The K-theory of the C^*-algebras associated to complex dynamical systems

Jeremy B. Hume

Submitted in fulfilment of the requirements for the
Degree of Doctor of Philosophy

School of Mathematics and Statistics
College of Science and Engineering
University of Glasgow

September 2024

Abstract

We compute the K-theory of the three C^*-algebras associated to a rational function R acting on the Riemann sphere, its Fatou set, and its Julia set. The latter C^*-algebra is a unital UCT Kirchberg algebra and is thus classified by its K-theory. The K-theory in all three cases is shown to depend only on the degree of R, the critical points of R, and the Fatou cycles of R. Our results yield new dynamical invariants for rational functions and a C^*-algebraic interpretation of the Density of Hyperbolicity Conjecture for quadratic polynomials. These calculations are possible due to new exact sequences in K-theory we induce from morphisms of C^*-correspondences.

Contents

Acknowledgements

This project has received funding from an NSERC Discovery Grant in Canada and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 817597). Early work on this project was done while the author was a MSc student at the University of Victoria, Canada, and he thanks Ian Putnam for many enlightening conversations about K-theory. He would also like to express thanks to Xin Li and Michael Whittaker for helpful comments on drafts of this project, and to the examiners Christian Voigt and Magnus Goffeng for their dilligent reading and numerous good suggestions improving the quality of the thesis.

Declaration

All work in this thesis was carried out by the author unless otherwise explicitly stated.

Chapter 1

Introduction

A rational function $R(z) = \frac{P(z)}{Q(z)}$, where $P(z)$, $Q(z)$ are polynomials with complex co-efficients, can be thought of as a holomorphic map on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We will assume the degree of R is larger than one, so that R is not a homeomorphism or a constant. $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ can be studied as a dynamical system by considering its forward iterations $\{R^{\circ n}\}_{n \in \mathbb{N}_0}$ by function composition.

Complex dynamical systems have been studied since the 19^{th} century, and we invite the reader to [1] and [2] for historical overviews. The modern theory was developed in the 1920's, primarily by Fatou [21] and Julia [24], with contributions from Lattés [35] and Ritt [51]. The standard reference for the modern theory is [39].

Fatou and Julia made the fundamental observation that the sphere partitions into two R-invariant domains exhibiting wildly different dynamical phenomena. The first such domain is the *Fatou set* F_R, where the dynamics is regular. Remarkably, every connected component of F_R is eventually mapped by R into a finite cycle of Fatou components, and there can only be four different types of behaviour of R when restricted to a Fatou cycle (see [39, Theorem 16.1]).

The complement of F_R in $\hat{\mathbb{C}}$ is the *Julia set* and is denoted J_R. This set is always non-empty ([39, Theorem 14.1]) and the dynamics restricted to J_R behaves chaotically ([39, Corollary 14.2]). There is no corresponding classification of the possible behaviours for $R : J_R \to J_R$, and these systems are notoriously hard to understand for seemingly simple families of rational functions.

An approach to understanding a dynamical system is to study its space of orbits. One quickly runs into trouble when trying to do this for a rational function $R : J_R \to J_R$, as each orbit $O(x) = \{y \in J_R : \exists\, n, m \in \mathbb{N} : R^{\circ n}(x) = R^{\circ m}(y)\}$ is dense in J_R and thus the space of orbits O_{R,J_R} is highly singular and contains little information. This is a common occurence for a dynamical system.

An insight from Connes' non-commutative geometry program [14] is to instead consider the "non-commutative" orbit space of a dynamical system, which is a C^*-algebra constructed from the dual action of the dynamical system, acting on the continuous complex-valued functions of its phase space. In general, the C^*-algebra of a dynamical system is well behaved, and whenever the orbit space is well behaved it is the same (up to strong Morita equivalence) as the C^*-algebra of continuous functions on the orbit space (see [57, Chapter 4]).

Kajiwara and Watatani in [28] were the first to study the C^*-algebra $\mathcal{O}_{R,J_R}$ of a rational function $R : J_R \to J_R$. They proved in [28] that $\mathcal{O}_{R,J_R}$ are unital UCT Kirchberg algebras and are therefore classified up to *-isomorphism (homeomorphism of non-commutative spaces) by their K-theory and the class of the unit in K_0 by the Kirchberg-Phillips Theorem [46]. Calculating the K-theory of $\mathcal{O}_{R,J_R}$ is therefore of fundamental importance in understanding these C^*-algebras.

Kajiwara and Watatani in [28] computed the K-theory of several examples of C^*-algebras associated to rational functions using special identifications of their dynamics and topology of their Julia sets, but they did not present a general method for calculation. Nekrashevych in [44] computed the K-theory of $\mathcal{O}_{R,J_R}$ in the case that R is hyperbolic and post-critically finite. While these are very interesting examples, there are only countably many distinct conjugacy classes of such R and they do not capture all of the diverse behaviour of rational dynamics. Moreover, the method Nekrashevych employs to calculate the K-theory does not extend past his example class, so new techniques must be developed.

In this thesis, we calculate the K-theory of $\mathcal{O}_{R,J_R}$ for a general rational function using an entirely different approach. We also calculate the K-theory of the C^*-algebras associated to the dynamics of R on $\hat{\mathbb{C}}$ ($\mathcal{O}_{R,\hat{\mathbb{C}}}$) and on F_R ($\mathcal{O}_{R,F_R}$). A corollary is that, for polynomials of a fixed degree, there are only finitely many isomorphism classes of the C^*-algebras $\mathcal{O}_{R,J_R}$ and two such C^*-algebras are isomorphic if and only if their corresponding rational functions have the same number of critical points inside their Julia sets and have the same number of Fatou cycles. In general, we express the K-theory of $\mathcal{O}_{R,J_R}$ in terms of the Fatou cycles and the kernel and co-kernel of a matrix associated to the oriented Herman cycles of R (a type of Fatou cycle) and the location of its critical points relative to these orientations. A new result from our calculations is that these kernel and co-kernel groups, as well as the Fatou cycle length data of R, are invariants for the topological conjugacy class of R restricted to its Julia set.

Our approach is to study a category **Cor** where the objects are C^*-correspondences and the morphisms are "intertwiners" between C^*-corrrespondences. A special sub-category of **Cor** has been studied by a number of authors, see for instance [38, Section 2.4] and the references therein. To a correspondence one can always associate a class in a KK^0 group.

We show for the first time this association is functorial. We are also the first to study exactness in this category and its KK-theoretic consequences. Our main tools in calculating the K-theory of these algebras are derived from these consequences.

Out of all families of rational functions, the quadratic family $f_c(z) = z^2 + c$, c in $\mathbb{C}$, has been studied the most extensively. By the work of Douady and Hubbard [18], we know that there is a deep relationship between the dynamical properties of f_c and the location of the parameter c relative to the Mandelbrot set $\mathcal{M} \subseteq \mathbb{C}$. For instance, by [18, Proposition 11(b)], the set of parameters c for which f_c is *structurally stable* on its Julia set (or J-stable) is precisely $\mathbb{C} \setminus \partial\mathcal{M}$. One of the most important conjectures of complex dynamics is that J-stability of f_c is equivalent to hyperbolicity of f_c. This conjecture is equivalent to the conjecture that hyperbolic quadratics are dense. See [36, p. 201] for a brief discussion of the importance of this conjecture or [37], [9] for surveys on progress made so far.

We specialize our K-theory calculations to the case of the quadratics and show there are only four isomorphism types of C^*-algebras, and the isomorphism type is dependent on the location of 0 relative to the filled Julia set K_c of f_c. A consequence of this is that the Density of Hyperbolicity Conjecture is equivalent to the density of quadratics with a certain C^*-algebra.

1.1 Statement of Results

1.1.1 K-theory of $\mathcal{O}_{R,\hat{\mathbb{C}}}$

The C^*-algebra $\mathcal{O}_{R,\hat{\mathbb{C}}}$ of a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the Exel crossed product of $C(\hat{\mathbb{C}})$ by the endomorphism $R^* : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ and the transfer operator Φ, defined for f in $C(\hat{\mathbb{C}})$ as

$$\Phi(f)(z) = \sum_{w:R(w)=z} \operatorname{ind}_R(w) f(w), \ \ z \text{ in } \hat{\mathbb{C}},$$

where $\operatorname{ind}_R(w)$ is the (positive) local winding number of $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ about w. The weightings ind_R are required so that Φ maps continuous functions to continuous functions. Kajiwara and Watatani first define $\mathcal{O}_{R,\hat{\mathbb{C}}}$ as the Cuntz-Pimsner algebra of the natural bi-module $E_{R,\hat{\mathbb{C}}}$ associated to R^* and Φ, but these are equivalent constructions ([28, Proposition 3.2]). The C^*-algebras $\mathcal{O}_{R,X}$ for $X = F_R, J_R$ are defined similarly and were also first studied in [28].

Our first new result is the calculation of the K-theory of $\mathcal{O}_{R,\hat{\mathbb{C}}}$. For each $X = F_R, \hat{\mathbb{C}}, J_R$, we shall denote by $C_{R,X}$ the critical points $\{c \in X : \operatorname{ind}_R(c) > 1\}$. These are finite sets. Denote $|C_{R,X}| = c_{R,X}$.

Theorem 1 (5.0.3). *Let R be a rational function of degree $d > 1$. Then,* $K_0(\mathcal{O}_{R,\hat{\mathbb{C}}}) \simeq \mathbb{Z}^{c_{R,\hat{\mathbb{C}}}+1}$, *with the class of the unit corresponding to a generator in a minimal generating set for* $\mathbb{Z}^{c_{R,\hat{\mathbb{C}}}+1}$*, and* $K_1(\mathcal{O}_{R,\hat{\mathbb{C}}}) \simeq \mathbb{Z}$.

There are rational functions with $J_R = \hat{\mathbb{C}}$. Some examples are considered by Lattès in [35], and Rees showed in [50] there is a "positive measure" subset of rational functions which have $J_R = \hat{\mathbb{C}}$. Thus, we have already determined the K-theory of $\mathcal{O}_{R,J_R}$ for a large class of examples.

1.1.2 K-theory of $\mathcal{O}_{R,F_R}$

Our second new result we present is the K-theory of $\mathcal{O}_{R,F_R}$. The *Fatou set* F_R is the maximal open set U such that the iterates $\{R^{\circ n}|_U\}_{n\in\mathbb{N}}$ are pre-compact in the compact-open topology on $C(U,\hat{\mathbb{C}})$. It can be shown to be *totally invariant* in the sense that $R^{-1}(F_R) = F_R$ (see [39, Lemma 4.3]). For this reason, $\mathcal{O}_{R,F_R}$ is an ideal of $\mathcal{O}_{R,\hat{\mathbb{C}}}$. A maximal connected component U of F_R is called a *Fatou component*. It is clear that R maps any Fatou component onto another such component. A finite collection P of Fatou components is a *Fatou cycle* if we can write $P = \{U_1, ..., U_n\}$, where $U_i \neq U_j$ for all $i \neq j$ and $R(U_i) = U_{i+1}$ for all $i \leq n$, with indices taken modulo n. We will denote the set of Fatou cycles of R by $\mathcal{F}_R$.

By [54, Corollary 2], $\mathcal{F}_R$ must be a finite set. By [56, Theorem 1], every Fatou component is eventually mapped onto an element in a Fatou cycle. Moreover, a Fatou cycle P is either an

(1) *attracting* cycle: for every U in P, $\{R^{\circ nk} : U \to U\}_{k\in\mathbb{N}}$ converges on compact sets to an attracting fixed point in U,

(2) *parabolic* cycle: for every U in P, $\{R^{\circ nk} : U \to U\}_{n\in\mathbb{N}}$ converges on compact sets to a parabolic (see [39, Section 10]) fixed point in ∂U,

(3) *Siegel* cycle: for every U in P, $R^{\circ n} : U \to U$ is conformally conjugate to an irrational rotation on $\mathbb{D}$, or

(4) *Herman* cycle: for every U in P, $R^{\circ n} : U \to U$ is conformally conjugate to an irrational rotation on $\mathbb{A}_r = \{z \in \mathbb{C} : 1 < |z| < r\}$ for some $r > 1$.

See [39, Theorem 16.1] for a proof of this fact. This is known as the Fatou-Julia-Sullivan classification of Fatou components, as it was proven in part by Fatou and Julia, and finished by Sullivan in [56] after he adapted the techniques of Kleinian group theory to the study of complex dynamical systems.

We shall denote the set of Herman cycles for R by $\mathcal{H}_R$. They play a very special role in the groups for $R : F_R \to F_R$ and $R : J_R \to J_R$, as we shall see. Denote $f_R = |\mathcal{F}_R|$ and $h_R = |\mathcal{H}_R|$.

Theorem 2 (6.0.6). *Let R be a rational function of degree $d > 1$. Then, $K_0(\mathcal{O}_{R,F_R}) \simeq \mathbb{Z}^{c_{R,F_R}+f_R+h_R}$ and $K_1(\mathcal{O}_{R,F_R}) \simeq \mathbb{Z}^{f_R+h_R}$.*

The proof uses the classification of Fatou components in an essential way. The Herman cycles are counted twice due to the non-triviality of K^{-1} of an annulus. In all other cycle types, any compact subset of the cycle will be eventually mapped into an "attractor" with trivial K^{-1} (see Corollary 2.5.17).

1.1.3 K-theory of $\mathcal{O}_{R,J_R}$

Our third new and most important result is the K-theory of $\mathcal{O}_{R,J_R}$. The *Julia set* J_R is the complement of the Fatou set. It is a compact uncountable subset of $\hat{\mathbb{C}}$ with no isolated points and is either the entire Riemann sphere, or it has no interior. Moreover, repelling periodic points of R are dense in J_R, and for every open set $V \subseteq J_R$, there is some number n such that $R^{\circ n}(V) = J_R$; see [39, Theorem 14.1] and [39, Corollary 14.2] for proofs of these assertions. Thus, the dynamics restricted to J_R behaves chaotically. The above dynamical properties also imply $\mathcal{O}_{R,J_R}$ is purely infinite and simple (see [28, Theorem 3.8]). For further properties of these algebras, see [23], [29] and [22].

We now present the K-theory of $\mathcal{O}_{R,J_R}$. Let Q be a Herman cycle of length n and U a Fatou component in Q. By [39, Lemma 15.7] the boundary of U has two connected components. We orient Q by first choosing a boundary component $\partial^+ U$ of ∂U. We then orient every other element V in Q (choosing a component $\partial^+ V$ of ∂V) by declaring R to be orientation preserving on Q, in the sense that $\partial^+ R^{\circ i}(U) = R^{\circ i}(\partial^+ U)$ for every $i \leq n$. Since $R^{\circ n} : U \to U$ is conjugate to an irrational rotation, it is homotopic to the identity. Therefore, $R^{\circ n}(\partial^+ U) = \partial^+ U$ and thus the orientation given to Q is well defined. Note also that there are only two possible orientations on Q defined in this way. We call Q with such a choice of orientation an *oriented Herman cycle.*

We will assume now that all of our Herman cycles in $\mathcal{H}_R$ are oriented. For x in J_R and Q in $\mathcal{H}_R$, we will let $H_Q(x)$ be the number of Fatou components U in Q for which x is in the connected component of $\hat{\mathbb{C}} \setminus U$ containing $\partial^+ U$. The function $H_Q : J_R \to \mathbb{Z}$ is continuous and satisfies the property that $H_Q - \Phi(H_Q)$ is constant (Corollary 7.0.6).

Moreover, one can show $\{H_Q\}_{Q\in\mathcal{H}_R} \cup \{1_{J_R}\}$ are linear independent over $\mathbb{Z}$ and generate the subgroup $\{f \in C(J_R,\mathbb{Z}) : f - \Phi(f) \text{ is constant}\}$ (Proposition 7.0.7).

We let H_R be the following matrix with rows indexed by elements in $C_{R,J_R} \cup \{u\}$ and columns indexed by $\mathcal{H}_R \cup \{u\}$ (u is an extra index we must add):

- $(H_R)_{c,Q} = H_Q(c)$ for all c in C_{R,J_R} and Q in $\mathcal{H}_R$,
- $(H_R)_{c,u} = 1$, for all c in C_{R,J_R},
- $(H_R)_{u,Q} = \Phi(H_Q) - H_Q$, for all Q in $\mathcal{H}_R$, and
- $(H_R)_{u,u} = \deg(R) - 1$.

H_R determines a group homomorphism $H_R : \mathbb{Z}[\mathcal{H}_R \cup \{u\}] \to \mathbb{Z}[C_{R,J_R} \cup \{u\}]$. Denote by ω_R the greatest common divisor of the Fatou cycle lengths $\{|P|\}_{P\in\mathcal{F}_R}$. If $\mathcal{F}_R = \emptyset$, then we set $\omega_R = 1$.

Theorem 3 (7.0.9). *Let R be a rational function of degree $d > 1$. Then, $K_1(\mathcal{O}_{R,J_R}) \simeq$ ker$(H_R) \oplus \mathbb{Z}/\omega_R\mathbb{Z} \oplus \mathbb{Z}^{|f_R-1|}$ and $K_0(\mathcal{O}_{R,J_R}) \simeq$ co-ker$(H_R) \oplus \mathbb{Z}^{|f_R+h_R-1|}$, with class of the unit corresponding to the class of u in co-ker(H_R).*

Interestingly, the matrix H_R, its kernel, or its co-kernel do not appear anywhere in the complex dynamics literature. Since these two groups are invariants for the topological conjugacy class of $R : J_R \to J_R$ (see Corollary 8.1.3), it would be worthwhile to compare it with other invariant combinatorial objects of R constructed from oriented Herman cycles, like its Shishikura tree [55].

When $\mathcal{H}_R = \emptyset$ (which happens, for instance, for all polynomials), we have $\ker(H_R) = 0$ and co-ker$(H_R) \simeq \mathbb{Z}^{c_{R,J_R}}$ if $C_{R,J_R} \neq \emptyset$ and co-ker$(H_R) \simeq \mathbb{Z}/(d-1)\mathbb{Z}$ otherwise, where d is the degree of R. In the first case, the class of the unit corresponds to a generator in a minimal generating set for $\mathbb{Z}^{c_{R,J_R}}$, while in the second case the unit generates $\mathbb{Z}/(d-1)\mathbb{Z}$.

Nekrashevych in [44, Theorem 6.6] computed the K-theory of $\mathcal{O}_{R,J_R}$ in the special case that R is hyperbolic and post-critically finite. This just means that every critical point is eventually mapped to an attracting periodic orbit. There are only countably many distinct conjugacy classes of such R by Thurston's Rigidity Theorem (see [8, Theorem 2.2]), and for quadratic polynomials there are only two: the class of z^2 and of $z^2 - 1$. With the extra structure afforded to such a rational function, Nekrashevych constructs a descending sequence of approximations $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of the Julia set, each with manageable K-theory, computes the K-theory for $R : \mathcal{C}_{n+1} \to \mathcal{C}_n$, then takes a limit as n approaches ∞ to get the K-theory for $\mathcal{O}_{R,J_R}$. We remark that this approximation method is not viable in general, due to the fact that the Fatou set can be more complicated (the components may no longer be simply connected, and all four of the cycle types might appear, not just attractors). We review our method and our main results on the C^*-correspondence category in Section 1.2.

Since a polynomial R always has an attracting cycle of length 1 at ∞, we always have $f_R \geq 1$ and $\omega_R = 1$. We therefore have a simple characterization for when two polynomials have isomorphic C^*-algebras:

Theorem 4 (8.2.2). *Let R and S be non-linear polynomials. Then, $\mathcal{O}_{R,J_R}$ is isomorphic to $\mathcal{O}_{S,J_S}$ if and only if either*

- $c_{R,J_R} = c_{S,J_S} = 0$*, deg*$(R) =$ *deg*(S)*, and* $f_R = f_S$*, or*
- $c_{R,J_R} = c_{S,J_S} \neq 0$ *and* $f_R = f_S$*.*

1.1.4 Applications

Now, we present applications. By the *Fatou cycle length data of R* we shall mean the tuple $L_R = (|P|)_{P \in \mathcal{F}_R}$, where the entries are ordered in non-decreasing order. Similarly, we define the *Herman cycle length data of R* to be the tuple $T_R = (|Q|)_{Q \in \mathcal{H}_R}$.

Corollary 5 (8.1.2). *Let R and S be rational functions. If R and S are topologically conjugate on their Julia sets, then $L_R = L_S$ and $T_R = T_S$.*

It is interesting that data from the dynamics of R on its Fatou set is an invariant for its dynamics on the Julia set - this result reflects the rigid nature of rational dynamics. The above corollary is proven by observing that if R and S are conjugate on their Julia sets, then $R^{\circ n}$ and $S^{\circ n}$ are conjugate on their Julia sets, for all $n \in \mathbb{N}$, and so our calculations in Chapter 7 imply $f_{R^{\circ n}} = f_{S^{\circ n}}$ and $h_{R^{\circ n}} = h_{S^{\circ n}}$. By a lemma concerning elementary number theory (Lemma 8.1.1), these sequences of numbers are equal if and only if $L_R = L_S$ and $T_R = T_S$. It is not clear how this result would be proven using a dynamical argument.

We now provide a more detailed description of the K-theory for a quadratic polynomial. Every quadratic polynomial is conjugate to one of the form $f_c(z) = z^2 + c$, z in $\hat{\mathbb{C}}$, for some c in $\mathbb{C}$. Note that its only critical points are 0 and ∞, with ∞ being a super-attracting fixed point inside the Fatou set. Denote $J_{f_c} = J_c$ and $\mathcal{O}_{f_c,J_c} = \mathcal{O}_{c,J}$. Define the *filled Julia set* of f_c to be the points z for which the orbit $\{f_c^{\circ n}(z)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{C}$ and denote this set K_c. It can be shown that $\partial K_c = J_c$ and that K_c is the union of J_c along with the bounded Fatou components of f_c (see [39, Lemma 9.4]).

Corollary 6 (8.2.4). *There are four isomorphism types for $\mathcal{O}_{c,J}$, dependent on the location of* 0 *relative to the filled Julia set.*

Case 0 $(0 \notin K_c)$: *Then,* $K_1(\mathcal{O}_{c,J}) = K_0(\mathcal{O}_{c,J}) = 0$.

Case 1 $(0 \in int(K_c))$: *Then,* $K_1(\mathcal{O}_{c,J}) \simeq K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$ *and* $[1_{J_c}] = 0$.

Case 2 $(0 \in \partial K_c = J_c,\ int(K_c) \neq \emptyset)$: *Then* $K_1(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$, $K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}^2$ *and* $[1_{J_c}]$ *is a generator in a minimal generating set for* $\mathbb{Z}^2$.

Case 3 $(0 \in \partial K_c = J_c,\ int(K_c) = \emptyset)$: *Then* $K_1(\mathcal{O}_{c,J}) = 0$, $K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$ *and* $[1_{J_c}]$ *is a generator.*

Case 0,1,3 above are realized by $c = 1, 0, -2$, respectively, while case 2 is realized by $c = \frac{e^{2\pi i\varphi}}{2}(1 + \frac{e^{2\pi i\varphi}}{2})$, where φ is the golden ratio.

We now identify the corresponding C^*-algebras to each case, and for more details see the discussion below Corollary 8.2.6. Case 0 corresponds to the Cuntz algebra $\mathcal{O}_2$ considered first in [15]. Case 1 corresponds to the 2-adic ring C^*-algebra $\mathcal{Q}_2$ studied by Larsen and Li in [34], and the author thanks Chris Bruce for this identification. It also appears in other contexts; see [34, remark 3.2]. To our knowledge, case 2 corresponds to a C^*-algebra that has not been studied in the literature. It can be shown it is a C^*-algebra of a certain graph, see the figure above Corollary 8.2.8. We denote it $\mathcal{Q}_{2,\infty}$ as it seems to share properties of both $\mathcal{Q}_2$ and $\mathcal{O}_\infty$. Case 3 corresponds to the Cuntz algebra $\mathcal{O}_\infty$ considered in [15] also.

A consequence of the above Corollary is that $[1_{J_c}] = 0$ if and only if f_c is either hyperbolic or parabolic. This allows us to re-state the Density of Hyperbolicity Conjecture for quadratics in terms of the K-theory for quadratics. This conjecture was first stated by Fatou in 1920 in [21, page 73] and is still being actively pursued - see [36, p. 201] for a brief discussion of the importance of this conjecture and [37], [9] for surveys on progress made so far.

Recall that the Mandelbrot set is defined as $\mathcal{M} = \{c \in \mathbb{C} : 0 \in K_c\}$ ($= \{c \in \mathbb{C} : K_0(\mathcal{O}_{c,J}) \neq 0\}$).

Corollary 7 (8.2.8). *The Density of Hyperbolicity Conjecture is true if and only if* $\mathcal{H}' := \{c \in \mathcal{M} : [1_{J_c}] = 0 \text{ in } K_0(\mathcal{O}_{c,J})\} = \{c \in \mathcal{M} : \mathcal{O}_{c,J} \simeq \mathcal{Q}_2\}$ *is dense in* $\mathcal{M}$.

We do not claim this makes the conjecture any easier, though it is interesting to see it in C^*-algebraic terms.

1.2 Methods

1.2.1 Motivation

We now motivate our approach to calculating the K-theory of $\mathcal{O}_{R,J_R}$. The situation is that we have a short exact sequence of Cuntz-Pimsner algebras

$$0 \longrightarrow \mathcal{O}_{R,F_R} \longrightarrow \mathcal{O}_{R,\hat{\mathbb{C}}} \longrightarrow \mathcal{O}_{R,J_R} \longrightarrow 0$$

and the Pimsner-Voiculescu 6-term exact sequence ([47, Theorem 4.9]) of $\mathcal{O}_{R,F_R}$ and $\mathcal{O}_{R,\hat{\mathbb{C}}}$ can be calculated. In these cases, the exact sequences split, so it determines for us the K-theory of $\mathcal{O}_{R,F_R}$ and $\mathcal{O}_{R,\hat{\mathbb{C}}}$. The problem is that this splitting is not natural, which makes calculating the K-theory of $\mathcal{O}_{R,J_R}$ from the 6-term exact sequence determined by the above extension of C^*-algebras rather difficult. Our observation is that not only do we have a short exact sequence of C^*-algebras, we also have a short exact sequence of their defining C^*-correspondences

$$0 \longrightarrow (E_{R,F_R}, \alpha_{F_R}) \longrightarrow (E_{R,\hat{\mathbb{C}}}, \alpha_{\hat{\mathbb{C}}}) \longrightarrow (E_{R,J_R}, \alpha_{J_R}) \longrightarrow 0,$$

and this will induce exact sequences at the level of the building blocks of K-theory determined by the Pimsner-Voiculescu 6-term exact sequences. Thus, we circumvent the need to work with an un-natural splitting. This is our motivation for considering the category of C^*-correspondences, which we now present along with our main results about it.

1.2.2 Morphisms of C^*-correspondences

Recall that if A_1, A_2 are C^*-algebras, then an A_1-A_2 *correspondence* ${}_{A_1}(E,\alpha)_{A_2}$ is a left action α of A_1 as endomorphisms of a Hilbert A_2-module E. Given such a correspondence, we can always restrict the left action α to the ideal $J_{(E,\alpha)}$ consisting of elements of A_1 which act by compact operators of E. After restriction this correspondence defines a class $K(E,\alpha)$ in $KK^0(J_{(E,\alpha)}, A_2)$. We define a category for which the assignment $(E,\alpha) \to K(E,\alpha)$ is functorial.

A *morphism of correspondences* is a triple $(\varphi_1, S, \varphi_2) : {}_{A_1}(E,\alpha)_{A_2} \to {}_{B_1}(F,\beta)_{B_2}$, where $\varphi_1 : A_1 \to A_2$, $\varphi_2 : A_2 \to B_2$ are $*$-homomorphisms and $S : E \to F$ is a linear map that satisfies compatibility criterion with φ_1, φ_2 comparable to that of a co-variant representation of a Hilbert bi-module; see Definition 3.0.1. The sub-category where $A_1 = A_2$, $B_1 = B_2$, and $\varphi_1 = \varphi_2$ has been studied by many authors, usually with applications to Cuntz-Pimsner algebras in mind. See for instance [38, Section 2.4] and the references therein. However, KK-theory consequences have not been considered and this is our main focus. Our first main tool is that the assignment $(E,\alpha) \to K(E,\alpha)$ is functorial, though we will not present it like this until Remark 3.0.4.

Proposition 8 (3.0.3). *Suppose $(\varphi_1, S, \varphi_2) : {}_{A_1}(E,\alpha)_{A_2} \to {}_{B_1}(F,\beta)_{B_2}$ is a morphism of correspondences. Then, $K(E,\alpha)\hat{\otimes}[\varphi_2] = [\varphi_1]\hat{\otimes}K(F,\beta)$, where $\hat{\otimes}$ is the Kasparov product and $[\varphi_i]$, for $i = 1, 2$, is the class of the $*$-homomorphism φ_i in KK^0.*

We now present our second main tool. If $(\varphi_1, S, \varphi_2) : {}_{A_1}(E,\alpha)_{A_2} \to {}_{B_1}(F,\beta)_{B_2}$ is a morphism, then it restricts to a morphism $(\varphi_1, S, \varphi_2) : {}_{J_{(E,\alpha)}}(E,\alpha)_{A_2} \to {}_{J_{(F,\beta)}}(F,\beta)_{B_2}$. We call this functor $J : \mathbf{Cor} \to \mathbf{Cor}$.

We say a sequence

$$ {}_{A_1}(E,\alpha)_{A_2} \xrightarrow{(\varphi_1,S,\varphi_2)} {}_{B_1}(F,\beta)_{B_2} \xrightarrow{(\psi_1,T,\psi_2)} {}_{C_1}(G,\gamma)_{C_2} $$

is *J-exact* if it is exact after applying the functor J (Definition 3.0.6).

Proposition 9 (3.0.8). *If*

$$ 0 \longrightarrow {}_{A_1}(E,\alpha)_{A_2} \xrightarrow{(i_1,I,i_2)} {}_{B_1}(F,\beta)_{B_2} \xrightarrow{(q_1,Q,q_2)} {}_{C_1}(G,\gamma)_{C_2} \longrightarrow 0 $$

is a J-exact sequence of correspondences, then $\delta_{J_{(F,\beta)}}\hat{\otimes}K(E,\alpha) = K(G,\gamma)\hat{\otimes}\delta_{B_2}$, where $\delta_{J_{(F,\beta)}}$, δ_{B_2} are the classes of the short exact sequences

$$ 0 \longrightarrow J_{(E,\alpha)} \xrightarrow{i_1} J_{(F,\beta)} \xrightarrow{q_1} J_{(G,\gamma)} \longrightarrow 0 $$

$$ 0 \longrightarrow A_2 \xrightarrow{i_2} B_2 \xrightarrow{q_2} C_2 \longrightarrow 0, $$

respectively, in KK^1.

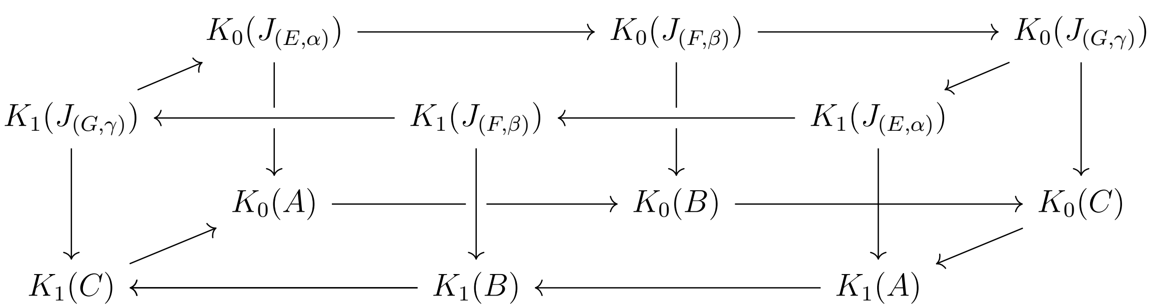
This result can be thought of as an extension of the usual naturality result for an extension class in KK^1 and is proven as such.

Applying Proposition 5 and Proposition 6 above to K-theory yields the following diagram, which is our main tool in this thesis.

Corollary 10 (3.0.9). *If*

$$ 0 \longrightarrow {}_{A}(E,\alpha)_{A} \xrightarrow{(i,I,i)} {}_{B}(F,\beta)_{B} \xrightarrow{(q,Q,q)} {}_{C}(G,\gamma)_{C} \longrightarrow 0 $$

is a J-exact sequence of correspondences, then the following diagram commutes



The top and bottom horizontal faces are the 6-term exact sequences of K-theory associated to the respective extensions of C^*-algebras, and the vertical maps are the respective maps of the form $\hat{\otimes}_i K(H,\eta)$, $(H,\eta) = (E,\alpha),(F,\beta),(G,\gamma)$.

The result in the thesis is more generally stated for non self-correspondences; see Corollary 3.0.9. We can also get similar diagrams for K-homology (with arrows reversed) and more generally for $KK^*(-,A)$, $KK^*(A,-)$, where A is any separable nuclear C^*-algebra.

We can extract exact sequences involving the K-theory building blocks of $\mathcal{O}_{R,F_R}$, $\mathcal{O}_{R,\hat{\mathbb{C}}}$ and $\mathcal{O}_{R,J_R}$ by applying the Snake Lemma strategically to certain vertical faces of the above diagram. This is done in Chapter 7, and it is how we calculate the K-theory of $\mathcal{O}_{R,J_R}$.

1.2.3 Branched functions

Inspired by Kajiwara and Watatani's construction of a C^*-correspondence from a rational function, we define C^*-correspondences from *branched functions*. A function $F : X \to Y$ is branched if at every point in its range, it admits a system of inverse branches; see Definition 4.0.1. This represents a broad and interesting class of functions, including all holomorphic functions between Riemann surfaces. In particular, we can for the first time define a C^*-algebra from an arbitrary complex dynamical system without excluding its branched points.

No similar definition of a branched function exists in the literature. Our reason for introducing branched functions here is that we will be working with holomorphic functions restricted to a variety of subspaces, and we would like a unified definition for C^*-correspondences associated to these to make applying our results on morphisms of C^*-correspondences to such restrictions easier.

We present two results that will be useful for us when calculating K-theory. Denote the C^*-correspondence of a branched function $F : X \to Y$ by $(E_{F,X}, \alpha_X)$.

Proposition 11 (4.0.7). *If $F : X \to Y$ is a branched function and $U \subseteq X$, $V \subseteq Y$ are open sets such that $F(U) \subseteq V$, then $F : U \to V$ is branched and the inclusion functions induce a morphism ${}_{C_0(U)}(E_{F,U}, \alpha_U)_{C_0(V)} \to {}_{C_0(X)}(E_{F,X}, \alpha_X)_{C_0(Y)}$*

We also characterize when a restriction of a branched function to a closed set yields a short J-exact sequence. For a branched function $F : X \to Y$, denote by $C_{F,X} = \{x \in X : F \text{ is not locally injective at } x\}$.

Proposition 12 (4.0.9). *Let $F : U \to V$ be a branched function and Y a closed subset of V. Denote $F^{-1}(Y) = X$. Then, inclusion and restriction induce morphisms $(E_{F,U\setminus X}, \alpha_{U\setminus X}) \to (E_U, \alpha_U)$ and $(E_U, \alpha_U) \to (E_X, \alpha_X)$, respectively. $C_{F,U} \cap X = C_{F,X}$ if and only if the sequence*

$$0 \longrightarrow (E_{F,U\setminus X}, \alpha_{U\setminus X}) \longrightarrow (E_{F,U}, \alpha_U) \longrightarrow (E_{F,X}, \alpha_X) \longrightarrow 0$$

is J-exact.

As a special case, any complex dynamical system $R : M \to M$, where M is a connected Riemann surface, will determine a J-exact sequence of correspondences via the restriction to its Julia set. See Section 4.1 for more information on these correspondences.

1.3 Organization

In Chapter 2, we provide the necessary background for this thesis. We review Hilbert modules, C^*-correspondences, Cuntz-Pimsner algebras, K-theory, KK-theory and complex dynamical systems. In Chapter 3 we define the category of C^*-correspondences and study its relation to KK-theory. We define branched functions in Chapter 4, construct their associated C^*-correspondences and study them within the context of our category. We specialize to branched functions that are restrictions of holomorphic functions in Section 4.1, prove some regularity properties about their C^*-correspondences and describe their action on K-theory. Chapters 5, 6 and 7 calculate the K-theory of $\mathcal{O}_{R,\hat{\mathbb{C}}}$, $\mathcal{O}_{R,F_R}$ and $\mathcal{O}_{R,J_R}$, respectively. Chapter 8 is devoted to proving applications of our K-theory results.

Chapter 2

Background

2.1 C*-algebras

We assume a level of familiarity with the basic terminology and theory of C^*-algebras. We recall the concepts pertaining to C^*-algebras we use in this thesis. None of these concepts are due to us, and the reader may consult the references provided for more details.

A map $\varphi : A \to B$ between C^*-algebras A and B is a **-homomorphism* if it is an algebra homomorphism and additionally $\varphi(a)^* = \varphi(a^*)$ for all $a \in A$. It is a remarkable fact that every $*$-homomorphism is norm contracting and therefore continuous [52, Section 1.1.4]. A $*$-homorphism $\varphi : A \to B$ is a *$*$-isomorphism* if it is a bijection. From the norm contracting property, it follows that such a φ is automatically isometric.

If B is a C^*-algebra, then $A \subseteq B$ is a *C^*-sub-algebra* if it is a C^*-algebra with respect to all the operations (addition, multiplication, involution, norm) inherited from B. In particular, A is closed.

As an example, if $\varphi : A \to B$ is a $*$-homomorphism, then $\varphi(A) \subseteq B$ is always a C^*-sub-algebra [52, Section 1.1.4].

