \varepsilon_n) < t_n + s.$$

This is a contradiction, which proves the uniqueness of $\gamma$. $\square$

Given a basepoint $b \in X$ we can now define a map $\Pi_b : \overline{X}^h \to \overline{X}^v_b$ by sending any $\xi \in \partial\overline{X}^h$ to the unique optimal geodesic $\gamma$ of $\xi$ with $\gamma(0) = b$, and by sending $h(x)$ to $x$ for any $x \in X$. This map is indeed an extension of the relation we had established for Busemann points in $\mathcal{B}(\partial\overline{X}^v)$, since if $\xi = B(\gamma)$ for $\gamma \in D_b$ then $\gamma$ is an optimal geodesic of $\xi$, giving us $\Pi_b(B(\gamma)) = \gamma$.

We will often write $\Pi$ instead of $\Pi_b$ whenever the basepoint is not relevant to the discussion. To prove that $\Pi$ is continuous, we first have to see the following result.

**Proposition 5.2.5.** *Let $(x_n) \subset X$ be a sequence converging to $\xi \in \partial\overline{X}^h$. Then, $(x_n)$ has a unique accumulation point in the visual compactification. Further, this accumulation point depends only on $\xi$.*

*Proof.* Since $\partial\overline{X}^v$ is compact, $(x_n)$ has accumulation points in the visual compactification. If $(x_n)$ has two accumulation points we can take two subsequences converging to two different geodesics, which by Lemma 5.2.3 are optimal geodesics, contradicting Proposition 5.2.4.

If there is another sequence $(y_n)$ converging to $\xi$ with a different accumulation point the result follows by merging both sequences and repeating the reasoning. $\square$

Hence, $\Pi$ can be alternatively defined by sending any $\xi \in \partial\overline{X}^h$ to the unique accumulation point in $\overline{X}^v$ of the sequences converging to $\xi$ in $\overline{X}^h$, and by sending $h(x)$ to $x$ for any $x \in X$.

By Proposition 5.2.5, this definition is equivalent to the previous one.

By this second definition of the map $\Pi$, we see how it is mostly related to the convergence of sequences, so using a diagonal sequence argument we can prove its continuity.

**Proposition 5.2.6.** *The map $\Pi$ is continuous.*

*Proof.* Take a sequence $(\xi_n) \subset \overline{X}^h$ converging to $\xi$. If $\xi \in h(X)$ we have that, as $h(X)$ is open, $\xi_n \in h(X)$ for $n$ big enough. Hence, $\Pi(\xi_n) = h^{-1}(\xi_n)$, which converges to $h^{-1}(\xi)$, as $h$ is a homeomorphism with its image.

If $\xi \in \partial\overline{X}^h$ we split the sequence into two subsequences, one contained in $h(X)$ and one contained in $\partial\overline{X}^h$. The one contained in $h(X)$ converges to $\xi$, so by definition of $\Pi$ and we have $\Pi(\xi) = \lim_{n\to\infty} h^{-1}(\xi_n)$.

Assume then that $(\xi_n) \subset \partial\overline{X}^h$ converges to $\xi$. We want to see that $\gamma_n = \Pi(\xi_n)$ converges to $\gamma = \Pi(\xi)$. For each $\xi_n$ we can take a sequence $(h(\gamma_n^m(t_n^m)))_m$ converging, as $m \to \infty$ to $\xi_n$. By Proposition 5.2.5 the sequence $\gamma_n^m(t_n^m)$ converges to $\gamma_n$. Let $\gamma'$ be an accumulation point of $\gamma_n$. Take a convergent subsequence of $\gamma_n$ converging to $\gamma'$, and relabel it as $\gamma_n$. Let $(V_n)$ be a nested sequence of open neighbourhoods of $\xi$ in $\overline{X}^h$ such that $\xi_n \in V_n$ and $\bigcap_n V_n = \{\xi\}$ and let $(W_n)$ be a nested sequence of open neighbourhoods of $\gamma'$ in $\overline{X}^v$ such that $\gamma_n \in W_n$ and $\bigcap_n W_n = \{\gamma'\}$. We can take such sequences of sets, as both spaces are metrizable.

For each $n$, there exists $m(n)$ big enough so that $\gamma_n^{m(n)} \in W_n$ and $h(\gamma_n^{m(n)}(t_n^{m(n)})) \in V_n$. By the first condition on $m(n)$, we have that $\gamma_n^{m(n)}$ converges to $\gamma'$. By the second condition, $h(\gamma_n^{m(n)}(t_n^{m(n)}))$ converges to $\xi$, so by the the definition of $\Pi$ and Proposition 5.2.5 the sequence $\gamma_n^{m(n)}$ converges to $\Pi(\xi) = \gamma$. Hence, $\gamma = \gamma'$, so the only accumulation point of $(\gamma_n)$ is $\gamma$ and by compactness of $\partial\overline{X}^v$ the sequence $(\gamma_n)$ converges to $\gamma$. $\square$

By combining Propositions 5.2.5 and 5.2.6 we get that $\Pi$ is the map announced at the introduction, giving us a proof of Theorem 5.1.1. As mentioned in the introduction, this map shows that the horofunction compactification is finer than the visual compactification. By using the Busemann map to insert the visual boundary inside the horoboundary, we can consider the map $\Pi$ as a projection.

One straightforward consequence of the continuity of $\Pi_b$ is as follows.

**Corollary 5.2.7.** *Let $\gamma$ be a geodesic ray, not necessarily starting at the basepoint $b \in X$. Then, $\gamma$ converges in the visual compactification of $X$ based at $b$.*

*Proof.* The ray $\gamma$ converges in the horofunction compactification to $B_\gamma$. Since $\Pi_b$ is continuous, the ray also converges in the visual compactification based at $b$ to $\Pi_b(B_\gamma)$. $\square$

For Teichmüller spaces with the Teichmüller metric this result was first proved by Walsh [Wal19, Theorem 7].

By Lemma 5.2.1, the visual compactification is finer than the horofunction compactification if and only if the Busemann map is continuous. Hence, since the horofunction compactification is always finer than the visual compactification, we obtain an isomorphism whenever this is the case, resulting in Proposition 5.1.4.

### 5.2.2 The fiber structure

To get a better picture of the shape of the horoboundary we shall study the shape of the preimages of the projection $\Pi$ restricted to the boundary. That is, for a given point $\gamma$ in the visual boundary we are interested in finding out information about the fiber $\Pi^{-1}(\gamma)$. We first prove the following lemma, which we will use to get bounds on the values of $\Pi^{-1}(\gamma)$.

**Lemma 5.2.8.** *Fix a geodesic ray $\gamma \in \partial\overline{X}^v$ and $p \in X$ not in the bi-infinite extension of the geodesic ray $\gamma$. Then, the function $h(\gamma(\cdot))(p)$, with domain $[0, \infty)$, is strictly decreasing.*

*Proof.* Take $t, s \geq 0$ with $s < t$. By the triangle inequality we have

$$d(\gamma(t), p) \leq d(\gamma(s), p) + d(\gamma(t), \gamma(s)) = d(\gamma(s), p) + t - s.$$

Further, we have strict inequality, as equality would give us two different paths with the same length between $\gamma(t)$ and $p$, with one of them being geodesic. Hence,

$$\begin{aligned} h(\gamma(t))(p) &= d(\gamma(t), p) - d(\gamma(t), b) \\ &< d(\gamma(s), p) + t - s - t \\ &= h(\gamma(s))(p). \end{aligned}$$

$\square$

The set $C(X)$ can be partially ordered by saying that $f \geq g$ whenever $f(x) \geq g(x)$ for all $x \in X$. If $f \geq g$ and $f \neq g$ then we write $f > g$. If $p = \gamma(r)$ for some $r$ and $s < t$ we have $h(\gamma(s))(p) = h(\gamma(t))(p) = -r$ for $r \leq s$ and $-s = h(\gamma(s))(p) > h(\gamma(t))(p) = -\min(r, t)$

otherwise. Hence, adding the previous lemma we have $h(\gamma(s)) > h(\gamma(t))$ whenever $s < t$. By attempting to extend this relation to the horofunction boundary we get that Busemann points are maximal in their fibers.

**Proposition 5.2.9.** *Let $\gamma \in \partial\overline{X}^v$ and $\xi \in \Pi^{-1}(\gamma)$. Then, $\xi \leq B(\gamma)$.*

*Proof.* Choose any sequence $(x_n) \subset X$ such that $h(x_n)$ converges to $\xi$. Since $\xi \in \Pi^{-1}(\gamma)$ the sequence $(x_n)$ converges to $\gamma$ in $\overline{X}^v$, so we can write $x_n = \gamma_n(t_n)$ with $t_n$ converging to infinity and $\gamma_n$ converging to $\gamma$.

Fix $p \in X$ and let $\varepsilon > 0$. Denote $s_n = \sup\{t : d(\gamma(t), \gamma_n(t)) < \varepsilon \text{ and } t < t_n\}$. The geodesics $\gamma_n$ converge to $\gamma$ uniformly on compact sets, so $s_n \to \infty$ as $n \to \infty$. Hence, by definition of the Busemann point and since $d(\gamma_n(s_n), \gamma(s_n)) < \varepsilon$,

$$B_\gamma(p) = \lim_{n\to\infty} h(\gamma(s_n))(p) \geq \limsup_{n\to\infty} h(\gamma_n(s_n))(p) - 2\varepsilon.$$

Furthermore, $s_n \leq t_n$, so by Lemma 5.2.8,

$$\xi(p) = \lim_{n\to\infty} h(\gamma_n(t_n))(p) \leq \limsup_{n\to\infty} h(\gamma_n(s_n))(p) \leq B_\gamma(p) + 2\varepsilon.$$

Since $\varepsilon$ can be arbitrarily small we get the proposition. □

While it might not be possible to get a similar unique minimum in each fiber, we can get the following result.

**Proposition 5.2.10.** *Let $\gamma \in \partial\overline{X}^v$ and $\xi \in \Pi^{-1}(\gamma)$. Furthermore, let $(x_n) \subset X$ be a sequence converging to $\xi$ with $x_n = \gamma_n(t_n)$. For any $p$, define $\eta(p) = \liminf_{n\to\infty} B(\gamma_n)(p)$. Then, $\xi \geq \eta$.*

*Proof.* The proof follows a similar reasoning as the last one.

Fix $p \in X$, choose a subsequence so $B(\gamma_n)(p)$ converges to $\eta(p)$ and let $(\varepsilon_m)$ be a sequence of positive numbers coverging to 0. For each $\varepsilon_m$, take $n(m)$ big enough so that $B(\gamma_{n(m)})(p) \geq \eta(p) - \varepsilon_m$. Further, take $s_m$ bigger than $t_{n(m)}$, and big enough so that

$$h(\gamma_{n(m)}(s_m))(p) \geq B(\gamma_{n(m)})(p) - \varepsilon_m.$$

Such an $s_m$ always exists by the definition of $B(\gamma_{n(m)})$. In particular, we have that

$$\liminf_{m\to\infty} h(\gamma_{n(m)}(s_m)(p) \geq \eta(p).$$

By Lemma 5.2.8 we have

$$\xi(p) = \lim_{m\to\infty} h(\gamma_{n(m)}(t_{n(m)})(p) \geq \liminf_{m\to\infty} h(\gamma_{n(m)}(s_m))(p) \geq \eta(p).$$

□

The intuition one might get from these propositions is that approaching $\gamma$ "through the boundary", that is, through the furthest way possible from the interior of $X$, gives a lower bound on the possible values of approaching through other angles, and approaching $\gamma$ in a straight way, that is, through the geodesic, gives an upper bound. Hence, when these two ways of approaching $\gamma$ are the same, every other possible angle of approach should also yield the same limit. Following this reasoning we get our next result, announced in the introduction.

**Proposition 5.1.5.** *Let $X$ be a proper, uniquely geodesic, straight metric space, $b \in X$ a basepoint, $B$ the corresponding Busemann map and furthermore, let $\gamma$ be a geodesic ray based at $b$. Then the following three statements are equivalent:*

1. *The Busemann map $B$ restricted to the boundary is continuous at $\gamma$.*
2. *The fiber $\Pi^{-1}(\gamma)$ is a singleton.*
3. *The Busemann map $B$ is continuous at $\gamma$.*

*Proof.* (1) $\implies$ (2): Take $\xi \in \Pi^{-1}(\gamma)$. By Proposition 5.2.9 we have $\xi \leq B(\gamma)$. Since $B$ is continuous at $\gamma$ when restricted to the boundary we have that for any $\gamma_n \to \gamma$ the horofunctions $B(\gamma_n)$ converge to $B(\gamma)$. Hence, by Proposition 5.2.10, $\xi \geq B(\gamma)$, so $\xi = B(\gamma)$ and we have (2).

(2) $\implies$ (3): Take then any $(x_n) \subset \overline{X}^v$ converging to $\gamma$, consider the sequence $(B(x_n)) \subset \overline{X}^h$ and let $\eta$ be an accumulation point. By the definition of $\Pi$ we have $\eta \in \Pi^{-1}(\gamma)$, so $\eta = B(\gamma)$ since we assumed that $\Pi^{-1}(\gamma)$ is a singleton. This shows that $B$ is continuous at $\gamma$.

Finally, it is clear that (3) $\implies$ (1). □

The relation obtained in Lemma 5.2.8 can be exploited further. Indeed, trying to carry it to the boundary in a more delicate manner we can see that the fibers are path connected.

**Proposition 5.2.11.** *Let $\gamma \in \partial\overline{X}^v$. For any $\xi \in \Pi^{-1}(\gamma)$ there exists a continuous path from $B(\gamma)$ to $\xi$ contained in $\Pi^{-1}(\gamma)$.*

*Proof.* Take a sequence $(x_n) \subset X$ converging to $\xi$ in the horofunction compactification, and write $x_n = \gamma_n(u_n)$. As we have seen in the proof of Proposition 5.2.10, we can take a sequence $(l_n) \subset \mathbb{R}$ with $\gamma_n(l_n)$ converging to $B_\gamma$ such that $l_n < u_n$ for all $n$. For each $n$ we have a path $\tilde{\alpha}^n(t)$ connecting $\gamma_n(l_n)$ and $\gamma_n(u_n)$ by setting $\tilde{\alpha}^n(t) = \gamma_n(tu_n + (1-t)l_n)$ for $t \in [0,1]$. We would like to carry this path to the limit, getting a path between $\xi$ and $B(q)$. However, directly taking such a limit might result in some discontinuities, so we have to choose a parametrization carefully.

To find a good parametrization we shall use a certain functional as a control. We want the functional to carry discontinuities and strict increases in the path of functions to discontinuities and strict increases in the value of the functional. Since $X$ is proper, it is separable, so let $(p_i)_{i\in\mathbb{N}}$ be a countable dense set in $X$. We define the functional $I : \overline{X}^h \to \mathbb{R}$ given by

$$I(f) = \sum_{i\in\mathbb{N}} \frac{f(p_i)}{2^i d(b,p_i)}.$$

Since $|f(x)| \leq d(b,x)$ for all $f \in \overline{X}^h$, the summation in the definition of $I(f)$ is absolutely convergent, so $I(f)$ is defined, finite, continuous with respect to $f$, and for any two $f, g \in \overline{X}^h$ we have $I(f+g) = I(f)+I(g)$. Furthermore, since $(p_n)$ is dense and we are taking continuous functions, we have that the functional translates strict inequalities. That is, $f > g$ implies $I(f) > I(g)$. Hence, if $I(f) = 0$ and $f \geq 0$ we have $f = 0$.

We define then the function $F_n(t) = I(h(\gamma_n(t))$. By continuity of $I$ this function is continuous, and by Lemma 5.2.8 it is strictly decreasing with respect to $t$. That is, we have continuous strictly decreasing functions $F_n : [l_n, u_n] \to [F_n(u_n), F_n(l_n)]$. Hence, we can define implicitly the continuous parametrizations $s_n : [0,1] \to [l_n, u_n]$ by taking the unique value $s_n(t)$ such that

$$F_n(s_n(t)) = (1-t)F_n(l_n) + tF_n(u_n).$$

Denote the $F_n(s_n(t))$ as $E_n(t)$. By the continuity of $I$ we have that $E_n(t)$ converges to $(1-t)I(B_\gamma) + tI(\xi)$ as $n \to \infty$, which we denote $E(t)$.

Take now a countable dense set $(t^k)_{k\in\mathbb{N}} \subset [0,1]$ containing 0 and 1. We are now ready to start defining the path $\alpha : [0,1] \to \Pi^{-1}(\gamma)$, and we begin defining it for the dense set $(t^k)$. For $k = 1$ we define $\alpha(t^1)$ as an accumulation point of $h(\gamma_n(s_n(t^1)))$. Denote $(\gamma_{m^1(n)})$ the subsequence of $\gamma_n$ such that $h(\gamma_{m^1(n)}(s_{m^1(n)}(t^1)))$ converges to $\alpha(t^1)$. Define inductively $\alpha(t^k)$ and $(\gamma_{m^k(n)})$ by taking an accumulation point and a corresponding converging subsequence of $h(\gamma_{m^{k-1}(n)}(s_{m^{k-1}(n)}(t^k)))$. By the continuity of $I$ we have

$$I(\alpha(t^k)) = \lim_{n\to\infty}(F_{m^k(n)}(s_{m^k(n)}(t^k))) = E(t^k).$$

For each pair $i > j$ we have that $m^i(n)$ is a subsequence of $m^j(n)$, so $h(\gamma_{m^i(n)}(s_{m^i(n)}(t^j)))$ converges to $\alpha(t^j)$. Assume $t^i > t^j$. By Lemma 5.2.8 we have that $h(\gamma_{m^i(n)}(s_{m^i(n)}(t^i))) < h(\gamma_{m^i(n)}(s_{m^i(n)}(t^j)))$, so $\alpha(t^i) \leq \alpha(t^j)$.

We now have to prove that the definition we have given for $\alpha$ on $(t^k)$ can be extended continuously to $[0,1]$. Fix any $t \notin (t^k)$ and take a subsequence of $t^k$, labeled $t^{k_n}$, such that $t^{k_n} \to t$. We shall now see that $\alpha(t^{k_n})$ converges to a function which does not depend on the chosen subsequence, and define $\alpha(t)$ as that limit. We can split and reorder the sequence $(t^{k_n})$ into $(t_n^+)$ and $(t_n^-)$ satisfying $t_n^+ > t_{n+1}^+ > t > t_{n+1}^- > t_n^-$. The associated $\alpha(t_n^\pm)$ are ordered, so for any $p \in X$ the sequence $\alpha(t_n^\pm)(p)$ is an increasing (or decreasing) sequence of of values in $\mathbb{R}$, bounded above (or below) by $\alpha(0)(p)$ (or $\alpha(1)(p)$). Hence, both sequences converge pointwise, which implies uniform convergence on compact sets, as these functions are 1-Liptschitz. Furthermore, these limits do not depend on the chosen sequence, since if we had any other we could intercalate them and the sequences would still converge. Denote then $\alpha^+$ the limit associated to $t_n^+$, and $\alpha^-$ the limit associated to $t_n^-$. Since $\alpha(t_n^+) < \alpha(t_m^-)$ for all $n, m$ we have $\alpha^+ \leq \alpha^-$. For each $\alpha(t^k)$ we have $I(\alpha(t^k)) = E(t^k)$. Hence by the continuity of $I$ we have that

$$I(\alpha^+) = E(t) = I(\alpha^-).$$

That is, we have

$$I(\alpha^- - \alpha^+) = 0.$$

Since $\alpha^-$ and $\alpha^+$ are continuous and $\alpha^- - \alpha^+ \geq 0$ we have $\alpha^- = \alpha^+$. We thus define $\alpha(t)$ to be either one. The same reasoning shows that $\alpha$ is continuous. $\square$

We would like to remark that several choices where made in the proof of the previous lemma, and the obtained path may not be unique.

We can use the previous result to observe that the horoboundary is connected if and only if the visual boundary is connected.

*Proof of Proposition 5.1.3.* Assume that the visual boundary is not connected. Then we have $U, V \subset \partial\overline{X}^v$ nonempty and open such that $U \cap V = \emptyset$ and $U \cup V = \partial\overline{X}^v$. As $\Pi$ is continuous, the sets $\Pi^{-1}(U)$ and $\Pi^{-1}(V)$ are open, so the horoboundary is not connected.

For the other implication, assume that the visual boundary is connected while the horoboundary is not connected. Then we have $U, V \subset \partial\overline{X}^h$ nonempty and open such that $U \cap V = \emptyset$ and $U \cup V = \partial\overline{X}^h$. Since fibers are path connected, each of them is contained in only one of $U$ or $V$, so $\Pi(U)$ and $\Pi(V)$ are disjoint. Since $U \cup V = \partial\overline{X}^h$ we have $\Pi(U) \cup \Pi(V) = \partial\overline{X}^v$, and since both $U$ and $V$ are nonempty, so are the images. Hence, both images cannot be open at the same time, as $\partial\overline{X}^v$ is connected. Therefore, these sets cannot be both closed. Assume $\Pi(U)$ is not closed. We then have a sequence $(\gamma_n) \subset \Pi(U)$ converging to a point in $\Pi(V)$. Again, since $U \cup V = \partial\overline{X}^h$, we have that $U = \Pi^{-1}\Pi(U)$ and $V = \Pi^{-1}\Pi(V)$. Hence, any lift of the sequence $(\gamma_n)$ to $\Pi^{-1}\Pi(U)$ is contained in $U$ and, since $\partial\overline{X}^h$ is compact, has accumulation points which, by the continuity of the projection map, are be contained in $\Pi^{-1}\Pi(V) = V$. Hence, $U$ is not closed and we get a contradiction. $\square$

### 5.2.3 An alternative definition of the horofunction compactification

Under what a priori seem to be more restrictive hypotheses on the space $X$ it is possible to characterize the horofunction compactification as a subset of the product of all of its visual compactifications. We detail the construction in this section.

The new extra hypotheses are both related to the differentiability of the distance function. We say a that a uniquely geodesic metric space $X$ is $C^1$ *along geodesics* if given a point $p \in X$ and a geodesic segment $\gamma$ that does not intersect $p$, the distance function $d(\gamma(t), p)$ is first differentiable and the value of the derivative depends continuously on both $t$ and $p$. Furthermore, the space $X$ has *constant distance variation* if for any two distinct geodesics $\gamma, \eta$ with $\gamma(0) = \eta(0)$ we have either

$$\frac{d}{dt} d(\gamma(t), \eta(s))\bigg|_{t=0} = \frac{d}{dt} d(\gamma(t), \eta(1))\bigg|_{t=0} \tag{5.2.1}$$

for all $s > 0$, or $\frac{d}{dt} d(\gamma(t), \eta(s))\big|_{t=0}$ does not exist for any $s > 0$.

