# An essay on price impact: How limit order book events and order flow affect price formation

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SUBMITTED IN FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Adam Smith Business School

College of Social Science

University of Glasgow

SEPTEMBER 2024

# Abstract

This thesis studies the price impact of limit order book events and their effects on price discovery. It consists of three independent essays that begin with examining the relationship between the shape of the limit order book and price impact.

Chapter 1 introduces the slope of the limit order book as a novel measure of price impact. By analyzing both the bid and ask sides of the high-frequency limit order book snapshot data, the study shows that there is a linear relationship between the cumulative size of liquidity and price impact in the limit order book. In addition, I find that if price impact admits a nonlinear functional form, under certain circumstances, a profitable round-trip arbitrage exists. I empirically show the minimum required trading volume for a profitable self-financing arbitrage and conditions that limit arbitrage.

Chapter 2 proposes a new approach to estimate the flow of this information and the price of that information (different from the stock price), and thus the total value of that information for each stock, and then sum up this value across all stocks, obtaining an estimate of the total value of the dynamic flow of information in the stock market as a whole. The results support the notion that the cross-correlation of price impact across stocks is consistent with the CAPM: there is a single systematic component of price impact, and this is driven by the volatility of the systematic component of the stock market. This result suggests that by separating the underlying information into two components, systematic and idiosyncratic, informed traders distinguish between productive assets that have a systematic impact on the economy and those that can be diversified.

Chapter 3 presents a two-period model of strategic interactions between a spoofer and a high-frequency trader (HFT) who employs pattern recognition algorithms to predict the incoming order. Detecting this strategy, the spoofer submits a spoofing order to mislead the HFT trader about the incoming order. The HFT protects itself by reducing its market participation. A pure strategy spoofing equilibrium exists and both spoofer and HFT make positive profits. It is shown that while spoofing delays price discovery in the short run, price dislocation will be so brief as to have few economic efficiency implications. Moreover, spoofing improves market liquidity and market welfare.

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# Acknowledgements

I would like to express my gratitude to my supervisors, Professor Bart Taub and Professor Christian Ewald, for their unwavering support and guidance throughout my PhD years. This thesis would not have been possible without Professor Bart Taub's invaluable critique and guidance. I acknowledge that a part of Chapter 2 and Chapter 3 is based on the joint work between me and Professor Bart Taub.

Furthermore, I would like to extend my thanks to all the employees of the Adam Smith Business School for their mental and research support. I am also grateful to the school for its financial support during my studies.

I am also appreciative to AlgoSeek Corporation for generously donating the data used in this study. I acknowledge with gratitude the allocation of computational resources from the XSEDE and ACCESS programs under the aegis of the National Science Foundation. I also thank Iacopo Mastromatteo, Professor Albert S. (Pete) Kyle (University of Maryland), Professor Johannes Muhle-Karbe (Imperial College London), and other participants during my presentation at The 12th World Congresses of the Bachelier Finance Society for their helpful comments on the thesis.

I wish to extend my gratitude to Capital Fund Management and the EconophysiX Lab for the opportunity to join them as a visiting PhD student. In particular, I am deeply grateful to Iacopo Mastromatteo, Michael Benzaquen, and Ruben Zakine for their warm welcome and generous support during my time at the EconophysiX Lab.

Finally, I would like to express my most profound appreciation for my family and friends whose encouragement, patience, and understanding kept me motivated through my PhD journey.

# Declaration

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution

**Hai Duong**

# Abbreviations

| | |
|---|---|
| Arca | NYSE Archipelago Exchange |
| CAPM | Capital Asset Pricing Model |
| CEA | Commodity Exchange Act |
| CFTC | Commodity Futures Trading Commission |
| DOJ | Department of Justice |
| FCA | Financial Conduct Authority |
| HFT | High Frequency Trader |
| IOC | Immediate-Or-Cancel |
| LOB | Limit Order Book |
| LSE | London Stock Exchange |
| OLS | Ordinary Least Squares |
| SEBI | Securities and Exchange Board of India |
| SEC | Securities and Exchange Commission |

# Introduction

Price impact refers to the effect of an incoming order on the price of an asset. It is the extent to which the price moves against the seller or buyer after their orders are executed. For large investors, price impact is one of the key considerations before any investment decision, as it accounts for the vast majority of execution costs. Therefore, monitoring and controlling impact has been one of the most active domains of research among practitioners and academics. Despite voluminous amounts of data and many theoretical papers, the interpretation of price impact is still a matter of debate among researchers.

From the market microstructure perspective, price impact is a result of some private information incorporated into the price. The arrival of new information encourages new trades, which causes other market participants to update their valuations, leading to price change. Kyle (1985) developed a model of strategic insider trading and showed that the insider would split their trade into small orders and let information gradually incorporate into the price. In this model, market impact is linear in trading volume and permanent in time. Keim and Madhaven (1996b) presented a model of the upstairs market with search cost and derived a power law temporary price impact function. Hasbrouck and Seppi (2001) empirically show that price impact is a concave function of meta-order size and it is approximately proportional to the square root of meta-order size across different markets. From the econophysics perspective, price impact is a statistical effect of order flow variations. J. D. Farmer (2005) assumed that order flow is a random process and showed that random fluctuations in supply and demand cause the price impact, and this effect is mechanical and transient.

Understanding the price impact has essential practical and regulatory implications. For practitioners, the impact of trades can adversely reduce their profit as it increases with trading size and places limits on fund size. By trading an excess size, large traders can turn a winning strategy into a losing strategy. This is particularly problematic for high-frequency trading and is the reason why price impact is one of the most rapidly expanding areas of research within trading firms. For regulators, price impact provides feedback on supply and demand, which is an essential component of price discovery. Understanding the price impact can help regulators design fair and efficient markets.

This thesis aims to undertake a study to shed some light on price impact mechanisms and reconcile different views from existing literature. First, an essential aspect of this study is the emphasis on understanding the price impact function form and its relationship with the limit order book. The functional form of price impact has been a puzzle in finance for a long time. However, specific function forms that give a good fit to data vary widely from study to study. Second, the study examines different factors that may drive the price impact of trades and the underlying information value. The aim is to partition price impacts into two components: systematic influence and idiosyncratic influence. Based on the findings of the first two chapters, the last chapter develops a theoretical model to explain the effects of spoofing on market welfare under the linear price impact setting. The thesis is organized into three chapters as follows.

The first chapter establishes the connections between the shape of the book and the price impact by using trades and quotes data from 82 Nasdaq-listed stocks. We introduce the slope of the limit order book as a new measure of price impact. By analyzing both bid and ask sides of high-frequency limit order book snapshot data, our study shows that there is a linear relationship between the cumulative size of liquidity and price impact in the limit order book. In addition, we find that if price impact admits a nonlinear functional form, under certain circumstances, a profitable round-trip arbitrage exists. We empirically show the minimum required trading volume for a profitable self-financing arbitrage and conditions that limit arbitrage. We find that arbitrage requires a large number of shares to the degree that it is impractical, thus providing evidence in support of Huberman and Stanzl.

The second chapter contributes to the literature by introducing a new method to estimate the value of the information. By drawing an analogy between the price impact $\lambda$ and the slope of the limit order book, our results confirm the validity of the Kyle [1985] model under the limit order book settings, which is under question as in the original paper, the market was modeled as a dealer market with only market orders.

Contrary to the traditional postulation of Kyle's $\lambda$, which suggests its estimation is derived from execution data, we propose a novel approach by positing that Kyle's $\lambda$ is expressed in the order book. Additionally, we estimated price and volume volatilities through an entirely different methodology by using execution orders. Although the price impact and volatility estimates are carried out using entirely different data and methods, they confirm the predictions of the Kyle model.

We find that the idiosyncratic shocks to fundamental asset values have little impact on cross-trades and any optimal cross-asset trading strategies would reduce to a diagonal matrix of trading intensity.

Chapter 3 investigates spoofing from economic, historical, and legal perspectives. By studying traders courtroom testimony and interviews, it is shown that practices resembling spoofing have existed for centuries. The recent introduction of electronic trading systems has increased the anonymity of trading, thus creating a perfect environment for spoofing to thrive, in turn increasing regulatory scrutiny. I study recent spoofing courtroom cases and find that the main victims of spoofing are HFTs: they get exploited because spoofers can easily detect and trick their algorithms. Based on those findings, I developed a two-period model of strategic interactions between a spoofer and a high-frequency trader (HFT) who employs pattern recognition algorithms to predict the incoming order. Detecting this strategy, the spoofer submits a spoofing order to mislead the HFT trader about the incoming order. The HFT protects itself by reducing its market participation. I show that that while spoofing delays price discovery in the short run, price dislocation will be so brief as to have few economic efficiency implications. Moreover, spoofing improves market liquidity and market welfare.

Chapter 1

# Arbitraging Nonlinear Price Impact: Testing Huberman and Stanzl

## 1.1 Literature review

### 1.1.1 Price impact function form

Current literature shows conflicting views on the form of price impact. From the market microstructure's perspective, Kyle [1985] shows that impact is both linear in the traded volume and permanent in time. In the model presented by Kyle [1985], informed traders and noise traders submit their orders to a market maker, and then the price is determined by the market maker by using a linear pricing rule. One of the most important characteristics of this model is that the trade signs are serially uncorrelated and symmetric between buys and sells.

In unrelated work, Huberman and Stanzl [2004] show that if there is permanent price impact, then it should be linear in order sizes under non quasi-arbitrage condition. They also point out that if a trade has a temporary price impact, only the permanent component must be linear, while the temporary one can be of more general forms. In the setting of this paper, a trader can buy and sell subsequently the same security at any amount. The trader can strategically earn a positive profit by employing such a strategy when the permanent price impact is not linear in trading volume. Built on Huberman and Stanzl [2004] 's result, Jusselin and Rosenbaum [2020] prove that the temporary component of impact function can only be of power-law type when the price is diffusive with rough volatility under no-arbitrage assumption. Rough volatility indicates the temporary impact function is driven by a (rough) fractional Brownian motion, which is of a short-memory nature. Unlike the aforementioned papers. Keim and Madhaven [1996b] propose a model of the upstairs market with search cost in which order size, beliefs, and prices are determined endogenously. The model setting leads to the power law price impact function of order size.

From the econophysics perspective, price impact is a statistical effect of order flow variations. Bouchaud, Farmer, and Lillo [2009] do empirical research on thousands of trades and point out that the autocorrelation of trade sides n decays extremely slowly with time, and the price fluctuation is persistent and predictable. Therefore, they argue that the price impact should not be linear and permanent. J. D. Farmer [2005] put forward a model of zero-intelligence agents with a budget constraint. The model considers price formation and order submission under the setting of a double auction. As a result, the model produces a highly concave function of price impact. The concavity of price impact comes from the fact that orders near the market price are cleared away more rapidly than those far from the market price. In an attempt to reconcile two perspectives, Kyle and Obizhaeva [2018] presents two methods to obtain price impact function form from a system of economic equations using trading volume and volatility. They point out that the price impact may be a linear or strictly concave function under certain conditions. Another attempt to extend Huberman and Stanzl [2004] 's and Bouchaud et al. [2009] work was Gatheral [2010] model of the price update function with market frictions. He argues that the trading cost should consist of two main components. The first component is market impact, which may decay over time. The second component is market friction, such as effective bid-ask spread that affects only the execution price, but he assumes that friction costs are negligible. Under the no-dynamic-arbitrage condition, where the expected cost of trading should be non-negative, the shape of the market impact function price should satisfy some conditions that make manipulation impossible.

The aforementioned literature on price impact has shown that the price impact is a function of the trade size and the evolution of limit order books. The most universal function form has been proposed as follows:

$$\Delta p = kv^{\beta} \tag{1.1}$$

Where $\Delta p$ is the price impact of a market order of size $v$, $k$ and $\beta$ are parameters of the function. They vary widely in different markets and at different time periods. Even though many empirical studies show that the market impact is a concave function of trade volume, there is a disagreement in the specific function form. The following table shows a comparison of empirical function forms proposed by different studies.

| | Function type | Function form | Market | Years |
|---|---|---|---|---|
| Hasbrouck (1991) | Concave | | NYSE | 1989 |
| Hausman et al. (1992) | Strongly concave | | NYSE | 1988 |
| Keim Madhavan (1996) | Concave | | Upstair market | 1985-1992 |
| Torre (1997) | $\Delta p = kv^{\beta}$ | $\beta = 0.5$ | NYSE | 1994 |
| Dufour Engle (2000) | Nonlinear | | NYSE (TORQ) | 1990-1991 |
| Gabaix et al. (2003) | $\Delta p = kv^{\beta}$ | $\beta = 0.5$ | NYSE | 1994-1995 |
| Farmer et al. (2005b) | $\Delta p = kv^{\beta}$ | $\beta = 0.25$ | LSE | 2001-2004 |
| Hopman (2007) | $\Delta p = kv^{\beta} + \epsilon$ | $\beta = 0.37$ | Paris Bourse | 1995-1999 |
| Bouchaud et al. (2009) | $\Delta p = kv^{\beta}$ | $\beta = 0.3$ | LSE | 2002 |
| Rama cont et al. (2014) | Linear | $\Delta p = kIFO + \epsilon$ | NYSE | 2010 |

Table 1.1: Comparison of impact functions

For these studies, price impact is typically measured by the change in mid-price before the order arrival and mid-price after the order has been executed. This measure of price impact tends to give an incomplete picture of the limit order book market in at least two ways. First, the price impact of market orders receives the same treatment regardless of their trading volumes. All market orders with order sizes smaller than the prevailing quoted depth at the best bid and ask prices are completely absorbed by the best bid and ask quotes. Therefore, they have no immediate price impact, and their delayed impact on the price is ignored. Second, other types of limit order book events than market orders have not been studied extensively or even ignored as only market orders are concerned. However, limit order book events should have indirect impacts on the price. For example, adding a sell limit order should exert extra downward pressure on the price, while canceling a sell limit order should ease this pressure.

Historically, most studies have considered price impact solely as a function of executed orders. However, the limit order book, including higher-order flows and the interplay between order flows, can also impact price. By analyzing the slope of the order book, we can capture these dynamics. For instance, orders at various levels can alter the depth of the limit order book differently, thereby affecting the slope in distinct ways.

Zoltán Eisler and Kockelkoren [2012] are among a few papers that provide a theoretical framework to study the impact of all order book events. The paper is built on the assumption that the price impact is a linear combination of the impact of all past trades.

$$p_t = \int_{t'<t} G(t-t')\epsilon_t v_t^{\theta} + \eta_t \, dt + p_{-\infty} \tag{1.2}$$

where $v_t$ is the volume of the trade at time t, $\epsilon_t$ is the sign of the trade (+ for buy and - for sell), $\eta_t$ is an independent noise term, $G(l)$ is the temporal evolution of the impact of a single trade. In order to incorporate all limit orders, the arrival of an event is considered a shock. The average price after the arrival of an event $\pi$ is defined by the following response function.

$$R(l) =< (p_{t+l} - p_t) \cdot \epsilon_t | \pi_t = \pi > \tag{1.3}$$

where $R(l)$ is the price response function at time $l$ as the result of the shock event $\pi$. It is the conditional expectation of the product of return between $t$ and $t+l$ and the sign of the trade.

### 1.1.2 Price impact asymmetry

Asymmetric price impacts have been studied in many previous papers. However, its existence and its cause are a matter of debate. On the one hand, empirical papers such as Kraus and Stoll [1972], Choe and Wood [1995], Busse and Green [2002], and Cohen, Frazzini, and Malloy [2008] find that buy-side impacts exceed sell-side impacts in the stock exchanges. More recently, Frino, Bjursell, Wang, and Lepone [2008] show similar findings for large trades in four Australian financial futures markets. In contrast, Keim and Madhaven [1996a] find that the price impact of sell orders in the upstairs market is larger than that of buys. Bikker, Spierdijk, and van der Sluis [2007] study the price impact of equity trading by one of the world's largest pension funds and finds a similar result. Brennan, Chordia, Subrahmanyam, and Tong [2012] show that sell-order liquidity is priced more strongly than buy-order liquidity.

Saar [2001] is among the first theoretical papers that give an explanation of why block buys have a larger permanent price impact than block sells. His paper hinges on four observations of institutional investors behaviors: 1) institutional investors invest substantial resources in gathering and analyzing information, 2) they only use money from shareholders to invest and are limited to using leverage, 3) mutual funds hold relatively diversified portfolios, 3) mutual funds cannot sell short as a matter of policy. The paper finds that the price impact is strictly positive when the stock doesn't experience significant price appreciation, and the longer the run-up, the less positive the price impact is. The argument is that initially, the probability that informed investors own the stock is low, and they cannot sell short of the stock; when new information arrives, all mutual funds buy and create a permanent price impact.

Reiss and Werner [2004] use unique data from the LSE to examine how trader anonymity and market liquidity in the inter-dealer market. They find evidence that price impact asymmetry exists but disagree with Saar on the cause of this phenomenon. London dealers have no problem taking extensive short positions, and short selling (stock loans) is inexpensive. Taking a different approach, Dierker, Kim, Lee, and Morck [2016] explain the price impact asymmetry based on information heterogeneity and risk aversion. They explain that informed traders have heterogeneous expectations of the value of the stocks. When new private information arrives, some investors may switch from one side of the market to the other, thus shifting their weight from one side of the market to the other. Both my model and the model by Dierker et al. [2016] consider inelastic demand and supply curves as a possible explanation for price impact asymmetry .On the other hand, J.Fleming, Mizrach, and Nguyen [2018] use GovPX data to assess the microstructure of the U.S. Treasury securities market and show contradicting findings. The paper points out that there is little evidence to corroborate the existence of price impact asymmetry. While they find that buy trades generally have a higher price impact than sell trades by a few percent, most of these differences are not statistically significant.

## 1.2 Data and measurement

### 1.2.1 Data

In our research, we collect data from several sources. First, we employ the proprietary database of US stocks that are trading on the NASDAQ exchange. The database contains message-level information of all stocks in February 2018. For each stock, there is a raw message file that contains all trading messages of one stock sent to the market at high speeds in milliseconds within a trading day. The file provides a comprehensive record of every trade and order book change of different stocks on the exchange. Therefore, limit order book reconstruction is needed. As the dataset records all events that led to state changes to the order book, we can reconstruct the limit order book for any stock at the full depth level for the specified period. The comprehensive and full-depth level data allow us to analyze different characteristics of price impact and its relationship with limit events with higher accuracy. Thus far, the empirical literature in this field has been limited to the use of pre-constructed LOB data such as Lobster with only the top of the book.

The message file contains every arriving market and limit orders as well as cancellations and updates of one stock. The information in the message file has 9 data fields.

1. "Date" provides information regarding the trading day
2. "Timestamp" All entries have timestamp of seconds after midnight with the precision of milliseconds.
3. "OrderNumber", each order has a unique ID. Zero reference orders correspond to a hidden limit market order.
4. "EventType" There are 11 types of market events in the data.
5. "Ticker" provides information regarding the trading stock
6. "Price" the price of the order
7. "Quantity" the quantity of the order
8. "MPID" provides information of Market Participant Identifier. This identifier is used by FINRA member firms to report trades.
9. "Exchange" There are 2 main exchanges ARCA and NASDAQ. All entries detail which exchanges the order was sent to.

Generally speaking, the order ID corresponds to the unique order reference number, which we can use to differentiate messages. However, there are some exceptions that may affect our limit order reconstruction.

1. All messages classified as "trade bid" and "trade ask" have zero reference orders. Those are hidden market orders with full information for all other fields except the order number. As they are market orders, they don't affect our limit order reconstruction, but we need to take them into consideration when we look at executed orders.

2. For big stocks that are trading across trading platforms, there are some order IDs corresponding to multiple different orders sent to different trading venues. One example is the messages with order ID 6168348 (TSLA, 08 Feb 2018). Essentially, the ID corresponds to 2 separate messages sent to different exchanges. The first order was a bid order at 08:20:56, which was sent to NASDAQ, then eventually got executed and filled later. The second order was an ask order at 09:42:40, which was sent to ARCA and then deleted eventually. In order to differentiate those different orders with the same reference number, we can look at the exchange and nature of the order. First, these orders were sent to different exchanges. We can use trading venues to find out and group all related orders. Second, we can use the nature, such as the order type and price, to map out all related orders. For example, "Add bid" orders should have related orders of type "execute bid" and "fill bid"; "Add ask" orders should have related orders of type "execute ask" and "fill ask."

Second, we obtain the data for the stock directory with market cap, $R^2$, $\beta_{CAPM}$, and variance from the NYSE and Zoonova. Our directory contains all active stocks during the period between 1 January 2021 and 31 December 2021. We collected the list of active tickers from January to December 2021, though the sampling period focuses on February 2018. The primary justification for this period is that February 2018 was relatively stable, falling outside the U.S. earnings season and unaffected by significant macroeconomic events. All stocks must meet three pre-screening criteria to be in the directory: (1) it is a common stock, (2) it is active on the first and last day during the sampling period. Active stocks refer to any stocks with trading activity on public exchanges during the sampling period. Out of over 6,500 tickers, some stocks were not listed or did not exist as of February 2018. (3) it has NASDAQ as the primary listing exchange. After filtering out all duplicates and erroneous entries, we are left with 6,481 stocks. There were 19 trading days in total, and each stock had approximately 10,000 to over 1,000,000 messages for one trading day. Therefore, the input file size can reach the region of 20 GB for one ticker day, thus posing technical changes in terms of computation and data storage. We employed stratified random sampling by partitioning all tickers into subpopulations. The sample stocks were chosen based on the following sampling characteristics: high $R^2$, low $R^2$, high $\beta$, low $\beta$, high market cap, low market cap and low variance, high variance. 12 tickers were randomly selected from each group, yielding an initial sample of 96 tickers. After removing duplicate entries, 83 unique tickers (including SPY) remained. The rationale behind this sampling method is that stocks have high variances in all those characteristics. Stratified random sampling allows us to effectively select stocks that represent a diverse range of groups. The statistical summary of those stocks is illustrated in the table below.

| | R-squared | Marketcap | Variance (yearly) | Beta |
|---|---|---|---|---|
| Mean | 0.1590 | 19,300,751,911 | 6.096 | 0.93 |
| Standard Error | 0.0300 | 6,811,346,968 | 1.335 | 0.18 |
| Median | 0.0299 | 353,644,000 | 0.970 | 0.85 |
| Minimum | 0.0001 | 23,198 | 0.063 | (2.47) |
| Maximum | 0.7253 | 343,970,000,000 | 35.490 | 4.81 |

Table 1.2: Descriptive statistics of the sample

### 1.2.2 Limit order book reconstruction

The message file contains every arriving market and limit orders as well as cancellations and updates of one stock. For every entry in the message file, we need to recreate a corresponding snapshot in the order book file that describes how the order book is updated immediately after the message file event. In summary, the information contained in the message files has the following properties

- All entries have timestamps of seconds after midnight with a precision of at least milliseconds, the order ID corresponding to the unique order reference number. Zero reference number corresponds to a hidden limit order
- There are 12 types of market events that are recorded in the data (see Table 2 below). In this study, a limit order book is the primary focus. Therefore, cross messages are filtered out as those transactions occurred in a dark pool or an auction. When a market order is matched against several limit orders, each matching is recorded separately Messages labeled as "FILL ASK" and "FILL BID" have missing price and quantity fields. We need to trace back to the original order and other orders of the same IDs to figure out missing pieces.

To reconstruct a limit order book from a raw message file, an initial snapshot of LOB is created, and then next snapshots will be updated when a new message arrives. After handling missing data and abnormal entries, limit order books are reconstructed with the ask price and the corresponding volume, as well as the bid price and its volume. The newly constructed limit order book has a full depth with price and volume at all levels. For "ADD ASK" and "ASK BID" message types, a new snapshot is updated by adding those new messages to the previous snapshot. For "CANCEL BID", "CANCEL ASK", "EXECUTE ASK", "EXECUTE BID" message types, the order ID and exchange of the message are matched against the orders in the previous snapshot to look for the outstanding order that should be updated by reducing its order size. For "DELETE BID", "DELETE ASK", "FILL ASK", "FILL BID" message types, the corresponding orders get processed completely. Therefore, the new snapshot is constructed by deleting all orders with the same IDs of incoming messages. At any time, there are only "ADD ASK" and "ADD BID" messages outstanding in a snapshot. Upon the creation of a snapshot, ask and bid order types are separated, sorted, and grouped by price. The final step is to filter out abnormal entries and then create the cumulative depths at each price level.

| Event Type | Description |
|---|---|
| ADD ASK | Submit a new ask order |
| ADD BID | Submit a new bid order |
| CANCEL BID | Cancel the bid order partly |
| CANCEL ASK | Cancel the ask order partly |
| CROSS | Dark pool transactions without price and quantity |
| DELETE ASK | Delete the whole ask order |
| DELETE BID | Delete the whole ask order |
| EXECUTE ASK | Execute the order partly |
| EXECUTE BID | Execute the order partly |
| FILL ASK | Fill the ask order completely |
| FILL BID | Fill the bid order completely |
| TRADE BID | Fill the bid order completely |

Table 1.3: Even Type in the message file

### 1.2.3 Defining the price impact and model setup

#### 1.2.3.1 Transaction costs

The trading costs generally consist of two main parts. The first component is the impact of an incoming order on the price of the stock which we refer to as the price impact in this paper. The second part comes from the market frictions such as effective ask-bid spread, brokerage fees, and trading fees which also affect our execution price and trading profit.

In this paper, the effective ask-bid spread is calculated as the difference between the best ask and best bid price in the limit order book after each message gets updated. To measure daily ask-bid spread, we take an average of ask-bid spreads across snapshots, but we exclude the snapshots with 30 minutes of market opening and closing.

$$Spread_t = a_t - b_t \tag{1.4}$$

where $Spread_t, a_t, b_t$ are the effective ask-bid spread, best ask price, and best bid price, respectively.

Another friction cost that we take into consideration in this paper is trading costs. There are many different types of trading costs; our main concern here is compulsory charges by the Nasdaq exchange. Membership is required to participate in most of the U.S. equities and options exchanges, including Nasdaq. For fixed costs, Nasdaq charges their members a membership fee of $1,200 per year and a trading rights fee of $200 per month. For variable costs, Nasdaq adopts the "taker-maker" fee model to attract order flow and encourage market participants to provide liquidity. The "taker-maker" model dates back to the late 1990s when electronic trading venues started becoming popular. In this payment scheme, a market generally pays its members a rebate to provide ("make") liquidity and charges them a fee to remove liquidity. As a result, market participants are incentivized to submit competitive quotes at the best bid prices, thus increasing the market's liquidity. By the mid-2000s, this model had become the standard pricing model for most US exchanges. The maker-taker fees in the equities markets are regulated by Rule 610 of Regulation NMS and get capped at $ 0.003 per share. Apart from maker-taker fees, market participants have to pay the SEC and FINRA Trading Activity Fee (TAF), which are regulatory fees charged on the sale of any security. These fees are automatically debited from the proceeds of any security sale. These fees year by year. Currently, SEC's fee rates applicable to most securities transactions will be set at \$8.00 per million dollars while FINRA fee is \$0.000145 per share for each sale of a covered equity security.

#### 1.2.3.2 Price impact

Kyle [1985] proposed a seminal model that provides insight into the mechanisms by which private information is gradually incorporated into prices in an efficient market. The model was then later extended by Kyle [1989] to take into account competing informed traders. In the original model, there are 3 main types of market participants: an informed trader with insider information that gives him greater predictive power about the future value of an asset than other players, noise traders, and market makers. Under this framework, Kyle suggested a single statistical measure of the impact that is both linear in traded volume (order size) and permanent in time. $\lambda$ is a measure of market impact cost from Kyle (1985), which can be interpreted as the cost of demanding a certain amount of liquidity over a given time period. $\lambda$ can also be used as a measure of market liquidity and can be estimated by the volume required to move the price of a security by one dollar. It can be interpreted as the elasticity of returns against net order flow or signed volume (volume times the sign of the return). Khandani and Lo [2011] estimate this measure on a daily basis by using all transactions during normal trading hours on each day. The authors obtain the sequences of intraday returns $(R_t)$ , prices $(p_t)$, and volume $(v_t)$ to estimate the following equations.

$$R_t = \alpha_t + \lambda Sgn(t) \ln(v_t p_t) + \epsilon_t \tag{1.5}$$

Where Sgn(t) is -1 or + 1 depending on the direction of the trade. $Sgn(t)$ is + 1 for buy orders, -1 for sell orders. Equation (1.5) implies that the intraday return is a signed logarithmic function of trading volume in dollars. Any interval with zero return receives the same sign as that of the most recent transaction with a non-zero return (using returns from the prior day, if necessary).

YakovAmihud [2002] proposes a price impact measure that captures the daily price response associated with one dollar of trading volume. Specifically, they use the following ratio.

$$Price\,impact = Average(\frac{|r_t|}{Volume}) \tag{1.6}$$

where $r_t$ is the stock return on day t and Volume is the dollar volume on day t. The average is calculated over all positive-volume days since the ratio is undefined for zero-volume days. Alternatively, following Hasbrouck (2009) and Goyenko, Holden, Trzcinka (2009), a representative coefficient is estimated as the $\lambda$ coefficient in the regression. Contrary to Kyle's lambda, this measure is a root square function of volume.

$$\Delta P = \lambda\,(Signed\,\sqrt{Dollar\,Volume}) + \epsilon \tag{1.7}$$

Using a different approach, Bouchaud et al. [2009], postulate that the simplest quantity, that can assist in the study of price changes, is the mean squared fluctuation of the prices between the given trade and the execution of the next one (correspondent to the execution of MOs in the context of LOB trading). They functionally define this degree of fluctuation as the difference of mid-price in 2 periods. Then, the price impact function is defined as the signed value of the average difference between the mid-price just before the arrival of an original market order and the mid-price just before the arrival of the next market order

$$D(l) = m_{t+1} - m_t \tag{1.8}$$

$$R(t) = \epsilon_t(m_{t+1} - m_t) \tag{1.9}$$

where $m_t$ is the mid-price at the period $t$. $\epsilon_t$ is the direction of the trade.

While there are a handful of studies that propose different measures for price impact, they construct price impacts of only *market orders* by using transaction-level data. The vast majority of the literature using the measure employs them on message-level (or finer) data for the limit order book. Kaniel and Liu [2006] study the private information flows into prices through both limit orders and market order executions and conclude that limit order events convey more information to stocks than market order executions. This means the usual way of measuring information content as the trade-correlated permanent component of price impact omits the information content of limit orders. Therefore, estimates of price impact by using market order executions may be understated. Furthermore, from practitioners' perspective, there is a lack of price impact measures that allow them to estimate their trading costs in a high-frequency trading environment even before they submit their orders.

In this study, we propose a new measure of price impact based on standard Kyle's $\lambda$. The new measure is the derivative of the cost of demanding a certain amount of liquidity when a new limit event is submitted. In other words, price impacts are the slopes of demand and supply curves implicit in the limit order book. We calculate this price impact ($\lambda$) as the coefficient of the regression.

$$P_{bid,i,t} = \lambda_{bid,t} V_{bid,i,t} + P_0 + \epsilon_t \tag{1.10}$$

$$P_{ask,i,t} = \lambda_{ask,t} V_{ask,i,t} + P_0 + \epsilon_t \tag{1.11}$$

Where $P_{bid,i,t}$ and $P_{ask,i,t}$ are bid and ask prices of the limit orders at the level $i^{th}$ at time $t$. $V_{bid,i,t}$ and $V_{ask,i,t}$ are cumulative depths at the level $i_{th}$ at time t, $\epsilon_t$ is the error term. $d_t^{a,i}$ is the market depth at the l-th best ask after the t-th event, $d_t^{b,i}$ is the market depth at the l-th best bid after the t-th event.

$$V_{bid,i,t} = \sum_{n=1}^{i} d_t^{b,i} \tag{1.12}$$

$$V_{ask,i,t} = \sum_{n=1}^{i} d_t^{a,i} \tag{1.13}$$

The proposed measure of price impact is the slopes of price against cumulative depths, thus making it fit well in Kyle's framework as the inverse of $\lambda$ in the limit is the depth at the specific price level.

$$\lim_{i \to j} \frac{1}{\lambda_i} = \lim_{i \to j} \frac{V_i - V_j}{P_i - P_j} \approx v^j \tag{1.14}$$

#### 1.2.3.3 Order flow imbalance and market depth

Order flow imbalance is the changes in supply and demand of a stock over a given period. In Kyle's model, the author suggests that the price change is driven by the order flow imbalance. The empirical implementation of order flow imbalance for a limit order book is a big challege. Hopman [2007] proposed two measures of order flow imbalance. The first natural measure is to add the volume of all buy orders and subtract all sell orders in a given period.

$$OFI_i = \sum v_i^{buy} - \sum v_i^{sell} \tag{1.15}$$

Where $OFI_i, v_i^{buy}, v_i^{sell}$ are the order flow imbalance, the volume of all buy orders, and the volume of all sell orders in the i period. Assuming that the price impact is a square root function of volume, Hopman [2007] used another SQRT measure to represent the order flow imbalance.

$$OFI_i = \sum (v_i^{buy})^{0.5} - \sum (v_i^{sell})^{0.5} \tag{1.16}$$

In Hopman [2007]'s paper, the author showed that most of the stock price changes can be explained by the imbalance between buy and sell orders. However, these measures have two shortcomings. First, they do not take other effects of limit order events on order flow imbalance. For example, cancellation of an order at the best ask can lead to a decrease in supply, while adding a new order at the best ask price may result in supply. Second, they require access to limit order data to ensure that order flow imbalance reflects the imbalance in investors intention to trade.

Cont [2014] proposed a model describing the relationship between order flow imbalance and price changes. Their study shows a linear relation between order flow imbalance and price changes, with a slope inversely proportional to the market depth. Given two consecutive snapshots of a limit order book, they define the bid $OF_n^{m.b}$ and ask order flows $OF_n^{m.a}$ at level $m$ at time $n$ as follows

$$OF_n^{m.b} := \begin{cases} q_n^{m.b} & \text{if } P_n^{m.b} \geq P_{n-1}^{m.b} \\ q_n^{m.b} - q_{n-1}^{m.b} & \text{if } P_n^{m.b} = P_{n-1}^{m.b} \\ -q_n^{m.b} & \text{if } P_n^{m.b} < P_{n-1}^{m.b} \end{cases} \tag{1.17}$$

$$OF_n^{m.a} := \begin{cases} -q_n^{m.a} & \text{if } P_n^{m.a} \geq P_{n-1}^{m.a} \\ q_n^{m.a} - q_{n-1}^{m.a} & \text{if } P_n^{m.a} = P_{n-1}^{m.a} \\ q_n^{m.b} & \text{if } P_n^{m.a} < P_{n-1}^{m.a} \end{cases} \tag{1.18}$$

where $P_n^{m,a}, q_n^{m,a}, P_n^{m,b}, q_n^{m,b}$ denote the ask price, ask size, the bid price, bid size at the level $m$ at time $n$, respectively. Essentially, the equations (1.17) and (1.18) state that the order flow is the change in the ask and bid size at each price level. The order flow imbalance is defined by.

$$OFI_n^{m,T} := \sum_{n=h}^{h+T} (OF_n^{m,a} - OF_n^{m,b}) \tag{1.19}$$

The order flow imbalance $OFI_n^{m,T}$ is defined by the sums of differences between the order flow of the ask and bid side at the level $m$. The shortcoming of this measure is that the ask and bid prices at level $m$ are different, and they grow wider as they move far away from the top of the book.

In this paper, we use the market depth dynamics as an indicator of the order flow. The full market depth with corresponding prices forms a snapshot (a state) of the limit order book. By studying the relationship between the market depth dynamic and the price, we assume that the price impact depends on both incoming orders and the state of the order book. Furthermore, the change in market depth at each level can fully capture the order flow imbalance at that specific price level. If we consider Cont [2014]'s model within a high-frequency time frame when there is at most one order sent to the market, the order flow imbalance by collapses into the order flow only from the bid or ask side. If there is only one order from the ask side, the order flow imbalance by is the change in cumulative market depth defined by (1.13).

$$OFI_{n,a}^{m,\delta T} := OF_n^{m,a} = \Delta V_{ask,m} \tag{1.20}$$

#### 1.2.3.4 Model specification

There is a consensus that the price impact should be a function of the trade size and the limited order book dynamics. The most popular function of price impact in the literature takes the form.

$$\Delta p = \alpha v^{\beta} \tag{1.21}$$

Where $\Delta p$ is the price impact of a market order of size $v$, $\alpha$ and $\beta$ are parameters of the function. In order to test the relationship between the price, cumulative depth, and price impact, we conjecture that the price change is a power function of a depth change.

$$|\Delta P_{i,t}| = \alpha |\Delta V_{i,t}|^{\beta} \tag{1.22}$$

Where $\Delta P_{i,t}$, $\Delta V_{i,t}$ are the differences of the bid price of the limit orders at the level $i^{th}$, cumulative depths at the level $i_{th}$ at time t and the best bid price, depth at the best bid price respectively.

The cumulative depth is only calculated at the price level where there is an active order. Therefore, zero changes in cumulative depth and price levels are trivial cases that are not possible in this scenario due to the model settings. Moreover, each limit order book event results in a non-zero change in cumulative depth at its price level. It is, therefore, reasonable to assume that $|\Delta V_{i,t}|$ and $|\Delta P_{i,t}|$ are positive. If we take logs on both sides (1.22), we have the empirically testable relationship.

$$\ln |\Delta P_{i,t}| = \ln \alpha + \beta \ln |\Delta V_{i,t}| \tag{1.23}$$

In order to test the linear relationship between the change in price and depth, we estimate the regression (1.22) and test the hypothesis $\beta = 1$.

### 1.2.4 Results

#### 1.2.4.1 Overview

This section reports the statistics summary result for average price impact cross stocks. Detailed results for each stock in our sample are presented in the Appendix A.1. For the first part of the report, we estimate the price impact as the coefficient of the regression given by (1.11), (1.10). Table 1.4 presents the summary of estimated $\lambda$ cross stocks for both bid and ask sides. Our study found that the coefficients $\lambda_{bid}$ and $\lambda_{ask}$ are statistically significant for more than 95% of the sample. The absolute value of $\lambda_{bid}$ and $\lambda_{ask}$ of the same snapshot show a positive relationship, and they are close to each other. The goodness of fit is surprisingly high for most stocks with

$R^2$, which is close to 0.89 on average. It is also notable that $\lambda$ varies significantly across stocks in different groups on both the bid and ask sides, indicating substantial variation in liquidity within the stock market.

| | $-\lambda_{bid}$ | $\lambda_{ask}$ | $R^2_{ask}$ | $R^2_{bid}$ |
|---|---|---|---|---|
| **Average** | 1.84E-03 | 1.27E-03 | 0.883 | 0.871 |
| **Median** | 2.72E-04 | 1.54E-04 | 0.892 | 0.921 |
| **Min** | 3.66E-06 | 4.38E-07 | 0.53 | 0.49 |
| **Max** | 4.81E-02 | 3.68E-02 | 0.973 | 0.987 |
| **Std** | 0.00681 | 0.00445 | 0.11 | 0.0845 |

Table 1.4: Descriptive Statistics of the average daily price impact

Table 1.5 provides an overview of the average $\beta$ and $\ln\alpha$ across stocks in Feb 2018. $\beta_{bid}$ and $\beta_{ask}$ are the average estimates of $\beta$ from the bid and ask sides respectively. Both of them covary positively in the sample across stocks and time in the same magnitude. $\beta_{bid}$ has an average cross-sectional value of 1.259 and a median of 1.11, while $\beta_{bid}$ has an average cross-sectional value of 1.14 and a median of 1.09. Both have means and medians surprisingly close to 1. Furthermore, the values of $R^2$ are significantly high for both bid and ask sides, and $R^2$ are greater than 80% for a majority of regressions. This supports the idea that price impact is a linear function of trading volume.

| | $\beta_{bid}$ | $\beta_{ask}$ | $\ln\alpha$ | $\ln_{\alpha}$ | $R^2_{ask}$ | $R^2_{bid}$ |
|---|---|---|---|---|---|---|
| Average | 1.259 | 1.136 | -12.918 | -10.802 | 0.865 | 0.881 |
| Median | 1.116 | 1.090 | -11.179 | -10.217 | 0.889 | 0.9118 |
| Variance | 0.151 | 0.169 | 90.801 | 16.336 | 0.0088 | 0.0071 |
| STD | 0.3887 | 0.411 | 9.52 | 4.041 | 0.0942 | 0.0845 |
| Max | 2.767 | 2.424 | -4.501 | -2.985 | 0.9636 | 0.9609 |
| Min | 0.723 | 0.3196 | -88.297 | -25.425 | 0.4657 | 0.4821 |
| Standard deviation | 0.388 | 0.411 | 9.528 | 4.041 | 0.0942 | 0.0846 |

Table 1.5: Descriptive Statistics of the average daily $\ln\alpha$ and $\beta$

Current literature shows conflicting views on the values of $\alpha$ and $\beta$. While Saar [2001] is among the first theoretical papers that support the price impact asymmetry, Huberman and Stanzl [2004] shows that the price impacts should be the same for both bid and ask sides; otherwise, traders can use arbitrage strategies to make a free-risk profit. In this section, we test whether the average $\beta_{bid}$ and $\beta_{ask}$ are different from each other and from 1 by running a t-test. Let $\beta' = \beta_{bid} - \beta_{ask}$, our null hypothesis is to test $\beta' = 0$ with the t-test:

$$t_{ask-bid} = \frac{\bar{\beta'}}{s(\beta')/\sqrt{n}} \tag{1.24}$$

$$t_{ask} = \frac{\bar{\beta}_{ask} - 1}{s(\beta_{ask})/\sqrt{n}} \tag{1.25}$$

Where $\bar{\beta}'$ is the average of cross-sectional differences between 2 parameters. $s(\beta')$ is the standard deviation of differences. In order to implement the test, we need to estimate daily bid and ask's $\beta$ across stock, then compute the pairwise differences. There are 82 stocks, with 19 trading days for each stock. We assume that differences are i.i.d. over time and across stocks and test whether the average difference is different from zero. The $t_{ask-bid}$ value is 2.3. So, $\beta_{bid}$ and $\beta_{ask}$ are not statistically different from each other. However, the $t_{ask}$ value is 3.3. In the next section, we will examine whether arbitrage is possible if the price impact admits a non-linear relationship.

