ESSAYS ON INFORMATION ECONOMICS UNDER AMBIGUITY

Hoang-Vu Le

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Abstract

This thesis consists of three chapters, each examining the impact of ambiguity on a specific economic problem.

Chapter 1, *Competitive Insurance Market Under Ambiguity*, extends the classic Stiglitz and Rothschild model to a setting where insurers face ambiguity about the composition of their customers. Using the epsilon-contamination framework, I characterize insurance contracts in a screening game under ambiguity, considering two scenarios: pooling equilibrium and separating equilibrium. Additionally, I provide a criterion that guarantees the existence of a separating equilibrium under ambiguity—an outcome not observed in the standard no-ambiguity model.

Chapter 2, *Moral Hazard Under Ambiguity*, examines the principal-agent problem in which both the principal and the agent face ambiguity about the stochastic relationship between the agent's effort and the project outcome. The chapter explores the optimal contract when effort is observable and the sub-optimal contract when effort is unobservable, within the context of ambiguity aversion. I specify conditions under which the principal's decision to induce high or low effort under ambiguity aligns with the decision in the absence of ambiguity.

Chapter 3, *Cheap Talk With Ambiguous Beliefs*, explores the cheap talk problem in the spirit of Crawford-Sobel (CS), introducing ambiguity by relaxing the assumption that both the sender and receiver know the actual distribution of the private message. The chapter examines CS-like partition equilibria in cases of small, complete, and intermediate ambiguity, using various frameworks to model ambiguity. It offers new insights into how ambiguity influences agents' behavior in strategic communication games.

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Declaration

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Hoang-Vu Le

Introduction

In most real-life decision-making situations, people tend to rely on probabilities. Since Blaise Pascal first laid the foundation of the probability theory (Ore (1960)), its power, efficiency, and elegant simplicity have guided decision-makers across various scientific fields where risk is involved, such as weather forecasting, finance, and computer science. In particular, economists have long employed probability theory in the context of subjective expected utility (SEU), where, under certain conditions, an agent's choices can be consistently explained by how she ranks acts based on a known prior of the states of the world (Savage (1954)). In this Bayesian-like approach, the decision-maker evaluates each act by considering the expectation of her utility over a set of possible events, each of which is assigned a forecasted probability.

Although SEU has long been a canonical framework for efficiently explaining human behavior, there are numerous real-life situations where the theory encounters significant challenges. For instance, in weather forecasting, an agent negatively impacted by rising sea levels may assess insurance as though sea level rise is highly probable, while another agent who benefits from it may behave as if climate change will have minimal impact. This inconsistency, where one individual is more pessimistic than another, is left unexplained by the SEU model (Ilut and Schneider (2022)). In economics, the well-known Ellsberg paradox also illustrates situations in which SEU predictions are violated.

In the most familiar version of Ellsberg's thought experiment, a decision-maker (she) is presented with an urn containing 90 balls in three colors: red, black, and yellow. The only exact information available is that there are 30 red balls, while no further information is provided about the number of black or yellow balls. The decision-maker is then asked to bet on the color of a ball drawn randomly from the urn. For example, an act **bet on red** refers to receiving \$100 if a red ball is drawn and \$0 otherwise. The experiment offers the following four options: (*I*) *bet on red*, (*II*) *bet on black*, (*III*) *bet on red or yellow*, (*IV*) *a bet on black or yellow*. The subject is asked to choose between bets (*I*) and (*II*) and between bets (*III*) and (*IV*) as shown in Table 1. Initially, one might consider applying the Principle of Insufficient Reason (Bernoulli (1954)) to assign equal probabilities to all possible outcomes. However, this principle has been subject to harsh criticism, most notably by Keynes (2004). The core of the disagreement lies in determining what constitutes an appropriate equal likelihood between drawing a black ball and a yellow ball. More precisely, the question arises whether we should assign equal probability to each type of ball, or if every possible configuration of black and yellow balls (ranging from 0 to 60) should be assigned an equal probability of $\frac{1}{61}$.

It turns out that most decision-makers prefer bet (*I*) over bet (*II*) and bet (*IV*) over bet (*III*). These choices violate the sure-thing principle, [1] which dictates that bet (*III*) should be preferred over bet (*IV*) if bet (*I*) is preferred over bet (II)as the two pairs differ only in the outcome when a yellow ball is drawn.

Table 1: Bets in Ellsberg's experiment

	R	B	Y = 60 - B
f_1	\$100	\$0	\$0
f_2	\$0	\$100	\$0
f_3	\$100	\$0	\$100
f_4	\$0	\$100	\$100

The unpredictability of the agent's behavior in the Ellsberg experiment suggests that decision-makers (DMs) with subjective probability beliefs need to be reconsidered. In the 1980s, Gilboa (1987) proposed an axiomatic model of rational choice that accounts for the Ellsberg experiment. Instead of estimating a single subjective probability distribution, DMs are assumed to hold a set of possible beliefs. The choice under ambiguity is then reduced to identifying the worst-case probability distribution for each belief and selecting the option that maximizes the DM's payoff. This mechanism is known as the maxmin expected utility (MEU) model.

1. In essence, the sure thing principle states that if an option is preferred in one scenario, it should still be preferred when an unrelated outcome is introduced.

It should be emphasized that the MEU model is not the only framework for capturing Knightian uncertainty. Various models, particularly under the assumption of ambiguity aversion, have been proposed in the literature. Ghirardato et al. (2010) consider MEU as a special case of the α-MEU, where an action f is evaluated based on a weighted average of the worst expected utility (with probability α) and the best expected utility (with probability $1-\alpha$). Although the α-MEU model appears to be more general than Gilboa's original model, it fails to disentangle ambiguity from ambiguity aversion Siniscalchi (2006)

Klibanoff et al. (2005), (KMM hereafter) propose the smooth model of ambiguity, which successfully separates ambiguity from the decision-maker's attitude toward it. In the KMM model, the decision-maker evaluates each Aumann-Ascombe act f using the following utility function:

$$U(f) = \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(h(\omega) dp(\omega)) \right) d\mu(p)$$

where μ reflects the probability measure on $\Delta(\Omega)$, $u : \Delta(C) \to R$ is a von Neumann-Morgenstern (vNM) utility function, and ϕ is concave under the assumption of ambiguity aversion. While KMM offers flexibility in modeling ambiguity, its foundations remain problematic, making it difficult to clearly distinguish its behavioral implications from those of the Gilboa model (Epstein and Schneider (2010)). Due to its solid axiomatic foundation and comprehensive nature, the MEU model, along with its variants, serves as the backbone of this thesis.

When the ability to gather information is limited, it is cognitively intuitive that decision-makers (DMs) form a set of probability distributions rather than a single one, as assumed in classical expected utility theory. In a general sense, an Aumann-Ascombe act is evaluated in the MEU framework using the following functional form:

$$V(f) = \min_{p \in P} \int_{\Omega} u(f) dp$$

where the DM selects the worst-case probability distribution from the set P, minimizing her expected utility over the set of priors for each act f. Unlike the smooth model (KMM), the MEU does not exclude kinks, as it allows for non-differentiability in the utility function.[2] The set P can take various forms, representing different degrees of ambiguity. In this thesis, we explore three classic problems in the presence of ambiguity, modeled using variants of the MEU framework, including the ε-contamination, maxmin expected utility, and multiplier utility models.

In the first chapter, a Rothschild-Stiglitz-like competitive insurance market is considered, where insurers are uncertain about the proportion of low-risk and high-risk customers. Ambiguity is modeled using the ε-contamination framework, where the degree of uncertainty in firms' subjective beliefs regarding the composition of customers is represented by ε. Under ambiguity, several novel findings, distinct from the RS model, emerge.

First, a pooling equilibrium, which guarantees full coverage for all high-risk customers, can exist under complete ambiguity. Second, when ε exceeds a certain threshold—independent of the firms' beliefs—the separating equilibrium becomes stable. These results are not observed in the original RS model. Finally, we show that the insurance policies for each customer type in the separating equilibrium under ambiguity closely resemble the separating policies in the RS model.

The second chapter focuses on examining the moral hazard problem when both the principal and the agent hold effort-dependent beliefs, represented as a simplex, about the probability of the project's success. We employ the MEU model following Gilboa and Schmeidler (1989), where both the principal and the agent choose actions that maximize the worst expected utility and expected profit over a set of prior beliefs.

2. See Chapter 2 for more details.

When the effort exerted is observable, we show that ambiguity reduces the principal's welfare but not the agent's. Although the principal can still incentivize the agent to exert the desired effort by offering a risk-free contract, his expected utility under the MEU model is lower than in the classic moral hazard (MH) model without ambiguity. Being ambiguity-averse, the principal assumes the project will succeed with the lowest possible probability, resulting in a lower expected payoff.

In the case of unobservable effort, ambiguity aversion produces an intriguing result. The principal's maxmin expected profit depends on the difference between the wages paid to the agent when the outcome is favorable and when it is unfavorable. This contrasts with the classic MH literature, where the principal's expected profit is determined in a unique way without ambiguity. Additionally, ambiguity aversion may lead to a situation where less effort is more profitable. When the sets of priors overlap, inducing the agent to exert less effort may be more advantageous than inducing greater effort, as the latter would require higher compensation for the agent, ultimately reducing the principal's profit.

The final chapter examines the problem of cheap talk following Crawford and Sobel (1982) (CS) within an ambiguity context. In the original CS model, the distribution of the private message is known to both the sender (S) and the receiver (R). We relax this condition by considering the situation where the receiver is uncertain about the probability distribution of the message.[3] Depending on ambiguity degree, the receiver may employ the MEU model (Gilboa and Schmeidler), or Multiplier Utility following Hansen and Sargent (2001) for intermediate cases. Due to the misalignment of interest between agents, the sender informs the receiver of an interval in which the private message lies, using a random signal. The set of informed intervals thus forms a partition equilibrium in all these models.

3. Li (2022) addresses a similar problem of cheap talk under ambiguity regarding the distribution of the private message. However, our approach differs from Li's, as will be discussed in detail in Chapter 3.

We demonstrate that, when the receiver's reference distribution is uniform, both R and S behave in the same manner under any level of ambiguity as they do in the Crawford-Sobel (CS) model in the absence of ambiguity, provided their preferences follow a quadratic-loss function. Therefore, the CS equilibrium remains robust under ambiguity when the receiver's reference probability is uniform. However, when the receiver's reference probability differs, her strategy under ambiguity can deviate from her behavior in the CS equilibrium, resulting in a shift of the partition equilibrium.

Overall, my research contributes to the ongoing discussion on the effects of ambiguity in information economics. Throughout this work, various models are employed to explain agents' behavior under ambiguity. While several conclusions align with those in the non-ambiguity context, ambiguity generally gives rise to market anomalies that deviate from expected utility theory.

In the context of the insurance market under ambiguity, Chapter 1 aligns with recent literature on screening (Spinnewijn (2013)) and insurance contracts under uncertainty (Birghila et al. (2023), Vergote (2010), Anwar and Zheng. (2012), Huang et al. (2015)). It is particularly related to Anwar and Zheng's work on competitive insurance markets. Although this chapter restricts the contamination of ambiguity to a simplex, it offers several novel insights compared to the standard Crawford-Sobel (CS) model. By adopting the ε-contamination model, we not only establish the existence of pooling equilibria but also demonstrate that the extent of ambiguity can outweigh the subjective probabilities of insurance firms. As a result, when ambiguity is sufficiently large, ambiguity aversion leads to the existence of separating equilibria, independent of the firms' beliefs.

Chapter 2 provides insights into how ambiguity aversion affects agents' decision-making in principal-agent problems under both symmetric and asymmetric information. While there is a substantial body of work on contract theory with imprecise information (**Lopomo**, Kellner and Riener (2014), Mastrolia and Possamaï (2018), Dumav and Khan (2018), Carroll (2015)), no study, to my knowledge, thoroughly examines the situation in which both parties face ambiguity regarding the effort-dependent distribution of the project's outcome. By employing the maxmin expected utility framework, both parties are concerned only with the degree of ambiguity, represented by probability ranges, rather than the structure of the imprecise belief. Thus, this chapter offers a general mechanism for determining the set of optimal and suboptimal contracts in an ambiguous setting.

While there is extensive literature on the effects of ambiguity in insurance markets and moral hazard, surprisingly, few studies have explored its impact on strategic communication. Some authors have extended the foundational work of Crawford-Sobel, but they focus primarily on cases where the receiver has no information about the true private message (Ishida and Shimizu (2019), Lai (2014), Chen and Gordon (2015)). To the best of my knowledge, aside from Li (2022), no research has examined the violation of the assumption regarding the private message distribution. However, Li's work is incomplete, lacking detailed technical explanations. Moreover, by modeling ambiguity solely through the multiplier preference framework, Li's approach becomes less intuitive and more technically complex. In contrast, we offer a simpler and more intuitive solution by modeling ambiguity using the ε-contamination approach in the small-case scenario, which is analytically less cumbersome. Our approach provides a natural way to study the effects of ambiguity on the CS model, from small levels of ambiguity to what can be considered ”higher-order uncertainty.” This research thus contributes to the growing body of literature on cheap talk and strategic communication.

Chapter 1

Competitive Insurance Market Under Ambiguity

We investigate the impact of ambiguity on a competitive insurance market when companies are uncertain about the risk profile distribution of customers. [1] By employing the epsilon-contamination framework, we show that, in contrast to the seminal work by Rothschild and Stiglitz (1976), a pooling equilibrium emerges if insurers face complete ambiguity. When ambiguity is less extreme, pooling contracts disappear, similar to Rothschild-Stiglitz's findings. Although ambiguity does not impact the terms of separating insurance policies, it affects their availability. When the ambiguity degree is sufficiently high, the existence of the separating equilibrium is guaranteed.

1.1 Introduction

The problem of adverse selection in a competitive insurance market with private information was initially considered by Rothschild and Stiglitz (1976) (RS hereafter). A notable outcome from their work is that, despite consumers concealing private information about their type, insurance companies can still offer a range of contracts that induce buyers to reveal their types if the composition of customers is known precisely.

1. A shorter version of this work was published in Le (2024).

However, RS' analysis proceeded under the assumption that agents' preferences can be expressed through the standard von Neumann-Morgenstern (vNM) expected utility, based on their judgment about the possibility of uncertain events. Similarly, actuarial firms often encounter difficulties in accurately estimating the market's customer composition or, equivalently, the percentage of individuals with a low probability of experiencing an accident and those with a higher probability. Consequently, it may be more plausible to consider errors that arise when insurance companies estimate this probability measure. In other words, insurers will inevitably grapple with uncertainty in their calculations of the proportion of each type of consumer.

Zheng et al. (2016) (hereafter referred to as ZWL) examined the feasible policies in a monopolistic insurance market with adverse selection when there is ambiguity about the composition of agents. However, their model only considered the case where a single monopolistic insurance company offers contracts to customers, leaving open the question of how ambiguity affects competition in the insurance market, as in the RS model. To the best of our knowledge, little is known about the impact of ambiguity on a competitive insurance market. Furthermore, there is limited research on how ambiguity influences the self-selection mechanism of insurance buyers.

This chapter addresses these gaps in the literature. First, we characterize two types of contracts—pooling and separating—in equilibrium when firms face ambiguity regarding the composition of insurees. Second, we examine how ambiguity affects the existence of separating equilibria.

Following ZWL, the degree of ambiguity is modeled by $\varepsilon-$contamination with a larger ε reflecting a larger degree of ambiguity, where $\varepsilon = 0$ is mapped to a single prior case as in the RS model, while $\varepsilon = 1$ corresponds to complete ignorance. All the intermediate cases, where $\varepsilon \in (0,1)$ are also examined in our paper.

Our work first establishes that when $\varepsilon = 1$, or in the case of complete ambiguity, a pooling insurance policy emerges in equilibrium, where high-risk customers are fully covered against their losses. Due to ambiguity aversion, insurance firms assume the most pessimistic scenario, in which there are no low-risk customers in the market, preventing them from offering any cream-skimming strategies to attract low-risk buyers. Although insurance companies may have initial conjectures about the composition of customers in the market, complete ambiguity nullifies these subjective priors, leading firms to behave as if low-risk customers are absent. This finding is novel compared to the RS model without ambiguity.

When $\varepsilon < 1$, insurers still face ambiguity regarding the proportion of each type of customer, but with less severity. In contrast to the case of full ambiguity, firms do not rule out the presence of low-risk customers in the market under ambiguity aversion. Therefore, for every pooling contract offered to customers, there is always an opportunity for a deviating firm to profit by attracting low-risk buyers. As a result, the pooling equilibrium disappears in this case, mirroring RS's conclusion in the absence of ambiguity ($\varepsilon = 0$).

In the context of ambiguity, insurance companies can still screen customer types by offering two distinguishable contracts that allow low-risk and high-risk customers to self-select, similar to the RS model. However, in the RS model, a separating equilibrium may not always exist when there are relatively few high-risk customers. In such cases, the separating equilibrium can be disrupted by a pooling contract that attracts low-risk customers, yielding positive profits for the company offering it. Under ambiguity aversion, we show that a separating equilibrium always exists when ε exceeds a certain threshold. This threshold depends only on the probabilities of accidents for low-risk and high-risk customers, as well as low-risk preferences, rather than on firms' subjective priors regarding the composition of buyers.

The paper is organized as follows: In Section 1.2, we review the related literature. In Section 1.3, we present the model of the screening game without ambiguity. Section1.4 characterizes equilibrium under ambiguity. Section 1.5 is for conclusion. All proofs are provided in Appendix 1.6.

1.2 Literature Review

The impacts of ambiguity on the financial market have been widely conducted systematically in recent years. It is the main reason that restricts participants in their decisions and reduces the availability of liquid assets in the market. Ambiguity is also considered a cause of adverse consequences towards risk distribution and volatility in equilibrium prices in the market of financial securities and derivatives.

Despite extensive examination by economists over the years, the co-existence of ambiguity aversion and private information is an unexplored and compelling area that warrants further attention. Kajii and Ui (2009) and Martins-da-Rocha (2010) exploited the definition of compatible priors to determine weakly interim efficient allocation in the uncertainty framework. Tallon (1998) and Condie and Ganguli (2011) indicated that the Grossman-Stiglitz paradox can be resolved by admitting ambiguity. Another significant conclusion from their research is that although information is included in the equilibrium price, agents in the financial market may agree to buy it with more extra cost.

One of the noticeable research on the characteristics of insurance policies under asymmetric information and imprecise probabilities is from Jeleva and Villeneuve (2004). Under some specified parameters, they obtained a pooling equilibrium when there is a monopolist company. However, insurance policies from their results are partial and can be considered inefficient.

Koufopoulos and Kozhan (2016)' work provided interesting results when many insurance companies compete to offer insurance contracts under adverse selection, and consumers are ambiguity averse and utility maximizers. The first important conclusion is that in some special range of parameters (the probability of getting an accident of a customer), any pooling equilibrium, if it exists, will require two types of consumers to buy full insurance policies. The next result is that a separating policy will create a situation where agents with a low probability of getting an accident will be underinsured. However, their coverage will be higher than the standard model in the absence of ambiguity. The last conclusion is that there is always a unique equilibrium that is interim incentive efficient due to the endogeneity.

While most research on ambiguity in the insurance market focused on the ambiguity about the distribution of loss, Zheng et al. (2016) provided a new viewpoint by considering the case when a monopolist insurer faces ambiguity concerning the proportion of different types of consumers. Employing the Choquet expected utility to incorporate the insurer's ambiguity about the proportion of different types of customers, these authors have proved the following significant results when insurers have to face uncertainty about the composition of customers under asymmetric information.

Firstly, under ambiguity (regardless of whether the insurer is ambiguity averse or ambiguity seeking), the high-type consumers acquire full coverage while the low-type consumers obtain less than full coverage. Secondly, for an ambiguity averse insurer, as the degree of ambiguity increases, the optimal menu of contracts moves toward a menu (the "attraction" menu) in which the profits the insurer earns from two types of customers are the same. The coverage of the low-type consumer can increase or decrease when ambiguity increases. When the attraction menu is reached, the optimal menu will no longer change even as the ambiguity continues to increase. An insurance company that is ambiguity averse may set the same menu of contracts (which is the attraction menu) for a range of prior beliefs. When the insurer is ambiguity-seeking, when the degree of ambiguity increases, the menu of the contract will move away from the menu in which two types of buyers give the same profits. Finally, their research suggests that when there are two types of consumers, an insurer who has the chance to learn and renegotiate the contract can effectively learn the distribution of consumers even if ambiguity about the proportion of consumers exists.

Our paper relates closely to ZWL's paper, as we also consider the case when insurers have to face the uncertainty of the proportion of each type of consumer. However, we extend the problem from monopolist insurance when market power exists to the competitive setting. The basic assumption that insurance companies can estimate exactly the probability of getting an accident for each type of customer is still intact. Hence, the indifferent curves of low-risk and high-risk type agents are the same as in the standard expected utility, which is convenient for our calculations.

1.3 Model Of Screening Game

This section illustrates the setting of the game and the equilibrium concept, which can be done before introducing ambiguity. Consider two insurance companies A and B, or insurers, competing to offer insurance policies, with one customer (she) selecting the best policy from those offered. The customer belongs to one of two types: the low-risk type ($L-$ type) with a low probability of accidents or the high-risk type ($H-$ type) with a higher probability of loss.

Both types of customers know exactly their probability of getting accidents, say π_L for $L-$ type, and π_H for $H-$ type. However, this information is unobservable to insurance companies. Therefore, insurers take steps to distinguish, or screen, individuals in the market to offer optimal insurance contracts. The problem can be stated as a three-step game:

Step 1. Two insurance companies, A and B, simultaneously offer a set of finite insurance policies to consumers. As both insurance companies only want to screen two types of customers, it would be enough for each firm to offer only one menu of policies, consisting of two policies for low-risk and high-risk types.

Specifically, the insurance company i ($i = A, B$) offers a menu in the form $\Psi^i = (\Psi^i_L, \Psi^i_H) = (B^i_L, p^i_L, B^i_H, p^i_H)$ where $\Psi^i_L = (B^i_L, p^i_L)$, $\Psi^i_H = (B^i_H, p^i_h)$ are the corresponding contracts for agents of low and high risk offered by firm i. B^i_L represents the coverage (benefit) that firm i will pay its customer in the event of an accident, and p^i_L is the premium for the insurance contract for the low-risk type, with similar notation for the high-risk type.

Step 2. Nature determines subsequently which type of buyer is offered the contract. Insurance firms, based on empirical observation, have a subjective belief that nature draws a low-risk consumer with probability α, and a high-risk consumer with probability $1-\alpha$. In the RS model without the presence of ambiguity, insurance companies know α precisely. Within ambiguity context, α is the best guess of firms regarding the proportion of the $L-$type customers .

Step 3. Each customer can only choose a single policy from either A or B. If a policy she wishes to choose is offered by both companies, the customer randomly selects between the two.

For a customer i, where i is $L-$type or $H-$type, we define a choice function $c_i(.)$ that specifies the insurance company she makes an agreement and its policy Ψ^i. If i receives less expected utility from the menus of the policy of both insurers than her utility without any insurance contract, she will not purchase any insurance policies, and choose the null policy $(0,0)$. Thus, $c_i(\Psi^A, \Psi^B) = (c, \Psi)$, where c is the chosen insurance company from which the customer will buy a policy, and Ψ can be Ψ^c_L, Ψ^c_H, or the null policy $(0,0)$.

As consumers show no signal to the insurance companies, the set of outcomes in the Perfect Bayesian Equilibrium is equivalent to the set of Subgame Perfect Nash Equilibrium outcomes. Therefore, the Subgame Perfect Nash Equilibrium can be justified to analyze our problem.

We define a pure Subgame Perfect Nash Equilibrium (SPNE) in the screening game as the set of strategies $(\Psi^A, \Psi^B, c_L(.), c_H(.))$ that satisfies the following:

Given the choice of B, A maximizes its expected profit:

$$\Pi_A\left(\Psi^A, \Psi^B, c_L(\Psi^A, \Psi^B), c_H(\Psi^A, \Psi^B)\right) \geq \Pi_A\left(\tilde{\Psi^A}, \Psi^B, c_L(\tilde{\Psi^A}, \Psi^B), c_H(\tilde{\Psi^A}, \Psi^B)\right)$$

for any other $\tilde{\Psi^A} \neq \Psi^A$.

Given the choice of A, B maximizes its expected profit:

$$\Pi_B\left(\Psi^A, \Psi^B, c_L(\Psi^A, \Psi^B), c_H(\Psi^A, \Psi^B)\right) \geq \Pi_B\left(\Psi^A, \tilde{\Psi^B}, c_L(\Psi^A, \tilde{\Psi^B}), c_H(\Psi^A, \tilde{\Psi^B})\right)$$

for any other $\tilde{\Psi^B} \neq \Psi^B$.

Each type of consumer chooses the choice function that gives her the maximum expected utility among those policies offered by both companies:

$u_L(\Psi) \geq u_L(\Psi')$, where $u_L(\Psi)$ is the expected utility of a $L-$type customer from the contract Ψ in her choice function $c_L(\Psi^A, \Psi^B)$. $u_L(\tilde{\Psi})$ is her expected utility corresponding to the contract Ψ' if she deviates. A similar definition can be applied to $H-$type consumers.

A Subgame Perfect Nash Equilibrium in the screening setting consists of the set of insurance policies offered by firms A and B, denoted by Ψ^A and Ψ^B. Additionally, it includes the choice functions of the low-risk and high-risk customers, represented as $c_L(\Psi^A, \Psi^B)$ and $c_H(\Psi^A, \Psi^B)$, respectively. Denote Ψ_L (Ψ_H) as the contract that the low-risk (or the

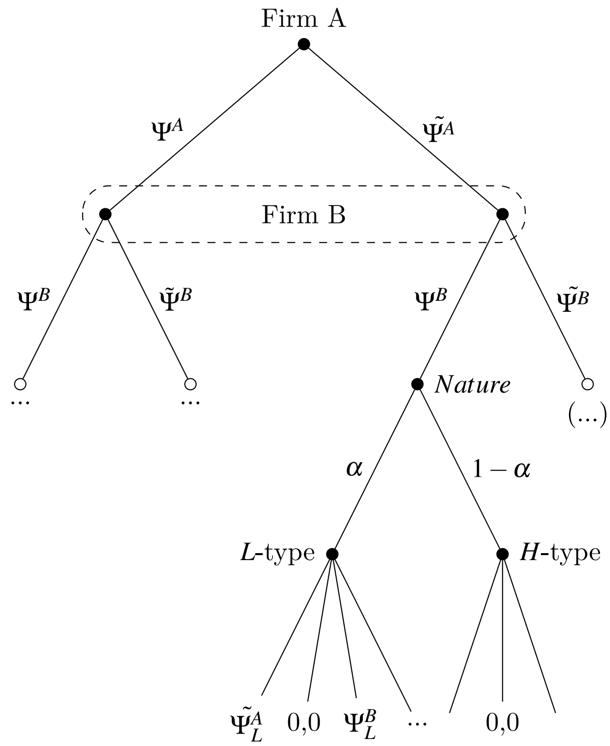

Figure 1.3.1: Screening game in the competitive insurance market

high-risk) consumer will choose from the choice function $c_L(\Psi^A, \Psi^B)$ $(c_H(\Psi^A, \Psi^B))$(note that $\Psi_L(\Psi_H)$ can be offered by firm A, firm B or both). The pure strategy SPNE is separating if $\Psi_L \neq \Psi_H$; otherwise, it is a pooling equilibrium. The game can be depicted by its extensive form in Figure 1.3.1.

Remark 1.3.1. *Since there are only two types of buyers in the market, insurance companies only need to offer at most two different insurance policies,* $(\Psi^A, \tilde{\Psi^A})$ *and* $(\Psi^B, \tilde{\Psi^B})$*, as illustrated in Figure 1.3.1.*

1.4 Main Results

1.4.1 Expected profit with ε contamination

Following Section 1.3, we exclusively consider the model where individuals are categorized as high-risk type and low-risk type. Initially, each individual is endowed with wealth w but if the accident occurs, she faces a loss L. The probabilities of being involved in an accident are π_L for low-risk customers and π_H for high-risk ones accordingly, where $0 < \pi_L < \pi_H < 1$. Each consumer then maximizes their von Neumann-Morgenstern expected utility when facing risks, using the utility function $u(.)$, which is strictly increasing, differentiable, continuous, and strictly concave.

Two competitive insurers will compete in a Bertrand game to offer a list of insurance policies to their customers. Throughout this paper, they are assumed to be ambiguity-averse. [2] An insurance policy takes the form of (B, p), where p is the premium paid to the insurer, and B is the payout in the event of a loss. Both insurers and consumers know the true values of π_L and π_H. Nonetheless, there is asymmetric information about the type of customers to which one belongs: buyers know exactly their probability of getting an accident, while companies can not observe this.

Without the presence of ambiguity, insurers have the same subjective prior belief $P_0 = (\alpha, 1-\alpha)$ [3] about the distribution of consumer types (or a proportion of low-risk type π_L): an individual is of low-type with proportion α and of high-type with probability $1-\alpha$. The estimation can be the insurer's best guess based on her experience and research.

2. According to Kunreuther et al. (1993), there is a large number of proofs from laboratory-controlled experiments that decision makers are ambiguity-averse. Therefore, we will limit the scope of this paper to the context of ambiguity aversion, with two insurance firms being profit maximizers and all customers being utility maximizers.
3. More precisely, α can be defined as a probability measure, $\mathscr{P}(S) \to [0,1]$, where S denotes the set containing the states L and H.

When ambiguity is taken into account, we introduce the notion of epsilon contamination, which has been extensively studied in Zheng et al. (2016) and Carlier et al. (2003). Following these, although insurers have formed a subjective belief regarding P_0, they assume it is incorrect with some probability ε in which α can take any possible probability measure.

More precisely, insurers perceive that with probability $(1-\varepsilon)$ the true distribution is P_0 and that with probability ε it can take any other probability law P_1. As such, ε can be interpreted as the insurer's uncertainty about whether P_0 accurately reflects reality. The true distribution is then modeled as a mixture of P_0 and P_1, controlled by ε. Note that under this interpretation, one insurance firm A will be more uncertainty-averse than another firm B if $\varepsilon_A > \varepsilon_B$. The contaminated distribution is given by

$$P = (1-\varepsilon)P_0 + \varepsilon P_1 \tag{1.4.1}$$

In this formula, P_0 represents the insurer's baseline belief about consumer risk types, and P_1 is a worst-case alternative distribution (e.g., skewed towards high-risk consumers). When there is no ambiguity, i.e., $\varepsilon = 0$, the problem reduces to the RS model. We therefore restrict the analysis to $\varepsilon \in (0,1]$.

Following Zheng et al. (2016) and formula 1.4.1, the maximum expected utility of an ambiguity-averse insurance company under ambiguity modeled by $\varepsilon-$contamination can be described by $(1-\varepsilon)E(u) + \varepsilon \min_{s \in S} u(s)$, with $u(.)$ denoting the vN-M utility function. In that fashion, the total expected profit from two ambiguity-averse insurers facing ambiguity can be defined as

$$\Pi = (1-\varepsilon)\left[\alpha \Pi_l + (1-\alpha)\Pi_H\right] + \varepsilon \min\left(\Pi_L, \Pi_H\right). \tag{1.4.2}$$

where Π_L, Π_H represent the expected profits from low-risk and high-risk consumers, respectively. In 1.4.2, each insurer perceives that with probability ε, her prior belief P is incorrect. The next section will exhibit the pure pooling and separating equilibria of the screening game under ambiguity.

