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Quantum many-body integrable systems and related algebraic structures

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Abstract

This thesis deals with various many-body quantum integrable Hamiltonian systems and algebraic structures related to them. More specifically, it discusses generalisations of Calogero–Moser–Sutherland (CMS) and Macdonald–Ruijsenaars (MR) type systems and their connections with the theory of double affine Hecke and related algebras.

Firstly, we consider the generalised CMS operators associated with the deformed root systems $BC(l, 1)$ and a CMS type operator associated with a planar configuration of vectors called AG_2, which is a union of the root systems A_2 and G_2. We construct suitably-defined (multidimensional) Baker–Akhiezer eigenfunctions for these operators, and we use this to prove a bispectral duality for each of these generalised CMS systems. In the case of AG_2, we give two corresponding dual difference operators of rational MR type in an explicit form, which we generalise to the trigonometric case as well by using the theory of double affine Hecke algebras (DAHAs). In the case of $BC(l, 1)$, the bispectral dual is a rational difference operator introduced by Sergeev and Veselov.

Secondly, we study systems with spin degrees of freedom. Quantum integrable spin CMS type systems with non-symmetric configurations of the singularities of the potential appeared in the rational case in the work of Chalykh, Goncharenko, and Veselov in 1999. In this thesis, we obtain various trigonometric spin CMS type systems by making use of the representation theory of degenerate DAHAs. Particular cases of our construction reproduce in the rational limit the examples discovered by Chalykh, Goncharenko, and Veselov.

Finally, inside the DAHA of type GL_n, which depends on two parameters q and τ, we define a subalgebra $\mathbb{H}^{\mathfrak{gl}_n}$ that may be thought of as a q-analogue of the degree zero part of the corresponding rational Cherednik algebra. We prove that the algebra $\mathbb{H}^{\mathfrak{gl}_n}$ is a flat τ-deformation of the crossed product of the group algebra of the symmetric group with the image of the Drinfeld–Jimbo quantum group $U_q(\mathfrak{gl}_n)$ under the q-oscillator (Jordan–Schwinger) representation. We find all the defining relations and an explicit PBW basis for the algebra $\mathbb{H}^{\mathfrak{gl}_n}$. We describe its centre and establish a double centraliser property. As an application, we obtain new integrable generalisations of Van Diejen's MR system in an external field.

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The support of my family has been vital to me throughout my studies, and I am especially grateful for my mother and grandfather František having instilled in me an interest and curiosity for mathematics and the natural sciences right from my childhood.

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To my parents

Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Abbreviations

AHA — Affine Hecke algebra
BA — Baker–Akhiezer
CMS — Calogero–Moser–Sutherland
DAHA — Double affine Hecke algebra
MR — Macdonald–Ruijsenaars
PBW — Poincaré–Birkhoff–Witt
RCA — Rational Cherednik algebra
TCA — Trigonometric Cherednik algebra

Chapter 1

Introduction

One broad aim of the study of integrability is to obtain exact information on the properties of various systems that arise within theoretical physics or are motivated by it. Integrable systems often reveal important underlying mathematical structures. Among the structures studied in this thesis are Cherednik's double affine Hecke algebras (DAHAs, or Cherednik algebras), which are a very active and diverse research area [35], and their various degenerations. On top of their numerous applications in mathematical physics, these algebraic objects are important from the perspective of several other branches of mathematics, having deep connections to geometry, special functions, combinatorics, and a rich representation theory.

The goal of this thesis is to investigate certain multi-particle quantum integrable Hamiltonian systems and related algebraic structures. The models discussed in this thesis are connected with the so-called Calogero–Moser–Sutherland (CMS) differential operators [44] and their difference (relativistic) version, the Macdonald–Ruijsenaars (MR) operators, which are related to the celebrated Macdonald polynomials [82]. In its original form, the CMS model describes particles confined to a line or a circle interacting pairwise via an interaction potential proportional to their inverse square distance.

The CMS models were among the earliest known examples of integrable multi-particle systems and they represented a landmark discovery in this respect. They have since been extensively generalised in different ways. For example, they admit generalisations associated with special collections of vectors, such as root systems of semisimple Lie algebras, with the original systems corresponding to the root system of type A [44]. Finding further integrable generalisations and developing algebraic tools to investigate these and similar models has also independent value for other areas of mathematics. Systems of MR type are intimately related to DAHAs [35], while CMS systems are connected to the degenerations of DAHAs known as rational and trigonometric Cherednik algebras (RCAs and TCAs, respectively). In geometry, some quantum systems of CMS type arise as radial parts of Laplace–Beltrami operators on symmetric spaces [6, 88], and the phase spaces of classical

CMS-type systems are interesting algebraic varieties [107]. These systems also have links to other important equations of theoretical physics; for example, they are related to the pole dynamics of solutions of the KdV and KP equations [1,36,78], and have close relations to the Knizhnik–Zamolodchikov equations from conformal field theory (see [35]). Further, RCAs and DAHAs have been related to Coulomb branches of certain supersymmetric gauge field theories (see [10] and references therein).

The main outcomes of this thesis are: 1.) a novel connection between a DAHA and a quantum group, obtained by introducing and studying a new algebra connected to both, which led us to new difference operators related to MR systems [57];
2.) advancement of the theory of generalised CMS and MR operators associated with non-reduced (i.e., containing collinear vectors) collections [54–56, 83]. In particular, progress has been achieved in the theory of their special eigenfunctions and the study of systems with spin degrees of freedom.

1.1 Overview and background

The topics in this thesis can be divided into three interrelated themes, which we introduce respectively in Sections 1.1.1, 1.1.2, and 1.1.3 below summarising the background and context, as well as the main results that we obtained, for each of these directions in turn. Theme 1.1.1 corresponds to Chapters 3 and 5, and themes 1.1.2 and 1.1.3 correspond to Chapters 4 and 6, respectively.

1.1.1 Calogero–Moser–Sutherland and Macdonald–Ruijsenaars systems

1.1.1.1 Calogero–Moser–Sutherland systems

The CMS models are an important example of integrable many-body Hamiltonian systems in one spatial dimension. Their study goes back to the works of Calogero [14], Sutherland [97], and Moser [85], who investigated systems of pairwise-interacting particles on a line (rational case) or a circle (trigonometric case) with an inverse square distance potential. After the work of Moser, an elliptic generalisation of these systems appeared as well [15].

Olshanetsky and Perelomov observed a connection between the original CMS Hamiltonians and the root system A_l ($l \in \mathbb{Z}_{>0}$), and they generalised these Hamiltonians to the case of arbitrary root systems of Weyl groups in [86, 87] (including the non-reduced root system BC_l) in a way that preserves integrability (in the rational case, integrability holds for root systems of arbitrary finite reflection groups).

There exist both classical and quantum versions of these systems. In this thesis, we focus on the quantum case throughout. A quantum system whose Hamiltonian is a differential operator L in an n-dimensional space is called (quantum) integrable if there exist n pairwise-commuting algebraically independent differential operators $L_1 = L,\ L_2 \dots, L_n$. Operators that commute with L are called its quantum integrals.

In comparison, a classical system whose Hamiltonian $\mathscr{L}$ is a function on a $2n$-dimensional phase space, which is a symplectic manifold, is called (Liouville) integrable if there exist n independent Poisson-commuting phase-space functions $\mathscr{L}_1 = \mathscr{L}, \mathscr{L}_2 \dots, \mathscr{L}_n$. Moser studied the integrability of the classical versions of Calogero's and Sutherland's systems through the method of Lax pairs [85]. The study of classical CMS-type systems through the geometric method of Hamiltonian reduction was initiated by Kazhdan, Kostant, and Sternberg [72]. A uniform construction of Lax pairs for the classical Olshanetsky–Perelomov systems was provided by Bordner, Corrigan, and Sasaki [8] (see also the work [40] by D'Hoker and Phong).

A uniform proof of the integrability of the quantum Olshanetsky–Perelomov systems using Dunkl operators was obtained by Heckman [62,63], which led to a connection between Cherednik algebras and CMS-type systems. We will discuss this connection more later. Let us just mention here that a way to uniformly construct quantum Lax pairs by using Dunkl operators was discovered in a recent work by Chalykh [17], and this also reproduces the above classical Lax pairs.

Chalykh, Feigin, and Veselov showed in [21,26,104] in the quantum case that the CMS models admit integrable generalisations related to other special configurations of vectors that are not root systems. The examples they discovered were certain one-parametric deformations $A(l,1)$ and $C(l,1)$ of the root systems A_{l+1} and C_{l+1}, respectively. Other examples have been discovered since then. For instance, one of them is a deformation $BC(l,l')$ of the root system $BC_{l+l'}$ that was first considered in [94] by Sergeev and Veselov (an elliptic version of the special case $BC(l,1)$ appeared earlier in [19]). Another more recent example is a configuration called AG_2, which was discovered by Fairley and Feigin [48], and which we studied in [55,56].

Hereafter, by CMS system, we will mean the trigonometric (or, equivalently, hyperbolic) kind, unless specified otherwise. The generalised CMS operator associated with a finite collection of vectors $\mathcal{A} \subset \mathbb{C}^n \setminus \{0\}$ with a multiplicity function $c\colon \mathcal{A} \to \mathbb{C}$, $\alpha \mapsto c_\alpha$ has the form

$$L = -\Delta + \sum_{\alpha \in \mathcal{A}} \frac{c_\alpha(c_\alpha + 2c_{2\alpha} + 1)\langle \alpha, \alpha \rangle}{\sinh^2 \langle \alpha, x \rangle}, \tag{1.1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ ($\partial_{x_i} = \partial/\partial x_i$) is the Laplacian on $\mathbb{C}^n$, $c_{2\alpha} \coloneqq 0$ if $2\alpha \notin \mathcal{A}$, and $\langle \cdot, \cdot \rangle$ denotes the $\mathbb{C}$-bilinear extension of the standard Euclidean inner product of $\mathbb{R}^n$. The case introduced by Olshanetsky and Perelomov corresponds to letting $\mathcal{A}$ in (1.1) be a positive half of a root system with a Weyl-invariant assignment of multiplicities c_α. Only for very exceptional collections $\mathcal{A}$ will the operator L be integrable.

1.1.1.2 Baker–Akhiezer functions

In general, the eigenfunctions of quantum integrable systems may be complicated. However, the generalised CMS operators associated with root systems admit as eigenfunctions various special multivariable polynomials. For example, for the CMS system of type A, they are the Jack polynomials. For its rational degeneration in the presence of an additional confining harmonic oscillator potential term, which ensures the spectrum is discrete, there are eigenfunctions expressed via multidimensional versions of the Hermite polynomials.

In the case when the multiplicity parameters have Weyl-invariant *integer* values, the generalised CMS operators associated with root systems admit as (singular) eigenfunctions so-called multidimensional Baker–Akhiezer (BA) functions, which are relatively elementary functions [28, 105]. They have the form

$$\psi(z, x) = P(z, x)e^{\langle z,x\rangle},$$

where P is a polynomial in the spectral variables $z \in \mathbb{C}^n$ whose coefficients depend on the variables x in which the CMS operators act. The function P has singularities in the variables x. In quantum mechanics, it is more usual to consider the action of a quantum Hamiltonian on a Hilbert space and look for eigenfunctions therein. However, the BA function is very useful for constructing quantum integrals of the system and for proving bispectral dualities, as we will see below. Moreover, in many cases, it can be related to orthogonal polynomial eigenfunctions and give new formulas for them, such as the BA function in type A for Jack polynomials (see [24]).

The function ψ can be characterised by its properties as a function of the variables z. Such an axiomatic definition of a multidimensional BA function was proposed by Chalykh, Styrkas, and Veselov for an arbitrary finite collection of non-collinear vectors with integer multiplicities in [105] — the case of (positive subsystems of) reduced root systems was considered earlier in [28] — and see also [49] for a weaker version of the axiomatics. For the (only) non-reduced root system BC_l, an axiomatic definition of the BA function was given by Chalykh and Veselov in [29] (see also [22, 24]). The key properties that the function needs to satisfy are quasi-invariance conditions of the form

$$\psi(z + s\alpha, x) = \psi(z - s\alpha, x) \tag{1.2}$$

at $\langle \alpha, z\rangle = 0$ for vectors α in the configuration, where s takes special integer values depending on the multiplicities.

Such a function can exist only for very special configurations. The corresponding generalised CMS operators have to be algebraically integrable, that is, contained in a large commutative ring of differential operators; more precisely, for an operator in n variables, algebraic integrability means that it is quantum integrable and, moreover, its algebra of quantum integrals cannot be generated by n operators [28, 46] (see also [29, 105] for a slightly stronger version of the definition of algebraic integrability for operators with constant highest symbols, and [47] for the difference case). And if the BA function exists, then it is a common eigenfunction for all the operators in this ring [49,105]. In addition to the root systems case, the previously known examples of configurations with a BA function included the aforementioned deformations from [21,26] of the root systems of type A and C in the case when the multiplicity parameters are integers. The corresponding BA functions were constructed in [24, 49], respectively. Another deformation $A_{l,2}$ of the root system of type A appeared in [30], and the BA function for it (satisfying the weakened axiomatics) was given in [49].

The generalised CMS operators associated with the configurations AG_2 and $BC(l, 1)$ are algebraically integrable in the case when all the multiplicities are integers [16,48], which suggested that in this case there might exist a BA function for them. These had been the only remaining known examples of algebraically integrable monodromy-free CMS-type operators for which a BA function had not been written down. One of the main results of this thesis is a construction in Chapter 3 of a suitably-defined BA function for AG_2 and $BC(l, 1)$. Our construction uses a method modelled on Chalykh's one from [24] (see [49] for further examples where such a technique has been applied). The construction uses certain difference operators, acting in the variables z, of rational MR type.

1.1.1.3 Macdonald–Ruijsenaars systems and bispectral dualities

Ruijsenaars and Schneider introduced a relativistic version of the CMS system of type A_{n-1} in the classical case in [93]. In the quantum case, the corresponding Hamiltonian and its quantum integrals, which are difference operators, were introduced by Ruijsenaars in [90]. His quantum Hamiltonian can be written as

$$D_R = \sum_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{q^{z_i - c} - q^{z_j}}{q^{z_i} - q^{z_j}} \right) T_i, \tag{1.3}$$

where q and c are parameters, and T_i is the (additive) shift operator acting on the variables $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ by $T_i(z_j) = z_j + \delta_{ij}$. Here δ_{ij} is the Kronecker delta. Generalisations of the Hamiltonian (1.3) for all reduced root systems were introduced by Macdonald [82], and for the root system BC by Koornwinder [76], in connection with the theory of orthogonal polynomials.

Ruijsenaars established a duality relation between the classical CMS system of type A

and a rational degeneration of its relativistic version, where the duality essentially swaps the action and angle variables of the two systems [91]. In the quantum case, he conjectured a bispectral duality (bispectrality) relation between the quantum CMS system of type A and the rational degeneration of the operator (1.3) obtained by taking the $q \to 1$ limit of its coefficients [92]. This conjecture, and its version for all root systems, was proved by Chalykh in [24]. Namely, there exists a function Ψ of the variables x and z with

$$L\Psi = \lambda\Psi, \quad D\Psi = \mu\Psi, \tag{1.4}$$

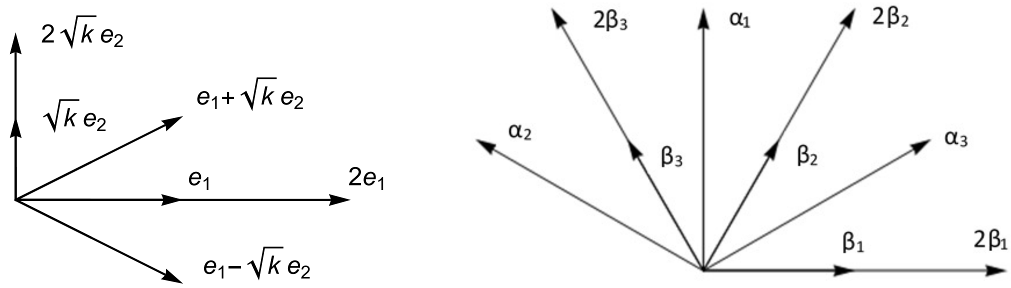
where $L = L(x, \partial_x)$ is the CMS Hamiltonian associated with any root system, or its quantum integral, D is a rational MR operator for the same root system, acting in the variables z, and $\lambda = \lambda(z)$, $\mu = \mu(x)$ are the respective eigenvalues. The bispectrality relations (1.4) are a multidimensional differential-difference analogue of the one-dimensional differential-differential bispectrality studied by Duistermaat and Grünbaum in [42].

In the case of integer (and Weyl-invariant) multiplicities, the function Ψ in (1.4) can be taken to be the BA function for the root system in question — and the case of non-integer multiplicities can be handled by an analytic continuation argument [24]. Moreover, Chalykh showed that the operator D can be used to explicitly construct the BA function itself. The key property needed for this from the operator D is the preservation of the space of analytic functions of z satisfying the quasi-invariance conditions (1.2).

A form of bispectrality may also be seen in terms of special families of multivariable orthogonal polynomials (rather than a single function Ψ depending on spectral parameters). In the case of the root system of type A, these are the Jack polynomials, and for other root systems they are the multivariable Jacobi polynomials, which admit Pieri-type formulas that can be interpreted as bispectrality between the CMS Hamiltonians and difference operators acting on the weights indexing the polynomials [24, 65, 81]. The relation (1.4) for the root system of type A and for a different function Ψ given by a Mellin–Barnes type integral was obtained recently by Kharchev and Khoroshkin in [73].

Let us also mention that a version of the notion of a BA function exists as well for the rational degeneration of the CMS Hamiltonians L with $\sinh\langle\alpha, x\rangle$ in (1.1) replaced by $\langle\alpha, x\rangle$. The rational BA function ψ satisfies, instead of conditions (1.2), relations of the form $\partial_\alpha^{2s-1}\psi(z, x) = 0$ at $\langle\alpha, z\rangle = 0$, where ∂_α is the directional derivative in z in the direction of the vector α. When suitably normalised, the rational BA function is symmetric under the exchange of x and z, and the rational version of L is bispectrally self-dual [105]. BA functions for the trigonometric MR operators and their bispectrality properties were investigated in [25].

The paper [24] proved also an analogue of the duality (1.4) for the deformed root system $A(l, 1)$ and the corresponding deformation D of the rational limit of Ruijsenaars' operators. For the deformed root systems $C(l, 1)$ and $A_{l,2}$, this was done (in the case of in-



(a) $BC(1,1)_+$

(b) $AG_{2,+}$

Figure 1.1: The configurations $BC(1,1)$ and AG_2.

teger multiplicity parameters) in [49]. In Chapter 3, we prove analogues of the duality (1.4) for the configurations AG_2 and $BC(l,1)$.

1.1.1.4 Configurations AG_2 and $BC(l,1)$

The configuration AG_2 is a planar collection of vectors obtained as a union of the root systems G_2 and A_2. It can be viewed as a non-reduced version of the root system G_2, where for each short root $\beta \in G_2$, the configuration contains 2β as well. It is, however, not itself a crystallographic root system (though, it is a trigonometric locus configuration [48]). A positive half $AG_{2,+}$ is shown in Figure 1.1. The multiplicity of the vectors $\{2\beta_i\colon i = 1,2,3\}$ coming from the root system A_2 is 1, and the multiplicities of the long and short roots from the root system G_2 are, respectively, m and $3m$, where $m \in \mathbb{C}$ is a parameter. The configuration AG_2 is invariant under the action of the Weyl group of type G_2.

The configuration $BC(l,1)$ is the non-reduced collection of vectors with a positive half

$$BC(l,1)_+ = \{e_i,\, 2e_i,\, e_i \pm \sqrt{k}e_{l+1}\colon 1 \le i \le l\} \cup \{\sqrt{k}e_{l+1},\, 2\sqrt{k}e_{l+1}\}$$
$$\cup\, \{e_i \pm e_j\colon 1 \le i < j \le l\} \subset \mathbb{C}^{l+1},$$

where $k \in \mathbb{C}^\times$ is a parameter, e_i are the standard orthogonal unit vectors in $\mathbb{R}^{l+1}$, and the multiplicities are required to satisfy $c_{e_i \pm \sqrt{k}e_{l+1}} = 1$, $c_{e_i \pm e_j} = k$, $c_{e_i} = kc_{\sqrt{k}e_{l+1}}$, and $2c_{2e_i}+1 = k(2c_{2\sqrt{k}e_{l+1}}+1)$ with $c_{\sqrt{k}e_{l+1}}, c_{2\sqrt{k}e_{l+1}} \in \mathbb{C}$ [19,94]. Figure 1.1 depicts this configuration in the case of $l = 1$. For $k = 1$, the configuration $BC(l,1)$ reduces to the root system BC_{l+1} with a Weyl-invariant assignment of multiplicities such that the vectors $e_i \pm e_j$ for $1 \le i < j \le l+1$ have multiplicity 1.

1.1.1.5 Main results

To cover the configurations AG_2 and $BC(l,1)$, we introduced the following extension of the axiomatic definition of BA functions from [22,24,29,105].

Definition (Definition 3.1)**.** Let $R \subset \mathbb{C}^n$ be a (not necessarily reduced) finite collection of non-isotropic vectors with a multiplicity map $c\colon R \to \mathbb{Z}_{\geq 0}$ possessing a subset $R_+ \subset R$ such that any collinear vectors in R_+ are of the form $\alpha, 2\alpha$, and $R = R_+ \coprod (-R_+)$. A function $\psi(z,x)$ $(z, x \in \mathbb{C}^n)$ is a BA function for R if it satisfies the conditions

1. $\psi(z,x) = P(z,x)e^{\langle z,x\rangle}$ for a polynomial P in z with highest-order term $\prod_{\alpha\in R_+}\langle \alpha, z\rangle^{c_\alpha}$;
2. $\psi(z+s\alpha, x) = \psi(z-s\alpha, x)$ at $\langle z,\alpha\rangle = 0$ for $s \in \{1,2,\ldots,c_\alpha\} \cup \{c_\alpha+2, c_\alpha+4, \ldots, c_\alpha+2c_{2\alpha}\}$ and $\alpha \in R_+$ with $\frac{1}{2}\alpha \notin R$, where $c_{2\alpha} := 0$ if $2\alpha \notin R$.

If in the above definition we put $R_+ = AG_{2,+}$ or $BC(l,1)_+$, we respectively get the notion of a BA function for AG_2 and $BC(l,1)$ with $\mathbb{Z}_{\geq 0}$-valued multiplicities. The case of reduced configurations from [105] corresponds to $c_{2\alpha}$ being 0 for all α in our above definition. In the case of $R = BC_l$, our definition coincides with that from [22, 29] (cf. also [24]).

We proved the following generalisation of analogous results given for reduced configurations and BC_l in [22, 24, 29, 105].

Theorem (Propositions 3.6 and 3.9, and Theorem 3.7)**.** *With notations as in the above definition, let $\mathcal{R}$ be the ring of polynomials in z satisfying condition 2. If the BA function $\psi(z,x)$ exists then it is unique, and for any $p(z) \in \mathcal{R}$, there is a differential operator $L_p(x,\partial_x)$ such that*

$$L_p(x,\partial_x)\psi(z,x) = p(z)\psi(z,x).$$

For any $p, q \in \mathcal{R}$, the operators L_p and L_q commute, and L_{-z^2} coincides with the generalised CMS operator (1.1) *for $\mathcal{A} = R_+$.*

Let $\mathcal{R}^a_R$ be the ring of all *analytic* functions in z satisfying condition 2 from the above definition of the BA function. In Section 3.3, we derive sufficient conditions for a rational difference operator of a quite general form to preserve the ring $\mathcal{R}^a_R$ (Theorem 3.11) under a symmetry assumption on the configuration R.

In Section 3.4, we first explain our proof from [55] of the integrability of the generalised CMS operator (1.1) with $\mathcal{A} = AG_{2,+}$ for any value of the multiplicity parameter m. Then we give a proof of the following results for the configuration AG_2 with $m \in \mathbb{Z}_{\geq 0}$. Denote by T_τ the (additive) shift operator acting on functions $f(z)$ by $T_\tau f(z) = f(z+\tau)$.

Theorem (Theorems 3.20, 3.24, 3.26, 3.27, and 3.28)**.** *The BA function $\psi(z,x)$ for $R = AG_2$ exists. There are two independent explicit commuting planar G_2-invariant difference operators preserving the ring $\mathcal{R}^a_{AG_2}$ of the form $\mathcal{D}_1 = \sum_{\tau : \frac{1}{2}\tau \in G_2} a_\tau(z)(T_\tau - 1)$ and $\mathcal{D}_2 = \sum_{\tau : \frac{1}{2}\tau \in AG_2} b_\tau(z)(T_\tau - 1)$ for some rational functions $a_\tau(z)$ and $b_\tau(z)$ such that*

$$\mathcal{D}_i\psi(z,x) = \mu_i(x)\psi(z,x)$$

for some functions $\mu_i(x)$ $(i = 1, 2)$.

For any polynomial $p(z) \in \mathcal{R}^a_{AG_2}$*, there is a difference operator* D_p *acting in* z *such that* $D_p\psi(z, x) = \mu_p(x)\psi(z, x)$ *for some functions* $\mu_p(x)$*. The operators* D_p *commute with* $\mathcal{D}_i$ *and with each other.*

We also prove that the generalised CMS operator for AG_2 is bispectrally dual to the above operators $\mathcal{D}_1$ and $\mathcal{D}_2$ for any $m \in \mathbb{C}$ (Theorem 3.55).

In Chapter 5, we derive (in Propositions 5.5 and 5.6) commuting trigonometric MR-type operators related to the configuration AG_2 that generalise the operators $\mathcal{D}_i$ to the trigonometric case (Propositions 5.7 and 5.8).

In Section 3.5, we prove the following results for the configuration $BC(l, 1)$ with multiplicities belonging to $\mathbb{Z}_{\geq 0}$.

Theorem (Theorems 3.35, 3.37, and 3.45)**.** *A Sergeev–Veselov rational difference operator* D *associated with* $BC(l, 1)$ *[95] preserves the ring* $\mathcal{R}^a_{BC(l,1)}$*. The BA function* $\psi(z, x)$ *for* $R = BC(l, 1)$ *exists, and* $D\psi(z, x) = \mu(x)\psi(z, x)$ *for some function* $\mu(x)$*. For any polynomial* $p(z) \in \mathcal{R}^a_{BC(l,1)}$*, we construct a difference operator* D_p *acting in* z *such that* $D_p\psi(z, x) = \mu_p(x)\psi(z, x)$ *for some functions* $\mu_p(x)$*. The operators* D_p *commute with* D *and with each other.*

We extend the bispectral duality statement for the generalised CMS operator of $BC(l, 1)$ and the above Sergeev–Veselov operator D to more general complex values of the multiplicities in Theorem 3.50.

1.1.2 Matrix-valued generalisation of CMS systems

A matrix (spin) version of the quantum CMS model appeared in the work of Ha and Haldane [59], who considered matrix differential operators acting on functions with values in an n-fold tensor product of $\mathbb{C}^m$. The integrability of this model was established by Minahan and Polychronakos using Polychronakos' version of Dunkl operators in [84], and independently at the same time by Hikami and Wadati in [67] by introducing a Lax pair. Subsequently, Bernard, Gaudin, Haldane, and Pasquier produced additional quantum integrals for this model in [7] by using a Yangian symmetry.

A matrix version of the Olshanetsky–Perelomov operators was considered by Cherednik in [33]. First deformed CMS-type matrix models with non-symmetric configurations of the singularities of the potential were introduced in the rational case by Chalykh, Goncharenko, and Veselov in [27]. They considered matrix differential operators of the form

$$L = \Delta - \sum_{\alpha \in \mathcal{A}} \frac{c_\alpha(c_\alpha - P_\alpha)\langle \alpha, \alpha \rangle}{\langle \alpha, x \rangle^2},$$

where $\mathcal{A}$ is a finite collection of non-zero vectors in a vector space $V \ni x$, P_α is a matrix acting on the vector space $\mathcal{U} \cong V$ as a reflection with respect to the hyperplane orthogonal to α, and c_α is an integer scalar multiplicity. One of the examples of configurations $\mathcal{A}$ that they considered is the deformed A_n-type system $A(n-1,1) \subset \mathbb{R}^{n+1}$ depending on a parameter m and consisting of the vectors $\{e_i - e_j \colon 1 \leq i < j \leq n\} \cup \{e_i - \sqrt{m} e_{n+1} \colon 1 \leq i \leq n\}$ with multiplicities $c_{e_i - e_j} = m$ and $c_{e_i - \sqrt{m} e_{n+1}} = 1$ (see [21]). Another example considered in [27] is the deformed C_n-type configuration.

Recall that the integrability of the *scalar* rational CMS operators associated with root systems was investigated with the use of Dunkl operators [43] by Heckman in [63], and in [62] in the trigonometric case. Generalised CMS systems related to non-symmetric configurations of vectors were investigated in the scalar rational case from the perspective of the representation theory of RCAs by Feigin in [50] using certain invariant parabolic submodules in the polynomial representation of RCAs. This allowed Feigin to derive many new integrable examples of such operators.

In [50], one starts with the polynomial representation $\mathcal{P}$ of an RCA, realised using the rational Dunkl operators. For special values of the parameters of the RCA, the representation $\mathcal{P}$ becomes reducible with a submodule $\mathcal{I}$ given by polynomials vanishing on the orbit of an intersection $\pi \subset V$ of Coxeter mirrors (a parabolic stratum). The allowed parabolic strata are described in terms of generalised Coxeter numbers related to the stratum and the multiplicity function. By restricting invariant combinations of Dunkl operators to the space of invariants $(\mathcal{P}/\mathcal{I})^W$, one obtains generalised rational CMS systems.

Another approach to (scalar) rational generalised CMS systems using RCAs was proposed recently by Berest and Chalykh in [5].

1.1.2.1 Main results

In Chapter 4, we use the representation theory of TCAs to obtain generalisations of spin CMS operators in the trigonometric case and for more general vector spaces $\mathcal{U}$ than were considered in [27].

We start with the polynomial representation $\mathcal{P}$ of the TCA (this is a slightly simplified account), realised with the help of Cherednik's commuting trigonometric Dunkl operators. For special values of the parameters, the representation $\mathcal{P}$ becomes reducible with a submodule $\mathcal{I}$ given by polynomials vanishing on π. We enlarge the quotient representation $\mathcal{P}/\mathcal{I}$ by taking the tensor product with a right-module U of the Weyl group W associated with the TCA, and make any element h of the TCA act on $U \otimes \mathcal{P}/\mathcal{I}$ by $\mathrm{id} \otimes h$. The action of W-invariant combinations of Dunkl operators on the *diagonal* invariants $(U \otimes \mathcal{P}/\mathcal{I})^W$ produces commuting matrix differential operators on π acting in the space of fixed points $\mathcal{U} = U^{W_0}$ of the action of the parabolic subgroup $W_0 \subset W$ corresponding to the subspace π.

The construction from [50] can be recovered by setting U to be the trivial representation and taking the rational limit of the resulting operators.

Let us explain our results in more detail in the case of $U = V$ being the reflection representation of W.

Let $R \subset V = \mathbb{C}^N$ be a reduced root system with associated Weyl group W and positive half R_+. Let $c\colon R \to \mathbb{C}$, $\alpha \mapsto c_\alpha$ be a W-invariant function. Let P be the weight lattice of R. Consider the TCA $\mathbb{H}_c^{\text{trig}}$ associated with R, defined by its faithful polynomial representation on the group algebra $\mathbb{C}[P] = \mathbb{C}[\{e^{\langle \alpha, x\rangle}\colon \alpha \in P\}]$; the TCA is generated by $e^{\langle\alpha,x\rangle}$ ($\alpha \in P$), W, and the trigonometric Dunkl operators

$$\nabla_\xi^{\text{trig}} = \partial_\xi - \sum_{\alpha\in R_+} \frac{c_\alpha \langle \alpha, \xi\rangle}{1 - e^{-\langle \alpha, x\rangle}}(1 - s_\alpha) + \langle \rho, \xi\rangle, \tag{1.5}$$

where $\xi \in V$, $\partial_\xi = \sum_{i=1}^N \langle \xi, e_i\rangle \partial_{x_i}$, $\rho = \frac{1}{2}\sum_{\alpha\in R_+} c_\alpha \alpha$, and s_α is the orthogonal reflection about the hyperplane orthogonal to α [31].

Let Γ be the Coxeter graph of R and Γ_0 an edge-preserving subgraph. Let Γ_0^v be the set of simple roots corresponding to the nodes of Γ_0. Let $W_0 = \langle s_\alpha\colon \alpha \in \Gamma_0^v\rangle$ be the corresponding parabolic subgroup. Define the space $\pi = \{x \in V\colon \langle \beta, x\rangle = 0,\ \forall \beta \in \Gamma_0^v\}$ $= V^{W_0}$ and the parabolic stratum $D = \cup_{w\in W} w(\pi)$.

For $x_0 \in \pi$, consider $\mathcal{C}_{Wx_0} = \bigoplus_{x\in Wx_0} \mathcal{C}_x$, where Wx_0 is the W-orbit of x_0, and $\mathcal{C}_x$ is the space of V-valued germs of analytic functions defined on V near the point $x \in Wx_0$. We explain that, for generic x_0, the TCA $\mathbb{H}_c^{\text{trig}}$ can act on $\mathcal{C}_{Wx_0}$, and under some assumptions on the parameters c_α, the subspace $\mathcal{I} \subset \mathcal{C}_{Wx_0}$ of those elements that vanish when restricted to D is preserved by $\mathbb{H}_c^{\text{trig}}$ (Theorem 4.2), which is a trigonometric version of [50, Theorem 1].

Assume that $\mathcal{I}$ is $\mathbb{H}_c^{\text{trig}}$-invariant. Then $\mathcal{C}_{Wx_0}/\mathcal{I}$ has a diagonal (left) W-action given by $(wF)(y) = F(w^{-1}y)\cdot w^{-1}$ for $F \in \mathcal{C}_x$, $x \in Wx_0$, and $y \in V$ near $w(x)$, where $\cdot$ denotes the action of W on its reflection representation V. We prove the following result.

Theorem (Theorems 4.3 and 4.4)**.** *If $\mathcal{I}$ is $\mathbb{H}_c^{\text{trig}}$-invariant, then the action of W-invariant polynomials in $\nabla_{e_1}^{\text{trig}}, \dots, \nabla_{e_N}^{\text{trig}}$ on the diagonal invariants $(\mathcal{C}_{Wx_0}/\mathcal{I})^W$ leads to commuting differential-reflection operators on π. Up to a gauge transformation, $\sum_{i=1}^N (\nabla_{e_i}^{\text{trig}})^2$ produces the generalised spin CMS operator*

$$\Delta_y - \sum_{\widehat{\alpha}\in\widehat{R}_+\setminus\{0\}} \frac{\widehat{c}_{\widehat{\alpha}}(\widehat{c}_{\widehat{\alpha}} + 2\widehat{c}_{2\widehat{\alpha}} + s_{\widehat{\alpha}})\langle\widehat{\alpha},\widehat{\alpha}\rangle}{4\sinh^2\left(\frac{\langle\widehat{\alpha},y\rangle}{2}\right)},$$

where $y = (y_1, \dots, y_n)$ are orthonormal coordinates on π, $\Delta_y = \sum_{i=1}^n \partial_{y_i}^2$, $\widehat{R}_+ = \{\widehat{\alpha}\colon \alpha \in R_+\}$ with $\widehat{\alpha}$ being the orthogonal projection of α onto π, $\widehat{c}_{\widehat{\alpha}} = \sum_{\substack{\gamma\in R_+\\ \widehat{\gamma}=\widehat{\alpha}}} c_\gamma$ and $\widehat{c}_{2\widehat{\alpha}} = 0$ when $2\widehat{\alpha} \notin \widehat{R}_+$, and we are assuming any collinear vectors in $\widehat{R}_+ \setminus \{0\}$ are of the form $\widehat{\alpha}$, $2\widehat{\alpha}$.

We also prove a more general statement when the above assumption on $\widehat{R}_+$ is not satisfied. We call $\mathcal{I}$ an invariant parabolic submodule for the TCA.

1.1.3 Special subalgebras of Cherednik algebras

DAHAs are a remarkable class of algebras associated with root systems. They have deep connections to integrable systems, geometry, and combinatorics. They were introduced by Cherednik as a powerful algebraic tool to solve a problem posed by Macdonald regarding the combinatorial properties of Macdonald polynomials [34]. A key relation between DAHAs and integrable systems is that the operator (1.3) arises via an action on Laurent polynomials of the commuting Cherednik elements Y_i of the DAHA $\mathbb{H}_n$ of type GL_n, which depends on two parameters q and $\tau = q^{-c/2}$ [35]. DAHAs of other types lead to versions of the operator (1.3) for other root systems.

The algebra $\mathbb{H}_n$ is generated by two commutative subalgebras of Laurent polynomials $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, $\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$, and the Hecke algebra of type A_{n-1} with generators T_k ($1 \le k \le n-1$). The latter satisfy the standard braid relations and the quadratic relations $(T_k - \tau)(T_k + \tau^{-1}) = 0$. The defining relations of $\mathbb{H}_n$ additionally include an action of T_k on the X and Y variables, and relations among X, Y.

RCAs, which are a degeneration of DAHAs investigated by Etingof and Ginzburg in the seminal paper [45], are flat deformations of the crossed product of a Weyl algebra (of differential operators with polynomial coefficients) with a finite Coxeter group W (see, e.g., [44]).

RCAs admit a faithful representation on a space of polynomials. In this representation, and in the case of the symmetric group $W = \mathfrak{S}_n$, the corresponding RCA $\mathcal{H}_n = \mathcal{H}_{n,c}$ ($c \in \mathbb{C}$) of type GL_n acts on $\mathbb{C}[x_1, \ldots, x_n]$, and it is generated by the transpositions $s_{ij} = (i,j) \in \mathfrak{S}_n$, multiplication operators x_i, and the rational Dunkl operators [43]

$$\nabla_i = \partial_{x_i} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{c}{x_i - x_j}(1 - s_{ij}) \qquad (\partial_{x_i} = \partial/\partial x_i,\ i = 1, \ldots, n).$$

The RCA $\mathcal{H}_n$ is a graded algebra, where the grading is determined by assigning degree 0 to the elements of the group $\mathfrak{S}_n$, degree 1 to the multiplication operators x_i, and degree -1 to Dunkl operators. The degree zero subalgebra $\mathcal{H}^{\mathfrak{gl}_n} = \mathcal{H}_c^{\mathfrak{gl}_n}$ is an interesting algebra in its own right from various perspectives. It is generated by $\mathfrak{S}_n$ and the operators $x_i \nabla_j$ ($i, j \in \{1, \ldots, n\}$).

The algebra $\mathcal{H}^{\mathfrak{gl}_n}$ enjoys, as the notation for it suggests, a link to Lie theory. More precisely, it is a flat c-deformation of the crossed product of the group algebra $\mathbb{C}\mathfrak{S}_n$ with a certain quotient $U(\mathfrak{gl}_n)/I$ of the universal enveloping algebra $U(\mathfrak{gl}_n)$ of the Lie algebra $\mathfrak{gl}_n$ over a two-sided ideal I, as was established by Feigin and Hakobyan in [51].

The quotient $U(\mathfrak{gl}_n)/I$ is the image of $U(\mathfrak{gl}_n)$ under the so-called oscillator (also known as Jordan–Schwinger) representation ρ_{JS} that maps the standard generators of $\mathfrak{gl}_n$ to the operators $x_i \partial_{x_j}$.

Similarly to the RCA itself, the algebra $\mathcal{H}^{\mathfrak{gl}_n}$ is a quadratic algebra of Poincaré–Birkhoff–Witt (PBW) type. In contrast to the RCA, the defining relations of $\mathcal{H}^{\mathfrak{gl}_n}$ include relations that are not of a commutator type. The associated graded algebra is the crossed product of $\mathbb{C}\mathfrak{S}_n$ with the algebra of polynomial functions on the space of $n \times n$ complex matrices of rank at most one [51].

The centre of the RCA is trivial [13], but the RCA has a commutative subalgebra which acts (in the polynomial representation) on symmetric polynomials as the rational CMS operator (albeit in a different gauge) and its quantum integrals [63]. On the other hand, the centre of the degree zero subalgebra $\mathcal{H}^{\mathfrak{gl}_n}$ is generated by the Euler operator eu, which can be related to the rational CMS operator with an additional harmonic potential term by an automorphism of the RCA [51]. The algebra $\mathcal{H}^{\mathfrak{gl}_n}$ and its centre are mutual centralisers inside the RCA, which is related to a deformation of the Howe dual pair $(\mathfrak{gl}_n, \mathfrak{gl}_1)$. The properties of the central quotient $\mathcal{H}^{\mathfrak{gl}_n}/(eu + \text{const})$ (quotient by the two-sided ideal generated by an element $eu + \text{const}$) and its '$t = 0$' (classical) version were studied recently in [4] in relation to deformations of symplectic singularities and nilpotent orbits in $\mathfrak{gl}_n$.

1.1.3.1 Main results

In Chapter 6, we generalise the main parts of the theory of $\mathcal{H}^{\mathfrak{gl}_n}$ to the q-deformed setting by introducing and studying a certain subalgebra $\mathbb{H}^{\mathfrak{gl}_n}$ inside the DAHA $\mathbb{H}_n$ of type GL_n. We note that even though the DAHA $\mathbb{H}_n$ has a natural grading, the subalgebra $\mathbb{H}^{\mathfrak{gl}_n} \subset \mathbb{H}_n$ is in general strictly smaller than the degree zero part. Another important difference with the RCA case is that the algebra $\mathbb{H}^{\mathfrak{gl}_n}$ contains the Y-elements of the DAHA. The main idea behind the definition of $\mathbb{H}^{\mathfrak{gl}_n}$ is to replace the role of $U(\mathfrak{gl}_n)$ by the Drinfeld–Jimbo quantum group $U_q(\mathfrak{gl}_n)$.

The algebra $U_q(\mathfrak{gl}_n)$ admits a representation ρ which is a q-multiplicative generalisation of the Jordan–Schwinger map ρ_{JS}. We consider the image $A = \rho(U_q(\mathfrak{gl}_n))$ and the algebra $\mathcal{A} = \mathbb{C}\mathfrak{S}_n \ltimes A$, where the symmetric group acts in a natural way. We then define inside the DAHA $\mathbb{H}_n$ a subalgebra $\mathbb{H}^{\mathfrak{gl}_n}$ whose generators are τ-deformations of those of $\mathcal{A}$. In a suitable $q \to 1$ limit, the algebra $\mathbb{H}^{\mathfrak{gl}_n}$ reduces to the degree zero part $\mathcal{H}^{\mathfrak{gl}_n}$ of the RCA. The following diagram summarises the relationships between the various algebras:

$$\begin{array}{ccc} \mathbb{H}^{\mathfrak{gl}_n} & \xrightarrow{\tau \to 1} & \mathcal{A} = \mathbb{C}\mathfrak{S}_n \ltimes \rho(U_q(\mathfrak{gl}_n)) \\ \Big\downarrow{\scriptstyle q\to 1} & & \Big\downarrow{\scriptstyle q\to 1} \\ \mathcal{H}^{\mathfrak{gl}_n} & \xrightarrow{c \to 0} & \mathbb{C}\mathfrak{S}_n \ltimes \rho_{\mathrm{JS}}(U(\mathfrak{gl}_n)), \end{array}$$

where in the left vertical arrow we also take the limit $\tau \to 1$ such that $\tau = q^{-c/2}$ and c does not depend on q.

We find all the defining relations of the algebra $\mathbb{H}^{\mathfrak{gl}_n}$, establish that it is an algebra of PBW type by explicitly constructing a PBW basis, and show that $\mathbb{H}^{\mathfrak{gl}_n}$ is a flat τ-deformation of the algebra $\mathcal{A}$ (Theorem 6.19).

We prove that the centre of $\mathbb{H}^{\mathfrak{gl}_n}$ is generated by a single invertible element $\widetilde{Y}$ (Theorem 6.23). When $q \to 1$, the central element $(1-q)^{-1}(1-\widetilde{Y})$ reduces to the generator eu of the centre $\mathcal{Z}(\mathcal{H}^{\mathfrak{gl}_n})$.

We also prove a double centraliser property (Theorem 6.30) — related to the $(\mathfrak{gl}_n, \mathfrak{gl}_1)$ Howe duality — that the algebra $\mathbb{H}^{\mathfrak{gl}_n}$ satisfies inside a subalgebra $\mathfrak{A} \subset \mathbb{H}_n$. The algebra $\mathfrak{A}$ may be thought of as a τ-deformation of the crossed product of $\mathbb{C}\mathfrak{S}_n$ with the q-Weyl algebra defined by Hayashi [61], and it is, as we explain, isomorphic to a particular cyclotomic DAHA (cyclotomic DAHAs were defined by Braverman, Etingof, and Finkelberg in [10]). The DAHA $\mathbb{H}_n$ contains pairwise-commuting elements D_i that can be thought of as a q-generalisation of rational Dunkl operators, and which we use to define the algebra $\mathbb{H}^{\mathfrak{gl}_n}$. Similar but different commuting elements appear in the definition of this cyclotomic DAHA in [10]. We show that the algebra $\mathbb{H}^{\mathfrak{gl}_n}$ is isomorphic to the subalgebra of degree zero elements of this cyclotomic DAHA.

We also consider pairwise-commuting elements $\mathcal{D}_i = \mathcal{D}_i^{(l_1,l_2)} \in \mathbb{H}_n$ of a more general form than D_i. The former depend on parameters $l_1, l_2 \in \mathbb{Z}_{\geq 0}$, and $a_j \in \mathbb{C}$ $(j = -l_1, \dots, l_2)$. In the case $l_2 = 0$, they are equivalent to certain generators of a general cyclotomic DAHA [10]. By looking at the action of symmetric combinations of $\mathcal{D}_i$ on the space of symmetric Laurent polynomials, we arrive at families of new commuting q-difference operators (Theorem 6.37) related to the MR system with a Morse term introduced by Van Diejen [101, 103].

For example, in the case $l_1 = l_2 = 1$, we obtain the following integrable Hamiltonian

$$M = \alpha \sum_{i=1}^{n} \frac{1}{X_i} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{\tau^2 X_i - X_j}{X_i - X_j} \right) t_i + \beta \sum_{i=1}^{n} \frac{1}{X_i} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{X_i - \tau^2 X_j}{X_i - X_j} \right) t_i^{-1} + \gamma \sum_{i=1}^{n} \frac{1}{X_i},$$

where t_i is the q-multiplicative shift operator in the variable X_i acting by $t_i(X_j) = q^{\delta_{ij}} X_j$, and α, β, γ are independent parameters.

Relations to known Hamiltonians are as follows. In the case of $\alpha = 0$ (corresponding to $l_2 = 0$ and $l_1 = 1$), the operator M appeared in the paper [3] by Baker and Forrester. A more general version of their q-difference operator was found earlier by Van Diejen without using q-Dunkl operators [101]. Van Diejen's operator has a limit to the operator M with an extra constraint on the parameters α, β, and γ [103]. Higher l_1 with $l_2 = 0$ generalisations of the Baker–Forrester operator were considered in [10], which recover as a special case

Chalykh's operators from [18]; see [20] for an explicit form of such a Hamiltonian for $l_1 = 2$.

Furthermore, MR operators of type A admit integrable generalisations to systems with two types of particles [24, 96]. They are related to submodules of the polynomial representation of DAHA at special values of the parameters [53]. We generalise Van Diejen's operator from [101] to a Hamiltonian involving two types of particles, and we explain how to obtain quantum integrals for it. This also leads to a generalisation of the above operator M for two types of particles.

1.2 Structure of the thesis

Chapter 2 summarises the relevant algebraic background material. We start with the theory of Coxeter and Weyl groups and their root systems. We then give an introduction to Hecke algebras associated with Coxeter groups. We also discuss their (double) affine generalisations, focusing specifically on type GL_n, which is needed for Chapter 6. Finally, we give an overview of the rational and trigonometric degenerations of DAHAs, used in Chapters 4 and 6.

Chapters 3–6 constitute the main parts of this thesis.

In Chapter 3, we study BA eigenfunctions of generalised CMS operators, we give a construction of the BA function for the configurations AG_2 and $BC(l, 1)$, and we discuss bispectral dualities and bispectral dual difference operators of rational MR type. This chapter is based on our papers [55, 56, 83].

In Chapter 4, we develop the theory of invariant parabolic submodules for TCAs and utilise them to construct generalisations of spin CMS operators. We explicitly work out numerous examples of our construction. This chapter is based on the preprint [54].

In Chapter 5, we use the theory of DAHAs to obtain two commuting trigonometric MR-type operators related to the configuration AG_2 that generalise the rational difference operators given for AG_2 in Chapter 3.

In Chapter 6, we define and study a subalgebra of a DAHA that realises inside the DAHA a deformation of the crossed product of the symmetric group with the image of the quantum group $U_q(\mathfrak{gl}_n)$ under its q-oscillator representation. We also obtain new integrable generalisations of Van Diejen's difference version of the rational CMS operator with a harmonic term and related systems. This chapter is based on the preprint [57].

In Chapter 7, we discuss possible questions for future further research stemming from the work presented in this thesis.

Chapter 2

Coxeter groups, Hecke and related algebras

We begin this chapter by summarising some relevant aspects of the theory of Coxeter groups. We then give an introduction to Hecke algebras, followed by a discussion of their (double) affine generalisations in type GL_n, and we end the chapter by recalling the definition and some properties of the rational and trigonometric Cherednik algebras.

2.1 Coxeter groups

Coxeter groups are an abstraction and generalisation of groups generated by reflections. Let E be a real Euclidean space, that is, a finite-dimensional $\mathbb{R}$-vector space with a positive-definite symmetric bilinear form $\langle \cdot, \cdot \rangle$. A linear transformation $s \in GL(E)$ is called a (Euclidean, or orthogonal) reflection if there is a vector $\alpha \in E \setminus \{0\}$ such that $s(\alpha) = -\alpha$ and such that s fixes pointwise the hyperplane $H_\alpha = \{x \in E \colon \langle \alpha, x \rangle = 0\}$ orthogonal to α. Let s_α denote such s. Note that $s_{\lambda\alpha} = s_\alpha$ for any non-zero scalar λ. Explicitly, the reflection s_α is given by the formula

$$s_\alpha(x) = x - \langle \alpha^\vee, x \rangle \alpha,$$

where $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$. It belongs to the group $\mathcal{O}(E)$ of orthogonal transformations of E.

Definition 2.1. A subgroup $W \leq \mathcal{O}(E)$ is a (finite) real reflection group if (it has finite order $|W| < \infty$ and) it is generated by reflections.

In 1934, Coxeter proved that every finite real reflection group admits a presentation of a particularly nice form.

Theorem 2.2. *[37] Let W be a finite group. If W is a real reflection group, then it has* *a presentation*

$$W = \langle s_{\alpha_i}\ (i \in I)\colon s_{\alpha_i}^2 = 1,\ (s_{\alpha_i}s_{\alpha_j})^{m(i,j)} = 1 \text{ for } i \neq j\rangle$$

for some finite set I and $m(i,j) = m(j,i) \in \mathbb{Z}_{\geq 2}$.

For a finite real reflection group $W \subset \mathcal{O}(E)$, the idea of the proof of Theorem 2.2 is to consider the set $\mathrm{Ref}(W)$ of all reflections in W, and to pick a connected component (a fundamental chamber) of the space

$$E^{\mathrm{reg}} = E \setminus \cup_{s_\alpha \in \mathrm{Ref}(W)} H_\alpha.$$

The walls of this chamber correspond to some of the hyperplanes H_α, and thus to some of the reflections in W, which can be taken as the generators s_{α_i} in the presentation in Theorem 2.2, and $\pi/m(i,j)$ is the angle between H_{α_i} and H_{α_j}. These reflections s_{α_i} are then called simple reflections.

This motivated the introduction of the following abstract definition.

Definition 2.3. Let I be a finite set and $m\colon I \times I \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ be such that $m(i,i) = 1$ for all $i \in I$, and $m(i,j) = m(j,i) \geq 2$ for $i \neq j \in I$. The associated Coxeter group is the group defined by the presentation

$$W = \langle s_i\ (i \in I)\colon (s_i s_j)^{m(i,j)} = 1 \text{ if } m(i,j) < \infty\rangle.$$

One often conflates the set of generators $\{s_i\}_{i \in I}$ with the indexing set I itself. The pair (W, I) is then called a Coxeter system, and the function m can be uniquely recovered from this data, as $m(i,j)$ is the order of $s_i s_j$ in W.

The data from the definition of a Coxeter group can be depicted with a so-called Coxeter graph, which is the labelled undirected graph with vertex set I such that

- if $m(i,j) = 2$, there is no edge between the vertices i and j;
- if $m(i,j) = 3$, there is an unlabelled edge between i and j;
- if $m(i,j) > 3$, there is an edge between i and j with label $m(i,j)$.

The defining relations of the Coxeter group mean, in particular, that $s_i^2 = 1$ for all $i \in I$, and consequently, the relations for s_i and s_j with $i \neq j$ and $m(i,j) < \infty$ can be equivalently rewritten in the form of braid relations

$$\underbrace{s_i s_j s_i s_j \cdots}_{m(i,j) \text{ terms}} = \underbrace{s_j s_i s_j s_i \cdots}_{m(i,j) \text{ terms}}. \tag{2.1}$$

Theorem 2.2 implies for any finite group that if it is a real reflection group then it is a finite Coxeter group. In 1935, Coxeter proved the converse of this statement.

Theorem 2.4. *[38] A finite group W is a Coxeter group if and only if it is a real reflection group.*

In other words, for finite groups, the notion of a real reflection group and Coxeter group essentially coincide, though this is not the case for infinite groups.

Example 2.5. The symmetric group $\mathfrak{S}_n$ ($n \geq 2$), the group of all permutations of the set $\{1, \dots, n\}$, is a Coxeter group with generators $s_i = (i, i+1)$ for $i = 1, \dots, n-1$, which are the simple transpositions that swap i and $i+1$, and a presentation

$$\mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \colon s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i \text{ if } |i-j| > 1 \rangle.$$

It can be realised as a real reflection group by putting $E = \mathbb{R}^n$ and letting s_i act as the reflection $s_{e_i - e_{i+1}}$, where $\{e_i\}_{i=1}^n$ is the standard orthonormal basis of $\mathbb{R}^n$. The resulting faithful action of $\mathfrak{S}_n$ on $\mathbb{R}^n$ is by permuting coordinates, and with respect to it, the set $\mathrm{Ref}(\mathfrak{S}_n)$ consists of all transpositions $(i, j) \in \mathfrak{S}_n$, acting respectively as $s_{e_i - e_j}$. The corresponding Coxeter graph is the Dynkin diagram of type A_{n-1}. The group $\mathfrak{S}_n$ is called the Coxeter group of type A_{n-1}.

Finite Coxeter groups have been classified [37,38]. They are finite direct products of the groups corresponding to the Coxeter systems of the following irreducible types (a Coxeter system is called irreducible if its associated Coxeter graph is connected): A_n ($n \geq 1$), $B_n = C_n$ ($n \geq 2$), D_n ($n \geq 4$), E_6, E_7, E_8, F_4, G_2, H_2, H_3, H_4, $I_2(m)$ ($m \geq 7$). Sometimes, the notation $I_2(3)$, $I_2(4)$, $I_2(5)$, $I_2(6)$ is used for A_2, B_2, H_2, and G_2, respectively. The reason is that then $I_2(m)$ for $m \geq 3$ are precisely the dihedral groups (the groups of symmetries of regular m-gons).

Each of the groups of types $A, \dots, G$ is a Weyl group (of a simple complex Lie algebra), that is, a *crystallographic* finite real reflection group, which are those that preserve a lattice in the Euclidean space E (the $\mathbb{Z}$-linear span of a basis of E). Types A, $B = C$, and D are called classical types, and the groups of types E_6, E_7, E_8, F_4, and G_2 are called exceptional Weyl groups. The remaining types are non-crystallographic.

2.1.1 Root systems

Considering the set of lines orthogonal to the reflecting hyperplanes for the elements $\mathrm{Ref}(W)$ of a finite real reflection group W leads to the notion of a root system. It turns out that W, when acting on the Euclidean space E, permutes the set of these lines. The definition of a root system can be axiomatised as follows (see, e.g., [69]).

Definition 2.6. A finite set $R \subset E \setminus \{0\}$ is a (reduced) root system if the following two conditions are satisfied for all $\alpha \in R$:

(i) $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$

(ii) $s_\alpha(R) = R$.

The elements of R are called roots. The root system is called *crystallographic* if $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in R$, which is known as the crystallographic condition. A *non-reduced* crystallographic root system is one that does not satisfy condition (i) but satisfies all the other conditions.

Given any root system R, the group $W = \langle s_\alpha \ (\alpha \in R) \rangle$, generated by reflections with respect to the roots $\alpha \in R$, is a finite Coxeter group, and it is a Weyl group if R is crystallographic. It turns out that $\mathrm{Ref}(W) = \{s_\alpha \colon \alpha \in R\}$. Let $I \subseteq \mathrm{Ref}(W)$ be a choice of simple reflections. The root system R is called irreducible if the Coxeter system (W, I) is irreducible.

Notice in Definition 2.6 that in the case when the crystallographic condition is not imposed, the relative lengths of roots that lie in different W-orbits do not matter.

A positive subsystem $R_+ \subset R$ is defined by choosing a generic $v \in E$ and letting $R_+ = \{\alpha \in R \colon \langle \alpha, v \rangle > 0\}$, where 'generic' means that $\langle \alpha, v \rangle \neq 0$ for any $\alpha \in R$. Note that $R = R_+ \coprod (-R_+)$.

Example 2.7. The root system of type A_{n-1} is

$$A_{n-1} = \{e_i - e_j \colon 1 \leq i \leq n, \ 1 \leq j \leq n, \ i \neq j\} \subset \mathbb{R}^n,$$

whose associated Coxeter group is the symmetric group $\mathfrak{S}_n$. A choice of a positive half is $A_{n-1,+} = \{e_i - e_j \colon 1 \leq i < j \leq n\}$.

Given any reduced root system R and a positive subsystem R_+, there is a unique subset $S \subseteq R_+$ of so-called (positive) simple roots such that S forms a basis for $\mathrm{Span}_{\mathbb{R}} R$, the $\mathbb{R}$-linear span of R in E, and such that every $\alpha \in R_+$ can be written as a linear combination of simple roots with non-negative coefficients (see, e.g., [69, Theorem 1.3]). The set S is called a simple subsystem, and $\{s_{\alpha_i} \colon \alpha_i \in S\}$ gives a set of simple reflections for the reflection group W associated with R. The cardinality of the set of simple roots is called the rank of R.

The classification of crystallographic root systems (up to 'isomorphism') is analogous to the classification of Weyl groups. (Two crystallographic root systems R, R' in Euclidean spaces E, E', respectively, are isomorphic if there is a vector space isomorphism $\phi\colon \mathrm{Span}_{\mathbb{R}} R \to \mathrm{Span}_{\mathbb{R}} R'$ such that $\phi(R) = R'$ and $\langle \phi(\alpha)^\vee, \phi(\beta) \rangle = \langle \alpha^\vee, \beta \rangle$ for all $\alpha, \beta \in R$.) Reduced irreducible crystallographic root systems come in types $A, \ldots, G$, just as the corresponding Weyl groups (see, e.g., [9]). The only difference is that there are crystallographic root systems

$$B_n = \{\pm e_i \colon 1 \le i \le n\} \cup \{\pm e_i \pm e_j \colon 1 \le i < j \le n\} \subset \mathbb{R}^n$$

and

$$C_n = \{\pm 2e_i \colon 1 \le i \le n\} \cup \{\pm e_i \pm e_j \colon 1 \le i < j \le n\} \subset \mathbb{R}^n$$

that are not isomorphic if $n > 2$, but have the same associated Weyl group. The standard realisations of all the reduced irreducible crystallographic root systems can be found in [9, Plate I–IX]. The only non-reduced irreducible crystallographic root system is

$$BC_n = \{\pm e_i,\ \pm 2e_i \colon 1 \le i \le n\} \cup \{\pm e_i \pm e_j \colon 1 \le i < j \le n\} \subset \mathbb{R}^n.$$

Its associated Coxeter group is the Weyl group of type B_n.

Given a crystallographic root system R, the set $R^\vee := \{\alpha^\vee \colon \alpha \in R\}$ is also a crystallographic root system, called the dual root system, and its elements are called coroots. The root systems R and $R^\vee$ generate the same Weyl group, but are not always isomorphic. Assume that R is reduced. Given a basis of simple roots $\alpha_1, \ldots, \alpha_k$, the lattice $Q = \bigoplus_{i=1}^k \mathbb{Z}\alpha_i$ is called the root lattice, and $\bigoplus_{i=1}^k \mathbb{Z}\alpha_i^\vee$ is called the coroot lattice, denoted $Q^\vee$. The weight lattice is defined by $P = \{\omega \in \operatorname{Span}_{\mathbb{R}} R \colon \langle \omega, \alpha^\vee \rangle \in \mathbb{Z},\ \forall \alpha \in R\}$. Its elements are called weights. The lattice of coweights is given by $\{\omega \in \operatorname{Span}_{\mathbb{R}} R \colon \langle \omega, \alpha \rangle \in \mathbb{Z},\ \forall \alpha \in R\}$ and is denoted $P^\vee$. Due to the crystallographic condition, we have $R \subset Q \subseteq P$ and $Q^\vee \subseteq P^\vee$. The lattices Q, $Q^\vee$, P, and $P^\vee$ are all preserved under the action of the Weyl group W.

The fundamental weights $\omega_1, \ldots, \omega_k \in P$ are defined as the basis of $\operatorname{Span}_{\mathbb{R}} R$ dual to $\{\alpha_i^\vee\}_{i=1}^k$, that is, they are defined by the condition $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ for all i, j.

A weight $\omega \in P$ is called minuscule if $-1 \le \langle \omega, \alpha^\vee \rangle \le 1$ for all $\alpha \in R$. The only reduced irreducible crystallographic root systems for which there does not exist any non-zero minuscule weight are E_8, F_4, and G_2 (see [9]). A quasiminuscule weight is an element $\omega \in R$ such that $-1 \le \langle \omega, \alpha^\vee \rangle \le 1$ for all $\alpha \in R \setminus \{\pm\omega\}$. A quasiminuscule weight exists in all cases, including E_8, F_4, and G_2 (see [9]).

2.2 Hecke algebras

Hecke algebras are certain deformations of group algebras of Coxeter groups. An alternative way to think of them is as particular quotients of group algebras of braid groups.

With notations as in Section 2.1, the braid group associated with a Coxeter system (W, I) is the group defined by the presentation

$$B_W = B_{(W,I)} = \langle \sigma_i \ (i \in I) \colon \underbrace{\sigma_i \sigma_j \sigma_i \cdots}_{m(i,j) \text{ terms}} = \underbrace{\sigma_j \sigma_i \sigma_j \cdots}_{m(i,j) \text{ terms}} \text{ if } m(i,j) < \infty \rangle.$$

That is, its defining relations are precisely of the form of the braid relations (2.1). Note that $\sigma_i^2 \neq 1$, whereas the generators s_i of the Coxeter group W satisfy $s_i^2 = 1$. For example, the braid group $B_{\mathfrak{S}_n}$ associated with the symmetric group (the Coxeter group of type A_{n-1}) is the standard braid group on n strands.

An expression $w = s_{i_1} \cdots s_{i_k} \in W$ is called reduced if w cannot be written as a product of s_i with fewer than k terms, and then k is called the length $\ell(w)$ of w. One can show that two reduced expressions are equal to the same element of W if and only if they are related by a sequence of braid relations (this is Matsumoto's theorem; see, e.g., [58]). It follows that the definition $\sigma_w \coloneqq \sigma_{i_1} \cdots \sigma_{i_k} \in B_W$ does not depend on the choice of reduced expression for w. Note that if $\ell(s_i w) > l(w)$, then $\sigma_i \sigma_w = \sigma_{s_i w}$. It follows that if $\ell(s_i w) < l(w)$ $(= l(s_i s_i w))$, then $\sigma_{s_i w} = \sigma_i^{-1} \sigma_w$.

Let $\tau \colon I \to \mathbb{C}^\times$, $i \mapsto \tau_i$ be a function such that $\tau_i = \tau_j$ if the Coxeter generators s_i and s_j are conjugate in W. The associated Hecke algebra can be defined as the quotient of the group algebra $\mathbb{C}B_W$ by the two-sided ideal generated by a set of quadratic relations called Hecke relations.

Definition 2.8. The Hecke algebra associated with (W, I) and τ is

$$\mathscr{H}_\tau(W) = \mathscr{H}_\tau(W, I) = \mathbb{C}B_W / ((\sigma_i - \tau_i)(\sigma_i + \tau_i^{-1}) \colon i \in I).$$

Let T_i $(i \in I)$ and T_w $(w \in W)$ be respectively the images of σ_i and σ_w under the quotient map $\mathbb{C}B_W \to \mathscr{H}_\tau(W)$.

The algebra $\mathscr{H}_\tau(W)$ is generated as a (unital, associative) $\mathbb{C}$-algebra by the elements $\{T_i\}_{i \in I}$, subject only to the braid relations and the Hecke relations $(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0$. The latter relation is equivalent to

$$T_i^{-1} = T_i + \tau_i^{-1} - \tau_i.$$

If the function τ is identically equal to 1, then this Hecke relation reduces to $T_i^2 = 1$, and thus $\mathscr{H}_1(W) \cong \mathbb{C}W$. The algebra $\mathscr{H}_\tau(W)$ for $\tau \neq 1$ is a deformation of $\mathbb{C}W$.

Remark 2.9. One may consider τ_i as formal commuting invertible variables, rather than complex parameters, in which case $\mathscr{H}_\tau(W)$ is considered as an algebra over the ring $\mathbb{C}[\tau^{\pm 1}]$ of Laurent polynomials in τ_i.

The Hecke algebra has a representation on $\mathbb{C}W = \bigoplus_{w \in W} \mathbb{C}e_w$, where e_w $(w \in W)$

denotes a basis for $\mathbb{C}W$, given by

$$T_i(e_w) = \begin{cases} e_{s_i w} & \text{if } \ell(s_i w) > \ell(w), \\ e_{s_i w} + (\tau_i - \tau_i^{-1}) e_w & \text{if } \ell(s_i w) < \ell(w). \end{cases} \tag{2.2}$$

See, e.g., [58] for a proof that this is a representation.

The representation (2.2), which may be thought of as a deformed version of the left regular representation of W, can be used to show the linear independence of the elements T_w $(w \in W)$. Indeed, we have $T_w(e_{\mathrm{id}}) = e_w$, so any linear dependence between T_w would contradict the linear independence of e_w. In fact, the following theorem takes place (see, e.g., [58, 69]).

Theorem 2.10. *The elements T_w $(w \in W)$ form a $\mathbb{C}$-linear basis for $\mathscr{H}_\tau(W)$, or if τ_i are formal variables then $\mathscr{H}_\tau(W)$ is a free $\mathbb{C}[\tau^{\pm 1}]$-module with this basis. In particular, $\mathscr{H}_\tau(W)$ is a* flat *deformation of $\mathbb{C}W$, as it is a flat module over $\mathbb{C}[\tau^{\pm 1}]$.*

Recall that a module M over a ring R is flat if the operation of taking the tensor product $\otimes_R M$ preserves exact sequences of R-modules, and that every free module is flat.

To see in Theorem 2.10 that the elements T_w span $\mathscr{H}_\tau(W)$, one uses that their span contains $1 = T_{\mathrm{id}}$ and is stable under left multiplication by T_i (indeed, if $\ell(s_i w) > \ell(w)$ then $T_i T_w = T_{s_i w}$ as $\sigma_i \sigma_w = \sigma_{s_i w}$; and if $\ell(s_i w) < \ell(w)$, then $\sigma_{s_i w} = \sigma_i^{-1} \sigma_w$, implying $T_{s_i w} = T_i^{-1} T_w = T_i T_w + (\tau_i^{-1} - \tau_i) T_w$), and that T_i generate $\mathscr{H}_\tau(W)$ as an algebra.

It follows from the preceding theorem that the representation (2.2) is faithful, and that it can be actually interpreted as an action of $\mathscr{H}_\tau(W)$ on itself, coinciding with the left regular representation of $\mathscr{H}_\tau(W)$.

Example 2.11. Let us consider as an example the type A_{n-1} Hecke algebra $\mathscr{H}_{n,\tau} = \mathscr{H}_\tau(\mathfrak{S}_n)$ associated with the symmetric group $\mathfrak{S}_n$. In this case, the function τ must be constant, so we may think of τ as an element of $\mathbb{C}^\times$. We have

$$\mathscr{H}_{n,\tau} \cong \langle T_1, \dots, T_{n-1} \colon (T_i - \tau)(T_i + \tau^{-1}) = 0,\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_i T_j = T_j T_i \text{ if } |i - j| > 1 \rangle.$$

The algebra $\mathscr{H}_{n,\tau}$ admits a faithful representation, called the polynomial representation, on the space of Laurent polynomials $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ given by

$$T_i \mapsto \tau s_i + \frac{\tau - \tau^{-1}}{X_i X_{i+1}^{-1} - 1}(s_i - 1), \tag{2.3}$$

where s_i acts by swapping X_i and X_{i+1}. The operators (2.3) are called Demazure–Lusztig operators (see the historical remarks in [35, p. 89]).

2.3 Affine Hecke algebra of type GL_n

The affine Hecke algebra (AHA) of type GL_n is an enlargement of the Hecke algebra $\mathscr{H}_{n,\tau}$ from Example 2.11. It will be convenient later to treat τ as a formal parameter. Let $\mathbb{C}_\tau = \mathbb{C}[\tau^{\pm 1}]$ denote the ring of Laurent polynomials in the variable τ. The AHA of type GL_n is the (unital, associative) $\mathbb{C}_\tau$-algebra $\widehat{\mathscr{H}}_{n,\tau}$ with generators T_k ($1 \le k \le n-1$), $X_i^{\pm 1}$ ($1 \le i \le n$), and the following relations [35, p. 76]:

$$(T_k - \tau)(T_k + \tau^{-1}) = 0, \quad T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}, \quad [T_k, T_l] = 0 \text{ if } |k-l| > 1,$$
$$T_k X_k T_k = X_{k+1}, \quad [T_k, X_i] = 0 \text{ for } i \ne k, k+1,$$
$$\text{and Laurent relations for } X_i^{\pm 1} \text{ (that is, } X_i X_i^{-1} = X_i^{-1} X_i = 1,\ [X_i, X_j] = 0,\ 1 \le j \le n),$$

where the bracket $[\cdot,\cdot]$ denotes the commutator. The elements T_k generate a subalgebra isomorphic to the non-affine Hecke algebra $\mathscr{H}_{n,\tau}$.

Similarly to the algebra $\mathscr{H}_{n,\tau}$, the AHA $\widehat{\mathscr{H}}_{n,\tau}$ has a faithful polynomial representation on $\mathbb{C}_\tau[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ in which T_k act via the Demazure–Lusztig operators (2.3), and $X_i^{\pm 1}$ act by multiplication (see, e.g., [35]).

In addition, the algebra $\widehat{\mathscr{H}}_{n,\tau}$ admits another different faithful representation on the same space $\mathbb{C}_\tau[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, called Cherednik's basic representation, defined as follows (see [35, Lemma 1.3.12]). Let $q \in \mathbb{C}^\times$ be not a root of unity. The basic representation $\beta = \beta_q$ sends T_k to the Demazure–Lusztig operators (2.3) and

$$X_i \mapsto \beta(T_{i-1} \cdots T_2 T_1) \pi \beta(T_{n-1}^{-1} \cdots T_{i+1}^{-1} T_i^{-1}),$$

where $\pi(X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}) = q^{-a_n} X_1^{a_n} X_2^{a_1} \cdots X_n^{a_{n-1}}$ for any $a_i \in \mathbb{Z}$. That is, π acts as the cyclic permutation $(1, 2, \dots, n) \in \mathfrak{S}_n$ followed by a scaling of the variable X_1 by q^{-1}.

The algebra $\widehat{\mathscr{H}}_{n,\tau}$ has a linear basis of Poincaré–Birkhoff–Witt (PBW) type consisting of the elements $T_w X_1^{m_1} \cdots X_n^{m_n}$ with $w \in \mathfrak{S}_n$ and $m_i \in \mathbb{Z}$. This follows from a similar statement about a PBW basis for DAHA of type GL_n (see Section 2.4 below) of which the AHA is a subalgebra. Here $\{T_w \colon w \in \mathfrak{S}_n\}$ is a basis of the non-affine Hecke algebra $\mathscr{H}_{n,\tau}$ (see Theorem 2.10 above).

The centre $\mathcal{Z}(\widehat{\mathscr{H}}_{n,\tau})$ of the algebra $\widehat{\mathscr{H}}_{n,\tau}$ is formed by symmetric Laurent polynomials in X-variables, $\mathcal{Z}(\widehat{\mathscr{H}}_{n,\tau}) = \mathbb{C}_\tau[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{\mathfrak{S}_n}$ (see, e.g., [35, Lemma 1.3.12] and a historical comment in [80]).

2.4 Double affine Hecke algebra of type GL_n

The idea behind the definition of the DAHA of type GL_n is, roughly speaking, to "glue together the polynomial and basic representations of the GL_n-type AHA $\widehat{\mathscr{H}}_{n,\tau}$ along the generators T_k". The precise definition is as follows.

Let $q \in \mathbb{C}^\times$ be not a root of unity. The DAHA $\mathbb{H}_n = \mathbb{H}_{n,q,\tau}$ of type GL_n is the (unital, associative) $\mathbb{C}_\tau$-algebra generated by T_k $(1 \le k \le n-1)$, $X_i^{\pm 1}$, and $Y_i^{\pm 1}$ $(1 \le i \le n)$ with the following relations [35, p. 100]:

$$(T_k - \tau)(T_k + \tau^{-1}) = 0, \tag{2.4}$$
$$T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}, \qquad [T_k, T_l] = 0 \text{ if } |k-l| > 1, \tag{2.5}$$
$$T_k X_k T_k = X_{k+1}, \qquad [T_k, X_i] = 0 \text{ for } i \ne k, k+1, \tag{2.6}$$
$$T_k^{-1} Y_k T_k^{-1} = Y_{k+1}, \qquad [T_k, Y_i] = 0 \text{ for } i \ne k, k+1, \tag{2.7}$$
$$\widetilde{Y} X_i = q X_i \widetilde{Y},$$
$$Y_2^{-1} X_1 Y_2 X_1^{-1} = T_1^2,$$
$$\text{and Laurent relations for } X_i^{\pm 1} \text{ and } Y_i^{\pm 1},$$

where $\widetilde{Y} = \prod_{i=1}^n Y_i$. Relations (2.7) imply that $\widetilde{Y}$ commutes with all T_k, which generate a subalgebra isomorphic to the Hecke algebra $\mathscr{H}_{n,\tau}$ of type A_{n-1}. By [35, Theorem 1.4.8], the map $X_i \mapsto Y_i^{-1}$, $Y_i \mapsto X_i^{-1}$, $T_k \mapsto T_k$ defines an anti-automorphism of $\mathbb{H}_n$. The subalgebras $\langle T_1, \dots, T_{n-1}, X_1^{\pm 1}, \dots, X_n^{\pm 1} \rangle$ and $\langle T_1, \dots, T_{n-1}, Y_1^{\pm 1}, \dots, Y_n^{\pm 1} \rangle$ are both isomorphic to the AHA $\widehat{\mathscr{H}_{n,\tau}}$ of type GL_n, and to each other via the map $T_k \mapsto T_k$, $Y_i \mapsto X_i^{-1}$.

As in [35, (1.4.57)], let $\pi = Y_1^{-1} T_1 \cdots T_{n-1}$. Relations (2.7) imply that

$$Y_i = T_i T_{i+1} \cdots T_{n-1} \pi^{-1} T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1} \tag{2.8}$$

for all $1 \le i \le n$ (for $i = 1$ and $i = n$, this is to be interpreted as $Y_1 = T_1 \cdots T_{n-1} \pi^{-1}$ and $Y_n = \pi^{-1} T_1^{-1} \cdots T_{n-1}^{-1}$, respectively).