If A is a C^*-algebra, then by an *ideal of* A we shall mean a closed (vector) subspace $I \subseteq A$ such that $I^* = I$ and $AI \subseteq I$. In particular, I is a C^*-sub-algebra of A.

Recall that if I is an ideal of A, then the quotient (as a Banach space) A/I has a unique C^*-algebra structure making the quotient map $q : A \to A/I$ a $*$-homomorphism.

A C^*-algebra A is *simple* if the only ideals of A are 0 and A.

An *approximate unit* for a C^*-algebra A is a net $(u_\lambda)_{\lambda\in\Lambda} \subseteq A$ of positive elements such that $\|u_\lambda\| \leq 1$ for all λ in Λ, $u_{\lambda'} \leq u_\lambda$ whenever $\lambda' \leq \lambda$, and $\lim_\lambda u_\lambda a = a$ for all a in A. By [43, Theorem 3.1.1], every C^*-algebra has an approximate unit.

Following [13], we say a C^*-sub-algebra A of a C^*-algebra B is *full* if A is not contained in any proper ideal of B. It is easy to see this is equivalent to A containing an approximate unit which is also an approximate unit for B.

A C^*-sub-algebra A of a C^*-algebra B is *hereditary* if a positive element b in B is in A whenever there is a in A such that $b \leq a$. By [13], if A is a full and hereditary C^*-sub-algebra of B, then A and B are *stably isomorphic*, a fact that we shall use once in Chapter 3.

A simple C^*-algebra A is *purely infinite* if for every positive elements a, b in A and $\epsilon > 0$, there is r in A such that $\|r^*ar - b\| \leq \epsilon$. Note that this property implies A is simple. There is a more general definition of purely infinite for non-simple C^*-algebras due to [26], but we will not need this for our purposes.

2.2 C^*-correspondences

We briefly review the concepts of Hilbert C^*-modules and C^*-correspondences, as well as some of their auxiliary notions. Our review of the standard definitions closely follow the exposition of Jensen and Thomsen in [27, Section 1]. See [33] for a more detailed treatment of the theory of Hilbert C^*-modules.

Let A be a C^*-algebra. A *right Hilbert A-module* is a complex vector space E with a right A-module structure, together with a right $\mathbb{C}$-linear map $\langle \cdot, \cdot \rangle : E \times E \to A$ that is also

(1) right A-linear: $\langle e_1, e_2 \cdot a \rangle = \langle e_1, e_2 \rangle b$, for all e_1, e_2 in E and a in A,

(2) Hermitian: $\langle e_1, e_2 \rangle^* = \langle e_2, e_1 \rangle$, for all e_1, e_2 in E,

(3) positive: $\langle e, e \rangle \geq 0$, for all e in E,

(4) faithful: $\langle e, e \rangle \neq 0$, for all non-zero e in E,

(5) and E is complete with respect to the norm $\|e\| := \|\sqrt{\langle e, e \rangle}\|$.

The fact that $\| \cdot \|$ is a norm requires some justification. Clearly $\| \cdot \|$ is homogeneous in the sense that $\|e \cdot z\| = \|e\||z|$ for any $e \in E$ and $z \in \mathbb{C}$. Sub-additivity of $\| \cdot \|$ will follow from the Cauchy-Schwarz inequality for Hilbert A-modules, which we prove for completeness. The proof follows that in [33, Proposition 1.1].

Proposition 2.2.1. *Let E be a right Hilbert A-module. Then, for any e_1, e_2 in E, we have*

$$\langle e_2, e_1 \rangle \langle e_1, e_2 \rangle \leq \|e_1\|^2 \langle e_2, e_2 \rangle.$$

Proof. By linearity of $\langle \cdot, \cdot \rangle$, it suffices to prove in the case where $\|e_1\| = 1$. By positivity, we have $0 \leq \langle e_1 \cdot a - e_2, e_1 \cdot a - e_2 \rangle$, where $a = \langle e_1, e_2 \rangle$. Expanding out by A-linearity of $\langle \cdot, \cdot \rangle$ and using the inequality $a^* \langle e_1, e_1 \rangle a \leq a^* a$ yields the proposition. $\square$

In particular, proposition 2.2.1 implies $\|\langle e_1, e_2\rangle\| \leq \|e_1\|\|e_2\|$. So, if e_1, e_2 are in E, then

$$\|e_1 + e_2\|^2 = \|\langle e_1 + e_2, e_1 + e_2\rangle\| \leq \|e_1\|^2 + 2\|e_1\|\|e_2\| + \|e_2\|^2 = (\|e_1\| + \|e_2\|)^2,$$

proving sub-additivity of $\|\cdot\|$.

We will call $\langle\cdot,\cdot\rangle$ an *A-inner product* if it satisfies the conditions $(1) - (5)$.

All C^*-algebras and Hilbert modules in this thesis will be assumed *seperable* (contains a countable dense subset).

We say a Hilbert A-module is *full* if the image of the B-inner product linearly spans a dense sub-*-algebra in A. In general, the closed linear span of $\langle\cdot,\cdot\rangle$ is an ideal in A - this is immediate from the Hermitian property and right A-linearity of $\langle\cdot,\cdot\rangle$. We shall denote this ideal by $\langle E\rangle$.

A right A-module endomorphism $T : E \to E$ is said to be *adjointable* if there exists another right A-module endomorphism $T^* : E \to E$ such that

$$\langle T(e_1), e_2\rangle = \langle e_1, T^*(e_2)\rangle \text{ for all } e_1, e_2 \text{ in } E.$$

T^* is called the *adjoint* of T. The collection of all bounded, adjointable right A-module endomorphisms is denoted $\mathcal{B}(E)$. It is a C^*-algebra with pointwise addition of endomorphisms as addition, composition as multiplication, the adjoint as the involution, and the operator norm induced from $(E, \|\cdot\|)$ as the norm. The only fact which is not immediately obvious is the C^*-identity $\|T^*T\| = \|T\|^2$. To show this, first observe from the inequality $\|\langle e_1, e_2\rangle\| \leq \|e_1\|\|e_2\|$ that $\|e\| = \sup\{\|\langle e, f\rangle : f \in E,\ \|f\| \leq 1\}$, for any e in E. Then, for any T in $\mathcal{B}(E)$, we have $\|T^*T\| = \sup\{\langle T(e), T(f)\rangle : e, f \in E,\ \|e\| \leq 1, \|f\| \leq 1\} = \|T\|^2$.

For each e_1, e_2 in E, define the endomorphism $\theta_{e_1,e_2} : E \to E$ as

$$\theta_{e_1,e_2}(e_3) = e_1 \cdot \langle e_2, e_3\rangle, \text{ for all } e_3 \text{ in } E.$$

The closed linear span of all such endomorphisms is denoted as $\mathcal{K}(E)$, the *compact operators of* E. Since $T \circ \theta_{e_1,e_2} = \theta_{T(e_1),e_2}$ and $(\theta_{e_1,e_2})^* = \theta_{e_2,e_1}$ for all T in $\mathcal{B}(E)$ and e_1, e_2 in E, it follows that $\mathcal{K}(E)$ is a closed ideal of $\mathcal{B}(E)$.

To a Hilbert A-module E, we may construct its *linking algebra*

$$\mathcal{L}(E) = \left\{ \begin{pmatrix} k & e \\ f & a \end{pmatrix} : k \in \mathcal{K}(E), e, f \in E, a \in \langle E\rangle \right\},$$

which is a C^*-algebra with involution $\begin{pmatrix} k & e \\ f & a \end{pmatrix}^* = \begin{pmatrix} k^* & f \\ e & a^* \end{pmatrix}$, multiplication

$$\begin{pmatrix} k_1 & e_1 \\ f_1 & a_1 \end{pmatrix} \cdot \begin{pmatrix} k_2 & e_2 \\ f_2 & a_2 \end{pmatrix} = \begin{pmatrix} k_1k_2 + \theta_{e_1,f_2} & k_1(e_2) + e_1a_2 \\ k_2^*(f_1) + f_2a_1^* & \langle f_1, e_2 \rangle + a_1a_2 \end{pmatrix},$$

and addition

$$\begin{pmatrix} k_1 & e_1 \\ f_1 & a_1 \end{pmatrix} + \begin{pmatrix} k_2 & e_2 \\ f_2 & a_2 \end{pmatrix} = \begin{pmatrix} k_1 + k_2 & e_1 + e_2 \\ f_1 + f_2 & a_1 + a_2 \end{pmatrix}.$$

See [49, Lemma 3.19] for a description of the C^*-norm on $\mathcal{L}(E)$. $\mathcal{K}(E)$ and $\langle E \rangle$ are full and hereditary C^*-sub-algebras of $\mathcal{L}(E)$ with respect to the obvious corner embeddings ([49, Theorem 3.19]).

Let E be a Hilbert A-module, F a Hilbert B-module, and $\varphi : A \to B$ a $*$-homomorphism. A *φ-twisted morphism from E to F* is a linear map $S : E \to F$ satisfying $\langle S(e), S(f) \rangle = \varphi(\langle e, f \rangle)$ for all e, f in E. This is a definition we make, but we do not claim originality. Certainly similar maps have been considered in the literature (see Remark 3.0.2).

It follows that $S(e \cdot a) = S(e) \cdot \varphi(a)$, for all e in E and a in A.

Proposition 2.2.2. *If $S : E \to F$ is a φ-twisted morphism of Hilbert Modules, then there is a well defined $*$-homomorphism $\hat{S} : \mathcal{K}(E) \to \mathcal{K}(F)$, determined by the equation*

$$\hat{S}(\sum_{i=1}^{n} \theta_{e_i,f_i}) = \sum_{i=1}^{n} \theta_{S(e_i),S(f_i)}, \text{ for any } \{e_i\}_{i=1}^n, \{f_i\}_{i=1}^n \subseteq E.$$

Proof. It suffices to show that $\hat{S}$ is well defined, as once this is verified, the fact that it is a $*$-homomorphism becomes clear. First, we show for any elements $\{e_i\}_{i=1}^n, \{f_i\}_{i=1}^n \subseteq E$, the norm of $\sum_{i=1}^n \theta_{e_i,f_i}$ is equal to the norm of the matrix $\sqrt{F}\sqrt{E}$ in $M_n(A)$, where the (i, j) entry of E, F, is $\langle e_i, e_j \rangle$, $\langle f_i, f_j \rangle$, respectively.

Suppose first $n = 1$, and lets show $\|\theta_{e,f}\| = \|\sqrt{\langle f, f \rangle}\sqrt{\langle e, e \rangle}\|$. Using the C^*-identity for the norms, this is equivalent to showing $\|\theta_{f\langle e,e \rangle,f}\| = \|\sqrt{\langle e, e \rangle}\langle f, f \rangle\sqrt{\langle e, e \rangle}\|$. By embedding $\theta_{f\langle e,e \rangle,f}$ into the linking algebra, we see that

$$\|\theta_{f\langle e,e \rangle,f}\| = \left\| \begin{pmatrix} 0 & 0 \\ e \cdot \sqrt{\langle f, f \rangle} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ e \cdot \sqrt{\langle f, f \rangle} & 0 \end{pmatrix}^* \right\| = \left\| \begin{pmatrix} 0 & e \cdot \sqrt{\langle f, f \rangle} \\ 0 & 0 \end{pmatrix} \right\|^2$$

and by embedding $\sqrt{\langle e, e \rangle}\langle f, f \rangle\sqrt{\langle e, e \rangle} = \left\langle f \cdot \sqrt{\langle e, e \rangle}, f \cdot \sqrt{\langle e, e \rangle} \right\rangle$ into the linking algebra, we see that

$$\Big\| \begin{pmatrix} 0 & e\cdot\sqrt{\langle f,f\rangle} \\ 0 & 0 \end{pmatrix} \Big\|^2 = \Big\| \begin{pmatrix} 0 & e\cdot\sqrt{\langle f,f\rangle} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & e\cdot\sqrt{\langle f,f\rangle} \\ 0 & 0 \end{pmatrix}^* \Big\| = \|\sqrt{\langle e,e\rangle}\langle f,f\rangle\sqrt{\langle e,e\rangle}\|.$$

The general case reduces to the one above by an amplification trick. Let $L = \mathcal{L}(E)$ be the linking algebra, and consider the Hilbert $M_n(L)$-module structure over L^n, where an element $\underline{b}$ in L^n is written in row notation $\underline{b} = (b_1, .., b_n)$ and thus $M_n(L)$ acts naturally on the right. The $M_n(L)$ inner product of $\underline{b} = (b_1, ..., b_n)$, $\underline{b'} = (b'_1, ..., b'_n)$ in L^n is the matrix with entries $(\langle \underline{b}, \underline{b'}\rangle)_{i,j} = b_i^* b'_j$ for $1 \le i, j \le n$. Identify $\mathcal{K}(L^n)$ with L by noticing the compact operator $\theta_{\underline{b},\underline{b'}}$ acts diagonally on L^n by $\sum_{i=1}^n b_i(b'_i)^*$.

Now, given $\{e_i\}_{i=1}^n, \{f_i\}_{i=1}^n \subseteq E$, set $l(e_i) = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}$, $\underline{e} = (l(e_1), .., l(e_n))$ and $l(f_i) = \begin{pmatrix} 0 & 0 \\ f_i & 0 \end{pmatrix}$, $\underline{f} = (l(f_1), ..., l(f_n))$. From above, $\|\sum_{i=1}^n \theta_{e_i,f_i}\| = \|\theta_{\underline{e},\underline{f}}\| = \|\sqrt{\langle \underline{f}, \underline{f}\rangle}\sqrt{\langle \underline{e}, \underline{e}\rangle}\|$. Note that $(\langle \underline{f}, \underline{f}\rangle)_{i,j} = \begin{pmatrix} 0 & 0 \\ 0 & (F)_{i,j} \end{pmatrix}$ and $(\langle \underline{e}, \underline{e}\rangle)_{i,j} = \begin{pmatrix} 0 & 0 \\ 0 & (E)_{i,j} \end{pmatrix}$. Therefore $\iota(\sqrt{F}\sqrt{E}) = \sqrt{\langle \underline{f}, \underline{f}\rangle}\sqrt{\langle \underline{e}, \underline{e}\rangle}$, where $\iota : M_n(\langle E\rangle) \to M_n(L)$ is the co-ordinate wise extension of the embedding $\langle E\rangle \to L$. Since ι is an injective $*$-homomorphism, we have $\|\sqrt{\langle \underline{f}, \underline{f}\rangle}\sqrt{\langle \underline{e}, \underline{e}\rangle}\| = \|\sqrt{F}\sqrt{E}\|$, proving the claim.

Now, to prove the proposition, it suffices to show $\|\sum_{i=1}^n \theta_{S(e_i),S(f_i)}\| \le \|\sqrt{F}\sqrt{E}\|$. From above, the norm of $\|\sum_{i=1}^n \theta_{S(e_i),S(f_i)}\|$ is equal to the norm of $\|\sqrt{F'}\sqrt{E'}\|$, where $(F')_{i,j} = \langle S(f_i)S(f_j)\rangle = \varphi(\langle f_i, f_j\rangle)$ and $(E')_{i,j} = \langle S(e_i)S(e_j)\rangle = \varphi(\langle e_i, e_j\rangle)$ for all $1 \le i, j \le n$. Therefore, $\sqrt{F'}\sqrt{E'} = \varphi_n(\sqrt{F}\sqrt{E})$, where $\varphi_n : M_n(A) \to M_n(B)$ is the co-ordinate wise extension of the $*$-homomorphism φ. Since φ_n is a $*$-homomorphism, it follows that $\|\varphi_n(\sqrt{F}\sqrt{E})\| \le \|\sqrt{F}\sqrt{E}\|$. This last inequality finishes the proposition. □

Now, it is easy to check that $\mathcal{L}(S) : \mathcal{L}(E) \to \mathcal{L}(F)$, defined for $\begin{pmatrix} k & e \\ f & a \end{pmatrix}$ in $\mathcal{L}(E)$ as

$$\mathcal{L}(S)\Big(\begin{pmatrix} k & e \\ f & a \end{pmatrix}\Big) = \begin{pmatrix} \hat{S}(k) & S(e) \\ S(f) & \varphi(a) \end{pmatrix}$$

is a $*$-homomorphism. It follows that any φ-twisted morphism S has closed image.

Let A and B be C^*-algebras. An A-B *C^*-correspondence* is a right Hilbert B-module E together with a $*$-homomorphism $\alpha : A \to \mathcal{B}(E)$. We will call α the *action of A on E* and will frequently use the notation $a \cdot e = \alpha(a)(e)$, a in A, e in E. A correspondence will be denoted ${}_A(E,\alpha)_B$. We will frequently write ${}_A(E,\alpha)_B = (E,\alpha)$ if the algebras A and B are specified. (E,α) is *faithful* if α is injective, *full* if E is full, and *non-degenerate* if the linear span of $A \cdot E$ is dense in E.

We shall denote $J_{(E,\alpha)} := \alpha^{-1}(\mathcal{K}(E))$. Note that this is an ideal of A.

We can compose an A-B C^*-correspondence (E,α) with a B-C C^*-correspondence (F,β) to obtain an A-C C^*-correspondence $(E \otimes_B F, \alpha \otimes_B \mathrm{id})$. This construction is called the *balanced (or internal) tensor product* and appears, for instance, in [27, Section 1.2.3]. We review the construction briefly.

Let $E \otimes F$ be the vector space tensor product of E and F. Clearly this is still a right C-module, and carries the left A action $\alpha \otimes \mathrm{id}$. We define a map $\langle \cdot,\cdot \rangle : E \otimes F \times E \otimes F \to C$ first on basic tensors as

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle := \langle f_1, \langle e_1, e_2 \rangle \cdot f_2 \rangle, \text{ for all } e_1, e_2 \text{ in } E, \ f_1, f_2 \text{ in } F,$$

and extend $\langle \cdot,\cdot \rangle$ linearly in the right-entry, anti-linearly in the left-entry to all of $E \otimes F \times E \otimes F$. $\langle \cdot,\cdot \rangle$ satisfies the axioms (1)-(3) for a C-inner product, but not necessarily (4) and (5).

However, if we quotient out by the C-sub-module $\{g \in E \otimes F : \langle g, g \rangle = 0\}$ and complete with respect to the quotient norm of $\|\cdot\| := \|\sqrt{\langle \cdot,\cdot \rangle}\|$, then we obtain a vector space $E \otimes_B F$ such that the right C-module structure and $\langle \cdot,\cdot \rangle$ on $E \otimes F$ pass down to a Hilbert C-module structure, and the left A action passes down to a $*$-homomorphism $\alpha \otimes_B \mathrm{id} : A \to \mathcal{B}(E \otimes_B F)$. $(E \otimes_B F, \alpha \otimes_B \mathrm{id})$ is therefore an A-C C^*-correspondence.

With this composition operation, C^*-algebras become the objects in a catgeory with morphisms (unitary equivalence classes of) C^*-correspondences. The "isomorphisms" in this category are the full and faithful A-B C^*-correspondences (E,α) such that $\phi(A) = \mathcal{K}(E)$, but we will not elaborate on this. See [49, Chapter 3] for more details. Such a correspondence is called an A-B *imprimitivity bi-module*. Notice that any Hilbert B-module E defines a $\mathcal{K}(E)$-$\langle E \rangle$ imprimitivity bi-module.

Following [33, Chapter 1], we will define the *orthogonal sum* of Hilbert B-modules $(E_n)_{n\in\mathbb{N}}$ to be the vector space

$$\sum_{n\in\mathbb{N}} E_n := \left\{ (e_n)_{n\in\mathbb{N}} : \ e_n \in E_n \ \forall n \in \mathbb{N} \text{ and } \sum_{n\in\mathbb{N}} \langle e_n, e_n \rangle \text{ converges in } B \right\}$$

with B-inner product defined for $\underline{e} = (e_n)_{n\in\mathbb{N}}$ and $\underline{f} = (f_n)_{n\in\mathbb{N}}$ in $\sum_{n\in\mathbb{N}} E_n$ as $\langle \underline{e}, \underline{f} \rangle = \sum_{n\in\mathbb{N}} \langle e_n, f_n \rangle$ and B-module structure defined for b in B as $\underline{e} \cdot b = (e_n \cdot b)_{n\in\mathbb{N}}$. Note that the sum in the inner product formula converges by the Cauchy-Schwarz inequality in Proposition 2.2.1. If each Hilbert module E_n carries a left action α_n of a (fixed) C^*-algebra A, then we can define a left action α_∞ on $\sum_{n\in\mathbb{N}} E_n$ as $\alpha_\infty(a)(\underline{e}) = (\alpha_n(a)(e_n))_{n\in\mathbb{N}}$, where a is in A and $\underline{e} = (e_n)_{n\in\mathbb{N}}$ is in $\sum_{n\in\mathbb{N}} E_n$. In this case, $(\sum_{n\in\mathbb{N}} E_n, \alpha_\infty)$ is an A-B C^*-correspondence.

2.3 Cuntz-Pimsner algebras

We now review the Cuntz-Pimsner algebra construction. We refer the reader to [47] for more details.

Let A be a C^*-algebra and (E, α) a faithful, A - A C^*-correspondence. Denote $E^{\otimes_A 0} = A$, $E^{\otimes_A 1} = E$, and define, for n in $\mathbb{N}$, $E^{\otimes_A n} := E \otimes_A E^{\otimes_A (n-1)}$. The *Fock module of* E is the orthogonal sum $\sum_{n=0}^{\infty} E^{\otimes_A n} = \mathcal{F}_E$. It is itself an A-A correspondence, and we shall denote the left action α_∞ of this orthogonal sum by α_∞.

For each e in E, we can define an adjointable, bounded Fock module endomorphism $T_e : \mathcal{F}_E \to \mathcal{F}_E$ as

$$T_e(g) = e \otimes_A g, \text{ for } g \text{ in } \mathcal{F}_E.$$

As an abuse of notation, for a in A, we shall write $\alpha_\infty(a) = a$. Then, note that $T_e^* T_f = \langle e, f\rangle$ for all e, f in E. Moreover, $a \cdot T_e = T_{a\cdot e}$, $T_e \cdot a = T_{e\cdot a}$, and $T_e + T_f = T_{e+f}$ for all e, f in E and a in A. The C^*-algebra generated by $\{T_e\}_{e\in E}$ and $\{a\}_{a\in A}$ is called the *Toeplitz algebra* of E, and is denoted $\mathcal{T}_E$. By [47, Theorem 3.4] (see also [25, Proposition 4.3]), $\mathcal{T}_E$ is the universal C^*-algebra with operators T_e, e in E, and a in A satisfying the above relations. Note also the operators T_x for x in $E^{\otimes_A n}$, defined as $T_x(g) = x \otimes_A g$ for all g in $\mathcal{F}_E$ are also in $\mathcal{T}_E$.

$\mathcal{T}_E$ carries a natural continuous action of $\mathbb{T}$ by $*$-isomorphisms. For $\underline{e} = (e_n)_{n\geq 0}$ in $\mathcal{F}_E$ and z in $\mathbb{T}$ let $U_z(\underline{e}) = (z^n e_n)_{n\geq 0}$. Then, $(U_z)_{z\in\mathbb{T}}$ is a collection of unitaries in $\mathcal{B}(\mathcal{F}_E)$ satisfying $U_z \circ U_w = U_{zw}$ for z, w in $\mathbb{T}$. For $\alpha_\infty(a) = a$, we have $U_z a = a U_z$ and for e in E, $U_z T_e = T_{ze} U_z$ for all z in $\mathbb{T}$. Hence, $(\gamma_z = U_z(-)U_z^*)_{z\in\mathbb{T}}$ defines an action of $\mathbb{T}$ by $*$-isomorphisms of $\mathcal{T}_E$ which is continuous in the sense that $z \to \gamma_z(T)$ is continuous, for all T in $\mathcal{T}_E$ (see the paragraph below [25, Definition 5.6]). This action is called the *gauge action of* $\mathcal{T}_E$. This action allows us to decompose a general operator T in $\mathcal{T}_E$ into its *Fourier coefficients*. For k in $\mathbb{Z}$, let

$$T_k = \int_{\mathbb{T}} \gamma_z(T) z^{-k} dz,$$

where dz is Lebesgue measure on the circle and the integral is made sense as a limit of Riemann sums. Then, T_k is in $\mathcal{T}_E$ and $T = \sum_{k\in\mathbb{Z}} T_k$, with the sum converging uniformly in norm. Each T_k is of the form $T_k = \sum T_x T_y^*$ for some collection of x in $E^{\otimes m}$ and y in $E^{\otimes n}$ satisfying $m - n = k$. Note also that $(TS)_k = \sum_{m-n=k} T_m S_n$ and $T_k^* = T_{-k}$ for all T, S in $\mathcal{T}_E$ and k in $\mathbb{Z}$.

The Cuntz-Pimsner algebra of (E, α) will be defined as a certain quotient of $\mathcal{T}_E$ by an ideal. Let $J_E = J_{(E,\alpha)}$ and consider the Hilbert J_E-module $G_E = \{x \in \mathcal{F}_E : \langle x, x\rangle \in J_E\}$.

Proposition 2.3.1. *[47, Theorem 3.13] Let (E,α) be a faithful A-A C^*-correspondence. Then, $\mathcal{K}(G_E)$ (considered as a C^*-sub-algebra of $\mathcal{K}(\mathcal{F}_E)$) is an ideal of $\mathcal{B}(\mathcal{F}_E)$.*

Proof. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for J_E. Then, x in $\mathcal{F}_E$ is contained in G_E if and only if $(x \cdot u_\lambda)_{\lambda\in\Lambda}$ converges to x. Therefore, $T(G_E) \subseteq G_E$ for all T in $\mathcal{B}(\mathcal{F}_E)$. Therefore $T \circ (\sum_{i=1}^n \theta_{x_i,y_i}) = \sum_{i=1}^n \theta_{T(x_i),y_i}$ is in $\mathcal{K}(G_E)$ for all $\{x_i\}_{i=1}^n$, $\{y_i\}_{i=1}^n \subseteq G_E$. It follows by density of the operators of the form $\sum_{i=1}^n \theta_{x_i,y_i}$ in $\mathcal{K}(G_E)$ that this C^*-algebra is an ideal of $\mathcal{B}(\mathcal{F}_E)$. □

We shall denote $\mathcal{K}(G_E)$ (when considered a C^*-sub-algebra of $\mathcal{B}(\mathcal{F}_E)$) by $\mathcal{I}_E$. We will need two other descriptions of this ideal, which we now develop.

First, the map sending, for each pair e, f in E, a compact operator $\theta_{e,f}$ in $\mathcal{K}(E)$ to $T_eT_f^*$ extends to a $*$-embedding $\phi : \mathcal{K}(E) \to \mathcal{T}_E$, so, for every a in J_E, $a - \phi(\alpha(a))$ is in $\mathcal{T}_E$, and is in fact in $\mathcal{I}_E$. Moreover, such operators generate the ideal $\mathcal{I}_E$, and this is the "standard" definition taken for $\mathcal{I}_E$ (see [47, Definition 1.1] and [25, Definition 3.5]).

Proposition 2.3.2. *[47, Theorem 3.13] Let (E,α) be a faithful A-A C^*-correspondence. Then, $\mathcal{I}_E$ is the ideal generated in $\mathcal{T}_E$ by the operators $a - \phi(\alpha(a))$, for a in J_E.*

Proof. Let the ideal generated in $\mathcal{T}_E$ by the operators $a - \phi(\alpha(a))$, for a in J_E be denoted $\mathcal{J}_E$. If a in J_E is positive, then we have $a - \phi(\alpha(a)) = \theta_{\sqrt{a},\sqrt{a}}$, where $\sqrt{a}$ is thought of as the sequence $(e_n)_{n\geq 0}$ in $\mathcal{F}_E$ with $e_0 = \sqrt{a}$ and $e_n = 0$ for $n \geq 1$. Therefore, $\mathcal{J}_E \subseteq \mathcal{I}_E$. Since ideals are hereditary, $x = (x_n)_{n\geq 0}$ in $\mathcal{F}_E$ is in G_E if and only if $\langle x_n, x_n\rangle$ is in J_E, for all $n \geq 0$. Therefore, to show $\mathcal{I}_E \subseteq \mathcal{J}_E$, it suffices to show θ_{x_n,y_m} is in $\mathcal{J}_E$ for any x_n in $E^{\otimes_A^n}$ and y_m in $E^{\otimes_A^m}$ such that $\langle x_n, x_n\rangle$, $\langle y_m, y_m\rangle$ are in J_E, where we think of x_n and y_m as in $\mathcal{F}_E$ in the obvious way. Since an element x is in the ideal of a C^*-algebra if and only if xx^* is, we can further reduce this to the case $x := x_n = y_m$. Let T_x be the operator defined as $T_x(y) = x \otimes_A y$ for y in $\mathcal{F}_E$. Then, T_x is clearly in $\mathcal{T}_E$, and if $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit in J_E, then we have $\theta_{x\cdot u_\lambda,x} = T_x(u_\lambda - \phi(\alpha(u_\lambda))T_x^*$, which is in $\mathcal{J}_E$ and converges to $\theta_{x,x}$. Therefore, $\mathcal{I}_E \subseteq \mathcal{J}_E$. □

Second, consider, for each k in $\mathbb{N}_0$ the projection $P_k : \mathcal{F}_E \to \mathcal{F}_E$ onto the Hilbert A-sub-module $\sum_{n=0}^k E^{\hat{\otimes} n}$. We say T in $\mathcal{B}(\mathcal{F}_E)$ is *finite rank* if there is k in $\mathbb{N}$ such that $P_kT = TP_k = T$, and write $\mathcal{I}_E^c$ for the collection of all such operators inside of $\mathcal{T}_E$.

The following description can be found as condition (4) above [47, Remark 3.9]

Proposition 2.3.3. *Let (E,α) be a faithful and non-degenerate A-A C^*-correspondence. Then, $\overline{\mathcal{I}_E^c} = \mathcal{I}_E$.*

Proof. It is easy to see that $T_e \circ T$ and $T \circ T_e$ are in $\mathcal{I}_E^c$ for all e in E and T in $\mathcal{I}_E^c$. Since $\mathcal{I}_E^c$ is additive and self-adjoint, it follows that $\overline{\mathcal{I}_E^c}$ is an ideal in $\mathcal{T}_E$. The generators $a - \phi(\alpha(a))$, a in J_E, are clearly inside $\mathcal{I}_E^c$, and thus $\mathcal{I}_E \subseteq \overline{\mathcal{I}_E^c}$. We show $\mathcal{I}_E^c \subseteq \mathcal{I}_E$.

Since each unitary U_z commutes with the projection P_k, if T is in $\mathcal{I}_E^c$, then all of its Fourier coefficients T_k (and hence T) are in $\mathcal{I}_E$ if $S = T_k^* T_k$ is in $\mathcal{I}_E$, for all k in $\mathbb{Z}$. Therefore, it suffices to show T is in $\mathcal{I}_E$ for T of the form $T = \sum_{n \geq 0} \phi_n(k_n)$, where $\phi_n : \mathcal{K}(E^{\otimes_A n}) \to \mathcal{T}_E$ is the $*$-homomorphism mapping the generators $\theta_{x,y}$, for x, y in $E^{\otimes_A n}$, to $T_x T_y^*$.

For such a T in $\mathcal{I}_E^c$, let n in $\mathbb{N} \cup \{0\}$ be the smallest number such that $P_n T = T P_n = T$. For any x, y in $E^{\otimes_A n}$, consider $T_x^* T T_y = \sum_{n \geq 0} \phi_n(k_n^{x,y})$, which is zero on $\sum_{n \geq 1} E^{\otimes_A n}$. Since the only summands of $T_x^* T T_y$ that can possibly be non-zero on E are $\phi_0(k_0^{x,y})$ and $\phi_1(k_1^{x,y})$, it follows that $\alpha(k_0^{x,y}) = -k_1^{x,y}$. Hence, $T_x^* T T_y = \alpha(k_0^{x,y}) - \phi(\alpha(k_0^{x,y}))$. Therefore, for any k_1, k_2 in $\mathcal{K}(E^{\otimes_A n})$, $\phi_n(k_1) T \phi_n(k_2)$ is in $\mathcal{I}_E$. Since (E, α) is non-degenerate, if $(k_\lambda)_{\lambda \in \Lambda}$ is an approximate unit in $\mathcal{K}(E^{\otimes_A n})$, then $(\phi_n(k_\lambda))_{\lambda \in \Lambda}$ converges strongly to $1 - P_{n-1}$. Hence, $(1 - P_{n-1})T$ is in $\mathcal{I}_E^c$. Now, $P_{n-1} T = T'$ is a new operator in $\mathcal{I}_E^c$ but with minimum number n' in $\mathbb{N} \cup \{0\}$ such that $P_{n'} T' = T' P_{n'} = T'$ satisfying $n' < n$. By repeating the argument above finitely many times, we see that T is in $\mathcal{I}_E$.

□

Notice that G_E defines an $\mathcal{I}_E$-$J_E \cap \langle E \rangle$ imprimitivity bi-module. We shall denote the ideal $J_E \cap \langle E \rangle$ by I_E. It will be useful when describing the K-theory of a general Cuntz-Pimsner algebra.

The *Cuntz-Pimsner algebra of* E is the quotient $\mathcal{T}_E / \mathcal{I}_E$, and is denoted $\mathcal{O}_E$. $A \subseteq \mathcal{O}_E$ is often called the *co-efficient algebra*. Since the gauge action $(\gamma_z)_{z \in \mathbb{T}}$ fixes $\mathcal{I}_E$, it passes down to a continuous action of $\mathbb{T}$ on $\mathcal{O}_E$ and the same Fourier decomposition holds.

Remark 2.3.4. *The definition of Cuntz-Pimsner algebra is a slight departure from the original construction Pimsner considers in [47] - he considers the C^*-algebra generated only by the image of the operators $(T_e)_{e \in E}$ in the quotient $\mathcal{T}_E / \mathcal{I}_E$ and calls this the "Cuntz-Krieger algebra of a Hilbert bi-module" [47, Definition 1.1]. His construction co-incides with the one we present above when the Hilbert module is full. In general, he refers to the algebra we consider as the "augmented Cuntz-Krieger algebra" and only makes brief remarks about its properties (See [47, Remark 1.1(3)] and [47, Remark 4.10(3)]).*

We will still refer heavily to the paper by Pimsner since in [47, Remark 4.10(3)] he makes a convincing argument that the K-theoretic properties of his algebra also hold for the augmented version. Proofs of the K-theoretic properties of the augmented version can be found, for instance, in [25].

2.4 K-theory and KK-theory

We will assume the reader has basic knowledge of K-theory for C^*-algebras, including the 6-term exact sequence associated to an extension. A good reference for the background required is [52]. We mention that no K-theory class in the K_0 or K_1 group of a non-commutative C^*-algebra is ever computed explicitly in this thesis (other than the class of the unit in K_0). For a locally compact Hausdorff space X, we shall write $K_i(C_0(X)) = K^{-i}(X)$, $i = 0, 1$, to keep certain notations from becoming too cumbersome, but we will still be using the operator K-theory picture for X. When A is a C^*-sub-algebra of B, we will often denote by j or i the inclusion map $A \to B$.

We briefly review the (un-graded) KK-theory that is used in this thesis. See [27] for more details about the general theory.

Let A and B be C^*-algebras, and recall that a *Kasparov* A *-* B *bi-module* is a triple (E, α, T), where (E, α) is an A - B correspondence and $T : E \to E$ is a bounded adjointable endomorphism such that, for all a in A, the operators $T\alpha(a) - \alpha(a)T$, $(T - T^*)\alpha(a)$, and $(T^2 - 1)\alpha(a)$ are in $\mathcal{K}(E)$.

It is clear that the structure of a Kasparov A - B bi-module is preserved under direct sum. There is a notion of *homotopy* between two Kasparov bi-modules such that, if we consider the set of all homotopy equivalence classes $KK^0(A, B)$, then this is an abelian group under the direct sum operation, with 0 being the class of the zero A - B bi-module. We shall denote the homotopy class of $\mathcal{E} = (E, \alpha, F)$ by $[\mathcal{E}]$.

There are natural isomorpishms $KK^i(\mathbb{C}, A) \simeq K_i(A)$ and $KK^i(A, \mathbb{C}) \simeq K^i(A)$, for $i = 0, 1$, making KK-theory a generalization of K-theory and K-homology.

A class $[\mathcal{E}]$ in $KK^0(A, B)$ determines mappings of K-theory $\hat{\otimes}_i[\mathcal{E}] : K_i(A) \to K_i(B)$, for $i = 0, 1$. This is a special case of the Kasparov product $\hat{\otimes} : KK^i(A, B) \times KK^j(B, C) \to KK^{i+j}(A, C)$, for i, j in $\{0, 1\}$, ($i + j$ is taken mod 2) which is why the image of a class g in $K_i(A)$ under $\hat{\otimes}_i[\mathcal{E}]$ will be denoted $g\hat{\otimes}_i[\mathcal{E}]$. For example, the induced map $\hat{\otimes}_i[\varphi]$ of a $*$-homomorphism φ is equal to the usual induced map φ_* on K-theory.

In this thesis, we will mostly only consider Kasparov bi-modules of the form $\mathcal{E} = (E, \alpha, 0)$, so $\alpha(A) \subseteq \mathcal{K}(E)$ necessarily. We now describe the operations we will have to consider on such bi-modules.