Many commonly studied metric spaces have constant distance variation. For example, spaces with bounded curvature, either above of below, have constant distance variation, as explained in the book by Burago–Burago–Ivanov [BBI01, Section 4]. Importantly to our case, Teichmüller spaces with the Teichmüller distance satisfy both hypotheses. Earle [Ear77] proved that the distance function is $C^1$ by providing a formula for its derivative. Applying the formula to (5.2.1) we get that the derivative depends only on the tangential vector to $\gamma$ at 0 and the unit area quadratic differential associated to $\eta$ at 0, so we also have constant distance variation. Furthermore, Teichmüller spaces with the Teichmüller distance are also straight and proper, so the results from this section can be applied to them.

Consider the product of all the possible visual compactifications obtained by changing the basepoint,

$$E = \prod_{b \in X} \overline{X}^v_b,$$

with the usual product topology. See the book by Munkres [Mun00, Chapters 2.19 and 5.37] for some background on infinite products of topological spaces. Denote $\pi_b$ the projection from $E$ to $\overline{X}^v_b$. By definition of the product topology, the diagonal inclusion $i : X \hookrightarrow E$ such that by $\pi_b(i(x)) = x$ for every $x, b \in X$ is continuous, and has continuous inverse restricted to $i(X)$ given by $\pi_b$. Hence, $i(X)$ is homeomorphic to $X$. That is, $i$ is an embedding. Furthermore, by Tychonoff's theorem the product is compact, as each factor of the product is compact. Hence the closure $\overline{i(X)}$, which we shall denote $\overline{X}^V$, is compact. The pair $(i, \overline{X}^V)$ is then a compactification of $X$, which tracks the information given by the visual boundary at each point. That is, a sequence in $X$ converges in the topology of $\overline{X}^V$ if and only if it converges for every possible visual compactification $\overline{X}^v_b$. The main interest of this compactification comes from the following result.

**Theorem 5.2.12.** *Let $X$ be a proper, uniquely geodesic, straight metric space which is $C^1$ along geodesics and has constant distance variation. Then $(i, \overline{X}^V)$ is isomorphic to $(h, \overline{X}^h)$.*

Denote $\Pi_b$ the continuous map from $\overline{X}^h$ to $\overline{X}^v_b$ given by Theorem 5.1.1. The isomorphism between $\overline{X}^h$ and $\overline{X}^V$ is defined by recording the value of each possible $\Pi_b$ within $\overline{X}^V$. That is, we define $\widetilde{\Pi} : \overline{X}^h \to \overline{X}^V$ in such a way that $\pi_b \circ \widetilde{\Pi} := \Pi_b$ for each $b \in X$. The only property required to prove that $\widetilde{\Pi}$ is an isomorphism not following directly from previous results is the injectivity. By Proposition 5.2.4 we know that if $f \in \Pi_b^{-1}(\gamma)$ then $\gamma$ is an optimal geodesic of $f$. That is, $f(\gamma(t)) - f(\gamma(s)) = -(t - s)$. Hence, if $f, g \in \Pi_b^{-1}(\gamma)$, then they differ by a constant along the geodesic $\gamma$. If $f$ and $g$ are horofunctions in the preimage of a point by $\widetilde{\Pi}$, then they differ by a constant along infinitely many geodesics, which cover $X$. However, the constant might depend on the geodesic, so we need a way to connect these constants. We proceed by strengthening Proposition 5.2.4 to show that any two functions in $\Pi_b^{-1}(\gamma)$ also have the same directional derivatives at points in $\gamma$, which allows us to connect the geodesics. Precisely, we prove the following.

**Proposition 5.2.13.** *Let $X$ be a proper, uniquely geodesic, straight metric space which is $C^1$ along geodesics and has constant distance variation. Furthermore, let $\gamma$ be a geodesic ray starting at $b$, and let $\alpha$ be a geodesic starting at some point on $\gamma$. Then, $\frac{d}{dt} f \circ \alpha(t)\big|_{t=0}$ exists and its value is the same for all $f \in \Pi_b^{-1}(\gamma)$.*

*Proof.* For any $b' \in \gamma$ we have that $\gamma$ is an optimal geodesic of $f$ passing through $b'$. Denoting $\gamma_{b'}$ the geodesic ray starting at $b'$ with the same bi-infinite extension as $\gamma$ we have that $f \in \Pi_{b'}^{-1}(\gamma_{b'})$, by Proposition 5.2.4. Hence, we can assume that $\alpha(0) = b$ by changing the basepoint if necessary. Let $x_n$ be a sequence converging to $f$. Furthermore, let $\eta_t^n$ be the geodesic from $\alpha(t)$ to $x_n$ and $g_n(t)$ be the value of $\frac{d}{ds} h(x_n) \circ \alpha(s)\big|_{s=t}$. By the definition of the map $h$ we have $g_n(t) = \frac{d}{ds} d(\alpha(s), x_n)\big|_{s=t}$. By the constant distant variation we have $g_n(t) = \frac{d}{ds} d(\alpha(s), \eta_t^n(1))\big|_{s=t}$, which since $X$ is $C^1$ along geodesics depends continuously on $\eta_t^n(1)$ and $t$.

By Proposition 5.2.5 the geodesics $\eta_t^n$ converge as $n \to \infty$ to some geodesics $\eta_t$, so $\eta_t^n(1)$ converges to $\eta_t(1)$. Since the space is $C^1$ along geodesics, the value of $\frac{d}{ds} d(\alpha(s), \eta_t^n(1))\big|_{s=t}$ depends continuously on $\eta_t^n(1)$, and so $g_n$ converges pointwise to $g(t) = \frac{d}{ds} d(\alpha(s), \eta_t(1))\big|_{s=t}$.

Take some $\delta > 0$ and assume the convergence is not uniform on $[-\delta, \delta]$. Then there is some $\varepsilon > 0$ such that for each $n$ there is at least one $t_n \in [-\delta, \delta]$ such that $|g_n(t_n) - g(t_n)| > \varepsilon$. Since $[-\delta, \delta]$ is compact we can take a converging subsequence such that $t_n$ converges to some $T \in [-\delta, \delta]$. Hence, the point $\eta_{t_n}^n(1)$ does not converge to $\eta_T(1)$, so by properness of $X$ we can take a subsequence such that $\eta_{t_n}^n(1)$ converges to some $p \in X$ different from $\eta_T(1)$. Let $\beta$ be the geodesic starting at $\alpha(T)$ passing through $p$. The geodesics $\eta_{t_n}^n$ converge uniformly to $\beta$, and $\beta \neq \eta_T$. For any fixed $t > 0$ we have, following the same reasoning than in the proof of Proposition 5.2.5,

$$f(\beta(t)) - f(\beta(0)) = \lim_{n \to \infty} d(x_n, \beta(t)) - d(x_n, \beta(0)) = -t.$$

Hence, $\beta$ is an optimal geodesic of $f$ passing through $\alpha(T)$. However, $f \in \Pi_{\alpha(T)}^{-1}(\eta_T)$, so $\eta_T$

is also an optimal geodesic passing through $\alpha(T)$, contradicting Proposition 5.2.4.

Hence, the convergence of $(h(x_n) \circ \alpha)' = g_n$ to $g$ is uniform on $[-\delta, \delta]$. Therefore, $f$ is differentiable and $f'(0) = g(0) = \frac{d}{ds} d(\alpha(s), \gamma(1))\big|_{s=0}$, which is the same for all $f \in \Pi^{-1}(\gamma)$. $\square$

*Proof of Theorem 5.2.12.* Each $\Pi_b$ is continuous, so by the definition of the product topology the map $\widetilde{\Pi}$ is continuous. Hence, by Lemma 2.1.1 to see that $\widetilde{\Pi}$ is an isomorphism it is enough to show that $\widetilde{\Pi}$ is injective.

Let $f, g \in \overline{X}^h$ be such that $\widetilde{\Pi}(f) = \widetilde{\Pi}(g)$. If there is some $b \in X$ such that $\pi_b \circ \widetilde{\Pi}(f) \in X$ then $f = h(\pi_b \circ \widetilde{\Pi}(f)) = g$. Assume then $\pi_b \circ \widetilde{\Pi}(f) \in \partial \overline{X}_b^v$ for all $b \in X$. By Proposition 5.2.13 they have the same directional derivatives at every point. Let $\alpha$ be a geodesic from a fixed basepoint $b$ to any other point. We have $(f \circ \alpha)' = (g \circ \alpha)'$, so $f - g$ is constant along $\alpha$, and hence everywhere, since any point can be connected to $b$ by a geodesic. Hence, $f$ and $g$ are the same horofunctions. $\square$

By the definition of the convergence in the product topology, this characterization gives us the following equivalence for the convergence to points in the horoboundary.

**Corollary 5.2.14.** *Let $X$ be a proper, uniquely geodesic, straight metric space, $C^1$ along geodesics and with constant distance variation. A sequence $(x_n) \subset X$ converges in the horofunction compactification if and only if the sequence converges in all the visual compactifications.*

# Chapter 6

# The horofunction compactification of Teichmüller spaces

## 6.1 Introduction

Let $S$ be a compact surface with (possibly empty) boundary and finitely many marked points, where we allow marked points to be on the boundary. Denote by $\mathcal{T}(S)$ its Teichmüller space equipped with the Teichmüller metric. Furthermore, for any quadratic differential $q$ based at some basepoint $b \in \mathcal{T}(S)$, denote by $R(q)$ the geodesic ray in $\mathcal{T}(S)$ starting at $b$ in the direction $q$, and $V(q)$ the vertical foliation associated to $q$, see Section 2.2 for a quick introduction or the book by Farb–Margalit [FM12] for a more in-depth explanation of these concepts. Recall that a measured foliation is *indecomposable* if it is either a thickened curve, or a component with a transverse measure that cannot be expressed as the sum of two projectively distinct non zero transverse measures. Furthermore, each measured foliation can be decomposed uniquely into finitely many indecomposable components (see Section 2.2.2 for detailed definitions). Walsh has shown the following characterization of the convergence of Busemann points in terms of the convergence of the associated quadratic differentials.

**Theorem 6.1.1** (Walsh [Wal19, Theorem 10]). *Let $(q_n)$ be a sequence of unit area quadratic differentials based at $b \in \mathcal{T}(S)$. Then, $B_{R(q_n)}$ converges to $B_{R(q)}$ if and only if both of the following hold:*

1. *$(q_n)$ converges to $q$ with respect to the $L^1$ norm on $T_b^*\mathcal{T}(S)$;*

2. *for every subsequence $(G^n)_n$ of indecomposable measured foliations such that, for each $n \in \mathbb{N}$, $G^n$ is a component of $V(q_n)$, we have that every limit point of $G^n$ is indecomposable.*

While Walsh's proof is done in the context of surfaces without boundary, it can be easily extended to our setting. In view of this theorem, we say that a sequence of quadratic differentials $(q_n)$ *converges strongly* to $q$ if it satisfies the two conditions of Theorem 6.1.1. Furthermore, we say that $q$ is *infusible* if every sequence of quadratic differentials converging to $q$ converges strongly. By Proposition 5.1.5, a quadratic differential $q$ is infusible if and only if the Busemann map is continuous at $R(q)$. In Section 6.3.1, and more precisely with Theorem 6.3.4, we derive a topological characterization of the vertical foliations of infusible quadratic differentials. This allows us to determine precisely which surfaces only admit infusible quadratic differentials, yielding the following result.

**Theorem 6.1.2.** *Let $S$ be a compact surface of genus $g$ with $b_m$ and $b_u$ boundary components with and without marked points respectively and $p$ interior marked points. Then the horofunction compactification of $\mathcal{T}(S)$ is isomorphic to the visual compactification if and only if $3g + 2b_m + b_u + p \leq 4$.*

This result had been previously proven by Miyachi [Miy08] for surfaces without boundary, that is, when $b_m = b_u = 0$. For the cases where we do not have an isomorphism Miyachi found non-Busemann points in the boundary. These points are in the closure of Busemann points, which prompted Liu–Su to ask the following question

**Question 6.1.3** (Liu–Su [LS14, Question 1.4.2])**.** *Is the set of Busemann points dense in the horofunction boundary?*

In Section 6.5.1 we give a negative answer to this question, summed up in the following result announced in the introduction.

**Theorem 1.3.1.** *Let $S$ be a closed surface of genus $g$ with $p$ marked points. Then the Busemann points are not dense in the horoboundary of $\mathcal{T}(S)$ whenever $3g + p \geq 5$.*

The result relies on Liu–Su's [LS14] and Walsh's [Wal19] characterization of the horofunction compactification as the Gardiner–Masur compactification. The latter compactification consists of certain real-valued functions on the space of measured foliations. Note that we use a slightly different but equivalent definition than usual for the Gardiner–Masur compactification, as the definition we use is more well suited for our computations, and more easily extendable to surfaces with boundary (see Section 6.2.2 for the precise definition). For each point in the horofunction compactification there is an associated real-valued function on the set of measured foliations. We show that the functions associated to elements in the closure of Busemann points are polynomials of degree 2 with respect to some variables (see Proposition 6.4.2 for the precise statement). We then show that the elements of the Gardiner–Masur boundary found by Fortier Bourque in [FB23] do not satisfy that condition. The main ingredient for this last part of the reasoning is the following result proven in Section 6.5.1, which shows that extremal length is not $C^2$ along certain smooth paths.

**Theorem 6.1.4.** *Let $S$ be a closed surface of genus $g$ with $p$ marked points and empty boundary satisfying $3g+p \geq 5$. Then there is a point $X \in \mathcal{T}(S)$ and a path $G_t$, $t \in [0, t_0]$, in the space of measured foliations on $X$, smooth with respect to the canonical piecewise linear structure of the space of measured foliations, such that* $\operatorname{Ext}(G_t)$ *is not* $C^2$.

The canonical piecewise linear structure of the space of measured foliations was developed by Bonahon [Bon96], [Bon97a] and [Bon97b]. The first derivative of the extremal length along such a path was determined by Miyachi [Miy13a], so our proof is based on finding cases where Miyachi's expression is not $C^1$. This follows from an explicit computation, whose complication is greatly reduced by using previous estimates established by Markovic [Mar18].

Another well-studied compactification of Teichmüller space was given by Thurston [Thu88]. The relation between the Thurston compactification and the horofunction compactification was studied by Miyachi [Miy13b]. He proves that, while neither Thurston's nor the horofunction compactification is finer than the other, there is map from the Thurston compactification to a subset of the horofunction compactification, which is bicontinuous in the union of $\mathcal{T}(S)$ and uniquely ergodic foliations in Thurstn's boundary. Masur showed [Mas82a] that this result can be interpreted to say that these two compactifications are the same almost everywhere according to the Lebesgue measure on Thurston's boundary. The image of uniquely ergodic foliations by the bicontinuous map is the set of Busemann points associated to uniquely ergodic foliations. As we show in Section 6.5.2, this set is nowhere dense within the horoboundary. Hence the map defined by Miyachi does not show that these two are the same almost everywhere according to any strictly positive measure on the horoboundary. In fact, any attempt to extend the identity map from the interior of the Thurston compactification to the interior of the horoboundary compactification to a set of full measure within the Thurston compactification results in the same problem, as we see in Section 6.5.2.

**Corollary 6.1.5.** *Let $\nu$ be any finite strictly positive measure on the horoboundary and let $\mu$ be the Lebesgue measure on the Thurston boundary. Furthermore, let $\phi$ be a map from the Thurston compactification to the horofunction compactification satisfying $\phi|_{\mathcal{T}(S)} = h$, where $h$ is the map used to define the horofunction compactification in Section 2.1.3. Then there is no subset $U$ of the Thurston boundary with full $\mu$-measure such that $\phi$ is continuous at every point in $U$ and $\phi(U)$ has full $\nu$-measure.*

As a straightforward result of the alternative definition of the horofunction compactification explained in Section 5.2.3, and the fact that the extra hypotheses are satisfied by Teichmüller spaces, as shown by Earle in [Ear77], we get the following characterization of converging sequences in the horofunction compactification.

**Corollary 6.1.6.** *A sequence $(x_n) \subset \mathcal{T}(S)$ converges in the horofunction compactification if and only if the sequence converges in all the visual compactifications.*

Considering the horocycles diverging in the horofunction compactification found by Fortier Bourque [FB23] we get that there is some visual compactification in which these horocycles do not converge.

**Corollary 6.1.7.** *Let $S$ be a closed surface of genus $g$ with $p$ marked points, such that $3g+p \geq 5$. There is a basepoint such that a horocycle diverges in the visual compactification based at that point.*

This contrasts with the behavior of Teichmüller rays, which converge in all visual compactifications [Wal19, Theorem 7].

The structure of the horoboundary provided by Theorem 5.1.1, as well as the path-connectivity of the fibers, allows us to prove in Section 6.6 the following path connectivity result, announced in the introduction.

**Theorem 1.3.2.** *The horoboundary of any Teichmüller space of real dimension at least 2 is path connected.*

Furthermore, we also prove that whenever the surface has empty boundary the map $\Pi$ restricted to the horoboundary admits a section, while it only admits a section for the simpler cases if the boundary is nonempty (see Theorem 6.6.1 of the same Section for details).

Finally, in Section 6.7 we see how Liu–Su's and Walsh's characterization of the horofunction compactification as the Gardiner–Masur compactification can be used to translate some of these findings to results regarding the asymptotic value of extremal length functions. For example, we get the following estimate.

**Theorem 6.1.8.** *Let $(q_n)$ be a sequence of unit quadratic differentials converging strongly to a unit quadratic differential $q$. Denote $G_j$ the components of the vertical foliation associated to $q$, and $H(q)$ the horizontal foliation. Then, for any $F \in \mathcal{MF}$ and sequence $(t_n)$ of real values converging to positive infinity we have*

$$\lim_{n\to\infty} e^{-2t_n} \operatorname{Ext}_{R(q_n;t_n)}(F) = \sum_j \frac{i(G_j, F)^2}{i(G_j, H(q))}.$$

This generalizes a previous result proven by Walsh in [Wal19, Theorem 1], where the same is shown for $q_n$ constant.

### 6.1.1 Disclosures

This chapter is based on the second part of work by the author [Aze21].

### 6.1.2 Outline of the chapter

The chapter is structured as follows. In Section 6.2 we give a short review of the necessary background on Teichmüller spaces. In Section 6.3 we determine which quadratic differentials are infusible, and find which surfaces admit infusible quadratic differentials, getting a proof of Theorem 6.1.2. In Section 6.4 we characterize the points in the closure of Busemann points, and get some bounds on the dimension of the fibers of the map $\Pi$. In Section 6.5 we show that Busemann points are not dense. In Section 6.6 we determine which surfaces result in the map $\Pi$ having a section, and prove that the horoboundary is path connected. Finally, in Section 6.7 we use the previous results to obtain estimates regarding asymptotic values of extremal lengths.

Some of the most dense parts of this chapter are due to the added complexity of considering surfaces with boundary. As such, the reader focused in surfaces with empty boundary might want to omit the corresponding sections on a first reading. One of the largest related parts starts after the remark following Theorem 6.3.4 and ends before the start of Section 6.3.2. The other sizable part starts with Proposition 6.6.3 and ends at the start of the proof of Theorem 1.3.2, where we note that the proof is significantly simpler in the case of surfaces without boundary.

## 6.2 Section specific background

### 6.2.1 The doubling trick

Let $X$ be a Riemann surface with nonempty boundary. Denote by $\overline{X}$ the mirror surface, obtained by composing each atlas of $X$ with the complex conjugation. Gluing $X$ to $\overline{X}$ along the corresponding boundary components we obtain the *conformal double* $X^d = X \cup \overline{X}/\sim$ of $X$. Note that $X^d$ has empty boundary. Given a foliation $F$ or a quadratic differential $q$ on $X$, we can repeat the same process, obtaining the corresponding conformal doubles $F^d$ and $q^d$ on $X^d$. For a more detailed treatment of this argument see [Abi80, Section II.1.5].

The main interest of the conformal doubles is that these are surfaces without boundary, so most of the results relating to Teichmüller theory of surfaces without boundary can be translated to surfaces with boundary. We have the following.

**Proposition 6.2.1.** *Let $X$ be a Riemann surface with boundary, and $F$ be a foliation on $X$. Then,*

$$\operatorname{Ext}_{X^d}(F^d) = 2\operatorname{Ext}_X(F).$$

*Proof.* We have $q_{F^d,X^d} = q^d_{F,X}$, so the result follows, as $\int_{X^d} |q^d_{F,X}| = 2\int_X |q_{F^d,X^d}|$. □

### 6.2.2 The Gardiner–Masur compactification of surfaces with boundary

Recall that for a surface $S$ with marked points and empty boundary we can embed $\mathcal{T}(S)$ into the space of continuous functions from the set $\mathcal{S}$ of simple closed curves on $S$ to $\mathbb{R}$ via

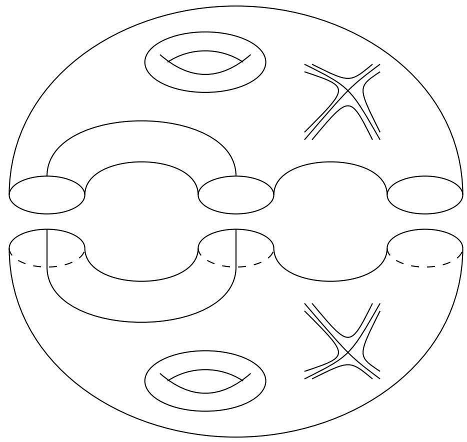


Figure 6.1: Visual representation of the doubling trick.

the map $\phi : \mathcal{T}(S) \to P(\mathbb{R}^{\mathcal{S}})$ defined by

$$\phi(X) = \left[\mathrm{Ext}_X(\alpha)^{1/2}\right]_{\alpha \in \mathcal{S}},$$

where the square brackets indicate a projective vector. Gardiner and Masur show [GM91] that this map is indeed an embedding, and that $\phi(\mathcal{T}(S))$ is precompact. The Gardiner–Masur compactification of a surface without boundary is then defined as the pair $(\phi, \overline{\phi(\mathcal{T}(S))})$.