The element π can be used to give an alternative presentation for the algebra $\mathbb{H}_n$ [35, p. 101]. Namely, $\mathbb{H}_n$ is isomorphic to the algebra with abstract generators T_k $(1 \le k \le n-1)$, $\pi^{\pm 1}$, and $X_i^{\pm 1}$ $(1 \le i \le n)$, and relations (2.4)–(2.6), Laurent relations for $X_i^{\pm 1}$, and the following relations involving π:

$$\begin{aligned} &\pi X_i = X_{i+1} \pi, && (i = 1, \dots, n-1) \\ &\pi^n X_i = q^{-1} X_i \pi^n, && (i = 1, \dots, n) \\ &\pi T_i = T_{i+1} \pi, && (i = 1, \dots, n-2) \\ &\pi^n T_i = T_i \pi^n, && (i = 1, \dots, n-1). \end{aligned} \tag{2.9}$$

It follows that $\pi X_n = q^{-1} X_1 \pi$ and $\pi^2 T_{n-1} = T_1 \pi^2$. We have $\pi^n = \widetilde{Y}^{-1}$ [35, p. 101], since relations (2.7) imply that $\pi^n = (Y_1^{-1} T_1 \cdots T_{n-1})^n = \widetilde{Y}^{-1} S_1 S_2 \cdots S_n$, where $S_i = T_1^{-1} T_2^{-1} \cdots T_{n-i}^{-1} T_{n-i+1} \cdots T_{n-1}$, and one can check using braid relations that $S_1 \cdots S_n = 1$. It is easy to see from relations (2.9) that the map $X_i \mapsto X_i$, $T_k \mapsto T_k$, $\pi \mapsto \pi^{-1}$ defines an anti-automorphism of $\mathbb{H}_n$.

The algebra $\mathbb{H}_n$ admits a faithful polynomial representation on the space of Laurent polynomials $\mathbb{C}_\tau[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ determined by

$$T_k \mapsto \tau s_k + \frac{\tau - \tau^{-1}}{X_k X_{k+1}^{-1} - 1}(s_k - 1), \tag{2.10}$$

$$\pi^{-1}(X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}) = q^{a_1} X_1^{a_2} \cdots X_{n-1}^{a_n} X_n^{a_1} \qquad (a_i \in \mathbb{Z}), \tag{2.11}$$

and the action of $X_i^{\pm 1}$ by multiplication [35, p. 101]. That is, T_k act as the Demazure–Lusztig operators, and Y_i act as $\beta(X_i^{-1})$, where β is Cherednik's basic representation from Section 2.3.

In particular, at $\tau = 1$, the element T_k acts as $s_k \in \mathfrak{S}_n$; and by equality (2.8) and formula (2.11), the element Y_i acts at $\tau = 1$ as the (multiplicative) q-shift operator $t_i = q^{X_i \partial_{X_i}}$, whose action is $t_i(X_j) = q^{\delta_{ij}} X_j$.

The DAHA $\mathbb{H}_n$ has a linear basis of PBW type formed by the elements $T_w X^{\mathbf{m}} Y^{\mathbf{p}}$ with $X^{\mathbf{m}} = X_1^{m_1} \cdots X_n^{m_n}$, $Y^{\mathbf{p}} = Y_1^{p_1} \cdots Y_n^{p_n}$, $w \in \mathfrak{S}_n$, and $\mathbf{m} = (m_1, \dots, m_n)$, $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n$. The fact that they span follows easily from the defining relations of $\mathbb{H}_n$, and the polynomial representation can be used to show that they are linearly independent (cf. [35, Theorem 3.2.1(ii)], which provides a proof of an analogous statement for the DAHA of type SL_n).

The algebra $\mathbb{H}_n$ admits a grading with $\deg T_k = \deg Y_i^{\pm 1} = 0$ and $\deg X_i^{\pm 1} = \pm 1$, for which $\widetilde{Y}$ plays the role of a "grading operator", in the sense that for an arbitrary monomial in the PBW basis of $\mathbb{H}_n$, we have

$$\widetilde{Y} T_w X^{\mathbf{m}} Y^{\mathbf{p}} = q^{m_1 + \cdots + m_n} T_w X^{\mathbf{m}} Y^{\mathbf{p}} \widetilde{Y},$$

where q's exponent $m_1 + \cdots + m_n = \deg(T_w X^{\mathbf{m}} Y^{\mathbf{p}})$. The degree zero part $\mathbb{H}_n^{(0)}$ is generated by the elements T_k, $Y_i^{\pm 1}$, and $X_i X_j^{-1}$, which can be seen by using the PBW basis.

2.5 Degenerations of Cherednik algebras

In this section, we discuss two types of algebras that can be obtained as degenerations of Cherednik's DAHAs. They are respectively called the rational and trigonometric Cherednik algebras. Their GL_n type (Section 2.5.3 below) will be used in Chapter 6. For Chapter 4, we need other types as well, and we cover them in Sections 2.5.1 and 2.5.2. Although RCAs can be defined for an arbitrary finite complex reflection group [45], the case of real reflection groups will be sufficient for our purposes. Similarly, we will not consider TCAs in the most general possible setting, as we will only need the case corresponding to reduced crystallographic root systems.

2.5.1 Rational Cherednik algebras

Assume that the Euclidean space E from Section 2.1 has dimension $\dim E = n$, so that we can identify it with $\mathbb{R}^n$. Let $V = \mathbb{C}^n$ be its complexification, and let the inner product $\langle \cdot, \cdot \rangle$ be extended $\mathbb{C}$-bilinearly to V. Let $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ denote the algebra of polynomial functions on V (also denoted as $\mathbb{C}[V]$ or $S(V^*)$, the latter meaning the symmetric algebra of the dual space V^*), where $x_i \in V^*$ form the dual basis of the standard basis $\{e_i\}_{i=1}^n$ of V. Let $R \subset \mathbb{R}^n$ be a (reduced, not necessarily crystallographic) root system, with associated Coxeter group W. The latter can naturally act on V and therefore on $\mathbb{C}[x]$. Let $c\colon R \to \mathbb{C}$, $\alpha \mapsto c(\alpha) = c_\alpha$ be a W-invariant function, called a multiplicity map. Let R_+ be a positive subsystem.

The associated RCA $\mathcal{H}_c = \mathcal{H}_c(W, V)$ can be defined by its faithful polynomial representation on $\mathbb{C}[x]$. Namely, it is the subalgebra of the endomorphism ring $\operatorname{End}_{\mathbb{C}} \mathbb{C}[x]$ generated by polynomials $p \in \mathbb{C}[x]$ (acting by multiplication), the reflection group W, and the rational Dunkl operators

$$\nabla_\xi = \partial_\xi - \sum_{\alpha \in R_+} \frac{c_\alpha \langle \alpha, \xi \rangle}{\langle \alpha, x \rangle} (1 - s_\alpha), \qquad (\xi \in V) \tag{2.12}$$

where $\partial_\xi = \sum_{i=1}^n \langle \xi, e_i \rangle \partial_{x_i}$ is the directional derivative along the vector ξ [45]. Let us note that one can consider a definition of the RCA with an extra parameter $t \in \mathbb{C}^\times$ by replacing ∂_ξ in (2.12) with $t\partial_\xi$. The resulting algebra is isomorphic to $\mathcal{H}_{c/t}$, so there is no loss of generality in choosing $t = 1$, as we do. A limit $t \to 0$ gives the so-called *classical* limit of the RCA.

It is well known that the Dunkl operators (2.12) commute among themselves, $[\nabla_\xi, \nabla_\eta] = 0$ for all $\xi, \eta \in V$ [43], and that they are W-equivariant, $w \nabla_\xi = \nabla_{w(\xi)} w$ for all $w \in W$. Dunkl operators preserve polynomials as $p - s_\alpha(p)$ is divisible by $\langle \alpha, x \rangle$ for any $p \in \mathbb{C}[x]$. They satisfy the following commutation relations

$$[\nabla_\xi, x_i] = \langle \xi, e_i \rangle - \sum_{\alpha \in R_+} c_\alpha \langle \alpha, \xi \rangle \langle \alpha^\vee, e_i \rangle s_\alpha$$

for $i = 1, \dots, n$. There is an automorphism of the RCA defined by $\nabla_{e_i} \mapsto x_i$, $x_i \mapsto -\nabla_{e_i}$, and $w \mapsto w$.

The algebra $\mathcal{H}_c$ has a linear basis of PBW type formed by $w x_1^{m_1} \cdots x_n^{m_n} \nabla_{e_1}^{p_1} \cdots \nabla_{e_n}^{p_n}$ for $w \in W$ and $m_i, p_i \in \mathbb{Z}_{\geq 0}$ [45]. The RCA $\mathcal{H}_c$ is a flat c-deformation of the crossed (equivalently, semi-direct or smash) product $\mathbb{C}W \ltimes D[V]$ of the group algebra $\mathbb{C}W$ with the Weyl algebra $D[V] = \mathbb{C}[x][\partial_{x_1}, \dots, \partial_{x_n}]$ of differential operators on V with polynomial coefficients.

The spherical subalgebra $S\mathcal{H}_c \subset \mathcal{H}_c$ is the subalgebra (with a different unit) of $\mathcal{H}_c$

defined by $S\mathcal{H}_c = e\mathcal{H}_c e$ using the idempotent symmetriser $e = |W|^{-1}\sum_{w\in W} w \in \mathbb{C}W$.

The RCA $\mathcal{H}_c$ admits a grading with $\deg x_i = 1$, $\deg w = 0$, and $\deg \nabla_\xi = -1$ for $w \in W$ and $\xi \in V$. Its degree zero subalgebra, generated by W and $x_i\nabla_{e_j}$ ($i, j \in \{1, \dots, n\}$), was studied in [51] (see also [4]).

2.5.2 Trigonometric Cherednik algebras

In this section, we continue to use the notations from Section 2.5.1, but R will now be a reduced crystallographic root system spanning the vector space V and W the corresponding Weyl group. The TCA (also known as degenerate DAHA) $\mathbb{H}_c^{\mathrm{trig}} = \mathbb{H}_c^{\mathrm{trig}}(R)$ associated with R can be defined by its faithful action on the group algebra $\mathbb{C}[P] = \mathbb{C}[\{e^{\langle\alpha,x\rangle} : \alpha \in P\}]$ of the weight lattice P of R. (Similarly to the RCA, one could consider a definition involving an additional parameter $t \in \mathbb{C}$, only needed to handle the classical limit.)

The algebra $\mathbb{H}_c^{\mathrm{trig}}$ is the (unital, associative) algebra over $\mathbb{C}$ generated by W (which preserves the weight lattice P, and hence acts naturally on $\mathbb{C}[P]$), multiplication operators $e^{\langle\alpha,x\rangle}$ ($\alpha \in P$), and Cherednik's commuting trigonometric Dunkl operators

$$\nabla_\xi^{\mathrm{trig}} = \partial_\xi - \sum_{\alpha\in R_+} \frac{c_\alpha\langle\alpha,\xi\rangle}{1 - e^{-\langle\alpha,x\rangle}}(1 - s_\alpha) + \langle\rho,\xi\rangle, \tag{2.13}$$

where $\xi \in V$ and $\rho = \rho_c = \frac{1}{2}\sum_{\alpha\in R_+} c_\alpha\alpha$ [31]. The operators (2.13) are a trigonometric generalisation of (2.12). These trigonometric Dunkl operators commute, $[\nabla_\xi^{\mathrm{trig}}, \nabla_\eta^{\mathrm{trig}}] = 0$ for all $\xi, \eta \in V$, but they are not W-equivariant. For any simple root $\alpha_i \in R_+$, they satisfy

$$s_{\alpha_i}\nabla_\xi^{\mathrm{trig}} - \nabla_{s_{\alpha_i}(\xi)}^{\mathrm{trig}} s_{\alpha_i} = c_{\alpha_i}\langle\alpha_i,\xi\rangle.$$

A different but related definition of trigonometric Dunkl operators such that they are W-equivariant, but do not commute, was given by Heckman in [62] — their definition is

$$\nabla_\xi^{\mathrm{H}} = \partial_\xi - \frac{1}{2}\sum_{\alpha\in R_+} \frac{c_\alpha\langle\alpha,\xi\rangle(1 + e^{-\langle\alpha,x\rangle})}{1 - e^{-\langle\alpha,x\rangle}}(1 - s_\alpha) = \nabla_\xi^{\mathrm{trig}} - \frac{1}{2}\sum_{\alpha\in R_+} c_\alpha\langle\alpha,\xi\rangle s_\alpha.$$

Their commutators are

$$[\nabla_\xi^{\mathrm{H}}, \nabla_\eta^{\mathrm{H}}] = -\frac{1}{4}\sum_{\alpha,\beta\in R_+} c_\alpha c_\beta\big(\langle\alpha,\xi\rangle\langle\beta,\eta\rangle - \langle\alpha,\eta\rangle\langle\beta,\xi\rangle\big)s_\alpha s_\beta$$

for all $\xi, \eta \in V$ (see, e.g., [64]).

The TCA $\mathbb{H}_c^{\mathrm{trig}}$ has a linear basis of PBW type formed by the elements $we^{\langle\alpha,x\rangle}p(\nabla^{\mathrm{trig}})$ for $w \in W$, $\alpha \in P$, and monomials $p \in \mathbb{C}[x]$, with $p(\nabla^{\mathrm{trig}}) = p(\nabla_{e_1}^{\mathrm{trig}}, \dots, \nabla_{e_n}^{\mathrm{trig}})$. This PBW property can be interpreted as a vector space isomorphism between $\mathbb{H}_c^{\mathrm{trig}}$ and $\mathbb{C}W^{ea} \otimes \mathbb{C}[x]$, where $W^{ea} = W \ltimes P$ is an extended affine Weyl group.

The spherical subalgebra $S\mathbb{H}_c^{\mathrm{trig}} \subset \mathbb{H}_c^{\mathrm{trig}}$ is defined similarly to the case of RCAs by $S\mathbb{H}_c^{\mathrm{trig}} = e\mathbb{H}_c^{\mathrm{trig}}e$.

The degenerate AHA (also known as the graded Hecke algebra) [41,80] is the subalgebra of $\mathbb{H}_c^{\mathrm{trig}}$ generated by W and $\nabla_\xi^{\mathrm{trig}}$ ($\xi \in V$). The centre of the degenerate AHA is equal to $\{p(\nabla^{\mathrm{trig}}) \colon p \in \mathbb{C}[x]^W\}$ (see, e.g., [64]), where $\mathbb{C}[x]^W$ denotes W-invariant polynomials. In particular, the operators $p(\nabla^{\mathrm{trig}})$ with $p \in \mathbb{C}[x]^W$ are W-invariant.

2.5.3 Type GL_n

In this section, we give an abstract definition of the RCA and TCA of type GL_n by generators and relations, as we need them in Chapter 6. Let $c \in \mathbb{C}$ be a parameter.

In the rational case, type GL_n refers to the RCA $\mathcal{H}_n = \mathcal{H}_{n,c} = \mathcal{H}_c(\mathfrak{S}_n, \mathbb{C}^n)$ associated with the A_{n-1}-type root system $R \subset E = \mathbb{R}^n$ (by contrast, type SL_n corresponds to choosing E to be the orthogonal complement of $\sum_{i=1}^n e_i$ in $\mathbb{R}^n$). Abstractly, the RCA $\mathcal{H}_n$ is the (unital, associative) algebra over $\mathbb{C}$ generated by the simple transpositions $s_k = (k, k+1) \in \mathfrak{S}_n$ ($1 \le k \le n-1$), and elements x_i, y_i ($1 \le i \le n$) subject to the following relations [45]:

$$\begin{aligned}
&[y_i, y_j] = 0 = [x_i, x_j], \\
&s_k x_k s_k = x_{k+1}, \qquad [s_k, x_i] = 0 \text{ for } i \neq k, k+1, \\
&s_k y_k s_k = y_{k+1}, \qquad [s_k, y_i] = 0 \text{ for } i \neq k, k+1, \\
&S_{ij} \coloneqq [y_i, x_j] = \begin{cases} 1 - c\sum_{l \neq i} s_{il} \text{ if } i = j, \\ c s_{ij} \text{ if } i \neq j. \end{cases}
\end{aligned} \tag{2.14}$$

Here $j, l \in \{1, \dots, n\}$, and s_{ij} denote the transpositions $(i, j) \in \mathfrak{S}_n$.

The algebra $\mathcal{H}_n$ admits a faithful polynomial representation on the space $\mathbb{C}[x_1, \dots, x_n]$. The elements s_k act by swapping x_k and x_{k+1}, the elements x_i act by multiplication, and y_i act as the rational Dunkl operators

$$\nabla_i = \partial_{x_i} - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{c}{x_i - x_j}(1 - s_{ij}). \tag{2.15}$$

The RCA $\mathcal{H}_n$ has a PBW-type basis formed by $w x_1^{m_1} \cdots x_n^{m_n} y_1^{p_1} \cdots y_n^{p_n}$ for $w \in \mathfrak{S}_n$, and $m_i, p_i \in \mathbb{Z}_{\ge 0}$.

For the A_{n-1}-type root system R, there exists a variation on the SL_n-type TCA $\mathbb{H}_c^{\mathrm{trig}}(R)$ where one takes $V = \mathbb{C}^n$ and replaces the weight lattice P with the lattice $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i \subset V$. This case is referred to as type GL_n. Abstractly, we let the TCA $\mathbb{H}_n^{\mathrm{trig}} = \mathbb{H}_{n,c}^{\mathrm{trig}}$ of type GL_n be the (unital, associative) algebra over $\mathbb{C}$ generated by $s_k \in \mathfrak{S}_n$ ($1 \leq k \leq n-1$), and elements $X_i^{\pm 1}$, $\widehat{y}_i$ ($1 \leq i \leq n$) subject to the following relations:

$$\begin{aligned}
&[\widehat{y}_i, \widehat{y}_j] = 0 = [X_i, X_j], && X_i X_i^{-1} = X_i^{-1} X_i = 1,\\
&s_k X_k s_k = X_{k+1}, && [s_k, X_i] = 0 \text{ for } i \neq k, k+1,\\
&s_k \widehat{y}_{k+1} - \widehat{y}_k s_k = c, && [s_k, \widehat{y}_i] = 0 \text{ for } i \neq k, k+1,\\
&(\widehat{y}_1 + \cdots + \widehat{y}_n) X_i = X_i (1 + \widehat{y}_1 + \cdots + \widehat{y}_n), &&\\
&\widehat{y}_2 - X_1 \widehat{y}_2 X_1^{-1} = c s_1, &&
\end{aligned}$$

where $1 \leq j \leq n$. Slightly different but equivalent sets of generators and relations are used in [2] (see also [98, Section 3]).

The above defining relations of $\mathbb{H}_n^{\mathrm{trig}}$ can be obtained by taking the relations of the DAHA $\mathbb{H}_n$ of type GL_n from Section 2.4 and performing a trigonometric degeneration by putting

$$Y_i = e^{\hbar \widehat{y}_i}, \quad q = e^{\hbar}, \quad \tau = e^{-\hbar c/2}, \quad T_k = s_k e^{-\hbar c s_k/2} \tag{2.16}$$

and taking a limit $\hbar \to 0$ (in other words $q \to 1$). More precisely, we consider $\mathbb{H}_n$ as an algebra over $\mathbb{C}[[\hbar]]$ via the assignments (2.16), and we put $\mathbb{H}_n^{\mathrm{trig}} = \mathbb{H}_n / \hbar \mathbb{H}_n$ Similarly, the relations of the RCA $\mathcal{H}_n$ given above can be obtained from those of the TCA $\mathbb{H}_n^{\mathrm{trig}}$ through a rational degeneration by putting $X_i = e^{\hbar x_i}$ and $y_i = \hbar \widehat{y}_i$, expanding the relations of $\mathbb{H}_n^{\mathrm{trig}}$ around $\hbar = 0$, and taking the first non-trivial relations that this imposes for s_k, x_i, and y_i.

The algebra $\mathbb{H}_n^{\mathrm{trig}}$ admits a faithful polynomial representation on the space of Laurent polynomials $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. The elements s_k act by swapping X_k and X_{k+1}, while $X_i^{\pm 1}$ act by multiplication, and $\widehat{y}_i$ act as the commuting trigonometric Dunkl operators

$$\nabla_i^{\mathrm{trig}} = X_i \partial_{X_i} - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{c}{1 - X_j X_i^{-1}} (1 - s_{ij}) - c \sum_{\substack{j=1\\ j>i}}^{n} s_{ij} = X_i \nabla_i^X - c \sum_{\substack{j=1\\ j>i}}^{n} s_{ij}, \tag{2.17}$$

where ∇_i^X is formally identical to the rational Dunkl operator ∇_i (2.15) from above *but* with x relabeled to X! To compare formula (2.17) to the formula for trigonometric Dunkl operators given in (2.13), one needs to put here $X_i = e^{x_{n-i+1}}$ (the reindexing is just due to a different convention in (2.13) for the choice of Dunkl operators), and then the operator (2.17) is equal (up to a constant) to the operator (2.13) with $R = A_{n-1}$ and $\xi = e_{n-i+1}$. Finally, note that in the rational degeneration, described in the preceding paragraph, where we put $X_i = e^{\hbar x_i}$, we have $\lim_{\hbar \to 0} \hbar \nabla_i^{\mathrm{trig}} = \nabla_i$, as expected.

The TCA $\mathbb{H}_n^{\mathrm{trig}}$ has a PBW-type basis formed by $w X_1^{m_1} \cdots X_n^{m_n} \widehat{y}_1{}^{p_1} \cdots \widehat{y}_n{}^{p_n}$ for $w \in \mathfrak{S}_n$, $m_i \in \mathbb{Z}$, and $p_i \in \mathbb{Z}_{\geq 0}$.

From the respective polynomial representations of the RCA $\mathcal{H}_n$ and the TCA $\mathbb{H}_n^{\mathrm{trig}}$ of type GL_n, and in particular formula (2.17) that relates ∇_i^{trig} and ∇_i^X, one sees that there is an algebra embedding of $\mathcal{H}_n$ into $\mathbb{H}_n^{\mathrm{trig}}$ given as follows [98, Proposition 4.1(i)]:

$$\begin{aligned} &s_k \mapsto s_k, \qquad x_i \mapsto X_i, \\ &y_i \mapsto X_i^{-1}\left(\widehat{y}_i + c\sum_{\substack{j=1\\ j>i}}^{n} s_{ij}\right) = X_i^{-1}(s_i s_{i+1}\cdots s_{n-1})\widehat{y}_n(s_{n-1}\cdots s_{i+1}s_i). \end{aligned} \tag{2.18}$$

The relationship between $\widehat{y}_i$ and $\widehat{y}_n$ follows from the fact that $Y_i = T_i \cdots T_{n-1} Y_n T_{n-1} \cdots T_i$ in the DAHA $\mathbb{H}_n$.

On the other hand, the degenerate AHA of type GL_n, which is the subalgebra of $\mathbb{H}_n^{\mathrm{trig}}$ generated by $\mathfrak{S}_n$ and $\mathbb{C}[\widehat{y}_1, \ldots, \widehat{y}_n]$, embeds into $\mathcal{H}_n$ via $s_k \mapsto s_k$, $\widehat{y}_i \mapsto x_i y_i - c\sum_{j>i} s_{ij}$ [45, Proposition 4.3(ii)] as follows from relation (2.17).

The algebras $\mathcal{H}_n$ and $\mathbb{H}_n^{\mathrm{trig}}$ both admit a grading, which will be discussed in Section 6.1 and where we will see that the form of the embedding (2.18) (in particular, the presence of X_i^{-1} in the image of y_i) is natural in view of these gradings.

The following diagram summarises the relationships between the (finite-type, affine, and double affine) Hecke algebras and their degenerations, in type GL_n, introduced in this chapter:

$$\mathscr{H}_{n,\tau} \subset \widehat{\mathscr{H}_{n,\tau}} \subset \mathbb{H}_{n,q,\tau} \xrightarrow{q\to 1} \mathbb{H}_{n,c}^{\mathrm{trig}} \longleftarrow \mathcal{H}_{n,c}$$

The following table summarises the notations for the generators (and their images in the respective polynomial representations, if denoted differently) of the DAHA, RCA, and TCA of type GL_n.

Algebra	Generators
$\mathbb{H}_{n,q,\tau}$	$T_k \;\; (1 \le k \le n-1), \quad X_i^{\pm 1}, \quad Y_i^{\pm 1} \;\; (1 \le i \le n)$
$\mathbb{H}_{n,c}^{\mathrm{trig}}$	$s_k \;\; (1 \le k \le n-1), \quad X_i^{\pm 1}, \quad \widehat{y}_i$ (or ∇_i^{trig}) $\;(1 \le i \le n)$
$\mathcal{H}_{n,c}$	$s_k \;\; (1 \le k \le n-1), \quad x_i, \quad y_i$ (or ∇_i) $\;(1 \le i \le n)$

Table 2.1: Notations for generators of DAHA, TCA, and RCA of type GL_n.

Chapter 3

Bispectrality of generalised Calogero–Moser–Sutherland systems

In this chapter, we extend the axiomatic definition of multidimensional BA functions from [22, 24, 28, 29, 105] to the case of configurations where arbitrary collinear vectors are allowed as long as all subsets of collinear vectors are of the form $\{\alpha, 2\alpha\}$, and we construct such a function for (the positive halves of) the configurations AG_2 and $BC(l, 1)$. This leads to bispectral duality statements for the corresponding generalised CMS quantum Hamiltonians associated with AG_2 and $BC(l, 1)$, respectively. In the case of AG_2, we present two corresponding dual difference operators of rational MR type in an explicit form. In the case of $BC(l, 1)$, we use a difference operator defined by Sergeev and Veselov. This chapter is based on our papers [55, 56, 83] (the initial parts of [56] were done in [106]).

The structure of this chapter is as follows. In Section 3.1, we recall the definition of generalised CMS operators associated with finite collections of vectors with prescribed multiplicities. In Section 3.2, we discuss their BA functions. In Section 3.3, we give an ansatz for a dual MR-type difference operator with rational coefficients of a particular form, and we find sufficient conditions for it to preserve a space of quasi-invariant analytic functions. We apply this machinery in Section 3.4, where we present an account of our construction from [56] of the BA function for the configuration AG_2 and the proof that it leads to a bispectral duality for the generalised CMS operator associated with AG_2. We studied the integrability of this operator in [55], and Section 3.4 includes a summary of the results of that paper. In Section 3.5, we give a construction of the BA function for the configuration $BC(l, 1)$ and prove a bispectral duality for the generalised CMS system of type $BC(l, 1)$, following our paper [83]. By an analytic continuation argument, we generalise in Section 3.6 the above two eigenfunctions from the case of non-negative integer values of the multiplicity parameters to more general complex values. In the case of AG_2, this further extends the investigations done in [56].

3.1 Generalised CMS Hamiltonians

Consider a finite set $\mathcal{A} \subset \mathbb{C}^n \setminus \{0\}$ and a multiplicity function $c\colon \mathcal{A} \to \mathbb{C}$ that assigns to each vector $\alpha \in \mathcal{A}$ a complex number c_α called its multiplicity. In most parts of this chapter, we will be specifically interested in the case where the multiplicities are contained in $\mathbb{Z}_{\geq 0}$.

The generalised CMS Hamiltonian associated with the collection of vectors $\mathcal{A}$ with prescribed multiplicities is the Schrödinger operator of the form

$$L = -\Delta + \sum_{\alpha \in \mathcal{A}} \frac{c_\alpha(c_\alpha + 2c_{2\alpha} + 1)\langle \alpha, \alpha \rangle}{\sinh^2 \langle \alpha, x \rangle}, \tag{3.1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ $(\partial_{x_i} = \partial/\partial x_i)$ is the Laplace operator on $\mathbb{C}^n$, and $c_{2\alpha} \coloneqq 0$ if $2\alpha \notin \mathcal{A}$. The convention of writing the couplings in the operator (3.1) as $c_\alpha(c_\alpha + 2c_{2\alpha} + 1)\langle \alpha, \alpha \rangle$ comes from the theory of symmetric spaces (see, e.g., [94]).

The above generalised CMS system is said to be of trigonometric (or, equivalently, hyperbolic) type. The rational version of these operators is obtained by replacing the function $\sinh\langle \alpha, x \rangle$ in formula (3.1) by $\langle \alpha, x \rangle$. It can be obtained as a limit of the operator (3.1) by introducing a scaling parameter $\omega > 0$, replacing each $\alpha \in \mathcal{A}$ by $\omega\alpha$, with the same multiplicity, and taking the limit $\omega \to 0$. There exists also an elliptic generalisation of CMS systems, which is the most general case, where the potential is expressed in terms of the Weierstrass $\wp$ function on an elliptic curve, but we will not deal with the elliptic case in this thesis. By CMS operators, we will mean the trigonometric type unless specified otherwise.

3.2 Baker–Akhiezer functions

In this section, we consider the axiomatic definition of BA functions proposed in [56]. We formulated the definition in such a way that it naturally extends the case of reduced configurations [28, 105], as well as the case of the root systems BC_n covered in [22, 24, 29]. We show that if a function satisfying this definition exists for some configuration of vectors, then it is an eigenfunction for the corresponding generalised CMS operator (3.1).

Let $R \subset \mathbb{C}^n$ be a finite collection of non-isotropic vectors. We assume there is a subset $R_+ \subset R$ such that any collinear vectors in R_+ are of the form α, 2α, and $R = R_+ \coprod (-R_+)$. Let $R^r = \{\alpha \in R\colon \frac{1}{2}\alpha \notin R\}$ and $R^r_+ = R^r \cap R_+$. Let $c\colon R \to \mathbb{Z}_{\geq 0}$ be a multiplicity map, and extend it to $c\colon R \cup 2R \to \mathbb{Z}_{\geq 0}$ by putting $c_{2\alpha} = 0$ if $2\alpha \notin R$ for some $\alpha \in R$. Without loss of generality, let us assume in this section that $c(R^r) \subset \mathbb{Z}_{>0}$. For any $\alpha \in R^r_+$, we define the set

$$A_\alpha \coloneqq \{1, 2, \dots, c_\alpha\} \cup \{c_\alpha + 2, c_\alpha + 4, \dots, c_\alpha + 2c_{2\alpha}\}.$$

Definition 3.1. We call a function $\psi(z,x)$ $(z,x\in\mathbb{C}^n)$ a BA function for R if

1. $\psi(z,x)=P(z,x)e^{\langle z,x\rangle}$ for a polynomial P in z with highest-order term $\prod_{\alpha\in R_+}\langle\alpha,z\rangle^{c_\alpha}$;
2. $\psi(z+s\alpha,x)=\psi(z-s\alpha,x)$ at $\langle z,\alpha\rangle=0$ for all $s\in A_\alpha$ and $\alpha\in R_+^r$.

For $\alpha\in\mathbb{C}^n$, we denote by δ_α the operator that acts on functions $f(z,x)$ by

$$\delta_\alpha f(z,x)=f(z+\alpha,x)-f(z-\alpha,x).$$

The second condition in Definition 3.1 admits the following equivalent characterisation.

Lemma 3.2. *Let $\alpha\in\mathbb{C}^n$ be non-isotropic, $c_\alpha\in\mathbb{Z}_{>0}$, and $c_{2\alpha}\in\mathbb{Z}_{\geq 0}$. A function $\psi(z,x)$ $(z,x\in\mathbb{C}^n)$ analytic in z satisfies $\psi(z+s\alpha,x)=\psi(z-s\alpha,x)$ at $\langle z,\alpha\rangle=0$ for all $s\in A_\alpha$ if and only if*

$$\left(\delta_\alpha\circ\frac{1}{\langle z,\alpha\rangle}\right)^{s-1}\delta_\alpha\psi(z,x)=0 \text{ at } \langle z,\alpha\rangle=0,\ s=1,\dots,c_\alpha, \tag{3.2}$$

and

$$\left(\delta_{2\alpha}\circ\frac{1}{\langle z,\alpha\rangle}\right)^{t}\circ\left(\delta_\alpha\circ\frac{1}{\langle z,\alpha\rangle}\right)^{c_\alpha-1}\delta_\alpha\psi(z,x)=0 \text{ at } \langle z,\alpha\rangle=0,\ t=1,\dots,c_{2\alpha}. \tag{3.3}$$

The proof of the preceding lemma follows from the one-dimensional statement in Lemma 3.3 below, whose proof can be found in our paper [56] (see also [105], where the corresponding statement in the case $c_{2\alpha}=0$ was stated). Let δ_r $(r\in\mathbb{Z}_{>0})$ be the difference operator that acts on functions $F(k)$ $(k\in\mathbb{C})$ by $\delta_rF(k)=F(k+r)-F(k-r)$. Write $\delta=\delta_1$ for short.

Lemma 3.3. *The following two properties are equivalent for any analytic function $F(k)$ $(k\in\mathbb{C})$ and any $n\in\mathbb{Z}_{>0}$, $m\in\mathbb{Z}_{\geq 0}$.*

1. *For all $s=1,\dots,n$,*
$$\left(\delta\circ\frac{1}{k}\right)^{s-1}\delta F(k)\bigg|_{k=0}=0,$$
and for all $t=1,\dots,m$,
$$\left(\delta_2\circ\frac{1}{k}\right)^{t}\circ\left(\delta\circ\frac{1}{k}\right)^{n-1}\delta F(k)\bigg|_{k=0}=0.$$
2. *$F(s)=F(-s)$ for all $s\in\{1,2,\dots,n\}\cup\{n+2,n+4,\dots,n+2m\}$.*

The next lemma is a generalisation of [49, Lemma 1] (see also [105, Proposition 1]), and its proof can be found in our paper [56]. This lemma is used to prove the uniqueness of the BA function when such a function exists.

Lemma 3.4. *Let $\psi(z,x) = P(z,x)e^{\langle z,x\rangle}$ $(z, x \in \mathbb{C}^n)$, where $P(z,x)$ is a polynomial in z. Suppose that ψ satisfies conditions* (3.2) *and* (3.3) *for some non-zero $\alpha \in \mathbb{C}^n$, $c_\alpha \in \mathbb{Z}_{>0}$, $c_{2\alpha} \in \mathbb{Z}_{\geq 0}$. Then $\langle \alpha, z\rangle^{c_\alpha + c_{2\alpha}}$ divides the highest-order term $P_0(z,x)$ of $P(z,x)$.*

Lemma 3.4 has the following consequence.

Lemma 3.5. *Let $\psi(z,x) = P(z,x)e^{\langle z,x\rangle}$ $(z, x \in \mathbb{C}^n)$ satisfy condition 2 in Definition 3.1, where $P(z,x)$ is a polynomial in z with highest-order term $P_0(z,x)$. Then $\prod_{\alpha\in R_+}\langle \alpha, z\rangle^{c_\alpha}$ divides $P_0(z,x)$.*

Indeed, Lemma 3.4 gives that $P_0(z,x)$ is divisible by $\langle z,\alpha\rangle^{c_\alpha+c_{2\alpha}}$ for all $\alpha \in R_+^r$. This is a constant multiple of $\langle z,\alpha\rangle^{c_\alpha}\langle z, 2\alpha\rangle^{c_{2\alpha}}$. The statement of Lemma 3.5 follows, since we are assuming that collinear vectors in R_+ are only of the form α, 2α.

Lemma 3.5 leads to the following uniqueness statement analogous to [105, Proposition 1] (cf. also [49, Proposition 1]) with an analogous proof.

Proposition 3.6. *[56] If a BA function satisfying Definition 3.1 exists, then it is unique.*

The next theorem generalises [105, Theorem 1] to the present context, and it is proved analogously to how that result is proved. It states that if the BA function satisfying Definition 3.1 exists, then it is a joint eigenfunction of a commutative ring of differential operators in the variables x. Let us first define an isomorphic ring of polynomials.

Let $\mathcal{R}$ be the ring of polynomials $p(z) \in \mathbb{C}[z_1,\dots,z_n]$ satisfying

$$p(z+s\alpha) = p(z-s\alpha) \text{ at } \langle z,\alpha\rangle = 0$$

for all $s \in A_\alpha$ and $\alpha \in R_+^r$ (notice the similarity with condition 2 in Definition 3.1). We have $z^2 = \langle z,z\rangle \in \mathcal{R}$. Indeed, for any $\gamma \in \mathbb{C}^n$ and $s \in \mathbb{Z}_{>0}$, we have $(z \pm s\gamma)^2 = z^2 \pm 2s\langle z,\gamma\rangle + s^2\gamma^2 = z^2 + s^2\gamma^2$ at $\langle z,\gamma\rangle = 0$. For any vector γ, we write γ^2 for $\langle \gamma,\gamma\rangle$ whenever convenient to save space.

For a polynomial $p(z) = p(z_1,\dots,z_n)$, by $p(\partial_x)$ we will mean $p(\partial_{x_1},\dots,\partial_{x_n})$. For example, if $p(z) = z^2 = z_1^2 + \dots + z_n^2$, then $p(\partial_x) = \Delta$ is the Laplacian.

The following statement takes place.

Theorem 3.7. *[56] If the BA function $\psi(z,x)$ satisfying Definition 3.1 exists, then for any $p(z) \in \mathcal{R}$ there is a differential operator $L_p(x,\partial_x)$ with highest-order term $p(\partial_x)$ such that*

$$L_p(x,\partial_x)\psi(z,x) = p(z)\psi(z,x).$$

For any $p, q \in \mathcal{R}$, the operators L_p and L_q commute.

The next lemma is used in the proof that the differential operator L_{z^2} from Theorem 3.7 corresponding to the polynomial $z^2 \in \mathcal{R}$ coincides with the generalised CMS

Hamiltonian (3.1) associated with the configuration $\mathcal{A} = R_+$. This lemma is a generalisation of [49, Lemma 2] (see also [105]), and its proof can be found in our paper [56].

Lemma 3.8. *Suppose $\psi(z,x) = P(z,x)e^{\langle z,x\rangle}$ satisfies Definition 3.1. Let $N = \sum_{\alpha\in R_+} c_\alpha$. Write $P(z,x) = \sum_{i=0}^N P_i(z,x)$ where $P_0(z,x) = \prod_{\alpha\in R_+}\langle \alpha, z\rangle^{c_\alpha}$ and P_i are polynomials homogeneous in z with $\deg P_i = N - i$. Then*

$$\frac{P_1(z,x)}{P_0(z,x)} = -\sum_{\alpha\in R_+} \frac{c_\alpha(c_\alpha + 2c_{2\alpha} + 1)\langle\alpha,\alpha\rangle}{2\langle\alpha, z\rangle}\coth\langle\alpha, x\rangle. \tag{3.4}$$

The following proposition has a completely analogous proof to [49, Proposition 2], it just uses Lemma 3.8 in place of [49, Lemma 2] (see also [105]).

Proposition 3.9. *[56] With notations and assumptions as in Theorem 3.7, the polynomial $p(z) = z^2 \in \mathcal{R}$ corresponds to the differential operator*

$$L_{z^2} = \Delta - \sum_{\alpha\in R_+} \frac{c_\alpha(c_\alpha + 2c_{2\alpha} + 1)\langle\alpha,\alpha\rangle}{\sinh^2\langle\alpha, x\rangle},$$

which coincides (up to sign) with the generalised CMS operator (3.1) *for $\mathcal{A} = R_+$.*

This then implies quantum integrability of the Hamiltonian L_{z^2}, as it provides a quantum integral L_p with highest term $p_0(\partial_x)$ for each $p(z) \in \mathcal{R}$, where p_0 is the highest homogeneous term of p, and $\mathcal{R}$ contains $Q(z)\prod_{\alpha\in R_+^r,\ s\in A_\alpha}(\langle\alpha, z\rangle^2 - s^2\langle\alpha,\alpha\rangle^2)$ for any polynomial $Q(z)$. Taking $Q(z)$ to be z_i for $i = 1, \dots, n$ gives n algebraically independent operators.

3.3 Ansatz for dual difference operators

In Section 3.4, we use a method for explicit construction of BA functions that was proposed by Chalykh [24] (see also [49] for further examples where this method is applied, and [26] for the differential case). The construction uses certain difference operators of rational MR type. The key element of the method is the preservation of a space of quasi-invariant analytic functions under the action of the difference operators. In this section, we find sufficient conditions for a (for our purposes) sufficiently general invariant difference operator to preserve such a ring of quasi-invariants.

We continue to use the notations from Section 3.2. Let $W = \langle s_\alpha \colon \alpha \in R\rangle$, where s_α is the orthogonal reflection about the hyperplane $\langle\alpha, x\rangle = 0$. We assume now that the collection R is W-invariant, that is, $w(R) = R$ for all $w \in W$, and that the multiplicity map is W-invariant, too. Let $u^\vee = 2u/\langle u, u\rangle$ for any $u \in \mathbb{C}^n$ such that $\langle u, u\rangle \neq 0$.

Let $\mathcal{R}^a$ be the ring of analytic functions $p(z)$ such that

$$p(z + t\alpha) = p(z - t\alpha) \text{ at } \langle \alpha, z\rangle = 0 \text{ for } t \in A_\alpha$$

for all $\alpha \in R^r_+$, where $A_\alpha \subset \mathbb{Z}_{>0}$ specifies the axiomatics that one wants to consider. For instance, it can be $A_\alpha = \{1, 2, \dots, c_\alpha\} \cup \{c_\alpha + 2, c_\alpha + 4, \dots, c_\alpha + 2c_{2\alpha}\}$ as before. We assume that $A_{|w\alpha|} = A_\alpha$ for all $w \in W$, $\alpha \in R^r_+$, where $|w\alpha| := w\alpha$ if $w\alpha \in R^r_+$ and $|w\alpha| := -w\alpha$ if $w\alpha \in (-R^r_+)$. For $\alpha \in R$, we let $\operatorname{sgn} \alpha = 1$ if $\alpha \in R_+$, and we let $\operatorname{sgn} \alpha = -1$ if $\alpha \in (-R_+)$.

Let $S \subset \mathbb{C}^n \setminus \{0\}$ be a W-invariant finite collection of vectors. Let $z \in \mathbb{C}^n$, and for any $\alpha \in \mathbb{C}^n$, let T_α be the (additive) shift operator that acts on functions $f(z)$ by $T_\alpha f(z) = f(z + \alpha)$. We are interested in difference operators D of the form

$$D = \sum_{\tau \in S} a_\tau(z)(T_\tau - 1), \tag{3.5}$$

where a_τ are rational functions with the following three properties:

(D_1) $\deg a_\tau = 0$.

(D_2) $a_\tau(z)$ has a simple pole at $\langle \alpha, z\rangle = c\alpha^2$ for some $\alpha \in R^r_+$ and $c \in \mathbb{C}$ if and only if $\lambda = s_\alpha(\tau) - 2c\alpha \in S \cup \{0\}$ and

$$\langle \tau + c\alpha, \alpha\rangle/\alpha^2 = c + \langle \tau, (2\alpha)^\vee\rangle \in A_\alpha \cup (-A_\alpha).$$

There are no other singularities in a_τ. Denote the set of all such pairs (α, c) for this τ by S_τ.

(D_3) $wa_\tau = a_{w\tau}$ for all $w \in W$.

Condition (D_2) implies that if a_τ has a singularity $\langle \alpha, z\rangle = c\alpha^2$, then for any such z the vectors $z + \tau$ and $z + \lambda$ are of the form $z + \tau = \widetilde{z} + t\alpha$ and $z + \lambda = \widetilde{z} - t\alpha$ for some $\widetilde{z}$ with $\langle \alpha, \widetilde{z}\rangle = 0$ and $t = c + \langle \tau, (2\alpha^\vee)\rangle \in A_\alpha \cup (-A_\alpha)$. We note also that $\lambda \neq \tau$ since $0 \notin A_\alpha$.

Note that if a_τ has a singularity $\langle \alpha, z\rangle = c\alpha^2$ and the corresponding $\lambda \neq 0$, then condition (D_2) implies that $a_\lambda(z)$ necessarily also has a singularity at $\langle \alpha, z\rangle = c\alpha^2$, since $s_\alpha(\lambda) - 2c\alpha = \tau \in S$ and $\langle \lambda + c\alpha, \alpha\rangle = \langle s_\alpha(\tau + c\alpha), \alpha\rangle = -\langle \tau + c\alpha, \alpha\rangle$. In other words, by condition (D_2), we have $(\alpha, c) \in S_\tau$ if and only if $(\alpha, c) \in S_\lambda$ for $\lambda = s_\alpha(\tau) - 2c\alpha$ provided that both $\tau, \lambda \neq 0$. We additionally observe the following.

Lemma 3.10. *For any $w \in W$, $(\alpha, c) \in S_\tau$ if and only if $(|w\alpha|, \operatorname{sgn}(w\alpha)c) \in S_{w\tau}$.*

Proof. Let $\varepsilon = \operatorname{sgn}(w\alpha)$. Since $s_{|w\alpha|} = ws_\alpha w^{-1}$ and $\varepsilon|w\alpha| = w\alpha$, we get that $s_{|w\alpha|}(w\tau) - 2\varepsilon c|w\alpha| = w(s_\alpha(\tau) - 2c\alpha)$ belongs to $S \cup \{0\}$ if and only if $s_\alpha(\tau) - 2c\alpha \in S \cup \{0\}$, due to W-invariance of S. Furthermore, $\langle w\tau + \varepsilon c|w\alpha|, |w\alpha|\rangle/|w\alpha|^2 = \pm\langle \tau + c\alpha, \alpha\rangle/\alpha^2$, and $A_\alpha = A_{|w\alpha|}$, by assumption. The statement follows. ∎

More explicitly, we are looking at operators of the form

$$D = \sum_{\tau \in S} P_\tau(z) \left(\prod_{(\alpha,c) \in S_\tau} \left(\langle \alpha, z \rangle - c\alpha^2\right)^{-1} \right) (T_\tau - 1)$$

for some polynomials $P_\tau(z)$ of degree $|S_\tau|$, ensuring that $\deg a_\tau = 0$, and such that condition (D_3) holds. We want to find some sufficient conditions that would ensure that D preserves the ring $\mathcal{R}^a$.

Theorem 3.11. *Suppose the operator* (3.5) *satisfies conditions* (D_2) *and* (D_3)*. Then for any* $\alpha \in R^r_+$ *and arbitrary* $p(z) \in \mathcal{R}^a$*, we have the following two properties.*

1. *$Dp(z)$ is non-singular at $\langle \alpha, z \rangle = 0$. Moreover, for any $c \neq 0$, provided that for all $\tau \in S$ such that $(\alpha, c) \in S_\tau$ and such that $\lambda = s_\alpha(\tau) - 2c\alpha \neq 0$ we have*

$$\operatorname{res}_{\langle \alpha, z \rangle = c\alpha^2}(a_\tau + a_\lambda) = 0,$$

then $Dp(z)$ is non-singular at $\langle \alpha, z \rangle = c\alpha^2$, where res *denotes residue.*

2. *Suppose, in addition to assumptions of part 1, that for all $\tau \in S$ and any $t \in A_\alpha$, the following is satisfied whenever $t + \langle \tau, (2\alpha)^\vee \rangle \notin A_\alpha \cup (-A_\alpha) \cup \{0\}$:*

   (a) $a_\tau(z + t\alpha) = 0$ at $\langle \alpha, z \rangle = 0$ (equivalently, $P_\tau(z)$ has a factor of $\langle \alpha, z \rangle - t\alpha^2$),

   or

   (b) letting $\lambda = s_\alpha(\tau) - 2t\alpha$, we have $\lambda \in S$ and $a_\lambda(z + t\alpha) = a_\tau(z + t\alpha)$ at $\langle \alpha, z \rangle = 0$.

   Then $Dp(z + t\alpha) = Dp(z - t\alpha)$ at $\langle \alpha, z \rangle = 0$ for all $t \in A_\alpha$.

Proof. 1. Let $c \in \mathbb{C}$. We want to show that the residue at $\langle \alpha, z \rangle = c\alpha^2$ of $Dp(z)$ is zero. Take any $\tau \in S$ such that $(\alpha, c) \in S_\tau$. Write $\tau + c\alpha = t\alpha + \gamma$, where $\langle \gamma, \alpha \rangle = 0$ and $t = \langle \tau + c\alpha, \alpha \rangle / \alpha^2$. Let $\lambda = s_\alpha(\tau) - 2c\alpha$. Then $\lambda + c\alpha = s_\alpha(\tau + c\alpha) = -t\alpha + \gamma$. At $\langle \alpha, z \rangle = c\alpha^2$, we have

$$p(z + \tau) = p((z - c\alpha + \gamma) + t\alpha) = p((z - c\alpha + \gamma) - t\alpha) = p(z + \lambda)$$

since $p(z) \in \mathcal{R}^a$, $\langle z - c\alpha + \gamma, \alpha \rangle = 0$, and $t \in A_\alpha \cup (-A_\alpha)$ by assumption (D_2). So, if $\lambda = 0$ then the simple pole at $\langle \alpha, z \rangle = c\alpha^2$ present in $a_\tau(z)$ is cancelled by $(T_\tau - 1)[p(z)] = p(z + \tau) - p(z)$. And if $\lambda \neq 0$, then the sum

$$a_\tau(z)(p(z + \tau) - p(z)) + a_\lambda(z)(p(z + \lambda) - p(z))$$

contributes zero to the residue provided that the residue of $a_\tau + a_\lambda$ is zero. For $c \neq 0$, the latter is satisfied by assumption. In the case of $c = 0$, we have $\lambda = s_\alpha(\tau)$, hence

$a_\tau(s_\alpha(z)) = a_\lambda(z)$ by the symmetry (D_3) of the operator, thus we get

$$\lim_{\langle\alpha,z\rangle\to 0}\langle\alpha,z\rangle a_\tau(z) = \lim_{\langle\alpha,z\rangle\to 0}\langle\alpha,s_\alpha(z)\rangle a_\tau(s_\alpha(z)) = -\lim_{\langle\alpha,z\rangle\to 0}\langle\alpha,z\rangle a_\lambda(z);$$

that is, the residue of $a_\tau(z)$ at $\langle\alpha,z\rangle = 0$ is minus that of $a_\lambda(z)$, as needed.

2. Fix $t \in A_\alpha$. By the symmetry (D_3) of the operator, we have that $a_\mu(z+s\alpha) = a_{s_\alpha(\mu)}(z-s\alpha)$ for all generic $s \in \mathbb{C}$ and generic $z \in \mathbb{C}^n$ with $\langle\alpha,z\rangle = 0$, $\mu \in S$. By using that $s_\alpha(S) = S$, we can thus write $Dp(z+t\alpha) - Dp(z-t\alpha)$ at $\langle\alpha,z\rangle = 0$ as

$$\lim_{s\to t}\sum_{\mu\in S} a_\mu(z+s\alpha)\Big(p(z+s\alpha+\mu) - p(z-s\alpha+s_\alpha(\mu)) - p(z+s\alpha) + p(z-s\alpha)\Big). \quad (3.6)$$

Firstly, let us consider any $\tau \in S$ for which $a_\tau(z+t\alpha)$ is non-singular at $\langle\alpha,z\rangle = 0$ (for generic z). Then the corresponding $\mu = \tau$ term in the sum (3.6) can be simplified to

$$a_\tau(z+t\alpha)\left(p(z+t\alpha+\tau) - p(z-t\alpha+s_\alpha(\tau))\right) \quad (3.7)$$

by using that $p(z) \in \mathcal{R}^a$. Let $\tau = b\alpha + \delta$, where $\langle\delta,\alpha\rangle = 0$ and $b = \langle\tau,(2\alpha)^\vee\rangle$. Then $s_\alpha(\tau) = -b\alpha + \delta$, and thus

$$p(z+t\alpha+\tau) - p(z-t\alpha+s_\alpha(\tau)) = p(z+\delta+(t+b)\alpha) - p(z+\delta-(t+b)\alpha), \quad (3.8)$$

where $\langle\alpha,z+\delta\rangle = 0$. Hence, if $t+b \in A_\alpha \cup (-A_\alpha) \cup \{0\}$, then (3.8) equals zero, and the whole term (3.7) vanishes. Else, we have by assumption two possibilities (cases (a) and (b)). If $a_\tau(z+t\alpha) = 0$ at $\langle\alpha,z\rangle = 0$, then (3.7) vanishes; and if $a_\tau(z+t\alpha) \neq 0$, then case (b) must apply, and so $\lambda \coloneqq s_\alpha(\tau) - 2t\alpha \in S \setminus \{\tau\}$ and $a_\lambda(z+t\alpha) = a_\tau(z+t\alpha)$. (Note that the fact that $t+b \notin A_\alpha \cup (-A_\alpha) \cup \{0\}$ implies that $\lambda \neq \tau$, and due to (D_2) also that $a_\lambda(z+t\alpha)$ is well-defined at $\langle\alpha,z\rangle = 0$ for generic z). In the latter case, the term corresponding to $\mu = \lambda$ in the sum (3.6) can be simplified to

$$\begin{aligned} &a_\lambda(z+t\alpha)\left(p(z+t\alpha+\lambda) - p(z-t\alpha+s_\alpha(\lambda))\right) \\ &\qquad = a_\tau(z+t\alpha)\left(p(z-t\alpha+s_\alpha(\tau)) - p(z+t\alpha+\tau)\right), \end{aligned}$$

which is the negative of (3.7), hence the terms corresponding to $\mu = \tau$ and $\mu = \lambda$ in (3.6) cancel out.

Secondly, let us consider any $\tau \in S$ for which $a_\tau(z+t\alpha)$ is singular at $\langle\alpha,z\rangle = 0$. Equivalently, $a_\tau(\widetilde{z})$ is singular at $\langle\alpha,\widetilde{z}\rangle = t\alpha^2$. Hence $(\alpha,t) \in S_\tau$ by assumption (D_2), in particular, $\lambda \in S \cup \{0\}$ and $t+b \in A_\alpha \cup (-A_\alpha)$. From the latter, it follows that the expression (3.8) vanishes. We can restate this as

$$p(z + s\alpha + \tau) - p(z - s\alpha + s_\alpha(\tau)) = (s - t)q(s)$$

for some analytic function $q(s)$ ($s \in \mathbb{C}$). Similarly, the property $p(z + t\alpha) = p(z - t\alpha)$ at $\langle \alpha, z\rangle = 0$ can be restated as

$$p(z + s\alpha) - p(z - s\alpha) = (s - t)r(s) \tag{3.9}$$

for some analytic function $r(s)$. Moreover, we also have

$$p(z+s\alpha+\lambda)-p(z-s\alpha+s_\alpha(\lambda)) = p(z-(2t-s)\alpha+s_\alpha(\tau))-p(z+(2t-s)\alpha+\tau) = (s-t)q(2t-s). \tag{3.10}$$

Suppose firstly that $\lambda \neq 0$. Then in the sum (3.6), the two terms corresponding to $\mu = \tau$ and $\mu = \lambda$ cancel out. Indeed, they equal

$$\lim_{s\to t} \Big(a_\tau(z + s\alpha)(s - t)(q(s) - r(s)) + a_\lambda(z + s\alpha)(s - t)(q(2t - s) - r(s))\Big)$$
$$= (q(t) - r(t)) \operatorname{res}_{\langle z,\alpha\rangle = t\alpha^2}(a_\tau + a_\lambda) = 0$$

because $\operatorname{res}_{\langle z,\alpha\rangle=t\alpha^2}(a_\tau + a_\lambda) = 0$ by the assumptions of part 1 with $c = t$. Suppose now that $\lambda = 0$, then $r(s) = q(2t - s)$ by equalities (3.9) and (3.10). Therefore, the term corresponding to $\mu = \tau$ in the sum (3.6) is $\lim_{s\to t} a_\tau(z + s\alpha)(s - t)\,(q(s) - q(2t - s)) = 0$.

It follows that the sum (3.6) vanishes, as required. ■

It follows that if conditions (D_2), (D_3), and the assumptions of both parts 1 and 2 of Theorem 3.11 are satisfied for all $\alpha \in R^r_+$, then D preserves the ring $\mathcal{R}^a$, that is, $Dp(z) \in \mathcal{R}^a$ for any $p(z) \in \mathcal{R}^a$.

Additionally, we can use the symmetry assumption (D_3) to reduce the number of conditions that we have to check to apply Theorem 3.11. The following statements take place.

Lemma 3.12. *Suppose that condition (D_3) holds. If $a_\tau + a_\lambda$ has zero residue at $\langle \alpha, z\rangle = c\alpha^2$, then $a_{w\tau} + a_{w\lambda}$ has zero residue at $\langle w\alpha, z\rangle = c\alpha^2$ for any $w \in W$.*

Proof. By the property (D_3), we have $a_{w\tau}(z) + a_{w\lambda}(z) = a_\tau(w^{-1}z) + a_\lambda(w^{-1}z)$, therefore

$$\operatorname{res}_{\langle w\alpha,z\rangle=c\alpha^2}(a_{w\tau}(z) + a_{w\lambda}(z)) = \lim_{\langle w\alpha,z\rangle\to c\alpha^2} (\langle w\alpha, z\rangle - c\alpha^2)(a_{w\tau}(z) + a_{w\lambda}(z))$$
$$= \lim_{\langle \alpha,w^{-1}z\rangle\to c\alpha^2} (\langle \alpha, w^{-1}z\rangle - c\alpha^2)(a_\tau(w^{-1}z) + a_\lambda(w^{-1}z))$$
$$= \operatorname{res}_{\langle \alpha,\widetilde{z}\rangle=c\alpha^2}(a_\tau(\widetilde{z}) + a_\lambda(\widetilde{z})) = 0.$$

■

By combining Lemmas 3.10, 3.12, and Theorem 3.11, we obtain the following.

Corollary 3.13. *Suppose the operator* (3.5) *satisfies conditions (D_2) and (D_3). If the assumptions of part 1 of Theorem 3.11 are satisfied for some $\alpha \in R^r_+$, then $Dp(z)$ is non-singular at $\langle w\alpha, z\rangle = c\alpha^2$ for all $w \in W$ and all $c \in \mathbb{C}$.*

Proof. By Theorem 3.11 part 1, it suffices to check that for any $\widetilde{\tau} \in S$ and $c \neq 0$ such that $(|w\alpha|, \operatorname{sgn}(w\alpha)c) \in S_{\widetilde{\tau}}$ and such that $\widetilde{\lambda} = s_{|w\alpha|}(\widetilde{\tau}) - 2cw\alpha \neq 0$, we have that the residue of $a_{\widetilde{\tau}} + a_{\widetilde{\lambda}}$ at $\langle w\alpha, z\rangle = c\alpha^2$ is zero. Since S is W-invariant, we can write $\widetilde{\tau} = w\tau$ for some $\tau \in S$. Lemma 3.10 then gives $(\alpha, c) \in S_\tau$. Note that $\widetilde{\lambda} = w\lambda$ for $\lambda = s_\alpha(\tau) - 2c\alpha$ (in particular, $\lambda \neq 0$ as $\widetilde{\lambda} \neq 0$). By assumption, part 1 of Theorem 3.11 holds for this (α, c), that is, $\operatorname{res}_{\langle\alpha,z\rangle=c\alpha^2}(a_\tau + a_\lambda) = 0$. Lemma 3.12 now gives what we need. ∎

Lemma 3.14. *Suppose the operator* (3.5) *satisfies conditions (D_2) and (D_3). If the assumptions of parts 1 and 2 of Theorem 3.11 are satisfied for some $\alpha \in R^r_+$, then the assumptions of part 2 are also satisfied for $w\alpha$ for all $w \in W$ such that $w\alpha \in R^r_+$.*

Proof. Note that $A_{w\alpha} = A_\alpha$. Thus we need to prove that whenever for some $t \in A_\alpha$ and $\widetilde{\tau} \in S$ we have $t + \langle\widetilde{\tau}, (2w\alpha)^\vee\rangle \notin A_\alpha \cup (-A_\alpha) \cup \{0\}$, then either $a_{\widetilde{\tau}}(z + tw\alpha) = 0$ at $\langle w\alpha, z\rangle = 0$, or else $\widetilde{\lambda} := s_{w\alpha}(\widetilde{\tau}) - 2tw\alpha$ satisfies $\widetilde{\lambda} \in S$ and $a_{\widetilde{\lambda}}(z + tw\alpha) = a_{\widetilde{\tau}}(z + tw\alpha)$ at $\langle w\alpha, z\rangle = 0$.

Suppose that $t + \langle\widetilde{\tau}, (2w\alpha)^\vee\rangle \notin A_\alpha \cup (-A_\alpha) \cup \{0\}$. Since S is invariant, we can write $\widetilde{\tau} = w\tau$ for some $\tau \in S$. Note that then $\widetilde{\lambda} = w(s_\alpha(\tau) - 2t\alpha) = w\lambda$. Note also that $(2w\alpha)^\vee = w(2\alpha)^\vee$. Therefore $t + \langle\tau, (2\alpha)^\vee\rangle = t + \langle\widetilde{\tau}, (2w\alpha)^\vee\rangle \notin A_\alpha \cup (-A_\alpha) \cup \{0\}$. By assumption, part 2 of Theorem 3.11 holds for this α. Suppose firstly (case (a)) that $a_\tau(\widetilde{z} + t\alpha) = 0$ at $\langle\alpha, \widetilde{z}\rangle = 0$. By the symmetry (D_3), at $\langle w\alpha, z\rangle = 0$ (or, equivalently, $\langle\alpha, w^{-1}z\rangle = 0$) we thus get $a_{\widetilde{\tau}}(z + tw\alpha) = a_\tau(w^{-1}z + t\alpha) = 0$, as required. Otherwise (case (b)), we have $\lambda \in S$, hence $\widetilde{\lambda} = w\lambda \in S$ by invariance, and at $\langle w\alpha, z\rangle = 0$ we get $a_{\widetilde{\lambda}}(z + tw\alpha) - a_{\widetilde{\tau}}(z + tw\alpha) = a_\lambda(w^{-1}z + t\alpha) - a_\tau(w^{-1}z + t\alpha) = 0$, as required. ∎

Remark 3.15. Let $\alpha \in R^r_+$. Suppose $w \in W$ satisfies $w\alpha = \alpha$. Then, for any $\tau \in S$, in part 2 of Theorem 3.11 it suffices to check the stated conditions for either τ or $w\tau$, as one implies the other. Indeed, we have $t + \langle w\tau, (2\alpha)^\vee\rangle = t + \langle\tau, (2\alpha)^\vee\rangle$. Also, $s_\alpha(w\tau) - 2t\alpha = w\lambda$, and at $\langle\alpha, z\rangle = 0$, using the symmetry (D_3), we have $a_{w\tau}(z + t\alpha) = a_\tau(w^{-1}z + t\alpha)$ and, in case (b), $a_{w\lambda}(z + t\alpha) = a_\lambda(w^{-1}z + t\alpha)$, where $\langle\alpha, w^{-1}z\rangle = 0$.

3.4 Configuration AG_2

In this section, we consider the generalised CMS operator L associated with the planar configuration of vectors called AG_2, which is a union of the root systems A_2 and G_2. We explain how to prove the integrability of L by means of an intertwining relation between L

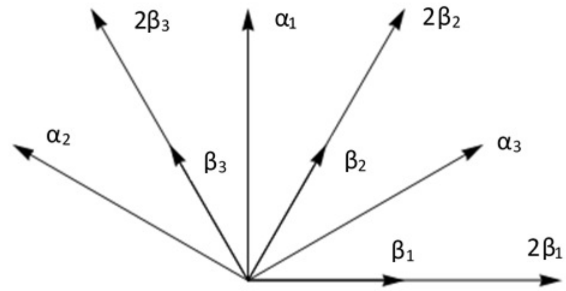


Figure 3.1: A positive half of the configuration AG_2.

and the CMS Hamiltonian for the root system G_2, giving an explicit quantum integral for L of order 6. We also establish the existence of a BA function for the operator L, give two constructions of this function, and show that L has a bispectrality property. We provide two corresponding dual difference operators of rational MR type in an explicit form.

The configuration AG_2 is a non-reduced collection of vectors in $\mathbb{R}^2$. A positive half $AG_{2,+} = G_{2,+} \cup A_{2,+}$ is shown in Figure 3.1, where $G_{2,+} = \{\alpha_i, \beta_i \colon i = 1,2,3\}$ and $A_{2,+} = \{2\beta_i \colon i = 1,2,3\}$. The indices of α_i's are assigned in such a way that $\langle \alpha_i, \beta_i \rangle = 0$ for all $i = 1,2,3$. The multiplicities assigned to the vectors α_i, β_i, and $2\beta_i$ are m, $3m$, and 1, respectively, where $m \in \mathbb{C}$ is a parameter.

We adopt a coordinate system where the vectors take the form

$$\begin{aligned} \alpha_1 = \omega\big(0, \sqrt{3}\big), \quad \alpha_2 = \omega\left(-\tfrac{3}{2}, \tfrac{\sqrt{3}}{2}\right), \quad \alpha_3 = \omega\left(\tfrac{3}{2}, \tfrac{\sqrt{3}}{2}\right), \\ \beta_1 = \omega\big(1, 0\big), \quad \beta_2 = \omega\left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right), \quad \beta_3 = \omega\left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right) \end{aligned} \tag{3.11}$$

for some scaling $\omega \in \mathbb{C}^\times$. These vectors satisfy $\beta_1 + \beta_3 = \beta_2$, $\alpha_2 + \alpha_3 = \alpha_1$,

$$\begin{aligned} &\beta_1 = 2\beta_2 - \alpha_1 = \alpha_1 - 2\beta_3 = \alpha_3 - \beta_2 = \beta_3 - \alpha_2, \\ &\alpha_1 = \tfrac{3}{2}\beta_2 + \tfrac{1}{2}\alpha_2 = \tfrac{3}{2}\beta_3 + \tfrac{1}{2}\alpha_3, \text{ and } \beta_1 = \tfrac{1}{2}\beta_2 - \tfrac{1}{2}\alpha_2 = -\tfrac{1}{2}\beta_3 + \tfrac{1}{2}\alpha_3. \end{aligned}$$

The configuration AG_2 is contained in the two-dimensional lattice $\mathbb{Z}\beta_1 \oplus \mathbb{Z}\alpha_2$. It is invariant under the Weyl group of type G_2, but it is not a crystallographic root system because, for example, the vectors β_1 and $2\beta_2$ have $2\langle \beta_1, 2\beta_2\rangle / \langle 2\beta_2, 2\beta_2\rangle = \frac{1}{2} \notin \mathbb{Z}$, so the crystallographic condition is not satisfied.

The structure of this section is as follows. We explain below in Section 3.4.1 that the corresponding generalised CMS quantum Hamiltonian (3.1) with $\mathcal{A} = AG_{2,+}$ is quantum integrable for any value of the parameter $m \in \mathbb{C}$, as we established in [55]. Moreover, by virtue of AG_2 being a locus configuration [48], the operator is algebraically integrable for $m \in \mathbb{Z}_{\geq 0}$, as follows from the general results presented in [16] (see also [55]). We then discuss the generalised CMS system for AG_2 in the special case of $m \in \mathbb{Z}_{\geq 0}$ further. Namely, we give in Section 3.4.2 a difference operator $\mathcal{D}_1$ related to the configuration AG_2 that satisfies the conditions from Section 3.3. We use this operator to prove in Section 3.4.3 that the BA function for the configuration AG_2 exists, and we express this function by iterated action of the operator. We show that the BA function is an eigenfunction of $\mathcal{D}_1$, which establishes a bispectral duality. In Section 3.4.4, we present another dual operator $\mathcal{D}_2$ for the configuration AG_2, and we establish the corresponding statements for this operator analogous to the ones for $\mathcal{D}_1$. The operators $\mathcal{D}_1$ and $\mathcal{D}_2$ commute. Finally, we consider in Section 3.4.5 the operator $\mathcal{D}_1$ at $m = 0$, which gives an MR operator for the root system A_2 with multiplicity 1. We show that a version of this operator for the root system A_1 can be decomposed into a sum of two non-symmetric commuting difference operators, which we relate with the standard MR operator for the minuscule weight of the root system A_1.

3.4.1 Integrability of the AG_2 generalised CMS system

Let L_0 be the CMS Hamiltonian for the root system G_2 with multiplicities m and $3m$ for the long and short roots, respectively. Let L be the Hamiltonian of the generalised CMS system associated with the above configuration AG_2. More precisely,

$$L_0 = -\Delta + \sum_{i=1}^{3} \left(v_i(x) + u_i(x)\right),$$

$$L = -\Delta + \sum_{i=1}^{3} \left(v_i(x) + \widetilde{u}_i(x)\right), \tag{3.12}$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$, $x = (x_1, x_2) \in \mathbb{C}^2$,

$$\begin{aligned} v_i(x) &= \frac{m(m+1)\langle\alpha_i, \alpha_i\rangle}{\sinh^2\langle\alpha_i, x\rangle}, \\ u_i(x) &= \frac{3m(3m+1)\langle\beta_i, \beta_i\rangle}{\sinh^2\langle\beta_i, x\rangle}, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \widetilde{u}_i(x) &= \frac{9m(m+1)\langle\beta_i, \beta_i\rangle}{\sinh^2\langle\beta_i, x\rangle} + \frac{8\langle\beta_i, \beta_i\rangle}{\sinh^2\langle 2\beta_i, x\rangle} \\ &= \frac{(3m+1)(3m+2)\langle\beta_i, \beta_i\rangle}{\sinh^2\langle\beta_i, x\rangle} - \frac{2\langle\beta_i, \beta_i\rangle}{\cosh^2\langle\beta_i, x\rangle}. \end{aligned}$$

We present below an intertwining relation between L and the integrable Hamiltonian L_0. This relation is valid for any value of the parameter m, including non-integer ones. This leads to integrability of L for all m, thus generalising integrability for integer m

known from [16,48]. The corresponding intertwining operator $\mathscr{D}$ has order 3. This, in turn, leads to a quantum integral for L of order 6. We note that the direct application of the results of [16] in the case of integer m leads to a higher-order intertwiner and a higher-order integral of L. The degree 6 for the integral of L is expected to be minimal possible. Indeed, for generic m, it follows from [99] that an independent integral for the rational version of L with constant highest term has to be of degree at least 6, since such highest term should be G_2-invariant.

The intertwining operator $\mathscr{D}$ has the form

$$\mathscr{D} = \partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3} - \sum_{\sigma\in A_3} f_{\sigma(1)}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}} + \sum_{i=1}^{3} g_i\partial_{\beta_i} - h, \tag{3.14}$$

where $A_3 = \{\mathrm{id},\ (1,2,3),\ (1,3,2)\}$ is the alternating group on 3 elements,

$$f_i = (3m+1)\langle\beta_i,\beta_i\rangle\coth\langle\beta_i,x\rangle + \langle\beta_i,\beta_i\rangle\tanh\langle\beta_i,x\rangle,$$

$$g_i = \prod_{j\neq i} f_j - \frac{\prod_{j\neq i}\langle\alpha_i,\beta_j\rangle}{\langle\alpha_i,\alpha_i\rangle}v_i - \frac{\prod_{j\neq i}\langle\beta_i,\beta_j\rangle}{\langle\beta_i,\beta_i\rangle}u_i$$

for $i = 1,2,3$, and

$$\begin{aligned} h = f_1f_2f_3 &- \sum_{i=1}^{3} f_i\left(\frac{\prod_{j\neq i}\langle\alpha_i,\beta_j\rangle}{\langle\alpha_i,\alpha_i\rangle}v_i + \frac{\prod_{j\neq i}\langle\beta_i,\beta_j\rangle}{\langle\beta_i,\beta_i\rangle}u_i\right) + \sum_{i=1}^{3}\frac{\prod_{j\neq i}\langle\beta_i,\beta_j\rangle}{\langle\beta_i,\beta_i\rangle}\partial_{\beta_i}(u_i) \\ &+ \frac{3m(3m+1)\langle\beta_1,\beta_1\rangle^3}{\sinh\langle\beta_1,x\rangle\sinh\langle\beta_2,x\rangle\sinh\langle\beta_3,x\rangle} + \frac{4(3m+1)\langle\beta_1,\beta_1\rangle^3}{\sinh\langle 2\beta_1,x\rangle\sinh\langle 2\beta_2,x\rangle\sinh\langle 2\beta_3,x\rangle}. \end{aligned}$$

The following theorem takes place. The proof is by a direct computation.

Theorem 3.16. *[55] The differential operator* (3.14) *satisfies*

$$L\mathscr{D} = \mathscr{D}L_0. \tag{3.15}$$

The quantum integrability of L and a quantum integral for it are obtained as a direct corollary by making use of a general statement from [23]. Let us recall the notion of the formal adjoint A^* of a differential operator A. It can be defined by the relations $\partial_{x_i}^* = -\partial_{x_i}$, $f^* = f$ for any function f, and $(AB)^* = B^*A^*$ for any differential operators A, B.

Theorem 3.17. *[55] Let $\mathscr{D}$ be given by* (3.14)*, and let $\mathscr{D}^*$ be the formal adjoint of $\mathscr{D}$. Let I be any differential operator such that the commutator $[I, L_0] = 0$. Then $\mathscr{D}I\mathscr{D}^*$ commutes with L. In particular, $[\mathscr{D}\mathscr{D}^*, L] = 0$.*

Indeed, taking the formal adjoint of the relation (3.15) gives $\mathscr{D}^*L = L_0\mathscr{D}^*$. Hence

$$L\mathscr{D}I\mathscr{D}^* = \mathscr{D}L_0I\mathscr{D}^* = \mathscr{D}IL_0\mathscr{D}^* = \mathscr{D}I\mathscr{D}^*L.$$

Note that for integer m the operator L_0 is algebraically integrable [105]. The above gives an alternative way to see algebraic integrability of the operator L for integer m. Indeed, it follows from the results of [105] that the operator L_0 for integer m admits an intertwiner $\mathscr{D}_0$ such that $L_0\mathscr{D}_0 = \mathscr{D}_0(-\Delta)$. Theorem 3.16 implies that $L\mathscr{D}\mathscr{D}_0 = \mathscr{D}\mathscr{D}_0(-\Delta)$, hence L is exactly solvable in the sense of [16, Definition 3.1], and so it is algebraically integrable by [16, Theorem 4.3]. The definition of algebraic integrability used in [16], based on [11], is that the space of common eigenfunctions for generic eigenvalues of the quantum integrals of the system is one-dimensional.

We also note that in the rational limit, the operator $\mathscr{D}\mathscr{D}^*$ reduces to a quantum integral for the rational CMS system associated with the root system G_2 with multiplicities m and $3m+1$ for the long and short roots, respectively [55].

Remark 3.18. An alternative proof of the integrability of the operator L follows from Section 4.4.1.2 below, which uses an invariant parabolic submodule for a TCA.

An interesting question is whether the classical analogue of this system is integrable. In the case of the root system G_2, Lax pairs for the corresponding CMS model were constructed in [40] (see also [8]), which may be a starting point for approaching a classical AG_2 CMS system. Another approach could be to investigate a classical version of the quantum integral $\mathscr{D}\mathscr{D}^*$. On the other hand, if we consider the operator $\hbar^2 L$ and take the limit $\hbar \to 0$, $m \to \infty$ such that $\hbar m \to$ const, then it is easy to see that the resulting classical Hamiltonian is the ordinary G_2 Hamiltonian. This suggests that the classical analogue of L with a potential as in the quantum case might not be integrable.