1.4.2 Equilibrium insurance policies

Before delving into the analysis of the impact of ambiguity on insurance companies, we first revisit the main findings from the RS model, where $\varepsilon = 0$:

i. There is no pooling equilibrium in competitive markets with asymmetric information.

ii. There is a unique separating equilibrium $\Psi_L^* = (B_L^*, p_L^*)$, $\Psi_H^* = (B_H^*, p_H^*)$ where $H-$types are fully covered and $L-$types are under-covered.

iii. However, the existence of a separating equilibrium is not guaranteed. When α is close to 1, the proportion of low-risk buyers is much higher than that of high-risk buyers, an insurance company can offer a pooling contract that entices low-risk buyers while generating positive profit. This scenario leads to the breakdown of the initial separating equilibrium.

We show that, on the one hand, the separating contract in (ii) is robust under ambiguity. On the other hand, ambiguity aversion leads to the existence of pooling equilibrium where insurers offer a unique policy that fully insures the $H-$type, thus (i) is violated. This is a novel result compared to the RS model.

Moreover, firms can still offer the separating equilibrium even when the subjective prior belief α, or the percentage of the $L-$types dominates that of the $H-$types if ε exceeds a certain threshold. This suggests that (iii) is no longer robust in the ambiguity context.

In the next step, we first set up the preliminaries. Assume that each type of consumer has the same vN-M utility preference $u(.)$ over their wealth, then the expected utility for each type of consumer, $L-$type and $H-$type from an insurance contract (B, p) will be:

$$u_L(B, p) = \pi_L u(w - L + B - p) + (1 - \pi_L)u(w - p) \tag{1.4.3}$$

$$u_H(B,p) = \pi_H u(w-L+B-p) + (1-\pi_H)u(w-p). \tag{1.4.4}$$

From the properties of $u(.)$ in 1.4.1, $u_L(B,p)$ and $u_H(B,p)$ are also continuous, differentiable, and strictly concave. Moreover, u_L and u_H are strictly increasing in B and strictly decreasing in p. The marginal rate of substitution in each contract (B,p) for low-risk customers is given by:

$$MRS_L(B,p) = \frac{\pi_L u'(w-L+B-p)}{\pi_L u'(w-L+B-p) + (1-\pi_L)u'(w-p)} \tag{1.4.5}$$

and for high-risk type:

$$MRS_H(B,p) = \frac{\pi_H u'(w-L+B-p)}{\pi_H u'(w-L+B-p) + (1-\pi_H)u'(w-p)}. \tag{1.4.6}$$

It can be observed from 1.4.5 and 1.4.6 that both MRS_L and MRS_H are increasing functions for π_L and π_H, and $MRS_L(B,p) < MRS_H(B,p)$ for all (B,p). This implies the single-crossing property, where the indifferent curves of two types of consumers intersect at most one time.

The expected profit from one $L-$type customer for a policy (B^*,p^*) is as follows:

$$\Pi_L = \pi_L(p^* - B^*) + (1-\pi_L)p^* = p^* - \pi_L B^* \tag{1.4.7}$$

Similarly, the expected profit from one $H-$type customer is:

$$\Pi_H = \pi_H(p^* - B^*) + (1-\pi_H)p^* = p^* - \pi_H B^* \tag{1.4.8}$$

The total expected profit of the market from $L-$types and $H-$types is determined according to 1.4.2:

$$\Pi = (1-\varepsilon)\Big[\alpha\Pi_L + (1-\alpha)\Pi_H\Big] + \varepsilon \min\Big(\Pi_L, \Pi_H\Big)$$

1.4.3 Insurance contracts in equilibrium under full ambiguity

We first consider the case when $\varepsilon = 1$. When both insurers are completely ambiguous about the composition of types of customers, the total gains from the two firms equals the lowest between the expected profits of $L-$types and of $H-$types:

$$\Pi = \min\left(\Pi_L, \Pi_H\right) \tag{1.4.9}$$

The following lemma shows that the expected profit of each insurer must be zero.

Lemma 1.4.1. *In every pure strategy Subgame Perfect Nash Equilibrium (SPNE) under full ambiguity, both insurance companies earn zero profit.*

Following 1.4.1, the pooling equilibrium where both types of buyer choose an identical insurance policy would also produce zero profit. In case of complete ambiguity, insurers anticipate the worst case when all buyers are high-risk type, thus ruling out the contribution expected from the low-risk type. Thus, a cream-skimming mechanism that upsets the pooling equilibrium as in the original context fails to exist in case of complete ambiguity.

Theorem 1.4.1. *There is a unique pure strategy pooling equilibrium under full ambiguity where the H-type is fully insured.*

The intuition of 1.4.1 is as follows. Under complete ambiguity, when one insurance firm offers the unique pooling contract that attracts only the $H-$type insurees, Ψ^*, the remaining firm has no incentive to offer a profitable one, as it assumes that the worst situation will not have $L-$ type customers. For any other pooling contract that attracts both types of buyers, since the expected profit of each firm is zero following the Lemma 1.4.1, it can be derived that the unique feasible policy is the null contract. Nevertheless, the null contract is easily upset once the other insurance firm offers a pooling policy better suited for $L-$ type, or implements cream-skimming.

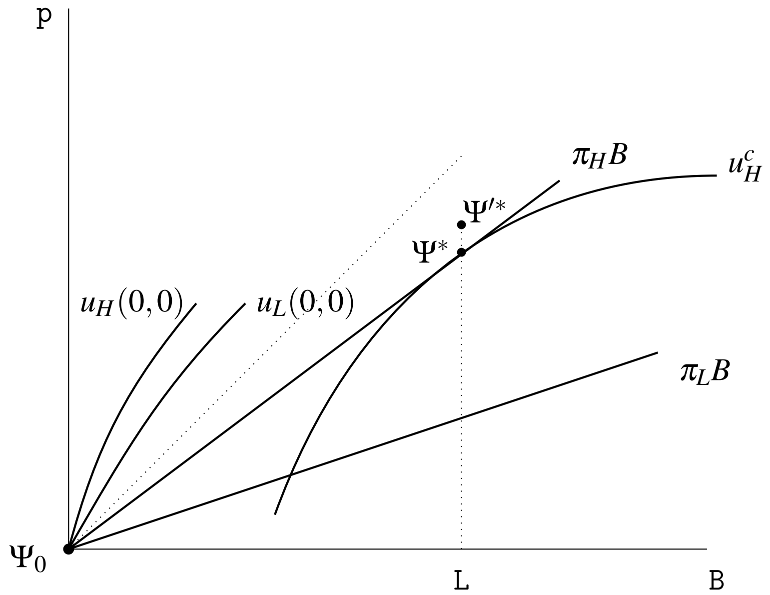

Figure 1.4.1: Pooling contract under complete ambiguity

Figure 1.4.1 illustrates the unique pooling policy Ψ^* in Theorem 1.4.1. As explained above, the null contract Ψ_0 is eliminated in equilibrium, since one firm can offer another pooling contract Ψ'^* which attracts two types of customers, leading to a positive profit upon deviating.

Next, we characterize the separating equilibrium under full ambiguity. Firstly, in every separating equilibrium Ψ_H^*, Ψ_L^* the $H-$type must gain the exact utility in the pooling contract Ψ^*. This can be shown by considering two cases. If the $H-$type is overinsured, then the profit gained from them is less than zero. Under full ambiguity and by ambiguity aversion, firms that offer the overinsurance policy for the $H-$type assume the worst scenario when there is no $L-$type in the market. Thus, their overall profits will be less than zero, which cannot happen in any equilibrium (since firms can always offer a null policy that at least guarantees the zero profit).

Next, we consider the remaining case if the $H-$type is underinsured in some equilibrium. Then in this equilibrium contract, the premium for the $H-$type will cost more than $\pi_H L$ in case he is reimbursed the full loss L. However, one firm can deviate and offer a policy $(L, \pi_H L + \varepsilon)$, with ε small enough so that the $H-$type buyers in this policy pay less for the premium than in the policy in equilibrium. As a result, this policy attracts

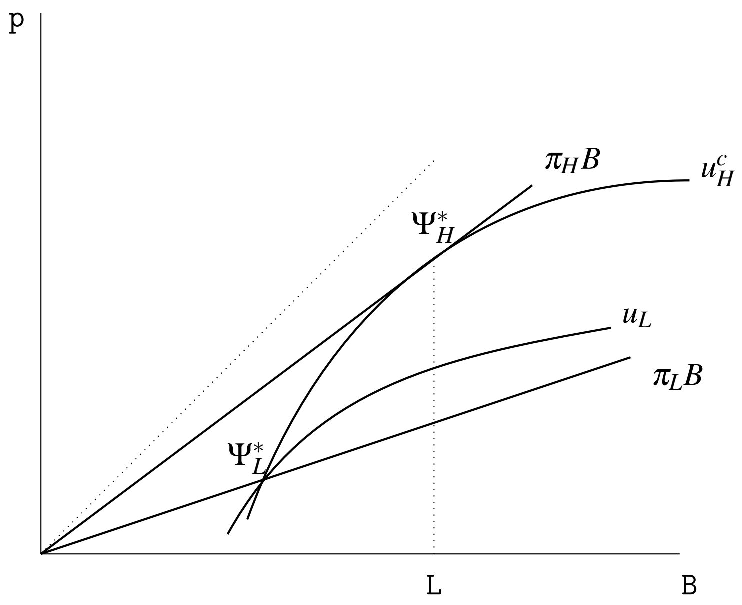

Figure 1.4.2: Separating contract under complete ambiguity

all $H-$types (if the policy also attracts $L-$types, it is obviously more profitable for the deviating company) and guarantees positive profits for the deviating firm, which is a contradiction. Thus, $\Pi_H = 0$ in the separating equilibrium, leading to $\Pi_L = 0$ by 1.4.1. By the incentive constraint, it can be derived that the $H-$type is fully insured while the $L-$type is underinsured. A detailed proof can be found in Jehle and Reny (2011), p 410-412.

Theorem 1.4.2. *The policies* (Ψ_H^*, Ψ_L^*) *selected by L-type and H-type customers in a pure strategy separating equilibrium are illustrated in Figure 1.4.2.*

The optimal contracts Ψ_L^* and Ψ_H^* in the separating equilibrium under ambiguity coincide with those in Rothschild and Stiglitz (1976)' paper. The reason is ambiguity only affects insurance companies' expected profit rather than customers' preferences. Moreover, Lemma 1.4.1 asserts that in any separating equilibrium under ambiguity, each firm earns only zero profit. Therefore, the separating equilibrium under full ambiguity is, in essence, no different from the RS model.

Existence of separating equilibrium under full ambiguity

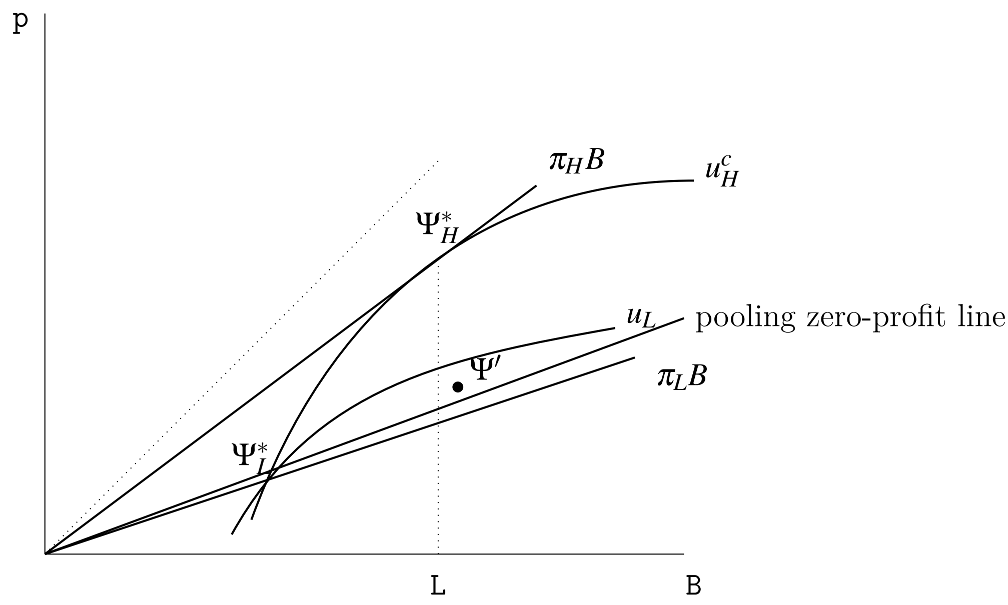

Figure 1.4.3: No separating equilibrium exists when $H-$types are outnumbered by $L-$types.

In the RS model without ambiguity, the separating equilibrium does not always exist. Consider the case when both insurers know precisely the probability of the low-risk type $\alpha(\varepsilon = 0)$. Ψ_H^* and Ψ_L^* are best and second-best contract in separating equilibrium in Figure 1.4.3. If one firm deviates by offering a pooling policy $\Psi' = (B', L')$ that lies below both u_H^c and u_L, then both $L-$types and $H-$types will prefer this policy to their former equilibrium contracts since it strictly improves their utilities. If the proportion of $L-$types dominates $H-$types, or $\alpha \approx 1$, the pooling zero profit line will be close enough to the zero profit line of $L-$types $\pi_L B$.

Thus one firm can earn strictly positive profit by offering $\Psi' = (B', p')$ that lies above the zero profit line of the $L-$type. As the proportion of low-risk buyers dominates high-risk buyers, the profit gained out of the former outweigh the loss caused by the latter. However, when both companies are fully ambiguous about α, or $\varepsilon = 1$, the separating equilibrium always exists since there can not be any pooling equilibrium that not only attracts both types of consumers but also guarantees positive profit to the firm offering it. Indeed, being completely ambiguous about the proportion of each type of customer, insurers assume the worst total expected profit, $p' - \pi_H B'$, when all customers are $H-$types. In other words, firms are prudent in the face of ambiguity as they are completely unsure about the composition of their customers. In light of that, the zero profit line for a pooling contract coincides with the zero profit line of $H-$types $\pi_H B$. Consequently, any pooling contract chosen by a $L-$type customer (and also by a $H-$type one) will lie below the zero profit line and return a negative profit, which is not chosen by any insurance firms. As such, the separating equilibrium exists.

1.4.4 Insurance contracts in equilibrium under intermediate ambiguity

When ambiguity is intermediate, i.e. $\varepsilon \in (0,1)$, the original conclusions in the standard model of RS are robust. The pooling equilibrium does not exist because one deviating firm can offer a new contract that would entice the low-risk type and guarantee a positive profit. Furthermore, the separating contract in equilibrium remains consistent with Theorem 1.4.2 where each type of consumer contributes zero to the firms' profit and there is no cross-subsidization in the model.

Lemma 1.4.2. *In every pure strategy SPNE under ambiguity with* $0 < \varepsilon < 1$*, both insurance companies earn zero profit.*

If there exists a pooling equilibrium when $\varepsilon \in (0,1)$, then following Lemma 1.4.2 the pooling contract must lie in the zero profit line $\pi_k B$ between $\pi_L B$ and $\pi_H B$, which gives rise to a cream-skimming situation when a deviating insurance firm can offer a different pooling contract which makes two types of the customer better off and gain a positive profit from low-risk type. Therefore, a pooling equilibrium also can not exist in this case. We have the following result:

Theorem 1.4.3. *There is no pure strategy pooling equilibria with* $\varepsilon \in (0,1)$.

Figure 1.4.4 illustrates Theorem 1.4.3. Indeed, a rational firm can strategically offer one policy in the region R and lies above $\pi_k B$ to appeal only to the low-risk consumer and earn a strictly positive profit.

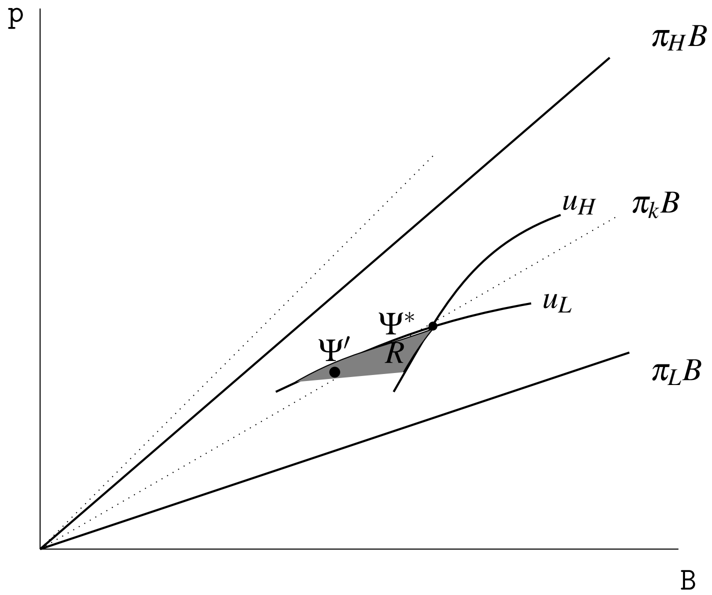

Figure 1.4.4: No pooling equilibrium exists when $\varepsilon \in (0,1)$.

We examine the existence of the separating equilibrium when $\varepsilon \in (0,1)$. Note that as analyzed in the case $\varepsilon = 1$, the separating contract is characterized by a pair of policies (Ψ^*_H, Ψ^*_L) as in 1.4.2. We have already shown that when insurers are completely ambiguous about the composition of customers ($\varepsilon = 1$), there always exists a separating equilibrium as firms are extremely cautious about the worst-case scenario when all consumers are high-risk type. Full ambiguity, thus, rules out the role of any prior subjective belief α when insurers only take into account the extreme case $\alpha = 0$. If insurance firms face insignificant ambiguity about the proportion of customers, or $\varepsilon \approx 0$, and the proportion of the low-risk type dominates the high-risk type ($\alpha \approx 1$), there will be a situation that the separating policy does not exist as in Figure 1.4.3. When the degree of ambiguity is significant, i.e. $\varepsilon \approx 1$, insurance firms possess almost no information regarding the true composition of low-risk customers α. Being ambiguity-averse, firms assume the worst scenario when there is a high chance that H-type customers dominate the market. In this case, no pooling equilibrium exists, and insurers offer separating policies in equilibrium, as analyzed in the case of full ambiguity.

When ε is not close to the two ends 0 and 1, we consider Figure 1.4.5. Firstly, we observe that the zero-profit line $\pi_k B$ concerning the ε-contamination lies above the zero profit line $\pi_p B$ when there is no ambiguity. Indeed, since π_p is defined by $\pi_p = \alpha\pi_L + (1-\alpha)\pi_H$, and $\pi_k = k\pi_L + (1-k)\pi_H = (1-\varepsilon)\alpha\pi_L + (1-(1-\varepsilon)\alpha)\pi_H$, then $\pi_k - \pi_p = \varepsilon\alpha(\pi_H - \pi_L) > 0$.

Consider the tangent $\pi_{u_L}B$ of the indifferent curve for the low-risk type u_L where the policy Ψ_L^* lies in. It can be seen that if the zero profit line $\pi_k B$ is lower than the tangent line $\pi_{u_L}B$, there is always a pooling policy Ψ' which can attract both types of customers and earn a strictly positive. Hence, to ensure the existence of a separating equilibrium, $\pi_k B$ must be higher than $\pi_{u_L}B$, or $\pi_k > \pi_{u_L}$. This is equivalent to $k\pi_L + (1-k)\pi_H > \pi_{u_L}$, hence $\pi_H - k(\pi_H - \pi_L) > \pi_{u_L}$, or $k < \dfrac{\pi_H - \pi_{u_L}}{\pi_H - \pi_L}$. Since $k = (1-\varepsilon)\alpha$, it implies that

$$\varepsilon > 1 - \frac{\pi_H - \pi_{u_L}}{\alpha(\pi_H - \pi_L)} = \varepsilon_\alpha \tag{1.4.10}$$

The intuition of 1.4.10 is as follows. When $\alpha \approx 0$, the RHS of the inequality is very small, thus 1.4.10 holds for every $\varepsilon > 0$. In other words, if firms have a small subjective belief regarding the proportion of $L-$ type, they always separate insurance policies regardless of the ambiguity degree. Ambiguity aversion, in this case, exacerbates the initial belief α of insurers. As α increases along the simplex, ε_α also moves upward. Therefore, the value of ε increases accordingly. This implies that although insurance firms hold a very high belief regarding the proportion of $L-$ type, they can offer the separating equilibrium if ambiguity is significant enough. Moreover, by taking $\alpha = 1$ in 1.4.10, we obtain the highest value of ε_α equal to $1 - \dfrac{\pi_H - \pi_{u_L}}{\pi_H - \pi_L} = \varepsilon_{\max}$. As $\varepsilon_{\max} \in (0,1)$ and it is the highest possible value of ε_α, insurance firms will offer the separating equilibrium when the ambiguity level exceeds it. Therefore, the separating equilibrium is always assured.

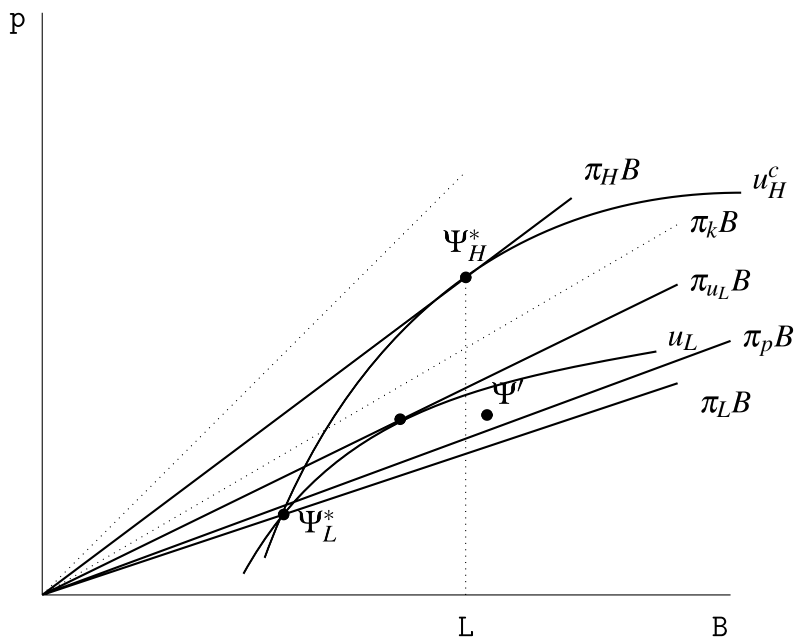

Figure 1.4.5: Existence of separating equilibrium when $\varepsilon < 1$.

1.5 Conclusion

This paper takes inspiration from the seminal work of Rothschild and Stiglitz (1976) where the authors concerned the competitive insurance market with adverse selection. We extend the analysis by introducing ambiguity with respect to the customer composition into the literature.

When insurance firms face complete ambiguity about the proportion of each customer type, a pooling equilibrium exists, as no company is able to attract low-risk customers due to ambiguity aversion. However, when ambiguity is present but not complete, the pooling equilibrium fails to exist, which aligns with the conclusions of the RS model.

Without ambiguity, the primitive model of RS predicts that the separating equilibrium is not guaranteed if the proportion of $H-$types is significantly outweighed by $L-$types. An entrant can strategically offer a more profitable contract that pulls all $L-$types customers out of the separating equilibrium, thus earning positive profit. However, when ambiguity is factored in, such a situation is eliminated. Our results show that under a significant degree of ambiguity, a separating equilibrium always exists, contributing a novel result compared to the RS model.

The study of ambiguity about the distribution of types of consumers can be further explored in a broader class of problems within the insurance market with adverse selection. For instance, investigating signaling models where insurers identify the type of buyers based on received signals under asymmetric information, or considering scenarios where both insurers and consumers face ambiguity presents interesting and complex avenues for future research. In summary, introducing ambiguity into the research of the insurance market poses numerous intriguing questions that will be the focus of future studies.

1.6 Appendix of Proofs

1.6.1 Proof of Lemma 1.4.1

If both $L-$type and $H-$type customers choose the same insurance policy $\Psi_L^* = \Psi_H^* = (B^*, p^*)$, the total expected profit from two insurance firms A and B from 1.4.9 will be $\Pi = \Pi_A + \Pi_B = \min\Big(\Pi_L, \Pi_H\Big) = \min\ (p^* - \pi_L B^*, p^* - \pi_H B^*) = p^* - \pi_H B^*$ since $\pi_L < \pi_H$. We use counterarguments to claim that profit Π_A, Π_B of each firms must be zero. Indeed, firstly both insurance companies' profits are positive ($\Pi_A, \Pi_B \geq 0$) because they can otherwise offer the contract $(0,0)$ and guarantee at least a null profit. Secondly, if there exists one company that gains strictly positive profit, for example, $\Pi_A > 0$, then we have $\Pi_B < \Pi_A + \Pi_B = p^* - \pi_H B^* = \Pi$. If firm B deviates by offering a pooling contract $\Psi_B' = (B^* + \varepsilon, p^*)$ with $\varepsilon > 0$ then both types of insuree will take this contract since they will be more covered with $B' = B + \varepsilon$ with the same premium. Then the expected profit for firm B is $\Pi'_B = \min(\Pi'_L, \Pi'_H) = \min(p^* - \pi_L(B^* + \varepsilon), p^* - \pi_H(B^* + \varepsilon)) = p^* - \pi_H(B^* + \varepsilon)$. When ε is small enough Π'_B can be close to Π arbitrarily. Specifically, there exists ε that makes firm B financially better off, which contradicts the fact that (B^*, p^*) is pure strategy SPNE.

Now consider the case each type of insuree takes a distinctive insurance policy, or $\Psi_L^* = (B_L^*, p_L^*) \neq \Psi_H^* = (B_H^*, p_H^*)$. Because low-risk buyers will choose (B_L^*, p_L^*) and high-risk buyer will choose (B_H^*, p_H^*) then the incentive compatibility constraints must be satisfied: $u_L(\Psi_L^*) \geq u_L(\Psi_H^*)(1)$, and $u_H(\Psi_H^*) \geq u_H(\Psi_L^*)(2)$. By single crossing property, these two inequalities can not happen simultaneously. Thus we can assume that $u_L(\Psi_L^*) > u_L(\Psi_H^*)$ WLOG. Then the total expected profits of firms A and B are $\Pi = \Pi_A + \Pi_B = \min(\Pi_L, \Pi_H)$ $= \min\ (p_L^* - \pi_L B_L^*, p_H^* - \pi_H B_H^*)$. Suppose that profit of firm A $\Pi_A > 0$ then $\Pi_B < \Pi$ and $\Pi = \Pi_A + \Pi_B > 0$. Because the total expected profit from two firms $\Pi = \min\ (p_L^* - \pi_L B_L^*, p_H^* - \pi_H B_H^*) > 0$, then we denote $\Pi = p_j - \pi_j B_j = \min\left(p_L^* - \pi_L B_L^*, p_H^* - \pi_L B_H^*\right) > 0$. Here $j = L$ or H and $\pi_j = \pi_L$ or π_H accordingly.

Consider the situation when insurance firm B proposes a pair of policies ($\Psi_L^\sigma = (B_L^* + \sigma, p_L^*)$) and ($\Psi_H^\beta = (B_H^* + \beta, p_H^*)$. Both types of consumers will prefer the new policy to the old one (Ψ_L, Ψ_H) due to the former's higher coverage. We show that one can choose σ and β small enough to satisfy the incentive compatibility condition (1) and (2) , which means only low-risk agents will choose Ψ_L^σ and only high-risk agents will choose Ψ_H^β. Indeed, since $u_L(\Psi_L^*) > u_L(\Psi_H^*)$, we can find β small enough such that $u_L(B_L^*, p_L^*) > u_L(B_H^* + \beta, p_H^*) = u_L(\Psi_H^\beta)(3)$. Now fix β, then there exists σ small enough such that $u_H(\Psi_H^\beta) = u_H(B_H^* + \beta, p_H^*) > u_H(B_L^* + \sigma, p_L^*) = u_H(\Psi_L^\sigma)(4)$. There is indeed σ such that (4) happens because $u_H(\Psi_H^\beta) > u_H(\Psi_H^*) \geq u_H(\Psi_L^*) = \lim_{\sigma \to 0} u_H(\Psi_L^\sigma)$. Take this σ to the contract Ψ_L^σ and from (3) we have the followings : $u_L(\Psi_L^\sigma) = u_L(B_L^* + \sigma, p_L^*) > u_L(B_L^*, p_L^*) > u_L(\Psi_H^\beta)(5)$. (4) and (5) show that $L-$types and $L-$types will choose contract Ψ_L^σ and Ψ_H^β.

Therefore, firm B will attract all types of consumers when it offers the pair of policies $(\Psi_L^\sigma, \Psi_H^\beta)$. By choosing the value of β and σ close enough to 0, firm B can still make $\Pi' = \min\left(p_L^* - \pi_L(B^* + \sigma), p_H^* - \pi_H(B_H^* + \beta)\right)$ close enough to total expected profit π_j and thus increase the expected profit of firm B. This is again a contradiction to our assumption that (Ψ_L^*, Ψ_H^*) is a SPNE. Hence both insurance companies must gain profit zero. ■

1.6.2 Proof of Theorem 1.4.1

Assume that both types of customers, choose the same policy $\Psi^* = (B^*, p^*)$ in equilibrium. The expected profit from a archetypal $L-$type is $\Pi_L = p^* - \pi_L B^*$ and from $H-$types is $\Pi_H = p^* - \pi_H B^*$. Then the total expected profits of two insurance companies under ambiguity is $\Pi = (1-\varepsilon)\left[\alpha \Pi_L + (1-\alpha)\Pi_H\right] + \varepsilon \min\left(\Pi_L, \Pi_H\right) = \min\left(\Pi_L, \Pi_H\right)$ (since $\varepsilon = 1$) $= p^* - \pi_H B^*$. From 1.4.1 both firms earn zero profits, therefore $\Pi = \Pi_A + \Pi_B = 0$, therefore it follows that $p^* - \pi_H B^* = 0$ and $\Pi_L \geq 0$. Now consider the case when $\Pi_L > 0$. As the total expected profit from two types is zero, insurance firms assume the worst case is when there are no $L-$ type customers. Thus, companies only offer policy for the $H-$type. The unique policy in equilibrium is $\Psi^* = (L, \pi_H L)$, which follows the standard result in the insurance market context.

We consider if $\Pi_L = 0$. From the expression of Π_L and Π_H, we can conclude that $(B^*, p^*) = (0,0)$ is the unique possible pooling contract. Note that with this insurance policy, both indifferent curves of two types of consumers will lie above the zero profit line $\pi_H B$ and $\pi_L B$ (we can also derive this by observing that $u_L(L, \pi_L L) > u_L(0,0)$ so policy $(L, \pi_L L)$ will lie below the utility of policy (0,0). A similar argument applies to high-risk buyers.