Alternatively, after choosing a basepoint $b \in \mathcal{T}(S)$, it is also possible to consider the map $\mathcal{E} : \mathcal{T}(S) \to C(\mathcal{MF})$ defined by

$$\mathcal{E}(X)(\cdot) := \left(\frac{\mathrm{Ext}_X(\cdot)}{K_{b,X}}\right)^{1/2},$$

This map is quite similar to the original map $\phi$, the differences being that $\mathcal{E}$ considers all measured foliations instead of just the closed curves, and normalizes instead of projectivizing. Walsh proves [Wal19] that, for surfaces without boundary, the map $\mathcal{E}$ defines a compactification in the same way that $\phi$ does, and in fact this compactification is isomorphic to the one defined by $\phi$.

The compactification defined by $\mathcal{E}$ fits better our goal, so we shall define the Gardiner–Masur compactification of surfaces with boundary as the one obtained by using $\mathcal{E}$. With this in mind, we first need the following result.

**Proposition 6.2.2.** *Let $S$ be a compact surface with possibly boundary and marked points.*

*Then the map $\mathcal{E} : \mathcal{T}(S) \to C(\mathcal{MF})$ is injective.*

*Proof.* Assume we have $x, y \in \mathcal{T}(S)$ with $\mathcal{E}(x)(F) = \mathcal{E}(y)(F)$ for all $F \in \mathcal{MF}$. Then,

$$K_{x,y} = \sup_{F \in P_b} \frac{\operatorname{Ext}_x(F)}{\operatorname{Ext}_y(F)} = \frac{K_{b,x}}{K_{b,y}}$$

and

$$K_{y,x} = \sup_{F \in P_b} \frac{\operatorname{Ext}_y(F)}{\operatorname{Ext}_x(F)} = \frac{K_{b,y}}{K_{b,x}} = K_{x,y}^{-1}.$$

However, $K_{y,x} = K_{x,y}$, since the Teichmüller distance is symmetric. Hence, $K_{x,y} = 1$ and, by Kerckhoff's formula, $d(x, y) = 1/2 \log K_{x,y} = 0$. □

Miyachi shows [Miy08] that the set $E(S) := \{\mathcal{E}(X) \mid X \in \mathcal{T}(S)\}$ is precompact when $S$ is a surface without boundary. Given a surface with boundary $S$, denote $\mathcal{MF}^d(S)$ the set of measured foliations on $S^d$ obtained by doubling the foliations $\mathcal{MF}(S)$. The set $E(S^d)|_{\mathcal{MF}^d(S)} = \{\mathcal{E}(X)|_{\mathcal{MF}^d(S)} \mid X \in \mathcal{T}(S^d)\}$, obtained by restricting the functions in $E(S^d)$ to $\mathcal{MF}^d$, is precompact. Furthermore, we can embed $E(S)$ into $E(S^d)|_{\mathcal{MF}^d(S)}$ by sending $f \in E(S)$ to $f^d \in E(S^d)|_{\mathcal{MF}^d(S)}$ defined by $f^d(F^d) = f(F)$. Hence, $E(S)$ is precompact.

We define the *Gardiner–Masur compactification* for a surface with boundary as the closure $\overline{E}$ of $E(S)$, together with the map $\mathcal{E}$. We shall be using the same characterization for surfaces without boundary.

One of the relevant features of the Gardiner–Masur compactification is that it coincides with the horofunction compactification. Indeed, Liu–Su [LS14] and Walsh [Wal19] prove that for surfaces without boundary these two compactifications are isomorphic. In the following, we shall extend the relevant results to surfaces with boundary. We begin with the driving theorem from Walsh's paper.

**Theorem 6.2.3** (Extension of [Wal19, Theorem 1] to surfaces with boundary)**.** *Let $R(q) : \mathbb{R}_+ \to \mathcal{T}(S)$ be the Teichmüller ray with initial unit-area quadratic differential $q$, and let $F$ be a measured foliation. Then,*

$$\lim_{t \to \infty} e^{-2t} \operatorname{Ext}_{R(q;t)}(F) = \sum_j \frac{i(G_j, F)^2}{i(G_j, H(q))},$$

*where the $\{G_j\}$ are the indecomposable components of the vertical foliation $V(q)$, and $H(q)$ is the horizontal foliation.*

*Proof.* If $S$ does not have boundary the result follows from Walsh's paper. Assume then that $S$ has boundary. Let $p$ be the number of proper arcs of $V(q)$, and reorder the components so $G_j$ is a proper arc for $j \leq p$. The conformal double $G_j^d$ is indecomposable whenever $G_j$ is a proper arc, and decomposes into two components otherwise, as it is not incident to the boundary of $S$. Denote $G_j^1$ and $G_j^2$ the two components of $G_j$ for $j > p$. We have

$$2 \lim_{t\to\infty} e^{-2t} \operatorname{Ext}_{Rqt}(F) = \lim_{t\to\infty} e^{-2t} \operatorname{Ext}_{Rq^dt}(F^d) = \sum_{j\leq p} \frac{i(G_j^d, F^d)^2}{i(G_j^d, H(q)^d)} + \sum_{i\in\{1,2\}} \sum_{j>p} \frac{i(G_j^i, F^d)^2}{i(G_j^i, H(q)^d)}.$$

For foliations $G, F \in \mathcal{MF}(S)$ we have $i(G^d, F^d) = 2i(G, F)$. Hence, $i(G_j^d, F^d) = 2i(G_j, F)$. Using the symmetry, $i(G_j^1, F^d) = i(G_j^2, F^d)$, so for $j > p$ we have $i(G_j^1, F^d) = i(G_j, F)$. Using these identities we get the result. □

Following the same reasoning we can extend as well the next result.

**Lemma 6.2.4** (Extension of [Wal19, Lemma 3] to surfaces with boundary)**.** *Let $q$ be a unit area quadratic differential. Then,*

$$e^{-2t} \operatorname{Ext}_{R(q;t)}(F) \geq \sum_j \frac{i(G_j, F)^2}{i(G_j, H(q))},$$

*where $t \in \mathbb{R}_+$ and $\{G_j\}$ are the indecomposable components of the vertical foliation $V(q)$.*

Most of the results in Walsh's paper use the previous theorem. In particular, we have the following.

**Corollary 6.2.5** (Extension of [Wal19, Corollary 1] to surfaces with boundary)**.** *Let $q$ be a quadratic differential and denote by $G_j$ the components of its vertical foliation. Then, the Teichmüller ray $R(q)$ converges in the Gardiner–Masur compactification to*

$$\left( \sum_j \frac{i(G_j, \cdot)^2}{i(G_j, H(q)} \right)^{1/2}.$$

The relation between the Gardiner–Masur compactification is given by the map $\Xi : \overline{E} \to \overline{\mathcal{T}(S)}^h$ defined by

$$\Xi(f)(x) := \frac{1}{2} \log \sup_{F\in\mathcal{P}} \frac{f(F)^2}{\operatorname{Ext}_x(F)}.$$

The following result can be extended to surfaces with boundary by repeating the proof found in Walsh's paper in this context.

**Theorem 6.2.6** (Extension of [Wal19, Lemma 21] to surfaces with boundary)**.** *The map* $\Xi$ *is an isomorphism between the compactifications* $(\mathcal{E}, \overline{E})$ *and* $(h, \overline{\mathcal{T}(S)}^h)$.

Directly from the definition of $\Xi$ we have the following

**Corollary 6.2.7.** *Let* $f, g \in \overline{E}$. *If* $f \geq g$ *then* $\Xi(f) \geq \Xi(g)$.

For a given quadratic differential $q \in Q^1(o)$, we shall denote by $B(q)$ the Busemann point obtained by following along the geodesic $R(q)$. Furthermore, the representation of the Busemann point $B(q)$ in the Gardiner–Masur compactification shall be denoted as $\mathcal{E}(q)$. By Corollary 6.2.5 we have an explicit representation of $\mathcal{E}(q)$. As we have seen in Propositions 5.1.4 and 5.1.5, the continuity of the Busemann map has some interesting implications, and it is enough to look for continuity of the map restricted to the boundary. Related to this question we have the following result, which can also be derived by the same proof found in Walsh's paper, applied to this context.

**Theorem 6.2.8** (Extension of [Wal19, Theorem 10] to surfaces with boundary)**.** *Let* $(q_n)$ *be a sequence of quadratic differentials based at* $b \in \mathcal{T}(S)$. *Then* $B(q_n)$ *converges to* $B(q)$ *if and only if both of the following hold:*

1. $(q_n)$ *converges to* $q$*;*
2. *for every subsequence* $(G^n)_n$ *of indecomposable elements of* $\mathcal{MF}$ *such that, for each* $n \in \mathbb{N}$, $G^n$ *is a component of* $V(q_n)$, *we have that every limit point of* $G^n$ *is indecomposable.*

In view of this theorem, we say that a sequence of quadratic differentials $(q_n)$ converges *strongly* to $q$ if it does so in the sense described by the theorem.

Finally, while the following result may be extendable to surfaces with boundary, we only use it in the context of surfaces without boundary, so we shall not be working on finding an extension.

**Theorem 6.2.9** ([Wal19, Theorem 3])**.** *For the Teichmüller space of a surface without boundary with the Teichmüller metric, for any basepoint* $X \in \mathcal{T}(S)$, *all Busemann points can be expressed as* $B(q)$ *for some quadratic differential* $q$ *based at* $X$.

## 6.3 Horoboundary convergence for Teichmüller spaces

### 6.3.1 Continuity of the Busemann map

We begin by using Proposition 5.1.5 to determine when the Busemann map is continuous. Recall that a sequence $(q_n)$ converges to $q$ strongly if and only if the sequence satisfies the conditions of Theorem 6.2.8. That is, a sequence $(q_n)$ converges to $q$ strongly if and only if the associated Busemann points $B(q_n)$ converge to $B(q)$. With this in mind we introduce the following notion.

**Definition 6.3.1.** *Let $q$ be a quadratic differential. We say that $q$ is* infusible *if any sequence of quadratic differentials converging to $q$ converges strongly. We say that $q$ is* fusible *if it is not infusible.*

In other words, we say that $q$ is fusible when it can be approached by a sequence of quadratic differentials $(q_n)$ such that there is some sequence $(G^n)$ of measured foliations with each $G^n$ being an indecomposable component of $V(q_n)$, with $(G^n)$ having at least one decomposable accumulation point. The following statement follows directly from this definition, Proposition 5.1.5 and Walsh's result.

**Proposition 6.3.2.** *Let $q$ be a unit area quadratic differential. The Busemann map $B$ is continuous at $q$ if and only if $q$ is infusible.*

*Proof.* If $q$ is fusible then we have a sequence converging to $q$ but not strongly. Hence, by Theorem 6.2.8 the sequence $(B(q_n))$ does not converge to $B(q)$, and so the Busemann map is not continuous at $q$.

If $q$ is infusible we have that any sequence $(q_n)$ converging to $q$ does so strongly, and so we have that $B(q_n)$ converges to $B(q)$, so $B$ is continuous at $q$ when restricted to the boundary. By Proposition 5.1.5 this implies that $B$ is continuous at $q$. □

We shall now find a criterion on the vertical foliation to determine when a unit area quadratic differential is infusible.

**Definition 6.3.3.** *Let $F$ be a measured foliation on a surface $S$ and let $G$ be one of its indecomposable components. We say that $G$ is a* boundary annulus *if it is an annulus parallel to a boundary with no marked points, and a* boundary component *if it is a boundary annulus or a proper arc. If $G$ is not a boundary component, we shall call it an* interior component.

*Each of the connected components of the surface obtained after removing the proper arcs shall be called* interior part*. If each of these interior parts has at most one interior component, then we say that $F$ is* internally indecomposable*. If $F$ is not internally indecomposable we say that it is* internally decomposable*.*

For surfaces without boundary, a foliation $F$ is internally indecomposable if and only if it is indecomposable, as we do not have boundary components. Given these definitions we can state our main result of this section

**Theorem 6.3.4.** *Let $q$ be a quadratic differential. Then $q$ is infusible if and only if its vertical foliation $V(q)$ is internally indecomposable.*

This result is somewhat straightforward whenever $S$ does not have boundary, as in order to have a sequence $(q_n)$ that converges to $q$ but not strongly we need a sequence of components of $V(q_n)$ converging to a decomposable component of $V(q)$, but if $S$ is closed and $V(q)$ is internally indecomposable, then $V(q)$ only has one indecomposable component. Conversely, if $V(q)$ has more than one indecomposable component, as $S$ does not have boundary $V(q)$ can be approached by a sequence of simple closed curves, so the associated sequence of quadratic differentials converges to $q$ but not strongly.

For surfaces with boundary the proof is more involved, as simple closed curves are no longer dense. However, the density of multicurves from Proposition 2.2.1 allows us to follow a slightly similar reasoning. We begin by proving some results regarding the shape that foliations have to take when approaching a foliation with boundary components, namely, boundary components have to be eventually included in the approaching foliations.

**Proposition 6.3.5.** *Let $(F_n)$ be a sequence of measured foliations converging to a measured foliation $F$, let $G$ be the union of the boundary components of $F$ and let $H$ be such that $F = H + G$. Then, for $n$ big enough, $F_n = H_n + a_n G$, with $a_n$ converging to $1$ and $H_n$ converging to $H$.*

In particular, the proper arcs of the limiting foliation have to be included in the approaching foliations. Hence, we will be able to separate the surface along these proper arcs into the interior parts of the limiting foliation, and study the convergence in each of these parts.

We say that a subset of a boundary component is a *boundary arc* if it is homeomorphic to an open interval or a circle, does not contain marked points and, if it is homeomorphic to an open interval, it is delimited by marked points.

Repeating the argument by Chen–Chernov–Flores–Fortier Bourque–Lee–Yang [CCF+18] to a more general setting we get the following characterization of foliations on simple surfaces, which we shall use to solve the simpler cases.

**Lemma 6.3.6.** *Let $S$ be a sphere with one boundary component possibly containing boundary marked points and one interior marked point. Then every indecomposable foliation on $S$ is a proper arc and there are finitely many distinct proper arcs.*

*Proof.* Assuming that there is some foliation $F$ with a recurrent leaf to some part of $S$ we get a contradiction, as explained in the proof of [CCF+18, Lemma 4.1]. Hence, each indecomposable foliation is a curve. Any closed curve in $S$ is contractible to the marked point. Hence, a each indecomposable foliation is a proper arc.

A proper arc in $S$ must have two endpoints, which must be contained in the boundary arcs in the boundary component of $S$. Denote $b_1$ and $b_2$ these two boundary arcs, which might be the same. We aim to show that there are at most two classes of arcs with endpoints in $b_1$ and $b_2$. Fix three proper arcs with endpoints on $b_1$ and $b_2$. Any intersection between these arcs can be removed by doing isotopies moving the endpoints along the arcs $b_1$ or $b_2$. Hence, these arcs can be isotoped to not intersect each other. Since there is only one interior marked point, two of these arcs delimit a rectangle with no marked interior marked points, so are isotopic. Hence, there are at most two different proper arcs between $b_1$ and $b_2$. There are finitely marked points in the boundary component, so there are finitely many boundary arcs. Therefore, there are finitely many pairs of boundary arcs, and since we have at most two proper arcs per pair, there are also finitely many different proper arcs. □

We shall first see the proposition for the case where $G$ contains a proper arc and we are approaching with a sequence of indecomposable foliations.

**Lemma 6.3.7.** *Let $S$ be a surface and let $(F_n)$ be a sequence of indecomposable foliations on $S$ converging to a measured foliation $G$. Then $G$ is either a multiple of a proper arc $\gamma$, in which case $F_n$ is also a multiple of $\gamma$ for $n$ big enough, or $G$ does not contain a proper arc.*

*Proof.* Assume $G$ contains a proper arc $\gamma$ with weight $w > 0$ and denote $b$ one of the boundary arcs where $\gamma$ is incident.

Our first step is seeing that, for $n$ big enough, $F_n$ intersects $b$. We shall do this by finding different test curves $\beta$ depending on the shape of $b$. If the boundary component containing

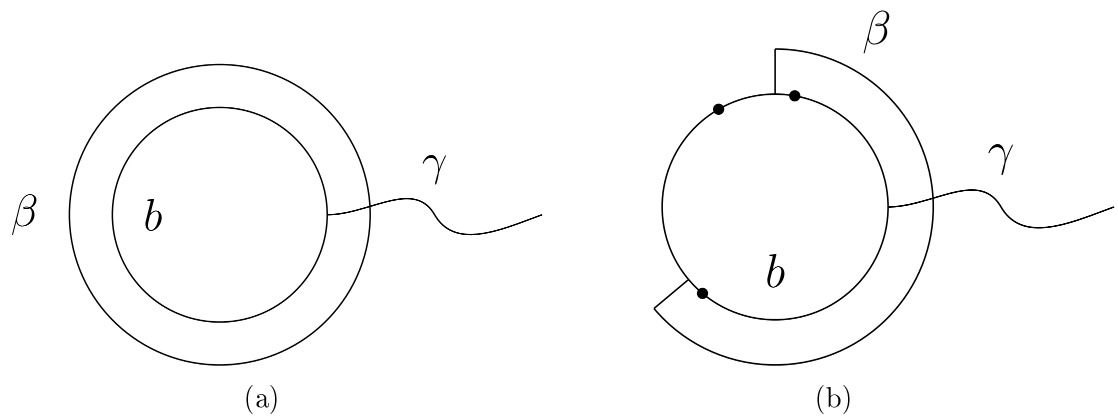


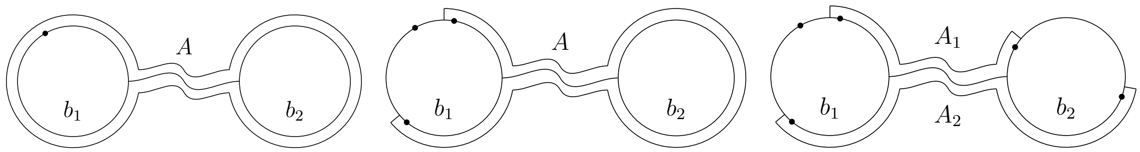

Figure 6.2: Sample curves used in the proof of Lemma 6.3.7

$b$ has at most one marked point, we consider $\beta$ to be a curve parallel to that boundary component as in Figure 6.2a. Otherwise we consider $\beta$ to be the curve defined by taking a small arc starting at the boundary arc next to $b$, concatenating with a curve parallel to $b$, and concatenating another segment with endpoint in the boundary arc after $b$, as shown in Figure 6.2b.

If the curve $\beta$ is contractible then $S$ is a sphere with one boundary component and at most one interior marked point, so by Lemma 6.3.6 the result follows. Assume then that $\beta$ is not contractible. We have $i(\gamma, \beta) > 0$, so $i(G, \beta) > 0$ and hence $i(F_n, \beta) > 0$ for $n$ big enough, which implies that $F_n$ intersects $b$. Hence, since $F_n$ is indecomposable, it is a weighted proper arc, which we denote $w_n\gamma_n$, where $w_n > 0$ is the weight at $\gamma_n$ is a proper arc.

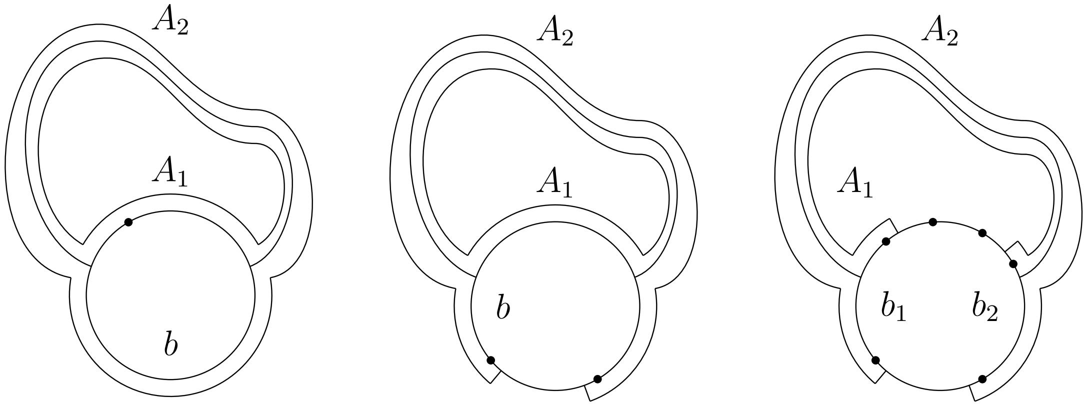
Denote $b_1$ and $b_2$ the boundary arcs where $\gamma$ has its endpoints, and denote $\beta_1$ and $\beta_2$ the associated test curves shown in Figure 6.2. If both endpoints are in the same boundary arc we set $b_2$ and $\beta_2$ as null curves. We shall now find a multicurve $A$ surrounding $\gamma$, $b_1$ and $b_2$ such that any leaf of $G$ intersecting $A$ but not $\gamma$ has an endpoint in either $b_1$ or $b_2$. The multicurve $A$ is chosen so that, together with the boundaries where $\gamma$ has its endpoints, delimits the smallest surface containing $\gamma$. The precise shape of $A$ depends on whether the endpoints of $\gamma$ are in the same boundary component or not, and the distribution of marked points in these boundaries.

If both endpoints of $\gamma$ are in different boundary components we proceed differently according to the distribution of marked points at these boundaries. If each of the boundaries




(a) (b) (c)

Figure 6.3: Construction of the curves $A_1$ and $A_2$ whenever $\gamma$ has endpoints in different boundary components in the proof of Lemma 6.3.7




(a) (b) (c)

Figure 6.4: Construction of the curves $A_1$ and $A_2$ whenever $\gamma$ has endpoints in the same boundary component in the proof of Lemma 6.3.7

contains at most one marked point then we define $A$ as the curve shown in Figure 6.3a. If one of the boundary components has two or more marked points, but the other has at most one marked point we define $A$ as the arc shown in Figure 6.3b. Finally, if each of the boundaries contains at least two marked marked points we define $A$ as the multicurve formed by the curves $A_1$ and $A_2$ as shown in Figure 6.3c.

If both endpoints $\gamma$ are in the same boundary we also proceed differently according to the distribution of marked points. In all cases $A$ is defined as a multicurve formed by two curves. If each possible segment within the boundary component joining the two endpoints has at most one marked points we proceed as in Figure 6.4a. If one of these segments has two or more marked points, while the other has at most one we proceed as in Figure 6.4b. Finally, if both of these segments have two or more marked points we proceed as in Figure 6.4c.