Suppose $\psi : B \to B'$ and $\varphi : A' \to A$ are $*$-homomorphism of C^*-algebras. The *pullback of* $\mathcal{E}$ *by* φ is the Kasparov A' - B bi-module $\varphi^*(\mathcal{E}) = (E, \alpha \circ \varphi, 0)$. The *pushforward of* $\mathcal{E}$ *by* ψ is the Kasparov A - B' bi-module $\psi_*(\mathcal{E}) = (E \otimes_B B', \alpha \otimes_B \mathrm{id}_{B'}, 0)$, where B' is regarded as the B - B' correspondence with inner product $\langle b_1, b_2 \rangle = b_1^* b_2$, right B'-action $b_1 \cdot b_2 = b_1 b_2$, defined for all b_1, b_2 in B', and the left B-action $b \cdot b' = \psi(b)b'$, defined for b in B and b' in

B'. See [27, Lemma 1.2.8] for the proof that $\psi_*(\mathcal{E})$ is a Kasparov bi-module. We will denote $E \otimes_B B' = E \otimes_\psi B'$ and $\alpha \otimes_B \mathrm{id} = \alpha \otimes_\psi \mathrm{id}$ so that the module structure is clear. It can be shown that $[\varphi^*(\mathcal{E})] = [\varphi]\hat{\otimes}[\mathcal{E}]$ and $[\psi_*(\mathcal{E})] = [\mathcal{E}]\hat{\otimes}[\psi]$.

We will say two Kasparov A - B bi-modules $\mathcal{E} = (E, \alpha, 0)$, $\mathcal{F} = (E, \beta, 0)$ are *isomorphic* if there is a unitary $U : E \to F$ such that $U\alpha U^* = \beta$.

When ψ as above is surjective, consider the quotient space $E/E^\psi =: E_\psi$ of E by the sub-module $E^\psi = \{e \in E : \psi(\langle e, e\rangle) = 0\}$. Let $q : E \to E_\psi$ denote the quotient map. It can be checked that $\psi_*\mathcal{E}$ is isomorphic to the bi-module structure on (E_ψ, α_ψ) satisfying $\langle q(e), q(f)\rangle = \psi(\langle e, f\rangle)$ and $\alpha_\psi(a)q(e)\psi(b) = q(\alpha(a)eb)$ for all e, f in E, a in A and b in B.

For t in $[0, 1]$, let $ev_t : C([0, 1], B) \to B$ be the $*$-homomorphism sending f to $ev_t(f) = f(t)$. Two bi-modules $\mathcal{E} = (E, \alpha, 0)$, $\mathcal{F} = (E, \beta, 0)$ are *homotopic* if there is a Kasparov A - $C([0, 1], B)$ bi-module $\mathcal{H} = (H, \eta, 0)$ such that $(ev_0)_*\mathcal{H}$ is isomorphic to $\mathcal{E}$ and $(ev_1)_*\mathcal{H}$ is isomorphic to $\mathcal{F}$.

As an example of a KK^0 class, an A - B correspondence (E, α) determines a Kasparov $J_{(E,\alpha)}$ - B bi-module $J(E, \alpha) := (E, \alpha|_{J_{(E,\alpha)}}, 0)$ and hence a class $K(E, \alpha) = [J(E, \alpha)]$ in $KK^0(J_{(E,\alpha)}, B)$.

The class of a short exact sequence of C^*-algebras

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ q\ } C \longrightarrow 0$$

in $KK^1(C, A)$ will be denoted δ_B. Recall that this class is natural in the sense that if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{i_1} & B_1 & \xrightarrow{q_1} & C_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A_2 & \xrightarrow{i_2} & B_2 & \xrightarrow{q_2} & C_2 & \longrightarrow & 0 \end{array}$$

is a commutative diagram of $*$-homomorphisms with exact rows, then $\delta_{B_1}\hat{\otimes}[\alpha] = [\gamma]\hat{\otimes}\delta_{B_2}$. This follows from [16, Remark 2.5.1], which explains how, under the natural identification $KK^1(C, A) \simeq KK^0(C_0((0, 1), C), A)$, the class δ_B can be written as a the product $[j]\hat{\otimes}[e]^{-1}$. Here, $[j]$ is the class of the $*$-homomorphism $j : C_0((0, 1), C) \to C_q$ sending f in $C_0((0, 1), C)$ to $j(f) = (0, f)$ in the mapping cone

$$C_q = \{(x, g) \in B \oplus C_0([0, 1), C) : q(x) = f(0)\}.$$

The element $[e]$ is the class of the $*$-homomorphism $e : A \to C_q$ sending a in A to $e(a) = (i(a), 0)$, which is shown to be invertible in $KK^0(A, C_q)$ [16, Theorem 2.1]. Then, $\delta_{B_1}\hat{\otimes}[\alpha] = [\gamma]\hat{\otimes}\delta_{B_2}$ follows from naturality of the identifications above, the fact that the Kasparov product of two classes of $*$-homomorphisms is equal to the class of their composition and the commutative diagrams

$$\begin{array}{ccc} C_0((0,1),C_1) & \xrightarrow{j_1} & C_{q_1} \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \zeta} \\ C_0((0,1),C_2) & \xrightarrow{j_2} & C_{q_2} \end{array} \qquad \begin{array}{ccc} A_1 & \xrightarrow{e_1} & C_{q_1} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \zeta} \\ A_2 & \xrightarrow{e_2} & C_{q_2}, \end{array}$$

where $\eta = (\mathrm{id}_{C_0(0,1))} \otimes \gamma)$, $\zeta = \beta \oplus (\mathrm{id}_{C_0([0,1))} \otimes \gamma)$.

When we consider the action of δ_B on K-theory, often we will denote $\hat{\otimes}_1 \delta_B =: \delta$ and call this the *index map*, and denote $\hat{\otimes}_0 \delta_B =: \exp$ and call this the *exponential map*. These are the usual index and exponential maps appearing in the 6-term exact sequence of K-theory associated to the above short exact sequence of C^*-algebras, which we now state. For a proof, see [52, Chapter 9].

Theorem 2.4.1 (6-term exact sequence associated to a short exact sequence)**.** *If*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} C \longrightarrow 0$$

is a short exact sequence of C^*-algebras, then the sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{i_*} & K_0(B) & \xrightarrow{q_*} & K_0(C) \\ {\scriptstyle \delta}\uparrow & & & & \downarrow{\scriptstyle exp} \\ K_1(C) & \xleftarrow[q_*]{} & K_1(B) & \xleftarrow[i_*]{} & K_1(A) \end{array}$$

is exact.

If (E,α) is a faithful and full A - A correspondence, we will let δ_E denote the class of the extension

$$0 \longrightarrow \mathcal{I}_E \longrightarrow \mathcal{T}_E \longrightarrow \mathcal{O}_E \longrightarrow 0$$

in $KK^1(\mathcal{O}_E, \mathcal{I}_E)$. Recall the Hilbert Module G_E defined in section 2.3 and our notation $J_E \cap \langle E \rangle = \langle G_E \rangle =: I_{(E,\alpha)}$. We think of G_E as an $\mathcal{I}_E$-$I_{(E,\alpha)}$ imprimitivity bi-module, defining an (invertible) $KK^0(\mathcal{I}_E, I_{(E,\alpha)})$ class $[G_E]$. Let $\delta_{E,PV} = \delta_E \hat{\otimes} [G_E]$ and set $I(E,\alpha) = (E, \alpha|_{I_{(E,\alpha)}}, 0)$

For a Cuntz-Pimsner algebra $\mathcal{O}_E$ Pimsner in [47] determined the following relationship between the K-theory of $I_{(E,\alpha)}$, A, and $\mathcal{O}_E$ which will be crucial in computing the K-theory of a rational function. See [47, Theorem 4.9] and [47, Remark 4.10(3)] for its validity in the case where the defining C^*-correspondence (E,α) is not full. One can also consult [25, Theorem 8.6] for a proof in the non-full case.

Proposition 2.4.2. *Let (E,α) be a faithful A-A correspondence. Let $i : A \to \mathcal{O}_E$ and $\iota : I_E \to A$ be the inclusions. Then, we have the following 6-term exact sequence of K-theory:*

$$\begin{array}{ccccc}
K_0(I_E) & \xrightarrow{\iota - \hat{\otimes}_0 [I(E,\alpha)]} & K_0(A) & \xrightarrow{i_*} & K_0(\mathcal{O}_E) \\
{\scriptstyle \hat{\otimes}_1 \delta_{E,PV}} \uparrow & & & & \downarrow {\scriptstyle \hat{\otimes}_0 \delta_{E,PV}} \\
K_1(\mathcal{O}_E) & \xleftarrow[i_*]{} & K_1(A) & \xleftarrow[\iota - \hat{\otimes}_1 [I(E,\alpha)]]{} & K_1(I_E).
\end{array}$$

The analogous 6-term exact sequence also holds for K-homology (with arrows reversed).

We will call this the *Pimsner-Voiculescu 6-term exact sequence of K-theory.* We will often denote $\hat{\otimes}_1 \delta_{E,PV} =: \delta_{PV}$ and $\hat{\otimes}_0 \delta_{E,PV} =: \exp_{PV}$.

2.5 Complex dynamical systems

Now, we review the theory of complex dynamical systems required to compute the K-theory of their associated C^*-algebras. Although we only compute the K-theory for rational functions, using the information provided in this section and our method, it is possible to compute the K-theory in the other interesting cases. This section is purely expository - we claim no originality.

For M and N locally compact Hausdorff with N metrizable we equip the set $C(M,N)$ of continuous functions from M to N with the compact-open topology. Choosing a metric d for N, it is shown in [39, Lemma 3.1] that a net $(f_\lambda)_{\lambda\in\Lambda}$ in $C(M,N)$ converges to f if and only if every compact set $K \subseteq M$, $(f_\lambda|_K)_{\lambda\in\Lambda}$ converges uniformly to $f|_K$ (relative to d).

If M and N are Riemann surfaces (which are automatically metrizable by [39, Corollary 2.2]), we denote $\mathrm{Hol}(M,N)$ the subspace of holomorphic functions from M to N equipped with the subspace topology from $C(M,N)$. Using the (local) Cauchy integral formula, it is easy to see $\mathrm{Hol}(M,N)$ is a closed subspace.

By a *complex dynamical system*, we shall mean a Riemann surface M equipped with a holomorphic self-map $h : M \to M$. For n in $\mathbb{N}$, denote by h^n the n-fold iterate of h. We say h is *stable at x* in M if there is an open neighbourhood U of x such that $\{h^n|_U\}_{n\in\mathbb{N}}$ is pre-compact in $\mathrm{Hol}(U,M) \cup \{U \to \infty\} \subseteq C(U, M \cup \{\infty\})$, where $M \cup \{\infty\}$ is the one point compactification of M. This means every sub-sequence of $\{h^n|_U\}_{n\in\mathbb{N}}$ has a further sub-sequence which either uniformly converges on compact sets in $\mathrm{Hol}(U,M)$ or uniformly diverges on compact sets to ∞. We call $F_h := \{x \in M : h \text{ is stable at } x\}$ the *Fatou set* of h.

Proposition 2.5.1. *Let $h : M \to M$ be a complex dynamical system. Then the Fatou set F_h is open and invariant in the sense that $h^{-1}(F_h) = F_h$.*

Proof. If h is constant, then $F_h = M$ and there is nothing to prove. So assume h is non-constant.

If U is an open set such that $\{h^n|_U\}_{n\in\mathbb{N}}$ is pre-compact, then $U \subseteq F_h$. Hence, F_h is open. Suppose $\{h^{n_k}|_U\}_{k\in\mathbb{N}}$ converges (diverges) uniformly on compact sets, then so does $\{h^{n_k+1}|_{h^{-1}(U)}\}_{n\in\mathbb{N}}$. This proves $h^{-1}(U) \subseteq F_h$.

Every non-constant holomorphic map is open, so if we denote $V = h(U)$, then using the fact (for open maps between locally compact Hausdorff spaces) that for every compact set $L \subseteq V$ there is a compact set $K \subseteq U$ such that $h(K) = L$, we see that if $\{h^{n_k}|_U\}_{k\in\mathbb{N}}$ converges (diverges) uniformly on compact sets, then so does $\{h^{n_k-1}|_V\}_{n\in\mathbb{N}}$. Hence $h(U) \subseteq F_h$. We have shown $h^{-1}(F_h) = F_h$. □

The *Julia set* of h is defined to be the complement $J_h := M \setminus F_h$ of the Fatou set. By Proposition 2.5.1, J_h is a closed set of M such that $h^{-1}(J_h) = J_h$. One of the basic goals of the theory of complex dynamical systems is to classify the possible dynamics on the Fatou set and understand the structure of the Julia set. We shall see now that the analysis largely depends on the ambient Riemann surface M.

By the Uniformization Theorem (see [39, Theorem 1.1]), every simply connected Riemann surface is conformally equivalent to the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the disk $\mathbb{D}$. It follows then (by the universal cover construction) that every Riemann surface M is conformally equivalent to $\tilde{M}/\Gamma$, where $\tilde{M}$ is one of the three simply connected Riemann surfaces above and $\Gamma = \pi_1(M)$ is a (discrete) group of conformal automorphisms of $\tilde{M}$ that acts freely and proper discontinuously on $\tilde{M}$.

When $\tilde{M} = \mathbb{D}$, M is said to be *hyperbolic*. In this case, if $h : M \to M$ is a complex dynamical system, then $F_h = M$, and the dynamics is classified as follows.

Theorem 2.5.2. *[39, Theorem 5.2] If $h : M \to M$ is a complex dynamical system on a hyperbolic Riemann surface, then one of the following possibilites occur:*

(1) there is a periodic orbit $\mathcal{O} = \{z_0, ..., z_{m-1}\}$ of period m such that $\{h^{nm}\}_{n\in\mathbb{N}}$ converges uniformly on compact sets to z_0,

(2) $\{h^n\}_{n\in\mathbb{N}}$ converges uniformly on compact sets to ∞,

(3) there is n in $\mathbb{N}$ such that $h^n = id_M$, or

(4) h is conformally conjugate to $z \to e^{2\pi i\alpha} z$ on the disk, punctured disk, or some annulus for some irrational number α.

Recall that there are only finitely many non-hyperbolic Riemann surfaces; Since $\hat{\mathbb{C}}$ is compact, the only group of automorphisms acting freely and proper discontinuously is the trivial group $\Gamma = \{\mathrm{id}_{\hat{\mathbb{C}}}\}$. Hence, if $\tilde{M} = \hat{\mathbb{C}}$, then $M \simeq \hat{\mathbb{C}}$. We will see shortly that the possible complex dynamics on $\hat{\mathbb{C}}$ can be quite interesting.

The last case to consider is $\tilde{M} = \mathbb{C}$. Recall a conformal automorphism of $\mathbb{C}$ is of the form $z \to \alpha z + \beta$. If $\alpha \neq 1$, then this automorphism has a fixed point and thus does not act freely. This forces $\alpha = 1$ and it is easy to show that a group Γ of conformal automorphisms acting freely and proper discontinuously on $\mathbb{C}$ is conjugate to either $\Gamma = \{z \to z+n : n \in \mathbb{Z}\}$ or $\Gamma_\tau = \{z \to z+n, z \to z+m\tau : n, m \in \mathbb{Z}\}$ for some τ in $\mathbb{C} \setminus \mathbb{R}$. In the first case, we can identify $\mathbb{C}/\Gamma$ with the punctured plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ via $\exp(2\pi i -) : \mathbb{C}/\Gamma \to \mathbb{C}^*$. For each τ in $\mathbb{C} \setminus \mathbb{R}$, $\mathbb{C}/\Gamma_\tau =: \mathbb{T}^2_\tau$ defines a 2-Torus. The 2-Torus case is the least complicated dynamically out of the non-hyperbolic cases, as the following theorem suggests. Let $\mathbb{Z} + \tau\mathbb{Z}$ denote the $\mathbb{Z}$-linear span of 1 and τ.

Theorem 2.5.3. *[39, Theorem 6.1] If $h : \mathbb{T}^2_\tau \to \mathbb{T}^2_\tau$ is a complex dynamical system, then there is α in $\mathbb{Z} + \tau\mathbb{Z}$ and β in $\mathbb{C}$ such that $\alpha(\mathbb{Z} + \tau\mathbb{Z}) \subseteq \mathbb{Z} + \tau\mathbb{Z}$ and $h(z) \equiv \alpha z + \beta \bmod \Gamma_\tau$.*

The number $d := |\alpha|^2$ is in $\mathbb{N}$ and is the degree of h. If $d \leq 1$, then $F_h = \mathbb{T}^2_\tau$. If $d > 1$, then $J_h = \mathbb{T}^2_\tau$, and h satisfies the property that for every open set $U \subseteq \mathbb{T}^2_\tau$ there is n in $\mathbb{N}$ such that $h^n(U) = \mathbb{T}^2_\tau$

It remains to describe dynamics on $\hat{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{C}^*$. We start by giving descriptions of the possible forms the functions can take.

If $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic, then by compactness of $\hat{\mathbb{C}}$, h can only have finitely many zeroes and poles (counting multiplicity). It follows, by an application of Louiville's theorem, that there are polynomials p and q such that $h(z) = \frac{p(z)}{q(z)}$, for all z in $\hat{\mathbb{C}}$, therefore h is a *rational function.* The number of solutions z to $h(z) = w$ (counting multiplicity) is known as the *degree* of h, and it is equal to the maximum of the degrees of p and q (assuming p and q have no common zeroes).

A holomorphic function $h : \mathbb{C} \to \mathbb{C}$ either extends continuously to $\hat{\mathbb{C}}$, and is therefore a polynomial, or has an essential singularity at ∞. In this case, h is called *transcendental.* The functions e^z, $\sin(z)$, $\cos(z)$ are the canonical examples.

Recall that if $h : \mathbb{C} \to \mathbb{C}$ is a holomorphic function such that 0 is not in the image of h, then $h(z) = e^{g(z)}$ for some entire holomorphic function g that is an anti-derivative of $\frac{h'}{h}$.

Proposition 2.5.4. *[11] If $h : \mathbb{C}^* \to \mathbb{C}^*$ is a complex dynamical system, then there is an entire function $g : \mathbb{C} \to \mathbb{C}$ such that $h(e^z) = e^{g(z)}$ for all z in $\mathbb{C}$ and $exp^{-1}(F_h) = F_g$.*

Now, we turn to describing the structure of the Julia set. By a repelling periodic point, we shall mean a point p in M such that $h^n(p) = p$ for some n and the Jacobian of h^n at p is greater than one.

Theorem 2.5.5. *If $h : M \to M$ is a non-invertible complex dynamical system and M is either $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{C}^*$, then the Julia set J_h is infinite and is the closure of repelling periodic points.*

Proof. For the proof of the case $M = \hat{\mathbb{C}}$, see [39, Theorem 13.1] and [39, Theorem 14.1]. The case $M = \mathbb{C}$ was proven by Baker [3]. The case $M = \mathbb{C}^*$ follows from Proposition 2.5.4, the case $M = \mathbb{C}$ and the fact that semi-conjugacies map periodic points to periodic points. □

If $h : M \to M$ is a complex dynamical system, then a point p in M is called *grand orbit finite* (see [39, Lemma 4.9]) if the set $\mathcal{O}(p) = \bigcup_{m,n\in\mathbb{N}} h^{-m}(h^n(p))$ is finite. The *exceptional points* of h is the collection $E = \{p \in M : p \text{ is grand orbit finite}\}$. Note that $h^{-1}(E) \subseteq E$ since $\mathcal{O}(p) \subseteq E$ for every p in E.

Proposition 2.5.6. *Let $h : M \to M$ be a non-invertible complex dynamical system.*

(1) If $M = \hat{\mathbb{C}}$, then $|E| \leq 2$ and $E \subseteq F_h$.

(2) If $M = \mathbb{C}$, then $|E| \leq 1$.

(3) If $M = \mathbb{C}^*$, then $|E| = 0$.

Proof. If M is either $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{C}^*$ and $F' = F \cup (\hat{\mathbb{C}} \setminus M)$ has at least four points, where $F \subseteq E$ is a finite set such that $h^{-1}(F) \subseteq F$, then $h : \hat{\mathbb{C}} \setminus F' \to \hat{\mathbb{C}} \setminus F'$ is hyperbolic. But then $J_h \subseteq F'$, a contradiction to Theorem 2.5.5.

Now, we show $E \subseteq F_h$ if $M = \hat{\mathbb{C}}$. Since h is surjective, we have $h^{-1}(E) = E$ and hence $h^2(e) = e$ for every e in E, and this is the only solution to $h^2(z) = e$. The degree of h is greater than one, and for a rational function R, the number of solutions (counting multiplicity) z to $R(z) = w$ must be equal to the degree of R. Therefore, $h^2(e) = e$ has multipicity greater than one. It follows that the Jacobian of h is zero at each point e in E, and hence h^2 shrinks distances locally at e in E. Therefore, for each e we can find a neighbourhood U_e of e such that $h^2(U_e) \subseteq U_e$. Then, since $h^2 : U_e \to U_e$ is a hyperbolic dynamical system, it follows that $U_e \subseteq F_{h^2} \subseteq F_h$. □

Remark 2.5.7. *In the above proposition for the case $M = \mathbb{C}$, the function $h(z) = 2ze^z$ provides an example where $E = \{0\}$ and 0 is in J_h. The function $h(z) = z^2$ provides an example where $E = \{0\}$ and 0 is in F_h.*

Off the exceptional points, inverse orbits accumulate to the Julia set.

Proposition 2.5.8. *If $h : M \to M$ is a non-invertible dynamical system on either $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{C}^*$, then for any p not in E, and z in J_h, there is sequence z_k in $h^{-n_k}(p)$, where $(n_k)_{k\in\mathbb{N}}$ diverges to ∞ and $(z_k)_{k\in\mathbb{N}}$ converges to z.*

Proof. If not, then there is a neighbourhood U of z and N such that $h^n(U)$ does not contain p for all $n \geq N$. Therefore, $h^n(U) \cap \bigcup_{j\leq k} h^{-j}(p) = \emptyset$ for all $n \geq N + k$. Since p is not an exceptional point, we can choose k large enough so that $\bigcup_{j\leq k} h^{-j}(p)$ contains at least three points. Then, $\mathcal{U} = \bigcup_{n\geq N+k} h^n(U)$ is a hyperbolic Riemann surface such that $h(\mathcal{U}) \subseteq \mathcal{U}$. Hence, $\mathcal{U} \subseteq F_h$, and in particular $h^{N+k}(z)$ is in F_h. It then follows from Proposition 2.5.1 that z is in F_h, a contradiction to z being in J_h. □

Corollary 2.5.9. *If $h : M \to M$ is a non-invertible dynamical system on either $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{C}^*$, then J_h contains no isolated points.*

Proof. If z is periodic, since J_h is infinite, we can choose p in J_h not in the orbit of z. Then, applying Proposition 2.5.8 to p we can find a sequence $(z_k)_{k\in\mathbb{N}}$ in J_h converging to z such that $z_k \neq z$ for infinitely many k in $\mathbb{N}$.

If z is not periodic, then choosing any p in J_h and applying Proposition 2.5.8 will suffice. □

Corollary 2.5.10. *If $h : M \to M$ is a complex dynamical system and M is either $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{C}^*$, then for every open set $U \subseteq M$ such that $U \cap J_h \neq \emptyset$ and compact set $K \subseteq M \setminus E$, there is n in $\mathbb{N}$ such that $K \subseteq h^n(U)$.*

Proof. Let z be in $U \cap J_h$. By Theorem 2.5.5 we may assume z is a repelling periodic point. By replacing h with $g := h^N$, where N is the period of z, we may assume z is a fixed point. Since z is repelling, we may choose an open set $V \subseteq U$ containing z such that $V \subseteq g(V)$. Note that the exceptional points E of h co-incide with the exceptional points of g.

By Proposition 2.5.8 for any x in $M \setminus E$, there is a sequence $(z_k)_{k\in\mathbb{N}} \subseteq \bigcup_{n\in\mathbb{N}} g^{-n}(x)$ converging to z, and therefore $g^{-k}(x) \cap V \neq \emptyset$ for some k in $\mathbb{N}$. It follows that $\bigcup_{n\in\mathbb{N}} g^n(V) = M \setminus E$. Since $K \subseteq M \setminus E$ is compact there is k in $\mathbb{N}$ such that $K \subseteq \bigcup_{n\leq k} g^n(V) = g^k(V) \subseteq h^{Nk}(U)$. □

Now, we study the extent of the Julia set in M.

Corollary 2.5.11. *If $h : M \to M$ is a complex dynamical system and M is either $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{C}^*$, then either $J_h = M$ or J_h has empty interior in M.*

Proof. If J_h contains an open set U of M then by Corollary 2.5.10 and invariance of J_h, we have that $M \setminus E \subseteq J_h$. Since J_h is closed and E is finite, it follows that $J_h = M$. □

If $h : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree d (i.e. ∞ is not an essential singularity), then there is $R > 1$ such that for all $|z| \geq R$, $|p(z)| \geq |z|^{d-\frac{1}{2}}$. It follows that the Fatou set contains a neighbourhood of ∞, and therefore J_p is a compact subset of $\mathbb{C}$. This is in contrast to the transcendental case.

Proposition 2.5.12. *If $h : \mathbb{C} \to \mathbb{C}$ has an essential singularity at ∞, then J_h is unbounded.*

Proof. By Picard's theorem [39, Theorem 2.6], in every neighbourhood at ∞, h attains all but finitely many values. Since J_h is infinite and $h^{-1}(J_h) = J_h$ (Theorem 2.5.8), it follows that J_h intersects every neighbourhood of ∞. □

A similar result holds for the case of dynamics on the punctured plane.

Proposition 2.5.13. *If $h : \mathbb{C}^* \to \mathbb{C}^*$ is a complex dynamical system with an essential singularity at 0 (resp. ∞), then the closure of J_h in $\hat{\mathbb{C}}$ contains 0 (resp. ∞).*

Proof. The proof is the same as above. □

We now turn to classifying the possible dynamical behaviours on the Fatou set. We will call a maximally connected component of F_h a *Fatou component.* A Fatou component U is *periodic* if there is N in $\mathbb{N}$ such that $h^N(U) \subseteq U$. The least such N is called the *period* of U. We will call the Fatou components $P = \{U_0, ..., U_{N-1}\}$ such that $U_0 = U$ and $h(U_i) \subseteq U_{i+1}$ a *Fatou cycle.*

We will say U is *pre-periodic* if there is some M in $\mathbb{N}$ such that $h^M(U)$ is contained in a periodic Fatou component.

If U is not pre-periodic, then it is called a *wandering domain.* Such a domain satisfies $h^m(U) \cap h^n(U) = \emptyset$ for any distinct m, n in $\mathbb{N}$. Sullivan proved in [56] that such domains do not exist for rational functions.

Theorem 2.5.14. *[56] If $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a complex dynamical system, then every Fatou component is pre-periodic.*

The first example of a transcendental function on $\mathbb{C}$ that admits a wandering domain was given in [4]. An example of such a domain for a complex dynamical system on $\mathbb{C}^*$ is given in [42].

There are only finitely many possible dynamical behaviours of h restricted to a periodic Fatou component, which we state below.

Theorem 2.5.15. *Let $h : M \to M$ be a complex dynamical system, where M is either $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{C}^*$. Suppose U is a periodic Fatou component of period N. Then, either*

(1) $\{h^{mN} : U \to U\}_{m\in\mathbb{N}}$ *converges uniformly on compact sets to a fixed point* z *in* U*,*

(2) $\{h^{mN} : U \to U\}_{m\in\mathbb{N}}$ *converges uniformly on compact sets to some* z *in* $\hat{\mathbb{C}} \setminus U$*, or*

(3) $h^N : U \to U$ *is conformally conjugate to an irrational rotation on either the disk or an annulus.*

Any other Fatou component V *in the Fatou cycle* P *generated by* U *will have the same behaviour as* U*.*

If in case (2) *the limit point* z *is in* M*, then* z *is a parabolic fixed point for* h^N*.*

Proof. For a proof for the case $M = \hat{\mathbb{C}}$, see [39, Theorem 16.1]. Note that the theorem in the reference is stated for only rational functions, but as noted in [10] the proof can be adapted for all cases. □

We will say a periodic Fatou cycle P is *attracting* if its components satisfy (1), *parabolic* (resp. *divergent*) if P satisfies (2) and z is in M (resp. in $\hat{\mathbb{C}} \setminus M$), a *Siegel cycle* if P satisfies (3) and its components are conjugate to the disk, and a *Herman cycle* if the components are conjugate to annuli.

Remark 2.5.16. *In all three cases of* M*, one can have attracting and parabolic Fatou cycles. For dynamics on* $\hat{\mathbb{C}}$ *and* $\mathbb{C}^*$*, one can have Siegel and Herman cycles.*

For dynamics on $\mathbb{C}$*, Herman cycles are not possible. This follows easily from the maximum modulus principle.*

The following proposition will be useful for our K-theory computations.

Corollary 2.5.17. *Let* $h : M \to M$ *be a complex dynamical system, where* M *is either* $\hat{\mathbb{C}}$*,* $\mathbb{C}$*, or* $\mathbb{C}^*$*. If* P *is an attracting, parabolic, divergent or Siegel cycle for* h*, then there is an open set* A_P *contained in a finite union* B *of pair-wise disjoint simply connected open subsets of* $\bigcup_{U\in P} U := U_P$ *such that* $h(A_P) \subseteq A_P$ *and for every compact set* $K \subseteq \bigcup_{x\in X_P} U_x$*, there is a* k *in* $\mathbb{N}$ *such that* $h^k(K) \subseteq A_P$*. Moreover,* B *can be chosen so that* $h^l(B) \subseteq A_P$ *for some* l *in* $\mathbb{N}$*.*

Proof. Let n denote the length of P. Write $P = \{U_1, ..., U_n\}$, where $h(U_i) \subseteq U_{i+1}$ mod n. First, suppose P is attracting, and let $\{z_1, ..., z_n\}$ be its attracting periodic orbit, where z_i is in U_i for $i \leq n$. By Koenig's linearization Theorem ([39, Theorem 8.2]) for the geometrically attracting case and Böttcher's Theorem for the super attracting case ([39, Theorem 9.1]), for any $i \leq n$, there is a simply connected, open, and pre-compact neighbourhood V_i of z_i contained in U_i such that $\overline{V_i} \subseteq U_i$ and $h^{\circ n}(V_i) \subseteq V_i$.

Let $A_P = \bigsqcup_{i=1}^{n} V_i \cap \bigcap_{j=1}^{n-1} h^{n-j}(V_{i+j})$. Then, A_P is contained in the disjoint union of simply connected open sets $\bigsqcup_{i=1}^{n} V_i = B$ and $h(A_P) \subseteq A_P$. Since A_P is a neighbourhood of the attracting periodic orbit, for every compact set $K \subseteq U_P$ there is k in $\mathbb{N}$ such that $h^{\circ k}(K) \subseteq A_P$. In particular, this is true for $K = \overline{B}$. Since any compact set of $\bigcup_{x \in X_P} U_x$ is eventually mapped into U_P, the result follows in this case.

If P is parabolic, then, by the Parabolic flower Theorem ([39, Theorem 10.7]), for every U in P there is an attracting *petal* (see [39, Definition 10.6]) $\mathcal{P}_U \subseteq U$ such that for every compact set $K \subseteq U$, there is k in $\mathbb{N}$ such that $h^{\circ nk}(K) \subseteq \mathcal{P}_U$. By definition, $\mathcal{P}_{U_1}$ is a simply connected open set which is mapped homeomorphically into itself by $h^{\circ n}$. Therefore, $A_P = \bigcup_{i=0}^{n-1} h^{\circ i}(\mathcal{P}_{U_1}) = B$ satisfies the conclusion of the Corollary.

If P is a Siegel cycle, then we just let $B = A_P = U_P$.

If P is a divergent cycle and $M = \mathbb{C}$, then [5, Theorem 3.1] shows every periodic Fatou component is simply connected, so we can let $B = A_P = U_P$. The case where $M = \mathbb{C}^*$ follows from [19, Lemma 5.2]. □

For a connected open set $U \subseteq \hat{\mathbb{C}}$, its *connectivity* number is the number of connected components of $\hat{\mathbb{C}} \setminus U$. Note that an open set U is simply connected if and only if its connectivity number is 1.

If U is a wandering Fatou component and n is in $\mathbb{N}$, we let U_n denote the Fatou component such that $h^n(U) \subseteq U_n$.

Proposition 2.5.18. *Let $h : M \to M$ be a complex dynamical system. If U is a wandering Fatou component, then there is N in $\mathbb{N}$ such that the connectivity of U_n is constant over all $n \geq N$. This constant can be*

(1) 1, 2 or ∞ if $M = \mathbb{C}$ or

(2) 1 if $M = \mathbb{C}^*$.

In the case of $M = \mathbb{C}$, each U_n is bounded and the restriction $h : U_n \to U_{n+1}$ is a finite degree branched covering, and the degree of $h^m : U_n \to U_{n+m}$ tends to ∞ as m tends to ∞

Proof. If $M = \mathbb{C}^*$ and h has a wandering Fatou component, then by [6, Theorem 1] there is at most one Fatou component with connectivity number greater than one.

For the case of $M = \mathbb{C}$, see [53, Theorem 2.5]

□

Chapter 3

Morphisms of C^*-correspondences

We show that C^*-correspondences are the objects of a category in which a morphism $(E, \alpha) \to (F, \beta)$ in this category intertwines (E, α) with (F, β) in a suitable sense. We show in Proposition 3.0.3 that such a morphism induces an intertwining of the KK^0-classes $K(E, \alpha)$ and $K(F, \beta)$. We then consider types of exactness in this category and show that a short J-exact sequence induces an intertwining of certain extension classes in KK-theory (Proposition 3.0.8). This can be thought of as a generalization of naturality of extension classes associated to exact sequences of C^*-algebras.

Definition 3.0.1. *A morphism from an A_1 - A_2 C^*-correspondence (E, α) to a B_1 - B_2 correspondence (F, β) is a triple $(\varphi_1, S, \varphi_2)$ of maps, where $\varphi_1 : A_1 \to B_1$, $\varphi_2 : A_2 \to B_2$ are *-homomorphisms, and $S : E \to F$ is a linear map satisfying the additional identities*

(1) $\langle S(e_1), S(e_2)\rangle = \varphi_2(\langle e_1, e_2\rangle)$ for all e_1, e_2 in E (i.e. S is a φ_2-twisted morphism),

(2) $S(a \cdot e) = \varphi_1(a) \cdot S(e)$, for all a in A_1, e in E, and

(3) $\hat{S}(\alpha(a)) = \beta(\varphi_1(a))$ for all a in $J_{(E,\alpha)}$.

Note that (3) implies $\varphi_1(J_{(E,\alpha)}) \subseteq J_{(F,\beta)}$, and Condition (3) above is automatically implied by (1), (2) and this corollary if S is surjective, or if $\alpha^{-1}(\mathcal{K}(E)) = \emptyset$.

Remark 3.0.2. *The special case of the above definition when $A_1 = A_2$, $B_1 = B_2$ and $\varphi_1 = \varphi_2$ appears often in the literature under the name "co-variant morphism" between self-correspondences, see [38, Section 2.4] and the references therein. The focus is usually on its functorial properties with respect to the Cuntz-Pimsner algebra construction, which we will review briefly, but here we will focus on its KK-theoretic consequences in the general case when A_1, A_2 and B_1, B_2 are not necessarily the same.*

It is easy to see that if $(\varphi_1, S, \varphi_2) : (E, \alpha) \to (F, \beta)$ and $(\psi_1, T, \psi_2) : (F, \beta) \to (G, \gamma)$ are morphisms, then their composition $(\psi_1, T, \psi_2) \circ (\varphi_1, S, \varphi_2) = (\psi_1 \circ \varphi_1, T \circ S, \psi_2 \circ \varphi_2)$ is a morphism, and that, given an A_1 - A_2 correspondence (E, α), $\mathrm{id}_{(E,\alpha)} = (\mathrm{id}_{A_1}, \mathrm{id}_E, \mathrm{id}_{A_2})$ is an identity morphism relative to this composition operation. Thus, C^*-correspondences are the objects of a category with morphisms as above.

Now, suppose (φ, S, φ) is a morphism from an A - A correspondence (E, α) to a B - B correspondence (F, β) (both assumed to be faithful). The universal property of Toeplitz algebras ([47, Theorem 3.4]) imply there is a unique $*$-homomorphism $\mathcal{T}(S) : \mathcal{T}_E \to \mathcal{T}_F$ defined by the equations

$$\mathcal{T}(S)(T_e) = T_{S(e)}, \text{ for all } e \text{ in } E \text{ and } \mathcal{T}(S)(a) = \varphi(a), \text{ for all } a \text{ in } A$$

Note that $\mathcal{T}(S) \circ \phi = \phi \circ \hat{S}$, so the fact that $\hat{S} \circ \alpha(a) = \beta(\varphi(a))$, for all a in J_E, implies

$\mathcal{T}(S)(a - \phi(\alpha(a))) = \varphi(a) - \phi(\beta(\varphi(a)))$, for all a in J_E. Therefore, $\mathcal{T}(S)(\mathcal{I}_E) \subseteq \mathcal{I}_F$, and hence $\mathcal{T}(S)$ passes down to a $*$-homomorphism at the level of Cuntz-Pimsner algebras, which we shall denote by $\mathcal{O}(S) : \mathcal{O}_E \to \mathcal{O}_F$.

We show now that a morphism between two correspondences intertwines their induced classes in KK^0.

Proposition 3.0.3. *Suppose $(\varphi_1, S, \varphi_2) : {}_{A_1}(E, \alpha)_{A_2} \to {}_{B_1}(F, \beta)_{B_2}$ is a morphism of correspondences. Then, $K(E, \alpha)\hat{\otimes}[\varphi_2] = [\varphi_1]\hat{\otimes}K(F, \beta)$.*

Proof. It is routine to see the mapping $T : E \otimes_{\phi_2} B_2 \to F$ defined on a basic tensor $e \otimes_{\phi_2} b$, for e in E and b in B_2, as $T(e \otimes_{\phi_2} b) = S(e)b$ is a unitary onto its image $G = \overline{\mathrm{span}\{S(E) \cdot B_2\}}$. Under T, the left action $\alpha \otimes_{\varphi_2} \mathrm{id}$ on $E \otimes_{\varphi_2} B_2$ is identified with the action defined, for all a in A_1 and g in G, as $\gamma(a) \cdot g := \beta(\varphi_1(a))g$.