3.4.2 Dual difference operator

Assume that $m \in \mathbb{Z}_{>0}$. In this section, we give a difference operator $\mathcal{D}_1$ satisfying the conditions from Theorem 3.11 for the configuration $R = AG_2$. The corresponding axiomatics is determined by the sets

$$A_\gamma = \{1, 2, \dots, c_\gamma\} \cup \{c_\gamma + 2c_{2\gamma}\} \tag{3.16}$$

for $\gamma \in G_{2,+}$. We define a difference operator acting in the variables $z \in \mathbb{C}^2$ of the form

$$\mathcal{D}_1 = \sum_{\tau\colon \frac{1}{2}\tau \in G_2} a_\tau(z)(T_\tau - 1). \tag{3.17}$$

Let W be the Weyl group of the root system G_2. For $\tau = 2\varepsilon\alpha_j$, $\varepsilon \in \{\pm 1\}$, $j \in \{1,2,3\}$, we define

$$a_{2\varepsilon\alpha_j}(z) = \prod_{\substack{\gamma\in W\beta_1 \\ \langle 2\varepsilon\alpha_j,(2\gamma)^\vee\rangle=3}} \left(1-\frac{(3m+2)\gamma^2}{\langle\gamma,z\rangle}\right)\left(1-\frac{(3m+1)\gamma^2}{\langle\gamma,z\rangle+\gamma^2}\right)\left(1-\frac{3m\gamma^2}{\langle\gamma,z\rangle+2\gamma^2}\right)$$
$$\times \prod_{\substack{\gamma\in W\alpha_1 \\ \langle 2\varepsilon\alpha_j,(2\gamma)^\vee\rangle=1}} \left(1-\frac{m\gamma^2}{\langle\gamma,z\rangle}\right)\times\left(1-\frac{m\alpha_j^2}{\langle\varepsilon\alpha_j,z\rangle}\right)\left(1-\frac{m\alpha_j^2}{\langle\varepsilon\alpha_j,z\rangle+\alpha_j^2}\right). \tag{3.18}$$

For $\tau = 2\varepsilon\beta_j$, we define

$$a_{2\varepsilon\beta_j}(z) = 3 \prod_{\substack{\gamma\in W\beta_1 \\ \langle 2\varepsilon\beta_j,(2\gamma)^\vee\rangle=1}} \left(1-\frac{(3m+2)\gamma^2}{\langle\gamma,z\rangle}\right)\left(1+\frac{3m\gamma^2}{\langle\gamma,z\rangle+2\gamma^2}\right)\left(1-\frac{(3m-1)\gamma^2}{\langle\gamma,z\rangle-\gamma^2}\right)$$
$$\times \prod_{\substack{\gamma\in W\alpha_1 \\ \langle 2\varepsilon\beta_j,(2\gamma)^\vee\rangle=1}} \left(1-\frac{m\gamma^2}{\langle\gamma,z\rangle}\right)\times\left(1-\frac{(3m+2)\beta_j^2}{\langle\varepsilon\beta_j,z\rangle}\right)\left(1-\frac{3m\beta_j^2}{\langle\varepsilon\beta_j,z\rangle+\beta_j^2}\right). \tag{3.19}$$

The following lemma shows that the functions $a_\tau(z)$ have G_2 symmetry.

Lemma 3.19. *Let $a_\tau(z)$ be defined as in* (3.18) *and* (3.19)*. Then for all $w \in W$, we have $wa_\tau = a_{w\tau}$.*

Proof. For any $w \in W$, we have $w(W\alpha_1) = W\alpha_1$, $w(W\beta_1) = W\beta_1$, and $\langle w\tau, w\gamma\rangle = \langle\tau,\gamma\rangle$ for all $\gamma,\tau \in \mathbb{C}^2$. The statement follows. ∎

Define the ring $\mathcal{R}^a_{AG_2}$ of analytic functions $p(z)$ satisfying the conditions

$$\begin{aligned} &p(z+s\alpha_j) = p(z-s\alpha_j) \text{ at } \langle\alpha_j,z\rangle = 0,\ s = 1,2,\dots,m,\\ &p(z+s\beta_j) = p(z-s\beta_j) \text{ at } \langle\beta_j,z\rangle = 0,\ s = 1,2,\dots,3m,3m+2 \end{aligned} \tag{3.20}$$

for all $j = 1,2,3$.

Theorem 3.20. *The operator* (3.17) *preserves the ring $\mathcal{R}^a_{AG_2}$.*

Proof. One can check that this operator has property (D_2) from Section 3.3 for $S = 2G_2$. Let $p(z) \in \mathcal{R}^a_{AG_2}$ be arbitrary. Without loss of generality, we put $\omega = \sqrt{2}$. We introduce new coordinates (A,B) on $\mathbb{C}^2$ given by $A = \langle\alpha_1,z\rangle$ and $B = \langle\beta_1,z\rangle$.

If $B = 4$ (equivalently, $\langle\beta_1,z\rangle = 2\beta_1^2$), then $\langle\beta_2,z\rangle = 2+\frac{1}{2}A$, $\langle\beta_3,z\rangle = -2+\frac{1}{2}A$, $\langle\alpha_2,z\rangle = -6+\frac{1}{2}A$, and $\langle\alpha_3,z\rangle = 6+\frac{1}{2}A$. The only terms singular at $B = 4$ are $a_{-2\beta_2}$, $a_{-2\alpha_3}$, $a_{2\beta_3}$, and $a_{2\alpha_2}$. Note that $s_{\beta_1}(-2\beta_2) - 4\beta_1 = -2\alpha_3$, and we compute that $\operatorname{res}_{B=4}(a_{-2\beta_2}) = -\operatorname{res}_{B=4}(a_{-2\alpha_3})$ equals

$$-3m(3m+2)(3m+4)(A-12)^{-1}(A-4)^{-1}A^{-3}(A+4)^{-1}(A+12)^{-1}(A-12-12m)$$

$$\times (A+6m)(A-4+12m)(A+12m)(A+4+12m)(A+12+12m)^2.$$

Since $s_{\alpha_1}(-2\beta_2) = 2\beta_3$ and $s_{\alpha_1}(-2\alpha_3) = 2\alpha_2$, by Lemma 3.12 with $w = s_{\alpha_1}$ we get that $a_{2\beta_3} + a_{2\alpha_2}$ has zero residue at $B = 4$, too. By Theorem 3.11 part 1, there is thus no singularity at $B = 4$ in $\mathcal{D}_1 p(z)$.

If $B = 2$ (equivalently, $\langle \beta_1, z\rangle = \beta_1^2$), then $\langle \beta_2, z\rangle = 1 + \frac{1}{2}A$, $\langle \beta_3, z\rangle = -1 + \frac{1}{2}A$, $\langle \alpha_2, z\rangle = -3 + \frac{1}{2}A$, and $\langle \alpha_3, z\rangle = 3 + \frac{1}{2}A$. The only $\tau \in 2G_2$ for which a_τ is singular at $B = 2$ and for which the corresponding $\lambda = s_{\beta_1}(\tau) - 2\beta_1 \neq 0$ are $\tau = 2\beta_2, 2\alpha_2, -2\beta_3, -2\alpha_3$. Note that $s_{\beta_1}(2\beta_2) - 2\beta_1 = 2\alpha_2$, and we compute that $\operatorname{res}_{B=2}(a_{2\beta_2}) = -\operatorname{res}_{B=2}(a_{2\alpha_2})$ equals

$$6(m+1)(3m-1)(3m+1)(A-6)^{-1}(A-2)^{-1}A^{-1}(A+2)^{-1}(A+6)^{-3}(A-10-12m)$$
$$\times (A-6-12m)(A-2-12m)(A+6-12m)^2(A-6m)(A+6+12m).$$

Since $s_{\alpha_1}(2\beta_2) = -2\beta_3$ and $s_{\alpha_1}(2\alpha_2) = -2\alpha_3$, by Lemma 3.12 we get that $a_{-2\beta_3} + a_{-2\alpha_3}$ has zero residue at $B = 2$, too. By Theorem 3.11 part 1, there is thus no singularity at $B = 2$ in $\mathcal{D}_1 p(z)$, nor at $B = 0$.

It follows from the above analysis and from the form of the coefficient functions (3.18) and (3.19) that there are no singularities in $\mathcal{D}_1 p(z)$ at $B = c$ for all $c \geq 0$. By Corollary 3.13, there is also no singularity in $\mathcal{D}_1 p(z)$ at $\langle \beta_i, z\rangle = c$ for all $i = 1, 2, 3$ and all $c \in \mathbb{C}$.

The only singularity at $A = \text{const} > 0$ present in the coefficients a_τ for some τ is at $A = 6$ (equivalently, $\langle \alpha_1, z\rangle = \alpha_1^2$) when $\tau = -2\alpha_1$. This singularity cancels in $\mathcal{D}_1 p(z)$ by Theorem 3.11 part 1, since the corresponding $\lambda = s_{\alpha_1}(-2\alpha_1) - 2\alpha_1 = 0$. By Corollary 3.13, there is also no singularity in $\mathcal{D}_1 p(z)$ at $\langle \alpha_i, z\rangle = c$ for all $i = 1, 2, 3$ and for all $c \in \mathbb{C}$. This completes the proof that $\mathcal{D}_1 p(z)$ is analytic.

Let us now show $\mathcal{D}_1 p(z)$ satisfies the axiomatics of $\mathcal{R}^a_{AG_2}$. We have $A_{\beta_i} = \{1, 2, 3, \ldots, 3m, 3m+2\}$ and $A_{\alpha_i} = \{1, 2, \ldots, m\}$ $(i = 1, 2, 3)$. Let us show firstly that $\mathcal{D}_1 p(z + t\beta_1) = \mathcal{D}_1 p(z - t\beta_1)$ at $\langle \beta_1, z\rangle = 0$ for all $t \in A_{\beta_1}$. To do so, we will check condition 2 in Theorem 3.11 with $\alpha = \beta_1$ for all $\tau \in 2G_2$.

Note that $(2\beta_1)^\vee = \frac{1}{2}\beta_1$. Let $\tau = 2\beta_1$. Then $|t + \langle \tau, (2\beta_1)^\vee\rangle| = t + 2$, which does not belong to $A_{\beta_1} \cup \{0\}$ if and only if $t = 3m - 1$ or $t = 3m + 2$. But

$$a_{2\beta_1}(z + (3m+2)\beta_1) = a_{2\beta_1}(z + (3m-1)\beta_1) = 0 \text{ at } \langle \beta_1, z\rangle = 0$$

because $a_{2\beta_1}(z)$ contains the factors $(1 - \frac{(3m+2)\beta_1^2}{\langle \beta_1, z\rangle})(1 - \frac{3m\beta_1^2}{\langle \beta_1, z\rangle + \beta_1^2})$.

Let now $\tau = -2\beta_1$. Then $|t + \langle \tau, (2\beta_1)^\vee\rangle| = |t - 2| \in A_{\beta_1} \cup \{0\}$ for all $t \in A_{\beta_1}$, as needed.

Let now $\tau = 2\beta_2$. Then $|t + \langle \tau, (2\beta_1)^\vee\rangle| = t + 1$, which does not belong to $A_{\beta_1} \cup \{0\}$ if and only if $t = 3m$ or $t = 3m + 2$. But

$$a_{2\beta_2}(z + (3m + 2)\beta_1) = a_{2\beta_2}(z + 3m\beta_1) = 0 \text{ at } \langle\beta_1, z\rangle = 0$$

because $a_{2\beta_2}(z)$ contains the factors $(1 - \frac{(3m+2)\beta_1^2}{\langle\beta_1,z\rangle})(1 - \frac{(3m-1)\beta_1^2}{\langle\beta_1,z\rangle-\beta_1^2})$.

Let now $\tau = -2\beta_2$. Then $|t + \langle\tau, (2\beta_1)^\vee\rangle| = t - 1$ which does not belong to $A_{\beta_1} \cup \{0\}$ if and only if $t = 3m + 2$. But $a_{-2\beta_2}(z + (3m + 2)\beta_1) = 0$ at $\langle\beta_1, z\rangle = 0$ because $a_{-2\beta_2}$ contains the factor $(1 + \frac{3m\beta_1^2}{-\langle\beta_1,z\rangle+2\beta_1^2})$. Since $s_{\alpha_1}(\beta_1) = \beta_1$, by Remark 3.15 there is nothing to check for $\tau = \pm 2\beta_3 = s_{\alpha_1}(\mp 2\beta_2)$.

For $\tau = \pm 2\alpha_1$, we get $|t + \langle\tau, (2\beta_1)^\vee\rangle| = t \in A_{\beta_1}$, as needed. Similarly for $\tau = 2\alpha_2$, $|t + \langle\tau, (2\beta_1)^\vee\rangle| = |t - 3| \in A_{\beta_1} \cup \{0\}$ for all $t \in A_{\beta_1}$, as needed.

Finally, let $\tau = -2\alpha_2$. Then $|t+\langle\tau, (2\beta_1)^\vee\rangle| = t+3 \notin A_{\beta_1}\cup\{0\}$ if and only if $t = 3m+2$, $t = 3m$ or $t = 3m - 2$, but $a_{-2\alpha_2}(z + t\beta_1) = 0$ at $\langle\beta_1, z\rangle = 0$ for those t because $a_{-2\alpha_2}$ contains the factors $(1 - \frac{(3m+2)\beta_1^2}{\langle\beta_1,z\rangle})(1 - \frac{(3m+1)\beta_1^2}{\langle\beta_1,z\rangle+\beta_1^2})(1 - \frac{3m\beta_1^2}{\langle\beta_1,z\rangle+2\beta_1^2})$. By Remark 3.15, there is nothing to check for $\tau = \pm 2\alpha_3 = s_{\alpha_1}(\mp 2\alpha_2)$.

Let us show next that $\mathcal{D}_1 p(z + t\alpha_1) = \mathcal{D}_1 p(z - t\alpha_1)$ at $\langle\alpha_1, z\rangle = 0$ for all $t \in A_{\alpha_1}$. For that, we will check condition 2 in Theorem 3.11 with $\alpha = \alpha_1$ for all $\tau \in 2G_2$.

Let $\tau = \pm 2\beta_1$. Then $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = t \in A_{\alpha_1}$, as needed.

Let now $\tau = 2\beta_2$. Note that $(2\alpha_1)^\vee = \frac{1}{6}\alpha_1$. Then $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = t + 1 \notin A_{\alpha_1} \cup \{0\}$ if and only if $t = m$. But $a_{2\beta_2}(z+m\alpha_1) = 0$ at $\langle\alpha_1, z\rangle = 0$ because $a_{2\beta_2}$ contains the factor $(1 - \frac{m\alpha_1^2}{\langle\alpha_1,z\rangle})$.

Let now $\tau = -2\beta_2$. Then $|t+\langle\tau, (2\alpha_1)^\vee\rangle| = t-1 \in A_{\alpha_1}\cup\{0\}$ for all $t \in A_{\alpha_1}$, as needed. Since $s_{\beta_1}(\alpha_1) = \alpha_1$, by Remark 3.15 there is nothing to check for $\tau = \pm 2\beta_3 = s_{\beta_1}(\pm 2\beta_2)$.

Let now $\tau = 2\alpha_1$. Then $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = t + 2 \notin A_{\alpha_1} \cup \{0\}$ if and only if $t = m$ or $t = m - 1$. But $a_{2\alpha_1}(z + t\alpha_1) = 0$ at $\langle\alpha_1, z\rangle = 0$ for those t because $a_{2\alpha_1}$ contains the factors $(1 - \frac{m\alpha_1^2}{\langle\alpha_1,z\rangle})(1 - \frac{m\alpha_1^2}{\langle\alpha_1,z\rangle+\alpha_1^2})$.

Let now $\tau = -2\alpha_1$. Then $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = |t - 2| \in A_{\alpha_1} \cup \{0\}$ for all $t \in A_{\alpha_1}$, as needed.

Let now $\tau = 2\alpha_2$. Then $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = t + 1 \notin A_{\alpha_1} \cup \{0\}$ if and only if $t = m$. But $a_{2\alpha_2}(z + m\alpha_1) = 0$ at $\langle\alpha_1, z\rangle = 0$ because $a_{2\alpha_2}$ contains the factor $(1 - \frac{m\alpha_1^2}{\langle\alpha_1,z\rangle})$.

Finally, for $\tau = -2\alpha_2$, we get $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = t - 1 \in A_{\alpha_1} \cup \{0\}$ for all $t \in A_{\alpha_1}$, as needed. By Remark 3.15 there is nothing to check for $\tau = \pm 2\alpha_3 = s_{\beta_1}(\pm 2\alpha_2)$.

Since all the vectors α_i, β_i are in the W-orbit of $\{\alpha_1\} \cup \{\beta_1\}$, the statement follows by Lemma 3.14. ∎

Let us now look at the expansion of the operator (3.17) as $\omega \to 0$. It produces the rational CMS operator in the potential-free gauge for the root system of type G_2 with multiplicity m for the long roots and multiplicity $3m + 1$ for the short roots, as the next proposition shows. Let $\widetilde{\beta}_j = \omega^{-1}\beta_j$ and $\widetilde{\alpha}_j = \omega^{-1}\alpha_j$ $(j = 1, 2, 3)$ with the same multiplicities as β_j and α_j, respectively, and let $c_{2\widetilde{\beta}_j} = c_{2\beta_j} = 1$, $c_{2\widetilde{\alpha}_j} = c_{2\alpha_j} = 0$.

Proposition 3.21. *We have*

$$\lim_{\omega\to 0} \frac{\mathcal{D}_1}{36\omega^2} = \Delta - \sum_{\gamma\in\{\widetilde{\beta}_i,\, \widetilde{\alpha}_i \,:\, i=1,2,3\}} \frac{2(c_\gamma + c_{2\gamma})}{\langle\gamma, z\rangle}\partial_\gamma,$$

where $\Delta = \partial_{z_1}^2 + \partial_{z_2}^2$ *and* ∂_γ *is the directional derivative in* z *in the direction of the vector* γ.

Proof. We have $T_{\pm 2\beta_j} - 1 = \pm\omega\partial_{2\widetilde{\beta}_j} + \frac{1}{2}\omega^2\partial^2_{2\widetilde{\beta}_j} + \ldots$, and similarly for the other shifts. The terms at order ω in the expansion around $\omega = 0$ of the operator $\mathcal{D}_1$ vanish. The terms that are of second order in derivatives in the coefficient at ω^2 in the expansion $\omega \to 0$ of the operator $\mathcal{D}_1$ are

$$3\sum_{j=1}^{3} \partial^2_{2\widetilde{\beta}_j} + \sum_{j=1}^{3} \partial^2_{2\widetilde{\alpha}_j} = 36\Delta.$$

Let us now consider the terms that are first-order in derivatives in the coefficient at ω^2. It is easy to see that such terms containing $\langle\widetilde{\beta}_1, z\rangle^{-1}$ are

$$-6(3m+1)\left(2\partial_{2\widetilde{\beta}_1} + \partial_{2\widetilde{\beta}_2} - \partial_{2\widetilde{\beta}_3} + \partial_{2\widetilde{\alpha}_3} - \partial_{2\widetilde{\alpha}_2}\right) = -72(c_{\beta_1} + c_{2\beta_1})\partial_{\widetilde{\beta}_1}.$$

Altogether, the term at ω^2 in the expansion of the operator $\mathcal{D}_1$ is as required. ∎

3.4.3 Construction of the BA function for AG_2

In this section, we employ a technique from [24] to construct the BA function for the configuration AG_2. Its BA function will be an eigenfunction for the difference operator $\mathcal{D}_1$ from Section 3.4.2, which establishes bispectrality of the AG_2 generalised CMS Hamiltonian.

The following lemma gives a useful way of expanding the functions a_τ in the operator (3.17).

Lemma 3.22. *Let* $a_\tau(z)$ *be defined as in* (3.18) *and* (3.19). *Then*

$$a_\tau(z) = \kappa_\tau - \kappa_\tau \sum_{\gamma\in G_{2,+}} \frac{\langle\tau,\gamma\rangle(c_\gamma + c_{2\gamma})}{\langle\gamma, z\rangle} + R_\tau(z), \tag{3.21}$$

where $\kappa_{\pm 2\beta_j} = 3$ *and* $\kappa_{\pm 2\alpha_j} = 1$, *and* $R_\tau(z)$ *is a rational function with* $\deg R_\tau \le -2$.

Proof. For the factors in a_τ that have shifted singularities at $\langle\gamma, z\rangle + c = 0$ with $c \neq 0$, we can use that

$$\frac{1}{\langle\gamma, z\rangle + c} = \frac{1}{\langle\gamma, z\rangle} - \frac{c}{(\langle\gamma, z\rangle + c)\langle\gamma, z\rangle},$$

which differs from $\langle\gamma, z\rangle^{-1}$ only by a rational function of degree -2 which cannot affect the coefficient at $\langle\gamma, z\rangle^{-1}$. The relation (3.21) is then obtained by multiplying out all the factors in each of the a_τ. ■

The next lemma is proved by a direct computation using Lemma 3.22. We will apply it in the proof of Theorem 3.24 below.

Lemma 3.23. *For $\gamma \in G_{2,+}$, let $n_\gamma \in \mathbb{Z}_{\geq 0}$ be arbitrary. Let $N = \sum_{\gamma \in G_{2,+}} n_\gamma$. Let*

$$\mu(x) = \sum_{\tau\colon \frac{1}{2}\tau \in G_2} \kappa_\tau (e^{\langle\tau, x\rangle} - 1), \tag{3.22}$$

where κ_τ are as in Lemma 3.22. Let $A(z) = \prod_{\gamma\in G_{2,+}} \langle\gamma, z\rangle^{n_\gamma}$. Write $(\mathcal{D}_1 - \mu)[A(z)e^{\langle z,x\rangle}] = R(z, x)e^{\langle z,x\rangle}$ for some rational function $R(z, x)$ in z, which has degree less than or equal to N. Then

$$R(z, x) = \sum_{\gamma\in G_{2,+}} (n_\gamma - c_\gamma - c_{2\gamma}) \left(\sum_{\tau\colon \frac{1}{2}\tau\in G_2} \kappa_\tau \langle\tau, \gamma\rangle e^{\langle\tau,x\rangle} \right) A(z)\langle\gamma, z\rangle^{-1} + S(z, x)$$

for some rational function $S(z, x)$ in z of degree less than or equal to $N - 2$.

In particular, for any *polynomial $B(z, x)$ in z, we have that $(\mathcal{D}_1 - \mu)[B(z, x)e^{\langle z,x\rangle}] = U(z, x)e^{\langle z,x\rangle}$ for a rational function $U(z, x)$ in z with $\deg U(z, x) \leq \deg B(z, x) - 1$.*

Proof. Making use of the expression for $a_\tau(z)$ given in Lemma 3.22, we get

$$\begin{aligned}
\mathcal{D}_1[A(z)e^{\langle z,x\rangle}] &= \sum_{\tau\colon \frac{1}{2}\tau\in G_2} a_\tau(z)(T_\tau - 1)[A(z)e^{\langle z,x\rangle}] \\
&= e^{\langle z,x\rangle} \sum_{\tau\colon \frac{1}{2}\tau\in G_2} a_\tau(z) \Big(e^{\langle\tau,x\rangle} \prod_{\gamma\in G_{2,+}} (\langle\gamma, z\rangle + \langle\tau, \gamma\rangle)^{n_\gamma} - A(z) \Big) \\
&= A(z)e^{\langle z,x\rangle} \sum_{\tau\colon \frac{1}{2}\tau\in G_2} \kappa_\tau \left(1 - \sum_{\gamma\in G_{2,+}} \langle\tau, \gamma\rangle (c_\gamma + c_{2\gamma}) \langle\gamma, z\rangle^{-1} + \text{l.o.t.} \right) \\
&\qquad \times \left(e^{\langle\tau,x\rangle} \left(1 + \sum_{\gamma\in G_{2,+}} n_\gamma \langle\tau, \gamma\rangle \langle\gamma, z\rangle^{-1} + \text{l.o.t.} \right) - 1 \right) \\
&= A(z)e^{\langle z,x\rangle} \left(\mu(x) + \sum_{\gamma\in G_{2,+}} (n_\gamma - c_\gamma - c_{2\gamma}) \left(\sum_{\tau\colon \frac{1}{2}\tau\in G_2} \kappa_\tau \langle\tau, \gamma\rangle e^{\langle\tau,x\rangle} \right) \langle\gamma, z\rangle^{-1} + \text{l.o.t.} \right),
\end{aligned}$$

l.o.t. denoting terms of lower degree in z, where we used that $\sum_{\tau\colon \frac{1}{2}\tau\in G_2} \kappa_\tau \langle\tau, \gamma\rangle = 0$ for all $\gamma \in G_{2,+}$, since if $\frac{1}{2}\tau \in G_2$ then also $-\frac{1}{2}\tau \in G_2$ and $\kappa_\tau = \kappa_{-\tau}$. ■

We are now ready to give the main result of this section.

Theorem 3.24. *Let* $M = \sum_{\gamma \in AG_{2,+}} c_\gamma = 12m + 3$. *Let*

$$b(x) = \frac{M!}{8} \prod_{\gamma \in G_{2,+}} \left(\sum_{\tau \colon \frac{1}{2}\tau \in G_2} \kappa_\tau \langle \tau, \gamma \rangle e^{\langle \tau, x \rangle} \right)^{c_\gamma + c_{2\gamma}}. \tag{3.23}$$

Define the polynomial

$$Q(z) = \prod_{\substack{\gamma \in G_{2,+} \\ s \in A_\gamma}} \left(\langle \gamma, z \rangle^2 - s^2 \langle \gamma, \gamma \rangle^2 \right), \tag{3.24}$$

where A_γ *is given by* (3.16). *Then the function*

$$\psi(z, x) = b(x)^{-1} (\mathcal{D}_1 - \mu(x))^M [Q(z) e^{\langle z, x \rangle}], \tag{3.25}$$

where $\mu(x)$ *is given by* (3.22)*, is the BA function for* AG_2*. Moreover,* ψ *is also an eigenfunction of the operator* $\mathcal{D}_1$ *with* $\mathcal{D}_1 \psi = \mu(x)\psi$*, thus bispectrality holds — the operator* $\mathcal{D}_1$ *is bispectrally dual to the* AG_2 *generalised CMS Hamiltonian.*

Proof. The operator $\mathcal{D}_1$ preserves the ring $\mathcal{R}^a_{AG_2}$ by Theorem 3.20. The function $Q(z)e^{\langle z,x \rangle}$ is contained in $\mathcal{R}^a_{AG_2}$ as it is analytic and satisfies conditions (3.20), since $Q(z + s\gamma) = Q(z - s\gamma) = 0$ at $\langle \gamma, z \rangle = 0$ for $s \in A_\gamma$, $\gamma \in G_{2,+}$. Since $\mathcal{D}_1$ preserves $\mathcal{R}^a_{AG_2}$, so does $\mathcal{D}_1 - \mu$, hence $\psi(z,x)$ given by (3.25) belongs to $\mathcal{R}^a_{AG_2}$. Its analyticity and the form of the functions a_τ imply that it equals $b^{-1}(x)P(z,x)e^{\langle z,x \rangle}$ for some polynomial $P(z,x)$ in z. To prove that $\psi(z,x)$ satisfies the definition of the BA function, it thus suffices to calculate the highest-degree term in $P(z,x)$.

The highest-degree term in $Q(z)$ is $Q_0(z) = \prod_{\gamma \in G_{2,+}} \langle \gamma, z \rangle^{2(c_\gamma + c_{2\gamma})}$ and $\deg Q_0 = 2M$. For $k \in \mathbb{Z}_{>0}$ with $k \leq M$, an analogous argument as above gives that $(\mathcal{D}_1 - \mu)^k [Q(z)e^{\langle z,x \rangle}]$ belongs to $\mathcal{R}^a_{AG_2}$ and is of the form $Q^{(k)}(z,x)e^{\langle z,x \rangle}$ for some polynomial $Q^{(k)}(z,x)$ in z. Let the highest-degree homogeneous component of $Q^{(k)}(z,x)$ be $Q_0^{(k)}(z,x)$. Lemma 3.23 allows to compute $Q_0^{(k)}(z,x)$.

Lemma 3.23 gives that after the first application of $\mathcal{D}_1 - \mu$ onto $Q(z)e^{\langle z,x \rangle}$ we get

$$Q_0^{(1)} = \sum_{\gamma \in G_{2,+}} (c_\gamma + c_{2\gamma}) \left(\sum_{\tau \colon \frac{1}{2}\tau \in G_2} \kappa_\tau \langle \tau, \gamma \rangle e^{\langle \tau, x \rangle} \right) \langle \gamma, z \rangle^{-1} Q_0(z).$$

The second application gives

$$Q_0^{(2)} = \sum_{\gamma \in G_{2,+}} (c_\gamma + c_{2\gamma})(c_\gamma + c_{2\gamma} - 1) \left(\sum_{\tau \colon \frac{1}{2}\tau \in G_2} \kappa_\tau \langle \tau, \gamma \rangle e^{\langle \tau, x \rangle} \right)^2 \langle \gamma, z \rangle^{-2} Q_0(z)$$

$$+ \sum_{\substack{\gamma,\delta\in G_{2,+}\\ \gamma\neq\delta}} (c_\gamma + c_{2\gamma})(c_\delta + c_{2\delta}) \left(\sum_{\tau:\, \frac{1}{2}\tau\in G_2} \kappa_\tau \langle \tau, \gamma\rangle e^{\langle \tau, x\rangle} \right)$$

$$\times \left(\sum_{\tau:\, \frac{1}{2}\tau\in G_2} \kappa_\tau \langle \tau, \delta\rangle e^{\langle \delta, x\rangle} \right) \langle \gamma, z\rangle^{-1} \langle \delta, z\rangle^{-1} Q_0(z).$$

By repeatedly applying Lemma 3.23, we get

$$Q_0^{(k)} = \sum_{\mathbf{n}} f_{\mathbf{n}}(x) Q_0(z) \prod_{\gamma\in G_{2,+}} \langle \gamma, z\rangle^{-n_\gamma}$$

where $\mathbf{n} = (n_\gamma)_{\gamma\in G_{2,+}}$ for $n_\gamma \in \mathbb{Z}_{\geq 0}$ such that n_γ add up to k, and where $f_{\mathbf{n}}(x)$ is non-zero only if $n_\gamma \leq c_\gamma + c_{2\gamma}$ for all γ. It follows that $\deg P \leq M$ and that the highest-degree term of $P(z, x)$ is

$$d(x) \prod_{\gamma\in G_{2,+}} \langle \gamma, z\rangle^{c_\gamma + c_{2\gamma}} = \frac{1}{8} d(x) \prod_{\gamma\in AG_{2,+}} \langle \gamma, z\rangle^{c_\gamma}$$

for some function $d(x)$. It also implies that the polynomial part of $(\mathcal{D}_1 - \mu)^{M+1}[Q(z)e^{\langle z,x\rangle}]$ has degree less than M, hence vanishes as a consequence of Lemma 3.5, giving $\mathcal{D}_1\psi = \mu\psi$. So, to complete the proof, we just need to verify that $b(x)$ given by (3.23) equals $\frac{1}{8}d(x)$.

To arrive at $\prod_{\gamma\in G_{2,+}} \langle \gamma, z\rangle^{c_\gamma + c_{2\gamma}}$ starting from $Q_0(z)$, we need to reduce the power of each of the factors $\langle \gamma, z\rangle$ by $c_\gamma + c_{2\gamma}$, and we do this by reducing the power of one of them by one at each step. The total number of possible orderings of doing that corresponds to the number of words of length M in the alphabet $G_{2,+}$ such that γ appears in the word $c_\gamma + c_{2\gamma}$ times for each $\gamma \in G_{2,+}$. This gives

$$\frac{M!}{\prod_{\gamma\in G_{2,+}} (c_\gamma + c_{2\gamma})!}$$

possibilities, and for each of them the total proportionality factor that we pick up equals

$$\prod_{\gamma\in G_{2,+}} (c_\gamma + c_{2\gamma})! \left(\sum_{\tau:\, \frac{1}{2}\tau\in G_2} \kappa_\tau \langle \tau, \gamma\rangle e^{\langle \tau, x\rangle} \right)^{c_\gamma + c_{2\gamma}}$$

by Lemma 3.23. It follows that $b(x)$ has the required form. ■

3.4.4 Another dual operator

In this section, we present another difference operator that preserves the ring of quasi-invariants $\mathcal{R}^a_{AG_2}$. We also give the corresponding second construction of the BA function for AG_2.

We define a difference operator acting in the variables $z \in \mathbb{C}^2$ of the form

$$\mathcal{D}_2 = \sum_{\tau\colon \frac{1}{2}\tau \in AG_2} a_\tau(z)(T_\tau - 1). \tag{3.26}$$

We now specify the functions $a_\tau(z)$. Let λ_τ be defined in terms of the multiplicity function of AG_2 by

$$\lambda_\tau = \frac{1}{4} c_{\frac{1}{2}\tau}(c_{\frac{1}{2}\tau} + 2c_\tau + 1)\tau^2.$$

This means $\lambda_{4\varepsilon\beta_j} = 8\beta_j^2$, $\lambda_{2\varepsilon\beta_j} = 9m(m+1)\beta_j^2$, and $\lambda_{2\varepsilon\alpha_j} = m(m+1)\alpha_j^2$ ($j = 1,2,3$, $\varepsilon \in \{\pm 1\}$, $m \in \mathbb{Z}_{>0}$). For $\tau = 2\varepsilon\alpha_j$, we define

$$\begin{aligned} a_{2\varepsilon\alpha_j}(z) = \lambda_{2\varepsilon\alpha_j} & \prod_{\substack{\gamma \in W\beta_1 \\ \langle 2\varepsilon\alpha_j, (2\gamma)^\vee \rangle = 3}} \left(1 - \frac{(3m+2)\gamma^2}{\langle \gamma, z\rangle}\right)\left(1 - \frac{(3m+1)\gamma^2}{\langle \gamma, z\rangle + \gamma^2}\right)\left(1 - \frac{3m\gamma^2}{\langle \gamma, z\rangle + 2\gamma^2}\right) \\ & \times \prod_{\substack{\gamma \in W\beta_1 \\ \langle 2\varepsilon\alpha_j, (2\gamma)^\vee \rangle = 0}} \left(1 - \frac{6\gamma^2}{\langle \gamma, z\rangle - \gamma^2}\right) \prod_{\substack{\gamma \in W\alpha_1 \\ \langle 2\varepsilon\alpha_j, (2\gamma)^\vee \rangle = 1}} \left(1 - \frac{m\gamma^2}{\langle \gamma, z\rangle}\right) \\ & \times \left(1 - \frac{m\alpha_j^2}{\langle \varepsilon\alpha_j, z\rangle}\right)\left(1 - \frac{m\alpha_j^2}{\langle \varepsilon\alpha_j, z\rangle + \alpha_j^2}\right). \end{aligned} \tag{3.27}$$

For $\tau = 4\varepsilon\beta_j$, we define

$$\begin{aligned} a_{4\varepsilon\beta_j}(z) = \lambda_{4\varepsilon\beta_j} & \prod_{\substack{\gamma \in W\alpha_1 \\ \langle 4\varepsilon\beta_j, (2\gamma)^\vee \rangle = 2}} \left(1 - \frac{m\gamma^2}{\langle \gamma, z\rangle}\right)\left(1 - \frac{m\gamma^2}{\langle \gamma, z\rangle + \gamma^2}\right) \\ \times & \prod_{\substack{\gamma \in W\beta_1 \\ \langle 4\varepsilon\beta_j, (2\gamma)^\vee \rangle = 2}} \left(1 - \frac{(3m+2)\gamma^2}{\langle \gamma, z\rangle}\right)\left(1 - \frac{3m\gamma^2}{\langle \gamma, z\rangle + \gamma^2}\right) \\ \times & \left(1 - \frac{(3m+2)\beta_j^2}{\langle \varepsilon\beta_j, z\rangle}\right)\left(1 - \frac{3m\beta_j^2}{\langle \varepsilon\beta_j, z\rangle + \beta_j^2}\right)\left(1 - \frac{(3m+2)\beta_j^2}{\langle \varepsilon\beta_j, z\rangle + 2\beta_j^2}\right)\left(1 - \frac{3m\beta_j^2}{\langle \varepsilon\beta_j, z\rangle + 3\beta_j^2}\right). \end{aligned} \tag{3.28}$$

For $\tau = 2\varepsilon\beta_j$, we define

$$\begin{aligned} a_{2\varepsilon\beta_j}(z) = \lambda_{2\varepsilon\beta_j} & \prod_{\substack{\gamma \in W\alpha_1 \\ \langle 2\varepsilon\beta_j, (2\gamma)^\vee \rangle = 0}} \left(1 - \frac{\frac{2}{3}\gamma^2}{\langle \gamma, z\rangle - \gamma^2}\right) \prod_{\substack{\gamma \in W\alpha_1 \\ \langle 2\varepsilon\beta_j, (2\gamma)^\vee \rangle = 1}} \left(1 - \frac{m\gamma^2}{\langle \gamma, z\rangle}\right) \\ \times & \prod_{\substack{\gamma \in W\beta_1 \\ \langle 2\varepsilon\beta_j, (2\gamma)^\vee \rangle = 1}} \left(1 - \frac{(3m+2)\gamma^2}{\langle \gamma, z\rangle}\right)\left(1 + \frac{3m\gamma^2}{\langle \gamma, z\rangle + 2\gamma^2}\right)\left(1 - \frac{(3m-1)\gamma^2}{\langle \gamma, z\rangle - \gamma^2}\right) \end{aligned}$$

$$\times\left(1-\frac{(3m+2)\beta_j^2}{\langle\varepsilon\beta_j,z\rangle}\right)\left(1-\frac{3m\beta_j^2}{\langle\varepsilon\beta_j,z\rangle+\beta_j^2}\right)\left(1+\frac{4\beta_j^2}{\langle\varepsilon\beta_j,z\rangle+3\beta_j^2}\right)\left(1-\frac{4\beta_j^2}{\langle\varepsilon\beta_j,z\rangle-\beta_j^2}\right). \tag{3.29}$$

The next lemma shows that the functions $a_\tau(z)$ have G_2 symmetry.

Lemma 3.25. *Let $a_\tau(z)$ be defined as in (3.27)–(3.29). Then for all $w \in W$, we have $wa_\tau = a_{w\tau}$.*

Proof. For any $w \in W$, $\lambda_{w\tau} = \lambda_\tau$ for all τ with $\frac{1}{2}\tau \in AG_2$, as the multiplicities are W-invariant. The statement follows as in the proof of Lemma 3.19. ■

Theorem 3.26. *The operator (3.26) preserves the ring $\mathcal{R}^a_{AG_2}$.*

Proof. One can check that the operator satisfies condition (D_2) from Section 3.3 for $S = 2AG_2$. Let $p(z) \in \mathcal{R}^a_{AG_2}$ be arbitrary. Without loss of generality, we put $\omega = \sqrt{2}$. We use the coordinates (A, B) on $\mathbb{C}^2$ given by $A = \langle\alpha_1, z\rangle$ and $B = \langle\beta_1, z\rangle$.

It follows from the form of the coefficient functions (3.27)–(3.29) and Theorem 3.11 that there are no singularities in $\mathcal{D}_2 p(z)$ at $B = c$ for $c \geq 0$ except possibly for $B = 2, 4, 6$. Let us consider each of these cases in turn.

If $B = 6$ (equivalently, $\langle\beta_1, z\rangle = 3\beta_1^2$), then $\langle\beta_2, z\rangle = 3 + \frac{1}{2}A$, $\langle\beta_3, z\rangle = -3 + \frac{1}{2}A$, $\langle\alpha_2, z\rangle = -9 + \frac{1}{2}A$, and $\langle\alpha_3, z\rangle = 9 + \frac{1}{2}A$. The only terms singular at $B = 6$ are $a_{-4\beta_1}$ and $a_{-2\beta_1}$. Note that $s_{\beta_1}(-4\beta_1) - 6\beta_1 = -2\beta_1$, and we compute that $\operatorname{res}_{B=6}(a_{-4\beta_1}) = -\operatorname{res}_{B=6}(a_{-2\beta_1})$ equals

$$48m(m+1)(3m+2)(3m+5)(A-18)^{-1}(A-6)^{-2}(A-2)^{-1}(A+2)^{-1}(A+6)^{-2}(A+18)^{-1}$$
$$\times(A-2-12m)(A-6-12m)(A-14-12m)(A-18-12m)(A+2+12m)$$
$$\times(A+6+12m)(A+14+12m)(A+18+12m).$$

Therefore, by Theorem 3.11 part 1, there is no singularity at $B = 6$ in $\mathcal{D}_2 p(z)$.

If $B = 4$ (equivalently, $\langle\beta_1, z\rangle = 2\beta_1^2$), then $\langle\beta_2, z\rangle = 2 + \frac{1}{2}A$, $\langle\beta_3, z\rangle = -2 + \frac{1}{2}A$, $\langle\alpha_2, z\rangle = -6 + \frac{1}{2}A$, and $\langle\alpha_3, z\rangle = 6 + \frac{1}{2}A$. The only $\tau \in 2AG_2$ for which a_τ is singular at $B = 4$ and for which the corresponding $\lambda = s_{\beta_1}(\tau) - 4\beta_1 \neq 0$ are $\tau = -2\beta_2, -2\alpha_3, 2\beta_3, 2\alpha_2$. Note that $s_{\beta_1}(-2\beta_2) - 4\beta_1 = -2\alpha_3$, and we compute that $\operatorname{res}_{B=4}(a_{-2\beta_2}) = -\operatorname{res}_{B=4}(a_{-2\alpha_3})$ equals

$$-18m^2(m+1)(3m+2)(3m+4)(A-12)^{-1}(A-8)^{-1}(A-4)^{-1}A^{-4}(A+4)^{-1}(A+12)^{-1}$$
$$\times(A-32)(A+24)(A-12-12m)(A+6m)(A-4+12m)(A+12m)$$
$$\times(A+4+12m)(A+12+12m)^2.$$

Since $s_{\alpha_1}(-2\beta_2) = 2\beta_3$ and $s_{\alpha_1}(-2\alpha_3) = 2\alpha_2$, by Lemma 3.12 the residue of $a_{2\beta_3} + a_{2\alpha_2}$

at $B = 4$ is also zero. Thus, by Theorem 3.11 part 1, there is no singularity at $B = 4$ in $\mathcal{D}_2 p(z)$.

If $B = 2$ (equivalently, $\langle \beta_1, z\rangle = \beta_1^2$), then $\langle \beta_2, z\rangle = 1+\frac{1}{2}A$, $\langle \beta_3, z\rangle = -1+\frac{1}{2}A$, $\langle \alpha_2, z\rangle = -3+\frac{1}{2}A$, and $\langle \alpha_3, z\rangle = 3+\frac{1}{2}A$. The only $\tau \in 2AG_2$ for which a_τ is singular at $B = 2$ and for which the corresponding $\lambda = s_{\beta_1}(\tau)-2\beta_1 \neq 0$ are $\tau = -4\beta_1, 2\beta_1, -4\beta_2, 4\beta_3, \pm 2\alpha_1, 2\beta_2, 2\alpha_2, -2\beta_3$, and $-2\alpha_3$. Note that $s_{\beta_1}(-4\beta_1) - 2\beta_1 = 2\beta_1$, and we compute that $\operatorname{res}_{B=2}(a_{-4\beta_1}) = -\operatorname{res}_{B=2}(a_{2\beta_1})$ equals

$$\begin{aligned}&144m(m+1)(3m-2)(3m+1)(A-6)^{-2}(A-2)^{-2}(A+2)^{-2}(A+6)^{-2}(A+6-12m)\\&\quad\times(A+2-12m)(A-6-12m)(A-10-12m)(A-6+12m)(A-2+12m)\\&\quad\times(A+6+12m)(A+10+12m).\end{aligned}$$

Similarly, $s_{\beta_1}(-4\beta_2) - 2\beta_1 = -2\alpha_1$, and we compute that $\operatorname{res}_{B=2}(a_{-4\beta_2}) = -\operatorname{res}_{B=2}(a_{-2\alpha_1})$ equals

$$\begin{aligned}&288m(m+1)(A-10)^{-1}(A-6)^{-4}(A-2)^{-2}A^{-1}(A+2)^{-1}(A+6)^{-1}(A-6+6m)\\&\quad\times(A+6m)(A-10+12m)(A-6+12m)^2(A-2+12m)(A+2+12m)\\&\quad\times(A+6+12m)^2(A+10+12m).\end{aligned}$$

Since $s_{\alpha_1}(-4\beta_2) = 4\beta_3$ and $s_{\alpha_1}(-2\alpha_1) = 2\alpha_1$, it follows by Lemma 3.12 that the residue of $a_{4\beta_3} + a_{2\alpha_1}$ at $B = 2$ is also zero. Next, note that $s_{\beta_1}(2\beta_2) - 2\beta_1 = 2\alpha_2$, and we compute that $\operatorname{res}_{B=2}(a_{2\beta_2}) = -\operatorname{res}_{B=2}(a_{2\alpha_2})$ equals

$$\begin{aligned}&36m(m+1)^2(3m-1)(3m+1)(A-6)^{-1}(A-2)^{-2}A^{-1}(A+2)^{-1}(A+6)^{-4}(A-26)(A+30)\\&\quad\times(A-10-12m)(A-6-12m)(A-2-12m)(A+6-12m)^2(A-6m)(A+6+12m).\end{aligned}$$

Since $s_{\alpha_1}(2\beta_2) = -2\beta_3$ and $s_{\alpha_1}(2\alpha_2) = -2\alpha_3$, it follows by Lemma 3.12 that the residue of $a_{-2\beta_3} + a_{-2\alpha_3}$ at $B = 2$ is also zero. Thus, by Theorem 3.11 part 1, there is no singularity at $B = 2$ in $\mathcal{D}_2 p(z)$.

Let us now consider possible singularities in $\mathcal{D}_2 p(z)$ at $A = c \geq 0$. By Theorem 3.11 part 1 and the form of the coefficients (3.27)–(3.29), it is sufficient to consider the case $A = 6$ (equivalently, $\langle \alpha_1, z\rangle = \alpha_1^2$). In this case $\langle \beta_2, z\rangle = \frac{1}{2}B + 3$, $\langle \beta_3, z\rangle = -\frac{1}{2}B + 3$, $\langle \alpha_2, z\rangle = -\frac{3}{2}B + 3$, and $\langle \alpha_3, z\rangle = \frac{3}{2}B + 3$. The only $\tau \in 2AG_2$ for which a_τ is singular at $A = 6$ and for which the corresponding $\lambda = s_{\alpha_1}(\tau) - 2\alpha_1 \neq 0$ are $\tau = -4\beta_2, -4\beta_3$, and $\pm 2\beta_1$. Note that $s_{\alpha_1}(-4\beta_2) - 2\alpha_1 = -2\beta_1$, and we compute that $\operatorname{res}_{A=6}(a_{-4\beta_2}) = -\operatorname{res}_{A=6}(a_{-2\beta_1})$ equals

$$\begin{aligned}&96m(m+1)(B-6)^{-2}(B-2)^{-4}B^{-1}(B+2)^{-2}(B+6)^{-1}(B-14-12m)(B-2-12m)\\&\quad\times(B-2+4m)(B+2+4m)(B-2+6m)(B+4+6m)(B-6+12m)(B+2+12m)\end{aligned}$$

$$\times (B + 6 + 12m)(B + 14 + 12m).$$

Since $s_{\beta_1}(-4\beta_2) = -4\beta_3$ and $s_{\beta_1}(-2\beta_1) = 2\beta_1$, it follows by Lemma 3.12 that the residue of $a_{-4\beta_3} + a_{2\beta_1}$ at $A = 6$ is also zero. By Theorem 3.11 part 1, there is thus no singularity at $A = 6$ in $\mathcal{D}_2 p(z)$.

By Corollary 3.13, it follows that $\mathcal{D}_2 p(z)$ has no singularities. The proof that $\mathcal{D}_2 p(z)$ satisfies the axiomatics of $\mathcal{R}^a_{AG_2}$ can be completed in an analogous way to how it was done for the operator (3.17) in the proof of Theorem 3.20. ■

We now give a second construction of the BA function for AG_2. The proof of the next theorem is similar to the proof of Theorem 3.24.

Theorem 3.27. *Let* $M = \sum_{\gamma\in AG_{2,+}} c_\gamma = 12m + 3$. *Let*

$$\widetilde{\mu}(x) = \sum_{\tau:\, \frac{1}{2}\tau\in AG_2} \lambda_\tau (e^{\langle \tau, x\rangle} - 1), \tag{3.30}$$

and let

$$\widetilde{b}(x) = \frac{M!}{8} \prod_{\gamma\in G_{2,+}} \left(\sum_{\tau:\, \frac{1}{2}\tau\in AG_2} \lambda_\tau \langle \tau, \gamma\rangle e^{\langle \tau, x\rangle} \right)^{c_\gamma + c_{2\gamma}}. \tag{3.31}$$

Then the function

$$\psi(z, x) = \widetilde{b}^{-1}(x)(\mathcal{D}_2 - \widetilde{\mu}(x))^M [Q(z)e^{\langle z, x\rangle}], \tag{3.32}$$

where the polynomial $Q(z)$ *is given by* (3.24), *is the BA function for* AG_2. *Moreover,* ψ *is also an eigenfunction of the operator* $\mathcal{D}_2$ *with* $\mathcal{D}_2\psi = \widetilde{\mu}(x)\psi$. *Thus, the operator* $\mathcal{D}_2$ *is bispectrally dual to the* AG_2 *generalised CMS Hamiltonian.*

We can extend the above bispectral duality statement as follows, in the spirit of analogous results from [24, 49] for other configurations. By Theorem 3.7, the BA function of AG_2 is a common eigenfunction for a large commutative ring of differential operators in x, and the following theorem states that a similar situation occurs in the variables z.

Theorem 3.28. *Let* $p(z) \in \mathcal{R}^a_{AG_2}$ *be a polynomial, and let* p_0 *be its highest-degree homogeneous term. Then the difference operator* D_p *acting in* z *given by*

$$D_p = \frac{1}{(\deg p)!} \operatorname{ad}^{\deg p}_{\mathcal{D}_1}(\widehat{p}),$$

where $\widehat{p}$ *is the operator of multiplication by* $p(z)$ *and* ad^r_A *is the* r*-th iteration of the operation* $\operatorname{ad}_A(B) = AB - BA$, *satisfies*

$$D_p\psi(z, x) = \mu_p(x)\psi(z, x)$$

for the function $\mu_p(x)$ obtained by substituting $\partial_{x_i}(\mu)$ in place of z_i $(i = 1, 2)$ into $p_0(z)$.

The operators $\mathcal{D}_1$ and $\mathcal{D}_2$ commute, and the operators D_p commute with $\mathcal{D}_i$ $(i = 1, 2)$ and with each other.

Proof. By Theorem 3.7, there is a differential operator $L_p(x, \partial_x)$ satisfying $L_p\psi = p(z)\psi$ with highest term $p_0(\partial_x)$. By Theorem 3.24, we have $\mathcal{D}_1\psi = \mu\psi$. The fact that ψ is an eigenfunction of the operators D_p follows by a standard argument about bispectral systems (see, e.g., the proof of [24, Theorem 4.1]) which gives that

$$D_p\psi = \frac{(-1)^{\deg p}}{(\deg p)!}\,\mathrm{ad}_\mu^{\deg p}(L_p)\psi.$$

Here $\mathrm{ad}_\mu^{\deg p}(L_p) = \mathrm{ad}_\mu^{\deg p}(p_0(\partial_x))$ is a zeroth-order operator (that is, an ordinary function of x), since each application of ad_μ decreases the order of a linear differential operator and $L_p = p_0(\partial_x)+$ lower terms. Explicitly, $\mathrm{ad}_\mu^{\deg p}(p_0(\partial_x)) = (-1)^{\deg p}(\deg p)!\, p_0(\partial_{x_1}(\mu), \partial_{x_2}(\mu))$, which can be proven by induction on the degree of p_0 (and note that it is enough to consider p_0 that are monomial). Indeed, it holds if $\deg p_0 = 1$; and assuming that it holds for some degree $A = a_1 + a_2 - 1 \geq 1$ $(a_1, a_2 \in \mathbb{Z}_{\geq 0})$, we get

$$\begin{aligned}
\mathrm{ad}_\mu^{A+1}(\partial_{x_1}^{a_1}\partial_{x_2}^{a_2}) &= \mathrm{ad}_\mu^{A}\left([\mu, \partial_{x_1}^{a_1}\partial_{x_2}^{a_2}]\right) = \mathrm{ad}_\mu^{A}\left(a_1[\mu, \partial_{x_1}]\partial_{x_1}^{a_1-1}\partial_{x_2}^{a_2} + a_2[\mu, \partial_{x_2}]\partial_{x_1}^{a_1}\partial_{x_2}^{a_2-1}\right)\\
&= -a_1\partial_{x_1}(\mu)\,\mathrm{ad}_\mu^{A}\left(\partial_{x_1}^{a_1-1}\partial_{x_2}^{a_2}\right) - a_2\partial_{x_2}(\mu)\,\mathrm{ad}_\mu^{A}\left(\partial_{x_1}^{a_1}\partial_{x_2}^{a_2-1}\right)\\
&= (-1)^{A+1}(A+1)!\,\partial_{x_1}(\mu)^{a_1}\partial_{x_2}(\mu)^{a_2},
\end{aligned}$$

as required, where we used the Leibniz rule, that $[\mu, \partial_{x_i}] = -\partial_{x_i}(\mu)$ is just a function, that ad_μ^A annihilates any operator of order less than A, and the induction hypothesis. This implies that $\mu_p(x)$ is as stated.

From formulas (3.17)–(3.19) for $\mathcal{D}_1$ and (3.22) for μ, it is clear that the BA function (3.25) can be expanded as

$$\psi = b(x)^{-1}e^{\langle z,x\rangle}\sum_{\nu\in 2\mathbb{Z}\beta_1\oplus 2\mathbb{Z}\alpha_2} b_\nu(z)e^{\langle \nu,x\rangle}$$

for some polynomials $b_\nu(z)$, where only finitely-many b_ν are non-zero. From there, it can be seen easily that if a finite difference operator $\widetilde{D}$ in z (with, say, rational coefficients) is such that $\widetilde{D}\psi = 0$ identically, then $\widetilde{D} = 0$. It follows that the operators D_p, $\mathcal{D}_1$, and $\mathcal{D}_2$ commute pairwise as ψ is their common eigenfunction. ∎

Remark 3.29. An alternative proof that the operators $\mathcal{D}_1$ and $\mathcal{D}_2$ commute is that they are the rational limit of certain trigonometric difference operators which we derive in Chapter 5 below and whose commutativity follows from the theory of DAHAs.

Remark 3.30. Using ideas similar to those in [24, Theorem 6.9], one can show that the AG_2 generalised CMS Hamiltonian is bispectrally dual to the operators $\mathcal{D}_1$ and $\mathcal{D}_2$ for non-integer values of the multiplicities as well, as we explain in Section 3.6.2 below.

3.4.5 Relation with A_2 and A_1 MR operators

In the case when $m = 0$, the configuration AG_2 reduces to the root system $A_2 = \{\pm 2\beta_i : i = 1, 2, 3\}$ with multiplicity 1 for all vectors. In this limit, the operator (3.26) reduces to the quasi-minuscule operator for (twice) this root system. Let us now consider the $m = 0$ limit of the operator (3.17). After a rescaling, this gives an operator of the form

$$D_0 = \sum_{\tau \in G_2} a_{\tau,0}(z)(T_\tau - 1) = -24 + \sum_{\tau \in G_2} a_{\tau,0}(z) T_\tau, \tag{3.33}$$

where for $\tau = \varepsilon\beta_j$, $\varepsilon \in \{\pm 1\}$, and $j = 1, 2, 3$, we have

$$a_{\tau,0}(z) = 3 \prod_{\substack{\gamma \in W\beta_1 \\ \langle \tau, (2\gamma)^\vee \rangle = \frac{1}{2}}} \left(1 - \frac{\frac{1}{2}\gamma^2}{\langle \gamma, z\rangle - \frac{1}{2}\gamma^2}\right) \prod_{\substack{\gamma \in W\beta_1 \\ \langle \tau, (2\gamma)^\vee \rangle = 1}} \left(1 - \frac{\gamma^2}{\langle \gamma, z\rangle}\right),$$

and for $\tau = \varepsilon\alpha_j$, we have

$$a_{\tau,0}(z) = \prod_{\substack{\gamma \in W\beta_1 \\ \langle \tau, (2\gamma)^\vee \rangle = \frac{3}{2}}} \left(1 - \frac{\frac{3}{2}\gamma^2}{\langle \gamma, z\rangle + \frac{1}{2}\gamma^2}\right).$$

Proposition 3.31. *The operator* (3.33) *preserves the ring of analytic functions $p(z)$ such that $p(z + \beta_i) = p(z - \beta_i)$ at $\langle \beta_i, z\rangle = 0$ for all $i = 1, 2, 3$.*

The proof is parallel to the proof of Theorem 3.20. In this case, though, condition 2(b) of Theorem 3.11 is needed while it does not play a role in the proofs of Theorems 3.20 and 3.26.

Let us rewrite the operator D_0 for the more standard realisation of the root system A_2 given by $A_2 = \{e_i - e_j : 1 \leq i \neq j \leq 3\} \subset \mathbb{R}^3$, where e_i are the standard basis vectors.

Proposition 3.32. *Define the set $S = S_1 \cup S_2$, where*

$$S_1 = \{3e_i : i = 1, 2, 3\} \cup \{2e_i + 2e_j - e_k : 1 \leq i < j \neq k \leq 3,\ i \neq k\},$$

$$S_2 = \{2e_i + e_j : 1 \leq i \neq j \leq 3\}.$$

Then the operator acting in the variables $z = (z_1, z_2, z_3) \in \mathbb{C}^3$ *given by*

$$\widetilde{D}_0 = 3 \sum_{\tau \in S_2} \left(\prod_{\substack{i \neq j \\ \langle \tau, e_i - e_j \rangle = 1}} \left(1 - \frac{1}{z_i - z_j - 1}\right) \prod_{\substack{i \neq j \\ \langle \tau, e_i - e_j \rangle = 2}} \left(1 - \frac{2}{z_i - z_j}\right) \right) T_\tau \\ + \sum_{\tau \in S_1} \left(\prod_{\substack{i \neq j \\ \langle \tau, e_i - e_j \rangle = 3}} \left(1 - \frac{3}{z_i - z_j + 1}\right) \right) T_\tau. \tag{3.34}$$

preserves the ring of analytic functions $p(z)$ *satisfying* $p(z + e_i - e_j) = p(z - e_i + e_j)$ *at* $z_i = z_j$ *for all* $i, j = 1, 2, 3$.

Notice that $S_1 = \{\tau \in S \colon |\langle \tau, e_i - e_j \rangle| \in \{0, 3\}$ for all $i, j = 1, 2, 3\}$ and that $S_2 = \{\tau \in S \colon |\langle \tau, e_i - e_j \rangle| \in \{0, 1, 2\}$ for all $i, j = 1, 2, 3\}$.

Let us now consider a version of the operator (3.34) for the root system A_1. Let $\sim$ denote equality of operators when acting on functions constant along the direction normal to the hyperplane $z_1 + z_2 = 0$.

Proposition 3.33. *Let* $S_1' = \{3e_1, 3e_2\}$ *and* $S_2' = \{2e_1 + e_2, e_1 + 2e_2\}$. *Then formula* (3.34) *after replacement of* S_i *with* S_i', $i = 1, 2$, *gives an operator* $\widehat{D}_0$ *acting in the variable* $z = (z_1, z_2) \in \mathbb{C}^2$ *that preserves the ring* $\mathcal{R}^a_{A_1}$ *of analytic functions* $p(z)$ *satisfying that* $p(z + e_1 - e_2) = p(z - e_1 + e_2)$ *at* $z_1 = z_2$. *Moreover, if we split the operator as* $\widehat{D}_0 = D_1 + D_2$, *where*

$$D_1 = 3\left(1 - \frac{1}{z_2 - z_1 - 1}\right) T_{e_1 + 2e_2} + \left(1 - \frac{3}{z_1 - z_2 + 1}\right) T_{3e_1}$$

and

$$D_2 = 3\left(1 - \frac{1}{z_1 - z_2 - 1}\right) T_{2e_1 + e_2} + \left(1 - \frac{3}{z_2 - z_1 + 1}\right) T_{3e_2},$$

then $D_i(\mathcal{R}^a_{A_1}) \subseteq \mathcal{R}^a_{A_1}$ *for* $i = 1, 2$. *The operators* D_i *satisfy the commutativity relations*

$$[D_1, D_2] = [D_1, D^{msl}] = [D_2, D^{msl}] = 0,$$

where D^{msl} *is the operator for the minuscule weight* $2e_1$ *of the root system* $2A_1$ *with multiplicity 1 given by*

$$D^{msl} = \left(1 - \frac{2}{z_1 - z_2}\right) T_{2e_1} + \left(1 - \frac{2}{z_2 - z_1}\right) T_{2e_2}.$$

We also have $\widehat{D}_0^2 \sim (D^{msl} + 2)^3$, *and* $D_1 D_2 \sim 3D^{qm} + 16 \sim 3(D^{msl})^2 + 4$, *where*

$$D^{qm} = \left(1 - \frac{2}{z_1 - z_2}\right)\left(1 - \frac{2}{z_1 - z_2 + 2}\right)(T_{4e_1} - 1)$$

$$+\left(1-\frac{2}{z_2-z_1}\right)\left(1-\frac{2}{z_2-z_1+2}\right)(T_{4e_2}-1)$$

is the operator for the quasi-minuscule weight $4e_1$ *of the root system* $2A_1$.

We note that the operators D_1 and D_2 are not symmetric under the swap of the variables z_1 and z_2. The operator D^{msl} is symmetric, and all three operators commute.

3.5 Deformed root system $BC(l,1)$

In this section, we show that a difference operator of rational MR type introduced by Sergeev and Veselov in [95] for the deformed root system $BC(l,1)$ preserves a ring of quasi-invariants in the case of non-negative integer values of the multiplicity parameters. We prove that in this case the operator admits a BA eigenfunction that, as a function of the spectral variables, is an eigenfunction for the generalised CMS Hamiltonian associated with $BC(l,1)$. By an analytic continuation argument, we generalise this eigenfunction later in Section 3.6.1 also to the case of more general complex values of the multiplicities.

This leads to a bispectral duality statement for the corresponding generalised CMS system of type $BC(l,1)$. In particular, for special integer values of the multiplicities, one recovers the results from [49] for the configuration $C(l,1)$. Another bispectrality property of the generalised CMS operator for $BC(l,1)$ (as well as for $BC(l,l')$) in terms of super-Jacobi polynomials was proved in [95].

Let us now describe the configuration $BC(l,1)$ in more detail. Recall that the root system BC_{l+1} has a positive half

$$BC_{l+1,+}=\{e_i,\,2e_i\colon 1\le i\le l+1\}\cup\{e_i\pm e_j\colon 1\le i<j\le l+1\},$$

where e_i denote the standard orthogonal unit vectors in $\mathbb{R}^{l+1}$. Its deformation $BC(l,1)$ has a positive half

$$\begin{aligned}BC(l,1)_+=\{e_i,\,2e_i,\,e_i\pm\sqrt{k}e_{l+1}\colon 1\le i\le l\}\cup\{\sqrt{k}e_{l+1},\,2\sqrt{k}e_{l+1}\}\\ \cup\{e_i\pm e_j\colon 1\le i<j\le l\}\subset\mathbb{C}^{l+1},\end{aligned}$$

where k is a non-zero complex parameter [19,94]. Let $BC(l,1)^r$ denote the reduced version of this system with a positive half

$$\begin{aligned}BC(l,1)^r_+&=\{\alpha\in BC(l,1)_+\colon \tfrac{1}{2}\alpha\notin BC(l,1)\}\\ &=\{e_i,\,e_i\pm\sqrt{k}e_{l+1}\colon 1\le i\le l\}\cup\{\sqrt{k}e_{l+1}\}\cup\{e_i\pm e_j\colon 1\le i<j\le l\}.\end{aligned}$$

The set $BC(l,1)_+$ has its multiplicity map given by $c_{e_i}=m$, $c_{2e_i}=n$, $c_{e_i\pm\sqrt{k}e_{l+1}}=1$,

$c_{e_i \pm e_j} = k$, $c_{\sqrt{k}e_{l+1}} = p$, and $c_{2\sqrt{k}e_{l+1}} = r$ for complex parameters m, n, p, r, subject to the constraint that $m = kp$ and $2n+1 = k(2r+1)$. For $m = p = 0$, the configuration $BC(l,1)$ reduces to the configuration $C(l,1)$, which was considered in [49]. For $k = 1$, the configuration $BC(l,1)$ reduces to the root system BC_{l+1} with a Weyl-invariant assignment of multiplicities such that the vectors $e_i \pm e_j$ for $1 \le i < j \le l+1$ have multiplicity 1. In Sections 3.5.1 and 3.5.2 below, we assume that $m, n, p, r \in \mathbb{Z}_{\ge 0}$, and if $l > 1$ then also that $k \in \mathbb{Z}_{>0}$.

If one puts $\mathcal{A} = BC(l,1)_+$ in the formula (3.1), one obtains the generalised CMS operator associated with the configuration $BC(l,1)$ [94].

3.5.1 Sergeev–Veselov difference operator for $BC(l,1)$

In this section, we recall the rational difference operator introduced by Sergeev and Veselov for the deformed root system $BC(l,1)$ [95], which deforms the rational version of Koornwinder's operator and also generalises an operator associated with $C(l,1)$ from [49]. We prove that, when all the multiplicity parameters are non-negative integers, the operator preserves the ring of quasi-invariants $\mathcal{R}^a_{BC(l,1)}$ consisting of those analytic functions $f\colon \mathbb{C}^{l+1} \to \mathbb{C}$ such that

$$f(z + s\alpha) = f(z - s\alpha) \text{ at } \langle z, \alpha \rangle = 0 \tag{3.35}$$

for all $\alpha \in BC(l,1)^r_+$ and $s \in A_\alpha = \{1, 2, \ldots, c_\alpha\} \cup \{c_\alpha + 2, c_\alpha + 4, \ldots, c_\alpha + 2c_{2\alpha}\}$.

Let $z = (z_1, \ldots, z_{l+1}) \in \mathbb{C}^{l+1}$. The difference operator for $BC(l,1)$ introduced in [95] has the form

$$\begin{aligned} D = \sum_{i=1}^{l} \big(a_{2e_i}(z)(T_{2e_i} - 1) + a_{-2e_i}(z)(T_{-2e_i} - 1)\big) \\ + a_{2\sqrt{k}e_{l+1}}(z)\big(T_{2\sqrt{k}e_{l+1}} - 1\big) + a_{-2\sqrt{k}e_{l+1}}(z)\big(T_{-2\sqrt{k}e_{l+1}} - 1\big), \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} a_{\pm 2e_i}(z) = \left(1 \mp \frac{m+2n}{z_i}\right)\left(1 \mp \frac{m}{z_i \pm 1}\right) \prod_{\substack{j=1 \\ j \ne i}}^{l} \left(1 \mp \frac{2k}{z_i - z_j}\right)\left(1 \mp \frac{2k}{z_i + z_j}\right) \\ \times \left(1 \mp \frac{2}{z_i + \sqrt{k}z_{l+1} \pm (1-k)}\right)\left(1 \mp \frac{2}{z_i - \sqrt{k}z_{l+1} \pm (1-k)}\right), \end{aligned}$$

$$a_{\pm 2\sqrt{k}e_{l+1}}(z) = \frac{1}{k}\left(1 \mp \frac{\sqrt{k}(p+2r)}{z_{l+1}}\right)\left(1 \mp \frac{\sqrt{k}p}{z_{l+1} \pm \sqrt{k}}\right)$$

$$\times \prod_{i=1}^{l} \left(1 \mp \frac{2k}{\sqrt{k}z_{l+1} + z_i \pm (k-1)}\right)\left(1 \mp \frac{2k}{\sqrt{k}z_{l+1} - z_i \pm (k-1)}\right).$$

Let us now assume that $m, n, p, r \in \mathbb{Z}_{\geq 0}$, and if $l > 1$ then also that $k \in \mathbb{Z}_{>0}$. In order to prove below in Theorem 3.35 that the operator D in this case preserves the ring $\mathcal{R}^a_{BC(l,1)}$, we first establish the following lemma.

Lemma 3.34. *Let $f \in \mathcal{R}^a_{BC(l,1)}$. Then Df is analytic.*

Proof. Based on the form of the functions $a_{\pm 2e_i}$ and $a_{\pm 2\sqrt{k}e_{l+1}}$, the only possible singularities of Df are potential simple poles at $z_i = 0$ for $1 \leq i \leq l+1$, as well as $z_{l+1} = \pm\sqrt{k}$, $z_i = \pm 1$, $z_i = \pm\sqrt{k}z_{l+1} + k - 1$, and $z_i = \pm\sqrt{k}z_{l+1} + 1 - k$ for $1 \leq i \leq l$, and also $z_i = \pm z_j$ for $1 \leq i < j \leq l$. The strategy is to show that at each of the possible poles, Df has zero residue. It will follow that Df is analytic everywhere. We describe the computation of the residue for most of the cases, the procedure for the remaining ones being analogous. Let us denote, as before, the residue of a function f at a simple pole $z_i = a \in \mathbb{C}$ by

$$\operatorname{res}_{z_i=a}(f) = \lim_{z_i \to a} (z_i - a) f(z).$$

We may assume, for simplicity, that $m \neq 0 \neq p$, as the case $m = p = 0$ was covered in [49]. Let us first compute the residue of $Df(z)$ at $z_i = 0$ for $1 \leq i \leq l$. We note that

$$a_{2e_i}(z) = a_{-2e_i}(s_{e_i}(z)), \tag{3.37}$$

which implies

$$\operatorname{res}_{z_i=0}(a_{2e_i}) = -\operatorname{res}_{z_i=0}(a_{-2e_i}).$$

Also, if $m \neq 1$ then

$$T_{2e_i} f(z)|_{z_i=0} = T_{-2e_i} f(z)|_{z_i=0},$$

where we use that $f \in \mathcal{R}^a_{BC(l,1)}$. Note that if $m = 1$ then $\operatorname{res}_{z_i=0}(a_{\pm 2e_i}) = 0$. It follows that for any $m \in \mathbb{Z}_{>0}$ the residue of $Df(z)$ at $z_i = 0$ is zero. Its residue at $z_{l+1} = 0$ can be shown to vanish in an analogous way.

Next, we consider $z_i = -1$. The only coefficient function in D that has a pole there is a_{2e_i}, and so the property $\operatorname{res}_{z_i=-1}(Df) = 0$ follows from the fact that

$$(T_{2e_i} f(z) - f(z))|_{z_i=-1} = 0,$$

where we again used that $f \in \mathcal{R}^a_{BC(l,1)}$. Similarly, the residues of $Df(z)$ at $z_i = 1$ and $z_{l+1} = \pm\sqrt{k}$ are zero.

Let now $1 \leq i < j \leq l$, and let us compute the residue of $Df(z)$ at $z_i = z_j$. We note that

$$a_{\pm 2e_i}(z) = a_{\pm 2e_j}(s_{e_i - e_j}(z)), \tag{3.38}$$

which implies

$$\operatorname{res}_{z_i=z_j}(a_{\pm 2e_i}) = -\operatorname{res}_{z_i=z_j}(a_{\pm 2e_j}).$$

Also, for any $u \in \mathbb{C}$ and $q = u+1$, we have

$$T_{2e_i} f(z)|_{z_i=z_j=u} = f(z+e_i-e_j)|_{z_i=z_j=q} = f(z-e_i+e_j)|_{z_i=z_j=q} = T_{2e_j} f(z)|_{z_i=z_j=u},$$

where in the penultimate equality we used that $f \in \mathcal{R}^a_{BC(l,1)}$. Similarly, $T_{-2e_i} f(z) = T_{-2e_j} f(z)$ at $z_i = z_j$. It follows that the residue of $Df(z)$ at $z_i = z_j$ is zero. The case of $z_i = -z_j$ is similar.

Finally, we consider $z_i = -\sqrt{k} z_{l+1} + k - 1$ for $1 \le i \le l$. We calculate the following residues:

$$\operatorname{res}_{z_i=-\sqrt{k}z_{l+1}+k-1}(a_{2e_i}) = -2\left(1+\frac{m+2n}{\sqrt{k}z_{l+1}+1-k}\right)\left(1+\frac{m}{\sqrt{k}z_{l+1}-k}\right)$$
$$\times\left(1+\frac{1}{\sqrt{k}z_{l+1}}\right)\prod_{\substack{j=1\\ j\neq i}}^{l}\left(1+\frac{2k}{\sqrt{k}z_{l+1}+z_j+1-k}\right)\left(1+\frac{2k}{\sqrt{k}z_{l+1}-z_j+1-k}\right),$$

$$\operatorname{res}_{z_i=-\sqrt{k}z_{l+1}+k-1}(a_{-2\sqrt{k}e_{l+1}}) = 2\left(1+\frac{\sqrt{k}(p+2r)}{z_{l+1}}\right)\left(1+\frac{\sqrt{k}p}{z_{l+1}-\sqrt{k}}\right)$$
$$\times\left(1+\frac{k}{\sqrt{k}z_{l+1}+1-k}\right)\prod_{\substack{j=1\\ j\neq i}}^{l}\left(1+\frac{2k}{\sqrt{k}z_{l+1}+z_j+1-k}\right)\left(1+\frac{2k}{\sqrt{k}z_{l+1}-z_j+1-k}\right).$$

In order to compare them, we observe that

$$\frac{\sqrt{k}p}{z_{l+1}-\sqrt{k}} = \frac{m}{\sqrt{k}z_{l+1}-k},$$

since $m = kp$; moreover, since $2n+1 = k(2r+1)$, we also have

$$\left(1+\frac{\sqrt{k}(p+2r)}{z_{l+1}}\right)\left(1+\frac{k}{\sqrt{k}z_{l+1}+1-k}\right)$$
$$= \left(\frac{\sqrt{k}z_{l+1}+m+2n+1-k}{\sqrt{k}z_{l+1}}\right)\left(\frac{\sqrt{k}z_{l+1}+1}{\sqrt{k}z_{l+1}+1-k}\right)$$
$$= \left(1+\frac{m+2n}{\sqrt{k}z_{l+1}+1-k}\right)\left(1+\frac{1}{\sqrt{k}z_{l+1}}\right).$$

Therefore,

$$\operatorname{res}_{z_i=-\sqrt{k}z_{l+1}+k-1}(a_{2e_i}) = -\operatorname{res}_{z_i=-\sqrt{k}z_{l+1}+k-1}(a_{-2\sqrt{k}e_{l+1}}).$$

Also, for any $u \in \mathbb{C}$ and $q = -u + \sqrt{k}$, we have

$$\begin{aligned} T_{2e_i} f(z)|_{z_i=-\sqrt{k}u+k-1,\, z_{l+1}=u} &= f(z + e_i + \sqrt{k}e_{l+1})|_{z_i=\sqrt{k}q,\, z_{l+1}=-q} \\ &= f(z - e_i - \sqrt{k}e_{l+1})|_{z_i=\sqrt{k}q,\, z_{l+1}=-q} = T_{-2\sqrt{k}e_{l+1}} f(z)|_{z_i=-\sqrt{k}u+k-1,\, z_{l+1}=u}, \end{aligned}$$

where in the penultimate equality we used that $f \in \mathcal{R}^a_{BC(l,1)}$. It follows that the residue of $Df(z)$ at $z_i = -\sqrt{k}z_{l+1} + k - 1$ is zero. The cases $z_i = -\sqrt{k}z_{l+1} + 1 - k$ and $z_i = \sqrt{k}z_{l+1} \pm (1 - k)$ are similar. This completes the proof. ■

Theorem 3.35. *Let $f \in \mathcal{R}^a_{BC(l,1)}$. Then $Df \in \mathcal{R}^a_{BC(l,1)}$.*

Proof. By Lemma 3.34, Df is analytic, so it only remains to show that Df satisfies the functional identities (3.35). Let $\alpha \in BC(l,1)^r_+$ and $s \in A_\alpha$.

Suppose $\alpha = e_i$ for $1 \leq i \leq l$. Then at $\langle \alpha, z\rangle = 0$, we have $a_{2e_i}(z + s\alpha) = a_{-2e_i}(z - s\alpha)$ by equality (3.37). And for all $j \neq i$, it is straightforward to see that $a_{\pm 2e_j}(z + s\alpha) = a_{\pm 2e_j}(z - s\alpha)$ and $a_{\pm 2\sqrt{k}e_{l+1}}(z + s\alpha) = a_{\pm 2\sqrt{k}e_{l+1}}(z - s\alpha)$ at $\langle \alpha, z\rangle = 0$. For $s \neq 1$, we have $a_{-2e_i}(z + s\alpha) = a_{2e_i}(z - s\alpha)$ at $\langle \alpha, z\rangle = 0$ by equality (3.37) (for $s = 1$, the functions $a_{-2e_i}(z + s\alpha)$ and $a_{2e_i}(z - s\alpha)$ are singular at $\langle \alpha, z\rangle = 0$ and we will deal with this case separately).