If one firm deviates from the null policy by offering $\Psi'^* = (B'^*, p'^*) = (L, \pi_H L + \varepsilon)$, the $H-$types will be pulled to the new contract since they receive higher utility. This new policy generates the strictly positive expected profit for the deviating, which is a contradiction. ■

1.6.3 Proof of Lemma 1.4.2

Firstly we prove the lemma for the pooling case. If both types of consumers choose the policy $\Psi_L = \Psi_H = (B^*, p^*)$, the total expected profit of two insurance firms under ambiguity with ε-contamination will be $\Pi = \Pi_A + \Pi_B = (1-\varepsilon)\Big[\alpha\Pi_L + (1-\alpha)\Pi_H\Big] + \varepsilon \min\Big(\Pi_L, \Pi_H\Big)$, where $\Pi_L = p^* - \pi_L B^*$ and $\Pi_H = p^* - \pi_H B^*$. Since $\pi_L < \pi_H$, we have $\Pi_H < \Pi_L$ and $\min\Big(\Pi_L, \Pi_H\Big) = \Pi_H$. We re-express the total expected profit of the market as $\Pi = (1-\varepsilon)\Big[\alpha\Pi_L + (1-\alpha)\Pi_H\Big] + \varepsilon\Pi_H = (1-\varepsilon)\alpha\Pi_L + (1-\varepsilon)\Pi_H - (1-\varepsilon)\alpha\Pi_H + \varepsilon\Pi_H = (1-\varepsilon)\alpha\Pi_L + (1-(1-\varepsilon)\alpha)\Pi_H = k\Pi_L + (1-k)\Pi_H(1)$, with $k = (1-\varepsilon)\alpha$. Since $0 < \varepsilon < 1$ and $0 < \alpha < 1$ we also have $0 < k < 1(2)$. If there is one firm gaining strictly positive profit, for example, $\Pi_A > 0$, then $\Pi_B < \Pi$. Now if firm B deviates to offer only policy $\Psi'_B = (B^* + \delta, p)$ with $\delta > 0$ then the new policy which covers more for accident $B^* + \delta$ will attract both types of players. The total expected profit from both types for firm B will be $\Pi' = (1-\varepsilon)\Big[\alpha\Pi'_L + (1-\alpha)\Pi'_H\Big] + \varepsilon \min\Big(\Pi'_L, \Pi'_H\Big)$. Here $\Pi'_L = p^* - \pi_L(B^* + \delta)$ and $\Pi'_H = p^* - \pi_H(B^* + \delta)$. Still $\min(\Pi'_L, \Pi'_H) = \Pi'_H$, thus we can use (1) to determine the total expected profit $\Pi'_B = k\Pi'_L + (1-k)\Pi'_H = k\Big(p^* - \pi_L(B^*+\delta)\Big) + (1-k)\Big(p^* - \pi_H(B^*+\delta)\Big) = k(p^* - \pi_L B^*) + (1-k)(p^* - \pi_H B^*) - \delta\Big(k\pi_L + (1-k)\pi_H\Big) = \Pi - \delta\Big(k\pi_L + (1-k)\pi_H\Big)$.

From (2) we have $k\pi_L + (1-k)\pi_H$ is a strictly positive constant, so for δ sufficiently small Π'_B is smaller than Π arbitrarily. Because $\Pi > \Pi_B$ there exists $\Pi'_B > \Pi_B$ and thus firm B is better off by deviating, which contradicts the assumption that (B^*, p^*) is a pure SPNE.

In the case when the separating equilibrium is established, by following the proof of Lemma 1.4.1, we also derive a similar result as in the pooling case which dictates that each insurance company earn zero profit. ■

1.6.4 Proof of Theorem 1.4.3

Assume that both types of customer choose the same insurance policy (B^*, p^*), then according to the proof of 1.4.2, the total expected profit of two firms will be $\Pi = k(p^* - \pi_L B^*) + (1-k)(p^* - \pi_H B^*) = p^* - (k\pi_L + (1-k)\pi_H)B^* = p^* - \pi_k B^* = 0(1)$, where the last equation comes from each firm will only earn zero profit following Lemma 1.4.2 (here we denote $\pi_k = k\pi_L + (1-k)\pi_H$). If $p^* = 0$ then we must have $B^* = 0$. This trivial case can be shown to be unstable because one firm can offer a contract that attracts all high-risk consumers and still guarantees a strictly positive. We only, therefore, consider the case when $p^* > 0$ and thus $B^* > 0$ from. Consider the situation when one firm in the shaded region R in Figure 1.4.4 offers a pooling policy $\Psi' = (B', p')$. The utility of low-risk customers will be improved whereas the utility of high-risk ones will be worse off. Consequently, all the low-risk type customers will move to a new policy and the high-risk one will not change their initial policy (B^*, p^*). Hence, the expected profit for the deviating firm is only from $L-$types: $\Pi' = p' - \pi_L B'$. Since we have denoted $\pi_k = k\pi_L + (1-k)\pi_H$ and $0 < k < 1$, $\pi_L < \pi_H$ then $\pi_L < \pi_k < \pi_H$. Hence $\Pi' = p' - \pi_L B' > p' - \pi_k B'(2)$.

But from (1) we already know that $p^* - \pi_k B^* = 0$. Hence, if another firm offers a contract $(B', p'\ )$ which is close enough to the initial contract (B^*, p^*) in region R which guarantees a positive profit from low-risk type, (2) will be strictly positive and $\Pi' > 0$, which is a contradiction because the deviating firm is financially better off. Henceforth the pooling equilibrium contract does not exist. ■

Chapter 2

Moral Hazard Under Ambiguity

We examine a two-outcome principal-agent problem where both the principal and the agent face ambiguity regarding the project's probability of success. We show that when effort is observable, the agent's welfare remains unaffected by ambiguity, while the principal's welfare declines relative to the non-ambiguous model. When effort is unobservable, ambiguity aversion leads to two distinct possible expected profits for the principal, which differs from traditional models without ambiguity. Consequently, ambiguity influences the principal's decision on whether to induce effort, diverging from the decisions made in the absence of ambiguity.

2.1 Introduction

The orthodox principal-agent problem assumes that while outcomes are random, their probability distributions are deterministic. This assumption leads to a unique first-best wage scheme when the principal can observe the agent's effort and a unique second-best wage scheme when effort is unobservable (Holmström (1979)). However, many argue that this assumption is overly restrictive (Weinschenk (2010)), as there are numerous situations where one or both parties cannot precisely evaluate the probability of a project's success. In this paper, we relax this assumption by considering the case where both the principal and the agent face ambiguity regarding the project's success probability.

Imprecise probabilities have been extensively documented across various areas of financial markets. For example, Koufopoulos and Kozhan (2016) examined competitive insurance contracts under ambiguity aversion, while Easley and O'Hara (2009) showed that ambiguity aversion can lead to non-participation in markets and influence asset prices. However, to the best of our knowledge, the impact of ambiguity on decision-making in the classic moral hazard framework has not been thoroughly explored.

By introducing ambiguity into the beliefs of both the principal and the agent, we employ the maxmin expected utility framework (Gilboa and Schmeidler (1989)) to examine the classic principal-agent problem under both symmetric and asymmetric information scenarios. In our model, both parties face ambiguity regarding the likelihood of the project's success, unlike much of the existing literature, where typically only the buyer is considered ambiguity-averse and the seller ambiguity-neutral (Koufopoulos and Kozhan (2016)). Introducing ambiguity aversion for both parties highlights the fact that their perceptions differ and may even be oppositional: the best-case scenario for one party could represent the worst-case scenario for the other.

Weinschenk (2010) also studies the principal-agent problem under ambiguity. However, their work differs from ours in two ways. First, their model does not assume ambiguity aversion for the principal. Second, the assumption that the set of performance distributions is effort-independent limits its flexibility. Our approach adopts a more flexible paradigm where the set of distributions depends on the manager's contribution. In this way, by manipulating the agent's effort, which adjusts the spectrum of possible priors for the project, the principal can control the worst-case scenario in the context of ambiguity aversion.

This paper contributes to the moral hazard literature by characterizing the optimal compensation schemes when effort is observable and unobservable under ambiguity. We first show that, if the manager's effort is observable, offering a risk-free contract that pays the agent a fixed wage, regardless of the project outcome, is optimal for the principal. This mirrors the best contract in Holmström (1979)'s non-ambiguity model, where the principal eliminates the risk of overpayment when the good outcome is realized. Due to ambiguity aversion, the principal assumes the worst probability of project success for any induced effort, lowering the principal's welfare under ambiguity, although the agent's utility remains unchanged when effort is observable.

When effort is unobservable, ambiguity aversion leads to multiple potential expected profit outcomes for the principal, with the worst-case scenario possibly occurring when the project's chance of success is highest. As a result, ambiguity can shift the principal's decision: while the principal may induce the agent to exert high effort if the true priors are known, they may opt for low effort under ambiguity to maximize expected profit, and vice versa. To our knowledge, this situation has not been explored in the existing moral hazard literature under ambiguity. Furthermore, we analyze conditions under which the principal's choice of effort remains consistent in both the presence and absence of ambiguity.

This paper is organized as follows. Section 2.2 introduces the model of moral hazard (MH) under ambiguity. The main results are presented in Section 2.3. Section 2.4 covers comparative statics, and Section 2.5 concludes the paper. All proofs are provided in the Appendix 2.6.

2.2 Model

With the exception of the ambiguity assumption, we adopt the setup of the classic principal-agent model. Consider a principal (he) who aims to design a contract to motivate an agent (she) to complete a one-time venture. Throughout this paper, we restrict our analysis to cases where the project outcome is binary: a good outcome or a bad outcome. Specifically, let $\mathscr{P} = (\pi_g, \pi_b) \in R^2_+$ represent the set of realized profits, where π_b denotes the lower payoff in the case of a bad outcome, and π_g represents the higher payoff in the case of a good outcome. We assume that $\pi_g > \pi_b$ in line with these definitions.[1] A compensation scheme (w_g, w_b) rewards the agent with a payment contingent on the project's outcome: she receives w_g in the case of a good result and w_b in the case of a bad result.[2]

The set of one-dimensional effort choices available to the agent is denoted by $E = \{e_0, e_1\}$, where e_0 and e_1 represent low and high effort, respectively. For each level of effort e implemented, the disutility of exerting the effort is denoted by $d(e)$. The agent's utility given by compensation w and the disutility c_e is defined by $u(w,e)$, where u satisfies $u_w(w,c_e) > 0$, $u_{ww}(w,e) \leq 0$ for all (w,e). The first inequality implies that the agent prefers more payment than less for the same level of effort implemented. The second implies that the agent is weakly risk-averse over income lotteries. Finally, to be more realistic, we impose that the manager prefers low effort to high effort for the same level of payment he or she receives.

Throughout this chapter, we only consider the special case $u(w,e) = v(w) - c(e)$, which has gained much of consideration in the literature. Literally, agent's utility is determined only by the difference between the utility from salary $v(.)$ and the disutility from implementing effort $c(e)$. Note that in this form of representation of $u(w,e)$, the disutility $c(e)$ has been completely separated from $v(.)$, implying that it does not depend on her utility function for the conpensation scheme. With assumptions on $u(w,e)$ defined earlier, it can be derived that $v'(w) > 0, v''(w) \leq 0$, and $c(e_1) > c(e_0)$ (we will use c_1 and c_0 for simplicity from onwards), indicating that exerting high effort incurs a higher cost than low effort.

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1. The outcomes in Mas-Colell et al. (1995) take values in $\underline{\pi}, \bar{\pi}$, where $\underline{\pi} < \bar{\pi}$. We can assume that $\underline{\pi} = \pi_b$ and $\bar{\pi} = \pi_g$, with only $\underline{\pi}$ and $\bar{\pi}$ in support of distribution.
2. We do not require output-based salary schemes to be positive. A negative compensation scheme (e.g., $w_b < 0$), can be interpreted as a situation where the agent bears responsibility for the bad outcome and faces a salary reduction to offset the project's losses. However, in the context of the insurance market, compensation schemes are typically required to be positive. See Koufopoulos and Kozhan (2016).

The agent needs the minimum utility (outside option) $\underline{u}$ to accept the contract the principal has offered. This is typically the utility the agent could achieve from their next best alternative, which could be doing nothing (a default utility), working for another principal, or pursuing another opportunity. The principal must offer the agent at least this level of utility, or else the agent will choose the outside option rather than engage in the contract.

The standard literature on the moral hazard problem typically assumes a stochastic relationship between the agent's effort and the project's performance. In our model, this assumption is relaxed, as both the principal and the agent are unable to precisely determine the effort-dependent probability of the project's success. For any effort $e \in E$, both parties face Knightian uncertainty regarding the prospect of the project outcome, resulting in a range of possible beliefs concerning the true likelihood of success. We outline the following assumptions to specify how the project's performance is statistically associated with the level of exerted effort under ambiguity.

Assumption 1. *Given that the agent implements effort e_i, $i \in \{0,1\}$, both the principal and the agent face a set of set of possible probabilities regarding the true probability of a good outcome, represented by the interval $[\underline{p_i}, \bar{p}_i]$. The true probability, from both parties' perspectives, could take any value $p \in [\underline{p_i}, \bar{p}_i]$.*

It is important to emphasize that an objective probability for the favorable outcome π_g which depends on the agent's effort, exists. Without ambiguity, the effort-dependent distribution of the project's good outcome is perfectly known to both the principal and the agent. However, the presence of external factors that create ambiguity prevents both parties from accurately identifying the exact distribution of outcomes, leading them to hold a spectrum of possible probabilities.[3]

3. Exogenous events that contribute to ambiguity have been examined in recent literature; see Etner and Spaeter (2010).

Assumption 2. *For every effort e_i, $\underline{p_i} < \bar{p_i}$. Moreover, $\underline{p_1} > \underline{p_0}$ and $\bar{p_1} > \bar{p_0}$.* [4]

The first inequality in Assumption 2 is obvious. The second inequality in Assumption 2 suggests that both parties, who are ambiguity-averse, assume that the lowest probability of project success when the agent exerts high effort is still higher than the lowest probability of success when the agent exerts low effort. Lastly, the third inequality in Assumption 2 states that implementing e_1 may result in a large likelihood of success that could not occur when implementing e_0. Otherwise, it will follow that $[\underline{p_1}, \bar{p_1}] \subset [\underline{p_0}, \bar{p_0}]$ if $\bar{p_1} \leq \bar{p_0}$ and $\underline{p_1} > \underline{p_0}$, making high effort e_1 indistinguishable from low effort e_0.

Remark 2.2.1. *From Assumption 2, it can be derived that for every effort e_i, $[\underline{p_i}, \bar{p_i}] \neq [0,1]$. For instance, if $[\underline{p_0}, \bar{p_0}] = [0,1]$, then $\bar{p_1} > \bar{p_0} = 1$ as the above argument, which is impossible. Similarly, if $[\underline{p_1}, \bar{p_1}] = [0,1]$, then $0 = \underline{p_1} > \underline{p_0}$, which cannot occur.*

Given that both the principal and the agent are ambiguity-averse,[5] their utility function follows the MEU criterion introduced by Gilboa and Schmeidler (1989). For each contract (w_g, w_b) offered by the principal, denote e^* as the effort the agent selects from $\{e_0, e_1\}$. To ensure the agent participates in the agreement, the contract must meet two constraints: the participation constraint (P.C.), which guarantees that the agent will be paid at least a required utility level, and the incentive constraint (I.C.), which ensures the principal can indirectly induce the agent to exert the desired effort. Therefore, the optimal incentive scheme (w_g, w_b) solves the following problem **(P)**:

$$\max_{(w_g, w_b)} \min_{p \in [\underline{p_e^*}, \bar{p_e^*}]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b) \tag{2.2.1}$$

4. Dumav and Khan (2018) shows that if an effort a_k is implementable, then for any other effort a_j with lower cost, $Q^A(a_j) \setminus Q^A(a_k) \neq \emptyset$, where $Q^A(a)$ represents a non-empty, compact, and convex set of the agent's effort-dependent distribution regarding technology. Although Assumption 2 is interpreted differently in our paper, it remains consistent with Dumav's findings.

5. We also examined the case where the principal is ambiguity-neutral, in which the problem becomes much simpler. Without ambiguity aversion, the principal faces a unique expected profit rather than a range of non-unique profit expectations, consistent with the classic moral hazard framework by Holmström. Beyond that, most of the results remain qualitatively similar to those presented in this paper.

subject to the participation constraint (P.C.) and the incentive constraint (I.C.) as follows (in that order):

$$\min_{p \in [\underline{p}_e^*, \bar{p}_e^*]} pv(w_g) + (1-p)v(w_b) \geq k_e \tag{2.2.2}$$

$$e^* = \arg\max_{e \in \{e_0, e_1\}} \min_{p \in [\underline{p_e}, \bar{p}_e]} pv(w_g) + (1-p)v(w_b) - c_e \tag{2.2.3}$$

Note that in the (P.C.) constraint 2.2.2, $k_e = c_e + \underline{u}$ for $e \in \{e_0, e_1\}$, which represents the agent's *reservation utility with respect to effort* e that she implements. Intuitively, the minimum expected payment for the agent under ambiguity, given that she implements effort e, must at least equal the disutility of exerting that effort plus her outside option.

In the following section, we derive the key results that highlight how ambiguity aversion affects the principal-agent relationship and agent effort decisions.

2.3 Main results

2.3.1 Preliminaries

Before presenting our results, we first outline several key facts that are essential to understanding the analysis in this chapter. The proofs can be found in the appendix.

Fact 2.3.1. *For every* $e \in E$, $\min_{p \in [\underline{p_e}, \bar{p}_e]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b)$ *is concave for* $(w_g, w_b) \in R^2$.

Fact 2.3.2. *In the domain where* $\underline{p_e} v(w_g) + (1 - \underline{p_e}) v(w_b)$ *satisfies the participation constraint, the expected compensation scheme* $pw_g + (1-p)w_b$ *for agent under any true distribution* $p \geq \underline{p_e}$ *will decrease if* w_b *increases or* w_g *decreases, and vice versa.*

The concavity of the objective function, as stated in Fact 2.3.1, supports the application of convex programming techniques, following Rockafellar (1970). The fact 2.3.2 further implies that in the region where the participation constraint binds at the lowest probability $\underline{p_e}$, the expected compensation to the agent when the principal reduces w_g and increases w_b for every $p \geq \underline{p_e}$ decreases. In other words, when the agent is incentivized to accept a contract under the lowest probability of success $\underline{p_e}$, the principal can lower the expected wage payment by offering a higher w_b in the scenario of a bad outcome and a lower w_g if a successful outcome occurs. Since $1 - p \leq 1 - \underline{p_e}$, the probability of the venture generating a bad outcome under p is lower than under $\underline{p_e}$. Consequently, by increasing the compensation in the event of a bad outcome, the principal effectively reduces the overall expected cost.

In the next section, we first examine the case where the principal can observe the agent's effort.

2.3.2 The optimal contract with observable effort

In the classic literature on symmetric information, the principal faces no ambiguity and can observe the effort exerted by the agent. An optimal contract provides the agent with a risk-sharing arrangement that ensures they receive at least their reservation utility level for all realizations of the project outcome. The principal then decides the observable managerial effort in E which generates higher expected profits after deducting the disutility incurred by implementing the effort.

When ambiguity is considered and effort is observable, the principal can still incentivize the agent to exert the desired level of effort by offering a fixed-wage scheme. This scheme fully insures the agent against any risk, similar to the model without ambiguity. Due to ambiguity aversion, the agent anticipates the worst-case scenario, in which the probability of a negative outcome is highest. As a result, the agent expects a lower payoff compared to the standard model. As a result, any payment policy that differentiates payments based on output becomes more costly for the principal than a fixed-wage scheme, since the agent would demand a higher payment for favorable outcomes. Let Π_e^{ob} denote the principal's MEP when effort is observable, and let $w_{e,g}^{ob}$ and $w_{e,b}^{ob}$ represent the payments when the outcome is good or bad, respectively, with the observable effort e under ambiguity. The optimal wage scheme, given ambiguity and observability, is specified as follows:

Proposition 2.3.1. *The optimal contract under ambiguity, when effort is observable, specifies that the principal chooses the effort level $e^* \in E$ that maximizes $\Pi_e^{ob} = \underline{p_e}\pi_g + (1 - \underline{p_e})\pi_b - v^{-1}(k_e)$. In this contract, the manager receives a fixed wage, $w_{e,g}^{ob} = w_{e,b}^{ob} = v^{-1}(k_e)$, regardless of the project outcome.*

The fixed contract in Proposition 2.3.1 ensures that the manager always receives a constant level of compensation, regardless of the project outcome, thereby maintaining identical reservation utility across all states of the world. However, the principal's expected profit for any induced effort e is lower in the presence of ambiguity compared to the Holmström (1979) model. Due to ambiguity aversion, the principal is less optimistic than in the no-ambiguity case, leading to a reduced expected profit, as they assume the worst possible likelihood of project success. Consequently, ambiguity aversion diminishes the principal's welfare, even when the agent's effort is fully observable, compared to the no-ambiguity scenario.

2.3.3 The optimal contract with unobservable effort

When the principal cannot observe the agent's effort, they can still indirectly control the agent's behavior by offering a contract that satisfies the incentive constraint. The principal can achieve the first-best contract by inducing the agent to exert low effort e_0, regardless of the effect of ambiguity. This compensation scheme eliminates any uncertainty about the agent's wage by offering a fixed wage, $w^{un}_{e_0,g} = w^{un}_{e_0,b} = v^{-1}(k_0)$, which we refer to as contract CT0. This risk-sharing contract satisfies the participation constraint, ensuring that the agent will choose low effort, as it yields a higher maxmin expected utility compared to high effort.

Moreover, this contract imposes the same expected cost on the principal as in the symmetric information case. Since the principal can never achieve a better outcome under asymmetry than under symmetry, [6] contract CT0 is the optimal choice for the principal when low effort is induced. Let $\Pi^{un}_{e_0}$ denote the principal's maxmin expected profit (MEP) from inducing the agent to exert e_0 under ambiguity and unobservable effort. We derive the following:

Proposition 2.3.2. *When effort is unobservable, the principal can induce the agent to implement low effort e_0 by offering the risk-sharing contract CT0. This payment scheme is optimal because it guarantees the same expected profit as in the observable case, which is $\Pi^{un}_{e_0} = \underline{p_0}\pi_g + (1-\underline{p_0})\pi_b - v^{-1}(k_0)$.*

The contract CT0 in Proposition 2.3.2 ensures that the agent will implement the project with low effort e_0 since their expected utility will always equal k_0, regardless of the amount of effort exerted. Therefore, the manager has no incentive to choose the higher effort e_1. The formal proof for Proposition 2.3.2 is omitted due to its simplicity.

When the principal aims to induce the agent to implement high effort e_1, the compensation must be sufficiently attractive. The share the agent receives in the case of a successful project (good outcome) must be lucrative enough to offset the lower payment in the event of a bad outcome, while still satisfying the participation constraint.

6. When effort is observable, the project owner can always offer the sub-optimal non-observability contract and allow the manager to choose the effort herself. Therefore, the principal can never do better with an unobservable contract. See Mas-Colell et al. (1995), Proposition 14.B.2.

Let $\Delta_v = v(w_g) - v(w_b)$ represent the difference in the utility of the wages corresponding to the good and bad outcomes when e_1 is implemented, and $\Delta_{c/\underline{p}} = \dfrac{c_1 - c_0}{\underline{p_1} - \underline{p_0}}$ represent the difference in effort costs relative to the lowest probability of success. Based on our assumptions, both $\Delta_v > 0$ and $\Delta_{c/\underline{p}} > 0$. We now state the necessary and sufficient conditions under which the agent is induced to implement the high effort:

Proposition 2.3.3. *The agent chooses to implement the high effort e_1 when effort is unobservable to the principal if and only if $\Delta_v \geq \Delta_{c/\underline{p}}$ and the participation constraint is satisfied.*

In a moral hazard problem without ambiguity, where the distribution of the project's possible outcomes is fully known, the second-best wage scheme that induces the agent to implement e_1 yields a unique expected profit for the principal. However, under ambiguity, this may not hold. The principal and the agent may interpret the worst-case scenario differently when high effort is implemented. For the agent, the worst-case scenario arises when the project succeeds with the lowest probability. In contrast, for the principal, the worst case could occur either when the project succeeds with the lowest probability or with the highest probability. The latter occurs because ambiguity introduces an equilibrium where the project has the highest likelihood of success. In such a scenario, the principal faces a higher probability of paying a substantial wage to the agent for a successful outcome. This becomes particularly detrimental for the principal when the profit margin between the good and bad outcomes is relatively small. Thus, ambiguity can result in two distinct expected payoffs for the principal.

We denote $w^{un}_{e_1,g}$ and $w^{un}_{e_1,b}$ as the payments made to the agent when the project outcome is good or bad, respectively, to induce the effort e_1 when the agent's effort is unobservable. The optimal contract is then specified as follows:

Theorem 2.3.1. *When effort is unobservable, under ambiguity aversion, the principal induces the agent to implement the high effort* e_1 *by offering the compensation scheme* $w^{un}_{e_1,g} = v^{-1}\left(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\right), w^{un}_{e_1,b} = v^{-1}\left(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\right)$*. His MEP is specified by the following:*

- $\Pi^{un}_{e_1,\bar{p_1}} = \bar{p_1}\pi_g + (1-\bar{p_1})\pi_b - \bar{p_1}v^{-1}\left(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\right) - (1-\bar{p_1})v^{-1}\left(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\right)$ *if* $v^{-1}\left(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\right) - v^{-1}\left(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\right) \geq \Delta_\pi$
- $\Pi^{un}_{e_1,\underline{p_1}} = \underline{p_1}\pi_g + (1-\underline{p_1})\pi_b - \underline{p_1}v^{-1}\left(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\right) - (1-\underline{p_1})v^{-1}\left(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\right)$ *if* $v^{-1}\left(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\right) - v^{-1}\left(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\right) < \Delta_\pi$

In theorem 2.3.1, $w^{un}_{e_1,g} = v^{-1}\left(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\right)$ and $w^{un}_{e_1,b} = v^{-1}\left(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\right)$ represent the payments contingent on the project's good and bad outcomes under ambiguity, respectively. Similar to the second-best contract in the classic model, both the participation constraint 2.2.3 and the incentive constraint **??** are binding. Due to ambiguity aversion, however, the agent assumes the worst-case scenario, where the project is successful with the lowest probability $\underline{p_1}$. The compensation scheme defined in theorem 2.3.1 is the unique solution that satisfies both the two constraints, given that the project succeeds with probability $\underline{p_1}$.

The intuition behind Theorem 2.3.1 is as follows: if the difference between the two output-based wages exceeds the difference between π_g and π_b, the principal considers the worst-case scenario to be when the project succeeds with the highest probability. In this situation, since the agent is compensated significantly if the project succeeds, the principal faces a scenario where they are more likely to pay a large wage to the agent, which is the least favorable outcome for the principal. A similar argument can be applied in the opposite direction. If the agent's wage for the good outcome is not significantly higher than the wage for the bad outcome, the principal prefers the scenario where the project succeeds with a high probability rather than with a low probability. However, since the principal is ambiguity-averse, their least expected scenario is when the project succeeds with the lowest probability.

2.4 Comparative Statics

How does ambiguity aversion affect the principal's decision regarding the level of effort to induce from the agent? To what extent does the final decision mirror the scenario without ambiguity? In this section, we first illustrate how ambiguity can alter the principal's choice. Specifically, while the principal may prefer to induce high effort in the absence of ambiguity, ambiguity aversion could prompt them to reconsider and opt for low effort when confronted with uncertainty.

We consider the case where effort is unobservable. If effort were observable, the principal could offer the contracts constructed in this example to the agent, which would result in the same behavior from the agent. Without ambiguity, both the principal and the agent know that the project succeeds with probability p_0 if the agent implements low effort, and with probability p_1 if she implements high effort ($p_0 < p_1$). In the classic model without ambiguity, the principal can offer a risk-free contract $w^{wa}_{e_0,g} = w^{wa}_{e_0,b} = v^{-1}(k_0)$ where $w^{wa}_{e_0,g}((w^{wa}_{e_0,b})$ represent the agent's wage when the outcome is good or bad, respectively, if he wishes to induce the agent to implement low effort e_0. The expected profit for the principal in this case is:

$$\Pi^{wa}_{e_0} = p_0\left(\pi_g - v^{-1}(k_0)\right) + (1-p_0)\left(\pi_b - v^{-1}(k_0)\right) = p_0\Delta_\pi + \pi_b - v^{-1}(k_0)$$

If the principal aims to induce the agent to contribute high effort e_1, from the results in the standard model without ambiguity, two constraints (I.C.) and (P.C.) must bind. [7] The optimal contract to induce the agent to implement e_1 is:

$$(w^{wa}_{e_1,g}, w^{wa}_{e_1,b}) = \left(v^{-1}(k_1 + (1-p_1)\Delta_{c/p}), v^{-1}(k_1 - p_1\Delta_{c/p})\right)$$

7. see Mas-Colell et al. (1995), Lemma 14.B.1

where $\Delta_{c/p} = \dfrac{c_1 - c_0}{p_1 - p_0}$. Similar to the low-effort case, the expected profit for the principal when the agent exerts high effort e_1 is:

$$\Pi_{e_1}^{wa} = p_1\Delta_\pi + \pi_b - p_1 v^{-1}\Big(k_1 + (1-p_1)\Delta_{c/p}\Big) - (1-p_1)v^{-1}\Big(k_1 - p_1\Delta_{c/p}\Big)$$

Without ambiguity, the agent is induced to work with the high effort e_1 if $\Pi_{e_1}^{wa} > \Pi_{e_0}^{wa}$, or equivalently: $p_1\Delta_\pi + \pi_b - p_1 v^{-1}\Big(k_1 + (1-p_1)\Delta_{c/p}\Big) - (1-p_1)v^{-1}\Big(k_1 - p_1\Delta_{c/p}\Big) > p_0\Delta_\pi + \pi_b - v^{-1}(k_0)$. This inequality can be rearranged as:

$$(p_1 - p_0)\Delta_\pi > p_1 v^{-1}\Big(k_1 + (1-p_1)\Delta_{c/p}\Big) + (1-p_1)v^{-1}\Big(k_1 - p_1\Delta_{c/p}\Big) - v^{-1}(k_0) \quad (2.4.1)$$

Within the ambiguity context, by Proposition 2.3.2 and Theorem 2.3.1 the principal prefers the low effort e_0 being implemented to the high effort e_1 if the maxmin expected profit gained from the former is higher than from the latter,i.e. $\Pi_{e_0}^{un} > \Pi_{e_1}^{un}$:

$$\underline{p_0}\pi_g + (1-\underline{p_0})\pi_b - v^{-1}(k_0) > \bar{p}_1\pi_g + (1-\bar{p}_1)\pi_b - \bar{p}_1 v^{-1}\Big(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\Big) - (1-\bar{p}_1)v^{-1}\Big(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\Big)$$

We reorganize the above inequality as:

$$(\bar{p}_1 - \underline{p_0})\Delta_\pi < \bar{p}_1 v^{-1}\Big(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\Big) + (1-\bar{p}_1)v^{-1}\Big(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\Big) - v^{-1}(k_0) \quad (2.4.2)$$

It follows from Theorem 2.3.1 that we require the condition under which the principal's worst-case distribution, given that e_1 is implemented, is $\bar{p}_1$, as expressed in 2.4.2:

$$v^{-1}\Big(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\Big) - v^{-1}\Big(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\Big) \geq \Delta_\pi \quad (2.4.3)$$

We provide an example to demonstrate that the set of solutions to the system of inequalities 2.4.1, 2.4.2, and 2.4.3 is non-empty. This result implies that, under ambiguity, the principal will induce the agent to implement low effort rather than high effort, which would have been the optimal choice in the absence of ambiguity.