#### 1.2.4.2 Round trip strategy

Following Huberman and Stanzl's definitions, we define $v_n$ as a round trip trade if its sum is zero or $\sum v_n = 0$. We assume that (1.22) is a correctly specified model of the price impact and depth. We consider the following round-trip strategies and study their opportunities.

**Strategy 1**: Buy M shares (M a big number) in 1 trade, then sell $\frac{M}{k}$ shares in k subsequent periods. M and k are integers.

**Strategy 2**: Buy $\frac{M}{k}$ shares (M a big number) in k consecutive trades, then sell M shares in the last period. M and k are integers.

The profit of a round trip trade is given by (1.26). By analogy with another of Huberman and Stanzls definitions, a risk-neutral manipulation is a round trip trade with a positive expected profit.

$$\pi(v_n) = \sum_{n=1}^{N} -p_n v_n \tag{1.26}$$

where $p_n, v_n$ are price and trading volume in the $n^{th}$ period.

Now we consider a limit order book in which the relationship between change in the limit order book depth and the price is described as $|\Delta P_t| = \alpha |\Delta V_t|^{\beta}$. This equation holds true for both the bid and ask sides. We assume that purchases are expected to have a positive impact on the price of the stock, and sell orders are expected to have a negative price on a stock. We denote $f(x)$ as the price impact function, where x is a change in the volume. Assume that a trader buy $x$ shares, that moves the price from $p_0$ to $p_1$. By the model's setup, the new price is given by

$p_1 = p_0 + \alpha x^\beta$. The trader needs to pay.

$$C(x) = \int_0^{p_1 - p_0} (p_0 + \Delta p)(f^{-1})'(\Delta p) d(\Delta p) \tag{1.27}$$

As the trader market buys x shares, he walks the book and depletes all outstanding limit orders between $p_0$ and $p_1$. His realized cost is the sum of all outstanding limit orders between $p_0$ and $p_1$. By the setting up of the model, the cumulative depth at the price level $p_i$ is $v_t = f^{-1}(\Delta p_i)$. Therefore, the depth or order size at that price level is the derivative of $f^{-1}(\Delta p_i)$.

Simplifying the equation (1.27), we can get the cost

$$C(x) = xP_0 + \frac{\alpha x^{\beta+1}}{\beta + 1} \tag{1.28}$$

Applying the equation (1.28) to the first strategy, the expected profit for the first round trip strategy is as follows

$$\pi_1 = -\frac{\alpha M^{\beta+1}}{\beta+1} + \left( \alpha M^\beta \frac{M}{k} k - \frac{(k-1)k}{2} \alpha \left(\frac{M}{k}\right)^\beta \frac{M}{k} \right) - k \frac{\alpha}{(1+\beta)} \left(\frac{M}{k}\right)^{\beta+1} - cM \tag{1.29}$$

Where $c$ is the trading cost per share, which includes the ask-bid spread and other trading fees. For the purpose of this paper, we estimate $c = spread + 0.006$, where \$ 0.006 comes from the Nasdaq fee per share.

| Order | No | Starting Price | Ending Price | V | Cost/profit |
|---|---|---|---|---|---|
| Buy | 1 | 0 | $\alpha M^\beta$ | $M$ | $\frac{\alpha M^{\beta+1}}{\beta+1}$ |
| Sell | 1 | $\alpha M^\beta$ | $\alpha M^\beta - \alpha \left(\frac{M}{k}\right)^\beta$ | $\frac{M}{k}$ | $(\alpha M^\beta)\frac{M}{k} - \alpha \left(\frac{M}{k}\right)^{\beta+1} \frac{1}{\beta+1}$ |
| Sell | 2 | $\alpha M^\beta - \alpha \left(\frac{M}{k}\right)^\beta$ | $\alpha M^\beta - 2\alpha \left(\frac{M}{k}\right)^\beta$ | $\frac{M}{k}$ | $(\alpha M^\beta - \alpha \left(\frac{M}{k}\right)^\beta)\frac{M}{k} - \alpha \left(\frac{M}{k}\right)^{\beta+1} \frac{1}{\beta+1}$ |
| Sell | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| Sell | $i$ | $\alpha M^\beta - (i-1)\alpha \left(\frac{M}{k}\right)^\beta$ | $\alpha M^\beta - i\alpha \left(\frac{M}{k}\right)^\beta$ | $\frac{M}{k}$ | $(\alpha M^\beta - (i-1)\alpha \left(\frac{M}{k}\right)^\beta)\frac{M}{k} - \alpha \left(\frac{M}{k}\right)^{\beta+1}$ [illegible] |

Table 1.6: Breakdown of cost/profit

The first term $-\frac{\alpha M^{\beta+1}}{\beta+1}$ comes from the cost of buying M share at once, thus pushing the price from $p_0$ to $p_1 = p_0 + f(M)$. On average, the trader pays $\frac{\alpha M^{\beta}}{\beta+1}$ more as the cost of price impact. The second term is the net revenue from selling $\frac{M}{k}$ shares over k periods. The first chunk of $\frac{M}{k}$ shares is sold at the price of $p_1$ and the $i^{th}$ chunk of $\frac{M}{k}$ shares is sold at the price of $(i-1)\alpha\frac{M^{\beta}}{k^{\beta}}$. The third term is the price impact of selling $\frac{M}{k}$ shares in k periods. By (1.28), the price impact cost of selling $\frac{M}{k}$ is $\frac{\alpha M^{\beta+1}}{(1+\beta)k^{\beta+1}}$. Therefore, the price impact of k sell orders of $\frac{M}{k}$ shares is $k\frac{\alpha M^{\beta+1}}{(1+\beta)k^{\beta+1}}$.

The trader chooses the number of trades $k$ to maximize his profit given by $\pi_1$. If $\beta < 1$, $\pi_1$ is a decreasing function in k. If If $\beta > 1$, $\pi_1$ is an increasing function in k. The first order condition of $\pi_1$ with respect to $k$ is given by

$$\pi_1' = \frac{\alpha M^{\beta+1}(\beta-1)}{2(1+\beta)k^{\beta+1}}((1+\beta)k - \beta) = 0 \tag{1.30}$$

There are two cases. In the case that $\beta < 1$, the derivative of $\pi_1$ with respect to $k$ is negative. Therefore, the profit function is decreasing in $k$. The maximum profit is $\pi_1(1)$, which means the trader should sell all shares in one trade. This makes sense as $\beta < 1$, the price impact function is concave, and the price impact of selling all shares is less than splitting up the order into small orders. The profit function at $k = 1$ is given by $\pi_1(0) = -\alpha M^{\beta+1}(\frac{1-\beta}{(1+\beta)}) - cM < 0$ which is negative. In other words, there are no arbitrage opportunities when $\beta < 1$. [1]

In the case that $\beta > 1$ the derivative of $\pi_1$ with respect to k is positive and the profit function is increasing in $k$. The profit function attains its maximum value at $k = M$. Therefore, the best strategy for the trader is to sell one share in each of M periods to minimize the price impact when $\beta > 1$. The profit function at $k = M$ is given by

$$\pi_1 = \frac{\alpha\beta M^{\beta+1}}{\beta+1} - \alpha M\left(\frac{1}{1+\beta} + \frac{M-1}{2}\right) - cM \tag{1.31}$$

Now, the trader chooses the optimal number of shares that he should trade. The derivative of $\pi_1$ with respect to $M$

$$\pi_1' = \alpha\left(\beta M^{\beta} - \frac{1}{1+\beta} - M + \frac{1}{2}\right) - c \tag{1.32}$$

1. In the case, we allow traders to trade futures and options in order to minimize the price impact, the cost of trading (excluding premiums) for one future contract or options contract is around $2, that is more than 10 times the average ask-bid spread in our sample.

The second derivative of $\pi_1$ with respect to $M$ is $\pi_1'' = \alpha(\beta^2 M^{\beta-1} - 1)$. As $\beta$ is strictly greater than one and $M > 1$, the second derivative of $\pi_1$ with respect to $M$ is strictly positive. Furthermore, we have $\pi_1'(0) = \frac{\alpha(\beta-1)}{2(1+\beta)} - c < 0$ as $\alpha$ is very small, and $\lim_{M \to +\infty} \pi_1'(M) = +\infty$. Therefore, $\pi_1'$ has a unique solution, denoted by $M_0$. The profit $\pi_1$ at $M_0$ is

$$\pi_1(M_0) = \alpha M_0 \frac{(1-\beta)((1+\beta)M_0 + 1 - \beta)}{2(1+\beta)^2} - \frac{\beta}{\beta+1} c M_0 \tag{1.33}$$

As $\beta > 1$, the profit $\pi_1$ attains its minimum at $M_0$ and $\pi_1(M_0)$ is negative.

Figures 1.1 - 1.3 illustrates the round trip strategy when $\beta > 1$. The trading costs, such as funding fees, ask-bid spread, and other trading costs, are excluded for simplicity. Each figure shows the evolution of the limit order book after a trade. Figure 1.1 presents the change in the limit order book after the trader buys $\Delta Q$ shares. The trade depletes the ask side and moves the price of the stock from $P_0$ to $P_1$. By our assumption, the trader needs to pay $C_{buy}$, which is the green area. After a big buy order, the trader gradually liquidates the position by splitting it into a series of small orders. Figure 1.2 shows the revenue, the yellow area, which the trader can obtain by selling $\Delta Q_1$ shares. Figure 1.3 presents the profit of the round trip strategy without trading costs when $\beta > 1$. The profit is the area between the curve $P_1Q_1$ and $P_1P_2P_3P_4Q_1$. When $\beta > 1$, $P_1P_2P_3P_4Q_1$ lies above $P_1Q_1$ and the trader makes a positive profit. When $\beta = 1$, $P_1P_2P_3P_4Q_1$ collapses into $P_1Q_1$ and the trader makes a zero profit. When $\beta < 1$, $P_1P_2P_3P_4Q_1$ lies below $P_1Q_1$ and the trader incurs a loss.

To achieve a positive profit, the number of M shares should exceed a threshold $M_1$ at which $\pi_1(M_1)$ is zero. As we cannot find a closed-form solution for $\pi_1$, we will solve it numerically by using our empirical estimations. Out of 83 stocks, there are 47 stocks with $\beta > 1$. The reason for this is that those stocks have most of their resting orders at the top of the book. Therefore, the shape of the book tends to be steep near the top of the book and becomes flat when moving far away from the best bid and ask. We solve $M_1$, the volume the trader needs to trade to break even for those stocks.

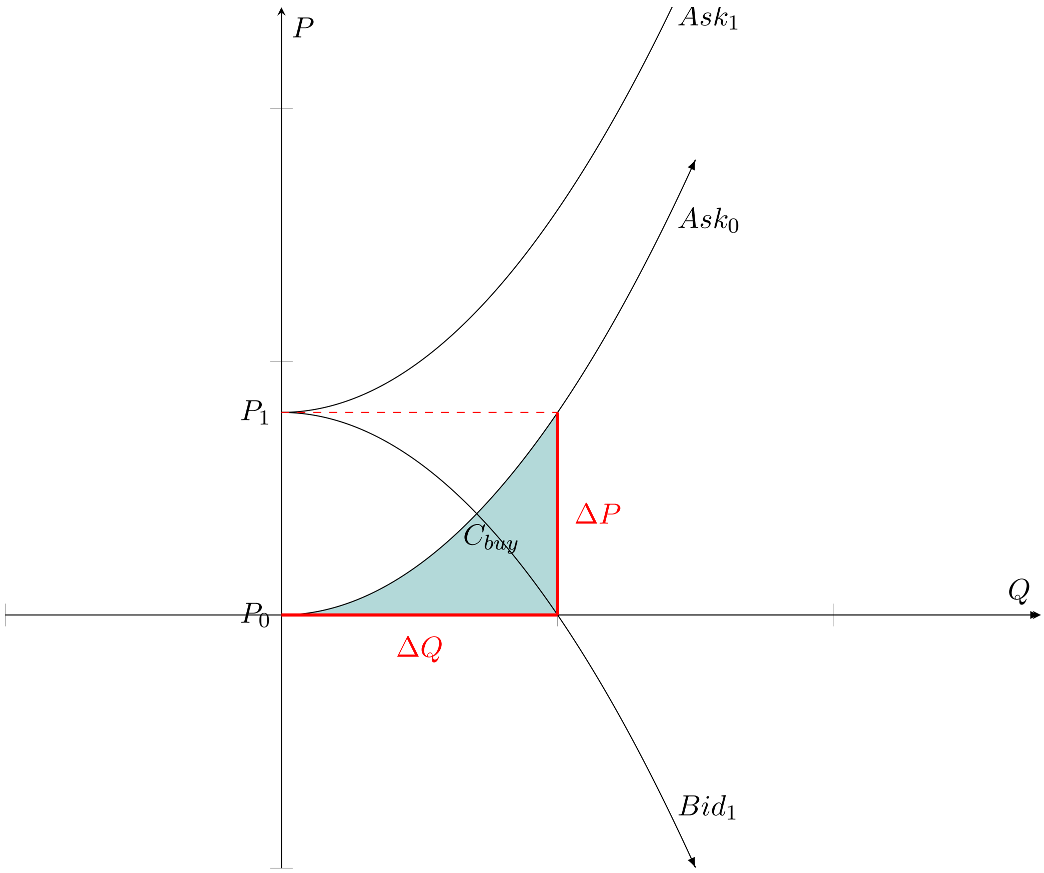



Figure 1.1: The cost of buying $\Delta Q$ when $\beta > 1$.

Table 7 provides an overview of the solutions $M_1$. The average $M_1$ is 11,488, the maximum value for $M_1$ is 238484. On average, the trader needs to trade 11488 shares with a trade value of $222,794 in order to break even. The number of trades required is about 27.3 times the daily average number of executed orders and 29 times the average size of executed orders. Even though there are arbitrage opportunities, there are some limits to arbitrage. First, in this model, we assume that the bid-ask spread is constant across time, requiring that the limit order book be highly resilient. However, when a big market order is submitted, the liquidity of one side gets depleted, and the spread between ask and bid sides widens. It may take a long time for the spread to return to a normal level. Second, the arbitrage strategy requires initially buying a large number of shares and subsequently selling one share at a time over many periods. The trader can fall victim to predatory trading if their strategy gets detected: other market participants can front-run their strategy. Furthermore, employing the strategy requires a sufficiently long horizon, thus exposing the trader to inventory risks if the price moves against his strategy.

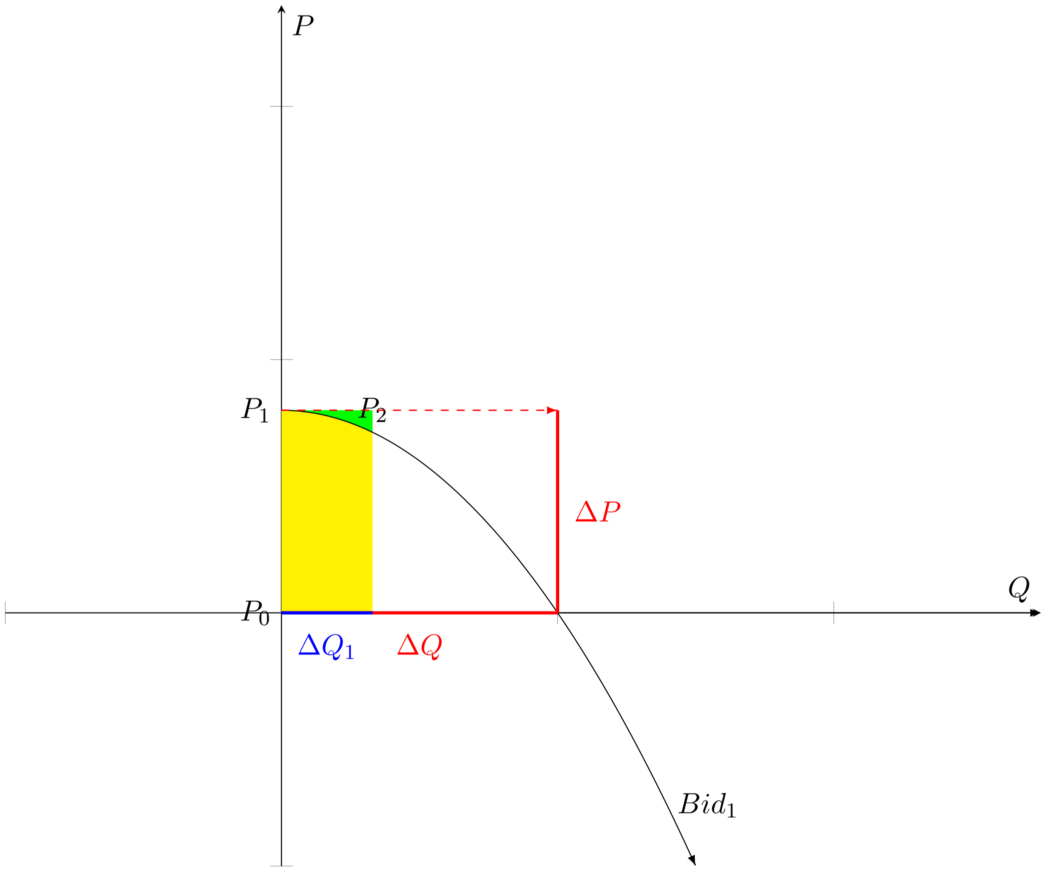



Figure 1.2: The revenue of selling $\Delta Q_1$ when $\beta > 1$

Now, we examine the self-financing strategy in which the trader borrows money at an interest rate ( Fed fund rate $r$ % daily) to finance his round-trip trades. The interest he needs to pay is $r(MP_0 + \frac{\alpha M^{\beta+1}}{\beta+1})$. When $\beta < 1$, the profit attains its maximum at $k = 1$ and $\pi_1(0) = -\alpha M^{\beta+1}(\frac{1-\beta}{(1+\beta)}) - cM - r(MP_0 + \frac{\alpha M^{\beta+1}}{\beta+1}) < 0$. As a result, there are no arbitrage opportunities. When $\beta > 1$ the profit function at k = M is given by

$$\pi_1 = \frac{\alpha\beta M^{\beta+1}}{\beta+1} - \alpha M\left(\frac{1}{1+\beta} + \frac{M-1}{2}\right) - cM - r\left(MP_0 + \frac{\alpha M^{\beta+1}}{\beta+1}\right) \tag{1.34}$$

Similarly, we solve $M_1^{sf}$, the minimum volume the trader needs to trade to break even. For simplicity, we calibrate the model with $r = 0.013\%$ and $P_0$ as the average executed price. We find that the trader needs to trade approximately 4.6% more shares than the baseline case to break even.

| | $M_1$ | $M_1^{sf}$ | Price | Volume | Trade value | M/ Exe. message | M/Volume |
|---|---|---|---|---|---|---|---|
| **Average** | 11488.2 | 12026.6 | 6552.7 | 170.3 | 222794.7 | 27.3 | 29.2 |
| **Median** | 1717.3 | 1920.8 | 2760.7 | 94.3 | 48916.0 | 0.3 | 15.1 |
| **Min** | 495.1 | 530.9 | 27.7 | 30.2 | 8354.1 | 0.1 | 6.8 |
| **Max** | 238,484 | 240,383 | 124741.7 | 985.7 | 5414125.0 | 850.8 | 937.6 |
| **Std** | 37937.3 | 38304.4 | 20876.9 | 148.0 | 1139721.6 | 148.4 | 136.9 |

Table 1.7: Descriptive Statistics of $M_1$

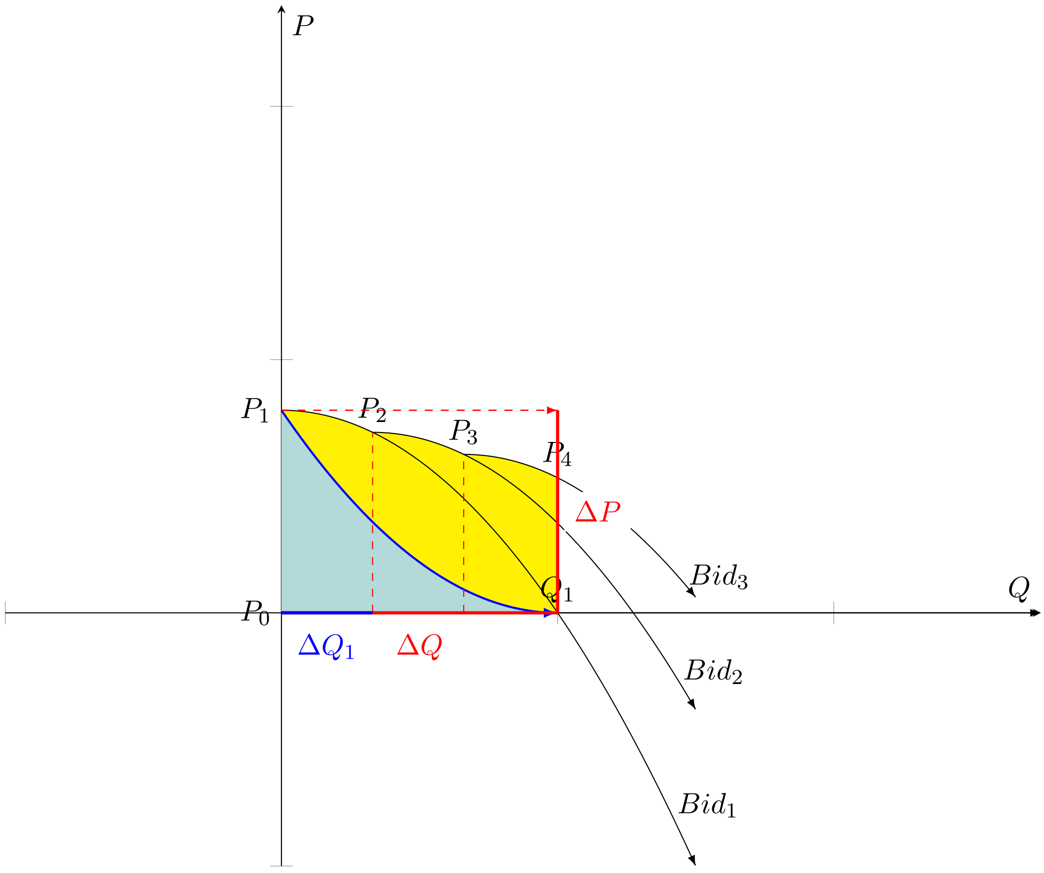



Figure 1.3: The profit of the strategy without the trading costs when $\beta > 1$

Applying the equation (1.28) to the second strategy, the expected profit for the second round trip strategy is as follows

$$\pi_2 = -\left(k\frac{\alpha}{(1+\beta)}\left(\frac{M}{k}\right)^{\beta+1} + \frac{(k-1)k}{2}\alpha\left(\frac{M}{k}\right)^{\beta}\frac{M}{k}\right) + k\alpha\left(\frac{M}{k}\right)^{\beta}M - \frac{\alpha M^{\beta+1}}{\beta+1} - cM \quad (1.35)$$

The trader chooses the number of trades $k$ to maximize his profit given by $\pi_2$. If $\beta < 1$, $\pi_1$ is an increasing function in k. If $\beta > 1$, $\pi_2$ is a decreasing function in k. The first order condition of $\pi_2$ with respect to $k$ is given by

$$\pi_2' = \frac{\alpha M^{\beta+1}(1-\beta)}{2(1+\beta)k^{\beta+1}}((1+\beta)k+\beta) = 0 \quad (1.36)$$

Similarly, there are two cases. When $\beta < 1$, the optimal value of $k$ is $M$ and the profit is

$$\pi_2(M) = \frac{\alpha M^{\beta+1}}{\beta+1} - \alpha M\left(\frac{1}{1+\beta} + \frac{M-1}{2}\right) - cM \quad (1.37)$$

Now, the trader chooses the optimal value of $M$ to maximize his profit. The derivative of $\pi_2'(M) = \alpha(M^\beta - M) + \frac{\beta-1}{2(1+\beta)} < 0$. The function $\pi_2(M)$ is decreasing in $M$ and $\pi_2(M) \leq \pi_2(0) = 0$. There are no arbitrage opportunities.

When $\beta > 1$, the optimal value of k is M and the profit is $\frac{\alpha(\beta-1)M^{\beta+1}}{\beta+1} - cM$. The optimal strategy for the trader is to buy all in one trade and then sell after that. However, if the trader sells right after the first trade, the ask-bid spread is high because after the trade, the spread increases by $\alpha M^\beta$. The expected profit $\pi_2 = \frac{\alpha(\beta-1)M^{\beta+1}}{\beta+1} - cM = \frac{\alpha(\beta-1)M^{\beta+1}}{\beta+1} - (spread + 0.006)M < \frac{\alpha(\beta-1)M^{\beta+1}}{\beta+1} - \alpha M^{\beta+1} < 0$. If the trader waits until the trader also needs to trade a high minimum threshold to break even, he faces the risk of the pricing moving against his strategy.

### 1.2.5 Conclusion

This paper investigates the relationship between the cumulative size of liquidity and the price impact of order book events by using trades and quotes data from 82 Nasdaq-listed stocks. We propose the slope of the limit order book as a new measure of price impact. By analyzing both bid and ask sides of high-frequency limit order book snapshot data, our study shows that there is a linear relationship between the cumulative size of liquidity and price impact in the limit order book. In addition, we find that if price impact admits a nonlinear functional form, under certain circumstances, a profitable round-trip arbitrage exists. We empirically show the minimum required trading volume for a profitable self-financing arbitrage and conditions that limit arbitrage. We find that arbitrage requires a large number of shares to the degree that it is impractical, thus providing evidence in support of Huberman and Stanzl.

Chapter 2

# The value of information flows in the stock market

## 2.1 Introduction

Capital must be allocated to myriad investments, properly balancing the relative risks and returns associated with each investment; this is done in stock markets, with information about each investment constantly flowing in and being weighed against information about competing investments.

This problem is important because the tens of thousands of individual traders in stock markets who have information have the incentive to keep that information private and to profit from it in trading, and it is important to understand whether their activities result in the efficient and proper allocation of investment resources when interacting in stock markets, as this determines the growth of the economy.

The literature has proposed a theory, the Kyle [1985] model, that shows how the equilibrium behavior of traders processing this information results in a relationship between the information, the price of assets, and the volatilities of the prices and trade volume of equities traded in the market. A fundamental quantity of interest in the Kyle [1985] model, $\lambda$, the slope of the supply curve reflects the fundamental forces driving the stock value; $\lambda$ reflects the marginal effect of trading on the price, and so is known as the price impact parameter.

While Kyle [1985] predicts how $\lambda$ is determined and quantitatively links it to the volatilities of price and trading volume, it was developed within the context of a dealership market without any reference to the limit order book. Furthermore, previous studies have not examined the relationship between $\lambda$ and the limit order book, typically estimating price impact using execution data rather than LOB data. By drawing an analogy between $\lambda$ and the slope of the order book, we observe that the limit order book has a visible structure: the set of unexecuted orders forms a pattern—essentially a supply curve—with a slope that is driven by the underlying incentives created by the information possessed by some of the traders.

In contrast to the traditional interpretation of Kyle's $\lambda$, which assumes its estimation originates from execution data, we propose an innovative approach by hypothesizing that Kyle's $\lambda$ is expressed within the order book. Furthermore, we estimated price and volume volatilities through a different methodology focused on execution orders. While the price impact and volatility estimates are carried out using entirely different data and methods, they confirm the predictions of the Kyle model.

To test the hypothesis that price impact—Kyle's $\lambda$—is embodied in the limit order book, we run a simple regression of price against quantity in each updated instance of the limit order book to recover an estimate of $\lambda$, and relate it to our estimates of the volatilities. But because of the immense number of messages and the hundreds of trading days for each stock this is in itself a huge task. Notably, previous studies estimating $\lambda$ from execution data did not incorporate limit order book data, making our data approach both novel and distinctive.

We carried out our analysis using a data set containing the complete set of transactions over a three-year period.[1] Order information is typically in extremely raw form: each order submitted by traders to buy or sell shares of stocks is recorded as a message in the so-called limit order book (LOB), with the stock ticker label, price, quantity, time of initiation, and time of execution or cancellation, and these orders and terminations occur on a rolling basis.

1. We thank AlgoSeek corporation for generously donating this data.

To carry out the analysis, the limit order book needs to be retrospectively reconstructed from this data, whilst handling the asynchrony of orders, that is, looking ahead and backward to link messages with their subsequent execution or cancellation. For a typical stock on a typical day, the number of messages will be in the tens of thousands, and for a few actively traded stocks, it is hundreds of thousands.

It is an important detail of market trading that the vast majority of messages—orders generated by computerized trading algorithms that are placed in the limit order book, waiting for a counterparty to agree to the trade a pre-specified price and quantity—are cancelled by those same algorithms before they are executed, often lasting just a few milliseconds. Using a small sample of 82 stocks over a one-month period, this starkly comes to the fore: 97 percent of all orders are cancelled before being filled. Theory predicts this; to test the theory it is essential to have the full limit order book data, as executed trades are only a small part of the story.

A key facet of the Kyle model, as outlined in Boulatov and Taub [2014], is that in addition to the original interpretation of $\lambda$ as a measure of price impact, the inverse of $\lambda$, $1/\lambda$, is a Lagrange multiplier for the constraint characterizing the how information is dynamically resolved for the optimization problem solved by traders with private information. It is, therefore, a shadow price, and when expressed in conjunction with the constraint to which it is associated, it is the shadow price of information.

Information has a precise definition in the Kyle model: it is the forecast error variance of the traders in the market who are not in possession of private information, and who must therefore glean information from the flow of orders that come to them in the market; these traders are designated as market makers in the Kyle model. Information is thus the ignorance of these market makers.

Combining the notion of information and a notion of the price of information, one can state the value of the information flowing into the market. This is our central aim, examined under both single-asset and multi-asset Kyle models.

### Multiple assets

Multi-asset extensions of the Kyle model show how informed traders use information about multiple assets to trade a specific asset, using information they have about the correlation of the fundamental characteristics of these assets. Uninformed traders—market-makers in the terminology of the Kyle model—are aware of the informed traders use of correlation information and price stocks with this in mind. This literature includes papers by Caballe and Krishnan [1994], Hasbrouck and Seppi [2001], Back, Cao, and Willard [2000], Seiler and Taub [2008] and Bernhardt and Taub [2008a].

The optimal strategies in these theoretical models boil down to a matrix of coefficients that are applied to the signals that traders observe: in the case of informed traders, the matrix is of trading intensities: for each stock, they choose an amount to trade based on the direct observation of the true value of the stock, but their trade is augmented by correlated information from the direct observation of other stocks. The pricing coefficient, $\Lambda$, similarly takes a matrix form, using information from order flow from correlated stocks when pricing each individual stock. In each instance of the literature, however, the cross-asset correlation structure was left abstract and unstructured.

There is a completely different theory, the capital asset pricing model (CAPM), which accounts for correlation—the correlation of returns—across assets. The central conclusion of the CAPM is that in equilibrium, the correlation across assets is due to a single factor, systematic risk, with all other returns being idiosyncratic and uncorrelated across all individual assets and thus fully diversifiable. It, therefore, makes sense to conjecture that a similar division can be made when characterizing the cross-asset correlation of the fundamentals of stocks in the Kyle model; if this conjecture is correct, then all cross-asset correlation would boil down to common systematic risk. If one could separate the systematic and idiosyncratic influences in the Kyle model, then it would be an immediate prediction that idiosyncratic shocks to fundamental asset values would be of no use in cross-asset trades, and so any optimal cross-asset trading strategies would reduce to diagonal matrix, and this would also be reflected in pricing that is, the equilibrium matrix describing $\Lambda$—the multi-asset version of Kyle's price impact measure $\lambda$—would be diagonal.

This is what we find. Our results support the notion that the cross-correlation of price impact across stocks is consistent with the CAPM: there is a single systematic component of price impact, and this is driven by the systematic component as captured by the volatility of the systematic component of the stock market.

### The nature of private information

The information in the Kyle model is the forecast error variance of the uninformed market makers. This information can be measured, and its value can also be measured; we provide the estimate of this value, which, when normalized, is consistent across stocks.

What is the purpose of the information flow? Our results confirming the single source of correlation, systematic risk, suggest that by separating the underlying information into two components, systematic and idiosyncratic, informed traders distinguish between productive assets that have a systematic impact on the economy and those that can be diversified. From a CAPM perspective, this is the only information that matters, as any non-systematic value can be diversified away.

The structure of this chapter is as follows. We first review relevant findings on the price impact (Section 2.2) and establish the framework for analyzing our order book event data (Section 2.3). We also relate the Kyle framework to dimensional analysis and invariance. Then, we present the details of the data and computation methods employed in our analysis (Section 2.4). Section 2.5 examines the price impact on the limit order book under both single-asset and multi-asset Kyle models. In Section 2.6, we explore the relationship between price impact and the value of information. The final section investigates the relationship between price impact and business cycles.

## 2.2 Related literature

There is a large literature focusing on price impact and, equivalently, liquidity. YakovAmihud [2002] used daily and monthly trading data of stocks traded in the New York Stock Exchange (NYSE) in the years 19631997 to examine the effect of illiquidity. In his paper, he measured the illiquidity ratios as the time-series average of the daily ratios of the absolute value of percentage returns to dollar volume and found out that expected stock returns are an increasing function of expected illiquidity. Alternatively, following Hasbrouck [2009] and Goyenko, Holden, and Trzcinka [2009], a representative coefficient is estimated as the $\lambda$ coefficient in the regression of the root square of dollar volume against the price. They show that their estimation and effective cost are moderately positively correlated. Bisias and Valavanis [2012] estimates this measure daily by using all transactions during normal trading hours on each day. The authors estimate $\lambda$ as the coefficient of regression of the natural logarithm of volume in dollars against the sequence of intraday returns.

Our study contributes to a growing body of literature on cross-asset price impact. The related theoretical work includes papers by Caballe and Krishnan [1994], Back et al. [2000], Seiler and Taub [2008] and Bernhardt and Taub [2008a]. The multi-asset Kyle model was first studied by Caballe and Krishnan [1994] under the general setting with $n$ assets and $m$ informed traders. The generality of the model makes only a partial analysis of the solution possible. Back et al. [2000] considers the univariate Kyle model with $n$ informed traders with correlated signals. They find that when signals of informed traders are perfectly correlated, there is no linear equilibrium. Bernhardt and Taub [2008a] presents a one-period model of $n$ risk-neutral informed traders and $m$ assets. They allow informed traders to internalize how their trades impact the prices and trades of other speculators. They show that the covariance structure of asset fundamentals is the driver of prices, while the covariance of liquidity trade drives that of order flows. Seiler and Taub [2008] extend the analysis of Bernhardt and Taub [2008a] to an infinite horizon model in which informed investors receive private long-lived information repeatedly.

A number of empirical studies show evidence of significant cross-price impact in stock markets, including Hasbrouck and Seppi [2001], Pasquariello and Vega [2015], Wang, Schafer, and Guhr [2016], Garcia del Molino, Mastromatteo, Benzaquen, and Bouchaud [2020], Mehdi Tomas and Benzaquen [2022]. Hasbrouck and Seppi [2001] decompose multi-asset order flows and returns, and find that two-thirds of the commonality in returns can be explained by commonality in order flows. Pasquariello and Vega [2015] investigate the trading activity in the New York Stock Exchange (NYSE) and the National Association of Securities Dealers Automated Quotation System (NASDAQ) stocks between 1993 and 2004 found that the cross-price impact is often negative and both direct and absolute cross-price impact are smaller when there are many speculators. Wang et al. [2016] use the intraday data of AAPL, GS, and XOM from the NASDAQ stock market in January 2008 and empirically show that cross-asset price impacts are small and appear to be transient instead of permanent. They also find the cross-correlation of the trade signs has a short memory.

Similar to our approach, Garcia del Molino et al. [2020] uses the multiple asset Kyle framework to estimate different price impact measures and shows that the Kyle estimator performs better in the market with heterogeneous volatility. Methodologically, our paper is closely related to papers that decompose the information in stock prices based on the CAPM, such as Hasbrouck and Seppi [2001]. However, unlike the prior papers that study information decomposition of stock returns, our paper aims to partition price impacts into two components: systematic influence and idiosyncratic influence. We find that the idiosyncratic shocks to fundamental asset values have little impact on cross-trades, and any optimal cross-asset trading strategies would reduce the trading intensity to a diagonal matrix.

Finally, this paper is consistent with an extensive body of literature studying the value of information flow and order flow in the stock market. Berk and van Binsbergen [2015] uses the Center for Research in Security Prices (CRSP) survivorship bias-free database of mutual funds to study the skill in the mutual fund industry. They find that each mutual fund has used its skills to generate about \$3.2 million per year and in the aggregate, mutual funds in the US markets made over \$19 billion per year. Yang and Zhu [2019] presents a model of the strategic interaction between fundamental investors and back-runners. They calibrate their model and estimate that the potential institutional investors daily profits in the U.S. equity market are in the order of \$150 million per day. The results of these papers are in line with our empirical result as the institutional investors profit is somewhat equivalent to the value of information. Even though our results are in the same order of magnitude as these estimations, we take a different approach. We extract the value of information separately for each stock from price and quote data, then find normalized information flow averaged across trading days.

## 2.3 LOB approach to price impact

Econophysics has gained traction by asserting that price impact is a consequence of executions and that this impact is inherently nonlinear. This view diverges sharply from traditional finance, primarily because econophysicists approach price impact in a purely phenomenological way, largely omitting the role of information and equilibrium from their models. Bouchaud et al. [2009] do empirical research on thousands of trades and point out that the autocorrelation of trade sides decays extremely slowly with time, and the price fluctuation is persistent and predictable. Therefore, they argue that price impact should not be linear and permanent. Jean-Philippe Bouchaud and Wyart [2004] demonstrate that order flow is autocorrelated: trades often cluster in the same direction, creating herding behaviors among traders. This autocorrelation magnifies price impact and contributes to increased volatility and substantial deviations from price equilibria assumed in classical models.

Another school of thought is the mainstream finance literature on market microstructurewhich views price impact with greater emphasis on informational trading, equilibrium, transaction costs, and arbitrage opportunities. Hasbrouck [2004] shows that price impact is not uniform across assets or time periods. He considers price impact as a dual indicator: temporary price impact (associated with liquidity costs) and permanent price impact (associated with informational effects). Hasbrouk [1991] introduced innovative econometric techniques to isolate the impact of informed trading.

A primary model in this traditional literature is the Kyle [1985] model, which assumes that total order flow, from the perspective of a market maker, represents pure noise without price impact. However, the Kyle model also predicts that price impact will emerge from the perspective of the informed trader, whose filtration of information allows them to predict and benefit from subsequent price movements. This assumption would require econophysicists to argue for an ex-post understanding of this informational filtrationsomething often unaddressed in their phenomenological framework.

The Kyle model also makes clearer predictions about how price impact manifests. Under the Kyle framework, price impact is in the minds of market makers, yet it is realized in the LOBs structure, specifically in the slope. By measuring the LOB's slope, we can quantify price impact, expecting it to exhibit stationarity and even constancy if underlying variances remain fixed and persistent. Despite shifts in the LOB, the price impact should appear linear across different timestamps. Additionally, this slope will likely differ across various tickers, reflecting how price impact can vary by asset characteristics.