Now observe that for $s \neq m - 1, m + 2n$, we have $s + 2 \in A_{e_i}$ and then since $f \in \mathcal{R}^a_{BC(l,1)}$, we have $T_{2e_i} f(z + se_i)|_{z_i=0} = T_{-2e_i} f(z - se_i)|_{z_i=0}$. On the other hand, if $s \in \{m - 1, m + 2n\}$ then $a_{2e_i}(z + se_i)|_{z_i=0} = 0 = a_{-2e_i}(z - se_i)|_{z_i=0}$. Also, since $\langle z \pm 2e_j, e_i\rangle = \langle z \pm 2\sqrt{k}e_{l+1}, e_i\rangle = z_i$, we have for any $s \in A_{e_i}$ that $T_{\pm 2e_j} f(z+se_i)|_{z_i=0} = T_{\pm 2e_j} f(z-se_i)|_{z_i=0}$ and $T_{\pm 2\sqrt{k}e_{l+1}} f(z + se_i)|_{z_i=0} = T_{\pm 2\sqrt{k}e_{l+1}} f(z - se_i)|_{z_i=0}$. And for $s \neq 1$, we have $s - 2 \in A_{e_i} \cup \{0\}$ and hence $T_{-2e_i} f(z + se_i)|_{z_i=0} = T_{2e_i} f(z - se_i)|_{z_i=0}$. It follows from this and the previous paragraph that the identities (3.35) are satisfied for Df for $\alpha = e_i$ for any $s \in A_{e_i} \setminus \{1\}$.

Let us now deal with the case $s = 1$. The property that

$$\left(f(z - e_i) - f(z + e_i)\right)|_{z_i=0} = 0$$

can be restated as $f(z - e_i) - f(z + e_i) = z_i g(z)$ for some analytic function $g(z)$. Thus, we have

$$\begin{aligned} \lim_{z_i\to 0} a_{-2e_i}(z + e_i)(T_{-2e_i} - 1)f(z + e_i) &= g(z)|_{z_i=0} \lim_{z_i\to 0} z_i a_{-2e_i}(z + e_i) \\ &\overset{(3.37)}{=} -g(z)|_{z_i=0} \lim_{z_i\to 0} z_i a_{2e_i}(z - e_i) = \lim_{z_i\to 0} a_{2e_i}(z - e_i)(T_{2e_i} - 1)f(z - e_i). \end{aligned}$$

It follows from this and the previous two paragraphs that the identity (3.35) for Df holds for $\alpha = e_i$ also when $s = 1$. The case of $\alpha = \sqrt{k}e_{l+1}$ can be dealt with in an analogous way.

Next, suppose $\alpha = e_i - e_j$ for $1 \leq i < j \leq l$. Then $a_{\pm 2e_i}(z + \widetilde{s}\alpha) = a_{\pm 2e_j}(z - \widetilde{s}\alpha)$ at $\langle \alpha, z\rangle = 0$ for $\widetilde{s} \in \{\pm s\}$ by equality (3.38). If $s \neq k$ then $s + 1 \in A_{e_i - e_j}$ and then for any $u \in \mathbb{C}$ and $q = u + 1$, we have

$$\begin{aligned} T_{2e_i} f(z + s(e_i - e_j))|_{z_i = z_j = u} &= f(z + (s+1)(e_i - e_j))|_{z_i = z_j = q} \\ &= f(z - (s+1)(e_i - e_j))|_{z_i = z_j = q} = T_{2e_j} f(z - s(e_i - e_j))|_{z_i = z_j = u}, \end{aligned}$$

where we used that $f \in \mathcal{R}^a_{BC(l,1)}$; and similarly $T_{-2e_j} f(z + s\alpha) = T_{-2e_i} f(z - s\alpha)$ at $\langle \alpha, z\rangle = 0$. On the other hand, if $s = k$ then $a_{2e_i}(z + s\alpha) = 0 = a_{-2e_j}(z + s\alpha)$ at $\langle \alpha, z\rangle = 0$. Moreover, for any $s \in A_{e_i - e_j}$, we have $s - 1 \in A_{e_i - e_j} \cup \{0\}$, which can be used to see that $T_{-2e_i} f(z + s\alpha) = T_{-2e_j} f(z - s\alpha)$ and $T_{2e_j} f(z + s\alpha) = T_{2e_i} f(z - s\alpha)$ at $\langle \alpha, z\rangle = 0$.

For all $t \neq i, j$ and $\langle \alpha, z\rangle = 0$, it is straightforward to see that $a_{\pm 2e_t}(z + s\alpha) = a_{\pm 2e_t}(z - s\alpha)$ and $a_{\pm 2\sqrt{k}e_{l+1}}(z + s\alpha) = a_{\pm 2\sqrt{k}e_{l+1}}(z - s\alpha)$; and that $T_{\pm 2e_t} f(z + s\alpha) = T_{\pm 2e_t} f(z - s\alpha)$ and $T_{\pm 2\sqrt{k}e_{l+1}} f(z + s\alpha) = T_{\pm 2\sqrt{k}e_{l+1}} f(z - s\alpha)$ since $f \in \mathcal{R}^a_{BC(l,1)}$ and $\langle z \pm 2e_t, \alpha\rangle = \langle z \pm 2\sqrt{k}e_{l+1}, \alpha\rangle = 0$.

It follows from the above two paragraphs that the identities (3.35) are satisfied for Df for $\alpha = e_i - e_j$. The case of $\alpha = e_i + e_j$ is analogous.

Finally, suppose $\alpha = e_i + \sqrt{k}e_{l+1}$ for $1 \leq i \leq l$. Note that $A_{e_i + \sqrt{k}e_{l+1}} = \{1\}$, so $s = 1$. For $\varepsilon \in \{\pm 1\}$,

$$a_{2\varepsilon e_i}(z + \varepsilon(e_i + \sqrt{k}e_{l+1}))|_{z_i = -\sqrt{k}z_{l+1}} = 0 = a_{2\varepsilon\sqrt{k}e_{l+1}}(z + \varepsilon(e_i + \sqrt{k}e_{l+1}))|_{z_i = -\sqrt{k}z_{l+1}}.$$

We also have

$$a_{-2\varepsilon e_i}(z + \varepsilon(e_i + \sqrt{k}e_{l+1}))|_{z_i = -\sqrt{k}z_{l+1}} = a_{2\varepsilon\sqrt{k}e_{l+1}}(z - \varepsilon(e_i + \sqrt{k}e_{l+1}))|_{z_i = -\sqrt{k}z_{l+1}}.$$

The latter can be seen upon rewriting m and n in terms of p and r, and using, in particular, that

$$\begin{aligned} \left(1 - \frac{\varepsilon(m + 2n)}{\sqrt{k}z_{l+1} - \varepsilon}\right)\left(1 - \frac{\varepsilon}{\sqrt{k}z_{l+1}}\right) &= \frac{\sqrt{k}z_{l+1} - \varepsilon k - \varepsilon k\,(p + 2r)}{\sqrt{k}z_{l+1}} \\ &= \left(1 - \frac{\varepsilon\sqrt{k}\,(p + 2r)}{z_{l+1} - \varepsilon\sqrt{k}}\right)\left(1 - \frac{\varepsilon\sqrt{k}}{z_{l+1}}\right). \end{aligned}$$

Moreover, we have

$$(T_{-2\varepsilon e_i} - 1) f(z + \varepsilon(e_i + \sqrt{k}e_{l+1}))|_{z_i = -\sqrt{k}z_{l+1}}$$

$$= \Big(f(z - \varepsilon(e_i - \sqrt{k}e_{l+1})) - f(z + \varepsilon(e_i + \sqrt{k}e_{l+1}))\Big)\,|_{z_i = -\sqrt{k}z_{l+1}}$$
$$= (T_{2\varepsilon\sqrt{k}e_{l+1}} - 1)f(z - \varepsilon(e_i + \sqrt{k}e_{l+1}))|_{z_i = -\sqrt{k}z_{l+1}},$$

where we used that $f \in \mathcal{R}^a_{BC(l,1)}$.

For $j \neq i$ and $z_i = -\sqrt{k}z_{l+1}$, we have $a_{2e_j}(z + e_i + \sqrt{k}e_{l+1}) = a_{2e_j}(z - e_i - \sqrt{k}e_{l+1})$, which can be seen by using that

$$\left(1 - \frac{2k}{z_j + \sqrt{k}z_{l+1} - 1}\right)\left(1 - \frac{2k}{z_j - \sqrt{k}z_{l+1} + 1}\right)\left(1 - \frac{2}{z_j + \sqrt{k}z_{l+1} + 1}\right)$$
$$\times\left(1 - \frac{2}{z_j - \sqrt{k}z_{l+1} + 1 - 2k}\right) = \frac{(z_j + \sqrt{k}z_{l+1} - 1 - 2k)(z_j - \sqrt{k}z_{l+1} - 1 - 2k)}{(z_j - \sqrt{k}z_{l+1} + 1)(z_j + \sqrt{k}z_{l+1} + 1)}$$
$$= \left(1 - \frac{2k}{z_j + \sqrt{k}z_{l+1} + 1}\right)\left(1 - \frac{2k}{z_j - \sqrt{k}z_{l+1} - 1}\right)\left(1 - \frac{2}{z_j + \sqrt{k}z_{l+1} + 1 - 2k}\right)$$
$$\times\left(1 - \frac{2}{z_j - \sqrt{k}z_{l+1} + 1}\right).$$

A similar calculation shows that $a_{-2e_j}(z+e_i+\sqrt{k}e_{l+1}) = a_{-2e_j}(z-e_i-\sqrt{k}e_{l+1})$. Moreover, $T_{2\varepsilon e_j}f(z + e_i + \sqrt{k}e_{l+1}) = T_{2\varepsilon e_j}f(z - e_i - \sqrt{k}e_{l+1})$ at $z_i = -\sqrt{k}z_{l+1}$ since $f \in \mathcal{R}^a_{BC(l,1)}$.

It follows that the identities (3.35) are satisfied for Df for $\alpha = e_i + \sqrt{k}e_{l+1}$. The case of $\alpha = e_i - \sqrt{k}e_{l+1}$ is analogous. This completes the proof. ■

3.5.2 Construction of the BA function for $BC(l,1)$

Even though the configuration $BC(l,1)_+$ is non-reduced, all of its subsets of collinear vectors are of the form $\{\alpha,\, 2\alpha\}$. Thus, we can apply Definition 3.1 to get the following notion of a BA function for $BC(l,1)$.

Definition 3.36. A function $\psi\colon \mathbb{C}^{l+1} \times \mathbb{C}^{l+1} \to \mathbb{C}$ is a BA function for the configuration $BC(l,1)$ with non-negative integer multiplicities if it satisfies the following conditions:

1. $\psi(z,x) = P(z,x)e^{\langle z,x\rangle}$ for some polynomial P in z whose highest-order term is $\prod_{\alpha\in BC(l,1)_+}\langle \alpha, z\rangle^{c_\alpha}$,
2. $\psi(z + s\alpha, x) = \psi(z - s\alpha, x)$ at $\langle z, \alpha\rangle = 0$ for all $\alpha \in BC(l,1)^r_+$ and $s \in A_\alpha$.

Note that condition 2 in Definition 3.36 is similar to the functional identities (3.35) satisfied by the elements of the ring $\mathcal{R}^a_{BC(l,1)}$, and indeed, for any $x \in \mathbb{C}^{l+1}$ for which $P(z,x)$ is non-singular, the function $\psi_x\colon z \mapsto \psi(z,x)$ belongs to $\mathcal{R}^a_{BC(l,1)}$.

It follows from the general results presented in Section 3.2 that if a function ψ satisfying Definition 3.36 exists then it is unique, and it is a joint eigenfunction for a large commutative ring of differential operators in x. By Proposition 3.9, this ring contains the Hamiltonian (3.1) for $\mathcal{A} = BC(l,1)_+$ with the non-negative integer values of the multiplicity parameters. Moreover, this ring also contains a complete set of quantum integrals for this Hamiltonian, as well as extra integrals that correspond to the algebraic integrability of this system. Namely, for every polynomial $p(z) \in \mathcal{R}^a_{BC(l,1)}$, there is a differential operator in x that commutes with the Hamiltonian and whose highest symbol is $p_0(\partial_x)$, where p_0 is the highest homogeneous term of p.

The following theorem gives an explicit construction of the BA function for $BC(l,1)$ using the Sergeev–Veselov difference operator D from Section 3.5.1. The BA function will be an eigenfunction for the operator D, which shows bispectrality of the generalised CMS Hamiltonian of $BC(l,1)$ for non-negative integer values of the multiplicity parameters (the case of non-integer multiplicities will be considered in the next section).

Theorem 3.37. *Let* $M = \sum_{\alpha \in BC(l,1)_+} c_\alpha = l(m+n+(l-1)k+2)+p+r$, *and let* $S = \{\pm 2e_i, \pm 2\sqrt{k}e_{l+1} \colon 1 \le i \le l\}$. *For* $x \in \mathbb{C}^{l+1}$, *let*

$$\mu(x) = \sum_{\tau \in S} \kappa_\tau (e^{\langle \tau, x\rangle} - 1),$$

and

$$b(x) = \frac{M!}{2^{ln+r}} \prod_{\alpha \in BC(l,1)^r_+} \left(\sum_{\tau \in S} \kappa_\tau \langle \tau, \alpha \rangle e^{\langle \tau, x \rangle} \right)^{c_\alpha + c_{2\alpha}}, \tag{3.39}$$

where $\kappa_\tau = k^{-1}$ *if* $\tau = \pm 2\sqrt{k}e_{l+1}$, *and* $\kappa_\tau = 1$ *otherwise. For* $z \in \mathbb{C}^{l+1}$, *let* $Q(z)$ *be the polynomial in* $\mathcal{R}^a_{BC(l,1)}$ *given by*

$$Q(z) = \prod_{\substack{\alpha \in BC(l,1)^r_+ \\ s \in A_\alpha}} \left(\langle \alpha, z \rangle^2 - s^2 \langle \alpha, \alpha \rangle^2 \right).$$

Then the function

$$\psi(z,x) = b(x)^{-1} \left(D - \mu(x)\right)^M \left[Q(z) e^{\langle z, x\rangle}\right] \tag{3.40}$$

is the BA function for $BC(l,1)$. *Moreover,* ψ *is an eigenfunction of the operator* D *with*

$$D\psi(z,x) = \mu(x)\psi(z,x).$$

The proof is analogous to the case of AG_2 in Theorem 3.24, it just uses the following two lemmas in place of Lemmas 3.22 and 3.23, respectively.

Lemma 3.38. *For all* $\tau \in S = \{\pm 2e_i, \pm 2\sqrt{k}e_{l+1} \colon 1 \le i \le l\}$, *the coefficient function* a_τ

in the operator D can be expanded as

$$a_\tau(z) = \kappa_\tau - \kappa_\tau \sum_{\alpha \in BC(l,1)^r_+} \frac{\langle \tau, \alpha \rangle (c_\alpha + c_{2\alpha})}{\langle \alpha, z \rangle} + R_\tau(z),$$

where κ_τ are as in Theorem 3.37, and R_τ is a rational function with $\deg R_\tau \le -2$.

The above way of expanding the coefficients of the operator D can be used to prove the following lemma (in a completely analogous way to how Lemma 3.23 is proved using Lemma 3.22).

Lemma 3.39. *For $\alpha \in BC(l,1)^r_+$, let $n_\alpha \in \mathbb{Z}_{\ge 0}$ be arbitrary. Let $N = \sum_{\alpha \in BC(l,1)^r_+} n_\alpha$. Let $\mu(x)$ be as in Theorem 3.37, and let $A(z) = \prod_{\alpha \in BC(l,1)^r_+} \langle \alpha, z \rangle^{n_\alpha}$. Then we have $(D-\mu)[A(z)e^{\langle z,x \rangle}] = R(z,x)e^{\langle z,x \rangle}$ for a rational function $R(z,x)$ in z of the form*

$$R(z,x) = \sum_{\alpha \in BC(l,1)^r_+} (n_\alpha - c_\alpha - c_{2\alpha}) \left(\sum_{\tau \in S} \kappa_\tau \langle \tau, \alpha \rangle e^{\langle \tau, x \rangle} \right) A(z) \langle \alpha, z \rangle^{-1} + S(z,x),$$

where $S(z,x)$ is a rational function in z of degree less than or equal to $N-2$.

In particular, for any *polynomial $B(z,x)$ in z, we have that $(D-\mu)[B(z,x)e^{\langle z,x \rangle}] = U(z,x)e^{\langle z,x \rangle}$ for a rational function $U(z,x)$ in z with $\deg U(z,x) \le \deg B(z,x) - 1$.*

Proof of Theorem 3.37. The idea of the proof is as follows. We have $Q(z)e^{\langle z,x \rangle} \in \mathcal{R}^a_{BC(l,1)}$ because $Q(z+s\alpha) = Q(z-s\alpha) = 0$ at $\langle z, \alpha \rangle = 0$ for all $\alpha \in BC(l,1)^r_+$ and $s \in A_\alpha$. The property that $(D-\mu)^M[Q(z)e^{\langle z,x \rangle}]$ satisfies condition 2 in Definition 3.36 thus follows from Theorem 3.35. Moreover, each repeated application of $D-\mu$ on $Q(z)e^{\langle z,x \rangle}$ gives a function of the form $R(z,x)e^{\langle z,x \rangle}$ with $R(z,x)$ a polynomial in z, which follows from the form of the operator D and Theorem 3.35. More specifically, for all $b \in \mathbb{Z}_{>0}$, we have that $(D-\mu)^b[Q(z)e^{\langle z,x \rangle}] = R_b(z,x)e^{\langle z,x \rangle}$ where $R_b(z,x)$ is a polynomial in z of degree at most $2M-b$ whose highest-order homogeneous component can be kept track of by using Lemma 3.39 (similarly to how this is done in the case of AG_2 in the proof of Theorem 3.24 by using Lemma 3.23). This makes it possible to see that $(D-\mu)^M[Q(z)e^{\langle z,x \rangle}]$ essentially satisfies also condition 1 in Definition 3.36, except that the highest-order term of its polynomial part has an extra factor of $b(x)$ given by formula (3.39). It follows that ψ defined by the expression (3.40) is the BA function. At the next application of $D-\mu$, we get $(D-\mu)^{M+1}[Q(z)e^{\langle z,x \rangle}] = 0$ as a consequence of Lemma 3.5, which implies that $D\psi = \mu\psi$. ■

In Section 3.6.1, we will need some further properties of the BA function (3.40), which we record in Propositions 3.42 and 3.44 below (cf. [24, Propositions 6.5 and 6.6], respectively). The following two lemmas will be useful for the proof of Proposition 3.42. Let $\propto$ denote proportionality by a constant factor.

Lemma 3.40. *The function $b(x)$ defined by formula (3.39) satisfies*

$$b(x) \propto \prod_{\alpha \in BC(l,1)_+^r} \sinh^{c_\alpha + c_{2\alpha}} \langle 2\alpha, x \rangle.$$

Proof. Using formula (3.39), we compute

$$\begin{aligned}
& b(x)(e^{2\sqrt{k}x_{l+1}} - e^{-2\sqrt{k}x_{l+1}})^{-p-r} \prod_{i=1}^{l} (e^{2x_i} - e^{-2x_i})^{-m-n} \propto \\
& \prod_{\substack{1 \le i \le l \\ \varepsilon \in \{\pm 1\}}} (e^{2x_i} - e^{-2x_i} + e^{2\varepsilon\sqrt{k}x_{l+1}} - e^{-2\varepsilon\sqrt{k}x_{l+1}}) \prod_{\substack{1 \le i < j \le l \\ \varepsilon \in \{\pm 1\}}} (e^{2x_i} - e^{-2x_i} + e^{2\varepsilon x_j} - e^{-2\varepsilon x_j})^k \\
& = \prod_{\substack{1 \le i \le l \\ \varepsilon \in \{\pm 1\}}} (e^{x_i + \varepsilon\sqrt{k}x_{l+1}} + e^{-x_i - \varepsilon\sqrt{k}x_{l+1}})(e^{x_i + \varepsilon\sqrt{k}x_{l+1}} - e^{-x_i - \varepsilon\sqrt{k}x_{l+1}}) \\
& \quad \times \prod_{\substack{1 \le i < j \le l \\ \varepsilon \in \{\pm 1\}}} (e^{x_i + \varepsilon x_j} + e^{-x_i - \varepsilon x_j})^k (e^{x_i + \varepsilon x_j} - e^{-x_i - \varepsilon x_j})^k,
\end{aligned}$$

where we used the identity $A^2 - A^{-2} + B^2 - B^{-2} = (AB^{-1} + A^{-1}B)(AB - A^{-1}B^{-1})$. Then by using the difference of two squares formula, we get

$$b(x) \propto \prod_{\alpha \in BC(l,1)_+^r} (e^{\langle 2\alpha, x \rangle} - e^{-\langle 2\alpha, x \rangle})^{c_\alpha + c_{2\alpha}} \propto \prod_{\alpha \in BC(l,1)_+^r} \sinh^{c_\alpha + c_{2\alpha}} \langle 2\alpha, x \rangle,$$

as required. ∎

Let us consider the function

$$\delta(x) = \prod_{\alpha \in BC(l,1)_+} (2 \sinh\langle \alpha, x \rangle)^{c_\alpha}. \tag{3.41}$$

The next lemma relates $\delta(x)$ to the function $b(x)$.

Lemma 3.41. *We have*

$$b(x) \propto \delta(x) \prod_{\alpha \in BC(l,1)_+^r} \cosh^{c_\alpha} \langle \alpha, x \rangle.$$

Proof. Note that

$$\delta(x) \propto \prod_{\alpha \in BC(l,1)_+^r} \sinh^{c_\alpha + c_{2\alpha}} \langle \alpha, x \rangle \cosh^{c_{2\alpha}} \langle \alpha, x \rangle. \tag{3.42}$$

The proof is thus completed by making use of Lemma 3.40. ∎

The proof of the next proposition is based on the ideas of the proof of [24, Proposition 6.5]. Let us define the lattice

$$\mathcal{L} = \mathcal{L}(k) = 2\mathbb{Z}e_1 \oplus \cdots \oplus 2\mathbb{Z}e_l \oplus 2\sqrt{k}\mathbb{Z}e_{l+1}. \tag{3.43}$$

Let $\mathcal{L}_+$ be the semigroup of $\nu = (\nu_1, \ldots, \nu_l, \sqrt{k}\nu_{l+1}) \in \mathcal{L}$ that have non-negative partial sums of ν_i, that is

$$\mathcal{L}_+ = \{\nu = (\nu_1, \ldots, \nu_l, \sqrt{k}\nu_{l+1}) \in \mathcal{L} \colon \textstyle\sum_{i=1}^r \nu_i \geq 0 \text{ for } r = 1, \ldots, l+1\} = \bigoplus_{i=1}^{l+1} \mathbb{Z}_{\geq 0}\alpha_i,$$

where $\alpha_1 = 2(e_1 - e_2)$, ..., $\alpha_{l-1} = 2(e_{l-1} - e_l)$, $\alpha_l = 2(e_l - \sqrt{k}e_{l+1})$, and $\alpha_{l+1} = 2\sqrt{k}e_{l+1}$. We note that $2\alpha \in \mathcal{L}_+$ for all $\alpha \in BC(l,1)_+$.

Proposition 3.42. *The function ψ defined by formula* (3.40) *can be expanded in the form*

$$\psi = \delta(x)^{-1} e^{\langle z-\rho, x\rangle} \sum_{\nu \in \mathcal{L}_+} c_\nu(z) e^{\langle \nu, x\rangle} \tag{3.44}$$

for some polynomials $c_\nu(z)$, where

$$\rho = \sum_{\alpha \in BC(l,1)_+} c_\alpha \alpha \tag{3.45}$$

and $\delta(x)$ is defined by formula (3.41).

Proof. By Theorem 3.7 and Proposition 3.9, the function ψ satisfies the eigenfunction equation $L\psi = -z^2\psi$ for the operator L given by formula (3.1) with $\mathcal{A} = BC(l,1)_+$. This operator can be rearranged as follows

$$L = -\Delta + \sum_{\alpha \in BC(l,1)_+^r} \frac{(c_\alpha + c_{2\alpha})(c_\alpha + c_{2\alpha} + 1)\langle \alpha, \alpha\rangle}{\sinh^2\langle \alpha, x\rangle} - \sum_{\alpha \in BC(l,1)_+^r} \frac{c_{2\alpha}(c_{2\alpha}+1)\langle \alpha, \alpha\rangle}{\cosh^2\langle \alpha, x\rangle}.$$

This form of the operator L makes it possible to see, by Laurent expanding the eigenfunction ψ in x around suitable hyperplanes, that ψ must have either a pole of order $c_\alpha + c_{2\alpha}$ or a zero of order $c_\alpha + c_{2\alpha} + 1$ along each of the hyperplanes $\sinh\langle \alpha, x\rangle = 0$ for $\alpha \in BC(l,1)_+^r$; and similarly a pole of order $c_{2\alpha}$ or a zero of order $c_{2\alpha} + 1$ along the hyperplanes $\cosh\langle \alpha, x\rangle = 0$ for $\alpha \in BC(l,1)_+^r$. The expression for $b(x)$ given in Lemma 3.40 suggests that the order of the poles of ψ at $\cosh\langle \alpha, x\rangle = 0$ might be higher than $c_{2\alpha}$, but the local expansion of ψ and the eigenvalue equation $L\psi = -z^2\psi$ imply that this cannot happen. It follows from the form of $b(x)$ that ψ cannot have any additional other singularities either. Hence $\delta(x)\psi$ is analytic (in both x and z variables) due to the property (3.42).

By construction, $\psi(z,x) = b(x)^{-1}\Phi(z,x)$ for the quasi-polynomial in z function

$$\Phi(z,x) = (D - \mu(x))^M [Q(z)e^{\langle z,x\rangle}].$$

From the fomulas for D and μ, it is clear that Φ is analytic in x and that it can be expanded as

$$\Phi = e^{\langle z,x\rangle} \sum_{\nu\in\mathcal{L}} b_\nu(z) e^{\langle \nu,x\rangle} \tag{3.46}$$

for some polynomials $b_\nu(z)$.

In view of Lemma 3.41, the analyticity of $\delta(x)\psi$ implies that the trigonometric polynomial in x given by (3.46) must be divisible by

$$\prod_{\alpha\in BC(l,1)_+^r} \cosh^{c_\alpha}\langle\alpha,x\rangle \propto e^{\langle\rho^r,x\rangle} \prod_{\alpha\in BC(l,1)_+^r} (e^{-2\langle\alpha,x\rangle}+1)^{c_\alpha} = e^{\langle\rho,x\rangle}\sum_{\nu\in\mathcal{L}} d_\nu e^{\langle\nu,x\rangle}, \tag{3.47}$$

where $d_\nu \in \mathbb{R}$ and

$$\rho^r := \sum_{\alpha\in BC(l,1)_+^r} c_\alpha \alpha,$$

and we used that $2\alpha \in \mathcal{L}$ for all $\alpha \in BC(l,1)_+^r$. The quotient of the function (3.46) by its divisor (3.47) will still be a trigonometric polynomial in x, and we get that

$$\psi = \delta(x)^{-1} e^{\langle z-\rho,x\rangle} \sum_{\nu\in\mathcal{L}} c_\nu(z) e^{\langle\nu,x\rangle}$$

for some polynomials $c_\nu(z)$. Let $\mathcal{P} = \{\nu\in\mathcal{L}\colon c_\nu \neq 0\}$. We need to show that $\mathcal{P}\subset\mathcal{L}_+$.

Let $y_i = e^{\langle\alpha_i,x\rangle}$ for $i = 1,\ldots,l{+}1$. The potential in the Hamiltonian L can be written as

$$\sum_{\alpha\in BC(l,1)_+} \frac{4c_\alpha(c_\alpha+2c_{2\alpha}+1)\langle\alpha,\alpha\rangle e^{2\langle\alpha,x\rangle}}{(e^{2\langle\alpha,x\rangle}-1)^2}. \tag{3.48}$$

By using that $2\alpha \in \mathcal{L}_+ = \bigoplus_{i=1}^{l+1}\mathbb{Z}_{\geq 0}\alpha_i$ for any $\alpha \in BC(l,1)_+$, we can rewrite the potential (3.48) in terms of the variables y_i and expand it (for small y_i, that is for x in $\{x\in\mathbb{C}^{l+1}\colon \operatorname{Re}\langle\alpha_i,x\rangle < 0,\ i=1,\ldots,l+1\}$) into a Taylor series in y_i, which will have no constant term, and thus obtain

$$L = -\Delta + \sum_{\mu\in\mathcal{L}_+\setminus\{0\}} g_\mu e^{\langle\mu,x\rangle}$$

for some constants g_μ. Similarly, one can expand the function

$$\delta(x)^{-1} = e^{\langle\rho,x\rangle} \prod_{\alpha\in BC(l,1)_+} (e^{\langle 2\alpha,x\rangle}-1)^{-c_\alpha} = e^{\langle\rho,x\rangle}\left((-1)^t + \sum_{\eta\in\mathcal{L}_+\setminus\{0\}} h_\eta e^{\langle\eta,x\rangle}\right),$$

where h_η are constants and $t = l(m+n)+p+r$. Thus, the eigenfunction equation $(L+z^2)\psi = 0$ gives that

$$\sum_{\nu\in\mathcal{P}} c_\nu(z)e^{\langle\nu,x\rangle}\Bigg(\left(z^2-(z+\nu)^2\right)(-1)^t + \sum_{\eta\in\mathcal{L}_+\setminus\{0\}} h_\eta \left(z^2-(z+\nu+\eta)^2\right) e^{\langle\eta,x\rangle} + \sum_{\mu\in\mathcal{L}_+\setminus\{0\}} (-1)^t g_\mu e^{\langle\mu,x\rangle} + \sum_{\mu,\eta\in\mathcal{L}_+\setminus\{0\}} g_\mu h_\eta e^{\langle\mu+\eta,x\rangle}\Bigg) = 0. \tag{3.49}$$

Since the set $\mathcal{P}$ is finite, it contains (one or several) minimal elements $\nu_{\min}$ with respect to the partial order on $\mathcal{L}$ defined by $\alpha > \beta$ if and only if $\alpha-\beta \in \mathcal{L}_+\setminus\{0\}$ for $\alpha,\beta\in\mathcal{L}$. The term $e^{\langle\nu_{\min},x\rangle}$ appears only once in the left-hand side of equality (3.49), and its coefficient must hence vanish. We get that $(z+\nu_{\min})^2 = z^2$ for generic z, and it follows that $\nu_{\min}=0$. Suppose that there is some $\nu\in\mathcal{P}\setminus\mathcal{L}_+$. Then $\nu\neq 0$, so it cannot be minimal in $\mathcal{P}$, hence there must be some $\nu_2\in\mathcal{P}\setminus\mathcal{L}_+$ with $\nu>\nu_2$. By iterating this argument, we get an infinite chain $\nu>\nu_2>\nu_3>\dots$ of elements in $\mathcal{P}$, contradicting the finiteness of this set. It follows that $\mathcal{P}\subset\mathcal{L}_+$, as required. ∎

Remark 3.43. By adapting the argument in the last paragraph of the proof of [24, Proposition 6.5], one could show that the set $\mathcal{P}$ from the proof of Proposition 3.42 is contained in the subset of $\mathcal{L}_+$ given by $\{2\sum_{\alpha\in BC(l,1)_+} t_\alpha\alpha \colon t_\alpha\in\mathbb{Z},\ 0\le t_\alpha\le c_\alpha\}$.

The proof of the following proposition is based on the ideas of the proof of [24, Proposition 6.6 (2)].

Proposition 3.44. *The polynomial* c_0 *in the expansion* (3.44) *is given by*

$$c_0(z) = (-1)^{l(m+n)+p+r}\ 2^{ln+r} \prod_{\substack{\alpha\in BC(l,1)^r_+\\ s\in A_\alpha}} (\langle\alpha,z\rangle + s\langle\alpha,\alpha\rangle).$$

Proof. The BA function (3.44) must satisfy condition 2 in Definition 3.36. This gives that

$$\sum_{\nu\in\mathcal{L}_+} c_\nu(z+s\alpha)e^{\langle\nu+s\alpha,x\rangle} = \sum_{\nu\in\mathcal{L}_+} c_\nu(z-s\alpha)e^{\langle\nu-s\alpha,x\rangle} \tag{3.50}$$

at $\langle z,\alpha\rangle = 0$ for $\alpha\in BC(l,1)^r_+$ and $s\in A_\alpha$. Since $-s\alpha\notin\frac{1}{2}\mathcal{L}_+$, while $\nu+s\alpha\in\frac{1}{2}\mathcal{L}_+$ for all $\nu\in\mathcal{L}_+$, we get that the term $e^{-\langle s\alpha,x\rangle}$ does not appear in the left-hand side of equality (3.50). Hence, it cannot appear in the right-hand side either, which means that $c_0(z-s\alpha)=0$ at $\langle z,\alpha\rangle=0$. In other words, the polynomial $c_0(z)$ must be divisible by

$$\prod_{\substack{\alpha\in BC(l,1)^r_+\\ s\in A_\alpha}} (\langle\alpha,z\rangle + s\langle\alpha,\alpha\rangle). \tag{3.51}$$

The quotient of $c_0(z)$ by the product (3.51) can only be some constant λ since the polynomial part of the function ψ is, by condition 1 in Definition 3.36, of the same degree in z as the polynomial (3.51). Moreover, the highest-degree term of the polynomial part of ψ is by definition $\prod_{\alpha \in BC(l,1)_+} \langle \alpha, z\rangle^{c_\alpha} = 2^{ln+r} \prod_{\alpha \in BC(l,1)^r_+, s \in A_\alpha} \langle \alpha, z\rangle$. Thus, denoting by c^0_ν the highest-degree term of c_ν, equality (3.44) implies

$$2^{ln+r}\delta(x) \prod_{\substack{\alpha \in BC(l,1)^r_+ \\ s \in A_\alpha}} \langle \alpha, z\rangle = \lambda e^{-\langle \rho, x\rangle} \prod_{\substack{\alpha \in BC(l,1)^r_+ \\ s \in A_\alpha}} \langle \alpha, z\rangle + e^{-\langle \rho, x\rangle} \sum_{\nu \in \mathcal{L}_+ \setminus \{0\}} c^0_\nu(z) e^{\langle \nu, x\rangle}.$$

By calculating the coefficient of $e^{-\langle \rho, x\rangle}$ in $\delta(x)$, we get $\lambda = 2^{ln+r} \prod_{\alpha \in BC(l,1)^r_+} (-1)^{c_\alpha + c_{2\alpha}}$, and the statement of the proposition follows. ■

We end this section with the following statement, whose proof is analogous to that of Theorem 3.28.

Theorem 3.45. *Let $p(z) \in \mathcal{R}^a_{BC(l,1)}$ be a polynomial, and let p_0 be its highest-degree homogeneous term. Then there is a difference operator D_p acting in z such that*

$$D_p \psi(z,x) = \mu_p(x) \psi(z,x),$$

$$D_p = \frac{1}{(\deg p)!} \operatorname{ad}_D^{\deg p}(\widehat{p}),$$

where ψ is the BA function (3.40) of $BC(l,1)$ and D is the Sergeev–Veselov operator (3.36). The eigenvalue $\mu_p(x)$ is obtained by substituting $4\sinh(2x_i)$ in place of z_i $(1 \le i \le l)$ and $4(\sqrt{k})^{-1}\sinh(2\sqrt{k}x_{l+1})$ in place of z_{l+1} into $p_0(z)$. The operators D_p commute with D and with each other.

3.6 Bispectrality for non-integer multiplicities

3.6.1 Case of $BC(l,1)$

In this section, we carry out an analytic continuation in the parameters of the BA function (3.40) in order to extend to more general complex values of the parameters the statement proved in Section 3.5.2 about the bispectrality of the Sergeev–Veselov operators for $BC(l,1)$. We do this by adapting to our present setting the approach developed in [24, Section VI.C]. The corresponding generalisation of the function $\psi(z,x)$ will be a deformation of a Heckman–Opdam multidimensional hypergeometric function [65].

We begin with a few preliminary results on the properties of the generalised CMS operator L (3.1) for $\mathcal{A} = BC(l,1)_+$, where we are now no longer assuming that the parameters c_α are integers. The next lemma gives a potential-free gauge-equivalent form of this operator. It is stated in [94]. We include a proof below for completeness.

Lemma 3.46. *[94] Consider the differential operator*

$$H = \Delta - \sum_{\alpha \in BC(l,1)_+} 2c_\alpha \coth\langle \alpha, x\rangle \partial_\alpha, \tag{3.52}$$

where $\partial_\alpha = \sum_{i=1}^{l+1} \langle \alpha, e_i\rangle \partial_{x_i}$ *is the directional derivative along* $\alpha \in \mathbb{C}^{l+1}$. *The operator* H *is gauge-equivalent to the operator* L *from above:*

$$H + \rho^2 = -\delta(x) \circ L \circ \delta(x)^{-1},$$

where ρ *is given by formula* (3.45) *and* $\delta(x)$ *by formula* (3.41).

Proof. We have

$$\partial_{x_i}[\delta(x)^{-1}] = -\delta(x)^{-1} \sum_{\alpha \in BC(l,1)_+} c_\alpha \alpha^{(i)} \coth\langle \alpha, x\rangle,$$

$$\begin{aligned}\partial^2_{x_i}[\delta(x)^{-1}] = \delta(x)^{-1} &\sum_{\alpha \in BC(l,1)_+} c_\alpha \left((c_\alpha + 1)\sinh^{-2}\langle \alpha, x\rangle + c_\alpha\right)(\alpha^{(i)})^2 \\ &+ \delta(x)^{-1} \sum_{\substack{\alpha,\beta \in BC(l,1)_+ \\ \alpha \neq \beta}} c_\alpha c_\beta \alpha^{(i)} \beta^{(i)} \coth\langle \alpha, x\rangle \coth\langle \beta, x\rangle\end{aligned}$$

for all $i = 1, \dots, l+1$, where $\alpha = (\alpha^{(1)}, \dots, \alpha^{(l+1)})$ and $\beta = (\beta^{(1)}, \dots, \beta^{(l+1)})$. Therefore, we get

$$\begin{aligned}-\delta(x) \circ L \circ \delta(x)^{-1} = &- \sum_{\alpha \in BC(l,1)_+} 2c_\alpha c_{2\alpha} \langle \alpha, \alpha\rangle \sinh^{-2}\langle \alpha, x\rangle + \sum_{\alpha \in BC(l,1)_+} c_\alpha^2 \langle \alpha, \alpha\rangle \\ &+ \sum_{\substack{\alpha,\beta \in BC(l,1)_+ \\ \alpha \neq \beta}} c_\alpha c_\beta \langle \alpha, \beta\rangle \coth\langle \alpha, x\rangle \coth\langle \beta, x\rangle - \sum_{\alpha \in BC(l,1)_+} 2c_\alpha \coth\langle \alpha, x\rangle \partial_\alpha + \Delta.\end{aligned} \tag{3.53}$$

By using that $\coth(u)\coth(2u) = \frac{1}{2}(\sinh^{-2}(u) + 2)$ for any $u \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}\pi i$, we have

$$\begin{aligned}&\sum_{\substack{\alpha,\beta \in BC(l,1)_+ \\ \alpha \neq \beta}} c_\alpha c_\beta \langle \alpha, \beta\rangle \coth\langle \alpha, x\rangle \coth\langle \beta, x\rangle \\ &= \sum_{\substack{\alpha,\beta \in BC(l,1)_+ \\ \alpha \notin \{\beta, 2\beta, \frac{1}{2}\beta\}}} c_\alpha c_\beta \langle \alpha, \beta\rangle \coth\langle \alpha, x\rangle \coth\langle \beta, x\rangle + \sum_{\alpha \in BC(l,1)_+} 2c_\alpha c_{2\alpha} \langle \alpha, \alpha\rangle (\sinh^{-2}\langle \alpha, x\rangle + 2).\end{aligned}$$

By substituting this into equality (3.53), we obtain that $-\delta(x)\circ L\circ\delta(x)^{-1}$ equals

$$H+\sum_{\substack{\alpha,\beta\in BC(l,1)_+\\ \alpha\notin\{\beta,2\beta,\frac{1}{2}\beta\}}} c_\alpha c_\beta\langle\alpha,\beta\rangle\coth\langle\alpha,x\rangle\coth\langle\beta,x\rangle + \sum_{\alpha\in BC(l,1)_+} 4c_\alpha c_{2\alpha}\langle\alpha,\alpha\rangle+\sum_{\alpha\in BC(l,1)_+} c_\alpha^2\langle\alpha,\alpha\rangle.$$

Since we can write

$$\rho^2=\sum_{\alpha\in BC(l,1)_+} c_\alpha^2\langle\alpha,\alpha\rangle+\sum_{\alpha\in BC(l,1)_+} 4c_\alpha c_{2\alpha}\langle\alpha,\alpha\rangle+\sum_{\substack{\alpha,\beta\in BC(l,1)_+\\ \alpha\notin\{\beta,2\beta,\frac{1}{2}\beta\}}} c_\alpha c_\beta\langle\alpha,\beta\rangle,$$

the proof is completed with the help of the following lemma.

Lemma 3.47. *[94, Equality (12)] We have*

$$\sum_{\substack{\alpha,\beta\in BC(l,1)_+\\ \alpha\notin\{\beta,2\beta,\frac{1}{2}\beta\}}} c_\alpha c_\beta\langle\alpha,\beta\rangle\coth\langle\alpha,x\rangle\coth\langle\beta,x\rangle=\sum_{\substack{\alpha,\beta\in BC(l,1)_+\\ \alpha\notin\{\beta,2\beta,\frac{1}{2}\beta\}}} c_\alpha c_\beta\langle\alpha,\beta\rangle. \tag{3.54}$$

Proof. We first indicate how to prove that the left-hand side of (3.54) is non-singular. One can show that singularities at $\sinh\langle\alpha,x\rangle=0$ for $\alpha\in\{e_i,2e_i\colon 1\le i\le l\}\cup\{\sqrt{k}e_{l+1},2\sqrt{k}e_{l+1}\}\cup\{e_i\pm e_j\colon 1\le i<j\le l\}$ cancel by using symmetry — namely that those α satisfy $s_\alpha(BC(l,1))=BC(l,1)$ — and using also that for those α we have $2\langle\alpha,\beta\rangle\langle\alpha,\alpha\rangle^{-1}\in\mathbb{Z}$ for all $\beta\in BC(l,1)$.

It remains to show that there are no singularities at $\sinh\langle\alpha,x\rangle=0$ for $\alpha=e_i\pm\sqrt{k}e_{l+1}$ $(1\le i\le l)$ either. Let $\alpha=e_i+\sqrt{k}e_{l+1}$. Then $\sinh\langle\alpha,x\rangle=0$ if and only if $x_i=i\pi d-\sqrt{k}x_{l+1}$ for some $d\in\mathbb{Z}$. We will show that the terms multiplying $\coth\langle\alpha,x\rangle$ in (3.54) go to 0 when $x_i\to i\pi d-\sqrt{k}x_{l+1}$. The terms in question are (up to a factor of 2) equal to

$$\begin{aligned}&m\coth(x_i)+2n\coth(2x_i)+kp\coth(\sqrt{k}x_{l+1})\\&\quad+2kr\coth(2\sqrt{k}x_{l+1})+(1-k)\coth(x_i-\sqrt{k}x_{l+1})\\&\quad+\sum_{\substack{j=1\\ j\neq i}}^{l}k\Big(\coth(x_i+x_j)+\coth(x_i-x_j)+\coth(x_j+\sqrt{k}x_{l+1})-\coth(x_j-\sqrt{k}x_{l+1})\Big).\end{aligned} \tag{3.55}$$

By using $i\pi$-periodicity of coth, the expression (3.55) simplifies at $x_i=i\pi d-\sqrt{k}x_{l+1}$ to

$$(kp-m)\coth(\sqrt{k}x_{l+1})+(2kr-2n+k-1)\coth(2\sqrt{k}x_{l+1})=0,$$

as required, where we used that $m = kp$ and $2n+1 = k(2r+1)$. The case of $\alpha = e_i - \sqrt{k}e_{l+1}$ can be handled similarly.

This completes the proof that the left-hand side of (3.54) is non-singular, and from its form, it is then clear that it can be rewritten in exponential variables $w_i = e^{2x_i}$ $(1 \le i \le l)$ and $w_{l+1} = e^{2\sqrt{k}x_{l+1}}$ as a non-singular rational function with zero degree, which therefore must be constant.

To calculate the constant, let us put $x = ((l+1)N, lN, \ldots, 2N, N/\sqrt{k})$, and take the limit $N \to \infty$ by using that $\langle \alpha, x\rangle \to \infty$ as $N \to \infty$ for all $\alpha \in BC(l,1)_+$ and that $\coth u \to 1$ when $u \to \infty$. This gives the right-hand side of equality (3.54). ■■

When x belongs to the region

$$B = B(k) = \{x \in \mathbb{C}^{l+1} \colon \operatorname{Re}\langle \alpha, x\rangle < 0 \text{ for all } \alpha \in BC(l,1)_+\},$$

the operator (3.52) can be expanded into a series as

$$H = \Delta + \sum_{\alpha \in BC(l,1)_+} 2c_\alpha \frac{1 + e^{2\langle \alpha, x\rangle}}{1 - e^{2\langle \alpha, x\rangle}} \partial_\alpha = \Delta + \sum_{\alpha \in BC(l,1)_+} 2c_\alpha \left(1 + 2\sum_{j=1}^{\infty} e^{2j\langle \alpha, x\rangle}\right) \partial_\alpha. \quad (3.56)$$

Let $\varphi = \varphi(z,x)$ be a solution of the equation

$$H\varphi = (z^2 - \rho^2)\varphi, \quad (3.57)$$

which is by Lemma 3.46 equivalent to the function $\delta(x)^{-1}\varphi$ being an eigenfunction of the Hamiltonian L with eigenvalue $-z^2$. In particular, if all $c_\alpha \in \mathbb{Z}_{\ge 0}$ then $\delta(x)\psi(z,x)$, where ψ is the BA function (3.40), satisfies equation (3.57).

Assume that there is a solution φ of the equation (3.57) of the particular form

$$\varphi = e^{\langle z-\rho, x\rangle} \sum_{\nu \in \mathcal{L}_+} q_\nu(z) e^{\langle \nu, x\rangle} \quad (3.58)$$

for some functions $q_\nu(z)$ with $q_0(z) = 1$. If all $c_\alpha \in \mathbb{Z}_{\ge 0}$ then by Proposition 3.42 the function $\delta(x)\psi(z,x)$ is of the form (3.58); it is just normalised differently since $c_0(z) = 2^{ln+r}\prod_{\alpha \in BC(l,1)^r_+,\, s \in A_\alpha}(-\langle \alpha, z\rangle - s\langle \alpha, \alpha\rangle) \ne 1$ by Proposition 3.44.

By substituting the series (3.58) into equation (3.57), using the expansion (3.56) for the operator H, and requiring that the respective coefficients of the terms $e^{\langle \nu, x\rangle}$ vanish for all $\nu \in \mathcal{L}_+$, we get recurrence equations for the functions q_ν. Namely, we get

$$\langle \nu, \nu + 2z\rangle q_\nu(z) + \sum_{\alpha \in BC(l,1)_+} 4c_\alpha \sum_{j=1}^{\infty} \langle \alpha, z - \rho + \nu - 2j\alpha\rangle q_{\nu - 2j\alpha}(z) = 0, \quad (3.59)$$

subject to the constraint that $q_0 = 1$, where we put $q_{\nu-2j\alpha} = 0$ if $\nu - 2j\alpha \notin \mathcal{L}_+$, meaning that the above sum over j is finite. For generic z, equations (3.59) (together with the normalisation choice $q_0 = 1$) determine all q_ν uniquely. Indeed, one can solve for them recursively by height of $\nu \in \mathcal{L}_+ = \bigoplus_{i=1}^{l+1} \mathbb{Z}_{\geq 0}\alpha_i$, since the above $\nu - 2j\alpha$ has a strictly lower height than ν, where for $\nu = \sum_{i=1}^{l+1} \nu^{(i)}\alpha_i$, its height is defined by $h(\nu) = \sum_{i=1}^{l+1} \nu^{(i)}$. We note that all $q_\nu(z)$ are rational functions of z, and their dependence on m, n, p, r, and $\sqrt{k}$ is also rational.

For k with $\operatorname{Re} k > 0$, the next lemma below, applied with x replaced by $x/2$, can be used to show that the resulting series (3.58) converges absolutely in the region B. Moreover, since it is a power series in $y_i = e^{\langle \alpha_i, x\rangle}$ $(i = 1, \dots, l+1)$, it converges uniformly in the open sets $\{x \in B \colon \operatorname{Re}\langle \alpha_i, x\rangle < \varepsilon < 0,\ i = 1, \dots, l+1\}$ for any $\varepsilon < 0$, and so φ is analytic in x on B. In the case of root systems, an analogue of the next lemma is proved in [66, Lemma 5.3].

Lemma 3.48. *Assume* $\operatorname{Re} k > 0$. *For* $\nu \in \mathcal{L}_+ \setminus \{0\}$, *let* $\Sigma_\nu = \{z \in \mathbb{C}^{l+1} \colon \langle \nu, \nu + 2z\rangle = 0\}$. *Suppose* $z \in \mathbb{C}^{l+1}$ *does not lie on any of the hyperplanes* Σ_ν, *and let* $x \in B$. *Then there exists a constant* $K = K(z, x) \in \mathbb{R}$ *(depending on* z *and* x *but not on* ν*) such that*

$$|q_\nu(z) e^{\langle \nu, x\rangle}| \leq K$$

for all $\nu \in \mathcal{L}_+$.

Proof. Let $\nu = \sum_{i=1}^{l+1} \nu^{(i)}\alpha_i \in \mathcal{L}_+$ and $\alpha \in BC(l,1)_+$ be arbitrary. We have

$$|\langle \nu - \rho + z, \alpha\rangle| \leq |\langle z - \rho, \alpha\rangle| + \sum_{i=1}^{l+1} \nu^{(i)} |\langle \alpha_i, \alpha\rangle| \leq \lambda_1 (h(\nu) + 1), \tag{3.60}$$

where we let $\lambda_1 = \lambda_1(z) > 0$ be the maximum (depending on z but not on ν nor α) of the finite set

$$\{|\langle z - \rho, \beta\rangle| \colon \beta \in BC(l,1)_+\} \cup \{|\langle \alpha_i, \beta\rangle| \colon \beta \in BC(l,1)_+,\ i = 1, \dots, l+1\}.$$

Further, we have

$$\operatorname{Re}\langle \nu, \nu\rangle = \sum_{i,j=1}^{l+1} \nu^{(i)}\nu^{(j)} \langle \widetilde{\alpha}_i, \widetilde{\alpha}_j\rangle = \|\widetilde{\nu}\|^2 \tag{3.61}$$

for $\widetilde{\nu} = \sum_{i=1}^{l+1} \nu^{(i)}\widetilde{\alpha}_i$, where $\widetilde{\alpha}_i \in \mathbb{R}^{l+1}$ are obtained from α_i by replacing $\sqrt{k}$ with $\sqrt{\operatorname{Re} k} \in \mathbb{R}$. Here $\|\cdot\|$ denotes the usual (real) Euclidean norm. Since $\widetilde{\alpha}_i$ form a basis of $\mathbb{R}^{l+1}$, the expression (3.61) is a positive-definite (real) quadratic form in the variables $\nu^{(i)}$ with associated symmetric matrix $A = (\langle \widetilde{\alpha}_i, \widetilde{\alpha}_j\rangle)_{i,j=1}^{l+1}$. By Sylvester's criterion, all leading principal minors M_i of A are positive, $M_i = \det A_i$, where A_i is the top left $i \times i$ corner of A. For any $c \in \mathbb{R}$, the expression $\operatorname{Re}\langle \nu, \nu \rangle - ch(\nu)^2$ is also a quadratic form in the variables $\nu^{(i)}$ with associated matrix $A^c = (\langle \widetilde{\alpha}_i, \widetilde{\alpha}_j \rangle - c)_{i,j=1}^{l+1}$. By the matrix determinant lemma, the leading principal minors M_i^c of A^c are $(1 - c\langle u_i, A_i^{-1} u_i \rangle) M_i$, where $u_i = (1, \ldots, 1) \in \mathbb{R}^i$. Since M_i are positive, M_i^c are positive for $c = \min\{2^{-1}\langle u_i, A_i^{-1} u_i \rangle^{-1} : i = 1, \ldots, l+1\} > 0$, and then Sylvester's criterion implies

$$|\langle \nu, \nu \rangle| \geq \operatorname{Re}\langle \nu, \nu \rangle \geq ch(\nu)^2.$$

We also have

$$|\langle \nu, z \rangle| \leq \sum_{i=1}^{l+1} \nu^{(i)} |\langle \alpha_i, z \rangle| \leq M h(\nu),$$

where we let $M = M(z) = \max\{|\langle \alpha_i, z \rangle| : i = 1, \ldots, l+1\} > 0$. Whenever $h(\nu) \geq 4M/c$, it follows that $|\langle \nu, z \rangle| \leq ch(\nu)^2/4$, and then by the reverse triangle inequality

$$|\langle \nu, \nu + 2z \rangle| \geq \big| |\langle \nu, \nu \rangle| - 2|\langle \nu, z \rangle| \big| \geq ch(\nu)^2/2.$$

Letting $\lambda_2 = \lambda_2(z) > 0$ be the minimum of $c/2$ and the finitely-many, positive values $|\langle \nu', \nu' + 2z \rangle| / h(\nu')^2$ for $\nu' \in \mathcal{L}_+ \setminus \{0\}$ with $h(\nu') < 4M/c$ (here we are using the assumption that $z \notin \Sigma_{\nu'}$), we thus get that

$$|\langle \nu, \nu + 2z \rangle| \geq \lambda_2 h(\nu)^2 \tag{3.62}$$

for any $\nu \in \mathcal{L}_+$.

By using inequalities (3.60), (3.62), and the recurrence relation (3.59), we get for all $\nu \in \mathcal{L}_+ \setminus \{0\}$ that

$$|q_\nu(z)| \leq 4\lambda_1 \lambda_2^{-1} h(\nu)^{-1} \sum_{\alpha \in BC(l,1)_+} |c_\alpha| \sum_{j=1}^{\infty} |q_{\nu - 2j\alpha}(z)|, \tag{3.63}$$

since $h(\nu - 2j\alpha) \leq h(\nu) - 1$. Let $\lambda = 4\lambda_1 \lambda_2^{-1}$.

Since $x \in B$ and the geometric series is absolutely convergent on the open unit disk in $\mathbb{C}$, there is $N_0 \in \mathbb{Z}_{>0}$ such that

$$\lambda \sum_{\alpha \in BC(l,1)_+} |c_\alpha| \sum_{j=1}^{\infty} |e^{2j\langle \alpha, x \rangle}| \leq N_0. \tag{3.64}$$

Let K be such that

$$|q_\eta(z) e^{\langle \eta, x \rangle}| \leq K \tag{3.65}$$

for those (finitely many) $\eta \in \mathcal{L}_+$ which have $h(\eta) \leq N_0$. One can prove that (3.65) holds for all $\eta \in \mathcal{L}_+$ by induction on $h(\eta)$. Indeed, assume that (3.65) holds for those η with

$h(\eta) < N$ for some integer $N > N_0$. Then for $\nu \in \mathcal{L}_+$ with $h(\nu) = N$, we have by inequalities (3.63), (3.64), and the induction hypothesis that

$$|q_\nu(z)| \leq \lambda N^{-1} \sum_{\alpha \in BC(l,1)_+} |c_\alpha| \sum_{j=1}^{\infty} K|e^{-\langle \nu - 2j\alpha, x\rangle}| \leq KN_0N^{-1}|e^{-\langle \nu, x\rangle}| < K|e^{-\langle \nu, x\rangle}|,$$

which completes the proof by induction. ■

As a corollary of Lemma 3.48, we have the following statement.

Proposition 3.49. *The series* (3.58) *defines an analytic function in* z*,* x*, and* k *on an open subset of* $\mathbb{C}^{2l+3}$*.*

Proof. Let $k_0 \in \mathbb{C}$ with $\operatorname{Re} k_0 > 0$. Since $\operatorname{Re}\langle \alpha_i, x\rangle$ is continuous in x and k, there exists an open ball $\mathcal{B}_{\mathfrak{K}}$ centred at k_0 and an open ball $\mathcal{B}_{\mathfrak{X}} \subset \mathbb{C}^{l+1}$ such that $\operatorname{Re} k > 0$ and $\mathcal{B}_{\mathfrak{X}} \subset B(k)$ for all $k \in \mathcal{B}_{\mathfrak{K}}$. Take any $z_0 \in \mathbb{C}^{l+1} \setminus \cup_{\nu \in \mathcal{L}(k_0)_+ \setminus \{0\}} \Sigma_\nu$. Consider the constant $c = c(k_0)$ from the proof of Lemma 3.48 for $k = k_0$. Note that $\langle u_i, A_i^{-1} u_i\rangle^{-1}$ is continuous in k at k_0. Consider also the constant $M = M(z_0, k_0)$ defined as in the proof of Lemma 3.48. Note that $|\langle \alpha_i, z\rangle|$ is continuous in z and k at (z_0, k_0). Therefore, there exists an open ball $\mathcal{B}'_{\mathfrak{K}} \subseteq \mathcal{B}_{\mathfrak{K}}$ centred at k_0 and an open ball $\mathcal{B}_{\mathfrak{Z}} \subset \mathbb{C}^{l+1}$ centred at z_0 such that for all $(z, k) \in \mathcal{B}_{\mathfrak{Z}} \times \mathcal{B}'_{\mathfrak{K}}$ we have $c(k) > c(k_0)/2$ and $M(z, k) < 2M(z_0, k_0)$.

Since $|\langle \nu(k_0), \nu(k_0) + 2z_0\rangle| > 0$ for all $\nu(k_0) \in \mathcal{L}(k_0)_+ \setminus \{0\}$, and $|\langle \nu(k), \nu(k) + 2z\rangle|$ is continuous in k and z, there exists an open ball $\mathcal{B}''_{\mathfrak{K}} \subseteq \mathcal{B}'_{\mathfrak{K}}$ and an open ball $\mathcal{B}'_{\mathfrak{Z}} \subseteq \mathcal{B}_{\mathfrak{Z}}$ such that for the *finitely-many* $\nu = \nu(k) \in \mathcal{L}(k)_+ \setminus \{0\}$ with $h(\nu) \leq 16M(z_0, k_0)/c(k_0)$ we have $|\langle \nu, \nu + 2z\rangle| > 0$ for all $k \in \mathcal{B}''_{\mathfrak{K}}$ and $z \in \mathcal{B}'_{\mathfrak{Z}}$. On the other hand, for those ν with $h(\nu) > 16M(z_0, k_0)/c(k_0)$, we have by the definition of the constants $c(k)$ and $M(z, k)$ that $|\langle \nu, \nu\rangle| \geq c(k)h(\nu)^2 > c(k_0)h(\nu)^2/2$ and $|\langle \nu, z\rangle| \leq M(z, k)h(\nu) < 2M(z_0, k_0)h(\nu) < c(k_0)h(\nu)^2/8$, and so by the reverse triangle inequality

$$|\langle \nu, \nu + 2z\rangle| \geq \big||\langle \nu, \nu\rangle| - 2|\langle \nu, z\rangle|\big| > c(k_0)h(\nu)^2/4 > 0.$$

In other words, for any $(z, x, k) \in U := \mathcal{B}'_{\mathfrak{Z}} \times \mathcal{B}_{\mathfrak{X}} \times \mathcal{B}''_{\mathfrak{K}}$, we have $\operatorname{Re} k > 0$, $z \in \mathbb{C}^{l+1} \setminus \cup_{\nu \in \mathcal{L}(k)_+ \setminus \{0\}} \Sigma_\nu$, and $x \in B(k)$. Then the sum $\varphi(z, x)$ of the series (3.58) is well-defined by the discussion preceding Lemma 3.48, and $\varphi(z, x)$ is on U the pointwise limit of a sequence of functions (the partial sums) that are analytic *jointly* in all the variables z, x, and k. As a consequence of (the multivariable version of) Osgood's Theorem [77, Theorem 8], there exists an open dense subset $V \subseteq U$ on which $\varphi(z, x)$ is analytic (jointly) in the variables z, x, and k, and on which the convergence is locally uniform (for the original, single-variable case of Osgood's Theorem, see [89, Theorem II]). ■

If all $c_\alpha \in \mathbb{Z}_{\geq 0}$, then by the uniqueness of the solution of the system (3.59), we must have that $\delta(x)\psi(z, x)$ is proportional to $\varphi(z, x)$ with the factor of proportionality being $c_0(z)$, that is

$$\psi(z,x) = 2^{ln+r}\delta(x)^{-1}\varphi(z,x)\prod_{\substack{\alpha\in BC(l,1)^r_+\\ s\in A_\alpha}}(-\langle\alpha,z\rangle - s\langle\alpha,\alpha\rangle).$$

For the case when c_α are not necessarily in $\mathbb{Z}_{\geq 0}$, but rather we have any $k,m,n,p,r\in\mathbb{C}$ with $\operatorname{Re} k>0$, $m=kp$, and $2n+1=k(2r+1)$, let us define the following function

$$\Psi(z,x) = C(z)\delta(x)^{-1}\varphi(z,x),$$

where we take any branch of $\delta(x)$, and where we defined the function

$$\begin{aligned}C(z) = &\prod_{\alpha\in\{\sqrt{k}e_{l+1},\, e_i\colon 1\leq i\leq l\}}\frac{\Gamma\left(-\langle\alpha,z\rangle\langle\alpha,\alpha\rangle^{-1}\right)\Gamma\left(-\frac{1}{2}\langle\alpha,z\rangle\langle\alpha,\alpha\rangle^{-1}-\frac{1}{2}c_\alpha\right)}{\Gamma\left(-\langle\alpha,z\rangle\langle\alpha,\alpha\rangle^{-1}-c_\alpha\right)\Gamma\left(-\frac{1}{2}\langle\alpha,z\rangle\langle\alpha,\alpha\rangle^{-1}-\frac{1}{2}c_\alpha - c_{2\alpha}\right)}\\ &\times\prod_{\alpha\in\{e_i\pm e_j\colon 1\leq i<j\leq l\}}\frac{\Gamma\left(-\langle\alpha,z\rangle\langle\alpha,\alpha\rangle^{-1}\right)}{\Gamma\left(-\langle\alpha,z\rangle\langle\alpha,\alpha\rangle^{-1}-k\right)}\prod_{\alpha\in\{e_i\pm\sqrt{k}e_{l+1}\colon 1\leq i\leq l\}}(-\langle\alpha,z\rangle - 1 - k).\end{aligned}$$

Here $\Gamma(u)$ is the classical gamma-function. Then $L\Psi = -z^2\Psi$, since $C(z)$ does not depend on x and φ solves equation (3.57) by construction. Moreover, $\Psi(z,x)$ coincides (up to a constant factor) with $\psi(z,x)$ if all $c_\alpha\in\mathbb{Z}_{\geq 0}$, since then $C(z)\propto c_0(z)$ as $\Gamma(u)/\Gamma(u-N)=\prod_{i=1}^N(u-i)$ for $u\in\mathbb{C}$, $N\in\mathbb{Z}_{\geq 0}$.

With that, the proof of the next theorem is then essentially the same as that of [24, Theorem 6.9]. It just uses Theorem 3.37 in place of [24, Theorem 6.2].

Theorem 3.50. *For any $k,m,n,p,r\in\mathbb{C}$ with $\operatorname{Re} k>0$, $m=kp$, and $2n+1=k(2r+1)$, the function $\Psi(z,x)$ satisfies*

$$\begin{aligned} L\Psi &= -z^2\Psi,\\ D\Psi &= \mu(x)\Psi. \end{aligned}\tag{3.66}$$

Proof. It only remains to show equality (3.66). If all $c_\alpha\in\mathbb{Z}_{\geq 0}$, it follows from Theorem 3.37. More generally, equation (3.66) is equivalent to φ satisfying $\widetilde{D}\varphi=\mu\varphi$ for the difference operator

$$\widetilde{D} = C(z)^{-1}\circ D\circ C(z) = \sum_{\tau\in S} a_\tau(z)\left(C(z)^{-1}C(z+\tau)T_\tau - 1\right),$$

where $S=\{\pm 2e_i, \pm 2\sqrt{k}e_{l+1}\colon 1\leq i\leq l\}$, since D was of the form $D=\sum_{\tau\in S}a_\tau(z)(T_\tau - 1)$. For any $\tau\in S$, the function $a_\tau(z)$ is rational in $z,m,n,p,r,\sqrt{k}$, and so is the function $C(z)^{-1}C(z+\tau)$, since $\frac{1}{2}\langle\alpha,\tau\rangle\langle\alpha,\alpha\rangle^{-1}\in\mathbb{Z}$ for $\alpha\in\{\sqrt{k}e_{l+1},\, e_i\colon 1\leq i\leq l\}$ and $\langle\alpha,\tau\rangle\langle\alpha,\alpha\rangle^{-1}\in\mathbb{Z}$ for $\alpha\in\{e_i\pm e_j\colon 1\leq i<j\leq l\}$.

By substituting the series (3.58) into the equation $\widetilde{D}\varphi=\mu\varphi$, the latter reduces (by looking at the coefficient of $e^{\langle \nu, x\rangle}$ for each $\nu \in \mathcal{L} \supset S$) to an infinite number of identities, each involving a finite number of the coefficients $q_\nu(z)$ and only involving rational functions of z, $m, n, p, r, \sqrt{k}$. Explicitly, these identities are

$$\sum_{\tau \in S}(\kappa_\tau - a_\tau(z))q_\nu(z) + \sum_{(\mu,\tau)\in\mathcal{L}_+\times S\colon \mu+\tau=\nu} \big(a_\tau(z)C(z)^{-1}C(z+\tau)q_\mu(z+\tau) - \kappa_\tau q_\mu(z)\big) = 0$$

for $\nu \in \mathcal{L}$, where we put $q_\nu = 0$ if $\nu \notin \mathcal{L}_+$. Theorem 3.37 implies that these identities hold in the case when all $c_\alpha \in \mathbb{Z}_{\geq 0}$, and then it follows that they hold in general. This completes the proof. ■

By analyticity, the bispectrality property of Theorem 3.50 holds in a bigger domain of analyticity of the function φ. To be more precise, by Proposition 3.49 we have on an open set V an analytic function φ that satisfies equation (3.57) and by the proof of Theorem 3.50 also $\widetilde{D}\varphi = \mu(x)\varphi$. Suppose $(z, x, k) \in \mathbb{C}^{l+1} \times \mathbb{C}^{l+1} \times \mathbb{C}$ is such that φ can be analytically extended to a function $\widetilde{\varphi}(z, x, k)$ on some neighbourhood W of (z, x, k) containing V. The function $H\widetilde{\varphi} - (z^2 - \rho^2)\widetilde{\varphi}$ is analytic in z, x, k away from the singularities of $\coth\langle\alpha, x\rangle$ for $\alpha \in BC(l, 1)_+$, and on V it is identically zero. Thus, it must be zero on all of its domain of analyticity. Similarly, $\widetilde{D}\widetilde{\varphi} - \mu(x)\widetilde{\varphi}$ is analytic in z, x, k away from the union P of the poles of the functions $a_\tau(z)$ and $C(z+\tau)C(z)^{-1}$, and it vanishes on the open set $V \setminus P$. Hence $\widetilde{D}\widetilde{\varphi} = \mu(x)\widetilde{\varphi}$ on $W \setminus P$. In terms of the function $\widetilde{\Psi} := C\widetilde{\varphi}/\delta$, this means that the following bispectrality relation is satisfied:

$$L\widetilde{\Psi} = -z^2\widetilde{\Psi},$$
$$D\widetilde{\Psi} = \mu(x)\widetilde{\Psi}.$$

3.6.2 Case of AG_2

Our bispectral duality statements for AG_2 in Theorems 3.24 and 3.27 can also be extended to the case of non-integer values of the multiplicities using ideas analogous to those in [24, Section VI.C]. Since we did the case of $BC(l, 1)$ in detail above, we just explain here those steps of the argument for AG_2 that differ from what we wrote for $BC(l, 1)$, and we state the results.

In the notations of Section 3.4, let $\mathcal{L} = 2\mathbb{Z}\beta_1 \oplus 2\mathbb{Z}\alpha_2$ (the root lattice of $2G_2$) and $\mathcal{L}_+ = 2\mathbb{Z}_{\geq 0}\beta_1 \oplus 2\mathbb{Z}_{\geq 0}\alpha_2$. We note that $2\gamma \in \mathcal{L}_+$ for all $\gamma \in AG_{2,+}$. We define

$$\delta(x) = \prod_{\gamma\in AG_{2,+}} (2\sinh\langle\gamma, x\rangle)^{c_\gamma} \propto \prod_{\gamma\in G_{2,+}} \sinh^{c_\gamma+c_{2\gamma}}\langle\gamma, x\rangle \cosh^{c_{2\gamma}}\langle\gamma, x\rangle. \tag{3.67}$$

In contrast to the case of $BC(l, 1)$, the function $b(x)$ (3.23) does not seem to fully factorise in a nice way (though, it is divisible by $\delta(x)\prod_{i=1}^3 \cosh^{3m}\langle\beta_i, x\rangle$, as can be checked), but it does not affect the proof of the next proposition.

Proposition 3.51. *The BA function ψ defined by formula* (3.25) *can be expanded as*

$$\psi = \delta(x)^{-1} e^{\langle z-\rho, x\rangle} \sum_{\nu \in \mathcal{L}_+} c_\nu(z) e^{\langle \nu, x\rangle} \tag{3.68}$$

for some polynomials $c_\nu(z)$, where

$$\rho = \sum_{\gamma \in AG_{2,+}} c_\gamma \gamma \tag{3.69}$$

and $\delta(x)$ is defined by formula (3.67).

Proof. Local analysis of the singularity structure of the function ψ (3.25) imposed by the eigenfunction equation $L\psi = -z^2\psi$, where L is the generalised CMS operator (3.1) for $\mathcal{A} = AG_{2,+}$, gives that $\delta(x)\psi$ is analytic in x (and z) variables.

By construction, $\psi(z, x) = b(x)^{-1}\Phi(z, x)$ for

$$\Phi(z, x) = (\mathcal{D}_1 - \mu(x))^M [Q(z) e^{\langle z, x\rangle}]$$

and the function $b(x)$ given by (3.23), which can be expanded as

$$b(x) = \sum_{\nu \in \mathcal{L}} f_\nu e^{\langle \nu, x\rangle} \tag{3.70}$$

for some $f_\nu \in \mathbb{R}$, where we used that $2\gamma \in \mathcal{L}$ for any $\gamma \in G_2$. Also, from formulas (3.17)–(3.19) for $\mathcal{D}_1$ and (3.22) for μ, it is clear that Φ is analytic in x and can be expanded as

$$\Phi = e^{\langle z, x\rangle} \sum_{\nu \in \mathcal{L}} b_\nu(z) e^{\langle \nu, x\rangle} \tag{3.71}$$

for some polynomials $b_\nu(z)$.

The analyticity of $\delta(x)\psi$ implies that $b(x)$ must divide

$$\delta(x)\Phi = e^{-\langle \rho, x\rangle}\Phi \prod_{\gamma \in AG_{2,+}} (e^{2\langle \gamma, x\rangle} - 1)^{c_\gamma} = e^{\langle z-\rho, x\rangle} \sum_{\nu \in \mathcal{L}} d_\nu(z) e^{\langle \nu, x\rangle} \tag{3.72}$$

where $d_\nu(z)$ are polynomials, and where we used equality (3.71) and that $2\gamma \in \mathcal{L}$ for all $\gamma \in AG_{2,+}$. The quotient of the function (3.72) by its divisor (3.70) will still be a trigonometric polynomial in x, and we get that

$$\psi = \delta(x)^{-1} e^{\langle z-\rho, x\rangle} \sum_{\nu \in \mathcal{L}} c_\nu(z) e^{\langle \nu, x\rangle}$$

for some polynomials $c_\nu(z)$. The proof that $\{\nu \in \mathcal{L}\colon c_\nu \neq 0\} \subset \mathcal{L}_+$ is completely analogous to the case of $BC(l,1)$ in the proof of Proposition 3.42 (the variables y_i should here be replaced by $y_1 = e^{2\langle \beta_1, x\rangle}$ and $y_2 = e^{2\langle \alpha_2, x\rangle}$). ■

By a completely analogous proof to that of Proposition 3.44, we get the following.

Proposition 3.52. *The polynomial c_0 in the expansion* (3.68) *is given by*

$$c_0(z) = -8 \prod_{\substack{\gamma \in G_{2,+} \\ s \in A_\gamma}} (\langle \gamma, z\rangle + s\langle \gamma, \gamma\rangle).$$

Let us now consider the multiplicity parameter $m \in \mathbb{C}$ not necessarily being an integer. The next lemma states a potential-free gauge-equivalent form of the generalised CMS Hamiltonian of AG_2.

Lemma 3.53. *The differential operator*

$$H = \Delta - \sum_{\gamma \in AG_{2,+}} 2c_\gamma \coth\langle \gamma, x\rangle \partial_\gamma \tag{3.73}$$

is gauge-equivalent to the operator L defined by formula (3.1) *with $\mathcal{A} = AG_{2,+}$:*

$$H + \rho^2 = -\delta(x) \circ L \circ \delta(x)^{-1},$$

where ρ is given by formula (3.69) *and $\delta(x)$ by formula* (3.67)*.*

In particular, this means that AG_2 satisfies the analogue of Lemma 3.47 with $BC(l,1)_+$ replaced by $AG_{2,+}$. Lemma 3.53 will follow from (the scalar case of) Theorems 4.3 and 4.4, Remark 4.6, and the discussions in Section 4.4.1.2 below, so we skip the proof here.

When x belongs to the region

$$\begin{aligned} B &= \{x \in \mathbb{C}^2\colon \operatorname{Re}\langle \gamma, x\rangle < 0 \text{ for all } \gamma \in AG_{2,+}\} \\ &= \{(x_1, x_2) \in \mathbb{C}^2\colon \operatorname{Re}(x_1) < 0,\ \operatorname{Re}(x_2) < \sqrt{3}\operatorname{Re}(x_1)\}, \end{aligned}$$

the operator (3.73) can be expanded like in (3.56), just with $BC(l,1)_+$ replaced by $AG_{2,+}$. In complete analogy with the case of $BC(l,1)$ in Section 3.6.1, let us formally define a function

$$\varphi = e^{\langle z-\rho, x\rangle} \sum_{\nu \in \mathcal{L}_+} q_\nu(z) e^{\langle \nu, x\rangle} \tag{3.74}$$

by the equation $H\varphi = (z^2 - \rho^2)\varphi$ and the normalisation condition $q_0(z) = 1$. It follows that $q_\nu(z)$ are rational in z and m. The next lemma can be used to show that the series (3.74) converges absolutely on B similarly to the case of $BC(l,1)$ above.

Lemma 3.54. *For $\nu \in \mathcal{L}_+ \setminus \{0\}$, let $\Sigma_\nu = \{z \in \mathbb{C}^2 \colon \langle \nu, \nu + 2z \rangle = 0\}$. Suppose*

$$z \in \mathbb{C}^2 \setminus \cup_{\nu \in \mathcal{L}_+ \setminus \{0\}} \Sigma_\nu$$

and $x \in B$. Then there exists a constant $K = K(z, x) \in \mathbb{R}$ independent of ν such that

$$|q_\nu(z) e^{\langle \nu, x \rangle}| \leq K$$

for all $\nu \in \mathcal{L}_+$.

The proof is essentially the same as that of Lemma 3.48 (but the configuration does not depend on any deformation parameter k this time, so the proof can be simplified in an obvious way).