By defining $v^{-1}\left(k_1+(1-\underline{p_1})\Delta_{c/\underline{p}}\right)=\underline{a_1}$, $v^{-1}\left(k_1-\underline{p_1}\Delta_{c/\underline{p}}\right)=\underline{b_1}$, $v^{-1}(k_0)=c_0$, $v^{-1}\left(k_1+(1-p_1)\Delta_{c/p}\right)=a_1$, $v^{-1}\left(k_1-p_1\Delta_{c/p}\right)=b_1$, we can then reformulate inequalities 2.4.1, 2.4.2, and 2.4.3 as follows:

$$(p_1-p_0)\Delta_\pi > p_1a_1+(1-p_1)b_1-c_0 \tag{2.4.4}$$

$$(\bar{p_1}-\underline{p_0})\Delta_\pi < \bar{p_1}\underline{a_1}+(1-\bar{p_1})\underline{b_1}-c_0 \tag{2.4.5}$$

$$\underline{a_1}-\underline{b_1}\geq\Delta_\pi \tag{2.4.6}$$

Note that $a_1, b_1, \underline{a_1}$, and $\underline{b_1}$ exhibit pairwise relationships, namely:

$$v(\underline{a_1})-v(\underline{b_1})=\Delta_{c/\underline{p}} \tag{2.4.7}$$

$$v(a_1)-v(b_1)=\Delta_{c/p} \tag{2.4.8}$$

$$v(\underline{b_1})-v(b_1)=p_1\Delta_{c/p}-\underline{p_1}\Delta_{c/\underline{p}} \tag{2.4.9}$$

The system of inequalities 2.4.1 to 2.4.3 will have a solution if there is a tuple $(a_1, b_1, \underline{a_1}, \underline{b_1})$ and a set of parameters $\boldsymbol{I}=(p_0, p_1, \underline{p_0}, \underline{p_1}, \bar{p_0}, \bar{p_1}, \Delta_\pi, \Delta_{c/\underline{p}}, \Delta_{c/p}, c_1, c_0)$ that satisfies 2.4.4 to 2.4.9. Thus, it is sufficient to construct at least one solution for 2.4.4 to 2.4.9.

The procedure is as follows: first, we choose $v(x)=5-e^{-x}$, then $v(.)$ is strictly increasing and concave. [8]

The set of input parameters includes $p_0=0.03$, $p_1=0.06$, $\underline{p_0}=0.01$, $\underline{p_1}=0.02$, $\bar{p_0}=0.04$, $\bar{p_1}=0.07$, and $c_1-c_0=0.006$ then $\Delta_{c/\underline{p}}=0.6, \Delta_{c/p}=0.2$. Note that the way of choosing probabilities satisfies Assumption 1 and Assumption 2, since $p_0<p_1$, $\underline{p_0}<\underline{p_1}$, and $\underline{p_0}<p_0<\bar{p_0}$, $\underline{p_1}<p_1<\bar{p_1}$. Next, we set $c_0=v^{-1}(k_0)=0.2$, then $k_0=4.181$, $k_1=k_0+c_1-c_0=4.187$. Finally, we set $\Delta_\pi=1$.

8. More precisely, we choose $v(x)=A-e^{-x}$ as it is an archetypal utility function in the literature, where A is some constant. For the purposes of our calculations, we set $A=5$, which keeps the output-based wages within a reasonable range.

Since $v^{-1}(y) = -\ln(5-y)$, the compensation schemes for the agent to implement high effort e_1 with respect to I, following Theorem 2.3.1 are as follows:

For the case where the principal faces ambiguity and effort is unobservable:

$$w^{un}_{e_1,g} = \underline{a_1} = v^{-1}\left(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\right) = 1.491\,,\; w^{un}_{e_1,b} = \underline{b_1} = v^{-1}\left(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\right) = 0.192$$

For the case without ambiguity and effort is unobservable:

$$w^{wa}_{e_1,g} = a_1 = v^{-1}\left(k_1 + (1-p_1)\Delta_{c/p}\right) = 0.47\,,\; w^{wa}_{e_1,b} = b_1 = v^{-1}\left(k_1 - p_1\Delta_{c/p}\right) = 0.192.$$

Remark 2.4.1. *Note that in our choice for the parameters in I, regardless of whether the principal has to face ambiguity, he offers the agent the same payments in the case of an unfavorable outcome given that the high effort is implemented (0.192). This is only for convenience purposes for our calculation.*

We make some observations from the compensation schemes in the cases where the agent is induced to implement either e_0 or e_1. Without ambiguity, the agent working with the low effort e_0 is rewarded the identical contract $w^{wa}_{e_0,g} = w^{wa}_{e_0,b} = v^{-1}(k_0) = c_0 = 0.2$. In contrast, the compensation scheme for implementing e_1 is $(w^{wa}_{e_1,g}, w^{wa}_{e_1,b}) = (0.47, 0.192)$. Therefore, if the project succeeds, the payment to the agent (0.47) is relatively modest, making it worthwhile for the principal to induce the agent to implement e_1 in exchange for a higher probability of earning a more significant profit ($\Delta_\pi = 1$).

Under ambiguity, when the principal motivates the agent to implement e_1, they must offer a significant payment of $w^{un}_{e_1,g} = 1.491$ if the project succeeds. In this case, Δ_π is not substantial enough to cover the agent's wage, which leads the principal to induce the agent to exert low effort instead.

The intuition behind our example is that when ambiguity is significant in the sense that $\underline{p_0}$ and $\underline{p_1}$ are close to zero and the effort is unobservable, the ambiguity-averse agent will only accept working with high effort if the discrepancy between the two output-based wages under ambiguity $w^{un}_{e_1,g}$ $(\underline{a_1})$ and $w^{un}_{e_1,b}$ $(\underline{b_1})$ is high enough, leading to inequalities 2.4.5 and 2.4.6 are satisfied. Since Δ_π is sufficiently attractive to the principal, as mentioned earlier, the principal prefers to induce the agent to exert low effort under ambiguity.

However, if the difference between the two project outcomes, Δ_π is significant enough for the principal, and he knows the probabilities p_0 and p_1 with certainty, he will induce the agent to implement high effort without ambiguity, provided that the cost of implementing low effort c_0 is not cheap. In this case, inequality 2.4.4 is satisfied.

It is evident that 2.4.4, 2.4.5, and 2.4.6 hold for the parameters in I selected above. By the nature of our construction, 2.4.7, 2.4.8, and 2.4.9 are satisfied. For inequality 2.4.4, we compare the left-hand side (LHS) and the right-hand side (RHS):

$$LHS = (p_1 - p_0)\Delta_\pi = 0.03 > RHS = p_1 a_1 + (1 - p_1) b_1 - c_0 = 0.009.$$

For inequality 2.4.5, we get $LHS = (\bar{p_1} - \underline{p_0})\Delta_\pi = 0.06 < RHS = \bar{p_1}\underline{a_1} + (1 - \bar{p_1})\underline{b_1} - c_0 = 0.083$.

For inequality 2.4.6, we have $\underline{a_1} - \underline{b_1} = 1.4916 - 0.1923 > \Delta_\pi = 1$.

Thus, all conditions from 2.4.4 to 2.4.9 are met. This confirms our example, demonstrating that the principal induces the agent to exert low effort under ambiguity, whereas high effort is induced in the absence of ambiguity.

Notice that, by the nature of our notation, $\underline{a_1}$, $\underline{b_1}$, and b_1 depends only on a_1. Specifically,

$$b_1 = v^{-1}\Big(v(a_1) - \Delta_{c/p}\Big)$$

$$\underline{a_1} = v^{-1}\left(v(a_1) + (1-\underline{p_1})\Delta_{c/\underline{p}} - (1-p_1)\Delta_{c/p}\right)$$

$$\underline{b_1} = v^{-1}\left(v(a_1) - \underline{p_1}\Delta_{c/\underline{p}} - (1-p_1)\Delta_{c/p}\right)$$

Let $a^{wa}_{e_1,p_1,p_0}$, $a^{am}_{e_1,\bar{p}_1,\underline{p_0}}$ be threshold values of inequalities 2.4.4 and 2.4.5, [9] and a_1^* be the threshold value of 2.4.6. These values can be interpreted as a level of payment to the agent that would make the principal indifferent between inducing high effort and low effort, both with and without ambiguity. We also denote $a^{am}_{e_1,\underline{p_1},\underline{p_0}}$ as a unique solution of $(\underline{p_1} - \underline{p_0})\Delta_\pi = \underline{p_1}\underline{a_1} + (1-\underline{p_1})\underline{b_1} - c_0$.

Since all these values exist uniquely, we now derive a condition under which the principal does not change his decision when facing ambiguity and effort is unobservable.

Theorem 2.4.1. *The Principal will make the same decision to induce the agent to implement high effort under ambiguity and without ambiguity if and only if either:*

$$a_1^* < v^{-1}\left(k_1 + (1-p_1)\Delta_{c/p}\right) < \min\left(a^{wa}_{e_1,p_1,p_0}, a^{am}_{e_1,\bar{p}_1,\underline{p_0}}\right) \text{ when } \underline{a_1} - \underline{b_1} \geq \Delta_\pi (i), \text{ or}$$

$$v^{-1}\left(k_1 + (1-p_1)\Delta_{c/p}\right) < \min\left(a^{wa}_{e_1,p_1,p_0}, a^{am}_{e_1,\underline{p_1},\underline{p_0}}, a_1^*\right) \text{ when } \underline{a_1} - \underline{b_1} < \Delta_\pi (ii).$$

The principal will make the same decision to induce the agent to implement low effort under ambiguity and without ambiguity if and only if either:

$$\max\left(a^{wa}_{e_1,p_1,p_0}, a^{am}_{e_1,\underline{p_1},\underline{p_0}}\right) < v^{-1}\left(k_1 + (1-p_1)\Delta_{c/p}\right) < a_1^* \text{ when } \underline{a_1} - \underline{b_1} < \Delta_\pi (iii), \text{ or}$$

$$v^{-1}\left(k_1 + (1-p_1)\Delta_{c/p}\right) > \max\left(a^{wa}_{e_1,p_1,p_0}, a^{am}_{e_1,\bar{p}_1,\underline{p_0}}, a_1^*\right) \text{ when } \underline{a_1} - \underline{b_1} \geq \Delta_\pi (iv).$$

9. A threshold value turns an inequality into an equality. Therefore, $a^{wa}_{e_1,p_1,p_0}$ is a solution of $(p_1 - p_0)\Delta_\pi = p_1 a_1 + (1-p_1)b_1 - c_0$ for the inequality 2.4.4, we have similar definitions for $a^{am}_{e_1,\bar{p}_1,\underline{p_0}}$ and other cases. Note that all threshold values reference a_1, since $\underline{a_1}$, $\underline{b_1}$, and b_1 are functions a_1.

Remark 2.4.2. *If inequality 2.4.4 has no threshold value, continuity implies that the inequality will be either strictly negative or positive. If* $(p_1 - p_0)\Delta_\pi < p_1 a_1 + (1-p_1)b_1 - c_0$ *holds for all* a_1*, the principal is unwilling to induce the agent to choose* e_1 *since he expects to pay a costly amount to the agent if the project succeeds. A similar argument applies to 2.4.5, where there exists a threshold value* a_1 *(as a function of* $\underline{a_1}$*) for* $(\bar{p}_1 - \underline{p_0})\Delta_\pi = \bar{p}_1 \underline{a_1} + (1-\bar{p}_1)\underline{b_1} - c_0$*. If* $(p_1 - p_0)\Delta_\pi > p_1 a_1 + (1-p_1)b_1 - c_0$ *for all* a_1*, condition* (i) *becomes unnecessary and can be ruled out.Therefore, we focus only on the cases where 2.4.4 and 2.4.5 have threshold values.*

Condition (i) predicts that even when the principal has to pay the agent a significant amount if the project succeeds, they may still induce high effort under ambiguity. This could occur if inequality 2.4.5 holds, for instance, when c_0 is large and $\underline{p_0}$ is very small. In other words, when exerting low effort requires a high reservation utility and the worst probability of project success with low effort is too low under ambiguity, the principal will direct the agent to exert high effort.

Conditions (ii) suggest that if the payments to the agent, both in the presence and absence of ambiguity, are below a certain threshold (specifically, the minimum threshold at which the agent is indifferent between exerting low or high effort), the principal will induce the agent to exert high effort.

The same arguments can be applied to conditions (iii) and (iv). In condition (iii), even if the cost of inducing high effort when the project succeeds is not too high, the principal may still prefer to induce low effort. This is likely when $\underline{p_0}$ is large, $\underline{p_0} \approx \underline{p_1}$, and exerting low effort requires a small reservation utility. In this case, inducing low effort is more beneficial for the principal. In condition (iv), when the cost of paying the agent for high effort exceeds a certain level under ambiguity or without ambiguity, the principal will induce low effort.

When the principal can observe the agent's contribution, we derive a set of conditions under which the principal can maintain their decision under ambiguity, although these conditions are less complex than those in Theorem 2.4.1.

If both parties know the effort-dependent distribution of the project's outcome with certainty, the principal will induce the agent to exert high effort if it results in a higher expected profit than low effort. Based on the standard contract in moral hazard when effort is observable and there is no ambiguity, we get:

$$p_1\left(\pi_g - v^{-1}(k_1)\right) + (1-p_1)\left(\pi_b - v^{-1}(k_1)\right) > p_0\left(\pi_g - v^{-1}(k_0)\right) + (1-p_0)\left(\pi_b - v^{-1}(k_0)\right),$$

or equivalently:

$$(p_1 - p_0)\Delta_\pi > v^{-1}(k_1) - v^{-1}(k_0),$$

which implies:

$$p_1 - p_0 > \frac{v^{-1}(k_1) - v^{-1}(k_0)}{\Delta_\pi}$$

Under ambiguity, and given the principal's ambiguity aversion, the principal's expected profit from inducing the agent to implement effort e is described as in Proposition 2.3.1. Although the manager is still offered a fixed wage, which eliminates uncertainty for them, the project owner assumes the worst-case scenario, where the chance of the project succeeding is minimized. In this case, the agent is incentivized to implement e_1 if:

$$\underline{p_1}\pi_g + (1-\underline{p_1})\pi_b - v^{-1}(k_1) > \underline{p_0}\pi_g + (1-\underline{p_0})\pi_b - v^{-1}(k_0)$$

By a similar transformation to the case without ambiguity, it can be derived that:

$$\underline{p_1} - \underline{p_0} > \frac{v^{-1}(k_1) - v^{-1}(k_0)}{\Delta_\pi}$$

The same reasoning applies when the agent is induced to contribute low effort e_0. The principal will stick to inducing e_0 with and without ambiguity if and only if:

$$p_1 - p_0 < \frac{v^{-1}(k_1) - v^{-1}(k_0)}{\Delta_\pi}, \text{ and } \underline{p_1} - \underline{p_0} < \frac{v^{-1}(k_1) - v^{-1}(k_0)}{\Delta_\pi}.$$

Summing up all derived conditions, we obtain the following:

Theorem 2.4.2. *If the principal can observe the agent's effort, they will induce the agent to implement high effort under ambiguity and without ambiguity if* $\min\left(p_1 - p_0, \underline{p_1} - \underline{p_0}\right) > \frac{v^{-1}(k_1) - v^{-1}(k_0)}{\Delta_\pi}$*, and implement low effort under both two cases if* $\max\left(p_1 - p_0, \underline{p_1} - \underline{p_0}\right) < \frac{v^{-1}(k_1) - v^{-1}(k_0)}{\Delta_\pi}$.

The intuition behind Theorem 2.4.2 is straightforward. In the first case, if both $p_1 - p_0$ and $\underline{p_1} - \underline{p_0}$ exceed a certain threshold, the principal is assured that the project's likelihood of success is high enough, regardless of ambiguity, when high effort is exerted. This "likelihood guarantee" mitigates ambiguity aversion, encouraging the principal to persuade the agent to implement e_1.

A similar reasoning applies when the principal induces the agent to exert low effort, both under ambiguity and without it: if $p_1 - p_0$ and $\underline{p_1} - \underline{p_0}$ are below a certain threshold, the upper bound of p_1 and p_1 ensures that the principal prefers the agent to choose low effort e_0. Since the principal is ambiguity-averse, they focus on the lower bound $\bar{p}_1$ rather than $\underline{p_1}$. As a result, the principal avoids the risk of inducing high effort because, in the worst-case scenario, high effort does not significantly increase the project's likelihood of success, while the compensation paid to the agent, $v^{-1}(k_1)$ is much higher than that in the low-effort case, $v^{-1}(k_0)$.

2.5 Conclusion

This chapter incorporates ambiguity into the moral hazard problem, where both the principal and the agent hold a set of prior beliefs regarding the probability of the project's success. We show that the effect of ambiguity aversion is two-fold. First, it reduces the principal's welfare compared to the non-ambiguity case when efforts are observable. Second, ambiguity introduces variability in the principal's expected profits, which may cause the principal to alter their choice of the agent's effort compared to the non-ambiguity scenario. Our results offer insights into seemingly counterintuitive behavior in the moral hazard problem, where ambiguity can make inducing high effort less desirable than inducing low effort. Although higher effort increases the probability of project success, the substantial payment required to incentivize the agent ultimately results in lower expected profit compared to implementing lower effort.

While we employed the Gilboa-Schmeidler maxmin expected utility framework to address the impact of ambiguity on the moral hazard problem, this is not the only approach available. Ambiguity can also be modeled through alternative frameworks, e.g., the smooth ambiguity model (Klibanoff et al. (2005)). Within the context of ambiguity aversion, it is reasonable to expect that these alternative models would yield similar conclusions to those derived using Gilboa's framework. Namely, the principal's expected profit under ambiguity is lower than in the non-ambiguity case, regardless of the specific model of ambiguity used. Exploring ambiguity through alternative frameworks may lead to further unexpected results beyond those presented in this chapter, and this will be the focus of future research.

2.6 Appendix of Proofs

2.6.1 Proof of Fact 2.3.1

Define $h(w_g, w_b) = \min_{p \in [\underline{p}, \bar{p}]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b)$, $(w_g, w_b) \in R^2$. The function $h(.)$ represents the minimum of a family of linear functions. According to a well-established result in convex optimization (Boyd and Vandenberghe (2004), Rockafellar (1970)), the minimum of a set of concave functions is also concave. Since linear functions are both concave and convex, we conclude that $h(.)$ is also concave. ■

2.6.2 Proof of Fact 2.3.2

Let $(w_g, w_b) \in R^2$ such that $\underline{p_e} v(w_g) + (1 - \underline{p_e}) v(w_b) = k_e$, we show that the compensation scheme function $q(w_g, w_b) = pw_g + (1-p)w_b$ is a decreasing function of w_b for any $p \geq \underline{p_e}$.

We define $u_g = v(w_g)$, $u_b = v(w_b)$, then $w_g = v^{-1}(u_g) = \phi(u_g)$ and $w_b = v^{-1}(u_b) = \phi(u_b)$. Since $v(.)$ is concave and strictly increasing, $\phi(.)$ is convex and increasing.

$q(w_g, w_b) = pw_g + (1-p)w_b = p\phi(u_g) + (1-p)\phi(u_b) = p\phi(u_g) + (1-p)\phi\left(\frac{k_e - \underline{p_e} u_g}{1 - \underline{p_e}}\right) =$ $h(u_g)$. Since then $h'(u_g) = p\phi'(u_g) - \frac{1-p}{1-\underline{p_e}} \underline{p_e} \phi'\left(\frac{k_e - \underline{p_e} u_g}{1 - \underline{p_e}}\right)$.

As $p \geq \underline{p_e}$, we have $p \geq \frac{1-p}{1-\underline{p_e}} \underline{p_e}$. Moreover, $\phi'(u_g) > \phi'(u_b) = \phi'(\frac{k_e - \underline{p_e} u_g}{1 - \underline{p_e}})$, given that Φ is convex and increasing. Therefore, $h'(u_g) \geq 0$, leading to $q(w_g, w_b)$ being an increasing (decreasing) function with respect to $w_g(w_b)$. This completes the claim in 2.3.2. ■

2.6.3 Proof of Proposition 2.3.1

Consider a compensate scheme (w_g, w_b) offered by the principal. Under the observability assumption, the incentive constraint 2.2.3 in problem **(P)** is unnecessary. We state the problem **(P)** only with 2.2.1 and 2.2.2 when the principal induces the agent to implement an observable effort e as follows:

$$\max_{(w_g,w_b)} \min_{p\in[\underline{p_e},\bar{p}_e]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b) \tag{2.6.1}$$

subject to

$$\min_{p\in[\underline{p_e},\bar{p}_e]} pv(w_g) + (1-p)v(w_b) \geq k_e(\mathbf{P.C}) \tag{2.6.2}$$

Now we consider two cases as follows:

a. If the principal offers any "unconventional" contract (w_g, w_b), that is, when $w_g \leq w_b$, from the P.C. constraint 2.6.2 we have $\min_{p\in[\underline{p_e},\bar{p}_e]} pv(w_g) + (1-p)v(w_b) = \min_{p\in[\underline{p_e},\bar{p}_e]} (v(w_g) - v(w_b)) + v(w_b) = \bar{p}_e(v(w_g) - v(w_b)) + v(w_b) \geq k_e$.

From 2.6.1 the worst scenario of expected profit for the principal given inducing the agent to implement the effort e is $E = \min_{p\in[\underline{p_e},\bar{p}_e]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b) = \min_{p\in[\underline{p_e},\bar{p}_e]} p(\Delta_\pi - \Delta_w) + \pi_b - w_b = \underline{p_e}(\Delta_\pi - \Delta_w) + \pi_b - w_b = \underline{p_e}\pi_g + (1-\underline{p_e})\pi_b - \big(\underline{p_e}w_g + (1-\underline{p_e})w_b\big)$, since $\Delta_\pi - \Delta_w > 0$, due to $\Delta_\pi > 0$ and $\Delta_w = w_g - w_b \leq 0$.

Hence, E will be maximized when the expected cost $\underline{p_e}w_g + (1-\underline{p_e})w_b$ is minimized. From the convexity of $v(.)$, $v(\underline{p_e}w_g + (1-\underline{p_e})w_b) \geq \underline{p_e}v(w_g) + (1-\underline{p_e})v(w_b) \geq \bar{p}_e v(w_g) + (1-\bar{p}_e)v(w_b) \geq k_e$, where the one before the penultimate comes from $w_g \leq w_b$ under the case we consider. Therefore $\underline{p_e}w_g + (1-\underline{p_e})w_b \geq v^{-1}(k_e)$. it implies that the optimal contract is when all the mentioned inequalities occur, i.e., $w_g = w_b = v^{-1}(k_e)$ The principal's expected profit in this case will be:

$$\max E = \underline{p_e}\pi_g + (1-\underline{p_e})\pi_b - v^{-1}(k_e) \tag{2.6.3}$$

b. Now we consider conventional contracts when $w_g > w_b$. In this domain, the participation constraint 2.6.2 becomes $\underline{p_e} v(w_g) + (1 - \underline{p_e}) v(w_b) \geq k_e$. The original optimal problem (P) can be reorganized as follows:

$$\max_{(w_g, w_b)} \min_{p \in [\underline{p_e}, \bar{p}_e]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b) \tag{2.6.4}$$

such that:

$$k_e - \underline{p_e} v(w_g) - (1 - \underline{p_e}) v(w_b) \leq 0 \tag{2.6.5}$$

$$w_b - w_g < 0 \tag{2.6.6}$$

As $\min_{p \in [\underline{p_e}, \bar{p}_e]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b)$ is concave following Fact 2.3.1, two constraints of the optimal problem are convex (given $v(.)$ is concave) and differentiable, the Rockefella theorem (Rockafellar 1970) is necessary and sufficient to find local maximizers of (P). The Lagrangian is defined as follows:

$$\begin{aligned} \mathscr{L} = \min_{p \in [\underline{p_e}, \bar{p}_e]} & \; p(\pi_g - w_g) + (1-p)(\pi_b - w_b) \\ & + \lambda \left(\underline{p_e} v(w_g) - (1 - \underline{p_e}) v(w_b) - k_e \right) \\ & + \mu (w_g - w_b) \end{aligned} \tag{2.6.7}$$

We consider the domain $D^1 = (w_g, w_b) | \{0 < \Delta_w = w_g - w_b < \Delta_\pi = \pi_g - \pi_b\}$. In D^1 we have $\min_{p \in [\underline{p_e}, \bar{p}_e]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b) = \min_{p \in [\underline{p_e}, \bar{p}_e]} p(\Delta_\pi - \Delta_w) + \pi_b - w_b = \underline{p_e}(\pi_g - w_g) + (1 - \underline{p_e})(\pi_b - w_b)$.

The two F.O.C conditions with respect to w_g and w_b are $\frac{\partial \mathscr{L}}{\partial w_g} = -\underline{p_e} + \lambda \underline{p_e} v'(w_g) + \mu = 0$, and $\frac{\partial \mathscr{L}}{\partial w_b} = -(1 - \underline{p_e}) + \lambda (1 - \underline{p_e}) v'(w_g) - \mu = 0$. Since $w_g > w_b$ and $\mu(w_g - w_b) = 0$, we have $\mu = 0$. But if we set this value of μ to the expression of two F.O.C conditions $\frac{\partial \mathscr{L}}{\partial w_g}$, $\frac{\partial \mathscr{L}}{\partial w_g}$, it can be seen that $v'(w_g) = v'(w_b) = 1/\lambda$, which can not happen since $w_g > w_b$. Therefore, there is no local maximizer (w_g, w_b) in D^1. Furthermore, $\sup_{(w_g, w_b)} \underline{p_e}(\pi_g - w_g) + (1 -$

$\underline{p_e})(\pi_b - w_b) = \underline{p_e}\pi_g + (1-\underline{p_e})\pi_b - \inf_{(w_g,w_b)}\left(\underline{p_e}w_g + (1-\underline{p_e})w_b\right)$. As $v(\underline{p_e}w_g + (1-\underline{p_e})w_b) \geq \underline{p_e}v(w_g) + (1-\underline{p_e})v(w_b) \geq k_e$, $\inf_{(w_g,w_b)}\left(\underline{p_e}w_g + (1-\underline{p_e})w_b\right) = v^{-1}(k_e)$. This implies that the principal can not be better off by offering any contract $\Delta_w < \Delta_\pi$, expectedly, in the domain D^1 than by offering a risk-sharing contract $w_g = w_b$ as in 2.6.3.

In the domain $D^2 = (w_g, w_b)|\{0 < \Delta_w = \Delta_\pi\}$, we apply the necessary condition of the Rockafella theorem for local maximizers to derive the following: [10]

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \begin{bmatrix} \mathscr{L}_{w_g} \\ \mathscr{L}_{w_b} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \min_{p\in[\underline{p}_e,\bar{p}_e]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b) + \\ \lambda(\underline{p}_e v(w_g) + (1-\underline{p}_e)v(w_b) - k_e) + \mu(w_g - w_b) \end{bmatrix}_{w_g} \\ \begin{bmatrix} \min_{p\in[\underline{p}_e,\bar{p}_e]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b) + \\ \lambda(\underline{p}_e v(w_g) + (1-\underline{p}_e)v(w_b) - k_e) + \mu(w_g - w_b) \end{bmatrix}_{w_b} \end{bmatrix} \tag{2.6.8}$$

Since $\min_{p\in[\underline{p}_e,\bar{p}_e]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b) = \underline{p}_e(\pi_g - w_g) + (1-\underline{p}_e)(\pi_b - w_b)$ if $\pi_g - w_g \geq \pi_b - w_b$, or $\bar{p}_e(\pi_g - w_g) + (1-\bar{p}_e)(\pi_b - w_b)$ if $\pi_g - w_g < \pi_b - w_b$, $\mu = 0$ since $w_g > w_b$, and $\underline{p}_e < \bar{p}_e$, the expression of 2.6.8 can be reduced to:

$$-\bar{p}_e + \lambda \underline{p}_e v'(w_g) \leq 0 \leq -\underline{p}_e + \lambda \underline{p}_e v'(w_g)$$

and

$$-(1-\underline{p}_e) + \lambda(1-\underline{p}_e)v'(w_b) \leq 0 \leq -(1-\bar{p}_e) + \lambda(1-\underline{p}_e)v'(w_g)$$

If $\lambda = 0$, then the first inequality implies that $-\bar{p}_e \leq 0 \leq -\underline{p}_e$, and the second inequality implies that $-(1-\underline{p}_e) \leq 0 \leq -(1-\bar{p}_e)$. This can only occur when $\underline{p}_e = 0$ and $\bar{p}_e = 1$, which is a contradiction following Remark 2.2.1. Therefore, $\lambda > 0$, and the participation constraint is binding, leading to the optimal policy (w_g^*, w_b^*), if it exists, being the (unique)

10. Notice that in the expression of the F.O.C, we used the sub-derivative as the objective function is non-smooth.

solution defined by:

$$\begin{aligned} \underline{p}_e v(w_g) + (1-\underline{p}_e)v(w_b) = k_e \\ w_g - w_b = \pi_g - \pi_b \end{aligned} \tag{2.6.9}$$

Notice that the uniqueness of the solution (w_g^*, w_b^*) in 2.6.9 comes from $v(.)$ being strictly increasing. The expected profit for the principal from offering (w_g^*, w_b^*) will then be $\pi_b - w_b^*$. However, this contract is worse than the identical one defined in 2.6.3. Indeed, one can verify that $\pi_b - w_b^* \leq \underline{p_e}\pi_g + (1-\underline{p_e})\pi_b - v^{-1}(k_e)$, which is equivalent to $\underline{p_e}\Delta_\pi + w_b^* > v^{-1}(k_e)$. But the last inequality holds because $v(\underline{p_e}\Delta_\pi + w_b^*) = v(\underline{p_e}(w_g^* - w_b^*) + w_b^*) = v(\underline{p_e}w_g^* + (1-\underline{p_e})v(w_b^*)) \geq \underline{p_e}v(w_g^*) + (1-\underline{p_e}v(w_b^*) = k_e$, due to the convexity of $v(.)$.

Now we consider the domain $D^3 = (w_g, w_b)|\{0 < \Delta_\pi < \Delta_w\}$. In this domain $\min_{p\in[\underline{p_e^*},\bar{p}_e^*]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b) = \bar{p}_e(\pi_g - w_g) + (1-\bar{p}_e)(\pi_b - w_b)$. The two F.O.C conditions with w_g and w_b are, respectively (notice that $\mu = 0$ as in the previous argument): $\frac{\partial \mathscr{L}}{\partial w_g} = -\bar{p}_e + \lambda \underline{p_e} v'(w_g) = 0$, $\frac{\partial \mathscr{L}}{\partial w_b} = -(1-\underline{p_e}) + \lambda(1-\underline{p_e})v'(w_g) = 0$. Given that $v(.)$ is strictly increasing, one can derive that $\lambda > 0$, $v'(w_g) = \frac{\bar{p}_e}{\lambda \underline{p_e}}$, and $v'(w_b) = \frac{1-\bar{p}_e}{\lambda(1-\underline{p_e})}$. But since $\frac{\bar{p}_e}{\lambda \underline{p_e}} > \frac{1-\bar{p}_e}{\lambda(1-\underline{p_e})}$ due to $\bar{p}_e > \underline{p_e}$, it leads to $v'(w_g) > v'(w_b)$, which is a contradiction since $w_g > w_b$ and $v(.)$ is concave. As the system of two equations of F.O.C conditions has no solution, there is no local maximizer in D^3. We will next show that $\sup\left(\min_{p\in[\underline{p_e^*},\bar{p}_e^*]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b)\right)$ cannot be larger the expected profit for the principal defined in 2.6.3 under constraints 2.6.5 and 2.6.6.