In any of the cases above if a component of $A$ is non essential we remove it from $A$. The following argument also applies whenever $A$ is a null curve. Put $A$ and $G$ in minimal position and denote $P$ the surface containing $\gamma$, delimited by $A$ and the boundary components where $\gamma$ has its endpoints. Let $\alpha$ be a connected component of a non critical leaf of $G$ restricted to $P$ intersecting $A$. Since $G$ contains $\gamma$ the proper arc $\alpha$ cannot intersect $\gamma$. Furthermore, by observing the possible configurations, if $\alpha$ has one endpoint in $A_1$, the other one cannot be in $A_2$, as whenever we have both $A_1$ and $A_2$, these are separated within $P$ by the proper arc $\gamma$. Furthermore, if both endpoints are in $A_1$ then $\alpha$ can be isotoped to not intersect $A$. Therefore, the other endpoint of $\alpha$ is in either $b_1$ or $b_2$. Hence, $i(G,\beta_1)+i(G,\beta_2) \geq i(G,A)+w\,i(\gamma,\beta_1)+w\,i(\gamma,\beta_2) > i(G,A)$. Since $w_n\gamma_n$ converges to $G$, this last inequality implies that for $n$ big enough,

$$i(\gamma_n,\beta_1)+i(\gamma_n,\beta_2) > i(\gamma_n,A).$$

Fix $n$ such that $\gamma_n$ satisfies the previous inequality. Assume $\gamma_n$ has just one endpoint inside $P$. Then, $i(\gamma_n,\beta_1)+i(\gamma_n,\beta_2)=1$, so $i(\gamma_n,A)=0$ and $\gamma_n$ cannot leave $P$. If $\gamma_n$ has both endpoints in $P$ then $i(\gamma_n,\beta_1)+i(\gamma_n,\beta_2)=2$. Furthermore, if $\gamma_n$ leaves $P$, then it has to reenter at some point, resulting in $i(\gamma_n,A_1+A_2)=2$. Hence, $\gamma_n$ stays inside $P$.

The weights $w_n$ do not converge to 0, as $w_n i(\gamma_n,\beta)$ converges to $i(G,\beta)$, but $i(\gamma_n,\beta)\leq 2$. Since $\gamma$ is contained in $G$ we have $i(G,\gamma)=0$. Therefore, for any $\varepsilon>0$ and $n$ big enough we have $w_n i(\gamma_n,\gamma)<\varepsilon$, so for $n$ big enough $i(\gamma_n,\gamma)=0$. Since $\gamma_n$ does not intersect $\gamma$ and stays inside $P$, $\gamma_n$ can be isotoped to stay inside one of the components obtained after removing $\gamma$ from $P$. Denote $C$ such component. The component $C$ has either one or two boundary components and no interior marked points or one boundary component and one interior marked point. By Lemma 6.3.6 the only case where we do not have finitely many different proper arcs is when $C$ has two boundary components. However, in that case one of the boundary components is associated to a curve in $A$, so $\gamma_n$ does not intersect it and that boundary can be treated as a marked point. Hence, in all cases there are finitely many possible proper arcs, and so $\gamma_n$ is a multiple of $\gamma$ for $n$ big enough. □

When the boundary component is an annulus we have to be a bit more careful, so we start by proving it for approaching curves.

**Lemma 6.3.8.** *Let $S$ be a surface and let $(w_n\gamma_n)$ be a sequence of weighted curves on $S$ converging to a foliation $G$, where $(w_n)$ are the weights and $(\gamma_n)$ are the curves. Then $G$ is either a multiple of a boundary annulus $\gamma$, in which case $\gamma_n$ is $\gamma$ for $n$ big enough, or $G$ does* *not contain a boundary annulus.*

*Proof.* If $S$ is a polygon with at most one interior marked point, then $G$ cannot contain a boundary annulus. If $S$ is a cylinder then, since we have a boundary annulus, at least one of the boundaries must not contain marked points. Hence, the number of curves is finite, as there is only one possible closed curve, and for counting the proper arcs we can consider the boundary without marked points as a marked point and apply Lemma 6.3.6. In that case, the conclusion follows.

Assume then that $S$ is neither a disk with at most one interior marked point nor a cyclinder with no interior marked points. Then there is a pair of pants $P$ in $S$ containing $\gamma$ where each boundary component of $P$ is either non contractible or contractible to a marked point. Denote $B_1$ the boundary component parallel to $\gamma$ and $B_2$ and $B_3$ the other two boundary components of $P$. Furthermore, assume that $G$ contains $\gamma$ with weight $w$.

Begin by assuming that $B_2$ and $B_3$ are not contractible to marked points. Let $C$ be the proper arc contained in $P$ with both endpoints in $B_1$. Put $B_2$, $B_3$ and $C$ in a minimal position with respect to $G$, and consider a connected component of a noncritical leaf of $G$ intersecting $C$ restricted to $P$. This noncritical leaf either is isotopic to $\gamma$, or to the curves $F$, $E$ and $D$ shown in Figure 6.5. Since the leaves of $G$ do not intersect, there cannot be leaves isotopic to $E$ and leaves isotopic to $D$ at the same time. Breaking symmetry, assume there are no leaves isotopic to $D$. Then, $i(C,G) = i(C,\gamma) + i(B_3,G) > i(B_3,G) \geq i(B_2,G)$. Doing the same reasoning assuming that there are no leaves isotopic to $E$ we get $i(C,G) > \max(i(B_2,G), i(B_3,G))$. Hence, since $w_n\gamma_n$ converges to $G$, $\gamma_n$ has to satisfy

$$i(C,\gamma_n) > \max(i(B_2,\gamma_n), i(B_3,\gamma_n))$$

for $n$ big enough.

For each $n$ put $B_3$, $B_2$ and $C$ in a minimal position with respect to $\gamma_n$, and consider the restriction of $\gamma_n$ to $P$. Assume $\gamma_n$ is not $\gamma$. Then, the curves on the restriction of $\gamma_n$ to $P$ intersecting $C$ are isotopic to either $E, F$ and $D$, but not $\gamma$. As before, this restriction cannot contain curves isotopic to $E$ and curves isotopic to $D$ for the same $n$, so assuming there are no curves isotopic to $D$ we have $i(C,\gamma_n) = i(B_3,\gamma_n)$ which is a contradiction. Doing the same reasoning assuming that there are no curves isotopic to $E$ also gives a contradiction. Hence, $\gamma_n$ is $\gamma$ for $n$ big enough.

If $B_2$ or $B_3$ are contractible to marked points we have $i(G,B_2)$ or $i(G,B_3)$ is 0, and a

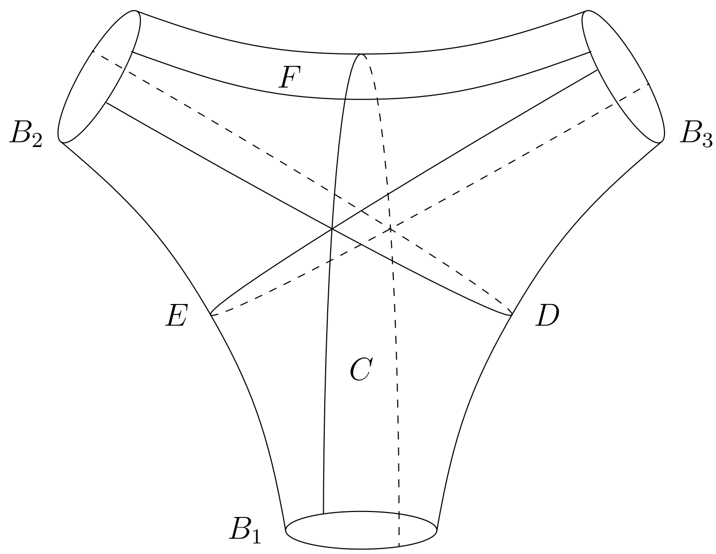



Figure 6.5: Curve labeling for the proof of Lemma 6.3.8

similar reasoning yields the same result. □

*Proof of Proposition 6.3.5.* Let $(F_n)$ be a sequence of measured foliations converging to $F$. As pointed out before, Proposition 2.2.1 can be extended to get sequences of weighted multicurves $(\gamma_n^m)_m$ converging to each $F_n$. Denote $\gamma_{n,1}^m, \gamma_{n,2}^m, \ldots, \gamma_{n,k(n,m)}^m$ the weighted curves of $\gamma_n^m$. For each $n$ we take a subsequence such that $k(n,m)$ is constant with respect to $m$, and $\gamma_{n,i}^m$ converges for each $i$ as $m \to \infty$. Denoting $F_{n,i}$ the limit of $\gamma_{n,i}^m$ as $m \to \infty$, we can write $F_n = \sum F_{n,i}$.

Denote $\beta_j$ the boundary components of $F$. That is, $\sum \beta_j = G$. Furthermore, denote $b_{n,j}$ and $b_{n,j}^m$ the weights of $\beta_j$ on $F_n$ and $\gamma_n^m$, where we set the weight to be 0 if $\beta_j$ is not contained in the foliation. It is clear that if $b_{n,j} = 0$ then $b_{n,j}^m \to 0$, as we must have $b_{n,j} \geq \liminf_{m\to\infty} b_{n,j}^m$. If $b_{n,j} > 0$ for some $n$, then $F_{n,i}$ contains $\beta_j$ for some $i$. Hence, by Lemmas 6.3.7 and 6.3.8 we have $F_{n,i}$ and $\gamma_{n,i}^m$ are both multiples of $\beta_j$ for $m$ big enough. Then, since each of the multicurves in $\gamma_n^m$ has to be different, $\beta_j$ is not contained in any other foliation $F_{n,i}$ for that given $n$, so $F_{n_i} = b_{n,j}\beta_j$ and $\gamma_{n,i}^m$ can be written as $b_{n,i}^m\beta_j$ for $m$ big enough, with $b_{n,i}^m$ converging to $b_{n,j}$ as $m \to \infty$.

Assume for some $j$ we have $b_{n,j}$ not converging to 1. We can then take a subsequence such that $b_{n,j}$ converges to some $\lambda \neq 1$. Denote $\delta = |1-\lambda|/2$. For each $n$, there exists some $m_0(n)$ big enough so that $|1 - b_{n,j}^m| > \delta$ for all $m \geq m_0(n)$. We can then take a diagonal sequence

$\gamma_n^{m(n)}$ converging to $F$ with $m(n) \geq m_0(n)$. However, following the previous reasoning we get that $\gamma_n^{m(n)}$ should contain $\beta_j$ for $n$ big enough, and the weight should converge to the weight in $G$, that is, to 1. However, $|1 - b_{n,j}^{m(n)}| > \delta$, giving us a contradiction. Hence, $b_{n,j}$ converges to 1 for all $j$. Let then $a_n = \min_j(b_{n,j})$. Since $b_{n,j} \geq a_n$ we can define $H_n = F_n - a_n G$ and we have $F_n = H_n + a_n G$. Finally, $a_n \to 1$ as $n \to \infty$, so the proposition is proved. □

**Proposition 6.3.9.** *Let $q$ be a unit area quadratic differential such that $V(q)$ is internally indecomposable. Then $q$ is infusible.*

*Proof.* Assume $q$ is fusible, that is, we have a sequence of quadratic differentials $(q_n)$ converging to $q$ but not strongly. Let $F_i^n$ be the indecomposable components of $V(q_n)$. To have non-strong convergence we must have at least one sequence of indecomposable components converging to a decomposable component $G$, which we assume is $(F_1^n)_n$. Let $\beta$ be a boundary component of $V(q)$. By Proposition 6.3.5 for $n$ big enough a multiple of $\beta$ must be contained in $V(q_n)$. Furthermore, $\beta$ cannot be contained in $G$. Since $G$ cannot contain boundary components, it must contain at least two interior components. On the other hand, since $V(q)$ is internally indecomposable, each interior part obtained by removing the proper arcs contains at most one interior component. Hence, for $n$ big enough $F_1^n$ must intersect at least two interior parts, that is, $F_1^n$ must cross at least one proper arc. However, for each proper arc $\gamma$ there is some $n$ big enough such that $\gamma$ is contained in the foliation $V(q_n)$, so $F_1^n$, a component of $V(q_n)$, intersects the foliation $V(q_n)$, giving us a contradiction. □

To prove the other direction we shall first see the following lemma.

**Lemma 6.3.10.** *Let $S$ be a compact surface with with possibly nonempty boundary and finitely many marked points, let $k \geq 2$ and let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a collection of non intersecting closed curves on $S$. Furthermore, let $p$ be the number of curves in $\alpha$ parallel to a boundary. Then there exists a collection of $\max(\lceil (p/2) \rceil), 1)$ non intersecting curves intersecting each $\alpha_i$.*

Our main interest in the lemma is that the amount of curves needed is strictly smaller than the amount of closed curves in $\alpha$. This will allow us, by doing Dehn twists along the closed curves in $\alpha$, to create a sequence of foliations converging to a foliation with strictly more components, which can be translated to a sequence of quadratic differentials that converge but not strongly. The proof of this lemma is based on a reasoning found in [FM12, Proposition 3.5].

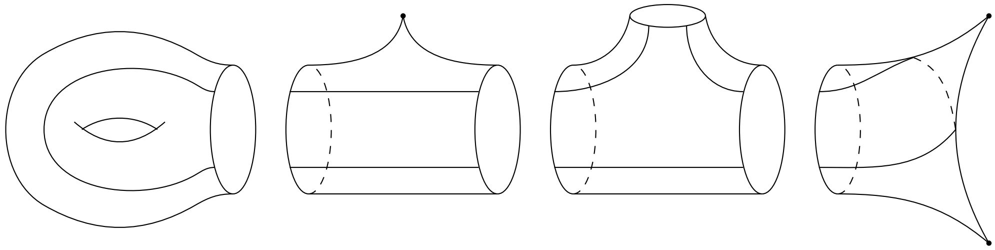


Figure 6.6: Laying out of curve segments for the proof of Lemma 6.3.10

*Proof.* We start by replacing all boundaries of $S$ without parallel curves in $\alpha$ by marked points. Let then $\alpha'$ be a completion of $\alpha$ to a pair of pants decomposition. Glue the remaining boundaries pairwise until we have at most one left. After cutting the surface along the closed curves that were not parallel to boundaries we get a collection of $\lceil p/2 \rceil$ tori with one boundary component and some spheres with $b$ boundary components and $n$ marked points, with $b + n = 3$ and $b \geq 1$. If $p$ is odd, one of these spheres has a boundary of $S$ as a boundary. We join the boundaries of each of these surfaces with non intersecting arcs, as shown in Figure 6.6, that is, in such a way that each boundary component has two arcs incident to it. We can then paste these surfaces back together in order to obtain a collection $\beta_1, \beta_2, \ldots, \beta_l$ of pairwise disjoint curves in $S$. If $p$ is odd this collection contains precisely one proper arc, as we only have two endpoints coming from the boundary we did not paste. If $p$ is even the collection does not contain any proper arc. By the bigon criterion each $\beta_j$ is in minimal position with respect to each $\alpha_i$, and each $\alpha_i$ intersects either one or two of the $\beta_j$. Furthermore, since we did not cut along the original boundaries we pasted from $S$, each $\alpha_i$ parallel to a boundary of $S$ intersects precisely one of the $\beta_j$. Suppose we have $\beta_j$ and $\beta_{j'}$ intersecting a curve $\kappa \in \alpha'$ and that $\beta_j$ and $\beta_{j'}$ are distinct. Since we have at most one proper arc, at least one of $\beta_j$ and $\beta_{j'}$ is a closed curve. Hence, doing a half twist about $\kappa$, $\beta_j$ and $\beta_{j'}$ become a single curve. Since this process does not create any bigons, the resulting collection is still in minimal position with $\alpha$. Continuing this way we obtain a single curve $\gamma$ intersecting each curve in $\kappa$. Furthermore, $\gamma$ intersects each pasted boundary once. Cutting along the pasted boundaries, we get the curves from the lemma. If $p$ is odd, $\beta$ is a proper arc, so each cut along a pasted boundary increases the curve count by one, totalling $(p+1)/2$ curves. If $p$ is even, $\beta$ is a closed curve, so the first cut transforms it into a proper arc, and the following ones increase the curve count by one, giving a total of $\max(p/2, 1)$ curves. □

**Proposition 6.3.11.** *Let $F$ be an internally decomposable measured foliation. Then, $F$ can* *be approached by a sequence of weighted multicurves with fewer components than $F$.*

*Proof.* By the extension to Proposition 2.2.1, we have a sequence of weighted multicurves $\gamma^n$ converging converging to $F$, with the only proper arcs being the ones contained in $F$. Cutting the surface along the proper arcs of $\gamma^n$ and quotienting these proper arcs to points we get $k$ many surfaces $Z_1, Z_2, \dots, Z_k$ with boundary. Let $\gamma_i^n$ be the restriction of $\gamma^n$ to $Z_i$, and let $F_i$ be the limit of $\gamma_i^n$. The foliation $F$ is the union of the foliations $F_i$ and the proper arcs.

Fix some $i$ such that $F_i$ is nonempty, and let $\alpha_1, \dots, \alpha_b$ be the closed curves parallel to the boundaries of $Z_i$. Let $a_1^n, \dots, a_b^n$ be the weights of $\alpha_1, \dots, \alpha_b$ in $\gamma_i^n$. We can take a subsequence such that $a_j^n$ converges for each $j$ to some $a_j$. If $a_j > 0$, the closed curve $\alpha_j$ is contained in $F_i$. If $a_j = 0$, then the weights $a_j^n$ can be set to 0 on the multicurves $\gamma_i^n$ while leaving the limit intact. Hence, we can assume that $a_j^n = 0$ for all $j$ such that $a_j = 0$. Let $p$ and $u$ be the number of closed curves with $a_j > 0$ parallel to boundaries with or without marked points respectively. Since we have removed all the closed curves with $a_j = 0$, the multicurve $\gamma_i^n$ contains precisely $p$ and $u$ closed curves parallel to boundaries with or without marked points for $n$ big enough. Denote by $B$ the set of closed curves parallel to boundary components without marked points. Applying Lemma 6.3.10 to the multicurve $\gamma_i^n$ minus $B$ we get $\max(\lceil (p/2) \rceil), 1)$ curves $\beta_i^n$ intersecting all closed curves in $\gamma_i^n$ except the ones parallel to boundaries without marked points. Doing the appropriate Dehn twists along the closed curves of $\gamma_i^n$ and rescaling to the curves $\beta_i^n$, and adding with the corresponding weights the curves in $B$, we get a sequence converging to $\gamma_i^n$ with $\max(\lceil (p/2) \rceil), 1) + u$ many components. As such, taking a diagonal sequence we can get a sequence of multicurves converging to $F_i$ with each multicurve containing $\max(\lceil (p/2) \rceil), 1) + u$ components.

Finally, since $F$ is internally decomposable, there is at least one $F_i$ with at least 2 interior components, so one of these multicurves has strictly less components than the limiting foliations, and we have non-strong convergence. □

Theorem 6.3.4 follows by combining Propositions 6.3.9 and 6.3.11.

We do not need $S$ to have a lot of topology to find internally decomposable foliations. In fact, determining which surfaces do not support internally decomposable foliations we get the following result.

**Proposition 6.3.12.** *Let $S_{g,b_m,b_u,p}$ be a surface of genus $g$ with $b_m$ and $b_u$ boundaries with and without marked points respectively and $p$ interior marked points. Then the Busemann* *map is continuous if and only if* $3g + 2b_m + b_u + p \leq 4$.

We shall split the proof in the following two lemmas

**Lemma 6.3.13.** *Let* $S_{g,b_m,b_u,p}$ *be a surface with* $3g + 2b_m + b_u + p > 4$. *Then it admits an internally decomposable foliation.*

*Proof.* A multicurve consisting of two interior closed curves generates an internally decomposable foliation, so we just have to find such a pair for each possible surface satisfying the hypothesis. If $S$ has genus at least 2 we can take a multicurve consisting of 2 non separating closed curves. If $S$ is a torus with at least 2 boundaries or marked points, or a boundary with marked points, we can take a non separating closed curve and a separating closed curve around 2 boundaries or marked points, or around a boundary with marked points. If $S$ is a sphere with at least 5 marked points or boundaries, we can take a closed curve around two interior points or boundaries, and a closed curve around two different interior points or boundaries. If $S$ is a sphere with 1 boundary with marked points and at least 3 other boundaries or interior points we can take a closed curve around the boundary with marked points, and a closed curve around two other interior points or boundaries. Lastly, if $S$ is a sphere with 2 boundaries with marked points and another interior marked point or boundary we take a closed curve around each boundary with marked points. $\square$

**Lemma 6.3.14.** *Let* $S_{g,b_m,b_u,p}$ *be a surface with* $3g + 2b_m + b_u + p \leq 4$. *Then every foliation on* $S$ *is internally indecomposable.*

*Proof.* Assume we have an internally decomposable foliation on $S_{g,b_m,b_u,p}$. Then we can get an internally decomposable foliation on $S_{g,0,0,b_u+p+2b_m}$ by removing the boundary components, replacing the boundaries without marked points with marked points and each boundary with marked points for 2 marked points. Furthermore, if we have at least one marked point, we can get an internally decomposable foliation in $S_{g,0,0,b_u+p+2b_m+k}$, $k \in \mathbb{N}$, by replacing a marked point with a $k+1$ marked points.

Hence, we only need to prove that a torus with one marked point and a sphere with 4 marked points do not admit internally decomposable foliations. However, since these do not have boundaries, a foliation being internally decomposable translates to a foliation having at least two indecomposable components.

Assume the torus with one marked point admits a foliation with two indecomposable components. We can replace the marked point with a boundary, and add to the foliation a boundary component parallel to that boundary. Considering the doubled surface explained in Section 6.2.1 we get a closed surface of genus 2 without boundaries nor marked points, with at least 5 indecomposable components. Recall that the maximum number of indecomposable components for a foliation on a surface of genus $g$ is $3g-3$, so for genus 2 the maximum is 3, giving us a contradiction. A similar process applies for the sphere with 4 marked points. $\square$

*Proof of Proposition 6.3.12.* The Busemann map is continuous at every point in the interior of Teichmüller space, as it is the identity when restricted in there and $\partial\overline{X}^v$ is closed. Hence, we only need to prove continuity or discontinuity at the points on the boundary. By Lemma 6.3.13 if $3g + 2b_m + b_u + p > 4$ then $S$ admits an internally decomposable foliation $F$, so by Theorem 6.3.4 the Hubbard–Masur quadratic differential associated to $F$ at the basepoint $X$ is fusible and hence the Busemann map is not continuous at that point. On the other hand, if $3g + 2b_m + b_u + p \leq 4$ then by Lemma 6.3.14 for any quadratic differential $q$, the vertical foliation $V(q)$ is internally indecomposable, so again by Theorem 6.3.4 every quadratic differential is infusible an $B$ is continuous at every boundary point. $\square$

By combining Proposition 6.3.12 with Proposition 5.1.4, we get the precise classification of surfaces with horofunction compactification isomorphic to visual compactification announced in Theorem 6.1.2 from the introduction.