When $m \in \mathbb{Z}_{\geq 0}$, then by using the uniqueness of the solution φ, Lemma 3.53, and Propositions 3.51 and 3.52, we get that the BA eigenfunction (3.25) of the operator L satisfies

$$\psi(z, x) = c_0(z) \delta(x)^{-1} \varphi(z, x) = -8 \delta(x)^{-1} \varphi(z, x) \prod_{\substack{\gamma \in G_{2,+} \\ s \in A_\gamma}} (\langle \gamma, z \rangle + s \langle \gamma, \gamma \rangle).$$

For the case when $m \in \mathbb{C}$ is not necessarily in $\mathbb{Z}_{\geq 0}$, let us generalise the function ψ to

$$\Psi(z, x) = C(z) \delta(x)^{-1} \varphi(z, x), \tag{3.75}$$

where we define

$$C(z) = \prod_{\gamma \in G_{2,+}} \frac{\Gamma(\langle \gamma, z \rangle \langle \gamma, \gamma \rangle^{-1} + c_\gamma + 1)}{\Gamma(\langle \gamma, z \rangle \langle \gamma, \gamma \rangle^{-1} + 1)} \prod_{i=1}^{3} (\langle \beta_i, z \rangle + (3m + 2) \langle \beta_i, \beta_i \rangle).$$

The function $\Psi(z, x)$ coincides (up to a constant factor) with $\psi(z, x)$ if $m \in \mathbb{Z}_{\geq 0}$, since then $C(z) \propto c_0(z)$. Now the proof of the next theorem is essentially the same as that of Theorem 3.50; it uses Theorems 3.24 and 3.27 and that $C(z)^{-1} C(z + \tau)$ is rational in z and m for all $\tau \in 2AG_2$, since $\langle \gamma, \tau \rangle \langle \gamma, \gamma \rangle^{-1} \in \mathbb{Z}$ for all $\gamma \in G_{2,+}$ (since G_2 is a crystallographic root system).

Theorem 3.55. *For any $m \in \mathbb{C}$, the function $\Psi(z, x)$ defined by formula (3.75) satisfies*

$$L \Psi = -z^2 \Psi,$$
$$\mathcal{D}_1 \Psi = \mu(x) \Psi,$$
$$\mathcal{D}_2 \Psi = \widetilde{\mu}(x) \Psi,$$

where L is the generalised CMS operator (3.1) for $\mathcal{A} = AG_{2,+}$, the function $\mu(x)$ is defined *by* (3.22) *and* $\widetilde{\mu}(x)$ *by* (3.30)*, and* $\mathcal{D}_i$ *(*$i = 1, 2$*) are the operators* (3.17) – (3.19) *and* (3.26) – (3.29)*, respectively.*

Chapter 4

Spin Calogero–Moser–Sutherland type systems

In this chapter, we make use of the representation theory of TCAs to construct generalisations of (trigonometric) spin CMS operators. This work constitutes one part of our paper in preparation [54]. That paper also includes the details of the rational version of this construction, including the case with an extra harmonic term in the potential. This is a generalisation of the work done in [50] by Feigin.

The structure of this chapter is as follows. In Section 4.1, we review the construction of scalar rational generalised CMS systems from [50]. Namely, we recall the definition of parabolic strata for finite Coxeter groups and the conditions that the strata have to satisfy for the ideal of polynomials vanishing on them to be invariant under the associated RCA. Then we recall how parabolic strata defining invariant ideals can be used to obtain operators of rational CMS type and quantum integrals for them. In Sections 4.2 and 4.3, we develop a similar construction for TCAs associated with reduced crystallographic root systems. We do this directly in the more general matrix case. This leads to generalised trigonometric spin CMS operators related to projections of these root systems. We apply this construction in Section 4.4 for exceptional root systems to derive several interesting explicit new examples of such operators. A systematic account of the case of projections of classical root systems and also of the non-reduced root system BC_n will appear in [54]. In Section 4.5, we list cases where the projection of a root system is itself a root system, as this allows one to obtain new examples of spin CMS operators associated with root systems.

Let us note that in [54], we additionally provide in deformed type A also extra quantum integrals for the corresponding deformed spin CMS operator by exploiting a Yangian symmetry.

4.1 Review of the scalar rational case

Let W be a finite real reflection group acting by orthogonal transformations in the complexified reflection representation $V = \mathbb{C}^N$. Let R and Γ be the corresponding (reduced) root system and Coxeter graph, respectively. Let $c\colon R \to \mathbb{C}$, $\alpha \mapsto c_\alpha$ be a W-invariant multiplicity function. Let $\mathcal{H}_c$ be the associated RCA acting faithfully on the space of polynomials $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_N]$. We assume that a positive subsystem $R_+ \subset R$ is chosen so that the vertices of Γ are identified with simple roots. Let $\Gamma_0 \subset \Gamma$ be a subgraph of the Coxeter graph, and let Γ_0^v denote the set of simple roots corresponding to the vertices of Γ_0. We denote by $W_0 \subset W$ the parabolic subgroup generated by the reflections with respect to the roots Γ_0^v.

Suppose Γ_0 is obtained by specifying some of the vertices of Γ and preserving all the edges between them. The vertices of Γ_0 determine the subspace

$$\pi = \pi_{\Gamma_0} = \{x \in V\colon \langle \beta, x\rangle = 0,\ \forall \beta \in \Gamma_0^v\}.$$

The associated parabolic stratum is defined as

$$D_{\Gamma_0} = \bigcup_{w \in W} w(\pi).$$

Let the corresponding parabolic ideal I_{Γ_0} be the set of polynomials vanishing on the stratum, $I_{\Gamma_0} = \{p \in \mathbb{C}[x]\colon p|_{D_{\Gamma_0}} = 0\}$. The following theorem, proved by Feigin, gives necessary and sufficient conditions for this ideal to be a submodule of the polynomial representation of the RCA.

Theorem 4.1. *[50] Let $\Gamma_0 = \coprod_{i=1}^{l} \Gamma_i$ be the decomposition of the subgraph Γ_0 into connected components. Then the parabolic ideal I_{Γ_0} is invariant under the RCA $\mathcal{H}_c$ if and only if the following relation is satisfied for all $i = 1, \dots, l$:*

$$\sum_{\alpha \in R \cap V_i} \frac{c_\alpha \langle \alpha, u\rangle \langle \alpha, v\rangle}{\langle \alpha, \alpha\rangle} = \langle u, v\rangle$$

for all $u, v \in V_i$, where V_i is the vector space spanned by the roots Γ_i^v.

We now explain how $\mathcal{H}_c$-invariant ideals I_{Γ_0} lead to generalised rational CMS operators and quantum integrals for them.

Suppose I_{Γ_0} is an invariant parabolic ideal as above. Let us define the quotient module $M = \mathbb{C}[x]/I_{\Gamma_0}$. Let $\{e_i\}_{i=1}^N$ be the standard orthonormal basis of V, and consider the Dunkl operators $\nabla_i = \nabla_{e_i}$ defined by formula (2.12) with $\xi = e_i$. Any polynomial function of ∇_i can act on the module M. Moreover, W-invariant polynomials of Dunkl operators preserve the space of invariants M^W. Consider the operators

$$H_p = \operatorname{Res} p(\nabla)$$

for W-invariant polynomials $p \in \mathbb{C}[x]^W$, $p(\nabla) = p(\nabla_1, \dots, \nabla_N)$, where Res denotes restriction to M^W. Let us note that M^W is, more generally, preserved by the centraliser $\mathcal{H}_c^W = C_{\mathcal{H}_c}(\mathbb{C}W)$ of the group algebra $\mathbb{C}W$ of the Coxeter group inside the RCA. In particular, it is a module for the spherical subalgebra $S\mathcal{H}_c \subset \mathcal{H}_c^W$, $S\mathcal{H}_c = e\mathcal{H}_c e$ with $e = |W|^{-1}\sum_{w\in W} w$.

To write down the operators H_p explicitly in local coordinates on π, it is convenient to consider action of $p(\nabla)$ on W-invariant (formal) sums of germs of analytic functions on W-orbits of small neighbourhoods on D_{Γ_0} of a generic point of π (see [50, Section 3] for the details), rather than on global functions. This way the operator H_p takes the form of a differential operator on π, which we denote $\operatorname{Res}_\pi p(\nabla)$. In particular, by using that

$$\sum_{i=1}^N \nabla_i^2 = \sum_{i=1}^N \partial_{x_i}^2 - \sum_{\alpha\in R_+} \frac{2c_\alpha}{\langle \alpha, x\rangle}\partial_\alpha + \sum_{\alpha\in R_+} \frac{c_\alpha \langle \alpha,\alpha\rangle}{\langle\alpha, x\rangle^2}(1 - s_\alpha),$$

one obtains the generalised rational CMS Hamiltonian in radial gauge

$$\operatorname{Res}_\pi \left(\sum_{i=1}^N \nabla_i^2\right) = \Delta_y - \sum_{\substack{\alpha\in R_+\\ \widehat{\alpha}\neq 0}} \frac{2c_\alpha}{\langle \widehat{\alpha}, y\rangle}\partial_{\widehat{\alpha}},$$

where $y = (y_1, \dots, y_n)$ are orthonormal coordinates on the space π, $\Delta_y = \partial_{y_1}^2 + \dots + \partial_{y_n}^2$ is the Laplacian on π, and $\widehat{\alpha}$ is the orthogonal projection of α onto π. The operators $\operatorname{Res}_\pi p(\nabla)$ for $p \in \mathbb{C}[x]^W$ give quantum integrals for this Hamiltonian.

4.2 Invariant parabolic submodules for TCAs

In this section, we will continue to use notations introduced in Section 4.1, but R will now be a reduced crystallographic root system, W its Weyl group, and P its weight lattice. Let $\mathbb{H}_c^{\mathrm{trig}}$ be the associated TCA. Let U be any complex vector space that is a right W-module and denote the action of $w \in W$ on $v \in U$ by $v \cdot w$.

Let x_0 be a generic point of π, meaning that if $e^{\langle \alpha, x_0\rangle} = 1$ for some $\alpha \in R$ then $\alpha \in \operatorname{Span}\Gamma_0^v$. Let $Wx_0 = \cup_{w\in W} w(x_0)$ be the W-orbit of x_0. Define the space

$$\mathcal{C}_{Wx_0}(U) = \mathcal{C}_{Wx_0}(D_{\Gamma_0}, U) = \bigoplus_{x\in Wx_0} \mathcal{C}_x(D_{\Gamma_0}, U),$$

where $\mathcal{C}_x(D_{\Gamma_0}, U)$ is the space of germs of analytic functions defined on D_{Γ_0} near the point

$x \in Wx_0$ with values in U. Note that $\mathcal{C}_{Wx_0}(U) \cong U \otimes \mathcal{C}_{Wx_0}(\mathbb{C})$. We would like to define an action of $\mathbb{H}_c^{\mathrm{trig}}$ on $\mathcal{C}_{Wx_0}(U)$.

Note that $\mathbb{H}_c^{\mathrm{trig}}$ can act on the space $\mathcal{C}_{Wx_0}(V,U) = \bigoplus_{x\in Wx_0} \mathcal{C}_x(V,U)$ of (formal) sums of U-valued analytic germs defined on V near the points Wx_0. The elements of the Weyl group $w \in W$ act by moving the germs at one point to another one. Namely, $w\colon \mathcal{C}_x(V,U) \to \mathcal{C}_{w(x)}(V,U)$ for any $x \in Wx_0$, with the action given by $(wF)(y) = F(w^{-1}y)$ for $y \in V$ near $w(x)$ and $F \in \mathcal{C}_x(V,U)$. The elements $e^{\langle\alpha,x\rangle} \in \mathbb{H}_c^{\mathrm{trig}}$ ($\alpha \in P$) act by multiplication. The trigonometric Dunkl operators act on a given sum of germs by formula (2.13) with each reflection $s_\alpha \in W$ acting as defined above. Note that the well-definedness of this action of $\nabla_\xi^{\mathrm{trig}}$ relies on the genericity of x_0.

The next theorem gives the conditions under which this $\mathbb{H}_c^{\mathrm{trig}}$-module $\mathcal{C}_{Wx_0}(V,U)$ has a submodule $\mathcal{I}_{\Gamma_0}$ consisting of those elements that vanish when restricted to D_{Γ_0} (note that the form of these conditions is the same as in the case of RCAs and U being the trivial representation from [50]). We call $\mathcal{I}_{\Gamma_0}$ a parabolic submodule, in analogy with the parabolic ideal I_{Γ_0} from Section 4.1.

Theorem 4.2. *Let $\Gamma_0 = \coprod_{i=1}^l \Gamma_i$ be the decomposition of the subgraph Γ_0 into connected components. Then $\mathcal{I}_{\Gamma_0}$ is invariant under the TCA $\mathbb{H}_c^{\mathrm{trig}}$ if and only if the following relation is satisfied for all $i = 1, \dots, l$:*

$$\sum_{\alpha\in R\cap V_i} \frac{c_\alpha \langle\alpha,u\rangle\langle\alpha,v\rangle}{\langle\alpha,\alpha\rangle} = \langle u,v\rangle$$

for all $u, v \in V_i = \operatorname{Span}\Gamma_i^v$.

Proof. Notice that it is sufficient to prove this statement in the case when $U = \mathbb{C}$, and it is enough to consider the elements $\nabla_\xi^{\mathrm{trig}} \in \mathbb{H}_c^{\mathrm{trig}}$. Let $V_0 = \operatorname{Span}\Gamma_0^v$ and $f \in \mathcal{I}_{\Gamma_0}$. We are going to consider first the condition that $(\nabla_\xi^{\mathrm{trig}} f)|_\pi = 0$. Since $f|_\pi = 0$, it can be written for $x \in V$ close to $x_0 \in \pi$ as

$$f = \sum_{\beta\in\Gamma_0^v} \langle\beta,x\rangle f_\beta(x)$$

for some analytic germs $f_\beta \in \mathcal{C}_{x_0}(V,\mathbb{C})$. For $\alpha \in R$, since $(s_\alpha f)|_\pi = f|_{s_\alpha\pi} = 0$ as $f|_{D_{\Gamma_0}} = 0$, we have that $(1-e^{-\langle\alpha,x\rangle})^{-1}(1-s_\alpha)f|_\pi = 0$ if $e^{\langle\alpha,x\rangle} \neq 1$, which by genericity of x_0 is satisfied if $\alpha \notin V_0$ and x is sufficiently close to x_0. We then write

$$\nabla_\xi^{\mathrm{trig}} f = \sum_{\beta\in\Gamma_0^v} \Bigg(\langle\beta,x\rangle\partial_\xi f_\beta(x) + \langle\beta,\xi\rangle f_\beta(x) - \sum_{\alpha\in R_+\cap V_0} \frac{c_\alpha\langle\alpha,\xi\rangle}{1-e^{-\langle\alpha,x\rangle}} \Big(\langle\beta,x\rangle\big(f_\beta(x) - f_\beta(s_\alpha x)\big) + \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\langle\alpha,x\rangle f_\beta(s_\alpha x)\Big)\Bigg) - \sum_{\substack{\alpha\in R_+\\ \alpha\notin V_0}} \frac{c_\alpha\langle\alpha,\xi\rangle}{1-e^{-\langle\alpha,x\rangle}}(1-s_\alpha)f + \langle\rho,\xi\rangle f.$$

Note that for $\alpha \in R_+ \cap V_0$ and $x \in \pi$ close to x_0,

$$\frac{f_\beta(x) - f_\beta(s_\alpha x)}{1 - e^{-\langle \alpha, x\rangle}}$$

is non-singular, $\langle \beta, x\rangle$ vanishes, and $(1 - e^{-\langle \alpha, x\rangle})^{-1}\langle \alpha, x\rangle f_\beta(s_\alpha x)$ becomes $f_\beta(x)$ since $\lim_{u\to 0}(1 - e^{-u})^{-1}u = 1$. Therefore, we get

$$(\nabla_\xi^{\mathrm{trig}} f)|_\pi = \sum_{\beta \in \Gamma_0^v} \left(\langle \beta, \xi\rangle - \sum_{\alpha \in R_+ \cap V_0} \frac{2c_\alpha \langle \alpha, \beta\rangle\langle \alpha, \xi\rangle}{\langle \alpha, \alpha\rangle} \right) f_\beta|_\pi.$$

This is zero for all $f \in \mathcal{I}_{\Gamma_0}$ if and only if

$$\langle u, v\rangle = \sum_{\alpha \in R_+ \cap V_0} \frac{2c_\alpha \langle \alpha, u\rangle\langle \alpha, v\rangle}{\langle \alpha, \alpha\rangle} \tag{4.1}$$

for all $u, v \in V_0$. Similarly to the proof of [50, Theorem 1], the proof is then completed by using that $V_0 = \bigoplus_{i=1}^{l} V_i$ and that the conditions $(\nabla_\xi^{\mathrm{trig}} f)|_{w(\pi)} = 0$ for $w \neq \mathrm{id}$ are equivalent to (4.1). ■

If $\mathcal{I}_{\Gamma_0}$ is $\mathbb{H}_c^{\mathrm{trig}}$-invariant, then the space $\mathcal{C}_{Wx_0}(U)$ can be identified with the quotient module $\mathcal{C}_{Wx_0}(V, U)/\mathcal{I}_{\Gamma_0}$ (cf. [50, Section 3] in the case of RCAs and U being the trivial representation), and thus becomes a $\mathbb{H}_c^{\mathrm{trig}}$-module. In other words, for an arbitrary element $f \in \mathcal{C}_{Wx_0}(U)$, it follows from the $\mathbb{H}_c^{\mathrm{trig}}$-invariance of $\mathcal{I}_{\Gamma_0}$ that for any $a \in \mathbb{H}_c^{\mathrm{trig}}$ and analytic extension $\overline{f} \in \mathcal{C}_{Wx_0}(V, U)$ of f to a W-invariant union of small neighbourhoods in the ambient space V of the points Wx_0, the result of the restriction $(a\overline{f})|_{D_{\Gamma_0}}$ does not depend on the choice of the extension $\overline{f}$ but only on f itself. Thus, we have a well-defined action of $\mathbb{H}_c^{\mathrm{trig}}$ on $\mathcal{C}_{Wx_0}(U)$.

4.3 Restricted spin CMS operators

Let $\mathcal{C}_{Wx_0}^W(U)$ be the subset of those elements of $\mathcal{C}_{Wx_0}(U)$ that are fixed by the *diagonal* (left) action of W determined by $(wF)(y) = F(w^{-1}y)\cdot w^{-1}$ for $F \in \mathcal{C}_x(D_{\Gamma_0}, U)$ and $y \in D_{\Gamma_0}$ near $w(x)$, $x \in Wx_0$. Then any element of $\mathcal{C}_{Wx_0}^W(U)$ is uniquely determined by the germ near the point x_0. In other words, $\mathcal{C}_{Wx_0}^W(U) \cong \mathcal{C}_{x_0}^{W_0}(D_{\Gamma_0}, U) \cong U^{W_0} \otimes \mathcal{C}_{x_0}(D_{\Gamma_0}, \mathbb{C})$ as vector spaces, where U^{W_0} is the subspace of vectors in U fixed under the action of $W_0 \subset W$.

Let $(\mathbb{H}_c^{\mathrm{trig}})^W$ be the centraliser of the group algebra $\mathbb{C}W$ inside the TCA. Assume that $\mathcal{I}_{\Gamma_0}$ is an invariant parabolic submodule for the TCA. Then in view of the discussions in Section 4.2, it follows that $\mathcal{C}_{Wx_0}^W(U)$ is an $(\mathbb{H}_c^{\mathrm{trig}})^W$-module. Thus, we can also treat $\mathcal{C}_{x_0}^{W_0}(D_{\Gamma_0}, U)$ as an $(\mathbb{H}_c^{\mathrm{trig}})^W$-module. We denote the action of an element $a \in (\mathbb{H}_c^{\mathrm{trig}})^W$ on this module by $\widetilde{\mathrm{Res}}_\pi a$.

Let $\nabla_i^{\mathrm{trig}} = \nabla_{e_i}^{\mathrm{trig}}$. Note that for $p \in \mathbb{C}[x]^W$, we have $p(\nabla^{\mathrm{trig}}) = p(\nabla_1^{\mathrm{trig}}, \dots, \nabla_N^{\mathrm{trig}}) \in (\mathbb{H}_c^{\mathrm{trig}})^W$ by [33] (see also [64]). We define the generalised spin CMS Hamiltonians

$$H_2 = \widetilde{\mathrm{Res}}_\pi \left(\sum_{i=1}^N (\nabla_i^{\mathrm{trig}})^2 \right).$$

They can be computed explicitly by using that

$$\sum_{i=1}^N (\nabla_i^{\mathrm{trig}})^2 = \sum_{i=1}^N \partial_{x_i}^2 - \sum_{\alpha \in R_+} c_\alpha \coth\left(\frac{\langle \alpha, x\rangle}{2}\right) \partial_\alpha + \sum_{\alpha \in R_+} \frac{c_\alpha \langle \alpha, \alpha \rangle}{4 \sinh^2\left(\frac{\langle \alpha, x\rangle}{2}\right)} (1 - s_\alpha) + \langle \rho, \rho \rangle. \tag{4.2}$$

We get the following theorem, where we use the same notations as in Section 4.1.

Theorem 4.3. *Assume that the stratum D_{Γ_0} defines an invariant parabolic submodule $\mathcal{I}_{\Gamma_0}$ for $\mathbb{H}_c^{\mathrm{trig}}$. Then the operator $\sum_{i=1}^N (\nabla_i^{\mathrm{trig}})^2$ restricted to $\mathcal{C}_{x_0}^{W_0}(D_{\Gamma_0}, U)$ has the generalised spin CMS form*

$$H_2 = \Delta_y - \sum_{\substack{\alpha \in R_+ \\ \widehat{\alpha} \neq 0}} c_\alpha \coth\left(\frac{\langle \widehat{\alpha}, y\rangle}{2}\right) \partial_{\widehat{\alpha}} + \sum_{\substack{\alpha \in R_+ \\ \widehat{\alpha} \neq 0}} \frac{c_\alpha \langle \alpha, \alpha \rangle}{4 \sinh^2\left(\frac{\langle \widehat{\alpha}, y\rangle}{2}\right)} (1 - P_\alpha) + \langle \rho, \rho \rangle, \tag{4.3}$$

where P_α denotes the action of the reflection $s_\alpha \in W$ on the vector space U. Moreover, for any $p \in \mathbb{C}[x]^W$, the operators $\widetilde{\mathrm{Res}}_\pi p(\nabla^{\mathrm{trig}})$ pairwise commute for different choices of invariant polynomials, and in particular, all of them commute with the operator (4.3).

Proof. The result follows immediately from equality (4.2) by similar arguments as in the proof of [50, Theorem 5]. ∎

We now rewrite the operator (4.3) in the potential gauge. Let $\widehat{R}_+ = \{\widehat{\alpha} \colon \alpha \in R_+\}$.

Theorem 4.4. *Define the generalised coupling constants*

$$\widehat{c}_{\widehat{\alpha}} = \sum_{\substack{\gamma \in R_+ \\ \widehat{\gamma} = \widehat{\alpha}}} c_\gamma.$$

The operator (4.3) *has the potential-gauge form*

$$f^{-1} H_2 f = \Delta_y - \sum_{\widehat{\alpha} \in \widehat{R}_+ \setminus \{0\}} \frac{\widehat{c}_{\widehat{\alpha}} \langle \widehat{\alpha}, \widehat{\alpha} \rangle \widehat{P}_{\widehat{\alpha}}}{4 \sinh^2\left(\frac{\langle \widehat{\alpha}, y\rangle}{2}\right)}$$
$$- \sum_{\widehat{\alpha} \in \widehat{R}_+ \setminus \{0\}} \sum_{\substack{\widehat{\beta} \in \widehat{R}_+ \setminus \{0\} \\ \widehat{\beta} \sim \widehat{\alpha}}} \frac{\widehat{c}_{\widehat{\alpha}} \widehat{c}_{\widehat{\beta}} \langle \widehat{\alpha}, \widehat{\beta} \rangle}{4} \coth\left(\frac{\langle \widehat{\alpha}, y\rangle}{2}\right) \coth\left(\frac{\langle \widehat{\beta}, y\rangle}{2}\right) + \lambda,$$

where $\sim$ denotes proportionality of vectors and

$$f = \prod_{\widehat{\alpha}\in\widehat{R}_+\setminus\{0\}} \left(\sinh\left(\frac{\langle\widehat{\alpha}, y\rangle}{2}\right)\right)^{\widehat{c}_{\widehat{\alpha}}},$$

$$\lambda = \langle\rho,\rho\rangle - \frac{1}{4}\sum_{\widehat{\alpha}\in\widehat{R}_+}\sum_{\substack{\widehat{\beta}\in\widehat{R}_+\\ \widehat{\beta}\nsim\widehat{\alpha}}} \widehat{c}_{\widehat{\alpha}}\widehat{c}_{\widehat{\beta}}\langle\widehat{\alpha},\widehat{\beta}\rangle,$$

$$\widehat{c}_{\widehat{\alpha}}\widehat{P}_{\widehat{\alpha}} = \widehat{c}_{\widehat{\alpha}} + \frac{1}{\langle\widehat{\alpha},\widehat{\alpha}\rangle}\sum_{\substack{\gamma\in R_+\\ \widehat{\gamma}=\widehat{\alpha}}} c_\gamma\langle\gamma,\gamma\rangle(P_\gamma - 1). \tag{4.4}$$

Proof. We compute that

$$f^{-1}H_2 f = \Delta_y - \sum_{\substack{\alpha\in R_+\\ \widehat{\alpha}\neq 0}} \frac{c_\alpha\big(\langle\widehat{\alpha},\widehat{\alpha}\rangle + \langle\alpha,\alpha\rangle(P_\alpha - 1)\big)}{4\sinh^2\left(\frac{\langle\widehat{\alpha},y\rangle}{2}\right)} \tag{4.5}$$
$$- \frac{1}{4}\sum_{\substack{\alpha\in R_+\\ \widehat{\alpha}\neq 0}}\sum_{\substack{\beta\in R_+\\ \widehat{\beta}\neq 0}} c_\alpha c_\beta\langle\widehat{\alpha},\widehat{\beta}\rangle\coth\left(\frac{\langle\widehat{\alpha},y\rangle}{2}\right)\coth\left(\frac{\langle\widehat{\beta},y\rangle}{2}\right) + \langle\rho,\rho\rangle.$$

Next, we simplify the last sum in (4.5). We will use a trigonometric analogue of [50, Equality (10)]. Namely, we claim for all $\alpha\in R_+$ that

$$\sum_{\substack{\beta\in R_+\\ \widehat{\beta}\nsim\widehat{\alpha}}} c_\beta\langle\widehat{\alpha},\widehat{\beta}\rangle\coth\left(\frac{\langle\widehat{\beta},y\rangle}{2}\right) = 0 \tag{4.6}$$

for $y\in\pi$ with $\langle\widehat{\alpha},y\rangle = 2i\pi k$ for $k\in\mathbb{Z}$, which can be seen as follows.

Let $W_\alpha = \langle W_0, s_\alpha\rangle$ be the group generated by s_α and the reflections about the roots in Γ_0^v. Let $S\subset R$ be the set of the roots $\beta\in R$ such that $\widehat{\beta}$ is not proportional to $\widehat{\alpha}$. Decompose S into W_α-orbits $S = \mathcal{O}_1\coprod\cdots\coprod\mathcal{O}_r$. We will show that

$$\sum_{\beta\in\mathcal{O}_i} c_\beta\langle\widehat{\alpha},\widehat{\beta}\rangle\coth\left(\frac{\langle\widehat{\beta},y\rangle}{2}\right) = 0 \tag{4.7}$$

for all i. Let $\beta_1,\beta_2\in\mathcal{O}_i$. Then $\coth(\frac{\langle\widehat{\beta}_1,y\rangle}{2}) = \coth(\frac{\langle\widehat{\beta}_2,y\rangle}{2})$. Indeed, this is evident if $\beta_1 = s_0\beta_2$ for $s_0\in W_0$; and if $\beta_1 = s_\alpha\beta_2$, then

$$\coth\left(\frac{\langle\widehat{s_\alpha\beta_2},y\rangle}{2}\right) = \coth\left(\frac{\langle s_\alpha\beta_2,y\rangle}{2}\right) = \coth\left(\frac{\langle\beta_2,s_\alpha y\rangle}{2}\right)$$

$$= \coth\left(\frac{\langle\widehat{\beta_2}, y\rangle}{2} - \pi i k \frac{2\langle\beta_2,\alpha\rangle}{\langle\alpha,\alpha\rangle}\right) = \coth\left(\frac{\langle\widehat{\beta_2}, y\rangle}{2}\right),$$

where we used that $\frac{2\langle\beta_2,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$. Now let $b_i = \sum_{\beta\in\mathcal{O}_i}\beta$. It satisfies the relations $s_0 b_i = b_i$ for $s_0 \in W_0$ and $s_\alpha b_i = b_i$. This translates into $\widehat{b_i} = b_i$ and $0 = \langle\alpha, b_i\rangle = \langle\widehat{\alpha}, \widehat{b_i}\rangle$, which implies (4.7) and hence (4.6).

Relation (4.6) implies that the expression

$$\sum_{\alpha\in R_+}\sum_{\substack{\beta\in R_+\\ \widehat{\beta}\nsim\widehat{\alpha}}} c_\alpha c_\beta \langle\widehat{\alpha},\widehat{\beta}\rangle \coth\left(\frac{\langle\widehat{\alpha},y\rangle}{2}\right)\coth\left(\frac{\langle\widehat{\beta},y\rangle}{2}\right)$$

has no poles; thus it is an entire bounded function that is constant due to Liouville's theorem. The constant can be calculated to be $4\langle\rho,\rho\rangle - 4\lambda$ by a limit at infinity in a suitable chamber such that $\coth\langle\widehat{\alpha}, y\rangle \to 1$ for all $\widehat{\alpha}$. Using this fact to simplify expression (4.5), the proof is then completed by using (4.4) and the definition of $\widehat{c}_{\widehat{\alpha}}$. ■

Remark 4.5. Let us note that the elements

$$S_{\widehat{\alpha}} = \sum_{\substack{\gamma\in R_+\\ \widehat{\gamma}=\widehat{\alpha}}} c_\gamma\langle\gamma,\gamma\rangle s_\gamma \in \mathbb{C}W$$

lie in the centraliser of $\mathbb{C}W_0$ inside the group algebra $\mathbb{C}W$. Indeed, let $\beta \in \Gamma_0^v$, then

$$[s_\beta, S_{\widehat{\alpha}}] = \sum_{\substack{\gamma\in R_+\\ \widehat{\gamma}=\widehat{\alpha}}} c_\gamma\langle\gamma,\gamma\rangle[s_\beta, s_\gamma] = \sum_{\substack{\gamma\in R_+\\ \widehat{\gamma}=\widehat{\alpha}}} c_\gamma\langle\gamma,\gamma\rangle(s_{s_\beta(\gamma)} - s_\gamma)s_\beta = 0,$$

where the last equality is obtained by changing the index of summation in the first term to $\widetilde{\gamma} = s_\beta(\gamma)$, which is possible due to the fact that the multiplicity function c_γ is W-invariant and $\beta \in \Gamma_0^v$.

Remark 4.6. Let us assume that any collinear vectors in $\widehat{R}_+ \setminus \{0\}$ are of the form $\widehat{\alpha}$, $2\widehat{\alpha}$. This is the case for the projections of all classical root systems. Then the operator $f^{-1}H_2 f$ from Theorem 4.4 becomes (up to a constant, namely, $\langle\rho,\rho\rangle - \langle\widehat{\rho},\widehat{\rho}\rangle$) equal to

$$L := \Delta_y - \sum_{\widehat{\alpha}\in\widehat{R}_+\setminus\{0\}} \frac{\widehat{c}_{\widehat{\alpha}}(\widehat{c}_{\widehat{\alpha}} + 2\widehat{c}_{2\widehat{\alpha}} + \widehat{P}_{\widehat{\alpha}})\langle\widehat{\alpha},\widehat{\alpha}\rangle}{4\sinh^2\left(\frac{\langle\widehat{\alpha},y\rangle}{2}\right)}, \tag{4.8}$$

where $\widehat{c}_{2\widehat{\alpha}} := 0$ when $2\widehat{\alpha} \notin \widehat{R}_+$. However, the above assumption may fail for exceptional root systems, for example, for G_2 and F_4 in some cases.

In the particular case where $U = V$ is the reflection representation of the Weyl group W, then $\widehat{P}_{\widehat{\alpha}}$ coincides with the reflection in the space π with respect to the projected root $\widehat{\alpha}$, as we prove in the next proposition.

Proposition 4.7. *Let $U = V$ be the reflection representation of the Weyl group W. For $\widehat{\alpha} \in \widehat{R}_+ \setminus \{0\}$, the operator*

$$\widehat{P}_{\widehat{\alpha}} = 1 + \frac{1}{\widehat{c}_{\widehat{\alpha}}\langle\widehat{\alpha}, \widehat{\alpha}\rangle} \sum_{\substack{\gamma\in R_+\\ \widehat{\gamma}=\widehat{\alpha}}} c_\gamma \langle\gamma, \gamma\rangle (P_\gamma - 1)$$

preserves the space π, and when acting on π, it is equal to the reflection with respect to $\widehat{\alpha}$.

Proof. Let us define

$$\widetilde{P}_{\widehat{\alpha}} = \sum_{\substack{\gamma\in R_+\\ \widehat{\gamma}=\widehat{\alpha}}} c_\gamma \langle\gamma, \gamma\rangle P_\gamma.$$

For any $y \in \pi$, we have

$$\widetilde{P}_{\widehat{\alpha}}(y) = \sum_{\substack{\gamma\in R_+\\ \widehat{\gamma}=\widehat{\alpha}}} c_\gamma \langle\gamma, \gamma\rangle y - 2 \sum_{\substack{\gamma\in R_+\\ \widehat{\gamma}=\widehat{\alpha}}} c_\gamma \langle\gamma, y\rangle \gamma. \tag{4.9}$$

To prove that the vector (4.9) belongs to the subspace π, we need to simplify the second sum in (4.9). We claim that

$$\sum_{\substack{\gamma\in R_+\\ \widehat{\gamma}=\widehat{\alpha}}} c_\gamma \langle\gamma, y\rangle \gamma = \widehat{c}_{\widehat{\alpha}} \langle\widehat{\alpha}, y\rangle \widehat{\alpha}. \tag{4.10}$$

Indeed, the set $S_y = \{c_\gamma\langle\gamma, y\rangle\gamma \colon \gamma \in R_+, \widehat{\gamma} = \widehat{\alpha}\}$ is W_0-invariant, where we use that W_0 is generated by reflections about simple roots Γ_0^v. Thus, the sum in the left-hand side of (4.10) is fixed by W_0, hence belongs to the subspace π, and relation (4.10) follows. Now, by using equalities (4.9) and (4.10), we get that the action of $\widehat{P}_{\widehat{\alpha}}$ on π is

$$\widehat{P}_{\widehat{\alpha}}(y) = y + \frac{1}{\widehat{c}_{\widehat{\alpha}}\langle\widehat{\alpha}, \widehat{\alpha}\rangle} \Bigg(\widetilde{P}_{\widehat{\alpha}}(y) - \sum_{\substack{\gamma\in R_+\\ \widehat{\gamma}=\widehat{\alpha}}} c_\gamma \langle\gamma, \gamma\rangle y \Bigg) = y - \frac{2\langle\widehat{\alpha}, y\rangle}{\langle\widehat{\alpha}, \widehat{\alpha}\rangle} \widehat{\alpha},$$

which is the formula for the reflection on π with respect to $\widehat{\alpha}$. ■

Particular cases of the operator (4.8) for R of type A and B are in the rational limit equivalent to the matrix Hamiltonians found in [27]. More details about this are given in [54].

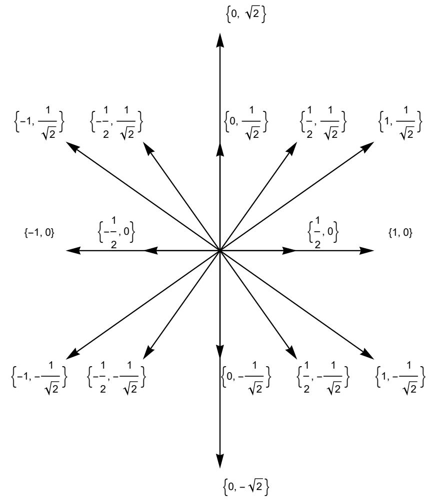


Figure 4.1: Projection of F_4 corresponding to a Coxeter subgraph of type A_1^2.

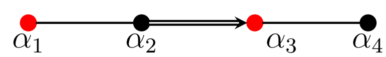
4.4 Examples from exceptional root systems

In this section, we consider all two-dimensional projections of the exceptional root systems of types E and F_4, and we write down explicitly various corresponding new examples of generalised spin CMS Hamiltonians.

4.4.1 Type F_4

4.4.1.1 $(F_4, A_1 \times A_1)$

The Dynkin diagram of the root system $F_4 \subset \mathbb{R}^4$ is



with the simple roots of F_4 being

$$\alpha_1 = e_2 - e_3,\ \alpha_2 = e_3 - e_4,\ \alpha_3 = e_4,\ \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4),$$

where $\{e_i\}_{i=1}^4$ is the standard orthonormal basis in $\mathbb{R}^4$. The red vertices indicate the chosen subgraph $\Gamma_0 \simeq A_1^2$. The corresponding plane π is given by the equations

$$x_2 = x_3,\ x_4 = 0,$$

and we require the multiplicities to be $c_1 = c_2 = \frac{1}{2}$, where $c_1 = c_{\alpha_1}$ and $c_2 = c_{\alpha_3}$. The corresponding projected system is shown in Figure 4.1, where the coordinates are in the basis formed by e_1 and $\widetilde{e}_2 = \frac{1}{\sqrt{2}}(e_2 + e_3)$. The multiplicities of the positive half are

$$\widehat{c}_{\frac{e_1}{2}} = 2, \quad \widehat{c}_{e_1} = \frac{3}{2}, \quad \widehat{c}_{\frac{\widetilde{e}_2}{\sqrt{2}}} = 3, \quad \widehat{c}_{\sqrt{2}\widetilde{e}_2} = \frac{1}{2}, \quad \widehat{c}_{e_1 \pm \frac{\widetilde{e}_2}{\sqrt{2}}} = \widehat{c}_{\frac{e_1}{2} \pm \frac{\widetilde{e}_2}{\sqrt{2}}} = 1.$$

By formula (4.8), the corresponding generalised spin CMS Hamiltonian is

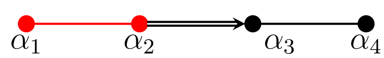
$$\begin{aligned} L = \partial_{y_1}^2 + \partial_{y_2}^2 &- \frac{5 + \widehat{P}_{\frac{e_1}{2}}}{8\sinh^2(\frac{y_1}{4})} - \frac{9 + 6\widehat{P}_{e_1}}{16\sinh^2(\frac{y_1}{2})} \\ &- \frac{12 + 3\widehat{P}_{\frac{\widetilde{e}_2}{\sqrt{2}}}}{8\sinh^2(\frac{y_2}{2\sqrt{2}})} - \frac{1 + 2\widehat{P}_{\sqrt{2}\widetilde{e}_2}}{8\sinh^2(\frac{y_2}{\sqrt{2}})} - \frac{3 + 3\widehat{P}_{\frac{\sqrt{2}e_1 + \widetilde{e}_2}{\sqrt{2}}}}{8\sinh^2(\frac{y_1}{2} + \frac{y_2}{2\sqrt{2}})} \\ &- \frac{3 + 3\widehat{P}_{\frac{\sqrt{2}e_1 - \widetilde{e}_2}{\sqrt{2}}}}{8\sinh^2(\frac{y_1}{2} - \frac{y_2}{2\sqrt{2}})} - \frac{3 + 3\widehat{P}_{\frac{e_1 + \sqrt{2}\widetilde{e}_2}{2}}}{16\sinh^2(\frac{y_1}{4} + \frac{y_2}{2\sqrt{2}})} - \frac{3 + 3\widehat{P}_{\frac{e_1 - \sqrt{2}\widetilde{e}_2}{2}}}{16\sinh^2(\frac{y_1}{4} - \frac{y_2}{2\sqrt{2}})}, \end{aligned}$$

where $y_1 = x_1$ and $y_2 = \frac{1}{\sqrt{2}}(x_2 + x_3)$. This operator is a trigonometric version with spin of the potential-gauge form of the operator [50, Formula (28)] for $m = \frac{7}{2}$, $n = 0$, and $\alpha = \sqrt{2}$.

Other choices of a subgraph $\Gamma_0 \simeq A_1^2$ in the Dynkin diagram of F_4 lead to equivalent projected configurations.

4.4.1.2 (F_4, A_2)

As another example, let us choose a subgraph A_2 in the Dynkin diagram of F_4 as indicated by the red vertices and edges in the following picture



with multiplicity $c_1 = \frac{1}{3}$ and c_2 being a free parameter. The corresponding plane π is defined by the equations

$$x_2 = x_3 = x_4.$$

The corresponding projected system is shown in Figure 4.2, where the coordinates are in the basis e_1 and $\widetilde{e}_2 = \frac{1}{\sqrt{3}}(e_2 + e_3 + e_4)$. These vectors and their multiplicities coincide with those of the configuration AG_2.

By formula (4.8), the corresponding generalised spin CMS Hamiltonian is

$$\begin{aligned} L = \partial_{y_1}^2 + \partial_{y_2}^2 &- \frac{c_2(c_2 + \widehat{P}_{e_1})}{4\sinh^2(\frac{y_1}{2})} - \frac{c_2\left(c_2 + \widehat{P}_{\frac{e_1}{2} + \frac{\sqrt{3}\widetilde{e}_2}{2}}\right)}{4\sinh^2(\frac{y_1}{4} + \frac{\sqrt{3}y_2}{4})} - \frac{c_2\left(c_2 + \widehat{P}_{\frac{e_1}{2} - \frac{\sqrt{3}\widetilde{e}_2}{2}}\right)}{4\sinh^2(\frac{y_1}{4} - \frac{\sqrt{3}y_2}{4})} \\ &- \frac{c_2\left(3c_2 + 2 + \widehat{P}_{\frac{\widetilde{e}_2}{\sqrt{3}}}\right)}{4\sinh^2(\frac{y_2}{2\sqrt{3}})} - \frac{c_2\left(3c_2 + 2 + \widehat{P}_{\frac{e_1}{2} + \frac{\widetilde{e}_2}{2\sqrt{3}}}\right)}{4\sinh^2(\frac{y_1}{4} + \frac{y_2}{4\sqrt{3}})} - \frac{c_2\left(3c_2 + 2 + \widehat{P}_{\frac{e_1}{2} - \frac{\widetilde{e}_2}{2\sqrt{3}}}\right)}{4\sinh^2(\frac{y_1}{4} - \frac{y_2}{4\sqrt{3}})} \\ &- \frac{1 + \widehat{P}_{\frac{2\widetilde{e}_2}{\sqrt{3}}}}{3\sinh^2(\frac{y_2}{\sqrt{3}})} - \frac{1 + \widehat{P}_{e_1 + \frac{\widetilde{e}_2}{\sqrt{3}}}}{3\sinh^2(\frac{y_1}{2} + \frac{y_2}{2\sqrt{3}})} - \frac{1 + \widehat{P}_{e_1 - \frac{\widetilde{e}_2}{\sqrt{3}}}}{3\sinh^2(\frac{y_1}{2} - \frac{y_2}{2\sqrt{3}})}, \end{aligned}$$

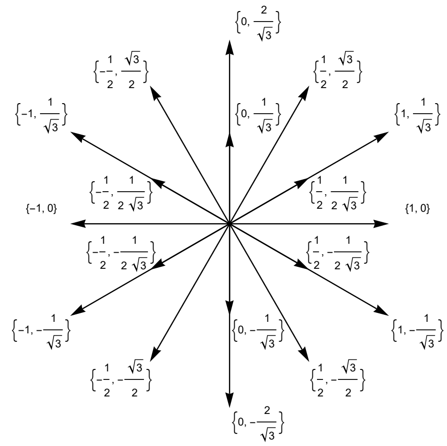


Figure 4.2: Projection of F_4 corresponding to a Coxeter subgraph of type A_2.

where $y_1 = x_1$ and $y_2 = \frac{1}{\sqrt{3}}(x_2 + x_3 + x_4)$. The scalar version of this operator, obtained by replacing all the occurrences of $\widehat{P}$ by the identity, reproduces (up to rescaling and rotating the configuration of vectors) the operator (3.12) with $m = c_2$. Thus, by using Theorem 4.3 with $U = \mathbb{C}$ being the trivial representation, we get an alternative proof of the integrability of the operator (3.12). Theorem 4.3 leads to a quantum integral of order 6 in this case, since the Weyl group of type F_4 has a basic invariant of degree 6.

The other possible choice of a subgraph $\Gamma_0 \simeq A_2$ in the Dynkin diagram of F_4 leads to G_2 with multiplicities $3c_1 + 1$ and c_1 for the short and long roots, respectively. The subgraph $\Gamma_0 \simeq B_2$ leads to BC_2 with multiplicities $4c_2$, $4c_1 + c_2$, and c_1 for the vectors e_i, $e_i \pm e_j$, and $2e_i$, respectively, where $2(c_1 + c_2) = 1$.

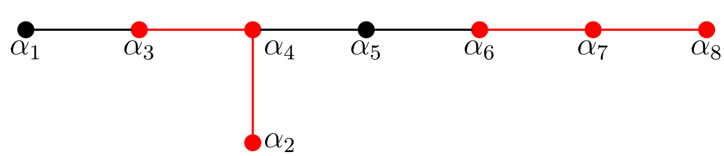
4.4.2 Type E

4.4.2.1 $(E_8, A_3 \times A_3)$

The root system $E_8 \subset \mathbb{R}^8$ has simple roots

$$\begin{gathered}\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8),\ \alpha_2 = e_1 + e_2,\\ \alpha_3 = e_2 - e_1,\ \alpha_4 = e_3 - e_2,\ \alpha_5 = e_4 - e_3,\ \alpha_6 = e_5 - e_4,\\ \alpha_7 = e_6 - e_5,\ \alpha_8 = e_7 - e_6,\end{gathered} \tag{4.11}$$

where $\{e_i\}_{i=1}^8$ is the standard orthonormal basis in $\mathbb{R}^8$. Its Dynkin diagram is



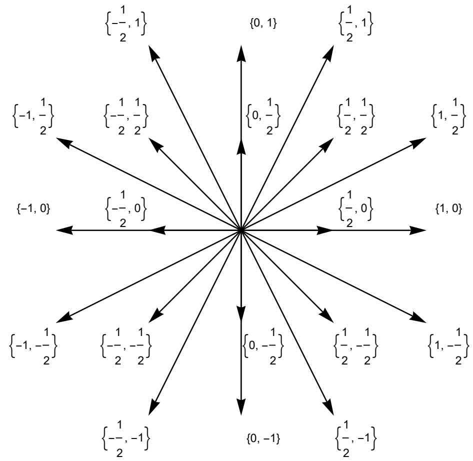


Figure 4.3: Projection of E_8 corresponding to a Coxeter subgraph of type A_3^2.

where the red vertices and edges indicate the chosen subgraph $\Gamma_0 \simeq A_3^2$. The corresponding plane π is defined by the equations

$$x_1 = x_2 = x_3 = 0, \ x_4 = x_5 = x_6 = x_7,$$

and the multiplicity must be $c = \frac{1}{4}$. The projected system is shown in Figure 4.3, where the coordinates are in the basis formed by $\widetilde{e}_1 = \frac{1}{2}(e_4 + e_5 + e_6 + e_7)$ and $\widetilde{e}_2 = e_8$. The multiplicities of the positive half are

$$\widehat{c}_{\frac{\widetilde{e}_i}{2}} = 6, \quad \widehat{c}_{\widetilde{e}_i} = \frac{3}{2}, \quad \widehat{c}_{\pm\frac{\widetilde{e}_1}{2}+\frac{\widetilde{e}_2}{2}} = 4, \quad \widehat{c}_{\pm\widetilde{e}_1+\frac{\widetilde{e}_2}{2}} = 1, \quad \widehat{c}_{\pm\frac{\widetilde{e}_1}{2}+\widetilde{e}_2} = 1.$$

By formula (4.8), the corresponding generalised spin CMS Hamiltonian is

$$\begin{aligned} L = \partial_{y_1}^2 + \partial_{y_2}^2 &- \sum_{i=1}^{2} \left(\frac{27 + 3\widehat{P}_{\frac{\widetilde{e}_i}{2}}}{8\sinh^2(\frac{y_i}{4})} + \frac{9 + 6\widehat{P}_{\widetilde{e}_i}}{16\sinh^2(\frac{y_i}{2})} \right) - \frac{4 + \widehat{P}_{\frac{\widetilde{e}_2 - \widetilde{e}_1}{2}}}{2\sinh^2(\frac{y_2 - y_1}{4})} \\ &- \frac{4 + \widehat{P}_{\frac{\widetilde{e}_1 + \widetilde{e}_2}{2}}}{2\sinh^2(\frac{y_1 + y_2}{4})} - \frac{5 + 5\widehat{P}_{\frac{\widetilde{e}_2}{2} - \widetilde{e}_1}}{16\sinh^2(\frac{y_2 - 2y_1}{4})} - \frac{5 + 5\widehat{P}_{\widetilde{e}_1 + \frac{\widetilde{e}_2}{2}}}{16\sinh^2(\frac{2y_1 + y_2}{4})} \\ &- \frac{5 + 5\widehat{P}_{\widetilde{e}_2 - \frac{\widetilde{e}_1}{2}}}{16\sinh^2(\frac{2y_2 - y_1}{4})} - \frac{5 + 5\widehat{P}_{\frac{\widetilde{e}_1}{2} + \widetilde{e}_2}}{16\sinh^2(\frac{y_1 + 2y_2}{4})}, \end{aligned}$$

where $y_1 = \frac{1}{2}(x_4 + x_5 + x_6 + x_7)$ and $y_2 = x_8$. This operator is a trigonometric version with spin of the potential-gauge form of the operator [50, Formula (28)] with $m = \frac{15}{2}$, $n = 4$, and $\alpha = (\sqrt{2n+1} + \sqrt{2(m+n+1)})/\sqrt{2m+1} = 2$.

The other possible choice of a subgraph $\Gamma_0 \simeq A_3^2$ in the Dynkin diagram of E_8 leads to an equivalent projected configuration.

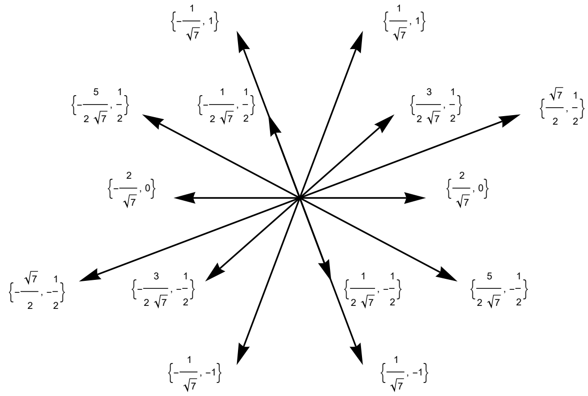
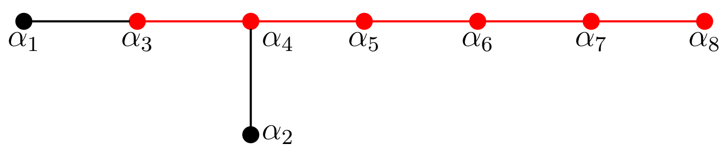


Figure 4.4: Projection of E_8 corresponding to a Coxeter subgraph of type A_6.

4.4.2.2 (E_8, A_6)

As another example, let us choose a subgraph A_6 in the Dynkin diagram of E_8 as indicated by the red vertices and edges in the following picture



The corresponding plane π is defined by the equations

$$x_1 = x_2 = \cdots = x_7,$$

with multiplicity $c = \frac{1}{7}$. The corresponding projected system is shown in Figure 4.4, where the coordinates are in the basis formed by $\widetilde{e}_1 = \frac{1}{\sqrt{7}}(e_1 + \cdots + e_7)$ and $\widetilde{e}_2 = e_8$.

This configuration resembles the configurations G_2 and AG_2, but it has exactly one line ℓ containing collinear vectors, and the configuration is scaled in the orthogonal direction to ℓ compared to G_2 and AG_2. The multiplicities are given by

$$\widehat{c}_{\frac{\widetilde{e}_2}{2}-\frac{5\widetilde{e}_1}{2\sqrt{7}}} = \widehat{c}_{\widetilde{e}_2 \pm \frac{\widetilde{e}_1}{\sqrt{7}}} = 1, \quad \widehat{c}_{\frac{\widetilde{e}_2}{2}-\frac{\widetilde{e}_1}{2\sqrt{7}}} = 5, \quad \widehat{c}_{\frac{3\widetilde{e}_1}{2\sqrt{7}}+\frac{\widetilde{e}_2}{2}} = \widehat{c}_{\frac{2\widetilde{e}_1}{\sqrt{7}}} = 3, \quad \widehat{c}_{\frac{\sqrt{7}\widetilde{e}_1}{2}+\frac{\widetilde{e}_2}{2}} = \frac{1}{7}.$$

By formula (4.8), the corresponding generalised spin CMS Hamiltonian is

$$L = \partial_{y_1}^2 + \partial_{y_2}^2 - \frac{35 + 5\widehat{P}_{\frac{\sqrt{7}\widetilde{e}_2 - \widetilde{e}_1}{2\sqrt{7}}}}{14\sinh^2(\frac{\sqrt{7}y_2 - y_1}{4\sqrt{7}})} - \frac{2 + 2\widehat{P}_{\widetilde{e}_2 - \frac{\widetilde{e}_1}{\sqrt{7}}}}{7\sinh^2(\frac{\sqrt{7}y_2 - y_1}{2\sqrt{7}})}$$
$$- \frac{2 + 2\widehat{P}_{\frac{\widetilde{e}_1}{\sqrt{7}} + \widetilde{e}_2}}{7\sinh^2(\frac{y_1 + \sqrt{7}y_2}{2\sqrt{7}})} - \frac{2 + 2\widehat{P}_{\frac{\sqrt{7}\widetilde{e}_2 - 5\widetilde{e}_1}{2\sqrt{7}}}}{7\sinh^2(\frac{\sqrt{7}y_2 - 5y_1}{4\sqrt{7}})} - \frac{9 + 3\widehat{P}_{\frac{2\widetilde{e}_1}{\sqrt{7}}}}{7\sinh^2(\frac{y_1}{\sqrt{7}})}$$

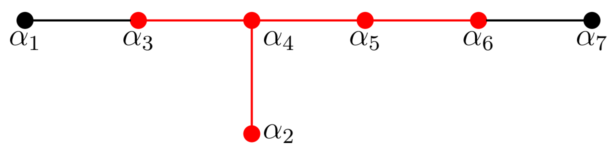
$$-\frac{1+7\widehat{P}_{\frac{\sqrt{7}\widetilde{e}_1+\widetilde{e}_2}{2}}}{98\sinh^2(\frac{\sqrt{7}y_1+y_2}{4})} - \frac{9+3\widehat{P}_{\frac{3\widetilde{e}_1+\sqrt{7}\widetilde{e}_2}{2\sqrt{7}}}}{7\sinh^2(\frac{3y_1+\sqrt{7}y_2}{4\sqrt{7}})},$$

where $y_1 = \frac{1}{\sqrt{7}}(x_1 + \cdots + x_7)$ and $y_2 = x_8$.

Other choices of a subgraph $\Gamma_0 \simeq A_6$ in the Dynkin diagram of E_8 lead to equivalent projected configurations. The subgraph $\Gamma_0 \simeq E_6$ leads to G_2 as in table [50, p. 272] (the multiplicity $\frac{23}{12}$ should be $\frac{27}{12}$). And $\Gamma_0 \simeq D_6$ leads to BC_2 with multiplicities $\frac{16}{5}$, $\frac{6}{5}$, and $\frac{1}{10}$ for the vectors e_i, $e_i \pm e_j$, and $2e_i$, respectively (cf. [50, p. 272]).

4.4.2.3 (E_7, D_5)

The Dynkin diagram of the root system E_7 is



with the simple roots being the first seven roots of (4.11). In this example, we choose a subgraph $\Gamma_0 \simeq D_5$ as shown above. The corresponding subspace π is given by the equations

$$x_1 = x_2 = x_3 = x_4 = x_5 = 0,$$

with multiplicity $c = \frac{1}{8}$. The corresponding projected system is shown in Figure 4.5, where the coordinates are with respect to $\widetilde{e}_1 = e_6$ and $\widetilde{e}_2 = \frac{1}{\sqrt{2}}(e_8 - e_7)$. The multiplicities are given by

$$\widehat{c}_{\widetilde{e}_1} = \frac{5}{4}, \quad \widehat{c}_{\sqrt{2}\widetilde{e}_2} = \frac{1}{8}, \quad \widehat{c}_{\frac{\widetilde{e}_2}{\sqrt{2}} \pm \frac{\widetilde{e}_1}{2}} = 2.$$

These vectors are as in a deformed C_2 configuration except for their multiplicities.

By formula (4.8), the corresponding generalised spin CMS Hamiltonian is

$$L = \partial_{y_1}^2 + \partial_{y_2}^2 - \frac{25+20\widehat{P}_{\widetilde{e}_1}}{64\sinh^2(\frac{y_1}{2})} - \frac{6+3\widehat{P}_{\frac{\widetilde{e}_1+\sqrt{2}\widetilde{e}_2}{2}}}{8\sinh^2(\frac{y_1+\sqrt{2}y_2}{4})}$$
$$-\frac{6+3\widehat{P}_{\frac{\sqrt{2}\widetilde{e}_2-\widetilde{e}_1}{2}}}{8\sinh^2(\frac{\sqrt{2}y_2-y_1}{4})} - \frac{1+8\widehat{P}_{\sqrt{2}\widetilde{e}_2}}{128\sinh^2(\frac{y_2}{\sqrt{2}})},$$

where $y_1 = x_6$ and $y_2 = \frac{1}{\sqrt{2}}(x_8 - x_7)$.

The scalar version of this operator coincides with the trigonometric degeneration (where the Weierstrass $\wp$ function is replaced by $\sinh^{-2}$) of the operator from [100, Theorem 1.4] for $a = \sqrt{2}$.

The other possible choice of a subgraph $\Gamma_0 \simeq D_5$ in the Dynkin diagram of E_7 leads to

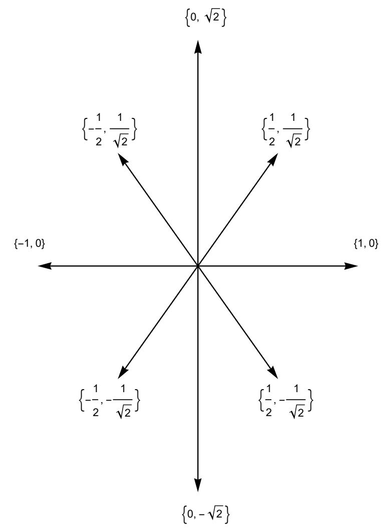
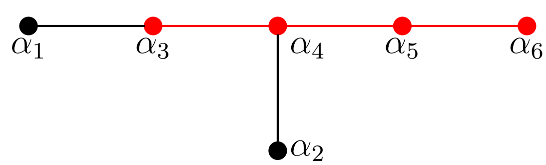


Figure 4.5: Projection of E_7 corresponding to a Coxeter subgraph of type D_5.

an equivalent projected configuration. Other allowed two-dimensional projections of E_7 are as in table [50, p. 273], except that here $(E_7, A_5)_1$ leads to the deformed BC_2 with deformation parameter $k = \frac{4}{3}$ and multiplicities $\frac{10}{3}$, $\frac{1}{6}$, $\frac{5}{2}$, 0, and 1 for the vectors e_1, $2e_1$, $\sqrt{k}e_2$, $2\sqrt{k}e_2$, and $e_1 \pm \sqrt{k}e_2$, respectively.

4.4.2.4 (E_6, A_4)

The Dynkin diagram of the root system E_6 is



with the simple roots being the first six roots of (4.11). Here we chose a subgraph $\Gamma_0 \simeq A_4$. The corresponding subspace π is given by the equations

$$x_1 = x_2 = x_3 = x_4 = x_5,$$

with multiplicity $c = \frac{1}{5}$. The corresponding projected system is shown in Figure 4.6, where the coordinates are with respect to $\widetilde{e}_1 = \frac{1}{\sqrt{3}}(e_8 - e_7 - e_6)$ and $\widetilde{e}_2 = \frac{1}{\sqrt{5}}(e_1 + \cdots + e_5)$. The multiplicities are given by

$$\widehat{c}_{\frac{2\widetilde{e}_2}{\sqrt{5}}} = \widehat{c}_{\frac{\sqrt{3}\widetilde{e}_1}{2} + \frac{\widetilde{e}_2}{2\sqrt{5}}} = 2, \quad \widehat{c}_{\frac{\sqrt{3}\widetilde{e}_1}{2} + \frac{\sqrt{5}\widetilde{e}_2}{2}} = \frac{1}{5}, \quad \widehat{c}_{\frac{\sqrt{3}\widetilde{e}_1}{2} - \frac{3\widetilde{e}_2}{2\sqrt{5}}} = 1.$$

These vectors are as in a deformed C_2 configuration except for their multiplicities.

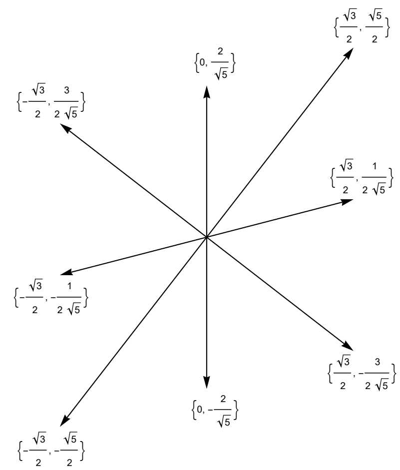


Figure 4.6: Projection of E_6 corresponding to a Coxeter subgraph of type A_4.

By formula (4.8), the corresponding generalised spin CMS Hamiltonian is

$$L = \partial_{y_1}^2 + \partial_{y_2}^2 - \frac{3 + 3\widehat{P}_{\frac{\sqrt{15}\tilde{e}_1 - 3\tilde{e}_2}{2\sqrt{5}}}}{10\sinh^2(\frac{\sqrt{15}y_1 - 3y_2}{4\sqrt{5}})} - \frac{4 + 2\widehat{P}_{\frac{2\tilde{e}_2}{\sqrt{5}}}}{5\sinh^2(\frac{y_2}{\sqrt{5}})}$$
$$- \frac{4 + 2\widehat{P}_{\frac{\sqrt{15}\tilde{e}_1 + \tilde{e}_2}{2\sqrt{5}}}}{5\sinh^2(\frac{\sqrt{15}y_1 + y_2}{4\sqrt{5}})} - \frac{1 + 5\widehat{P}_{\frac{\sqrt{3}\tilde{e}_1 + \sqrt{5}\tilde{e}_2}{2}}}{50\sinh^2(\frac{\sqrt{3}y_1 + \sqrt{5}y_2}{4})},$$

where $y_1 = \frac{1}{\sqrt{3}}(x_8 - x_7 - x_6)$ and $y_2 = \frac{1}{\sqrt{5}}(x_1 + \cdots + x_5)$.

The scalar version of this operator coincides with the trigonometric degeneration of the operator from [100, Theorem 1.4] for $a = \sqrt{3/5}$.

Other choices of a subgraph $\Gamma_0 \simeq A_4$ in the Dynkin diagram of E_6 lead to equivalent projected configurations. All other allowed two-dimensional projections of E_6 are as in table [50, p. 273].

Remark 4.8. With a view towards generalising the above two operators L corresponding to the restrictions (E_7, D_5) and (E_6, A_4), a natural question is whether there exists a spin version of the one-parametric family of integrable operators from [100, Theorem 1.4]. In the trigonometric limit, the Hamiltonian should have the form

$$L = \partial_{x_1}^2 + \partial_{x_2}^2 - \frac{(3-a^2)\left(3 - a^2 + 4a^2\widetilde{P}_{e_1}\right)}{4a^2\sinh^2(2ax_1)} - \frac{(3a^2-1)\left(3a^2 - 1 + 4\widetilde{P}_{e_2}\right)}{4\sinh^2(2x_2)}$$
$$- \frac{2(a^2+1)\left(2 + \widetilde{P}_{ae_1+e_2}\right)}{\sinh^2(ax_1 + x_2)} - \frac{2(a^2+1)\left(2 + \widetilde{P}_{-ae_1+e_2}\right)}{\sinh^2(-ax_1 + x_2)} \tag{4.12}$$

for some matrices $\widetilde{P}$. Note that in the cases $a = \sqrt{2}$ and $a = \sqrt{3/5}$ seen above, the operator (4.12) has a degree 6 quantum integral since the Weyl groups of type E_7 and E_6 have basic invariants of degree 6. Note also that the scalar version of the operator (4.12) has a quantum integral of degree 6 for any a by [100].

4.5 Projections that give root systems

Table 4.1 lists cases where the projected configuration is a root system. For the exceptional root systems, we only list their projections of rank at least 2. In each case, the list of multiplicities is ordered by the length of the vectors (in increasing order). We denote the multiplicity of the roots $e_i \pm e_j$ in types B, C, and F by c_1, and we denote by c_2 the multiplicity of e_i in types B and F, and the multiplicity of $2e_i$ in type C.

Formula (4.8) gives the Hamiltonian corresponding to each of these projections with multiplicities $\widehat{c}$ as given in Table 4.1 and with the matrices $\widehat{P}_\alpha$ defined in terms of a representation of the Weyl group corresponding to Γ by using formula (4.4).

In the rational case, there is additionally the stratum $(H_4, I_2(5))$ leading to a projected root system of type $I_2(10)$, and the corresponding generalised rational spin CMS Hamiltonian can be obtained by using the general results in [54].

Example 4.9. Let a positive half of the root system B_N be $B_{N,+} = \{e_i \pm e_j \colon 1 \leq i < j \leq N\} \cup \{e_i \colon i = 1, \dots, N\}$. Let $\mathcal{B}_N$ denote the associated Weyl group. Let us consider the case (B_3, B_1) from Table 4.1. In the notations of the table, $m = 2$, $k = 1$, and $l = 1$, so we impose $c_2 = \frac{1}{2}$, and c_1 is a free parameter. Let U be the two-dimensional irreducible representation of $\mathcal{B}_3$ with basis $x_1^2 - x_2^2$ and $x_2^2 - x_3^2$, with the natural action of $\mathcal{B}_3$ on polynomials. Then $U^{W_0} = U$ as $W_0 = \langle s_{e_3} \rangle$, and the projected configuration is $\widehat{R}_+ \setminus \{0\} = B_{2,+}$. By formula (4.8), the corresponding restricted Hamiltonian is

$$L = \partial_{x_1}^2 + \partial_{x_2}^2 - \sum_{i=1}^{2} \frac{\widehat{c}_{e_i}(\widehat{c}_{e_i} + \widehat{P}_{e_i})}{4\sinh^2(\frac{x_i}{2})} - \sum_{\varepsilon \in \{\pm 1\}} \frac{c_1(c_1 + \widehat{P}_{e_1+\varepsilon e_2})}{2\sinh^2(\frac{x_1+\varepsilon x_2}{2})},$$

where $\widehat{c}_{e_i} = 2c_1 + \frac{1}{2}$, and

$$\widehat{P}_{e_1} = \tfrac{1}{4c_1+1}\begin{pmatrix} 1-4c_1 & -8c_1 \\ -8c_1 & 1-4c_1 \end{pmatrix}, \qquad \widehat{P}_{e_2} = \tfrac{1}{4c_1+1}\begin{pmatrix} 4c_1+1 & 0 \\ 8c_1 & 1-12c_1 \end{pmatrix},$$

$$\widehat{P}_{e_1 \pm e_2} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This Hamiltonian L has a 4th-order quantum integral $\widetilde{\operatorname{Res}}_\pi(\sum_{i=1}^3 (\nabla_i^{\mathrm{trig}})^4)$, where ∇_i^{trig} are trigonometric Dunkl operators for the root system B_3.

We note that for generic c_1, there is no representation φ of $\mathcal{B}_2$ on $\mathbb{C}^2$ and multiplicities

$m, n \in \mathbb{C}$ such that the above L would coincide with the standard spin B_2 CMS operator

$$\partial_{x_1}^2 + \partial_{x_2}^2 - \sum_{i=1}^{2} \frac{m(m + \varphi(s_{e_i}))}{4\sinh^2(\frac{x_i}{2})} - \sum_{\varepsilon \in \{\pm 1\}} \frac{n(n + \varphi(s_{e_1+\varepsilon e_2}))}{2\sinh^2(\frac{x_1+\varepsilon x_2}{2})}.$$

(Γ, Γ_0)	Projection	Multiplicities
(A_{mk-1}, A_{k-1}^m), $m, k \in \mathbb{Z}_{\geq 2}$	A_{m-1}	k
$(B_{mk+l}, A_{k-1}^m \times B_l)$, $m, k \in \mathbb{Z}_{\geq 1}$, $l \in \mathbb{Z}_{\geq 0}$	BC_m	$(2c_1 l + c_2)k$, $c_1 k^2$, $\frac{k-1}{2}$ If $l > 0$, $c_2 = \frac{1}{2} - c_1(l-1)$, and if $k > 1$, $c_1 = \frac{1}{k}$.
$(C_{mk+l}, A_{k-1}^m \times C_l)$, $m, k \in \mathbb{Z}_{\geq 1}$, $l \in \mathbb{Z}_{\geq 0}$	BC_m	$2c_1 kl$, $c_1 k^2$, $\frac{k-1}{2} + c_2 k$ If $l > 0$, $c_2 = \frac{1}{2} - c_1(l-1)$, and if $k > 1$, $c_1 = \frac{1}{k}$.
$(D_{mk+l}, A_{k-1}^m \times D_l)$, $m \in \mathbb{Z}_{\geq 1}$, $k \in \mathbb{Z}_{\geq 2}$, $l = 0$, or k is even and $l = \frac{k}{2} + 1$	BC_m	$2l$, k, $\frac{k-1}{2}$
(D_{m+l}, D_l), $m \in \mathbb{Z}_{\geq 1}$, $l \in \mathbb{Z}_{\geq 2}$	B_m	$\frac{l}{l-1}$, $\frac{1}{2(l-1)}$
(F_4, A_2) with $\Gamma_0^v = \{\alpha_3, \alpha_4\}$	G_2	$3c_1 + 1$, c_1
(F_4, B_2)	BC_2	$4c_2$, $4c_1 + c_2$, c_1 with $2(c_1 + c_2) = 1$
(E_6, A_2^2)	G_2	3, $\frac{1}{3}$
(E_6, D_4)	A_2	$\frac{4}{3}$
(E_7, A_1^3) with $\Gamma_0^v = \{\alpha_2, \alpha_5, \alpha_7\}$	F_4	2, $\frac{1}{2}$
(E_7, D_4)	C_3	$\frac{4}{3}$, $\frac{1}{6}$
(E_7, A_5) with $\Gamma_0^v = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$	G_2	$\frac{5}{2}$, $\frac{1}{6}$
(E_8, D_4)	F_4	$\frac{4}{3}$, $\frac{1}{6}$
(E_8, D_6)	BC_2	$\frac{16}{5}$, $\frac{6}{5}$, $\frac{1}{10}$
(E_8, E_6)	G_2	$\frac{27}{12}$, $\frac{1}{12}$

Table 4.1: Pairs (Γ, Γ_0) for which the projected configuration is a root system.

Chapter 5

Macdonald–Ruijsenaars type system for AG_2

In this chapter, we recall a construction of generalised MR operators from [102] in the case of the root system F_4. We then apply a restriction procedure from [53] to obtain a pair of planar commuting difference operators related to the configuration AG_2 such that their rational degeneration reproduces the difference operators from Sections 3.4.2 and 3.4.4. The restriction procedure from [53] is a q-difference (DAHA) version of the construction from [50] and of the scalar case of our construction in Section 4.3 above.

5.1 A pair of commuting F_4 operators

Let us recall the operator, which we will denote $\widetilde{E}_1$, given by formula (3.2a) in [102] in the case of the root system

$$R = S = (2F_4)^\vee = \{\pm e_i \colon 1 \le i \le 4\} \cup \left\{\pm \tfrac{1}{2}(e_i \pm e_j) \colon 1 \le i < j \le 4\right\}$$
$$\cup \left\{\pm \tfrac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\right\} \subset \mathbb{C}^4$$

and the small weight $\omega_1 = 2(e_1 + e_2)$ of $S^\vee$. We will also need the Macdonald operator E_2 for the quasi-minuscule weight $\omega_2 = 2e_1$ of $S^\vee$ [82].

Let $t \colon R \to \mathbb{C}$, $\alpha \mapsto t(\alpha) = t_\alpha$ be a function invariant under the Weyl group W associated with R. Let $v_\alpha(z) = t_\alpha^{-1/2}(1 - t_\alpha z)/(1 - z)$. Let $q \in (0, 1)$. Let $\Lambda = \{\pm 2e_i \colon 1 \le i \le 4\} \cup \{\pm e_1 \pm e_2 \pm e_3 \pm e_4\}$. The operator E_2 has the form

$$E_2 = \sum_{\nu \in \Lambda} V_\nu (T_\nu - 1), \tag{5.1}$$

where

$$V_\nu = V_\nu(z) = \prod_{\substack{\alpha\in R\\ \langle\alpha,\nu\rangle>0}} v_\alpha(q^{\langle\alpha,z\rangle}) \prod_{\substack{\alpha\in R\\ \langle\alpha,\nu\rangle=2}} v_\alpha(q^{\langle\alpha,z\rangle+1}), \tag{5.2}$$

$z\in\mathbb{C}^4$. Note that the second product in (5.2) contains a single term for all $\nu\in\Lambda$.

Let $R_\nu = \{\alpha\in R\colon \langle\alpha,\nu\rangle = 0\}$ for $\nu\in\mathbb{C}^4$. The generalised MR operator from [102] has the form

$$\begin{aligned}\widetilde{E}_1 = &\sum_{1\le i<j\le 4}\sum_{\varepsilon_i,\varepsilon_j\in\{\pm1\}} V_{2\varepsilon_i e_i+2\varepsilon_j e_j} T_{2\varepsilon_i e_i+2\varepsilon_j e_j} + \sum_{\substack{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\in\{\pm1\}\\ \nu=\sum_{k=1}^4\varepsilon_k e_k}} V_\nu \sum_{1\le i<j\le 4} U^\nu_{2\varepsilon_i e_i+2\varepsilon_j e_j} T_\nu\\ &+\sum_{i=1}^4\sum_{\varepsilon\in\{\pm1\}} V_{2\varepsilon e_i}\sum_{\substack{j=1\\ j\ne i}}^4\sum_{\delta\in\{\pm1\}} U^{2\varepsilon e_i}_{2\varepsilon e_i+2\delta e_j} T_{2\varepsilon e_i} + \sum_{\eta\in W\omega_1} U^0_\eta,\end{aligned} \tag{5.3}$$

where

$$U^\nu_\eta = U^\nu_\eta(z) = \prod_{\substack{\alpha\in R_\nu\\ \langle\alpha,\eta\rangle>0}} v_\alpha(q^{\langle\alpha,z\rangle}) \prod_{\substack{\alpha\in R_\nu\\ \langle\alpha,\eta\rangle=2}} v_\alpha(q^{-\langle\alpha,z\rangle-1}). \tag{5.4}$$

Remark 5.1. The operator (5.3) is invariant under the Weyl group W of type F_4 (see [102]). It has shifts in the direction of the elements of the weight lattice of $R^\vee = 2F_4$, and the denominators of its coefficients have the form $1 - rq^{\langle\alpha,z\rangle}$ with $\alpha\in R$ and $r\in\{1, q^{\pm1}\}$. Since the operator (5.3) commutes with the quasi-minuscule operator E_2 (see [102]), it follows from [79] that $\widetilde{E}_1$ is a restriction to W-invariants of a W-invariant element of the DAHA $\mathbb{H}$ associated with the pair of root systems $(R, R^\vee)$ (case **a.** in [79]).