Indeed, in domain D^3 with $\Delta_\pi < \Delta_w$ and $w_g > w_b$, $\sup\left(\min_{p\in[\underline{p_e^*},\bar{p}_e^*]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b)\right) = \sup\left(\bar{p}_e(\pi_g - w_g) + (1-\bar{p}_e)(\pi_b - w_b)\right) = \bar{p}_e\pi_g + (1-\bar{p}_e)\pi_b - \inf\left(\bar{p}_e w_g + (1-\bar{p}_e)w_b\right)$. Let's $C = \bar{p}_e w_g + (1-\bar{p}_e)w_b$. For the lower bound of C, we first observe that 2.6.5 must be binding because otherwise, the principal can always lower w_b to reduce the total expected cost C while still being able to induce the agent to implement the chosen effort e, i.e. both 2.6.5 and 2.6.6 are satisfied.

Now we consider $\inf C$ under the binding constraint 2.6.5 $\underline{p_e}v(w_g)+(1-\underline{p_e})v(w_b)=k_e$ and $w_g - w_b > \Delta_\pi$. Following Fact 2.3.2, the expected total cost C will be lowered if the wage paid for the agent in the case of good outcome w_g is reduced. Therefore, $\inf C$ happens when $w_g - w_b = \Delta_\pi$, which is similar to the proof in domain D^2 above. This verifies that no contract in domain D^3 is better for the principal than the identical one in 2.6.3. Combined with the results in D^1 and D^2, we completely show that without the presence of ambiguity, the optimal incentive scheme for the principal is the unique one defined by 2.6.3.

2.6.4 Proof of Proposition 2.3.3

From the incentive constraint, the principal can induce the agent to exert a high effort e_1 if and only if:

$$\min_{p\in[\underline{p_1},\bar{p_1}]} pv(w_g)+(1-p)v(w_b)-c_1 \geq \min_{p\in[\underline{p_0},\bar{p_0}]} pv(w_g)+(1-p)v(w_b)-c_0$$

which is equivalent to:

$$\min_{p\in[\underline{p_1},\bar{p_1}]} pv(w_g)+(1-p)v(w_b)-\min_{p\in[\underline{p_0},\bar{p_0}]} pv(w_g)+(1-p)v(w_b)\geq c_1-c_0$$

By eliminating $v(w_b)$, we can get the following:

$$\min_{p\in[\underline{p_1},\bar{p_1}]} p\Big(v(w_g)-v(w_b)\Big)-\min_{p\in[\underline{p_0},\bar{p_0}]} p\Big(v(w_g)-v(w_b)\Big)\geq c_1-c_0$$

If $w_g \leq w_b$ then $v(w_g)\leq v(w_b)$, thus $\min_{p\in[\underline{p_1},\bar{p_1}]} p(v(w_g)-v(w_b))=\bar{p_1}(v(w_g)-v(w_b))$, $\min_{p\in[\underline{p_0},\bar{p_0}]} p(v(w_g)-v(w_b))=\bar{p_0}(v(w_g)-v(w_b))$, so the LHS is negative or zero since $\bar{p_1}>\bar{p_0}$, which cannot be the case. Henceforth, we must have $w_g > w_b$ and $v(w_g)>v(w_b)$. Then the expression of LHS equals $(\underline{p_1}-\underline{p_0})(v(w_g)-v(w_b))\geq c_1-c_0$, which completes the proof. ■

2.6.5 Proof of Theorem 2.3.1

When the principal induces the agent to implement the high effort e_1 under ambiguity, from Proposition 2.3.3 we have $\Delta_v \geq \Delta_{c/\underline{p}}$. Consequently, and sensibly, the principal must compensate a higher payment when the project runs successfully than the payment when it otherwise does not, i.e. $w_g > w_b$.

Now let's consider again problem **(P)**. As the induced effort is e_1 and $w_g > w_b$, the participation constraint (P.C) will be $\underline{p_1} v(w_g) + (1 - \underline{p_1}) v(w_b) \geq \underline{u} + c_1 = k_1$. It can be observed that for every feasible contract (w_g, w_b) offered by the principal, and for every possible value of prior p, the principal can always be financially better off by lowering w_b if (P.C) is not binding. Indeed, by offering a slightly lower $w'_b < w_b$, the principal can reduce the total expected payment for the agent, while sufficiently encouraging the manager to make a high effort since both (PC) and (IC) are satisfied. Thus, 2.2.2 must be binding.

The above arguments imply that we consider the following problem **(P')**:

$$\max_{(w_g, w_b)} \min_{p \in [\underline{p_1}, \bar{p_1}]} p(\pi_g - w_g) + (1 - p)(\pi_b - w_b)$$

under two constraints:

$$\underline{p_1} v(w_g) + (1 - \underline{p_1}) v(w_b) = k_1 \tag{a}$$

$$v(w_g) - v(w_b) \geq \Delta_{c/\underline{p}}. \tag{b}$$

Define $D' = (w_g, w_b)$ so that constraints (a) and (b) are satisfied. Notice that D' is not empty since there is always at least w_g, w_b satisfying the system of equations of equality with (a) and (b).

We consider the following three cases as follows.

i. First, we consider the following situation:

$$v^{-1}\left(k_1+(1-\underline{p_1})\Delta_{c/\underline{p}}\right)-v^{-1}\left(k_1-\underline{p_1}\Delta_{c/\underline{p}}\right)>\Delta_\pi \tag{c}$$

We claim that for every (w_g, w_b) that satisfies constraints (a) and (b), we must have $w_g - w_b > \Delta_\pi$ under (c).

This is done by using counterarguments. Assume that $w_g - w_b \leq \Delta_\pi$, from (a) and (b) we have $v(w_g) \geq k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}$, and $v(w_b) \leq k_1 - \underline{p_1}\Delta_{c/\underline{p}}$. Since $v(.)$ is strictly increasing, $w_g \geq v^{-1}(k_1+(1-\underline{p_1})\Delta_{c/\underline{p}})$, and $w_b \leq v^{-1}(k_1-\underline{p_1}\Delta_{c/\underline{p}})$. It leads to $w_g - w_b \geq v^{-1}(k_1+(1-\underline{p_1}\Delta_{c/\underline{p}}))-v^{-1}(k_1-(1-\underline{p_1}\Delta_{c/\underline{p}})>\Delta_\pi$, which is a contradiction with our assumption that $w_g - w_b \leq \Delta_\pi$. Therefore, $w_g - w_b > \Delta_\pi$.

Set up the Lagrangian as follows:

$$\begin{aligned}\mathscr{L} = \min_{p\in[\underline{p_1},\bar{p_1}]} & \; p(\pi_g-w_g)+(1-p)(\pi_b-w_b)\\ & +\lambda_1\left(\underline{p_1}v(w_g)+(1-\underline{p_1})v(w_b)-k_1\right)\\ & +\lambda_2\left(v(w_g)-v(w_b)-\Delta_{c/\underline{p}}\right)\end{aligned} \tag{2.6.10}$$

Since $w_g - w_b > \Delta_\pi$, $\min_{p\in[\underline{p_1},\bar{p_1}]} p(\pi_g-w_g)+(1-p)(\pi_b-w_b) = \bar{p_1}(\pi_g-w_g)+(1-\bar{p_1})(\pi_b-w_b)$. Taking the F.O.C we obtain:

$$\mathscr{L}_{w_g} = -\bar{p_1}+\lambda_1\underline{p_1}v'(w_g)+\lambda_2 v'(w_g)=0 \tag{a'}$$

$$\mathscr{L}_{w_b} = -(1-\bar{p_1})+\lambda_1(1-\underline{p_1})v'(w_b)-\lambda_2 v'(w_b)=0 \tag{b'}$$

$$\lambda_2\left(v(w_g)-v(w_b)-\Delta_{c/\underline{p}}\right)=0 \tag{c'}$$

If $\lambda_2 = 0$, then from (a') and (b') we have $v'(w_g) = \dfrac{\bar{p_1}}{\lambda_1 \underline{p_1}}$ and $v'(w_b) = \dfrac{1-\bar{p_1}}{\lambda_1(1-\underline{p_1})}$. Since $\dfrac{\bar{p_1}}{\underline{p_1}} > \dfrac{1-\bar{p_1}}{1-\underline{p_1}}$ we derive $v'(w_g) > v'(w_b)$, which is a contradiction because $w_g > w_b$ and $v(.)$ is concave, therefore $v'(w_g)$ must be less than $v'(w_b)$. Henceforth, $\lambda_2 > 0$, and the constraint (b) is binding, or $v(w_g) - v(w_b) = \Delta_{c/\underline{p}}$. Combining with the participation constraint (a), the optimal wage scheme is $(w^{un}_{e_1,g}, w^{un}_{e_1,b}) = \left(v^{-1}\big(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\big), v^{-1}(k_1 - \underline{p_1}\Delta_{c/\underline{p}})\right)$.

The maxmin expected profit for the principal in this case is

$$\begin{aligned}
\Pi^{un}_{e_1,\bar{p_1}} &= \bar{p_1}(\pi_g - w^{un}_{e_1,g}) + (1-\bar{p_1})(\pi_b - w^{un}_{e_1,b}) \\
&= \bar{p_1}\pi_g + (1-\bar{p_1})\pi_b - \bar{p_1}v^{-1}\left(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\right) \\
&\quad - (1-\bar{p_1})v^{-1}\left(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\right)
\end{aligned}$$

Remark 2.6.1. *Provided $(w^{un}_{e_1,g}, w^{un}_{e_1,b})$ specified above, it can be verified that λ_1, λ_2 are strictly positive from F.O.C conditions (a') to (c'). With a slight abuse of notation, we use w_g and w_b instead of $w^{un}_{e_1,g}$ and $w^{un}_{e_1,b}$.*

Rewrite (a') as $\lambda_1 \underline{p_1} v'(w_g) = \bar{p_1} - \lambda_2 v'(w_g)$ and multiply both sides with $1-\bar{p_1}$ gives $\lambda_1 \underline{p_1}(1-\bar{p_1})v'(w_g) = (1-\bar{p_1})\left(\bar{p_1} - \lambda_2 v'(w_g)\right)$. Similarly for (b'), we have $\lambda_1(1-\underline{p_1})\bar{p_1}v'(w_b) = \bar{p_1}\left((1-\bar{p_1}) + \lambda_2 v'(w_b)\right)$. Subtracting two equations leads to $\lambda_1\left[(\underline{p_1} - \underline{p_1}\bar{p_1})v'(w_g) - (\bar{p_1} - \underline{p_1}\bar{p_1})v'(w_b)\right] = -\lambda_2\left[(1-\bar{p_1})v'(w_g) + \bar{p_1}v'(w_b)\right]$ (r.m.1). Now notice that $(\underline{p_1} - \underline{p_1}\bar{p_1})v'(w_g) - (\bar{p_1} - \underline{p_1}\bar{p_1})v'(w_b) < 0$ since $v'(w_g) < v'(w_b)$ since $v(.)$ is concave, $\underline{p_1} < \bar{p_1}$, $v'(w_g), v'(w_b) > 0$ then from (r.m.1) λ_1 and λ_2 are both positive are negative. But if they are all negative, then from (a') $\bar{p_1} < 0$, which cannot happen. Therefore, $\lambda_1, \lambda_2 > 0$.

ii. Consider the case when

$$v^{-1}\left(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\right) - v^{-1}\left(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\right) < \Delta_\pi \tag{d}$$

We first claim that there is no local maximizer of the problem (P') in the sub-domain $D'_1 \in D', D'_1 = (w_g, w_b) | \{w_g - w_b > \Delta_\pi\}$. Indeed, this case is analogous to the previous part (i), which derives a unique optimal policy $(w_g^{op}, w_b^{op}) \equiv (w^{un}_{e_1,g}, w^{un}_{e_1,b})$, but $w^{un}_{e_1,g} - w^{un}_{e_1,b} = v^{-1}\left(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\right) - v^{-1}\left(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\right) < \Delta_\pi$ as our assumption, which is a contradiction since $(w_g^{op}, w_b^{op}) \in D'_1$ thus $w_g^{op} - w_b^{op} > \Delta_\pi$.

Let's consider the sub-domain $D'_2 = D' \setminus D'_1 = \{(w_g, w_b) \mid w_g - w_b \leq \Delta_\pi\}$. We show that the optimal contract does exist in this domain.

Let $D'_3 = \{(w_g, w_b) \in D'_2 \mid w_g - w_b = \Delta_\pi\}$, it must be that the incentive constraint (b) slacks in D'_3, i.e. $v(w_g) - v(w_b) > \Delta_{c/\underline{p}}$. Because if (b) is binding with an optimal solution (w_g^{op}, w_b^{op}), then from (a) and (b), $(w_g^{op}, w_b^{op}) \equiv (w^{un}_{e_1,g}, w^{un}_{e_1,b})$, but in this case $w_g^{op} - w_b^{op} = v^{-1}\left(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\right) - v^{-1}\left(k_1 - (1-\underline{p_1}\Delta_{c/\underline{p}})\right) = \Delta_\pi$, which is inconsistent with the preliminary assumption of (ii).

Therefore, from now on we can consider only the case when $v(w_g) - v(w_b) > \Delta_{c/\underline{p}}, w_g - w_b = \Delta_\pi$, and $\underline{p_1}v(w_g) + (1-\underline{p_1})v(w_b) = k_1$ for $(w_g, w_b) \in D'_3$. The last two conditions define a unique contract (w'_g, w'_b) due to continuity and strictly increasing of $v(.)$. Nevertheless, it does not serve as an optimal contract. Indeed, since $v(w_g) - v(w_b) > \Delta_{c/\underline{p}}$, the principal can offer a contract (w''_g, w''_b) specifying a slightly lower wage for the manager in the case of a good outcome, and slightly higher wage in case of a bad outcome, i.e., $w''_g < w'_g$, $w''_b > w'_b$. The refined contract satisfies both the participate constraint (a) $\underline{p_1}v(w''_g) + (1-\underline{p_1})v(w''_b) = k_1$, the incentive constraint (b) $v(w''_g) - v(w''_b) \geq \Delta_{c/\underline{p}}$, and $w''_g - w''_b < \Delta_\pi$. Thus, $(w''_g, w''_b) \in D'_2$ and produce a higher expected profit for the principal according to 2.3.2. Consequently, the principal only needs to consider contracts in D'_2 when $w_g - w_b < \Delta_\pi$. [11]

Now consider domain $D'_4 = D'_2 \setminus D'_3 = \{(w_g, w_b) \mid w_g - w_b < \Delta_\pi\}$. We show that (w^i_g, w^i_b) is the optimal contract on D'_4, which is also the optimal one for the principal over the whole domain D' satisfying (d).

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11. Note that there always exists at least a contract (w''_g, w''_b) as mentioned, as we can take $(w''_g, w''_b) = (w^{un}_{e_1,g}, w^{un}_{e_1,b})$. This contract lies in D'_2 and is consistent with the preliminary condition (d) in (ii).

Reconsider the Lagrangian in 2.6.10, we have $\min_{p\in[\underline{p_1},\bar{p_1}]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b) = \underline{p_1}(\pi_g - w_g) + (1-\underline{p_1})(\pi_b - w_b)$ since $w_g - w_b < \Delta_\pi$.

The F.O.C conditions with w_g, w_b are $\mathscr{L}_{w_g} = -\underline{p_1} + \lambda_1 \underline{p_1} v'(w_g) + \lambda_2 v'(w_g) = 0$ (d'), $\mathscr{L}_{w_b} = -(1-\underline{p_1}) + \lambda_1(1-\underline{p_1})v'(w_b) - \lambda_2 v'(w_b) = 0$ (e'), and $\lambda_2\Big(v(w_g) - v(w_b) - \Delta_{c/\underline{p}}\Big) = 0$ (f').

The incentive constraint in (f') must be binding. If not, or $\lambda_2 = 0$, from (d') and (e') $\lambda_1 \underline{p_1} v'(w_g) = \underline{p_1}$, and $\lambda_1(1-\underline{p_1})v'(w_b) = (1-\underline{p_1})$, but it then implies $v'(w_g) = v'(w_b)$, which is a contradiction.

Therefore, $v(w_g) - v(w_b) = \Delta_{c/\underline{p}}$, together with the binding of the participation constraint, we can derive the optimal contract, which is $(w^{un}_{e_1,g}, w^{un}_{e_1,b})$. [12]

Since $w_g - w_b < \Delta_\pi$, the worst scenario for the principal is that the project succeeds with the lowest probability $\underline{p_1}$. Therefore, the maxmin expected profit is as follows

$$\Pi^{un}_{e_1,\underline{p_1}} = \underline{p_1}\pi_g + (1-\underline{p_1})\pi_b - \underline{p_1}v^{-1}\Big(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\Big) - (1-\underline{p_1})v^{-1}\Big(k_1 - \underline{p_1}\Delta_{c/\underline{p}}\Big)$$

iii. Lastly, we consider the case when $v^{-1}\Big(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}\Big) - v^{-1}\Big(k_1 - \underline{p_1}\Delta_{c/\underline{p}} = \Delta_\pi$.

From two constraints (a) and (b), any $(w_g, w_b) \in D'$ satisfies $v(w_g) \geq k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}$ and $v(w_b) \leq k_1 - \underline{p_1}\Delta_{c/\underline{p}}$, then $w_g - w_b \geq v^{-1}(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}}) - v^{-1}(k_1 - \underline{p_1}\Delta_{c/\underline{p}}) = \Delta_\pi$. Henceforth, $\min_{p\in[\underline{p_1},\bar{p_1}]} p(\pi_g - w_g) + (1-p)(\pi_b - w_b) = \bar{p_1}(\pi_g - w_g) + (1-\bar{p_1})(\pi_b - w_b)$. Therefore, the MEP for the principal happens if $\bar{p_1}w_g + (1-\bar{p_1}w_b)$ is minimized. By Fact 2.3.2, the principal can always reduce the expected cost by increasing the payment if the outcome is bad and reducing the payment if the outcome is good. Therefore, the optimal contract for the principal is $w_b = v^{-1}(k_1 - \underline{p_1}\Delta_{c/\underline{p}})$, $w_g = v^{-1}(k_1 + (1-\underline{p_1})\Delta_{c/\underline{p}})$. Finally, the MEP for the principal coincides with part (i), which is $\Pi^{un}_{e_1,\bar{p_1}}$.■

12. An analogous argument to Remark 2.6.1 can be applied to show λ_1, λ_2 are positive with this contract.

2.6.6 Proof of Theorem 2.4.1

We prove for the part of theorem where $\underline{a_1} - \underline{b_1} \geq \Delta_\pi$. An analogous reasoning can be used in proving the part where $\underline{a_1} - \underline{b_1} < \Delta_\pi$.

When effort is unobservable, following 2.4.1, the principal will induce high effort e_1 without the presence of ambiguity if:

$$(p_1 - p_0)\Delta_\pi > p_1 a_1 + (1 - p_1) b_1 - c_0 \tag{a}$$

From 2.4.2, the principal will induce the agent to implement e_1 under ambiguity if:

$$(\bar{p_1} - \underline{p_0})\Delta_\pi > \bar{p_1}\underline{a_1} + (1 - \bar{p_1})\underline{b_1} - c_0 \tag{b}$$

Moreover, we have to include condition 2.4.3 that aligns with the expression of the principal's expected profit in (b), which is $\underline{a_1} - \underline{b_1} \geq \Delta_\pi$ (c).

Now consider (a), since $v(a_1) - v(b_1) = \Delta^0_{c/p}$, we have $p_1 a_1 + (1 - p_1) b_1 = p_1 a_1 + (1 - p_1)\left(v^{-1}(v(a_1) - \Delta^0_{c/p})\right) = q(a_1)$, which is strictly increasing, since

$$q'(a_1) = p_1 + (1 - p_1)\frac{v'(a_1)}{v'\left(v^{-1}(v(a_1) - \Delta^0_{c/p})\right)} > 0$$

Therefore, $a_1 < a^{wa}_{e_1}$, the threshold value of (a). Similarly, from (b) $a_1 < a^{am}_{e_1}$ with $a^{am}_{e_1}$ being the threshold value of (b) as a function of a_1.

Now consider condition $\underline{a_1} - \underline{b_1} \geq \Delta_\pi$, which is equivalent to $\underline{a_1} - v^{-1}\left(v(\underline{a_1}) - \Delta_{c/\underline{p}}\right)) \geq \Delta_\pi$.

The derivative of LHS with respect to $\underline{a_1}$ equals $1-\dfrac{v'(a_1)}{v'\left(v^{-1}\left(v(\underline{a_1})-\Delta_{c/p}\right)\right)}$. Since $v^{-1}\left(v(\underline{a_1})-\Delta_{c/p}\right)=\underline{b_1}$, and $\underline{a_1}>\underline{b_1}$, $v(.)$ is concave, the sign of the derivative is positive. Therefore, $\underline{a_1}$ that satisfies (c) must be at least equal to its threshold value. As $v(\underline{a_1})=v(a_1)+(1-\underline{p_1})\Delta_{c/\underline{p}}-(1-p_1)\Delta_{c/p}$, it implies that $a_1>a_1^*$, which is the threshold value of (c) as a function of a_1. In short, the system of (a), (b) and (c) has a solution if $a_1^*<a_1=v^{-1}\left(k_1+(1-p_1)\Delta_{c/p}\right)<\min(a_{e_1}^{wa},a_{e_1}^{am})$. Conversely, if $a_1^*<v^{-1}\left(k_1+(1-p_1)\Delta_{c/p}\right)<\min(a_{e_1}^{wa},a_{e_1}^{am})$, it can be seen that the principal will induce the agent to implement e_1 under ambiguity.

Now consider if $\underline{a_1}-\underline{b_1}<\Delta_\pi$, from 2.4.1 the expected profit for the principal under ambiguity upon inducing the agent to implement e_1 is:

$$\Pi_{e_1}^{am}=\underline{p_1}\pi_g+(1-\underline{p_1})\pi_b-\underline{p_1}v^{-1}(k_1+(1-\underline{p_1})\Delta_{c/\underline{p}})-(1-\underline{p_1})v^{-1}(k_1-\underline{p_1}\Delta_{c/\underline{p}})$$

The condition that leads to implementing e_1 under ambiguity is $\Pi_{e_1}^{am}>\Pi_{e_0}^{am}=\underline{p_0}\pi_g+(1-\underline{p_0})\pi_b-v^{-1}(k_0)$, or equivalently:

$$(\underline{p_1}-\underline{p_0})\Delta_\pi>\underline{p_1}\underline{a_1}+(1-\underline{p_1})\underline{b_1}-c_0 \tag{d}$$

Similar to the previous case, as $a_{e_1,\underline{p_1},\underline{p_0}}^{am}$ is a unique threshold value of (d), we must have $a_1<a_{e_1,\underline{p_1},\underline{p_0}}^{am}$. Combined with the condition of $a_1<a_{e_1}^{am}$ in (a) and $a_1<a_1^*$ when $\underline{a_1}-\underline{b_1}<\Delta_\pi$, we come to the conclusion that $a_1=v^{-1}(k_1+(1-p_1)\Delta_{c/p})<\min(a_{e_1}^{wa},a_{e_1,\underline{p_1},\underline{p_0}}^{am},a_1^*)$.

If the principal aims to induce the agents to implement e_0 and $\underline{a_1}-\underline{b_1}\geq\Delta_\pi$, the inequality sign in (a) and (b) will be reversed:

$$(p_1-p_0)\Delta_\pi<p_1a_1+(1-p_1)b_1-c_0(e)$$

$$(\bar{p_1}-\underline{p_0})\Delta_\pi<\bar{p_1}\underline{a_1}+(1-\bar{p_1})\underline{b_1}-c_0(f)$$

Analogously, we can derive from (e) and (f) that $a_1 > a^{wa}_{e_1}$, and $a_1 > a^{am}_{e_1,\bar{p}_1,\underline{p_0}}$. Combined with $a_1 > a^*_1$ from $\underline{a_1} - \underline{b_1} \geq \Delta_\pi$, it can be seen that $a_1 > \max(a^{wa}_{e_1}, a^{am}_{e_1,\bar{p}_1,\underline{p_0}}, a^*_1)$. Finally, the case $\underline{a_1} - \underline{b_1} < \Delta_\pi$ leads to $\max(a^{wa}_{e_1}, a^{am}_{e_1,\underline{p_1},\underline{p_0}}) < a^*_1$. ■

Chapter 3

Cheap Talk With Ambiguous Beliefs

This chapter investigates strategic information transmission in the presence of ambiguous language. Extending the classic cheap talk model introduced by Crawford and Sobel (1982), we explore a setting where the uninformed receiver is uncertain about the distribution of the sender's private information. We find that the partition equilibria, under varying levels of distributional ambiguity, mirror that of the Crawford-Sobel standard model when the receiver's reference distribution is uniform, assuming both parties' preferences are characterized by a quadratic-loss function. However, when the receiver's reference distribution deviates from uniformity, intermediate ambiguity alters her optimal strategy compared to the original model, leading to an adjustment in the Crawford-Sobel partition equilibrium.

3.1 Introduction

The cheap talk problem, as proposed by the pioneering work of Crawford and Sobel (CS hereafter), provides a consistent framework for analyzing communication strategies between an informed sender (S) and an uninformed receiver (R). Since its introduction, extensive literature has explored extending the original problem to various settings. One such extension incorporates ambiguity into the CS model, where one or both parties face Knightian uncertainty regarding some of the assumptions in the original framework.

The implications of ambiguity on the outcomes of the cheap talk problem have garnered significant interest in recent years. Blume et al. (2007) studied the game where a noisy communication channel introduces randomness in the messages received by ***R***. Their findings show that channel noise fosters communication by forcing the pooling of sender types, which makes ***R*** less reactive to messages and, in turn, encourages S to reveal more information. In Blume's partition equilibria, ***R***'s expectation, given a message, is a weighted average of the conditional expectation without transmission error and the ex-ante mean. ***R***'s expectation, when given a low message, is distorted upward, implying a reduction in de facto preference alignment.

Gordon and Nöldeke (2015) finds that S employs truth-distorting techniques, such as exaggeration or understatement. Lipman (2009) analyzes cheap talk with aligned preferences, concluding that vagueness is only efficient when the informed party holds "vague views of the world." Kellner and Le Quement (2018) examines a cheap talk game where the sender can adopt a Knightian uncertainty strategy. Under ambiguous randomization, any standard influential communication equilibrium has a Pareto-dominant counterpart, consistent with ex-ante utilities and strategic planning.

A critical assumption in cheap talk problems is that both the sender and receiver are fully aware of the exact distribution function governing message generation. On one hand, this assumption is reasonable, as it simplifies the analysis of posterior beliefs, which, in turn, influences agents' decisions in equilibrium. On the other hand, empirical evidence suggests that decision-makers often lack precise knowledge of the message-generating process, thus introducing uncertainty.

We propose a cheap talk game where the distribution of the private message observed by ***S*** is ambiguous to ***R***, who is ambiguity-averse, as in previous chapters.

Since ambiguity only affects how R evaluates actions and does not influence the preferences of either the principal or agent, the misalignment between the interests of R and S remains similar to the standard CS model. As a result, all equilibria take a partition form, where S pools his signals before sending them to R.

In this chapter, depending on how R perceives the degree of ambiguity, she judges actions according to the corresponding ambiguity model, rather than selecting the action that maximizes conditional expected utility, as in the CS model.

When R faces complete ambiguity, the true distribution of the message can take any form. As a result, R's posterior belief after Bayesian updating becomes irrelevant: being ambiguity-averse, R assumes the worst-case scenario, where the private message's distribution degenerates, placing the message furthest from her optimal action. In this context, we apply the maxmin expected utility (MEU; Gilboa and Schmeidler (1989)) framework to characterize R's behavior. The Gilboa-Schmeidler model, in its simplest form, offers a straightforward mechanism to rank actions under ambiguity without relying on the information R has learned about the true distribution.

However, when R faces some ambiguity, but not extreme, the MEU mechanism, which excludes the information R acquires regarding the true distribution of the private message, may no longer be efficient for evaluating R's responses. This is because R's best response, under intermediate ambiguity, should reflect the information she has about the private message. Intuitively, when ambiguity is intermediate, R's action should be closer to the private message than when she faces complete ambiguity. To address the shortcomings of the MEU in this context, we adopt the multiplier utility following Andersson et al. (2003), which measures the divergence between R's best guess about the actual distribution and the worst-case misspecified distribution. As a result, R can narrow down the set of potential candidates for the actual distribution, leading to an optimal response that incorporates some information about the uninformed type.

The first contribution of this chapter is that it demonstrates and proves the existence of partition equilibria across varying levels of ambiguity, including small, complete, and intermediate cases. In contrast to the CS model, where ***R***'s Bayesian optimal action is consistently determined in equilibrium from the updated prior, there is no single functional form to describe ***R***'s behavior across different ambiguity models. However, we show that the partition equilibrium exists as long as ***R***'s behavior satisfies continuous differentiability and monotonicity, both of which hold in the three models discussed in the text.

Secondly, I show that when ***R*** faces complete ambiguity, she chooses the midpoint of the informed interval as her best response, regardless of her reference belief, or ***S***'s strategy. Thus, all equilibria under complete ambiguity are essentially equivalent. In the intermediate ambiguity case, modeled by the multiplier preference framework, I find that ***R*** also selects the midpoint as her optimal action if her reference distribution is uniform. Although this result appears similar to the complete ambiguity case, the underlying rationale differs. When ***R*** faces complete ambiguity, her action is an anti-ignorance strategy, balancing between two extreme scenarios at the ends of the informed interval. In contrast, her behavior under intermediate ambiguity resembles the CS model, assuming ***S***'s strategy is uniform.

However, if ***R***'s reference distribution is non-uniform, I show that her optimal action under intermediate ambiguity can deviate from both cases including no ambiguity (CS) and the complete ambiguity. Therefore, the original partition equilibrium in the CS model is robust under ambiguity when ***R***'s reference distribution is uniform but may be altered when ***R*** holds a different belief. Finally, I present examples comparing the partition equilibrium in the CS model and the ambiguity model, illustrating how different degrees of ambiguity impact the outcomes.

In the context of cheap talk under ambiguity, I discovered that Li (2022) had explored a closely related problem after I had already developed the core idea that the receiver faces ambiguity regarding the distribution of the private message. Although I adopt Li's approach of using the multiplier utility (MU) model, leading to some overlapping results, my work differs from his in several significant ways.

First, Li's approach, which relies heavily on the MU model and the use of relative entropy between distributions, is analytically complex. In contrast, when ambiguity is relatively small, I introduce an alternative approach using the ε-contamination model. This model avoids the need for relative entropy and the functional extrema found in Li's model, simplifying the analysis. Additionally, I provide a more detailed and formal proof, along with clearer explanations of Li's results within the MU framework. Furthermore, I demonstrate that the receiver's (R's) strategy under ambiguity generally differs from that in the original CS model and reveals more information than when R is entirely ignorant of the true distribution of the private message. These key distinctions are absent from Li's work.

Secondly, Li does not formally prove the existence of equilibrium under ambiguity, a crucial aspect that my paper addresses. Establishing equilibrium existence is vital for understanding how ambiguity impacts the structure of partition equilibria. Additionally, Li's work does not explicitly define the strategy profiles for both agents under ambiguity. In contrast, I show that when ambiguity is modeled using the MU framework, the partition equilibria exhibit a fundamental structural equivalence to those in the CS model. These critical points are not covered in Li's research.

Lastly, I provide illustrative examples and figures that showcase how partition equilibria shift under ambiguity, offering clear visual insights that further distinguish my contribution from Li's.