*Proof of Theorem 6.1.2.* As shown in Proposition 5.1.4, the visual compactification and the horofunction compactification are isomorphic if and only if the Busemann map is continuous, so the theorem follows by applying Proposition 6.3.12. $\square$

### 6.3.2 Criteria for convergence

One straightforward consequence of the horofunction compactification being finer than the visual compactification is the following criterion regarding the convergence of sequences in the horofunction compactification.

**Corollary 6.3.15.** *Let $(x_n) \subset \mathcal{T}(S)$ be a sequence. If $(x_n)$ converges to a quadratic differential $q$ in the visual compactification, then all accumulation points of $(x_n)$ in the horofunction compactification are contained in $\Pi^{-1}(q)$. In particular, if $V(q)$ is internally indecomposable, then $(x_n)$ converges in the horofunction compactification.*

*Furthermore, if $(x_n)$ does not converge in the visual compactification, then it does not converge in the horofunction compactification.*

*Proof.* If $x_n$ converges in the visual compactification to a quadratic differential $q$ then by the continuity of $\Pi$ all its accumulation points are in $\Pi^{-1}(q)$. If $V(q)$ is internally indecomposable, then by Theorem 6.3.4 the quadratic differential $q$ is infusible, so the Busemann map is continuous at $q$ and by Proposition 5.1.5 the fiber $\Pi^{-1}(q)$ is a singleton. Therefore $x_n$ converges to $\Pi^{-1}(q)$, as that is the only accumulation point of $x_n$ and the horofunction compactification is compact.

On the other hand, if $x_n$ converges to $\xi$ in the horofunction compactification, by continuity of $\Pi$, $x_n$ converges to $\Pi(\xi)$ in the visual compactification. □

A frequent topic in the study of compactifications of Teichmüller spaces is the convergence of certain measure-preserving paths. We shall see now how the previous results can be applied in that study.

Let $X \in \mathcal{T}(S)$ be a point in Teichmüller space and $q$ be a unit quadratic differential based at $X$. It is a well known fact that there exists a unique orientation-preserving isometric embedding $\iota : \mathbb{H} \to \mathcal{T}(S)$ from the hyperbolic plane $\mathbb{H}$ to the Teichmüller space such that $\iota(i) = X$ and $\iota^*(q) = i$, see the work of Herrlich–Schmithüsen [HS07] for a detailed explanation. The path $\iota(i+t)$ for $t \in \mathbb{R}_+$ is called the *horocycle* generated by $q$. Since $\iota$ is an isometric embedding, $h(X)(p) = d(\iota^{-1}X, \iota^{-1}p) - d(\iota^{-1}X, \iota^{-1}b)$ for $X, b, p \in \iota(\mathbb{H})$. That is, if we restrict the evaluations of horofunctions to the image of the Teichmüller disc, the value coincides with the values in the hyperbolic plane. Hence, since the path $i+t$ is a horocycle of the Busemann point obtained by moving along the geodesic $e^t i$ along the hyperbolic plane, the path $\iota(i+t)$ is also a horocycle of the corresponding Busemann point $B(q)$, obtained by moving along the geodesic $\iota(e^t i)$.

Since $\iota$ is an isometric embedding, the geodesic between $X$ and $\iota(i+t)$ is contained in $\iota(\mathbb{H})$. Furthermore, the pushforward and pullback maps are continuous, so denoting $q_t$ the unit quadratic differential spawning the geodesic between $X$ and $\iota(i+t)$, we have $\lim_{t\to\infty} \iota^*(q_t) = i$, and $\iota_*(i) = q$, so $\lim_{t\to\infty} q_t = q$. The distance between $\iota(i+t)$ and $X$ grows to infinity, so any horocycle path generated by some $q$ based at $X$ converges to $q$ in the visual compactification based at $X$. Hence, horocycles generated by infusible quadratic differentials converge in the horofunction compactification, which had been previously shown by Jiang–Su [JS16] and Alberge [Alb16] in the context of surfaces without boundary.

**Corollary 6.3.16.** *Let $S$ be a compact surface with possibly nonempty boundary and finitely many marked points and let $q$ be an infusible quadratic differential based at any $X \in \mathcal{T}(S)$.*

*Then the horocycle generated by $q$ converges in the horofunction compactification.*

*Proof.* The horocycle path converges to $q$ in the visual compactification based at $X$, so by Corollary 6.3.15 all accumulation points in the horofunction compactification are contained in $\Pi_X^{-1}(q)$. Furthermore, since $q$ is infusible, $\Pi_X^{-1}(q)$ is a singleton, so the horocycle path has a unique accumulation point in the horofunction compactification, and hence it converges. $\square$

On the other hand, Fortier Bourque found some diverging horocycles in the horofunction compactification.

**Theorem 6.3.17** (Fortier Bourque [FB23, Theorem 1.1])**.** *Let $S$ be a closed surface of genus $g$ with $p$ marked points, such that $3g+p \geq 5$. Then there is some fusible quadratic differential $q$ based at some basepoint $X \in \mathcal{T}(S)$ such that the associated horocycle path does not converge in the horofunction compactification.*

Corollary 6.3.15 gives an upper limit on the set of accumulation points, as it has to be contained in $\Pi_X^{-1}(q)$.

Furthermore, by Corollary 5.2.14 we have that a path converges in the horofunction compactification if and only if it converges in each visual compactification. Hence, such a divergent horocycle also diverges in some visual compactification. That is, we get Corollary 6.1.7. This contrasts with the behavior of Teichmüller rays, which by Corollary 5.2.7 or [Wal19, Theorem 7] converge in all visual compactifications.

## 6.4 Dimension of the fibers

Our first approach in determining the shape of the fibers is looking at the limits of Busemann points, which by Proposition 5.2.10 give us bounds on the elements of $\Pi^{-1}(q)$. For a given quadratic differential $q$ and a foliation $G$ we define $\mathcal{W}^q(G)$ as the map from measured foliations to $\mathbb{R}$ given by

$$\mathcal{W}^q(G) = \frac{i(G,\cdot)^2}{i(G,H(q))},$$

if $i(G,H(q)) > 0$, and $\mathcal{W}^q(G) = 0$ otherwise. By the extension of Walsh's Corollary 6.2.5 describing Busemann points in the Gardiner–Masur compactification, we see that the element $\mathcal{E}_q = \Xi^{-1}B_q$ has the form $\sqrt{\sum_i \mathcal{W}^q(V_i)}$, where $V_i$ are the indecomposable components of $V(q)$. Hence, a reasonable path to follow for understanding the limits of Busemann points is understanding the limits of $\mathcal{W}^q$ as $q$ varies.

**Lemma 6.4.1.** *Let $q_n$ be a sequence of quadratic differentials on $X$ converging to $q$, and let $V_j^n$, $0 < i \leq c(n)$ be the indecomposable components of $V(q_n)$. Let $G^n$ be a sequence of non zero measured foliations of the form $\sum \alpha_j^n V_j^n$, converging to a measured foliation $G$. Then*

$$\lim_{n\to\infty} \mathcal{W}^{q_n}(G^n) = \mathcal{W}^q(G)$$

*if $G$ is non zero and $\lim_{n\to\infty} \mathcal{W}^{q_n}(G^n) = 0$ if $G$ is zero, where the convergence is pointwise in both cases.*

*Proof.* For any measured foliation $F$ we have $\mathcal{W}^{q_n}(G^n)(F) = \frac{i(G^n,F)^2}{i(G^n,H(q_n))}$, so if $G$ is non zero the lemma follows by continuity of the intersection number.

If $G$ is zero the result follows from applying the same proof than in [Wal19, Lemma 27]. □

Denote $\mathcal{B}$ the set of Busemann points, $\overline{\mathcal{B}}$ its closure and $\overline{\mathcal{B}}(q)$ the intersection $\overline{\mathcal{B}} \cap \Pi^{-1}(q)$. We can use the previous lemma to show that the elements of $\overline{\mathcal{B}}(q)$ satisfy certain properties.

**Proposition 6.4.2.** *Let $S$ be a closed surface with possibly marked points, $\xi \in \overline{\mathcal{B}}(q)$ and $V_i$, $i \in \{1, \dots, k\}$ be the indecomposable components of $V(q)$. Denote $x_i = \frac{i(V_i,\cdot)}{i(V_i,H(q))}$. Then, the square of the representation of $\xi$ in the Gardiner–Masur compactification, $(\Xi^{-1}\xi)^2$, is a homogeneous polynomial of degree 2 in the variables $x_i$, whose coefficients sum to 1.*

Recall that we are using a normalized version of the Gardiner–Masur compactification. Under the projectivized version the sum of the coefficients cannot have any fixed value.

*Proof.* Since the surface does not have boundary, all Busemann points are of the form $B(q')$ for some quadratic differential of unit area $q'$. Consider a sequence $(q_n)$ such that $B(q_n)$ converges to $\xi$ and $q_n$ converges to $q$. Let $c(n)$ be the number of indecomposable vertical components of $V(q_n)$, and let $V_j^n$, $0 < j \leq c(n)$ be those components. We know that $c(n)$ is bounded by some number depending on the topology of the surface. Take a subsequence such that $c(n)$ is equal to some constant $c$ and $V_j^n$ converges for each $j$. The sum $\sum_{j=1}^c V_j^n$ converges as $n \to \infty$ to $\sum_{i=1}^k V_i$, so the limit of each $V_j^n$ has to be of the form $\sum_{i=1}^k \alpha_j^i V_i$. Furthermore, $\sum_{j=1}^c \alpha_j^i = 1$, since

$$\sum_{i=1}^k V_i = V(q) = \lim_{n\to\infty} V(q_n) = \lim_{n\to\infty} \sum_{j=1}^c V_j^n = \sum_{j=1}^c \sum_{i=1}^k \alpha_j^i V_i = \sum_{i=1}^k \left( \sum_{j=1}^c \alpha_j^i \right) V_i.$$

The element associated to the Busemann point $B(q_n)$ in the Gardiner–Masur compactification satisfies

$$\mathcal{E}_{q_n}^2 = \sum_{j=1}^{c} \mathcal{W}^{q_n}(V_j^n).$$

Hence, applying Lemma 6.4.1 we get the following expressions for the square of the limit of Busemann points:

$$(\Xi^{-1}\xi)^2 = \sum_{j=1}^{c} \mathcal{W}^q \left( \sum_{i=1}^{k} \alpha_j^i V_i \right) = \sum_{j=1}^{c} \frac{\left( \sum_{i=1}^{k} \alpha_j^i i(V_i, H(q)) x_i \right)^2}{\sum_{i=1}^{k} \alpha_j^i i(V_i, H(q))}.$$

That is, we get a homogeneous polynomial of degree 2 in the variables $x_i$. Since $q$ has unit area, the sum of the coefficients is

$$\sum_{j=1}^{c} \sum_{i=1}^{k} \alpha_j^i i(V_i, H(q)) = \sum_{i=1}^{k} i(V_i, H(q)) = 1,$$

which completes our claim. □

By Proposition 5.2.9, the Busemann point $B(q)$ gives an upper bound on all functions in $\Pi^{-1}(q)$. While Proposition 5.2.10 does not give us a lower bound directly, we can use Lemma 2.1.1 to get one. For a unit area quadratic differential $q$, let $Z_j$ be the interior parts of $V(q)$, and denote $G_j$ the union of interior indecomposable components within $Z_j$. Further, let $P_i$ be the boundary components of $V(q)$. We define the *minimal point* at $q$ as

$$M(q) = \Xi \left( \sum_i \mathcal{W}^q(P_i) + \sum_j \mathcal{W}^q(G_j) \right)^{1/2}.$$

**Proposition 6.4.3.** *Let $q$ be a quadratic differential. Then, for any $\xi \in \Pi^{-1}(q)$, we have*

$$\Xi^{-1}\xi \geq \Xi^{-1} M(q)$$

*in the Gardiner–Masur compactification. Furthermore, $M(q) \in \Pi^{-1}(q)$ whenever each $G_j$ has at most two annuli parallel to the boundaries of $Z_j$ with marked points.*

In the context of surfaces without boundary the previous result has been also proven by

Liu–Shi in [LS22, Lemma 3.10]. In such context we have $M(q) = \Xi i(V(q), \cdot)^2$, which by the proposition is always contained in $\Pi^{-1}(q)$.

The minimality is essentially derived from the following well-known inequality.

**Lemma 6.4.4** (Titu's lemma)**.** *For any positive reals $a_1, \dots, a_n$ and $b_1, \dots, b_n$ we have*

$$\sum_j \frac{a_j^2}{b_j} \geq \frac{\left(\sum_j a_j\right)^2}{\sum_j b_j}.$$

*Proof.* The inequality can be written as

$$\sum_i b_i \sum_j \frac{a_j^2}{b_j} \geq \left(\sum_j a_j\right)^2,$$

so the result follows after applying the Cauchy–Schwartz inequality. □

The implication this lemma has for our discussion is that $\mathcal{W}^q(\cdot)$ is convex, in the sense that for any $G = \sum_i G_i$ and any measured foliation $F$ we have

$$\sum_i \mathcal{W}^q(G_i)(F) \geq \mathcal{W}^q(G)(F).$$

*Proof of Proposition 6.4.3.* If $q$ is infusible then each $G_j$ is indecomposable, so $M(q) = B(q)$, the fiber $\Pi^{-1}(q)$ has one point and the proposition is satisfied.

Consider then $q$ fusible and $\xi \in \Pi^{-1}(q)$. Let $(x_n) = (R(q_n; t_n)) \subset \mathcal{T}$ converging to $\xi$. By Lemma 6.2.4 we have $\Xi^{-1}(h(x_n)) \geq \Xi^{-1} B(q_n)$. Hence, $\Xi^{-1}\xi \geq \liminf_{n\to\infty} \Xi^{-1} B(q_n)$.

Given a measured foliation $F$, take a subsequence so that

$$\liminf_{n\to\infty} \Xi^{-1} B(q_n)(F) = \lim_{n\to\infty} \Xi^{-1} B(q_n)(F).$$

The foliations $V(q_n)$ converge to $V(q)$, so by Proposition 6.3.5 for $n$ big enough all boundary components $P_i$ are contained within $V(q_n)$. Hence, for $n$ big enough the foliations $V(q_n)$ can be split to the interior parts $Z_j$ by cutting along the proper arcs. Denote $G_j^n$ the interior components of the foliation $V(q_n)$ restricted to $Z_j$. Let $G_{j,k}^n$ be the indecomposable components of $G_j^n$. The sequence $G_j^n$ converges to $G_j$, so we can take a subsequence such that each $G_{j,k}^n$ converges to some foliation $G_{j,k}$ with $\sum_k G_{j,k} = G_j$. Applying Lemma 6.4.1

we have

$$\lim_{n\to\infty} \Xi^{-1}B(q_n)(F) = \lim_{n\to\infty} \sum_i^n \mathcal{W}^{q_n}(P_i) + \sum_j \sum_k \mathcal{W}^{q_n}\left(G^n_{j,k}\right) = \sum_i \mathcal{W}^q(P_i) + \sum_j \sum_k \mathcal{W}^q\left(G_{j,k}\right).$$

Hence, applying Lemma 6.4.4 to the second sum we get the first part of the proposition.

To observe that the limit is actually reached we can repeat the proof of Proposition 6.3.11 and observe that a proper arc for each interior part is enough to approach the foliation whenever each interior part of the foliation has at most two annuli parallel to boundaries with marked points. □

By Corollary 6.2.7 this lower bound is carried to the horofunction representation and by Proposition 5.2.9 we have an upper bound. Hence, we have the chain of inequalities

$$M(q) \leq \xi \leq B(q),$$

for any $\xi \in \Pi^{-1}(q)$. As we see in the next proposition, this chain can be translated as well to the Gardiner–Masur compactification.

**Proposition 6.4.5.** *Let $\xi \in \Pi^{-1}(q)$. Then,*

$$\Xi^{-1}\xi \leq \Xi^{-1}B(q).$$

*Proof.* We have a sequence of points $R(q_n;t_n)$ converging to $\xi$, with $q_n$ converging to $q$. By Lemma 5.2.3 we have $\xi(R(q;t)) = -t$. Further, $R(q_n;t_n)$ converges in the Gardiner–Masur compactification to the function $f(G)^2 = \lim_{n\to\infty} e^{-2t_n} \operatorname{Ext}_{R(q_n;t_n)}(G)$, and we have $\Xi f(x) = \xi(x)$. Hence,

$$\frac{1}{2}\log \frac{f(F)}{\operatorname{Ext}_{R(q;t)}(F)} \leq \frac{1}{2}\log \sup_{G\in P} \frac{f(G)}{\operatorname{Ext}_{R(q;t)}(G)} = -t.$$

Upon exponentiating and reordering the terms, we get

$$\lim_{n\to\infty} e^{-2t_n} \operatorname{Ext}_{R(q_n;t_n)}(F) = f^2(F) \leq e^{-2t} \operatorname{Ext}_{R(q;t)}(F)$$

for all $t$. Letting $t \to \infty$, the right hand side converges to $(\Xi^{-1}B(q)(F))^2$, so we get the proposition. □

Using these bounds we can further refine the characterization of points in $\Xi^{-1}\Pi^{-1}(q)$.

**Proposition 6.4.6.** *Let $q$ be a quadratic differential, let $V_i$, $i \in \{1, \dots, k\}$ be the indecomposable components of $V(q)$ and let $x_i(F) = \frac{i(V_i,F)}{i(V_i,H(q))}$. Given $f \in \Xi^{-1}\Pi^{-1}(q)$ and $c > 0$ we have, for all $F \in \mathcal{MF}$,*

$$f^2(F) = c^2 + 2c\sum_i i(V_i, H(q))(x_i(F) - c) + \sum_{i,j} O\left((x_i(F) - c)(x_j(F) - c)\right).$$

*In particular, as a function of the values $x_i(F)$ at the point $x_i = c$ for all $i$, $f^2(x_1, \dots, x_k)$ takes value $c^2$, is differentiable and satisfies $\frac{\partial}{\partial x_i} f^2(x_1, \dots, x_k) = 2c\, i(V_i, H(q))$.*

*Proof.* We have that $\left(\Xi^{-1}M(q)\right)^2 \leq f^2 \leq \left(\Xi^{-1}B(q)\right)^2$. Denoting $a_i = i(V_i, H(q))$ and $x_i = x_i(\cdot)$ we have by Lemmas 6.4.4 and 6.4.3 that $\left(\sum a_i x_i\right)^2 \leq \left(\Xi^{-1}M(q)\right)^2$. Writing the bounds on $f^2$ in terms of the variables $x_i$, we obtain

$$\left(\sum a_i x_i\right)^2 \leq f^2 \leq \sum a_i x_i^2.$$

Adding that $\sum a_i = 1$, we have that $f^2$ is bounded below by the arithmetic mean, and above by the quadratic mean. Rewritting both sides as a polynomial in $x_i - c$, we get

$$c^2 + 2c\sum a_i(x_i - c) + \left(\sum a_i(x_i - c)\right)^2 \leq f^2 \leq c^2 + 2c\sum a_i(x_i - c) + \sum a_i(x_i - c)^2,$$

so the first part of the proposition is satisfied. Subbing in the value $x_i(F) = c$ we get the second part. $\square$

By Propositions 5.2.4 and 5.2.13 all members of $\Pi^{-1}(q)$ share their values along $R(q; \cdot)$, as well as the directional derivatives at the points of the geodesic. For a given $q$ we have $x_i(\lambda H(q)) = \lambda$ for all $i$ and all $\lambda > 0$. Hence, Proposition 6.4.6 shows a similar relation for the representations of the elements of $\Pi^{-1}(q)$ in the Gardiner–Masur compactification, as they share their value, as well as some derivatives, at all foliations of the form $\lambda H(q)$.

As shown by Fortier Bourque [FB23], the Gardiner–Masur boundary contains extremal length functions, so we can use Proposition 6.4.6 to get some information on the differentials of these functions. Namely, we recover in a more restricted setting the following result, proven in [Miy13a, Theorem 1.1].

**Theorem 6.4.7** (Miyachi)**.** *Let $G_t$, $t \in [0, t_0]$ be a path in the space of measured foliations on $X$ which admits a tangent vector $\dot{G}_0$ at $t = 0$ with respect to the canonical piecewise linear structure. Then, the extremal length* $\operatorname{Ext}(G, X)$ *is right-differentiable at $t = 0$ and satisfies*

$$\frac{d}{dt^+} \operatorname{Ext}(G_t, X)\bigg|_{t=0} = 2i(\dot{G}_0, F_{G_0,X}),$$

*where $F_{G_0,X}$ is the horizontal foliation of the Hubbard–Masur differential associated to $G_0$ on $X$.*

The concrete extremal length functions in the Gardiner–Masur boundary we are going to use are given by the following theorem.

**Theorem 6.4.8** (Fortier Bourque)**.** *Let $\{w_1, \ldots, w_k\}$ be weights with $w_i > 0$, let $\phi_n = \tau_1^{\lfloor nw_1 \rfloor} \circ \cdots \circ \tau_k^{\lfloor nw_k \rfloor}$ be a sequence of Dehn multitwist around a multicurve $\{\alpha_1, \ldots, \alpha_k\}$ in a surface $S$ and let $X \in \mathcal{T}(S)$. Then the sequence $\phi_n(X)$ converges to*

$$\left[\operatorname{Ext}^{1/2}\left(\sum_{i=1}^{k} w_i i(F, \alpha_i)\alpha_i, X\right)\right]_{F \in \mathcal{MF}(S)}$$

*in the projective Gardiner–Masur compactification as $n \to \infty$.*

The precise statement of this result is slightly weaker [FB23, Corollary 3.4], but the same proof yields this extension.