By the construction of difference operators as restrictions of W-invariant elements of a DAHA (see, e.g., [79]), it follows that these operators map the space of functions independent of z to itself. Hence, it follows by Remark 5.1 that the operator $\widetilde{E}_1$ is equal up to an additive constant to the following operator E_1:

$$\begin{aligned}E_1 = &\sum_{1\le i<j\le 4}\sum_{\varepsilon_i,\varepsilon_j\in\{\pm1\}} V_{2\varepsilon_i e_i+2\varepsilon_j e_j} (T_{2\varepsilon_i e_i+2\varepsilon_j e_j} - 1) + \sum_{\substack{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\in\{\pm1\}\\ \nu=\sum_{k=1}^4\varepsilon_k e_k}} V_\nu \sum_{1\le i<j\le 4} U^\nu_{2\varepsilon_i e_i+2\varepsilon_j e_j} (T_\nu - 1)\\ &+\sum_{i=1}^4\sum_{\varepsilon\in\{\pm1\}} V_{2\varepsilon e_i}\sum_{\substack{j=1\\ j\ne i}}^4\sum_{\delta\in\{\pm1\}} U^{2\varepsilon e_i}_{2\varepsilon e_i+2\delta e_j} (T_{2\varepsilon e_i} - 1).\end{aligned} \tag{5.5}$$

Let us introduce parameters $c_\alpha\in\mathbb{C}$ such that $t_\alpha = q^{-c_\alpha}$, where t_α depends on q so that c_α does not. Then in the rational limit $q\to 1$, we have

$$v_\alpha(q^{\langle\alpha,z\rangle}) \to 1 - \frac{c_\alpha}{\langle\alpha,z\rangle}, \quad v_\alpha(q^{\pm(\langle\alpha,z\rangle+1)}) \to 1 \mp \frac{c_\alpha}{\langle\alpha,z\rangle+1}.$$

Let $m = c_{e_i}$ and $n = c_{\frac{1}{2}(e_i\pm e_j)}$. The operators E_1 and E_2 can be simplified in the rational limit as the next proposition below demonstrates. Firstly, we need the following lemma.

Lemma 5.2. *Let $U_\eta^{\nu,r}$ be the rational limit of the function U_η^ν given by* (5.4)*. For $\nu = \pm 2e_i$, define $\Theta_\nu = \{\nu + 2\delta e_j \colon \delta \in \{\pm 1\},\, j \neq i\}$. For $\nu = \sum_{k=1}^4 \varepsilon_k e_k$ ($\varepsilon_k \in \{\pm 1\}$), define $\Theta_\nu = \{2\varepsilon_i e_i + 2\varepsilon_j e_j \colon 1 \leq i < j \leq 4\}$. Then for all $\nu \in W(2e_1)$, we have*

$$\sum_{\eta \in \Theta_\nu} U_\eta^{\nu,r} = \frac{m(m+1)}{2n(n+1)} \prod_{\alpha \in We_1 \cap R_\nu} \left(1 - \frac{2n}{\langle \alpha, z \rangle - 1}\right) - \frac{m(m+1)}{2n(n+1)} + 6, \tag{5.6}$$

where $m = c_{e_i}$ and $n = c_{\frac{1}{2}(e_i \pm e_j)}$.

Proof. By W-invariance of the operator (5.3), it is enough to consider the case $\nu = 2e_1$. Since the left-hand side of equality (5.6) is invariant under the group $B_3 \subset W$ that stabilises e_1, we get that this sum cannot have first-order poles at hyperplanes passing through the origin, hence it is non-singular at all such mirrors. It follows from the structure of the second product in formula (5.4) that the only possible singularities of $U_\eta^{\nu,r}$ are at the affine hyperplanes $z_i = \pm 1$, where $2 \leq i \leq 4$.

Observe that for $z_i = 2n+1$, the left-hand side of (5.6) is non-singular. Indeed, let us consider a possible singularity at $z_j = \pm 1$, $j \neq 1, i$. Note that the term in the rational limit of the first product of formula (5.4) with $\alpha = \frac{1}{2}(e_i \mp e_j)$ vanishes at $z_j = \pm 1$. Similarly, for $z_i = -(2n+1)$, the term with $\alpha = -\frac{1}{2}(e_i \pm e_j)$ vanishes at $z_j = \pm 1$, hence there is also no singularity at $z_i = -(2n+1)$.

Observe that each summand $U_\eta^{\nu,r}$ is a degree 0 rational function in the variables z_2, z_3, z_4. Moreover, specialisation to $z_i = \pm(2n+1)$ gives a degree 0 rational function as well, and hence the specialisation of the left-hand side of (5.6) is such a function, too. Since it has no singularities, it follows that it is equal to some constant $A_i^\pm \in \mathbb{C}$. By invariance of the operator (5.3), it follows that all these constants are equal to some constant $A \in \mathbb{C}$.

Let us consider the function $Q = \sum_{\eta \in \Theta_\nu} U_\eta^{\nu,r} - A$. This is a rational function of degree 0 with simple poles at $z_i = \pm 1$. By the above, it has zeros at $z_i = \pm(2n+1)$. It follows that $Q = B \prod_{2 \leq i \leq 4} \frac{z_i \pm (2n+1)}{z_i \pm 1}$ for some $B \in \mathbb{C}$. By considering the residue at $z_2 = 1$ and the behaviour at infinity, we see that the constants A and B match formula (5.6). ∎

In the limit $q \to 1$, the quasi-minuscule operator E_2 tends to the operator E_2^r given by

$$E_2^r = \sum_{\nu \in W(2e_1)} \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha \rangle > 0}} \left(1 - \frac{c_\alpha}{\langle \alpha, z \rangle}\right) \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha \rangle = 2}} \left(1 - \frac{c_\alpha}{\langle \alpha, z \rangle + 1}\right)(T_\nu - 1). \tag{5.7}$$

It appears that a suitable linear combination of the rational limits E_1^r, E_2^r of the operators E_1, E_2 has coefficients which can be factorised explicitly. More precisely, the following statement holds.

Proposition 5.3. *The linear combination* $2n(n+1)E_1^r + (m(m+1) - 12n(n+1))E_2^r$ *equals*

$$\sum_{\nu \in 2F_4} a_\nu(z)(T_\nu - 1),$$

where for $\nu \in W(2e_1 + 2e_2)$*, we have*

$$a_\nu(z) = 2n(n+1) \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha \rangle > 0}} \left(1 - \frac{c_\alpha}{\langle \alpha, z \rangle}\right) \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha \rangle = 2}} \left(1 - \frac{c_\alpha}{\langle \alpha, z \rangle + 1}\right), \tag{5.8}$$

and for $\nu \in W(2e_1)$*, we have*

$$a_\nu(z) = m(m+1) \prod_{\alpha \in W e_1 \cap R_\nu} \left(1 - \frac{2n}{\langle \alpha, z \rangle - 1}\right) \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha \rangle > 0}} \left(1 - \frac{c_\alpha}{\langle \alpha, z \rangle}\right) \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha \rangle = 2}} \left(1 - \frac{c_\alpha}{\langle \alpha, z \rangle + 1}\right). \tag{5.9}$$

Proof. The rational limit of the operator E_1 has the form

$$\begin{aligned} E_1^r = & \sum_{\nu \in W(2e_1+2e_2)} \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha \rangle > 0}} \left(1 - \frac{c_\alpha}{\langle \alpha, z \rangle}\right) \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha \rangle = 2}} \left(1 - \frac{c_\alpha}{\langle \alpha, z \rangle + 1}\right)(T_\nu - 1) \\ & + \sum_{\nu \in W(2e_1)} \sum_{\eta \in \Theta_\nu} U_\eta^{\nu,r} \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha \rangle > 0}} \left(1 - \frac{c_\alpha}{\langle \alpha, z \rangle}\right) \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha \rangle = 2}} \left(1 - \frac{c_\alpha}{\langle \alpha, z \rangle + 1}\right)(T_\nu - 1), \end{aligned} \tag{5.10}$$

where Θ_ν is defined as in Lemma 5.2. The statement follows by using formula (5.7) and Lemma 5.2. ■

5.2 Two-dimensional restriction

The process of restriction of operators from the DAHA to special planes was developed in [53]. It corresponds to the action of the operators in a quotient of the polynomial representation. This quotient exists for special values of the parameters of the DAHA. Let us apply this construction in the case of the DAHA $\mathbb{H}$ from Section 5.1, the plane

$$\pi = \left\{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \colon z_2 - z_3 = z_3 - z_4 = \tfrac{2}{3}\right\},$$

and the parameter $n = c_{\frac{1}{2}(e_1 - e_2)} = \frac{1}{3}$.

Let us consider a W-invariant operator

$$D = \sum_{\nu \in \mathcal{V}} a_\nu T_\nu, \tag{5.11}$$

where $\mathcal{V} \subset \mathbb{C}^4$, and a_ν are some functions. Let π_0 be the linear plane parallel to π, $\pi_0 = \{z \in \mathbb{C}^4 \colon z_2 = z_3 = z_4\}$. For any $\nu \in \mathbb{C}^4$, let $\overline{\nu}$ be its orthogonal projection to π_0. Let us define the following operator $\overline{D}$ on π_0:

$$\overline{D} = \sum_{\nu' \in \overline{\mathcal{V}}} f_{\nu'} T_{\nu'}, \tag{5.12}$$

where $f_{\nu'}(z) = \sum_{\substack{\nu \in \mathcal{V} \\ \overline{\nu} = \nu'}} a_\nu(z + \delta)$, $z \in \pi_0$, and $\delta = (0, \frac{2}{3}, 0, -\frac{2}{3})$ is the normal vector to the plane π_0 whose endpoint belongs to the plane π.

Suppose that two operators D_1, D_2 of the form (5.11) commute and are obtained as restrictions to W-invariants of W-invariant elements of the DAHA $\mathbb{H}$. Then it follows from [53] that their restrictions $\overline{D_i}$ also commute.

We are going to compute the restricted operators for the pair of commuting operators E_1 and E_2 from Section 5.1. Firstly, we observe that these restricted operators have certain invariance properties.

Let P be the parabolic subgroup of W that fixes π_0 pointwise. Let us consider the normaliser N_P of this subgroup in W. It has the form $N_P = GP$, where G is the subgroup of W of type G_2 [68]. The group N_P consists of those elements of W that preserve π_0, and the root system R_G of the group G is a subset in π_0. We can assume that the root system R_G contains the vectors $\pm e_1, \pm\frac{1}{2}(e_1 \pm (e_2 + e_3 + e_4)) \in R$.

Proposition 5.4. *Suppose an operator D of the form* (5.11) *is W-invariant. Suppose also that its coefficients a_ν are generically well-defined on π, and hence the restricted operator $\overline{D}$ given by* (5.12) *is generically well-defined on π_0. Then the operator $\overline{D}$ is invariant under the group G.*

Proof. Let us consider an element $\alpha \in R \cap R_G$ and the corresponding reflection $s_\alpha \in G \subset W$. For a shift vector ν in the operator D, if we have $s_\alpha(\nu) = \nu$, then we also have for the projection $\nu' \in \pi_0$ that $s_\alpha(\nu') = \nu'$. Since $s_\alpha(a_\nu) = a_\nu$ and $s_\alpha(\delta) = \delta$, we get that the term $a_\nu(z+\delta)T_{\nu'}$ in $\overline{D}$ is s_α-invariant.

The other terms $a_\nu T_\nu$ in the operator D split into pairs $a_{\nu_1}T_{\nu_1}$, $a_{\nu_2}T_{\nu_2}$ such that $s_\alpha(\nu_1) = \nu_2$. Since for the projections we have $s_\alpha(\nu_1') = \nu_2'$, we get that the corresponding sum $a_{\nu_1}(z+\delta)T_{\nu_1'} + a_{\nu_2}(z+\delta)T_{\nu_2'}$ in $\overline{D}$ is s_α-invariant.

The group G is generated by the reflections s_α with $\alpha \in R \cap R_G$ and the transformation $j = -\mathrm{id}$. We are left to prove the invariance of the operator $\overline{D}$ under the map j. Let us first consider pairs of terms $a_{\mu_1}T_{\mu_1}$, $a_{\mu_2}T_{\mu_2}$ in D such that $s_\gamma \circ j(\mu_1) = \mu_2$ for $\gamma = e_2 - e_4$. By the W-invariance of the operator D, we have $s_\gamma \circ j(a_{\mu_1}) = a_{\mu_2}$. This relation implies for any $z \in \pi_0$ that

$$j(a_{\mu_1}(z+\delta)) = a_{\mu_1}(-z+\delta) = a_{\mu_1}(s_\gamma(-z)+\delta) = a_{\mu_1}(s_\gamma \circ j(z+\delta)) = a_{\mu_2}(z+\delta).$$

Since $\mu_1' = -\mu_2'$, we get that the combination $a_{\mu_1}(z+\delta)T_{\mu_1'} + a_{\mu_2}(z+\delta)T_{\mu_2'}$ in the operator $\overline{D}$ is invariant under j.

Let us also consider the terms in D of the form $a_\mu T_\mu$ such that $s_\gamma \circ j(\mu) = \mu$. Then $s_\gamma \circ j(a_\mu) = a_\mu$ and $\mu' = 0$. This implies that $j(a_\mu(z+\delta)) = a_\mu(z+\delta)$ for any $z \in \pi_0$, and hence the corresponding term $a_\mu(z+\delta)T_0$ is also invariant under the map j. ■

We introduce coordinates on π_0 by choosing the orthonormal basis $f_1 = \frac{1}{\sqrt{3}}(e_2 + e_3 + e_4)$, $f_2 = e_1$. We consider the configuration $AG_2 \subset \pi_0$ given in the basis f_1, f_2 by the coordinates (3.11), where we fix $\omega = \frac{1}{\sqrt{3}}$. In this realisation, $\alpha_1 = (0,1) = f_2$ and $\beta_1 = \frac{1}{\sqrt{3}}(1,0) = \frac{1}{\sqrt{3}} f_1$. We also have $\alpha_2 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ and $\alpha_3 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$. The orthogonal projection of a vector $v = \sum_{i=1}^4 v_i e_i$ onto π_0 is $\overline{v} = \frac{1}{\sqrt{3}}(v_2+v_3+v_4)f_1 + v_1 f_2$.

Let us now find the restricted operators $\overline{E_1}$ and $\overline{E_2}$ explicitly. The following statements take place.

Proposition 5.5. *The restriction $\overline{E_2}$ of the quasi-minuscule operator E_2 has the form*

$$\overline{E_2} = \sum_{j=1}^{3} \sum_{\varepsilon \in \{\pm 1\}} \left(a_{2\varepsilon\alpha_j}(T_{2\varepsilon\alpha_j} - 1) + a_{2\varepsilon\beta_j}(T_{2\varepsilon\beta_j} - 1) \right), \tag{5.13}$$

where

$$\begin{aligned} a_{2\varepsilon\alpha_j}(z) = q s^{-5} &\prod_{\substack{\gamma \in (2G\beta_1)^\vee \\ \langle 2\varepsilon\alpha_j, \gamma\rangle = 3}} \left(\frac{1 - s q^{\frac{\langle \gamma, z\rangle}{3} - \frac{2}{3}}}{1 - q^{\frac{\langle \gamma, z\rangle}{3}}} \right) \left(\frac{1 - s q^{\frac{\langle \gamma, z\rangle}{3}}}{1 - q^{\frac{\langle \gamma, z\rangle}{3} + \frac{1}{3}}} \right) \left(\frac{1 - s q^{\frac{\langle \gamma, z\rangle}{3} + \frac{2}{3}}}{1 - q^{\frac{\langle \gamma, z\rangle}{3} + \frac{2}{3}}} \right) \\ &\times \prod_{\substack{\gamma \in (2G\alpha_1)^\vee \\ \langle 2\varepsilon\alpha_j, \gamma\rangle = 1}} \left(\frac{1 - s q^{\langle \gamma, z\rangle}}{1 - q^{\langle \gamma, z\rangle}} \right) \times \left(\frac{1 - s q^{\langle \varepsilon\alpha_j, z\rangle}}{1 - q^{\langle \varepsilon\alpha_j, z\rangle}} \right) \left(\frac{1 - s q^{\langle \varepsilon\alpha_j, z\rangle + 1}}{1 - q^{\langle \varepsilon\alpha_j, z\rangle + 1}} \right), \end{aligned} \tag{5.14}$$

$$\begin{aligned} a_{2\varepsilon\beta_j}(z) = (1 + q^{1/3} + q^{2/3}) q^{1/3} s^{-5} &\prod_{\substack{\gamma \in (2G\beta_1)^\vee \\ \langle 2\varepsilon\beta_j, \gamma\rangle = 1}} \left(\frac{1 - s q^{\frac{\langle \gamma, z\rangle}{3} - \frac{2}{3}}}{1 - q^{\frac{\langle \gamma, z\rangle}{3}}} \right) \left(\frac{1 - s q^{-\frac{\langle \gamma, z\rangle}{3} - \frac{2}{3}}}{1 - q^{-\frac{\langle \gamma, z\rangle}{3} - \frac{2}{3}}} \right) \left(\frac{1 - s q^{\frac{\langle \gamma, z\rangle}{3}}}{1 - q^{\frac{\langle \gamma, z\rangle}{3} - \frac{1}{3}}} \right) \\ &\times \prod_{\substack{\gamma \in (2G\alpha_1)^\vee \\ \langle 2\varepsilon\beta_j, \gamma\rangle = 1}} \left(\frac{1 - s q^{\langle \gamma, z\rangle}}{1 - q^{\langle \gamma, z\rangle}} \right) \times \left(\frac{1 - s q^{\langle \varepsilon\beta_j, z\rangle - \frac{2}{3}}}{1 - q^{\langle \varepsilon\beta_j, z\rangle}} \right) \left(\frac{1 - s q^{\langle \varepsilon\beta_j, z\rangle + \frac{1}{3}}}{1 - q^{\langle \varepsilon\beta_j, z\rangle + \frac{1}{3}}} \right), \end{aligned} \tag{5.15}$$

and $s = t_{e_1}$, $z \in \pi_0$.

Proof. It is easy to see that $\overline{\Lambda}$ consists of the vectors $\pm 2\alpha_j, \pm 2\beta_j$. Let us firstly establish formula (5.14). We have $\overline{2e_1} = 2\alpha_1$, and it is easy to see that no other vector from Λ is projected to $2\alpha_1$. By formula (5.2), we have

$$V_{2e_1}(z) = v_{e_1}(q^{\langle e_1, z\rangle + 1}) \prod_{\substack{\gamma \in R \\ \langle \gamma, e_1\rangle > 0}} v_\gamma(q^{\langle \gamma, z\rangle})$$

$$= v_{\alpha_1}(q^{\langle\alpha_1,z\rangle+1})v_{\alpha_1}(q^{\langle\alpha_1,z\rangle})v_{\alpha_2}(q^{\langle\alpha_2,z\rangle})v_{\alpha_3}(q^{\langle\alpha_3,z\rangle})\prod_{\gamma\in C_1\cup C_2} v_\gamma(q^{\langle\gamma,z\rangle}),$$

where

$$C_1 = \{\tfrac12(e_1-e_2), \tfrac12(e_1-e_3), \tfrac12(e_1-e_4), \tfrac12(e_1+e_2-e_3-e_4), \tfrac12(e_1-e_2+e_3-e_4), \tfrac12(e_1-e_2-e_3+e_4)\}$$

and

$$C_2 = \{\tfrac12(e_1+e_2), \tfrac12(e_1+e_3), \tfrac12(e_1+e_4), \tfrac12(e_1+e_2+e_3-e_4), \tfrac12(e_1+e_2-e_3+e_4), \tfrac12(e_1-e_2+e_3+e_4)\}.$$

Recall that $t_{\frac12(e_1-e_2)} = q^{-1/3}$. For $z \in \pi$, we have

$$v_{\frac12(e_1-e_2)}(q^{\frac12\langle e_1-e_2,z\rangle})v_{\frac12(e_1-e_3)}(q^{\frac12\langle e_1-e_3,z\rangle})v_{\frac12(e_1-e_4)}(q^{\frac12\langle e_1-e_4,z\rangle}) = q^{1/2}\frac{1-q^{-1/3}q^{\frac12(z_1-z_2)}}{1-q^{2/3}q^{\frac12(z_1-z_2)}},$$

and hence

$$\prod_{\gamma\in C_1} v_\gamma(q^{\langle\gamma,z\rangle}) = \frac{q^{1/2}s^{-3/2}(1-sq^{\frac12(z_1-z_2)+1})(1-sq^{\frac12(z_1-z_2)+\frac13})(1-sq^{\frac12(z_1-z_2)-\frac13})}{(1-q^{\frac12(z_1-z_2)+\frac23})(1-q^{\frac12(z_1-z_2)+1})(1-q^{\frac12(z_1-z_2)+\frac13})}. \quad (5.16)$$

Similarly,

$$\prod_{\gamma\in C_2} v_\gamma(q^{\langle\gamma,z\rangle}) = \frac{q^{1/2}s^{-3/2}(1-sq^{\frac12(z_1+z_2)-1})(1-sq^{\frac12(z_1+z_2)-\frac13})(1-sq^{\frac12(z_1+z_2)+\frac13})}{(1-q^{\frac12(z_1+z_2)})(1-q^{\frac12(z_1+z_2)+\frac13})(1-q^{\frac12(z_1+z_2)-\frac13})}. \quad (5.17)$$

Let now $z = (z_1, z_2, z_2, z_2) \in \pi_0$. Then $z + \delta \in \pi$, and by equality (5.16), we get

$$\prod_{\gamma\in C_1} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) = \frac{q^{1/2}s^{-3/2}(1-sq^{\frac12(z_1-z_2)+\frac23})(1-sq^{\frac12(z_1-z_2)})(1-sq^{\frac12(z_1-z_2)-\frac23})}{(1-q^{\frac12(z_1-z_2)+\frac13})(1-q^{\frac12(z_1-z_2)+\frac23})(1-q^{\frac12(z_1-z_2)})}.$$

The vector $\beta_3 \in \pi_0$ has the form $\beta_3 = -\frac{1}{2\sqrt3}f_1 + \frac12 f_2 = \frac12 e_1 - \frac16(e_2+e_3+e_4)$, hence $(2\beta_3)^\vee = \frac32 e_1 - \frac12(e_2+e_3+e_4)$, and so $\frac13\langle(2\beta_3)^\vee, z\rangle = \frac12(z_1-z_2)$. Therefore, we have

$$\prod_{\gamma\in C_1} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) = \frac{q^{1/2}s^{-3/2}(1-sq^{\frac13\langle(2\beta_3)^\vee,z\rangle+\frac23})(1-sq^{\frac13\langle(2\beta_3)^\vee,z\rangle})(1-sq^{\frac13\langle(2\beta_3)^\vee,z\rangle-\frac23})}{(1-q^{\frac13\langle(2\beta_3)^\vee,z\rangle+\frac13})(1-q^{\frac13\langle(2\beta_3)^\vee,z\rangle+\frac23})(1-q^{\frac13\langle(2\beta_3)^\vee,z\rangle})}.$$

Similarly, (5.17) gives

$$\prod_{\gamma\in C_2} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) = \frac{q^{1/2}s^{-3/2}(1-sq^{\frac12(z_1+z_2)-\frac23})(1-sq^{\frac12(z_1+z_2)})(1-sq^{\frac12(z_1+z_2)+\frac23})}{(1-q^{\frac12(z_1+z_2)+\frac13})(1-q^{\frac12(z_1+z_2)+\frac23})(1-q^{\frac12(z_1+z_2)})},$$

and $\frac{1}{3}\langle(2\beta_2)^\vee, z\rangle = \frac{1}{2}(z_1+z_2)$. Hence, altogether, we have

$$\prod_{\gamma\in C_1\cup C_2} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) = qs^{-3}\prod_{\substack{\gamma\in(2G\beta_1)^\vee\\ \langle 2\alpha_1,\gamma\rangle=3}}\left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}-\frac{2}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}}}\right)\left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}+\frac{1}{3}}}\right)\left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}+\frac{2}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}+\frac{2}{3}}}\right).$$

To complete the proof of equality (5.14) for $j=1$ and $\varepsilon=1$, we note that $(2\alpha_2)^\vee = \alpha_2$ and $(2\alpha_3)^\vee = \alpha_3$, therefore

$$v_{\alpha_1}(q^{\langle\alpha_1,z+\delta\rangle+1})\prod_{i=1}^{3} v_{\alpha_i}(q^{\langle\alpha_i,z+\delta\rangle}) = s^{-2}\left(\frac{1-sq^{\langle\alpha_1,z\rangle}}{1-q^{\langle\alpha_1,z\rangle}}\right)\left(\frac{1-sq^{\langle\alpha_1,z\rangle+1}}{1-q^{\langle\alpha_1,z\rangle+1}}\right)$$
$$\times\prod_{\substack{\gamma\in(2G\alpha_1)^\vee\\ \langle 2\alpha_1,\gamma\rangle=1}}\left(\frac{1-sq^{\langle\gamma,z\rangle}}{1-q^{\langle\gamma,z\rangle}}\right),$$

as required. Equality (5.14) in general follows from the G-invariance of the operator $\overline{E_2}$ established in Proposition 5.4.

Let us now prove formula (5.15). We have $\overline{2e_2} = \overline{2e_3} = \overline{2e_4} = 2\beta_1$, and no other vector from Λ is projected to $2\beta_1$. For $z \in \pi$, we have $V_{2e_2}(z) = 0$ since $v_{\frac{1}{2}(e_2-e_3)}(q^{\frac{1}{2}\langle e_2-e_3,z\rangle}) = 0$, and we have $V_{2e_3}(z) = 0$ since $v_{\frac{1}{2}(e_3-e_4)}(q^{\frac{1}{2}\langle e_3-e_4,z\rangle}) = 0$. By formula (5.2), we have

$$\begin{aligned} V_{2e_4}(z) &= v_{e_4}(q^{\langle e_4,z\rangle+1})\prod_{\substack{\gamma\in R\\ \langle\gamma,e_4\rangle>0}} v_\gamma(q^{\langle\gamma,z\rangle}) \\ &= v_{e_4}(q^{\langle e_4,z\rangle+1})v_{-\alpha_2}(q^{-\langle\alpha_2,z\rangle})v_{\alpha_3}(q^{\langle\alpha_3,z\rangle})\prod_{\gamma\in C_1'\cup C_2'\cup C_3'\cup C_4'} v_\gamma(q^{\langle\gamma,z\rangle}), \end{aligned}$$

where

$$\begin{aligned} C_1' &= \{\tfrac{1}{2}(e_4-e_1), \tfrac{1}{2}(e_4-e_1-e_2+e_3), \tfrac{1}{2}(e_4-e_1+e_2-e_3), \tfrac{1}{2}(e_4+e_1-e_2-e_3)\},\\ C_2' &= \{\tfrac{1}{2}(e_4+e_1), \tfrac{1}{2}(e_4+e_1-e_2+e_3), \tfrac{1}{2}(e_4+e_1+e_2-e_3), \tfrac{1}{2}(e_4-e_1-e_2-e_3)\},\\ C_3' &= \{e_4, \tfrac{1}{2}(e_4+e_2), \tfrac{1}{2}(e_4+e_3)\}, \quad \text{and} \quad C_4' = \{\tfrac{1}{2}(e_4-e_2), \tfrac{1}{2}(e_4-e_3)\}. \end{aligned}$$

Let now $z = (z_1, z_2, z_2, z_2) \in \pi_0$. Then we have

$$\begin{aligned} \prod_{\gamma\in C_1'} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) &= \frac{q^{1/6}s^{-3/2}(1-sq^{\frac{1}{2}(z_2-z_1)-\frac{2}{3}})(1-sq^{\frac{1}{2}(z_2-z_1)})(1-sq^{\frac{1}{2}(z_1-z_2)-\frac{2}{3}})}{(1-q^{\frac{1}{2}(z_2-z_1)-\frac{1}{3}})(1-q^{\frac{1}{2}(z_2-z_1)})(1-q^{\frac{1}{2}(z_1-z_2)-\frac{2}{3}})} \\ &= \frac{q^{1/6}s^{-3/2}(1-sq^{-\frac{1}{3}\langle(2\beta_3)^\vee,z\rangle-\frac{2}{3}})(1-sq^{-\frac{1}{3}\langle(2\beta_3)^\vee,z\rangle})(1-sq^{\frac{1}{3}\langle(2\beta_3)^\vee,z\rangle-\frac{2}{3}})}{(1-q^{-\frac{1}{3}\langle(2\beta_3)^\vee,z\rangle-\frac{1}{3}})(1-q^{-\frac{1}{3}\langle(2\beta_3)^\vee,z\rangle})(1-q^{\frac{1}{3}\langle(2\beta_3)^\vee,z\rangle-\frac{2}{3}})}. \end{aligned} \tag{5.18}$$

Similarly, for the product over $\gamma \in C_2'$, we get expression (5.18) with β_3 replaced by $-\beta_2$, and therefore

$$\prod_{\gamma\in C_1'\cup C_2'} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) = q^{1/3}s^{-3}\prod_{\substack{\gamma\in(2G\beta_1)^\vee\\ \langle 2\beta_1,\gamma\rangle=1}}\left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}-\frac{2}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}}}\right)\left(\frac{1-sq^{-\frac{\langle\gamma,z\rangle}{3}-\frac{2}{3}}}{1-q^{-\frac{\langle\gamma,z\rangle}{3}-\frac{2}{3}}}\right)\left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}-\frac{1}{3}}}\right).$$

Further, we have

$$\prod_{\gamma\in C_4'} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) = q^{-1/3}(1+q^{1/3}+q^{2/3}).$$

Next, we have

$$\begin{aligned} v_{e_4}(q^{\langle e_4,z+\delta\rangle+1})\prod_{\gamma\in C_3'} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) &= q^{1/3}s^{-1}\frac{(1-sq^{z_2+\frac{1}{3}})(1-sq^{z_2-\frac{2}{3}})}{(1-q^{z_2+\frac{1}{3}})(1-q^{z_2})} \\ &= q^{1/3}s^{-1}\left(\frac{1-sq^{\langle\beta_1,z\rangle-\frac{2}{3}}}{1-q^{\langle\beta_1,z\rangle}}\right)\left(\frac{1-sq^{\langle\beta_1,z\rangle+\frac{1}{3}}}{1-q^{\langle\beta_1,z\rangle+\frac{1}{3}}}\right), \end{aligned}$$

where we used that the vector $\beta_1\in\pi_0$ has the form $\beta_1=\frac{1}{\sqrt{3}}f_1=\frac{1}{3}(e_2+e_3+e_4)$, hence $\langle\beta_1,z\rangle=z_2$. To complete the proof of equality (5.15) for $j=1$ and $\varepsilon=1$, we note that

$$v_{-\alpha_2}(q^{-\langle\alpha_2,z+\delta\rangle})v_{\alpha_3}(q^{\langle\alpha_3,z+\delta\rangle}) = s^{-1}\prod_{\substack{\gamma\in(2G\alpha_1)^\vee\\ \langle 2\beta_1,\gamma\rangle=1}}\left(\frac{1-sq^{\langle\gamma,z\rangle}}{1-q^{\langle\gamma,z\rangle}}\right),$$

as required. Equality (5.15) in general follows from Proposition 5.4 on the G-invariance of the operator $\overline{E_2}$. ■

Proposition 5.6. *The restriction $\overline{E_1}$ of the operator* (5.5) *satisfies*

$$\overline{E_1}+c\overline{E_2} = \sum_{j=1}^{3}\sum_{\varepsilon\in\{\pm1\}}\left(a_{4\varepsilon\beta_j}(T_{4\varepsilon\beta_j}-1)+a_{2\varepsilon\alpha_j}(T_{2\varepsilon\alpha_j}-1)+a_{2\varepsilon\beta_j}(T_{2\varepsilon\beta_j}-1)\right), \quad (5.19)$$

where

$$c = s^{-1}(1+q^{1/3}+q^{2/3})\frac{q^{7/3}}{1-q^{7/3}}\left(\frac{q^{1/3}(1-sq^{-8/3})(1-sq^{5/3})}{1-q^{8/3}}+\frac{(1-sq^{-3})(1-sq^{2})}{1-q^{2}}\right),$$

and the functions $a_{4\varepsilon\beta_j}$, $a_{2\varepsilon\alpha_j}$, and $a_{2\varepsilon\beta_j}$ are given by

$$
\begin{aligned}
a_{4\varepsilon\beta_j}(z) =&(1+q^{1/3}+q^{2/3})qs^{-6} \prod_{\substack{\gamma\in(2G\alpha_1)^\vee\\ \langle 4\varepsilon\beta_j,\gamma\rangle=2}} \left(\frac{1-sq^{\langle\gamma,z\rangle}}{1-q^{\langle\gamma,z\rangle}}\right)\left(\frac{1-sq^{\langle\gamma,z\rangle+1}}{1-q^{\langle\gamma,z\rangle+1}}\right)\\
&\times \prod_{\substack{\gamma\in(2G\beta_1)^\vee\\ \langle 4\varepsilon\beta_j,\gamma\rangle=2}} \left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}-\frac{2}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}}}\right)\left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}+\frac{1}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}+\frac{1}{3}}}\right) \qquad (5.20)\\
&\times\left(\frac{1-sq^{\langle\varepsilon\beta_j,z\rangle-\frac{2}{3}}}{1-q^{\langle\varepsilon\beta_j,z\rangle}}\right)\left(\frac{1-sq^{\langle\varepsilon\beta_j,z\rangle+\frac{1}{3}}}{1-q^{\langle\varepsilon\beta_j,z\rangle+\frac{1}{3}}}\right)\left(\frac{1-sq^{\langle\varepsilon\beta_j,z\rangle}}{1-q^{\langle\varepsilon\beta_j,z\rangle+\frac{2}{3}}}\right)\left(\frac{1-sq^{\langle\varepsilon\beta_j,z\rangle+1}}{1-q^{\langle\varepsilon\beta_j,z\rangle+1}}\right),
\end{aligned}
$$

$$
\begin{aligned}
a_{2\varepsilon\alpha_j}(z) = d&\left(\frac{1-sq^{\langle\varepsilon\alpha_j,z\rangle}}{1-q^{\langle\varepsilon\alpha_j,z\rangle}}\right)\left(\frac{1-sq^{\langle\varepsilon\alpha_j,z\rangle+1}}{1-q^{\langle\varepsilon\alpha_j,z\rangle+1}}\right) \prod_{\substack{\gamma\in(2G\alpha_1)^\vee\\ \langle 2\varepsilon\alpha_j,\gamma\rangle=1}} \left(\frac{1-sq^{\langle\gamma,z\rangle}}{1-q^{\langle\gamma,z\rangle}}\right)\\
\times \prod_{\substack{\gamma\in(2G\beta_1)^\vee\\ \langle 2\varepsilon\alpha_j,\gamma\rangle=3}}& \left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}-\frac{2}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}}}\right)\left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}+\frac{1}{3}}}\right)\left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}+\frac{2}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}+\frac{2}{3}}}\right) \prod_{\substack{\gamma\in(2G\beta_1)^\vee\\ \langle 2\varepsilon\alpha_j,\gamma\rangle=0}} \left(\frac{1-q^{\frac{\langle\gamma,z\rangle}{3}-\frac{7}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}-\frac{1}{3}}}\right),
\end{aligned}
\qquad (5.21)
$$

with

$$
d = \frac{q^{16/3}s^{-6}(1-sq^{-1})(1-s)}{(1-q^{1/3})(1-q^{8/3})(1-q^{1/3}+q^{2/3})},
$$

and

$$
\begin{aligned}
a_{2\varepsilon\beta_j}(z) = &(1+q^{1/3}+q^{2/3})q^{1/3}s^{-5}\\
&\times \prod_{\substack{\gamma\in(2G\beta_1)^\vee\\ \langle 2\varepsilon\beta_j,\gamma\rangle=1}} \left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}-\frac{2}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}}}\right)\left(\frac{1-sq^{-\frac{\langle\gamma,z\rangle}{3}-\frac{2}{3}}}{1-q^{-\frac{\langle\gamma,z\rangle}{3}-\frac{2}{3}}}\right)\left(\frac{1-sq^{\frac{\langle\gamma,z\rangle}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}-\frac{1}{3}}}\right)\\
&\times\left(\frac{1-sq^{\langle\varepsilon\beta_j,z\rangle-\frac{2}{3}}}{1-q^{\langle\varepsilon\beta_j,z\rangle}}\right)\left(\frac{1-sq^{\langle\varepsilon\beta_j,z\rangle+\frac{1}{3}}}{1-q^{\langle\varepsilon\beta_j,z\rangle+\frac{1}{3}}}\right) \prod_{\substack{\gamma\in(2G\alpha_1)^\vee\\ \langle 2\varepsilon\beta_j,\gamma\rangle=1}} \left(\frac{1-sq^{\langle\gamma,z\rangle}}{1-q^{\langle\gamma,z\rangle}}\right)\\
\times\Bigg[&c+s^{-1}\left(\frac{1-sq^{\langle g\alpha_1,z\rangle}}{1-q^{\langle g\alpha_1,z\rangle}}\right)\left(\frac{1-sq^{-\langle g\alpha_1,z\rangle-1}}{1-q^{-\langle g\alpha_1,z\rangle-1}}\right)\left(\frac{1-q^{\langle g\beta_2,z\rangle-\frac{1}{3}}}{1-q^{\langle g\beta_2,z\rangle+\frac{1}{3}}}\right)\left(\frac{1-q^{-\langle g\beta_3,z\rangle+\frac{2}{3}}}{1-q^{-\langle g\beta_3,z\rangle}}\right)\\
&+s^{-1}\left(\frac{1-sq^{-\langle g\alpha_1,z\rangle}}{1-q^{-\langle g\alpha_1,z\rangle}}\right)\left(\frac{1-sq^{\langle g\alpha_1,z\rangle-1}}{1-q^{\langle g\alpha_1,z\rangle-1}}\right)\left(\frac{1-q^{-\langle g\beta_3,z\rangle-\frac{1}{3}}}{1-q^{-\langle g\beta_3,z\rangle+\frac{1}{3}}}\right)\left(\frac{1-q^{\langle g\beta_2,z\rangle+\frac{2}{3}}}{1-q^{\langle g\beta_2,z\rangle}}\right)\\
&+(1+q^{-1/3})s^{-1}\left(\frac{1-sq^{\langle g\beta_1,z\rangle-\frac{1}{3}}}{1-q^{\langle g\beta_1,z\rangle-\frac{1}{3}}}\right)\left(\frac{1-sq^{-\langle g\beta_1,z\rangle-\frac{2}{3}}}{1-q^{-\langle g\beta_1,z\rangle-\frac{1}{3}}}\right)\left(\frac{1-q^{\langle g\beta_2,z\rangle+\frac{2}{3}}}{1-q^{\langle g\beta_2,z\rangle+\frac{1}{3}}}\right)\left(\frac{1-q^{-\langle g\beta_3,z\rangle+\frac{2}{3}}}{1-q^{-\langle g\beta_3,z\rangle+\frac{1}{3}}}\right)\\
&+(1+q^{-1/3})s^{-1}\left(\frac{1-sq^{\langle g\beta_1,z\rangle}}{1-q^{\langle g\beta_1,z\rangle+\frac{1}{3}}}\right)\left(\frac{1-sq^{-\langle g\beta_1,z\rangle-1}}{1-q^{-\langle g\beta_1,z\rangle-1}}\right)\left(\frac{1-q^{-\langle g\beta_2,z\rangle+\frac{1}{3}}}{1-q^{-\langle g\beta_2,z\rangle}}\right)\left(\frac{1-q^{\langle g\beta_3,z\rangle+\frac{1}{3}}}{1-q^{\langle g\beta_3,z\rangle}}\right)\Bigg],
\end{aligned}
\qquad (5.22)
$$

where $g = g(\varepsilon, j) \in G$ is (either one of the two elements of G) such that $g\beta_1 = \varepsilon\beta_j$, $s = t_{e_1}$, $z \in \pi_0$.

Proof. Any linear combination of the operators E_1, E_2 given by formulas (5.5), (5.1) can contain shifts only along the vectors from $R^\vee$. It is easy to see that the non-zero vectors in $\overline{R^\vee}$ are $\pm 2\alpha_j, \pm 2\beta_j, \pm 4\beta_j$. Let us firstly establish formula (5.20).

We have $\overline{2e_2 + 2e_3} = \overline{2e_2 + 2e_4} = \overline{2e_3 + 2e_4} = 4\beta_1$, and none of the other vectors from $R^\vee$ project to $4\beta_1$. For $z \in \pi$, we have $V_{2e_2+2e_3}(z) = 0$ since $v_{\frac{1}{2}(e_3-e_4)}(q^{\frac{1}{2}\langle e_3-e_4,z\rangle}) = 0$, and $V_{2e_2+2e_4}(z) = 0$ since $v_{\frac{1}{2}(e_2-e_3)}(q^{\frac{1}{2}\langle e_2-e_3,z\rangle}) = 0$. By formula (5.2), we have

$$V_{2e_3+2e_4}(z) = \prod_{\gamma \in \{-\alpha_2, \alpha_3\}} v_\gamma(q^{\langle \gamma, z\rangle}) v_\gamma(q^{\langle \gamma, z\rangle + 1}) \prod_{\gamma \in B_0 \cup B_1 \cup B_2 \cup B_3} v_\gamma(q^{\langle \gamma, z\rangle}) \prod_{\gamma \in B_1' \cup B_2' \cup B_3'} v_\gamma(q^{\langle \gamma, z\rangle + 1}),$$

where

$$\begin{aligned}
&B_0 = \{\tfrac{1}{2}(e_4 - e_2), \tfrac{1}{2}(e_3 - e_2)\},\\
&B_1' = \{e_3, e_4, \tfrac{1}{2}(e_3 + e_4)\}, \quad B_1 = B_1' \cup \{\tfrac{1}{2}(e_2 + e_3), \tfrac{1}{2}(e_2 + e_4)\}\\
&B_2' = \{\tfrac{1}{2}(e_1 - e_2 + e_3 + e_4)\}, \quad B_2 = B_2' \cup \{\tfrac{1}{2}(e_1 + e_3), \tfrac{1}{2}(e_1 + e_4)\},\\
&B_3' = \{\tfrac{1}{2}(-e_1 - e_2 + e_3 + e_4)\}, \quad \text{and} \quad B_3 = B_3' \cup \{\tfrac{1}{2}(e_3 - e_1), \tfrac{1}{2}(e_4 - e_1)\}.
\end{aligned}$$

Let now $z = (z_1, z_2, z_2, z_2) \in \pi_0$. Then

$$\begin{aligned}
\prod_{\gamma \in B_2} v_\gamma(q^{\langle \gamma, z+\delta\rangle}) \prod_{\gamma \in B_2'} v_\gamma(q^{\langle \gamma, z+\delta\rangle + 1}) &= \frac{q^{1/3}s^{-1}(1 - sq^{\frac{1}{2}(z_1+z_2)+\frac{1}{3}})(1 - sq^{\frac{1}{2}(z_1+z_2)-\frac{2}{3}})}{(1 - q^{\frac{1}{2}(z_1+z_2)})(1 - q^{\frac{1}{2}(z_1+z_2)+\frac{1}{3}})}\\
&= \frac{q^{1/3}s^{-1}(1 - sq^{\frac{1}{3}\langle (2\beta_2)^\vee, z\rangle + \frac{1}{3}})(1 - sq^{\frac{1}{3}\langle (2\beta_2)^\vee, z\rangle - \frac{2}{3}})}{(1 - q^{\frac{1}{3}\langle (2\beta_2)^\vee, z\rangle})(1 - q^{\frac{1}{3}\langle (2\beta_2)^\vee, z\rangle + \frac{1}{3}})}.
\end{aligned} \tag{5.23}$$

Similarly, for the products over $\gamma \in B_3$ and $\gamma \in B_3'$, we get expression (5.23) with β_2 replaced by $-\beta_3$, and therefore

$$\prod_{\gamma \in B_2 \cup B_3} v_\gamma(q^{\langle \gamma, z+\delta\rangle}) \prod_{\gamma \in B_2' \cup B_3'} v_\gamma(q^{\langle \gamma, z+\delta\rangle + 1}) = q^{2/3}s^{-2} \prod_{\substack{\gamma \in (2G\beta_1)^\vee \\ \langle 4\beta_1, \gamma\rangle = 2}} \left(\frac{1 - sq^{\frac{\langle \gamma, z\rangle}{3} - \frac{2}{3}}}{1 - q^{\frac{\langle \gamma, z\rangle}{3}}}\right)\left(\frac{1 - sq^{\frac{\langle \gamma, z\rangle}{3} + \frac{1}{3}}}{1 - q^{\frac{\langle \gamma, z\rangle}{3} + \frac{1}{3}}}\right).$$

For the product over B_1, we have

$$\prod_{\gamma \in B_1} v_\gamma(q^{\langle \gamma, z+\delta\rangle}) = \frac{q^{1/2}s^{-1}(1 - sq^{z_2})(1 - sq^{z_2 - \frac{2}{3}})}{(1 - q^{z_2})(1 - q^{z_2 + \frac{1}{3}})}, \tag{5.24}$$

and for the product over B_1', we have

$$\prod_{\gamma\in B_1'} v_\gamma(q^{\langle\gamma,z+\delta\rangle+1}) = \frac{q^{1/6}s^{-1}(1-sq^{z_2+\frac{1}{3}})(1-sq^{z_2+1})}{(1-q^{z_2+\frac{2}{3}})(1-q^{z_2+1})}. \tag{5.25}$$

Since $\langle\beta_1, z\rangle = z_2$, the product of expressions (5.24) and (5.25) is

$$q^{2/3}s^{-2}\left(\frac{1-sq^{\langle\beta_1,z\rangle-\frac{2}{3}}}{1-q^{\langle\beta_1,z\rangle}}\right)\left(\frac{1-sq^{\langle\beta_1,z\rangle+\frac{1}{3}}}{1-q^{\langle\beta_1,z\rangle+\frac{1}{3}}}\right)\left(\frac{1-sq^{\langle\beta_1,z\rangle}}{1-q^{\langle\beta_1,z\rangle+\frac{2}{3}}}\right)\left(\frac{1-sq^{\langle\beta_1,z\rangle+1}}{1-q^{\langle\beta_1,z\rangle+1}}\right).$$

Next, we have

$$\prod_{\gamma\in B_0} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) = q^{-1/3}(1+q^{1/3}+q^{2/3}). \tag{5.26}$$

To complete the proof of equality (5.20) for $j=1$ and $\varepsilon=1$, we note that

$$\prod_{\gamma\in\{-\alpha_2,\alpha_3\}} v_\gamma(q^{\langle\gamma,z\rangle})v_\gamma(q^{\langle\gamma,z\rangle+1}) = s^{-2}\prod_{\substack{\gamma\in(2G\alpha_1)^\vee\\ \langle4\beta_1,\gamma\rangle=2}}\left(\frac{1-sq^{\langle\gamma,z\rangle}}{1-q^{\langle\gamma,z\rangle}}\right)\left(\frac{1-sq^{\langle\gamma,z\rangle+1}}{1-q^{\langle\gamma,z\rangle+1}}\right),$$

as required. Equality (5.20) in general follows from the G-invariance of the operators $\overline{E_1}$ and $\overline{E_2}$ established in Proposition 5.4.

Let us now prove formula (5.21). We have $\overline{2e_1} = 2\alpha_1$, and no other vector from $R^\vee$ is projected to $2\alpha_1$. By formulas (5.1) and (5.5), the coefficient at T_{2e_1} in the operator $E_1 + cE_2$ is

$$V_{2e_1}\left(c + \sum_{j=2}^{4}\sum_{\delta\in\{\pm1\}} U^{2e_1}_{2e_1+2\delta e_j}\right). \tag{5.27}$$

For $z\in\pi_0$, $V_{2e_1}(z+\delta)$ is given by the right-hand side of formula (5.14). Let us compute the second factor of the expression (5.27). For $z\in\pi$, by formula (5.4), we have $U^{2e_1}_{2e_1-2e_3}(z) = U^{2e_1}_{2e_1+2e_2}(z) = 0$ since $v_{\frac{1}{2}(e_2-e_3)}(q^{\frac{1}{2}\langle e_2-e_3,z\rangle}) = 0$; and $U^{2e_1}_{2e_1-2e_4}(z) = U^{2e_1}_{2e_1+2e_3}(z) = 0$ since $v_{\frac{1}{2}(e_3-e_4)}(q^{\frac{1}{2}\langle e_3-e_4,z\rangle}) = 0$. Furthermore,

$$U^{2e_1}_{2e_1-2e_2}(z) = v_{-e_2}(q^{\langle e_2,z\rangle-1})\prod_{\gamma\in B_0\cup C_1} v_\gamma(q^{\langle\gamma,z\rangle}) = s^{-1/2}\frac{1-sq^{z_2-1}}{1-q^{z_2-1}}\prod_{\gamma\in B_0\cup C_1} v_\gamma(q^{\langle\gamma,z\rangle}), \tag{5.28}$$

where $C_1 = \{-\frac{1}{2}(e_2+e_3), -\frac{1}{2}(e_2+e_4), -e_2\}$ and, recall, $B_0 = \{\frac{1}{2}(e_4-e_2), \frac{1}{2}(e_3-e_2)\}$. Let now $z=(z_1,z_2,z_2,z_2)\in\pi_0$. Then we have

$$\prod_{\gamma\in C_1} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) = q^{1/3}s^{-1/2}\frac{1-sq^{-z_2-\frac{2}{3}}}{1-q^{-z_2}}. \tag{5.29}$$

Therefore, we get from relations (5.26), (5.28), and (5.29) that

$$U^{2e_1}_{2e_1-2e_2}(z+\delta) = s^{-1}(1+q^{1/3}+q^{2/3})\frac{(1-sq^{\langle\beta_1,z\rangle-\frac{1}{3}})(1-sq^{-\langle\beta_1,z\rangle-\frac{2}{3}})}{(1-q^{\langle\beta_1,z\rangle-\frac{1}{3}})(1-q^{-\langle\beta_1,z\rangle})}.$$

Similarly, we compute

$$U^{2e_1}_{2e_1+2e_4}(z+\delta) = s^{-1}(1+q^{1/3}+q^{2/3})\frac{(1-sq^{\langle\beta_1,z\rangle-\frac{2}{3}})(1-sq^{-\langle\beta_1,z\rangle-\frac{1}{3}})}{(1-q^{-\langle\beta_1,z\rangle-\frac{1}{3}})(1-q^{\langle\beta_1,z\rangle})}.$$

Observe that the sum $U^{2e_1}_{2e_1-2e_2}(z+\delta)+U^{2e_1}_{2e_1+2e_4}(z+\delta)$ is not singular at $q^{\langle\beta_1,z\rangle}=1$, and

$$\begin{aligned} c+\sum_{j=2}^{4}\sum_{\delta\in\{\pm1\}} U^{2e_1}_{2e_1+2\delta e_j} &= q^{-1}s^5 d\left(\frac{1-q^{\langle\beta_1,z\rangle-\frac{7}{3}}}{1-q^{\langle\beta_1,z\rangle-\frac{1}{3}}}\right)\left(\frac{1-q^{-\langle\beta_1,z\rangle-\frac{7}{3}}}{1-q^{-\langle\beta_1,z\rangle-\frac{1}{3}}}\right)\\ &= q^{-1}s^5 d\prod_{\substack{\gamma\in(2G\beta_1)^\vee\\ \langle 2\alpha_1,\gamma\rangle=0}}\left(\frac{1-q^{\frac{\langle\gamma,z\rangle}{3}-\frac{7}{3}}}{1-q^{\frac{\langle\gamma,z\rangle}{3}-\frac{1}{3}}}\right),\end{aligned}$$

which completes the proof of (5.21) for $j=1$ and $\varepsilon=1$. Equality (5.21) in general follows from the G-invariance of the operators $\overline{E_1}$ and $\overline{E_2}$ established in Proposition 5.4.

Let us now prove formula (5.22). We have $\overline{2e_2}=\overline{2e_3}=\overline{2e_4}=2\beta_1$, and no other vector from $R^\vee$ is projected to $2\beta_1$. For $z\in\pi$, we have $V_{2e_2}(z)=0$ since $v_{\frac{1}{2}(e_2-e_3)}(q^{\frac{1}{2}\langle e_2-e_3,z\rangle})=0$, and we have $V_{2e_3}(z)=0$ since $v_{\frac{1}{2}(e_3-e_4)}(q^{\frac{1}{2}\langle e_3-e_4,z\rangle})=0$, so the coefficients at T_{2e_2} and T_{2e_3} in the operator E_1+cE_2 are both zero. By formulas (5.1) and (5.5), the coefficient at T_{2e_4} in the operator E_1+cE_2 is

$$V_{2e_4}\left(c+\sum_{j=1}^{3}\sum_{\delta\in\{\pm1\}} U^{2e_4}_{2e_4+2\delta e_j}\right). \tag{5.30}$$

For $z\in\pi_0$, $V_{2e_4}(z+\delta)$ is given by the right-hand side of formula (5.15). Let us compute the second factor of (5.30). For $z\in\pi$, by formula (5.4), we have $U^{2e_4}_{2e_4-2e_3}(z)=U^{2e_4}_{2e_2+2e_4}(z)=0$ since $v_{\frac{1}{2}(e_2-e_3)}(q^{\frac{1}{2}\langle e_2-e_3,z\rangle})=0$. We have

$$U^{2e_4}_{2e_1+2e_4}(z) = v_{\alpha_1}(q^{-\langle\alpha_1,z\rangle-1})v_{\alpha_1}(q^{\langle\alpha_1,z\rangle})\prod_{\gamma\in C_2\cup C_3} v_\gamma(q^{\langle\gamma,z\rangle}),$$

where $C_2=\{\frac{1}{2}(e_1+e_2),\frac{1}{2}(e_1+e_3)\}$ and $C_3=\{\frac{1}{2}(e_1-e_2),\frac{1}{2}(e_1-e_3)\}$. Let now $z=(z_1,z_2,z_2,z_2)\in\pi_0$. Then

$$\prod_{\gamma\in C_2} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) = q^{1/3}\frac{1-q^{\frac{1}{2}(z_1+z_2)-\frac{1}{3}}}{1-q^{\frac{1}{2}(z_1+z_2)+\frac{1}{3}}} = q^{1/3}\frac{1-q^{\langle\beta_2,z\rangle-\frac{1}{3}}}{1-q^{\langle\beta_2,z\rangle+\frac{1}{3}}}. \tag{5.31}$$

Similarly,

$$\prod_{\gamma\in C_3} v_\gamma(q^{\langle\gamma,z+\delta\rangle}) = q^{1/3}\frac{1-q^{\langle\beta_3,z\rangle-\frac{2}{3}}}{1-q^{\langle\beta_3,z\rangle}}.$$

Finally,

$$v_{\alpha_1}(q^{-\langle\alpha_1,z+\delta\rangle-1})v_{\alpha_1}(q^{\langle\alpha_1,z+\delta\rangle}) = s^{-1}\left(\frac{1-sq^{\langle\alpha_1,z\rangle}}{1-q^{\langle\alpha_1,z\rangle}}\right)\left(\frac{1-sq^{-\langle\alpha_1,z\rangle-1}}{1-q^{-\langle\alpha_1,z\rangle-1}}\right).$$

Therefore,

$$U^{2e_4}_{2e_1+2e_4}(z+\delta) = q^{2/3}s^{-1}\left(\frac{1-sq^{\langle\alpha_1,z\rangle}}{1-q^{\langle\alpha_1,z\rangle}}\right)\left(\frac{1-sq^{-\langle\alpha_1,z\rangle-1}}{1-q^{-\langle\alpha_1,z\rangle-1}}\right)\left(\frac{1-q^{\langle\beta_2,z\rangle-\frac{1}{3}}}{1-q^{\langle\beta_2,z\rangle+\frac{1}{3}}}\right)\left(\frac{1-q^{\langle\beta_3,z\rangle-\frac{2}{3}}}{1-q^{\langle\beta_3,z\rangle}}\right).$$

Similarly, we compute

$$U^{2e_4}_{2e_4-2e_1}(z+\delta) = q^{-2/3}s^{-1}\left(\frac{1-sq^{-\langle\alpha_1,z\rangle}}{1-q^{-\langle\alpha_1,z\rangle}}\right)\left(\frac{1-sq^{\langle\alpha_1,z\rangle-1}}{1-q^{\langle\alpha_1,z\rangle-1}}\right)\left(\frac{1-q^{\langle\beta_2,z\rangle+\frac{2}{3}}}{1-q^{\langle\beta_2,z\rangle}}\right)\left(\frac{1-q^{\langle\beta_3,z\rangle+\frac{1}{3}}}{1-q^{\langle\beta_3,z\rangle-\frac{1}{3}}}\right),$$

$$\begin{aligned}U^{2e_4}_{2e_4-2e_2}(z+\delta) = (1+q^{1/3})s^{-1}&\left(\frac{1-sq^{\langle\beta_1,z\rangle-\frac{1}{3}}}{1-q^{\langle\beta_1,z\rangle-\frac{1}{3}}}\right)\left(\frac{1-sq^{-\langle\beta_1,z\rangle-\frac{2}{3}}}{1-q^{-\langle\beta_1,z\rangle-\frac{1}{3}}}\right)\left(\frac{1-q^{\langle\beta_2,z\rangle+\frac{2}{3}}}{1-q^{\langle\beta_2,z\rangle+\frac{1}{3}}}\right)\\ &\times\left(\frac{1-q^{\langle\beta_3,z\rangle-\frac{2}{3}}}{1-q^{\langle\beta_3,z\rangle-\frac{1}{3}}}\right),\end{aligned}$$

$$\begin{aligned}U^{2e_4}_{2e_3+2e_4}(z+\delta) = (1+q^{1/3})s^{-1}&\left(\frac{1-sq^{\langle\beta_1,z\rangle}}{1-q^{\langle\beta_1,z\rangle+\frac{1}{3}}}\right)\left(\frac{1-sq^{-\langle\beta_1,z\rangle-1}}{1-q^{-\langle\beta_1,z\rangle-1}}\right)\left(\frac{1-q^{\langle\beta_2,z\rangle-\frac{1}{3}}}{1-q^{\langle\beta_2,z\rangle}}\right)\\ &\times\left(\frac{1-q^{\langle\beta_3,z\rangle+\frac{1}{3}}}{1-q^{\langle\beta_3,z\rangle}}\right).\end{aligned}$$

This completes the proof of (5.22) for $j = 1$ and $\varepsilon = 1$. Equality (5.22) in general follows from the G-invariance of the operators $\overline{E_1}$ and $\overline{E_2}$ established in Proposition 5.4. ■

In the rational limit, we recover the operators from Sections 3.4.2 and 3.4.4.

Proposition 5.7. *In the rational limit $q \to 1$, the operator* (5.13) *tends to the operator* (3.17).

Proof. Recall that $s = t_{e_1} = q^{-m}$, where m is independent of q. Hence,

$$\lim_{q\to 1} qs^{-5} = 1 \quad\text{and}\quad \lim_{q\to 1}(1+q^{1/3}+q^{2/3})q^{1/3}s^{-5} = 3.$$

The functions (5.14) tend to

$$\prod_{\substack{\gamma\in(2G\beta_1)^\vee \\ \langle 2\varepsilon\alpha_j,\gamma\rangle=3}}\left(1-\frac{3m+2}{\langle\gamma,z\rangle}\right)\left(1-\frac{3m+1}{\langle\gamma,z\rangle+1}\right)\left(1-\frac{3m}{\langle\gamma,z\rangle+2}\right)$$
$$\times\prod_{\substack{\gamma\in(2G\alpha_1)^\vee \\ \langle 2\varepsilon\alpha_j,\gamma\rangle=1}}\left(1-\frac{m}{\langle\gamma,z\rangle}\right)\times\left(1-\frac{m}{\langle\varepsilon\alpha_j,z\rangle}\right)\left(1-\frac{m}{\langle\varepsilon\alpha_j,z\rangle+1}\right),$$

which equals the corresponding functions (3.18), where in the last two terms we use that $\alpha_j^2=1$ here. Similarly, the functions (5.15) tend to

$$3\prod_{\substack{\gamma\in(2G\beta_1)^\vee \\ \langle 2\varepsilon\beta_j,\gamma\rangle=1}}\left(1-\frac{3m+2}{\langle\gamma,z\rangle}\right)\left(1+\frac{3m}{\langle\gamma,z\rangle+2}\right)\left(1-\frac{3m-1}{\langle\gamma,z\rangle-1}\right)$$
$$\times\prod_{\substack{\gamma\in(2G\alpha_1)^\vee \\ \langle 2\varepsilon\beta_j,\gamma\rangle=1}}\left(1-\frac{m}{\langle\gamma,z\rangle}\right)\times\left(1-\frac{m+\frac{2}{3}}{\langle\varepsilon\beta_j,z\rangle}\right)\left(1-\frac{m}{\langle\varepsilon\beta_j,z\rangle+\frac{1}{3}}\right),$$

which equals the corresponding functions (3.19), where in the last two terms we use that $\beta_j^2=\frac{1}{3}$ here. ■

Proposition 5.8. *In the rational limit $q\to 1$, the operator* (5.19) *tends to a scalar multiple of the operator* (3.26)*.*

Proof. We have $s=t_{e_1}=q^{-m}$, where m is independent of q. Hence,

$$\begin{aligned}&\lim_{q\to1}(1+q^{1/3}+q^{2/3})qs^{-6}=\lim_{q\to1}(1+q^{1/3}+q^{2/3})q^{1/3}s^{-5}=3,\\&\lim_{q\to1}c=\tfrac{9}{8}m(m+1)-6,\ \text{ and}\\&\lim_{q\to1}d=\tfrac{9}{8}m(m+1).\end{aligned}$$

The functions (5.20) tend to

$$3\prod_{\substack{\gamma\in(2G\alpha_1)^\vee \\ \langle 4\varepsilon\beta_j,\gamma\rangle=2}}\left(1-\frac{m}{\langle\gamma,z\rangle}\right)\left(1-\frac{m}{\langle\gamma,z\rangle+1}\right)\prod_{\substack{\gamma\in(2G\beta_1)^\vee \\ \langle 4\varepsilon\beta_j,\gamma\rangle=2}}\left(1-\frac{3m+2}{\langle\gamma,z\rangle}\right)\left(1-\frac{3m}{\langle\gamma,z\rangle+1}\right)$$
$$\times\left(1-\frac{m+\frac{2}{3}}{\langle\varepsilon\beta_j,z\rangle}\right)\left(1-\frac{m}{\langle\varepsilon\beta_j,z\rangle+\frac{1}{3}}\right)\left(1-\frac{m+\frac{2}{3}}{\langle\varepsilon\beta_j,z\rangle+\frac{2}{3}}\right)\left(1-\frac{m}{\langle\varepsilon\beta_j,z\rangle+1}\right),$$

which is $\frac{9}{8}$ times the corresponding functions (3.28), where we used that $\beta_j^2=\frac{1}{3}$ here.

The functions (5.21) tend to

$$\frac{9}{8}m(m+1)\prod_{\substack{\gamma\in(2G\alpha_1)^\vee\\ \langle 2\varepsilon\alpha_j,\gamma\rangle=1}}\left(1-\frac{m}{\langle\gamma,z\rangle}\right)\times\left(1-\frac{m}{\langle\varepsilon\alpha_j,z\rangle}\right)\left(1-\frac{m}{\langle\varepsilon\alpha_j,z\rangle+1}\right)$$
$$\times\prod_{\substack{\gamma\in(2G\beta_1)^\vee\\ \langle 2\varepsilon\alpha_j,\gamma\rangle=3}}\left(1-\frac{3m+2}{\langle\gamma,z\rangle}\right)\left(1-\frac{3m+1}{\langle\gamma,z\rangle+1}\right)\left(1-\frac{3m}{\langle\gamma,z\rangle+2}\right)$$
$$\times\prod_{\substack{\gamma\in(2G\beta_1)^\vee\\ \langle 2\varepsilon\alpha_j,\gamma\rangle=0}}\left(1-\frac{6}{\langle\gamma,z\rangle-1}\right),$$

which is $\frac{9}{8}$ times the corresponding functions (3.27), where we used that $\alpha_j^2=1$ here.

Assume $\varepsilon=j=1$ (the general case will follow by G-invariance of the operators). Then the function (5.22) tends to

$$3\prod_{\substack{\gamma\in(2G\alpha_1)^\vee\\ \langle 2\beta_1,\gamma\rangle=1}}\left(1-\frac{m}{\langle\gamma,z\rangle}\right)\prod_{\substack{\gamma\in(2G\beta_1)^\vee\\ \langle 2\beta_1,\gamma\rangle=1}}\left(1-\frac{3m+2}{\langle\gamma,z\rangle}\right)\left(1+\frac{3m}{\langle\gamma,z\rangle+2}\right)\left(1-\frac{3m-1}{\langle\gamma,z\rangle-1}\right)$$
$$\times\left(1-\frac{m+\frac{2}{3}}{\langle\beta_1,z\rangle}\right)\left(1-\frac{m}{\langle\beta_1,z\rangle+\frac{1}{3}}\right)$$
$$\times\Bigg[\frac{9}{8}m(m+1)-6+\left(1-\frac{m}{\langle\alpha_1,z\rangle}\right)\left(1+\frac{m}{\langle\alpha_1,z\rangle+1}\right)\left(1-\frac{2}{3\langle\beta_2,z\rangle+1}\right)\left(1-\frac{2}{3\langle\beta_3,z\rangle}\right)$$
$$+\left(1+\frac{m}{\langle\alpha_1,z\rangle}\right)\left(1-\frac{m}{\langle\alpha_1,z\rangle-1}\right)\left(1+\frac{2}{3\langle\beta_2,z\rangle}\right)\left(1+\frac{2}{3\langle\beta_3,z\rangle-1}\right)$$
$$+2\left(1-\frac{m}{\langle\beta_1,z\rangle-\frac{1}{3}}\right)\left(1+\frac{m+\frac{1}{3}}{\langle\beta_1,z\rangle+\frac{1}{3}}\right)\left(1+\frac{1}{3\langle\beta_2,z\rangle+1}\right)\left(1-\frac{1}{3\langle\beta_3,z\rangle-1}\right)$$
$$+2\left(1-\frac{m+\frac{1}{3}}{\langle\beta_1,z\rangle+\frac{1}{3}}\right)\left(1+\frac{m}{\langle\beta_1,z\rangle+1}\right)\left(1-\frac{1}{3\langle\beta_2,z\rangle}\right)\left(1+\frac{1}{3\langle\beta_3,z\rangle}\right)\Bigg],$$

which can be shown to be $\frac{9}{8}$ times the corresponding functions (3.29) (where we are using, in particular, that $\beta_1^2=\frac{1}{3}$ and $\alpha_1^2=1$). Indeed, the expression in the square brackets above can be shown to equal

$$\frac{9}{8}m(m+1)\left(1-\frac{2/3}{\langle\alpha_1,z\rangle-1}\right)\left(1+\frac{2/3}{\langle\alpha_1,z\rangle+1}\right)\left(1-\frac{4}{3\langle\beta_1,z\rangle-1}\right)\left(1+\frac{4/3}{\langle\beta_1,z\rangle+1}\right),$$

as required. ■

Corollary 5.9. *The operators* (3.17) *and* (3.26) *commute.*

Chapter 6

A subalgebra of DAHA and Van Diejen's operators

Inside the DAHA $\mathbb{H}_n = \mathbb{H}_{n,q,\tau}$ of type GL_n from Section 2.4, we define a subalgebra $\mathbb{H}^{\mathfrak{gl}_n}$ that may be thought of as a q-analogue of the degree zero part of the corresponding RCA. We prove that the algebra $\mathbb{H}^{\mathfrak{gl}_n}$ is a flat τ-deformation of the crossed product of the group algebra of the symmetric group $\mathfrak{S}_n$ with the image of the Drinfeld–Jimbo quantum group $U_q(\mathfrak{gl}_n)$ under the q-oscillator (Jordan–Schwinger) representation. We find all the defining relations and an explicit PBW basis for the algebra $\mathbb{H}^{\mathfrak{gl}_n}$. We describe its centre and establish a double centraliser property that $\mathbb{H}^{\mathfrak{gl}_n}$ satisfies as a subalgebra of a cyclotomic DAHA. As an application, we also obtain new generalisations of certain Hamiltonians introduced by Van Diejen. This chapter follows our paper [57].

The structure of this chapter is as follows. In Section 6.1, we recall the properties of the degree zero subalgebra of the RCA of type GL_n. In Section 6.2, we recall the definition of the quantum group $U_q(\mathfrak{gl}_n)$, its Jordan–Schwinger representation ρ, and we study the properties of the algebras $A = \rho(U_q(\mathfrak{gl}_n))$ and $\mathbb{C}\mathfrak{S}_n \ltimes A$. In Section 6.3, we define the algebra $\mathbb{H}^{\mathfrak{gl}_n}$. In Section 6.3.1, we study the properties of certain commuting elements $D_i \in \mathbb{H}_n$, which are used in the definition of $\mathbb{H}^{\mathfrak{gl}_n}$. In Section 6.3.2, which is the most technical part of this chapter, we give all the defining relations of $\mathbb{H}^{\mathfrak{gl}_n}$ and a linear basis for it. In Section 6.3.3, we give its centre and the double centraliser property. In Section 6.4, we derive new generalisations of Van Diejen's and related systems.

6.1 Degree zero part of an RCA

The RCA $\mathcal{H}_n$ of type GL_n (see Section 2.5.3) admits a grading in which $\deg s_k = 0$, $\deg x_i = 1$, and $\deg y_i = -1$. The subalgebra $\mathcal{H}^{\mathfrak{gl}_n} = \mathcal{H}_n^{(0)}$ of those elements that have degree zero was studied in [51]. It is generated by the elements s_k and the products

$E_{ij} \coloneqq x_i y_j$ $(i, j \in \{1, \dots, n\})$ with relations

$$s_k E_{ij} = E_{s_k(i), s_k(j)} s_k,$$

$$E_{ij} E_{kl} - E_{il} E_{kj} = E_{il} S_{jk} - E_{ij} S_{lk}, \tag{6.1}$$

$$E_{ij} E_{kl} - E_{kj} E_{il} = S_{jk} E_{il} - S_{ji} E_{kl}. \tag{6.2}$$

Here S_{ij} is defined by formula (2.14). Equivalently, the third set of relations could be replaced by the commutator-type relations

$$[E_{ij}, E_{kl}] = E_{il} S_{jk} - S_{il} E_{kj} + [S_{kl}, E_{ij}].$$

Note that S_{ij} is symmetric in i and j, it has degree 0, and it c-deforms the Kronecker delta δ_{ij}, so the above commutator relation c-deforms the relations of the standard generators of the Lie algebra $\mathfrak{gl}_n$.

The algebra $\mathcal{H}^{\mathfrak{gl}_n}$ admits a filtration determined by assigning degree 1 to the generators E_{ij} and degree 0 to s_k. The associated graded algebra is $\operatorname{gr} \mathcal{H}^{\mathfrak{gl}_n} = \mathbb{C}\mathfrak{S}_n \ltimes \mathbb{C}[\mathcal{M}]$, where

$$\mathcal{M} = \{M \in \mathrm{Mat}_n(\mathbb{C})\colon\ \operatorname{rank} M \leq 1\},$$

that is, $\mathcal{M}$ is the space of $n \times n$ complex matrices of rank at most one.

The algebra $\mathcal{H}^{\mathfrak{gl}_n}$ admits a PBW-type basis consisting of the monomials

$$w E_{i_1 j_1}^{k_1} \cdots E_{i_r j_r}^{k_r},$$

where $w \in \mathfrak{S}_n$, $r \in \mathbb{Z}_{\geq 0}$, $k_u \in \mathbb{Z}_{>0}$, $1 \leq i_1 \leq \cdots \leq i_r \leq n$, and $1 \leq j_1 \leq \cdots \leq j_r \leq n$ with $i_u = i_{u+1} \Rightarrow j_u < j_{u+1}$. The algebra $\mathcal{H}^{\mathfrak{gl}_n}$ is a flat c-deformation of $\mathbb{C}\mathfrak{S}_n \ltimes \rho_{\mathrm{JS}}(U(\mathfrak{gl}_n))$, where ρ_{JS} is the oscillator (also known as Jordan–Schwinger) representation of the universal enveloping algebra $U(\mathfrak{gl}_n)$ mapping the standard basis of the Lie algebra $\mathfrak{gl}_n$ to the operators $x_i \partial_{x_j}$ $(i, j \in \{1, \dots, n\})$. The algebra $\mathcal{H}^{\mathfrak{gl}_n}$ is an example of a non-homogeneous quadratic algebra over $\mathbb{C}\mathfrak{S}_n$ of PBW type (cf. [12]).

The element $eu = \sum_{i=1}^n x_i y_i - c \sum_{i<j} s_{ij}$ (which in the polynomial representation of the RCA is up to a constant equal to the Euler operator $\sum_{i=1}^n x_i \partial_{x_i}$) generates the centre $\mathcal{Z}(\mathcal{H}^{\mathfrak{gl}_n})$.

The TCA $\mathbb{H}_n^{\mathrm{trig}}$ of type GL_n (see Section 2.5.3) also has a grading, given by $\deg s_k = \deg \widehat{y}_i = 0$ and $\deg X_i^{\pm 1} = \pm 1$ [10]. The embedding (2.18) preserves the respective gradings. The degree zero part $\mathbb{H}_n^{\mathrm{trig},\,(0)}$ is generated by the elements s_k, $\widehat{y}_i$, and $X_i X_j^{-1}$. The algebra $\mathcal{H}^{\mathfrak{gl}_n} = \mathcal{H}_n^{(0)}$ embeds into $\mathbb{H}_n^{\mathrm{trig},\,(0)}$ via a restriction of the mapping (2.18).

6.2 Quantum group $U_q(\mathfrak{gl}_n)$

Let $q \in \mathbb{C}^\times$ be not a root of unity. The quantum group $U_q(\mathfrak{gl}_n)$ is the (unital, associative) algebra over $\mathbb{C}$ generated by $g_i^{\pm 1}$ $(1 \le i \le n)$ and e_k, f_k $(1 \le k \le n-1)$ subject to the following relations [75, p. 163–164] (cf. [70]):

$$\begin{aligned}
&g_i g_i^{-1} = g_i^{-1} g_i = 1, \qquad && [g_i, g_j] = 0,\\
&g_i e_k g_i^{-1} = q^{\delta_{ik} - \delta_{i,k+1}} e_k, && g_i f_k g_i^{-1} = q^{\delta_{i,k+1} - \delta_{ik}} f_k,\\
&[e_k, f_l] = \delta_{kl} \frac{g_k g_{k+1}^{-1} - g_k^{-1} g_{k+1}}{q - q^{-1}}, &&\\
&[e_k, e_l] = 0 = [f_k, f_l] \text{ if } |k-l| > 1, &&\\
&e_k^2 e_l - (q+q^{-1}) e_k e_l e_k + e_l e_k^2 = 0 \text{ if } |k-l| = 1, &&\\
&f_k^2 f_l - (q+q^{-1}) f_k f_l f_k + f_l f_k^2 = 0 \text{ if } |k-l| = 1, &&
\end{aligned}$$

$(1 \le i, j \le n,\ 1 \le k, l \le n-1)$. It follows from these relations that $\prod_{i=1}^n g_i$ lies in the centre $\mathcal{Z}(U_q(\mathfrak{gl}_n))$ of this algebra, and [70, Proposition 4] provides some further central elements.

In this section, we recall a representation of $U_q(\mathfrak{gl}_n)$ on the space of Laurent polynomials $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Let us firstly set up some notations. Let $t_i = q^{X_i \partial_{X_i}}$ be the q-shift operator which acts on functions f by $(t_i f)(X_1, \dots, X_n) = f(X_1, \dots, qX_i, \dots, X_n)$. Let us also consider the following operator

$$d_i = \frac{X_i^{-1}(t_i - t_i^{-1})}{q - q^{-1}}. \tag{6.3}$$

In the $q \to 1$ limit, it satisfies $d_i \to \partial_{X_i}$.