The rest of the paper is organized as follows. Section 3.2 introduces the model of cheap talk under ambiguity. Section 3.3 established the existence of the CS-like equilibrium under ambiguity. Sections 3.4 and 3.5 characterize partition equilibrium following the $\varepsilon-$contamination model and max-min expected utility model in two cases: small ambiguity and complete ambiguity. Section 3.6 studies agents' responses under intermediate ambiguity modeled by the multiplier utility framework Hansen and Sargent (2001). The conclusion is presented in Section 3.7 and proofs are included in Appendix 3.8.

3.2 Model

We follow the notation of the standard CS model. There are two players: a sender (S, he), who is the only one capable of sending a costless, non-verifiable message to a receiver (R, she). Occasionally, we will refer to S as the expert and R as the decision maker (see Barreda (2010)). The sender observes the value of a random variable $\theta \in \Theta = [0,1]$, which represents a payoff-relevant state of the world. In the CS model, θ is drawn from the uniform distribution over $[0,1]$, and this distribution is known by both parties. However, in this chapter, we relax this assumption by introducing ambiguity regarding the true distribution f that generates θ, which is ambiguous to the receiver.[1] As the sender's payoff is determined solely by the receiver's actions, only the receiver's attitude toward ambiguity is relevant. For consistency with the assumptions made in previous chapters, we assume that R is an ambiguity-averse decision-maker.

The utilities of S and R, $U^S(y,\theta,b)$ and $U^R(y,\theta)$, depend on the private information θ and the action y taken by R after observing the message m from S. The parameter b captures the misalignment between R and S. We assume that S incurs no cost in sending any message to R. Following the assumption in CS, I assume that $U^i_{12} > 0$, and $U^i_{11} < 0$ for $i = R,S$. It is important to note that we do not require the agents' preferences to take the quadratic-loss form as in the original CS model at this stage.

1. Since the sender can only observe the private message and takes no action, the sender's prior does not influence the CS equilibrium. Therefore, we assume ambiguity only affects the receiver.

The game proceeds as follows, after observing random information (or *type*) θ, S sends a signal (message) $m \in Z$ to R, where Z denotes the signal space. The signal space Z is sufficiently rich such that, in the most informative mode of transmission where S truthfully reports his type to R, Z can accommodate all possible values of θ. This ensures that R can distinguish between any two arbitrary signals in Z. Therefore, without loss of generality (WLOG), we assume $Z = \Theta = [0,1]$.

Our equilibrium concept follows the Bayesian Nash equilibrium. At each information set where an agent is called upon to act, R's optimal action is determined based on her prior belief and the sender's signaling rule in equilibrium. Since S's is not influenced by ambiguity, he only chooses a strategy to send the message m that maximizes his expected payoff contingent on R subsequent action. Meanwhile, R's strategy depends on S's signaling rule and how she incorporates ambiguity into her updated beliefs.

Formally, a cheap talk problem under ambiguity is defined as follows:

(i) S's problem: given private information θ, S's strategy involves selecting a state-dependent distribution of messages $\sigma(m|\theta)$ over Z. In the absence of ambiguity, R updates his prior belief $\mu(\theta|m)$ according to Bayes' law upon observing the signal m :

$$\mu(\theta|m) = \frac{\sigma(m|\theta)f(\theta)}{\int_0^1 \sigma(m,t)f(t)dt} \tag{3.2.1}$$

In the setting of ambiguity, however, R is uncertain about $f(.)$, which leads to the posterior belief $\mu(\theta|m)$ also being vague. Despite this, R forms a conjecture about the actual distribution of the private message. [2] With a slight abuse of notation, we refer to R's reference distribution as $f(\theta)$, implicitly indicating that it represents her conjecture rather than the actual distribution of θ. Throughout this chapter, I assume that f is continuously differentiable. An analogous notation is applied to the posterior belief $\mu(\theta|m)$ within ambiguity context.

2. This is rather similar to the problem of the competitive insurance market in Chapter 1, where actuarial firms hold a subjective belief regarding the proportion of low-risk and high-risk customers.

Let $\Delta(\Theta)$ denote the set of probability distributions on Θ. For each reference distribution $f(\theta)$, let $K_{\Delta(\Theta)}$ represent a non-empty, compact, convex subset of $\Delta(\Theta)$ that contains $f(\theta)$ and all potential candidates for the true distribution under ambiguity associated with f. In the following sections, we will examine different versions of $K_{\Delta(\Theta)}$, corresponding to cases of small, complete, and intermediate ambiguity.

(ii) R's problem: upon observing the signal m from S, R selects an action rule $y(m)$ that maximizes his expected utility, given the ambiguous posterior $\mu(\theta|m)$ in 3.2.1. The process by which R handles the ambiguous posterior and find the optima will be studied in next sections. As $U^R(.)$ is a concave function, t follows from Jensen's inequality that R never employs a mixed strategy.

We define the equilibrium in the context of ambiguity based on conditions **(i)** and **(ii)** as follows.

Definition 3.2.1. A strategy profile $(\sigma(m|\theta), y(m))$ is called a Crawford-Sobel equilibrium under ambiguity (CSUA) if the following conditions hold:

a. Given R' s optimal action $y(m)$, S' s strategy is optimal, i.e., if m is in support of $\sigma(m|\theta)$, then $y(m) = \underset{\hat{m} \in Z}{\text{argmax}}\, U^S(y(\hat{m}), \theta)$, here $\hat{m}$ is any feasible message in Z.

b. Given S' s strategy $\sigma(m, \theta)$, R chooses the action $y(m)$ that maximizes the minimum expected utility of $U^R(y(m), \theta)$ over a set of probability measures. [3] Note that the minimum expected utility of $U^R(y(m), \theta)$ is evaluated within the framework that accounts for ambiguity when accessing actions.

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3. R is assumed ambiguity-averse, as in previous chapters, ensuring that every model examined in the following sections satisfies the uncertainty aversion axiom (Epstein and Schneider (2010)).

In this chapter, the case of small ambiguity is modeled using the $\varepsilon-$contamination framework. Complete ambiguity is represented by the max-min expected utility framework of Gilboa and Schmeidler, as in Chapter 2. Finally, the multiplier utility framework of Hansen and Sargent (2001) is used to model cases of intermediate ambiguity. In the next section, similarly to Theorem 1 in CS, we state the general result asserting the existence of the CSUA equilibrium when R's optimal strategy meets the continuous differentiability and monotonicity conditions.

3.3 Characterization of equilibria under ambiguity

For any observed information θ, we define S and R' desired action as $y^S(\theta,b) = \underset{y}{\text{argmax}} U^S(y,\theta,b)$, and $y^R(\theta) = \underset{y}{\text{argmax}} U^R(y,\theta)$. Following the misalignment assumption in CS, I assume that $y^S(\theta,b) \neq y^R(\theta)$ for every θ. [4] It is worth noting that the conflict of interest condition just mentioned rules out separating equilibria, in which the expert fully discloses his type.

By an argument analogous to Lemma 1 in the CS model, it can be verified that any equilibrium under ambiguity also has a partition structure. First of all, the misalignment between $y^S(\theta,b)$ and $y^R(\theta)$ for every θ implies that there exists ε such that $|y^S(\theta,b) - y^R(\theta)| \geq \varepsilon$ for all θ (i). Now let's consider two actions u and v induced in equilibrium. As S prefers u to v for the $S-type$ that induces u and vice versa, there exists an $\bar{\theta} \in [0,1]$ that makes S indifferent between inducing these two actions, i.e. $U^S(u,\bar{\theta},b) = U^S(v,\bar{\theta},b)$. The conditions $U^S_{11} < 0$ and $U^S_{12} > 0$ then implies that $u < y^S(\bar{\theta},b) < v$(ii), u can only be induced by some $\theta \leq \bar{\theta}$ and v can only be induced by some $\theta \geq \bar{\theta}$. Combining these last two implications with $U^R_{12} > 0$, we derive that $u \leq y^R(\bar{\theta}) \leq v$ (iii). From (i),(ii), and (iii), we conclude that $v - u \geq \varepsilon$, which means that the set of induced actions in equilibrium is finite. Given that the set of induced actions in any equilibrium is bounded by $y^R(0)$ and $y^R(1)$ since $U^R_{12} > 0$, we conclude that the set of induced actions in equilibrium is

4. Although not specified at this step, the reader can assign the quadratic-loss utility functions for $U^S(.)$ and $U^R(.)$. The misalignment assumption then naturally follows. There are, of course, other functional forms of non-quadratic loss (for example, see Krishna and Morgan (2004), Barreda (2010)). Nevertheless, the misalignment assumption remains a crucial and widely adopted feature in the cheap talk literature.

finite. Due to the continuity of $U^R(.)$, the decision maker will choose a unique action for signals received from some open interval. Thus, the CSUA equilibrium takes the form of a partition over the space of $S-$ type $[0,1]$. S informs R the exact interval where the private information lies by sending a random signal m within that interval.

Lemma 3.3.1. *Any CSUA equilibrium, if it exists, takes the form of a partition of* $[0,1]$*, wherein all* S*-types within each open interval induce a unique action for* R*.*

Roughly speaking, although ambiguity may shift R's response compared to the original model, it does not affect the misalignment between S's and R's desired actions, which results in a minimum gap between any two actions induced by R in equilibrium.

For every CSUA equilibirum, denote $a_0 = 0, a_1, ...a_N = 1$ be a partition of $[0,1]$. S observes the private type θ and then informs R the exact interval $[a_i, a_{i+1}]$ where θ lies in. If θ falls on some threshold a_i, S can choose either $[a_{i-1}, a_i]$ or $[a_i, a_{i+1}]$ to inform R.

Let us define $y(a_i, a_i) = y^R(a_i)$ for all i and $y(a_i, a_{i+1})$ as the best response of R upon receiving the signal in (a_i, a_{i+1}). Since $U^R_{11} < 0$, it can be shown that $y(a_i, a_{i+1})$ lies within (a_i, a_{i+1}). This implies that $a_{i+1} - a_{i-1}$ is bounded below since there exists a minimum gap between $y(a_i, a_{i+1})$ and $y(a_{i-1}, a_i)$, leading to the number of cutoffs a_i in equilibrium being finite.

Corollary 3.3.1. *For each* b*, there exists a maximum number* $N(b)$ *of cutoffs* a_i *in the partitioned CSUA equilibrium.*

We omitted the proof of Lemma 3.3.1 and Corollary 3.3.1. Next, we characterize the necessary condition for the existence of a partitioned CSUA equilibrium under ambiguity. Similar to the CS equilibrium, the arbitrage condition is necessary to ensure that S is indifferent between sending signals from either (a_{i-1}, a_i) or (a_i, a_{i+1}) when the true observed message is at the cutoff a_i. This condition ensures that R acts consistently upon receiving any signal within (a_i, a_{i+1}).

Theorem 3.3.1. *Suppose that for each informed interval* $[a_i, a_{i+1}]$*, R's optimal response* $y(a_i, a_{i+1})$ *the following conditions:*

i. $y(a_i, a_{i+1})$ *is continuous differentiable.*

ii. $y(a_i, a_{i+1})$ *is monotonic, i.e.,* $y(a_i, \alpha) \leq y(a_i, \beta)$ *when* $\alpha \leq \beta$ *and* $y(\alpha, a_i) \leq y(\beta, a_i)$ *when* $\alpha \leq \beta$ *for all* i*. Moreover,* $y(a_i, a_i) = y^R(a_i)$ *for all* i*.*

Then, for every $N \leq N(b)$ *there exists a partition* $0 = a_0 < a_1 < ... < a_N = 1$ *that satisfies the arbitrage condition, that is,*

$$U^S(y(a_{i-1}, a_i), a_i, b) = U^S(y(a_i, a_{i+1}), a_i, b)(C)$$

Remark 3.3.1. *Theorem 3.3.1 establishes the existence of partial equilibrium under ambiguity. Although it closely resembles Theorem 1 in the CS model, there is an important difference. The first thing is with the best response* $y(a_i, a_{i+1})$*. In the CS model, the best response* y *is uniquely derived from the conditional expectation of* θ *after* R *updates her belief, and this expectation remains consistent across contexts. In contrast, since our model incorporates various frameworks for ambiguity,* $y(m)$ *in our model is not necessarily specified in a unique way based on the posterior belief. Therefore, the existence of partial equilibria depends solely on the generic properties of* y*, specified by conditions (i) and (ii).*

Remark 3.3.2. *In CS model, the distribution function* $f(m)$*, is not generally assumed to be continuous. Therefore, the continuity of* R*'s optimal response* $y(a_i, a_{i+1})$ *is not guaranteed. However, the assumption of differentiability is crucial for proving the existence of equilibrium cutoffs* a_i *under condition* (C)*. Lastly, the monotonicity condition (ii) can be interpreted that the closer* a_i *or* a_{i+1} *moves to 1,* R*'s optimal choice* $y(a_i, a_{i+1})$ *will increase accordingly.*

The key idea behind the proof of Theorem 3.3.1 is as follows. First, we construct a partition equilibrium that satisfies the arbitrage condition (C) with the terminal $a_N < 1$. Then, by the implicit theorem, if we vary a_1 continuously, there exists some a_1 such that $a_N = 1$. Note that this can be achieved since the sequence $\{a_i\}$ does not "stop" at any i, or $a_i = a_{i+1}$, under conditions (i) and (ii). [5]

The argument is outlined more explicitly as follows. For each y there exists $N(b)$ such that the maximum number of thresholds in one partition equilibrium satisfying the arbitrage condition and $a_N \leq 1$ does not exceed $N(b)$, as established in Crawford and Sobel (1982). If the last element of the largest partition equilibrium $a_N = 1$ ($N = N(b)$), the proposition is true. Otherwise, there exists a sequence $0 < a_1 < .. < a_N < 1$ that satisfies condition (C). If a solution for condition (C) already exists when $a_N < 1$, then by the implicit function theorem, there also exists a neighborhood of a_1 where condition (C) continues to hold. Therefore, if the initial point a_1^* is the largest element satisfying condition (C), the corresponding last element a_N^* must be one. Otherwise, a neighborhood of a_1^* can still be found where condition C holds, contradicting the assumption that a_1^* is the supremum element.

In the following sections, we first investigate the CSUA equilibrium for the complete ambiguity in section 3.5, using the ε-contamination model and the max-min expected utility (MEU) framework proposed by Gilboa and Schmeidler (1989). Following Li (2022), we model the intermediate ambiguity case by employing the framework developed by Hansen and Sargent (2001), which offers a more convenient approach for modeling the divergence between distributions. This model allows us not only to study the CSUA equilibrium when the preference distribution is uniform but also to derive R's strategy in equilibrium for a broader range of distributions. The results of this analysis are presented in Section 3.6.

5. See Appendix 3.8.1.

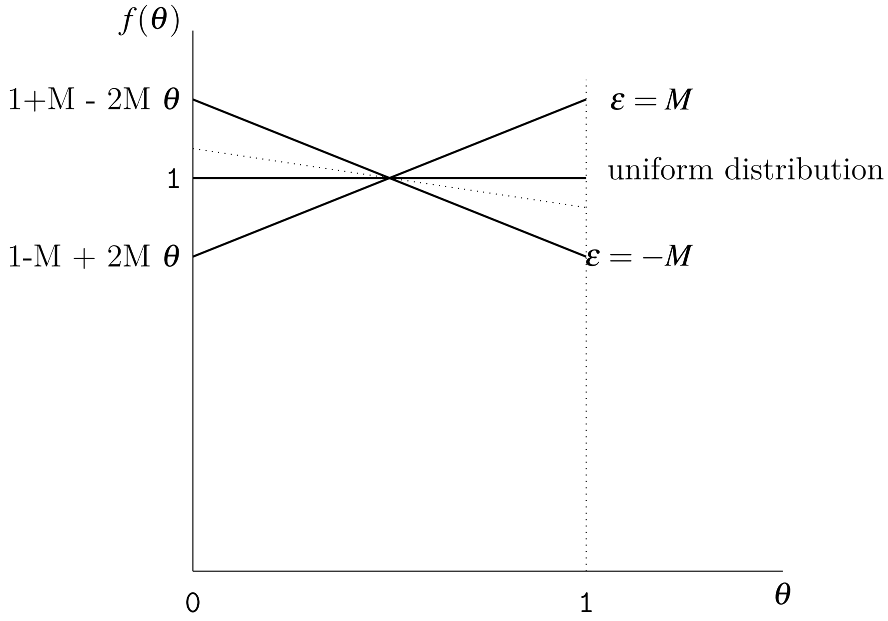

Figure 3.4.1: The set of R's possible distributions under small ambiguity.

3.4 Small ambiguity equilibrium

In the canonical example in CS model, the probability density function (pdf) of the private message is assumed to be uniform, specifically, $f(\theta) = 1$ for every θ. When facing small ambiguity regarding f, R has a set of priors about the actual distribution. However, since the degree of ambiguity is small, R knows that the true pdf is close enough to the uniform distribution, in the sense that it still takes a linear form but deviates slightly from the uniform distribution. Specifically, ambiguity is modeled using the concept ε-contamination, which results in a set of linear forms for the density function: $f(\theta, \varepsilon) = 2\varepsilon\theta + (1 - \varepsilon)$ for $|\varepsilon| \leq M$ for some small M.[6]

In Figure 3.4.1, if $\varepsilon = 0$ then $f(\theta, \varepsilon) = 1$, which corresponds exactly to the uniform distribution in the example of CS. For sufficiently small M, the true density function $f(.)$ from R's perspective, can be represented by any line lying between the two boundaries $2M\theta + (1 - M)$ and $-2M\theta + (1 + M)$. [7]

---

6. Intuitively, M can take value one as its upper bound. In this case, the set of true distributions lies between two extreme ones, -2θ and 2θ. However, we can lower this bound so that $1 + M - 2M\theta > 0$ every θ. (see the proof of Theorem 3.4.1). In this sense, we can establish a better upper bound $M \leq 0.5$.
7. It can be easily verified that both of these linear distributions (and any other symmetric, linear distributions lying between them, as shown in Figure 3.4.1 are proper distribution functions.

With a slight abuse of notation, let $N(b)$ denote the maximum number of thresholds in the original CS equilibrium, which depends on the degree of misalignment between S and R.

Let $y(m)$ denote the optimal strategy of R' when informed of the interval containing the message m in equilibrium. We first present a general result for a class of $U^R(.)$ preferences, which are formed by a linear combination of quadratic-loss functions.

Theorem 3.4.1. *In the small ambiguity case, for* $k_i \in N$ *for every* i*, if* $U^S(y,\theta,b) = -(y-(\theta+b)^2)$ *and* $U^R(y,\theta) = -\sum_{i=1}^{l} \alpha_i (y-\theta)^{2k_i} + c_i$*, where* $(\alpha_1, ..\alpha_l) \in R^l_+$*, then for every positive integer* $N \leq N(b)$*, there exists at least one equilibrium* $(y(m), \sigma(m,\theta))$ *where* $\sigma(m,\theta)$ *is a uniform distribution, supported on* $[a_i, a_{i+1}]$ *if* $\theta \in (a_i, a_{i+1})$ *satisfying:*

i. $U^S(y(a_{i-1},a_i),a_i,b) = U^S(y(a_i,a_{i+1}),a_i,b)$

ii. $y(m) = \dfrac{a_i + a_{i+1}}{2}$.

Since R's preference in the CS model is a special case of the general preference defined in Theorem 3.4.1, with $l = 1, \alpha_i = 1, c_i = 0$ for all i, we derive the following result.

Corollary 3.4.1. *When* S *and* R*' preferences are defined as in the CS model, the CSUA equilibrium is characterized by* $(y(m), \sigma(m,\theta)$*, which satisfies:*

i. $U^S(y(a_{i-1},a_i),a_i,b) = U^S(y(a_i,a_{i+1}),a_i,b)$

ii. $y(m) = \dfrac{a_i + a_{i+1}}{2}$ *for all* $m \in (a_i, a_{i+1})$

Corollary 3.4.1 states that, under small ambiguity, R's best response coincides with its strategy in the CS model. Although R's strategy appears similar in both cases, the underlying mechanism is different. In the CS model, R simply chooses the strategy y that maximizes her conditional expected utility, which is a straightforward one-step process.

However, under ambiguity, R cannot directly maximize the conditional expected utility to determine the optimal action, as the distribution of the message is uncertain. Given that R is ambiguity-averse, she must consider the worst-case scenarios associated with each possible action y.

The intuition behind Corollary 3.4.1 is as follows: when y lies in the right half of the signaled interval $[a_i, \frac{a_i + a_{i+1}}{2}]$, R perceives the worst-case scenario when the distribution of θ is as concentrated as possible on the left, represented by $f_1(\theta) = -2M\theta + (1+M)$. Conversely, when y lies in the left half of $[\frac{a_i + a_{i+1}}{2}, a_{i+1}]$,the worst-case scenario for R is when the distribution of θ is concentrated towards the right, represented by $f_2(\theta) = 2M\theta + (1-M)$. As y approaches the midpoint of $[a_i, a_{i+1}]$, R's expected payoffs improve with respect to both extreme distributions, $f_1(\theta)$ and $f_2(\theta)$. Given that R preference is symmetric, the midpoint of $[a_i, a_{i+1}]$ maximizes receiver's expected utility in both of worst-case scenario, thus constitutes a unique response of R.

With the case of small ambiguity addressed, we now move on to analyze the equilibrium under complete ambiguity.

3.5 Complete ambiguity equilibrium

This section examines the case where R faces complete ambiguity regarding the true probability of the private message θ. Consequently, R has no certainty toward her best guess distribution $f(\theta)$. From R's perspective, the true distribution could take any possible distribution. As a result, under the assumption of complete ambiguity, the reference probability distribution $f(\theta)$ provides no advantage in determining R's optimal strategy. In other words, R is completely ignorant of the actual distribution of the private message. The set of possible candidates for the true $f(\theta)$ now spans the entire domain, or $K_{\Delta(\Theta)} = \Delta(\Theta)$. Maintaining the assumptions of quadratic utility functions and the misalignment between R's and S's best ex-post actions, and following the arguments in the CS model, every equilibrium in this case remains a partition equilibrium.

Being ambiguity-averse, R ranks actions in the informed interval in equilibrium under complete ambiguity following Gilboa and Schmeidler (1989):

$$y(m) = \underset{y}{\text{argmax}} \min_{\mu(\theta|m)\in\Delta(\Theta)} \left[\int_{a_i}^{a_{i+1}} U^R(y,\theta)\mu(\theta|m)d\theta \right] \tag{3.5.1}$$

In 3.5.1, $\mu(\theta|m)$ is the posterior belief with respect to the reference probability $f(\theta)$ as in 3.2.1. Since f carries no information in the case of complete ambiguity, $\mu(\theta|m)$ could take any posterior belief. The following result characterizes the partition equilibrium in the case of complete ambiguity:

Proposition 3.5.1. *Under complete ambiguity, the set of equilibria strategy profiles* $(y(m), \sigma(m,\theta))$ *contains a signaling rule* $\sigma(m,\theta)$ *supported on* $[a_i, a_{i+1}]$ *if* $\theta \in (a_i, a_{i+1})$, *and* $y(m)$ *that satisfies:*

i. $U^S(y(a_{i-1},a_i),a_i,b) = U^S(y(a_i,a_{i+1}),a_i,b)$

ii. $y(m) = \dfrac{a_i + a_{i+1}}{2}$ *for all* $m \in (a_i, a_{i+1})$

We can provide an intuition for Preposition 3.5.1 as follows: R is informed the true state θ is in $[a_i, a_{i+1}]$, however, the true pdf f is completely ambiguous to her. Being ambiguity-averse, R assumes the worst situation under which the distribution function of θ is a degenerate distribution that minimizes R's expected payoff over the set of possible priors

$\Delta(\Theta)$. In the worst case, the minimum value of R's expected utility is when the selected action y lies in one half of the interval (a_i, a_{i+1}), and θ lies in the opposite half with probability equal to 1. Therefore, the optimal action for the R is the position on (a_i, a_{i+1}) being indifferent to both these two ends. That explains his choice $y(m) = \frac{a_i + a_{i+1}}{2}$.

Since R always responds to the message m under the assumption that the true distribution of the message is degenerate and the true state θ can be either a_i or a_{i+1}, it is not necessary for S to send the message in uniform distribution as in CS. Rather than that, she can chooses any strategy $\sigma(m, \theta)$, as long as it guarantees $m \in (a_i, a_{i+1})$.

Remark 3.5.1. *in Proposition 3.5.1, the sender can choose an arbitrary signaling rule in equilibrium. To put it another way, we have a continumm of equilibria where* $\sigma(m, \theta)$ *can take any distribution in* $\Delta(\Theta)$. *As* R *faces complete under ambiguity, she only evaluates actions by assuming the worst situation where the true information* θ *is furthest from her response. In equilibrium,* R *only chooses the action at the midpoint of* $[a_i, a_{i+1}]$ *and do not consider the guess distribution. Thus* S*'s strategy is not necessarily equivalent to the uniform distribution as in the CS model.*

3.6 Multiplier preference model

This section is devoted to examining the case where the receiver faces some level of ambiguity in the distribution f of θ, but not complete as in Section 3.5. R still faces ambiguity regarding the actual distribution f, but he has more information about the probability distribution than the case of complete ambiguity.

Denote $\mu(\theta|m)$ as the updated prior of R after observing the signal m as in 3.2.1. Although $\mu(\theta|m)$ is ambiguous to R, the receiver may still capture some information about the true distribution f, or a valid "guess" of $f(\theta)$. Therefore, the agent can conjecture some reference distribution for $\mu(\theta|m)$, leading to the need to accommodate her guess in her strategy, rather than only concerning the worst scenario as in the multiple-prior framework. Following Hansen and Sargent (2001), we employ the multiplier utility (MU) model to represent R's preference under ambiguity:

$$U^R_{MU}(y) = \min_{P\in\Delta(\Omega)} \left[\int_{a_i}^{a_{i+1}} U^R(y,\theta)dP + KR(P||Q) \right] \tag{3.6.1}$$

In the above equation, firstly R finds the worst misspecification P for his best guess of the true distribution Q for each action y. Then y is chosen to maximize R's expected pay-off corresponding to the worst distribution P.

The second term on the right-hand side of equation 3.6.1, $R(P||Q)$ is called the Kullback-Leibler divergence, [8] which measures the difference between the considered probability distribution P and the reference probability distribution Q. For distributions P and Q of a continuous variable as in our model, R is defined as follows: [9]

$$R(p||q) = \int_{-\infty}^{\infty} p(x) log\left(\frac{p(x)}{q(x)}\right) dx \tag{3.6.2}$$

8. R(p||q) is also called Relative Entropy, or I-divergence in some other sources

9. More rigorously, $R(p||q)$ can be defined as $\int_{\theta\in\Theta} log\left(\frac{p(d\theta)}{q(d\theta)}\right) p(d\theta)$, where p and q are two probability measures on a measurable space Θ, where $\frac{p(d\theta)}{q(d\theta)}$ is the Radon–Nikodym derivative of p for q. However, introducing measure theory to the model is unnecessary as it does not change the main results in our paper. Employing the definition of the probability distribution of a continuous variable, as in 3.6.2 is sufficient for our purposes.

In 3.6.2, $p(\theta)$ and $q(\theta)$ are pdfs of P and Q. Note that as we only restrict the set of message space being $\Theta = [0,1]$, the two boundaries of the integral in 3.6.2 are zero and unity. Also in 3.6.1, K reflects the level of ambiguity associated with R's best guess of distribution f. More specifically, an increase in K^{-1} leads to R being more ambiguity-averse (Epstein and Schneider (2010)).If $K=0$, K^{-1} goes to infinity, which reflects the case where R is completely ambiguous about the precise distribution f, thus her guess is uninformative in choosing the best action in equilibrium. If K tends to infinity, R gains substantial information about the distribution law. In this case, $P=Q$, and R's guess coincides with the true distribution of f, as in the CS model.

Replace Q in 3.6.1 by the expression of the reference distribution $\mu(\theta|m)$, and U^R by the quadratic-loss utility, R's optimal strategy under MU framework given message m can be interpreted as follows:

$$y(m) = \underset{y}{\text{argmax}} \inf_{p(\theta)} \left[\int_{a_i}^{a_{i+1}} -(y-\theta)^2 p(\theta) d\theta + K p(\theta) log\left(\frac{p(\theta)}{\mu(\theta|m)}\right) d\theta \right] \tag{3.6.3}$$

In 3.6.3, upon receiving the signal m, for each response y, the receiver assumes the most pessimistic scenario where the actual prior $p(\theta)$ is most divergent from the conjectured prior $\mu(\theta|m)$.

We consider $K>0$ (the case $K=0$ is the complete ambiguity case as in 3.5). Moreover, to be able to find an analytical solution for $p(\theta)$ in 3.6.3, we further assume that both $p(\theta)$ and $\mu(\theta|m)$ are continuously differentiable. The worst distribution satisfies the following:

Proposition 3.6.1. $p^*(\theta) = \exp\left(A + \frac{(y-\theta)^2}{K}\right)\mu(\theta|m)$, *where A is the unique real number normalizing* $p(\theta)$ *as a probability density distribution, i.e.* $\int_{a_i}^{a_{i+1}} p^*(\theta) d\theta = 1$.

The result of Proposition 3.6.1 can be illustrated in figures below. Given that K represents the degree of ambiguity in the best estimate of the distribution of R', the shift from the graph of the conjectured distribution $\mu(\theta|m)$ to the graph of the worst distribution $p^*(\theta)$ can be observed accordingly. In Figure 3.6.1, $\mu(\theta|m)$, purple in color, is assumed to be a beta distribution. When the ambiguity is significant, respectively $K = 0.05$, the blue diagram of $p^*(\theta)$ shifts farthest and has a relatively different shape from the original beta distribution. When R can access a considerable amount of information on the true distribution $\mu(\theta|m)$ $(K = 1)$, it can be seen that $p^*(\theta)$ is very close to $\mu(\theta|m)$. Similar observations can be seen in Figures 3.6.2 and 3.6.3. Note that in Figure 3.6.2, the shape of $p^*(\theta)$ is similar to the normal distribution due to its symmetry, which can be seen in proposition 3.6.1.

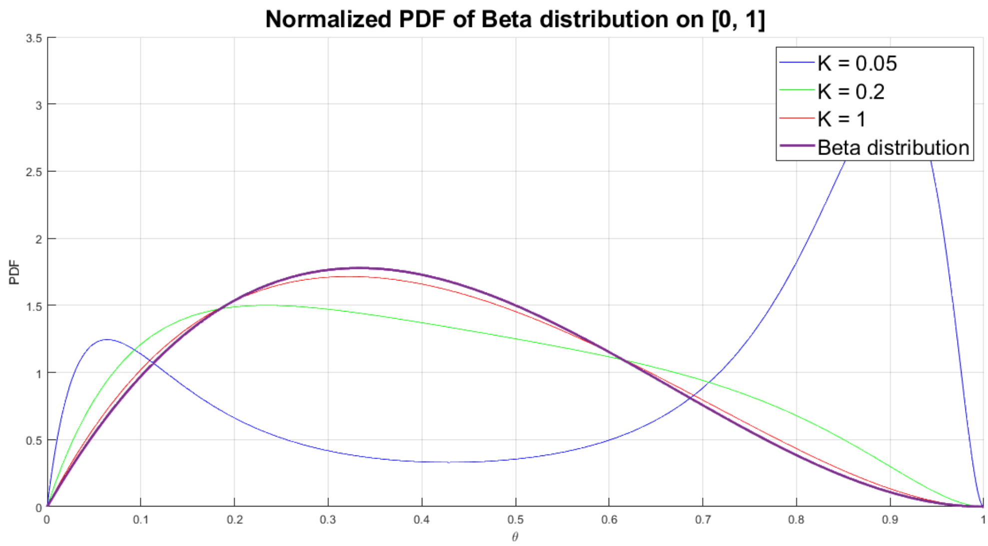

Figure 3.6.1: Beta distribution and $p^*(\theta)$ with different values of K.