Fix a multicurve $\{\alpha_1, \ldots, \alpha_k\}$, weights $\{w_1, \ldots, w_k\}$ and denote $\alpha = \sum w_i \alpha_i$. Furthermore, normalize the weights $\{w_1, \ldots, w_k\}$ so that there is a unit area quadratic differential $q$ such that $V(q) = \alpha$. Denote $V_i$ the vertical components of $V(q)$. That is, $V_i = w_i \alpha_i$. We are able to recover Miyachi's formula when $i(V_i, H(q)) = w_i$ for all $i$. The sequence $\phi_n(X)$ converges in the visual compactification based at $X$ to $q \in T_X \mathcal{T}(S)$. By Theorem 6.4.8 the function $f(F) = \lambda^{1/2} \operatorname{Ext}^{1/2}\left(\sum_{i=1}^{k} w_i i(F, \alpha_i)\alpha_i, X\right)$ is in $\Xi^{-1}\Pi^{-1}(q)$ for some $\lambda > 0$. We have $i(F, \alpha_i) = x_i(F) i(V_i, H(q))/w_i$. So, assuming $i(V_i, H(q)) = w_i$ we can write

$$f^2(F) = \lambda \operatorname{Ext}\left(\sum_{i=1}^{k} x_i(F) V_i, X\right).$$

We have $x_i(H(q)) = 1$ for all $i$, so by Proposition 6.4.6 the value of $\lambda$ satisfies

$$f^2(H(q)) = \lambda \operatorname{Ext}(V(q), X) = 1.$$

Since $q$ has unit area, $\operatorname{Ext}(V(q), X) = 1$, so $\lambda = 1$. Let $I$ be any foliation such that $H(q)+I$ is well defined, and let $F_t = H(q) + tI$. We have

$$f^2(F_t) = \operatorname{Ext}\left(\sum_i V_i + t\sum_i x_i(I)V_i, X\right).$$

Hence, denoting $J = \sum x_i(I)V_i$ and $G_t = V(q) + tJ$ we can apply Proposition 6.4.6 to get

$$\frac{d}{dt^+}\operatorname{Ext}(G_t, X)\bigg|_{t=0} = \sum_i \frac{dx_i}{dt}\bigg|_{t=0} \frac{\partial f^2}{\partial x_i}\bigg|_{x_i=1} = \sum_i \frac{i(V_i, I)}{i(V_i, H(q))} \cdot 2i(V_i, H(q)) = 2i(V(q), I).$$

On the other hand, applying Miyachi's Theorem 6.4.7 directly we get

$$\begin{aligned}\frac{d}{dt^+}\operatorname{Ext}(G_t, X)\bigg|_{t=0} &= 2i(H(q), J) = 2\sum_i i(H(q), V_i)x_i(I) \\ &= 2\sum_i i(H(q), V_i)\frac{i(V_i, I)}{i(H(q), V_i)} = 2i(V(q), I),\end{aligned}$$

so both expressions coincide, and we have recovered Theorem 6.4.7 in this rather restricted setting. We would like to note that Proposition 6.4.6 also gives some information for finding the second derivatives around the point $H(q)$. Namely, the second derivatives cannot diverge to infinity as we approach $H(q)$.

Combining Proposition 6.4.6 with Proposition 6.4.2 we get fairly restrictive necessary conditions for the points in $\overline{\mathcal{B}}(q)$ for surfaces without boundary. We shall be using these conditions in Section 6.5 to prove that Busemann points are not dense in the horoboundary. Now we prove a more straightforward consequence. For a topological space $U$, denote $\dim(U)$ its Lebesgue dimension. See the book by Munkres [Mun00, Chapter 5.80] for some background on basic dimension theory. Given an embedding $U \hookrightarrow V$ we have $\dim(U) \leq \dim(V)$, so the conditions for the points on $\overline{\mathcal{B}}(q)$ gives us the following result.

**Corollary 6.4.9.** *Let $S$ be a surface without boundary. Let $q$ be a quadratic differential such* *that $V(q)$ has $n$ indecomposable components. Then,*

$$\dim(\overline{\mathcal{B}}(q)) \leq \frac{n(n-1)}{2}.$$

*Proof.* By Proposition 6.4.2 we have an embedding of $\overline{\mathcal{B}}(q)$ into the space of homogeneous polynomials of degree 2. For a given $\xi \in \overline{\mathcal{B}}(q)$, let $b^{\xi}_{i,j}$ be the coefficient of $x_i x_j$. Adding the restriction $b_{i,j} = b_{j,i}$ we have a coefficient for each possible pair, so the dimension of homogeneous polynomials of degree 2 is equal to the number of possible pairs, that is, $\frac{n(n+1)}{2}$. Furthermore, by Proposition 6.4.6 we know the value of the first derivatives at $x_i = c$ for all $i$. For each $i$ this gives us the linear equation $\sum_{j \neq i} b^{\xi}_{i,j} + 2b^{\xi}_{i,i} = 2i(V_i, H(q))$. These $n$ equations are linearly independent, as $b^{\xi}_{i,i}$ is only contained on the equation related to $x_i$. As such, the dimension of the coefficients is at most $\frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$.

We note that the sum of the coefficients being 1 is the equation we get when summing the $n$ equations given by the derivatives, so we cannot use that to restrict further the dimension. □

Recall that the number of indecomposable components $n$ is bounded in terms of the topology of the surface. Hence, the previous corollary gives us a uniform upper bound on the dimension of $\overline{\mathcal{B}}(q)$. More interestingly, we can also get a lower bound for the dimension of $\overline{\mathcal{B}}(q)$. This allows us to get a lower bound on the dimension of $\Pi^{-1}(q)$. Furthermore, as this is a lower bound, we do not need to restrict ourselves to surfaces without boundary, as the set of Busemann points always contains the set of Busemann points of the form $B(q)$. The bound is obtained by finding a dimensionally big set of different ways to approach a certain $q$ along the boundary and showing that each of these different approaches results in different limits for the associated Busemann points.

**Theorem 6.4.10.** *Let $S$ be a surface of genus $g$ with $b_m$ and $b_u$ boundaries with and without marked points respectively and $p$ interior marked points. Then there is some unit quadratic differential $q$ such that*

$$\dim(\overline{\mathcal{B}}(q)) \geq 2\left\lfloor \frac{g+b_m}{2} + \frac{b_u+p}{4} - \sigma(g, b_u+p) \right\rfloor,$$

*where $\sigma$ has value*

- *0 if $g \geq 2$,*

- *1/4 if* $g = 1$ *and* $b_u + p \geq 1$*,*
- *1/2 if* $g = 1$ *and* $b_u + p = 0$ *or* $g = 0$ *and* $b_u + p \geq 2$*,*
- *3/4 if* $g = 0$ *and* $b_u + p = 1$ *and*
- *1 if* $g = 0$ *and* $b_u + p = 0$*.*

*Proof.* For simplicity we shall first do the proof in the case where $b_m = b_u = p = 0$, and $g \geq 2$. Let $q$ be the quadratic differential such that $V(q)$ is the union of the closed curves $V_1, \ldots, V_{3C}$ shown in Figure 6.7, where $C = \lfloor g/2 \rfloor$. Let $U \subset \mathbb{R}^{3C}$ be the space of vectors $(\alpha_1, \alpha_2, \ldots, \alpha_{3C})$ with positive coefficients and such that

$$\alpha_{3k+1} + \alpha_{3k+2} + \alpha_{3k+3} = \frac{1}{C}. \tag{6.4.1}$$

Each independent linear restriction reduces the dimension of the set $U$ by 1, so $\dim U = 2C$. Hence, to prove the simplest case of the theorem it suffices to build an injective continuous map from $U$ to $\overline{\mathcal{B}}(q)$.

Choose $\alpha \in U$ and consider the multicurve $\gamma^\alpha = \sum \alpha_i G_i$, where $G_i$ are as in Figure 6.7. We will shortly show that by applying Dehn twists about the closed curves $V_i$ to $\gamma^\alpha$ we can get a sequence of multicurves approaching $V(q)$. We can then take the sequences of associated Busemann points, which as we will see converge to distinct points in $\Pi^{-1}(q)$. We will define the injective continuous map from $U$ to $\Pi^{-1}(q)$ by setting it as the limit of the associated sequence of Busemann points, giving us the theorem.

Let $\tau_i$ be the Dehn twist around $V_i$, and let $w_i^\alpha$ be such that

$$w^\alpha_{3k+1}(\alpha_{3k+2} + \alpha_{3k+3}) = w^\alpha_{3k+2}(\alpha_{3k+3} + \alpha_{3k+1}) = w_{3k+3}(\alpha_{3k+1} + \alpha_{3k+2}) = \frac{1}{3C}. \tag{6.4.2}$$

Define $\phi_n^\alpha = \tau_1^{\lfloor w_1^\alpha n \rfloor} \circ \tau_2^{\lfloor w_2^\alpha n \rfloor} \circ \cdots \circ \tau_{3C}^{\lfloor w_{3C}^\alpha n \rfloor}$. For $1 \leq k \leq C$ and $j \in \{1,2,3\}$ Denote $F^\alpha_{k,j} = \sum_{i \in \{1,2,3\} - j} w^\alpha_{3k+i} V_{3k+i}$. By counting the intersections between the curves $V_i$ and $G_i$ we have that there is some sequence $\lambda_n$ such that $\lambda_n \phi_n^\alpha G_{3k+j}$ converges to $F^\alpha_{k,j}$ for all $k, j$ as $n \to \infty$. By the conditions on the weights, $\lambda_n \phi_n^\alpha \gamma^\alpha$ converges to $V(q)$. Let $q_n^\alpha$ be the quadratic differential associated to $\lambda_n \phi_n^\alpha \gamma^\alpha$. Since $\lambda_n \phi_n^\alpha \gamma^\alpha$ converges to $V(q)$, we have that $q_n$ converges to $q$, so all accumulation points of $(B(q_n))$ are in $\Pi^{-1}(q)$. We know that

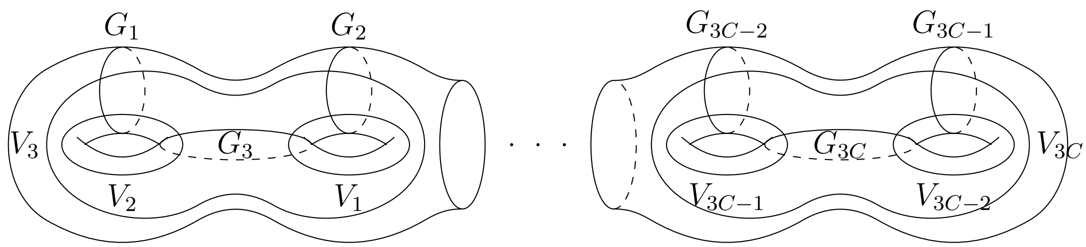



Figure 6.7: Labelling of the curves when the surface has no boundaries nor marked points. If $g$ is odd then there is an unused handle.

$(\Xi^{-1}B(q_n^\alpha))^2 = \sum_i \mathcal{W}^q(\alpha_i\lambda_n\phi_n^\alpha G_i)$, so by Lemma 6.4.1 we have

$$(\xi^\alpha)^2 = \lim_{n\to\infty}(\Xi^{-1}B(q_n^\alpha))^2 = \sum_{k=0}^{C-1}\sum_{j\in\{1,2,3\}} \alpha_{3k+j}\mathcal{W}^q(F_{k,j}^\alpha).$$

Define then the map from $U$ to $\Pi^{-1}(q)$ sending $\alpha \in U$ to $\Xi\xi^\alpha \in \Pi^{-1}(q)$. As before, we shall denote $x_i := \frac{i(V_i,\cdot)}{i(V_i,H(q))} = 3Ci(V_i,\cdot)$. With this notation we have

$$\mathcal{W}^q(F_{k,j}^\alpha) = \frac{i(F_{k,j}^\alpha,\cdot)^2}{i(F_{k,j}^\alpha, H(q))} = \frac{\left(\sum_{i\notin\{1,2,3\}-j} w_{3k+i}^\alpha x_{3k+i}\right)^2}{3C\sum_{i\notin\{1,2,3\}-j} w_{3k+i}^\alpha}.$$

That is, given $\alpha$ we know precisely the shape of the polynomial $\xi^\alpha$. Since $\alpha$ has positive coefficients, each of the $w_i^\alpha$ depends continuously on $\alpha$, so $\xi^\alpha$ depends continuously on $\alpha$.

It remains to show injectivity. Let $\beta \in U$ be such that $\xi^\alpha = \xi^\beta$. While we have equated two polynomials, we cannot conclude directly that the coefficients are equal, as these cannot be evaluated for arbitrary values. However, we can evaluate at elements of the form $b_1G_{3k+1} + b_2G_{3k+2} + b_3G_{3k+3}$ for $b_1, b_2, b_3 \geq 0$, which is enough to prove that $\xi^\alpha$ and $\xi^\beta$ have the same coefficients.

Equating then the coefficients for $x_{3k+1}x_{3k+2}$, $x_{3k+2}x_{3k+3}$ and $x_{3k+1}x_{3k+3}$ we get

$$\begin{aligned}
\frac{\alpha_{3k+1} w^{\alpha}_{3k+2} w^{\alpha}_{3k+3}}{w^{\alpha}_{3k+2} + w^{\alpha}_{3k+3}} &= \frac{\beta_{3k+1} w^{\beta}_{3k+2} w^{\beta}_{3k+3}}{w^{\beta}_{3k+2} + w^{\beta}_{3k+3}}, \\
\frac{\alpha_{3k+2} w^{\alpha}_{3k+1} w^{\alpha}_{3k+3}}{w^{\alpha}_{3k+1} + w^{\alpha}_{3k+3}} &= \frac{\beta_{3k+2} w^{\beta}_{3k+1} w^{\beta}_{3k+3}}{w^{\beta}_{3k+1} + w^{\beta}_{3k+3}} \qquad \text{and} \\
\frac{\alpha_{3k+3} w^{\alpha}_{3k+1} w^{\alpha}_{3k+2}}{w^{\alpha}_{3k+1} + w^{\alpha}_{3k+2}} &= \frac{\beta_{3k+3} w^{\beta}_{3k+1} w^{\beta}_{3k+2}}{w^{\beta}_{3k+1} + w^{\beta}_{3k+2}}.
\end{aligned}$$

Dividing these equalities and using equations (6.4.1) and (6.4.2) we get

$$\begin{aligned}
\frac{\alpha_{3k+1}}{\alpha_{3k+2}} \frac{(1/C + \alpha_{3k+2})}{(1/C + \alpha_{3k+1})} &= \frac{\beta_{3k+1}}{\beta_{3k+2}} \frac{(1/C + \beta_{3k+2})}{(1/C + \beta_{3k+1})}, \\
\frac{\alpha_{3k+2}}{\alpha_{3k+3}} \frac{(1/C + \alpha_{3k+3})}{(1/C + \alpha_{3k+2})} &= \frac{\beta_{3k+2}}{\beta_{3k+3}} \frac{(1/C + \beta_{3k+3})}{(1/C + \beta_{3k+2})} \qquad \text{and} \\
\frac{\alpha_{3k+3}}{\alpha_{3k+1}} \frac{(1/C + \alpha_{3k+1})}{(1/C + \alpha_{3k+3})} &= \frac{\beta_{3k+3}}{\beta_{3k+1}} \frac{(1/C + \beta_{3k+1})}{(1/C + \beta_{3k+3})}.
\end{aligned}$$

Rearranging the first equality we have

$$\frac{\alpha_{3k+1}}{\beta_{3k+1}} \frac{\beta_{3k+2}}{\alpha_{3k+2}} = \frac{(1/C + \alpha_{3k+1})}{(1/C + \beta_{3k+1})} \frac{(1/C + \beta_{3k+2})}{(1/C + \alpha_{3k+1})}. \tag{6.4.3}$$

If $\frac{\alpha_{3k+1}}{\beta_{3k+1}} < 1$ we have $\frac{(1/C+\alpha_{3k+1})}{(1/C+\beta_{3k+1})} > \frac{\alpha_{3k+1}}{\beta_{3k+1}}$, and if $\frac{\alpha_{3k+2}}{\beta_{3k+2}} > 1$ we have $\frac{(1/C+\alpha_{3k+1})}{(1/C+\beta_{3k+1})} < \frac{\alpha_{3k+1}}{\beta_{3k+1}}$. Assume then that $\alpha_{3k+1} < \beta_{3k+1}$. One of the factors of the left hand side of the product in Equation (6.4.3) is replaced in the right hand side by a larger value. Hence, the other factor has to be replaced by a smaller value. That is, the inequality $\alpha_{3k+2} < \beta_{3k+2}$ has to be satisfied. Similarly, if $\alpha_{3k+2} < \beta_{3k+2}$ we have $\alpha_{3k+3} < \beta_{3k+3}$. Equation (6.4.1) leads to

$$\frac{1}{C} = \alpha_{3k+1} + \alpha_{3k+2} + \alpha_{3k+3} < \beta_{3k+1} + \beta_{3k+2} + \beta_{3k+3} = \frac{1}{C},$$

which is a contradiction. Similarly, $\alpha_{3k+1} > \beta_{3k+1}$ leads to another contradiction, so $\alpha_{3k+1} = \beta_{3k+1}$, which leads to $\alpha = \beta$. Therefore, $\dim(\overline{\mathcal{B}}(q)) \geq \dim(U) = 2\left\lfloor \frac{g}{2} \right\rfloor$.

Assume now that $g \geq 2$ and there are some marked points or boundaries. For each pair of marked points or unmarked boundaries, or for each marked boundary we can repeat the

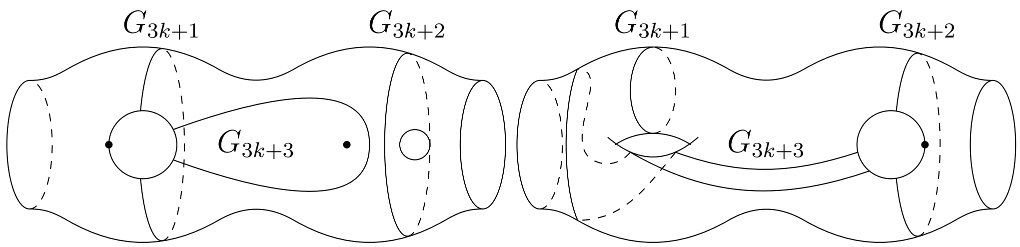



Figure 6.8: Each pair of marked points and boundary components without marked points can replace a genus, as well as each boundary with marked points.

proof with an extra genus, by replacing the curves $G_i$ by the curves shown in Figure 6.8, and halving the associated weights for $w_i$, as the curves intersect now twice the vertical components instead of once.

If $g = 1$ we need to place at least one feature at one of the ends to prevent the curve $G_1$ from being contractible or parallel to a unmarked boundary, so if we have marked points or boundaries without marked points we place these, as boundaries with marked points are more effective at increasing the dimension. In this way we get that if $b_u + p \geq 1$ then

$$\dim(\overline{\mathcal{B}}(q)) \geq 2 \left\lfloor \frac{g + b_m}{2} + \frac{b_u + p - 1}{4} \right\rfloor$$

and if $b_u + p = 0$ then

$$\dim(\overline{\mathcal{B}}(q)) \geq 2 \left\lfloor \frac{g + b_m - 1}{2} \right\rfloor .$$

Lastly, if $g = 0$ we need to place two elements, one at each end. Using the same choice as we took for $g = 1$ we get

$$\dim(\overline{\mathcal{B}}(q)) \geq 2 \left\lfloor \frac{b_m}{2} + \frac{b_u + p - 2}{4} \right\rfloor \text{ for } b_u + p \geq 2,$$

$$\dim(\overline{\mathcal{B}}(q)) \geq 2 \left\lfloor \frac{b_m - 1}{2} \right\rfloor \text{ for } b_u + p = 1 \text{ and}$$

$$\dim(\overline{\mathcal{B}}(q)) \geq 2 \left\lfloor \frac{b_m - 2}{2} \right\rfloor \text{ for } b_u + p = 0.$$

□

We would like to note that this lower bound is does not look optimal to us. Furthermore, the method used is restricted to getting to the dimension of the closure of Busemann points, so the dimension of the whole fiber may be significantly larger than what could be achieved by refining the strategy from the proof.

## 6.5 Non density of the Busemann points

### 6.5.1 Busemann points are not dense in the horoboundary

By Proposition 6.4.2 we know that points in the closure of Busemann points are smooth in the Gardiner–Masur representation with respect to certain variables. By showing that at least one point in the horoboundary is not smooth with respect to the corresponding variables we will prove that Busemann points are not dense. The points we use for this analysis are once again the ones found by Fortier Bourque in Theorem 6.4.8.

Following Fortier Bourque's reasoning, we shall first prove the non density for the sphere with five marked points, and then lift to general closed surfaces by using the branched coverings given by the following Lemma, found in [GM20, Lemma 7.1].

**Lemma 6.5.1** (Gekhtman–Markovic)**.** *Let $S$ be a closed surface of genus $g$ with $p$ marked points, such that $3g + p \geq 5$. Then there is a branched cover $\overline{S_{g,p}} \to \overline{S_{0,5}}$ that branches at all preimages of marked points that are not marked and induces an isometric embedding $\mathcal{T}(S_{0,5}) \hookrightarrow \mathcal{T}(S_{g,p})$.*

The particular conformal structure given to $S_{0,5}$ is obtained as follows. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and let $C = S^1 \times [-1, 1]$. We obtain a sphere $\Sigma$ by sealing the top and bottom of $C$ via the relation $(x, y) \sim (-x, y)$ for all $(x, y) \in S^1 \times \{-1, 1\}$. Let $P$ be set consisting of the five points $(0, \pm 1)$, $(1/2, \pm 1)$ and $(0, 0)$. The pair $S = (\Sigma, P)$, where we view $\Sigma$ as a topological space, is the sphere with five marked points. We get a point $X$ in $\mathcal{T}(S)$ by considering the complex structure on $\Sigma$ obtained by the construction, using the identity map as our marking.