Let us show $J_{(E,\alpha)} \subseteq J_{(G,\gamma)}$. By the definition of morphism, for every a in $J_{(E,\alpha)}$ and $\epsilon > 0$, there are $\{e_i\}_{i=1}^n, \{e_i'\}_{i=1}^n \subseteq E_1$ such that $\|\beta(\varphi_1(a)) - \sum_{i=1}^n \theta_{S(e_i),S(e_i')}\|_{\mathcal{B}(E_2)} \leq \epsilon$. If $\{u_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit in B_2, then there is λ_0 in Λ such that $\| \sum_{i=1}^n \theta_{S(e_i),S(e_i')} - \sum_{i=1}^n \theta_{S(e_i)\sqrt{u_{\lambda_0}},S(e_i')\sqrt{u_{\lambda_0}}}\|_{\mathcal{B}(F)} \leq \varepsilon$. Therefore,

$$\|\gamma(a) - \sum_{i=1}^n \theta_{S(e_i)\sqrt{u_{\lambda_0}},S(e_i')\sqrt{u_{\lambda_0}}}\|_{\mathcal{B}(G)} \leq \|\alpha_2(\varphi(a)) - \sum_{i=1}^n \theta_{S(e_i)\sqrt{u_{\lambda_0}},S(e_i')\sqrt{u_{\lambda_0}}}\|_{\mathcal{B}(F)} \leq 2\varepsilon.$$

As $\sum_{i=1}^n \theta_{S(e_i)u_{\lambda_0},S(e_i')u_{\lambda_0}}$ is in $\mathcal{K}(G)$ and ϵ was arbitrary, we may conclude that a is in $J_{(G,\gamma)}$. Hence, $J_{(E,\alpha)} \subseteq J_{(G,\gamma)}$.

Consider the Hilbert $C([0,1], B_2)$-module $H = \{h \in C([0,1], F) : h(0) \in G\}$ with operations defined point-wise as

(1) $g \cdot b(t) := g(t)b(t)$, for all h in H, b in B_2 and t in $[0,1]$,

(2) $\langle h_1, h_2\rangle(t) := \langle h_1(t), h_2(t)\rangle$, for all h_1, h_2 in H and t in $[0,1]$.

Since $\beta(\varphi_1(A_1)) \cdot G \subseteq G$, we may define the left action η on H as $\eta(a) \cdot h(t) := \beta(\varphi(a)) \cdot h(t)$, for all a in A_1, h in H, and t in $[0,1]$.

G embeds into H as the constant functions and η restricted to G is equal to γ, so $\mathcal{K}(G) \subseteq \mathcal{K}(H)$ and $J_{(G,\gamma)} \subseteq J_{(H,\eta)}$. Therefore, $J_{(E,\alpha)} \subseteq J_{(H,\eta)}$.

Thus, $(H, \eta|_{J_{(E,\alpha)}})$ is a Kasparov $J_{(E,\alpha)} - C([0,1], B_2)$ module such that $(ev_0)_*\mathcal{E}_H \simeq (G,\gamma) \simeq (\varphi_2)_*(E, \alpha|_{(J_{(E,\alpha)}})$ and $(ev_1)_*\mathcal{E}_H \simeq \varphi_1^*(F, \beta|_{J_{(F,\beta)}})$, proving the Proposition. □

Remark 3.0.4. *Suppose a is in $KK(A_1, A_2)$, b is in $KK(B_1, B_2)$. We will say a pair (x_1, x_2) in $KK(A_1, B_1) \times KK(A_2, B_2)$ intertwines a with b if $a\hat{\otimes}x_2 = x_1\hat{\otimes}b$. We call (x_1, x_2) an intertwiner and write $(x_1, x_2) : a \to b$ suggestively. Clearly if $(x_1, x_2) : a \to b$ and $(y_1, y_2) : b \to c$ are intertwiners, then $(x_1, x_2)\hat{\otimes}(y_1, y_2) := (x_1\hat{\otimes}y_1, x_2\hat{\otimes}y_2) : a \to c$ is an intertwiner. Proposition 3.0.3 can be re-stated as saying that if $(\varphi_1, S, \varphi_2) : (E,\alpha) \to (F,\beta)$ is a morphism, then $([\varphi_1], [\varphi_2]) : K(E,\alpha) \to K(F,\beta)$ is an intertwiner. Moreover, it is easy to see that the assignment $(\varphi_1, S, \varphi_2) \to ([\varphi_1], [\varphi_2])$ is functorial, but we shall not make use of this fact.*

Let us specialize to the case of a morphism between faithful self correspondences and show that such a morphism induces intertwinings of the KK-classes involved in the Pimsner-Voicelescu 6-term exact sequences of the respective Cuntz-Pimsner algebras (Proposition 2.4.2).

Proposition 3.0.5. *Let $(\varphi, S, \varphi) : {}_A(E,\alpha)_A \to {}_B(F,\beta)_B$ be a morphism between two faithful C^*-correspondences. Then,*

(1) $[\varphi]\hat{\otimes}(\iota - [I(F,\beta)])) = (\iota - [I(E,\alpha)])\hat{\otimes}[\varphi]$,

(2) $[\varphi]\hat{\otimes}[i] = [i]\hat{\otimes}[\mathcal{O}(S)]$ *and*

(3) $[\mathcal{O}(S)]\hat{\otimes}\delta_{PV} = \delta_{PV}\hat{\otimes}[\varphi]$.

Proof. $(\varphi, \mathcal{O}(S), \mathcal{O}(S)) : i \to i$ and $(\varphi, \varphi, \varphi) : \iota \to \iota$ and $(\varphi, S, \varphi) : (E, \alpha|_{\langle E\rangle}) \to (F, \beta|_{\langle F\rangle})$ are morphisms, so (1) and (2) follow from Proposition 3.0.3.

The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_E & \longrightarrow & \mathcal{T}_E & \longrightarrow & \mathcal{O}_E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathcal{T}(S)} & & \downarrow{\scriptstyle \mathcal{T}(S)} & & \downarrow{\scriptstyle \mathcal{O}(S)} & & \\ 0 & \longrightarrow & \mathcal{I}_F & \longrightarrow & \mathcal{T}_F & \longrightarrow & \mathcal{O}_F & \longrightarrow & 0 \end{array}$$

commutes, so naturality of δ implies $\delta_E \hat{\otimes} [\mathcal{T}(S)] = [\mathcal{O}(S)] \hat{\otimes} \delta_F$.

Since (φ, S, φ) is a morphism, it is easy to see that $(\mathcal{T}(S), S^\infty, \varphi) : G_E \to G_F$ is a morphism, where $S^\infty : \mathcal{F}_E \to \mathcal{F}_F$ is defined on a basic tensor $t = e_1 \otimes_A e_2 ... \otimes_A e_n$ in $E^{\otimes_A n}$ as $S^\infty(t) = S(e_1) \otimes_B S(e_2) \otimes ... \otimes S(e_n)$. Proposition 3.0.3 then implies $[\mathcal{T}(S)] \hat{\otimes} [G_F] = [G_E] \hat{\otimes} [\varphi]$. Combining this equality with naturality of the index map and $\delta_{H,PV} = \delta_H \hat{\otimes} [G_H]$, for $H = E, F$, we have that $[\mathcal{O}(S)] \hat{\otimes} \delta_{F,PV} = \delta_{E,PV} \hat{\otimes} [\varphi]$. □

We now formulate a type of exactness in this category.

Definition 3.0.6. *Let (E, α) be an $A_1 - A_2$ correspondence, (F, β) a $B_1 - B_2$ correspondence and (G, γ) a $C_1 - C_2$ correspondence. Morphisms $(\varphi_1, S, \varphi_2) : (E, \alpha) \to (F, \beta)$ and $(\psi_1, T, \psi_2) : (F, \beta) \to (G, \gamma)$ are said to be J-exact if*

$$J_{(E,\alpha)} \xrightarrow{\varphi_1} J_{(F,\beta)} \xrightarrow{\psi_1} J_{(G,\gamma)}$$

$$E \xrightarrow{S} F \xrightarrow{T} G$$

$$A_2 \xrightarrow{\varphi_2} B_2 \xrightarrow{\psi_2} C_2$$

are exact.

Remark 3.0.7. *Given a morphism $(\varphi_1, S, \varphi_2) : (E, \alpha) \to (F, \beta)$ from an A_1-A_2 correspondence to a B_1-B_2 correspondence, its image is the $\varphi_1(A_1)$ - $\varphi_2(A_2)$ correspondence $im(\varphi_1, S, \varphi_2) = (S(E), \beta|_{\varphi_1(A_1)})$ and its kernel is the $ker(\varphi_1)$ - $ker(\varphi_2)$ correspondence $ker(\varphi_1, S, \varphi_2) = (ker(S), \alpha|_{ker(\varphi_1)})$.*

It is easy to see that the inclusion of the kernel and image correspondences into (E, α), (F, β), respectively and the restriction of the co-domain of $(\varphi_1, S, \varphi_2)$ to the image are morphisms. Moreover, they satisfy the universal property for images and kernels of morphisms in a category. The categorically correct notion of exactness is therefore a pair of morphisms $(\varphi_1, S, \varphi_2) : (E, \alpha) \to (F, \beta)$ and $(\psi_1, T, \psi_2) : (F, \beta) \to (G, \gamma)$ such that $im(\varphi_1, S, \varphi_2) = ker(\psi_1, T, \psi_2)$. This is the case if and only if

$$A_1 \xrightarrow{\varphi_1} B_1 \xrightarrow{\psi_1} C_1$$

$$E \xrightarrow{S} F \xrightarrow{T} G$$

$$A_2 \xrightarrow{\varphi_2} B_2 \xrightarrow{\psi_2} C_2$$

are exact.

Unfortunately an exact pair of morphisms is not necessarily J-exact. Suppose B is a C^*-algebra with a non-trivial ideal I and equip I, B and B/I with their identity Hilbert module structures. Let $\alpha : B \to \mathcal{M}(I)$ and $\beta : B \to \mathcal{M}(B)$ be the standard actions by multipliers and let $\gamma : 0 \to \mathcal{M}(B/I)$. Then, denoting $i : I \to B$ the inclusion and $q : B \to B/I$ the quotient map,

$$(I, \alpha) \xrightarrow{(id_B, i, i)} (B, \beta) \xrightarrow{(0, q, q)} (B/I, \gamma)$$

is exact in the categorical sense, but is not J-exact.

The following result is an extension of naturality of the index class of extensions.

Proposition 3.0.8. *If*

$$0 \longrightarrow {}_{A_1}(E, \alpha)_{A_2} \xrightarrow{(i_1, I, i_2)} {}_{B_1}(F, \beta)_{B_2} \xrightarrow{(q_1, Q, q_2)} {}_{C_1}(G, \gamma)_{C_2} \longrightarrow 0$$

is a J-exact sequence of correspondences, then $\delta_{J_{(F,\beta)}} \hat{\otimes} K(E, \alpha) = K(G, \gamma) \hat{\otimes} \delta_{B_2}$.

Proof. In the notation of remark 3.0.4, we must show $(K(G, \gamma), K(E, \alpha)) : \delta_{J_{(F,\beta)}} \to \delta_{B_2}$ is an intertwiner. By the Rieffel correspondence ([49, Proposition 3.24]), exactness of

$$0 \longrightarrow E \xrightarrow{I} F \xrightarrow{Q} G \longrightarrow 0$$

implies exactness of

$$0 \longrightarrow \mathcal{K}(E) \xrightarrow{\hat{I}} \mathcal{K}(F) \xrightarrow{\hat{Q}} \mathcal{K}(G) \longrightarrow 0$$

$$0 \longrightarrow \langle E \rangle \xrightarrow{i_2} \langle F \rangle \xrightarrow{q_2} \langle G \rangle \longrightarrow 0.$$

Hence, the following diagram commutes and has exact rows (the vertical maps labelled with

i are the respective inclusions):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_{(E,\alpha)} & \xrightarrow{i_1} & J_{(F,\beta)} & \xrightarrow{q_1} & J_{(G,\gamma)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \mathcal{K}(E) & \xrightarrow{\hat{I}} & \mathcal{K}(F) & \xrightarrow{\hat{Q}} & \mathcal{K}(G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i^E_{K\to L}} & & \downarrow{\scriptstyle i^F_{K\to L}} & & \downarrow{\scriptstyle i^G_{K\to L}} & & \\
0 & \longrightarrow & \mathcal{L}(E) & \xrightarrow{\mathcal{L}(I)} & \mathcal{L}(F) & \xrightarrow{\mathcal{L}(Q)} & \mathcal{L}(G) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i^E_{\langle\rangle\to L}} & & \uparrow{\scriptstyle i^F_{\langle\rangle\to L}} & & \uparrow{\scriptstyle i^G_{\langle\rangle\to L}} & & \\
0 & \longrightarrow & \langle E\rangle & \xrightarrow{i_2} & \langle F\rangle & \xrightarrow{q_2} & \langle G\rangle & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i^{A_2}} & & \downarrow{\scriptstyle i^{B_2}} & & \downarrow{\scriptstyle i^{C_2}} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{i_2} & B_2 & \xrightarrow{q_2} & C_2 & \longrightarrow & 0.
\end{array}$$

Therefore, naturality of δ implies

(1) $([\gamma],[\alpha]) : \delta_{J_{(F,\beta)}} \to \delta_{\mathcal{K}(F)}$,

(2) $([i^G_{K\to L}],[i^E_{K\to L}]) : \delta_{\mathcal{K}(F)} \to \delta_{\mathcal{L}(F)}$,

(3) $([i^G_{\langle\rangle\to L}],[i^E_{\langle\rangle\to L}]) : \delta_{\langle F\rangle} \to \delta_{L(F)}$, and

(4) $([i^{C_2}],[i^{A_2}]) : \delta_{\langle F\rangle} \to \delta_{B_2}$

are all intertwiners.

For $H = E, F, G$, the inclusion $i^H_{\langle\rangle\to L} : \langle H\rangle \to \mathcal{L}(H)$ is full and hereditary, and therefore its class $[i^H_{\langle\rangle\to L}]$ in KK^0 is invertible. Therefore, $([i^G_{\langle\rangle\to L}]^{-1},[i^E_{\langle\rangle\to L}]^{-1}) : \delta_{L(F)} \to \delta_{\langle F\rangle}$ is an intertwiner. Composing all the intertwiners above yields an intertwiner $(g,e) : \delta_{J_{(F,\beta)}} \to \delta_{B_2}$, where $g = [\gamma]\hat{\otimes}[i^G_{K\to L}]\hat{\otimes}[i^G_{\langle\rangle\to L}]^{-1}\hat{\otimes} i^{C_2}$ and $e = [\alpha]\hat{\otimes}[i^E_{K\to L}]\hat{\otimes}[i^E_{\langle\rangle\to L}]^{-1}\hat{\otimes} i^{A_2}$. Let us show $e = K(E,\alpha)$ and $g = K(G,\gamma)$.

For $H = E, F, G$, denote by $M(H)$ the $\mathcal{K}(H)$ - $\langle H\rangle$ Kasparov bi-module H. $(i^H_{\langle\rangle\to L})_* M(H)$ is isomorphic as a bi-module to $\begin{pmatrix}\mathcal{K}(H) & H\\ 0 & 0\end{pmatrix}$ with left action of k in $\mathcal{K}(H)$ by the matrix $i^H_{K\to L}(k) = \begin{pmatrix}k & 0\\ 0 & 0\end{pmatrix}$ and right $\mathcal{L}(H)$-module structure inherited by the identity Hilbert module structure on $\mathcal{L}(H)$. The isomorphism $U : H\hat{\otimes}_{i^H_{\langle\rangle\to L}}\mathcal{L}(H) \to \begin{pmatrix}\mathcal{K}(H) & H\\ 0 & 0\end{pmatrix}$ on a basic tensor $t = e_1\hat{\otimes}\begin{pmatrix}k & e_2\\ f & a\end{pmatrix}$ is $U(t) = \begin{pmatrix}0 & e_1\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}k & e_2\\ f & a\end{pmatrix} = \begin{pmatrix}\theta_{e_1,f} & e_1\cdot a\\ 0 & 0\end{pmatrix}$.

Let $D = \{f \in C([0,1], \mathcal{L}(H)) : f(1) \in \begin{pmatrix} \mathcal{K}(H) & H \\ 0 & 0 \end{pmatrix}\}$ with the right $C([0,1], \mathcal{L}(H))$-module structure inherited by the identity Hilbert module structure on $C([0,1], \mathcal{L}(H))$. Let $\eta : \mathcal{K}(H) \to \mathcal{K}(D)$ be the embedding that sends k in $\mathcal{K}(H)$ to the constant function $\eta(k)(t) = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$, t in $[0,1]$. Then, (D, η) is a Kasparov $\mathcal{K}(H)$-$C([0,1], \mathcal{L}(H))$ bi-module such that $(ev_0)_*(D, \eta) \simeq (\mathcal{L}(H), i^H_{K \to L})$ and $(ev_1)_*(D, \eta) \simeq (\begin{pmatrix} \mathcal{K}(H) & H \\ 0 & 0 \end{pmatrix}, i^H_{K \to L}) \simeq (i^H_{\langle\rangle \to L})_* M(H)$. Hence, $[M(H)] \hat{\otimes} [i^H_{\langle\rangle \to L}] = [i^H_{K \to L}]$, so that $[M(H)] = [i^H_{\langle\rangle \to L}] \hat{\otimes} [i^H_{K \to L}]^{-1}$.

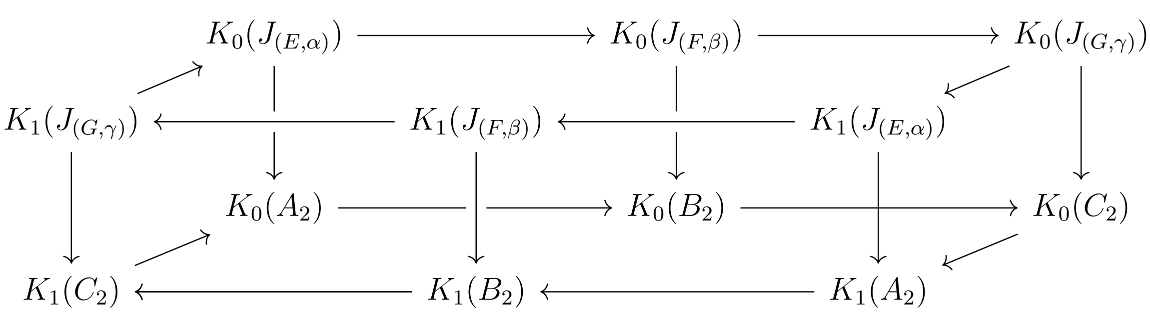
Now, it is routine to see that $[\alpha] \hat{\otimes} [M(E)] \hat{\otimes} [i^{A_2}] = K(E, \alpha)$ and $[\gamma] \hat{\otimes} [M(F)] \hat{\otimes} [i^{C_2}] = K(F, \gamma)$, so the calculation directly above implies $e = K(E, \alpha)$ and $g = K(G, \gamma)$. □

The intertwinings we have proven above yield commutative diagrams of their induced maps on K-theory and K-homology (and more generally on $KK^*(A, -)$ and $KK^*(-, A)$, A nuclear). We record the diagram that will be of use to us in this thesis.

Corollary 3.0.9. *If*

$$0 \longrightarrow {}_{A_1}(E, \alpha)_{A_2} \xrightarrow{(i_1, I, i_2)} {}_{B_1}(F, \beta)_{B_2} \xrightarrow{(q_1, Q, q_2)} {}_{C_1}(G, \gamma)_{C_2} \longrightarrow 0$$

is a J-exact sequence of correspondences, then the following diagram commutes



The top and bottom horizontal faces are the 6-term exact sequences of K-theory associated to the respective extensions of C^*-algebras, and the vertical maps are the respective maps of the form $\hat{\otimes}_i K(H, \eta)$, $(H, \eta) = (E, \alpha), (F, \beta), (G, \gamma)$.

Proof. This is a direct application of Proposition 3.0.3 and 3.0.8. □

Corollary 3.0.10. *If*

$$0 \longrightarrow {}_{A}(E, \alpha)_{A} \xrightarrow{(i, I, i)} {}_{B}(F, \beta)_{B} \xrightarrow{(q, Q, q)} {}_{C}(G, \gamma)_{C} \longrightarrow 0$$

is a J-exact sequence of morphisms, then so is

$$0 \longrightarrow {}_{\langle E\rangle}(E,\alpha)_A \xrightarrow{(i,I,i)} {}_{\langle F\rangle}(F,\beta)_B \xrightarrow{(q,Q,q)} {}_{\langle C\rangle}(G,\gamma)_C \longrightarrow 0,$$

And so in the commuting diagram of Corollary 3.0.9, we have $J_{(H,\eta|_{\langle H\rangle})} = I_{(H,\eta)}$ with the vertical maps of the form $\iota - \hat{\otimes}_i[I(H,\eta)]$, where $\iota : K_i(I_{(H,\eta)}) \to K_i(D)$ is the map induced by inclusion, and $(H,\eta) = (E,\alpha),(F,\beta),(G,\gamma)$.

Proof. As in the proof of Proposition 3.0.8, we have exactness of

$$0 \longrightarrow \langle E\rangle \xrightarrow{i} \langle F\rangle \xrightarrow{q} \langle G\rangle \longrightarrow 0,$$

and exactness of

$$0 \longrightarrow J_{(E,\alpha)} \xrightarrow{i} J_{(F,\beta)} \xrightarrow{q} J_{(G,\gamma)} \longrightarrow 0,$$

is by assumption. Since $J_{(H,\eta|_{\langle H\rangle})} = \langle H\rangle \cap J_{(H,\eta)}$, the exactness of these two sequences yields J-exactness of

$$0 \longrightarrow {}_{\langle E\rangle}(E,\alpha)_A \xrightarrow{(i,I,i)} {}_{\langle F\rangle}(F,\beta)_B \xrightarrow{(q,Q,q)} {}_{\langle C\rangle}(G,\gamma)_C \longrightarrow 0.$$

□

Let us show a J-exact sequence of self-correspondences induces a short exact sequence of Cuntz-Pimsner algebras.

Proposition 3.0.11. *If*

$$0 \longrightarrow {}_A(E,\alpha)_A \xrightarrow{(i,I,i)} {}_B(F,\beta)_B \xrightarrow{(q,Q,q)} {}_C(G,\gamma)_C \longrightarrow 0$$

is a J-exact sequence of morphisms of faithful correspondences such that (E,α) is non-

degenerate, then the following diagram commutes and has exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_E & \xrightarrow{\mathcal{T}(I)} & \mathcal{I}_F & \xrightarrow{\mathcal{T}(Q)} & \mathcal{I}_G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_E} & & \downarrow{\scriptstyle i_F} & & \downarrow{\scriptstyle i_G} & & \\
0 & \longrightarrow & \mathcal{T}_E & \xrightarrow{\mathcal{T}(I)} & \mathcal{T}_F & \xrightarrow{\mathcal{T}(Q)} & \mathcal{T}_G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q_E} & & \downarrow{\scriptstyle q_F} & & \downarrow{\scriptstyle q_G} & & \\
0 & \longrightarrow & \mathcal{O}_E & \xrightarrow{\mathcal{O}(I)} & \mathcal{O}_F & \xrightarrow{\mathcal{O}(Q)} & \mathcal{O}_G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Proof. We first show the sequence of Toeplitz algebras is exact. For T in $\mathcal{T}_E$, $\mathcal{T}(I)(T)$ restricted to the A-module $\sum_{n=0}^{\infty} I(E)^{\otimes_B n} \simeq \mathcal{F}_E$ is conjugate to T. Therefore, $\mathcal{T}(I)$ is injective.

$\mathcal{T}(Q)$ surjects the generators $T_F = \{T_f : f \in F\}$ onto the generators T_G, and is therefore surjective.

We show $\ker(\mathcal{T}(Q)) = \operatorname{im}(\mathcal{T}(I))$. By functoriality of $\mathcal{T}(-)$, we have that $\mathcal{T}(Q) \circ \mathcal{T}(I) = \mathcal{T}(Q \circ I) = 0$. It remains to show $\mathcal{T}(I)(\mathcal{T}_E)$ is an ideal and $\mathcal{T}(Q) : \mathcal{T}_F/\mathcal{T}(I)(\mathcal{T}_E) \to \mathcal{T}_G$ is injective.

First, if a is in A and f is in F, then $f \cdot i(a)$ and $i(a) \cdot f$ are in $I(E)$ because $Q(i(a) \cdot f) = q(i(a)) \cdot Q(f) = 0$ $Q(f \cdot i(a)) = Q(f) \cdot q(i(a)) = 0$. Second, if e is in E and f is in F, then $\langle I(e), f\rangle$ is in $i(\langle E\rangle) \subseteq \mathcal{T}(I)(\mathcal{T}_E)$, $T_f T_{I(e)}^*$ is in $\hat{I}(\mathcal{K}(E)) \subseteq \mathcal{T}(I)(\mathcal{T}_E)$ and $I(e) \otimes_B f$, $f \otimes_B I(e)$ are in $I(E^{\otimes_A 2})$. These facts follow from the fact that if $(a_\lambda)_{\lambda\in\Lambda}$ is an approximate unit in $\langle E\rangle$, then for ψ in $F^{\otimes_B n}$, if $(\psi \cdot i(a_\lambda))_{\lambda\in\Lambda}$ converges to ψ, then ψ is in $I(E^{\otimes_A n})$ (this follows from the first fact and induction on n).

It follows from above that for generators (as algebras) a in $\mathcal{T}_E$ and b in $\mathcal{T}_F$, we have that $\mathcal{T}(I)(a)b$ is in $\mathcal{T}(I)(\mathcal{T}_E)$, and this finishes the proof that $\mathcal{T}(I)(\mathcal{T}_E)$ is an ideal in $\mathcal{T}_F$.

For T in $\mathcal{T}_F$, denote by $\overline{T}$ its image in $\mathcal{T}_F/\mathcal{T}(I)(\mathcal{T}_E)$. For c in C and g in G, let b in B and f in F be such that $q(b) = c$ and $Q(f) = g$. Let $s(c) = \bar{b}$ and $\pi(g) = \overline{T_f}$. $s : C \to \mathcal{T}_F/\mathcal{T}(I)(\mathcal{T}_E)$ and $\pi : G \to \mathcal{T}_F/\mathcal{T}(I)(\mathcal{T}_E)$ are well defined since $\mathcal{T}(I)(\mathcal{T}_E) \cap B = i(A)$ and $\mathcal{T}(I)(\mathcal{T}_E) \cap T_F = T_{I(E)}$, and a different choice of representative for $s(c)$, $\pi(g)$ differs by an element of $i(A)$, $\mathcal{T}_{I(E)}$, respectively.

It is easy to see that (π, s) is a Toeplitz representation. By the universal property of Toeplitz algebras, (π, s) induces a $*$-homomorphism $\hat{\pi} : \mathcal{T}_G \to \mathcal{T}_F/\mathcal{T}(I)(\mathcal{T}_E)$ that is the inverse of $\mathcal{T}(Q) : \mathcal{T}_F/\mathcal{T}(I)(\mathcal{T}_E) \to \mathcal{T}_G$.

Now, we show the top row sequence is exact. Suppose T in $\mathcal{T}_E$ is such that $\mathcal{T}(I)(T) =: S$ is in $\mathcal{I}_F^c$, meaning there is k in $\mathbb{N}$ such that the projection P_k^F onto the sub-module $\sum_{n=0}^{k} F^{\otimes_B n}$ of $\mathcal{F}_F$ satisfies $P_k^F S = SP_k^F = S$. The restriction of S and P_k^F onto the invariant sub-module $\sum_{n=0}^{\infty} I(E^{\otimes_A n}) \simeq \mathcal{F}_E$ are naturally identified with T and P_k^E, respectively, and hence $P_k^E T = TP_k^E = T$. Therefore, T is in $\mathcal{I}_E^c$.

It follows that $\mathcal{T}(I)^{-1}(\overline{\mathcal{I}_F^c}) \subseteq \overline{\mathcal{I}_E^c}$. Since (E, α) is non-degenerate, we have $\overline{\mathcal{I}_E^c} = \mathcal{I}_E$ by Proposition 2.3.3. Hence, $\mathcal{T}(I)^{-1}(\mathcal{I}_F) \subseteq \mathcal{I}_E$ The reverse inclusion is obvious using the first characterization of $\mathcal{I}_E$ (Proposition 2.3.2).

The equality $\mathcal{T}(I)^{-1}(\mathcal{I}_F) = \mathcal{I}_E$ together with exactness of the middle row implies $\mathcal{T}(I)(\mathcal{I}_E) = \ker(\mathcal{T}_Q) \cap \mathcal{I}_F$.

It remains to show $\mathcal{T}(Q) : \mathcal{I}_F \to \mathcal{I}_G$ is surjective. By J-exactness, $q : J_{(F,\beta)} \to J_{(E,\alpha)}$ is surjective. Therefore, for every generator for $\mathcal{I}_G$ of the form $c - \phi(\gamma(c))$, c in $J_{(G,\gamma)}$, there is b in $J_{F,\beta)}$ such that $q(b) = c$. Hence $\mathcal{T}(Q)(b - \phi(\beta(b))) = c - \phi(\gamma(c))$. Therefore, $\mathcal{T}(Q) : \mathcal{I}_F \to \mathcal{I}_G$ is surjective.

The above commutative diagram has exact columns by definition and the top two rows are exact from what we have shown. Exactness of the bottom row then follows from a diagram chase. $\square$

Chapter 4

Correspondences from branched functions

Along the way to proving our main result, we will be working with the dynamics of holomorphic functions restricted to a variety of subspaces, so we need to abstract the C^*-correspondence construction of Kajiwara and Watatani [28] to allow for these restrictions. It turns out that the only essential fact for constructing correspondences in a similar fashion is that the function has a system of inverse branches, which we now make precise.

Definition 4.0.1. *Let $F : X \to Y$ be a continuous function between locally compact Hausdorff spaces X and Y. We will say F is a branched function if there is a function $ind_F : X \to \mathbb{N}$ such that for every x in X, there is a neighbourhood U of x, a neighbourhood V of $F(x)$, and functions $\{F_i^{-1} : V \to U\}_{i=1}^{ind_F(x)}$ satisfying*

(1) $F \circ F_i^{-1} = id_V$, for all $i \leq ind_F(x)$,

(2) $F_i^{-1}(F(x)) = x$ and F_i^{-1} is continuous at $F(x)$, for all $i \leq ind_F(x)$,

(3) $U = \bigcup_{i=1}^{ind_F(x)} F_i^{-1}(V)$, and

(4) for all u in U, $ind_F(u) = |\{i \leq ind_F(x) : u \in F_i^{-1}(V)\}|$.

ind_F is called the index function and $\{F_i^{-1} : V \to U\}_{i=1}^{d}$ are called inverse branches of F centered at x.

We remark that branched functions are in abundance. For instance, take a sheet of paper (allowing for arbitrary shape and cuts) and fold it in such a way that the crease lines and boundary are on the boundary of the folded paper. This is a branched function.

A special case of this construction is to take a Sierpinksi triangle and fold the outer three equilateral triangles into the middle. We may rotate the resulting Sierpinksi triangle by 180 degrees and dilate by 3 to get a surjective self map if we wish. The index function will be 1 everywhere except 3 at the intersection points of the three outer triangles.

Our definition is different than the notion of a branched covering $p : Y \to X$ in [45], where they only require p to be an open surjection of finite degree. Here, we require $F : X \to Y$ to be "locally finite degree", not necessarily surjective, and open is a consequence of the existence of inverse branches.

As in [28, Lemma 2.1], these branching properties can be used to show a canonical transfer operator of F maps $C_c(X)$ functions to $C_c(Y)$ functions.

First, note that if $\{F_i^{-1} : V \to U\}_{i=1}^{\mathrm{ind}_F(x)}$ are inverse branches centered at x in X, then $U \cap F^{-1}(F(x)) = \{x\}$. Hence, $F^{-1}(y)$ is discrete, for all y in Y.

Proposition 4.0.2. *Let $F : X \to Y$ be a branched function. If f is in $C_c(X)$, then the function $\Phi(f) : Y \to \mathbb{C}$ defined, for y in Y, as*

$$\Phi(f)(y) = \sum_{x \in F^{-1}(y)} ind_F(x) f(x)$$

is in $C_c(Y)$.

Proof. Let $\mathrm{supp}(f) = K$ and y in Y. We show $\Phi(f)$ is continuous at y. Since F is branched, $F^{-1}(y)$ is discrete and hence $F^{-1}(y) \cap K$ is finite. Hence, $\Phi(f)(y) < \infty$. Write $F^{-1}(y) \cap K = \{x_1, ..., x_k\}$ and let $\{U_i\}_{i=1}^k$ be a collection of pairwise disjoint open sets such that $U_i \cap F^{-1}(F(x)) = \{x_i\}$ and U_i is a co-domain of inverse branches $\{F_{ij}^{-1} : F(U_i) \to U_i\}_{j=1}^{\mathrm{ind}(x_i)}$ centered at x_i, for $i \leq k$. By a compactness argument, there is an open neighbourhood $V \subseteq \bigcap_{i=1}^k F(U_i)$ of y such that $F^{-1}(V) \cap K = \bigcup_{i=1}^k F^{-1}(V) \cap U_i \cap K$. By property (3) of the inverse branches, we have $\bigcup_{i=1}^k F^{-1}(V) \cap U_i \cap K = \bigcup_{i=1}^k \bigcup_{j=1}^{\mathrm{ind}(x_i)} F_{ij}^{-1}(V) \cap K$

Hence, for every v in V, we have

$$\sum_{x \in F^{-1}(v)} \mathrm{ind}_F(x) f(x) = \sum_{x \in \bigcup_{i=1}^k \bigcup_{j=1}^{\mathrm{ind}(x_i)} F_{ij}^{-1}(v)} \mathrm{ind}_F(x) f(x).$$

By property (4) of the inverse branches, we have

$$\sum_{x \in \bigcup_{i=1}^k \bigcup_{j=1}^{\mathrm{ind}(x_i)} F_{ij}^{-1}(v)} \mathrm{ind}_F(x) f(x) = \sum_{i=1}^{k} \sum_{j=1}^{\mathrm{ind}_F(x_i)} f \circ F_{ij}^{-1}(v).$$

Since F_{ij}^{-1} is continuous at y for all $i \leq k$ and $j \leq \mathrm{ind}_F(x_i)$, it follows that $\Phi(f)|_V = \sum_{i=1}^k \sum_{j=1}^{\mathrm{ind}_F(x_i)} f \circ F_{ij}^{-1}$ is continuous at y.

As y in Y was arbitrary, we may conclude $\Phi(f)$ is continuous. Note that $\mathrm{supp}(\Phi(f)) = F(\mathrm{supp}(f))$, and therefore $\Phi(f)$ is in $C_c(Y)$. $\square$

To a branched function $F : X \to Y$, we can associate a $C_0(X) - C_0(Y)$ correspondence $(E_{F,X}, \alpha_X)$ in a similar way as in [28]. We show this now. Let $\tilde{E}_{F,X} = C_c(X)$, equipped with a right $C_0(Y)$-module structure defined, for ψ in $\tilde{E}_{F,X}$ and g in $C_0(Y)$, as

$$(\psi \cdot g)(x) = \psi(x)g(F(x)), \ x \text{ in } X.$$

$\tilde{E}_{F,X}$ is equipped with the $C_0(Y)$-valued inner product defined, for ψ, φ in $\tilde{E}_{F,X}$, as

$$\langle \psi, \varphi \rangle = \Phi(\overline{\psi}\varphi).$$

For all ψ in $\tilde{E}_{F,X}$ we have $\|\psi\| \leq \|\psi\|_2$, where $\|\cdot\|$ denotes the sup norm and $\|\psi\|_2 := \|\sqrt{\langle \psi, \psi \rangle}\|$. If $\deg(F) := \sup_{y \in Y} |F^{-1}(y)| < \infty$, then we also have $\|\psi\|_2 \leq \sqrt{\deg(\tilde{F})}\|\psi\|$. In this case, the Hilbert $C_0(Y)$-module completion of $\tilde{E}_{F,X}$ is $E_{F,X} = C_0(X)$ and the inner product formula and right action extend naturally.

In general, the completion $E_{F,X}$ is a strict subspace of $C_0(X)$ that is invariant under the right action of $C_0(Y)$ defined above and equipped with the above inner product. We identify this now. First, for $i = 1, 2$, we let

$$C_i(X) = \{\psi \in C_0(X) : \Phi(|\psi|^i) \in C_0(Y)\}.$$

Lemma 4.0.3. *Let $F : X \to Y$ be a branched function. Then, $C_1(X)$ is a subspace of $C_0(X)$ that is hereditary in the sense that if φ, ψ are in $C_0(X)$ with ψ in $C_1(X)$ and $0 \leq \varphi \leq \psi$, then φ is in $C_1(X)$. Moreover, $C_1(X)$ is complete with respect to the norm $\|\cdot\|_1 := \|\Phi(|\cdot|)\|$.*

Proof. The fact that $C_1(X)$ is a subspace will follow from the hereditary property. Suppose $0 \leq \varphi \leq \psi$, ψ is in $C_1(X)$ and φ is in $C_0(X)$. First, we show $\Phi(\varphi)$ is continuous.