The hypothesis that price impact is expressed in the LOB rather than purely in transaction data (TAQ), presents a substantial econometric challenge. The sheer volume of data in the LOB vastly exceeds that in transaction logs, as approximately 97% of all LOB orders are ultimately canceled without execution. The Kyle model and the hypothesis that $\lambda$ is realized in the LOB suggest that orders get canceled as new information arrives or as prices shift. This reflects traders' adjustments based on updated information or perceived changes in asset value. This abundance of order activity in the LOB, combined with the relative sparsity of executed trades, requires sophisticated statistical techniques to parse meaningful price impact data.

### 2.3.1 The elementary Kyle model

In this section, we analyze the basic static Kyle model. The model has three ingredients: (i) the variance of the fundamental value per share of the security that is being traded, $\Sigma$, for a Gaussian distributed value and from which the realized value is drawn; (ii) the variance of the order flow of the so-called "noise" traders, $\sigma^2$; and (iii) the price impact of any trade on the price, $\lambda$, which is determined by rational traders—"market makers"—who are not informed about the true value of the security as determined by fundamentals, and who are in competition with each other. The realized value of the security is privately observed by a single trader who then exploits this information in his trade. The noise traders' trades are treated as entirely exogenous; that is, they do not react to observations about price in any way.

In equilibrium, the following relationship holds:

$$\lambda = \frac{1}{2}\frac{\sqrt{\Sigma}}{\sigma} \tag{2.1}$$

This formula concerns the underlying structure of the model, but $\Sigma$ and $\sigma^2$ are not directly observable. Defining $\Sigma_P$ and the variance of observable price and $\sigma_V^2$ as the volatility of executed transaction volume, the following proposition establishes that formula (2.1) can be restated in terms of observables:

**Proposition 2.3.1.**

$$\lambda = \frac{1}{2}\frac{\sqrt{\Sigma}}{\sigma} = \frac{\sqrt{\Sigma_P}}{\sigma_V} \tag{2.2}$$

*Proof.* See Appendix B.1. □

### A brief aside concerning dimensional analysis and invariance

In any model in which the stock price and trading volume are functions of the volatilities of prices and volumes, then they must satisfy a homogeneity property. This is an instance of Buckingham's theorem and is also reflected in the papers of Kyle and Obizhaeva [2016] and Obezhayeva and Kyle [2017].

The argument is straightforward. Suppose that we posit that the price impact of trades is as follows:

$$\lambda \equiv \frac{\Delta P}{\Delta V} = f(\sigma_P, \sigma_V)$$

Because the volatilities are constructed from the differences of the levels of price and volume, if we apply a multiplicative factor $k$ to the price, as would occur for example, in a stock split, then the volatilities must also reflect this scale factor:

$$\frac{k\Delta P}{\Delta V} = f(k\sigma_P, \sigma_V)$$

Now, choose the scale factor to, in fact, equal the inverse of the volatility:

$$\frac{\frac{\Delta P}{\sigma_P}}{\Delta V} = f(1, \sigma_V)$$

The same argument holds for the volume:

$$\frac{\frac{\Delta P}{\sigma_P}}{\frac{\Delta V}{\sigma_V}} = f(1, 1)$$

Thus, any statistical test would reasonably construct the left-hand side, which we can think of as normalized price impact, and then look for a constant on the right-hand side (if the theory has no further ingredients beyond price impact and volatilities). In the Kyle model, the predicted right-hand side constant is 1 (applying Proposition 2.3.1 to the Kyle model).

An additional observation is that if the underlying true model is linear, which the log of Kyle's ratio is, then it will be entirely vacuous to obtain regression coefficients of $1/2$ and $-1/2$ (using logs of the variances on the right-hand side). The only relevant result in such a regression is the intercept term.

Does this reasoning, that is, that it is trivial that the estimated coefficients are $1/2$, apply to the limit order book? In the limit order book model, the estimated volatilities come from *executed* prices and volumes, whilst, in the limit order book, the price impact is derived from the shape of the limit order book and need not be driven by the pattern of executions; most of the orders in the limit order book messages are, in fact, never executed. There is no theoretical a priori reason to expect the price impact in the limit order book to follow the Kyle model structure.

## 2.4 Data and computational details

We collected data from several sources. Our primary data source is a proprietary database of US stocks that are trading on the NASDAQ exchange. The database contains message-level information of all stocks from 2016 to 2018. For each stock, there is a raw message file that contains all trading messages of one stock sent to the market at high speeds in milliseconds within a trading day. The file provides a comprehensive record of every trade and order book change of all stocks on the exchange; there are approximately 6,500 stocks in all.

The first step in the empirical analysis is the reconstruction of the limit order book, moment by moment. As the dataset records all events that led to state changes to the order book, we can reconstruct limit order book for any stock at each moment in the trading day and at full depth for the specified period. The comprehensive and full-depth level data allow us to analyze different characteristics of price impact and its relationship with limit events with high accuracy.[2]

The message file contains every arriving market and limit orders as well as cancellations and updates of one stock. The information of the message file has 9 data fields.

1. "Date" provides information regarding the trading day
2. "Timestamp" All entries have a timestamp of seconds after midnight with the precision of milliseconds.
3. "Ordernumber", each order has a unique ID; subsequent actions such as execution, deletion or partial execution are indexed by the same number. Zero reference orders correspond to a hidden limit market order.
4. "EventType" There are 11 types of market events in the data. Provided details in a table in the additional appendix, Appendix B.3.1]
5. "Ticker" provides information regarding the trading stock
6. "Price" the price of the order
7. "Quantity" the quantity of the order
8. "MPID" provides information of Market Participant Identifier. This identifier is used by FINRA member firms to report trades.

2. Thus far, the empirical literature in this field has been limited to the use of pre-constructed LOB data such as Lobster with only a few layers of top-of-the-book information.

9. "Exchange" There are two main exchanges, ARCA (the electronic order book of the NYSE) and NASDAQ. All entries detail which exchanges the order was sent to.

Generally speaking, the order ID corresponds to the unique order reference number, which we can use to differentiate messages. However, there are some exceptions that may affect our limit order reconstruction.

1. All messages classified as "trade bid" and "trade ask" have zero reference orders. Those are hidden market orders with full information for all other fields except the order number. As they are market orders, they don't affect our limit order reconstruction, but we need to take them into consideration when we look at executed orders.
2. For big stocks that are trading across trading platforms, there are some order IDs corresponding to multiple different orders sent to different trading venues. One example is the messages with order ID 6168348 (TSLA, 08 Feb 2018). Essentially, the ID corresponds to 2 separate messages sent to different exchanges. The first order was a bid order at 08:20:56, which was sent to NASDAQ, then eventually got executed and filled later. The second order was an ask order at 09:42:40, which was sent to ARCA, then deleted eventually. To differentiate those different orders with the same reference number, we can look at the exchange and nature of the order. First, these orders were sent to different exchanges. We can use trading venues to find out and group all related orders. Second, we can use the nature, such as the order type and price, to map out all related orders. For example, "Add bid" orders should have related orders of type "execute bid" and "fill bid"; "Add ask" orders should have related orders of type "execute ask" and "fill ask."

Second, we obtain the data for the stock directory with market cap, $R^2_{CAPM}$, $\beta_{CAPM}$ and variance from the NYSE [3] and Zoonova [4].

Our sample data directory contains all active stocks during the period between 1 January 2021 and 31 December 2021. All stocks must meet three pre-screening criteria to be in the directory: (1) it is a common stock (2) it is active on the first and last day during the sampling period. Active stocks refer to any stocks with trading activity on public exchanges during the sampling period. Out of over 6,500 tickers, some stocks were not listed or did not exist as of February 2018, (3) it has NASDAQ as the primary listing exchange. After filtering out all duplicates and erroneous

3. Stock directory, https://www.nyse.com/listings_directory/stock
4. Stock market watch,https://www.zoonova.com/Home/Markets

entries, we are left with 6,481 stocks. In our initial study, we obtained a sample from February 2018; there were 19 trading days in total for each stock, and each trading day had between approximately 10,000 to over 10,000,000 messages for one stock. The primary justification for selecting this period is that February 2018 was a calm month, falling outside the U.S. earnings season and unaffected by major macroeconomic events.Therefore, the input file size can reach the region of 20 GB for one ticker on each trading day, thus posing technical challenges in terms of computation and data storage. We employed stratified random sampling by partitioning all tickers into subpopulations. The sample stocks were chosen based on the following sampling characteristics: high $R^2$, low $R^2$, high $\beta$, low $\beta$, high market cap, low market cap and low variance, high variance. 12 tickers were randomly selected from each group, yielding an initial sample of 96 tickers. After removing duplicate entries, 82 unique tickers remained. The rationale behind this sampling method is that stocks have high variances in all those characteristics. Stratified random sampling allows us to effectively select stocks that represent a diverse range of groups. The statistical summary of those stocks is illustrated in Table 2.1.

| | R-squared | Marketcap | Yearly Price Variance | Beta |
|---|---|---|---|---|
| Mean | 0.1590 | 19,300,751,911 | 6.096 | 0.93 |
| Standard Error | 0.0300 | 6,811,346,968 | 1.335 | 0.18 |
| Median | 0.0299 | 353,644,000 | 0.970 | 0.85 |
| Minimum | 0.0001 | 23,198 | 0.063 | (2.47) |
| Maximum | 0.7253 | 343,970,000,000 | 35.490 | 4.81 |

Table 2.1: Descriptive statistics of the sample

## 2.5 Testing Kyle model

This section empirically tests the Kyle model framework using the LOB data introduced in Section 2.4. We also discuss the relationship between price impact, volatilities, and the slope of the order book within both the static single-asset model and the multi-asset model.

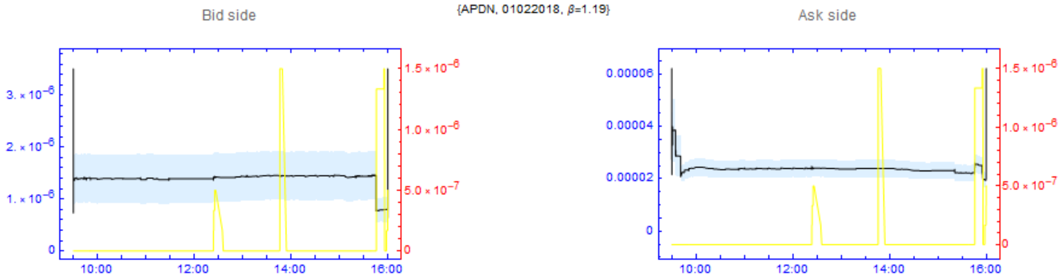
### 2.5.1 Testing the univariate static model

To carry out tests of the model, we estimated three fundamental quantities: $\lambda$, $\sigma_P$ and $\sigma_V$.

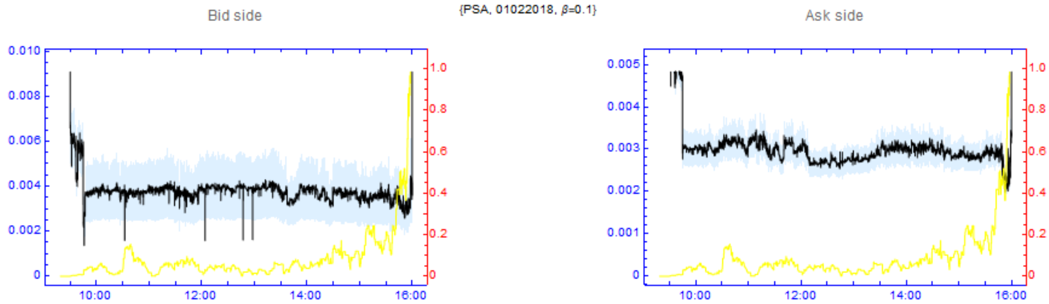
We estimate $\lambda$ by calculating the slope of the LOB, yielding an estimate $\hat{\lambda}$, using a sample of 82 stocks for the 19 trading days in February 2018. The algorithm reconstructs the sequence of limit order books by parsing and sorting the raw file of messages for the days trades for a single stock. Each trading message results in an update of the limit order book; each updated limit order book is called a snapshot. Each stock typically has tens of thousands of messages, so consequently, there are tens of thousands of snapshots; for the most heavily traded stocks, there are hundreds of thousands of snapshots.

We denote the collection of snapshots a ticker-day. These snapshots for each ticker-day are then statistically analyzed, with three key estimates being generated. First, the slope of each ticker is estimated with an OLS regression; we separately estimate the bid side (downward-sloping demand curve) and the ask side (upward-sloping supply curve); theory predicts that these slopes should be the same in absolute value. We re-estimate $\lambda$ for each snapshot using OLS, compiling a list of estimated $\hat{\lambda}$s for later averaging. We ignore the spread at the top of the book. Figure 2.1 depicts the estimated $\lambda$s (back lines) of an example of 4 tickers APDN, PSA, PZZA, THS on 01 September 2018 for both ask and bid sides. The blue bands are 95% confidence intervals of estimated $\lambda$s. The yellow lines are price volatilities during the trading day, estimated as price quadratic variations by minute. Among all stocks, estimated $\lambda$s are higher at the beginning and the end of the trading day. The main reason behind this trend is that at the beginning of the trading daily, the market makers start making the market, and at the end of the trading day, all market participants cancel their resting orders. Therefore, the LOB liquidity is low, and the price impact is bigger. If excluding the first half hour of the trading day and the last half hour of the trading day, the estimated $\lambda$s are highly stable for all stocks.

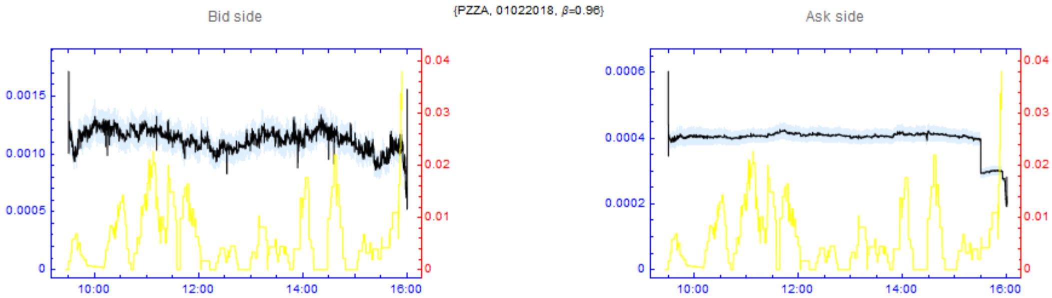
Second, we estimate the variances of executed price and of executed order flow. Estimating the volatilities is challenging because trades occur at random times. We compute $(\Delta P_t)^2$ for each interval, normalize by dividing by $\Delta t$ for that interval, and then take the moving average to estimate each variance.



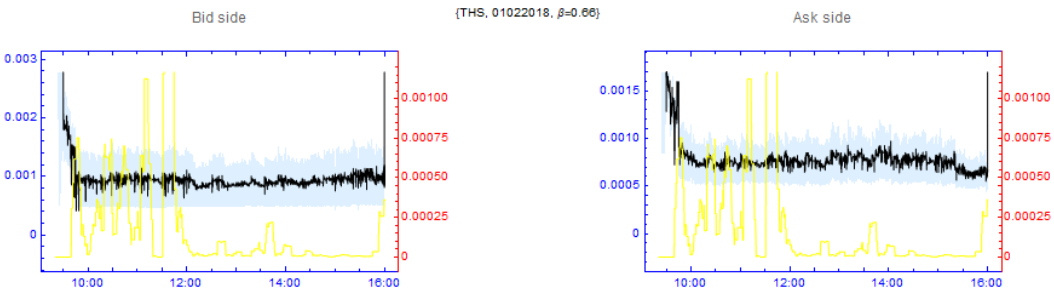

APDN




PSA




PZZA




PSA

Figure 2.1: The estimated $\lambda$s of APDN, PSA, PZZA, THS on 01/02/2018

### Empirical tests

Our first test implements the notion that invariance holds, even though any price impact that appears in the limit order book need not follow the invariance requirement. If invariance holds, then as we articulated in (2.3.1), the normalized price impact must equal a constant; for the Kyle model specifically, the normalized price impact is equal to 1. In our first test of the model, conjecturing that the Kyle invariance relationship holds in the limit order book, not just in executed prices, we calculated the normalized price impact ratio $\frac{\sigma_P}{\sigma_V}/\lambda$ using our $\lambda$ estimates from the slope of the order book and the volatility estimates, obtaining values of .7 (bid side) and 1.1 (ask side) from a sample of about 82 tickers.

### Comparisons with direct price impact

Conceptually, price impact concerns the impact of *executed* orders on price, that is, direct price impact, as this is a central concern of real traders. It is conceivable that direct price impact is driven by structure outside of the ken of the Kyle model. It, therefore, behooves us to compare the direct price impact with the impact we have measured in the limit order book.

In order to measure direct price impact we calculated the ratio $\frac{P_t - P_{t-1}}{y_t - y_{t-1}}$ for each instance $t$ of an executed trade, and calculated the average for each ticker for our February 2018 sample. The resulting value, calculated in the same way as in the discussion of invariance, is 2.04, double that of the ratio using the slope of the order book to estimate price impact.

In scrutinizing this result further, we found that the results were widely scattered: some tickers had a direct price impact as high as 80 times as high as the LOB estimate of the ratio, some were a fraction, but the average ratio was 11.3 times higher in our sample. It is apparent that the tickers with the highest discrepancy were thinly traded, while for heavily traded stocks, the direct price impact and the LOB estimated $\lambda$s were essentially the same.

To test this observation, we used market capitalization as a proxy for trading intensity, and indeed, we found that low-capitalization stocks have higher direct-versus-LOB price impact ratios. We also measured trading intensity directly, as the volume of executed order flow per minute for our sample of stocks, with identical results: high-order flow stocks had significantly lower executed price impact. In a regression of the ratio of executed price impact to LOB estimated $\lambda$ versus the log of executed order flow per minute. the estimated coefficient for the dependent variable is $-1.1$ ($R^2$ : .18, t-statistic: $-4.4$.)

Our explanation is as follows. For thinly traded stocks, executions occur only infrequently relative to LOB order messages; therefore, for thinly traded stocks, there is significant LOB activity in between executions. The LOB activity reflects up-to-date information, whilst executions happen after a long evolution of information; therefore, execution is more likely to reflect new private information, and pricing reflects this.

More specifically, consider the LOB at times $t$, $t+1$, and $t+2$. At times t, the LOB has a specific slope, driven by the dictates of the private information volatilities underlying the fundamentals of the stock. At $t+1$ new information arrives and, if it is positive information, moves the *entire* LOB up; however, reflecting the thin trading, no execution takes place. At $t+2$ there is an execution—for the sake of discussion, a buy. The starting point for this execution is the new top of the book, which has moved up due to the cumulative arrival of information at time $t+1$ and also at $t+2$, but then, in addition, the order walks up the book. The walk up the book has the impact of the $\lambda$ from the slope of the book, but the effect of the prior movement of the entire book is added to the impact, thus seeming to magnify it. The *cumulative* effect of the earlier arrival of the new information is combined with the book walk—so the price impact is bigger. The effect is bigger for thinly traded stocks because there are longer delays between executed trades. A technical argument is provided in Appendix B.3.3.

**Regression tests**

Our next statistical test consists of a simple linear regression of the logarithm of the averaged $\hat{\lambda}$ on the logarithmic transform of the formula (2.2) for 82 stocks, with the estimates of $\lambda$, $\Sigma_P$ and $\sigma_V$ averaged over 19 trading days in February 2018, treating each ticker as an observation. This yielded the results in Table 2.2:

| $R^2 = .88$ N=82 F: 315.7 | Predicted value | Coefficient | Standard error | $t$-statistic | $P$-value |
|---|---|---|---|---|---|
| Intercept | 0 | 1.13 | 0.41 | 2.75 | .007 |
| $\ln(\Sigma)$ | $\frac{1}{2}$ | .57 | .023 | 24.56 | 9.61E-39 |
| $\ln(\sigma^2)$ | $-\frac{1}{2}$ | -.336 | .032 | -10.40 | 1.83E-16 |

Table 2.2: Basic univariate static model regression results. (82 stocks, 19 trading days 2018)

The predicted values of the coefficients are 0 for the intercept term, $\frac{1}{2}$ and $-\frac{1}{2}$ respectively for the price volatility and volume volatility; basic statistical theory suggests that when explanatory variables are measured with error, as $\hat{\lambda}$, must be, the estimated coefficients are biased toward zero; the coefficients here thus reflect this bias; the intercept term is less successful; however, the $P$-value is weak. These results thus strongly support the Kyle model formulation and, more generally, an informational interpretation of stock market trading.

Given the panel structure of the dataset, a more robust approach than averaging $\lambda$ over 19 trading days is to apply panel regression analysis. To determine whether fixed or random effects are more appropriate, a Hausman test is conducted, with the null hypothesis favoring the random effects model over the alternative, fixed effects model. The resulting p-value of $5.8 \times 10^{-16}$ strongly suggests that the fixed effects model is more consistent. Additionally, to assess the presence of time effects, we perform a Lagrange Multiplier Test, which yields a p-value of 0.008416, indicating significant time effects.

We then estimated the model with both individual ticker and time effects, and the results are presented in Table 2.3

| $R^2 = .891$<br>N =82<br>T = 19<br>F: 5965.7 | Predicted value | Coefficient | Standard error | $t$-statistic | P-value |
|---|---|---|---|---|---|
| $\ln(\Sigma)$ | $\frac{1}{2}$ | .523 | .0055 | 94.518 | 2.2E-16 *** |
| $\ln(\sigma^2)$ | $-\frac{1}{2}$ | -.2267 | .00132 | -22.141 | 2.2E-16 *** |

Table 2.3: Static model panel regression results. (82 stocks, 19 trading days 2018)

In comparison to the previous model, the coefficient of $\ln(\Sigma)$ shows a slight decrease, approaching the theoretical value of $\frac{1}{2}$, while the coefficient of $\ln(\sigma^2)$ increases to -0.23. Overall, the estimated coefficients remain statistically significant and closely align with the predicted values. These findings provide further support for the validity of the Kyle model framework

### 2.5.2 Cross-asset correlation and the CAPM

As discussed in the introduction, a number of theoretical extensions of the Kyle model explore cross-asset effects. This literature does not ascribe the cross-asset correlations to particular causes, however the logic of the CAPM would point to a single cause of correlation, with stocks otherwise uncorrelated.

The CAPM perspective thus leads to a sharp prediction: that the cross-asset effects in the pricing matrix $\Lambda$ are driven only by systematic factors. If there is a way to filter out these systematic factors, then the residual $\Lambda$ matrix should be diagonal. We test this idea in two distinct ways.

#### First method: extraction from regression

For the first test of the correlation structure we enhance the regression of the logarithm of $\hat{\lambda}$ on the logarithms of the volatilities as in Table 2.2 above by including the CAPM $R^2$, displayed in Table 2.4:

| $R^2 = .89$<br>N=82<br>F: 223.70 | Predicted value | Coefficient | Standard error | $t$-statistic | $P$-value |
|---|---|---|---|---|---|
| Intercept | 0 | 0.44 | 0.50 | 0.88 | 0.38 |
| $\ln(\Sigma)$ | $\frac{1}{2}$ | 0.53 | 0.03 | 17.65 | 5.62E-29 |
| $\ln(\sigma^2)$ | $-\frac{1}{2}$ | -0.34 | 0.03 | -10.78 | 3.96E-8 |
| CAPM $R^2$ | | 0.9 | 0.39 | 2.30 | .024 |

Table 2.4: Basic univariate static model including CAPM $R^2$.

These results are still in accord with the underlying model in the sense that the coefficients on the volatility terms are consistent with the values of $\frac{1}{2}$ and $-\frac{1}{2}$ predicted by theory; the CAPM $R^2$ coefficient is statistically significant.

### Second method: covariance matrices

The second method uses cross-asset information. As shown in Bernhardt and Taub [2008a], one can express the equilibrium matrix $\Lambda$ in terms of the cross-asset price and volume covariance matrices:

$$\Lambda = \Sigma^{1/2} \cdot \sigma^{-1}$$

where $\Sigma^{1/2}$ and $\sigma$ are the Cholesky factors for the cross-asset price and volume covariance matrices, respectively; we can generalize this as with Proposition 2.3.1, we can demonstrate that the relationship holds for the covariance matrices of price and executed volume.[5]

One of the assets included in the portfolio of assets (again, the 19 trading days in February 2018) is SPY, the index fund tracking the S&P500, which is a widely accepted proxy for the systematic asset. We eliminate the rows and columns corresponding to SPY from the price and covariance matrices and calculate the resulting $\Lambda$ for the remaining tickers. The resulting estimate of $\Lambda$ should, in principle, have the influence of SPY removed and, if the CAPM intuition is correct, be driven solely by heterogenous fundamentals across the remaining assets. If the CAPM intuition is correct, the resulting residual matrix should be diagonal, as cross-asset information is irrelevant. The diagonal of the matrix is then the proper estimate of the $\Lambda$ associated with idiosyncratic firm value.

5. The derivations for the two-asset version of the model are set out in Appendix B.6.

### Norm comparisons

If the hypothesis that the cross-correlation between the $\lambda$ values is due entirely to correlation with the systematic market process, then when we remove the SPY rows and columns from the $\Lambda$ matrix, the matrix should be essentially diagonal, that is, each stock's $\lambda$ value should not be affected by any other stock, other than SPY. Therefore, the matrix norm of the reduced $\Lambda$ matrix should be driven solely by the diagonal. Carrying out this experiment using the trace norm yields a sum of the absolute values of the eigenvalues of the idiosyncratic matrix of 0.0395, whereas the similar sum with the diagonal removed is 0.000545, that is, essentially zero.

Alternatively, we can compare the matrix norms of the two matrices, where the matrix norm is the maximum singular value; in this case, the norms with and without the diagonals are 0.021576 and 0.0153497 respectively, again demonstrating that the off-diagonal correlation is reduced relative to the diagonal. These calculations support the hypothesis of CAPM-driven correlation.

### Comparing the two methods

One can roughly calculate the correlation of the idiosyncratic $\lambda$ values calculated using the first method and the second method. We carried this out with the sample of 82 tickers, again limited to the 19 trading days in February 2018. Using the first method, we calculated a predicted-$\lambda$ series in which the effect of the $R^2$ term was dropped; intuitively, this series would roughly capture the non-systematic element of the variances. We then calculated the correlation between the forecasted idiosyncratic series with the diagonal of the $\Lambda$ matrix with the SPY elements removed. The correlation between these two measures is .52; that is, there is a significant degree of correlation, suggesting that both methods at least partially succeed in isolating and extracting the idiosyncratic component of $\Lambda$.

The conclusion we draw is that, using two different approaches, that there appears to be a single factor driving cross-asset correlation, and that the two approaches yield measures of the correlation that are very closely correlated.

## 2.6 Measuring information flows

New information is constantly brought to the market. Traders keep the information private to preserve their advantage. Nevertheless, traders impart their information, and prices reflect it after trading. The Kyle model quantifies this process. In Kyle's framework, price impact is a result the incorporation of private information into asset prices.

Can this information be *measured*? Yes, it is the market makers' *forecast error variance* $\Sigma_t$. It is related to the variance of price.

Does the information have a price, and can it be measured? Yes: it is related to $\lambda$: the shadow price is $1/\lambda$. Boulatov and Taub [2014] provides the theoretical justification for price interpretation: $1/\lambda$ is the Lagrange multiplier for the constraint facing the informed trader, expressing how the market maker's forecast error variance decays as a result of trade, with the "income" in the constraint equal to the forecast error variance. Therefore, we can use the estimates of $\lambda$, and also the price volatility estimates, to estimate the quantity and price of information.

Using the Boulatov-Taub interpretation of the inverse $1/\lambda$ as the shadow value of information and the variance of price $\Sigma$ as a proxy for the market makers forecast error variance as a measure of the information, one can then calculate the value of information on a per-share basis as the product of these two quantities.

Using the basic structure of the Kyle model, this information value flow can be shown to be equivalent to the profit for the informed trader, which is equal to the product of the volatilities of price and order flow.

$$\frac{\Sigma}{\lambda} = \sqrt{\Sigma}\sigma$$

Using the estimates of $\lambda$ and volatilities for 82 tickers in February 2018 from the previous section, the correlation between these two measures is 0.944.

### Normalized information flow

Using 82 tickers over the month of February 2018, we can analyze information flow value. The information flow is per unit of value for each ticker; thus, by dividing the value of information by the value of shares traded (price times volume, each averaged over the trading day), one obtains the normalized information flow; this average value is about 0.024, and the median value is of 0.0021, but with a wide variance. Another way to calculate the average normalized information value is to divide the total value of information of all stocks by the total trading volumes. Using this method, we yield the normalized information flow of $7.5 \times 10^{-5}$. For the longitudinal data for Wednesday trading over three years, 2016, 2017, and 2018, the estimated normalized information flow values are $1.6 \times 10^{-4}$, $1.6 \times 10^{-4}$, and $1.79 \times 10^{-4}$ respectively.

Define this information flow parameter as $\omega$. We can multiply the normalized flow by the value of all stocks traded to obtain an estimate of the value of all information flowing in the economy, divided into systematic and idiosyncratic elements. Multiplying the total daily Nasdaq trading value of about \$300 billion [6] by the normalized information flow $\omega \sim .00016$, you obtain \$48 million per day. This is in close accord with the estimate of Yang and Zhu [2019] as discussed in the introduction.

## 2.7 Business cycle effects

Given the support for the CAPM-driven correlation hypothesis in section 2.5, a natural conjecture is that the systematic component of $\lambda$ might be correlated with the business cycle. This influence could be driven by changes in the systematic part of the volatility of fundamental asset values, that is, volatility of returns seems to rise in recessions.

6. Nasdaq, https://www.nasdaqtrader.com/Trader.aspx?id=DailyMarketSummary

To test this hypothesis, we calculated the value of information flows over the business cycle. We again used a sample of stocks for every Wednesday spanning the three years 2016-2018. After winnowing the sample to exclude tickers that did not span the whole period,[7] out of an initial sample of 49 tickers, this left a sample of 29 tickers.

We then carried out the similar exercise of estimating the slope of the bid side of the LOB for each ticker on each day by averaging the calculated OLS slopes for each snapshot and also estimating the price volatility and executed-volume volatility for each ticker-day. We then calculated the normalized information value flow, $\frac{\frac{\Sigma_P}{\lambda}}{PV}$ that is, the value of information flow relative to the average value of trade for that ticker and day. This yields a dimensionless constant. We regressed this constant against the volatility of the SPY.[8] The coefficients, $t$-statistics, $R^2$, and $P$-values were then averaged. The results are displayed in Table 2.5.

| | Coefficient estimate | $t$-statistic average | $\lvert t\text{-statistic}\rvert$ |
|---|---|---|---|
| Intercept average | $8.60 \times 10^{-4}$ | 1.8 | 2.78 |
| $\lvert\text{Intercept}\rvert$ average | $1.34 \times 10^{-3}$ | | |
| $\Sigma_{SPY}$ average | $8.86 \times 10^{-2}$ | 4.06 | 4.1 |
| $R^2$ average | .12 | | |

Table 2.5: Normalized information flow versus $\Sigma_{SPY}$. (Mean($\Sigma_{SPY}$) = .0078)

The average of the absolute value of the $t$-statistics is included to reflect the fact that many of the estimated intercept terms in the regressions are negative, whereas most of the estimated coefficients of the SPY volatility are positive. Thus, there appears to be a systematic component of normalized information flow value.

The magnitudes of the information flow value from the systematic part, which can be roughly estimated as the product of the coefficient on $\Sigma_{\text{SPY}}$ with the mean of $\Sigma_{\text{SPY}}$, yields $0.0886 \times .0078 = 6.91 \times 10^{-4}$; the magnitude from the absolute values of the idiosyncratic parts are $1.34 \times 10^{-3}$, which is of similar magnitude. The average of the signed values of the intercept terms is $8.6 \times 10^{-4}$, an even smaller number.

7. This potentially biases the results due to survivorship biases.
8. The volatility of the SPY is similar in spirit to the VIX volatility index for the S&P500 index. However, the VIX is the volatility of the *return* to the S&P500 index, whereas the SPY volatility is the volatility of the *level* of the index.

We can conclude that the value of information increases during times of high systematic return volatility, that is, during recessions, as reflected in the countercyclicality of the VIX and the SPY volatility.

### Information flow and CAPM variables

Given the correlation of the normalized information flow with the volatility of the SPY (analog of the VIX), is there a relationship with the CAPM? We regressed the normalized information value, averaged over the three years for each ticker, against the CAPM beta and the CAPM $R^2$ for each ticker. The result is in Table 2.6. (The regressions on the CAPM beta did not yield significant results.)

| $R^2$: .22 | Coefficient value | $t$-statistic | $P$-value |
|---|---|---|---|
| Intercept | $6.2 \times 10^{-4}$ | 0.97 | .337 |
| CAPM $R^2$ | 0.000976 | 2.055 | 0.049 |

Table 2.6: Normalized information flow (three-year average, each ticker) versus CAPM $R^2$.

The results strongly support the hypothesis that systematic information flows are strongly correlated with the business cycle, as high-$R^2$ stocks have higher information flows.

There is an additional conclusion: because, like the VIX, the $\Sigma_{SPY}$ is strongly correlated with the business cycle, it is a proxy for the underlying systematic process. CAPM reasoning suggests that investors and traders care only about the systematic component of the value of any stock. The statistical significance of the coefficient on the $\Sigma_{SPY}$ suggests that the only information that matters for traders for any stock is the systematic component, and the value of the information is the value of unearthing and isolating the information about the systematic part of the information.

## 2.8 Conclusion

By treating the Kyle model's price impact parameter, $\lambda$, as the slope of the limit order book, our results strongly support the validity of the Kyle [1985] model.

While the theory literature has developed a number of models analyzing how cross-asset correlation of the underlying fundamental asset values influences the cross-asset correlation of the corresponding price impacts, our results support the notion that the cross-correlation of price impact across stocks is consistent with the CAPM: there is a single systematic component of price impact, and this is driven by the systematic component as captured by the volatility of the systematic component of the stock market, that is, the SPY volatility, and this systematic component of the underlying value is responsible for any cross-asset correlation, and any concomitant correlation of the price impact measures, that is, the $\lambda$s.

The information in the Kyle model is the forecast error variance of the uninformed market makers. This information can be measured, and its value can also be measured. When normalized by the value of trade in each ticker, the value of the information flow per unit of value is on the order of .0005. This number accords well with the overall income of firms engaging in stock market trading.

The normalized information flow value is strongly countercyclical, that is, it is strongly correlated with the volatility of the overall market. The connection of the information flow with the SPY volatility is strongly confirmed by the strong correlation of the normalized information flow, averaged over time, with the degree to which the stock is influenced by the aggregate market, that is, the CAPM $R^2$ value of the stock.

What is the purpose of the information flow? By separating the underlying information into two components, systematic and idiosyncratic, the traders distinguish between productive assets that have a systematic impact on the economy and those that can be diversified. From a CAPM perspective, this is the only information that matters, as any non-systematic value can be diversified away.

# Chapter 3

# Spoofing and market confidence

## 3.1 Introduction

Over the last decade, the US Commodity Futures Trading Commission (CFTC), the U.S. Securities and Exchange Commission (SEC), and the Department of Justice (DOJ) have stepped up their efforts to crack down on the type of disruptive trading called "spoofing". This emphasis coincides with a similarly increasing focus by the UK Financial Conduct Authority (FCA). Over 50 cases involving spoofing have been filed against individuals and companies by US regulators, while over 5 enforcement actions have been taken in the UK. One of the largest fines was JP-Morgan Chase's case [1] in which they entered an agreement to pay regulators USD 920 million as part of a settlement admitting to spoofing precious metals futures and US government bonds. Spoofing is the practice of submitting big limit orders to the markets with the intention of avoiding their completion by canceling them before they are executed. Spoofing is considered illegal in many jurisdictions. On the topic concerning spoofing, Aitan Goelman [2], the CFTCs Director of Enforcement, commented: "Spoofing seriously threatens the integrity and stability of futures markets because it discourages legitimate market participants from trading". The question of whether spoofing is harmful to market integrity and efficiency is a matter of debate among regulators and industry practitioners.

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1. Press release 8260-20,https://www.cftc.gov/PressRoom/PressReleases/8260-20
2. Press release 7264-15, https://www.cftc.gov/PressRoom/PressReleases/7264-15

In this paper, we investigate the effects of spoofing from historical, regulatory, and market microstructure perspectives. We find that spoofing is not a disruptive practice; the spoofing resemblance strategies have existed for centuries. By looking at different legal cases on spoofing, we point out that the primary victims of spoofing are HFTs that employ order anticipation strategies. Contrary to criticism expressed by regulators and industry practitioners, we show that while spoofing delays price discovery in a short horizon, price divergence will be so brief as to have little economic efficiency implications. Furthermore, spoofing improves market liquidity and fosters uninformed traders' welfare.

Our investigation into this matter builds on a two-period Kyle model with two additional market participants: a spoofer and an anticipatory trader. The informed trader trades on proprietary information regarding the value of the traded asset, but his order is delayed by one period. The order anticipation HFT uses pattern recognition algorithms to detect the incoming order and trades on this signal. However, the spoofer detects his trading strategy. The spoofer exploits the anticipatory trader by pretending to be a large trader and submitting two orders, one real order and another big spoofing order from the opposite side of the market, then cancels the spoofing order. In this way, the spoofer can add more noise to the anticipatory trader's signal and give a false sense of market demand.

The irrationality of the anticipatory trader primarily drives our results. Order anticipators study trades and quotes to find traces of informed trades, then trade ahead of such orders to profit from expected price changes. Computers play an important role in the successful implementation of order anticipation strategies because they can often perform pattern recognition faster and more accurately than humans do. However, these algorithmic trading strategies follow a rigid set of rules, thus making them vulnerable to other market participants in the ever-changing markets. Exploiting this feature, spoofers can trick those trading algorithms and make profits.

To study the effects of spoofing, we consider 4 different economies based on Kyle's model with: a spoofer and an order anticipator, only a spoofer, only an order anticipation HFT, and a standard Kyle model. We find that without an order anticipation algorithm, the spoofer falls victim to his own strategy and incurs a loss as he is uninformed and loses money to the informed trader. In the economy with both traders, a spoofing equilibrium exists, and both the spoofer and the anticipatory trader make profits. When the spoofer increases his trading intensity, the signal of the anticipation HFT becomes noisier. The anticipatory trader strategically responds to the spoofer by reducing his participation. Therefore, he becomes less active in the market. As a result, the uninformed traders benefit from spoofing.

## 3.2 An overview of spoofing

This section provides a regulatory and market structure overview of spoofing. We will discuss the history of spoofing, how spoofing works, and how it is regulated under different jurisdictions.

### 3.2.1 What constitutes spoofing

While there is no universally accepted definition of what constitutes spoofing, some common practices are generally considered as spoofing. A simple spoofing scheme involves a trader placing one or more highly visible orders but has no **intention** of keeping. It is designed to create a false sense of investor demand in the market, thereby changing the behavior of other traders and allowing the spoofer to profit from these changes. Apart from simple spoofing, there are some other popular spoofing resemblance practices [3].

- **Layering**: A trader places a small order on the intent side of the market and orders at multiple price levels on the spoof side of the market to increase the depth of the spoof side.
- **Vacuuming**: A trader places a small order on one side of the market and a larger order on the same side of the market. The larger spoofing order is then canceled to entice market movement toward the smaller order.
- **Collapsing of layers**: A trader places a small order on one side of the market and several spoof orders at different price points on the other side of the market. The spoof orders are then changed into a single price point to give the appearance of a large volume.

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3. Automated spoofing,https://library.tradingtechnologies.com/tt-score/inv-automated-spoofing.html

- **Flipping**: A trader places orders on one side of the market with the intent of switching, or flipping, to the other side of the market.
- **Spread squeezing**: A trader places an order on the spoof side at successively higher or lower prices with the spread to squeeze it in one direction, enticing other market participants to join or beat the newly established top of the book. The trader then switches sides and executes against those participants.

Chapter 1 studies the dynamics of the limit order book and the information value of order flow. By examining the limit orders of over 80 US stocks in February 2018, we find that over 90% of all limit orders got canceled eventually. Therefore, it is challenging to distinguish between normal cancellation orders and spoofing orders. The most important aspect of classifying whether it constitutes spoofing is the trader's intention to cancel the order before execution, which is hard to identify. Based on previous cases in the US, enforcement authorities may offer the following evidence.