Let $\alpha(t) = (t, 1/2)$ and $\beta(t) = (t, -1/2)$ for $t \in S^1$. Denote $\tau_\alpha$ and $\tau_\beta$ the Dehn twists along $\alpha$ and $\beta$. By Fortier Bourque's theorem, the sequence $(X_n) = ((\tau_\alpha \circ \tau_\beta)^n X)$ converges to a multiple of $\mathrm{Ext}^{1/2}(i(\alpha, \cdot)\alpha + i(\beta, \cdot)\beta, X))$ in the Gardiner–Masur compactification. Furthermore, the sequence $(X_n)$ converges in the visual compactification based at $X$ to the geodesic spawned by the quadratic differential $q_{\alpha+\beta,X}$. Indeed, as detailed in [FB23, Section 4], the elements $(X_n)$ diverge to infinity along the horocycle defined by the quadratic differ-

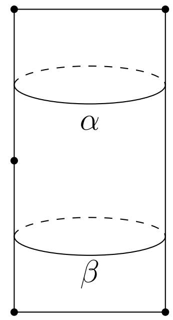



Figure 6.9: Sphere with five marked points, with curves $\alpha$ and $\beta$. We show that the extremal length is not $C^2$ along the path $\alpha + t\beta$, $t \in [0, t_0]$.

ential $q_{\alpha+\beta,X}$. Hence, inside embedded hyperbolic plane associated to $q_{\alpha+\beta,X}$, the sequence $(X_n)$ converges in the visual boundary to the geodesic spawned by $q_{\alpha+\beta,X}$, and so the same occurs in the ambient space. That is, $\Xi\,\mathrm{Ext}^{1/2}(i(\alpha,\cdot)\alpha + i(\beta,\cdot)\beta, X) \in \Pi^{-1}(q_{\alpha+\beta,X})$, so by Proposition 6.4.2 if we show that $\mathrm{Ext}(i(\alpha,\cdot)\alpha + i(\beta,\cdot)\beta, X)$ is not smooth with respect to the values of $i(\alpha,\cdot)$ and $i(\beta,\cdot)$, then $\Xi\,\mathrm{Ext}^{1/2}(i(\alpha,\cdot)\alpha + i(\beta,\cdot)\beta, X) \notin \overline{\mathcal{B}}(q_{\alpha+\beta,X})$, and hence it is also not in $\overline{\mathcal{B}}$.

**Lemma 6.5.2.** *Let $X \in \mathcal{T}(S_{0,5})$ and $G_t$, $t \in [0, t_0]$ be the foliation $\alpha + t\beta$ on $S_{0,5}$. The map $f(t) := \mathrm{Ext}(G_t, X)$ is not $C^2$.*

*Proof.* By Miyachi's Theorem 6.4.7 we have

$$\frac{d}{dt}\,\mathrm{Ext}(G_t, X) = 2i(\beta, F_{G_t,X}),$$

where we remind that $F_{G_t,X}$ is the horizontal foliation of the unique Hubbard–Masur differential associated to $G_t$ on $X$. Hence, the Lemma is equivalent to proving that $g(t) = i(\beta, F_{G_t,X})$ is not $C^1$.

For a general surface finding a precise expression of $F_{G,X}$ is a complicated problem, as the relation established by Hubbard and Masur is not explicit. However, in our case the surface is topologically simple, and one can use Schwartz–Christoffel maps to get a map from $G$ to $F_{G,X}$. In particular, it is possible to show that the sphere with 5 marked points is conformally equivalent to the Riemannian surface obtained by doubling an $L$-shaped polygon, marking the inner angles as shown in Figure 6.10 and setting certain values for $a, b$ and $l$. Furthermore, the quadratic differential obtained by $dz^2$ has $\alpha$ and $\beta$ as vertical foliations, with weights $a$ and $b$. Hence $q_{G_t,X}$ is $dz^2$ on the $L$-shaped pillowcase where $a = 1$ and $b = t$,

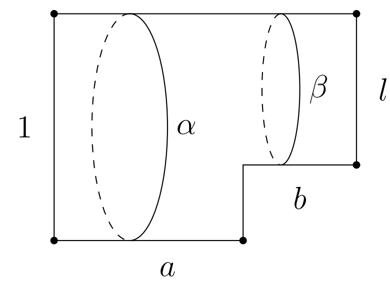



Figure 6.10: Doubling of the $L$-shaped polygon together with the curves $\alpha$ and $\beta$.

so $i(\beta, F_{G_t,X}) = 2l$. Markovic estimated in [Mar18, Section 9] the values of $a, b$ and $l$ around $b = 0$ depending on a common parameter $r$. Up to rescaling, these values are given by

$$\begin{aligned} a(r) =& a(0) + D_1 r + O(r^2), \\ b(r) =& D_2 r + O(r^2) \text{ and} \\ l(r) =& l(0) + D_3 r \log \frac{1}{r} + o\left( r \log \frac{1}{r} \right), \end{aligned}$$

where $A(r) = B(r) + O(f(r))$ means $\frac{|A(r)-B(r)|}{f(r)}$ is bounded around $r = 0$, and $A(r) = B(r) + o(f(r))$ means $\frac{|A(r)-B(r)|}{f(r)}$ converges to 0 as $r$ converges to 0.

Rescaling the pillowcase by $1/a(r)$ we see that the parameter $t$ can be expressed as $t(r) = b(r)/a(r)$, and $g(t(r)) = i(\beta, F_{G_t,X}) = 2l(r)/a(r)$. Observing that $t(0) = 0$, we can evaluate the first derivative of $g(t)$ at 0 by evaluating the limit

$$\lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{r \to 0} \frac{g(t(r)) - g(0)}{t(r)} = \lim_{r \to 0} \frac{2l(r)/a(r) - 2l(0)/a(0)}{b(r)/a(r)} =$$
$$2 \lim_{r \to 0} \frac{l(r) - l(0)a(r)/a(0)}{b(r)} = 2 \lim_{r \to 0} \frac{D_3 r \log(\frac{1}{r}) + o(r \log(\frac{1}{r})) - \frac{l(0)D_1}{a(0)} r}{D_2 r + O(r^2)} = \infty.$$

And so, $g(t)$ is not differentiable at $t = 0$, and hence $f(t)$ is not $C^2$. □

Repeating Fortier Bourque's reasoning we can lift this example to any surface of genus $g$ with $p$ marked points as long as $3g + p \geq 5$. Besides Gekhtman–Markovic's Lemma 6.5.1, the other key ingredient for the lifting is the following result.

**Lemma 6.5.3** (Fortier Bourque). *Let $\pi : S_{g,p} \to S_{0,5}$ be a branched cover of degree $d$ and let $\iota : \mathcal{T}(S_{0,5}) \hookrightarrow \mathcal{T}(S_{g,p})$ be the induced isometric embedding. For any measured foliation $F$*

*on $S_{0,5}$ and any $X \in \mathcal{T}(S_{0,5})$, we have the identity*

$$\operatorname{Ext}(\pi^{-1}(F), \iota(X)) = d \operatorname{Ext}(F, X).$$

*Proof.* Recall that $q_{F,X}$ is the Hubbard–Masur differential associated to $\gamma$. We have that $\pi^* q_{F,X} = q_{\pi^{-1}(F),\iota(X)}$, so

$$\operatorname{Ext}(\pi^{-1}(F), \iota(X)) = \int_{\iota(X)} |q_{\pi^{-1}(F),\iota(X)}| = d \int_X |q_{F,X}| = d \operatorname{Ext}(F, X).$$

□

Lifting the foliation $G_t$ from Lemma 6.5.2 we get an upper bound for the smoothness of the extremal length.

**Theorem 6.5.4.** *Let $S$ be a closed surface of genus $g$ with $p$ marked points, such that $3g + p \geq 5$. Then there exist two non intersecting multicurves $\hat{\alpha}$, $\hat{\beta}$ and some $X \in \mathcal{T}(S)$ such that the map $f(t) := \operatorname{Ext}(\hat{\alpha} + t\hat{\beta}, X)$, $t \in [0, t_0]$ is not $C^2$.*

*Proof.* Since $3g+p \geq 5$ we have a map $\pi : S_{g,p} \to S_{0,5}$, with an induced isometric embedding $\iota : \mathcal{T}(S_{0,5}) \hookrightarrow \mathcal{T}(S_{g,p})$. By Lemma 6.5.2 we have two curves $\alpha, \beta \in S_{0,5}$ such that, for any $X \in \mathcal{T}(S_{0,5})$ the map $t \to \operatorname{Ext}(\alpha+t\beta, X)$ is not $C^2$. Let $\hat{\alpha} = \pi^{-1}(\alpha)$ and $\hat{\beta} = \pi^{-1}(\beta)$. We have $\hat{\alpha}+t\hat{\beta} = \pi^{-1}(\alpha+t\beta)$, so applying Lemma 6.5.3 we get $\operatorname{Ext}(\hat{\alpha}+t\hat{\beta}, i(X)) = d \operatorname{Ext}(\alpha+t\beta, X)$. By Lemma 6.5.2 the function $\operatorname{Ext}(\alpha + t\beta, X)$ is not $C^2$, so we get the theorem. □

Theorem 6.1.4 is essentially a rephrasing of the previous theorem. Finally, we are able to prove that Busemann points are not dense.

*Proof of Theorem 1.3.1.* Let $\alpha$ and $\beta$ be as in Lemma 6.5.2. Furthermore, let $\pi : S_{g,p} \to S_{0,5}$ and $\iota : \mathcal{T}(S_{0,5}) \hookrightarrow \mathcal{T}(S_{g,p})$ be as in Lemma 6.5.1. For the $X \in \mathcal{T}(S_{0,5})$ described before Lemma 6.5.2 the sequence $(X_n) = (\tau_\beta \circ \tau_\alpha)^n X$ is contained in the horocycle generated by $q_{\alpha+\beta,X}$ and the distance $d(X_n, X)$ goes to infinity. Therefore $(X_n)$ converges in $\overline{\mathcal{T}(S_{0,5})}^v_X$ to the geodesic spawned by $q_{\alpha+\beta,X}$. Following Fortier Bourque's reasoning in the proof of [FB23, Theorem 1.1], using half translation structures, applying the Dehn twist $\tau_\alpha \circ \tau_\beta$ to $X$ is equivalent to applying the shearing transformation

$$h_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

to the half translation structure defined by $q_{\alpha+\beta,X}$. This action commutes with the pull-back coming from the branched cover, so the elements $(X_n)$ are associated with the half translation structure defined by $h_n\pi^*(q_{\alpha+\beta,X})$. These points diverge to infinity along the horocycle defined by $\pi^*(q_{\alpha+\beta,X})$, and so converge in $\overline{\mathcal{T}(S_{g,p})}^v_{\iota(X)}$ to the geodesic spawned by $q_{\pi^{-1}(\alpha)+\pi^{-1}(\beta),\iota(X)}$.

Let $c_i$, $1 \leq i \leq k$, be the components of the half translation structure associated to $\pi^{-1}(\alpha+\beta, X)$. Each $c_i$ covers either $\alpha$ or $\beta$ with some degree $d_i \in \mathbb{N}$. Hence, each component $c_i$ corresponds to a curve and is a cylindrical with height 1 and circumference $d_i$. Therefore, if $m$ is the common multiple between all $d_i$, and $\gamma_i$ is the curve associated to the component $c_i$, shifting the flat metric via the matrix $h_m$ is equivalent to performing $m/d_i$ Dehn twists around each curve $\gamma_i$. Letting $\phi$ be the composition of such Dehn twists, we have $\iota(X_{mn}) = \phi^n\iota(X)$. Hence, by Fortier Bourque's Theorem 6.4.8, in the Gardiner–Masur compactification the sequence $(\iota(X_{mn}))_n$ converges, as $n \to \infty$, to

$$\xi = \left[\operatorname{Ext}^{1/2}\left(\sum_{i=1}^{k} \frac{1}{d_i} i(F, \gamma_i)\gamma_i, \iota(X)\right)\right]_{F\in\mathcal{MF}(S_{g,n})}.$$

Therefore, $\Xi\xi \in \Pi^{-1}(q_{\pi^{-1}(\alpha)+\pi^{-1}(\beta),\iota(X)})$. To see that $\Xi\xi$ is not in $\overline{\mathcal{B}}$ it remains to see that it is not in $\overline{\mathcal{B}}(q_{\pi^{-1}(\alpha)+\pi^{-1}(\beta),\iota(X)})$. We have, $i(c, H(q_{\pi^{-1}(\alpha)+\pi^{-1}(\beta),\iota(X)}) = d_i$, so by Proposition 6.4.2 it remains to prove that there is some path of foliations $G_t$ such that the functions $x_i = \frac{i(\gamma_i, G_t)}{d_i}$ vary smootly, while the function $f(x_1, \ldots, x_k) = \operatorname{Ext}\left(\sum_{i=1}^{k} \frac{1}{d_i} x_i\gamma_i, \iota(X)\right)$ does not. Reorder the curves so there is some $p \geq 1$ such that $\pi^{-1}\alpha = \gamma_1 + \ldots + \gamma_p$ and $\pi^{-1}\beta = \gamma_{p+1} + \ldots + \gamma_k$. It follows from Dehn-Thurston's coordinates that for any natural numbers $n_j$, $1 \leq j \leq l$ there is a multicurve $G_{(n_j)}$ such that $i(G_{(n_j)}, \gamma_i) = n_i$. See, for example, the book by Penner–Harer [PH92, Theorem 1.2.1]. Allowing renormalizations of the multicurves we get that $n_j$ can be any non-negative rationals. Finally, doing a limit argument in the space of projective measured foliations we can take $n_j$ to be any non-negative real numbers. That is, for any $t \geq 0$ there exists a measured foliation $G_t$ such that $i(G_t, \gamma_i) = d_i$ for $i \leq p$, and $i(G_t, \gamma_i) = td_i$ otherwise. Hence, along such foliations we have $x_i = 1$ for $i \leq p$ and $x_i = t$ otherwise. Therefore, along this path,

$$f(1, \ldots, 1, t, \ldots, t) = \operatorname{Ext}\left(\pi^{-1}(\alpha) + t\pi^{-1}(\beta), \iota(X)\right),$$

which by Theorem 6.5.4 is not smooth, as $\pi^{-1}(\alpha)$ and $\pi^{-1}(\beta)$ are the curves used in the proof of the Theorem. $\square$

### 6.5.2 Busemann points with one indecomposable component are nowhere dense

The Thurston compactification can be build in a similar way as the Gardiner–Masur compactification, by using the hyperbolic length of the curves instead of the extremal length. Let $\phi$ be the map between $\mathcal{T}(S)$ and $P\mathbb{R}_+^{\mathcal{S}}$ defined by sending $X \in \mathcal{T}(S)$ to the projective vector $[\ell(\alpha, X)]_{\alpha\in\mathcal{S}}$. The pair $(\phi, \overline{\phi(\mathcal{T}(S))})$ defines a compactification, and the boundary is given by the space of projective measured foliations, denoted $\mathcal{PMF}$.

As explained by Miyachi [Miy13b], neither the Thurston nor the horofunction compactification is finer than the other one. However, it is possible to get some relation. Let $\mathcal{PMF}^{UE} \subset \mathcal{PMF}$ be the set of uniquely ergodic foliations. Following the work of Masur [Mas82a], $\mathcal{PMF}^{UE}$ has full Lebesgue measure within $\mathcal{PMF}$. Miyachi [Miy13b, Corollary 1] shows that the mapping $\phi$ on $\mathcal{T}(S)$ can be extended to an homeomorphism $f$ between $\phi(\mathcal{T}(S)) \cup \mathcal{PMF}^{UE}$ and $h(\mathcal{T}(S)) \cup B_{UE}$ such that for $x \in \mathcal{T}(S)$ we have $f(\phi(x)) = h$, where $B_{UE}$ are the Busemann points associated to quadratic differentials whose vertical foliation is uniquely ergodic. One might understand this result as stating that the two compactifications are the same almost everywhere with respect to the Lebesgue measure on $\mathcal{PMF}$. As we shall see, the same does not follow with respect to any strictly positive measure on the horoboundary.

The homeomorphism $f$ described by Miyachi is obtained by first defining a map between the boundaries. For a given $x \in \mathcal{T}(S)$, the map on the boundary is denoted $\mathcal{G}_x$, and by its definition we have $\mathcal{G}_x(F) = B(q_{F,x})$, where we recall that $q_{F,x}$ is the quadratic differential on $x$ with $V(q_{F,x}) = F$. Denote $\mathcal{B}_1$ the set of Busemann points associated to foliations with one indecomposable component. We have $\mathcal{G}_x(\mathcal{PMF}^{UE}) = \mathcal{B}_{UE} \subset \mathcal{B}_1$. However, the following is also satisfied.

**Theorem 6.5.5.** *Let $S$ be a closed surface of genus $g$ with $p$ marked points, such that $3g + p \geq 5$. Then the set $\mathcal{B}_1$ is nowhere dense in the horoboundary.*

*Proof.* The action of $\mathrm{MCG}(S)$ on $\mathcal{T}(S)$ is extended to the projectivized version of the Gardiner–Masur compactification by $\psi[f(\alpha)]_{\alpha\in\mathcal{S}} = [f(\psi\alpha)]_{\alpha\in\mathcal{S}}$. For any $q$ such that $V(q)$ is an indecomposable measured foliation, $\mathcal{E}_q = \Xi^{-1}B(q) = [i(V(q),\alpha)]_{\alpha\in\mathcal{S}}$, and hence $\psi\mathcal{E}_q = [i(V(q),\psi(\alpha))]_{\alpha\in\mathcal{S}} = [i(\psi^1(V(q)),\alpha)]_{\alpha\in\mathcal{S}}$. Hence, $\psi\mathcal{E}_q$ is equal to the representation of the Busemann point in the Gardiner–Masur compactification associated to the quadratic differential with vertical foliation $\psi^{-1}V(q)$, which also is an indecomposable measured foliation. Therefore, $\mathcal{B}_1$ is invariant under the action of $\mathrm{MCG}(S)$, and since $\mathrm{MCG}(S)$ acts by homeomorphisms, the complement of the closure is also invariant.

Let $q_0$ be a quadratic differential such that there is some $f \in \Xi^{-1}\Pi^{-1}(q_0)$ not in $\Xi^{-1}\overline{\mathcal{B}}$. Such a quadratic differential exists, by Theorem 1.3.1. By the proof of the theorem, we can assume that $V(q_0)$ is a multicurve. Furthermore, let $q$ be a quadratic differential such that $V(q)$ and $H(q)$ are the stable and unstable foliations respectively of some pseudo-Anosov element $\phi \in MCG(S)$. It is well known [FLP12, Expose 12] that for any closed curve $\alpha$ we have that $\lambda^{-n}\phi^n(\alpha)$ converges to $\frac{i(\alpha,V(q))}{i(H(q),V(q))}H(q)$, where $\lambda$ is the stretch factor of $\phi$. For any foliation $F$ we have that $\Pi^{-1}M(q_0)(F) = 0$ if and only if $i(V(q_0),F) = 0$, where $M(q_0)$ is the minimal point defined in Section 6.4. Hence, since $H(q)$ is the unstable foliation of a pseudo-Anosov element and $V(q_0)$ is a multicurve, we have $i(V(q_0),H(q)) \neq 0$, and so $f(H(q)) \geq \Pi^{-1}M(q_0)(H(q)) > 0$. We have $\phi^n[f(\alpha)]_{\alpha\in\mathcal{S}} = [f(\phi^n(\alpha))]_{\alpha\in\mathcal{S}}$. Taking limits and using that the functions in the Gardiner–Masur compactification are homogeneous of degree 1, we get that

$$\lim_{n\to\infty}[\phi^n f(\alpha)]_{\alpha\in\mathcal{S}} = \left[i(\alpha,V(q))f\left(\frac{H(q)}{i(V(q),H(q))}\right)\right]_{\alpha\in\mathcal{S}} = [i(\alpha,V(q))]_{\alpha\in\mathcal{S}},$$

Hence, in the normalized version, $\phi^n f$ converges to $i(\cdot,V(q)) = \Xi^{-1}B(q)$, as $V(q)$ is uniquely ergodic and therefore indecomposable. That is, $B(q)$ can be approached through a sequence of elements contained in the complement of the closure of $\mathcal{B}_1$.

Let $B(q')$ be any element in $\mathcal{B}_1$, where $q'$ is any quadratic differential such that $V(q)$ has one indecomposable component. The set of pseudo-Anosov foliations is dense in $\mathcal{MF}(S)$, so we have a sequence of quadratic differentials $(q_n)$ converging to $q'$ with $V(q_n)$ being a pseudo-Anosov foliation. Since $q'$ has one indecomposable component, the convergence is strong, and so $B(q_n)$ converges to $B(q)$. Each $B(q_n)$ can be approached through a sequence of elements contained in the complement of the closure of $\mathcal{B}_1$, so taking a diagonal sequence the same can be said for $B(q)$. $\square$

**Corollary 6.5.6.** *Let $S$ be a closed surface of genus $g$ with $p$ marked points, such that*

$3g + p \geq 5$. *Then, for any finite strictly positive measure $\nu$ on the horoboundary, the set $\overline{\mathcal{B}}_1$ does not have full $\nu$-measure.*

*Proof.* By Theorem 6.5.5, the complement of $\overline{\mathcal{B}}_1$ is open and nonempty, so it must have positive $\nu$-measure. $\square$

This last result tells us that the image of Miyachi's homeomorphism does not have full measure within the horoboundary. However, as announced in the introduction, any attempt to extend the identity from the Thurston compactification to the horoboundary compactification to a set of full measure within the Thurston compactification results in the same problem. We restate here the result as we shall use the notation for the proof.

**Corollary 6.1.5.** *Let $\nu$ be any finite strictly positive measure on the horoboundary and let $\mu$ be the Lebesgue measure on the Thurston boundary. Furthermore, let $\phi$ be a map from the Thurston compactification to the horofunction compactification satisfying $\phi|_{\mathcal{T}(S)} = h$, where $h$ is the map used to define the horofunction compactification in Section 2.1.3. Then there is no subset $U$ of the Thurston boundary with full $\mu$-measure such that $\phi$ is continuous at every point in $U$ and $\phi(U)$ has full $\nu$-measure.*

*Proof.* Assume such a $U$ exists. Choose then a basepoint $x \in \mathcal{T}(S)$ and let $U' = U \cap \mathcal{PMF}^{UE}$. For each element of $F \in U'$ the associated Hubbard–Masur quadratic differential $q_{F,x}$ satisfies $R(q_{F,x}; t) \to F$ as $t \to \infty$. Hence, since $\phi$ is continuous at $F$ we have $\phi(F) = B(q_{F,x})$. That is, $\phi(U') \subset \mathcal{B}_1$.