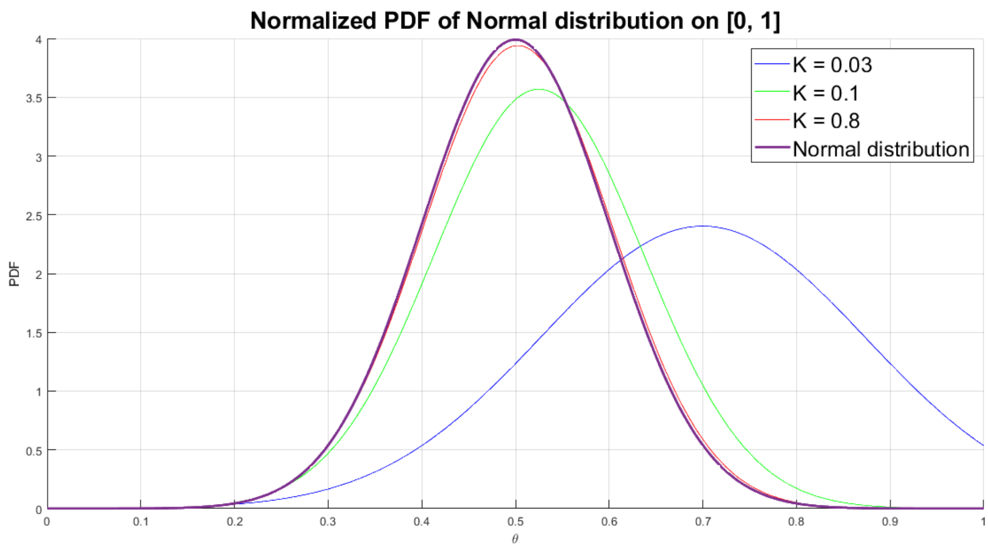

Figure 3.6.2: Normal distribution and $p^*(\theta)$ with different values of K.

Given Proposition 3.6.1, there is a unique worst scenario of the probability distribution $p^*(\theta)$ for each response y of R. The DM's strategy, thus, can be derived accordingly:

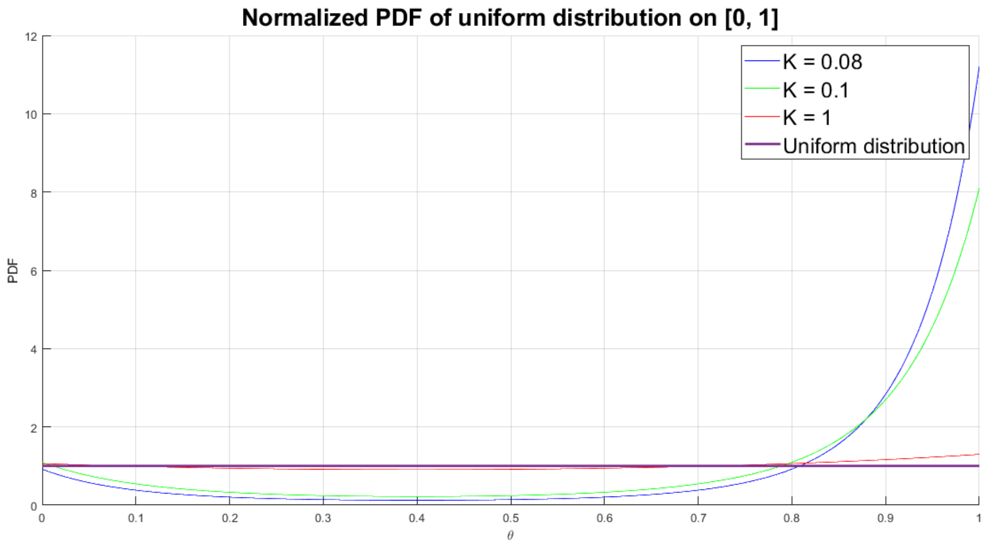

Figure 3.6.3: Uniform distribution and $p^*(\theta)$ with different values of K.

Corollary 3.6.1. *Upon observing the signal in* $[a_i, a_{i+1}]$*, R's best action is the unique solution* $y(a_i, a_{i+1}) \in [a_i, a_{i+1}]$ *satisfying* $\int_{a_i}^{a_{i+1}} (\theta - y) \exp\left(\frac{(y-\theta)^2}{K}\right) f(\theta) d\theta = 0$*. There exists at least one equilibirum* $(y(m), \sigma(m|\theta))$*, where* $\sigma(m|\theta)$ *is uniform, supported on* $[a_i, a_{i+1}]$ *if* $\theta \in (a_i, a_{i+1})$ *that satisfies the condition (C). Moreover, any equilibrium is essentially equivalent to one in this class.*

The corollary 3.6.1 identifies the best strategy for R when observing the signal m in the interval $[a_i, a_{i+1}]$ given that his best guess of the true probability distribution is f, and the posterior belief corresponding is $\mu(\theta|m)$ defined by 3.2.1. If $f(.)$ is a uniform distribution, one can derive $y(a_i, a_{i+1}) = \frac{a_i + a_{i+1}}{2}$, which coincides with the best action of R in previous sections. When $f(.)$ is different from the uniform distribution, $y(a_i, a_{i+1}) \neq \frac{a_i + a_{i+1}}{2}$ in general. Moreover, the integral in 3.6.1 is hardly solvable. However, given that $\mu(\theta|m)$ is continuously differentiable, by Implicit Theorem it can be verified that the solution $y(a_i, a_{i+1})$ is also continuously differentiable, which satisfies the conditions of theorem 3.3.1. Thus, the partition equilibrium exists in the multiplier preference model.

Corollary 3.6.2. *R's optimal action identified from 3.6.1 satisfies all the conditions of Theorem 3.3.1, therefore, the CSUA partition equilibrium exists.*

A direct implication can be observed from 3.6.1: when R gains a lot of information from f, or K goes to infinity, one can show that y will converge to the conditional expected value of θ on (a_i, a_{i+1}) under the assumed distribution $\mu(\theta|m)$. This is exactly the response of R in the CS model. In particular, when $\mu(\theta|m)$ is a uniform distribution, y converges to the midpoint of $[a_i, a_{i+1}]$ regardless of the value of K. Hence, R's action in the MU model when her guess distribution is uniform accommodates the complete ambiguity in section 3.5as a special case. The consistency in R's choice of the midpoint of $[a_i, a_{i+1}]$ under ambiguity can be attributed to her preference's quadratic form and the uniform distribution's symmetry.

Corollary 3.6.3. $y = \dfrac{a_i + a_{i+1}}{2}$ *when the reference distribution* $f(\theta)$ *is uniform for* $K \in (0, +\infty]$.

For an arbitrary $\mu(\theta|m)$, when R is completely uncertain about the distribution of the private message, that is, $K = 0$, the best response is $y^{full\ ambiguity} = \dfrac{a_i + a_{i+1}}{2}$. When R does not face ambiguity in the distribution law, namely $K = \infty$, it is straightforward to show that $y^{no\ ambiguity} = y^{CS} = \displaystyle\int_{a_i}^{a_{i+1}} \theta f(\theta) d\theta$, which coincides with the optimal action in the expected utility model of CS.

When R holds certain ambiguity about the true distribution of the private message but is not too extreme, i.e., $K \in (0, \infty)$, in general, her optimal action is neither $y^{full\ ambiguity}$ nor $y^{no\ ambiguity}$. Although proving this in the general case seems challenging, it is constructive to provide a few examples to illustrate it.

Firstly, we give an example to show that for some $K > 0$, $y(a_i, a_{i+1}) \neq y^{CS}$. We take $K = 0.1$ and consider the distribution $f(\theta) = 2\theta$. By setting the informed interval is the whole domain $[0, 1]$, we get $y^{CS} = \displaystyle\int_0^1 \theta 2\theta d\theta = \frac{2}{3}$, $y(a_i, a_{i+1})$ is the solution of $\displaystyle\int_0^1 (\theta - y) \exp\left(\frac{(\theta - y)^2}{K}\right) 2\theta d\theta = 0$, so $y(a_i, a_{i+1}) \approx 0.60 \neq y^{CS}$.

We then show that, for a class of motonotic distributions, $y(a_i, a_{i+1}) \neq y^{full\ ambiguity}$. Notice that the distribution $f(\theta) = 2\theta$ defined above also satisfies this property.

Proposition 3.6.2. *If $f(\theta)$ is weakly increasing (decreasing), R's action in CSUA is strictly higher (lower) than her solution in the case of full ambiguity.*

The intuition behind proposition 3.6.2 is as follows: when R has a best guess that the true distribution is increasing (decreasing), or getting higher to the right (left) of the informed interval, he will choose her action to the right (left) of the completely ambiguous action $y = \frac{a_i + a_{i+1}}{2}$ when she is the most uncertain about f. Since R has some information related to f, she is not completely uncertain, and thus her action must be closer to private information than when she is completely uncertain about f. As the distribution of $f(\theta)$ is increasing (decreasing), there is a higher chance that the true θ lies in the half-right (left) interval than in the half-right (right) of (a_i, a_{i+1}), which explains the choice of R.

Finally, we illustrate R's optimal responses and equilibrium cutoffs in the standard CS and ambiguity models for cases where $N = 2$ or $N = 3$, and where K is either small or large.

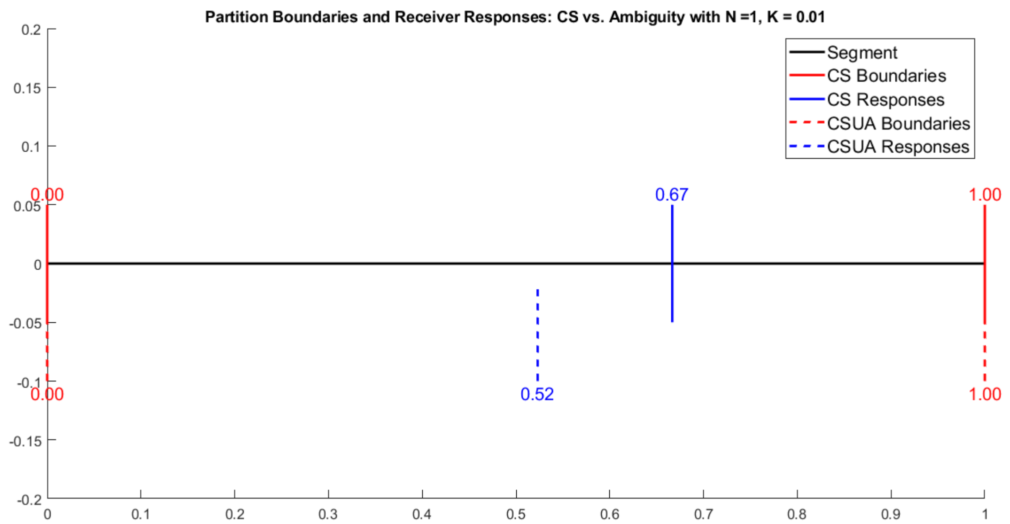

Figure 3.6.4: Equilibrium cutoffs in the CS and in CSUA models where $N = 1$ under large ambiguity.

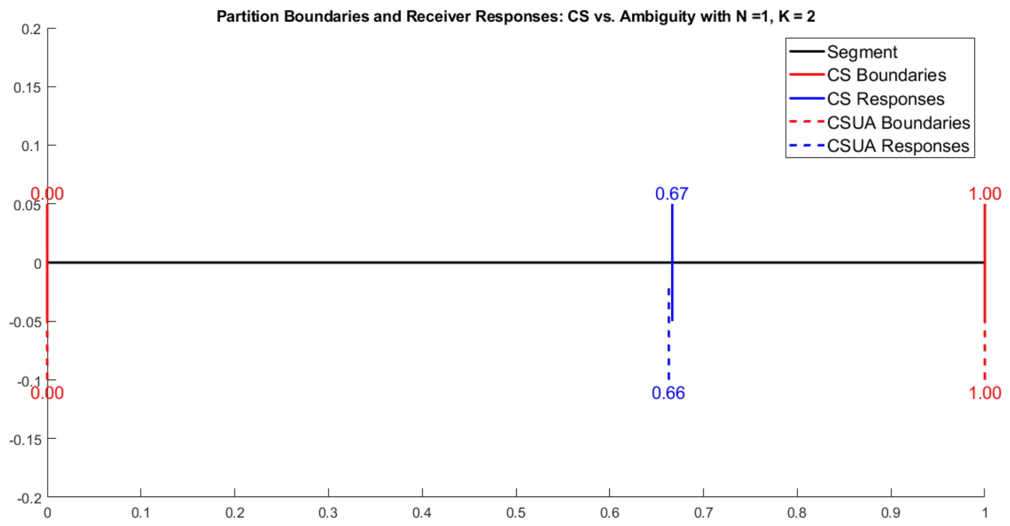

Figure 3.6.5: Equilibrium cutoffs in the CS and in CSUA models where $N = 1$ under small ambiguity.

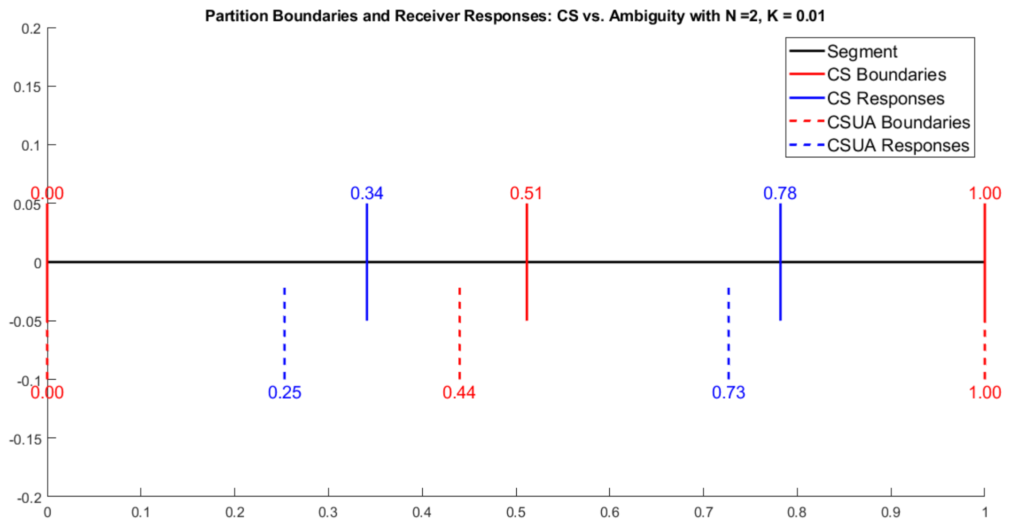

Figure 3.6.6: Equilibrium cutoffs in the CS and in CSUA models where $N = 2$ under large ambiguity.

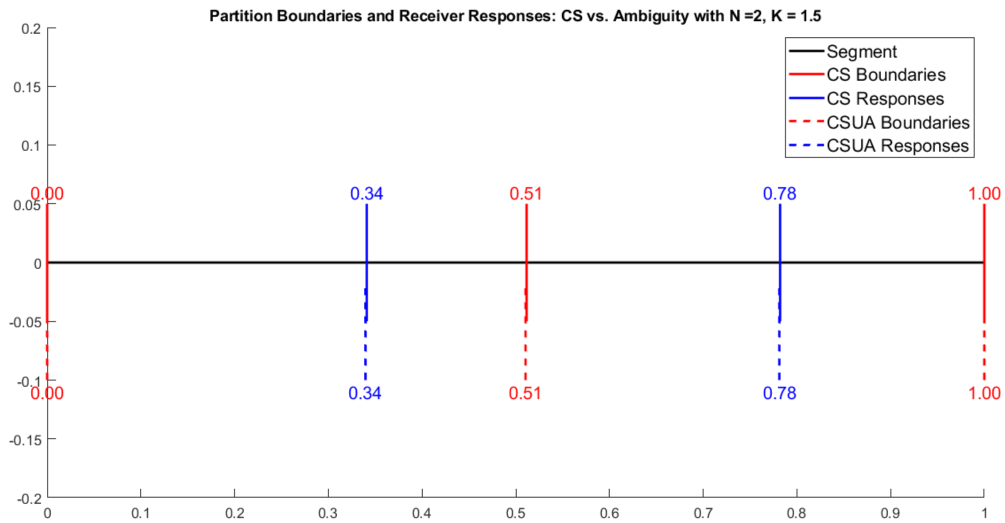

Figure 3.6.7: Equilibrium cutoffs in the CS and in CSUA models where $N = 2$ under small ambiguity.

3.7 Conclusion

This chapter examines the strategic communication problem based on the canonical Crawford-Sobel model, considering the case where the receiver faces ambiguity regarding the distribution of private information. Several key conclusions are drawn under this ambiguity context: the sender transmits a signal from the interval in which his type lies and the receiver selects an action that maximizes her payoff under ambiguity. The set of signals that induces the same response forms intervals, partitioning the entire signal domain. Consequently, every equilibrium takes the form of a partition. Additionally, the receiver follows a strategy similar to that of the CS model if her reference distribution is uniform.

However, ambiguity significantly impacts the receiver's behavior: under the assumption of ambiguity aversion, the receiver consistently selects the midpoint of the informed interval. This strategy ensures that her action in equilibrium is not too far removed from the unobserved information. When ambiguity is less extreme, and the receiver's reference distribution deviates from uniformity, her response to each received signal may shift from its original position in the CS model. As a result, the entire partition equilibrium shifts accordingly.

Although ambiguity can shift the CS partition equilibrium, this raises several interesting questions: does this shift imply that information can be transmitted more efficiently under ambiguity? How does ambiguity affect the maximum number of steps in equilibrium? These questions remain unanswered in this research, and I plan to address them in future studies.

3.8 Appendix

3.8.1 Proof of Theorem 3.3.1

For simplicity, we do not write b in the expression of the utility function $U^S(.)$. The first part of the proof is devoted to verifying the existence of the CSUA equilibrium, or condition C. In essence, (C) specifies a second-order condition with an initial point a_1 and a terminal point $a_N = 1$.

Define $M(a_1) = \sup\{N | \exists$ partition size N that satisfies C and $a_N \leq 1\}$. We can assume that $M(a_1) \geq 2$. For the case of an uninformative equilibrium when $M(a_1) \leq 1$, no cutoff exists, and condition C holds vacuously.

Given that $y^S(\theta, b) \neq y^R(\theta)$ for every m, WLOG supposes that $y^S(\theta) > y^R(\theta)$ (I suppress $y^S(\theta, b)$ by $y^S(\theta)$ for simplicity) for every θ, we show that the gap between the two adjacent thresholds must be bounded below. The claim is made by counterargument, assuming that there is no minimum gap between each step. Then in any partition equilibrium, for every ε there always exists a_n, a_{n+1} such that $|a_{n+1} - a_n| < \varepsilon$. Since the number of intervals in the partition equilibrium is limited, there is some i such that $|a_{i+1} - a_i| < \varepsilon$ for all small enough ε. Therefore, $a_i = a_{i+1}$ for some i.

Due to the arbitrage condition, $U^S(y(a_{i-1}, a_i), a_i) = U^S(y(a_i, a_{i+1}), a_i) = U^S(y(a_i, a_i), a_i) = U^S(y^R(a_i), a_i)$. Therefore, $U^S(y(a_{i-1}, a_i), a_i) = U^S(y^R(a_i), a_i)$. From the monotonicity $y(a_{i-1}, a_i) < y(a_i, a_i) = y^R(a_i)$, it follows that $y(a_{i-1}, a_i) < y^S(a_i) < y^R(a_i)$, which is a contradiction since $y^S(\theta) > y^R(m)$ for every θ. Next, we prove the following lemma.

Lemma 1. Assume that the arbitrage condition has a solution $0 = a_0 < a_1 < ... < a_n < 1$, then there exist an open neighborhood V of a_1 and a set of continuously differentiable functions $\xi_i : V \to R$ such that: $\xi_i(a_1) = a_i$ for every $i = 0, n$, and $(0, a_1^{'}, \xi_2(a_1^{'}), ...\xi_n(a_1^{'}))$ satisfies the arbitrage condition for every $a_1^{'} \in V$.

Proof: The proof is carried out by employing the implicit function theorem (IFT) [10] under the induction assumption. Without loss of generalization, we can assume that $\xi_0(a_1) = 0$ and set $\xi_1(a_1^{'}) = a_1^{'}$. We will show that if the lemma is true under $n-1$ and $n-2$, it is also justified for every n.

In fact, consider a partition equilibrium $0 = a_0 < a_1 < ... < a_n < 1$. Since the arbitrage condition holds, it is obvious that for every $i = 1,..n-1$, the induction assumption is still satisfied. Then by the induction assumption, there exist continuously differentiable $\xi_0(a_1') = 0, \xi_1(a_1') = a_1', ..\xi_{n-2}(a_1')$ defined in some neighborhood V of a_1 such that $\xi_i(a_1) = a_i$ for $i = 0,..,n-1$, and

$$U^S(y(\xi_{i-1}(a_1^{'}), \xi_i(a_1^{'})), \xi_i(a_1^{'})) = U^S(y(\xi_i(a_1^{'}), \xi_{i+1}(a_{i+1}^{'})), \xi_i(a_1^{'}))(1)$$

for $i = 1,..,n-2$.

Now we have the arbitrage condition at $n-1$:

$$U^S(y(a_{n-2}, a_{n-1}), a_{n-1}) = U^S(y(a_{n-1}, a_n), a_{n-1})$$

By the induction assumption, it is equivalent to:

$$U^S(y(\xi_{n-2}(a_1), \xi_{n-1}(a_1)), \xi_{n-1}(a_1)) = U^S(y(\xi_{n-1}(a_1), a_n), \xi_{n-1}(a_1))$$

This equation defines a_n as a function of a_1. Note that by our assumptions, $U(.)$, $y(.)$, $\xi(.)$ are all continuously differentiable. By the implicit function theorem, there exists ξ_n on a neighborhood V' of a_1 such that $\xi_n(a_1) = a_n$, and

$$U^S(y(\xi_{n-2}(a_1^{'}), \xi_{n-1}(a_1^{'})), \xi_{n-1}(a_1^{'})) = U^S(y(\xi_{n-1}(a_1^{'}), \xi_n(a_1^{'})), \xi_{n-1}(a_1^{'}))$$

for every $a_1^{'} \in V^{'}(2)$.

10. Kono and Kandori (2019) provides a proof of Lemma 1 without employing IFT. However, as R's optimal action in the multiplier preference model is continuously differentiable (see Corollary 3.6.2), we apply this continuous differentiability to the assumption $y(a_i, a_{i+1})$. This justifies our use of the IFT in the proof.

From (1), (2), it can be seen that in the neighborhood $V \cap V'$, the family of $\xi_i(a_1)$ combined with a_1' defines a solution for the arbitrage condition, which completes the proof.

Now we will use Lemma 1 to complete the proof. More specifically, we want to show that if $y^S(m) > y^R(m)$ then the CS equilibrium exists when R's optimal response satisfies all the conditions in 3.3.1. Consider any solution $0 < a_1 < ... < a_n < 1$ which meets the arbitrage condition, we claim that there exist another partition equilibrium with the same size where the terminal point $a_n' = 1$. Note that, for any $n \leq N(b)$, if the terminal $a_n = 1$ then Theorem 3.3.1 is obvious. Therefore, we only consider the cases where $a_n < 1$.

Let $D = \{a_1' \in (0,1) |$ there exists $0 < a_1' < a_2' < ... < a_n' \leq 1\}$. Since at least $(0, a_1, ..., a_n) \in D$, hence D is non-empty. Denote $a_1^* = \sup D$, then there exists a sequence $\{a_{1i}'\}_{i \in N}$ in D which converges to a_1^*. Each $\{a_{1i}'\}$ defines the associated solution of the arbitrage condition $a_{2i}', ..., a_{ni}'$. Now consider the sequence of associated solutions $(a_{1i}', a_{2i}', ..., a_{ni}')_{i \in N}$. As the sequence is inside the compact set $[0,1]^n$, there exists a subsequence $(a_{1ik}', a_{2ik}', ..., a_{nik}')_{k \in N}$ that converges to some $(a_1^*, a_2^*, ..., a_n^*)$, note that every subsequence a_{1ik}' all converges to a_1^*. As all the subsequence are in D, the arbitrage condition yields:

$$U^S(y(a_{j-1,ik}', a_{j,ik}'), a_{j,ik}') = U^S(y(a_{j,ik}', a_{j+1,ik}'), a_{j,ik}')$$

for every $j = 1, ..n - 1$

By taking limit of both sides we have:

$$U^S(y(a_{j-1}^*, a_j^*), a_j^*) = U^S(y(a_j^*, a_{j+1}^*), a_j^*)$$

under assumptions that U and y are continuous.

The above expression shows that $(a_1^*, a_2^*, ..., a_n^*)$ also satisfied the arbitrage condition since $a_1^* = \sup D$ which leads to $a_1^* \geq a_1 > 0$, and all a_i^*, a_j^* are different. As $a_i^* = \sup D$, then a_n^* must be 1. Indeed, if $a_n^* < 1$, then from Lemma 1 there exists a neighborhood V of a_1^* such that for any $a_1^{**} \in V$, particularly $a_1^{**} > a_1^*$, $a_1^{**} \in D$. This is a contradiction as $a_1^* = \sup D$. Therefore, there always exists an arbitrage solution with $a_n = 1$.

For the case $y^S(m) < y^R(m)$, all arguments above can be still applied in the same fashion under the initial condition $a_{n-1} < a_n = 1$. Therefore, the existence of the CS equilibrium is warranted under both cases when $y^S(m) < y^R(m)$, and when $y^S(m) > y^R(m)$.

Provided that $y^S(m) \neq y^R(m)$ for every m, it implies that either $y^S(m) < y^R(m)$ or $y^S(m) > y^R(m)$. Otherwise, there exists some m' such that $y^S(m') = y^R(m')$ due to the continuity of $y^S(.)$ and $y^R(.)$. In either case, the existence of the CS equilibrium is always guaranteed following the above arguments.

Now given the existence of the partition equilibrium defined in (C) with $a_0 = 0$ and $a_1 = 1$, and R's law of action $y(a_i, a_{i+1})$ for any signal received in (a_i, a_{i+1}), we show that S's optimal strategy is also sending a random signal in the considered interval. To show S does not have an incentive to send signals from other intervals, it is sufficient to verify that:

$$U^S(y(a_i, a_{i+1}), \theta, b) \geq U^S(y(a_j, a_{j+1}), \theta, b)$$

for all $j \neq i$ and $\theta \in (a_i, a_{i+1})$.

It can be seen from the arbitrage condition:

$$U^S(y(a_{i-1}, a_i), a_i, b) = U^S(y(a_i, a_{i+1}), a_i, b)$$

For every $k < i-1$, following the intermediate value theorem, there exists $q \in (y(a_k, a_{k+1}), y(a_{i-1}, a_i))$ such that:

$$U^S(y(a_{i-1}, a_i), a_i, b) - U^S(y(a_k, a_{k+1}), a_i, b) = U_1^S(q, a_i, b)(a)$$

Similarly, there is $q' \in (y(a_{i-1},a_i), y(a_i,a_{i+1}))$ such that:

$$U^S(y(a_i,a_{i+1}),a_i,b) - U^S(y(a_{i-1},a_i),a_i,b) = 0 = U_1^S(q',a_i,b)(b)$$

Here notice that we have used the fact that $y(a_i,a_{i+1}) > y(a_{i-1},a_i) > y^(a_k,a_{k+1})$ from R's best response.

From (a) and (b), $q < q'$ and $U_{11}^S < 0$, it can be seen $U^S(y(a_{i-1},a_i),a_i,b) - U^S(y(a_k,a_{k+1}),a_i,b) = U_1^S(q,a_i,b) > U_1^S(q',a_i,b) = 0$ (c). The analogous argument is applied for $j > i+1$, i.e.

$$U^S(y(a_i,a_{i+1}),a_i,b) - U^S(y(a_j,a_{j+1}),a_i,b) > 0$$

Therefore, when the true state is $\theta = a_i$, the S's signaling rule is to send a signal $m \in (a_{i-1},a_i)$ or $m \in (a_i,a_{i+1})$.

Now, for any other $\theta \in (a_i,a_{i+1})$, using the fact that $U_{12}^S > 0$, we can derive the following inequalities: Now, for any other $\theta \in (a_i,a_{i+1})$, using the fact that $U_{12}^S > 0$, we can derive the following inequalities:

$$(U^S(y(a_i,a_{i+1}),\theta,b) - U^S(y(a_k,a_{k+1}),\theta,b) \geq U^S(y(a_i,a_{i+1}),a_i,b) - U^S(y(a_k,a_{k+1}),a_i,b) \geq 0$$

for $k < i-1$ (d), and a similar inequality for $j > i+1$:

$$(U^S(y(a_i,a_{i+1}),\theta,b) - U^S(y(a_j,a_{j+1}),\theta,b) \geq U^S(y(a_i,a_{i+1}),a_{i+1},b) - U^S(y(a_j,a_{j+1}),a_{i+1},b) \geq 0$$

(e).

Indeed, as $U^S(.)$ is differentiable, by applying Cauchy's mean value theorem, there exists some $\gamma \in (y(a_i,a_{i+1}), y(a_k,a_{k+1}))$ such that:

$$\frac{U^S((y(a_i,a_{i+1}),\theta,b) - U^S((y(a_k,a_{k+1}),\theta,b)}{U^S((y(a_i,a_{i+1}),a_i,b) - U^S((y(a_k,a_{k+1}),a_i,b)} = \frac{U_1^S(\gamma,\theta)}{U_1^S(\gamma,a_i)}$$

Since $a_i < \theta$ and $U^S_{12} > 0$, we have $U^S_1(\gamma,\theta) \geq U^S_1(\gamma,a_i)$. Therefore (d) is proved, the same argument can be applied to (e). These two inequalities guarantee that for R's best response $y(m)$, S's best response is to send a message in (a_i,a_{i+1}). ■

3.8.2 Proof of Theorem 3.4.1

We only give the proof for part (ii); part (i) is similar to Theorem 3.3.1.

Consider R, from 3.2.1 and from S's strategy is uniform; her updated belief when hearing a signal in (a_i,a_{i+1}) is given by:

$$\mu(\theta|m) = \frac{f(\theta,\varepsilon)}{\displaystyle\int_{a_i}^{a_{i+1}} \sigma(m,t)f(t,\varepsilon)dt}$$

$$= \frac{2\varepsilon\theta+(1-\varepsilon)}{\displaystyle\int_{a_i}^{a_{i+1}} 2\varepsilon t+1-\varepsilon dt} = \frac{2\varepsilon\theta+(1-\varepsilon)}{(a_{i+1}-a_i)\Big(1-\varepsilon+\varepsilon(a_i+a_{i+1})\Big)} = \frac{2\varepsilon\theta+(1-\varepsilon)}{\varepsilon C+a_{i+1}-a_i}$$

, where $C = (a_i+a_{i+1}-1)(a_{i+1}-a_i)$. Notice that if $\varepsilon = 0$ then $\mu(\theta|m) = \dfrac{1}{a_{i+1}-a_i}$, which resonates the uniform posterior as in the CS model.