- For algorithmic trading, the contents of the algorithms are examined for evidence of intent.
- For manual trading, emails, instant messages, and phone recordings may help to establish intent. Witness testimony may be offered.
- For some cases, trading data is used to identify an individual's trading pattern and then compare it to the market trend.

### 3.2.2 Who are victims of spoofing

Based on previous cases, victims of spoofing are principally high-frequency trading firms that used price quotes for their trading strategies. In the Igor B. Oystacher case (2016), CFTC claimed that Mr. Oystacher used spoofing to create false book pressure as he knew that algorithmic trading firms like CGTA and Citadel had programmed their algorithms to rely primarily upon book pressure when making trading decisions in particular markets. During the trial, The CFTC presented 2 victim witnesses [4], Richard May of Citadel and Matthew Wasko of HTG Capital Partners. Mr May testified that.

4. Case: 1 15−cv−09196, U.S. District Court - Northern District of Illinois, https://www.govinfo.gov/content/pkg/USCOURTS−ilnd−_15−cv−09196/pdf/USCOURTS−ilnd−1_15−cv−09196−.pdf

"Specifically, Citadels trading strategies tend to look at three key factors: 1) relative value, 2) book pressure, and 3) trade flow...In 2013, Mr. May and his team observed what they believed was spoofing in the ES market. Around this time, they noticed a significant decline in Citadels profitability...Immediately, Citadel **scaled back** its participation in the ES market by over fifty percent."

Mr. Matthew Wasko gave his testimony that "Beginning around July 2012, Mr. Wasko and his team began to observe suspicious trading they believed was spoofing in the ES market. **After noticing a decline in profitability**, they began reviewing historical trading data from market replays of losing trades...This trading activity was detrimental to CGTAs trading because the initial large orders appeared to its algorithms as genuine interest from multiple market participants, leading CGTA to **join that movement and enter orders it intended to trade.**"

At the recent spoofing trial in which two ex-Deutsche Bank traders were prosecuted for spoofing, the government testified on behalf of just two victims of the alleged spoofing by the two defendants on trial: one was a representative from Citadel Securities, and the other was a company called Quantlab. Both of them are among the most secretive and highly profitable high-frequency trading firms.

Obviously, some HFTs are vulnerable to spoofers. However, HFT strategies vary considerably, and only predatory HFTs can fall victim to spoofers easily. Following Harris [2013], there are three main types of HFTs.

- **Valuable HFT** High-frequency traders who use dealing and arbitrage strategies that make markets more liquid. Spoofers have little effect on this type of HFT trader as spreads across markets, or exchanges are their concerns.
- **Harmful HFT** High-frequency traders use computers to monitor and interpret electronic news feeds. Obviously, with information acquisition, the HFTs become informed traders and know the true value of the stocks. Therefore, spoofers cannot influence their strategies.
- **Very Harmful HFT** A few high-frequency traders examine trades and quotes (book pressure and order flow) to detect when traders are using algorithms to split up large orders that will move the market. They then trade ahead of such orders to profit from expected price changes. Some market participants refer to this practice as "front-running,". But

this conduct is legal and different from "traditional front-running," which is defined as entering a trade with advance knowledge of a block transaction that will influence the price of the asset and the trader improperly obtain such information. In this case, HFTs use public information; they are just faster and better at transmitting and processing data. To differentiate it from "traditional front-running", I refer to this practice as "order anticipation or anticipation strategy." This type of strategy is highly vulnerable to spoofers as spoofers can add more noise to quotes which are used as their main indicators.

From the witnesses' testimony and our reasonings, we can deduce that victims of spoofing are primarily HFTs which used order anticipation strategies. Their strategies were detected and exploited by spoofers. They only noticed that their strategies were exploited when they saw a significant decline in profitability and immediately scaled back their participation. This resonates with our results in the next section, as in equilibrium, with the spoofer, the anticipatory traders make less profit and scale back their trading activities when there is a spoofer.

### 3.2.3 Brief history of spoofing

Spoofing may have been occurring since the establishment of formal financial markets in Europe during the 1600s-1700s. The earliest recording of spoofing incidents was from Daniel Defoe in his essay "Anatomy of Exchange Alley. In a passage recounting the trading practice of Sir Josiah Child, an English economist, merchant, and governor of the East Indian Company, Daniel Defoe gave a glimpse into a spoofing-resemblance trick.

*"If Sir Josiah had a mind to buy, the first thing he did was to commission his brokers to look sower, shake their heads, suggest bad news from India; and at the bottom, it followed, I have commission from Sir Josiah to sell out whatever I can, and perhaps they would actually sell ten, perhaps twenty thousand pounds. Immediately, the Exchange was full of sellers; nobody would buy a shilling", "till perhaps the stock would fall six, seven, eight, ten percent, sometimes more; then the cunning jobber had another set of men employed on purpose to buy."*

Hundreds of years later, spoofing became a common practice in the trading pits during the twentieth century. MacKenzie [2022] interviewed different floor traders and recorded their re-counts of spoofing "It sounds like a normal day in the pit. We spoofed all the time.". On the trading floor, traders and brokers communicated by shouting out bids and offers or using hand signals to indicate the prices and quantities. Their behaviors were under the scrutiny of all other traders. Many brokers wanted to hide their trading intentions. When they want to buy, they might shout or hand signal offers to sell without an intention of selling. Such practice would be labeled as spoofing nowadays but well regarded as "good brokerage" among market participants. However, those employing this practice might face reputation risks. Even though verbal deals were not legally enforceable, constant spoofing might freeze the traders out of future trades as other traders knew who often regened their deals.

The adoption of electronic trading systems in the late 1990s and early 2000s created a perfect environment for spoofing to thrive. A computer-powered system was first introduced in the financial markets in 1969 but did not take off until the late 1990s. The new advent of technology brought in an anonymous trading mechanism, thus eradicating the need for social interactions and the reputation risks of spoofing in the pits. Zaloom [2003] documented the trading activities in the early 2000s "The most recurring character was called the "Spoofer." The Spoofer used large quantities of bids or offers to create the illusion that there was more demand to buy or pressure to sell than the "true" bids and offers represented." Most market participants considered this practice legitimate. "Although there would be nothing illegal about the Spoofer's maneuver of supplementing the numbers with the weight of his bid or off...". With the changes from face-to-face trading to electronic trading, there was a gradual shift in moral and regulatory treatments of spoofing. From a highly regarded practice, spoofing became a serious criminal conduct when New Jersey trader Michael Coscia became the first person to be convicted of spoofing and sentenced to 3 years in prison in 2016. Since then, regulators in many countries have intensified their crackdowns on this type of practice.

### 3.2.4 Regulations of spoofing

#### 3.2.4.1 United States

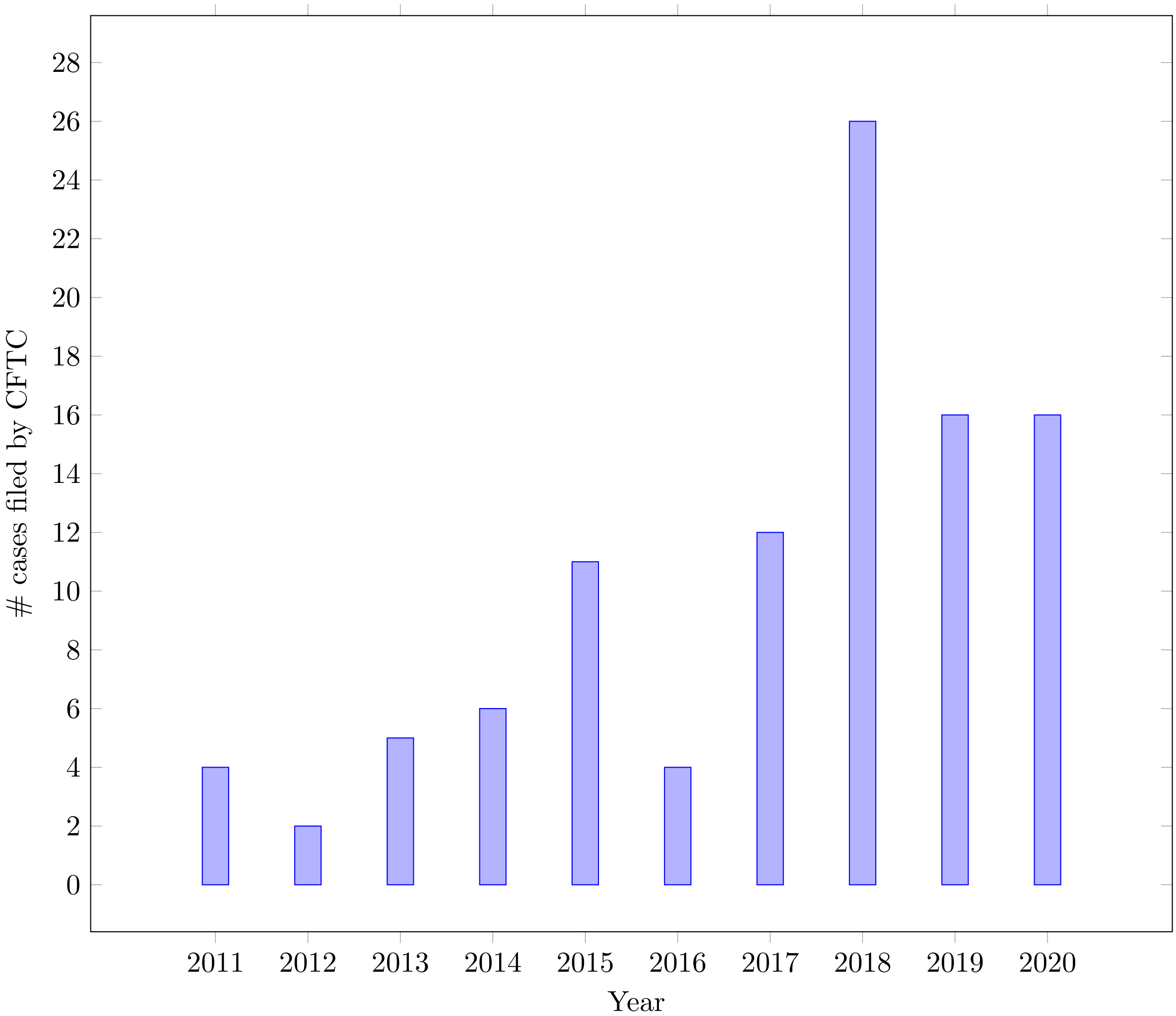
Spoofing is prosecuted according to civil and criminal law in the US. Regulators are required to prove the traders' intention to spoof the market by canceling the orders before execution. Depending on markets and the severity of the cases, the SEC, CFTC, FINRA, and the DOJ (Department of Justice) enforce spoofing under different laws. In the US, regulators must provide evidence of traders' intention to cancel bids or ask before execution. Civil cases can brought in case of act with with some degree of intent, or scienter, beyond recklessness. while criminal cases are for individuals who knowingly engage in Spoofing.

In the commodity markets, manipulative conduct is enforceable by the CFTC. Before the enactment of the Dodd-Frank Act in 2010, the CFTCs authority to regulate spoofing was limited to the Commodity Exchange Act (CEA) (Section 6c 9(a)(2)). Section 6(c) of the CEA gave the CFTC authority to take administrative enforcement action against traders who manipulated or attempted to manipulate the market price, while Section 9(a)(2) made it unlawful to manipulate or attempt to manipulate the price of a commodity or future. This rule vaguely defines what"market manipulation" is, thus making it almost impossible to prove manipulation. Therefore, the agency is believed to have successfully brought only one market manipulation case to final judgment from 1975 to 2010. After the Dodd-Frank Act in 2010, Section 747 of the Dodd-Frank Act added Section 4c(a)(5)(C) to the Commodity Exchange Act (CEA) to ban three types of transactions labeled as disruptive trading. One of those transactions is spoofing in commodity markets and this is the first time spoofing was expressly prohibited by a federal statute. Not until 2013 did the CFTC take the first enforcement action under the amended CEA by settling with Panther Energy Trading, LLC (Panther). Since then, the CFTC has stepped up enforcement against spoofing. The 2020 Division of Enforcement Annual Report [5] showed that the CFTC has intensified its crackdowns on spoofing. Nearly 10% number of cases filed by CFTC in 2020 involved spoofing.

5. FY 2020 Division of Enforcement Annual Report https://www.cftc.gov/media/5321/DOE_FY2020_AnnualReport_120120/

Unlike the Commodity Exchange Act, the federal securities statutes do not expressly prohibit spoofing by name. Instead, the Securities and Exchange Commission (SEC) has taken action against spoofing by characterizing it as a manipulative practice. The SEC has been investigating and prosecuting alleged spoofing in the securities markets since the early 2000s. The full lists of civil and criminal cases against spoofing are in Appendix C.9.

Number of CFTC enforcement involving manipulative conduct or spoofing.



#### 3.2.4.2 The UK

UK law does not include any specific anti-spoofing provisions; rather, spoofing behavior is generally construed to be a form of market manipulation that may result in civil or criminal liability. The UKs MAR is modeled on the EUs Regulation No 596/2014 (Reg 596), which was passed on April 16, 2014. While the FCA made some changes to UK MAR as it adopted the regulation following Brexit, the UK regime is still primarily based on the EU Market Abuse Regulation.

In Europe and the UK, prosecutions can be made where the regulator deems that Spoofing took place, and they don't need to prove that spoofers have the intention of canceling the orders. While there have not yet been any criminal spoofing cases in Britain, the Financial Conduct Authority (FCA) and Office of Gas and Electricity Markets (Ofgem)22 used their powers to impose stiff fines on those who engage in market manipulation practices in the UK. This intensified crackdown coincides with a similarly increasing focus by the US Commodity Futures Trading Commission and the US Department of Justice. In 2015, the Financial Conduct Authority (FCA) fined Da Vinci Invest Ltd, Mineworld Ltd, Mr Szabolcs Banya, Mr Gyorgy Szabolcs Brad and Mr Tamas Pornye č7,570,000 for spoofing. The case was instigated in 2011. The defendants were accused of using manipulative behavior, which consisted of an abusive trading strategy known as layering, involving the entering and trading of orders in relation to shares traded on the electronic trading platform of the London Stock Exchange (LSE) and multi-lateral trading facilities (MTFs)[2] in such a way as to create a false or misleading impression as to the supply and demand for those shares and enabling them to trade those shares at an artificial price.

#### 3.2.4.3 Europe

Article 12 of EU MAR gives definitions of what constitutes market manipulations, including entering into a transaction, placing an order to trade or any other behavior which gives, or is likely to give, false or misleading signals as to the supply of, demand for or price of a financial instrument; or secures, or is likely to secure, the price of one or several financial instruments at an abnormal or artificial level. Article 15, thereafter states that a person shall not engage in or attempt to engage in market manipulation such as those defined in Article 12.

While EU MAR doesnt provide any specific provisions regarding spoofing, the regulation clearly defines the indicators that firms should monitor and detect potential market manipulation. One of the indicators is for Spoofing. MAR Article 16 highlights that firms must have effective arrangements, systems, and procedures to prevent and detect insider dealing, market manipulation, and attempted insider dealing and market manipulation.

#### 3.2.4.4 Asia

While regulations on spoofing take different forms across Asia, spoofing is deemed illegal in most countries. There is a big gap between emerging and developed markets in spoofing regulation and enforcement processes. While developed markets have clear regulations and streamlined enforcement processes, emerging markets lack clarity in spoofing definitions and strict enforcement actions against market manipulators.

We study the regulatory measure and enforcement system in 2 advanced markets, namely Japan and South Korea, and Hong Kong. In Japan, a market surveillance system has been implemented to oversight the market. Trading data from 2 exchanges, the spot market (Tokyo Stock Exchange) and derivatives market (Osaka Exchange), is analyzed daily to detect any abnormal activities. Any transactions that are suspected of spoofing are reported to the Securities and Exchange Surveillance Commission. Spoofing is classified as a market manipulation under Article 159, Paragraph 2, Item 1 of the Financial Instruments and Exchange Act[6] and the Securities and Exchange Surveillance Commission. Spoofing has issued administrative fines to some market participants for their alleged spoofing activity. In 2019, the market regulator fined Citi Group $ 1.2 million for spoofing in the future market. In 2022, an administrative penalty order was issued to Atlantic Trading London Limited for their involvement in spoofing 10-year Japanese Government Bond Futures. In South Korea, the regulating authority is The Financial Service Commission (FSC), which is responsible for overseeing the securities and futures industry. Regulation on market disturbances was introduced on December 30, 2014, and went into effect on July 1, 2015, which is intended to enhance regulations on market manipulations. Article 178-2 of this provision bans market participants from engaging in "An act that adversely affects, or is likely to, adversely affect the market price by submitting a large volume of asking prices at

6. The Financial Instruments and Exchange Act,https://www.fsa.go.jp/common/law/fie01.pdf

which deals are unlikely to be concluded, or by repeatedly correcting or canceling asking prices after submitting them" [7]. Using these regulations, The Financial Service Commission has imposed a fine of 11.88 billion won on Citadel Securities for their distortion of stock prices by using immediate-or-cancel (IOC) buy market orders to exhaust the best ask prices and submitting buy limit orders on any remaining unfilled quantity, then cancel these orders.

For comparison, we examine the regulations governing spoofing in two Asian emerging markets, namely, India and China. In the case of India, the regulatory body which is in charge of the development and supervision of the Indian capital market is the Securities and Exchange Board of India. Section 12A of the Securities and Exchange Board of India Act, 1992 (SEBI Act) bans market participants from engaging in fraudulent and unfair trade practices through the use of any manipulative device, insider trading. However, there was a lack of clarity regarding spoofing until SEBI issued a circular on Order-based Surveillance Method-Persistent Noise Creators. [8] to address the issue of excessive cancellation of orders in 2021. The circular proposed a surveillance mechanism to deter excessive order modifications and cancellations with the intent to avoid execution. Various parameters, such as order-to-trade ratio and cancellation ratio, are examined daily to detect any potential market manipulation. In 2023, the Securities and Exchange Board of India issued an order [9] to investigate the trading activities of Nimi Enterprises for alleged engagement of spoofing. This was the first time, the term spoofing was introduced in a regulatory document in India to describe the actions undertaken by Nimi Enterprises. In the case of China, the stock market was closed in 1950 and reopened in December 1990. The official regulation of market manipulation began in 1993. Spoofing came into the spotlight in 2015 when Chinas securities regulator targeted high-frequency traders following the stock market turbulence. Article 55 of the Securities Law of the People's Republic of China (2019 Revision) [10] officially banned any person from "placing and canceling orders frequently or in large numbers, not for the purpose of the consummation of trades."

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7. Disturbance of capital market, http://www.koreanlii.or.kr/w/index.php/Disturbance_of_capital_market?ckattempt=2
8. Order Based Surveillance Measure: Persistent Noise Creators, https://www.bseindia.com/markets/MarketInfo/DispNewNoticesCirculars.aspx?page=20210326-55
9. Order in the matter of trading activities of Nimi Enterprises, https://www.sebi.gov.in/enforcement/orders/apr-2023/order-in-the-matter-of-trading-activities-of-nimi-enterprises_70718.html
10. Securities Law of the People's Republic of China (2019 Revision), https //www.lawinfochina.com/display.aspx?id=31925&lib=law

## 3.3 Literature review

There is a paucity of social science literature on spoofing. To our knowledge, our paper is the first paper to show that spoofing restricts the market participation of very harmful HFTs and doesn't impede price discovery. Our finding is in contrast to that of Williams and Skrzypacz [2020], who studies spoofing equilibrium under the Glosten-Milgron framework. They show that spoofing can occur in equilibrium, slowing price discovery and raising spreads and volatility. A novel prediction is that the prevalence of equilibrium spoofing is single-peaked in the measure of informed traders. However, they only allow spoofers to trade one unit of share in each period, which deviates from the true sense of spoofing, in which traders trade high volumes to give a false picture of supply and demand. In our model, spoofers use large orders to mislead other traders.

Our paper is closely related to papers that study front-running in the market. We adopt similar two-period settings to Bernhardt and Taub [2008b], Xu and Cheng [2023]. However, our paper allows traders not to trade and cancel their orders. Compared to front-running papers, another type of trader was added: a spoofer who can submit a big order to manipulate other HFTs' beliefs. Without a spoofer, our model collapses into a front-running model. Therefore, the results of these papers and our results are complementary.

Most empirical papers have difficulties in identifying traders' intentions and spoofing activities. Lee and Park [2013] use the complete intraday order and trade data of the Korea Exchange (KRX) (data with customer number) to study the possibility of spoofing activities in general. They define a spoofing order as a bid/ask with a size at least twice the previous day's average order size and with an order price at least 6 ticks away from the market price, followed by an order on the opposite side of the market, and subsequently followed by the withdrawal of the first order. They show that price disclosure leads to a dramatic decrease in spoofing frequency. In contrast, Kong and Wang [2014] investigate a specific spoofing case in the Chinese stock market. Using a unique dataset of a spoofing case, they found that spoofing affects investors' behaviors in the short term, but this effect disappears rapidly in the long run. Their findings are consistent with our result that the impact of spoofing is short-lived as the spoofer is a short-term trader, and their net trading position is zero over a long time horizon.

## 3.4 Modelling spoofing

In this section, we present a variant of the dynamic Kyle model. As mentioned in the second section's analysis, spoofer and their "victims" are either HFTs or really fast traders who can take advantage of their speeds to capture short-term movements of the price. Sometimes, they are labeled as "scalpers" by other market participants. We modeled them as fast traders.

### 3.4.1 Traders and market

Similar to Kyle [1985], the model has two types of slow traders who can submit orders to the market with latency: i) noise traders whose trades are treated as entirely exogenous, that is, they do not react to observations about price in any way. Their trade is normally distributed $u \sim N(0, \sigma_u^2)$ and $\sigma_u > 0$; ii) one monopolistically informed trader who has private information of the stock $v \sim N(0, \sigma^2)$. The realized value of the security is privately observed by the informed trader, who then exploits this information in his trade.

Apart from slow traders, there are three types of fast traders who employ different high-frequency strategies. First, a spoofer who can observe the trading activities with a low latency uses spoofing strategies. Second, an HFT who depends on a pressure book to anticipate other traders' strategies is labeled as an "anticipatory trader". Third, competitive market makers set the price to absorb the order flow imbalance and make zero expected profit.

There are trading periods that are denoted by $t = 1, 2$. In the period, the informed trader submits an order of volume x with a latency. The spoofer can submit an order of $-z_1$ with low latency to get an immediate execution and an order of $z_2$ with high latency with an option to cancel the order just before the period 2 execution. The anticipatory trader can observe a noisy private signal about the incoming big order flow in the second period $\tilde{i} + z_2 = x + z_2 + \epsilon$ with, and he can trade m shares. In the second period, the spoofer can cancel the order of $z_2$ and submit an order of $z_1$. The order of the informed trader arrives in the market. The anticipatory trader trades an order of $-m$ to liquidate all his positions. We allow the possibility that spoofers and anticipatory traders choose not to trade.

The noise orders during the period 1 and 2 are respectively denoted by $u_1$ and $u_2$. Both of them have the same distribution as u, and they are independent of each other and other random variables. The noise term $\epsilon$ is normally distributed, $\epsilon \sim N(0, \sigma_\epsilon^2)$ and independent of other random variables. $z_2$ is independent of each other and other random variables and $z_2 \sim N(0, \sigma_{z_2}^2)$. This assumption implies that the spoofer is an uninformed trader; he has no private signal, and his spoofing order is uncorrelated with all other random variables.

### 3.4.2 Information sets and model discussion

No matter what type of spoofing strategies the spoofers employ, the main tenet is to give a false sense of supply and demand (false signal) to the anticipatory trader. In this way, spoofers can manipulate HFTs which use anticipatory algorithms. In this model, we model it as the signal $\tilde{i}$, which is analogous to "pressure book". One interpretation of i is that the anticipatory trader can observe the limit order book. By spotting the big limit orders, the trader can deduce the incoming order flow from the big traders (informed traders) in the second section. Spoofers may have used trading data or "ping orders" to detect anticipatory traders. For example, Mizuho Bank[11] was fined a $250,000 civil monetary penalty by the CFTC for using spoofing strategies to test the markets reaction to his spoof orders. When anticipatory traders' strategies get detected, the spoofer pretends to be an informed trader and sends a big order of $z_2$ to mislead the anticipatory trader. The big order is canceled immediately after the real order $z_1$ gets executed.

Both the spoofer and anticipatory trader hold no inventory at the end of the second period. The main reason for this assumption is that both of them are short-term traders and have no information about the fundamental value of the stock. Their strategy is to capture short-term movements of the price. Therefore, they are risk-averse to holding inventory and tend to hold no inventory at the end of the day. Recounting the Navinder Sarao's trading strategy, Liam Vaughan[12] gave a short description in his book "At the end of almost every session, he made sure he had no outstanding positionsthat he was flat, in the idiom of the trader. The next day, he started afresh." This assumption is also consistent with empirical findings of Kirilenko, Kyle, Samadi, and Tuzun [2017].

11. CFTC press release 7800-18,https://www.cftc.gov/PressRoom/PressReleases/7800-18
12. Liam Vaughan, Flash Crash: a trading Savant, a Global Manhunt and the Most Mysterious Market crash in History

Informed trader faces execution latency. There are several ways to interpret this assumption. First, the informed trader may have a private signal about the asset fundamentals, but he is a slow trader. Second, informed trades tend to be a big order and can move the market equilibrium. The informed trader may chop his meta order into many small orders, thus slowing down his execution. Third, the informed trader may submit only limit orders and wait for a better price instead of market orders. Therefore, he faces a delay in execution.

We capture spoofing by allowing spoofers to submit and cancel under the Kyle [1985] framework, which in practice usually takes place with limit orders rather than market orders, as we do not find any existing limit order book models that allow tractable modeling of spoofing. However, in Duong and Taub [2023], we draw an analogy between the limit order book and Kyle [1985] model. The limit order book has a visible structure: the set of resting orders forms a pattern, essentially a supply curve, with a slope that is driven by the underlying incentives created by the information possessed by some of the traders. A fundamental theory, the Kyle [1985] model, explains this structure and predicts that the slope, $\lambda$, of the supply curve reflects the fundamental forces driving the stock value; $\lambda$ reflects the marginal effect of trading on the price, and so is known as the price impact parameter. This analogy plays an important role in our model estimation in the next section.

Under the high-frequency setting, the time horizon is short. Therefore, we assume that once all market participants choose their trading strategies they are committed to their strategies at the beginning of the first periods. The timeline of the two-period model is as follows.

| t=1 | t=2 |
|---|---|
| • Informed trader observes $v$ and submits a slow order $x$.<br>• The spoofer sells $-z_1$ shares and submits a slow spoofing order of $z_2$.<br>• The anticipatory trader observes $\tilde{i}+z_2$ and submit order flow $m$.<br>• Noise traders submit order flow $u_1$ | • Order flow $x$ arrives.<br>• The spoofer cancels a slow spoofing order of $z_2$ and submits order flow $z_1$.<br>• The anticipatory trader liquidates his position $m$.<br>• Noise traders submit order flow $u_2$ |

Table 3.1: Model timeline

The information available to different market participants is as follows:

- **Informed trader** can observe a private signal of the true fundamental value of the asset $v$.
- **Spoofer** he's aware of the anticipatory HFT and his strategies, but he doesn't know whether the anticipatory HFT will trade or not.
- **Anticipatory trader** can observe a private signal of $\tilde{i}$, but he's not able to distinguish between the true signal and the spoofed volume. One of the interpretations of $\tilde{i}$ is that the front runner can observe the limit order book. By spotting the big limit orders, the trader can deduce the incoming order flow from the big traders (informed traders). Knowingly, the spoofer submits a spoofing trader $z_2$ to manipulate the anticipatory trader's belief. Therefore, instead of observing the true signal of $\tilde{i}$, the anticipatory only observes $\tilde{i} + z_2$.
- **Market maker** is aware of the informed trader, spoofer, and anticipatory trader. He can observe the total order flow for each period, but he doesn't know exactly how much informed trader, spoofer, or anticipatory trader trade.

## 3.5 Model equilibrium

We use the subscripts I, A, S, M, U for the variables or parameters of informed trader, anticipatory trader, spoofer, market maker, and uninformed traders. We denote the order flows of the first and second periods, $y_1, y_2$ respectively. Let the strategy functions of the spoofer, anticipatory trader, and informed trader be $S(.), A(.), I(.)$, and the market maker commits to a pricing function $P(.)$. The equilibrium is defined by four functions $S(.), A(.), I(.), P(.)$ such that the following conditions hold:

1. *Informed trader's profit maximization.* Given $I(.), A(.), P(.)$ and his signal about the true value of the assets $v$, he chooses $x^*$ to maximize his expected profit $\pi_I = x(v - p_2)$.

$$x^* = X(v, S(.), A(.), P(.)) = \arg\max_x E[\pi_I | v, S(.), A(.), P(.)] \tag{3.1}$$

   Where $p_2$ is the execution price in the second period.

2. *The spoofer's profit maximization.* The optimal strategy of the spoofer, $S(.)$ is a set of real-valued functions $S(.) = \{(P_S(.)Z_1(.))\}$, $P_S$ is the probability that the anticipatory trader chooses not to trade, $Z_1(.)$ is the optimal trading volume function if he decides to trade. The optimal strategy of the anticipatory trader. Given $I(.), A(.), P(.)$, his signal $\tilde{i}$ about the informed trades and spoofing order $z_2$ , he chooses $(p_S^*, z_1^*)$ to maximize his expected profit $\pi_S = z_1(p_1 - p_2)$.

$$(p_S^*, z_1^*) = S(v, I(.), A(.), P(.)) = \arg\max_{p_S, z_1} E[\pi_S | I(.), A(.), P(.)] \tag{3.2}$$

3. *Anticipatory trader's profit maximization.* The optimal strategy of the anticipatory trader, $A(.)$ is a set of real-valued functions $A(.) = \{(P_A(.)M(.))\}$, $P_A$ is the probability that the anticipatory trader chooses not to trade, $M(.)$ is the optimal trading volume function if he decides to trade. Given $I(.), P(.)$ and his signal $\tilde{i} + z_2$, he chooses $(p_A^*, m^*)$ to maximize his expected profit $\pi_A = m(p_2 - p_1)$.

$$(p_A^*, m^*) = \arg\max_{p_A, m} E[\pi_A | \tilde{i} + z_2, I(.), P(.)] \tag{3.3}$$

4. *Market efficiency.* By the model's setting, the market maker observes only the total order flow at each period $y_1, y_2$. Given the strategies of the spoofer, anticipatory trader, and inform trader, the market maker sets the price $p_1, p_2$, equal to the posterior expectations of $v$

$$p_1 = E[v | y_1, S(.), A(.), I(.)] \tag{3.4}$$

$$p_2 = E[v | y_1, y_2, S(.), A(.), I(.)] \tag{3.5}$$

Note that in the perfect Bayesian equilibrium, we allow mixed strategies, that is, in principle $p_A, p_S$ are distributions over strategies of the anticipatory trader and the spoofer (Trade, Not trade) respectively. These 4 possibilities: (Trade, Trade), (Not Trade, Trade), (Trade, Not trade),(Not trade, Not trade), which represent different economies. The first economy corresponds to one in which the spoofer can trick the anticipatory trader into trading a big order. The second possibility represents one in which the anticipatory trader can extract signals about informed orders from the order book. The third one is the model in which the spoofer falls victim to his own strategy. The fourth possibility is the standard Kyle model. We use superscripts AS, "0S", "A0" and Kyle to indicate these economies

Obviously, the strategy function $S(.), A(.), I(.), P(.)$ can take any forms. For the model's tractability, we will focus on linear equilibria, i.e., the trading strategies and pricing functions are linear. Formally, a *linear equilibrium* is defined as a perfect Bayesian equilibrium in which there exist constants $(p_A, p_S)$ and 4 sets of $(\lambda_1, \lambda_{12}, \lambda_{22}, \beta, \beta_1, \beta_2)$ corresponding to 4 above possibilities, such that:

$$p_1 = \lambda_1 y_1 \tag{3.6}$$

$$p_2 = \lambda_{12} y_1 + \lambda_{22} y_2 \tag{3.7}$$

$$x = \beta v \tag{3.8}$$

$$m = \beta_1(\tilde{i} + z_2) \tag{3.9}$$

$$z_1 = \beta_2 z_2 \tag{3.10}$$

Following Bernhardt and Taub [2008b]'s setting, we allows the possibility that $\lambda_1 \neq \lambda_{12}$. In this way, the market maker can reevaluate the information content of period-1 order flow in period-2 pricing.

The values of the net order flow depend primarily on the strategies of the spoofer and the anticipatory trader. We only allow these two types of traders not to trade. This assumption comes from the fact that the noise traders trades are treated as entirely exogenous in the model, and they always trade. At the same time, the private signal of informed traders is short-lived. His optimal is to trade on his information's advantages. He makes zero profit if not trade, while he makes a positive profit if he chooses to trade.

If both the spoofer and anticipatory trader choose not to trade, the model turns into the standard Kyle [1985] model. In the first period, there are only noise traders. As there is no informed trade, the market maker sets the price $p_1 = v_0 = 0$. In the second period, there are only the informed trader and noise traders.

If both spoofer and anticipatory trader decide to trade, they need to choose optimal trading volume to maximize their expected profits. The total net order flow $y_1$ and $y_2$ executed at t = 1 and t = 2 are

$$y_1 = -z_1 + m + u_1 \tag{3.11}$$

$$y_2 = z_1 - m + x + u_2 \tag{3.12}$$

If only the anticipatory trader chooses to trade, the model collapses into the Xu and Cheng [2023] model. The total net order flow $y_1$ and $y_2$ executed at t = 1 and t = 2 are

$$y_1 = m + u_1 \tag{3.13}$$

$$y_2 = -m + x + u_2 \tag{3.14}$$

If only the spoofer chooses to trade, the total net order flow $y_1$ and $y_2$ executed at t = 1 and t = 2 are

$$y_1 = -z_1 + u_1 \tag{3.15}$$

$$y_2 = z_1 + x + u_2 \tag{3.16}$$

### 3.5.1 Anticipatory trader's problem

The anticipatory trader is a short-term fast trader who has speed advantages over other slow traders. Due to these advantages, he can trade twice in the model. He opens his position in the first period and liquidates it entirely in the second period. As a short term trader, he holds no inventory at the end. Unlike the informed trader, whose profit is determined by the difference between the entry price and the fundamental value, the anticipatory trader's profit depends on the difference between his entry and exit prices. Therefore, his main focus is to predict the short-term price dynamics based on his signal, not the fundamental value of the asset.

If the anticipatory trader does not trade, his profit is zero. Therefore, we only need to consider his optimization problem if he decides to trade. Given the strategies of the informed trader, the spoofer, and the pricing rule of the market maker, the anticipatory trader chooses trading volume $m$ to maximize his profit. In the anticipatory trader's belief, there may be only the informed trader and uninformed traders, or he may be aware of the spoofer but do not know the spoofer's strategy. In this case, we assume that the order anticipation HFT misinterprets the spoofing order as the noise in his signal. This assumption is consistent with Richard May's testimony [13] "In 2013,

13. Case: 1 15−cv−09196, U.S. District Court - Northern District of Illinois, https://www.govinfo.gov/content/pkg/USCOURTS−ilnd−_15−cv−09196/pdf/USCOURTS −ilnd−1_15−cv−09196−.pdf

Mr. May and his team observed what they believed was spoofing in the ES market. Around this time, they noticed a significant decline in Citadels profitability...Immediately, Citadel scaled back its participation in the ES market by over fifty percent. Mr. Mays team began investigating the market data to determine why Citadel was experiencing such a decline...Eventually, Mr. Mays team developed a program to detect this behavior on a more automated basis in an effort to determine whether this was a new phenomenon or something that had always been there that [they] hadnt previously seen". Anticipatory trader is only aware of the existence of spoofing when their profit declines. Therefore, in his optimization problem, the total net order flow $y_1$ and $y_2$ executed at t = 1 and t = 2 are

$$y_1 = m + u_1 \tag{3.17}$$

$$y_2 = -m + x + u_2 \tag{3.18}$$

If the anticipatory trader wants to trade, his optimization problem is to choose $m$ to maximize his expected profit.

$$\max_m E[m(p_2 - p_1)|\tilde{i} + z_2, P(.), I(.)] \tag{3.19}$$

Plug equations (3.6), (3.7), (3.17), (3.18) into the equation (3.19) and simplify

$$\max_m E[m(\lambda_{22}x - (\lambda_1 + \lambda_{22} - \lambda_{12})m))|\tilde{i} + z_2, I(.)] \tag{3.20}$$

For tractability, we also conjecture that the informed trader's strategy admits a linear function of his signal $x = \beta v$. By using the projection formula, we can obtain $E[x|\tilde{i}+z_2] = \frac{\beta^2\sigma^2}{\beta^2\sigma^2+\sigma_\epsilon^2+\sigma_{z_2}^2}(\tilde{i}+z_2)$ .Therefore, the first-order condition for the anticipatory problem is

$$m = \frac{\lambda_{22}}{2(\lambda_1 + \lambda_{22} - \lambda_{12})} \frac{\beta^2\sigma^2}{\beta^2\sigma^2 + \sigma_\epsilon^2 + \sigma_{z_2}^2}(x + \epsilon + z_2) \tag{3.21}$$

As the second order condition is $\lambda_1 + \lambda_{22} - \lambda_{12} > 0$. Compare equation (3.21) and the conjectured strategy (3.9), we have

$$\beta_1 = \frac{\lambda_{22}}{2(\lambda_1 + \lambda_{22} - \lambda_{12})} \frac{\beta^2\sigma^2}{\beta^2\sigma^2 + \sigma_\epsilon^2 + \sigma_{z_2}^2} \tag{3.22}$$

In the next section, we will also consider the problem in which the anticipatory trader can extract the informed trade perfectly from the noisy signal even when the spoofer adds more noise to the market.

### 3.5.2 Spoofer's problem

Similar to the anticipatory trader, the spoofer is a fast trader with a short-term trading horizon. He also has no inventory holding at the end of the second period. The main difference between the anticipatory and the spoofer lies in their trading strategies. While the anticipatory trader uses pattern recognition algorithms to examine trades and quotes to extract trading signals, the spoofer's focus is to add more noise to the limit order book by submitting a big spoofing order. In this way, the spoofer can mislead the anticipatory trader.

When the spoofer opts to trade, he needs to choose to submit a real order flow $z_1$ and a spoofing order $z_2$ to maximize his expected profit. For the model's simplicity, in this section, we only allow the spoofer to choose the optimal $z_1$ explicitly. Even though only $z_1$ is chosen optimally in the spoofer's optimization problem, $z_2$ implicitly faces some constraints. First, $z_2$ faces the upper bound of his inventory of the asset or his ability to borrow the asset. Second, the spoofer is a fast trader who does not want to hold inventory at the end. The higher the spoofing order $z_2$ is, the higher volume is exposed to the risk of execution. Even though the spoofing order only rests for a short period in the limit order book, it faces the same risk of execution as other open orders when it is there. The execution risk can leave the spoofer holding the inventory at the end of the second period, thus deviating from his trading strategy. The spoofer's signal optimization problem is presented in the next section.

We only need to consider the spoofer's optimization problem when he opts to trade. Formally, the spoofer chooses $z_1$ to maximize his expected profit, given his signal, strategies of the anticipatory trader, the informed trader, and the market maker.

$$\max_{z_1} E[z_1(p_1 - p_2)|z_2, I(.), A(.), P(.)] \tag{3.23}$$

Inserting (3.11), (3.12), (3.6), (3.7) into (3.23) yielding

$$\max_{z_1} E[z_1(\lambda_1(m - z_1) - \lambda_{12}(m - z_1) - \lambda_{22}(x + u_2 - m + z_1))|z_2, I(.), A(.), P(.)] \tag{3.24}$$

By our assumption, $z_1 = \beta_2 z_2$ and $z_2$ is mutually independent of $u_1, u_2, x$. Therefore, $E[z_1 x] = 0, E[z_1 u_2] = 0, E[z_1 m] = \beta_2 z_2^2$. Substituting this expression into (3.24), we obtain

$$\max_{\beta_2} \beta_2(\lambda_1 + \lambda_{22} - \lambda_{12})(\beta_1 - \beta_2)z_2^2 \tag{3.25}$$

Taking the first-order-condition (FOC) results in the solution as follows:

$$\beta_2 = \frac{1}{2}\beta_1 \tag{3.26}$$

The second-order condition for the spoofer's problem is the same as the order anticipation HFT's problem $\lambda_1 + \lambda_{22} - \lambda_{12} > 0$.