The next lemma collects some properties of d_i and t_i, which can be checked by a direct computation. We use the notation $[a, b]_\lambda$ for $\lambda \in \mathbb{C}$ to mean $ab - \lambda ba$.

Lemma 6.1. *For all $i, j \in \{1, \dots, n\}$, we have*

(1) $[t_i, X_j]_{q^{\delta_{ij}}} = 0$,

(2) $[d_i, d_j] = 0 = [t_i, t_j]$,

(3) $[d_i, t_j]_{q^{\delta_{ij}}} = 0$,

(4) $d_i X_i = (q - q^{-1})^{-1}(q t_i - q^{-1} t_i^{-1})$, *and* $[d_i, X_j]_{q^{\pm\delta_{ij}}} = \delta_{ij} t_i^{\mp 1}$.

In terms of d_i, t_i, and the multiplication operators X_i, one can write down a representation of $U_q(\mathfrak{gl}_n)$ on $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ as follows. It is called the Jordan–Schwinger or q-oscillator representation.

Proposition 6.2. *[61] There is a representation ρ of $U_q(\mathfrak{gl}_n)$ given on the generators by $\rho\colon g_i^{\pm 1} \mapsto t_i^{\pm 1}$, $e_k \mapsto X_k d_{k+1}$, and $f_k \mapsto X_{k+1} d_k$.*

In fact, this representation ρ has a submodule $\mathbb{C}[X_1, \dots, X_n]$, which we revisit later in Section 6.3.3.

Let us consider the algebra $A = \rho(U_q(\mathfrak{gl}_n))$, the image of $U_q(\mathfrak{gl}_n)$ under the representation ρ:

$$A = \langle t_i^{\pm 1} \ (1 \le i \le n),\ X_k d_{k+1},\ X_{k+1} d_k \ (1 \le k \le n-1) \rangle.$$

By the isomorphism theorems, A is isomorphic to $U_q(\mathfrak{gl}_n)/I_q$ for $I_q = \ker(\rho)$. We next describe the algebra A abstractly by (a different set of) generators and relations.

Let $E_{ij}^q = X_i d_j$ $(i, j \in \{1, \dots, n\})$. Then $E_{ii}^q = (q - q^{-1})^{-1}(t_i - t_i^{-1}) \in A$. The operators E_{ij}^q for $|i - j| > 1$ are related to Jimbo's analogue of the non-simple root vectors of the Lie algebra $\mathfrak{gl}_n$ from [70, Proposition 1]. The following formulas hold for all $1 \le i < j \le n-1$ [39, (3.3)]:

$$\begin{aligned} E_{i,j+1}^q &= \rho\left([e_i, [e_{i+1}, \cdots [e_{j-1}, e_j]_q \cdots]_q]_q \ g_{i+1} g_{i+2} \cdots g_j\right), \\ E_{j+1,i}^q &= \rho\left([f_j, [f_{j-1}, \cdots [f_{i+1}, f_i]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} g_{i+1}^{-1}\ g_{i+2}^{-1} \cdots g_j^{-1}\right). \end{aligned}$$

It follows that $E_{ij}^q \in A$ for all i, j, and that $A = \langle t_i^{\pm 1},\ E_{ij}^q\ (i \neq j) \rangle$ as an algebra.

In the $q \to 1$ limit, the representation ρ recovers the oscillator representation of the universal enveloping algebra $U(\mathfrak{gl}_n)$, since $\lim_{q \to 1} E_{ij}^q = X_i \partial_{X_j}$ for all i, j.

The next two propositions describe relations satisfied by the generators E_{ij}^q and t_i. Let us introduce the notation

$$S_{ij}^q := [d_i, X_j] = \begin{cases} (q+1)^{-1}\left(q t_i + t_i^{-1}\right) & \text{if } i = j, \\ 0 \text{ if } i \neq j, \end{cases}$$

where we used Lemma 6.1 (4). Note that S_{ij}^q is symmetric in i and j, and at $q = 1$, it reduces to δ_{ij}. The following proposition is a straightforward consequence of the definitions and the commutativity of the operators d_i. (It will also follow from the $\tau = 1$ limit of the more general discussions presented in the next section.)

Proposition 6.3. *For all $i, j, k, l \in \{1, \dots, n\}$,*

$$\begin{aligned} E_{ij}^q E_{kl}^q - E_{il}^q E_{kj}^q &= E_{il}^q S_{jk}^q - E_{ij}^q S_{lk}^q, \\ E_{ij}^q E_{kl}^q - E_{kj}^q E_{il}^q &= S_{jk}^q E_{il}^q - S_{ji}^q E_{kl}^q. \end{aligned} \tag{6.4}$$

Notice that the form of the relations (6.4) is exactly the same as that of relations (6.1) and (6.2). The following statement holds as a result of Lemma 6.1 (1) and (3).

Proposition 6.4. *For all $i, j, k \in \{1, \ldots, n\}$,*

$$t_i E_{jk}^q t_i^{-1} = q^{\delta_{ij} - \delta_{ik}} E_{jk}^q. \tag{6.5}$$

The preceding two propositions lead to a PBW-type basis and a presentation for the algebra A.

Proposition 6.5. *The algebra A has a linear basis formed by elements*

$$(E_{i_1 j_1}^q)^{k_1} \cdots (E_{i_r j_r}^q)^{k_r} \prod_{l=1}^{n} t_l^{m_l}, \tag{6.6}$$

where $r \in \mathbb{Z}_{\geq 0}$, $k_u \in \mathbb{Z}_{>0}$, $m_l \in \mathbb{Z}$, $1 \leq i_1 \leq \cdots \leq i_r \leq n$, $1 \leq j_1 \leq \cdots \leq j_r \leq n$ with $i_u = i_{u+1} \Rightarrow j_u < j_{u+1}$, and none of the indices i_u equal any of the indices j_v.

The algebra A has a presentation by generators $t_i^{\pm 1}$, E_{ij}^q $(i \neq j)$ and relations (6.4) *with $i \neq j$ and $k \neq l$,* (6.5) *with $j \neq k$, and the Laurent relations for $t_i^{\pm 1}$, namely $t_i t_i^{-1} = t_i^{-1} t_i = 1$ and $[t_i, t_j] = 0$ for all i, j.*

Proof. It follows from relations (6.4) and (6.5) that any element of A can be written as a linear combination of elements of the form (6.6), thus they span A. We now show that they are linearly independent over $\mathbb{C}$ as operators on $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

For any $k \in \mathbb{Z}_{>0}$ and $i \neq j$, we get by using Lemma 6.1 that

$$(E_{ij}^q)^k = (q - q^{-1})^{-k} X_i^k X_j^{-k} \prod_{l=0}^{k-1} (q^{-l} t_j - q^l t_j^{-1}).$$

More generally, for elements of the form (6.6) we have

$$(E_{i_1 j_1}^q)^{k_1} \cdots (E_{i_r j_r}^q)^{k_r} \prod_{l=1}^{n} t_l^{m_l} \propto X_{i_1}^{k_1} \cdots X_{i_r}^{k_r} X_{j_1}^{-k_1} \cdots X_{j_r}^{-k_r} t_{j_1}^{k_1} \cdots t_{j_r}^{k_r} \prod_{l=1}^{n} t_l^{m_l} + \ldots,$$

where $\ldots$ denotes terms in which the overall sum of the exponents on the t_i's is lower than in the above leading term, and $\propto$ denotes proportionality by a non-zero factor, which may depend on q.

Assume a non-trivial linear dependence of some terms of the form (6.6). This implies a non-trivial linear dependence of their corresponding leading (with highest degree in t_i's) terms

$$X_{i_1}^{k_1} \cdots X_{i_r}^{k_r} X_{j_1}^{-k_1} \cdots X_{j_r}^{-k_r} t_{j_1}^{k_1} \cdots t_{j_r}^{k_r} \prod_{l=1}^{n} t_l^{m_l}. \tag{6.7}$$

By the assumptions on the indices of the monomials (6.6), their leading terms (6.7) are different, and since $\prod_{l=1}^{n} X_l^{n_l} \prod_{l=1}^{n} t_l^{n_l'}$ $(n_l, n_l' \in \mathbb{Z})$ are linearly independent over $\mathbb{C}$ as operators on $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ (we are using here that q is not a root of unity), we get a contradiction. The statement follows. ∎

The above basis can be used to give the following proof of what the centre of A is.

Proposition 6.6. *The centre $\mathcal{Z}(A)$ is generated by $(\prod_{i=1}^n t_i)^{\pm 1}$.*

Proof. For the monomial (6.6), if $i_1 = i_2 = \cdots = i_{\widetilde{r}} < i_{\widetilde{r}+1} \leq \cdots \leq i_r$ then

$$t_{i_1}(E^q_{i_1 j_1})^{k_1} \cdots (E^q_{i_r j_r})^{k_r} \left(\prod_{l=1}^n t_l^{m_l}\right) t_{i_1}^{-1} = q^{k_1+k_2+\cdots+k_{\widetilde{r}}}(E^q_{i_1 j_1})^{k_1} \cdots (E^q_{i_r j_r})^{k_r} \prod_{l=1}^n t_l^{m_l}.$$

Together with our assumption that q is not a root of unity, this implies for any $f \in \mathcal{Z}(A)$ that its expansion in the PBW basis from Proposition 6.5 cannot involve any basis elements for which $r > 0$.

Similarly, since we have for all $1 \leq k \leq n-1$ that

$$\left(\prod_{l=1}^n t_l^{m_l}\right) E^q_{k,k+1} = q^{m_k - m_{k+1}} E^q_{k,k+1} \prod_{l=1}^n t_l^{m_l},$$

we get that the expansion of f can only contain terms of the form $\prod_{l=1}^n t_l^{m_l}$ where all m_l are equal. Conversely, all such terms do belong to the centre. The statement follows. ∎

Additionally, let us consider the crossed product algebra $\mathcal{A} := \mathbb{C}\mathfrak{S}_n \ltimes A$. As a vector space, $\mathcal{A} \cong \mathbb{C}\mathfrak{S}_n \otimes A$, and its algebra structure is defined by the natural action of the symmetric group $\mathfrak{S}_n$ on A given by

$$s_k t_i^{\pm 1} = t_{s_k(i)}^{\pm 1} s_k, \qquad s_k E^q_{ij} = E^q_{s_k(i), s_k(j)} s_k, \tag{6.8}$$

$1 \leq k \leq n-1$, where $s_k = (k, k+1) \in \mathfrak{S}_n$. This action is well-defined as it preserves the defining relations of A given in Proposition 6.5. The algebra $\mathcal{A}$ has a presentation analogous to that of A, just with the extra generators s_k and the extra relations (6.8) along with the Coxeter relations among s_k that hold in $\mathfrak{S}_n$. The algebra $\mathcal{A}$ has a basis of PBW type consisting of the elements

$$w(E^q_{i_1 j_1})^{k_1} \cdots (E^q_{i_r j_r})^{k_r} \prod_{l=1}^n t_l^{m_l} \qquad (w \in \mathfrak{S}_n) \tag{6.9}$$

with the same restrictions on the indices as above in Proposition 6.5.

Proposition 6.7. *The centre of $\mathcal{A}$ satisfies $\mathcal{Z}(\mathcal{A}) = \mathcal{Z}(A) = \langle (\prod_{i=1}^n t_i)^{\pm 1} \rangle$.*

Proof. Since all elements of $\mathcal{Z}(A)$, described in Proposition 6.6, are $\mathfrak{S}_n$-invariant, we have that $\mathcal{Z}(A) \subseteq \mathcal{Z}(\mathcal{A})$. Since $A \subset \mathcal{A}$, we have $\mathcal{Z}(\mathcal{A}) \cap A \subseteq \mathcal{Z}(A)$. It is now sufficient to show that $\mathcal{Z}(\mathcal{A}) \subseteq A$.

Denote the elements of the basis (6.9) schematically as wET where

$$E = (E^q_{i_1 j_1})^{k_1} \cdots (E^q_{i_r j_r})^{k_r}$$

and $T = \prod_{l=1}^n t_l^{m_l}$. We have $t_i wET \propto wETt_{w^{-1}(i)}$, where the proportionality factor is a power of q. For any $f \in \mathcal{Z}(\mathcal{A})$, let us group in its expansion in the basis (6.9) the terms that have the same w and E parts. Each such group has the form $wE \sum_k \alpha_k T^{(k)}$, where $\alpha_k \in \mathbb{C}$ and $T^{(k)} = \prod_{l=1}^n t_l^{m_{kl}}$. Any t_i has to commute individually with each of the groups. Suppose $w \neq \mathrm{id}$. Take i, j such that $j = w^{-1}(i) \neq i$. We have $t_j E = q^a E t_j$ for some a. Commutativity requires

$$wE \sum_k \alpha_k T^{(k)} t_i = t_i wE \sum_k \alpha_k T^{(k)} = q^a wE \sum_k \alpha_k T^{(k)} t_j,$$

or equivalently, $wE(q^a - t_i t_j^{-1}) \sum_k \alpha_k T^{(k)} = 0$, which forces $\sum_k \alpha_k T^{(k)} = 0$.

This showed for any $f \in \mathcal{Z}(\mathcal{A})$ that its expansion in the basis (6.9) cannot involve any basis elements for which $w \neq \mathrm{id}$. This completes the proof. ■

In the next section, we define inside the DAHA of type GL_n a subalgebra that deforms the algebra $\mathcal{A}$ in a natural way.

6.3 Subalgebra $\mathbb{H}^{\mathfrak{gl}_n}$

In this section, we describe a subalgebra, which we denote $\mathbb{H}^{\mathfrak{gl}_n}$, of the DAHA $\mathbb{H}_n$. As we explain, this subalgebra is a q-deformation of the degree zero part $\mathcal{H}^{\mathfrak{gl}_n}$ of the RCA of type GL_n, and it is a τ-deformation of the algebra $\mathcal{A} \cong \mathbb{C}\mathfrak{S}_n \ltimes (U_q(\mathfrak{gl}_n)/I_q)$ from Section 6.2.

We will use throughout the following shorthand notations

$$T^+_{ij} = \begin{cases} T_i T_{i+1} \cdots T_j \text{ if } i \leq j, \\ 1 \text{ if } i > j, \end{cases} \qquad (T^{-1})^+_{ij} = \begin{cases} T_i^{-1} T_{i+1}^{-1} \cdots T_j^{-1} \text{ if } i \leq j, \\ 1 \text{ if } i > j, \end{cases}$$

$$T^-_{ij} = \begin{cases} T_i T_{i-1} \cdots T_j \text{ if } i \geq j, \\ 1 \text{ if } i < j, \end{cases} \qquad (T^{-1})^-_{ij} = \begin{cases} T_i^{-1} T_{i-1}^{-1} \cdots T_j^{-1} \text{ if } i \geq j, \\ 1 \text{ if } i < j, \end{cases}$$

$$(\mathcal{R}^\varepsilon)^\pm_{ij} := \begin{cases} (T^\varepsilon)^-_{i-1,j+1} T_j^{\pm 2} (T^{-\varepsilon})^+_{j+1,i-1} \text{ if } i > j, \\ 1 \text{ if } i \leq j, \end{cases} \qquad \text{with } \varepsilon \in \{1, -1\}.$$

We write $\mathcal{R}$ for $\mathcal{R}^1$ and T for T^1. We note that $(\mathcal{R}^\varepsilon)^-_{ij}(\mathcal{R}^\varepsilon)^+_{ij} = 1 = T^+_{ij}(T^{-1})^-_{ji}$. Notice that $(\mathcal{R}^\varepsilon)^\pm_{ij}$, when thought of as elements of the braid group, belong to (and generate) the pure braid group, which is the kernel of the quotient map $B_{\mathfrak{S}_n} \to \mathfrak{S}_n$.

Let $D_n = (q-q^{-1})^{-1}X_n^{-1}(Y_n - Y_n^{-1})$, and let

$$D_i = T^+_{i,n-1}D_nT^-_{n-1,i} = (q-q^{-1})^{-1}X_i^{-1}(T^{-1})^+_{i,n-1}(Y_n - Y_n^{-1})T^-_{n-1,i} \tag{6.10}$$

for $1 \leq i \leq n-1$. With the assignments (2.16), upon performing the trigonometric degeneration $q \to 1$, we get $D_i \to y_i$, where we implicitly use the embedding (2.18). At $\tau = 1$, the elements D_i act in the polynomial representation of the DAHA as the operators d_i from Section 6.2.

Let $e_{ij} = X_iD_j$ $(i, j \in \{1, \dots, n\})$. We now define the main object of this chapter. Inside $\mathbb{H}_n$, we define $\mathbb{H}^{\mathfrak{gl}_n} = \mathbb{H}^{\mathfrak{gl}_n}_{q,\tau}$ as the following subalgebra:

$$\mathbb{H}^{\mathfrak{gl}_n} = \langle T_k,\, Y_i^{\pm 1},\, e_{ij} \colon 1 \leq k \leq n-1,\, 1 \leq i, j \leq n,\, i \neq j \rangle \subset \mathbb{H}_n.$$

Note that, by equality (6.10), we have

$$e_{ii} = (q-q^{-1})^{-1}(T^{-1})^+_{i,n-1}(Y_n - Y_n^{-1})T^-_{n-1,i} \in \mathbb{H}^{\mathfrak{gl}_n}.$$

At $\tau = 1$, the generators T_k, $Y_i^{\pm 1}$, and e_{ij} of $\mathbb{H}^{\mathfrak{gl}_n}$ act (in the polynomial representation) respectively as s_k, $t_i^{\pm 1}$, and E^q_{ij}, which generate the algebra $\mathcal{A}$ from Section 6.2. In the trigonometric limit $q \to 1$, we get $T_k \to s_k$, $Y_i^{\pm 1} \to 1$, and $e_{ij} \to x_iy_j$ for all i, j, which are the generators of $\mathcal{H}^{\mathfrak{gl}_n}$, where we implicitly use the embedding (2.18).

We note that $\mathbb{H}^{\mathfrak{gl}_n} \subset \mathbb{H}_n^{(0)}$, where $\mathbb{H}_n^{(0)}$ is the degree zero subalgebra of $\mathbb{H}_n$ (see Section 2.4); however, $\mathbb{H}^{\mathfrak{gl}_n} \neq \mathbb{H}_n^{(0)}$ for $n \geq 2$. Indeed, in the limit $q \to 1$, we do not get, for example, the elements $X_iX_j^{-1}$ for $i \neq j$.

In the next remark, we explain that the algebra $\mathbb{H}^{\mathfrak{gl}_n}$ is isomorphic to a subalgebra of a cyclotomic DAHA introduced in [10].

Remark 6.8. Elements similar to but different from D_i appear in the definition of the cyclotomic DAHA $HH^l_{n,t}(Z, q^{-1})$ for $l = 2$, $Z_1 = 1$, $Z_2 = -1$, $Z = (Z_1, Z_2)$ [10, Section 3.6], where we assume that t is a formal parameter and q is numerical. Let us make the relation more precise. The following elements $D_i^{\mathrm{BEF}} \equiv D_i^{(2)}$ were considered in [10]:

$$D_i^{\mathrm{BEF}} = (T^{-1})^-_{i-1,1}X_1^{-1}(Y_1^2 - 1)(T^{-1})^+_{1,i-1}. \tag{6.11}$$

The DAHA $HH_{n,t}(q)$ considered in [10] is isomorphic to the DAHA $\mathbb{H}_n$ considered in this thesis via an isomorphism $g\colon HH_{n,t}(q) \to \mathbb{H}_n$ given by

$$g(T_k) = T_k, \quad g(X_i) = Y_i^{-1}, \quad g(Y_i) = X_i, \quad g(\mathbf{t}) = \tau,$$

where $t = \mathbf{t}^2$, and the parameter q from [10] corresponds to our q. According to [10], there is an isomorphism $\varphi\colon HH_{n,t}(q^{-1}) \to HH_{n,t}(q)$ given by

$$\varphi(T_k) = T_k^{-1}, \quad \varphi(X_i) = Y_i^{-1}, \quad \varphi(Y_i) = X_i^{-1}, \quad \varphi(\mathbf{t}) = \mathbf{t}^{-1}.$$

Also, it is straightforward to check that the DAHA $\mathbb{H}_n$ has an automorphism h given by

$$h(T_k) = T_{n-k}, \quad h(X_i) = X_{n-i+1}^{-1}, \quad h(Y_i) = Y_{n-i+1}^{-1}, \quad h(\tau) = \tau.$$

By combining these morphisms and applying them to $D_{n-i+1}^{\mathrm{BEF}} \in HH_{n,t}^2((1,-1), q^{-1})$, we get

$$\begin{aligned}(h \circ g \circ \varphi)(D_{n-i+1}^{\mathrm{BEF}}) &= T_{i,n-1}^{+} X_n^{-1}(Y_n^{-2} - 1) T_{n-1,i}^{-} \\ &= (q^{-1} - q) D_i Y_i^{-1} T_{i,n-1}^{+} T_{n-1,i}^{-}. \end{aligned} \tag{6.12}$$

Equivalently, $D_i = (q^{-1} - q)^{-1}(h \circ g \circ \varphi)(D_{n-i+1}^{\mathrm{BEF}})(T^{-1})_{i,n-1}^{+} Y_n T_{n-1,i}^{-}$. It follows from this that $(\varphi^{-1} \circ g^{-1} \circ h)(D_i) \in HH_{n,t}^2((1,-1), q^{-1})$. This implies that $\mathbb{H}^{\mathfrak{gl}_n}$ is isomorphic to a subalgebra of $HH_{n,t}^2((1,-1), q^{-1}) \subset HH_{n,t}^1(1, q^{-1})$. Recall that a general cyclotomic DAHA $HH_{n,t}^l(Z, q)$ for $Z = (Z_1, \dots, Z_l) \in \mathbb{C}^l$ and $l \in \mathbb{Z}_{\geq 1}$ was defined in [10] as the subalgebra of $HH_{n,t}(q)$ generated by T_k $(1 \leq k \leq n-1)$, X_i, $Y_i^{\pm 1}$, and

$$D_i^{(l)} = (T^{-1})_{i-1,1}^{-} X_1^{-1}(Y_1 - Z_1) \cdots (Y_1 - Z_l)(T^{-1})_{1,i-1}^{+} \qquad (1 \leq i \leq n).$$

The choice (6.10) of the elements D_i is needed in order to be able to make the connection of the subalgebra $\mathbb{H}^{\mathfrak{gl}_n}$ with the quantum group $U_q(\mathfrak{gl}_n)$. We now derive some properties of D_i for later use.

6.3.1 Properties of D_i

We begin by some technical preliminaries. The following lemma and its corollary record some braid group identities.

Lemma 6.9. *For all $n-1 \geq k > j \geq i \geq 1$ and $\varepsilon \in \{\pm 1\}$, we have*

$$T_{j+1}^{\varepsilon} T_{ik}^{+} = T_{ik}^{+} T_j^{\varepsilon}, \qquad T_{ki}^{-} T_{j+1}^{\varepsilon} = T_j^{\varepsilon} T_{ki}^{-}.$$

Proof. Using the braid relations (and their versions with some of the generators inverted), we get

$$T_{j+1}^{\varepsilon} T_{ik}^{+} = T_{i,j-1}^{+} T_{j+1}^{\varepsilon} T_j T_{j+1} T_{j+2,k}^{+} = T_{i,j-1}^{+} T_j T_{j+1} T_j^{\varepsilon} T_{j+2,k}^{+} = T_{ik}^{+} T_j^{\varepsilon},$$

as required. Similarly for the other relation. ∎

The following is a straightforward corollary of the preceding lemma.

Corollary 6.10. *For all $n \geq j > i \geq 1$ and $\varepsilon \in \{\pm 1\}$, we have*

(i) $(T^\varepsilon)^+_{j,n-1} T^+_{i,n-1} = T^+_{i,n-1} (T^\varepsilon)^+_{j-1,n-2}$,

(ii) $(T^\varepsilon)^-_{n-1,j} T^+_{i,n-1} = T^+_{i,n-1} (T^\varepsilon)^-_{n-2,j-1}$,

(iii) $T^-_{n-1,i} (T^\varepsilon)^-_{n-1,j} = (T^\varepsilon)^-_{n-2,j-1} T^-_{n-1,i}$,

(iv) $T^-_{n-1,i} (T^\varepsilon)^+_{j,n-1} = (T^\varepsilon)^+_{j-1,n-2} T^-_{n-1,i}$.

The next lemma gives some identities for $(\mathcal{R}^\varepsilon)^\pm_{ji}$ in the Hecke algebra.

Lemma 6.11. *For all $n \geq j > i \geq 1$ and $\varepsilon \in \{\pm 1\}$, we have*

$$(\mathcal{R}^\varepsilon)^\pm_{ji} = (T^{-\varepsilon})^+_{i,j-2} T^{\pm 2}_{j-1} (T^\varepsilon)^-_{j-2,i}.$$

Proof. Let $\varepsilon = -1$. The claim trivially holds if $j = i+1$, so let $j > i+1$. We want to show for $\delta \in \{\pm 1\}$ that

$$T_i^{2\delta} T^+_{i+1,j-1} T^+_{i,j-2} = T^+_{i+1,j-1} T^+_{i,j-2} T^{2\delta}_{j-1}. \tag{6.13}$$

Since $T_i^{2\delta} = 1 + \delta(\tau - \tau^{-1}) T_i^\delta$, the left-hand side of equality (6.13) equals

$$T^+_{i+1,j-1} T^+_{i,j-2} + \delta(\tau - \tau^{-1}) T_i^\delta T^+_{i+1,j-1} T^+_{i,j-2}. \tag{6.14}$$

By the braid relations, for any $n \geq j > l > i \geq 1$, we have

$$(T^+_{i,l-2} T^\delta_{l-1} T^+_{l,j-1}) T_{l-1} = T_l (T^+_{i,l-1} T^\delta_l T^+_{l+1,j-1})$$

(proved similarly to Lemma 6.9), which upon repeated application (for $l = i+1, \ldots, j-1$) gives that $T_i^\delta T^+_{i+1,j-1} T^+_{i,j-2} = T^+_{i+1,j-1} T^+_{i,j-2} T^\delta_{j-1}$. Hence the expression (6.14) equals

$$T^+_{i+1,j-1} T^+_{i,j-2} + \delta(\tau - \tau^{-1}) T^+_{i+1,j-1} T^+_{i,j-2} T^\delta_{j-1} = T^+_{i+1,j-1} T^+_{i,j-2} T^{2\delta}_{j-1},$$

as required. The case when $\varepsilon = 1$ can be proved similarly. ■

The next lemma is an analogue of relations (2.6) and (2.7) for T_k and D_i.

Lemma 6.12. *We have $[T_k, D_i] = 0$ for $i \neq k, k+1$, and $T_k^{-1} D_k T_k^{-1} = D_{k+1}$.*

Proof. The fact that $T_k^{-1} D_k T_k^{-1} = D_{k+1}$ is clear from the definition. If $i \neq k$, $k+1$, then either $i \geq k+2$, in which case $[T_k, D_i] = 0$ because T_k commutes with X_n^{-1}, $Y_n^{\pm 1}$, and with both $T^+_{i,n-1}$ and $T^-_{n-1,i}$; or $i < k$, in which case $T_k T^+_{i,n-1} = T^+_{i,n-1} T_{k-1}$ and $T_{k-1} T^-_{n-1,i} = T^-_{n-1,i} T_k$ by Lemma 6.9, and so $[T_k, D_i] = 0$ follows, as $[T_{k-1}, X_n^{-1}] = [T_{k-1}, Y_n^{\pm 1}] = 0$. ■

The next lemma is a τ-deformed version of Lemma 6.1.

Lemma 6.13. *The following relations are satisfied.*

(1) (Relations between Y_i and X_j) For $n \geq i \neq j \geq 1$, $n \geq l \geq 1$, we have

$$Y_i \mathcal{R}^+_{ij} X_j = X_j (\mathcal{R}^{-1})^-_{ji} Y_i,$$
$$Y_l X_l = q T^+_{l,n-1} T^-_{n-1,l} X_l Y_l T^-_{l-1,1} T^+_{1,l-1}.$$

(2) $[D_i, D_j] = 0$ for all i, j.

(3) (Relations between Y_i and D_j) For $n \geq i \neq j \geq 1$, $n \geq l \geq 1$, we have

$$Y_i D_j = (\mathcal{R}^{-1})^+_{ji} D_j Y_i \mathcal{R}^+_{ij},$$
$$Y_l T^-_{l-1,1} T^+_{1,l-1} D_l = q^{-1} D_l (T^{-1})^+_{l,n-1} (T^{-1})^-_{n-1,l} Y_l.$$

(4) (Relations between X_i and D_j) For $n \geq l \geq 1$, we have

$$X_l D_l = (q - q^{-1})^{-1} (T^{-1})^+_{l,n-1} (Y_n - Y_n^{-1}) T^-_{n-1,l}, \tag{6.15}$$
$$D_l X_l = (q - q^{-1})^{-1} (T^{-1})^-_{l-1,1} (q Y_1 - q^{-1} Y_1^{-1}) T^+_{1,l-1}. \tag{6.16}$$

For $n \geq j > i \geq 1$, we have

$$[D_j, X_i] = \frac{\tau^{-1} - \tau}{q - q^{-1}} (T^{-1})^+_{i,n-1} (T^{-1})^-_{j-2,1} (q Y_1 + Y_n^{-1}) T^-_{n-1,j} T^+_{1,i-1}, \tag{6.17}$$
$$[D_i, X_j] = \frac{\tau^{-1} - \tau}{q - q^{-1}} (T^{-1})^-_{i-1,1} (T^{-1})^+_{j,n-1} (q^{-1} Y_1^{-1} + Y_n) T^+_{1,j-2} T^-_{n-1,i}. \tag{6.18}$$

The trigonometric degeneration $q \to 1$ of relations (6.15)–(6.18) recovers the commutator relation (2.14) that holds in the RCA $\mathcal{H}_n$. We now proceed to prove each part of Lemma 6.13 in turn. Another proof of the commutativity of D_i will follow from the proof of Proposition 6.32 below. The important thing for us about the form of relations (6.15)–(6.18) will be that their right-hand sides depend only on the generators T and Y.

Proof of Lemma 6.13 (1). This follows from [35, (1.4.64) and (1.4.68)] and the duality between X and Y described in [35, Theorem 1.4.8]. ∎

Proof of Lemma 6.13 (2). As $Y_{n-1} X_n = X_n T^{-2}_{n-1} Y_{n-1}$ by part (1), we have

$$\begin{aligned}(Y_n - Y_n^{-1}) T_{n-1} X_n^{-1} &= T^{-1}_{n-1} Y_{n-1} X_n^{-1} - T_{n-1} Y^{-1}_{n-1} T^2_{n-1} X_n^{-1} \\ &= T_{n-1} X_n^{-1} (Y_{n-1} - Y^{-1}_{n-1}).\end{aligned} \tag{6.19}$$

Next, we have

$$[(Y_{n-1}-Y_{n-1}^{-1})(Y_n-Y_n^{-1}),T_{n-1}]=0. \tag{6.20}$$

Indeed, this follows from the fact that $[T_{n-1},(Y_nY_{n-1})^{\pm 1}]=0$ and that

$$\begin{aligned} T_{n-1}(Y_{n-1}^{-1}Y_n+Y_{n-1}Y_n^{-1}) &= Y_n^{-1}T_{n-1}^{-1}Y_n+(\tau-\tau^{-1})Y_{n-1}Y_n^{-1}+Y_nT_{n-1}Y_n^{-1}\\ &= Y_n^{-1}Y_{n-1}T_{n-1}^{-1}+(\tau-\tau^{-1})Y_{n-1}Y_n^{-1}+Y_nY_{n-1}^{-1}T_{n-1}=(Y_{n-1}^{-1}Y_n+Y_{n-1}Y_n^{-1})T_{n-1}, \end{aligned}$$

where we used several times that $T_{n-1}^{\pm 1}=T_{n-1}^{\mp 1}\pm(\tau-\tau^{-1})$.

Using relations (6.19), (6.20), and that $D_{n-1}=T_{n-1}D_nT_{n-1}$, we get

$$\begin{aligned} (q-q^{-1})^2[D_{n-1},D_n] &= [T_{n-1}X_n^{-1}(Y_n-Y_n^{-1})T_{n-1},X_n^{-1}(Y_n-Y_n^{-1})]\\ &\overset{(6.19)}{=} X_{n-1}^{-1}X_n^{-1}(Y_{n-1}-Y_{n-1}^{-1})(Y_n-Y_n^{-1})\\ &\quad - X_n^{-1}X_{n-1}^{-1}T_{n-1}^{-1}(Y_{n-1}-Y_{n-1}^{-1})(Y_n-Y_n^{-1})T_{n-1} \overset{(6.20)}{=} 0. \end{aligned}$$

For $1\le i\le n-2$, since $[D_n,T_{i,n-2}^{+}]=[D_n,T_{n-2,i}^{-}]=0$, we get

$$[D_i,D_n]=[T_{i,n-2}^{+}D_{n-1}T_{n-2,i}^{-},D_n]=T_{i,n-2}^{+}[D_{n-1},D_n]T_{n-2,i}^{-}=0.$$

For $n-1\ge j>i\ge 1$, $[D_i,T_{j,n-1}^{+}]=[D_i,T_{n-1,j}^{-}]=0$ by Lemma 6.12, hence

$$[D_i,D_j]=[D_i,T_{j,n-1}^{+}D_nT_{n-1,j}^{-}]=T_{j,n-1}^{+}[D_i,D_n]T_{n-1,j}^{-}=0.$$

This completes the proof. ∎

Proof of Lemma 6.13 (3). Let $n\ge j>i\ge 1$. Firstly, using equality (6.10), Lemma 6.13 (1), and that $[Y_i,(T^{-1})_{j,n-1}^{+}]=[Y_i,T_{n-1,j}^{-}]=0$, we get

$$\begin{aligned} (q-q^{-1})Y_iD_j &= Y_iX_j^{-1}(T^{-1})_{j,n-1}^{+}(Y_n-Y_n^{-1})T_{n-1,j}^{-}\\ &= (\mathcal{R}^{-1})_{ji}^{+}X_j^{-1}Y_i(T^{-1})_{j,n-1}^{+}(Y_n-Y_n^{-1})T_{n-1,j}^{-}\\ &= (q-q^{-1})(\mathcal{R}^{-1})_{ji}^{+}D_jY_i, \end{aligned}$$

as required. Secondly, by using equality (6.10), Lemmas 6.13 (1) and 6.11, and that $Y_j(T^{-1})_{i,j-2}^{+}T_{j-1}(T^{-1})_{j,n-1}^{+}=(T^{-1})_{i,n-1}^{+}Y_{j-1}$, we get

$$\begin{aligned} (q-q^{-1})Y_jD_i &= Y_jX_i^{-1}(T^{-1})_{i,n-1}^{+}(Y_n-Y_n^{-1})T_{n-1,i}^{-}\\ &= X_i^{-1}Y_j(T^{-1})_{i,j-2}^{+}T_{j-1}(T^{-1})_{j,n-1}^{+}(Y_n-Y_n^{-1})T_{n-1,i}^{-}\\ &= X_i^{-1}(T^{-1})_{i,n-1}^{+}(Y_n-Y_n^{-1})Y_{j-1}T_{n-1,i}^{-}\\ &= (q-q^{-1})D_i(T^{-1})_{i,n-1}^{+}Y_{j-1}T_{n-1,i}^{-}\\ &= (q-q^{-1})D_iY_j(T^{-1})_{i,j-2}^{+}T_{j-1}^{2}T_{j-2,i}^{-}=(q-q^{-1})D_iY_j\mathcal{R}_{ji}^{+}, \end{aligned}$$

as required. Thirdly, by a similar calculation

$$\begin{aligned}(q-q^{-1})Y_l T^-_{l-1,1}T^+_{1,l-1}D_l &= Y_l T^-_{l-1,1}T^+_{1,l-1}X_l^{-1}(T^{-1})^+_{l,n-1}(Y_n-Y_n^{-1})T^-_{n-1,l}\\ &= q^{-1}X_l^{-1}(T^{-1})^+_{l,n-1}(T^{-1})^-_{n-1,l}Y_l(T^{-1})^+_{l,n-1}(Y_n-Y_n^{-1})T^-_{n-1,l}\\ &= q^{-1}X_l^{-1}(T^{-1})^+_{l,n-1}(Y_n-Y_n^{-1})(T^{-1})^-_{n-1,l}Y_l\\ &= q^{-1}(q-q^{-1})D_l(T^{-1})^+_{l,n-1}(T^{-1})^-_{n-1,l}Y_l,\end{aligned}$$

as required. ∎

Proof of Lemma 6.13 (4). Relation (6.15) follows from equality (6.10).

Next, using Lemma 6.13 (1) with $l=n$, we compute

$$\begin{aligned}(q-q^{-1})D_1X_1 &= T^+_{1,n-1}X_n^{-1}(Y_n-Y_n^{-1})X_n(T^{-1})^-_{n-1,1}\\ &= T^+_{1,n-1}\left(qY_nT^-_{n-1,1}T^+_{1,n-1}-q^{-1}(T^{-1})^-_{n-1,1}(T^{-1})^+_{1,n-1}Y_n^{-1}\right)(T^{-1})^-_{n-1,1}\\ &= qT^+_{1,n-1}Y_nT^-_{n-1,1}-q^{-1}(T^{-1})^+_{1,n-1}Y_n^{-1}(T^{-1})^-_{n-1,1}=qY_1-q^{-1}Y_1^{-1},\end{aligned} \tag{6.21}$$

which proves relation (6.16) for $l=1$. For $2\le l\le n$, we have

$$D_lX_l=(T^{-1})^-_{l-1,1}D_1(T^{-1})^+_{1,l-1}X_l=(T^{-1})^-_{l-1,1}D_1X_1T^+_{1,l-1},$$

which combined with equality (6.21) completes the proof of relation (6.16).

Next, using Lemma 6.13 (1) with $i=n$, $j=1$, and that $\mathcal{R}^+_{n,1}=1+(\tau-\tau^{-1})T^-_{n-1,1}(T^{-1})^+_{2,n-1}$, we get

$$\begin{aligned}(q-q^{-1})[D_n,X_1] &= X_n^{-1}[Y_n-Y_n^{-1},X_1]\\ &= X_n^{-1}\left(Y_n(1-\mathcal{R}^+_{n,1})X_1+(1-\mathcal{R}^+_{n,1})X_1Y_n^{-1}\right)\\ &= (\tau^{-1}-\tau)X_n^{-1}\left(Y_nT^-_{n-1,1}(T^{-1})^+_{2,n-1}X_1+T^-_{n-1,1}(T^{-1})^+_{2,n-1}X_1Y_n^{-1}\right).\end{aligned} \tag{6.22}$$

Here $T^-_{n-1,1}(T^{-1})^+_{2,n-1}X_1=X_n(T^{-1})^-_{n-1,1}(T^{-1})^+_{2,n-1}$, and we apply Lemma 6.13 (1) with $l=n$ to get that the expression (6.22) equals

$$\begin{aligned}&(\tau^{-1}-\tau)(qY_nT^-_{n-1,1}(T^{-1})^+_{2,n-1}+(T^{-1})^-_{n-1,1}(T^{-1})^+_{2,n-1}Y_n^{-1})\\ &=(\tau^{-1}-\tau)(T^{-1})^-_{n-1,1}(T^{-1})^+_{2,n-1}(qY_1+Y_n^{-1}).\end{aligned} \tag{6.23}$$

We then note that, for $n\ge j>i\ge 1$, we have

$$[D_j,X_i]=[T^+_{j,n-1}D_nT^-_{n-1,j},T^-_{i-1,1}X_1T^+_{1,i-1}]=T^-_{i-1,1}T^+_{j,n-1}[D_n,X_1]T^-_{n-1,j}T^+_{1,i-1}, \tag{6.24}$$

since $[D_n,T_k]=0$ if $k\le n-2$, and $[X_1,T_k]=0$ if $k\ge 2$. Relation (6.17) then follows from (6.23), (6.24), and the fact that $(T^{-1})^-_{j-1,2}(T^{-1})^+_{1,n-1}=(T^{-1})^+_{1,n-1}(T^{-1})^-_{j-2,1}$. For the latter, we use the inverse of one of the relations in Lemma 6.9 to move successively $T_2^{-1}, \dots, T_{j-1}^{-1}$ to the right of $(T^{-1})^+_{1,n-1}$.

It now only remains to prove relation (6.18).

By Corollary 6.10 (i) and (iii) (with $i = 1$, $j = 2$), we respectively get $(T^+_{1,n-1})^2 = T^+_{2,n-1}T^+_{1,n-2}T^2_{n-1}$ and $(T^-_{n-1,1})^2 = T^2_{n-1}T^-_{n-2,1}T^-_{n-1,2}$. We will additionally use that $T^2_{n-1} = 1 + (\tau - \tau^{-1})T_{n-1}$, that $[D_n, T^-_{n-2,1}] = 0 = [D_n, T^+_{1,n-2}]$, and that $[X_1, T^-_{n-1,2}] = 0 = [X_1, T^+_{2,n-1}]$. We also apply Corollary 6.10 (ii), (iv), relations (6.23), (6.15) (with $l = 1$), and (6.21). We get

$$\begin{aligned}
[D_1, X_n] &= [T^+_{1,n-1}D_nT^-_{n-1,1}, T^-_{n-1,1}X_1T^+_{1,n-1}] \\
&= T^+_{1,n-1}D_nT^2_{n-1}T^-_{n-2,1}T^-_{n-1,2}X_1T^+_{1,n-1} - T^-_{n-1,1}X_1T^+_{2,n-1}T^+_{1,n-2}T^2_{n-1}D_nT^-_{n-1,1} \\
&= T^-_{n-1,2}T^+_{1,n-1}[D_n, X_1]T^-_{n-1,1}T^+_{2,n-1} \\
&\quad + (\tau^{-1} - \tau)\big(T^+_{1,n-1}T^-_{n-2,1}X_1D_1 - D_1X_1T^-_{n-1,1}T^+_{2,n-1}\big) \\
&= \frac{\tau^{-1} - \tau}{q - q^{-1}}(q^{-1}Y_1^{-1} + Y_n)T^-_{n-1,1}T^+_{2,n-1}.
\end{aligned} \tag{6.25}$$

We then note that, for $n \geq j > i \geq 1$, we have

$$\begin{aligned}
[D_i, X_j] &= [(T^{-1})^-_{i-1,1}D_1(T^{-1})^+_{1,i-1}, (T^{-1})^+_{j,n-1}X_n(T^{-1})^-_{n-1,j}] \\
&= (T^{-1})^-_{i-1,1}(T^{-1})^+_{j,n-1}[D_1, X_n](T^{-1})^-_{n-1,j}(T^{-1})^+_{1,i-1},
\end{aligned} \tag{6.26}$$

as $[D_1, (T^{-1})^+_{j,n-1}] = [D_1, (T^{-1})^-_{n-1,j}] = 0$ by Lemma 6.12, and $[X_j, (T^{-1})^-_{i-1,1}] = 0 = [X_j, (T^{-1})^+_{1,i-1}]$. Relation (6.18) then follows from (6.25) and (6.26) because we have $T^-_{n-1,1}T^+_{2,j-1} = T^+_{1,j-2}T^-_{n-1,1}$, which is seen by using a relation from Lemma 6.9 to move successively $T_2, \dots, T_{j-1}$ to the left of $T^-_{n-1,1}$. ∎

6.3.2 A presentation of $\mathbb{H}^{\mathfrak{gl}_n}$ and a basis

We begin this section by describing relations among the generators of $\mathbb{H}^{\mathfrak{gl}_n}$. We write them in a form that makes it apparent that they τ-deform the relations of the algebra $\mathcal{A}$ from Section 6.2.

Define $S^\tau_{ij} := [D_i, X_j]$. Explicit formulas for S^τ_{ij} follow from Lemma 6.13 (4). In particular, one can see that $S^\tau_{ij} \in \mathbb{H}^{\mathfrak{gl}_n}$. The following statement is a consequence of the commutativity of the elements D_i. These relations in the special case of $\tau = 1$ were given earlier in Proposition 6.3. The relations below look formally the same as those in (6.4) just with E^q_{ij} replaced by e_{ij} and S^q_{ij} by S^τ_{ij}.

Proposition 6.14. *For all $1 \leq i \neq j \leq n$, $1 \leq k \neq l \leq n$, we have*

$$e_{ij}e_{kl} - e_{il}e_{kj} = e_{il}S^\tau_{jk} - e_{ij}S^\tau_{lk},$$

$$e_{ij}e_{kl} - e_{kj}e_{il} = S^\tau_{jk}e_{il} - S^\tau_{ji}e_{kl}.$$

Proof. The second relation is proved similarly to the first. For the first one, we have

$$e_{ij}e_{kl} = e_{ij}\left(D_l X_k - S^\tau_{lk}\right) = e_{il}D_j X_k - e_{ij}S^\tau_{lk} = e_{il}\left(e_{kj} + S^\tau_{jk}\right) - e_{ij}S^\tau_{lk}. \qquad \blacksquare$$

Further relations are as follows. One can move T_k to the left through $Y_i^{\pm 1}$ thanks to relations (2.7) of the DAHA $\mathbb{H}_n$. The relations that enable us to move T_k to the left through e_{ij} $(i \neq j)$ are given in the next proposition. These relations at $\tau = 1$ coincide with those from (6.8) between s_k and E^q_{ij}.

Proposition 6.15. *For $i, j, k \in \{1, \dots, n\}$ with $i \neq n$, $j \neq i, i+1$, and $k \neq j, i, i+1$, we have*

$$T_i e_{i,i+1} T_i = e_{i+1,i} + (\tau^{-1} - \tau)(q - q^{-1})^{-1}(T^{-1})^+_{i+1,n-1}(Y_n - Y_n^{-1})T^-_{n-1,i},$$
$$T_i e_{ij} T_i = e_{i+1,j}, \qquad T_i e_{j,i+1} T_i = e_{ji}, \qquad e_{jk}T_i = T_i e_{jk}.$$

Proof. For $1 \leq i \leq n-1$, we have

$$T_i e_{i,i+1} T_i = T_i X_i T_i^{-1} D_i = X_{i+1} D_i + (\tau^{-1} - \tau) T_i X_i D_i,$$

and the first relation follows. For $j \in \{1, \dots, n\} \setminus \{i, i+1\}$, Lemma 6.12 gives $[D_j, T_i] = 0$, and so $T_i e_{ij} T_i = T_i X_i T_i D_j = X_{i+1} D_j = e_{i+1,j}$, as required. The third relation is proved similarly, since $[X_j, T_i] = 0$ and $T_i D_{i+1} T_i = D_i$. If also $k \in \{1, \dots, n\} \setminus \{i, i+1\}$, then $[e_{jk}, T_i] = 0$ as $[D_k, T_i] = 0$, too. $\blacksquare$

The relations that enable us to move $Y_i^{\pm 1}$ to the right through e_{jk} $(j \neq k)$ can be split into three cases: $Y_i^{\pm 1}$ with e_{ij} for $i \neq j$; $Y_i^{\pm 1}$ with e_{ji} for $i \neq j$; and $Y_i^{\pm 1}$ with e_{jk} for $i \neq j \neq k \neq i$. We have the following statement, which at $\tau = 1$ reproduces (parts of) Proposition 6.4. The only thing that we will use about the form of the expressions C_1 and C_2 below is that they only depend on the generators T and Y, not their precise form.

Proposition 6.16. *(1) For $n \geq i \neq j \geq 1$, we have*

$$Y_i e_{ij} (T^{-1})^-_{i-1,1} (T^{-1})^+_{1,i-1} Y_i^{-1} = q T^+_{i,n-1} T^-_{n-1,i} e_{ij} + (q - q^{-1})^{-1}(\tau - \tau^{-1}) C_1,$$

where

$$C_1 = \begin{cases} q T^+_{i,n-1}(T^{-1})^-_{j-2,i}(Y_n - Y_n^{-1})T^-_{n-1,j} & \text{if } j > i, \\ T^+_{i,n-1}(T^{-1})^-_{j-1,1}(T^{-1})^+_{1,n-2} Y^{-1}_{n-1}(Y_n^2 - 1)T^-_{n-1,j}(T^{-1})^-_{n-1,i} & \text{if } i > j. \end{cases}$$

(2) *For* $n \geq i \neq j \geq 1$, *we have*

$$Y_i T^-_{i-1,1} T^+_{1,i-1} e_{ji} Y_i^{-1} = q^{-1} e_{ji} (T^{-1})^+_{i,n-1} (T^{-1})^-_{n-1,i} + (q-q^{-1})^{-1}(\tau^{-1}-\tau) C_2,$$

where

$$C_2 = \begin{cases} q^{-1}(T^{-1})^+_{j,n-1}(Y_n - Y_n^{-1}) T^+_{i,j-2}(T^{-1})^-_{n-1,i} & \text{if } j > i, \\ T^+_{i,n-1}(T^{-1})^+_{j,n-1} Y_{n-1}(1-Y_n^{-2}) T^-_{n-2,1} T^+_{1,j-1}(T^{-1})^-_{n-1,i} & \text{if } i > j. \end{cases}$$

(3) *For all* $i, j, k \in \{1, \dots, n\}$ *with* $i \neq j \neq k \neq i$, *we have*

$$Y_i \mathcal{R}^+_{ij} e_{jk} \mathcal{R}^-_{ik} Y_i^{-1} = (\mathcal{R}^{-1})^+_{ki} e_{jk} (\mathcal{R}^{-1})^-_{ji}.$$

Throughout the proof of Proposition 6.16, we freely use Lemmas 6.12 and 6.13.

Proof of Proposition 6.16 (1). For $n \geq j > i \geq 1$, we have

$$\begin{aligned} Y_i e_{ij} (T^{-1})^-_{i-1,1} (T^{-1})^+_{1,i-1} Y_i^{-1} &= Y_i X_i (T^{-1})^-_{i-1,1} (T^{-1})^+_{1,i-1} D_j Y_i^{-1} \\ &= q T^+_{i,n-1} T^-_{n-1,i} X_i Y_i D_j Y_i^{-1} = q T^+_{i,n-1} T^-_{n-1,i} X_i (\mathcal{R}^{-1})^+_{ji} D_j \\ &= q T^+_{i,n-1} T^-_{n-1,i} e_{ij} + q(\tau - \tau^{-1}) T^+_{i,n-1} T^-_{n-1,i} X_i (T^{-1})^-_{j-1,i+1} T^+_{i,j-1} D_j, \end{aligned} \tag{6.27}$$

where we used that $(\mathcal{R}^{-1})^+_{ji} = 1 + (\tau - \tau^{-1})(T^{-1})^-_{j-1,i+1} T^+_{i,j-1}$. Now,

$$\begin{aligned} T^-_{n-1,i} X_i (T^{-1})^-_{j-1,i+1} T^+_{i,j-1} D_j &= T^-_{n-1,i} (T^{-1})^-_{j-1,i+1} X_i D_i (T^{-1})^+_{i,j-1} \\ &= T^-_{n-1,i} (T^{-1})^-_{j-1,i+1} (T^{-1})^+_{i,n-1} X_n D_n T^-_{n-1,j} = (T^{-1})^-_{j-2,i} X_n D_n T^-_{n-1,j}, \end{aligned} \tag{6.28}$$

since $T^-_{n-1,i}(T^{-1})^-_{j-1,i+1} = (T^{-1})^-_{j-2,i} T^-_{n-1,i}$ by using a relation from Lemma 6.9 to move successively $T^{-1}_{j-1}, \dots, T^{-1}_{i+1}$ to the left of $T^-_{n-1,i}$. Relations (6.27) and (6.28) imply the claim of Proposition 6.16 (1) for $j > i$.

Let us now establish the claim for $i = n$, $j = n-1$. We compute

$$\begin{aligned} & Y_n e_{n,n-1} (T^{-1})^-_{n-1,1} (T^{-1})^+_{1,n-1} Y_n^{-1} \\ &= q X_n Y_n T^-_{n-1,1} T^+_{1,n-2} T^2_{n-1} D_n (T^{-1})^-_{n-2,1} (T^{-1})^+_{1,n-1} Y_n^{-1} \\ &= q X_n Y_n T_{n-1} D_n T^{-1}_{n-1} Y_n^{-1} + (\tau - \tau^{-1}) X_n D_n Y_n (T^{-1})^-_{n-2,1} (T^{-1})^+_{1,n-1} Y_n^{-1}, \end{aligned} \tag{6.29}$$

where we used that $T^2_{n-1} = 1 + (\tau - \tau^{-1}) T_{n-1}$, and then this is equal to

$$\begin{aligned} & q X_n Y_n D_{n-1} T^{-2}_{n-1} Y_n^{-1} + (\tau - \tau^{-1}) (T^{-1})^-_{n-2,1} (T^{-1})^+_{1,n-2} X_n D_n Y_n T^{-1}_{n-1} Y_n^{-1} \\ &= q X_n D_{n-1} + (\tau - \tau^{-1}) (T^{-1})^-_{n-2,1} (T^{-1})^+_{1,n-2} X_n D_n Y_n Y^{-1}_{n-1} T_{n-1}, \end{aligned} \tag{6.30}$$

from which the claim for $i = n$, $j = n-1$ follows.

Then for all $1 \le j \le n-1$,

$$\begin{aligned} &Y_n e_{nj} (T^{-1})^-_{n-1,1} (T^{-1})^+_{1,n-1} Y_n^{-1} \\ &= Y_n X_n T^+_{j,n-2} D_{n-1} T^-_{n-2,j} (T^{-1})^-_{n-1,1} (T^{-1})^+_{1,n-1} Y_n^{-1} \\ &= T^+_{j,n-2} Y_n e_{n,n-1} (T^{-1})^-_{n-1,1} (T^{-1})^+_{1,n-1} Y_n^{-1} T^-_{n-2,j}, \end{aligned} \tag{6.31}$$

where we used that $T^-_{n-2,j}$ commutes with $(T^{-1})^-_{n-1,1}(T^{-1})^+_{1,n-1}$ as a consequence of Corollary 6.10 (i), (iv). Now we use the form (6.30) for the expression (6.29) to rearrange expression (6.31) as

$$q e_{nj} + (\tau - \tau^{-1}) (T^{-1})^-_{j-1,1} (T^{-1})^+_{1,n-2} X_n D_n Y_n Y_{n-1}^{-1} T^-_{n-1,j},$$

which completes the proof of the claim for $i = n$.

For $n \ge i > j \ge 1$, we have

$$(T^{-1})^-_{n-1,i} Y_i e_{ij} (T^{-1})^-_{i-1,1} (T^{-1})^+_{1,i-1} Y_i^{-1} T^+_{i,n-1} = Y_n e_{nj} (T^{-1})^-_{n-1,1} (T^{-1})^+_{1,n-1} Y_n^{-1}. \tag{6.32}$$

The proof is completed by combining equality (6.32) with the claim for $i = n$, and using that $e_{nj}(T^{-1})^-_{n-1,i} = T^-_{n-1,i} e_{ij}$. ■

Proof of Proposition 6.16 (2). For $n \ge j > i \ge 1$, we have

$$\begin{aligned} &Y_i T^-_{i-1,1} T^+_{1,i-1} e_{ji} Y_i^{-1} = Y_i X_j T^-_{i-1,1} T^+_{1,i-1} D_i Y_i^{-1} = X_j (\mathcal{R}^{-1})^-_{ji} Y_i T^-_{i-1,1} T^+_{1,i-1} D_i Y_i^{-1} \\ &= q^{-1} X_j (\mathcal{R}^{-1})^-_{ji} D_i (T^{-1})^+_{i,n-1} (T^{-1})^-_{n-1,i} = q^{-1} e_{ji} (T^{-1})^+_{i,n-1} (T^{-1})^-_{n-1,i} \\ &+ q^{-1} (\tau^{-1} - \tau) X_j (T^{-1})^-_{j-1,i} T^+_{i+1,j-1} D_i (T^{-1})^+_{i,n-1} (T^{-1})^-_{n-1,i}, \end{aligned} \tag{6.33}$$

where we used that $(\mathcal{R}^{-1})^-_{ji} = 1 + (\tau^{-1} - \tau)(T^{-1})^-_{j-1,i} T^+_{i+1,j-1}$. We also have

$$\begin{aligned} &X_j (T^{-1})^-_{j-1,i} T^+_{i+1,j-1} D_i (T^{-1})^+_{i,n-1} = T^-_{j-1,i} T^+_{i+1,j-1} X_i D_i (T^{-1})^+_{i,n-1} \\ &= T^-_{j-1,i} T^+_{i+1,j-1} (T^{-1})^+_{i,n-1} X_n D_n = (T^{-1})^+_{j,n-1} X_n D_n T^+_{i,j-2}, \end{aligned} \tag{6.34}$$

since $T^+_{i+1,j-1}(T^{-1})^+_{i,n-1} = (T^{-1})^+_{i,n-1} T^+_{i,j-2}$ by using the inverse of a relation in Lemma 6.9 to move successively $T_{j-1}, \ldots, T_{i+1}$ to the right of $(T^{-1})^+_{i,n-1}$. Relations (6.33) and (6.34) imply the claim of Proposition 6.16 (2) for $j > i$.

Let us now establish the claim for $i = n$, $j = n-1$. We compute

$$\begin{aligned} &Y_n T^-_{n-1,1} T^+_{1,n-1} e_{n-1,n} Y_n^{-1} = Y_n T^2_{n-1} X_{n-1} T^-_{n-1,1} T^+_{1,n-2} T^{-1}_{n-1} D_n Y_n^{-1} \\ &= X_{n-1} Y_n T^-_{n-1,1} T^+_{1,n-2} T^{-1}_{n-1} D_n Y_n^{-1} = X_{n-1} Y_n T^-_{n-1,1} T^+_{1,n-1} D_n Y_n^{-1} \\ &+ (\tau^{-1} - \tau) X_{n-1} Y_n T^-_{n-1,1} T^+_{1,n-2} D_n Y_n^{-1} \end{aligned} \tag{6.35}$$

where we used that $T_{n-1}^{-1} = T_{n-1} + \tau^{-1} - \tau$, and then this is equal to

$$\begin{aligned} &q^{-1} X_{n-1} D_n + (\tau^{-1} - \tau) X_{n-1} T_{n-1}^{-1} Y_{n-1} D_n Y_n^{-1} T_{n-2,1}^{-} T_{1,n-2}^{+} \\ &= q^{-1} e_{n-1,n} + (\tau^{-1} - \tau) T_{n-1}^{-1} X_n D_n Y_{n-1} Y_n^{-1} T_{n-2,1}^{-} T_{1,n-2}^{+}, \end{aligned} \tag{6.36}$$

from which the claim for $i = n$, $j = n-1$ follows.

Then for all $1 \leq j \leq n-1$,

$$\begin{aligned} &Y_n T_{n-1,1}^{-} T_{1,n-1}^{+} e_{jn} Y_n^{-1} \\ &\quad = Y_n T_{n-1,1}^{-} T_{1,n-1}^{+} (T^{-1})_{j,n-2}^{+} X_{n-1} (T^{-1})_{n-2,j}^{-} D_n Y_n^{-1} \\ &\quad = (T^{-1})_{j,n-2}^{+} Y_n T_{n-1,1}^{-} T_{1,n-1}^{+} e_{n-1,n} Y_n^{-1} (T^{-1})_{n-2,j}^{-} \end{aligned} \tag{6.37}$$

since $(T^{-1})_{j,n-2}^{+}$ commutes with $T_{n-1,1}^{-} T_{1,n-1}^{+}$ as a consequence of Corollary 6.10 (i) and (iv). Now we use the form (6.36) for the left-hand side of equality (6.35) to rearrange expression (6.37) as

$$q^{-1} e_{jn} + (\tau^{-1} - \tau)(T^{-1})_{j,n-1}^{+} X_n D_n Y_{n-1} Y_n^{-1} T_{n-2,1}^{-} T_{1,j-1}^{+},$$

which completes the proof of the claim for $i = n$.

For $n \geq i > j \geq 1$, we have

$$(T^{-1})_{n-1,i}^{-} Y_i T_{i-1,1}^{-} T_{1,i-1}^{+} e_{ji} Y_i^{-1} T_{i,n-1}^{+} = Y_n T_{n-1,1}^{-} T_{1,n-1}^{+} e_{jn} Y_n^{-1}. \tag{6.38}$$

The proof is completed by combining equality (6.38) with the claim for $i = n$, and using that $T_{i,n-1}^{+} e_{jn} = e_{ji} (T^{-1})_{i,n-1}^{+}$. ■

Proof of Proposition 6.16 (3). For $j > i$, $k > i$, $j \neq k$, we have

$$Y_i e_{jk} Y_i^{-1} = X_j (\mathcal{R}^{-1})_{ji}^{-} Y_i D_k Y_i^{-1} = X_j (\mathcal{R}^{-1})_{ji}^{-} (\mathcal{R}^{-1})_{ki}^{+} D_k. \tag{6.39}$$

By Lemma 6.11 applied to $(\mathcal{R}^{-1})_{ji}^{-}$ and $(\mathcal{R}^{-1})_{ki}^{+}$, if $k > j$ then the right-hand side of equality (6.39) equals

$$\begin{aligned} &X_j T_{i,j-2}^{+} T_{j-1}^{-1} T_{j,k-2}^{+} T_{k-1}^{2} (T^{-1})_{k-2,i}^{-} D_k \\ &\quad = T_{i,k-2}^{+} T_{k-1}^{2} (T^{-1})_{k-2,j}^{-} X_{j-1} (T^{-1})_{j-1,i}^{-} D_k \\ &\quad = T_{i,k-2}^{+} T_{k-1}^{2} (T^{-1})_{k-2,j-1}^{-} X_j T_{j-1}^{-2} (T^{-1})_{j-2,i}^{-} D_k \\ &\quad = T_{i,k-2}^{+} T_{k-1}^{2} (T^{-1})_{k-2,j-1}^{-} e_{jk} T_{j-1}^{-2} (T^{-1})_{j-2,i}^{-} \\ &\quad = T_{i,k-2}^{+} T_{k-1}^{2} (T^{-1})_{k-2,i}^{-} e_{jk} T_{i,j-2}^{+} T_{j-1}^{-2} (T^{-1})_{j-2,i}^{-} = (\mathcal{R}^{-1})_{ki}^{+} e_{jk} (\mathcal{R}^{-1})_{ji}^{-}, \end{aligned}$$

as required. Similarly if $j > k$, the right-hand side of equality (6.39) equals

$$\begin{aligned} & X_j T^+_{i,j-2} T^{-2}_{j-1} (T^{-1})^-_{j-2,k} T_{k-1} (T^{-1})^-_{k-2,i} D_k \\ & \quad = X_j T^+_{i,k-1} D_{k-1} T^+_{k,j-2} T^{-2}_{j-1} (T^{-1})^-_{j-2,i} \\ & \quad = X_j T^+_{i,k-2} T^2_{k-1} D_k T^+_{k-1,j-2} T^{-2}_{j-1} (T^{-1})^-_{j-2,i} \\ & \quad = T^+_{i,k-2} T^2_{k-1} e_{jk} T^+_{k-1,j-2} T^{-2}_{j-1} (T^{-1})^-_{j-2,i} \\ & \quad = T^+_{i,k-2} T^2_{k-1} (T^{-1})^-_{k-2,i} e_{jk} T^+_{i,j-2} T^{-2}_{j-1} (T^{-1})^-_{j-2,i} = (\mathcal{R}^{-1})^+_{ki} e_{jk} (\mathcal{R}^{-1})^-_{ji}, \end{aligned}$$

as required. Next, for $i > j$, $i > k$, $j \neq k$, we have

$$Y_i \mathcal{R}^+_{ij} e_{jk} \mathcal{R}^-_{ik} Y_i^{-1} = X_j Y_i D_k \mathcal{R}^-_{ik} Y_i^{-1} = e_{jk} Y_i \mathcal{R}^+_{ik} \mathcal{R}^-_{ik} Y_i^{-1} = e_{jk},$$

as required. Next, for $k > i > j$, we have

$$Y_i \mathcal{R}^+_{ij} e_{jk} Y_i^{-1} = X_j Y_i D_k Y_i^{-1} = X_j (\mathcal{R}^{-1})^+_{ki} D_k = (\mathcal{R}^{-1})^+_{ki} e_{jk},$$

since $[X_j, (\mathcal{R}^{-1})^+_{ki}] = 0$, as required. Finally, for $j > i > k$, we have

$$Y_i e_{jk} \mathcal{R}^-_{ik} Y_i^{-1} = X_j (\mathcal{R}^{-1})^-_{ji} Y_i D_k \mathcal{R}^-_{ik} Y_i^{-1} = X_j (\mathcal{R}^{-1})^-_{ji} D_k = e_{jk} (\mathcal{R}^{-1})^-_{ji},$$

since $[D_k, (\mathcal{R}^{-1})^-_{ji}] = 0$, as required. This covered all the possibilities. ■

In Proposition 6.16, the relations in cases

$$(1); (2) \text{ for } i = 1; \text{ and } (3) \text{ for } j > i \tag{6.40}$$

have the elements Y_i in the left-hand side placed immediately before the corresponding elements e_{jk}. On the other hand, the relations in cases (2) when $i \neq 1$, and (3) for $i > j$ have Hecke algebra elements T_l in between the Y_i and e_{jk} in the left-hand side.

In order to be able to move an arbitrary Y_i to the right through an arbitrary directly adjacent e_{jk}, we thus also need the following Lemma 6.17. Whenever we encounter Y_i directly adjacent to some e_{jk} with their indices not falling into one of the cases (6.40), we can expand such a monomial into a sum of terms each of which can be handled, in the sense of moving Y's to the right. The case $Y_i e_{jk}$ with $i > j$ and $j \neq k \neq i$ can be dealt with by Lemma 6.17 (ii) and Proposition 6.16 (1), (3). The case $Y_i e_{ji}$ with $j > i \geq 2$ can be dealt with by Lemma 6.17 (i) and Proposition 6.16 (2), (3). The final case to consider is $Y_i e_{ji}$ with $i > j$. The first step is to apply Lemma 6.17 (i). Then by applying Proposition 6.16 (2), we are left to consider terms $Y_k e_{ji}$ with $k < i$. For the terms with $k < j$, we apply Proposition 6.16 (3); for the term with $k = j$, we apply Proposition 6.16 (1), and we deal with the terms with $k > j$ by applying Lemma 6.17 (ii) and Proposition 6.16 (1), (3).

Lemma 6.17. *(i) For $n \geq i \geq 1$, we have*

$$Y_i = Y_i T^-_{i-1,1} T^+_{1,i-1} + (\tau^{-1} - \tau) \sum_{k=1}^{i-1} (T^{-1})^-_{i-1,k} T^+_{k+1,i-1} Y_k.$$

(ii) For $n \geq i > j \geq 1$, we have

$$Y_i = Y_i \mathcal{R}^+_{ij} + (\tau^{-1} - \tau)(T^{-1})^-_{i-1,j}(T^{-1})^+_{j+1,i-1} Y_j.$$

Proof. (i) The claim is trivial for $i = 1$, so suppose $i > 1$. By using $T_1^2 = 1 + (\tau - \tau^{-1})T_1$, we get

$$\begin{aligned} Y_i T^-_{i-1,1} T^+_{1,i-1} &= Y_i T^-_{i-1,2} T^+_{2,i-1} + (\tau - \tau^{-1}) Y_i T^-_{i-1,1} T^+_{2,i-1} \\ &= Y_i T^-_{i-1,2} T^+_{2,i-1} + (\tau - \tau^{-1})(T^{-1})^-_{i-1,1} T^+_{2,i-1} Y_1. \end{aligned} \tag{6.41}$$

If $i = 2$, the desired relation follows by rearranging equality (6.41) for $Y_i T^-_{i-1,2} T^+_{2,i-1} = Y_i$. Assume $i > 2$, then we iterate $i - 2$ times the manipulation in (6.41). Thus, at the next step, we use $T_2^2 = 1 + (\tau - \tau^{-1})T_2$ to get that $Y_i T^-_{i-1,1} T^+_{1,i-1}$ equals

$$\begin{aligned} &Y_i T^-_{i-1,3} T^+_{3,i-1} + (\tau - \tau^{-1}) Y_i T^-_{i-1,2} T^+_{3,i-1} + (\tau - \tau^{-1})(T^{-1})^-_{i-1,1} T^+_{2,i-1} Y_1 \\ &= Y_i T^-_{i-1,3} T^+_{3,i-1} + (\tau - \tau^{-1}) \sum_{k=1}^{2} (T^{-1})^-_{i-1,k} T^+_{k+1,i-1} Y_k, \end{aligned}$$

and so forth until we obtain the desired relation.

(ii) By using $T_j^2 = 1 + (\tau - \tau^{-1})T_j$, we get

$$\begin{aligned} Y_i \mathcal{R}^+_{ij} &= Y_i + (\tau - \tau^{-1}) Y_i T^-_{i-1,j} (T^{-1})^+_{j+1,i-1} \\ &= Y_i + (\tau - \tau^{-1})(T^{-1})^-_{i-1,j} (T^{-1})^+_{j+1,i-1} Y_j, \end{aligned}$$

and the desired relation follows. ■

In order to be able to move Y_i^{-1} to the right past e_{jk}, we use the relations from Proposition 6.16 multiplied by Y_i^{-1} from the left and rearranged to find an expression for the first term in the right-hand side. We also need the next lemma that serves an analogous purpose as Lemma 6.17 (proved similarly, too) to deal with the cases where we end up with Hecke algebra elements T_l in between the Y_i^{-1} and e_{jk}.

Lemma 6.18. *(i) For $n \geq i \geq 1$, we have*

$$Y_i^{-1} = Y_i^{-1} T_{i,n-1}^{+} T_{n-1,i}^{-} + (\tau^{-1} - \tau) \sum_{k=0}^{n-i-1} (T^{-1})_{i,n-k-1}^{+} T_{n-k-2,i}^{-} Y_{n-k}^{-1}.$$

(ii) For $n \geq k > i \geq 1$, we have

$$Y_i^{-1} = Y_i^{-1} (\mathcal{R}^{-1})_{ki}^{+} + (\tau^{-1} - \tau)(T^{-1})_{k-1,i+1}^{-} (T^{-1})_{i,k-1}^{+} Y_k^{-1}.$$

In view of the relations presented above, we arrive at a $\mathbb{C}_\tau$-basis for the algebra $\mathbb{H}^{\mathfrak{gl}_n}$. It may be thought of as a PBW-type basis. At $\tau = 1$, it reduces to the one given by formula (6.9) for the algebra $\mathcal{A}$. The following theorem also implies that $\mathbb{H}^{\mathfrak{gl}_n}$ is a flat τ-deformation of the algebra $\mathcal{A}$ from Section 6.2.

Theorem 6.19. *The algebra $\mathbb{H}^{\mathfrak{gl}_n}$ has a free basis over $\mathbb{C}_\tau$ consisting of the monomials*

$$T_w e_{i_1 j_1}^{k_1} \cdots e_{i_t j_t}^{k_t} \prod_{l=1}^{n} Y_l^{m_l}, \tag{6.42}$$

where $w \in \mathfrak{S}_n$, $t \in \mathbb{Z}_{\geq 0}$, $k_s \in \mathbb{Z}_{>0}$, $m_l \in \mathbb{Z}$, $1 \leq i_1 \leq \cdots \leq i_t \leq n$, $1 \leq j_1 \leq \cdots \leq j_t \leq n$ with $i_s = i_{s+1} \Rightarrow j_s < j_{s+1}$, and none of the indices i_r equal any of the indices j_s. Here T_w ($w \in \mathfrak{S}_n$) is the standard basis of the Hecke algebra of type A_{n-1}.

The algebra $\mathbb{H}^{\mathfrak{gl}_n}$ has a presentation by generators T_k ($1 \leq k \leq n-1$), $Y_i^{\pm 1}$, e_{ij} ($1 \leq i \neq j \leq n$) and relations (2.4), (2.5), (2.7), *Laurent relations for $Y_i^{\pm 1}$, and relations from Propositions 6.14–6.16. Further, $\mathbb{H}^{\mathfrak{gl}_n}/(\tau - 1)\mathbb{H}^{\mathfrak{gl}_n} \cong \mathcal{A}$.*

Proof. Consider any monomial in the generators T_k, $Y_i^{\pm 1}$, e_{jl} ($j \neq l$). In step one, we apply the relations from Proposition 6.16 and Lemmas 6.17 and 6.18 to move $Y_i^{\pm 1}$ to the right past any directly adjacent e_{jl}. All of those relations are linear in e's, so this does not increase the number of e's in any single term. In step two, we move all T_k's completely to the left end of each term by using Proposition 6.15 and relations (2.7). This does not increase in any single term the number of e's, and in those terms where the number of e's stayed the same, this did not increase the number of Y's. We then repeat steps one and two until all Y's are to the right of any e's and all T's are at the left end of each monomial term. We achieve this in finitely many steps. Since T_w ($w \in \mathfrak{S}_n$) form a basis of the Hecke algebra of type A_{n-1} (see Theorem 2.10), we can now assume that all monomial terms take the form T_w(product of e's) $\prod_{l=1}^{n} Y_l^{m_l}$.

Next, we use the relations from Proposition 6.14 for the product of e's to order them in accordance with the conditions on the indices as in (6.42). We can handle by induction the terms where the number of e's has decreased after an application of a relation from Proposition 6.14, so we care only about the quadratic terms in e in those relations, and these do not introduce any new Y's nor T's. This proves that the monomials (6.42) span the algebra $\mathbb{H}^{\mathfrak{gl}_n}$.

It remains to show that these monomials are linearly independent over $\mathbb{C}_\tau$. It suffices to show that for $\tau = 1$ they are linearly independent over $\mathbb{C}$. This holds because at $\tau = 1$ they coincide with the PBW basis for the algebra $\mathcal{A}$ from Section 6.2 (see formula (6.9)).

It is straightforward to see that under the correspondence $T_k \leftrightarrow s_k$, $Y_i^{\pm 1} \leftrightarrow t_i^{\pm 1}$, $e_{ij} \leftrightarrow E_{ij}^q$, the defining relations of $\mathbb{H}^{\mathfrak{gl}_n}$ are just deformations of those of the algebra $\mathcal{A}$ (relations from Proposition 6.5 together with relations (6.8) and the Coxeter relations), and reduce to them when $\tau = 1$. The last part of the statement follows. ■

Remark 6.20. One may also consider a non-formal version of the algebra $\mathbb{H}^{\mathfrak{gl}_n}$ where $\tau = \lambda \in \mathbb{C}^\times$, that is, we consider the $\mathbb{C}$-algebra

$$\mathbb{H}^{\mathfrak{gl}_n}/(\tau - \lambda)\mathbb{H}^{\mathfrak{gl}_n} \cong \mathbb{H}^{\mathfrak{gl}_n} \otimes_{\mathbb{C}_\tau} \mathbb{C},$$

where we use the ring homomorphism $\varphi\colon \mathbb{C}_\tau \to \mathbb{C}$ given by $\varphi(\tau) = \lambda$. Then, it follows from Theorem 6.19 that (the coset representatives of) the elements (6.42) give a $\mathbb{C}$-basis of this algebra.

6.3.3 Centre and a double centraliser property

In this section, we consider the DAHA $\mathbb{H}_n$ and its subalgebra $\mathbb{H}^{\mathfrak{gl}_n}$ defined in an analogous way as in Sections 2.4 and 6.3, respectively, but over the field $\mathbb{C}(\tau)$ of rational functions in the variable τ instead of $\mathbb{C}_\tau$. An analogous proof shows that this version of $\mathbb{H}^{\mathfrak{gl}_n}$ satisfies the direct analogue of Theorem 6.19. We now study some further properties of this algebra.

Proposition 6.21. *The element $\widetilde{Y} = \prod_{i=1}^n Y_i$ belongs to the centre $\mathcal{Z}(\mathbb{H}^{\mathfrak{gl}_n})$.*

Proof. From the defining relations of $\mathbb{H}_n$, it follows that $\widetilde{Y}$ commutes with all $Y_i^{\pm 1}$ and T_k. Also, $\widetilde{Y} X_i = q X_i \widetilde{Y}$, hence $[\widetilde{Y}, X_i X_j^{-1}] = 0$, and it follows that $\widetilde{Y}$ commutes with all

$$e_{ij} = (q - q^{-1})^{-1} X_i X_j^{-1} (T^{-1})^+_{j,n-1} (Y_n - Y_n^{-1}) T^-_{n-1,j}.$$

Thus, $\widetilde{Y}$ commutes with all the generators of $\mathbb{H}^{\mathfrak{gl}_n}$, as required. ■

Remark 6.22. We have

$$\lim_{q \to 1} \frac{1 - \widetilde{Y}}{1 - q} = \sum_{i=1}^n x_i y_i - c \sum_{i<j} s_{ij} = eu,$$

which is the generator of the centre $\mathcal{Z}(\mathcal{H}^{\mathfrak{gl}_n})$ from Section 6.1.