Now, for $U^R(y,\theta) = -\sum_{i=1}^{l} \alpha_i(y-\theta)^{2k_i}+c_i$, the following properties are true:

a. $U^R(a_i,\theta)$ is decreasing in θ, $U^R(a_{i+1},\theta)$ is increasing in θ.

b. $\displaystyle\int_{a_i}^{a_{i+1}} U^R(\frac{a_i+a_{i+1}}{2},\theta)(2\theta-a_i-a_{i+1})\,d\theta \leq 0$

c. $\displaystyle\int_{a_i}^{a_{i+1}} U^R_1(\frac{a_i+a_{i+1}}{2},\theta)\,d\theta \leq 0$

The proof is provided in the following steps.

Step 1. We show that there exists a unique action y^* such that $\int_{a_i}^{a_{i+1}} U^R(y,\theta)\mu(\theta|m)\,d\theta$ is indifferent for every ε.

It suffices to show that there exists $y^* \in (a_i, a_{i+1})$ such that the integral

$I(y^*,\varepsilon) = \int_{a_i}^{a_{i+1}} U^R(y^*,\theta)\frac{2\varepsilon\theta + (1-\varepsilon)}{\varepsilon C + a_{i+1} - a_i}\,d\theta$ does not depend on ε.

Taking the first derivative w.r.t ε yields:

$I'_\varepsilon(y^*,\varepsilon) = \frac{(a_{i+1}-a_i)}{2(\varepsilon C + a_{i+1} - a_i)^2}\int_{a_i}^{a_{i+1}} U^R(y^*,\theta)(2\theta - (a_i + a_{i+1}))\,d\theta$ (i).

It reduces to show that there exists $y^* \in (a_i, a_{i+1})$ such that:

$\int_{a_i}^{a_{i+1}} U^R(y^*,\theta)(2\theta - (a_i + a_{i+1}))\,d\theta = 0$ (ii)

Define $h(t) = \int_{a_i}^{a_{i+1}} U^R(t,\theta)(2\theta - (a_i + a_{i+1}))\,d\theta$, for any $t \in (a_i, a_{i+1})$, we have:

$h(a_i) = \int_{a_i}^{a_{i+1}} U^R(a_i,\theta)(2\theta - (a_i + a_{i+1}))\,d\theta$, $h(a_{i+1}) = \int_{a_i}^{a_{i+1}} U^R(a_i,\theta)(2\theta - (a_i + a_{i+1}))\,d\theta$.

From (a), $U^R(a_i,\theta)$ is a decreasing function of θ, $2\theta - (a_i + a_{i+1})$ is a strictly increasing function of θ, the Chebyshev's sum inequality for integral implies:

$$\begin{aligned}\frac{1}{a_{i+1}-a_i}\int_{a_i}^{a_{i+1}} U^R(a_i,\theta)(2\theta - (a_i + a_{i+1}))\,d\theta &< \left(\frac{1}{a_{i+1}-a_i}\int_{a_i}^{a_{i+1}} U^R(a_i,\theta)\,d\theta\right) \\ &\quad \left(\frac{1}{a_{i+1}-a_i}\int_{a_i}^{a_{i+1}} (2\theta - (a_i + a_{i+1}))\,d\theta\right) \\ &= 0, \quad \text{since} \quad \int_{a_i}^{a_{i+1}} (2\theta - (a_i + a_{i+1}))\,d\theta = 0.\end{aligned}$$

Notice that equality cannot hold in the inequality above, as $U^R(a_i,\theta)$ and $2\theta - (a_i + a_{i+1})$ are not constant functions. Analogously, we can show that:

$$\frac{1}{a_{i+1}-a_i}\int_{a_i}^{a_{i+1}} U^R(a_{i+1},\theta)(2\theta - (a_i + a_{i+1}))\,d\theta > 0$$

Therefore $h(a_i) < 0 < h(a_{i+1})$, due to the continuity of $h(.)$ there exists $y^* \in (a_i, a_{i+1})$ such that $h(y^*) = 0$. Next, we show that $h(t)$ is increasing, leading to the uniqueness of y^*.

Consider $h'(t) = \int_{a_i}^{a_{i+1}} U_1^R(t,\theta)(2\theta - (a_i + a_{i+1}))\,d\theta$, since $U_{12}^R > 0$ we see that $U_1^R(t,\theta)$ is an increasing function of θ. We apply again the Chebyshev's inequality for two increasing functions $U_1^R(t,\theta)$ and $2\theta - (a_i + a_{i+1})$ to derive:

$$\int_{a_i}^{a_{i+1}} U_1^R(t,\theta)(2\theta-(a_i+a_{i+1}))\,d\theta > \frac{1}{a_{i+1}-a_i}\int_{a_i}^{a_{i+1}} U_1^R(t,\theta)\,d\theta \int_{a_i}^{a_{i+1}} (2\theta-(a_i+a_{i+1}))\,d\theta = 0.$$

Thus, $\int_{a_i}^{a_{i+1}} U^R(t,\theta)(2\theta - (a_i + a_{i+1})d\theta$ is an increasing function of t. This implies the uniqueness of y^*.

Step 2. We claim that for action y^* satisfying $h(y^*) = 0$, $\int_{a_i}^{a_{i+1}} U_1^R(y^*,\theta)d\theta \leq 0$.

Observe that $y^* \geq \frac{a_i + a_{i+1}}{2}$. In fact, we have $h(y^*) = 0$ and from (c), $h(\frac{a_i + a_{i+1}}{2}) \leq 0$, given that $h(t)$ is a strictly increasing function of t from Step 1, we have $y^* \geq \frac{a_i + a_{i+1}}{2}$. Since $\int_{a_i}^{a_{i+1}} U_1^R(\frac{a_i + a_{i+1}}{2},\theta)d\theta \leq 0$, $U_{11}^R < 0$, $y^* \geq \frac{a_i + a_{i+1}}{2}$, it implies that

$$\int_{a_i}^{a_{i+1}} U_1^R(y^*,\theta)d\theta \leq \int_{a_i}^{a_{i+1}} U_1^R(\frac{a_i + a_{i+1}}{2},\theta)d\theta \leq 0.$$

Step 3. We claim that for any action $y \in (y^*, a_{i+1}]$, the receiver's expected payoff under the most extreme cases of small ambiguity, i.e. $\varepsilon = -M$ (or M) is less than that of action y^*. Specifically, we show that:

$$\int_{a_i}^{a_{i+1}} U^R(y^*,\theta)\frac{1+M-2M\theta}{a_i+a_{i+1}-MC}d\theta \geq \int_{a_i}^{a_{i+1}} U^R(y,\theta)\frac{1+M-2M\theta}{a_i+a_{i+1}-MC}d\theta.$$

The inequality is reorganized as follows:

$$\int_{a_i}^{a_{i+1}} \Big(U^R(y^*,\theta) - U^R(y),\theta)\Big)\frac{1+M-2M\theta}{a_i+a_{i+1}-MC}d\theta \geq 0$$

By the mean-values theorem, the L.H.S can be expressed as:

$\int_{a_i}^{a_{i+1}} (y^* - y) U_1^R(y', \theta) \frac{1 + M - 2M\theta}{a_i + a_{i+1} - MC} d\theta \geq 0$ for some $y' \in (y^*, y)$, or equivalently $\int_{a_i}^{a_{i+1}} U_1^R(y', \theta)(1 + M - 2M\theta) d\theta \leq 0$ when we drop $y^* - y < 0$ and $a_{i+1} - a_i - MC = (a_{i+1} - a_i)(1 + M - M(a_i + a_{i+1)})) > 0$.

Now, since $U_{12}^R < 0$, $U_1^R(y', \theta)$ is increasing in θ, $1 + M - 2M\theta$ is a decreasing function in θ. Applying Chebyshev's inequality for these two increasing and decreasing functions yields:

$\int_{a_i}^{a_{i+1}} U_1^R(y', \theta)(1 + M - 2M\theta) d\theta \leq \frac{1}{a_{i+1} - a_i} \int_{a_i}^{a_{i+1}} U_1^R(y', \theta) d\theta \int_{a_i}^{a_{i+1}} (1 + M - 2M\theta)) d\theta < 0$ since $\int_{a_i}^{a_{i+1}} U_1^R(y', \theta) d\theta < \int_{a_i}^{a_{i+1}} U_1^R(y^*, \theta) d\theta \leq 0$ following Step 2, and $\int_{a_i}^{a_{i+1}} (1 + M - 2M\theta)) d\theta > 0$ for M small enough. Therefore, Step 3 is verified.

Step 4. we show that the optimal action y^* satisfying Step 1 is indeed R' optimal action following the max-min expected utility under ambiguity.

For each action $y(m)$, R assumes that the worst case occurs with some value of ε.

$W(y) = \min_{\varepsilon} \int_{a_i}^{a_{i+1}} U^R(y, \theta) \frac{2\varepsilon\theta + (1 - \varepsilon)}{\varepsilon K + a_{k+1} - a_k} d\theta = \min_{\varepsilon} I(y, \varepsilon)$ where $I(y, \varepsilon)$ is the integral in the expression.

Taking the derivative with respect to ε of the integral yields:

$$I'_\varepsilon(y, \varepsilon) = (a_{i+1} - a_i) \int_{a_i}^{a_{i+1}} U^R(y, \theta) \frac{2\theta - (a_i + a_{i+1})}{(\varepsilon K + a_{i+1} - a_i)^2} d\theta$$

The above expression shows that the sign of the derivative $I'_\varepsilon(y, \varepsilon)$ depends only on the sign of $\int_{a_i}^{a_{i+1}} U^R(y, \theta)(2\theta - (a_i + a_{i+1}) d\theta$. Since $\int_{a_i}^{a_{i+1}} U^R(y, \theta)(2\theta - (a_i + a_{i+1}) d\theta$ is an increasing function of y and y^* satisfies $\int_{a_i}^{a_{i+1}} U^R(y^*, \theta)(2\theta - (a_i + a_{i+1}) d\theta = 0$ according to step 1, it holds that if $y > y^*$ then $I'_\varepsilon(y, \varepsilon) > 0$, and if $y < y^*$ then $I'_\varepsilon(y, \varepsilon) < 0$.

Now consider action $y \geq y^*$, $I'_\varepsilon(y,\varepsilon) > 0$ means that the worst case is when $\varepsilon = -M$, but from Step 3 we know that the best action for the receiver in the domain $[y^*, a_{i+1}]$ is y^* and the max-min expected utility is $\int_{a_i}^{a_{i+1}} U^R(y^*,\theta)\frac{1+M-2M\theta}{MC+a_{i+1}-a_i}d\theta$. Similarly, over the domain $[a_i, y^*]$, $I'_\varepsilon(y,\varepsilon) \leq 0$, so the worst case is when $\varepsilon = M$ and the max-min expected utility is $\int_{a_i}^{a_{i+1}} U^R(y^*,\theta)\frac{1-M+2M\theta}{-MC+a_{i+1}-a_i}d\theta$. From the definition of y^*, the two integrals $I(y^*,M)$ and $I(y^*,-M)$ have the same value. This implies that y^* is the best action for the receiver. ■

3.8.3 Proof of Proposition 3.5.1

We first calculate R's best response in the case of complete ambiguity. Upon receiving a signal $m \in (a_i, a_{i+1})$ from S, R updates his prior belief and solves the MEU with the posterior $\mu(\theta|m)$ defined in 3.2.1:

$$\max_{y(m)}\min_{\mu(.)} \int_{a_i}^{a_{i+1}} U^R(y,\theta)\mu(\theta|m)\,d\theta$$

Being ambiguity-averse, R calculates the worst expected utility associated with some posterior belief μ under ambiguity for each response strategy y,:

$$W(y) = \min_{\mu(.)} \int_{a_i}^{a_{i+1}} U^R(y,\theta)\mu(\theta|m)\,d\theta \tag{3.8.1}$$

It is obvious that $U^R(y(m),\theta)$ is exactly the distance between the true state θ and the best guess y of R. In any case, once R knows for sure that θ lies in $[a_i, a_{i+1}]$, any other option y outside the interval would be dominated by a_i or a_{i+1}. Therefore, we can assume R's optimal action $y(m)$ lies in $[a_i, a_{i+1}]$.

For each $y \in [a_i, a_{i+1}]$, denote by $\theta(y)$ the true state that minimizes $U^R(y,\theta)$. Being ambiguity-averse, the worst scenario for R is when the probability distribution of θ degenerates with support at $\theta(y)$:

$$\min_{\mu(.)} \int_{a_i}^{a_{i+1}} U^R(y,\theta)\mu(\theta|m)\,d\theta = U^R(y,\theta(y))$$

Now, since $U^R(y,\theta(y)) = \min_{\theta} U^R(y,\theta) = \min_{\theta} -(y-\theta)^2$ for $\theta \in [a_i, a_{i+1}]$, $\theta(y)$ will maximize $(y-\theta)^2$. Consider $h(\theta) = (y-\theta)^2$, then $h'(\theta) = -2(y-\theta)$ for $a_i \leq y \leq a_{i+1}$.

It can be seen that if $\theta \in [a_i, y)$ then $h'(\theta) \leq 0$ therefore h is maximized at $\theta = a_i$. If $\theta \in (y, a_{i+1}]$ then $h'(\theta) \geq 0$ henceforth h is maximized at $\theta = a_{i+1}$. In short, when $y \in [a_i, a_{i+1}]$ the worst scenario for R is either $(y-a_i)^2$ or $(y-a_{i+1})^2$. By direct calculation, if $y > \frac{a_i + a_{i+1}}{2}$ then the worst case occurs when $\theta = a_{i+1}$ (3.5.2.1), if $y < \frac{a_i + a_{i+1}}{2}$ then the worst case happens when $\theta = a_i$ (3.5.2.2).

From (3.5.2.1) and (3.5.2.2), the worst scenario for R when he chooses an action y upon hearing signal $m \in [a_i, a_{i+1}]$ is the following:

- if $y > \frac{a_i + a_{i+1}}{2}$ then $W(y) = -(y - a_{i+1})^2$

- if $y < \frac{a_i + a_{i+1}}{2}$ then $W(y) = -(y - a_i)^2$

- if $y = \frac{a_i + a_{i+1}}{2}$ then $W(y) = -\frac{(a_{i+1} - a_i)^2}{2}$ and the worst distribution is the Bernoulli distribution with $\mu(a_i|m) = k$ and $\mu(a_{i+1}|m) = 1-k$ for some $k \in [0,1]$.

From the three observations above, it is clear that R's best strategy is to choose y at which she is indifferent between two possible worst scenarios, or $y(m) = \frac{a_i + a_{i+1}}{2}$ ■

3.8.4 Proof of Proposition 3.6.1

The proof is based on two steps. In the first step, we show that the worst distribution $p(\theta)_{\min}$ is defined as in the expression of proposition 3.6.1. In the second step, we show the uniqueness of A, which makes $p(\theta)_{\min}$ a probability density.

Firstly, for each response y of R, he identifies the worst distribution that differs from his best guess $\mu(\theta|m)$. More specifically, R solves the following:

$$\min_{p(\theta)} \int_{a_i}^{a_{i+1}} -(y-\theta)^2 p(\theta) + Kp(\theta)\log\left(\frac{p(\theta)}{\mu(\theta|m)}\right) d\theta$$

$$= \min_{p(\theta)} \int_{a_i}^{a_{i+1}} -(y-\theta)^2 p(\theta) + Kp(\theta)\log(p(\theta)) - Kp(\theta)\log(\mu(\theta|m))\, d\theta$$

Given the assumption that $p(\theta)$ is continuous, we have $\int_{a_i}^{a_{i+1}} -(y-\theta)^2 p(\theta) d\theta = \int_{a_i}^{a_{i+1}} -(y-\theta)^2 dP(\theta) = -(y-\theta)^2 P(\theta)\Big|_{a_i}^{a_{i+1}} - \int_{a_i}^{a_{i+1}} P(\theta) d - (y-\theta)^2 = C + \int_{a_i}^{a_{i+1}} P(\theta)(2\theta - 2y) d\theta$, where $C = -(y-\theta)^2 P(\theta)\Big|_{a_i}^{a_{i+1}}$.

Therefore, the worst distribution $p(\theta)$ can be solved from the equivalent problem:

$\min_{p(\theta)} \int_{a_i}^{a_{i+1}} P(\theta)(2\theta-2y) + Kp(\theta)log(p(\theta)) - Kp(\theta)log(\mu(\theta|m))d\theta$ (b).

We define $L(P, p, \theta) = P(\theta)(2\theta-2y) + Kp(\theta)log(p(\theta)) - Kp(\theta)log(\mu(\theta|m))$. Then in (b) we need to find the distribution $P(\theta) \in C^2[0,1]$ that extremizes the integral on RHS of (b).

$J[P] = \int_{a_i}^{a_{i+1}} L(P, p, \theta) d\theta$, with $p(\theta) = \frac{dP}{d\theta}$. The Euler-Lagrange equation yields the following:

$\frac{\partial L(P, p, \theta)}{\partial P} - \frac{d}{d\theta}\frac{\partial L}{\partial p} = 0$. Equivalently, $(2\theta - 2y) - K\frac{p'(\theta)}{p(\theta)} - K\frac{\mu'(\theta|m)}{\mu(\theta|m)} = 0$. By changing the side of the equation, we derive $\frac{p'(\theta)}{p(\theta)} = \frac{\mu'(\theta|m)}{\mu(\theta|m)} + \frac{2(y-\theta)}{K}$ (c).

Now taking the integral for both sides of (c), we obtain $\log(p(\theta)) = \log(\mu(\theta|m)) + \frac{(y-\theta)^2}{K} + C$, which finally yields $p(\theta) = \exp(A + \frac{(y-\theta)^2}{K})\mu(\theta|m)$ for A being a constant such that $\int_{a_i}^{a_{i+1}} p(\theta) d\theta = 0$. ■

3.8.5 Proof of Corollary 3.6.1

By replacing the expression of the worst $p^*(\theta)$ in 3.6.1 to 3.6.2, we derive:

$$y(a_i, a_{i+1}) = \underset{y}{\operatorname{argmax}} \int_{a_i}^{a_{i+1}} -(y-\theta)^2 p(\theta) + K \exp\left(A + \frac{(y-\theta)^2}{K}\right) \mu(\theta|m) \left(A + \frac{(y-\theta)^2}{K}\right) d\theta$$
$$= \underset{y}{\operatorname{argmax}} AK \int_{a_i}^{a_{i+1}} \exp\left(A + \frac{(y-\theta)^2}{K}\right) \mu(\theta|m) d\theta = \underset{y}{\operatorname{argmax}} AK$$

, since the value of the integral is unity. Therefore, R needs to find y such that $A(y)$ is maximized under the condition $\int_{a_i}^{a_{i+1}} \exp(A(y) + \frac{(y-\theta)^2}{K})\mu(\theta|m)d\theta = 1$.

Denote $F(A, y)$ as the expression on L.H.S. The implicit function theorem and the first-order condition yield the following:

$\frac{dA}{dy} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial A}} = -\int_{a_i}^{a_{i+1}} \frac{2(y-\theta)}{K} \exp\left(A + \frac{(y-\theta)^2}{K}\right) \mu(\theta|m) d\theta = 0$, which implies the optimal condition for $y(a_i, a_{i+1})$:

$$\int_{a_i}^{a_{i+1}} (y-\theta) \exp\left(A + \frac{(y-\theta)^2}{K}\right) \mu(\theta|m) d\theta = 0 \qquad \text{(i)}$$

By replacing the expression of $\mu(\theta|m)$, (i) is equivalent to:

$$\int_{a_i}^{a_{i+1}} (y-\theta) \exp\left(A + \frac{(y-\theta)^2}{K}\right) \frac{\sigma(m|\theta) f(\theta)}{\int_0^1 \sigma(m,t) f(t) dt} d\theta = 0$$

, or equivalently we have:

$$\int_{a_i}^{a_{i+1}} (y-\theta) \exp\left(A + \frac{(y-\theta)^2}{K}\right) \sigma(m|\theta) f(\theta) d\theta = 0 \qquad \text{(ii)}$$

, since $\int_0^1 \sigma(m,t) f(t) dt$ is well defined for each signal m.

Now because $y(a_i, a_{i+1})$ is the action induced in equilibrium, let $M_i \equiv \{m|y(m) = y(a_i, a_{i+1})\}$ then R's action must be optimized for all sigmal $m' \in M_i$. Therefore (ii) occurs for all signal m' induced equilibrium, which leads to:

$$\int_{M_i}\int_{a_i}^{a_{i+1}} (y-\theta)\exp\left(A + \frac{(y-\theta)^2}{K}\right)\sigma(m'|\theta)f(\theta)dm'd\theta = 0 \tag{iii}$$

Since y is constant over the range of integration and conditional densities integrate to unity over signal m', we derive:

$$\int_{a_i}^{a_{i+1}} (y-\theta)\exp\left(A + \frac{(y-\theta)^2}{K}\right)f(\theta)d\theta = 0 \tag{iv}$$

(iv) shows that in equilibrium, R's optimal action depends on her best guess of the actual distribution $f(\theta)$ and not on S's strategy $\sigma(m|\theta)$. Particularly, it can be easily checked that (iv) also specified R's best action when S' signaling rule is uniform, namely $\sigma(m|\theta) = \frac{1}{a_{i+1} - a_i}$ for all m. As a result, all equilibria are essentially equivalent to the uniform distribution.

Finally, we shall verify the existence and uniqueness of R's response in equilibrium.

Consider $w(y) = \int_{a_i}^{a_{i+1}} (\theta - y)\exp(\frac{(y-\theta)^2}{K})f(\theta)d\theta = 0$. It can be seen that $w(a_i) < 0$ and $w(a_{i+1} > 0)$, therefore there exists a solution between (a_i, a_{i+1}) followed by the mean value theorem. Next, $w'(y) = \int_{a_i}^{a_{i+1}} \left(-\exp(\frac{(y-\theta)^2}{K}) - \frac{2(\theta-y)^2}{K}\exp(\frac{(y-\theta)^2}{K})\right)f(\theta)d\theta < 0$, thus w is strictly decreasing implying the uniqueness of $y(a_i, a_{i+1})$. ■

3.8.6 Proof of Proposition 3.6.2

From 3.6.1, $y(a_i, a_{i+1})$ is a solution of the following:

$$Q(a_i, a_{i+1}, y) = \int_{a_i}^{a_{i+1}} (\theta - y) \exp\left(\frac{(y-\theta)^2}{K}\right) f(\theta) d\theta = 0$$

As $f(\theta)$ assumed continuously differentiable, the Implicit Function Theorem yields $\frac{\partial y}{\partial a_i} = -\frac{\frac{\partial Q}{\partial a_i}}{\frac{\partial Q}{\partial y}}$ and $\frac{\partial y}{\partial a_{i+1}} = -\frac{\frac{\partial Q}{\partial a_{i+1}}}{\frac{\partial Q}{\partial y}}$. By direct calculation we get $\frac{\partial Q}{\partial a_i} < 0$, $\frac{\partial Q}{\partial a_i} < 0$, and $\frac{\partial Q}{\partial y} > 0$. Moreover, all these terms are continuously differentiable from the assumption of f. Thus both partial derivatives of y with respect to a_i, a_{i+1} are positive and also continuously differentiable. Henceforth, $y(a_i, a_{i+1})$ is monotonic and continuously differentiable as required. ■

3.8.7 Proof of Proposition 3.6.3

Substitute $y = \frac{a_i + a_{i+1}}{2}$ into the equation that determines $y(a_i, a_{i+1})$ in Corollary 3.6.1. Notice that $f(\theta)$ is constant since it is uniform, it is straightforward to verify y satisfies the expression:

$$\int_{a_i}^{a_{i+1}} \left(\theta - (\frac{a_i + a_{i+1}}{2})\right) \exp\left(\frac{(\frac{a_i + a_{i+1}}{2} - \theta)^2}{K}\right) d\theta = 0$$

, which completes the proof. ■

3.8.8 Proof of Proposition 3.6.2

We claim that $y(a_i, a_{i+1}) \geq \frac{a_i + a_{i+1}}{2}$, where $y(a_i, a_{i+1})$ is the unique solution identified from corollary 3.6.1 under the assumption that the "best guess" distribution $f(\theta)$ is increasing. If $f(\theta)$ is decreasing, a similar result could be derived, namely $y(a_i, a_{i+1}) \leq \frac{a_i + a_{i+1}}{2}$.

Consider $h(y)=\int_{a_i}^{a_{i+1}}(\theta-y)\exp(\frac{(y-\theta)^2}{K})f(\theta)d\theta=0$, for $y\in(a_i,a_{i+1})$. The derivative of h is $h'(y)=-\int_{a_i}^{a_{i+1}}\left(1+\frac{2(\theta-y)^2}{K}\right)\exp(\frac{(y-\theta)^2}{K})f(\theta)d\theta<0$, therefore $h(y)$ is strictly decreasing. Consequently, we can show that R's response $y(a_i,a_{i+1})$ will be higher than the midpoint of the informed interval, or her action under complete ambiguity, if:

$$h(\frac{a_i+a_{i+1}}{2})=\int_{a_i}^{a_{i+1}}(\theta-\frac{a_i+a_{i+1}}{2})\exp(\frac{(\frac{a_i+a_{i+1}}{2}-\theta)^2}{K})f(\theta)d\theta>0.$$

We can see that $\frac{d(\theta-\frac{a_i+a_{i+1}}{2})\exp(\frac{(\frac{a_i+a_{i+1}}{2}-\theta)^2}{K})}{d\theta}=\exp(\frac{(\frac{a_i+a_{i+1}}{2}-\theta)^2}{K})+\frac{2}{K}(\theta-\frac{a_i+a_{i+1}}{2})^2\exp(\frac{(\frac{a_i+a_{i+1}}{2}-\theta)^2}{K})>0$, hence $(\theta-\frac{a_i+a_{i+1}}{2})\exp(\frac{(\frac{a_i+a_{i+1}}{2}-\theta)^2}{K})$ is increasing in θ. Given $f(\theta)$ is increasing from our assumption, we can apply the Chebyshev inequality for integral which yields:

$$h(\frac{a_i+a_{i+1}}{2})\geq\frac{1}{a_{i+1}-a_i}\int_{a_i}^{a_{i+1}}(\theta-\frac{a_i+a_{i+1}}{2})\exp(\frac{(\frac{a_i+a_{i+1}}{2}-\theta)^2}{K})d\theta\int_{a_i}^{a_{i+1}}f(\theta)d\theta$$

$=\frac{K}{2(a_{i+1}-a_i)}\int_{a_i}^{a_{i+1}}\exp(\frac{(\theta-\frac{a_i+a_{i+1}}{2})^2}{K})d(\frac{(\theta-\frac{a_i+a_{i+1}}{2})^2}{K})\int_{a_i}^{a_{i+1}}f(\theta)d\theta=0$, since the first integral is equal to zero. Notice that the equality does not hold since both functions under integrals are not constant and $K>0$. Thus $h(\frac{a_i+a_{i+1}}{2})>0$, combined with $h(y(a_i,a_{i+1})=0$ and h is decreasing, we derive that $\frac{a_i+a_{i+1}}{2}<y(a_i,a_{i+1})$. ■

Chapter 4

Conclusion

This thesis studies three information economics problems under ambiguity with different objectives. In the first chapter, we investigate the competitive insurance market where actuarial firms have ambiguous information regarding the composition of their customer base. The chapter contains both of an extended version of the Rothschild-Stiglitz model under Knightian uncertainty and Zheng et al. 2016's work on insurance market under ambiguity regarding the proportion of low-risk and high-risk customers.

Under the assumption of ambiguity aversion, the expected profit of companies facing ambiguity in the composition of customers is modeled using ε-contamination, distinguishing it from the RS model. In this setup, the total expected profit depends not only on the composition of customers but also on the degree of ambiguity. In complete ambiguity case, insurance firms' subjective beliefs about the market's composition carry no information. Due to their ambiguity-averse nature, pessimistic insurers prepare for the worst-case scenario, which involves only gaining profits from the high-risk type, resulting in the creation of a unique pooling equilibrium that fully insures the H-types. Note that in the Rothschild-Stiglitz model without ambiguity, the pooling equilibrium does not exist once firms hold some (positive) subjective belief about the composition of the market. Thus, ambiguity has prevented a cream-skimming mechanism where one deviating firm can entice L-type customers from others. On the other hand, the separating equilibrium is unaffected by ambiguity and therefore agrees with the RS model.

When the ambiguity level is less than one, a pooling equilibrium does not exist because there is always a possibility of cream-skimming L-type consumers by a deviating company. Nevertheless, the separating equilibrium always exists if the degree of ambiguity exceeds a certain threshold. This is a novel result compared to the standard model.

In Chapter 2, we examine the moral hazard problem when both the principal and the agent are uncertain about the effort-dependent distribution of the project's outcome. Using the maxmin expected utility (MEU) framework, as introduced by Gilboa-Schmeidler, we find that if the agent's effort is observable, ambiguity reduces the principal's expected profit compared to the standard model, as the principal adopts a more pessimistic view of the project's likelihood of success. When the agent's effort is unobservable, inducing high effort leads to two possible expected profit outcomes, depending on the initial parameters of the model. Specifically, the principal anticipates a high expected profit if the compensation for the agent in the event of success is not significantly different from the compensation in the event of failure. However, when the reward for success is considerably higher than for failure, the principal expects a lower profit due to ambiguity aversion. This result differs from the standard moral hazard model, where the distribution of the project's outcome, conditional on the agent's effort, is assumed to be unambiguous.

The final chapter examines the Crawford-Sobel (CS) cheap talk problem in the context of a decision maker facing ambiguity about the distribution of private information. When ambiguity is small, $\boldsymbol{R}$'s evaluation of actions is modeled using the $\boldsymbol{\varepsilon}$-contamination framework. For cases of extreme ambiguity, $\boldsymbol{R}$'s behavior admits the maxmin expected utility framework by Gilboa-Schmeidler. In situations of intermediate ambiguity, the multiplier utility framework by Hansen and Sargent is applied.

We demonstrate that CS-like partition equilibrium exists at all levels of ambiguity. In the presence of ambiguity, $\boldsymbol{R}$ consistently selects the midpoint of the informed interval as her optimal response when her reference distribution is uniform and her preferences follow a quadratic loss function, consistent with the findings in the CS model. However, when the updated reference distribution deviates from the uniform case, $\boldsymbol{R}$'s strategy generally diverges from the CS model. Ambiguity can influence $\boldsymbol{R}$'s actions by shifting her optimal response either upward or downward, resulting in adjusted cutoffs in the new equilibrium.

Overall, my thesis contributes to the ongoing discourse on decision making under ambiguity by highlighting its impact on the competitive insurance market, the moral hazard problem, and the cheap talk problem. The thesis suggests some promising directions for future work, given the ubiquity of ambiguity in both practice and research when decision makers have limited access to the source of true information. In Chapter 1, for instance, it would be more advantageous to employ the $\boldsymbol{\varepsilon}-$contamination framework and analyze the problem for a general class of compact, convex subsets of distributions, rather than restricting it to a simplex. Similarly, in Chapter 2, the moral hazard problem can be analyzed when both the principal and the agent face ambiguity over the distribution of the project's outcome in non-binary states. Finally, while Chapter 3 has shown that ambiguity can influence the receiver's optimal action within the informed interval, it remains uncertain whether this shift is strictly confined to the range between the CS optimal action and the complete-ambiguity action. Investigating whether the receiver's action under ambiguity can extend beyond these boundaries offers a compelling direction for future research.

This thesis concludes, but the journey of research continues. Every discovery sparks new questions, leaving ample room for further exploration.

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