### 3.5.3 Market maker's problem

From the equilibrium definition, the market maker sees the aggregate order flow in each period and sets the prices efficiently. However, these aggregate order flows vary according to the spoofer and the anticipatory's strategies. There are 4 possibilities:

1. *Both traders choose to trade.* By combining equations (3.9),(3.10), (3.11), (3.12), and (3.26), we can obtain the total net order flow $y_1$ and $y_2$ executed at t = 1 and t = 2 as follows:

$$y_1 = -\frac{1}{2}\beta_1 z_2 + \beta_1(\tilde{i} + z_2) + u_1 = \beta_1\tilde{i} + \frac{1}{2}\beta_1 z_2 + u_1 \tag{3.27}$$

$$y_2 = x - \beta_1\tilde{i} - \frac{1}{2}\beta_1 z_2 + u_2 \tag{3.28}$$

2. *Only the anticipatory trader trades.* In this case, the real order $z_1$ and spoofing order $z_2$ of the spoofer are both zero. By inserting the equation (3.9) into (3.13), (3.14), we arrive at the aggregate order flows

$$y_1 = \beta_1\tilde{i} + u_1 \tag{3.29}$$

$$y_2 = x - \beta_1\tilde{i} + u_2 \tag{3.30}$$

3. *Only the spoofer trader trades.* As the anticipatory opts out of the trade, his trading volume is zero. By using the equations (3.26) into (3.15), (3.16), we arrive at the aggregate order flows

$$y_1 = -\frac{1}{2}\beta_1 z_2 + u_1 = -\frac{1}{2}\beta_1 z_2 + u_1 \tag{3.31}$$

$$y_2 = x + \frac{1}{2}\beta_1 z_2 + u_2 \tag{3.32}$$

4. *If both traders do not trade.* In the first period, there are only noise traders. Therefore $p_1 = p_0 = 0$. In the second period, the economy collapses into the standard Kyle model.

We can rewrite these aggregate order flows under different circumstances into a unified general form

$$y_1 = a_1\tilde{i} + t + u_1 \tag{3.33}$$

$$y_2 = a_2 x - a_1\epsilon - t + u_2 \tag{3.34}$$

Where we denote $a_1 = k\beta_1, a_2 = 1 - a_1$ and $t$ is independent of $u_1, u_2, v, \epsilon$. $k$ and $t$ for each case are as follows:

$$(k,t) = \begin{cases} (1, \frac{1}{2}\beta_1 z_2) & \text{Both trade} \\ (1,0) & \text{Only the anticipatory trades} \\ (0, -\frac{1}{2}\beta_1 z_2) & \text{Only the spoofer trades} \\ (0,0) & \text{Both do not trade} \end{cases}$$

In the first period, the market maker can observe the order flow $y_1$ and set the price $p_1 = E[v|y_1]$. By using the project theorem, we can derive $\lambda_1$

$$\lambda_1 = \frac{Cov(v, y_1)}{Var(y_1)} = \frac{a_1\beta\sigma^2}{a_1^2(\beta^2\sigma^2 + \sigma_\epsilon^2) + \frac{1}{4}\beta_1^2\sigma_{z_2}^2 + \sigma_u^2} \tag{3.35}$$

Similarly, the market maker see the aggregate order flow $y_1, y_2$ in the second period and set the price $p_2 = E[v|y_1, y_2]$. By combining equations (3.32), (3.33), (3.7) and applying the projection theorem, we obtain:

$$\begin{aligned} \lambda_{12} &= \frac{Cov(y_1, v)Var(y_2) - Cov(y_1, y_2)Cov(v, y_2)}{Var(y_1)Var(y_2) - Cov^2(y_1, y_2)} \\ &= \frac{\sigma^2\beta(a_1^2\sigma_\epsilon^2 + \frac{1}{4}\beta_1^2\sigma_{z_2}^2 + a_1\sigma_u^2)}{\sigma_u^2(2a_1^2\sigma_\epsilon^2 + \frac{1}{2}\beta_1^2\sigma_{z_2}^2 + \sigma_u^2) + \sigma^2\beta^2(a_1^2\sigma_\epsilon^2 + \frac{1}{4}\beta_1^2\sigma_{z_2}^2 + (a_1^2 + a_2^2)\sigma_u^2)} \end{aligned} \tag{3.36}$$

$$\begin{aligned} \lambda_{22} &= \frac{Cov(y_2, v)Var(y_1) - Cov(y_1, y_2)Cov(v, y_1)}{Var(y_1)Var(y_2) - Cov^2(y_1, y_2)} \\ &= \frac{\sigma^2\beta(a_1^2\sigma_\epsilon^2 + \frac{1}{4}\beta_1^2\sigma_{z_2}^2 + a_2\sigma_u^2)}{\sigma_u^2(2a_1^2\sigma_\epsilon^2 + \frac{1}{2}\beta_1^2\sigma_{z_2}^2 + \sigma_u^2) + \sigma^2\beta^2(a_1^2\sigma_\epsilon^2 + \frac{1}{4}\beta_1^2\sigma_{z_2}^2 + (a_1^2 + a_2^2)\sigma_u^2)} \end{aligned} \tag{3.37}$$

### 3.5.4 Informed trader's problem

Unlike the anticipatory trader and the spoofer, the informed trader is a slow trader who can only trade in the second period. He submits his order in the first period, and the order only arrives in the exchange in the second period. The latency allows the anticipatory trader to use pattern recognition algorithms to detect the informed trader. Even though our model only allows market orders, we can interpret the latency under the limit order settings in the following way. The informed trader may have submitted a big limit order, and it will take time until the order gets executed. During that time, by using algorithms, the anticipatory trader can detect informed trading intentions.

Based on his signal about the true value of the assets $v$, The informed trader chooses $x^*$ to maximize his expected profit. Using equations (3.33), (3.34), we can obtain his expected profit.

$$E[x(v - p_2)|v, I(.), A(.), P(.)] = x(v - (a_1\lambda_{12} + a_2\lambda_{22})x) \tag{3.38}$$

Taking the first-order-condition (FOC) results in the solution as follows:

$$x = \frac{1}{2(a_1\lambda_{12} + a_2\lambda_{22})}v \tag{3.39}$$

The second order condition is $2(a_1\lambda_{12} + a_2\lambda_{22}) > 0$. Combining with the conjectured strategy, we have

$$\beta = \frac{1}{2(a_1\lambda_{12} + a_2\lambda_{22})} \tag{3.40}$$

Inserting the equations (3.37), (3.36) into (3.40) to obtain:

$$\beta^2 = \frac{\sigma_u^2(2a_1^2\sigma_\epsilon^2 + \frac{1}{2}\beta_1^2\sigma_{z_2}^2 + \sigma_u^2)}{\sigma^2(a_1^2\sigma_\epsilon^2 + \frac{1}{4}\beta_1^2\sigma_{z_2}^2 + (a_1^2 + a_2^2)\sigma_u^2)} \tag{3.41}$$

### 3.5.5 Equilibrium Characterization and Properties

We denote $\theta_\epsilon = \frac{\sigma_\epsilon^2}{\sigma_u^2}$ and $\theta_{z_2} = \frac{\sigma_{z_2}^2}{\sigma_u^2}$.From the above analysis, there are 4 possibilities depending on the strategies of the spoofer and the anticipatory trader. The following proposition formally specifies a linear equilibrium when both traders opt to trade.

**Proposition 3.5.1.** *In the economy where both the spoofer and the anticipatory trader choose to trade, there exists a unique linear strategy equilibrium. The equilibrium is characterized by a tuple of* $(\lambda_1, \lambda_{12}, \lambda_{22}, \beta, \beta_1, \beta_2)$ *through the system of equations:*

$$\lambda_{12} = \frac{\sigma^2\beta(a_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + a_1)}{2\sigma_u^2(2a_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)} \tag{3.42}$$

$$\lambda_{22} = \frac{\sigma^2\beta(a_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + a_2)}{2\sigma_u^2(2a_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)} \tag{3.43}$$

$$\lambda_1 = \frac{a_1\beta\sigma^2}{a_1^2(\beta^2\sigma^2 + \sigma_\epsilon^2) + \frac{1}{4}\beta_1^2\sigma_{z_2}^2 + \sigma_u^2} \tag{3.44}$$

$$\beta_2 = \frac{1}{2}\beta_1 \tag{3.45}$$

$$\beta^2 = \frac{\sigma_u^2(2a_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)}{\sigma^2(a_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + (a_1^2 + a_2^2))} \tag{3.46}$$

$$\beta_1 = \frac{\lambda_{22}}{2(\lambda_1 + \lambda_{22} - \lambda_{12})}\frac{\beta^2\sigma^2}{\beta^2\sigma^2 + \sigma_\epsilon^2 + \sigma_{z_2}^2} \tag{3.47}$$

*Where* $a_1 = \beta_1, a_2 = 1 - a_1$. *Thus, the profit of each trader is given by*

$$e[\pi_A^{AS}] = E[m(p_2 - p_1)] = (\lambda_1^{AS} + \lambda_{22}^{AS} - \lambda_{12}^{AS})(\beta_1^{AS})^2(\frac{3}{2}\sigma_{z_2}^2 + (\beta^{AS})^2\sigma^2 + \sigma_\epsilon^2) \tag{3.48}$$

$$E[\pi_S^{AS}] = E[z_1(p_1 - p_2)] = \frac{\lambda_1^{AS} + \lambda_{22}^{AS} - \lambda_{12}^{AS}}{4}(\beta_1^{AS})^2\sigma_{z_2}^2 \tag{3.49}$$

$$E[\pi_I^{AS}] = E[x(v - p_2)] = \frac{\beta^{AS}}{2}\sigma^2 \tag{3.50}$$

In the cases of both traders trading, the second order condition from the optimization problem indicates that $\lambda_1^{AS} + \lambda_{22}^{AS} - \lambda_{12}^{AS} > 0$. Therefore, $E[\pi_A^{AS}] > 0$ and $E[\pi_A^{AS}] > 0$. In other words, in the economy where both the spoofer and the anticipatory trader choose to trade, both of them make positive profits.

Proposition 3.5.1 reveals that in equilibrium, the spoofer only uses part of the signal that he sent to the anticipatory trader. The anticipatory trader loses money from trading against the spoofer but makes a positive profit from anticipating the informed order. As the anticipatory trader can deduce the informed order, he protects himself from the spoofer by reducing the trading intensity when there is the spoofer who adds more noise to the anticipatory trader's signal. On average, the anticipatory trader still makes a positive profit as the loss from the spoofer is compensated by profit from exploiting the informed trader.

Now, we consider the economy where only the anticipatory trader trades. Similarly, the equilibrium of this economy is characterized by a tuple of $(\lambda_1, \lambda_{12}, \lambda_{22}, \beta, \beta_1, \beta_2)$. As the spoofer opts out of the market, his real and spoofing orders are zero. Therefore, we do not need to consider the spoofer problem or $\beta_2 = 0$. The following proposition formally specifies a linear strategy equilibrium of this economy.

**Proposition 3.5.2.** *In an economy where only the anticipatory trader trades, there exists a unique linear strategy equilibrium. The equilibrium is characterized by a tuple of* $(\lambda_1, \lambda_{12}, \lambda_{22}, \beta, \beta_1, \beta_2)$ *through the system of equations:*

$$\lambda_{12} = \frac{\sigma^2 \beta (a_1^2 \theta_\epsilon + a_1)}{2\sigma_u^2 (2a_1^2 \theta_\epsilon + 1)} \tag{3.51}$$

$$\lambda_{22} = \frac{\sigma^2 \beta (a_1^2 \theta_\epsilon + a_2)}{2\sigma_u^2 (2a_1^2 \theta_\epsilon + 1)} \tag{3.52}$$

$$\lambda_1 = \frac{a_1 \sigma^2 \beta (a_1^2 \theta_\epsilon + (a_1^2 + a_2^2))}{a_1^2 \sigma_u^2 (2a_1^2 \theta_\epsilon + 1) + \sigma_u^2 (a_1^2 \theta_\epsilon + 1)(a_1^2 \theta_\epsilon + (a_1^2 + a_2^2))} \tag{3.53}$$

$$\beta_2 = 0 \tag{3.54}$$

$$\beta^2 = \frac{\sigma_u^2 (2a_1^2 \theta_\epsilon + 1)}{\sigma^2 (a_1^2 \theta_\epsilon + (a_1^2 + a_2^2))} \tag{3.55}$$

$$\beta_1 = \frac{\lambda_{22}}{2(\lambda_1 + \lambda_{22} - \lambda_{12})} \frac{\beta^2\sigma^2}{\beta^2\sigma^2 + \sigma_\epsilon^2} \tag{3.56}$$

*Where* $a_1 = \beta_1, a_2 = 1 - a_1$. *Thus, the profit of each trader is given by*

$$E[\pi_A^{A0}] = E[m(p_2 - p_1)] = (\lambda_1 + \lambda_{22} - \lambda_{12})(\beta_1^{A0})^2((\beta^{A0})^2\sigma^2 + \sigma_\epsilon^2) \tag{3.57}$$

$$E[\pi_S^{A0}] = 0 \tag{3.58}$$

$$E[\pi_I^{A0}] = E[x(v - p_2)] = \frac{\beta^{A0}}{2}\sigma^2 \tag{3.59}$$

The result is immediate using the proof of the proposition 3.5.1. The proposition 3.5.2 is the special case of the the proposition 3.5.1 with $\theta_{z_2} = 0$. Similarly, the expected profit $E[\pi_A^{A0}]$ of the anticipatory trader is positive.

If the order anticipation HFT does not participate in the market, his trading volume is zero or $\beta_1 = 0$. However, the spoofer still expects the HFT's order. We denote $\tilde{\beta_1}$ the trading intensity of the HFT under the spoofer's belief. As a result of the false anticipation of HFT's strategy, the spoofer loses money. The following proposition formally specifies a linear strategy equilibrium of this economy.

**Proposition 3.5.3.** *When only spoofer opts to trade, there exists a unique linear strategy equilibrium* $(\lambda_1, \lambda_{12}, \lambda_{22}, \beta, \beta_1, \beta_2)$ *specified by the system of equations:*

$$\lambda_{12} = \frac{\sigma^2\beta\frac{1}{4}\tilde{\beta_1}^2\theta_{z_2}}{2\sigma_u^2(\frac{1}{2}\tilde{\beta_1}^2\theta_{z_2} + 1)} \tag{3.60}$$

$$\lambda_{22} = \frac{\sigma^2\beta(\frac{1}{4}\tilde{\beta_1}^2\theta_{z_2} + 1)}{2\sigma_u^2(\frac{1}{2}\tilde{\beta_1}^2\theta_{z_2} + 1)} \tag{3.61}$$

$$\lambda_1 = 0 \tag{3.62}$$

$$\beta_2 = \frac{1}{2}\tilde{\beta_1} \tag{3.63}$$

$$\beta^2 = \frac{\sigma_u^2(\frac{1}{2}\tilde{\beta_1}^2\theta_{z_2} + 1)}{\sigma^2(\frac{1}{4}\tilde{\beta_1}^2\theta_{z_2} + 1)} \tag{3.64}$$

$$\tilde{\beta}_1 = \frac{\lambda_{22}}{2(\lambda_1 + \lambda_{22} - \lambda_{12})} \frac{\beta^2\sigma^2}{\beta^2\sigma^2 + \sigma_\epsilon^2 + \sigma_{z_2}^2} \tag{3.65}$$

$$\beta_1 = 0 \tag{3.66}$$

*Thus, the profit of each trader is given by*

$$E[\pi_A^{0S}] = 0 \tag{3.67}$$

$$E[\pi_S^{0S}] = E[z_1(p_1 - p_2)] = -\frac{\lambda_1 + \lambda_{22} - \lambda_{12}}{4}(\tilde{\beta}_1{}^{0S})^2\sigma_{z_2}^2 \tag{3.68}$$

$$E[\pi_I^{0S}] = E[x(v - p_2)] = \beta^{0S}\sigma^2 \tag{3.69}$$

When there is only the spoofer, all orders in the first period are uninformed. Therefore, the market maker sets $\lambda_1 = 0$. However, in the second period, he adjusts the price impact $\lambda_{12}$ of the first-period order flow as he learns that order flows of two periods are correlated. As there is no anticipatory trader to be preyed upon, the spoofer suffers from a loss to the informed trader.

**Proposition 3.5.4.** *There exists a unique linear pure strategy equilibrium in which both the spoofer and the anticipatory trader use pure strategies and make positive profits.*

The payoff matrix for both players is presented in table 3.2. Obviously, "Trade" is the dominant strategy for the anticipatory trader as he makes zero profit if he chooses not to trade. When the anticipatory trader plays "trade," the optimal strategy for the spoofer is to trade. Therefore, in equilibrium, both traders opt to trade and make positive profits. Order anticipation strategies are profitable against traditional orders entered by big players. But with spoofers in the mix, the game looks quite different. When the order anticipation HFT wants to jump ahead of the spoofer, the HFT falls prey to the spoofer and loses money. In short, spoofing poses the risk of making order anticipation strategies unprofitable. However, spoofing is only profitable if order anticipation algorithms are active. When the anticipatory traders choose not to trade, the spoofer gets fooled by his own strategy and loses money. Zaloom [2003] documented the incidence in which the spoofer falls prey to his own strategy and gets fooled by other traders. " Traders learned to identify a spoofer by watching changes in the aggregate number of bids or offers on the screen, creating a novel strategy for profit. By riding the tail of a spoofer, a small trader could make money in the market direction. Traders who dealt in large contract sizes aspired

Table 3.2: Payoff matrix for anticipatory trader and spoofer.

| | | Spoofer: Trade | Spoofer: No trade |
|---|---|---|---|
| **Anticipatory trader** | Trade | $(E[\pi_A^{AS}], E[\pi_S^{AS}])$ | $(E[\pi_A^{A0}], 0)$ |
| | No trade | $(0, E[\pi_S^{0S}])$ | $(0, 0)$ |

to "take out" the Spoofer by calling his bluff, selling into his bid, and waiting for him to balk. There was great symbolic capital attached to "taking out" a spoofer by matching wits with this high-risk player. Taking out the Spoofer showed the prowess of a trader in one-to-one combat". The spoofer and the anticipatory trader are two sides of the same coin; the existence of one keeps the other in check.

**Proposition 3.5.5.** *In equilibrium, the optimal intensities of the anticipatory trader and the informed trader decrease with $\theta_{z_2}$ and $\theta_\epsilon$. Mathematically,*

$$\frac{\partial \beta}{\partial \theta_{z_2}} < 0, \frac{\partial \beta}{\partial \theta_\epsilon} < 0 \tag{3.70}$$

$$\frac{\partial \beta_1}{\partial \theta_{z_2}} < 0, \frac{\partial \beta_1}{\partial \theta_\epsilon} < 0 \tag{3.71}$$

The proposition 3.5.5 shows that the anticipatory trader strategically responds to the spoofer by reducing his participation when the spoofer increases spoofing intensity. When $\theta_{z_2}$ is higher, the signal of the anticipation HFT becomes noisier. Therefore, he becomes less active in the market. This result is consistent with the testimony of Mr.May in the second section. Surprisingly, spoofing affects the informed trader unfavorably. However, the informed trader is less responsive to spoofing than the anticipatory trader. In other words, spoofing only has indirect effects on the informed trader's strategy as a result of changes in other traders' strategies. This argument is clearly illustrated in Section 8.

## 3.6 The spoofer's signal optimization problem

In the previous section, $z_2$ is treated as a given random variable. In this section, the spoofer is allowed to optimally choose $z_2$ to maximize his ex-ante expected profit. $z_2$ is characterized by the variance $\sigma_{z_2}^2$, which can be interpreted as the spoofing intensity of the spoofer. From the previous section, $\sigma_{z_2}^2$ has a mixed effect on the spoofer's profit. The higher the spoofing intensity, the lower the trading intensity of the anticipatory trader, thus reducing the real trading volume of the spoofer. However, the higher the spoofing intensity, the higher the profit per share. From the equation (3.49), the spoofer's expected profit is given by.

$$E[\pi_S] = E[z_1(p_1 - p_2)] = \frac{\lambda_1 + \lambda_{22} - \lambda_{12}}{4}(\beta_1)^2\sigma_{z_2}^2 \tag{3.72}$$

The spoofer's signal optimization problem is to choose $\sigma_{z_2}^2$ to maximize his expected profit $E[\pi_S]$ subject to constraints (3.42), (3.43), (3.44), (3.45), (3.46), (3.47), and $\lambda_1 + \lambda_{22} - \lambda_{12} > 0$.

**Proposition 3.6.1.** *Given $\sigma^2, \sigma_\epsilon^2, \sigma_u^2$, the spoofer's signal optimization problem has a global maximum.*

From the proof in the Appendix C.5, we can see that not spoofing or $\sigma_z = 0$ is not the optimal solution to the spoofer's signal optimization problem as he makes a zero profit. Too high spoofing variance $\sigma_z$ is also not optimal, as the higher the spoofing intensity is, the lower the anticipatory trader order flow is. When $\sigma_{z_2}$ approaches infinity, $\beta_1$ goes to zero as the order anticipation HFT protects himself by reducing his market participation. As a result, the profit of the spoofer approaches zero.

## 3.7 Market quality

### 3.7.1 Market efficiency

Two important aspects of market efficiency are price accuracy (or price discovery) and the liquidity of the market. First, liquidity is a multi-dimensional concept, with most measures only capturing one of its many aspects. Under the Kyle framework, market liquidity is defined as the inverse of the Kyle lambdas $\lambda_1$, $\lambda_{12}$, $\lambda_{22}$, which are price impacts of trading. Those $\lambda$s measure how much the price moves with one unit of share. The lower the price impact, the deeper and more liquid the market is. Second, price discovery is measured by how much information is incorporated into the price of an asset. More accurate pricing stocks can generate more efficient capital allocations and foster investor's sense of fairness. For Kyle's setting, price discovery is measured by the market maker's forecast error variance.

$$\Sigma_1 = E[(v - p_1)^2] \tag{3.73}$$

$$\Sigma_2 = E[(v - p_2)^2] \tag{3.74}$$

**Proposition 3.7.1.** *In equilibrium, the price impacts of the first period $\lambda_1$ and $\lambda_{12}$ are decreasing in $\sigma_{z_2}$ while the price impact of the second period of the second period is increasing in $\sigma_{z_2}$. Mathematically*

$$\frac{\partial \lambda_1}{\partial \theta_{z_2}} \leq 0, \frac{\partial \lambda_{12}}{\partial \theta_{z_2}} \leq 0 \tag{3.75}$$

$$\frac{\partial \lambda_{22}}{\partial \theta_{z_2}} \geq 0, \tag{3.76}$$

In the first period, price impacts are decreasing in the spoofing intensity. The more the spoofing variance, the lower the price impact. As the spoofer adds more noise to the order anticipation HFT's signal, the anticipatory trader reduces his trading activities in the first period. This makes aggregate order in the first period less informed. Therefore, spoofing leads to a lower price impact in the first period. However, in the second period, spoofing increases the price impact. If we consider each period separately, spoofing has a mixed effect on liquidity. But if we combine them, the best measure is $\lambda_1 + \lambda_{22}$, which is the proxy for the welfare of uninformed traders. We will consider this one in the next section.

**Proposition 3.7.2.** *In equilibrium, the price discovery measure of the first period* $\Sigma_1$ *is increasing in* $\sigma_z$ *while the price discovery measure of the second period* $\Sigma_2$ *is the same for all different models (AS, A0, 0S, Kyle). Mathematically*

$$\frac{\partial \Sigma_1}{\partial \theta_{z_2}} \leq 0 \tag{3.77}$$

$$\Sigma_2 = \frac{\sigma^2}{2} \tag{3.78}$$

In the first period, the market maker's forecast error variance is decreasing in spoofing intensity. As the result of a proposition 3.7.2, we can have $\Sigma_1^{Kyle} \geq \Sigma_1^{AS} \geq \Sigma_1^{A0}$. Compared to the standard Kyle model, both spoofing and order anticipation speed up the price discovery. However, the improvement of price discovery is at the expense of the informed trader in the form of information leakage. In a short duration, spoofing delays price discovery by adding more noise to the order anticipation HFTs' signal, thus reducing information leakage. In the second period, the market maker's forecast error variances are the same across models. The reason is that both the spoofer and anticipatory trader are short-term traders; they tend to close their positions within a short timeframe, and their net positions are zero within two periods.

### 3.7.2 Wealth transfer and market welfare

Trading is a zero-sum game, so if someone has expected profits from the trade, the other has to suffer the loss. To understand how spoofing affects market welfare, we need to study how this practice affects the wealth positions of all market participants and the implications of these effects.

**Proposition 3.7.3.** *In equilibrium, the expected profit of the anticipatory trader and the loss to uninformed traders are decreasing in* $\sigma_z$*. Mathematically*

$$\frac{\partial E[\pi_I]}{\partial \theta_{z_2}} \leq 0 \tag{3.79}$$

$$\frac{\partial E[\pi_U]}{\partial \theta_{z_2}} \geq 0 \tag{3.80}$$

For the uninformed trader, the higher the spoofing intensity, the lower the loss to the uninformed trader. From proposition 3.5.5, when the spoofer increases the spoofing variance, the trading intensities of the anticipatory trader and the informed trader decrease. As a result, uninformed traders are less likely to be exploited by other traders. In other words, uninformed traders indirectly benefit from spoofing.

## 3.8 Model calibration

The main purpose of this section is to simulate the model numerically to ensure that the calibrated model is consistent with our findings. From the previous section, we have proved that the model can be solved numerically if a set of $\sigma^2, \sigma_u^2, \sigma_\epsilon^2, \sigma_{z_2}^2$ is given. We interpret the traded asset as a typical stock in the US stock market. Specifically, we choose SPY (SPDR S&P 500 ETF Trust), as it tracks the S&P 500 index. Chapter 2 used trading data in February 2018 and estimated $\sigma^2 = 0.00039$ and $\sigma_u^2 = 731,957$. In order to reduce the computation, we convert the unit of $\sigma_u$ to thousand shares, $\sigma_u = 0.8555$. The only remaining parameter $\sigma_\epsilon^2$ which is hard to observe but an important one that determines the nature of the equilibrium. We start our analysis with $\theta_\epsilon = \frac{\sigma_\epsilon^2}{\sigma_u^2} = 0.4$, then we explore the variation in $\theta_\epsilon$ in subsequent analysis. The optimal signal $\sigma_{z_2}^2$ can be recovered from the spoofer's signal optimization problem.

| **Parameter** | $\sigma^2$ | $\sigma_u^2$ | $\sigma_\epsilon^2$ |
|---|---|---|---|
| **Value** | 0.00039 | 0.732 | 0.2927 |
| **Unit** | Dollar squared | Thousand shares squared | Thousand shares squared |

Table 3.3: Parameter values

Using these parameters, we solve 2 separate models numerically. The baseline model is the economy with only the anticipatory trader $\sigma_{z_2} = 0$. The second model is the economy with both the spoofer and the order anticipation HFT. In this model, we allow the spoofer to choose its optimal spoofing strategy. The solutions of the two models are given as follows

| | Baseline (A0) | Spoofer's optimization |
|---|---|---|
| $\sigma_{z_2}$ | 0 | 1.072 |
| $\beta_1$ | 0.3181 | 0.1797 |
| $\beta$ | 57.83 | 51.96 |
| $\lambda_1$ | 0.0083 | 0.00464 |
| $\lambda_{12}$ | 0.0051 | 0.0027 |
| $\lambda_{22}$ | 0.01029 | 0.0111 |

Table 3.4: Solutions to AS and A0 models

The solutions to AS and A0 models are consistent with the findings of Chapter 2. In that paper, we measure the slope of the book by running a simple regression of price against quantity in each snapshot of the limit order book and recover an estimate of $\lambda$. The estimated $\lambda = 0.0000394$/share or 0.0394/thousands shares for SPY. It is in the same magnitude as our calibrated $\lambda_{22}$. Furthermore, Chapter 2 also presents that the average and median volumes of SPY per message are 169.5 shares and 100 shares. The optimal spoofing deviation for the spoofer is 1072 shares, which is over 10 times the median volume per message and 6.3 times the average volume per message. This scale is in line with the spoofing orders recorded in many spoofing cases. For example, in the complaint against Igor B.Oystacher [14], the CFTC presented evidence that Igor B.Oystacher used big orders to give the false sense of market depth. At 8:02:34.360 a.m. on November 30, 2012, he was alleged to have opened a short position of 10 futures contracts in natural gas while placing seven visible orders of 103 contracts. His strategy led to an 11 times increase in the visible market depth. Unsurprisingly, even though the spoofer needs to send a big order to mislead the order anticipation HFT, it is not optimal to send an order that is too big. The optimal variance of the spoofing order, in this case, is about 1.23 times more than the variance of noise trade. The big order exposes the spoofer to the risk of execution and detection by other traders.

In order to help intuition, we study the variation of $\sigma^2_{z_2}$ and its effects on other traders' strategies. We also compare the results with the baseline model above.

Figure 3.1 presents the numerical solutions to different models for the various values of $\sigma_{z_2}$. The green dashed line is the outcome for the economy (AS) with both traders and the spoofer maximizing his signal. The red dashed line represents the solutions to the baseline model with only the anticipatory trader. The solid blue line is the outcome of the AS models with various

14. Complaint Case: 1:15-cv-09196 https://www.cftc.gov/sites/default/files/idc/groups/public/@lrenforcementactions/documents/legalpleading/enfigorcomplnt101915.pdf

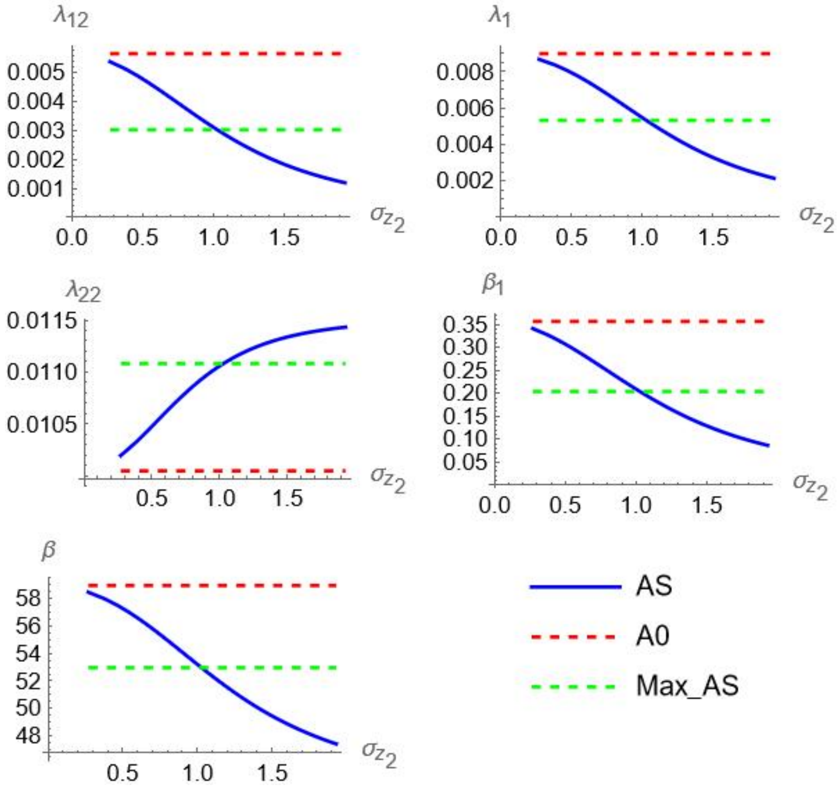


Figure 3.1: Numerical solutions to the models when $\theta\epsilon = 0.4$

values of $\sigma_{z_2}$. Looking across panels of Figure 1, it is obvious that $\lambda_1, \lambda_{12}, \beta_1, \beta$ are decreasing in $\sigma_{z_2}$ and lie below the red line of the baseline model which indicates those values of AS models is less than those of the baseline model. It can be explained that when the spoofer adds more noise to the market, there is more buffer liquidity in the market, thus leading to a decrease in the price impact of the first period. At the same time, the spoofer makes the anticipatory trader's signal less accurate. The order anticipation HFT protects itself by reducing its trading intensity. Contrarily, $\lambda_{22}$ is an increasing function of $\sigma_{z_2}$ and lies above the baseline line. It is notable that $\lambda_{22}, \beta$ are also relatively insensitive to changes in $\sigma_{z_2}$ while $\sigma_{z_2}$ variations affects $\beta_1, \lambda_1, \lambda_{12}$ significantly. This is due to the fact that the spoofer directly influences the strategies of the anticipatory HFT but has indirect effects on the informed trader. In all panels of Figure 3.1, the green dashed lines cross the blue line at the optimal value of $\sigma_{z_2} = 1.072$.

Figure 3.2 shows the ex-ante profits(loss) of all traders in the model. In the case of the spoofer, his profit is a concave function in $\sigma_z$. The green dashed line is tangent to his profit curve at the optimal value of $\sigma_{z_2} = 1.072$. The profits of the informed trader and the anticipatory trader are decreasing in spoofing intensity and lie below the red dashed line. However, the profit of the anticipatory HFT is more sensitive to $\sigma_z$ than that of the informed trader. It is understandable

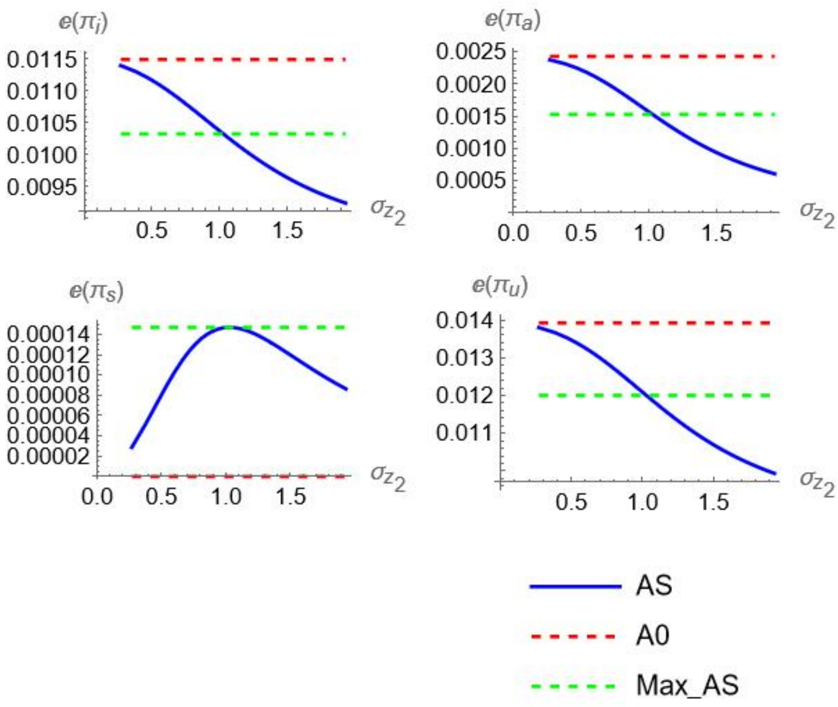


Figure 3.2: Ex-ante profits of traders when $\theta\epsilon = 0.4$

as the order anticipation trader has a noisier signal as the spoofer increases his spoofing intensity. The anticipatory trader reduces his trading intensity rapidly to avoid the loss, thus leading to the sensitivity of his trading strategy to the spoofer's strategy. In the case of the informed trader, he has private information therefore, he is less sensitive to the spoofer's strategy.

Figure 3.3 and figure 3.4 present how the model solution is sensitive to variations in $\theta_\epsilon$. Looking across panels of Figure 3, it is obvious that the dependencies of $\lambda_1, \lambda_{12}.\beta_1, \beta, \lambda_{22}$ on $\theta_\epsilon$ is similar to those dependencies on $\theta_{z_2}$. It is understandable as both $\theta_\epsilon$ and $\theta_{z_2}$ are noises to the anticipatory's trader. The only difference between them is the sources of noise. The optimal spoofing deviations for the spoofer range from above 1000-1700 shares, which are over 10 times the median volume per message. This scale is in line with the spoofing orders recorded in many spoofing cases we presented above. If we examine Figure 3, the profit of the spoofer is decreasing in $\theta_\epsilon$. As $\theta_\epsilon$ increases, the signal of the anticipatory traders becomes noisier. As a result, he scales back his trading participation, thus reducing the profit of the spoofer. In other words, the more accurate the signal of the anticipatory trader, the more easily he can be exploited.

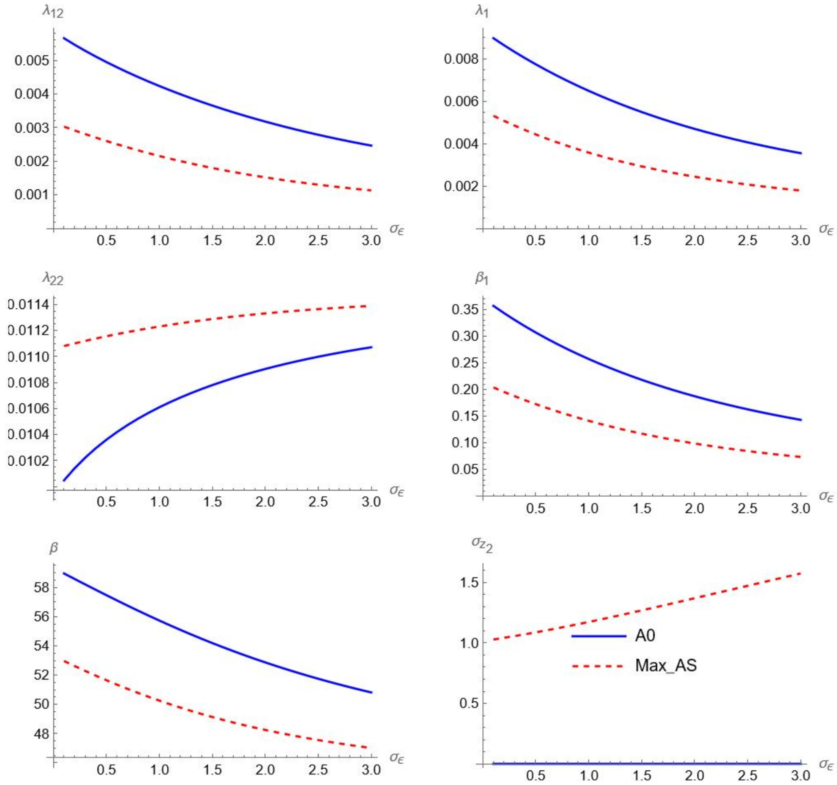


Figure 3.3: Numerical solution for different values of $\theta\epsilon$

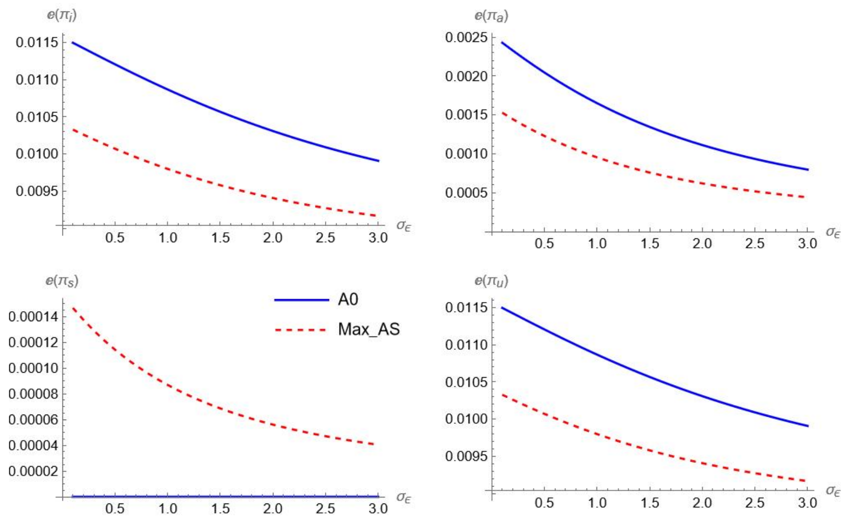


Figure 3.4: Ex-ante profits of traders for different values of $\theta\epsilon$

## 3.9 Policy discussion

There are many reasons that regulators cited to justify the prohibition of spoofing. However, the most common ones are to protect market integrity and fairness. The DOJ[15] has stated that spoofing “poses a significant risk of eroding confidence in U.S. markets” and “ protecting the integrity of our markets remains a significant priority in our fight against economic crime.” James McDonald, the CFTCs Director of Enforcement, also has commented, “Spoofing undermines the integrity of our markets and gives those engaging in the unlawful conduct an unfair advantage over law-abiding market participants”[16]. Even though market integrity is one of the main missions of securities and commodities regulators, it is poorly defined by regulators. Although this approach can give regulators more flexibility to interpret what they perceive as challenges when they arise, the lack of clarifications makes it impossible to assess the progress of securities regulators toward achieving these goals. This also leads to different interpretations of market integrity by market participants. Austin [2017] presents different definitions of what market integrity should encompass. In a narrow sense, the market integrity is often defined as the ability of investors to transact in a fair and informed market where prices reflect information. This definition is close to market efficiency. Austin [2017] also suggests that market integrity and market fairness may be equivalent. Shefrin and Statman [1993] defined market fairness as a claim to 7 entitlements: freedom from coercion, freedom from misrepresentation, equal information, equal processing power, freedom from impulse, efficient prices, and equal bargaining power. Apart from market efficiency, this definition extends to equal access to information and equal information processing. In this paper, we adopt the framework from Fox, Glosten, and Guan [2022] to evaluate spoofing through 3 main aspects: efficiency considerations, wealth transfer from an ex-ante perspective, and fairness considerations.