Fix y in Y and $\epsilon > 0$. Since $\Phi(\psi)(y) < \infty$, there is k in $\mathbb{N}$ and $\{x_i\}_{i=1}^k \subseteq F^{-1}(y)$ such that $x_i \neq x_j$ for all $i \neq j \leq k$ and

$$|\Phi(\psi)(y) - \sum_{i=1}^{k} \mathrm{ind}_F(x_i)\psi(x_i)| < \epsilon/7.$$

For each $i \leq k$, let $\{F_{ij}^{-1} : V \to U_i\}_{j=1}^{\mathrm{ind}_F(x_i)}$ be inverse branches of F centered at x_i. By continuity of these branches and $\Phi(\psi)$ at y, we may assume the neighbourhood V of y is chosen such that, for all $\tilde{y}$ in V, we have

$$|\sum_{i=1}^{k}\mathrm{ind}_F(x_i)\psi(x_i)-\sum_{i=1}^{k}\sum_{j=1}^{\mathrm{ind}_F(x_i)}\psi\circ F_{ij}^{-1}(\tilde{y})|<\epsilon/7,$$

$$|\Phi(\psi)(y)-\Phi(\psi)(\tilde{y})|<\epsilon/7,$$

and

$$|\sum_{i=1}^{k}\sum_{j=1}^{\mathrm{ind}_F(x_i)}\varphi\circ F_{ij}^{-1}(y)-\sum_{i=1}^{k}\sum_{j=1}^{\mathrm{ind}_F(x_i)}\varphi\circ F_{ij}^{-1}(\tilde{y})|<\epsilon/7. \qquad (*)$$

Let $U=\bigcup_{i=1}U_i$. For any f in $C_0(X)$ and $\tilde{y}$ in V, we have

$$\sum_{x\in F^{-1}(\tilde{y})\cap U}\mathrm{ind}_F(x)f(x)=\sum_{i=1}^{k}\sum_{j=1}^{\mathrm{ind}_F(x_i)}f\circ F_{ij}^{-1}(\tilde{y})$$

. Therefore, the first three inequalities imply, by the triangle inequality, that

$$\sum_{x\in F^{-1}(\tilde{y})\setminus U}\mathrm{ind}_F(x)\psi(x)=|\Phi(\psi)(\tilde{y})-\sum_{x\in F^{-1}(\tilde{y})\cap U}\mathrm{ind}_F(x)\psi(x)|<3\epsilon/7,$$

for all $\tilde{y}$ in V.

Since $0\le\varphi\le\psi$, it follows that

$$|\Phi(\varphi)(y)-\sum_{x\in F^{-1}(\tilde{y})\cap U}\mathrm{ind}_F(x)\varphi(x)|\le\sum_{x\in F^{-1}(\tilde{y})\setminus U}\mathrm{ind}_F(x)\psi(x)<3\epsilon/7, \qquad (**)$$

for all $\tilde{y}$ in V. So, by the triangle inequality and the inequalities (*) and (**), we have

$$|\Phi(\varphi)(y)-\Phi(\varphi)(\tilde{y})|<\epsilon,$$

for all $\tilde{y}$ in V. This shows $\Phi(\varphi)$ is continuous.

$0\le\Phi(\varphi)\le\Phi(\psi)$ and the fact that $\Phi(\psi)$ vanishes at infinity implies $\Phi(\varphi)$ does too. Therefore, $\Phi(\varphi)$ is in $C_0(Y)$.

We now show $C_1(X)$ is complete with respect to $\|\cdot\|_1$. Let $\{\psi_n\}_{n\in\mathbb{N}}\subset C_0(X)$ such that $\sum_{n\in\mathbb{N}}\|\psi_n\|_1<\infty$. Since $\|\cdot\|\le\|\cdot\|_1$, $\sum_{n\in\mathbb{N}}\psi_n$ and $\sum_{n\in\mathbb{N}}|\psi_n|$ converge (with respect to $\|\cdot\|$) to elements in $C_0(X)$.

$0\le|\sum_{n\in\mathbb{N}}\psi_n|\le\sum_{n\in\mathbb{N}}|\psi_n|$ so, by the hereditary property of $C_1(X)$, to prove that $\sum_{n\in\mathbb{N}}\psi_n$ is in $C_1(X)$ it suffices to show $\sum_{n\in\mathbb{N}}|\psi_n|$ is in $C_1(X)$.

$\|\sum_{n\le k}\Phi(|\psi_n|)-\sum_{n\in\mathbb{N}}\Phi(|\psi_n|)\|\le\sum_{n>k}\|\psi_n\|_1$ and so the hypothesis implies $\Phi(\sum_{n\in\mathbb{N}}|\psi_n|)=\sum_{n\in\mathbb{N}}\Phi(|\psi_n|)$ is in $C_0(Y)$. Hence, $C_1(X)$ is complete. □

Proposition 4.0.4. *Let $F : X \to Y$ be a branched function. Then, the Hilbert $C_0(Y)$-module completion of $\tilde{E}_{F,X}$ is isomorphic to $C_2(X)$.*

Proof. Let us first show $C_2(X)$ is complete. Let $\{\psi_n\}_{n\in\mathbb{N}} \subset C_0(X)$ such that $\sum_{n\in\mathbb{N}} \|\psi_n\|_2 < \infty$. Since $\|\cdot\| \leq \|\cdot\|_2$, $\sum_{n\in\mathbb{N}} \psi_n$ and $\sum_{n\in\mathbb{N}} |\psi_n|$ converge (with respect to $\|\cdot\|$) to elements in $C_0(X)$.

We must show $|\sum_{n\in\mathbb{N}} \psi_n|^2$ is in $C_1(X)$. By the Cauchy-Schwarz inequality, $\|\overline{\psi_n}\psi_m\|_1 \leq \|\psi_n\|_2\|\psi_m\|_2$ for all n, m in $\mathbb{N}$. Hence, $|\sum_{n\in\mathbb{N}} \psi_n|^2 = \sum_{n,m\in\mathbb{N}} \overline{\psi_n}\psi_m$ converges absolutely with respect to $\|\cdot\|_1$. Lemma 4.0.3 then implies $|\sum_{n\in\mathbb{N}} \psi_n|^2$ is in $C_1(X)$.

Lastly, we must show $C_c(X)$ is dense in $C_2(X)$ relative to $\|\cdot\|_2$. It suffices to show positive elements in $C_2(X)$ can be approximated arbitrarily by elements in $C_c(X)$.

Let $0 \leq \varphi$ be such that $\psi := \varphi^2$ is in $C_1(X)$. Given $\epsilon > 0$, let $K \subseteq Y$ be a compact set such that $\Phi(\psi)(\tilde{y}) < \epsilon^2/2$ for all $\tilde{y}$ in $X \setminus K$. As in Lemma 4.0.3, For each y in K there is a pre-compact open set $U_y \subseteq X$ such that y is in $F(U_y) =: V_y$ and

$$|\Phi(\psi)(\tilde{y}) - \sum_{x\in F^{-1}(\tilde{y})\cap U_y} \mathrm{ind}_F(x)\psi(x)| < \epsilon^2/2$$

for all $\tilde{y}$ in V_y. Let $\{y_i\}_{i=1}^d \subseteq K$ be such that $K \subseteq \bigcup_{i=1}^d V_{y_i}$. Then, $U := \bigcup_{i=1}^d U_{y_i}$ is a pre-compact open set such that

$$|\Phi(\psi)(\tilde{y}) - \sum_{x\in F^{-1}(\tilde{y})\cap U} \mathrm{ind}_F(x)\psi(x)| < \epsilon^2$$

for all $\tilde{y}$ in Y.

Choose ϕ in $C_c(X)$ such that $0 \leq \phi \leq 1$ and $\phi(x) = 1$ for all x in U. From the above inequality, we have

$$|\Phi(|\varphi - \sqrt{\phi}\varphi|^2)(\tilde{y})| \leq \sum_{x\in F^{-1}(\tilde{y})\setminus U} \mathrm{ind}_F(x)\psi(x) < \epsilon^2$$

for all $\tilde{y}$ in Y. Hence, $\|\varphi - \sqrt{\phi}\varphi\|_2 \leq \epsilon$. □

$E_{F,X} = C_2(X)$ has a left action α_X of $C_0(X)$ by Hilbert module endomorphisms given, for f in $C_0(X)$ and ψ in $E_{F,X}$, as

$$(f \cdot \psi)(x) = f(x)\psi(x),\ x \text{ in } X.$$

The pair $(E_{F,X}, \alpha_X)$ is a $C_0(X)$-$C_0(Y)$ correspondence. It is always injective and non-degenerate. If F is surjective, it is full.

If $F : X \to Y$ and $G : Y \to Z$ are branched functions with index functions ind_F and ind_G, then $G \circ F$ is a branched function with index function $\mathrm{ind}_{G\circ F} := (\mathrm{ind}_G \circ F)\mathrm{ind}_F$; the inverse branches of $G \circ F$ are the composite of inverse branches for F and G.

For $F : X \to Y$ a continuous function, the *critical points of* F will be the set of points in X for which F is *not* a local homeomorphism at, in the sense that there is no open neighbourhood U about x such that $F(U)$ is open in Y and $F : U \to F(U)$ is a homeomorphism. We shall denote this set by $C_{F,X}$. If $F : X \to Y$ is an open map (which is the case for a branched function), then this is a closed subset.

We can think of a branched function $F : X \to Y$ and its canonical transfer operator Φ as defining a topological quiver in the sense of [41]; in the notation of that paper, $E^1 = X$, $E^0 = Y$ and $r = s = F$, with r-system $\{\Phi(-)(y)\}_{y\in Y}$. Then, the Hilbert $C_0(Y)$-module constructed in [41, Section 3.1] is isomorphic to $E_{F,X}$. [41, Theorem 3.11] then can be applied to show $J_{(E_{F,X},\alpha_X)} = C_0(X \setminus C_{F,X})$.

Let us show this with a more direct proof generalizing [28, Proposition 2.5].

Proposition 4.0.5. *If $F : X \to Y$ is a branched function, then $J_{(E_{F,X},\alpha_X)} = C_0(X \setminus C_{F,X})$.*

Proof. We first show $C_0(X \setminus C_{F,X}) \subseteq \phi^{-1}(\mathcal{K}(E_{F,X}))$. Suppose $f \geq 0$ in $C_c(X)$ has $\mathrm{supp}(f) \subseteq U$, where U is an open neighbourhood of x not in $C_{F,X}$ such that $F(U)$ is open and $F : U \to F(U) = V$ is a homeomorphism and let $\{F_i^{-1} : V \to U\}_{i=1}^{\mathrm{ind}_F(x)}$. Then, by property (*) of ind_F and the fact that $F : U \to V$ is a homeomorphism, we have $\mathrm{ind}_F(y) = \mathrm{ind}_F(x)$ for all y in U. Therefore, $f = \frac{1}{\mathrm{ind}_F(x)}\theta_{\sqrt{f},\sqrt{f}}$. The containment $C_0(X \setminus C_{F,X}) \subseteq \phi^{-1}(\mathcal{K}(E_{F,X}))$ then follows by density of the (algebraic) span of functions f in $C_0(X \setminus C_{F,X})$ of the type above.

Now, we show $\phi(f)$ is not in $\mathcal{K}(E_{F,X})$ if $f(x) \neq 0$ for some x in $C_{F,X}$. Let $D := \mathrm{ind}_F(x)$. Suppose on the contrary that $\phi(f)$ is in $\mathcal{K}(E_{F,X})$. Then, $\frac{|f|}{|f(x)|}$ is in $\mathcal{K}(E_{F,X})$. By the hereditary property of ideals, there is then a function g in $C_c(X)$ such that $g(x) = 1$, $0 \leq g \leq 1$ and $\mathrm{supp}(g) \subseteq U$, where U is the co-domain of an inverse branch system $\{F_i^{-1} : F(U) \to U\}_{i=1}^{D}$ at x. Since x is in $C_{F,X}$ there are two nets $\{x_\lambda^1\}_{\lambda\in\Lambda}, \{x_\lambda^2\}_{\lambda\in\Lambda} \subseteq U$ converging to x such that $x_\lambda^1 \neq x_\lambda^2$ and $F(x_\lambda^1) = F(x_\lambda^2) =: y_\lambda$ for all λ in Λ. By choosing a sub-net if required, we may assume that there are $i, j \leq D$ such that $F_i^{-1}(y_\lambda) = x_\lambda^1$ and $F_j^{-1}(y_\lambda) = x_\lambda^2$ for all λ in Λ. Therefore, by property $(*)$ of ind_F, we have $\mathrm{ind}_F(x_\lambda^1) \leq D - 1$ for all λ in Λ. By taking a further sub-sequence, we may assume there is $d < D$ such that $\mathrm{ind}_F(x_\lambda^1) = d$ for all λ in Λ.

Note that if $\|\phi(g)^{1/3} - \sum_{i=1}^N \theta_{e_i,f_i}\| \leq \epsilon$, then $\|\phi(g) - \sum_{i=1}^N \theta_{g^{1/3}e_i,g^{1/3}f_i}\| = \|\phi(g)^{1/3}(\phi(g)^{1/3} - \sum_{i=1}^N \theta_{e_i,f_i})\phi(g)^{1/3}\| \leq \epsilon$. So, for every $\epsilon > 0$ there are functions $\{e_i\}_{i=1}^N, \{f_i\}_{i=1}^N \subseteq C_c(U)$ such that $\|\phi(f) - \sum_{i=1}^N \theta_{e_i,f_i}\| \leq \epsilon$.

Choose, for each λ in Λ a function h_λ in $C_c(U)$ such that $0 \leq h \leq 1$, $h_\lambda(x_\lambda^1) = 1$ and $\mathrm{supp}(h_\lambda) \cap F^{-1}(F(x_\lambda^1)) = \{x_\lambda^1\}$. Then, $\|h\|_2 \leq \sqrt{D}$

$$|h(x^1_\lambda)-d\sum_{i=1}^{N}e_i(x^1_\lambda)\overline{f_i}(x^1_\lambda)| \leq \|\phi(g)h-\sum_{i=1}^{N}e_i(\Phi(\overline{f_i}h)\circ F)\| \leq \|\phi(g)(h)-\sum_{i=1}^{N}\theta_{e_i,f_i}(h)\|_2 \leq \epsilon\sqrt{D}.$$

Taking the limit, we therefore have $|1-d\sum_{i=1}^{N}e_i(x)\overline{f_i}(x)| \leq \sqrt{D}\epsilon$. However, if we choose $\tilde{h}$ in $C_c(U)$ such that $0 \leq \tilde{h} \leq 1$ and $\tilde{h}(x)=1$, we have

$$|1-D\sum_{i=1}^{N}e_i(x)\overline{f_i}(x)| \leq \|\phi(g)h-\sum_{i=1}^{N}e_i(\Phi(\overline{f_i}\tilde{h})\circ F)\| \leq \|\phi(g)(\tilde{h})-\sum_{i=1}^{N}\theta_{e_i,f_i}(\tilde{h})\|_2 \leq \epsilon\sqrt{D}.$$

Combining these two inequalities, we see that
$|\frac{1}{d}-\frac{1}{D}| \leq \epsilon\sqrt{D}(\frac{1}{D}+\frac{1}{d})$. So, by choosing ϵ small enough, we arrive at a contradiction. □

Let us determine a sufficient condition for when the C^*-algebra $\mathcal{O}_{F,X}$ associated to a branched functions $F : X \to X$ is simple and purely infinite. The proof is based on that found in [28] for the case of a rational function restricted to its Julia set.

Proposition 4.0.6. *Let $F : X \to X$ be a branched function such that X is infinite. If for every compact set $K \subseteq X$ and open set $U \subseteq X$, there is n in $\mathbb{N}$ such that $K \subseteq F^n(U)$, then $\mathcal{O}_{F,X}$ is simple and purely infinite. Moreover, $\mathcal{O}_{F,X}$ is a Kirchberg algebra that is unital if and only X is compact.*

Proof. Let $0 \leq u \leq 1$ be a non-zero function in $C_c(X)$. We show for every $a \geq 0$ in $\mathcal{O}_{F,X}$ and $\epsilon > 0$, there is $r \in \mathcal{O}_{F,X}$ such that $\|r^*ar-u\| \leq \epsilon$. This will complete the proof since $C_0(X)$ contains an approximate unit for $\mathcal{O}_{F,X}$ (the left action α on E_X is non-degenerate).

For every $\delta > 0$, there is c in the algebraic span of $C_0(X)$ and $\{S_e\}_{e\in E_{F,X}}$ such that $b=c^*c$ satisfies $\|a-b\| \leq \delta$ We can write b as a finite sum $b=\sum_{j=-M}^{M}b_j$ of its Fourier co-efficients. Further, we can write $b_0=\sum_{k=0}^{N}S_{e_k}S^*_{f_k}$, where e_k, f_k are in $E^{n_k}_{F,X}$ and $n_k \leq n_{k+1}$ for all $0 \leq k \leq N$. Therefore, we can view b_0 (isometrically) as an operator in $\mathcal{B}(E^n_{F,X})$, where $n_N = n$. Hence, there is f in $E^n_{F,X}$ with $\|f\|_2 = 1$ and x in X such that $\langle b_0(f), f\rangle(x) \geq \|b_0\|-\delta$. By the hypothesis, in every neighbourhood U of x, there is x' such that $\{F^j(x')\}_{j=0}^{M}$ are distinct. Therefore, by choosing U small enough, we can find ϕ in $C_c(X)$ such that $0 \leq \phi \leq 1$, $\phi(\phi\circ F^j)=0$ for all $0 \leq j \leq M$, and $\|x^*bx\| = \|\phi\langle b_0(f), f\rangle\phi\| \geq \|b_0\|-2\delta$, where $x = S_f\phi$. Note that $xb_jx^*\phi(\phi\circ F^j)S^*_fb_jS_f = 0$ for $0 \leq j \leq M$ and similarly $xb_jx^* = 0$ for $-M \leq j < 0$, so we have $x^*bx = x^*b_0x$. Set $d = x^*b_0x$, and let V be a open set such that $\|b_0\|-3\delta \leq d \leq \|b_0\|$. Let y in $C_c(V)$ be such that $0 \leq y \leq 1$ and $y = 1$ on a neighbourhood contained in V. By the hypothesis, there is K in $\mathbb{N}$ such that $\Phi^K(y^2) \geq u$, so if we let

$g = y\frac{\sqrt{u}\circ F^n}{\sqrt{\Phi^K(y^2)\circ F^n}}$ and view it in $E^K_{F,X}$, then $S_g^* S_g = u$. Since g is supported in V, we have $(\|b_0\| - 3\delta)u \leq S_g^* d S_g \leq \|b_0\| u$. With $r = S_f \phi S_g$, we have $\|r\| \leq 1$ and $\|r^* a r - \|b_0\| u\| \leq 4\delta$. Since the mapping $T \in \mathcal{O}_{F,X} \to T_0$ is continuous and non-zero on positive elements, we can choose $b \in \mathcal{O}_{F,X}$ to also satisfy $\|a_0 - b_0\| \leq \delta$. Therefore, there is r in $\mathcal{O}_{F,X}$ such that $\|r^* a r - u\| \leq \frac{5}{\|a_0\|}\delta$. Choosing $\delta = \epsilon \frac{\|a_0\|}{5}$ completes the proof that $\mathcal{O}_{F,X}$ is simple and purely infinite.

The algebras $\mathcal{O}_{F,X}$ are always nuclear and satisfy the UCT by an argument similar to [28, Proposition 3.9]. Therefore, $\mathcal{O}_{F,X}$ is a Kirchberg algebra. If X is unital, then the unit 1 in $C(X)$ is a unit for $\mathcal{O}_{F,X}$. In the reverse direction, if $\mathcal{O}_{F,X}$ is unital, then since the inclusion $C_0(X) \to \mathcal{O}_{F,X}$ is non-degenerate, an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ for $C_0(X)$ is an approximate unit for $\mathcal{O}_{F,X}$. Since $\mathcal{O}_{F,X}$ is unital, $(u_\lambda)_{\lambda\in\Lambda}$ is convergent in $\mathcal{O}_{F,X}$ to 1. Using the fact that the image of $C_0(X)$ inside $\mathcal{O}_{F,X}$ is closed, we see that 1 is in $C_0(X)$ and hence X is compact. $\square$

We will denote the trivially graded Kasparov $C_0(X \setminus C_{F,X}) - C_0(Y)$ bi-module $(E_{F,X}, \alpha_X|_{C_0(X\setminus C_{F,X})}, 0)$ by $\mathcal{E}_{F,X}$, and denote its class in $KK^0(C_0(X \setminus C_{F,X}), C_0(Y))$ by $[\mathcal{E}_{F,X}]$. We shall denote the class of the inclusion $i : C_0(X \setminus C_{F,X}) \to C_0(X)$ in $KK^0(C_0(X \setminus C_{F,X}), C_0(X))$ by ι.

We remark that there is an ambiguity in our notations $E_{F,X}$ and $\mathcal{E}_{F,X}$, since we do not specify the co-domain of F, but it will be clear from the context what it is.

We now show that the class $[\mathcal{E}_{F,X}]$ behaves well with restrictions of F. If U is an open set in X, let $i_U = i : C_0(U) \to C_0(X)$, $\iota_U = i : C_0(U \setminus C_{F,X}) \to C_0(X \setminus C_{F,X})$ denote the respective inclusions. For an open set V of Y such that $F(U) \subseteq V$ it is clear that $F : U \to V$ is also a branched function.

Proposition 4.0.7. *If $F : X \to Y$ is a branched function and U is an open set in X, then $(\iota_U)^*[\mathcal{E}_{F,X}] = (i_V)_*[\mathcal{E}_{F,U}]$, where V is any open set in Y such that $F(U) \subseteq V$. Moreover, $(i_U, i_U, i_V) : (E_{F,U}, \alpha_U) \to (E_{F,X}, \alpha_X)$ is a morphism of correspondences.*

Proof. The proof that $(i_U, i_U, i_V) : (E_{F,U}, \alpha_U) \to (E_{F,X}, \alpha_X)$ is a morphism of correspondences is straightforward, and we omit it. Proposition 3.0.3 implies $(\iota_U)^*[\mathcal{E}_{F,X}] = (i_V)_*[\mathcal{E}_{F,U}]$. $\square$

If $X = Y$ and $F : X \to X$ is a branched function, we will denote the corresponding Cuntz-Pimsner algebra and its Toeplitz algebra extension by $\mathcal{O}_{F,X}$ and $\mathcal{T}_{F,X}$, respectively. These algebras are conjugacy invariants for the pair (F, ind_F).

Proposition 4.0.8. *Suppose $F : X \to X$ and $G : Y \to Y$ are branched functions, and $\varphi : X \to Y$ is a homeomorphism such that $\varphi \circ F = G \circ \varphi$ and $ind_G \circ \varphi = ind_F$. Then,*

$(\varphi^*, \varphi^*, \varphi^*) : E_{G,Y} \to E_{F,X}$ is an isomorphism of correspondences. In particular, the induced $*$-homomorphisms $\mathcal{T}(\varphi^*) : \mathcal{T}_{G,Y} \to \mathcal{T}_{F,X}$, $\mathcal{O}(\varphi^*) : \mathcal{O}_{G,Y} \to \mathcal{O}_{F,X}$ are $*$-isomorphisms.

Proof. The proof is straightforward, and we omit it. □

We now consider exact sequences of branched functions. If $F : U \to V$ is branched and $Y \subseteq V$ is closed, letting $F^{-1}(Y) = X$, $F|_X : X \to Y$ is also branched, with $\mathrm{ind}_{F|_X} = \mathrm{ind}_F|_X$ and inverse branches equal to the inverse branches of F restricted to Y.

Proposition 4.0.9. *Let $F : U \to V$ be a branched function, and Y a closed subset of V. Denote $F^{-1}(Y) = X$, $i_1 : C_0(U \setminus X) \to C_0(U)$, $i_2 : C_0(V \setminus Y) \to C_0(V)$ the inclusions and $r_1 : C_0(U) \to C_0(X)$, $r_2 : C_0(V) \to C_0(Y)$ the restrictions. Then, $(i_1, i_1, i_2) : (E_{F,U\setminus X}, \alpha_{U\setminus X}) \to (E_U, \alpha_U)$ and $(r_1, r_1, r_2) : (E_U, \alpha_U) \to (E_X, \alpha_X)$ are morphisms. $C_{F,U} \cap X = C_{F,X}$, if and only if the sequence*

$$0 \longrightarrow (E_{F,U\setminus X}, \alpha) \xrightarrow{(i_1,i_1,i_2)} (E_{F,U}, \alpha) \xrightarrow{(r_1,r_1,r_2)} (E_{F,X}, \alpha) \longrightarrow 0$$

is J-exact.

Proof. Since $F^{-1}(Y) = X$, it is clear that

(1) $i_1(f \cdot \psi) = i_1(f) \cdot i_1(\psi)$ for all f in $C_0(U \setminus X)$, ψ in $E_{F,U\setminus X}$,

(2) $\langle i_1(\psi), i_1(\varphi)\rangle = i_2(\langle \psi, \varphi\rangle)$, for all ψ, φ in $E_{F,U\setminus X}$, and

(3) $i_1(\psi \cdot g) = i_1(\psi) \cdot i_2(g)$, for all ψ in $E_{F,U\setminus X}$ and g in $C_0(V \setminus Y)$.

Recall that for a branched function $G : A \to B$, $J_{(E_{G,A}, \alpha_A)} = C_0(A \setminus C_{G,A})$. Since $U \setminus X$ is open in U, we have that $C_{F,U\setminus X} = C_{F,U} \cap (U \setminus X)$ and therefore $i(C_0((U \setminus X) \setminus C_{F,U\setminus X})) \subseteq C_0(U \setminus C_{F,U})$. To finish proving (i_1, i_1, i_2) is a morphism, it suffices to show $\hat{i_1}(\alpha_{U\setminus X}(f)) = \alpha_U(i_1(f))$ for all f in $C_0((U \setminus X) \setminus C_{F,U\setminus X})$.

It suffices to prove this for a positive function f compactly supported in an open domain W of U such that $W \cap (X \cup C_{F,U}) = \emptyset$ and $F : W \to F(W)$ is injective. In this case, $\alpha_{U\setminus X}(f) = \theta_{\sqrt{f},\sqrt{f}}$ and $\alpha_U(i_1(f)) = \theta_{\sqrt{i_1(f)},\sqrt{i_1(f)}}$. By definition, $\hat{i_1}(\theta_{\sqrt{f},\sqrt{f}}) = \theta_{i_1(\sqrt{f}),i_1(\sqrt{f})}$. Hence, $\hat{i_1}(\alpha_{U\setminus X}(f)) = \alpha_U(i(f))$.

Similarly, it is clear from $F^{-1}(Y) = X$ that

(1) $r_1(f \cdot \psi) = r_1(f) \cdot r_1(\psi)$

(2) $\langle r_1(\psi), r_1(\varphi)\rangle = r_2(\langle \psi, \varphi\rangle)$, for f in $C_0(U)$, ψ, φ in $E_{F,U}$, and

(3) $r_1(\psi \cdot g) = r_1(\eta) \cdot r_2(g)$, for all ψ in $E_{F,U}$ and g in $C_0(V)$.

Therefore, $r_1 : C_2(U) \to C_2(X)$ is a r_2-twisted morphism of Hilbert modules. Every twisted morphism of Hilbert modules has closed image (it extends isometrically to a *-homomorphism of the linking algebra, which has closed image), and therefore $r_1(C_c(U)) = C_c(X)$ implies $r_1(C_2(U)) = C_2(X)$.

It is general that $C_{F,X} \subseteq C_{F,U} \cap X$, so we always have $r(C_0(U \setminus C_{F,U})) \subseteq C_0(X \setminus C_{F,X})$. By surjectivity of $r_1 : C_2(U) \to C_2(X)$ it is immediate that $\hat{r_1}(\alpha_U(f)) = \alpha_X(r_1(f))$

J-exactness of

$$0 \longrightarrow (E_{F,U\setminus X}, \alpha) \xrightarrow{(i_1,i_1,i_2)} (E_{F,U}, \alpha) \xrightarrow{(r_1,r_1,r_2)} (E_{F,X}, \alpha) \longrightarrow 0$$

is equivalent to exactness of the sequences

$$0 \longrightarrow C_0(U \setminus X) \xrightarrow{i_1} C_0(U) \xrightarrow{r_1} C_0(X) \longrightarrow 0$$

$$0 \longrightarrow C_0(U \setminus (X \cup C_{F,U\setminus X})) \xrightarrow{i_1} C_0(U \setminus C_{F,U}) \xrightarrow{r_1} C_0(X \setminus C_{F,X}) \longrightarrow 0$$

$$0 \longrightarrow C_2(U \setminus X) \xrightarrow{i_1} C_2(U) \xrightarrow{r_1} C_2(X) \longrightarrow 0$$

$$0 \longrightarrow C_0(V \setminus Y) \xrightarrow{i_2} C_0(V) \xrightarrow{r_2} C_0(Y) \longrightarrow 0.$$

Exactness of the top and bottom sequence is by hypothesis. Since $r_1(C_2(U)) = C_2(X)$ and $C_2(U) \cap C_0(U \setminus X) = C_2(U \setminus X)$, the second sequence from the bottom one is exact. Exactness of the sequence second from the top is equivalent to $C_{F,U} \cap X = C_{F,X}$. □

4.1 Correspondences from holomorphic functions

We now consider branched functions which are the restrictions of holomorphic functions and prove some extra regularity properties about them.

Let M and N be Riemann surfaces and $X \subseteq M$, $Y \subseteq N$ be closed subspaces. A function $F : X \to Y$ is a *holomorphic branched function* if there is a holomorphic function $\tilde{F} : M \to N$ such that $\tilde{F}|_X = F$ and $\tilde{F}^{-1}(Y) = X$. F is necessarily an open map, since $\tilde{F}$ is an open map and $\tilde{F}^{-1}(Y) = X$.

Moreover, because $\tilde{F}$ is holomorphic, for every u in M, there is a neighbourhood U of u, bi-holomorphisms $\phi : U \to \mathbb{D}_r$, $\psi : F(U) \to \mathbb{D}_{r^n}$, for some $r > 0$ and n in $\mathbb{N}$, such that $\psi \circ \tilde{F} \circ \phi^{-1}(z) = z^n$ for all z in $\mathbb{D}_r$. Since z^n is branched, it follows that $\tilde{F}$ is branched, with

$n = \mathrm{ind}_{\tilde{F}}(u)$. Hence, $\tilde{F}|_X = F$ is branched. Note that $\{u \in M : \mathrm{ind}_{\tilde{F}}(u) > 1\} = C_{\tilde{F},M}$ and this set is countable and discrete.

We will now show J-exactness of extensions of holomorphic branched functions is automatic, provided that its domain contains no isolated points. First, we need a lemma.

Lemma 4.1.1. *Let $F : X \to Y$ be a holomorphic branched function and assume Y contains no isolated points. For x in X, let $d_F(x) = d$ be the maximal number for which there exists a sequence $\{x_{n,j}\}_{n\in\mathbb{N},j\leq d}$ satisfying the properties*

(1) $\lim_{n\to\infty} x_{n,i} = x$ for any $i \leq d$,

(2) $F(x_{n,i}) = F(x_{n,j}) := x'_n \neq F(x)$ for any $i, j \leq d$, n in $\mathbb{N}$,

(3) $F^{-1}(x'_n) = \bigcup_{i=1}^{d}\{x_{n,i}\}$, for any n in $\mathbb{N}$, and

(4) $x_{n,i} \neq x_{n,j}$ for any two distinct $i, j \leq d$, for any n in $\mathbb{N}$.

Then, $\mathrm{ind}_F(x) = d_F(x)$.

Proof. For x in X, let $\{F_i^{-1} : V \to U\}_{i=1}^{\mathrm{ind}_F(x)}$ be inverse branches centered at x. Since Y contains no isolated points, there is a sequence x'_n in $V \setminus F(x)$ converging to $F(x)$. Since $\{x \in X : \mathrm{ind}_F(x) > 1\}$ is discrete, we may assume U is small enough so that $\mathrm{ind}_F(u) = 1$ for all u in $U \setminus \{x\}$. Hence, $\tilde{F}_i^{-1}(x'_n) := x_{n,i}$ must satisfy $x_{n,i} \neq x_{n,j}$ for all $i \neq j \leq \mathrm{ind}_F(x)$ and n in $\mathbb{N}$. By continuity of the inverse branches at $F(x)$, we have $\lim_{n\to\infty} x_{n,i} = x$ for all $i \leq \mathrm{ind}_F(x)$. Lastly, we have $\bigcup_{i=1}^{\mathrm{ind}_F(x)} x_{n,i} = F^{-1}(x'_n)$, so that the collection $\{x_{n,i}\}_{i\leq \mathrm{ind}_F(x)}$ satisfy (1)-(4) in the hypothesis of the lemma. Hence, $\mathrm{ind}_F(x) \leq d_F(x)$.

If $\{x_{n,i}\}_{i\leq d_F(x)}$ is a collection satisfying (1)-(4) above with $F(x_{n,i}) := x'_n$, then $\bigcup_{i=1}^{d_F(x)} x_{n,i} \subseteq U \setminus x$ eventually, so that $F^{-1}(x'_n) = \{F_i^{-1}(x'_n)\}_{i=1}^{\mathrm{ind}_F(x)} \subseteq \bigcup_{i=1}^{d_F(x)} x_{n,i} = F^{-1}(x'_n)$ eventually. Since $x_{n,i} \neq x_{n,j}$ for all $i \neq j \leq d_F(x)$, it follows that $d_F(x) \leq \mathrm{ind}_F(x)$. □

Corollary 4.1.2. *Let $F : M \to N$ be a holomorphic branched function, and suppose $Y \subseteq N$ is a closed set that contains no isolated points. Let $F^{-1}(Y) =: X$. Then, $C_{F,M} \cap X = C_{F,X}$, so that the sequence*

$$0 \longrightarrow (E_{F,M\setminus X}, \alpha) \xrightarrow{(i_1,i_1,i_2)} (E_{F,M}, \alpha) \xrightarrow{(r_1,r_1,r_2)} (E_{F,X}, \alpha) \longrightarrow 0$$

is J-exact.

Proof. If x is in $C_{F,M} \cap X$, then $\mathrm{ind}_F(x) > 1$. By Lemma 4.1.1, $d_{F|_X}(x) = \mathrm{ind}_{F|_X}(x) := \mathrm{ind}_F(x) > 1$. It is easy to see that $\{x \in X : d_{F|_X}(x) > 1\} = C_{F,X}$, so that x is in $C_{F,X}$. Hence, $C_{F,X} = C_{F,M} \cap X$. Proposition 4.0.9 then implies the above sequence of correspondences is exact. □

Corollary 4.1.3. *If $F : X \to X$ and $G : Y \to Y$ are holomorphic branched functions for which X and Y contain no isolated points and $\varphi : X \to Y$ is a homeomorphism such that $G \circ \varphi = \varphi \circ F$, then $ind_G \circ \varphi = ind_F$ and hence $(\varphi^*, \varphi^*, \varphi^*) : (E_{G,Y}, \alpha_Y) \to (E_{F,X}, \alpha_X)$ is an isomorphism.*

Proof. It is easy to see that $d_G \circ \varphi = d_F$, so Lemma 4.1.1 and Proposition 4.0.8 imply the Corollary. □

Corollary 4.1.4. *Let $h : M \to M$ be a complex dynamical system, where M is either $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{C}^*$. Then,*

$$0 \longrightarrow \mathcal{O}_{h,F_h} \xrightarrow{i} \mathcal{O}_{h,M} \xrightarrow{r} \mathcal{O}_{h,J_h} \longrightarrow 0$$

is exact, where i is induced from the inclusion $C_0(F_h) \to C_0(M)$ and r is induced from the restriction $C_0(M) \to C_0(J_h)$.

$\mathcal{O}_{h,J_h}$ is unital if and only if h extends to a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, in which case $\mathcal{O}_{h,J_h} = \mathcal{O}_{R,J_R}$. If $M = \hat{\mathbb{C}}$ or $\mathbb{C}^*$, then $\mathcal{O}_{h,J_h}$ is purely infinite and simple. If $M = \mathbb{C}$, then $\mathcal{O}_{h,J_h}$ is purely infinite and simple if and only if J_h does not contain the exceptional point (if it exists).

If J_h contains the exceptional point e and e is not a critical point, then point evaluation $C_0(J_h) \to \mathbb{C}$ at e induces a $*$-homomorphism $\mathcal{O}_{h,J_h} \to \mathbb{C}$ with kernel $\mathcal{O}_{h,J_h \setminus \{e\}}$. The ideal $\mathcal{O}_{h,J_h \setminus \{e\}}$ is purely infinite and simple.

If J_h contains the exceptional point e and e is a critical point, then point evaluation $C_0(J_h) \to \mathbb{C}$ at e induces a non-zero trace τ on $\mathcal{O}_{h,J_h}$.

Proof. The exact sequence follows from Corollary 4.1.2 and the fact that J_h contains no isolated points (Corollary 2.5.9). The characterization of the existence of a unit follows from Proposition 4.0.6. The purely infinite and simple properties in the case $M = \hat{\mathbb{C}}$ and $M = \mathbb{C}^*$ and $M = \mathbb{C}$ when the exceptional point e is not contained in J_h follow from Proposition 4.0.6, Corollary 2.5.10 and Proposition 2.5.6.

Suppose $M = \mathbb{C}$ and the exceptional point e is contained in J_h. If e is not a critical point, then the short exact sequence

$$0 \longrightarrow C_0(J_h \setminus \{e\}) \xrightarrow{i} C_0(J_h) \xrightarrow{\mathrm{ev}_e} \mathbb{C} \longrightarrow 0$$

induces a J-exact sequence of correspondences

$$0 \longrightarrow (E_{h,J_h \setminus \{e\}}, \alpha) \xrightarrow{(i,i,i)} (E_{h,J_h}, \alpha) \xrightarrow{(\mathrm{ev}_e,\mathrm{ev}_e,\mathrm{ev}_e)} (\mathbb{C}, \mathrm{id}_{\mathbb{C}}) \longrightarrow 0$$

And so the Corollary follows from Proposition 3.0.11 and the fact that $h : J_h \setminus \{e\} \to J_h \setminus \{e\}$ is conjugate to the dynamics of a holomorphic map of the punctured plane restricted to its Julia set.