Take any element $0 \neq f \in \mathcal{Z}(\mathbb{H}^{\mathfrak{gl}_n})$. We can expand f in the $\mathbb{C}(\tau)$-basis of monomials from Theorem 6.19. That is, there exist finitely many monomials $M_1, M_2, \dots, M_N$ of the form (6.42), and some $\lambda_1(\tau), \dots, \lambda_N(\tau) \in \mathbb{C}(\tau) \setminus \{0\}$ such that

$$f = \lambda_1(\tau) M_1 + \cdots + \lambda_N(\tau) M_N.$$

We can assume that λ_i do not have a pole at $\tau = 1$ and that $\lambda_i(1) \neq 0$ for some i. We have

$$\lambda_1(1) M_1^{\tau=1} + \cdots + \lambda_N(1) M_N^{\tau=1} \in \mathcal{Z}(\mathcal{A}) = \langle \prod_{i=1}^n t_i, \ \prod_{i=1}^n t_i^{-1} \rangle,$$

where we used that the centre of the algebra $\mathcal{A}$ was given in Proposition 6.7. Here $M_i^{\tau=1}$ are the monomials M_i with Y_l replaced by t_l, T_w by w, and e_{jl} by E_{jl}^q.

Thanks to the PBW basis of monomials (6.9) for the algebra $\mathcal{A}$, we can conclude that in the expansion of f, the monomials M_i for which $\lambda_i(1) \neq 0$ must have the form $\widetilde{Y}^m$ ($m \in \mathbb{Z}$). By subtracting those terms from f and repeating the above argument, we arrive at the following theorem.

Theorem 6.23. *The centre $\mathcal{Z}(\mathbb{H}^{\mathfrak{gl}_n})$ is generated by $\widetilde{Y}^{\pm 1}$.*

Let us now consider the subalgebra $\mathfrak{A}$ of $\mathbb{H}_n$ generated by

$$\mathbb{C}[X_1, \dots, X_n], \qquad \mathbb{C}[D_1, \dots, D_n], \qquad \mathbb{C}[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}],$$

and $T_1, \dots, T_{n-1}$. We note that $\mathbb{H}^{\mathfrak{gl}_n} \subset \mathfrak{A}$. In the limit $\tau = 1$, those generators of $\mathfrak{A}$ that are not T_k reduce to the generators of the q-Weyl algebra considered by Hayashi in [61]. The algebra $\mathfrak{A}$ can be thought of also as a q-analogue of the RCA $\mathcal{H}_n$. Indeed, in the trigonometric limit $q \to 1$ given by (2.16), the algebra $\mathfrak{A}$ reduces to $\mathcal{H}_n \subset \mathbb{H}_n^{\mathrm{trig}}$.

Remark 6.24. By [10, Section 3.7], the cyclotomic DAHA $HH_{n,t}^2(Z, q^{-1})$ for $Z = (1, -1)$ is the subalgebra of the DAHA $HH_{n,t}(q^{-1})$ generated by T_k ($1 \leq k \leq n-1$), X_i, $Y_i^{\pm 1}$, and D_i^{BEF} ($1 \leq i \leq n$) given by (6.11). By using the isomorphism $h \circ g \circ \varphi \colon HH_{n,t}(q^{-1}) \to \mathbb{H}_n$ from Remark 6.8, we get that this subalgebra $HH_{n,t}^2(Z, q^{-1}) \subset HH_{n,t}(q^{-1})$ is isomorphic to the subalgebra of $\mathbb{H}_n$ generated by T_k, X_i, $Y_i^{\pm 1}$, and

$$D_i Y_i^{-1} T_{i,n-1}^{+} T_{n-1,i}^{-}$$

(see equality (6.12)), which coincides with the algebra $\mathfrak{A}$. That is, $\mathfrak{A} \cong HH_{n,t}^2((1, -1), q^{-1})$.

We need the following basis of the algebra $\mathfrak{A}$. Another basis of $\mathfrak{A} \cong HH_{n,t}^2((1, -1), q^{-1})$ was considered in [10] (see paragraph above Proposition 3.32 therein).

Proposition 6.25. *The algebra $\mathfrak{A}$ has a $\mathbb{C}(\tau)$-basis consisting of the monomials*

$$T_w M_X M_D M_Y, \tag{6.43}$$

where $w \in \mathfrak{S}_n$, M_X is a monomial in X_i, M_Y a monomial in $Y_i^{\pm 1}$, and M_D a monomial in D_i such that for all i, M_X does not contain X_i or M_D does not contain D_i.

Proof. Consider any monomial M in the generators T_k, X_i, D_i, $Y_i^{\pm 1}$. Firstly, we will show that we can write M as a linear combination of terms of the form (6.43). We will proceed recursively based on the total power of X's that appear in M. We apply the following procedure to M.

In step one, we use relations from Lemma 6.13 (1) and (3), and Lemmas 6.17 and 6.18 to move $Y_i^{\pm 1}$ to the right past any directly adjacent X_j or D_j. In step two, we move all T_k's completely to the left end of each term by using Lemma 6.12 and relations (2.6) and (2.7). We then repeat steps one and two until in each monomial term all Y's are to the right of any X's and D's, and all T's are at each term's left end. We achieve this in finitely many steps. At the end, all the monomial terms that were produced have the same total power of X's (and of D's) as the original monomial M, and we can assume each of the terms to be of the form T_w(product of X's and D's)M_Y for some (not necessarily the same) T_w and M_Y.

We now apply Lemma 6.13 (4). It gives that $S_{ij}^{\tau} = [D_i, X_j]$ can be expressed in terms of Y and T variables, hence we can commute D with X up to a term with a lower total power of X's (which we can handle by recursion). Furthermore, whenever we encounter $X_i D_i$, we can replace it with an expression containing Y and T only. It follows that the monomials (6.43) span the algebra $\mathfrak{A}$.

It remains to show that these monomials are linearly independent over $\mathbb{C}(\tau)$. It suffices to show that for $\tau = 1$ they are linearly independent over $\mathbb{C}$. We will work with the faithful polynomial representation of the DAHA. Recall that at $\tau = 1$, the elements D_i act as the operators d_i from Section 6.2, while T_k and $Y_i^{\pm 1}$ act as s_k and $t_i^{\pm 1}$, respectively. Therefore, for any $a_i, b_i \in \mathbb{Z}_{\geq 0}$ and $c_i \in \mathbb{Z}$, we have

$$T_w \prod_{i=1}^{n} X_i^{a_i} \prod_{i=1}^{n} D_i^{b_i} \prod_{i=1}^{n} Y_i^{c_i} \Big|_{\tau=1} \propto w \prod_{i=1}^{n} X_i^{a_i - b_i} \prod_{i=1}^{n} t_i^{b_i + c_i} + \ldots, \tag{6.44}$$

where $\ldots$ denotes terms in which the overall sum of the exponents on the t_i's is lower than in the above leading term, and $\propto$ denotes proportionality by a non-zero factor, which may depend on q.

Assume a non-trivial linear dependence of some monomials (6.43) at $\tau = 1$. This implies a non-trivial linear dependence of their corresponding leading (with highest degree in t_i) terms, whose form is shown in the right-hand side of (6.44). By the assumptions on the monomials (6.43), either $a_i = 0$ or $b_i = 0$, hence different monomials (6.43) lead to distinct leading terms. However, operators $w\prod_{i=1}^n X_i^{n_i}\prod_{i=1}^n t_i^{n_i'}$ for different $n_i, n_i' \in \mathbb{Z}$ and $w \in \mathfrak{S}_n$ are linearly independent as operators on $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. We arrived at a contradiction, which completes the proof. ■

Lemma 6.26. *Any monomial* (6.43) *with* $\deg M_X = \deg M_D$ *belongs to* $\mathbb{H}^{\mathfrak{gl}_n}$.

Proof. We permute the elements X_i and D_j in such a monomial so as to pair them up into a product of elements e_{ij}. This can be done up to a combination of terms of lower degree in X with equal degree in D, since $S_{ij}^\tau = [D_i, X_j]$ can be expressed in terms of Y and T by Lemma 6.13. By re-expressing these lower degree terms via the basis (6.43), the statement follows inductively by degree in X. ■

Remark 6.24 enables us to prove the following proposition.

Proposition 6.27. *The algebra* $\mathfrak{A}$ *has an irreducible representation* ψ *on the space of polynomials* $\mathbb{C}(\tau)[X_1, \dots, X_n]$ *given by*

$$\begin{aligned}
&\psi(T_k) = \tau s_k + \frac{(\tau - \tau^{-1})X_{k+1}}{X_k - X_{k+1}}(s_k - 1),\\
&\psi(X_i) = X_i, \quad \psi(\tau) = \tau,\\
&\psi(Y_i) = \tau^{n-1}\psi(T_{i,n-1}^+)\pi^{-1}\psi((T^{-1})_{1,i-1}^+),\\
&\psi(D_i) = (q - q^{-1})^{-1}X_i^{-1}\psi\left((T^{-1})_{i,n-1}^+(Y_n - Y_n^{-1})T_{n-1,i}^-\right),
\end{aligned} \tag{6.45}$$

where $\pi^{-1} = (n, \dots, 2, 1)t_1 = t_n(n, \dots, 2, 1)$ *(see formula* (2.11)*).*

Proof. By Remark 6.24, the algebra $\mathfrak{A}$ is isomorphic via $\varphi^{-1}\circ g^{-1}\circ h$ to $HH_{n,t}^2((1,-1), q^{-1})$. The latter has by [10, Proposition 3.6] an action on $\mathbb{C}(\mathbf{t})[X_1, \dots, X_n]$ via $\rho_{\mathrm{BEF}} \equiv \rho$ defined in [10, Proposition 3.3] (with q replaced by q^{-1}). Let $\psi' = \rho_{\mathrm{BEF}}\circ\varphi^{-1}\circ g^{-1}\circ h$, which then gives an action of $\mathfrak{A}$ on $\mathbb{C}(\mathbf{t})[X_1, \dots, X_n]$. We have

$$\begin{aligned}
&\psi'(T_k) = \rho_{\mathrm{BEF}}(T_{n-k}^{-1}) = \mathbf{t}^{-1}s_{n-k} + \frac{(\mathbf{t}^{-1} - \mathbf{t})X_{n-k}}{X_{n-k+1} - X_{n-k}}(s_{n-k} - 1),\\
&\psi'(X_i) = X_{n-i+1}, \quad \psi'(\tau) = \mathbf{t}^{-1},\\
&\psi'(Y_i) = \rho_{\mathrm{BEF}}(Y_{n-i+1}^{-1}) = \mathbf{t}^{1-n}\rho_{\mathrm{BEF}}((T^{-1})_{n-i,1}^-)(1, \dots, n)t_n\rho_{\mathrm{BEF}}(T_{n-1,n-i+1}^-),\\
&\psi'(D_i) = (q - q^{-1})^{-1}X_{n-i+1}^{-1}\psi'\left((T^{-1})_{i,n-1}^+(Y_n - Y_n^{-1})T_{n-1,i}^-\right).
\end{aligned}$$

The representation (6.45) of $\mathfrak{A}$ is obtained from the module $\mathbb{C}(\mathbf{t})[X_1, \dots, X_n]$ by relabelling $\mathbf{t}$ to τ^{-1} and X_i to X_{n-i+1}.

The proof of irreducibility is similar to that of [61, Proposition 2.1]. Let V be a non-trivial submodule, and choose in it a non-zero element $v = \sum_{\mathbf{m}} a_{\mathbf{m}}X_1^{m_1}\cdots X_n^{m_n}$, where $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$ and $a_{\mathbf{m}} \in \mathbb{C}(\tau)$. We can assume that those $a_{\mathbf{m}}$ with maximal

$\sum_{i=1}^n m_i$ among $\{\mathbf{m} \in \mathbb{Z}_{\geq 0}^n : a_{\mathbf{m}} \neq 0\}$ do not have a pole at $\tau = 1$, and that at least one of them is non-zero at $\tau = 1$, say for $\mathbf{m}' = (m_1', \ldots, m_n')$. Since the action of $\psi(D_i)$ reduces the degree of a polynomial, we get that $\psi(D_1^{m_1'} \cdots D_n^{m_n'})v \in \mathbb{C}(\tau)$ and is well-defined at $\tau = 1$. Moreover, it must be a non-zero element of $\mathbb{C}(\tau)$ because at $\tau = 1$ it equals

$$d_1^{m_1'} \cdots d_n^{m_n'}(v) = a_{\mathbf{m}'}(1)[m_1']!_q \cdots [m_n']!_q,$$

which belongs to $\mathbb{C}^\times$ as q is not a root of unity. Here we use for $m \in \mathbb{Z}_{\geq 0}$ the notation

$$[m]!_q = [m]_q[m-1]_q \cdots [2]_q[1]_q, \qquad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}},$$

and the operators d_i are given by formula (6.3). It follows that $1 \in V$, and by acting on 1 by combinations of $\psi(X_i)$, we get that $V = \mathbb{C}(\tau)[X_1, \ldots, X_n]$. ∎

Corollary 6.28. *The subalgebra $\mathbb{H}^{\mathfrak{gl}_n} \subset \mathfrak{A}$ acts on $\mathbb{C}(\tau)[X_1, \ldots, X_n]$. Moreover, this action preserves for all $k \in \mathbb{Z}_{\geq 0}$ the subspace $\mathbb{C}(\tau)[X_1, \ldots, X_n]^{(k)}$ of homogeneous polynomials of degree k, and this is an irreducible $\mathbb{H}^{\mathfrak{gl}_n}$-module.*

Proof. Irreducibility is proved similarly to the proof of Proposition 6.27. Using the same notation, this time we have $\sum_{i=1}^n m_i' = k$. An arbitrary monomial $X_1^{a_1} \cdots X_n^{a_n} \in \mathbb{C}(\tau)[X_1, \ldots, X_n]^{(k)}$ can be obtained as

$$\psi(cX_1^{a_1} \cdots X_n^{a_n} D_1^{m_1'} \cdots D_n^{m_n'})v$$

for suitable $c \in \mathbb{C}(\tau) \setminus \{0\}$, where $X_1^{a_1} \cdots X_n^{a_n} D_1^{m_1'} \cdots D_n^{m_n'} \in \mathbb{H}^{\mathfrak{gl}_n}$ by Lemma 6.26 since $\sum_{i=1}^n a_i = k = \sum_{i=1}^n m_i'$. ∎

The preceding corollary generalises the fact that the polynomial representation of the algebra $\mathcal{H}^{\mathfrak{gl}_n}$ preserves the space $\mathbb{C}[X_1, \ldots, X_n]^{(k)}$, which is an irreducible module for it, and that this space is also preserved by the algebra $\mathcal{A}$ from Section 6.2 (for $\tau = 1$, cf. also [61, Theorem 4.1(A)]).

Remark 6.29. The assignments (6.45) almost coincide with those of the polynomial representation of the DAHA $\mathbb{H}_n$ given in Section 2.4 above, except that the image of Y_i in (6.45) has an extra factor of τ^{n-1} (the action of Y_i in the polynomial representation can be deduced from relations (2.8)). A way to think about this is that the operators from the polynomial representation on $\mathbb{C}(\tau)[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ in Section 2.4 formally preserve also the space

$$\left(\prod_{i=1}^n X_i\right)^{\log_q \tau^{n-1}} \mathbb{C}(\tau)[X_1^{\pm 1}, \ldots, X_n^{\pm 1}],$$

which induces another action of $\mathbb{H}_n$ on $\mathbb{C}(\tau)[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ under which the subalgebra $\mathfrak{A}$ preserves the subspace $\mathbb{C}(\tau)[X_1, \ldots, X_n]$ and acts as given in Proposition 6.27.

We are now going to show that

$$\begin{aligned} \mathbb{H}^{\mathfrak{gl}_n} &= C_{\mathfrak{A}}(\widetilde{Y}), \\ C_{\mathfrak{A}}(\mathbb{H}^{\mathfrak{gl}_n}) &= \langle \widetilde{Y}, \widetilde{Y}^{-1} \rangle, \end{aligned} \tag{6.46}$$

where $C_A(B) = \{a \in A \colon [a, b] = 0, \forall b \in B\}$ denotes the centraliser. This statement is a q-generalisation of the property that

$$\begin{aligned} \mathcal{H}^{\mathfrak{gl}_n} &= C_{\mathcal{H}_n}(eu), \\ C_{\mathcal{H}_n}(\mathcal{H}^{\mathfrak{gl}_n}) &= \langle eu \rangle. \end{aligned}$$

The first of the latter equalities follows from the fact that the RCA $\mathcal{H}_n$ has a natural grading such that its faithful polynomial representation is a graded one. The element eu acts (up to a constant) as the grading operator $\sum_{i=1}^n x_i \partial_{x_i}$, hence it only commutes with the degree zero part of $\mathcal{H}_n$, which is precisely $\mathcal{H}^{\mathfrak{gl}_n}$. The second equality follows from the fact that $\deg eu = 0$, hence $eu \in \mathcal{H}^{\mathfrak{gl}_n}$, so the previous sentence implies that $C_{\mathcal{H}_n}(\mathcal{H}^{\mathfrak{gl}_n}) = \mathcal{Z}(\mathcal{H}^{\mathfrak{gl}_n})$, which equals $\langle eu \rangle$ by [51].

The fact that eu is essentially the grading operator has a q-counterpart in the following property of $\widetilde{Y}$. Since $\widetilde{Y} = \pi^{-n}$ [35, p. 101], we get by using formula (2.11) that

$$\widetilde{Y}(X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}) = q^{\sum_{i=1}^n a_i} X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}. \tag{6.47}$$

That is, $\widetilde{Y}$ acts in the polynomial representation as a grading operator.

Let us now provide a proof of relations (6.46). From Theorem 6.23, it follows that $\mathbb{H}^{\mathfrak{gl}_n} \subseteq C_{\mathfrak{A}}(\widetilde{Y})$. We now prove the reverse inclusion. Let $f \in C_{\mathfrak{A}}(\widetilde{Y})$. Since $\widetilde{Y} D_i = q^{-1} D_i \widetilde{Y}$, we have

$$\widetilde{Y} T_w M_X M_D M_Y = q^{\deg M_X - \deg M_D} T_w M_X M_D M_Y \widetilde{Y}.$$

This implies that the expansion of f in the basis of $\mathfrak{A}$ given in Proposition 6.25 can contain only those monomials where $\deg M_X = \deg M_D$, as q is not a root of unity. Hence, $f \in \mathbb{H}^{\mathfrak{gl}_n}$ by Lemma 6.26. We have proved that $\mathbb{H}^{\mathfrak{gl}_n} = C_{\mathfrak{A}}(\widetilde{Y})$.

Suppose now that $f \in C_{\mathfrak{A}}(\mathbb{H}^{\mathfrak{gl}_n})$. Then it must, in particular, commute with $\widetilde{Y} \in \mathbb{H}^{\mathfrak{gl}_n}$. Thus, by the same argument as above, we get $f \in \mathbb{H}^{\mathfrak{gl}_n}$. Therefore, $C_{\mathfrak{A}}(\mathbb{H}^{\mathfrak{gl}_n}) = \mathcal{Z}(\mathbb{H}^{\mathfrak{gl}_n}) = \langle \widetilde{Y}, \widetilde{Y}^{-1} \rangle$ by Theorem 6.23, as required.

Thus, we have established the following theorem.

Theorem 6.30. *We have* $C_{\mathfrak{A}}(\widetilde{Y}) = \mathbb{H}^{\mathfrak{gl}_n}$ *and* $C_{\mathfrak{A}}(\mathbb{H}^{\mathfrak{gl}_n}) = \langle \widetilde{Y}, \widetilde{Y}^{-1} \rangle$.

This theorem implies that $\mathbb{H}^{\mathfrak{gl}_n}$ coincides with the degree zero part of the algebra $\mathfrak{A}$, where the grading on $\mathfrak{A}$ is inherited from the DAHA.

Related to the previous considerations, let $\mathfrak{g} = \langle \widetilde{Y}, \widetilde{Y}^{-1} \rangle \cong \mathbb{C}(\tau)[x^{\pm 1}]$. We have

$\mathfrak{g} \subset \mathbb{H}^{\mathfrak{gl}_n} \subset \mathfrak{A}$. From Corollary 6.28, it follows that $V = \mathbb{C}(\tau)[X_1, \ldots, X_n]$ is a $(\mathfrak{g}, \mathbb{H}^{\mathfrak{gl}_n})$-bimodule, which by Proposition 6.27 is an irreducible $\mathfrak{A}$-module. It admits the decomposition

$$V = \bigoplus_{k=0}^{\infty} U_k \otimes_{\mathbb{C}(\tau)} W_k,$$

where $W_k = \mathbb{C}(\tau)[X_1, \ldots, X_n]^{(k)}$, which by Corollary 6.28 is an irreducible module of $\mathbb{H}^{\mathfrak{gl}_n}$, and $U_k = \mathbb{C}(\tau)$ is the irreducible (one-dimensional) module of $\mathfrak{g}$ determined by $\widetilde{Y} \mapsto q^k$ (this is by formula (6.47) the action of $\widetilde{Y}$ on W_k). If $k \neq l \in \mathbb{Z}_{\geq 0}$, then $W_k \not\cong W_l$, because their dimensions as vector spaces differ, and $U_k \not\cong U_l$ since q is not a root of unity.

6.4 Related integrable systems

In Section 6.3.1, we considered a family of pairwise-commuting elements D_i. We now introduce certain pairwise-commuting $\mathcal{D}_i^{(l_1,l_2)}$ of a more general form depending on additional parameters $l_1, l_2 \in \mathbb{Z}_{\geq 0}$, $a_j \in \mathbb{C}$ $(j = -l_1, \ldots, l_2)$. The action of symmetric combinations of $\mathcal{D}_i^{(l_1,l_2)}$ on the space of symmetric Laurent polynomials $\mathbb{C}_\tau[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\mathfrak{S}_n}$ will lead to families of commuting q-difference operators related to operators of MR and Van Diejen types. We will assume $a_{-l_1} \neq 0 \neq a_{l_2}$. We recover $D_i = (q - q^{-1})^{-1}\mathcal{D}_i^{(1,1)}$ for $a_{-1} = -1$, $a_0 = 0$, and $a_1 = 1$.

We define

$$\mathcal{D}_n = \mathcal{D}_n^{(l_1,l_2)} = X_n^{-1} \sum_{j=-l_1}^{l_2} a_j Y_n^j,$$

and for $1 \leq i \leq n-1$, we define

$$\begin{aligned} \mathcal{D}_i = \mathcal{D}_i^{(l_1,l_2)} &= T_{i,n-1}^{+} \mathcal{D}_n T_{n-1,i}^{-} \\ &= X_i^{-1} (T^{-1})_{i,n-1}^{+} \left(\sum_{j=-l_1}^{l_2} a_j Y_n^j \right) T_{n-1,i}^{-}. \end{aligned} \tag{6.48}$$

We have $T_k^{-1} \mathcal{D}_k T_k^{-1} = \mathcal{D}_{k+1}$, and $[T_k, \mathcal{D}_i] = 0$ for $i \neq k, k+1$ by an analogous proof as for Lemma 6.12. In Proposition 6.32 below, we prove that $\mathcal{D}_i$ pairwise commute. Let us first develop some tools to be used in the proof.

Let $\mathbb{H}_n^-$ be the (unital, associative) $\mathbb{C}_\tau$-algebra generated by T_k $(1 \leq k \leq n-1)$ and $\mathbb{C}_\tau[Z_1, \ldots, Z_n]$, $\mathbb{C}_\tau[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$ subject to the following relations:

$$(T_k - \tau)(T_k + \tau^{-1}) = 0, \quad T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}, \quad [T_k, T_l] = 0 \text{ if } |k - l| > 1,$$
$$T_k^{-1} Z_k T_k^{-1} = Z_{k+1}, \qquad [T_k, Z_i] = 0 \text{ for } i \neq k, k+1, \tag{6.49}$$
$$T_k^{-1} Y_k T_k^{-1} = Y_{k+1}, \qquad [T_k, Y_i] = 0 \text{ for } i \neq k, k+1,$$
$$\widetilde{Y} Z_i = q^{-1} Z_i \widetilde{Y}, \tag{6.50}$$

$$Y_2 Z_1 = Z_1 Y_2 T_1^2, \tag{6.51}$$

where $\widetilde{Y} = \prod_{i=1}^n Y_i$.

There is an algebra homomorphism $\phi\colon \mathbb{H}_n^- \to \mathbb{H}_n$ given by

$$\phi(T_k) = T_k, \quad \phi(Z_i) = X_i^{-1}, \quad \phi(Y_i^{\pm 1}) = Y_i^{\pm 1},$$

whose image contains the elements $\mathcal{D}_i$.

The next proposition gives a family of endomorphisms of the algebra $\mathbb{H}_n^-$.

Proposition 6.31. *Let $f(z) \in \mathbb{C}[z, z^{-1}]$ be an arbitrary single-variable Laurent polynomial. There is an algebra endomorphism $\theta = \theta_f$ of $\mathbb{H}_n^-$ determined by $\theta(T_k) = T_k$, $\theta(Y_i) = Y_i$, and*

$$\theta(Z_i) = T^+_{i,n-1} Z_n f(Y_n) T^-_{n-1,i} \qquad \left(= Z_i (T^{-1})^+_{i,n-1} f(Y_n) T^-_{n-1,i}\right).$$

Proof. It suffices to check that θ preserves relations (6.49)–(6.51). Firstly,

$$\theta(T_k^{-1} Z_k T_k^{-1}) = T^+_{k+1,n-1} Z_n f(Y_n) T^-_{n-1,k+1} = \theta(Z_{k+1}),$$

as required. Suppose now that $i \neq k, k+1$. Then either $i > k+1$, in which case it is easy to see that $\theta(T_k)$ commutes with $\theta(Z_i)$. Or $i < k$, in which case using Lemma 6.9 twice and that $[T_{k-1}, Z_n] = 0 = [T_{k-1}, f(Y_n)]$, we get

$$\begin{aligned} \theta(T_k Z_i) &= T_k T^+_{i,n-1} Z_n f(Y_n) T^-_{n-1,i} = T^+_{i,n-1} T_{k-1} Z_n f(Y_n) T^-_{n-1,i} \\ &= T^+_{i,n-1} Z_n f(Y_n) T_{k-1} T^-_{n-1,i} = T^+_{i,n-1} Z_n f(Y_n) T^-_{n-1,i} T_k = \theta(Z_i T_k). \end{aligned}$$

This completes the proof that θ preserves relations (6.49).

Secondly, we have

$$\theta(\widetilde{Y} Z_i) = T^+_{i,n-1} \widetilde{Y} Z_n f(Y_n) T^-_{n-1,i} = q^{-1} T^+_{i,n-1} Z_n f(Y_n) T^-_{n-1,i} \widetilde{Y} = q^{-1} \theta(Z_i \widetilde{Y}),$$

hence θ preserves relations (6.50).

Finally, since $\theta(Z_1) = Z_1 (T^{-1})^+_{1,n-1} f(Y_n) T^-_{n-1,1}$, we see due to relation (6.51) that it will follow that $\theta(Y_2^{-1} Z_1 Y_2) = \theta(Z_1 T_1^{-2})$ if we show that

$$T_1^{-1} Y_2^{-1} (T^{-1})^+_{1,n-1} f(Y_n) T^-_{n-1,1} Y_2 = (T^{-1})^+_{2,n-1} f(Y_n) T^-_{n-1,2} T_1^{-1}.$$

The left-hand side of the latter equality can be rearranged as

$$Y_1^{-1} (T^{-1})^+_{2,n-1} f(Y_n) T^-_{n-1,1} Y_2 = (T^{-1})^+_{2,n-1} f(Y_n) T^-_{n-1,2} Y_1^{-1} T_1 Y_2$$

$$= (T^{-1})^{+}_{2,n-1} f(Y_n) T^{-}_{n-1,2} T_1^{-1},$$

as required. Thus, θ preserves the relation (6.51) as well. ■

The next proposition proves that the elements $\mathcal{D}_i \in \mathbb{H}_n$ defined by (6.48) commute.

Proposition 6.32. *We have $[\mathcal{D}_i, \mathcal{D}_j] = 0$ for all i, j (for fixed values of the parameters l_1, l_2, and a_k).*

Proof. Let $f(z) = \sum_{j=-l_1}^{l_2} a_j z^j$. The pairwise-commuting elements $Z_i \in \mathbb{H}_n^-$ satisfy $Z_i = T^{+}_{i,n-1} Z_n T^{-}_{n-1,i}$, and

$$\theta_f(Z_n) = Z_n \sum_{j=-l_1}^{l_2} a_j Y_n^j \in \mathbb{H}_n^-,$$

hence $(\phi \circ \theta_f)(Z_i) = \mathcal{D}_i$. It follows that $[\mathcal{D}_i, \mathcal{D}_j] = (\phi \circ \theta_f)([Z_i, Z_j]) = 0$. ■

Remark 6.33. Commutativity of $\mathcal{D}_i$ in the special case of $l_2 = 0$ was proved in [10, Corollary 3.22 (i)] by a different method. Indeed, the elements $D_i^{(l)}$ considered in that paper satisfy $(h \circ g \circ \varphi)(D^{(l_1)}_{n-i+1}) = a^{-1}_{-l_1} \mathcal{D}_i$ for $l_2 = 0$ and Z_i expressed in terms of a_i. Here, h, g, and φ are the isomorphisms from Remark 6.8.

Remark 6.34. The algebra $\mathbb{H}^{\mathfrak{gl}_n}$ is the subalgebra of $\mathbb{H}_n$ generated by T_k, $Y_i^{\pm 1}$, and $X_i \mathcal{D}_j$ $(i \neq j)$ for $l_1 = l_2 = 1$, and $a_{-1} = -1$, $a_0 = 0$, $a_1 = 1$, since in that case $\mathcal{D}_n = X_n^{-1}(Y_n - Y_n^{-1}) = (q - q^{-1}) D_n$. It would be interesting to see if the subalgebra of $\mathbb{H}_n$ generated by T_k, $Y_i^{\pm 1}$, and $X_i \mathcal{D}_j$ $(i \neq j)$ for more general l_1, l_2, and a_j — equivalently, the degree zero subalgebra of a general cyclotomic DAHA — has good properties as well.

Recall the polynomial representation of $\mathbb{H}_n$ on $\mathbb{C}_\tau[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, mentioned in Section 2.4, in which the element π^{-1} acts according to formula (2.11) as $(n, \ldots, 1) t_1 = t_n (n, \ldots, 1)$, the action of $X_i^{\pm 1}$ is by multiplication, and the Hecke generators T_k act according to formula (2.10) as

$$\tau s_k + \frac{\tau - \tau^{-1}}{X_k X_{k+1}^{-1} - 1}(s_k - 1) = \frac{\tau^{-1} X_{k+1} - \tau X_k}{X_{k+1} - X_k} s_k + \frac{(\tau - \tau^{-1}) X_{k+1}}{X_{k+1} - X_k}.$$

It follows that the elements $T_k^{-1} = T_k + \tau^{-1} - \tau$ act as

$$\frac{\tau^{-1} X_{k+1} - \tau X_k}{X_{k+1} - X_k} s_k + \frac{(\tau - \tau^{-1}) X_k}{X_{k+1} - X_k}. \tag{6.52}$$

By combining relations (6.48) and (2.8), we get

$$\mathcal{D}_i = X_i^{-1} \left(\sum_{j=1}^{l_2} a_j \left((T^{-1})^{+}_{i,n-1} \pi^{-1} (T^{-1})^{+}_{1,i-1} \right)^j + \sum_{j=1}^{l_1} a_{-j} \left(T^{-}_{i-1,1} \pi T^{-}_{n-1,i} \right)^j + a_0 \right). \tag{6.53}$$

We now prove that the action of symmetric combinations of $\mathcal{D}_i$ preserves the subspace $\mathbb{C}_\tau[X_1^{\pm1}, \dots, X_n^{\pm1}]^{\mathfrak{S}_n}$. Let $\mathbb{C}[\mathcal{D}_1, \dots, \mathcal{D}_n]^{\mathfrak{S}_n}$ denote the set of all symmetric combinations of $\mathcal{D}_i$, where $\mathfrak{S}_n$ acts by permuting the indices. We will make use of the following lemma.

Lemma 6.35. *We have $[T_k, D] = 0$ for any $D \in \mathbb{C}[\mathcal{D}_1, \dots, \mathcal{D}_n]^{\mathfrak{S}_n}$ and for all k.*

Proof. The subalgebra $\langle T_1, \dots, T_{n-1}, Y_1^{\pm1}, \dots, Y_n^{\pm1}\rangle \subset \mathbb{H}_n$ is a GL_n-type AHA, whose centre contains $\mathbb{C}[Y_1, \dots, Y_n]^{\mathfrak{S}_n}$. We have $[\mathcal{D}_i, \mathcal{D}_j] = 0$ by Proposition 6.32; also, recall that $T_k^{-1}\mathcal{D}_k T_k^{-1} = \mathcal{D}_{k+1}$ and $[T_k, \mathcal{D}_i] = 0$ for $i \neq k, k+1$. Thus, there is an epimorphism from the subalgebra $\langle T_1, \dots, T_{n-1}, Y_1, \dots, Y_n\rangle$ to the subalgebra $\langle T_1, \dots, T_{n-1}, \mathcal{D}_1, \dots, \mathcal{D}_n\rangle$ given by $T_k \mapsto T_k$, $Y_i \mapsto \mathcal{D}_i$. The claim follows since T_k commute with any element of $\mathbb{C}[Y_1, \dots, Y_n]^{\mathfrak{S}_n}$. ■

Proposition 6.36. *Let $D \in \mathbb{C}[\mathcal{D}_1, \dots, \mathcal{D}_n]^{\mathfrak{S}_n}$. Then the action of D on $\mathbb{C}_\tau[X_1^{\pm1}, \dots, X_n^{\pm1}]$ preserves the space of invariants $\mathbb{C}_\tau[X_1^{\pm1}, \dots, X_n^{\pm1}]^{\mathfrak{S}_n}$.*

Proof. From formula (2.10), it follows that $p \in \mathbb{C}_\tau[X_1^{\pm1}, \dots, X_n^{\pm1}]$ is $\mathfrak{S}_n$-invariant if and only if $T_k(p) = \tau p$ for all k. The claim thus follows from the fact that D commutes with all T_k by Lemma 6.35. ■

Let f be any operator on $\mathbb{C}_\tau[X_1^{\pm1}, \dots, X_n^{\pm1}]$ of the form

$$f = \sum_{\substack{i \in \{1, \dots, n\} \\ j \in \mathbb{Z},\, w \in \mathfrak{S}_n}} g_{i,j,w} t_i^j w, \qquad g_{i,j,w} \in \mathbb{C}_\tau(X_1, \dots, X_n).$$

For instance, the action of any $D \in \mathbb{C}[\mathcal{D}_1, \dots, \mathcal{D}_n]^{\mathfrak{S}_n}$ can be written in this form. The operator $\mathrm{Res}(f)$ is defined by

$$\mathrm{Res}(f) := \sum_{\substack{i \in \{1, \dots, n\} \\ j \in \mathbb{Z},\, w \in \mathfrak{S}_n}} g_{i,j,w} t_i^j.$$

Thus, $\mathrm{Res}(f)$ is a q-difference operator with rational coefficients. On the elements of the space $\mathbb{C}_\tau[X_1^{\pm1}, \dots, X_n^{\pm1}]^{\mathfrak{S}_n}$, it acts identically to f. In particular, if the latter preserves this space then so does $\mathrm{Res}(f)$.

We note that the elements D are not invariant with regard to the action of the symmetric group $\mathfrak{S}_n$, but the operators $\mathrm{Res}(D)$ are.

Theorem 6.37. *The operators $\mathrm{Res}(D)$ for $D \in \mathbb{C}[\mathcal{D}_1, \dots, \mathcal{D}_n]^{\mathfrak{S}_n}$ are pairwise-commuting, $\mathfrak{S}_n$-invariant, and preserve the space $\mathbb{C}_\tau[X_1^{\pm1}, \dots, X_n^{\pm1}]^{\mathfrak{S}_n}$. Furthermore, an algebraic basis $p_1, \dots, p_n \in \mathbb{C}[X_1, \dots, X_n]^{\mathfrak{S}_n}$ gives n algebraically independent operators $\mathrm{Res}\, p_i(\mathcal{D}_1, \dots, \mathcal{D}_n)$.*

Proof. Preservation of $\mathbb{C}_\tau[X_1^{\pm1}, \dots, X_n^{\pm1}]^{\mathfrak{S}_n}$ follows from Proposition 6.36.

Let $D, \widetilde{D} \in \mathbb{C}[\mathcal{D}_1, \ldots, \mathcal{D}_n]^{\mathfrak{S}_n}$ and $p \in \mathbb{C}_\tau[X_1^{\pm1}, \ldots, X_n^{\pm1}]^{\mathfrak{S}_n}$. By using Proposition 6.32, we get $\operatorname{Res}(D)\operatorname{Res}(\widetilde{D})p = D\widetilde{D}p = \widetilde{D}Dp = \operatorname{Res}(\widetilde{D})\operatorname{Res}(D)p$. Thus, $\operatorname{Res}(D)$ and $\operatorname{Res}(\widetilde{D})$ commute when restricted to $\mathbb{C}_\tau[X_1^{\pm1}, \ldots, X_n^{\pm1}]^{\mathfrak{S}_n}$, which implies the commutativity of $\operatorname{Res}(D)$ and $\operatorname{Res}(\widetilde{D})$ (for the statement see [74, Theorem 4.5], [34, Theorem 3.3], for a proof see [71, Proposition 3.2], and for the additive case [32, Lemma 3.7]).

For any $w \in \mathfrak{S}_n$, we have $w\operatorname{Res}(D)w^{-1}p = Dp$ thanks to Proposition 6.36. Thus $w\operatorname{Res}(D)w^{-1}$ and $\operatorname{Res}(D)$ are equal as operators on $\mathbb{C}_\tau[X_1^{\pm1}, \ldots, X_n^{\pm1}]^{\mathfrak{S}_n}$. As in the preceding paragraph, it follows that $w\operatorname{Res}(D)w^{-1} = \operatorname{Res}(D)$.

The final claim follows by specialisation to $\tau = 1$, which reduces $\mathcal{D}_i$ to an operator in the variable X_i. ∎

Explicitly, for $l_1 = l_2 = 1$, and for the symmetric combination $\sum_{i=1}^n \mathcal{D}_i$ of degree one, we get the following formula for the corresponding integrable Hamiltonian.

Proposition 6.38. *With $a = a_1$, $b = a_{-1}$, and $c = a_0$, we have*

$$M_{a,b,c} := \operatorname{Res}\left(\sum_{i=1}^n \mathcal{D}_i^{(1,1)}\right) = a\tau^{1-n}\sum_{i=1}^n \frac{1}{X_i}\left(\prod_{\substack{j=1\\ j\neq i}}^n \frac{\tau^2 X_i - X_j}{X_i - X_j}\right) t_i$$
$$+ b\tau^{1-n}\sum_{i=1}^n \frac{1}{X_i}\left(\prod_{\substack{j=1\\ j\neq i}}^n \frac{X_i - \tau^2 X_j}{X_i - X_j}\right) t_i^{-1} + c\sum_{i=1}^n \frac{1}{X_i}. \tag{6.54}$$

The proof will follow from the next lemma. Let

$$\mathcal{D}_i^+ = X_i^{-1}(T^{-1})^+_{i,n-1}\pi^{-1}(T^{-1})^+_{1,i-1},$$
$$\mathcal{D}_i^- = X_i^{-1}T^-_{i-1,1}\pi T^-_{n-1,i},$$

so that by relation (6.53) we have

$$\operatorname{Res}\left(\sum_{i=1}^n \mathcal{D}_i^{(1,1)}\right) = a\operatorname{Res}\left(\sum_{i=1}^n \mathcal{D}_i^+\right) + b\operatorname{Res}\left(\sum_{i=1}^n \mathcal{D}_i^-\right) + c\sum_{i=1}^n \frac{1}{X_i}.$$

Then the following statement holds.

Lemma 6.39 (cf. [3, Lemma 5.3])**.** *For all $m \in \{1, \ldots, n\}$, let*

$$E_m^+ = \tau^{1-n}\sum_{i=m}^n \frac{1}{X_i}A_{i,m}t_i, \quad \text{where} \quad A_{i,m} = \prod_{\substack{j=m\\ j\neq i}}^n \frac{\tau^2 X_i - X_j}{X_i - X_j}.$$

Then

$$\mathrm{Res}\left(\sum_{i=m}^{n} \mathcal{D}_i^+\right) = E_m^+. \tag{6.55}$$

Furthermore,

$$\mathrm{Res}\left(\sum_{i=1}^{m} \mathcal{D}_i^-\right) = E_m^-, \tag{6.56}$$

where

$$E_m^- = \tau^{n-2m+1} \sum_{i=1}^{m} \frac{1}{X_i} B_{i,m} t_i^{-1}, \qquad \text{where} \qquad B_{i,m} = \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{X_i - \tau^2 X_j}{X_i - X_j}.$$

The proof is analogous to that of [3, Lemma 5.3]. For convenience, we indicate here how to adapt that proof in our context.

Proof. We give the proof of equality (6.56), since (6.55) works similarly. By using formulas (2.10) and (2.11), we get

$$\mathrm{Res}(\mathcal{D}_i^-) = \tau^{n-i}\,\mathrm{Res}(X_i^{-1} T_{i-1,1}^- t_1^{-1}) = \tau^{n-i}\,\mathrm{Res}\left((T^{-1})_{i-1,1}^- X_1^{-1} t_1^{-1}\right).$$

In particular, $\mathrm{Res}(\mathcal{D}_1^-) = \tau^{n-1} X_1^{-1} t_1^{-1}$, from which equality (6.56) for $m = 1$ follows. Thus, it now suffices to show that we have for all $m = 1, \dots, n-1$ that

$$\mathrm{Res}(\mathcal{D}_{m+1}^-) = E_{m+1}^- - E_m^-. \tag{6.57}$$

For $i \neq m+1$, we have

$$B_{i,m+1} = \frac{X_i - \tau^2 X_{m+1}}{X_i - X_{m+1}} \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{X_i - \tau^2 X_j}{X_i - X_j} = \left(1 + \frac{(1-\tau^2) X_{m+1}}{X_i - X_{m+1}}\right) B_{i,m}.$$

Hence, relation (6.57) is equivalent to

$$\mathrm{Res}(\mathcal{D}_{m+1}^-) = \frac{\tau^{n-2m-1}}{X_{m+1}} B_{m+1,m+1} t_{m+1}^{-1} + \sum_{i=1}^{m} \frac{\tau^{n-2m-1}(1-\tau^2)}{X_i - X_{m+1}} B_{i,m} t_i^{-1}. \tag{6.58}$$

Let the right-hand side of equality (6.58) be the definition of R_{m+1} for $m = 0, 1, \dots, n-1$. We trivially have $R_1 = \mathrm{Res}(\mathcal{D}_1^-)$. We note that

$$\mathrm{Res}(\mathcal{D}_{m+1}^-) = \mathrm{Res}(T_m^{-1} \mathcal{D}_m^- T_m^{-1}) = \tau^{-1}\,\mathrm{Res}(T_m^{-1} \mathcal{D}_m^-).$$

Thus, to prove equality (6.58) for all $m = 1, \dots, n-1$, it suffices to prove that $\mathrm{Res}(T_m^{-1} R_m) =$

τR_{m+1}. Indeed, we will then get that

$$\operatorname{Res}(\mathcal{D}_2^-) = \tau^{-1}\operatorname{Res}(T_1^{-1}\mathcal{D}_1^-) = \tau^{-1}\operatorname{Res}(T_1^{-1}\operatorname{Res}(\mathcal{D}_1^-)) = \tau^{-1}\operatorname{Res}(T_1^{-1}R_1) = R_2,$$

as required; and similarly for $\operatorname{Res}(\mathcal{D}_3^-)$, etc.

By using formula (6.52) for the action of T_m^{-1}, we compute

$$\begin{aligned}\operatorname{Res}(T_m^{-1}R_m) = &\frac{\tau^{n-2m}}{X_{m+1}}B_{m+1,m+1}t_{m+1}^{-1} + \frac{\tau^{n-2m}(1-\tau^2)}{X_m - X_{m+1}}B_{m,m}t_m^{-1}\\ &+\sum_{i=1}^{m-1}\frac{\tau^{n-2m}(1-\tau^2)(X_{m+1}-\tau^2X_m)}{(X_{m+1}-X_m)(X_i-X_{m+1})}B_{i,m-1}t_i^{-1}\\ &-\sum_{i=1}^{m-1}\frac{\tau^{n-2m}(1-\tau^2)^2X_m}{(X_{m+1}-X_m)(X_i-X_m)}B_{i,m-1}t_i^{-1}.\end{aligned}$$

The proof that $\operatorname{Res}(T_m^{-1}R_m) = \tau R_{m+1}$ is completed by using in the preceding equality that

$$\begin{aligned}&\frac{1}{X_{m+1}-X_m}\left(\frac{X_{m+1}-\tau^2X_m}{X_i - X_{m+1}} - \frac{(1-\tau^2)X_m}{X_i-X_m}\right)B_{i,m-1}\\ &\qquad = \frac{X_i - \tau^2X_m}{(X_i-X_{m+1})(X_i-X_m)}B_{i,m-1} = \frac{1}{X_i - X_{m+1}}B_{i,m}\end{aligned}$$

for $i \neq m$. This completes the proof of the lemma. ■

Remark 6.40. The operator $M_{a,b,c}$ given by (6.54) for a special choice of the parameters a, b, c can be related to a particular limit of the operator (3.13a) from [103] as follows. In the latter operator, let us make a translation of the center-of-mass of the form $q^{x_j} \to \kappa^{-1}q^{x_j}$ ($j = 1, ..., n$) for a constant κ, make the change of variables $X_j = q^{-x_j}$ (in particular, the additive shift operators T_j, T_j^{-1} become respectively t_j^{-1}, t_j in our notation), put $t = \tau^2$, and multiply the whole operator by κ. Then in the limit $\kappa \to 0$, one obtains the operator (6.54) for $a = -\tau^{n-1}\widehat{t}_0$, $b = -\tau^{1-n}\widehat{t}_1$, and $c = \widehat{t}_0 + \widehat{t}_1$. Further specialisation of this operator at $\widehat{t}_1 = 0$ appeared in [10, Example 3.24]. The operator $M_{a,0,c}$ is gauge-equivalent to the standard MR operator with an extra term proportional to $\sum_{i=1}^n X_i^{-1}$ (see [10]).

For more general values of l_1 and l_2, and the degree one symmetric combination $\sum_{i=1}^n \mathcal{D}_i$, the following proposition takes place.

Proposition 6.41. *We have*

$$\operatorname{Res}\left(\sum_{i=1}^n \mathcal{D}_i^{(l_1,l_2)}\right) = \tau^{l_2(1-n)}a_{l_2}\sum_{i=1}^n\frac{1}{X_i}\left(\prod_{k=0}^{l_2-1}\prod_{\substack{j=1\\ j\neq i}}^n\frac{q^k\tau^2X_i - X_j}{q^kX_i - X_j}\right)t_i^{l_2}$$

$$+\tau^{l_1(1-n)}a_{-l_1}\sum_{i=1}^{n}\frac{1}{X_i}\left(\prod_{k=0}^{l_1-1}\prod_{\substack{j=1\\j\neq i}}^{n}\frac{X_i-q^k\tau^2X_j}{X_i-q^kX_j}\right)t_i^{-l_1}+\ldots,$$

where ... denotes "non-leading terms", that is, terms with shifts $\prod_{j=1}^{n}t_j^{k_j}$ such that $-l_1 < k_j < l_2$ for all j. Moreover, in each term, either all k_j are non-negative with $\sum_{j=1}^{n}k_j \leq l_2$, or all k_j are non-positive with $\sum_{j=1}^{n}k_j \geq -l_1$.

The proof is similar to the calculation of the leading term of a general Macdonald operator, polynomial in Y variables of the DAHA, from [34, Proposition 3.4].

Proof. By using equality (6.53), we get

$$\operatorname{Res}\left(\sum_{i=1}^{n}\mathcal{D}_i^{(l_1,l_2)}\right)=\operatorname{Res}\left(\sum_{i=1}^{n}X_i^{-1}\sum_{j=1}^{l_2}a_j\left((T^{-1})^+_{i,n-1}t_n(n,\ldots,1)(T^{-1})^+_{1,i-1}\right)^j\right.$$
$$\left.+\sum_{i=1}^{n}X_i^{-1}\sum_{j=1}^{l_1}a_{-j}\left(T^-_{i-1,1}t_1^{-1}(1,\ldots,n)T^-_{n-1,i}\right)^j+a_0\sum_{i=1}^{n}X_i^{-1}\right).$$

From there, we see due to formula (6.52) that the term containing $t_1^{l_2}$ can only come from

$$\operatorname{Res}\left(a_{l_2}X_1^{-1}\left((T^{-1})^+_{1,n-1}t_n(n,\ldots,1)\right)^{l_2}\right).$$

Hence, using formula (6.52), we can compute this $t_1^{l_2}$ term to be

$$\tau^{l_2(1-n)}a_{l_2}\frac{1}{X_1}\left(\prod_{j=2}^{n}\frac{\tau^2X_1-X_j}{X_1-X_j}t_1\right)^{l_2}$$
$$=\tau^{l_2(1-n)}a_{l_2}\frac{1}{X_1}\left(\prod_{k=0}^{l_2-1}\prod_{j=2}^{n}\frac{q^k\tau^2X_1-X_j}{q^kX_1-X_j}\right)t_1^{l_2}.$$

We can use $\mathfrak{S}_n$-invariance (see Theorem 6.37) to deduce the coefficient at $t_i^{l_2}$ for any i. Similarly, one can compute explicitly the coefficient at $t_n^{-l_1}$, and then use $\mathfrak{S}_n$-invariance again to complete the proof of the proposition. ■

For example, for $l_1 = 1$ and $l_2 = 2$, we get the following integrable Hamiltonian

$$\operatorname{Res}\left(\sum_{i=1}^{n}\mathcal{D}_i^{(1,2)}\right)=\tau^{2(1-n)}a_2\sum_{i=1}^{n}\frac{1}{X_i}\left(\prod_{\substack{j=1\\j\neq i}}^{n}\frac{(\tau^2X_i-X_j)(q\tau^2X_i-X_j)}{(X_i-X_j)(qX_i-X_j)}\right)t_i^2$$

$$+ q\tau^{2(1-n)}a_2 \sum_{1\le i<j\le n} \frac{(\tau^2-1)(\tau^2-q)(X_i+X_j)}{(qX_i-X_j)(qX_j-X_i)} \left(\prod_{\substack{l=1\\ l\neq i,j}}^n \frac{(\tau^2 X_i - X_l)(\tau^2 X_j - X_l)}{(X_i-X_l)(X_j-X_l)} \right) t_i t_j$$

$$+ \tau^{1-n} a_1 \sum_{i=1}^n \frac{1}{X_i} \left(\prod_{\substack{j=1\\ j\neq i}}^n \frac{\tau^2 X_i - X_j}{X_i - X_j} \right) t_i + \tau^{1-n} a_{-1} \sum_{i=1}^n \frac{1}{X_i} \left(\prod_{\substack{j=1\\ j\neq i}}^n \frac{X_i - \tau^2 X_j}{X_i - X_j} \right) t_i^{-1} + a_0 \sum_{i=1}^n \frac{1}{X_i}$$

which is a generalisation of the operator [20, (5.18)] to which it reduces for $a_{-1} = 0$.

6.4.1 Differential limits

Let us explain the relationship of the operator $M_{a,b,c}$ (6.54) to various differential operators.

6.4.1.1 Relation to type A CMS operator in an external field

The operator $M_{a,b,c}$ with $a = \tau^{n-1}q^{-1/2}$ and $b = 0$ can be gauged by the function $\exp(\sum_{i=1}^n \frac{(\log X_i)^2}{2\log q})$ (see [10]) to the operator

$$M = \sum_{i=1}^n \left(\prod_{\substack{j=1\\ j\neq i}}^n \frac{\tau^2 X_i - X_j}{X_i - X_j} \right) t_i + c \sum_{i=1}^n \frac{1}{X_i},$$

which is the standard MR operator with an additional $\sum_{i=1}^n X_i^{-1}$ term. If we put $X_i = e^{x_i}$, $q = e^{\hbar}$, $\tau = e^{-\hbar m/2}$, $c = \lambda\hbar^2/2$, and expand around $\hbar = 0$, we get

$$M = \sum_{i=1}^n \left(\prod_{\substack{j=1\\ j\neq i}}^n \left(1 - \frac{m e^{x_i}\hbar}{e^{x_i} - e^{x_j}} + \frac{m^2 e^{x_i}\hbar^2}{2(e^{x_i} - e^{x_j})} + O(\hbar^3)\right) \right) \mathcal{T}_{x_i}^{\hbar} + \frac{\lambda\hbar^2}{2} \sum_{i=1}^n e^{-x_i},$$

where $\mathcal{T}_{x_i}^{\hbar} = 1 + \hbar\partial_{x_i} + \frac{1}{2}\hbar^2\partial_{x_i}^2 + O(\hbar^3)$ is the (additive) shift operator defined by $x_j \mapsto x_j + \delta_{ij}\hbar$. Thus, the expansion of the operator M in powers of $\hbar$ gives

$$M = n + \hbar \left(\sum_{i=1}^n \partial_{x_i} - \frac{mn(n-1)}{2} \right)$$
$$+ \frac{\hbar^2}{2} \left(\Delta - \sum_{i<j} \frac{2m}{e^{x_i} - e^{x_j}} (e^{x_i}\partial_{x_i} - e^{x_j}\partial_{x_j}) + \lambda \sum_{i=1}^n e^{-x_i} + \frac{m^2(2n-1)n(n-1)}{6} \right) + O(\hbar^3).$$

The term at $\hbar^2/2$ is, up to an additive constant, equal to

$$\Delta - \sum_{i<j} m \coth\left(\frac{x_i - x_j}{2}\right)(\partial_{x_i} - \partial_{x_j}) + \lambda \sum_{i=1}^{n} e^{-x_i} - m(n-1)\sum_{i=1}^{n} \partial_{x_i}, \tag{6.59}$$

which is related to the standard CMS operator with an additional $\sum_{i=1}^{n} e^{-x_i}$ term (this corresponds to Block 2 in [101, Figure 1] with $g_1 = g_2 = g_3 = 0$ and $\alpha = \frac{1}{2}$). Note that the term $\sum_{i=1}^{n} \partial_{x_i}$ (which does not commute with the operator (6.59) unless $\lambda = 0$) can be removed from (6.59) by conjugating the operator (6.59) by $\exp(\frac{1}{2}m(n-1)\sum_{i=1}^{n} x_i)$.

In the rational limit — obtained by putting $x_i = \omega y_i$, $\lambda = \gamma\omega^{-3}$, multiplying the operator (6.59) by ω^2, subtracting $\gamma n \omega^{-1}$, and taking the limit $\omega \to 0$ —, we get

$$\sum_{i=1}^{n} \partial_{y_i}^2 - \sum_{i<j} \frac{2m}{y_i - y_j}(\partial_{y_i} - \partial_{y_j}) - \gamma \sum_{i=1}^{n} y_i,$$

which is the standard rational CMS operator in radial gauge with an additional $\sum_{i=1}^{n} y_i$ term (this corresponds to Block 4 in [101, Figure 1] with $g_1 = g_2 = g_3 = 0$).

6.4.1.2 Relation to type B rational CMS operator

Let us consider the operator $M_{a,b,c}$ (6.54) with $X_i = e^{x_i}$, $q = e^{\hbar}$, $\tau = e^{-\hbar m/2}$, $a = \tau^{n-1}\alpha$, $b = \tau^{n-1}\beta$, and $c = \lambda\hbar^2/2 + m(n-1)\beta\hbar - \alpha - \beta$. Then the expansion of $M_{a,b,c}$ around $\hbar = 0$ (similar to Section 6.4.1.1) gives

$$\hbar(\alpha - \beta)\sum_{i=1}^{n} e^{-x_i}\partial_{x_i} + \frac{\hbar^2}{2}\Bigg((\alpha+\beta)\sum_{i=1}^{n} e^{-x_i}\partial_{x_i}^2 - \sum_{i<j}\frac{2m\alpha}{e^{x_i} - e^{x_j}}(\partial_{x_i} - \partial_{x_j})$$
$$- \sum_{i<j}\frac{2m\beta}{e^{x_i} - e^{x_j}}(e^{x_j - x_i}\partial_{x_i} - e^{x_i - x_j}\partial_{x_j}) + \big(\lambda + m^2(n-1)^2\beta\big)\sum_{i=1}^{n} e^{-x_i}\Bigg) + O(\hbar^3).$$

The term at order $\hbar$ (which does not commute with the term at order $\hbar^2$) can be removed by choosing $\alpha = \beta$. Letting $x_i = 2z_i$, $X_i = e^{z_i}$, and $\gamma = 4(\lambda + m^2(n-1)^2\beta)(\alpha+\beta)^{-1}$, the term at order $\hbar^2$ is proportional to

$$\sum_{i=1}^{n}\frac{1}{X_i^2}(X_i\partial_{X_i})^2 - \sum_{i<j}\frac{4m\alpha(\alpha+\beta)^{-1}}{X_i^2 - X_j^2}(X_i\partial_{X_i} - X_j\partial_{X_j})$$
$$- \sum_{i<j}\frac{4m\beta(\alpha+\beta)^{-1}}{X_i^2 - X_j^2}(X_j^2X_i^{-1}\partial_{X_i} - X_i^2X_j^{-1}\partial_{X_j}) + \gamma\sum_{i=1}^{n}\frac{1}{X_i^2}$$
$$= \sum_{i=1}^{n}\partial_{X_i}^2 + \sum_{i=1}^{n}\frac{1 + 4m(n-1)\beta(\alpha+\beta)^{-1}}{X_i}\partial_{X_i} - \sum_{i<j}\frac{2m}{X_i - X_j}(\partial_{X_i} - \partial_{X_j})$$

$$-\sum_{i<j}\frac{2m}{X_i+X_j}(\partial_{X_i}+\partial_{X_j})+\gamma\sum_{i=1}^{n}\frac{1}{X_i^2},$$

which is gauge-equivalent to a type B rational CMS operator.

6.4.2 MR system with two types of particles in an external field

In this subsection, we obtain a generalisation of the MR system with a Morse term [103, (2.1)] (a particular limit of which, [103, (3.13a)], was mentioned in Remark 6.40 above). This system was introduced by Van Diejen in [101] and studied further by Van Diejen and Emsiz in [103]. Our generalisation introduces into the system a second, different set of particles interacting with each other and also with the original set of particles. In the case of MR systems, such two-types-of-particles generalisations were considered in [24, 96].

To obtain such a generalisation, we take the following approach. The operator [103, (2.1)] can be obtained from the Koornwinder operator (i.e., the operator of MR type for the root system BC_n) by a limit in which the centre of mass is sent to infinity [101]. We now take the generalised Koornwinder operator [53, (5.12)] introduced by Feigin and Silantyev, and apply to it an analogous centre-of-mass-to-infinity limit. As a by-product, we also get a generalisation of the operators [103, (3.13a)] and $M_{a,b,c}$ (6.54) (for particular a, b, c) to the case of two types of particles. Another way to obtain the latter would be to make use of the representation theory of the DAHA of type GL_n and the elements $\mathcal{D}_i^{(1,1)}$ defined in Section 6.4 (see Remark 6.42 below).

In the operator [53, (5.12)], let us make the substitutions $x_i \to x_i + R$, $y_i \to y_i + R + \log(qs^{-1})$, $a \to ae^R$, $b \to be^R$, $c \to ce^{-R}$, $d \to de^{-R}$, and then let $R \to \infty$. In order to make connection with the notations used in [103] (recall that $q = e^{\hbar/2}$ and $s = e^{\xi/2}$ in [53]), let us in the resulting limit make the replacements $q \to q^{-1/2}$, $s = t^{1/2}$ (so that now $q = e^{-\hbar}$ and $t = e^{\xi}$), $a = t_1$, $b = t_2$, $c = t_0^{-1}$, $d = t_3^{-1}$, $x_i \to \hbar x_i$, and $y_i \to \hbar y_i$ (so that the shift operators $\mathcal{T}_{x_i}^{\varepsilon\hbar}$ and $\mathcal{T}_{y_i}^{\varepsilon\xi}$ for $\varepsilon \in \{\pm 1\}$ become $\mathcal{T}_{x_i}^{\varepsilon}$ and $\mathcal{T}_{y_i}^{-\varepsilon \log(t)/\log(q)}$, respectively). Then we get the following Hamiltonian

$$H_{t_0,t_1,t_2,t_3} = \sum_{i=1}^{N_1}(1-t_1q^{x_i})(1-t_2q^{x_i})\left(\prod_{\substack{j=1\\j\neq i}}^{N_1}\frac{t^{-1}-q^{x_i-x_j}}{1-q^{x_i-x_j}}\right)\left(\prod_{j=1}^{N_2}\frac{q-q^{x_i-y_j}}{1-q^{x_i-y_j}}\right)(\mathcal{T}_{x_i}-1)$$
$$+\frac{t_1t_2}{qt_0t_3}\sum_{i=1}^{N_1}(1-t_0q^{x_i})(1-t_3q^{x_i})\left(\prod_{\substack{j=1\\j\neq i}}^{N_1}\frac{t-q^{x_i-x_j}}{1-q^{x_i-x_j}}\right)\left(\prod_{j=1}^{N_2}\frac{t-q^{x_i-y_j}}{qt-q^{x_i-y_j}}\right)(\mathcal{T}_{x_i}^{-1}-1)$$
$$+\frac{1-q}{1-t^{-1}}\sum_{i=1}^{N_2}(1-t_1q^{y_i})(1-t_2q^{y_i})\left(\prod_{j=1}^{N_1}\frac{t^{-1}-q^{y_i-x_j}}{1-q^{y_i-x_j}}\right)$$

$$
\begin{aligned}
&\times \left(\prod_{\substack{j=1\\ j\neq i}}^{N_2} \frac{q-q^{y_i-y_j}}{1-q^{y_i-y_j}}\right) (\mathcal{T}_{y_i}^{-\log(t)/\log(q)}-1)\\
&+\frac{t_1t_2(1-q^{-1})}{qt_0t_3(1-t)} \sum_{i=1}^{N_2} (1-t_0tq^{y_i+1})(1-t_3tq^{y_i+1}) \left(\prod_{j=1}^{N_1} \frac{q^{-1}-q^{y_i-x_j}}{q^{-1}t^{-1}-q^{y_i-x_j}}\right)\\
&\times \left(\prod_{\substack{j=1\\ j\neq i}}^{N_2} \frac{q^{-1}-q^{y_i-y_j}}{1-q^{y_i-y_j}}\right) (\mathcal{T}_{y_i}^{\log(t)/\log(q)}-1). \qquad (6.60)
\end{aligned}
$$

For $N_2 = 0$ and $N_1 = n$, the operator (6.60) reduces to the Van Diejen–Emsiz operator [103, (2.1)] up to a factor of $\sqrt{qt_0t_3/(t_1t_2)}$. Also, if in (6.60) we put $n = N_1 + N_2$, $y_i = x_{N_1+i}$, and $t = q^{-1}$ (that is, $\xi = \hbar$), then we get the operator [103, (2.1)] with $t = q^{-1}$.

By applying the same limiting procedure to the set of quantum integrals of the generalised Koornwinder operator found in [53, Proposition 5.6], we get quantum integrals for the Hamiltonian (6.60).

We also obtain a generalisation to the case of two types of particles of the Hamiltonian [103, (3.13a)]. Indeed, if in (6.60) we put $t_3 = 1$, define $\widehat{t_i}$ ($i = 0, 1, 2$) by $t_0 = q^{-1}\widehat{t_1}\widehat{t_2}$, $t_1 = \widehat{t_0}\widehat{t_2}$, and $t_2 = \widehat{t_0}\widehat{t_1}$ following [103, (3.12b)], and then take the limit $\widehat{t_2} \to 0$, we get

$$
\begin{aligned}
H_{\widehat{t_0},\widehat{t_1}} &= \sum_{i=1}^{N_1} (1-\widehat{t_0}\widehat{t_1}q^{x_i}) \left(\prod_{\substack{j=1\\ j\neq i}}^{N_1} \frac{t^{-1}-q^{x_i-x_j}}{1-q^{x_i-x_j}}\right) \left(\prod_{j=1}^{N_2} \frac{q-q^{x_i-y_j}}{1-q^{x_i-y_j}}\right) (\mathcal{T}_{x_i}-1)\\
&+\widehat{t_0}^2 \sum_{i=1}^{N_1} (1-q^{x_i}) \left(\prod_{\substack{j=1\\ j\neq i}}^{N_1} \frac{t-q^{x_i-x_j}}{1-q^{x_i-x_j}}\right) \left(\prod_{j=1}^{N_2} \frac{t-q^{x_i-y_j}}{qt-q^{x_i-y_j}}\right) (\mathcal{T}_{x_i}^{-1}-1)\\
&+\frac{1-q}{1-t^{-1}} \sum_{i=1}^{N_2} (1-\widehat{t_0}\widehat{t_1}q^{y_i}) \left(\prod_{j=1}^{N_1} \frac{t^{-1}-q^{y_i-x_j}}{1-q^{y_i-x_j}}\right) \left(\prod_{\substack{j=1\\ j\neq i}}^{N_2} \frac{q-q^{y_i-y_j}}{1-q^{y_i-y_j}}\right) \qquad (6.61)\\
&\qquad \times (\mathcal{T}_{y_i}^{-\log(t)/\log(q)}-1)\\
&+\frac{\widehat{t_0}^2(1-q^{-1})}{1-t} \sum_{i=1}^{N_2} (1-tq^{y_i+1}) \left(\prod_{j=1}^{N_1} \frac{q^{-1}-q^{y_i-x_j}}{q^{-1}t^{-1}-q^{y_i-x_j}}\right) \left(\prod_{\substack{j=1\\ j\neq i}}^{N_2} \frac{q^{-1}-q^{y_i-y_j}}{1-q^{y_i-y_j}}\right)\\
&\qquad \times (\mathcal{T}_{y_i}^{\log(t)/\log(q)}-1).
\end{aligned}
$$

For $N_2 = 0$ and $N_1 = n$, this reduces to the operator [103, (3.13a)] up to a factor of $\widehat{t_0}$.

Let us now consider an analogous limit of the operator (6.61) as the limit described in Remark 6.40, just additionally making also the change $q^{y_i} \to \kappa^{-1} q^{y_i}$ and introducing the variables $Y_i = q^{-y_i}$. Let us denote a q-multiplicative shift operator in the variable X_i by $t^q_{X_i}$ and analogously for Y_i (so that $\mathcal{T}^{\pm 1}_{x_i}$ and $\mathcal{T}^{\pm \log(t)/\log(q)}_{y_i}$ become respectively $t^{q^{\mp 1}}_{X_i}$ and $t^{\tau^{\mp 2}}_{Y_i}$). Then, in this limit, we obtain the multiplicative operator

$$
\begin{aligned}
\widetilde{H}_{\widehat{t}_0,\widehat{t}_1} = \widetilde{a} \sum_{i=1}^{N_1} \frac{1}{X_i} \left(\prod_{\substack{j=1 \\ j \neq i}}^{N_1} \frac{\tau^2 X_i - X_j}{X_i - X_j} \right) \left(\prod_{j=1}^{N_2} \frac{\tau^2 X_i - Y_j}{q\tau^2 X_i - Y_j} \right) t^q_{X_i} \\
+ \frac{\widetilde{a}(1-q)}{1-\tau^{-2}} \sum_{i=1}^{N_2} \frac{1}{Y_i} \left(\prod_{j=1}^{N_1} \frac{q^{-1} Y_i - X_j}{q^{-1}\tau^{-2} Y_i - X_j} \right) \left(\prod_{\substack{j=1 \\ j \neq i}}^{N_2} \frac{q^{-1} Y_i - Y_j}{Y_i - Y_j} \right) t^{\tau^{-2}}_{Y_i} \\
+ \widetilde{b} \sum_{i=1}^{N_1} \frac{1}{X_i} \left(\prod_{\substack{j=1 \\ j \neq i}}^{N_1} \frac{\tau^{-2} X_i - X_j}{X_i - X_j} \right) \left(\prod_{j=1}^{N_2} \frac{q X_i - Y_j}{X_i - Y_j} \right) t^{q^{-1}}_{X_i} \\
+ \frac{\widetilde{b}(1-q)}{1-\tau^{-2}} \sum_{i=1}^{N_2} \frac{1}{Y_i} \left(\prod_{j=1}^{N_1} \frac{\tau^{-2} Y_i - X_j}{Y_i - X_j} \right) \left(\prod_{\substack{j=1 \\ j \neq i}}^{N_2} \frac{q Y_i - Y_j}{Y_i - Y_j} \right) t^{\tau^2}_{Y_i} \\
+ \widetilde{c} \sum_{i=1}^{N_1} \frac{1}{X_i} + \frac{\widetilde{c}(1-q)}{1-\tau^{-2}} \sum_{i=1}^{N_2} \frac{1}{Y_i},
\end{aligned}
\tag{6.62}
$$

where $\widetilde{a} = -\widehat{t}_0^{\,2}$, $\widetilde{b} = -\widehat{t}_0 \widehat{t}_1$, and $\widetilde{c} = \widehat{t}_0(\widehat{t}_0 + \widehat{t}_1)$, and where we twice used the following polynomial identity (which generalises to two types of particles the identity from the top of [103, p. 1621]):

$$
\begin{aligned}
\sum_{i=1}^{N_1} z_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{N_1} \frac{t z_j - z_i}{z_j - z_i} \right) \prod_{j=1}^{N_2} \frac{s w_j - z_i}{w_j - z_i} \\
+ \frac{1-s}{1-t} \sum_{i=1}^{N_2} w_i \left(\prod_{j=1}^{N_1} \frac{t z_j - w_i}{z_j - w_i} \right) \prod_{\substack{j=1 \\ j \neq i}}^{N_2} \frac{s w_j - w_i}{w_j - w_i} = \sum_{i=1}^{N_1} z_i + \frac{1-s}{1-t} \sum_{i=1}^{N_2} w_i.
\end{aligned}
$$

We used it once with $z_i = X_i^{-1}$, $w_i = Y_i^{-1}$, $t = \tau^{-2}$, $s = q$, and the second time with $z_i = X_i^{-1}$, $w_i = q\tau^2 Y_i^{-1}$, $t = \tau^2$, and $s = q^{-1}$.

The operator (6.62) generalises the Hamiltonian $M_{a,b,c}$ given by (6.54), for special values of a, b, c, to a Hamiltonian of a system containing two types of particles. Indeed, if we put $N_2 = 0$, $N_1 = n$ in (6.62), we recover $M_{a,b,c}$ with $a = -\tau^{n-1}\widehat{t}_0^{\,2}$, $b = -\tau^{1-n}\widehat{t}_0\widehat{t}_1$, and $c = \widehat{t}_0(\widehat{t}_0 + \widehat{t}_1)$.

Remark 6.42. An alternative way to arrive at the operator (6.62) and also a version of it with arbitrary parameters $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ is to apply to the Hamiltonian (6.54) a restriction procedure, similar to those considered in [53], for a suitable submodule of the polynomial representation of the GL_n-type DAHA. Moreover, this approach should lead to integrable generalisations of the Hamiltonians $\mathrm{Res}\left(\sum_{i=1}^n \mathcal{D}_i^{(l_1,l_2)}\right)$ for general l_1, l_2 to the case of two types of particles.

Chapter 7

Concluding remarks

Let us describe a few possible directions for further research related to the present work. One broad direction is about investigating (and obtaining further new) MR-type systems through representation-theoretic methods and the lens of dualities, and the study of their eigenfunctions. Another is to construct and investigate natural analogues of the algebra $\mathbb{H}^{\mathfrak{gl}_n}$ from Chapter 6 that would establish an intriguing interplay between quantum groups and DAHAs. The resulting algebraic structures would be expected to lead to novel integrable difference operators of MR flavour and to have connections to the geometry of symplectic singularities. More specifically, some possible research problems are as follows.

It would be interesting to extend our construction from Chapter 4 to the q-difference setting, in other words, to devise a spin generalisation of the construction of MR-type operators developed in [53] that uses special submodules in the polynomial representation of DAHAs. Another important problem is to construct eigenfunctions for any new integrable operators thus obtained, which could lead to interesting new special functions. For example, the standard MR operator of type A is diagonalised by the Macdonald polynomials, and its generalisation associated with a deformation of the A-type root system has the super-Macdonald polynomials as eigenfunctions [96]. Further, it would be interesting to extend the constructions from [50] and Chapter 4 to systems of elliptic type in the differential, q-difference, and matrix-valued settings. This could, in particular, lead to an integrable elliptic version of the generalised CMS operator for AG_2 studied in Chapter 3 (cf. [52]).

We would also like to extend the techniques from the paper [25] to study the bispectrality properties of the generalised MR operators associated with AG_2 introduced in Chapter 5 as well as the trigonometric generalisation of the Sergeev–Veselov difference operator for $BC(l, 1)$, due to Feigin and Silantyev [53]. In particular, this would yield compact formulas for BA eigenfunctions for these generalised MR operators.

In addition to the algebra $\mathcal{H}^{\mathfrak{gl}_n}$ discussed in Chapter 6, the paper [51] introduced and studied also an $\mathfrak{so}_n$ version $\mathcal{H}^{\mathfrak{so}_n}$ — the Dunkl angular momenta algebra (DAMA). From an integrability perspective, the DAMA is the natural algebraic structure for studying the angular part of the rational CMS operator. The angular part admits many quantum integrals which can be realised inside the spherical subalgebra of the DAMA. The algebra $\mathcal{H}^{\mathfrak{so}_n}$ has a link to the Lie algebra $\mathfrak{so}_n$ analogous to the link of $\mathcal{H}^{\mathfrak{gl}_n}$ to $\mathfrak{gl}_n$. In type A, it is the subalgebra of the RCA $\mathcal{H}_n$ generated by $\mathbb{C}\mathfrak{S}_n$ and deformations — by means of Dunkl operators — of the quantum angular momentum generators $x_i\partial_{x_j} - x_j\partial_{x_i}$ $(1 \leq i \neq j \leq n)$ defining the Jordan–Schwinger representation of $\mathfrak{so}_n$. The DAMA is a flat c-deformation of the crossed product of $\mathbb{C}\mathfrak{S}_n$ with the image of $U(\mathfrak{so}_n)$ under this Jordan–Schwinger map. It was recently shown in [4] that $\mathcal{H}^{\mathfrak{so}_n}$ coincides with the subalgebra of the RCA invariant under an action of $SL_2(\mathbb{C})$. Geometrically, $\mathcal{H}^{\mathfrak{so}_n}$ is conjecturally related to a deformation of a symplectic singularity given as the quotient by $\mathfrak{S}_n$ of the closure of the minimal special nilpotent orbit of $\mathfrak{so}_n$. For the symplectic Lie algebra $\mathfrak{sp}_{2n}$, Hakobyan constructed in [60] an analogous algebra — the Dunkl symplectic algebra (DSA). It would be natural and interesting to try to construct and study q-deformed generalisations of the DAMA and DSA inside a DAHA, similarly to how the algebra $\mathbb{H}^{\mathfrak{gl}_n} \subset \mathbb{H}_n$ in Chapter 6 is a q-analogue of $\mathcal{H}^{\mathfrak{gl}_n}$. They are expected to be related to suitable quantum algebras for $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$, respectively, and they might potentially lead to new integrable difference operators.

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