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15. Press release. U.S. Dept of Justice, Eight Individuals Charged with Deceptive Trading Practices Executed on U.S. Commodities Markets (Jan. 29, 2018) https://www.justice.gov/opa/pr/eight-individuals-charged-deceptive-trading-practices-executed-us-commodities-markets
16. Release Number 7686-18, https://www.cftc.gov/PressRoom/PressReleases/7686-18

### 3.9.1 Efficiency

For price accuracy, we find that in a very short time frame, spoofing delays the price discovery. The main driver of this result is that spoofing makes the anticipatory trader less active in the market, thus delaying the information dissipation in the market. The improvement in price discovery caused by the order anticipation strategies is at the expense of informed traders in the form of information leakage. while order anticipation strategies can foster price discovery in the short term, the widespread order anticipation HFTs will harm price discovery in the long run as they discourage informed traders from researching and finding mispricing assets. Informed traders will have less incentive to create information. With spoofing, the participation of anticipatory traders can be kept in check. In the longer timeframe, both spoofing and order anticipation strategies have a limited effect on price discovery, as both spoofers and order anticipators are short-term traders. Their daily net positions are usually zero.

Contrary to Fox et al. [2022]'s arguments, our above model shows that spoofing has positive impacts on liquidity and market welfare. From our above analysis, with spoofing both anticipatory traders and informed traders reduce their trading intensities. As an indirect effect, uninformed traders suffer less loss, and the market liquidity improves.

Another aspect is to examine how spoofing affects market confidence. Our results are consistent with Fox et al. [2022]'s arguments that spoofing does not decrease the wealth position of ordinary investors, and any additional risk-related spoofing can be diversified away. However, we disagree with their arguments that misperceptions that spoofing occurs may harm ordinary investors can reduce their participation.

### 3.9.2 Wealth transfer

To evaluate spoofing, we need to consider how spoofing affects different members of our society. From the above analysis, spoofing has no or little impact on HFTs that use arbitrage and market-making strategies. For informed traders, they benefit from spoofing as spoofing reduces the participation of anticipatory traders. Uninformed traders also indirectly benefit from spoofing as they suffer less loss. The only victims of spoofing, in this case, are those order anticipation HFTs. The higher the spoofing intensity is, the lower the profit order anticipators can make.

### 3.9.3 Fairness considerations

From our above analysis, legalizing spoofing doesn't harm informed traders, and they actually benefit from spoofing. For informed traders, even though they reduce their trading intensity, they also benefit from less active anticipatory traders.

To study the fairness of spoofing, we examine the arguments put forward by the Department of Justice in the cases against Andre Flotron [17] and B. Oystacher. First, regulators tend to paint HFTs as the innocent targets of spoofing, at least in part, to give the jury a reason to care about the crime it was trying to prove. However, if we examine HFTs' testimony, we find that the only harmful HFTs that use order anticipation strategies are vulnerable to spoofing. Other good HFTs using market-making, arbitrage, and news feed strategies are not harmed by spoofing. Order anticipation HFTs are sophisticated traders with pattern recognition algorithms. They only fall victim to spoofing as their algorithms get detected. Therefore, the argument that HFTs are the innocent targets of spoofing seems far from the truth. Second, a spoofer is claimed to have conducted fraudulent misrepresentation of the price. According to regulators, a spoofer fraudulently induces other traders into filling its real orders using the spoof order. For example, if an HFT trades against a spoofer, the terms of that transaction are fully and accurately disclosed in the market. No one forces the HFT to trade, and the transactions are executed with

17. Court Docket No.:3:17-cr-00220-JAM, https://www.justice.gov/criminal/criminal-vns/united-states-v-andre-flotron

the exact terms disclosed. Furthermore, the limit order book is the second order information. There is no law to require traders to fully disclose their trade intentions. One of these examples is iceberg orders are legal in most jurisdictions. The responsibility for that error should come with anticipatory algorithms, not spoofers.

## 3.10 Conclusion

In recent years, spoofing has become a main target of manipulative crackdowns by regulators. This paper provides a two-period model of strategic interactions between a spoofer and an anticipatory trader who employs pattern recognition algorithms to predict the incoming order. Detecting this strategy, the spoofer submits a spoofing order to mislead the anticipatory trader about the incoming order. The order anticipation HFT protects itself by reducing its market participation. A pure strategy spoofing equilibrium exists and both traders make positive profits.

We show that while spoofing delays price discovery in a short horizon, price dislocation will be so brief as to have little economic efficiency implications. Moreover, spoofing improves market liquidity and market welfare. By studying different recounts of traders, we find that spoofing resemblance practice has existed for centuries. The introduction of electronic trading systems has altered regulators' ethical judgments of spoofing. Furthermore, we study different legal cases on spoofing and find that the main victim of spoofing is order anticipation HFTs. They get exploited because their algorithms are detected and easy to get tricked by spoofers.

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# Appendices

## A Appendix for chapter 1

### A.1 Cross-section estimated $\beta$s

Table 6 represents the daily average values of the parameters $\beta_{bid}$ , $\beta_{ask}$, $\ln\alpha_{bid}$, $\ln\alpha_{ask}$, $t_{\beta_{bid}}$ , $t_{\beta_{ask}}$ ,$t_{\ln\alpha_{bid}}$ ,$t_{\ln\alpha_{ask}}$, $R^2_{bid}$ ,$R^2_{ask}$ . The result comes from estimating the regression (1.23) both ask and bid sides snapshot by snapshot for all stocks across 19 trading days, then take average cross snapshot and trading days.

| Tickers | $\beta_{bid}$ | $\beta_{ask}$ | $\ln\alpha_{bid}$ | $\ln\alpha_{ask}$ | $t_{\beta_{bid}}$ | $t_{\beta_{ask}}$ | $t_{\ln\alpha_{bid}}$ | $t_{\ln\alpha_{ask}}$ | $R^2_{bid}$ | $R^2_{ask}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| ACN | 1.35 | 1.46 | -9.93 | -10.71 | -24.58 | -28.51 | 27.31 | 31.01 | 0.92 | 0.94 |
| ADT | 1.18 | 1.15 | -11.84 | -11.66 | -20.63 | -22.98 | 19.75 | 22.09 | 0.89 | 0.90 |
| AEM | 1.68 | 1.69 | -14.34 | -14.25 | -28.19 | -33.35 | 29.28 | 35.33 | 0.93 | 0.95 |
| AGD | 1.38 | 1.80 | -11.96 | -15.70 | -6.40 | -6.04 | 6.32 | 6.01 | 0.74 | 0.74 |
| AME | 1.44 | 1.24 | -10.99 | -9.87 | -23.42 | -20.12 | 24.16 | 20.37 | 0.94 | 0.91 |
| AMZN | 1.10 | 1.14 | -7.98 | -8.45 | -162.57 | -46.26 | 268.62 | 71.91 | 0.95 | 0.83 |
| APDN | 0.96 | 0.79 | -11.39 | -9.42 | -11.15 | -24.87 | 9.23 | 21.13 | 0.72 | 0.91 |
| APH | 1.32 | 1.37 | -9.91 | -10.56 | -23.30 | -25.53 | 24.80 | 26.21 | 0.94 | 0.94 |
| AWR | 1.42 | 1.38 | -10.46 | -10.21 | -19.74 | -18.83 | 21.56 | 20.24 | 0.91 | 0.90 |
| AWX | 1.08 | 0.93 | -9.21 | -7.67 | -5.82 | -6.84 | 4.66 | 5.44 | 0.63 | 0.66 |
| BBW | 0.85 | 0.80 | -8.32 | -7.83 | -8.57 | -8.56 | 8.00 | 8.02 | 0.77 | 0.75 |
| BDR | 1.92 | 0.57 | -19.97 | -6.82 | -8.93 | -18.40 | 6.80 | 11.57 | 0.81 | 0.87 |
| BIOC | 1.60 | 0.71 | -20.84 | -11.29 | -10.21 | -20.55 | 8.53 | 15.34 | 0.87 | 0.93 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| BLCM | 0.79 | 0.97 | -8.57 | -9.85 | -33.00 | -26.60 | 31.56 | 25.19 | 0.92 | 0.91 |
| CDXC | 0.86 | 0.79 | -9.49 | -8.72 | -26.74 | -36.74 | 24.83 | 35.28 | 0.90 | 0.94 |
| CEI | 1.26 | 1.26 | -88.30 | -18.96 | -4.77 | -11.41 | 4.20 | 8.71 | 0.87 | 0.93 |
| CERN | 1.49 | 1.04 | -12.84 | -9.01 | -43.88 | -38.29 | 47.20 | 45.34 | 0.94 | 0.92 |
| CL | 1.55 | 1.50 | -13.13 | -12.52 | -35.16 | -37.66 | 35.24 | 38.70 | 0.95 | 0.96 |
| CLWT | 0.84 | 0.46 | -7.86 | -4.53 | -11.74 | -16.38 | 10.77 | 15.04 | 0.70 | 0.73 |
| CLX | 1.16 | 1.35 | -8.59 | -9.87 | -28.65 | -29.80 | 33.12 | 33.63 | 0.92 | 0.94 |
| COGT | 1.04 | 1.00 | -12.46 | -11.70 | -8.39 | -11.25 | 7.62 | 10.73 | 0.78 | 0.79 |
| DG | 1.36 | 1.37 | -10.57 | -10.56 | -33.29 | -34.10 | 36.61 | 36.21 | 0.94 | 0.94 |
| DGLY | 0.72 | 0.53 | -7.98 | -5.84 | -10.47 | -14.64 | 8.86 | 12.48 | 0.81 | 0.85 |
| DLPN | 1.15 | 0.32 | -9.81 | -2.99 | -5.22 | -5.74 | 4.40 | 4.72 | 0.47 | 0.48 |
| DVN | 1.68 | 2.00 | -16.15 | -18.84 | -38.20 | -31.00 | 37.54 | 31.01 | 0.94 | 0.92 |
| EXPR | 1.36 | 1.37 | -14.20 | -14.30 | -15.70 | -16.51 | 14.38 | 15.31 | 0.87 | 0.88 |
| EYES | 1.05 | 0.97 | -13.13 | -11.77 | -21.78 | -29.22 | 19.39 | 25.91 | 0.87 | 0.92 |
| FAT | 1.03 | 0.77 | -8.47 | -6.46 | -10.63 | -18.79 | 10.22 | 20.14 | 0.78 | 0.91 |
| FB | 1.08 | 1.05 | -11.28 | -11.02 | -120.28 | -116.98 | 148.24 | 144.01 | 0.93 | 0.94 |
| FNGD | 0.97 | 0.94 | -4.50 | -4.41 | -6.00 | -5.61 | 8.82 | 8.45 | 0.90 | 0.90 |
| FTFT | 1.08 | 0.71 | -11.28 | -7.50 | -15.18 | -20.83 | 13.36 | 19.04 | 0.84 | 0.88 |
| GBR | 0.94 | 1.26 | -9.13 | -12.73 | -13.59 | -16.38 | 10.56 | 13.41 | 0.85 | 0.86 |
| GME | 1.35 | 1.46 | -14.01 | -14.70 | -28.39 | -49.62 | 26.87 | 49.97 | 0.92 | 0.96 |
| GOOGL | 1.03 | 1.11 | -6.55 | -7.28 | -90.99 | -65.36 | 155.83 | 103.44 | 0.95 | 0.94 |
| GORO | 1.12 | 1.09 | -11.40 | -11.39 | -16.53 | -16.49 | 14.63 | 14.84 | 0.88 | 0.86 |
| GSAT | 1.31 | 1.37 | -16.78 | -17.54 | -7.01 | -11.47 | 5.92 | 10.03 | 0.66 | 0.76 |
| HRL | 1.58 | 1.64 | -14.87 | -15.28 | -41.63 | -38.69 | 41.37 | 39.09 | 0.96 | 0.95 |
| INTU | 1.58 | 1.53 | -12.32 | -11.97 | -51.59 | -37.65 | 60.61 | 42.49 | 0.95 | 0.94 |
| ISIG | 0.82 | 0.81 | -9.61 | -8.83 | -10.95 | -18.95 | 8.48 | 15.85 | 0.74 | 0.86 |
| JAKK | 0.90 | 0.66 | -10.77 | -7.80 | -8.24 | -11.16 | 7.03 | 9.84 | 0.83 | 0.86 |
| JNJ | 1.63 | 1.59 | -13.91 | -13.45 | -44.26 | -46.99 | 49.05 | 52.08 | 0.94 | 0.95 |
| JPM | 1.82 | 1.86 | -17.06 | -17.33 | -44.02 | -28.40 | 46.82 | 28.79 | 0.94 | 0.89 |
| JRJC | 1.00 | 0.87 | -10.83 | -9.61 | -16.34 | -24.67 | 14.23 | 22.14 | 0.83 | 0.90 |
| MA | 1.57 | 1.37 | -11.80 | -10.22 | -39.14 | -29.13 | 44.11 | 31.28 | 0.95 | 0.92 |
| MARA | 1.03 | 0.73 | -11.41 | -8.72 | -23.11 | -31.50 | 20.73 | 28.54 | 0.89 | 0.91 |
| MCO | 1.24 | 1.31 | -8.25 | -8.98 | -21.92 | -22.59 | 26.10 | 25.53 | 0.93 | 0.93 |
| MMP | 1.10 | 1.03 | -8.71 | -7.38 | -17.75 | -27.53 | 20.54 | 34.43 | 0.85 | 0.93 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| MS | 1.91 | 1.86 | -18.18 | -17.85 | -42.20 | -44.49 | 42.84 | 44.82 | 0.95 | 0.95 |
| MTG | 1.79 | 1.67 | -18.59 | -17.49 | -16.49 | -23.09 | 15.61 | 22.20 | 0.87 | 0.91 |
| MTR | 0.84 | 0.52 | -5.91 | -3.28 | -5.16 | -6.21 | 5.44 | 7.26 | 0.56 | 0.57 |
| NAKD | 2.01 | 1.47 | -21.54 | -17.18 | -33.37 | -55.86 | 27.57 | 46.06 | 1.00 | 1.00 |
| NBRV | 0.92 | 0.76 | -9.13 | -7.19 | -18.18 | -20.62 | 16.69 | 20.03 | 0.87 | 0.90 |
| NCTY | 0.97 | 0.91 | -13.10 | -11.85 | -19.81 | -22.21 | 15.98 | 18.50 | 0.91 | 0.88 |
| NNDM | 1.07 | 1.53 | -10.38 | -14.11 | -12.82 | -10.23 | 11.04 | 8.82 | 0.85 | 0.82 |
| NRP | 0.87 | 0.94 | -6.17 | -6.94 | -6.30 | -8.61 | 7.38 | 10.03 | 0.77 | 0.82 |
| NURO | 1.08 | 0.81 | -12.22 | -9.36 | -13.31 | -31.00 | 10.91 | 26.49 | 0.82 | 0.92 |
| NVDA | 1.00 | 0.92 | -9.41 | -8.77 | -165.61 | -64.30 | 224.05 | 84.01 | 0.95 | 0.88 |
| NVFY | 0.85 | 0.58 | -9.41 | -6.99 | -16.53 | -37.30 | 13.66 | 32.38 | 0.85 | 0.92 |
| OTEX | 1.92 | 1.88 | -17.63 | -16.96 | -28.01 | -31.01 | 28.76 | 32.13 | 0.91 | 0.94 |
| PSA | 1.19 | 1.21 | -7.78 | -7.92 | -23.05 | -27.73 | 28.32 | 32.77 | 0.92 | 0.93 |
| PZZA | 1.54 | 1.30 | -12.56 | -10.57 | -36.57 | -43.57 | 39.59 | 50.93 | 0.93 | 0.94 |
| RHE | 2.77 | 0.89 | -30.71 | -11.25 | -11.48 | -11.61 | 6.47 | 7.07 | 0.86 | 0.86 |
| RIOT | 0.92 | 0.94 | -9.45 | -9.65 | -43.23 | -49.38 | 45.88 | 52.72 | 0.91 | 0.92 |
| RSG | 1.40 | 1.45 | -11.02 | -11.70 | -28.25 | -28.12 | 29.44 | 28.24 | 0.95 | 0.94 |
| SAFE | 1.29 | 1.19 | -10.58 | -9.60 | -12.96 | -12.12 | 12.77 | 12.09 | 0.89 | 0.87 |
| SCKT | 1.63 | 1.09 | -16.51 | -11.03 | -13.76 | -13.96 | 12.72 | 12.96 | 0.79 | 0.82 |
| SGOC | 0.75 | 0.58 | -9.00 | -6.98 | -17.87 | -17.31 | 13.81 | 13.66 | 0.84 | 0.82 |
| SIEB | 1.00 | 0.93 | -8.79 | -8.32 | -11.74 | -12.50 | 11.46 | 12.45 | 0.81 | 0.82 |
| SLGN | 1.49 | 1.49 | -12.98 | -12.92 | -26.70 | -28.97 | 25.66 | 28.18 | 0.94 | 0.94 |
| SLP | 0.92 | 0.74 | -8.31 | -7.07 | -14.14 | -15.03 | 14.42 | 15.05 | 0.84 | 0.84 |
| SNOA | 1.11 | 0.66 | -10.57 | -6.22 | -11.25 | -12.48 | 9.85 | 12.07 | 0.81 | 0.83 |
| SOGO | 1.11 | 1.08 | -12.06 | -11.30 | -25.36 | -38.49 | 23.87 | 38.40 | 0.90 | 0.93 |
| SPCB | 0.93 | 0.68 | -9.31 | -6.86 | -11.46 | -15.48 | 9.70 | 13.94 | 0.79 | 0.86 |
| SPY | 2.52 | 2.42 | -26.46 | -25.43 | -51.18 | -25.91 | 55.77 | 27.29 | 0.86 | 0.75 |
| SQBG | 1.00 | 1.02 | -12.32 | -12.19 | -16.31 | -27.31 | 13.82 | 24.05 | 0.84 | 0.92 |
| THS | 1.28 | 1.37 | -10.27 | -10.91 | -22.62 | -20.85 | 22.83 | 20.97 | 0.94 | 0.93 |
| TNK | 1.10 | 1.14 | -14.60 | -14.78 | -9.11 | -14.29 | 7.61 | 12.46 | 0.74 | 0.84 |
| TSLA | 0.97 | 0.92 | -7.74 | -7.58 | -109.67 | -82.45 | 159.57 | 118.92 | 0.95 | 0.92 |
| TSM | 1.86 | 2.00 | -17.69 | -18.85 | -53.91 | -34.93 | 56.16 | 35.64 | 0.96 | 0.94 |
| USAU | 0.94 | 0.73 | -11.18 | -8.29 | -33.31 | -28.91 | 30.03 | 25.61 | 0.92 | 0.91 |
| V | 1.34 | 1.25 | -11.67 | -11.16 | -62.03 | -42.46 | 71.73 | 47.07 | 0.95 | 0.93 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Average | 1.26 | 1.14 | -12.92 | -10.80 | -28.66 | -27.38 | 32.47 | 28.91 | 0.87 | 0.88 |
| Median | 1.12 | 1.09 | -11.18 | -10.22 | -19.81 | -24.87 | 19.75 | 24.05 | 0.89 | 0.91 |

Table 6: Daily average cross-sectional results

### A.2 Threshold $M_1$

Table 7 presents the minimum thresholds the trader needs to trade to break even. $M_1$, $M_1^{sf}$ are the thresholds for the baseline model and the self-financing model. "Ex. msg", "A. vol", "MP", "M/Ex. msg", "M/A.vol" are the daily average number of executed messages, the average order size, the minimum threshold in dollars, the ratio between $M1$ and the daily average number of executed messages, the ratio between $M1$ and the average order size

| Tickers | $M_1$ | $M_1^{sf}$ | Ex. msg | A. vol | $MP$ | M/ Ex. msg | M/A. vol |
|---|---|---|---|---|---|---|---|
| ACN | 582.5 | 639.0 | 4214.7 | 63.5 | 92548.4 | 0.1 | 9.2 |
| ADT | 1794.8 | 1920.8 | 1648.2 | 143.2 | 21674.9 | 1.1 | 12.5 |
| AEM | 962.7 | 1033.4 | 2516.8 | 86.1 | 41276.7 | 0.4 | 11.2 |
| AGD | 2681.5 | 2696.7 | 27.7 | 101.5 | 29248.8 | 96.9 | 26.4 |
| AME | 636.9 | 703.8 | 2898.8 | 70.7 | 48207.1 | 0.2 | 9.0 |
| AMZN | 3703.8 | 4354.0 | 47931.3 | 38.3 | 5357025.2 | 0.1 | 96.6 |
| APH | 553.6 | 608.7 | 2612.1 | 65.5 | 49503.1 | 0.2 | 8.5 |
| AWR | 712.3 | 732.7 | 533.7 | 62.1 | 38059.9 | 1.3 | 11.5 |
| CEI | 113154.3 | 113342.2 | 374.6 | 985.7 | 12419.1 | 302.1 | 114.8 |
| CERN | 1311.6 | 1648.7 | 7848.6 | 86.3 | 83761.2 | 0.2 | 15.2 |
| CL | 694.7 | 800.1 | 5531.3 | 86.2 | 48916.0 | 0.1 | 8.1 |
| CLX | 511.9 | 563.2 | 3482.6 | 61.4 | 66464.6 | 0.1 | 8.3 |
| COGT | 238484.1 | 240383.4 | 280.3 | 254.4 | 767430.2 | 850.8 | 937.6 |
| DG | 617.0 | 676.4 | 5383.8 | 66.7 | 60013.3 | 0.1 | 9.2 |
| DVN | 2217.1 | 2444.7 | 12576.0 | 106.5 | 77511.8 | 0.2 | 20.8 |
| EXPR | 2734.0 | 2834.0 | 1235.3 | 119.4 | 19211.7 | 2.2 | 22.9 |
| FB | 12020.1 | 16390.5 | 60643.4 | 101.8 | 2171755.7 | 0.2 | 118.0 |
| GBR | 5549.8 | 5560.9 | 37.7 | 179.0 | 8354.1 | 147.3 | 31.0 |
| GME | 2211.6 | 2381.0 | 3347.4 | 116.4 | 35395.8 | 0.7 | 19.0 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| GOOGL | 1611.6 | 1846.9 | 25574.8 | 30.2 | 1761867.4 | 0.1 | 53.3 |
| GORO | 3717.5 | 3788.5 | 301.6 | 119.3 | 16368.4 | 12.3 | 31.2 |
| GSAT | 19965.2 | 20135.9 | 1133.7 | 327.1 | 18916.8 | 17.6 | 61.0 |
| HRL | 1428.4 | 1598.3 | 4595.9 | 94.5 | 47361.6 | 0.3 | 15.1 |
| INTU | 986.4 | 1083.8 | 7946.5 | 67.3 | 163393.5 | 0.1 | 14.7 |
| JNJ | 1120.9 | 1309.2 | 12605.8 | 82.0 | 147597.4 | 0.1 | 13.7 |
| JPM | 1997.7 | 2542.0 | 26781.5 | 98.8 | 228220.7 | 0.1 | 20.2 |
| MA | 606.1 | 680.9 | 9923.1 | 65.4 | 104173.3 | 0.1 | 9.3 |
| MCO | 561.1 | 595.1 | 2760.7 | 54.4 | 91537.0 | 0.2 | 10.3 |
| MMP | 2341.1 | 2414.5 | 1062.6 | 74.2 | 156668.6 | 2.2 | 31.5 |
| MS | 2306.1 | 2703.1 | 14756.2 | 109.5 | 126713.8 | 0.2 | 21.1 |
| MTG | 4157.5 | 4423.7 | 2154.4 | 130.1 | 59606.2 | 1.9 | 31.9 |
| NAKD | 25625.4 | 25676.3 | 51.1 | 232.6 | 35927.6 | 501.4 | 110.2 |
| NNDM | 3706.4 | 3713.0 | 98.8 | 294.8 | 9861.5 | 37.5 | 12.6 |
| OTEX | 1648.3 | 1777.6 | 2668.7 | 94.3 | 58334.7 | 0.6 | 17.5 |
| PSA | 495.1 | 530.9 | 3380.7 | 50.0 | 93841.4 | 0.1 | 9.9 |
| PZZA | 801.8 | 862.1 | 4188.5 | 81.8 | 46653.8 | 0.2 | 9.8 |
| RSG | 543.0 | 611.2 | 4000.6 | 80.4 | 35455.7 | 0.1 | 6.8 |
| SAFE | 1040.5 | 1056.5 | 193.6 | 85.9 | 17482.3 | 5.4 | 12.1 |
| SCKT | 4355.8 | 4399.5 | 88.4 | 307.7 | 16971.4 | 49.3 | 14.2 |
| SLGN | 819.7 | 884.3 | 1932.3 | 91.9 | 23473.1 | 0.4 | 8.9 |
| SOGO | 2520.4 | 3540.2 | 937.9 | 178.3 | 23689.4 | 2.7 | 14.1 |
| SPY | 20000.0 | 13096.5 | 124741.7 | 169.5 | 5414125.0 | 0.2 | 118.0 |
| SQBG | 25329.9 | 35724.2 | 262.8 | 176.8 | 44388.7 | 96.4 | 143.3 |
| THS | 584.9 | 664.0 | 2312.7 | 84.2 | 24127.4 | 0.3 | 6.9 |
| TNK | 17529.1 | 25822.5 | 219.1 | 356.9 | 20670.0 | 80.0 | 49.1 |
| TSM | 1717.3 | 2373.0 | 6555.7 | 116.7 | 74187.3 | 0.3 | 14.7 |
| V | 1291.9 | 1681.0 | 14818.5 | 77.0 | 155975.9 | 0.1 | 16.8 |
| **Average** | 11488.2 | 12026.6 | 6552.7 | 170.3 | 222794.7 | 27.3 | 29.2 |
| **Median** | 1717.3 | 1920.8 | 2760.7 | 94.3 | 48916.0 | 0.3 | 15.1 |

Table 7: The minimum required trading volume $M_1$

# B Appendix for Chapter 2

## B.1 Proofs and derivations

Proof of Proposition 2.3.1. Notation:

| | |
|---|---|
| $V$ | underlying value of asset |
| $x$ | informed trader's trade $= \beta V$ |
| $u$ | "noise" trade |
| $y$ | total order flow $x + u$ |
| $\beta$ | informed trader's trading intensity with solution $\frac{1}{2\lambda}$ |

*Proof.*

$$\begin{aligned} \text{var}(y) &= \text{var}(x+u) = \text{var}(x) + \text{var}(u) \\ &= \beta^2\Sigma + \sigma^2 = \frac{1}{4\lambda^2}\Sigma + \sigma^2 \\ &= \frac{4\sigma^2}{4\Sigma}\Sigma + \sigma^2 = 2\sigma^2 \end{aligned}$$

Also,

$$\text{var}(P) = \text{var}(\lambda y) = \lambda^2 2\sigma^2 = \frac{\Sigma}{4\sigma^2}2\sigma^2 = \frac{1}{2}\Sigma$$

$$\rightarrow \sqrt{\frac{\text{var(executed price)}}{\text{var(executed volume)}}} = \sqrt{\frac{\frac{1}{2}\Sigma}{2\sigma^2}} = \frac{1}{2}\frac{\sqrt{\Sigma}}{\sigma} = \lambda$$

□

## B.2 Cross-section estimated $\lambda$s

Table 7 presents the estimated values for $\lambda$ across different stocks. $\lambda_{bid}$ , $\lambda_{ask}$, P. V , Vol. V, A. Vol, M. Vol, Ex. rate are the estimated $\lambda$ for the bid side, the estimated $\lambda$ for the ask side, price variance, volume variance, average volume, median volume, and execution rate, respectively. The execution rate is calculated by the ratio between the total number of executed messages and the total number of messages.

| Ticker | $\lambda_{bid}$ | $\lambda_{ask}$ | P. V | Vol. V | A. Vol | M. Vol | Ex. rate |
|---|---|---|---|---|---|---|---|
| APDN | 4.30E-05 | 2.60E-05 | 5.76E-08 | 7.72E+02 | 200 | 100 | 4% |
| ADT | 1.36E-04 | 9.47E-05 | 3.18E-06 | 7.70E+03 | 143 | 100 | 3% |
| AGD | 4.32E-04 | 4.54E-04 | 4.06E-07 | 6.71E+00 | 101 | 100 | 3% |
| AWX | 3.08E-04 | 2.78E-04 | 3.07E-08 | 2.22E+01 | 189 | 100 | 1% |
| AWR | 2.84E-03 | 1.87E-03 | 2.40E-05 | 9.89E+01 | 62 | 65 | 4% |
| BBW | 1.23E-04 | 1.18E-04 | 2.15E-06 | 2.75E+02 | 102 | 100 | 2% |
| BDR | 4.87E-05 | 3.11E-05 | 2.82E-08 | 2.08E+02 | 269 | 100 | 1% |
| HRL | 1.99E-04 | 2.58E-04 | 6.47E-06 | 1.51E+03 | 95 | 100 | 3% |
| AEM | 8.60E-04 | 7.12E-04 | 1.63E-05 | 1.11E+03 | 86 | 100 | 1% |
| PZZA | 1.87E-03 | 6.82E-04 | 8.13E-05 | 4.66E+03 | 82 | 100 | 5% |
| BIOC | 4.36E-06 | 6.49E-07 | 5.30E-08 | 2.42E+05 | 742 | 200 | 7% |
| BLCM | 4.10E-05 | 5.64E-05 | 2.83E-06 | 5.44E+03 | 132 | 100 | 5% |
| CDXC | 3.35E-05 | 1.96E-05 | 1.68E-06 | 4.56E+03 | 156 | 100 | 5% |
| NVFY | 1.02E-04 | 3.03E-05 | 3.56E-07 | 8.72E+02 | 174 | 100 | 5% |
| CEI | 3.64E-06 | 4.40E-07 | 2.46E-08 | 8.16E+04 | 986 | 200 | 4% |
| CLX | 1.82E-03 | 1.80E-03 | 1.13E-04 | 7.09E+02 | 61 | 54.5 | 7% |
| COGT | 2.95E-05 | 2.36E-05 | 1.31E-06 | 2.75E+03 | 254 | 100 | 5% |
| DGLY | 2.31E-04 | 5.38E-05 | 1.21E-06 | 6.91E+02 | 237 | 100 | 4% |
| DLPN | 2.51E-04 | 7.59E-05 | 1.10E-06 | 4.97E-01 | 89 | 100 | 1% |
| EYES | 1.47E-05 | 1.18E-05 | 4.46E-07 | 1.05E+04 | 318 | 100 | 1% |
| FAT | 4.94E-04 | 2.12E-04 | 1.86E-06 | 1.23E+02 | 88 | 55 | 2% |
| FTFT | 1.18E-04 | 5.54E-05 | 4.22E-06 | 2.44E+03 | 186 | 100 | 9% |
| GBR | 1.18E-04 | 1.03E-04 | 4.96E-07 | 6.05E+02 | 179 | 100 | 4% |
| GORO | 6.42E-05 | 5.94E-05 | 4.89E-07 | 5.01E+02 | 119 | 100 | 2% |
| GSAT | 5.66E-06 | 3.61E-06 | 1.42E-07 | 4.11E+04 | 327 | 100 | 3% |
| ISIG | 3.67E-05 | 4.12E-05 | 5.83E-08 | 1.25E+02 | 209 | 100 | 3% |
| JAKK | 1.39E-04 | 2.52E-05 | 2.51E-07 | 2.12E+03 | 280 | 100 | 2% |
| DVN | 4.06E-04 | 4.36E-04 | 2.37E-05 | 1.05E+04 | 106 | 100 | 2% |
| JRJC | 5.03E-05 | 4.30E-05 | 4.14E-07 | 1.62E+03 | 305 | 100 | 4% |
| MARA | 8.07E-05 | 1.70E-05 | 1.39E-06 | 4.84E+03 | 217 | 100 | 6% |
| MCO | 2.91E-03 | 2.41E-03 | 2.77E-04 | 2.89E+02 | 54 | 50 | 7% |
| MTG | 9.55E-05 | 6.43E-05 | 1.51E-06 | 3.79E+03 | 130 | 100 | 1% |
| MMP | 9.65E-04 | 1.27E-03 | 3.04E-05 | 3.98E+02 | 74 | 69 | 5% |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| MTR | 1.50E-03 | 1.36E-03 | 4.47E-06 | 3.67E+00 | 106 | 100 | 1% |
| NAKD | 5.14E-05 | 2.68E-05 | 1.80E-07 | 3.61E+02 | 233 | 100 | 3% |
| NBRV | 8.32E-05 | 8.85E-05 | 1.98E-06 | 6.38E+03 | 123 | 100 | 5% |
| NNDM | 1.11E-04 | 1.50E-04 | 1.18E-06 | 3.96E+03 | 295 | 100 | 4% |
| NRP | 1.23E-03 | 1.19E-03 | 9.68E-06 | 4.10E+01 | 123 | 50 | 1% |
| PSA | 2.72E-03 | 4.45E-03 | 2.71E-04 | 3.94E+02 | 50 | 40 | 9% |
| RHE | 1.58E-05 | 9.35E-06 | 4.84E-08 | 1.73E+03 | 490 | 100 | 3% |
| RIOT | 1.80E-04 | 8.71E-05 | 6.71E-05 | 2.09E+04 | 157 | 100 | 14% |
| SAFE | 6.25E-04 | 6.90E-04 | 3.31E-06 | 1.29E+02 | 86 | 100 | 4% |
| SCKT | 1.54E-04 | 1.23E-04 | 6.04E-07 | 1.78E+03 | 308 | 100 | 8% |
| SGOC | 2.63E-05 | 2.89E-05 | 1.16E-07 | 2.26E+02 | 203 | 100 | 2% |
| SIEB | 4.92E-04 | 3.89E-04 | 2.60E-04 | 1.82E+02 | 118 | 100 | 5% |
| SLGN | 3.23E-04 | 3.61E-04 | 5.03E-06 | 1.63E+03 | 92 | 100 | 4% |
| SLP | 4.52E-04 | 1.45E-04 | 5.40E-06 | 1.82E+02 | 86 | 94 | 2% |
| SNOA | 2.67E-04 | 1.07E-04 | 1.72E-06 | 3.38E+02 | 133 | 100 | 3% |
| SOGO | 4.03E-05 | 5.36E-05 | 3.45E-06 | 6.49E+03 | 178 | 100 | 2% |
| SPCB | 9.88E-05 | 6.03E-05 | 4.06E-07 | 8.08E+02 | 158 | 100 | 3% |
| SQBG | 1.17E-05 | 1.23E-05 | 1.78E-07 | 1.94E+03 | 177 | 100 | 5% |
| THS | 6.93E-04 | 8.31E-04 | 2.72E-05 | 9.20E+02 | 84 | 100 | 6% |
| TNK | 4.11E-06 | 3.95E-06 | 9.00E-08 | 5.64E+03 | 357 | 100 | 2% |
| USAU | 3.95E-05 | 4.05E-05 | 4.45E-07 | 2.14E+03 | 286 | 100 | 6% |
| AME | 1.23E-03 | 5.23E-04 | 3.65E-05 | 5.88E+02 | 71 | 100 | 5% |
| EXPR | 5.32E-05 | 5.27E-05 | 1.32E-06 | 9.62E+02 | 119 | 100 | 2% |
| APH | 1.21E-03 | 9.53E-04 | 3.60E-05 | 5.61E+02 | 65 | 88 | 4% |
| ACN | 1.90E-03 | 1.84E-03 | 3.22E-04 | 9.26E+02 | 63 | 69 | 4% |
| DG | 1.36E-03 | 1.24E-03 | 7.97E-05 | 1.15E+03 | 67 | 79 | 4% |
| CERN | 9.12E-04 | 3.41E-04 | 3.99E-05 | 2.64E+03 | 86 | 100 | 5% |
| CL | 1.73E-03 | 1.28E-03 | 2.15E-05 | 1.58E+03 | 86 | 100 | 3% |
| RSG | 8.97E-04 | 6.39E-04 | 1.62E-05 | 1.57E+03 | 80 | 100 | 7% |
| GME | 1.70E-04 | 1.17E-04 | 4.06E-06 | 5.56E+03 | 116 | 100 | 0% |
| INTU | 2.70E-03 | 1.91E-03 | 3.97E-04 | 1.93E+03 | 67 | 70 | 6% |
| MA | 5.14E-03 | 4.16E-03 | 4.87E-04 | 2.30E+03 | 65 | 65 | 5% |
| OTEX | 4.95E-04 | 4.00E-04 | 9.62E-06 | 1.39E+03 | 94 | 100 | 2% |
| FB | 2.83E-04 | 2.27E-04 | 5.12E-04 | 1.41E+05 | 102 | 100 | 7% |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| TSLA | 3.97E-03 | 3.72E-03 | 2.87E-03 | 2.03E+04 | 61 | 40 | 10% |
| TSM | 3.58E-04 | 3.63E-04 | 1.69E-05 | 1.09E+04 | 117 | 100 | 2% |
| V | 1.19E-03 | 7.25E-04 | 2.39E-04 | 6.90E+03 | 77 | 100 | 5% |
| JNJ | 2.12E-03 | 1.44E-03 | 2.02E-04 | 5.90E+03 | 82 | 100 | 4% |
| NAKD | 5.14E-05 | 2.68E-05 | 1.80E-07 | 3.61E+02 | 233 | 100 | 3% |
| FNGD | 1.01E-02 | 8.91E-03 | 1.63E-04 | 1.64E+01 | 177 | 100 | 1% |
| CLWT | 1.79E-04 | 1.11E-04 | 4.38E-07 | 6.39E+01 | 214 | 100 | 2% |
| NCTY | 5.57E-06 | 4.66E-06 | 1.45E-07 | 7.30E+03 | 569 | 200 | 5% |
| NURO | 2.55E-05 | 2.81E-05 | 2.88E-07 | 3.14E+03 | 287 | 100 | 6% |
| MS | 5.73E-04 | 3.39E-04 | 3.63E-05 | 1.72E+04 | 110 | 100 | 2% |
| AMZN | 3.93E-02 | 1.46E-02 | 1.40E-01 | 1.07E+04 | 38 | 15 | 9% |
| GOOGL | 5.01E-02 | 4.40E-02 | 2.45E-02 | 2.82E+03 | 30 | 10 | 3% |
| JPM | 1.01E-03 | 5.77E-04 | 1.68E-04 | 3.93E+04 | 99 | 100 | 2% |
| NVDA | 1.42E-03 | 8.04E-04 | 3.12E-03 | 1.24E+05 | 80 | 54 | 9% |
| SPY | 5.80E-04 | 3.94E-04 | 3.91E-04 | 7.32E+05 | 170 | 100 | 4% |
| Average | 1.87E-03 | 1.36E-03 | 2.13E-03 | 1.99E+04 | 1.71E+02 | 9.47E+01 | 4% |

Table 8: The daily estimated $\lambda$s

### B.3 Details of the message data

#### B.3.1 Data structure

Twelve types of market events are recorded in the data (see Table 6 below). As the limit order book is a primary focus, “cross messages that occurred in a dark pool or an auction are filtered out. When a market order is matched against several limit orders, each matching is recorded separately. Messages labeled as “FILL ASK” and “FILL BID” have missing price and quantity fields. We need to trace back to the original order of the same IDs to figure out the missing pieces.