Suppose $M = \mathbb{C}$ and the exceptional point e is a critical point contained in J_h. By [32, Theorem 2.5], the point evaluation (thought of as a trace) $\tau : C_0(J_h) \to \mathbb{C}$ extends to a trace on the Cuntz-Pimsner algebra $\mathcal{O}_{h,J_h}$ if and only if $\tau(f) = \mathrm{Tr}_\tau(f) := \sup_I \sum_{\xi \in I} \tau(\langle \xi, f \cdot \xi \rangle)$ for all f in $J_{(E_{h,J_h},\alpha)}$, where the supremum runs over all families I of vectors in E_{h,J_h} such that $\sum_{\xi \in I} \theta_{\xi,\xi} \leq 1$. Note that $J_{(E_{h,J_h},\alpha)} \subseteq C_0(J_h \setminus \{e\})$ and (due to $h^{-1}(e) \subseteq \{e\}$) $\langle \xi, f\eta \rangle$ is in $C_0(J_h \setminus \{e\})$, for all ξ, η in E_{h,J_h}, whenever f is in $J_{(E_{h,J_h},\alpha)}$. Therefore, $\tau(f) = 0 = \mathrm{Tr}_\tau(f)$ for all f in $J_{(E_{h,J_h},\alpha)}$ and τ extends to a trace on $\mathcal{O}_{h,J_h}$. Therefore, $\mathcal{O}_{h,J_h}$ is not purely infinite. □

4.1.1 K-theoretic properties of holomorphic branched functions

In this section we study the induced mapping on K-theory associated to the C^*-correspondence of a holomorphic branched function. The results here will be crucial later in Chapter 7.

Our main example of a holomorphic branched function will be a complex dynamical system $R : M \to M$ and its restriction to its Fatou set F_R and its Julia set J_R. Recall from Section 2.5 that when M is either $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{C}^*$ and $\deg(R) > 1$, J_R is non-empty, totally invariant ($R^{-1}(J_R) = J_R$), contains no isolated points and is a closed subset of M. Corollary 4.1.2 implies we have a J-exact sequence

$$0 \longrightarrow (E_{R,F_R}, \alpha_{F_R}) \xrightarrow{(i,i,i)} (E_{R,M}, \alpha_M) \xrightarrow{(r,r,r)} (E_{R,J_R}, \alpha_{J_R}) \longrightarrow 0.$$

We now provide a concrete description of $\iota - \hat{\otimes}[\mathcal{E}_{R,J_R}]$ acting on K^0.

For a locally compact Hausdorff space W, let $\mathrm{Tr} : K^0(W) \to C_0(W, \mathbb{Z})$ be the homomorphism (of additive groups) defined for g in $K^0(W)$ and w in W as $\mathrm{Tr}(g)(w) = (ev_w)_*(g)$, where $ev_w : C_0(W) \to \mathbb{C}$ is evaluation at w. We can identify $K_0(\mathbb{C})$ with $\mathbb{Z}$ via the trace map.

We first show $\hat{\otimes}[\mathcal{E}_{R,J_R}]$ acting on K^0 can be identified with the transfer operator Φ using the trace map Tr.

Proposition 4.1.5. *If $F : X \to Y$ is a holomorphic branched function and X, Y are closed and proper subsets of either $\mathbb{C}^*$, $\mathbb{C}$ or $\hat{\mathbb{C}}$, then $Tr : K^0(X \setminus C_{F,X}) \to C_0(X \setminus C_{F,X}, \mathbb{Z})$, $Tr : K^0(Y) \to C(Y, \mathbb{Z})$ are isomorphisms, and $Tr \circ (\hat{\otimes}_0[\mathcal{E}_{F,X}]) = (\Phi) \circ Tr$.*

Proof. When X is a compact, connected proper subspace of the Riemann sphere, $\hat{\mathbb{C}} \setminus X$ is a non-empty disjoint union of simply connected open sets, and so $K^{-1}(\hat{\mathbb{C}} \setminus X) = 0$ and

$i_* : K^0(\hat{\mathbb{C}} \setminus X) \to K^0(\hat{\mathbb{C}})$ maps onto $\mathbb{Z} \cdot \beta_{\hat{\mathbb{C}}}$ (by Corollary 4.1.12). Therefore, the 6-term exact sequence of K-theory associated to

$$0 \longrightarrow C_0(\hat{\mathbb{C}} \setminus X) \longrightarrow C(\hat{\mathbb{C}}) \longrightarrow C(X) \longrightarrow 0$$

implies $K^0(X) = \mathbb{Z}[1_X] \simeq \mathbb{Z}$, and in particular $\mathrm{Tr} : K^0(X) \to C(X, \mathbb{Z})$ is an isomorphism.

Now, when X is a finite disjoint union of compact connected sets in $\hat{\mathbb{C}}$, the above result implies $\mathrm{Tr} : K^0(X) \to C(X, \mathbb{Z})$ is an isomorphism.

In general, if X is a compact, proper set of $\hat{\mathbb{C}}$, then we can write $X = \bigcap_{n \in \mathbb{N}} X_n$, where, for every n in $\mathbb{N}$, X_n is a finite disjoint union of compact connected sets such that $X_{n+1} \subseteq X_n$. The diagram

$$\begin{array}{ccc} K^0(X_n) & \xrightarrow{r_*} & K^0(X_{n+1}) \\ \mathrm{Tr}\downarrow & & \downarrow\mathrm{Tr} \\ C(X_n, \mathbb{Z}) & \xrightarrow{r} & C(X_{n+1}, \mathbb{Z}) \end{array}$$

commutes, for all n in $\mathbb{N}$, where r is the restriction map, and the vertical maps are isomorphisms. Therefore, the limit map $\mathrm{Tr} : K^0(X) \to C(X, \mathbb{Z})$ is an isomorphism.

Now, suppose C is a finite set contained in X. $K^{-1}(C) = 0$, so the following diagram has exact rows and commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K^0(X \setminus C) & \longrightarrow & K^0(X) & \longrightarrow & K^0(C) \\ & & \downarrow\mathrm{Tr} & & \downarrow\mathrm{Tr} & & \downarrow\mathrm{Tr} \\ 0 & \longrightarrow & C_0(X \setminus C, \mathbb{Z}) & \longrightarrow & C(X, \mathbb{Z}) & \longrightarrow & C(C, \mathbb{Z}). \end{array}$$

Since the two right-most vertical maps are isomorphisms, a diagram chase implies $\mathrm{Tr} : K^0(X \setminus C) \to C_0(X \setminus C, \mathbb{Z})$ is an isomorphism.

Now if X and Y are closed and proper subsets of either $\mathbb{C}^*$, $\mathbb{C}$ or $\hat{\mathbb{C}}$, then $X = \overline{X} \setminus F_X$ and $Y = \overline{Y} \setminus F_Y$, where the closure is in $\hat{\mathbb{C}}$ and F_X, F_Y are finite sets. By properness of X and Y, we have $\overline{X} \neq \hat{\mathbb{C}}$ and $\overline{Y} \neq \hat{\mathbb{C}}$, so the above result applies to see that $\mathrm{Tr} : K^0(X) \to C_0(X, \mathbb{Z})$ and $\mathrm{Tr} : K^0(Y) \to C_0(Y, \mathbb{Z})$ are isomorphisms.

Since $C_{F,X}$ is closed and discrete, we have $K^{-1}(C_{F,X}) = 0$ and $\mathrm{Tr} : K^0(C_{F,X}) \to C_0(C_{F,X}, \mathbb{Z})$ is an isomorphism. By a similar diagram chase to that above, we have that $\mathrm{Tr} : K^0(X \setminus C_{F,X}) \to C_0(X \setminus C_{F,X}, \mathbb{Z})$ is an isomorphism.

Note that every element f in $C_0(X, \mathbb{Z})$ is compactly supported, so that $\Phi(f)$ is well defined. For y in Y, $(\mathrm{ev}_y)_* \mathcal{E}_{F,X}$ is represented by the Hilbert $\mathbb{C}$-module with orthogonal basis $\{\delta_x\}_{x \in F^{-1}(y)}$ and inner product satisfying $\langle \delta_x, \delta_x \rangle = \mathrm{ind}_F(x)$, for all x in $F^{-1}(y)$. The left action is defined for f in $C_0(X \setminus C_{F,X})$ and x in $F^{-1}(y)$ as $f \cdot \delta_x = f(x)\delta_x$. Therefore,

$(\mathrm{ev}_y)_* \mathcal{E}_{F,X} = \sum_{x \in F^{-1}(y)} (\mathrm{ev}_x)_*$, where $(ev_x)_*$ is thought of as a class in $KK^0(C_0(X \setminus C_{F,X}), \mathbb{C})$ and the sum is the direct sum operation of Kasparov bi-modules. When $\mathrm{ind}_F(x) > 1$, x is in $C_{F,X}$ and consequently $(ev_x)_* = 0$ in $KK^0(C_0(X \setminus C_{F,X}), \mathbb{C})$. So, we can write $(ev_y)_* \mathcal{E}_{F,X}$ as $\sum_{x \in F^{-1}(y)} \mathrm{ind}_F(x)(ev_x)_*$ and the diagram

$$\begin{array}{ccc} K^0(X \setminus C_{F,X}) & \xrightarrow{\hat{\otimes}_0 [\mathcal{E}_{F,X}]} & K^0(Y) \\ \mathrm{Tr}\downarrow & & \downarrow\mathrm{Tr} \\ C_0(X \setminus C_{F,X}, \mathbb{Z}) & \xrightarrow{\Phi} & C_0(Y, \mathbb{Z}). \end{array}$$

commutes. □

For a holomorphic branched function $F : X \to Y$ with X, Y closed and proper subsets of either $\mathbb{C}^*$, $\mathbb{C}$ or $\hat{\mathbb{C}}$, we will identify $K^0(X \setminus C_{F,X})$, $K^0(X)$ and $K^0(Y)$ with $C_0(X \setminus C_{F,X}, \mathbb{Z})$, $C_0(X, \mathbb{Z})$ and $C_0(Y, \mathbb{Z})$, respectively.

If $R : M \to M$ is a holomorphic function such that $J_R \neq M$ and M is either $\mathbb{C}^*$, $\mathbb{C}$ or $\hat{\mathbb{C}}$, by Corollary 3.0.9 and Proposition 4.1.5, we have a commutative diagram

$$\begin{array}{ccc} C_0(J_R \setminus C_{R,J_R}, \mathbb{Z}) & \xrightarrow{\exp} & K^{-1}(F_R \setminus C_{R,F_R}) \\ \Phi\downarrow & & \downarrow\hat{\otimes}_1[\mathcal{E}_{R,F_R}] \\ C_0(J_R, \mathbb{Z}) & \xrightarrow{\exp} & K^{-1}(F_R). \end{array}$$

The left-most vertical map extends to the group homomorphism $\Phi : C_0(J_R, \mathbb{Z}) \to C_0(J_R, \mathbb{Z})$ and the top horizontal map extends to the exponential map $\exp : C_0(J_R, \mathbb{Z}) \to K^{-1}(F_R \setminus C_{R,F_R})$ from the short exact sequence

$$0 \longrightarrow C_0(F_R \setminus C_{R,F_R})) \xrightarrow{i} C_0(M \setminus C_{R,F_R}) \xrightarrow{r} C_0(J_R) \longrightarrow 0.$$

We will show the extension of the diagram above commutes. This result will be used to compute the kernel and co-kernel of $\mathrm{id} - \Phi : C(J_R, \mathbb{Z}) \to C(J_R, \mathbb{Z})$ when R is rational (Proposition 7.0.7). First, we we make a definition and prove two lemmas.

If U is a simply connected open proper subset of $\hat{\mathbb{C}}$ and z is in U, the short exact sequence

$$0 \longrightarrow C_0(U \setminus \{z\})) \longrightarrow C_0(U) \xrightarrow{\mathrm{ev}_z} \mathbb{C} \longrightarrow 0$$

yields an isomorphism $\exp : \mathbb{Z} = K^0(\{z\}) \to K^{-1}(U \setminus z)$. If 1_z denotes the characteristic function on the point $\{z\}$, we will let $v_z =: \exp(1_z)$

Let $F_d : \mathbb{D} \to \mathbb{D}$ be the mapping defined, for z in $\mathbb{D}$, as $F_d(z) = z^d$.

Lemma 4.1.6. *$\hat{\otimes}_1[\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}] : K^{-1}(\mathbb{D}\setminus\{0\}) \to K^{-1}(\mathbb{D}\setminus\{0\})$ maps v_0 to $d\cdot v_0$.*

Proof. If $d = 1$, then $F_d = \mathrm{id}_{\mathbb{D}}$ and the lemma follows.

Assume $d > 1$. Denote $(F_d)^* : C_0(\mathbb{D}) \to C_0(\mathbb{D})$ by φ_d. By naturality of exp, the diagram

$$\begin{array}{ccc} K^0(\{0\}) & \xrightarrow{(\varphi_d)_*} & K^0(\{0\}) \\ \big\downarrow \mathrm{exp} & & \big\downarrow \mathrm{exp} \\ K^{-1}(\mathbb{D}\setminus\{0\}) & \xrightarrow{(\varphi_d)_*} & K^{-1}(\mathbb{D}\setminus\{0\}) \end{array}$$

commutes. The top horizontal map is equal to the identity. Hence, $(\varphi_d)_*(v_0) = v_0$. Therefore, to prove the lemma, it suffices to show $\varphi_d^*[\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}]$ is equal to $d\cdot \mathrm{id}_{C_0(\mathbb{D}\setminus\{0\})}$ in $KK^0(C_0(\mathbb{D}\setminus\{0\}), C_0(\mathbb{D}\setminus\{0\}))$.

Let ω be a d^{th} root of unity, and consider for $0 \leq j \leq d-1$ the linear map $P_j : C_0(\mathbb{D}\setminus\{0\}) \to C_0(\mathbb{D}\setminus\{0\})$ defined for ψ in $C_0(\mathbb{D}\setminus\{0\})$ as $P_j(\psi)(z) = \frac{1}{d}\sum_{k=1}^{d-1}\omega^{-jk}\psi(\omega^k z)$, z in $\mathbb{D}\setminus\{0\}$. If φ is a function in the image of P_j, then it satisfies $\varphi(\omega z) = \omega^j\varphi(z)$, for all z in $\mathbb{D}\setminus\{0\}$, so $P_j(\varphi) = \varphi$, and hence $P_j^2 = P_j$.

We also have, for any z in $\mathbb{D}\setminus\{0\}$, that $\sum_{j=0}^{d-1} P_j(\psi)(z) = \sum_{k=0}^{d-1}(\frac{1}{d}\sum_{j=0}^{d-1}\omega^{-jk})\psi(\omega^k z)$. $\frac{1}{d}\sum_{j=0}^{d-1}\omega^{-jk} = 1$ if $j = 0$, and is zero otherwise. Hence, $\sum_{j=0}^{d-1} P_j = \mathrm{id}$.

We show P_j is a Hilbert $C_0(\mathbb{D}\setminus\{0\})$-module endomorphism of $E_{F_d,\mathbb{D}\setminus\{0\}}$ commuting with the left action $\alpha_{\mathbb{D}\setminus\{0\}}\circ\varphi_d$. Since $\omega^d = 1$, we have, for any a in $C_0(\mathbb{D}\setminus\{0\})$, ψ in $E_{F_d,\mathbb{D}\setminus\{0\}}$ and z in $\mathbb{D}\setminus\{0\}$, that

$$P_j(\psi\cdot a)(z) = \frac{1}{d}\sum_{k=1}^{d-1}\omega^{-jk}\psi(\omega^k z)a((\omega^k z)^d) = (\frac{1}{d}\sum_{k=1}^{d-1}\omega^{-jk}\psi(\omega^k z))a(z^d) = (P_j(\psi)\cdot a)(z).$$

We also have $P_j(a\cdot\psi) = a\cdot P_j(\psi)$, since the left and right actions are equal for $\varphi_d^*\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}$.

Since $\sum_{j=0}^{d-1} P_j = \mathrm{id}$, to show that $P_j^* = P_j$ (hence P_j is adjointable), we only need to show that for all ψ_1, ψ_2 in $E_{F_d,\mathbb{D}\setminus\{0\}}$ and $i \neq j$, we have $\langle P_i(\psi_1), P_j(\psi_2)\rangle = 0$. Write $\varphi_1 = P_i(\psi_1)$ and $\varphi_2 = P_j(\psi_2)$. For z in $\mathbb{D}\setminus\{0\}$, we have $\langle\varphi_1,\varphi_2\rangle(F_d) = \sum_{k=0}^{d-1}\overline{\varphi_1}(\omega^k z)\varphi_2(\omega^k z)$. Since $\varphi_1(\omega^k z) = \omega^{ik}\varphi_1(z)$ and $\varphi_2(\omega^k z) = \omega^{jk}\varphi_1(z)$, we have that $\sum_{k=0}^{d-1}\overline{\varphi_1}(\omega^k z)\varphi_2(\omega^k z) = (\sum_{k=0}^{d-1}\omega^{(j-i)k})\overline{\varphi_1}(z)\varphi_2(z) = 0$.

We have shown that $\{P_i\}_{i=0}^{d-1}$ is a collection of mutually orthogonal projections of the Kasparov $C_0(\mathbb{D}\setminus\{0\}) - C_0(\mathbb{D}\setminus\{0\})$ bi-module $\varphi_d^*\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}$ that sum to id. Hence, $[\varphi_d^*\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}] = \sum_{i=0}^{d-1}[P_i\varphi_d^*\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}]$.

Consider the unitary U of $\varphi_d^*\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}$ defined for ψ in $E_{F_d,\mathbb{D}\setminus\{0\}}$ as $U(\psi)(z) = \frac{z}{|z|}\psi(z)$, z in $\mathbb{D}\setminus\{0\}$. Then, one checks that $UP_iU^* = P_{i+1}$ for $0 \leq i \leq d-1$ ($P_d = P_0$). Therefore, $[\varphi_d^*\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}] = d[P_0\varphi_d^*\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}]$. We show $[P_0\varphi_d^*\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}] = \mathrm{id}$.

Note that $[\mathcal{E}_{F_0,\mathbb{D}\setminus\{0\}}] = [\mathrm{id}_{C_0(\mathbb{D}\setminus\{0\})}]$, so it suffices to show $[P_0\varphi_d^*\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}] = [\mathcal{E}_{F_0,\mathbb{D}\setminus\{0\}}]$. Consider the map $S = \frac{1}{\sqrt{d}}(F_d)^* : E_{z,\mathbb{D}\setminus\{0\}} \to E_{F_d,\mathbb{D}\setminus\{0\}}$. It is obvious S is $C_0(\mathbb{D}\setminus\{0\})$ linear with respect to the left and right actions on each bi-module. For ψ_1, ψ_2 in $E_{z,\mathbb{D}\setminus\{0\}}$, we have $\langle S(\psi_1), S(\psi_2)\rangle(z) = \frac{1}{d}\sum_{w:w^d=z}\overline{\psi_1}(w^d)\psi_2(w^d) = \overline{\psi_1}(z)\psi_2(z) = \langle\psi_1,\psi_2\rangle(z)$. So, S preserves the inner products.

Every function φ in the image of S satisfies $\varphi(\omega z) = \varphi(z)$ for all z in $\mathbb{D}\setminus\{0\}$, so $\mathrm{im}(S) \subseteq \mathrm{im}(P_0)$. For φ in $E_{F_d,\mathbb{D}\setminus\{0\}}$, let $\Phi(\varphi)(z) = \frac{1}{\sqrt{d}}\sum_{w:w^d=z}\varphi(w)$. Then, $S\Phi = P_0$, so if φ is in $\mathrm{im}(P_0)$, we have $S\Phi(\varphi) = \varphi$. Therefore, $\mathrm{im}(S) = \mathrm{im}(P_0)$ and $S : E_{F_0,\mathbb{D}\setminus\{0\}} \to P_0E_{F_d,\mathbb{D}\setminus\{0\}}$ is an isomorphism of Kasparov bi-modules. Hence, $[P_0\varphi_d^*\mathcal{E}_{F_d,\mathbb{D}\setminus\{0\}}] = [\mathcal{E}_{F_0,\mathbb{D}\setminus\{0\}}]$. □

Lemma 4.1.7. *Let M be either $\mathbb{C}^*$, $\mathbb{C}$ or $\hat{\mathbb{C}}$ and X a proper closed subset of M. Then, for every non-zero v in $K^{-1}(M\setminus X)$, there is x in X such that $(i_x)_*(v) \neq 0$, where $i_x : C_0(M\setminus X) \to C_0(M\setminus x)$ is the inclusion.*

Proof. We may assume $|X| \geq 2$, otherwise the lemma is trivial. If $M = \hat{\mathbb{C}}$, by applying a rotation, we may assume without loss of generality that ∞ is in X. Then, $M\setminus X = \mathbb{C}\setminus(X\setminus\{\infty\})$ and $X\setminus\{\infty\}$ is closed in $\mathbb{C}$. This reduces the case of $M = \hat{\mathbb{C}}$ to the case where $M = \mathbb{C}$.

If $M = \mathbb{C}^*$, then $X\cup\{0\} =: X_0$ is closed in $\mathbb{C}$ and for x in X, $(i_x)_* = (j_x)_*$, where $j_x : C_0(\mathbb{C}\setminus X_0) \to C_0(\mathbb{C}\setminus\{x,0\})$ is the inclusion. Let $k_x : C_0(\mathbb{C}\setminus X_0) \to C_0(\mathbb{C}\setminus\{x\})$ and $l_x : C_0(\mathbb{C}\setminus\{x,0\}) \to C_0(\mathbb{C}\setminus\{x\})$ denote the inclusions. For x in X, we have $l_x\circ j_x = k_x$ and $k_0 = l_0\circ j_x$. By functorality of K-theory it follows that if $(k_x)_*(v)\neq 0$ for some v in $K^{-1}(\mathbb{C}^*\setminus X))$, then we must have $(j_x)_*(v)\neq 0$. This reduces the case of $M = \mathbb{C}^*$ to the case $M = \mathbb{C}$.

So, assume X is a closed and proper subset of $\mathbb{C}$. For x in X, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(\mathbb{C}\setminus X) & \longrightarrow & C_0(\mathbb{C}) & \xrightarrow{r_X} & C_0(X) & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle i_x} & & \Big\| & & \Big\downarrow{\scriptstyle \mathrm{ev}_x} & & \\
0 & \longrightarrow & C_0(\mathbb{C}\setminus\{x\}) & \longrightarrow & C_0(\mathbb{C}) & \longrightarrow & \mathbb{C} & \longrightarrow & 0
\end{array}$$

commutes and has exact rows. Therefore, by naturality of exp, the diagram

$$\begin{array}{ccc}
K^{-1}(\mathbb{C}\setminus X) & \xleftarrow{\exp} & C_0(X,\mathbb{Z})\\
\Big\downarrow{\scriptstyle (i_x)_*} & & \Big\downarrow{\scriptstyle \mathrm{ev}_x}\\
K^{-1}(\mathbb{C}\setminus x) & \xleftarrow{\exp} & K^{-1}(x) = \mathbb{Z}.
\end{array}$$

Since $K^{-1}(\mathbb{C}) = 0$ the horizontal maps are surjections. $K^0(\mathbb{C}) \to K^0(x)$ and $(r_X)_* : K^0(\mathbb{C}) \to C_0(X,\mathbb{Z})$ both are the zero maps and hence the horizontal maps above are also injections.

For every $f \neq 0$ in $C_0(X, \mathbb{Z})$ there is x in X such that $ev_x(f) \neq 0$. Using this and that the horizontal maps above are isomorphisms, the lemma follows. □

Proposition 4.1.8. *Let $R : M \to M$ be a non-invertible complex dynamical system, where M is either $\mathbb{C}^*$, $\mathbb{C}$ or $\hat{\mathbb{C}}$. If $J_R \neq M$ then the diagram*

$$\begin{array}{ccc} C_0(J_R, \mathbb{Z}) & \xrightarrow{exp} & K^{-1}(F_R \setminus C_{R,F_R}) \\ \Phi \downarrow & & \downarrow \hat{\otimes}_1[\mathcal{E}_{R,F_R}] \\ C_0(J_R, \mathbb{Z}) & \xrightarrow{exp} & K^{-1}(F_R) \end{array}$$

commutes.

Proof. First, we show for every x in J_R, the diagram

$$\begin{array}{ccc} C_0(R^{-1}(x), \mathbb{Z}) & \xrightarrow{\Phi_x} & \mathbb{Z} = K^0(x) \\ \downarrow \exp & & \downarrow \exp \\ K^{-1}(M \setminus (R^{-1}(x) \cup C_{R,M})) & \xrightarrow{\hat{\otimes}_1[\mathcal{E}_{R,M\setminus R^{-1}(x)}]} & K^{-1}(M \setminus \{x\}) \end{array}$$

commutes, where $\Phi_x : C(R^{-1}(F), \mathbb{Z}) \to \mathbb{Z}$ is the group homomorphism sending 1_z, z in $R^{-1}(x)$, to $\mathrm{ind}_R(z) 1_x$.

$R^{-1}(x)$ and $C_{R,M}$ are closed and discrete, so for every z in $R^{-1}(x)$, there is a simply connected neighbourhood U_z of z such that $U_z \cap (R^{-1}(x) \cup C_{R,M}) = \{x\}$, $V_z := R(U_z)$ is simply connected, with bi-holomorphisms $\varphi : U_z \to \mathbb{D}$, $\psi : V_z \to \mathbb{D}$ such that $\varphi(z) = 0$, $\psi(x) = 0$ and, for all w in $\mathbb{D}$, $\psi \circ R \circ \varphi^{-1}(w) = w^{\mathrm{ind}_R(z)}$.

Lemma 4.1.6 implies $v_0 \hat{\otimes}_1[\mathcal{E}_{\psi\circ R\circ\varphi^{-1},\mathbb{D}\setminus\{0\}}] = \mathrm{ind}_R(z) \cdot v_0$.

Naturality of exp implies $((\varphi^{-1})^*)_*(v_z) = v_0$ and $(\psi^*)_*(v_0) = v_x$, so $v_z \hat{\otimes}_1[\mathcal{E}_{R,U_z\setminus\{z\}}] = ((\psi^{-1})^*)_*(v_0 \hat{\otimes}_1[\mathcal{E}_{\psi\circ R\circ\varphi^{-1},\mathbb{D}\setminus\{0\}}]) = \mathrm{ind}_R(z) \cdot v_x$.

Naturality of exp and Proposition 4.0.7 then imply $\exp(1_z)\hat{\otimes}_1[\mathcal{E}_{R,M\setminus R^{-1}(x)}] = \mathrm{ind}_R(z) v_x = \exp(\Phi_x(1_x))$. As z in $R^{-1}(x)$ was arbitrary, the above identity implies the diagram

$$\begin{array}{ccc} C_0(R^{-1}(x), \mathbb{Z}) & \xrightarrow{\Phi_x} & \mathbb{Z} = K^0(x) \\ \downarrow \exp & & \downarrow \exp \\ K^{-1}(M \setminus (R^{-1}(x) \cup C_{R,M})) & \xrightarrow{\hat{\otimes}_1[\mathcal{E}_{R,U\setminus R^{-1}(F)}]} & K^{-1}(M \setminus \{x\}) \end{array}$$

commutes, as claimed.

Let x be in J_R, and let $i_x : K^{-1}(M \setminus (J_R \cup C_{R,F_R})) \to K^{-1}(M \setminus (R^{-1}(x) \cup C_{R,F_R}))$ and $j_x : K^{-1}(M \setminus J_R) \to K^{-1}(M \setminus \{x\})$ be the maps induced by the respective inclusions. Let $r_x : C(J_R, \mathbb{Z}) \to C(R^{-1}(x), \mathbb{Z})$ and $ev_x : C(J_R, \mathbb{Z}) \to \mathbb{Z}$ be the restriction/evaluation maps. By the above commutative diagram, for every g in $C(J_R, \mathbb{Z})$, we have

$0 = \exp(r_x(g))\hat{\otimes}_1[\mathcal{E}_{R,M\setminus R^{-1}(x)}] - \exp(\Phi_x(r_x(g)))$. Note that $\Phi_x \circ r_x = ev_x \circ \Phi$, and naturality of exp implies $\exp \circ r_x = i_x \circ \exp$ and $\exp \circ ev_x = j_x \circ \exp$.

Therefore,

$$\exp(r_x(g))\hat{\otimes}_1[\mathcal{E}_{R,M\setminus R^{-1}(x)}] - \exp(\Phi_x(r_x(g))) = i_x(\exp(g))\hat{\otimes}_1[\mathcal{E}_{R,M\setminus R^{-1}(x)}] - j_x(\exp(\Phi(g))).$$

Proposition 4.0.7 implies $i_x(\exp(g))\hat{\otimes}_1[\mathcal{E}_{R,M\setminus R^{-1}(x)}] = j_x(\exp(g)\hat{\otimes}_1[\mathcal{E}_{R,F_R}])$. So, putting these equalities together, we have $0 = j_x(\exp(g)\hat{\otimes}_1[\mathcal{E}_{R,M\setminus J_R}] - \exp(\Phi(g)))$. As x in J_R was arbitrary, Lemma 4.1.7 implies $\exp(g)\hat{\otimes}_1[\mathcal{E}_{R,F_R}] = \exp(\Phi(g))$. □

4.1.2 Appendix: Bott projections

To compute certain maps in the exact sequences we will work with in this thesis, it will be useful to "orient" the K^0 (and K^{-1}) groups for special subsets of $\hat{\mathbb{C}}$ by finding minimal generating sets that behave well with maps like inclusion and bi-holomorphisms. In this section, given a connected open set U of $\hat{\mathbb{C}}$, we describe such a canonical generator β_U for $K^0(U)$, the *Bott projection of* U. When $U = \mathbb{C}$ it is the usual Bott projection. We could not find these results in the literature, but we do not claim any originality as they are likely folklore. The reader may wish to skip this section if they believe Corollary 4.1.12.

Our method is to construct the Bott projection for the open unit disk $\mathbb{D}$, prove some properties about it, then bootstrap up to the general construction (Corollary 4.1.12).

First, let U be a simply connected open set of the complex plane $\mathbb{C}$, and let γ be a Jordan curve inside it. Denote by U_γ^- the connected component of $U \setminus \mathrm{im}(\gamma)$ whose closure in $\mathbb{C}$ intersects the boundary ∂U, and denote the other component by U_γ^+.

For a continuous function $a : \mathrm{im}(\gamma) \to U_\gamma^+$, let $u_{a,\gamma} : \mathrm{im}(\gamma) \to S^1$ be defined, for z in $\mathrm{im}(\gamma)$, as $u_{a,\gamma}(z) = \frac{z-a(z)}{|z-a(z)|}$. Any two functions $a, b : \mathrm{im}(\gamma) \to U_\gamma^+$ are homotopic, since U_γ^+ is homeomorphic to $\mathbb{D}$. It follows then, by compactness of $\mathrm{im}(\gamma)$, that $u_{a,\gamma}$ and $u_{b,\gamma}$ are homotopic as elements in $C(\mathrm{im}(\gamma), S^1)$. Hence, the class $u_\gamma := [u_{a,\gamma}]$ in $K^{-1}(\mathrm{im}(\gamma))$ is independent of a.

Let $\delta_\gamma : K^{-1}(\mathrm{im}(\gamma)) \to K^0(U_\gamma^+)$ be the index map from the 6-term exact sequence associated to the short exact sequence

$$0 \longrightarrow C_0(U_\gamma^+) \longrightarrow C_0(\overline{U_\gamma^+}) \longrightarrow C(\mathrm{im}(\gamma)) \longrightarrow 0.$$

Let $\iota_\gamma : K^0(U_\gamma^+) \to K^0(U)$ be the map induced from the inclusion $i : C_0(U_\gamma^+) \to C_0(U)$.

Proposition 4.1.9. *For every simply connected open set U properly contained in $\hat{\mathbb{C}}$, there is a unique generator β_U in $K^0(U) \simeq \mathbb{Z}$ such that*

(1) if $U \subseteq \mathbb{C}$, then $\beta_U = \iota_\gamma \delta_\gamma(u_\gamma)$ for any Jordan curve γ in U,

(2) if $T : U \to V$ is a bi-holomorphism, where V is an open subset of $\hat{\mathbb{C}}$, then $((T^{-1})^*)_* \beta_U = \beta_V$, and

(3) if $U \subseteq V$, where V is a simply connected open set properly contained in $\hat{\mathbb{C}}$, and $i : C_0(U) \to C_0(V)$ is the inclusion, then $i_* \beta_U = \beta_V$.

Proof. We first prove (1) and (2) of the Proposition for $U = \mathbb{D} = V$, and then bootstrap to the general case.

First, we show $\iota_\gamma \delta_\gamma(u_\gamma) = \iota_\eta \delta_\eta(u_\eta)$ for any two Jordan curves in $\mathbb{D}$. To prove this, it suffices to show $\iota_\gamma \delta_\gamma(u_\gamma) = \iota_{\varepsilon_r} \delta_{\varepsilon_r}(u_{\varepsilon_r})$, where $\varepsilon_r(t) = re^{2\pi i t}$, for t in $[0, 1]$, for any $r < 1$ such that $|\gamma| < r$.

Let $\mathbb{A}_{r,\gamma} = \mathbb{D}_r \setminus \overline{\mathbb{D}_\gamma^+}$. Let $\delta_{\mathbb{A}} : K^{-1}(\mathrm{im}(\gamma)) \oplus K^{-1}(\mathrm{im}(\varepsilon_r)) \to K^0(\mathbb{A}_{r,\gamma})$ be the index map from the 6-term exact sequence associated to

$$0 \longrightarrow C_0(\mathbb{A}_{r,\gamma}) \xrightarrow{\ i\ } C_0(\overline{\mathbb{A}_{r,\gamma}}) \xrightarrow{\ r\ } C(\mathrm{im}(\gamma) \sqcup \mathrm{im}(\varepsilon_r)) \longrightarrow 0.$$

Choose a point z_0 in $\mathbb{D}_\gamma^+$, and define $u : \overline{\mathbb{A}_{r,\gamma}} \to S^1$, for z in $\overline{\mathbb{A}_{r,\gamma}}$, as $u(z) = \frac{z - z_0}{|z - z_0|}$. Since $\mathbb{D}_\gamma^+ \subseteq \mathbb{D}_{\varepsilon_r}^+ = \mathbb{D}_r$, it follows that $r_*[u] = [u_{z_0,\gamma}] \oplus [u_{z_0,\varepsilon_r}]$. Hence, by exactness, $\iota_{\mathbb{A}} \delta_{\mathbb{A}}(u_\gamma) = -\iota_{\mathbb{A}} \delta_{\mathbb{A}}(u_{\varepsilon_r})$, where $\iota_{\mathbb{A}} : K^0(\mathbb{A}_{r,\gamma}) \to K^0(\mathbb{D})$ is the map induced from inclusion.

Let $\delta : K^{-1}(\mathrm{im}(\gamma)) \to K^0(\mathbb{D}_\gamma^+) \oplus K^0(\mathbb{A}_{r,\gamma})$ be the index map from the 6-term exact sequence associated to

$$0 \longrightarrow C_0(\mathbb{D}_\gamma^+ \sqcup \mathbb{A}_{r,\gamma}) \longrightarrow C_0(\mathbb{D}_r) \longrightarrow C(\mathrm{im}(\gamma)) \longrightarrow 0.$$

By naturality of the index map, we have that $\delta(u_\gamma) = \delta_\gamma(u_\gamma) \oplus \delta_{\mathbb{A}}(u_\gamma)$, and so exactness implies $\iota_\gamma \delta_\gamma(u_\gamma) + \iota_{\mathbb{A}} \delta_{\mathbb{A}}(u_\gamma) = 0$. Hence, $\iota_\gamma \delta_\gamma(u_\gamma) = \iota_{\mathbb{A}} \delta_{\mathbb{A}}(u_{\varepsilon_r})$. By naturality of the index map, $\iota_{\mathbb{A}} \delta_{\mathbb{A}}(u_{\varepsilon_r}) = \iota_{\varepsilon_r} \delta_{\varepsilon_r}(u_{\varepsilon_r})$, proving the claim.

Now, we show $\beta_{\mathbb{D}} := \iota_{\varepsilon_r} \delta_{\varepsilon_r}(u_{\varepsilon_r})$ generates $K^0(\mathbb{D})$. Since $C_0(\mathbb{D} \setminus \mathbb{D}_r)$ is contractible to 0, the 6-term exact sequence of K-theory implies the map $\iota_{\varepsilon_r} : K^0(\mathbb{D}_r) \to K^0(\mathbb{D})$ is an isomorphism. Similarly, $C_0(\overline{\mathbb{D}_r} \setminus \{0\})$ being contractible to 0 implies $K^0(\overline{\mathbb{D}}_r) = \mathbb{Z}[1_{\overline{\mathbb{D}_r}}]$. Hence, $i_* : K^0(\mathbb{D}_r) \to K^0(\overline{\mathbb{D}_r})$ must be zero. It then follows by exactness that $\iota_{\varepsilon_r} \delta_{\varepsilon_r} : K^0(\mathrm{im}(\varepsilon_r)) \to K^0(\mathbb{D})$ is an isomorphism. Clearly the class of $u_{0,\varepsilon_r}(z) = \frac{z}{r}$, z in rS^1, generates $K^{-1}(\mathrm{im}(\varepsilon_r))$, and therefore $\beta_{\mathbb{D}} = \iota_{\varepsilon_r} \delta_{\varepsilon_r}(u_{\varepsilon_r})$ generates $K^0(\mathbb{D})$. Hence, (1) is proven for $U = \mathbb{D}$.