Let $G \in U$. The set $\mathcal{PMF}^{UE}$ has full $\mu$ measure, so $U' = \mathcal{PMF}^{UE} \cap U$ also has full measure. Hence, since the Lebesgue measure is strictly positive, $U'$ is dense within $\mathcal{PMF}$. Therefore $G$ can be accessed through a sequence $(F_n) \subset U'$. Hence, since $\phi$ is continuous in $G$ we have $\phi(G) = \lim \phi(F_n)$, so $\phi(U) \subset \overline{\mathcal{B}}_1$ and $\phi(U)$ can not have full $\nu$-measure. $\square$

Another natural family of measures on the boundary is obtained by considering harmonic measures, as explained in Section 2.3.1. As a reminder, given a non-elementary measure $\mu$ on $\mathrm{MCG}(S)$ it is possible to define a random walk $(w_n)$ as the sequence of random variables defined by

$$w_n = g_0 g_1 g_2 \cdots g_n,$$

where $g_i$ are independent, identically distributed random variables on $\mathrm{MCG}(S)$ sampled according to the distribution $\mu$. As proven by Kaimanovich and Masur in [KM96, Theorem

2.2.4], random walks generated by a non-elementary probability measure converges almost surely in Thurston's compactification, so we can define the hitting measure $\nu$ in $\mathcal{PMF}$. Furthermore, the walk converges almost surely to uniquely ergodic projective foliations, so we can translate this result to the horofunction compactification in the following way.

**Corollary 6.5.7.** *Let $\mu$ be a non-elementary measure on* $\mathrm{MCG}(S)$*. Then the associated harmonic measure on the horoboundary is supported in a nowhere dense set.*

*Proof.* For any $x \in \mathcal{T}(S)$ the sequence $(w_n x)$ converges almost surely in Thurston compactification to some $F \in \mathcal{PMF}^{UE}$. Hence, by [Miy13b, Corollary 1], the sequence $(w_n x)$ converges almost surely to the Busemann point generated by a quadratic differential $q$ with $V(q)$ being a multiple of $F$. Hence, the support of the harmonic measure is contained in $\mathcal{B}_1$, which is nowhere dense by Theorem 6.5.5. □

## 6.6 Topology of the Horoboundary

In this section we make some progress towards determining the global topology of the horoboundary. We begin by showing that the minimal point $M(q)$ introduced in Proposition 6.4.3 serves as a section for the map $\Pi$ whenever $S$ does not have a boundary. Our main goal for this section is proving the following Theorem.

**Theorem 6.6.1.** *Let $S$ be a surface of genus $g$ with $b_m$ and $b_u$ boundaries with and without marked points respectively and $p$ interior marked points. Then, the map $\Pi$ restricted to the boundary has a global continuous section $\partial\overline{\mathcal{T}}^v \to \partial\overline{\mathcal{T}}^h$ if and only if at least one of the two following conditions is satisfied:*

- $b_m = b_u = 0$ *or*
- $2g + 2b_m + b_u + p - \max(1 - b_u, 0) \leq 4.$

*The section is given by sending the ray in the direction of $q$ to the point $M(q)$ defined before Proposition* 6.4.3.

*Furthermore, if the map does not admit a global section, then it does not admit any local section around some points.*

We begin by proving the theorem for surfaces without boundary, as it is significantly easier to prove.

**Proposition 6.6.2.** *Let $S$ be a surface without boundary. Then the projection map $\Pi$ restricted to the boundary admits a global section, given by the map $M : \partial\overline{\mathcal{T}}^v \to \partial\overline{\mathcal{T}}^h$.*

*Proof.* By Proposition 6.4.3 every preimage $\Pi^{-1}(q)$ contains $M(q)$. We have $M(q) = \Xi(i(V(q), \cdot))$, which is continuous, as the map $\Xi$ is continuous. □

The rest of the cases of Theorem 6.6.1 require a more careful analysis.

**Proposition 6.6.3.** *Let $S$ be either*

- *a torus with up at most two unmarked boundaries or interior marked points,*
- *a torus with one marked boundary and one interior marked point,*
- *a sphere with one marked boundary and up to three interior marked points or*
- *a sphere with two marked boundaries and interior marked point.*

*Then the projection map $\Pi$ restricted to the boundary admits a global section, given by the map $M : \partial\overline{\mathcal{T}}^v \to \partial\overline{\mathcal{T}}^h$.*

*Proof.* We shall build the section in the same way we built it in Proposition 6.6.2, that is, sending $q$ to $M(q)$.

Our first step in the proof is seeing that if $V(q)$ contains a separating proper arc then only one of the two parts separated by the proper arc admit interior components. We shall do this by inspecting each possible case. Assume then that $V(q)$ has a separating proper arc.

If $S$ is a torus with up to two unmarked boundaries or marked points or a torus with one marked boundary and one marked point, then the separating proper arc splits the surface into a torus with a marked boundary and a sphere with a marked boundary and a marked point or unmarked boundary. The latter does not admit an interior component.

If $S$ is a sphere with one marked boundary and up to three boundaries then the separating proper arc splits the surface into two spheres, both with one marked boundary, one of them with two marked points and the other one with one marked point. Again, the latter does not admit an interior component.

Finally, if $S$ is a sphere with two marked boundaries and one marked point or unmarked boundary, the proper arc splits the surface into one sphere with two marked boundaries and a sphere with one marked boundary and one marked point, which again does not admit an interior component.

Take then a sequence of unit quadratic differentials $(q_n)$ converging to $q$. Let $P_i$, $i \in \{1, \ldots, c\}$ be the boundary components of $V(q)$. Furthermore, denote $G$ the union of the interior components. By the first part of the proof, all the interior components are contained in the same interior part. We thus have

$$\Xi^{-1} M(q) = \left( \sum_i \mathcal{W}^q(P_i) + \mathcal{W}^q(G) \right)^{1/2}.$$

By Proposition 6.3.5 all boundary components of $V(q)$ are contained in $V(q_n)$ for $n$ big enough, and all other boundary components of $V(q_n)$, denoted $P^n$, vanish in the limit. Denote $G^n$ the union of the interior components of $V(q_n)$. As before, each indecomposable component of $G^n$ is contained in the same interior part, so we have

$$\Xi^{-1} M(q_n) = \left( \sum_i \mathcal{W}^{q_n}(\alpha_i^n P_i) + \mathcal{W}^{q_n}(P^n) + \mathcal{W}^{q_n}(G^n) \right)^{1/2},$$

which converges to $\Xi^{-1} M(q)$. □

**Proposition 6.6.4.** *Let $S$ be either*

- *a surface of genus at least two and at least one boundary;*
- *a torus with at least one boundary and two more boundaries or interior marked points;*
- *a torus with at least two boundaries, one being marked, and possibly interior marked points;*
- *a sphere with at least one boundary, and four more boundaries or interior marked points;*
- *a sphere with at least two boundaries, one being marked, and two interior marked points or*
- *a sphere with at least three boundaries, two being marked, and possibly interior marked points.*

*Then the projection map* $\Pi$ *restricted to the boundary does not admit a local section around some points.*

*Proof.* We shall prove this by finding a quadratic differential $q$ and sequences $(q_n^1)$ and $(q_n^2)$ converging to $q$ such that their preimages by $\Pi$ are singletons, but such that $\Pi^{-1}(q_n^1)$ and $\Pi^{-1}(q_n^2)$ converge to different points in $\Pi^{-1}(q)$. If we had a section around $q$, then its value at $q_n^1$ and $q_n^2$ would be $\Pi^{-1}(q_n^1)$ and $\Pi^{-1}(q_n^2)$ respectively, giving us a contradiction.

In all cases the construction will be similar. For $q_n^1$ we build a foliation with a separating proper arc $P$ such that each of the parts has precisely one interior component consisting of a closed curve, which we denote $G_1$ and $G_2$. Letting the weight of the proper arc diminish to 0 we can get a sequence of quadratic differentials $(q_n^1)$ converging to a quadratic differential $q$ such that $V(q) = G_1 + G_2$. Let $F_n^1 = P + nG_1 + nG_2$, $A_n^1$ and $A$ the area of the Hubbard–Masur differentials $q_{F_n^1,X}$ and $q_{G_1+G_2,X}$ respectively. Denote $\frac{1}{\sqrt{A_n^1}} q_{F_n^1,X}$ as $q_n^1$. These quadratic differentials have unit area, and converge to $\frac{1}{\sqrt{A}} q_{G_1+G_2,X}$, which we denote $q$. By construction, $V(q_n^1)$ is internally indecomposable, so $\Pi^{-1}(q_n^1)$ is a singleton, and $\Xi^{-1}\Pi^{-1}(q_n^1) = \left\{ \left( \frac{\mathcal{W}^{q_n^1}(P) + n\mathcal{W}^{q_n^1}(G_1) + n\mathcal{W}^{q_n^1}(G_2)}{\sqrt{A_n^1}} \right)^{1/2} \right\}$. The sequences $\frac{P}{\sqrt{A_n^1}}$, $\frac{nG_1}{\sqrt{A_n^1}}$ and $\frac{nG_2}{\sqrt{A_n^1}}$ converge respectively to 0, $\frac{G_1}{\sqrt{A}}$ and $\frac{G_2}{\sqrt{A}}$. Hence, by Lemma 6.4.1 the sequence $\Pi^{-1}(q_n^1)$ converges to $\left\{ \left( \frac{\mathcal{W}^{q}(G_1) + \mathcal{W}^{q}(G_2)}{\sqrt{A}} \right)^{1/2} \right\}$.

For building $q_n^2$ we take a curve $\gamma$ intersecting $G_1$ and $G_2$ at $b_1$ and $b_2$ times, where $b_1, b_2 \in \{1, 2\}$. Denote $\tau_1$ and $\tau_2$ the Dehn twists around $G_1$ and $G_2$. Let $F_n^2 = \tau_1^{2n/b_1} \tau_2^{2n/b_2} \gamma$ and $A_n^2$ the area of the Hubbard–Masur differential $q_{F_n^2,X}$. As before. Denote $\frac{1}{\sqrt{A_n^2}} q_n^2$ the quadratic differentials $\frac{1}{\sqrt{A_n^2}} q_{F_n^2,X}$. These quadratic differentials have unit area, and converge to $q$. Furthermore, each $V(q_n^2)$ is a singleton and $\Xi^{-1}\Pi^{-1}(q_n^2) = \left\{ \left( \frac{\mathcal{W}^{q_n^2}((\tau_1\tau_2)^n \gamma)}{\sqrt{A_n^2}} \right)^{1/2} \right\}$. The sequence $\frac{(\tau_1\tau_2)^n \gamma}{\sqrt{A_n^2}}$ converges to $\frac{G_1+G_2}{A}$, so by Lemma 6.4.1 the sequence $\Xi^{-1}\Pi^{-1}(q_n^2)$ converges to $\left\{ \left( \frac{\mathcal{W}^{q}(G_1+G_2)}{\sqrt{A}} \right)^{1/2} \right\}$, which is different than the limit of $\Xi^{-1}\Pi^{-1}(q_n^1)$.

It remains then to find such a multicurve. For genus at least two we take $P$ to be a separating proper arc such that each of the parts is of genus at least one, and $G_1$ and $G_2$ to be non contractible curves, not parallel to unmarked boundaries on each part, as shown in Figure 6.11a.

For the torus we take $P$ to be a separating proper arc with both endpoints in the unmarked

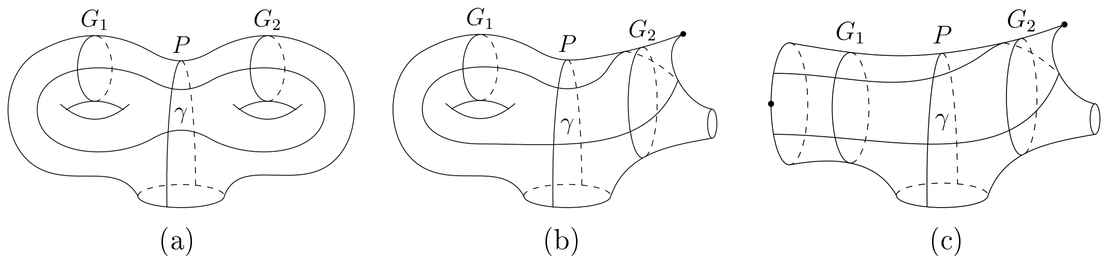



Figure 6.11: Curves chosen in the proof of Proposition 6.6.4

boundary, or a marked boundary if there are no unmarked boundaries. Further, we choose the proper arc such that, after cutting along the arc, one part is a torus with one boundary. That is, every other feature of the surface lies in the other part. Then we let $G_1$ and $G_2$ be non contractible curves on each part, as shown in Figure 6.11b.

Finally, for the sphere we let $P$ be a separating proper arc with both endpoints on an unmarked boundary, or a marked boundary if there are no boundaries without marked points. Further, we choose the arc such that each interior part has at least either a combination of two marked points or boundaries without marked points, or a boundary with marked points. Hence, each interior part supports an interior component formed by a curve, as shown in Figure 6.11c.

□

*Proof of Theorem 6.6.1.* This is a combination of the results from Propositions 6.6.2, 6.6.3 and 6.6.4. □

By Proposition 5.1.3 we know that the horoboundary is connected whenever the real dimension of Teichmüller space is at least 2. In the following result we go a bit further, by showing that it is actually path connected.

*Proof of Theorem 1.3.2.* Let $x, y \in \partial\overline{\mathcal{T}(S)}^h$. If $S$ does not have boundary then $\Pi$ has a global section, so we can lift any path between $\Pi(x)$ and $\Pi(y)$ to a path between $M(\Pi(x))$ and $M(\Pi(y))$. Then, since $\Pi^{-1}(x)$ and $\Pi^{-1}(y)$ are path connected, we can connect $x$ to $M(\Pi(x))$ and $y$ to $M(\Pi(y))$ via paths.

If $S$ has boundary we might have to be a bit more careful, as we might not have a global section. However, as we shall see, we can take a path $q_t$ between $\Pi(x)$ and $\Pi(y)$ such that $B(q_t)$ has finitely many discontinuities. Then, since each of the preimages is path connected these discontinuities can be fixed by using paths in the fibers, so we will have a path between $x$ and $y$.

Choose a boundary component of $S$, denote $b$ a curve parallel to that boundary and let $F_x = V(\Pi(x))$. If $F_x$ contains $b$ then all the expressions of the form $(1-t)F_x + tb$ with $t \in [0,1]$ correspond to foliations on $S$, which we denote $F_t$. Denote $q_t$ the unit area quadratic differential such that $V(q_t)$ is a multiple of $F_t$. This defines a continuous path joining $\Pi(x)$ and the unit area quadratic differential associated to a multiple of $b$. Let $V_i$ be the vertical components of $F_x$ that are not $b$, and let $w_0$ be the weight of $b$ in $F_x$. Then, $B(q_t)^2 = \frac{1}{\sqrt{\mathrm{Area}(q_{F_t,X})}}((1-t)\sum \mathcal{W}^{q_t}(V_i) + (t+(1-t)w_0)\mathcal{W}^{q_t}(b))$, which gives a continuous path from $B(q_0) \in \Pi^{-1}\Pi(x)$ to $B(q_1) \in \Pi^{-1}(q_1)$. If $F_x$ does not contain $b$, but $b$ can be added to the foliation then we proceed just as before. Hence, if both $x$ and $y$ result in foliations where $b$ can be added, we create a path by concatenating the paths between $x$, the Busemann point in $\Pi^{-1}\Pi(x)$, the Busemann point associated to $b$, the Busemann point in $\Pi^{-1}\Pi(y)$ and $y$.

If $b$ cannot be added to the foliation $F_x$ then there must be some set $P$ of proper arcs in $F_x$ incident to the boundary component associated to $b$. Let $F'_x$ be the foliation $F_x$ without the proper arcs $P$ and assume $F'_x$ is nonempty. Denote $F_t$ the foliations $(1-t)P + (1+t)F'_x$, $t \in [0,1]$, and $q_t$ the unit area quadratic differentials such that $V(q_t)$ is a multiple of $F_t$. Denoting $V_i$ the vertical components of $F'_x$, and $P_j$ the proper arcs incident to the boundary component associated to $b$, we have $B(q_t)^2 = \frac{1}{\sqrt{\mathrm{Area}(q_{F_t,X})}}\Big((1-t)\sum_j \mathcal{W}^{q_t}(P_j) + (1+t)\sum \mathcal{W}^{q_t}(V_i)\Big)$ for $t < 1$, which is continuous. Furthermore, $\lim_{t\to 1} B(q_t) \in \Pi^{-1}(q_1)$. Hence, we can concatenate a paths between $x$, the Busemann point in $\Pi^{-1}\Pi(x)$, the limit $\lim_{t\to 1} B(q_t)$, the Busemann point $B(q_1)$ and Busemann point associated to $b$.

If $F'_x$ is empty we want to add some other components to $F_x$. If it admits some other component $k$ then we repeat the previous reasoning with $F_t = (1-\frac{t}{2})F_x + \frac{t}{2}k$, which does not result in any discontinuity. If $F_x$ does not admit any other component then there must be at least 2 proper arcs incident to the boundary component associated to $b$, so we choose one of them, denoted $p$, and repeat the previous reasoning with $F_t = (1-t)F_x + tp$, which does not result in any discontinuity. Finally, we concatenate this last path with the previous paths. $\square$

## 6.7 Formulas for limits of extremal lengths

We finish by reframing the bounds we got for the elements of $\Xi^{-1}\Pi^{-1}(q)$ as results regarding limits of extremal lengths, getting in this way some extensions of [Wal19, Theorem 1].

**Proposition 6.7.1.** *Let $F$ be a measured foliation, $(q_n)$ be a sequence of unit area quadratic differentials converging to a quadratic differential $q$ and $(t_n)$ be a sequence of real numbers converging to infinity. Then,*

$$\left(\Xi^{-1}M(q)\right)^2 \leq \liminf_{n\to\infty} e^{-2t_n}\operatorname{Ext}_{R(q_n;t_n)}(F) \leq \limsup_{n\to\infty} e^{-2t_n}\operatorname{Ext}_{R(q_n;t_n)}(F) \leq \left(\Xi^{-1}B(q)\right)^2$$

*Proof.* Take a subsequence such that $e^{-2t_n}\operatorname{Ext}_{R(q_n;t_n)}(F)$ converges to the liminf. Furthermore, take a subsequence such that $R(q_n;t_n)$ converge to a point $\xi \in \Pi^{-1}(q)$. By Proposition 6.4.3 we have $(\Xi^{-1}M(q))^2 \leq \xi^2$. Since $e^{-2t_n}\operatorname{Ext}_{R(q_n;t_n)}(F)$ converges to $\xi^2(F)$ we have the lower bound. For the upper bound we repeat the process taking the limsup and using Proposition 6.4.5. □

By noting that $\Xi^{-1}M(q)(F)$ and $\Xi^{-1}B(q)(F)$ evaluate to 0 if and only if $i(V(q),F)=0$, we get the following corollary, which has also been proven for surfaces without boundary by Liu–Shi in [LS22, Corollary 3.11].

**Corollary 6.7.2.** *Let $(q_n)$ be a sequence of unit area quadratic differentials converging to a quadratic differential $q$, and $(t_n)$ be a sequence of real numbers converging to infinity. Then,*

$$\liminf_{n\to\infty} e^{-2t_n}\operatorname{Ext}_{R(q_n;t_n)}(F) = 0 \iff i(V(q),F)=0.$$

Proposition 6.7.1 can be strengthened slightly in the following manner.

**Proposition 6.7.3.** *Let $(q_n)$ be a sequence of unit area quadratic differentials converging to a quadratic differential $q$. Furthermore, denote $V_i^n$ the indecomposable components of $q_n$. If the vertical components can be reordered so that for each $i$ we have that $V_i^n$ converges to a foliation $V_i$, then*

$$\liminf_{n\to\infty} e^{-2t_n}\operatorname{Ext}_{R(q_n;t_n)}(F) \geq \sum_i \mathcal{W}^q(V_i).$$

*Proof.* Take a sequence such that the limit is equal to the liminf, and such that we have convergence in the Gardiner–Masur compactification. Let $\xi$ be the limit in the horofunction compactification. By Lemma 6.2.4 we have $e^{-2t_n} \operatorname{Ext}_{R(q_n;t_n)}(F) \geq (\Xi^{-1}B(q_n))^2$, and by Corollary 6.2.5 we have $(\Xi^{-1}B(q_n))^2 = \sum_i \mathcal{W}^{q_n}(V_i^n)$. Hence, by Lemma 6.4.1, taking limits on both sides we get the proposition. $\square$

If we have strong convergence the upper bound from Proposition 6.7.1 and the lower bound from Proposition 6.7.3 coincide, so adding Walsh's formula for the Busemann points [Wal19, Theorem 1] we have a proof of Theorem 6.1.8.

Finally, the path connectedness of the fibers can be translated to the following result.

**Proposition 6.7.4.** *Let $(q_n)$ be a sequence of unit quadratic differentials converging to $q$, and $(t_n)$ be a sequence of times converging to infinity. Further, for any $F \in \mathcal{MF}$ denote $L(F) := \liminf_{n\to\infty} \operatorname{Ext}_{R(q_n;t_n)}(F)$. Then, for any $s \in [L(F), \mathcal{E}_q^2(F)]$ there is a subsequence of $q_{n_k^s}$ and a sequence $(t_k^s)$ of times such that, for any $G \in \mathcal{MF}$ the limit*

$$\lim_{k\to\infty} e^{-2t_k^s} \operatorname{Ext}_{R(q_{n_k^s};t_k^s)}(G)$$

*is defined, and if $G = F$ it has value $s$.*

*Proof.* We can take a subsequence such that $\lim_{n\to\infty} \operatorname{Ext}_{R(q_n;t_n)}(F)$ converges to the liminf, and a further subsequence such that we have convergence in the Gardiner–Masur compactification to a point $\Xi^{-1}\xi \in \Xi^{-1}\Pi^{-1}(q)$. By Theorem 5.2.11 we have a path between $\xi$ and $B(q)$ contained in $\Pi^{-1}(q)$, and hence a path $\gamma$ between $\Xi^{-1}\xi$ and $\Xi^{-1}B(q)$ contained in $\Xi^{-1}\Pi^{-1}(q)$. By continuity there is a point in that path such that $\gamma_t(F) = \sqrt{s}$, and by the way we constructed $\gamma_t$, it is reached by taking a subsequence of $(q_{n_k^s})$ and a sequence $(t_k^s)$ of times converging to infinity. Finally, since $\gamma_t$ is a point in the Gardiner–Masur compactification approached by $R(q_{n_k^s}^s; t_k^s)$, the value of $\gamma_t(G)^2$ is equal to the limit from the proposition. $\square$

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