To reconstruct a limit order book from a raw message file we follow the following procedure.

| Event Type | Description |
|---|---|
| ADD ASK | Submit a new ask order |
| ADD BID | Submit a new bid order |
| CANCEL BID | Cancel the bid order partly |
| CANCEL ASK | Cancel the ask order partly |
| CROSS | Dark pool transactions without price and quantity |
| DELETE ASK | Delete the whole ask order |
| DELETE BID | Delete the whole ask order |
| EXECUTE ASK | Execute the order partly |
| EXECUTE BID | Execute the order partly |
| FILL ASK | Fill the ask order completely |
| FILL BID | Fill the bid order completely |
| TRADE BID | Fill the bid order completely |

Table 9: Even Type in the message file

1. **Step 1**. Eliminate abnormal messages that aren't with the active region. As we observed, messages with a price above or under 1.5 times the average price were normally not within the active region. We filtered out those messages out of the sample. We also handle missing data from "FILL ASK" and "FILL BID" as mentioned above.
2. **Step 2** Construct a first snapshot with only the first order book event.
3. **Step 3** Iterate over all new events to construct all snapshots and store them in an array. The newly constructed limit order book snapshot has a full depth with price and volume at all levels. For "ADD ASK" and "ASK BID" message types, a new snapshot is updated by adding those new messages to the previous snapshot. For the "CANCEL BID", "CANCEL ASK", "EXECUTE ASK", and "EXECUTE BID" message types, the order ID and exchange of the message are matched against the orders in the previous snapshot to look for the outstanding order that should be updated by reducing its order size. For the "DELETE BID", "DELETE ASK", "FILL ASK", and "FILL BID" message types, the corresponding orders get processed completely. Therefore, the new snapshot is constructed by deleting all orders with the same IDs of incoming messages. At any time, there are only "ADD ASK" and "ADD BID" messages outstanding in a snapshot. Upon the creation of a snapshot, ask and bid order types are separated, sorted, and grouped by price. The final step is to filter out abnormal entries and then create the cumulative depths at each price level.

To illustrate the above procedure, we assume that the initial snapshot of a stock (ACN) has 2 outstanding orders as follows.

| Timestamp | OrderNumber | EventType | Ticker | Price | Quantity | Exchange |
|---|---|---|---|---|---|---|
| 00:00.0 | 120 | ADD BID | ACN | 60.02 | 50 | ARCA |
| 00:00.0 | 129 | ADD ASK | ACN | 68 | 4 | ARCA |

In the next period, a new "ADD BID" message of 40 shares arrives at the price of 61, the snapshot will be updated by adding the new message to the new snapshot.

| Timestamp | OrderNumber | EventType | Ticker | Price | Quantity | Exchange |
|---|---|---|---|---|---|---|
| 00:00.0 | 120 | ADD BID | ACN | 60.02 | 50 | ARCA |
| 00:00.0 | 129 | ADD ASK | ACN | 174.7 | 4 | ARCA |
| 00:00.0 | 138 | ADD BID | ACN | 62 | 40 | ARCA |

Right after, the trader of order ID 120 wants to reduce his order size, so he submits a 'CANCEL BID" message of 20 shares. The new snapshot will updated by reducing his outstanding order size by 20 shares.

| Timestamp | OrderNumber | EventType | Ticker | Price | Quantity | Exchange |
|---|---|---|---|---|---|---|
| 00:00.0 | 120 | ADD BID | ACN | 60.02 | 30 | ARCA |
| 00:00.0 | 129 | ADD ASK | ACN | 174.7 | 4 | ARCA |
| 00:00.0 | 138 | ADD BID | ACN | 62 | 40 | ARCA |

### B.3.2 Sample covariance matrices

For the univariate model, we use quadratic variations for volume and price variance to capture all market variances. For multi-asset models, each asset has a different execution pattern and time frame. In order to calculate the quadratic covariance matrices for price and volume, we divide the trading into a uniform grid in time $t_0, ..., t_n$ with a timescale $t_k - t_{k-1} = 600$ seconds. In this way, price and volume changes of all assets have the same dimensions. The price at $t_k$ ($p_{t_k}$)is defined as the executed price of the closet execution order before $t_k$. The volume at $t_k$ ($v_{t_k}$)is defined as the cumulative volume between $t_{k-1}$ and $t_k$.

### B.3.3 Direct price impact

We assume that between 2 executed orders at times $t_0 = 0, t_{n+1} = T$, there are n limit order book events at $0 < t_1 < ... < t_n < T$ with the volume $\Delta V_{t_i}$. The price change between 0 and T can be defined as

$$\Delta p = p_T - p_0 = \sum_{i=1}^{n+1}(p_{t_i} - p_{t_{i-1}}) = \sum_{i=1}^{n+1}\Delta p_{t_i} \tag{81}$$

Where $p_{t_i}$ is the contribution components of events $i$ to the direct price impact.

In the case of no execution order, we can interpret the $\Delta p_{t_i}$ as the change of the shadow price or the change in the fundamental values of the asset because of the arrival of the limit order event at $t_i$. If the event at $t_i$ is an executed order, we have $\Delta p_{t_i} = \Delta V_{t_i}\lambda$. If the event at $t_i$ is not an executed order, we define $\Delta p_{t_i} = \alpha_{t_i}\Delta V_{t_i}\lambda$. The reason for this definition is that an $\Delta p_{t_i}$ increasing function of $\Delta V_{t_i}$. For example, a market maker should react more strongly to the big order at the top of the book. Second, we can justify this assumption by considering $\alpha_{t_i}$ as a function of the probability of execution. Another way is to interpret $\alpha_{t_i}$ as a discount factor of the information content of the order. If we substitute those equations into the equation (81) and obtain.

$$\frac{p_T - p_0}{\Delta V_T} = \sum_{i=1}^{n}\alpha_{t_i}\frac{\Delta V_{t_i}}{V_T}\lambda + \lambda = (\sum_{i=1}^{n}\alpha_{t_i}\frac{\Delta V_{t_i}}{V_T} + 1)\lambda \tag{82}$$

If we define $a_{t_i} = \alpha_{t_i}\frac{\Delta V_{t_i}}{V_T}$, we can rearrange and arrive at the following expression.

$$\frac{\frac{p_T - p_0}{\Delta V_T}}{\lambda} = (\sum_{i=1}^{n} a_{t_i} + 1) \tag{83}$$

The absolute value of the right-hand side of the above equation is greater than 1 if $\sum_{i=1}^{n} a_{t_i} \geq 0$. If we assume the limit order book events are symmetric. For any limit order book event type on the ask side, there is a corresponding type on the bid side. For example, “add bids” and “add ask” are corresponding pairs. The effects of these 2 corresponding orders on the price impact are exactly opposite. If the assumption that the limit order events are symmetric holds, on average $a_i = -a_{-i}$ for all $(i, -i)$ which are corresponding event types. In other words, the average of $\frac{\frac{p_T - p_0}{\Delta V_T}}{\lambda} = 1$. Therefore, the effects of all limit order events converge to the direct price impact for a sufficiently long time.

### B.4 Converting CAPM returns to prices

The paper Boulatov and Taub [2014] sets out a dynamic version of the Kyle model in which there are multiple stocks, the underlying value of which, and also the prices, can be correlated. There is a completely separate literature on the correlation across stocks, the CAPM, but this is a theory of stock *returns*, not prices. The purpose of this note is to demonstrate that one can compute the correlations of stock prices if one is given the $\beta$s of the stocks, and importantly, also the $\mathbf{R}^2$ attached to the stock by the CAPM structure.

In the CAPM the correlation across stocks is driven entirely by the market return, which they share, as the residuals in the CAPM return equation are inherently mutually independent. The magnitude of the correlation is then determined by the $\beta$s and the $\mathbf{R}^2$s, but the magnitude requires some calculations, which are presented here.

The calculations use an approximation result as a key step, and this approximation result is outlined in Appendix B.5.

## Main derivations

From the CAPM we have the following characterization of the *returns* for asset $i$:

$$R_t^i = r + \beta^i (R_t^M - r) + e_t^i \tag{84}$$

We want to convert this equation into an equation relating the *prices* to the aggregate prices. Begin by taking logs:

$$\ln\left(\frac{P_t^i}{P_{t-1}^i}\right) = r + \beta^i\left(\ln\left(\frac{S_t}{S_{t-1}}\right) - r\right) + e_t^i \tag{85}$$

It is worth noting that even though $e_t^i$ and $e_t^j$ are independent, they don't necessarily have the same variance, that is, we need to keep in mind that $\sigma_{ei}^2 \neq \sigma_{ej}^2$ is possible.

Taking the exponential yields

$$P_t^i = e^{(1-\beta_t^i)r + e_t^i}\left(\frac{S_t}{S_{t-1}}\right)^{\beta^i} P_{t-1}^i \tag{86}$$

The market price itself is a process:

$$\frac{S_t}{S_{t-1}} = \frac{S_{t-1} + dS}{S_{t-1}} = 1 + R^M dt + \sigma^M dZ_t \tag{87}$$

where $dZ_t$ is the systematic risk process. The price level equation becomes

$$P_t^i = e^{(1-\beta_t^i)r + e_t^i}\left(1 + R^M dt + \sigma^M dZ_t\right)^{\beta^i} P_{t-1}^i \tag{88}$$

Expressing this in level terms yields

$$P_t^i = e^{(1-\beta_t^i)r + e_t^i}\left(1 + R^M + \sigma^M \zeta_t\right)^{\beta^i} P_{t-1}^i \tag{89}$$

where $\zeta_t$ is the innovation of the systematic return process. This can now be decomposed into idiosyncratic and systematic parts. Taking logs,

$$\begin{aligned} \ln(P_t^i) &= \ln(P_{t-1}^i) + (1-\beta_t^i)r + e_t^i + \beta^i\left(1 + R^M + \sigma^M\zeta_t\right) \\ &= \ln(P_{t-1}^i) + (1-\beta_t^i)r + \beta^i\left(1 + R^M\right) + e_t^i + \beta^i\sigma^M\zeta_t \end{aligned} \tag{90}$$

However the systematic coefficient is a mixture of the idiosyncratic and systematic shocks.

The next question is how the decomposition of the shocks into idiosyncratic and systematic parts translates into the multi-asset Kyle model. But this has a known answer from the model in Seiler and Taub [2008]. That paper does not decompose the value shocks into idiosyncratic and systematic parts, however it does treat correlation across prices. The main issue however is the fact that equation (90) is in logs, whereas the model is in terms of levels. The correlation structure can however be calculated by using equation (89).

$$\left(P_t^i, P_t^j\right) = \frac{\left(e^{e_t^i}\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}, e^{e_t^j}\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^j}\right)}{\text{var}\left(e^{e_t^i}\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}\right)^{1/2}\text{var}\left(e^{e_t^j}\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^j}\right)^{1/2}} \tag{91}$$

(Note that the terms $P_{t-1}^i$ and $P_{t-1}^j$ cancel in the correlation formula.) Thus, if we can estimate the CAPM elements for a particular stock ticker, and also estimate the systematic $\zeta_t$ process, then we can calculate the correlation and apply using the model from Boulatov and Taub [2014].

Thus, we can develop the correlation simply from the CAPM residuals for the tickers. The variances of the residuals can in turn be calculated from the $R^2$ statistics of the CAPM equations, which are given along with the $\beta^i$ coefficients for each of the tickers. Specifically, we have

$$\text{var}\, R_t^i = \text{var}\left(r+\beta^i(R^M-r)+e^i\right) = \beta_i^2\sigma_M^2+\sigma_{e_i}^2 \rightarrow \mathbf{R}_i^2 = \frac{\beta_i^2\sigma_M^2}{\beta_i^2\sigma_M^2+\sigma_{e_i}^2} \tag{92}$$

Therefore

$$\sigma_{e_i}^2 = \frac{\beta_i^2\sigma_M^2}{\mathbf{R}_i^2} - \beta_i^2\sigma_M^2 = \frac{(1-\mathbf{R}_i^2)}{\mathbf{R}_i^2}\beta_i^2\sigma_M^2 \tag{93}$$

and this can then be used to calculate var $\left(e^{e_t^j}\right)$. (Notice that because the $\beta_i$ and $\mathbf{R}_i^2$ are different across tickers, the expected values can also differ.) Specifically,

$$E\left[e^{e_i}\right] = e^{\frac{1}{2}\sigma_{e_i}^2} \qquad \text{var}\left[e^{e_i}\right] = E\left[e^{2e_i}\right] - \left(E\left[e^{e_i}\right]\right)^2 = \left(e^{\sigma_{e_i}^2}-1\right)e^{\sigma_{e_i}^2}$$

Similarly,

$$\left(e^{e_t^i}, e^{e_t^j}\right) = E\left[e^{e_i+e_j}\right] - \left(E\left[e^{e_i}\right]E\left[e^{e_i}\right]\right) = 0$$

To calculate the covariance we need to calculate the expected value of the product

$$E\left[\left(e^{e_t^i}\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}\right)\left(e^{e_t^j}\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^j}\right)\right]$$

and subtract the product of the expectations. The product of the expectations is direct from the calculations already done. The expected value of the product is

$$\begin{aligned}
&E\left[\left(e^{e_t^i}\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}\right)\left(e^{e_t^j}\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^j}\right)\right]\\
=&E\left[e^{e_t^i}e^{e_t^j}\right]E\left[\left(\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}\right)\left(\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^j}\right)\right]\\
=&E\left[e^{e_t^i}\right]E\left[e^{e_t^j}\right]E\left[\left(\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}\right)\left(\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^j}\right)\right]\\
=&\left[e^{\frac{1}{2}\sigma_{e_i}^2}e^{\frac{1}{2}\sigma_{e_j}^2}\right]E\left[\left(\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}\right)\left(\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^j}\right)\right]\\
\approx&\left[e^{\frac{1}{2}\left(\sigma_{e_i}^2+\sigma_{e_j}^2\right)}\right]e^{(\beta^i+\beta^j)R^M}e^{\frac{1}{2}(\beta^i+\beta^j)^2\sigma_M^2}
\end{aligned}$$

where the first equality follows from the independence of the $e_i$ from $\zeta$, and the second equality comes from the independence of the $e_i$, and finally the approximation result from the appendix is used.

The product of the expectations is more straightforward:

$$\begin{aligned}
&E\left[\left(e^{e_t^i}\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}\right)\right]\\
=&\left[e^{\frac{1}{2}\sigma^2}\right]E\left[\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}\right]\\
\approx&\left[e^{\frac{1}{2}\sigma^2}\right]\left[e^{\beta^iR^M+\frac{1}{2}\beta_i^2\sigma_M^2}\right]
\end{aligned}$$

Yielding the product

$$\begin{aligned}
&\left[e^{\frac{1}{2}\sigma_{e_i}^2}\right]\left[e^{\beta^iR^M+\frac{1}{2}\beta_i^2\sigma_M^2}\right]\left[e^{\frac{1}{2}\sigma_{e_j}^2}\right]\left[e^{\beta^jR^M+\frac{1}{2}\beta_j^2\sigma_M^2}\right]\\
=&\left[e^{\frac{1}{2}\left(\sigma_{e_i}^2+\sigma_{e_j}^2\right)}\right]\left[e^{(\beta_i+\beta_j)R^M+\frac{1}{2}(\beta_i^2+\beta_j^2)\sigma_M^2}\right]
\end{aligned}$$

The covariance is then the difference

$$\begin{aligned}
&\left[e^{\frac{1}{2}\left(\sigma_{e_i}^2+\sigma_{e_j}^2\right)}\right]e^{(\beta^i+\beta^j)R^M}e^{\frac{1}{2}(\beta^i+\beta^j)^2\sigma_M^2}-\left[e^{\frac{1}{2}\left(\sigma_{e_i}^2+\sigma_{e_j}^2\right)}\right]\left[e^{(\beta^i+\beta^j)R^M+\frac{1}{2}(\beta_i^2+\beta_j^2)\sigma_M^2}\right]\\
=&e^{\frac{1}{2}\left(\sigma_{e_i}^2+\sigma_{e_j}^2\right)}e^{(\beta^i+\beta^j)R^M}e^{\frac{1}{2}(\beta^i+\beta^j)^2\sigma_M^2}\left(1-e^{(-\beta_i\beta_j)\sigma_M^2}\right)\\
=&e^{\frac{1}{2}\left(\sigma_{e_i}^2+\sigma_{e_j}^2\right)}e^{(\beta^i+\beta^j)R^M}e^{\frac{1}{2}(\beta_i^2+\beta_j^2)\sigma_M^2}\left(e^{(\beta_i\beta_j)\sigma_M^2}-1\right)
\end{aligned}$$

The variance is not a simple variation on the covariance. The expectation of the product is

$$\begin{aligned}
&E\left[\left(e^{e_t^i}\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}\right)^2\right] \\
=&E\left[e^{2e_t^i}\right]E\left[\left(\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}\right)^2\right] \\
\approx& e^{2\sigma^2_{e_i}}\left[e^{2\beta^i R^M+2\beta_i^2\sigma_M^2}\right]
\end{aligned}$$

The square of the expectation is

$$e^{\sigma^2_{e_i}}e^{2(\beta^i R^M+\frac{1}{2}\beta_i^2\sigma_M^2)}$$

So the variance is the difference

$$\begin{aligned}
&e^{2\sigma^2_{e_i}}e^{2\beta^i R^M+2\beta_i^2\sigma_M^2}-e^{\sigma^2_{e_i}}e^{2(\beta^i R^M+\frac{1}{2}\beta_i^2\sigma_M^2)} \\
=&e^{\sigma^2_{e_i}}e^{2\beta^i R^M}e^{\beta_i^2\sigma_M^2}\left(e^{\sigma^2_{e_i}}e^{\beta_i^2\sigma_M^2}-1\right)
\end{aligned}$$

Thus, there is an interaction between the two variances such that the variance does not cleave into two separate parts.

Combining to form the correlation, and using the approximation of the ratio for the terms involving $\zeta_t$ from the appendix, we have the reduced expression

$$\begin{aligned}
\left(P_t^i,P_t^j\right)=&\frac{e^{\frac{1}{2}\left(\sigma^2_{e_i}+\sigma^2_{e_j}\right)}e^{(\beta^i+\beta^j)R^M}e^{\frac{1}{2}(\beta^i+\beta^j)^2\sigma_M^2}\left(1-e^{(-\beta_i\beta_j)\sigma_M^2}\right)}{\left(e^{\sigma^2_{e_i}}e^{2\beta^i R^M}e^{\beta_i^2\sigma_M^2}\left(e^{\sigma^2_{e_i}}e^{\beta_i^2\sigma_M^2}-1\right)\right)^{1/2}\left(e^{\sigma^2_{e_j}}e^{2\beta^j R^M}e^{\beta_j^2\sigma_M^2}\left(e^{\sigma^2_{e_j}}e^{\beta_j^2\sigma_M^2}-1\right)\right)^{1/2}} \\
=&\frac{e^{(\beta^i+\beta^j)R^M}e^{\frac{1}{2}(\beta^i+\beta^j)^2\sigma_M^2}\left(1-e^{(-\beta_i\beta_j)\sigma_M^2}\right)}{\left(e^{2\beta^i R^M}e^{\beta_i^2\sigma_M^2}\left(e^{\sigma^2_{e_i}}e^{\beta_i^2\sigma_M^2}-1\right)\right)^{1/2}\left(e^{2\beta^j R^M}e^{\beta_j^2\sigma_M^2}\left(e^{\sigma^2_{e_j}}e^{\beta_j^2\sigma_M^2}-1\right)\right)^{1/2}} \\
=&\frac{e^{\frac{1}{2}(\beta^i+\beta^j)^2\sigma_M^2}\left(1-e^{(-\beta_i\beta_j)\sigma_M^2}\right)}{\left(e^{\beta_i^2\sigma_M^2}\left(e^{\sigma^2_{e_i}}e^{\beta_i^2\sigma_M^2}-1\right)\right)^{1/2}\left(e^{\beta_j^2\sigma_M^2}\left(e^{\sigma^2_{e_j}}e^{\beta_j^2\sigma_M^2}-1\right)\right)^{1/2}} \\
=&\frac{e^{\frac{1}{2}(\beta_i^2+\beta_j^2)\sigma_M^2}\left(e^{(\beta_i\beta_j)\sigma_M^2}-1\right)}{\left(e^{\beta_i^2\sigma_M^2}\left(e^{\sigma^2_{e_i}}e^{\beta_i^2\sigma_M^2}-1\right)\right)^{1/2}\left(e^{\beta_j^2\sigma_M^2}\left(e^{\sigma^2_{e_j}}e^{\beta_j^2\sigma_M^2}-1\right)\right)^{1/2}} \\
=&\frac{\left(e^{(\beta_i\beta_j)\sigma_M^2}-1\right)}{\left(e^{\sigma^2_{e_i}}e^{\beta_i^2\sigma_M^2}-1\right)^{1/2}\left(e^{\sigma^2_{e_j}}e^{\beta_j^2\sigma_M^2}-1\right)^{1/2}}=\frac{\left(e^{(\beta_i\beta_j)\sigma_M^2}-1\right)}{\left(e^{\sigma^2_{e_i}+\beta_i^2\sigma_M^2}-1\right)^{1/2}\left(e^{\sigma^2_{e_j}+\beta_j^2\sigma_M^2}-1\right)^{1/2}}
\end{aligned} \tag{94}$$

where the last equality emphasizes that the $\sigma^2_{e_i}$ term is added in the exponent, not multiplied. Evidently the $e_i$ terms reduce the correlation, which is intuitively sensible in that the idiosyncratic error reduces the effect of the systematic risk, equivalent to reducing the $\mathbf{R}^2$.

### B.5 Approximation

We want to compute

$$\text{var}\left[\left(1 + R^M + \sigma^M \zeta_t\right)^{\beta^i}\right]$$

Use an approximation:

$$\begin{aligned} E\left[\left(1 + R^M + \sigma^M \zeta_t\right)^{\beta^i}\right] &= E\left[e^{\beta^i \ln\left(1+R^M+\sigma_M \zeta_t\right)}\right] \\ &\approx E\left[e^{\beta^i \left(R^M+\sigma_M \zeta_t\right)}\right] \\ &= e^{\beta^i R^M + \frac{1}{2}\beta_i^2 \sigma_M^2} \end{aligned}$$

Thus the variance approximation is the expectation of the square minus the squared expectation:

$$e^{2\beta^i R^M + 4\frac{1}{2}\beta_i^2\sigma_M^2} - e^{2\left(\beta^i R^M + \frac{1}{2}\beta_i^2\sigma_M^2\right)} = e^{2\beta^i R^M + \beta_i^2\sigma_M^2}\left(e^{\beta_i^2\sigma_M^2} - 1\right)$$

The covariance approximation calculations will be similar. A reminder that

$$(x, y) = E[xy] - E[x]E[y]$$

Thus,

$$\begin{aligned} E\left[\left(1 + R^M + \sigma^M \zeta_t\right)^{\beta^i}\left(1 + R^M + \sigma^M \zeta_t\right)^{\beta^j}\right] &= E\left[e^{\beta^i \ln\left(1+R^M+\sigma_M \zeta_t\right)} e^{\beta^j \ln\left(1+R^M+\sigma_M \zeta_t\right)}\right] \\ &\approx E\left[e^{\beta^i \left(R^M+\sigma_M \zeta_t\right)} e^{\beta^j \left(R^M+\sigma_M \zeta_t\right)}\right] \\ &= e^{(\beta^i+\beta^j)R^M} E\left[e^{(\beta^i+\beta^j)\sigma_M \zeta_t}\right] \\ &= e^{(\beta^i+\beta^j)R^M} e^{\frac{1}{2}(\beta^i+\beta^j)^2\sigma_M^2} \end{aligned}$$

Thus,

$$\begin{aligned} (x, y) &\approx e^{(\beta^i+\beta^j)R^M} e^{\frac{1}{2}(\beta^i+\beta^j)^2\sigma_M^2} - e^{\beta^i R^M + \frac{1}{2}\beta_i^2\sigma_M^2} e^{\beta^j R^M + \frac{1}{2}\beta_j^2\sigma_M^2} \\ &= e^{(\beta^i+\beta^j)R^M}\left(e^{\frac{1}{2}(\beta^i+\beta^j)^2\sigma_M^2} - e^{\frac{1}{2}\left(\beta_i^2+\beta_j^2\right)\sigma_M^2}\right) \\ &= e^{(\beta^i+\beta^j)R^M} e^{\frac{1}{2}\left(\beta_i^2+\beta_j^2\right)\sigma_M^2}\left(e^{\beta^i\beta^j\sigma_M^2} - 1\right) \end{aligned}$$

The correlation ratio is then

$$\frac{e^{(\beta^i+\beta^j)R^M}e^{\frac{1}{2}\left(\beta_i^2+\beta_j^2\right)\sigma_M^2}\left(e^{\beta^i\beta^j\sigma_M^2}-1\right)}{\left(e^{2\beta^iR^M+\beta_i^2\sigma_M^2}\left(e^{\beta_i^2\sigma_M^2}-1\right)e^{2\beta^jR^M+\beta_j^2\sigma_M^2}\left(e^{\beta_j^2\sigma_M^2}-1\right)\right)^{1/2}}$$

$$=\frac{e^{\frac{1}{2}\left(\beta_i^2+\beta_j^2\right)\sigma_M^2}\left(e^{\beta^i\beta^j\sigma_M^2}-1\right)}{\left(e^{\beta_i^2\sigma_M^2}\left(e^{\beta_i^2\sigma_M^2}-1\right)e^{\beta_j^2\sigma_M^2}\left(e^{\beta_j^2\sigma_M^2}-1\right)\right)^{1/2}}$$

$$=\frac{\left(e^{\beta^i\beta^j\sigma_M^2}-1\right)}{\left(\left(e^{\beta_i^2\sigma_M^2}-1\right)\left(e^{\beta_j^2\sigma_M^2}-1\right)\right)^{1/2}}$$

Notice that this is equal to 1 if $\beta^i=\beta^j$.

A more precise calculation can be carried out using the Taylor series approximation of $\left(1+R^M+\sigma^M\zeta_t\right)^{\beta^i}$.

### B.6 Two-asset correlation model

Bernhardt and Taub [2008a] sets out a static model of a multi-asset Kyle model in which asset values are cross-correlated, exploring how informed speculators with differential information about the spectrum of assets exploit that information in trading correlated assets. Informed speculators use cross-asset information to trade strategically if they can observe prices. If prices are unobserved before trade, they do not use the information.

The purpose of this note is to translate the cross-asset speculation models (CAM) into a slightly simpler setting in which there are just two assets, one of which is the systematic asset (in practice, the S&P 500 index fund, SPY), and the other of which is an ordinary stock with positive correlation driven by CAPM considerations.

### Main derivations

There are $N$ informed traders and $M$ assets. In the basic model of interest, $N = 1$ and $M = 2$. The value of asset 1 is

$$\begin{aligned} v_1 &= v_{11}e_1 \\ v_2 &= v_{21}e_1 + v_{22}e_2 \end{aligned} \tag{95}$$

so

$$v = Ve, \qquad V \equiv \begin{pmatrix} v_{11} & 0 \\ v_{21} & v_{22} \end{pmatrix} \tag{96}$$

Thus, $v_1$ is the systematic asset value, and $v_2$ is the heterogeneous asset; moreover with this interpretation,

$$v_{11} = 1 \tag{97}$$

To maintain the spirit of the basic static Kyle model, we can assume that there is only a single informed speculator, and so the signal structure is

$$\begin{pmatrix} s_1^1 \\ s_2^1 \end{pmatrix} = \begin{pmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{98}$$

Because it is a CAPM-driven model we can assume that the informed trader has full information about the systematic asset, that is,

$$A = \begin{pmatrix} 1 & 0 \\ A_{21}^1 & A_{22}^1 \end{pmatrix} \tag{99}$$

which implies that he has full information about the second asset as well after netting out the systematic part, leaving

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{100}$$

The trading strategy for the informed trader is

$$\begin{aligned} x_1^1 &= b_{11}^1 s_1^1 + b_{12}^1 s_2^1 + B_{11}^1(X_1 + u_1) + B_{12}^1(X_2 + u_2) \\ x_2^1 &= b_{21}^1 s_1^1 + b_{22}^1 s_2^1 + B_{21}^1(X_1 + u_1) + B_{22}^1(X_2 + u_2) \end{aligned} \tag{101}$$

or

$$\begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \begin{pmatrix} b_{11}^1 & b_{12}^1 \\ b_{21}^1 & b_{22}^1 \end{pmatrix} \begin{pmatrix} s_1^1 \\ s_2^1 \end{pmatrix} + \begin{pmatrix} B_{11}^1 & B_{12}^1 \\ B_{21}^1 & B_{22}^1 \end{pmatrix} \begin{pmatrix} X_1 + u_1 \\ X_2 + u_2 \end{pmatrix} \tag{102}$$

The pricing rule is given by

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} X_1 + u_1 \\ X_2 + u_2 \end{pmatrix} \tag{103}$$

Define

$$\Gamma \equiv I + \sum_{k=1}^{N} \gamma^k$$

Recalling that $N$ is the number of informed traders, in the basic Kyle model $N = 1$; here we are assuming two assets, so $M = 2$.

### Equilibrium formulas

Defining the total order flow covariance matrix

$$\Psi \equiv \begin{pmatrix} bA & I \end{pmatrix} \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} A'b' \\ I \end{pmatrix} \tag{104}$$

and skipping to Proposition 2 in Bernhardt and Taub [2008a], we have

$$\begin{aligned} b^i &= A^1 \Sigma_e V'(I + \gamma^i) \\ \gamma^{i'} &= -\Psi^{-1} bA\Sigma_e A^{i'} b^{i'} \\ \Gamma'\lambda' &= \Psi^{-1} bA\Sigma_e V' \end{aligned} \tag{105}$$

where $\Sigma_e$ is the variance-covariance matrix of the fundamentals, and $V$ is the vector of realized asset fundamental values.

Next, we can state Proposition 4, which provides a formula for the direct trading intensities:

$$b \sim (A\Sigma_e A')^{-1/2} \Sigma_u^{1/2} \tag{106}$$

Notice that this reduces to the fundamental static Kyle model formula if there is one trader and one asset,

$$b = \frac{\sigma_u}{\Sigma_0^{1/2}} \tag{107}$$

However, the key measurable quantity is $\lambda$:

$$\lambda = V\Sigma_e A b' \Psi^{-1\prime} \Gamma^{-1} \tag{108}$$

The quantities on the right-hand side need to be related to the observables, namely price and total order flow. First, substituting from (106),

$$\lambda \sim V\Sigma_e A \Sigma_u^{1/2} (A\Sigma_e A')^{-1/2} \Psi^{-1\prime} \Gamma^{-1} \tag{109}$$

(noting the "$\sim$" rather than "$=$").We can also substitute from (106) into (104):

$$\Psi \sim \begin{pmatrix} (A\Sigma_e A')^{-1/2} \Sigma_u^{1/2} A & I \end{pmatrix} \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} A' \Sigma_u^{1/2} (A\Sigma_e A')^{-1/2} \\ I \end{pmatrix} \tag{110}$$

so that

$$\lambda \sim V\Sigma_e A \Sigma_u^{1/2} (A\Sigma_e A')^{-1/2} \left( \begin{pmatrix} (A\Sigma_e A')^{-1/2} \Sigma_u^{1/2} A & I \end{pmatrix} \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} A' \Sigma_u^{1/2} (A\Sigma_e A')^{-1/2} \\ I \end{pmatrix} \right)^{-1} \Gamma^{-1} \tag{111}$$

Also, we can reduce $\Gamma$:

$$\begin{aligned} \Gamma &= I + \sum_{k=1}^{N} \gamma^k = I - \sum_{k=1}^{N} bA\Sigma_e A^{i\prime} b^{i\prime} \Psi^{-1} \\ &= I - \sum_{k=1}^{N} (A\Sigma_e A')^{-1/2} \Sigma_u^{1/2} A \Sigma_e A^{i\prime} b^{i\prime} \left( \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_u \end{pmatrix} \begin{pmatrix} A' \Sigma_u^{1/2} (A\Sigma_e A')^{-1/2} \\ I \end{pmatrix} \right)^{-1} \end{aligned} \tag{112}$$

where $b^i$ has been left unreduced.

The covariance matrix of prices is as follows:

$$\lambda \Gamma \Psi \Gamma' \lambda' \tag{113}$$

Because $\Psi$ is the covariance matrix of total order flow, we can, in principle, recover $\lambda\Gamma$ by factoring the observed $\Psi$ and also calculating the price covariance matrix. The tricky part is $\Gamma$. From Proposition 7, the covariance matrix of total order flow is $\Sigma_u \Gamma'$ [See proof of Proposition 7, p. 41 of Bernhardt and Taub [2008a].]

### Multiple-asset trading

The logic of the cross-asset paper Bernhardt and Taub [2008a] presupposes an environment in which there are just a few relevant stocks. However, there are thousands of stocks, and to capture the appropriate $\lambda\Gamma$ for a single stock $i$, we would need to add up the cross-asset $\lambda_{ij}\Gamma$ for all stocks $j$. What makes more sense is to treat SPY as the cross-asset ticker and isolate the effect of SPY on the $\lambda\Gamma$ of each of the smaller stocks.

By subtracting the influence of SPY, we can isolate the trade on private information unique to each ticker. This information, and also its value, can then be added up.

# C Appendix for Chapter 3

## C.1 Proof of Proposition 3.5.1

Combining the equations (3.44) and (3.46), we can arrive at the following expression:

$$\lambda_1 = \frac{a_1\sigma^2\beta(a_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + (a_1^2 + a_2^2))}{a_1^2\sigma_u^2(2a_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1) + \sigma_u^2(a_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + 1)(a_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + (a_1^2 + a_2^2))} \tag{114}$$

Inserting this expression and equations (3.42) and (3.43) and (3.46) into (3.47), we can obtain the following equation.

$$F(\beta_1, \theta_\epsilon, \theta_{z_2}, k) = 0 \tag{115}$$

Given $(\theta_\epsilon, \theta_{z_2}, k)$, the function $F(\beta_1, \theta_\epsilon, \theta_{z_2}, k)$ is a $6^{th}$ polynomial defined as follows:

$$
\begin{aligned}
F(\beta_1, \theta_\epsilon, \theta_{z_2}, k) = {} & 32 + \beta_1(-64 - 64k - 64\theta_\epsilon - 64\theta_{z_2}) + \beta_1^2(64k + 96k^2 + 192k\theta_\epsilon + 128k^2\theta_\epsilon + \\
& 32\theta_{z_2} + 192k\theta_{z_2}) + \beta_1^3(-128k^2 - 576k^2\theta_\epsilon - 192k^3\theta_\epsilon - 128k^2\theta_\epsilon^2 - 48\theta_{z_2} - \\
& 48k\theta_{z_2} - 384k^2\theta_{z_2} - 32\theta_\epsilon\theta_{z_2} - 128k^2\theta_\epsilon\theta_{z_2} - 32\theta_{z_2}^2) + \beta_1^4(384k^3\theta_\epsilon + 320k^4\theta_\epsilon + \\
& 64k^3\theta_\epsilon^2 + 160k^4\theta_\epsilon^2 + 80k^2\theta_{z_2} + 384k^3\theta_{z_2} + 16k\theta_\epsilon\theta_{z_2} + 80k^2\theta_\epsilon\theta_{z_2} + 64k^3\theta_\epsilon\theta_{z_2} + \\
& 10\theta_{z_2}^2 + 16k\theta_{z_2}^2) + \beta_1^5(-512k^4\theta_\epsilon - 128k^4\theta_\epsilon^2 - 128k^5\theta_\epsilon^2 - 64k^4\theta_\epsilon^3 - 64k^2\theta_{z_2} - \\
& 256k^4\theta_{z_2} - 64k^2\theta_\epsilon\theta_{z_2} - 64k^3\theta_\epsilon\theta_{z_2} - 32k^2\theta_\epsilon^2\theta_{z_2} - 64k^4\theta_\epsilon^2\theta_{z_2} - 8\theta_{z_2}^2 - \\
& 8k\theta_{z_2}^2 - 4\theta_\epsilon\theta_{z_2}^2 - 32k^2\theta_\epsilon\theta_{z_2}^2 - 4\theta_{z_2}^3) + \beta_1^6(-512k^5\theta_\epsilon^2 + 256k^6\theta_\epsilon^2 - 128k^5\theta_\epsilon^3 + \\
& 64k^6\theta_\epsilon^3 - 192k^3\theta_\epsilon\theta_{z_2} + 128k^4\theta_\epsilon\theta_{z_2} - 256k^5\theta_\epsilon\theta_{z_2} - 64k^3\theta_\epsilon^2\theta_{z_2} + 48k^4\theta_\epsilon^2\theta_{z_2} - \\
& 128k^5\theta_\epsilon^2\theta_{z_2} - 16k\theta_{z_2}^2 + 16k^2\theta_{z_2}^2 - 64k^3\theta_{z_2}^2 - 8k\theta_\epsilon\theta_{z_2}^2 + 12k^2\theta_\epsilon\theta_{z_2}^2 - 64k^3\theta_\epsilon\theta_{z_2}^2 + \\
& \theta_{z_2}^3 - 8k\theta_{z_2}^3)
\end{aligned} \tag{116}
$$

Assume that $\beta_1 \leq 0$. From the second order condition of the spoofer problem, we have $\lambda_1 + \lambda_{22} - \lambda_{12} > 0$. Combining this condition and equation (3.47), $\lambda_{22}$ is not positive. From the equation (3.43), $a_2 \leq 0$ or $\beta_1 > 1$, contradicting with our assumption. So $\beta_1 > 0$. Now, we will prove that $F(\beta_1, \theta_\epsilon, \theta_{z_2}, 1)$ have a unique equation $\beta_1$ in $(0, \frac{1}{2})$. We have

$$
F(0, \theta_\epsilon, \theta_{z_2}, 1) = 32 > 0 \tag{117}
$$

$$
\begin{aligned}
F(\frac{1}{2}, \theta_\epsilon, \theta_{z_2}, 1) = {} & -8 - 20\theta_\epsilon - 14\theta_\epsilon^2 - 3\theta_\epsilon^3 - 17\theta_{z_2} - 19\theta_\epsilon\theta_{z_2} - \frac{21\theta_\epsilon^2\theta_{z_2}}{4} - \frac{31\theta_{z_2}^2}{8} - \frac{33\theta_\epsilon\theta_{z_2}^2}{16} - \\
& \frac{15\theta_{z_2}^3}{64} < 0
\end{aligned} \tag{118}
$$

As $F(\beta_1, \theta_\epsilon, \theta_{z_2}, 1)$ is a continuous function in $\beta_1$, there must exist at least one solution of (115) in $(0, \frac{1}{2})$. Now, we take first derivative of $F(\beta_1, \theta_\epsilon, \theta_{z_2}, 1)$ and obtain:

$$
\begin{aligned}
F'(\beta_1, \theta_\epsilon, \theta_{z_2}, 1) = {} & -28 - (10 - 16\beta_1)^2 - 128\beta_1^2 - 64\theta_\epsilon(1 - 10\beta_1 + 36\beta_1^2 - 44\beta_1^3 + 40\beta_1^4) - \\
& 128\theta_\epsilon^2\beta_1^2(3 - 7\beta_1 + 10\beta_1^2) - 1536\beta_1^5\theta_\epsilon^2 - 320\beta_1^4\theta_\epsilon^3 - 384\beta_1^5\theta_\epsilon^3 - 32\theta_{z_2}(2 - \\
& 14\beta_1 + 45\beta_1^2 - 56\beta_1^3 + 50\beta_1^4) - 160\theta_{z_2}\beta_1^2\theta_\epsilon(2 + (1 - 2\beta_1)^2) - (1920\beta_1^5\theta_\epsilon + \\
& 480\beta_1^4\theta_\epsilon^2 + 864\beta_1^5\theta_\epsilon^2)\theta_{z_2} - 8\theta_{z_2}^2\beta_1^2(12 - 13\beta_1 + 10\beta_1^2) - (384\beta_1^5 + 180\beta_1^4\theta_\epsilon + \\
& 360\beta_1^5\theta_\epsilon)\theta_{z_2}^2 - (20\beta_1^4 + 42\beta_1^5)\theta_{z_2}^3
\end{aligned} \tag{119}
$$

For any $\beta_1$, $F'(\beta_1, \theta_\epsilon, \theta_{z_2}, 1) < 0$. Therefore, $F(\beta_1, \theta_\epsilon, \theta_{z_2}, 1)$ is an decreasing function in $\beta_1$ on $(0, +\infty)$. So $F(\beta_1, \theta_\epsilon, \theta_{z_2}, 1) = 0$ has unique solution $\beta_1$ and $0 < \beta_1 < \frac{1}{2}$.

## C.2 Proof of Proposition 3.5.2

When the spoofer does not trade, his real trading volume and spoofing volume are both zero. Therefore, $\beta_2 = 0$ and $\sigma_z^2 = 0$. This problem is a special case of the proposition 3.5.1 with $\sigma_z^2 = 0$.