Every bi-holomorphism $T : \mathbb{D} \to \mathbb{D}$ is of the form $T(z) = e^{2\pi i\theta}\frac{z-a}{1-\bar{a}z}$, for any z in $\mathbb{D}$, for some a in $\mathbb{D}$, θ in $\mathbb{R}$, and is therefore homotopic to the identity. Therefore, $((T^{-1})^*)_*\beta_{\mathbb{D}} = \beta_{\mathbb{D}}$. This proves (2) in the case that $U = V = \mathbb{D}$.

Now we prove the Proposition in the case that U and V are properly contained in $\mathbb{C}$. By the Riemann mapping Theorem, it follows that there is a bi-holomorphism $T : \mathbb{D} \to U$. By naturality of the index map, the diagram

$$\begin{array}{ccc} K^{-1}(\mathrm{im}(\gamma)) & \xrightarrow{T_*} & K^{-1}(\mathrm{im}(T(\gamma))) \\ \downarrow{\scriptstyle \delta_\gamma} & & \downarrow{\scriptstyle \delta_{T(\gamma)}} \\ K^0(\mathbb{D}^+_\gamma) & \xrightarrow{T_*} & K^0(U^+_\gamma) \\ \downarrow{\scriptstyle \iota_\gamma} & & \downarrow{\scriptstyle \iota_{T(\gamma)}} \\ K^0(\mathbb{D}) & \xrightarrow{T_*} & K^0(U) \end{array}$$

commutes. Hence, to prove (1) and $((T^{-1})^*)_*\beta_{\mathbb{D}} = \beta_U$, it suffices to prove $T_*u_\gamma = u_{T(\gamma)}$ for any Jordan curve γ in $\mathbb{D}$.

Since T is a bi-holomorphism and $\mathrm{im}(\gamma)$ is compact, for every $\epsilon > 0$, there is a continuous map $b : \mathrm{im}(T(\gamma)) \to U^+_{T(\gamma)}$ such that

$$\left| \frac{T^{-1}(z) - T^{-1}(b(z))}{z - b(z)} \cdot \frac{|z - b(z)|}{|T^{-1}(z) - T^{-1}(b(z))|} - \frac{(T^{-1})'(z)}{|(T^{-1})'(z)|} \right| < \epsilon, \text{ for all } z \text{ in } \mathrm{im}(T(\gamma)).$$

Let $a = T^{-1} \circ b \circ T$. Note that, for all w in $\mathrm{im}(T(\gamma))$, $u_{a,\gamma} \circ T^{-1}(w) = \frac{T^{-1}(w) - T^{-1}(b(w))}{|T^{-1}(w) - T^{-1}(b(w))|}$ and $u_{b,T(\gamma)}(w) = \frac{w-b(w)}{|w-b(w)|}$. Hence, $\|(u_{a,\gamma} \circ T^{-1}) \cdot (u_{b,T(\gamma)})^{-1} - \frac{(T^{-1})'}{|(T^{-1})'|}\| < \epsilon$. So, if we choose $\epsilon < 2$, then, by [52, Lemma 2.1.3 (iii)], $(u_{a,\gamma} \circ T^{-1}) \cdot (u_{b,T(\gamma)})^{-1}$ is homotopic to $\frac{(T^{-1})'}{|(T^{-1})'|}$ as elements in $C(\mathrm{im}(T(\gamma)), S^1)$.

The domain of $\frac{(T^{-1})'}{|(T^{-1})'|} : \mathrm{im}(T(\gamma)) \to S^1$ extends continuously to $U^+_{T(\gamma)}$ and is therefore homotopic, as an element in $C(\mathrm{im}(T(\gamma)), S^1)$, to a constant. Hence, $T_*u_\gamma - u_{T(\gamma)} = [(u_{a,\gamma} \circ T^{-1})(u_{b,T(\gamma)})^{-1}] = 0$ in $K^{-1}(\mathrm{im}(T(\gamma)))$.

Now, suppose U, V are simply connected proper open sets of $\mathbb{C}$. Let $T_U : \mathbb{D} \to U$ and $T_V : \mathbb{D} \to V$ be bi-holomorphisms. If $T : U \to V$ is a bi-holomorphism, then $T_*\beta_U = (T_V)_*(T_V^{-1})_*T_*(T_U)_*\beta_{\mathbb{D}} = (T_V)_*\beta_{\mathbb{D}} = \beta_V$, proving (2) in this case.

If $U \subseteq V$, let γ be a Jordan curve in U. Denote γ by γ_U, γ_V when thinking of it as a curve in U, V, respectively. The commutative diagram

$$\begin{array}{ccc} K^{-1}(\mathrm{im}(\gamma_U)) & = & K^{-1}(\mathrm{im}(\gamma_V)) \\ \downarrow{\scriptstyle \delta_{\gamma_U}} & & \downarrow{\scriptstyle \delta_{\gamma_V}} \\ K^0(U^+_{\gamma_U}) & = & K^0(V^+_{\gamma_V}) \\ \downarrow{\scriptstyle \iota_{\gamma_U}} & & \downarrow{\scriptstyle \iota_{\gamma_V}} \\ K^0(U) & \xrightarrow{i_*} & K^0(V) \end{array}$$

along with the fact that $\iota_{\gamma_U}\delta_{\gamma_U}(u_{\gamma_U}) = \beta_U$ and $\iota_{\gamma_V}\delta_{\gamma_V}(u_{\gamma_V}) = \beta_V$ implies $i_*\beta_U = \beta_V$. This proves (3) for such U and V.

The only case remaining for (1) is when $U = \mathbb{C}$. Since $C_0(\mathbb{C})$ is the inductive limit of the proper simply connected open sets of $\mathbb{C}$ (ordered by inclusion) and $i_*\beta_U = \beta_V$ when $U \subseteq V$, by continuity of K^0, we have that $\beta_{\mathbb{C}} := i_*\beta_U$ generates $K^0(\mathbb{C})$ and is independent of U, where U is any simply connected proper open set of $\mathbb{C}$. Every Jordan curve γ in $\mathbb{C}$ is eventually contained in a proper simply connected open set U of $\mathbb{C}$, so naturality of the index map, and the fact that $i_*\beta_U = \beta_{\mathbb{C}}$, implies $\iota_\gamma\delta_\gamma(u_\gamma) = \beta_{\mathbb{C}}$. This proves (1), and (2) in the case that $V = \mathbb{C}$.

If $T : \mathbb{C} \to \mathbb{C}$ is a bi-holomorphism, it is affine linear, and hence homotopic to the identity. Therefore, $T_*\beta_{\mathbb{C}} = \beta_{\mathbb{C}}$, and so (3) is proven when $U = V = \mathbb{C}$.

Now, for an arbitrary open simply connected set U properly contained in $\hat{\mathbb{C}}$, let $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Mobius transformation such that $T^{-1}(U) \subseteq \mathbb{C}$ and define $\beta_U = T_*\beta_{T^{-1}(U)}$. By the properties of $\beta_{T^{-1}(U)}$ proven above, it is clear that β_U is independent of T, and that (2) and (3) of the Proposition are satisfied. □

Corollary 4.1.10. *$\beta_{\hat{\mathbb{C}}} := i_*(\beta_U)$ in $K^0(\hat{\mathbb{C}})$ is independent of the choice of simply connected proper open set U in $\hat{\mathbb{C}}$. $\beta_{\hat{\mathbb{C}}}$ and $[1_{\hat{\mathbb{C}}}]$ form a minimal generating set for $K^0(\hat{\mathbb{C}}) \simeq \mathbb{Z}^2$.*

Proof. Let U and V be simply connected open sets properly contained in $\hat{\mathbb{C}}$. Choose simply connected sets $U' \subseteq U$ and $V' \subseteq V$ such that $U' \cup V'$ is contained in a simply connected proper open set W of $\hat{\mathbb{C}}$. The commutative diagram

$$\begin{array}{ccccc} K^0(U') & \longrightarrow & K^0(U) & & \\ & \searrow & & \searrow & \\ & & K^0(W) & \longrightarrow & K^0(\hat{\mathbb{C}}) \\ & \nearrow & & \nearrow & \\ K^0(V') & \longrightarrow & K^0(V) & & \end{array}$$

and Proposition 4.1.9 (3) implies $i_*(\beta_U) = i_*(\beta_V) := \beta_{\hat{\mathbb{C}}}$.

The short exact sequence

$$0 \longrightarrow K^0(\mathbb{C}) \xrightarrow{i_*} K^0(\hat{\mathbb{C}}) \longrightarrow K^0(\infty) \longrightarrow 0$$

implies $\beta_{\hat{\mathbb{C}}}$ and $[1_{\hat{\mathbb{C}}}]$ form a minimal generating set for $K^0(\hat{\mathbb{C}}) \simeq \mathbb{Z}^2$. □

Proposition 4.1.11. *Let U be a connected open set properly contained in $\hat{\mathbb{C}}$. Then, $K^0(U) \simeq \mathbb{Z}$, with a unique generator $\beta_U := i_*(\beta_V)$ in $K^0(U)$, for any simply connected open set $V \subseteq U$.*

Proof. Since $U \neq \hat{\mathbb{C}}$, and $((T^{-1})^*)_*\beta_V = \beta_{T(V)}$ for any fractional linear transformation T and simply connected open set V of $\hat{\mathbb{C}}$, by applying a Mobius transformation to U, it suffices to prove the Proposition in the case that U is an open set in $\mathbb{C}$.

First, we show that $K^{-1}(\hat{\mathbb{C}} \setminus U) = 0$ and $q_* : K^0(\hat{\mathbb{C}}) \to K^0(\hat{\mathbb{C}} \setminus U)$ sends $\beta_{\hat{\mathbb{C}}}$ to 0, where q is the restriction map.

We claim that U can be written as a countable union $U = \bigcup_{n\in\mathbb{N}} U_n$ where, for each n in $\mathbb{N}$, U_n is a finite union of bounded open rectangles $\mathcal{U}_n = \{R_1^n, ..., R_{k_n}^n\}$ such that

(1) U_n is connected,

(2) The closure of U_n in $\mathbb{C}$ is contained in U,

(3) $U_n \subseteq U_{n+1}$, and

(4) for any $i, j \leq k_n$, $\overline{R_i^n} \cap \overline{R_j^n}$ is either empty or a closed rectangle.

Finding a collection $\mathcal{U}_n = \{R_1^n, ..., R_{k_n}^n\}_{n\in\mathbb{N}}$ satisfying (1) and (2) is easy. Suppose $\mathcal{U}_l$ also satisfy (3) and (4) for $l \leq n-1$.

Since $\overline{U_n}$ is compact and contained in U, for each $i \leq k_n$ we can dilate R_i^n to a larger rectangle $\tilde{R}_i^n$ so that $\{\tilde{R}_1^n, ..., \tilde{R}_{k_n}^n\} = \mathcal{U}_n'$ satisfies (1), (2) and (4). Denote $U_n' = \bigcup_{i\leq k_n} \tilde{R}_i^n$.

Now, for each $i \leq k_n$, cover $\tilde{R}_i^n \setminus \overline{R_i^n}$ by a finite collection of open rectangles $\mathcal{S}_i = \{S_{i,1}, ..., S_{i,m_i}\}$ in U, and, for $l \geq n+1$, let $U_l' = U_l \cup \bigcup_{i,j} S_{i,j}$ and $\mathcal{U}_l' = \mathcal{U}_l \cup \bigcup_i \mathcal{S}_i$. $\mathcal{U}_{n+1}'$ satisifes (1) and (2), and U_n satisfies (3). So, we may induct this process to replace a collection $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ satisfying (1) and (2) with a collection satisfying (1) − (4).

Fix such a collection $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$. Let X be a maximal connected component of $\hat{\mathbb{C}} \setminus U_n$. We show that X has a (non-empty) simply connected interior with a boundary that is a Jordan curve.

For each $i \leq k_n$, write $\partial R_i^n = \bigcup_{j=1}^4 L_{i,j}$, where $L_{i,j}$ is a maximal horizontal or vertical line. Let x be in ∂X. Then, x is in a line $L_{i,j}$ for some i, j. Let $\gamma_x : [0, t] \to \partial X$ be the unit speed curve following the line $L_{i,j}$ in the counter-clockwise direction, starting at $\gamma_x(0) = x$

and ending once the line intersects a new line at $\gamma_x(t)$. By condition (4) of $\mathcal{U}_n$, this new line is unique. Continue defining γ_x on this new line, and inductively on the next new lines until γ_x intersects itself at $\gamma_x(t_*)$. By condition (4) of $\mathcal{U}_n$, this first intersection point of γ_x must be x. Since $\gamma_x : [0, t_*] \to \partial X$ is piece-wise linear with no intersections other than at the endpoints, it is a Jordan curve.

By the Jordan curve Theorem, γ_x separates $\hat{\mathbb{C}}$ into two simply connected components, one which contains U_n, and the other, denoted A_x, which does not. By maximality of X, we must have $\overline{A_x} \subseteq X$. By (4) and connectedness of ∂X (which follows from (1)), we must have $\partial X = \gamma_x([0, t_*]) = \partial \overline{A_x}$ and hence $\overline{A_x} = X$. We have shown X has a non-empty simply connected interior with boundary that is a Jordan curve.

Therefore, by Carathéodory's Theorem, there is a homeomorphism $\varphi : X \to \overline{\mathbb{D}}$, which is holomorphic on the interior of X. Hence $K^{-1}(X) = 0$. Since $\hat{\mathbb{C}} \setminus U_n$ is a finite disjoint union of its maximal connected components, it follows that $K^{-1}(\hat{\mathbb{C}} \setminus U_n) = 0$. By Corollary 4.1.10, $i_* : K^0(A) \to K^0(\hat{\mathbb{C}})$ sends β_A, where A is a simply connected component as above, to $\beta_{\hat{\mathbb{C}}}$, and so $q_*(\beta_{\hat{\mathbb{C}}}) = j_*(\beta_A)$ in the direct summand $K^0(X)$ of $K^0(\hat{\mathbb{C}} \setminus U_n)$, where $j = i : C_0(A) \to C(X)$ is the inclusion. Since φ is a bi-holomorphism on A, Proposition 4.1.9 (2) implies $((\varphi^{-1})^*)_* j_*(\beta_A) = j_*(\beta_{\mathbb{D}})$, which is 0 in $K^0(\overline{\mathbb{D}})$. Hence, $q_*(\beta_{\mathbb{C}}) = 0$ in $K^0(\hat{\mathbb{C}} \setminus U_n)$.

$C(\hat{\mathbb{C}} \setminus U)$ is the inductive limit of the restriction maps $r_n : C(\hat{\mathbb{C}} \setminus U_n) \to C(\hat{\mathbb{C}} \setminus U_{n+1})$, so, by continuity of K^{-1} and K^0, it follows that $K^{-1}(\hat{\mathbb{C}} \setminus U) = 0$ and $q_*(\beta_{\hat{\mathbb{C}}}) = 0$ in $K^0(\hat{\mathbb{C}} \setminus U)$. Note also that $q_*([1_{\hat{\mathbb{C}}}]) = [1_{\hat{\mathbb{C}} \setminus U}]$.

Hence, the 6 term exact sequence of K-theory associated to

$$0 \longrightarrow C_0(U) \xrightarrow{i} C(\hat{\mathbb{C}}) \xrightarrow{q} C(\hat{\mathbb{C}} \setminus U) \longrightarrow 0$$

implies that $i_* : K^0(U) \to K^0(\hat{\mathbb{C}})$ is an isomorphism onto $\mathbb{Z}\beta_{\hat{\mathbb{C}}}$. Let β_U be the unique generator of $K^0(U)$ such that $i_*(\beta_U) = \beta_{\hat{\mathbb{C}}}$.

By Corollary 4.1.10, $j_*(\beta_V) = \beta_{\hat{\mathbb{C}}}$ for any simply connected open set $V \subseteq U$, where $j : C_0(V) \to C(\hat{\mathbb{C}})$ is the inclusion. Hence, $i_*(\beta_V) = \beta_U$. □

Corollary 4.1.12. *For any connected open set U of $\hat{\mathbb{C}}$, there is a generator β_U in $K^0(U)$ such that, for $V \subseteq \hat{\mathbb{C}}$ another connected open set,*

(1) if $T : U \to V$ is a bi-holomorphism, then $((T^{-1})^*)_* \beta_U = \beta_V$, and

(2) if $U \subseteq V$ and $i : C_0(U) \to C_0(V)$ is the inclusion, then $i_* \beta_U = \beta_V$.

Moreover, if U is not entirely $\hat{\mathbb{C}}$, then $K^0(U) = \mathbb{Z}\beta_U \simeq \mathbb{Z}$. If $U = \hat{\mathbb{C}}$, then $\beta_{\hat{\mathbb{C}}}$ and $[1_{\hat{\mathbb{C}}}]$ form a minimal generating set for $K^0(\hat{\mathbb{C}}) \simeq \mathbb{Z}^2$.

Proof. The generating properties of β_U are contained in Corollary 4.1.10 and Proposition 4.1.11.

Let $U' \subseteq U$ be a simply connected open set, $U \subseteq V$ and $T : U \to T(U)$ a bi-holomorphism. By Proposition 4.1.11, the maps induced by inclusion send $\beta_{U'}$ to β_U, $\beta_{U'}$ to β_V, and $\beta_{T(U')}$ to $\beta_{T(U)}$. Hence, the commutative diagrams

$$\begin{array}{ccc} K^0(U') & \xrightarrow{T_*} & K^0(T(U')) \\ \downarrow & & \downarrow \\ K^0(U) & \xrightarrow{T_*} & K^0(T(U)) \end{array} \qquad \begin{array}{ccc} K^0(U') & & \\ \downarrow & \searrow & \\ K^0(U) & \xrightarrow{i_*} & K^0(V) \end{array}$$

and Proposition 4.1.9 (2) imply the Corollary. □

Chapter 5

The K-theory of a rational function acting on the Riemann sphere

We now compute the K-theory for an arbitrary rational function acting on the Riemann sphere, which follows easily from Corollary 4.1.12, Proposition 4.0.7, and the Pimsner-Voiculescu 6-term exact sequence.

We first record an easy lemma that will be used in this section as well as in Chapter 6.

Lemma 5.0.1. *Let R be a rational function and U an open set such that $\tilde{R} = R : U \to R(U) = V$ is a homeomorphism. Then, $\beta_U \hat{\otimes}_0 [\mathcal{E}_{\tilde{R},U}] = \beta_V$.*

Proof. For f in $C_0(U)$, ψ, φ in $E_{\tilde{R},U}$, and g in $C_0(V)$, we have $f \cdot \psi = f\psi$, $\langle \psi, \varphi \rangle = (\overline{\psi}\varphi) \circ (\tilde{R})^{-1}$, and $\varphi \cdot g = \varphi(g \circ \tilde{R})$. Therefore, the class of $\mathcal{E}_{\tilde{R},U}$ in $KK^0(C_0(U), C_0(V))$ is equal to the class of the *-isomorphism $(\tilde{R}^{-1})^*$.

$\tilde{R} : U \to V$ is a bi-holomorphism, so by Corollary 4.1.12, we have $((\tilde{R}^{-1})^*)_* \beta_U = \beta_V$. □

We now compute one of the maps appearing in the Pimsner-Voiculescu 6-term exact sequence of K-theory for $\mathcal{O}_{R,\hat{\mathbb{C}}}$.

Proposition 5.0.2. *Let R be a rational function. Then, $\hat{\otimes}[\mathcal{E}_{R,\hat{\mathbb{C}}}] = \iota$ as mappings $K^0(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}) \to K^0(\hat{\mathbb{C}})$.*

Proof. Let U be a connected open set in $\hat{\mathbb{C}}$ such that $U \cap C_{R,\hat{\mathbb{C}}} = \emptyset$ and $\tilde{R} = R : U \to R(U) =: V$ is a homeomorphism. By Proposition 4.0.7, $(\iota_U)^*[\mathcal{E}_{R,\hat{\mathbb{C}}}] = (i_V)_*[\mathcal{E}_{\tilde{R},U}]$.

Corollary 4.1.12 implies that $(\iota_U)_* : K^0(U) \to K^0(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$ is an isomorphism. Therefore, to prove the Proposition, it suffices to show $\hat{\otimes}(\iota_U)^*[\mathcal{E}_{R,\hat{\mathbb{C}}}] = \hat{\otimes}(\iota_U)^*\iota$ as mappings $K^0(U) \to K^0(\hat{\mathbb{C}})$.

Note that

(1) $\hat{\otimes}(\iota_U)^*\iota = (i_U)_*$, and

(2) $\hat{\otimes}(\iota_U)^*[\mathcal{E}_{R,\hat{\mathbb{C}}}] = (i_V)_* \circ (\hat{\otimes}_0[\mathcal{E}_{\tilde{R},U}])$ (from above).

By Corollary 4.1.12, both $(i_U)_*$ and $(i_V)_* \circ (\hat{\otimes}_0[\mathcal{E}_{\tilde{R},U}])$ send β_U to $\beta_{\hat{\mathbb{C}}}$. □

The above calculation is enough information to determine the K-theory of $\mathcal{O}_{R,\hat{\mathbb{C}}}$.

Theorem 5.0.3. *$\delta_{PV} : K_1(\mathcal{O}_{R,\hat{\mathbb{C}}}) \to K^0(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$ is an isomorphism, and*

$$0 \longrightarrow K^0(\hat{\mathbb{C}}) \xrightarrow{i_*} K_0(\mathcal{O}_{R,\hat{\mathbb{C}}}) \xrightarrow{exp_{PV}} K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}) \longrightarrow 0$$

is a short exact sequence. Consequently, $K_0(\mathcal{O}_{R,\hat{\mathbb{C}}}) \simeq \mathbb{Z}^{|C_{R,\hat{\mathbb{C}}}|+1}$, with the class of the unit a generator in a minimal generating set for $K_0(\mathcal{O}_{R,\hat{\mathbb{C}}})$, and $K_1(\mathcal{O}_{R,\hat{\mathbb{C}}}) \simeq \mathbb{Z}$.

Proof. By Proposition 5.0.2, and the fact that $K^{-1}(\hat{\mathbb{C}}) = 0$, we may fill in the Pimsner-Voiculescu 6-term exact sequence as follows:

$$\begin{array}{ccccc} K^0(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})) & \xrightarrow{0} & K^0(\hat{\mathbb{C}}) & \xrightarrow{i} & K_0(\mathcal{O}_{R,\hat{\mathbb{C}}}) \\ \delta\uparrow & & & & \downarrow exp \\ K_1(\mathcal{O}_{R,\hat{\mathbb{C}}}) & \longleftarrow & 0 & \longleftarrow & K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}). \end{array}$$

Exactness of the above diagram imply the first two claims of the Proposition.

$K_1(\mathcal{O}_{R,\hat{\mathbb{C}}}) \simeq \mathbb{Z}$ then follows from Corollary 4.1.12. By the paragraph preceding Proposition 7.0.1 (this forward reference won't cause any circular arguments), we have that $K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}) \simeq \mathbb{Z}^{|C_{R,\hat{\mathbb{C}}}|-1}$, and so the short exact sequence in the Corollary splits. Hence, by Corollary 4.1.12 in the case $U = \hat{\mathbb{C}}$, we have $K_0(\mathcal{O}_{R,\hat{\mathbb{C}}}) \simeq K^0(\hat{\mathbb{C}}) \oplus K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}) \simeq \mathbb{Z}^{|C_{R,\hat{\mathbb{C}}}|+1}$, with the class of the unit a generator in a minimal generating set. □

Chapter 6

The K-theory of a rational function acting on its Fatou set

In this section we compute the K-theory of a rational function R acting on its Fatou set. As in the previous section, we do so by calculating the kernel and co-kernel of $\iota - \hat{\otimes}_i[\mathcal{E}_{R,F_R}]$, $i = 0, 1$.

The case for $i = 0$ follows similar techniques as in the previous section once we understand how R permutes its Fatou components. Most of the section will be dedicated to $i = 1$. The fact that any compact set of F_R is eventually mapped into a global "attractor" with manageable K^{-1} (essentially Corollary 2.5.17) makes this calculation possible.

Let's write $F_R = \bigcup_{x \in X} U_x$, where U_x are the maximal connected components of F_R and X is a (countable) indexing set. Since $R^{-1}(F_R) = F_R$, it is easy to see that R maps U_x onto another component $U_{\sigma(x)}$, for every x in X.

For each x in X, denote $\beta_{U_x} = f_x$ and $\beta_{U_x \setminus C_{R,F_R}} = e_x$. By Corollary 4.1.12 we can (and will) identify $K^0(F_R \setminus C_{R,F_R})$ and $K^0(F_R)$ with $\bigoplus_{x \in X} \mathbb{Z}[e_x]$ and $\bigoplus_{x \in X} \mathbb{Z}[f_x]$, respectively, via the inclusion maps.

Proposition 6.0.1. *$\hat{\otimes}_0[\mathcal{E}_{R,F_R}]$ sends e_x to $f_{\sigma(x)}$, for all x in X.*

Proof. For each x in X, let U be an open set in U_x such that $\tilde{R} = R : U \to R(U) = V$ is a homeomorphism. Then, by Proposition 4.0.7, we have $(\iota_U)^*[\mathcal{E}_{R,U_x}] = (i_V)_*[\mathcal{E}_{\tilde{R},U}]$, where $\iota_U = i : C_0(U) \to C_0(U_x \setminus C_{R,F_R})$ and $i_V = i : C_0(V) \to C_0(U_{\sigma(x)})$ are the inclusions.

Hence, $e_x \hat{\otimes}_0[\mathcal{E}_{R,U_x}] = ((\iota_U)_* \beta_U) \hat{\otimes} [\mathcal{E}_{R,U_x}] = (i_V)_*(\beta_U \hat{\otimes} \mathcal{E}_{\tilde{R},U})$. By Lemma 5.0.1, $\beta_U \hat{\otimes} \mathcal{E}_{\tilde{R},U}$ $= \beta_V$, and, by its definition, $f_{\sigma(x)} = (i_V)_* \beta_V$. Therefore, $e_x \hat{\otimes}_0[\mathcal{E}_{R,U_x}] = f_{\sigma(x)}$ for all x in X.

Since $(\iota_x)^* \mathcal{E}_{R,F_R} = (i_{\sigma(x)})_* \mathcal{E}_{R,U_x}$ for all x in X, where $\iota_x = i : C_0(U_x \setminus C_{R,\hat{\mathbb{C}}}) \to C_0(F_R)$ and $i_{\sigma(x)} = i : C_0(U_{\sigma(x)}) \to C_0(F_R)$ are the inclusions, we have that, under the identifications made previous to the Corollary, $\hat{\otimes}_0[\mathcal{E}_{R,F_R}] = \oplus_{x \in X} \hat{\otimes}_0[\mathcal{E}_{R,U_x}]$ as mappings from $\bigoplus_{x \in X} \mathbb{Z}[e_x]$

to $\bigoplus_{x\in X}\mathbb{Z}[f_x]$, proving the result. □

We will call a finite subset $P = \{x_1, ..., x_n\} \subseteq X$ a *cycle* if $\sigma(P) = P$ and $\sigma : P \to P$ is minimal, and so, equivalently, P is a periodic orbit of $\sigma : X \to X$. It easy to see that two distinct cycles are disjoint. We will denote the collection of cycles to be $\mathcal{F}_R$.

By [54, Corollary 2], $\mathcal{F}_R$ must be a finite set. By [56, Theorem 1], for every x in X, there is k in $\mathbb{N}$ and a cycle P in $\mathcal{F}_R$ such that $\sigma^k(x)$ is in P. So, $X = \bigsqcup_{P\in\mathcal{F}_R}\bigcup_{n\in\mathbb{N}}\sigma^{-n}(P)$.

Corollary 6.0.2. *The mapping $\eta = \iota - \hat{\otimes}_0[\mathcal{E}_{R,F_R}] : K^0(F_R \setminus C_{R,\hat{\mathbb{C}}}) \to K^0(F_R)$ has kernel generated by the elements $e_P := \sum_{x\in P} e_x$, where P is a cycle in $\mathcal{F}_R$.*

For each cycle P, choose an x_P in P. The subgroup of $K^0(F_R)$ generated by the elements $\{f_{x_P}\}_{P\in\mathcal{F}_R}$ maps isomorphically onto the co-kernel of η, via the quotient map.

Proof. For each $P \in \mathcal{F}_R$, denote $X_P = \bigcup_{n\in\mathbb{N}}\sigma^{-n}(P)$, and E_P, F_P the subgroups of $K^0(F_R \setminus C_{R,\hat{\mathbb{C}}})$, $K^0(F_R)$ generated by $\{e_x\}_{x\in X_P}$, $\{f_x\}_{x\in X_P}$, respectively. By Corollary 4.1.12 and Proposition 6.0.1, $\eta(e_x) = f_x - f_{\sigma(x)}$ for all x in X, so $\eta^{-1}(F_P) = E_P$ for all P in $\mathcal{F}_R$. Denote $\eta_P = \eta : E_P \to F_P$. By the above commentary we have $X_P \cap X_{P'} = \emptyset$ for distinct cycles P, P' and $\bigsqcup_{P\in\mathcal{F}_R} X_P = X$, so (canonically) $K^0(F_R \setminus C_{R,\hat{\mathbb{C}}}) \simeq \bigoplus_{P\in\mathcal{F}_R} E_P$, $K^0(F_R) \simeq \bigoplus_{P\in\mathcal{F}_R} F_P$ and $\eta \simeq \oplus_{P\in\mathcal{F}_R}\eta_P$. Hence, it suffices to show $\ker(\eta_P) = \mathbb{Z}[e_P]$ and $\mathbb{Z}[f_x]$ maps isomorphically onto co-ker(η_P) via the quotient map, for any cycle P in $\mathcal{F}_R$ and any x in P.

Since $\sigma : P \to P$ is a bijection, we have that $\eta(e_P) = \sum_{x\in P} f_x - \sum_{x\in P} f_{\sigma(x)} = 0$. Suppose $g = \sum_{x\in F} a_x e_x$, where F is a finite set of X_P containing P, and $0 = \eta(g) = \sum_{x\in F} a_x f_x - \sum_{x\in F} a_x f_{\sigma(x)}$. Then, $\sum_{x\in F\setminus\sigma(F)} a_x f_x = 0$, so we may conclude that $F' = \{x \in F : a_x \neq 0\}$ satisfies $\sigma(F') = F'$. P is the only non-empty cycle contained in X_P, so either $F' = \emptyset$ $(g = 0)$ or $F' = P$. Let's assume the latter, and write $P = \{x_1, ..., x_n\}$, where $\sigma(x_i) = x_{i+1} \mod n$, for all $i \leq n$. We have that $0 = \eta(g) = \sum_{x\in P} a_x f_x - \sum_{x\in P} a_x f_{\sigma(x)} = \sum_{i=1}^{n}(a_i - a_{i-1}) f_{x_i}$, so $a_i - a_{i-1} = 0 \mod n$. Hence, $g = a \cdot e_P$ for $a = a_1$. Therefore, $\ker(\eta_P) = \mathbb{Z}[e_P]$.

Now, it remains to show for any x in P and n in $\mathbb{Z} \setminus \{0\}$, nf_x is not in $\mathrm{im}(\eta_P)$, and $\mathbb{Z}[f_x] + \mathrm{im}(\eta_P) = F_P$. Let $\varphi : F_P \to \mathbb{Z}$ be the homomorphism satisfying $\varphi(f_y) = 1$ for all y in X_P. Since $\varphi \circ \eta_P(e_y) = 0$ for all y in X_P, we have $\varphi(\mathrm{im}(\eta_P)) = 0$. Since $0 \neq n = \varphi(nf_x)$, we have that nf_x is not in $\mathrm{im}(\eta_P)$. For any y in X_P, $0 = \eta(e_y) = f_y - f_{\sigma(y)} \mod \mathrm{im}(\eta_P)$, so $f_y = f_{\sigma^k(y)} \mod \mathrm{im}(\eta_P)$ for all k in $\mathbb{N}$. By definition of X_P, there is a k in $\mathbb{N}$ such that $\sigma^k(y) = x$. Hence, $\mathbb{Z}[f_x] + \mathrm{im}(\eta_P) = F_P$. □

We will now determine the kernel and co-kernel of $\iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}]$ acting on K^{-1}.

First, we must explain how we orient the components of a Herman cycle. This will be crucial later on to calculate the kernel and co-kernel of the connecting maps between the groups associated to $R : F_R \to F_R$ and that of $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which we will need to know if we are to calculate the groups associated to $R : J_R \to J_R$ from our exact sequences relating all three. For this reason, we must describe part of the kernel and co-kernel of $\iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}] : K^{-1}(F_R \setminus C_{R,F_R}) \to K^{-1}(F_R)$ in terms of this orientation, which we now define.

Let R be a rational function, and suppose Q is a Herman cycle for R. Choose a component x_Q in Q. The boundary ∂U_{x_Q} has two connected components ([39, Lemma 15.7]); choose one to denote $\partial^+ U_{x_Q}$ and call it the *interior boundary.* Denote the other component $\partial^- U_{x_Q}$ and call it the *exterior boundary.*

From the 6-term exact sequence of K-theory associated to

$$0 \longrightarrow C_0(U_{x_Q}) \longrightarrow C(\overline{U_{x_Q}}) \longrightarrow C(\partial^+ U_{x_Q} \bigsqcup \partial^- U_{x_Q}) \longrightarrow 0,$$

there is a group homomorphism $\exp : K^0(\partial^+ U_{x_Q}) \oplus K^0(\partial^- U_{x_Q}) \to K^{-1}(U_{x_Q})$. Since $\overline{U_{x_Q}}$, $\partial^+ U_{x_Q}$, and $\partial^- U_{x_Q}$ are connected, compact and proper subsets of $\hat{\mathbb{C}}$, by Proposition 4.1.5, their K^0 groups are isomorphic to $\mathbb{Z}$ and are generated by $[1_{\overline{U_{x_Q}}}], [1_{\partial^+ U_{x_Q}}], [1_{\partial^- U_{x_Q}}]$, respectively. Hence, the kernel of exp is generated by $[1_{\partial^+ U_{x_Q}}] + [1_{\partial^- U_{x_Q}}]$, the image is generated by $\exp([1_{\partial^+ U_{x_Q}}])$, and $\exp([1_{\partial^+ U_{x_Q}}]) = -\exp([1_{\partial^- U_{x_Q}}])$. U_{x_Q} is homeomorphic to an open annulus, so $K^{-1}(U_{x_Q}))$ is isomorphic to $\mathbb{Z}$, and the image of exp is isomorphic to $n \cdot \mathbb{Z}$ for some $n > 0$. From the 6-term exact sequence associated to

$$0 \longrightarrow C_0(\hat{\mathbb{C}} \setminus \overline{U_{x_Q}}) \longrightarrow C(\hat{\mathbb{C}}) \longrightarrow C(\overline{U}_{x_Q}) \longrightarrow 0,$$

it is easy to see $K^{-1}(\overline{U_{x_Q}})$ contains no torsion. Therefore, exp must be surjective, and so $u_{x_Q} := \exp([1_{\partial^+ U_{x_Q}}])$ is a choice of generator for $K^{-1}(U_{x_Q})$.

We now orient the boundaries of the rest of the cycle elements as follows. Suppose Q is length n, and $0 \leq k \leq n-1$. Let $\partial^+ U_{\sigma^k(x_Q)} = R^k(\partial^+ U_{x_Q})$, $\partial^- U_{\sigma^k(x_Q)} = R^k(\partial^- U_{\sigma^k(x_Q)})$, and $u_{\sigma^k(x_Q)} = \exp([1_{\partial^+ U_{\sigma^k(x_Q)}}])$.

$R^{\circ n} : U_{x_Q} \to U_{x_Q}$ is conjugate to an irrational rotation, and is therefore homotopic to the identity. Hence, the induced map $(R^{\circ n*})_*$ on $K^{-1}(U_{x_Q})$ is equal to id. Therefore, by naturality of exp, we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}[1_{\partial^+ U_{x_Q}}] & \xleftarrow{(R^{\circ n*})_*} & \mathbb{Z}[1_{R^{\circ n}(\partial^+ U_{x_Q})}] \\
\big\downarrow \text{exp} & & \big\downarrow \text{exp} \\
K^{-1}(U_{x_Q}) & \xleftarrow[\text{id}]{} & K^{-1}(U_{x_Q}).
\end{array}$$

Hence, $R^{\circ n}(\partial^+ U_{x_Q}) = \partial^+ U_{x_Q}$.

Similarly, for any x in Q, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(U_x) & \longrightarrow & C_0(\overline{U_x}) & \longrightarrow & C(\partial^+ U_x \bigsqcup \partial^- U_x) & \longrightarrow & 0 \\ & & \uparrow R^* & & \uparrow R^* & & \uparrow R^* & & \\ 0 & \longrightarrow & C_0(U_{\sigma(x)}) & \longrightarrow & C_0(\overline{U_{\sigma(x)}}) & \longrightarrow & C(\partial^+ U_{\sigma(x)} \bigsqcup \partial^- U_{\sigma(x)}) & \longrightarrow & 0 \end{array}$$

implies, by naturality of exp, a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[1_{\partial^+ U_x}] & \xleftarrow{(R^*)_*} & \mathbb{Z}[1_{\partial^+ U_{\sigma(x)}}] \\ \downarrow \exp & & \downarrow \exp \\ K^{-1}(U_x) & \xleftarrow[(R^*)_*]{} & K^{-1}(U_{\sigma(x)}), \end{array}$$

and so $((R^{\circ -1})^*)_* u_x = u_{\sigma(x)}$.