## C.3 Proof of Proposition 3.5.3

Similar to the proof of the proposition 3.5.1, $\tilde{\beta_1}$ is the solution to the following equation

$$F(\beta_1, \theta_\epsilon, \theta_{z_2}, 0) = 0 \tag{120}$$

Inserting $k = 0$ into equation (116) to arrive at the following $5^{th}$ polynomial

$$\begin{aligned} F(\beta_1, \theta_\epsilon, \theta_{z_2}, 0) = 32 + \beta_1(-64 - 64\theta_\epsilon - 64\theta_{z_2}) + 32\beta_1^2\theta_{z_2} + 10\beta_1^4\theta_{z_2}^2 + \beta_1^6\theta_{z_2}^3 + \beta_1^3(-48\theta_{z_2} - \\ 32\theta_\epsilon\theta_{z_2} - 32\theta_{z_2}^2) + \beta_1^5(-8\theta_{z_2}^2 - 4\theta_\epsilon\theta_{z_2}^2 - 4\theta_{z_2}^3) \end{aligned} \tag{121}$$

Taking derivatives of equation (121) and obtaining

$$\begin{aligned} F'(\beta_1, \theta_\epsilon, \theta_{z_2}, 0) = -64 - 64\theta_\epsilon + (-64 + 64\beta_1 - 144\beta_1^2 - \beta_1^2\theta_\epsilon)\theta_{z_2} + (-96\beta_1^2 + 40\beta_1^3 - 40\beta_1^4 - \\ 20\beta_1^4\theta_\epsilon)\theta_{z_2}^2 + (-20\beta_1^4 + 6\beta_1^5)\theta_{z_2}^3 \end{aligned} \tag{122}$$

We have $F(0, \theta_\epsilon, \theta_{z_2}, 0) = 32$, and $F(\frac{1}{2}, \theta_\epsilon, \theta_{z_2}, 0) = -32\theta_\epsilon - 30\theta_{z_2} - 4\theta_\epsilon\theta_{z_2} - \frac{29\theta_{z_2}^2}{8} - \frac{\theta_\epsilon\theta_{z_2}^2)}{8} - \frac{7\theta_{z_2}^3}{64} \leq 0$. It is easy to see that $F'(\beta_1, \theta_\epsilon, \theta_{z_2}, 0) \leq 0$ for all $\beta_1$ in (0,1). Therefore, $F(\beta_1, \theta_\epsilon, \theta_{z_2}, 0)$ has unique solution in $(0, \frac{1}{2})$.

## C.4 Proof of Proposition 3.5.5

Using equation (115) with the case in which both traders choose to trade or $k = 1$

$$F(\beta_1, \theta_\epsilon, \theta_{z_2}, 1) = 0 \tag{123}$$

From the proposition 3.5.1, , the optimal $\beta_1$ is an implicit function in $\theta_\epsilon$ and $\theta_{z_2}$. Taking derivative of (123) with respect to $\theta_{z_2}$.

$$\frac{\partial F}{\partial \beta_1}\frac{\partial \beta_1}{\partial \theta_{z_2}} + \frac{\partial F}{\partial \theta_{z_2}} = 0 \tag{124}$$

The partial derivative of $F(\beta_1, \theta_\epsilon, \theta_{z_2}, 1)$ is given by the following equation

$$\begin{aligned}\frac{\partial F}{\partial \theta_{z_2}} = & -16\beta_1(4-14\beta_1+30\beta_1^2-29\beta_1^3+20\beta_1^4)-32\beta_1^3\theta_\epsilon(5-5\beta_1+4\beta_1^2)-(320\beta_1^6\theta_\epsilon+ \\ & 96\beta_1^5\theta_\epsilon^2 + 144\beta_1^6\theta_\epsilon^2) - 4\theta_{z_2}((4-2\beta_1)^2 + 3\beta_1 + 2\beta_1^2) - (128\beta_1^6 + 72\beta_1^5\theta_\epsilon + \\ & 120\beta_1^6\theta_\epsilon)\theta_{z_2} - (12\beta_1^5 + 21\beta_1^6)\theta_{z_2}^2\end{aligned} \tag{125}$$

As $4-14\beta_1+30\beta_1^2-29\beta_1^3+20\beta_1^4 > 0$ for any $\beta_1$ and $\beta_1 > 0$ , $\frac{\partial F}{\partial \theta_{z_2}} < 0$. Combining this condition with (3.71) and $\frac{\partial F}{\partial \beta_1} < 0$, we can conclude that $\frac{\partial \beta_1}{\partial \theta_{z_2}} < 0$.

Inserting $a_1 = \beta_1, a_2 = 1 - \beta_1$ into the equation (3.47) and obtaining

$$\beta^2 = \frac{\sigma_u^2(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)}{\sigma^2(\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + (\beta_1^2 + (1-\beta_1)^2))} \tag{126}$$

We denote the right side of equation (126) as $G(\beta_1, \theta_{z2})$. Taking derivative both sides of (126)

$$\frac{\partial \beta^2}{\partial \theta_{z_2}} = \frac{\partial G}{\partial \beta_1}\frac{\partial \beta_1}{\partial \theta_{z_2}} + \frac{\partial G}{\partial \theta_{z_2}} \tag{127}$$

Combining equations (127) and (124) to arrive at the following expression

$$\frac{\partial F}{\partial \beta_1}\frac{\partial \beta^2}{\partial \theta_{z_2}} = -\frac{\partial G}{\partial \beta_1}\frac{\partial F}{\partial \theta_{z_2}} + \frac{\partial G}{\partial \theta_{z_2}}\frac{\partial F}{\partial \beta_1} \tag{128}$$

We need to prove that $\frac{\partial \beta^2}{\partial \theta_{z_2}} \leq 0$. This condition is equivalent to:

$$4(1-2\beta_1)^2(\beta_1^2)\frac{\partial F}{\partial \beta_1} + 8(-1+2\beta_1)(4+\beta_1(4\theta_\epsilon + \theta_{z_2}))\frac{\partial F}{\partial \theta_{z_2}} \geq 0 \tag{129}$$

Inserting equations (125) and (119) into the above the inequality and simplify

$$-16\beta_1(-1+2\beta_1)(N_0 + N_1\theta_{z_2} + N_2\theta_{z_2}^2 + N_3\theta_{z_2}^3) \geq 0 \tag{130}$$

Where $N_0, N_1, N_2, N_3$ are defined as follows:

$$N_3 = \beta_1^5 + 10\beta_1^6 + 21\beta_1^7 \tag{131}$$

$$N_2 = 8\beta_1^3 + 72\beta_1^4 - 14\beta_1^5 + 8\beta_1^6 + 192\beta_1^7 + 15\beta_1^5\theta_\epsilon + 102\beta_1^6\theta_\epsilon + 180\beta_1^7\theta_\epsilon \tag{132}$$

$$N_1 = 16\beta_1 + 160\beta_1^2 - 448\beta_1^3 + 1016\beta_1^4 - 912\beta_1^5 + 800\beta_1^6 + (88\beta_1^3 + 360\beta_1^4 - 112\beta_1^5 + 256\beta_1^6 + 960\beta_1^7)\theta_\epsilon + (72\beta_1^5 + 336\beta_1^6 + 432\beta_1^7)\theta_\epsilon^2 \tag{133}$$

$$N_0 = 128 - 480\beta_1 + 1104\beta_1^2 - 1184\beta_1^3 + 832\beta_1^4 + (112\beta_1 + 64\beta_1^2 - 256\beta_1^3 + 1184\beta_1^4 - 768\beta_1^5 + 1280\beta_1^6)\theta_\epsilon + (224\beta_1^3 + 288\beta_1^4 - 224\beta_1^5 + 896\beta_1^6 + 768\beta_1^7)\theta_\epsilon^2 + (112\beta_1^5 + 352\beta_1^6 + 192\beta_1^7)\theta_\epsilon^3 \tag{134}$$

As $\beta_1$ lies in $[0, \frac{1}{2}]$, $-1 + 2\beta_1 \leq 0$ and it easy to see that $N_0, N_1, N_2, N_3$ are non-negative. Therefore, $\beta$ is decreasing in $\sigma_{z_2}$.

## C.5 Proof of Proposition 3.6.1

We can rewrite the optimization into another form by inserting equation (3.47) into equation (3.49) to obtain

$$E[\pi_S] = \frac{\lambda_1 + \lambda_{22} - \lambda_{12}}{4}(\beta_1)^2\sigma_{z_2}^2 = \sigma_{z_2}^2\frac{\lambda_{22}}{8}\frac{\beta_1\beta^2\sigma^2}{\beta^2\sigma^2 + \sigma_\epsilon^2 + \sigma_{z_2}^2} \tag{135}$$

Combining equations (3.43) and (135) to arrive at the following expression:

$$E[\pi_S] = \theta_{z_2}\frac{\sigma^2\beta(\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + 1 - \beta_1)}{16(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)}\frac{\beta_1\beta^2\sigma^2}{\beta^2\sigma^2 + (\theta\epsilon + \theta_{z_2})\sigma_u^2} \tag{136}$$

We denote $H(\beta_1, \theta_\epsilon, \theta_{z_2}) = \frac{\beta^2\sigma^2}{\beta^2\sigma^2 + (\theta\epsilon + \theta_{z_2})\sigma_u^2}$. Substituting equation (3.46) into this expression to get

$$H(\beta_1, \theta_\epsilon, \theta_{z_2}) = \frac{(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)}{(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1) + (\theta\epsilon + \theta_{z_2})(\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + (\beta_1^2 + (1 - \beta_1)^2))} \tag{137}$$

With $a_1 = \beta_1, a_2 = 1 - a_1$, we have the inequality $a_1^2 + a_2^2 = a_1^2 + (1 - a_1)^2 \geq \frac{1}{2}$. By using this inequality and (3.46), we arrive at the following upper bound of $\beta$

$$\beta^2 \leq \frac{2\sigma_u^2}{\sigma^2} \tag{138}$$

Combining this inequality and $\beta_1$ in $(0, \frac{1}{2})$, we have

$$\frac{\sigma^2\beta(\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + 1 - \beta_1)}{16(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)} \leq \frac{\sigma_u\sigma}{8\sqrt{2}} \tag{139}$$

Using the same inequality $a_1^2 + a_2^2 = a_1 + (1-a_1)^2 \geq \frac{1}{2}$, we can have the following inequality

$$H(\beta_1, \theta_\epsilon, \theta_{z_2}) \leq \frac{2}{2 + \theta_\epsilon + \theta_{z_2}} \tag{140}$$

Combining the inequalities (139), (140) and equation (136) to obtain

$$E[\pi_S] \leq \frac{\sigma_u\sigma}{4\sqrt{2}}\frac{\theta_{z_2}}{2 + \theta_\epsilon + \theta_{z_2}}\beta_1 \leq \frac{\sigma_u\sigma}{4\sqrt{2}}\beta_1 \tag{141}$$

From the proposition 3.5.5, $\beta_1$ is a decreasing continuous function of $\sigma_{z_2}$ and $\beta_1$ in $(1, \frac{1}{2})$, there exists a constant $\beta_0$ in $[0, 1]$ such that $\lim_{\sigma_{z_2}\to+\infty}\beta_1 = \beta_0$ and $\beta_0$ is the lower bound of $\beta_1$. We consider the case $\beta_0 > 0$. From the proposition 3.5.1, we have

$$\begin{aligned} F(\beta_1, \theta_\epsilon, \theta_{z_2}, 1) = {} & 32 - 128\beta_1 + 160\beta_1^2 - 128\beta_1^3 - 64\beta_1\theta_\epsilon + 320\beta_1^2\theta_\epsilon - 768\beta_1^3\theta_\epsilon + 704\beta_1^4\theta_\epsilon - \\ & 512\beta_1^5\theta_\epsilon - 128\beta_1^3\theta_\epsilon^2 + 224\beta_1^4\theta_\epsilon^2 - 256\beta_1^5\theta_\epsilon^2 - 256\beta_1^6\theta_\epsilon^2 - 64\beta_1^5\theta_\epsilon^3 - 64\beta_1^6\theta_\epsilon^3 + \\ & (-64\beta_1 + 224\beta_1^2 - 480\beta_1^3 + 464\beta_1^4 - 320\beta_1^5 - 160\beta_1^3\theta_\epsilon + 160\beta_1^4\theta_\epsilon - 128\beta_1^5\theta_\epsilon - \\ & 320\beta_1^6\theta_\epsilon - 96\beta_1^5\theta_\epsilon^2 - 144\beta_1^6\theta_\epsilon^2)\theta_{z_2} + (-32\beta_1^3 + 26\beta_1^4 - 16\beta_1^5 - 64\beta_1^6 - 36\beta_1^5\theta_\epsilon - \\ & 60\beta_1^6\theta_\epsilon)\theta_{z_2}^2 + (-4\beta_1^5 - 7\beta_1^6)\theta_{z_2}^3 = 0 \end{aligned} \tag{142}$$

If $\beta_0 > 0$, the left side of (138) goes to infinity when $\sigma_{z_2}$ goes to infinity. This contradicts with the equation. So, $\lim_{\sigma_{z_2}\to+\infty}\beta_1 = 0$. If we take limit the both sides of (140), we have

$$0 \leq \lim_{\sigma_{z_2}\to+\infty} E[\pi_S] \leq \lim_{\sigma_{z_2}\to+\infty} \frac{\sigma_u\sigma}{4\sqrt{2}}\beta_1 = 0 \tag{143}$$

So, we have $\lim_{\sigma_{z_2}\to+\infty} E[\pi_S] = 0$. When $\sigma_{z_2} = 0$, then $E[\pi_S] = 0$. From the proposition 3.5.1, we can deduce $E[\pi_S]$ is a continuous function in $\sigma_{z_2}$, $0 \leq E[\pi_S] \leq \frac{\sigma_u\sigma}{8\sqrt{2}}$ and $\lim_{\sigma_{z_2}\to+\infty} E[\pi_S] = \lim_{\sigma_{z_2}\to 0} E[\pi_S] = 0$. Therefore, $E[\pi_S]$ has the global maximum.

### C.6 Proof of Proposition 3.7.1

From the equation (3.42), we have the equation for $\lambda_{12}$ as follows

$$\lambda_{12} = \frac{\sigma^2\beta(\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + \beta_1)}{2\sigma_u^2(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)} \tag{144}$$

We denote the right side of equation (144) as $R_{12}(\beta, \beta_1, \theta_{z2})$. Taking derivative both sides of (144)

$$\frac{\partial\lambda_{12}}{\partial\theta_{z_2}} = \frac{\partial R_{12}}{\partial\beta_1}\frac{\partial\beta_1}{\partial\theta_{z_2}} + \frac{\partial R_{12}}{\partial\beta}\frac{\partial\beta}{\partial\theta_{z_2}} + \frac{\partial R_{12}}{\partial\theta_{z_2}} \tag{145}$$

The partial derivatives of $R_{12}(\beta, \beta_1, \theta_{z2})$ are given by

$$\frac{\partial R_{12}}{\partial\beta_1} = \frac{\beta\sigma^2(2 + \beta_1(1-\beta_1)(4\theta_\epsilon + \theta_{z_2}))}{\sigma_u^2(2 + \beta^2(4\theta_\epsilon + \theta_{z_2}))^2} \tag{146}$$

$$\frac{\partial R_{12}}{\partial\beta} = \frac{\sigma^2(\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + \beta_1)}{2\sigma_u^2(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)} \tag{147}$$

$$\frac{\partial R_{12}}{\partial\theta_{z_2}} = -\frac{\beta\beta_1^2(-1 + 2\beta_1)\sigma^2}{2\sigma_u^2(2 + \beta_1^2(4\theta_\epsilon + \theta_{z_2}))} \tag{148}$$

Based on the results of the proposition 3.5.1, $0 \leq \beta_1 \leq \frac{1}{2}$. Therefore, $\frac{\partial R_{12}}{\partial\beta_1} \geq 0$, $\frac{\partial R_{12}}{\partial\beta} \geq 0$ and $\frac{\partial R_{12}}{\partial\theta_{z_2}} \leq 0$. Combining with the proposition 3.5.5, $\frac{\partial\lambda_{12}}{\partial\theta_{z_2}} \leq 0$.

From the equation (3.43), we have the equation for $\lambda_{22}$ as follows

$$\lambda_{22} = \frac{\sigma^2\beta(\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + 1 - \beta_1)}{2\sigma_u^2(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)} \tag{149}$$

Squaring both sides of equation (149), then combining with equation (3.47)

$$\lambda_{22}^2 = \frac{\sigma^2(\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + 1 - \beta_1)^2}{4\sigma_u^2(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)(\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + \beta_1^2 + (1-\beta_1)^2)} \tag{150}$$

We denote the right side of equation (150) as $R_{22}(\beta_1, \theta_{z2})$. Taking derivative both sides of (149)

$$\frac{\partial\lambda_{22}^2}{\partial\theta_{z_2}} = \frac{\partial R_{22}}{\partial\beta_1}\frac{\partial\beta_1}{\partial\theta_{z_2}} + \frac{\partial R_{22}}{\partial\theta_{z_2}} \tag{151}$$

We need to prove that $\frac{\partial \lambda_{22}^2}{\partial \theta_{z_2}} \geq 0$. Substituting the partial derivatives of $R_{22}$, then simplify to obtain the following condition

$$-2\frac{\partial F}{\partial \theta_{z_2}}(-16 - 16\beta_1^2\theta_\epsilon^2 + 4(-1 + 3\beta_1 - 8\beta_1^2 + 4\beta_1^3)\theta_{z_2} - \beta_1^2\theta_{z_2}^2 + 8\theta_\epsilon(-2 + 6\beta_1 + 8\beta_1^3 - \beta_1^2(16 + \theta_{z_2}))) + \frac{\partial F}{\partial \beta_1}\beta_1(-1 + 2\beta_1)\left(4 - 4\beta_1 + 2\beta_1^3(4\theta_\epsilon + \theta_{z_2}) + \beta_1^2(8 + 4\theta_\epsilon + \theta_{z_2})\right) \leq 0 \tag{152}$$

Inserting equations (125) and (119) into the above the inequality and simplify

$$M_0 + M_1\theta_{z_2} + M_2\theta_{z_2}^2 + +M_3\theta_{z_2}^3 + M_4\theta_{z_2}^4 \leq 0 \tag{153}$$

Where $M_0, M_1, M_2, M_3, M_4$ are defined as follows

$$M_0 = -1536\beta_1 + 4352\beta_1^2 - 7936\beta_1^3 + 3072\beta_1^4 + 1024\beta_1^5 - 6144\beta_1^6 + (-1792\beta_1 + 9984\beta_1^2 - 39936\beta_1^3 + 80128\beta_1^4 - 119808\beta_1^5 + 93184\beta_1^6 - 61440\beta_1^7)\theta_\epsilon + (-5376\beta_1^3 + 16896\beta_1^4 - 48640\beta_1^5 + 43008\beta_1^6 - 43008\beta_1^7 - 16384\beta_1^8 - 24576\beta_1^9)\theta_\epsilon^2 + (-5376\beta_1^5 + 3840\beta_1^6 - 15360\beta_1^7 - 13312\beta_1^8 - 8192\beta_1^9 - 24576\beta_1^{10})\theta_\epsilon^3 + (-1792\beta_1^7 - 3072\beta_1^8 - 5120\beta_1^9 - 6144\beta_1^{10})\theta_\epsilon^4 \tag{154}$$

$$M_1 = -256\beta_1 + 768\beta_1^2 - 3072\beta_1^3 + 64\beta_1^4 + 6144\beta_1^5 - 22016\beta_1^6 + 18432\beta_1^7 - 15360\beta_1^8 + (-2112\beta_1^3 + 5760\beta_1^4 - 22784\beta_1^5 + 21504\beta_1^6 - 39168\beta_1^7 + 16384\beta_1^8 - 39936\beta_1^9)\theta_\epsilon + (-3456\beta_1^5 + 2688\beta_1^6 - 20736\beta_1^7 - 7680\beta_1^8 - 9216\beta_1^9 - 36864\beta_1^{10})\theta_\epsilon^2 + (-1600\beta_1^7 - 2304\beta_1^8 - 8960\beta_1^9 - 15360\beta_1^{10})\theta_\epsilon^3 \tag{155}$$

$$M_2 = -192\beta_1^3 + 384\beta_1^4 - 2656\beta_1^5 + 2688\beta_1^6 - 7104\beta_1^7 + 5120\beta_1^8 - 8448\beta_1^9 + (-720\beta_1^5 + 624\beta_1^6 - 7488\beta_1^7 - 1344\beta_1^8 - 3072\beta_1^9 - 13824\beta_1^{10})\theta_\epsilon + (-528\beta_1^7 - 576\beta_1^8 - 4800\beta_1^9 - 9216\beta_1^{10})\theta_\epsilon^2 \tag{156}$$

$$M_3 = -48\beta_1^5 + 48\beta_1^6 - 816\beta_1^7 - 64\beta_1^8 - 320\beta_1^9 - 1536\beta_1^{10} + (-76\beta_1^7 - 48\beta_1^8 - 1040\beta_1^9 - 2112\beta_1^{10})\theta_\epsilon \tag{157}$$

$$M_4 = -4\beta_1^7 - 80\beta_1^9 - 168\beta_1^{10} \tag{158}$$

For $0 \leq \beta_1 \leq \frac{1}{2}$ and $\theta_\epsilon \geq 0$, it is easy to prove that $M_0, M_1, M_2, M_3, M_4$ are non-positive. Therefore the inequality (153) holds or $\frac{\partial \lambda_{22}^2}{\partial \theta_{z_2}} \geq 0$.

From the proposition 3.7.3, $\lambda_1 + \lambda_{22}$ is decreasing in $\sigma_\epsilon$ while $\lambda_{22}$ is increasing in $\sigma_\epsilon$ . Therefore $\lambda_1$ is decreasing in $\sigma_\epsilon$.

## C.7 Proof of Proposition 3.7.2

By the definition, the forecast error variance of the market maker is defined by

$$\Sigma_1 = E[(v - p_1)^2] = (1 - \lambda_1\beta_1\beta)\sigma^2 \tag{159}$$

From the propositions (3.5.5), (3.7.1), $\lambda_1, \beta_1, \beta$ are decreasing in $\sigma_u$. Therefore, $\Sigma_1$ is increasing in $\sigma_u$. Similarly, the forecast error variance of the market maker in the second period is defined by

$$\Sigma_2 = E[(v - p_2)^2] = \frac{1}{2}\sigma^2 \tag{160}$$

## C.8 Proof of proposition 3.7.3

The expected profit to uninformed traders is given by $E[\pi_U] = -(\lambda_1 + \lambda_{22})\sigma_u^2$. Now we will prove that $\lambda_1 + \lambda_{22}$ is decreasing in $\sigma_{z_2}$. Similar to the proof of 3.7.1, we consider $(\lambda_1 + \lambda_{22})^2$

$$(\lambda_1 + \lambda_{22})^2 = B_3(B_1 + B_2)^2\frac{\sigma^2}{\sigma_u^2} \tag{161}$$

Where $B_1, B_2, B_3$ are defined as follows

$$B_1 = \frac{\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + 1 - \beta_1}{2(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)} \tag{162}$$

$$B_2 = \frac{\beta_1(\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + (\beta_1^2 + (1-\beta_1)^2))}{\beta_1^2(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1) + (\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + 1)(\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + (\beta_1^2 + (1-\beta_1)^2))} \tag{163}$$

$$B_3 = \frac{2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1}{\beta_1^2\theta_\epsilon + \frac{1}{4}\beta_1^2\theta_{z_2} + (\beta_1^2 + (1-\beta_1)^2)} \tag{164}$$

We denote $K(\beta_1, \theta_\epsilon, \theta_{z_2}) = B_3(B_1 + B_2)^2$. We can factorize K into $K(\beta_1, \theta_\epsilon, \theta_{z_2}) = (B_1 + B_2)(B_3B_1 + B_3B_2)$ . Taking the first derivative of $K$ with respect to $\theta_{z_2}$, we have the following expression.

$$\frac{\partial(\lambda_1 + \lambda_{22})^2}{\partial\theta_{z_2}} = (\frac{\partial K}{\partial\beta_1}\frac{\partial\beta_1}{\partial\theta_{z_2}} + \frac{\partial K}{\partial\theta_{z_2}})\frac{\sigma^2}{\sigma_u^2} \tag{165}$$

If we rearrange the terms of $B_1$, we have

$$B_1 = \frac{1}{4} + \frac{1 - 2\beta_1}{4(2\beta_1^2\theta_\epsilon + \frac{1}{2}\beta_1^2\theta_{z_2} + 1)} \tag{166}$$

As $0 \leq \beta_1 \leq \frac{1}{2}$, $1-2\beta_1 \geq 0$. Therefore, given $\beta_1$, $B_1$ is decreasing in $\theta_{z_2}$ or the partial derivative of $B_1$ with respect to $\theta_{z_2}$ is nonpositive. Now we consider the partial derivatives of $B_1B_3, B_1B_2, B_2$ with respect to $\theta_{z_2}$

$$\frac{\partial(B_1B_3)}{\partial\theta_{z_2}} = -\frac{2\beta_1^3(1 - 2\beta_1)}{(4 - 8\beta_1 + \beta_1^2(8 + 4\theta_\epsilon + \theta_{z_2}))^2} \tag{167}$$

$$\frac{\partial(B_2B_3)}{\partial\theta_{z_2}} = -\frac{8\beta_1^4(16 - 16\beta_1 + 4x\beta_1 + \beta_1^3x^2)}{(16 - 32\beta_1 - 8\beta_1^3 + 8\beta_1^2(6 + x) + \beta_1^4(x + 16)x)^2} \tag{168}$$

$$\frac{\partial(B_2)}{\partial\theta_{z_2}} = -\frac{4\beta_1^3(16(1 - 2\beta_1)^2 + 8\beta_1^2x(1 - 2\beta_1) + 8\beta_1^2(1 + (3 - 4\beta_1)^2) + \beta_1^4x(16 + x))}{(16 - 32\beta_1 - 8\beta_1^3 + 8\beta_1^2(6 + x) + \beta_1^4(x + 16)x)^2} \tag{169}$$

Where $x = 4\theta_\epsilon + \theta_{z_2}$ With $0 \leq \beta_1 \leq \frac{1}{2}$, it is easy to see that $\frac{\partial(B_2)}{\partial\theta_{z_2}} \leq 0$, $\frac{\partial(B_2B_3)}{\partial\theta_{z_2}} \leq 0$ and $\frac{\partial(B_1B_3)}{\partial\theta_{z_2}} \leq 0$. Combining with $B_1, B_2, B_3$ are non negative, we arrive at the conclusion $\frac{\partial K}{\partial\theta_{z_2}}) \leq 0$.

Similarly, now we will prove that $\frac{\partial K}{\partial\beta_1} \geq 0$. Using $x = 4\theta_\epsilon + \theta_{z_2}$. The partial derivative of K with respect to $\theta_{z_2}$ can be written as

$$\frac{\partial K}{\partial\beta_1} = \frac{T_0 + T_1x + T_2x^2 + T_3x^3 + T_4x^4 + T_5x^5 + T_6x^6 + T_7x^7 + T_8x^8 + T_9x^9}{(2(2 + \beta_1^2x)^2(4 - 8\beta_1 + \beta_1^2(8 + x))^2(16 - 32\beta_1 - 8\beta_1^3x + 8\beta_1^2(6 + x) + \beta_1^4x(16 + x))^3} \tag{170}$$

Where $T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9$ are defined as follows

$$T_0 = 524288 - 3407872\beta_1 + 10223616\beta_1^2 - 18612224\beta_1^3 + 20709376\beta_1^4 - 13369344\beta_1^5 + 1310720\beta_1^6 + 3932160\beta_1^7 - 3407872\beta_1^8 \tag{171}$$

$$T_1 = -65536\beta_1 + 1835008\beta_1^2 - 9895936\beta_1^3 + 26476544\beta_1^4 - 44367872\beta_1^5 + 45350912\beta_1^6 - 26542080\beta_1^7 - 1310720\beta_1^8 + 11927552\beta_1^9 - 10485760\beta_1^{10} + 1310720\beta_1^{11} \tag{172}$$

$$T_2 = -131072\beta_1^3 + 2228224\beta_1^4 - 10108928\beta_1^5 + 22790144\beta_1^6 - 32145408\beta_1^7 + 24215552\beta_1^8 - 5980160\beta_1^9 - 13385728\beta_1^{10} + 14155776\beta_1^{11} - 9895936\beta_1^{12} + 1310720\beta_1^{13} \tag{173}$$

$$T_3 = -114688\beta_1^5 + 1359872\beta_1^6 - 5124096\beta_1^7 + 9158656\beta_1^8 - 10444800\beta_1^9 + 4734976\beta_1^{10} - 45056\beta_1^{11} - 4046848\beta_1^{12} + 1228800\beta_1^{13} - 1245184\beta_1^{14} - 262144\beta_1^{15} \tag{174}$$

$$T_4 = -57344\beta_1^7 + 462848\beta_1^8 - 1420288\beta_1^9 + 1786880\beta_1^{10} - 1719296\beta_1^{11} + 367616\beta_1^{12} - 839680\beta_1^{13} + 487424\beta_1^{14} - 1343488\beta_1^{15} + 458752\beta_1^{16} - 262144\beta_1^{17} \tag{175}$$

$$T_5 = -17920\beta_1^9 + 84992\beta_1^{10} - 210176\beta_1^{11} + 101376\beta_1^{12} - 114944\beta_1^{13} - 126976\beta_1^{14} - 87040\beta_1^{15} - 122880\beta_1^{16} - 16384\beta_1^{17} - 65536\beta_1^{18} \tag{176}$$

$$T_6 = -3584\beta_1^{11} + 5632\beta_1^{12} - 12992\beta_1^{13} - 25280\beta_1^{14} + 11776\beta_1^{15} - 59648\beta_1^{16} + 21504\beta_1^{17} - 36864\beta_1^{18} \tag{177}$$

$$T_7 = -448\beta_1^{13} - 704\beta_1^{14} + 240\beta_1^{15} - 5568\beta_1^{16} + 2368\beta_1^{17} - 5888\beta_1^{18} \tag{178}$$

$$T_8 = -32\beta_1^{15} - 152\beta_1^{16} + 48\beta_1^{17} - 368\beta_1^{18} \tag{179}$$

$$T_9 = -\beta_1^{17} - 8\beta_1^{18} \tag{180}$$

With $0 \le \beta_1 \le \frac{1}{2}$, it is easy to see that the denominator of (170) is positive. We only need to prove that the numerator of(170) which we denote as $D(\beta_1, x)$ is positive. From the equation (116), we have $F(\beta_1, \theta_{z_2}, \theta_\epsilon, 1) = 0$. With $\theta_{z_2} = x - 4\theta_\epsilon$, we can write the equation in the following way.

$$32 - 128\beta_1 + 160\beta_1^2 - 128\beta_1^3 - 64\beta_1 x + 224\beta_1^2 x - 480\beta_1^3 x + 464\beta_1^4 x - 320\beta_1^5 x - 32\beta_1^3 x^2 + 26\beta_1^4 x^2 - 16\beta_1^5 x^2 - 64\beta_1^6 x^2 - 4\beta_1^5 x^3 - 7\beta_1^6 x^3 + \theta_\epsilon(48\beta_1(2-3\beta_1)^2 + 144\beta_1^3(5-8\beta_1) + 768\beta_1^5 + 96\beta_1^3 x - 48\beta_1^4 x + 192\beta_1^6 x + 12\beta_1^5 x^2 + 24\beta_1^6 x^2) = 0 \tag{181}$$

With $0 \le \beta_1 \le \frac{1}{2}$, it is obvious that $48\beta_1(2-3\beta_1)^2 + 144\beta_1^3(5-8\beta_1) + 768\beta_1^5 + 96\beta_1^3 x - 48\beta_1^4 x + 192\beta_1^6 x + 12\beta_1^5 x^2 + 24\beta_1^6 x^2 \ge 0$. As the coefficient of $\theta_\epsilon$ is nonnegative , We substitute $\frac{x}{4} \ge \theta_\epsilon$ into the equation (180) and simplify to obtain the following inequality.

$$32 - 128\beta_1 + 160\beta_1^2 - 128\beta_1^3 + (-16\beta_1 + 80\beta_1^2 - 192\beta_1^3 + 176\beta_1^4 - 128\beta_1^5)x + (-8\beta_1^3 + 14\beta_1^4 - 16\beta_1^5 - 16\beta_1^6)x^2 + (-\beta_1^5 - \beta_1^6)x^3 \ge 0 \tag{182}$$

As $0 \le \beta_1 \le \frac{1}{2}$, we can verify that all coefficients of $x, x^2, x^3$ are not positive. Therefore we can have 2 below inequalities.

$$32 - 128\beta_1 + 160\beta_1^2 - 128\beta_1^3 + (-16\beta_1 + 80\beta_1^2 - 192\beta_1^3 + 176\beta_1^4 - 128\beta_1^5)x \ge 0 \tag{183}$$

$$32 - 128\beta_1 + 160\beta_1^2 - 128\beta_1^3 \geq 0 \tag{184}$$

Solving the inequality (184), we have a stricter condition for $\beta_1$

$$\beta_1 \leq \frac{1}{12}\left(5 - \frac{23}{\sqrt[3]{\left(12\sqrt{87} - 19\right)}} + \sqrt[3]{\left(12\sqrt{87} - 19\right)}\right) = \beta_1^* \tag{185}$$

Using the inequality (183), we can rewrite the coefficient of $x$ in the right side of (183) as follows

$$-16\beta_1 + 80\beta_1^2 - 192\beta_1^3 + 176\beta_1^4 - 128\beta_1^5 = -5\beta_1 - 11\beta_1 + 80\beta_1^2 - 192\beta_1^3 + 176\beta_1^4 - 128\beta_1^5 \leq -5\beta_1 \tag{186}$$

As $0 \leq \beta_1 \leq \beta_1^*$, we can verify that $-11\beta_1 + 80\beta_1^2 - 192\beta_1^3 + 176\beta_1^4 - 128\beta_1^5 \leq 0$. Therefore, the inequality (186) holds. Combining inequalities (183), (184),(186), we have the constraint for $x$ as follows

$$32 - 128\beta_1 + 160\beta_1^2 - 128\beta_1^3 - 5\beta_1 x \geq 0 \tag{187}$$

We can computationally verify that $T_4, T_5, T_6, T_7, T_8, T_9$ are not positive for all $0 < \beta_1 \leq \beta_1^*$. Now, we consider $T_2, T_3$. Using the equations (173), (174), we can obtain the following expressions.

$$T_2 - 50000\beta_1^3 = 16\beta_1^3(-11317 + 139264\beta_1 - 631808\beta_1^2 + 1424384\beta_1^3 - 2009088\beta_1^4 + 1513472\beta_1^5 - 373760\beta_1^6 - 836608\beta_1^7 + 884736\beta_1^8 - 618496\beta_1^9 + 81920\beta_1^{10}) \leq 0 \tag{188}$$

$$T_3 - 20000\beta_1^5 = -32\beta_1^5(4209 - 42496\beta_1 + 160128\beta_1^2 - 286208\beta_1^3 + 326400\beta_1^4 - 147968\beta_1^5 + 1408\beta_1^6 + 126464\beta_1^7 - 38400\beta_1^8 + 38912\beta_1^9 + 8192\beta_1^{10}) \leq 0 \tag{189}$$

Similarly, we can verify that both inequalities (186) and (187) hold for all $0 < \beta_1 \leq \beta_1^*$. Therefore, we have the following inequality

$$D(\beta_1, x) \geq T_0 + T_1 x + (T_2 - 50000\beta_1^3)x^2 + (T_3 - 20000\beta_1^5)x^3 + T_4 x^4 + T_5 x^5 + T_6 x^6 + T_7 x^7 + T_8 x^8 + T_9 x^9 \tag{190}$$

If $\beta_1 = 0$, $D = 524288 > 0$, now we only need to consider the case when $\beta_1 > 0$. We denote the right side of (190) as $D_1(\beta_1, x)$ We consider 2 following cases:

1. $T_1 < 0$

When $T_1 < 0$, it is obvious that $D_1(\beta_1, x)$ is a polynomial of $x$ with almost all of coefficients $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9$ nonpositive. Combining with the inequality (187), we have that

$$D_1(\beta_1, x) \geq D_1(\beta_1, \frac{32 - 128\beta_1 + 160\beta_1^2 - 128\beta_1^3}{5\beta_1}) \tag{191}$$

We can verify that $D_1(\beta_1, \frac{32-128\beta_1+160\beta_1^2-128\beta_1^3}{5\beta_1}) > 0$ for all $0 < \beta_1 \leq \beta_1^*$ .

2. $T_1 \geq 0$, we have the following inequality

$$D(\beta_1, x) \geq T_0 + (T_2 - 50000\beta_1^3)x^2 + (T_3 - 20000\beta_1^5)x^3 + T_4x^4 + T_5x^5 + T_6x^6 + T_7x^7 + T_8x^8 + T_9x^9 \tag{192}$$

We denote the right side of (192) as $D_2(\beta_1, x)$. Similarly, we can see that $D_2(\beta_1, x)$ is a polynomial of $x$ with almost all of coefficients except $T_0$ nonpositive. Combining with the inequality (187), we have that

$$D_2(\beta_1, x) \geq D_2(\beta_1, \frac{32 - 128\beta_1 + 160\beta_1^2 - 128\beta_1^3}{5\beta_1}) \tag{193}$$

We can verify that $D_2(\beta_1, \frac{32-128\beta_1+160\beta_1^2-128\beta_1^3}{5\beta_1}) > 0$ for all $0 < \beta_1 \leq \beta_1^*$.

So we can conclude that the loss to uninformed traders is decreasing in $\sigma_{z_2}$.

### C.9 List of legal cases on spoofing

| Defendant | Year | Market | Spoofing period | Victim | Penalty |
|---|---|---|---|---|---|
| Joseph R. Blackwell | 2001 | Stocks | 2000 | | Fined $3,212.67 |
| Robert Monski | 2001 | Stocks | 1997 | | Fined $15,000, |
| Igor Oystacher | 2016 | E-mini S&P 500 | 2011-2014 | Citadel, HTG Capital | Fined $2.5 million |
| Michael Coscia | 2015 | CME. NYMEX | 2011 | D.E. Shaw, Citadel | 36 months |
| Navinder Sarao | 2017 | E-mini S&P 500 | 2010-2014 | | Home confinement |
| David Liew | 2017 | Metal contracts | 2009-2012 | | Pending |
| Jiongsheng Zhao | 2018 | E-mini S&P 500 | 2012 -2016 | | Time served |
| Andre Flotron | 2018 | Metal contracts | | | Acquitted |
| John Edmonds | 2018 | Metal contracts | 2009-2015 | | Pending |
| Kamaldeep Gandhi | 2018 | EMini S&P 500 | 2012-2014 | | Pending |
| Krishna Mohan | 2018 | EMini S&P 500 | 2012-2014 | | Pending |
| Edward Bases | 2018 | COMEX. NYMEX | 2008-2014 | | 1 year in prison |
| John Pacilio | 2018 | COMEX. NYMEX | 2008-2014 | | 1 year in prison |
| Jitesh Thakkar | 2019 | E-mini S&P | 2011-2015 | | Dismissed |
| Corey Flaum | 2019 | Metal contracts | 2007-2016 | | Pending |
| Christian Trunz | 2019 | Metal contracts | 2006-2010 | | Pending |
| Xiasong Wang | 2019 | Stocks | | | Pending |
| Jiali Wang | 2019 | Stocks | | | Pending |
| Gregg Smith | 2019 | Metal contracts | 2008 -2016 | Citadel , Quantlab | 2 years in prison |
| Michael Nowak | 2019 | Metal contracts | 2008 -2016 | Citadel , Quantlab | 1 year in prison |
| Christopher Jordan | 2019 | Metal contracts | 2006-2010 | Citadel , Quantlab | Pending |
| Jeffrey Ruffo | 2019 | Metal contracts | 2008 -2016 | Citadel , Quantlab | Acquitted |
| James Vorley | 2020 | Metal contracts | 2008-2013 | Quantlab Financial | Pending |
| Cedric Chanu | 2020 | Metal contracts | 2008-2013 | Quantlab Financial | Pending |
| Merrill Lynch Commodities | 2019 | Metal contracts | 2008-2014 | | $25 million |
| Tower Research Capital LLC | 2019 | E-Mini S&P 500 | 2012-2013 | | $67.4 million |
| Propex Derivatives Pty Ltd. | 2020 | EMini S&P 500 | 2012-2016 | | $1 million |
| Bank of Nova Scotia | 2020 | Metal contracts | 2008-2016 | | $60.4 million |
| JP Morgan Chase & Co. | 2020 | Metal contracts | 2008 -2016 | Citadel , Quantlab | $920 million |
| Nielsen | 2020 | Arrayit | 2020 | | 3 years supervision |
| Nicholas Mejia Scrivener | 2020 | Stocks | 2015-2016 | | $205,270 |
| Xuepeng Xie | 2021 | Stocks | | | $2,708,778 |