We will call the choice of generators $\{u_x\}_{x\in Q}$ for $K^{-1}(U_Q) = \bigoplus_{x\in Q} K^{-1}(U_x)$ (or equivalently a choice of boundary components) an *orientation for* Q, and denote $u_Q = \sum_{x\in Q} u_x$. Note that there are only two possible choices of such an orientation for Q. We will call Q equipped with a choice of orientation an *oriented Herman cycle*.

We will also need to understand the relationship between $K^{-1}(F_R \setminus C_{R,F_R})$ and $K^{-1}(F_R)$, which is the domain and co-domain of $\iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}]$, respectively. The following lemma is all we need.

Lemma 6.0.3. *Let U be a proper open set of $\hat{\mathbb{C}}$ and $D \subseteq U$ a finite set. Then, there are open sets V and W contained in U such that*

(1) $V \cap D = \emptyset$, and V contains any connected component of U not intersecting D,

(2) W is a disjoint union of simply connected open sets which contain D,

(3) $i_* : K^{-1}(V) \to K^{-1}(U)$ is an isomorphism, and

(4) $i_* + i_* : K^{-1}(W \setminus \tilde{D}) \oplus K^{-1}(V) \to K^{-1}(U \setminus \tilde{D})$ is an isomorphism, for any $\tilde{D} \subseteq D$.

Denote the image of $i_* : K^{-1}(W \setminus D) \to K^{-1}(U \setminus D)$ to be $G(U, D)$. Assuming $|\hat{\mathbb{C}} \setminus U| \geq 2$, $G(U, D)$ has the property that if $\tilde{W}$ is any open set in U that contains D and is the disjoint union of simply connected open sets, then $i_* : K^{-1}(\tilde{W} \setminus D) \to K^{-1}(U \setminus D)$ maps isomorphically onto $G(U, D)$.

Proof. Let $U_1, ..., U_k$ be the maximal connected components of U that contain D. For each $i \leq k$, let L_i be the image of a smooth non-self-intersecting curve $\gamma_i : [0, 1] \to \hat{\mathbb{C}}$ which passes through $U_i \cap D$ and $L_i \cap U_i = \gamma_i([0, 1))$. We may also assume $\gamma_i(0)$ is in D, for all $i \leq k$. Let W_i be a simply connected open neighbourhood of $\gamma_i([0, 1))$ in U_i (which can be found, for instance, by applying the tubular neighbourhood Theorem to the embedding $\gamma_i((0,1)) \subseteq U_i$). Then, $W_i' = W_i \setminus \gamma_i([0,1))$ is also simply connected ($\hat{\mathbb{C}} \setminus W_i'$ and W_i' are both connected).

Let $V = U \setminus (\bigcup_{i=1}^k L_i)$, $W = \bigcup_{i=1}^k W_i$, and $W' = \bigcup_{i=1}^k W_i'$. Then, $V \cap D = \emptyset$, V contains all connected components of U not intersecting D, and $D \subseteq W$.

Since $U \setminus V$ is homeomomorphic to the disjoint union of k half-open intervals, $C_0(U \setminus V)$ is contractible to 0, so by the 6-term exact sequence of K-theory associated to the extension

$$0 \longrightarrow C_0(V) \longrightarrow C_0(U) \longrightarrow C_0(U \setminus V)) \longrightarrow 0,$$

we have that $i_* : K^{-1}(V) \to K^{-1}(U)$ is an isomorphism.

Let $\tilde{D}$ be a subset of D. Since $V \cup (W \setminus \tilde{D}) = U \setminus \tilde{D}$ and $V \cap (W \setminus \tilde{D}) = W'$, by [30, Theorem 4.19], we have the following exact sequence:

$$\begin{array}{ccccc} K^0(W') & \xrightarrow{i_* \oplus -i_*} & K^0(W \setminus \tilde{D}) \oplus K^0(V) & \xrightarrow{i_* + i_*} & K^0(U \setminus \tilde{D}) \\ \uparrow & & & & \\ K^{-1}(U \setminus \tilde{D}) & \xleftarrow[i_* + i_*]{} & K^{-1}(W \setminus \tilde{D}) \oplus K^{-1}(V) & \xleftarrow[i_* \oplus -i_*]{} & K^{-1}(W'). \end{array}$$

By Corollary 4.1.12, $K^0(W')$, $K^0(W \setminus \tilde{D})$ are free abelian groups generated by $\{\beta_{W_i'}\}_{i=1}^k$, $\{\beta_{W_i \setminus \tilde{D}}\}_{i=1}^k$, and $i_*(\beta_{W_i'}) = \beta_{W_i \setminus \tilde{D}}$ for all $i \leq k$. Therefore, the left-most horizontal map in the above diagram is injective. Exactness then implies that $i_* + i_* : K^{-1}(W \setminus \tilde{D}) \oplus K^{-1}(V) \to K^{-1}(U \setminus \tilde{D})$ is surjective. Since W' is the disjoint union of simply connected open sets, $K^{-1}(W') = 0$. Exactness of the diagram then implies $i_* + i_* : K^{-1}(W \setminus \tilde{D}) \oplus K^{-1}(V) \to K^{-1}(U \setminus \tilde{D})$ is injective.

We now show $G(U, D)$ has the stated property below Lemma 6.0.3.

If W_1 and W_2 are open proper subsets of $\hat{\mathbb{C}}$ that are the disjoint union of simply connected sets and D is a finite set such that $D \subseteq W_1 \subseteq W_2$, then $K^{-1}(W_j) = 0$, $j = 0, 1$, and by Corollary 4.1.12, $i_* : K^0(W_j \setminus D) \to K^0(W_j)$ is an isomorphism, for $j = 0, 1$. Therefore, by naturality of the exponential maps associated to the short exact sequences

$$0 \longrightarrow C_0(W_j \setminus D) \longrightarrow C_0(W_j) \longrightarrow C(D) \longrightarrow 0,$$

we have a commutative diagram

$$\begin{array}{ccc} K^0(D) & = & K^0(D) \\ \downarrow \exp & & \downarrow \exp \\ K^{-1}(W_1 \setminus D) & \xrightarrow{i_*} & K^{-1}(W_2 \setminus D) \end{array}$$

with vertical arrows being isomorphisms. Hence, $i_* : K^{-1}(W_1 \setminus D)) \to K^{-1}(W_2 \setminus D)$ is an isomorphism.

Now, suppose $\tilde{W}$ is the disjoint union of simply connected open sets contained in U and containing D. Since $|\hat{\mathbb{C}} \setminus U| \geq 2$, we may regard W and $\tilde{W}$ as the disjoint unions of simply connected, proper open sets of the complex plane $\mathbb{C}$. Therefore, $W \cap \tilde{W}$ is also the disjoint union of simply connected open sets contained in U and containing D ($\hat{\mathbb{C}} \setminus W$ and $\hat{\mathbb{C}} \setminus \tilde{W}$ are connected, contain the common point ∞, and hence $\hat{\mathbb{C}} \setminus W \cap \tilde{W} = \hat{\mathbb{C}} \setminus W \cup \hat{\mathbb{C}} \setminus \tilde{W}$ is connected). We have a commututative diagram

$$\begin{array}{ccc} K^{-1}(\tilde{W} \cap W \setminus D)) & \longrightarrow & K^{-1}(W \setminus D) \\ \downarrow & & \downarrow \\ K^{-1}(\tilde{W} \setminus D) & \longrightarrow & K^{-1}(U \setminus D). \end{array}$$

The top horizontal and left-most vertical map have been shown to be isomorphisms, so commutativity implies $i_* : K^{-1}(\tilde{W} \setminus D) \to K^{-1}(U \setminus D)$ maps injectively onto $G(U, D)$. □

We can now describe the kernel and co-kernel of $\iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}] : K^{-1}(F_R \setminus C_{R,F_R}) \to K^{-1}(F_R)$.

Proposition 6.0.4. *Let R be a rational function, denote $\mathcal{H}_R$ to be the set of Herman cycles for R, and fix an orientation for every P in $\mathcal{H}_R$. The mapping $\gamma = \iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}] : K^{-1}(F_R \setminus C_{R,F_R}) \to K^{-1}(F_R)$ has kernel generated by the elements $u_P := \sum_{x \in P} u_x$, where P is a Herman cycle, and the subgroup $G(F_R, C_{R,F_R})$ of $K^{-1}(F_R \setminus C_{R,F_R})$.*

For each P in $\mathcal{H}_P$, choose an x_P in P. The subgroup of $K^{-1}(F_R)$ generated by the elements $\{u_{x_P}\}_{P \in \mathcal{H}_P}$ maps isomorphically onto the co-kernel of γ via the quotient map.

Proof. For each cycle P in $\mathcal{F}_R$, We will denote $U_P = \bigcup_{x \in P} U_x$, $X_P = \bigcup_{n \in \mathbb{N}_0} \sigma^{-n}(P)$, and $F(P) = \bigcup_{x \in X_P} U_x$. Then, $F(P) \cap F(P') = \emptyset$ for distinct cycles P, P', $F_R = \bigcup_{P \in \mathcal{F}_R} F(P)$, and $R^{-1}(F(P)) = F(P)$ for any cycle P. In particular, we can identify $K^{-1}(F_R)$, $K^{-1}(F_R \setminus C_{R,F_R})$ with $\bigoplus_{P \in \mathcal{F}_R} K^{-1}(F(P))$, $\bigoplus_{P \in \mathcal{F}_R} K^{-1}(F(P) \setminus C_{R,F(P)})$, respectively, via the inclusion maps, and (by Proposition 4.0.7) $(\iota_{F(P)})^*[\mathcal{E}_{R,F_R}] = (i_{F(P)})_*[\mathcal{E}_{R,F(P)}]$ for any cycle P in $\mathcal{F}_R$. Therefore, to prove Proposition 6.0.4, it suffices to show the mapping $\gamma_P = \iota - \hat{\otimes}_1[\mathcal{E}_{R,F(P)}] : K^{-1}(F(P) \setminus C_{R,F(P)}) \to K^{-1}(F(P))$ has kernel generated by $G(F(P), C_{R,F(P)})$, along with u_P if P is a Herman cycle, and co-kernel generated by u_{x_P} if P is a Herman cycle, with co-kernel 0 otherwise.

Let A_P and B be as in Proposition 2.5.17, and denote $B = A_P = U_P$ when P is a Herman cycle. Let $\{V_n'\}_{n \in \mathbb{N}}$ be a family of pre-compact open sets in $F(P)$ such that $\overline{V_n'} \subseteq F(P)$, $V_n' \subseteq V_{n+1}'$ for all n in $\mathbb{N}$, and $\bigcup V_n' = F(P)$. By Proposition 2.5.17, for every n in $\mathbb{N}$ there is a k_n in $\mathbb{N}$ such that $R^{k_n}(V_n' \cup B) \subseteq A_P$. Define $V_n = (\bigcup_{i=0}^{k_n} R^i(V_n' \cup B)) \cup A_P$ for all n in $\mathbb{N}$. Then, $\bigcup_{n \in \mathbb{N}} V_n = F(P)$ and, for all n, V_n has the properties

(1) $A_P \subseteq B \subseteq V_n \subseteq V_{n+1}$,

(2) $R(V_n) \subseteq V_n$, and

(3) $R^{k_n}(V_n) \subseteq A_P$.

By (1) above, $K^{-1}(F(P) \setminus C_{R,F(P)})$ and $K^{-1}(F(P))$ are the inductive limits of the (maps induced by) the inclusions $(\iota_n)^* : K^{-1}(V_n \setminus C_{R,F(P)}) \to K^{-1}(V_{n+1} \setminus C_{R,F(P)})$, and $(i_n)_* : K^{-1}(V_n) \to K^{-1}(V_{n+1})$, respectively. Denote $R_n = R : V_n \to V_n$. By Proposition 4.0.7, we have that $(\iota_n)^*[\mathcal{E}_{R_{n+1},V_{n+1}}] = (i_n)_*[\mathcal{E}_{R_n,V_n}]$ for all n in $\mathbb{N}$, and $(\nu_n)^*[\mathcal{E}_{R,F(P)}] = (\mu_n)_*[\mathcal{E}_{R_n,V_n}]$, where $\nu_n = i : C_0(V_n \setminus C_{R,F(P)}) \to C_0(F(P) \setminus C_{R,F(P)})$ and $\mu_n = i : C_0(V_n) \to C_0(F(P))$, for all n in $\mathbb{N}$. Therefore, γ_P is the inductive limit of the maps $\gamma_n = \iota - \hat{\otimes}_1[\mathcal{E}_{R_n,V_n}] : K^{-1}(V_n \setminus C_{R,F(P)}) \to K^{-1}(V_n)$, so it suffices to show, for all n in $\mathbb{N}$, that γ_n has kernel generated by $G(V_n, C_{R,F(P)} \cap V_n)$, along with u_P if P is a Herman cycle, and co-kernel generated by u_{x_P} if P is a Herman cycle, with co-kernel 0 otherwise.

First, we prove the following lemma.

Lemma 6.0.5. *for any open set A of F_R, and finite set $D \subseteq A$, $G(A, D \cup (A \cap C_{R,F_R}))$ is in the kernel of $\hat{\otimes}_1[\mathcal{E}_{R,A\setminus D}] : K^{-1}(A \setminus (D \cup C_{R,F_R})) \to K^{-1}(F_R)$.*

Proof. Denote $D' = D \cup (A \cap C_{R,F_R})$ Since R is a rational function, for every c in D', we can find local co-ordinates about ϕ, ψ about c, $R(c)$, respectively such that $\phi(c) = 0$, $\psi(R(c)) = 0$, and $\psi(R(\phi^{-1}(z))) = z^m$, for all z in a neighbourhood of 0, for some m in $\mathbb{N}$. Therefore, for every c in D', there is a simply connected open set W_c containing c and contained in A such that $W_c \cap W_{c'} = \emptyset$ for all distinct c, c' in D' and $R(W_c)$ is simply connected.

By Lemma 6.0.3, $\sum (j_c)_* : \bigoplus_{c \in D'} K^{-1}(W_c \setminus c) \to G(A, D')$ is an isomorphism, where $j_c = i : C_0(W_c \setminus c) \to C_0(A \setminus D')$. By Proposition 4.0.7, we have $(j_c)^*[\mathcal{E}_{R,A\setminus D}] = (i_{R(W_c)})_*[\mathcal{E}_{R,W_c}]$. Since $K^{-1}(R(W_c)) = 0$, it follows that $\hat{\otimes}(i_{R(W_c)})_*[\mathcal{E}_{R,W_c}] = 0$ for all c in D. This proves the lemma. □

Denote $C_n = V_n \cap C_{R,F(P)}$. As a special case of Lemma 6.0.5, we have that $G(V_n, C_n)$ is in the kernel of $\hat{\otimes}_1[\mathcal{E}_{R_n,V_n}]$. The diagram

$$\begin{array}{ccc} G(V_n, C_n) & \longrightarrow & K^{-1}(V_n) \\ \uparrow & & \uparrow \\ K^{-1}(W \setminus C_n) & \longrightarrow & K^{-1}(W) \end{array}$$

commutes, where W is any disjoint union of simply connected open sets containing C_n and contained in V_n, and the maps are induced by inclusion. The left vertical map is an isomorphism by Lemma 6.0.3, and the bottom right group is zero. Hence $G(V_n, C_n)$ is also in the kernel of ι, so that $G(V_n, C_n) \subseteq \ker(\gamma_n)$.

We now determine the rest of the kernel, but first we set some notation. For $0 \leq i \leq k_n$, let $V^i = V_n \setminus \bigcup_{j=0}^{k_n - i} R^{-j}(C_n)$. Then, $R(V_i) \subseteq V_{i+1}$, for all $0 \leq i \leq k_n - 1$. let $\tilde{R}_i = R : V^i \to V^{i+1}$.

Denote by V the (same labelled) open set from Lemma 6.0.3 applied in the case that $U = V_n$, $D = \bigcup_{i=0}^{k_n} R^{-1}(C_n)$, and, for every $0 \leq i \leq k_n$, let $\varepsilon_i : K^{-1}(V) \to K^{-1}(V^i)$ be the (map induced by) inclusion. Then, by Lemma 6.0.3, $\varepsilon_i + j_* : K^{-1}(V) \oplus G(V_n, \bigcup_{j=0}^{k_n - i} R^{-j}(C_n)) \to K^{-1}(V^i)$ is an isomorphism, for every $0 \leq i \leq k_n$.

For every $0 \leq i \leq k_n$, we shall denote $\Psi = [\mathcal{E}_{\tilde{R}_i, V^i}]$ and $\iota = j_* : K^{-1}(V^i) \to K^{-1}(V_n)$. It will always be clear the domain of these maps, so no confusion from this notation ambiguity will arise. Moreover, for i, m in $\mathbb{N}$ such that $i + m \leq k_n$, we will denote $\Psi^m =$
$[\mathcal{E}_{\tilde{R}_i, V^i}] \hat{\otimes} [\mathcal{E}_{\tilde{R}_{i+1}, V^{i+1}}] ... \hat{\otimes} [\mathcal{E}_{\tilde{R}_{i+m-1}, V^{i+m-1}}]$.

Now, by Proposition 4.0.7, we have, for every $0 \leq i \leq k_n$ and u in $K^{-1}(V)$, that $\varepsilon_{k_n}(u) \hat{\otimes}_1 [\mathcal{E}_{R_n, V_n}] = \iota(\varepsilon_i(u) \hat{\otimes}_1 \Psi)$. So, if $\varepsilon_{k_n}(u)$ is in the kernel of γ_n, then $\iota(\varepsilon_i(u) - \varepsilon_{i-1}(u) \hat{\otimes} \Psi)$ $= 0$ for all $1 \leq i \leq k_n$. It is easy to see from Lemma 6.0.3 that, for every $0 \leq i \leq k_n$, the kernel of $\iota : K^{-1}(V^i) \to K^{-1}(V_n)$ is precisely $G(V_n, \bigcup_{j=0}^{k_n - i} R^{-j}(C_n))$. So, we may conclude that $\varepsilon_i(u) - \varepsilon_{i-1}(u) \hat{\otimes} \Psi$ is in $G(V_n, \bigcup_{j=0}^{k_n - i} R^{-j}(C_n))$.

We now prove that $\varepsilon_i(u) - \varepsilon_0(u) \hat{\otimes} \Psi^i$ is in $G(V_n, \bigcup_{j=0}^{k_n - i} R^{-j}(C_n))$ for all $1 \leq i \leq k_n$. From directly above, we know this is true for $i = 1$. Suppose we know it to be true for $i \leq k_n - 1$. By Lemma 6.0.5, we then have that $0 = (\varepsilon_i(u) - \varepsilon_0(u) \hat{\otimes} \Psi^i) \hat{\otimes} \Psi = \varepsilon_i(u) \hat{\otimes} \Psi - \varepsilon_0(u) \hat{\otimes} \Psi^{i+1}$. From directly above, we then know that $(\varepsilon_{i+1}(u) - \varepsilon_i(u) \hat{\otimes} \Psi) + (\varepsilon_i(u) \hat{\otimes} \Psi - \varepsilon_0(u) \hat{\otimes} \Psi^{i+1}) = \varepsilon_{i+1}(u) - \varepsilon_0(u) \hat{\otimes} \Psi^{i+1}$ is in $G(V_n, \bigcup_{j=0}^{k_n - i - 1} R^{-j}(C_n))$. By induction, the result holds. In particular, $\varepsilon_{k_n}(u) - \varepsilon_0(u) \hat{\otimes} \Psi^{k_n}$ is in $G(V_n, C_n)$.

We will now show for any g in $K^{-1}(V^{k_n})$, there is b in $K^{-1}(B \setminus C_n)$ such that $g \hat{\otimes} \Psi^{k_n} = i_*(b)$, where j is the inclusion map.

Denote $X_i = R^i(V_n) \setminus \bigcup_{j=0}^{k_n - i} R^{-j}(C_n)$ for $0 \leq i \leq k_n$, $F_i = R : X_i \to X_{i+1}$ for $0 \leq i \leq k_n - 1$, and $\Delta^i = [\mathcal{E}_{F_0, X_0}] \hat{\otimes} [\mathcal{E}_{F_1, X_1}] \hat{\otimes} ... \hat{\otimes} [\mathcal{E}_{F_{i-1}, X_{i-1}}]$ for $1 \leq i \leq k_n$. We claim $j_* \Delta^m = \Psi^m$, where $j = i : C_0(X_m) \to C_0(V^m)$ for all $1 \leq m \leq k_n$.

We prove this by induction. Note that $X_0 = V^0$. By Proposition 4.0.7, $\Psi = j_* \Delta$. Now suppose the claim is true for $m \leq k_n - 1$. Then, $\Psi^{m+1} = (j_* \Delta^m) \hat{\otimes} [\mathcal{E}_{\tilde{R}_m, V^m}]$. By the properties of the Kasparov product (see Section 2.4), $(j_* \Delta^m) \hat{\otimes} [\mathcal{E}_{\tilde{R}_m, V^m}] = \Delta^m \hat{\otimes} [j^* \mathcal{E}_{\tilde{R}_m, V^m}]$. Since $R(X_m) \subseteq X_{m+1} \subseteq V^{m+1}$, by Proposition 4.0.7, we have that $j^*[\mathcal{E}_{\tilde{R}_m, V^m}] = j_*[\mathcal{E}_{F_m, X_m}]$. Therefore $\Psi^{m+1} = \Delta^m \hat{\otimes} j_* \Delta = j_* \Delta^{m+1}$. By induction, we have proven the claim.

In particular, $\Psi^{k_n} = j_* \Delta^{k_n}$ for $j = i : C_0(X_{k_n}) \to C_0(V_n \setminus C_n)$. Since $X_{k_n} \subseteq A_P \setminus$

$C_n \subseteq B \setminus C_n$, we can factor j_* as $j_* = i_* \circ \tilde{i}_*$, for $\tilde{i}_* = K^{-1}(X_{k_n}) \to K^{-1}(B \setminus C_n)$ and $i_* = K^{-1}(B \setminus C_n) \to K^{-1}(V_n \setminus C_n)$. Therefore, for any g in $K^{-1}(V^{k_n})$, $b = g \hat{\otimes} \tilde{i}_* \Delta^{k_n}$ satisfies $g \hat{\otimes} \Psi^{k_n} = i_*(b)$.

So far, we have shown for every u in $K^{-1}(V)$ such that $\varepsilon_{k_n}(u)$ is in $\ker(\gamma_n)$, there is b in $K^{-1}(B \setminus C_n)$ such that $\varepsilon_{k_n}(u) - i_*(b)$ is in $G(V_n, C_n)$. We must now separate the analysis into two cases, the case when P is a Herman cycle, and the case when it isn't.

Suppose P is not a Herman cycle. Then, for $i_* : K^{-1}(B \setminus C_n) \to K^{-1}(V_n \setminus C_n)$, we can factor $\iota \circ i_*$ as $\iota \circ i_* = (i_1)_*(i_2)_*$, where $i_2 = i : C_0(B \setminus C_n) \to C_0(B)$. Since in this case B is the disjoint union of simply connected sets, $K^{-1}(B) = 0$. Hence, $\iota(i_*(b)) = 0$, which implies $\iota(\varepsilon_{k_n}(u)) = 0$. By Lemma 6.0.3, $\iota \circ \varepsilon_{k_n} = i : K^{-1}(V) \to K^{-1}(V_n)$ is injective, and therefore $u = 0$. So, when P is not a Herman cycle, the kernel of γ_n is equal to $G(V_n, C_n)$.

Now, suppose P is a Herman cycle. Then $B \setminus C_n = U_P$. Moreover, U_P does not intersect $D = \bigcup_{i=0}^{k_n} F^{-i}(C_n)$ (otherwise an irrational rotation would contain a critical point), and so by (1) of Lemma 6.0.3, $U_P \subseteq V$. Therefore, $i_*(b) = \varepsilon_{k_n}(j_*(b))$, where $j : C_0(U_P) \to C_0(V)$ is the inclusion. So, we can write $\varepsilon_{k_n}(u) - i_*(b) = \varepsilon_{k_n}(u - j_*(b))$. Since the intersection of the image of ε_{k_n} and $G(V_n, C_n)$ is zero, it follows that $\varepsilon_{k_n}(u) = i_*(b)$. We can write $i_*(b) = \sum_{x \in P} a_x u_x$, for some a_x in $\mathbb{Z}$, x in P. Since $R_P = R : U_P \to U_P$ is a homeomorphism, the class of $[\mathcal{E}_{R_P, U_P}]$ is equal to the class of $(R_P^{-1})^*$. Hence, by the definition of u_x, for x in P, we have $u_x \hat{\otimes}_1 [\mathcal{E}_{R_n, V_n}] = u_{\sigma(x)}$ for all x in P. Therefore, $0 = \iota(\varepsilon_{k_n}(u)) - \varepsilon_{k_n}(u) \hat{\otimes}_1 [\mathcal{E}_{R_n, V_n}] = \sum_{x \in P} (a_x - a_{\sigma(x)}) u_x$. Hence, $a_x = a_y =: a$ for all x, y in P. Therefore, the kernel of γ_n in the Herman cycle case is equal to $G(V_n, C_n) + \mathbb{Z} \cdot u_P$.

We will now determine the co-kernel of γ_n.

First, note that for any $0 \leq i \leq k_n - 1$ and g in $K^{-1}(V^i)$, we have $\iota(g \otimes_i \Psi) = j_*(g) \hat{\otimes}_1 [\mathcal{E}_{R_n, V_n}]$, where $j : C_0(V^i) \to C_0(V_n \setminus C_n)$ is the inclusion. Therefore, for any $0 \leq i \leq k_n - 1$ and g in $K^{-1}(V^i)$, $\iota(g - g \hat{\otimes} \Psi)$ is in the image of γ_n.

Let v be in $K^{-1}(V_n)$. Then by Lemma 6.0.3, there is u in $K^{-1}(V)$ such that $\iota(\varepsilon_{k_n}(u)) = v$. By the above note, $\iota(\varepsilon_{k_n - i}(u) \hat{\otimes} \Psi^i - \varepsilon_{k_n-(i+1)}(u) \Psi^{(i+1)})$ is in the image of γ_n for all $0 \leq i \leq k_n - 1$. Therefore, v is equal to $\iota(\varepsilon_{k_n}(u)) + \sum_{i=0}^{k_n-1} \iota(\varepsilon_{k_n-(i+1)}(u) \Psi^{(i+1)} - \varepsilon_{k_n - i} \hat{\otimes} \Psi^i) = \iota(\varepsilon_0(u) \hat{\otimes} \Psi^{k_n})$ modulo the image of γ_n.

We have already shown while describing the kernel that $\iota(\varepsilon_0(u) \hat{\otimes} \Psi^{k_n}) = 0$ if P is not a Herman cycle. Therefore, in this this case, the co-kernel of γ_n is zero.

If P is a Herman cycle, then $\iota(\varepsilon_0(u) \hat{\otimes} \Psi^{k_n}) = \sum_{x \in P} a_x u_x$, for some a_x in $\mathbb{Z}$, x in P. Since $\gamma_n(u_x) = u_x - u_{\sigma(x)}$ for all x in P, it follows that $\sum_{x \in P} a_x u_x$ (and hence v) is equivalent to $a \cdot u_{x_P}$ modulo the image of γ_n, where $a := \sum_{x \in P} a_x$.

We now show that for any a in $\mathbb{Z}$, $a \cdot u_{x_P}$ is not in the image of γ_n, which will complete the proof of this Proposition.

First, assume u is an element of $K^{-1}(V \setminus U_P)$ such that $\iota(\varepsilon_i(u) - \varepsilon_{i-1}(u)\hat{\otimes}\Psi) = w$ for some w in $K^{-1}(U_P)$. Therefore, for every $0 \leq i \leq k_n - 1$, there is h_i in $G(V_n, \bigcup_{j=0}^{i} R^{-j}(C_n))$ such that $\varepsilon_{k_n-i}(u) - \varepsilon_{k_n-(i+1)}(u)\hat{\otimes}\Psi + h_i = w$. Since $K^{-1}(U_P)$ is invariant under the action of Ψ, it follows that $\varepsilon_{k_n}(u) - \varepsilon_0(u)\hat{\otimes}\Psi^{k_n} + h_0 = \sum_{i=0}^{k_n-1} \varepsilon_{k_n-i}(u)\hat{\otimes}\Psi^i - \varepsilon_{k_n-(i+1)}(u)\Psi^{(i+1)} = w'$ for some w' in $K^{-1}(U_P)$. $\varepsilon_0(u)\hat{\otimes}\Psi^{k_n}$ is also in $K^{-1}(U_P)$, and therefore $\varepsilon_{k_n}(u)$ is in $(K^{-1}(U_P) + G(V_n, C_n)) \cap \varepsilon_{k_n}(K^{-1}(V \setminus U_P)) = \{0\}$. Hence, $u = 0$ whenever u is in $K^{-1}(V \setminus U_P)$ and $\iota(\varepsilon_i(u) - \varepsilon_{i-1}(u)\hat{\otimes}\Psi)$ is in $K^{-1}(U_P)$.

This implies that if $\iota(g) - g\hat{\otimes}_1[\mathcal{E}_{R_n,V_n}] = a \cdot u_{x_P}$ for some g in $K^{-1}(V_n \setminus C_n)$ and a in $\mathbb{Z}$, then $\iota(g) - g\hat{\otimes}_1[\mathcal{E}_{R_n,V_n}] = \sum_{x \in P}(a_x - a_{\sigma(x)})u_x$ for some a_x in $\mathbb{Z}$, x in P. Therefore, $a = \sum_{x \in P}(a_x - a_{\sigma(x)}) = 0$. □

We can now compute the K-theory of R acting on F_R.

Theorem 6.0.6. *Let R be a rational function. Denote by $\mathcal{F}_R$ the set of Fatou cycles for R and $\mathcal{H}_R$ the set of Herman cycles for R. Let $G(F_R, C_{R,F_R}) \subseteq K^{-1}(F_R \setminus C_{R,F_R})$ be the group in Lemma 6.0.3 applied to the case $U = F_R$, $D = C_{R,F_R}$.*

Let $\delta_{PV} : K_1(\mathcal{O}_{R,F_R}) \to K^0(F_R \setminus C_{R,F_R})$ and $exp_{PV} : K_0(\mathcal{O}_{R,F_R}) \to K^{-1}(F_R \setminus C_{R,F_R})$ be the same-labelled maps appearing in the Pimsner-Voiculescu 6-term exact sequence for $R : F_R \to F_R$. Then, we have short exact sequences

$$0 \longrightarrow \bigoplus_{P \in \mathcal{F}_R} \mathbb{Z}[f_{x_P}] \xrightarrow{i_*} K_0(\mathcal{O}_{R,F_R}) \xrightarrow{exp_{PV}} G(F_R, C_{R,F_R}) \oplus \bigoplus_{Q \in \mathcal{H}_R} \mathbb{Z} \cdot u_Q \longrightarrow 0$$

$$0 \longrightarrow \bigoplus_{Q \in \mathcal{H}_R} \mathbb{Z} \cdot u_{x_P} \xrightarrow{i_*} K_1(\mathcal{O}_{R,F_R}) \xrightarrow{\delta_{PV}} \bigoplus_{P \in \mathcal{F}_R} \mathbb{Z}[e_P] \longrightarrow 0,$$

where x_P is a choice of an element in the cycle P, for all P in $\mathcal{F}_R$ (or $\mathcal{H}_R$). Hence, $K_0(\mathcal{O}_{R,F_R}) \simeq \mathbb{Z}^{|\mathcal{F}_R|+|\mathcal{H}_R|+|C_{R,F_R}|}$ and $K_1(\mathcal{O}_{R,F_R}) \simeq \mathbb{Z}^{|\mathcal{F}_R|+|\mathcal{H}_R|}$.

Proof. By Corollary 6.0.2 and Proposition 6.0.4, we can fill in the Pimsner-Voiculescu 6-term exact sequence as follows:

$$\begin{array}{ccccc}
\bigoplus_{P \in \mathcal{F}_R} \mathbb{Z}[e_P] & \xrightarrow{0} & \bigoplus_{P \in \mathcal{F}_R} \mathbb{Z}[f_{x_P}] & \xrightarrow{i_*} & K_0(\mathcal{O}_{R,F_R}) \\
\delta \uparrow & & & & \downarrow \exp \\
K_1(\mathcal{O}_{R,F_R}) & \xleftarrow[i_*]{} & \bigoplus_{Q \in \mathcal{H}_R} \mathbb{Z} \cdot u_{x_Q} & \xleftarrow[0]{} & G(F_R, C_{R,F_R}) \oplus \bigoplus_{Q \in \mathcal{H}_R} \mathbb{Z} \cdot u_Q
\end{array}$$

Exactness of the above diagram concludes the proof that we have short exact sequences as claimed.

The 2nd paragraph of Chapter 7 implies $G(F_R, C_{R,F_R}) \simeq \mathbb{Z}^{|C_{R,F_R}|}$, and so both the short exact sequences above are split exact (this forward reference won't cause a circular argument).

$\square$

Chapter 7

The K-theory of a rational function acting on its Julia set

In this section we compute the kernel and co-kernel of $\iota - \hat{\otimes}[\mathcal{E}_{R,J_R}]$ (in both degrees), as well as some related groups. As a Corollary, we will have determined the K-theory of a rational function acting on its Julia set. First, we orient some K^{-1} groups.

Let W be a union of pairwise disjoint open simply connected proper sets of $\hat{\mathbb{C}}$, and $D \subseteq W$ a finite set. By Corollary 4.1.12, $i_* : K^0(W \setminus D) \to K^0(W)$ is an isomorphism. Since W is a disjoint union of simply connected open sets, we have that $K^{-1}(W) = 0$. These two facts, imply the exponential map $\exp : K^0(D) \to K^{-1}(W \setminus D)$ from the 6-term exact sequence of K-theory associated to

$$0 \longrightarrow C_0(W \setminus D) \longrightarrow C_0(W) \longrightarrow C(D) \longrightarrow 0$$

is an isomorphism. For each d in D, we will denote $\exp([1_d]) = v_d$. The free basis $\{v_d\}_{d \in D}$ for $K^{-1}(W \setminus D)$ will be our canonical choice of generators, or "orientation".

Similarly, the short exact sequence

$$0 \longrightarrow C_0(\hat{\mathbb{C}} \setminus D) \longrightarrow C_0(\hat{\mathbb{C}}) \longrightarrow C(D) \longrightarrow 0$$

gives a surjection $\exp : K^0(D) \to K^{-1}(\hat{\mathbb{C}})$. In this case Corollary 4.1.12 implies $\mathbb{Z}[1_{\hat{\mathbb{C}}}]$ maps isomorphically onto the co-kernel of $i_* : K^0(\hat{\mathbb{C}} \setminus D) \to K^0(\hat{\mathbb{C}})$, via the quotient map. Therefore, the kernel of exp is $i_*[1_{\hat{\mathbb{C}}}] = \sum_{d \in D}[1_d]$. Denote $\exp([1_d]) = v_d$, for d in D. $K^{-1}(\hat{\mathbb{C}} \setminus D)$ is then the group generated by $\{v_d\}_{d \in D}$ satisfying the relation $\sum_{d \in D} v_d = 0$. Therefore, $\{v_d\}_{d \in D \setminus \{d'\}}$ is a free basis for $K^{-1}(\hat{\mathbb{C}} \setminus D)$, for any d' in D, but we will prefer to work with the whole generating set modulo its relation.

These generators behave well with respect to inclusion in the following sense.

Proposition 7.0.1. *Let W_1, W_2 be unions of pairwise disjoint simply connected (not necessarily proper) open sets in $\hat{\mathbb{C}}$ such that $W_1 \subseteq W_2$, and suppose D, K, C are finite sets such that $D \subseteq W_1$, $K \subseteq W_1 \setminus D$, and $C \subseteq W_2 \setminus W_1$. Then, $i_* : K^{-1}(W_1 \setminus (K \cup D)) \to K^{-1}(W_2 \setminus (D \cup C))$ sends v_d to v_d, for all d in D, and sends v_k to 0, for all k in K.*

Proof. By naturality of exp, we have a commutative diagram

$$\begin{array}{ccc} K^0(K \cup D) & \xrightarrow{\quad\varphi\quad} & K^0(D \cup C) \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ K^{-1}(W_1 \setminus (K \cup D)) & \xrightarrow{i_*} & K^{-1}(W_2 \setminus (D \cup C)), \end{array}$$

where φ sends 1_d to 1_d, for all d in D, and 1_k to 0, for all k in K. □

For c_1 and c_2 in C_{R,J_R}, we will write $c_1 \sim c_2$ if $\{c_1, c_2\}$ is contained in a connected subset of J_R. Clearly $\sim$ is an equivalence relation. For c in C_{R,J_R}, we will denote its equivalence class by $[c]$ and the group element $\sum_{d \in C_{R,J_R}: d \sim c} v_d$ in $K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$ by $v_{[c]}$. The collection of distinct equivalence classes will be denoted $[C_{R,J_R}]$.

Proposition 7.0.2. *Let R be a rational function. The image of $i_* : K^{-1}(F_R \setminus C_{R,F_R}) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,J_R})$ is generated by $\{v_c\}_{c \in C_{R,F_R}}$ together with $\{v_{[c]}\}_{[c] \in [C_{R,J_R}]}$.*

Proof. By Lemma 6.0.3 in the case that $U = F_R$ and $D = C_{R,F_R}$ there are open subsets W and V of F_R such that W is a disjoint union of simply connnected open sets containing C_{R,F_R}, V is disjoint from C_{R,F_R}, and the mappings $i_* + i_* : K^{-1}(W \setminus C_{R,F_R}) \oplus K^{-1}(V) \to K^{-1}(F_R \setminus C_{R,F_R})$, $i_* : K^{-1}(V) \to K^{-1}(F_R)$ are isomorphisms. Therefore, the image of $i_* : K^{-1}(F_R \setminus C_{R,F_R}) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$ is equal to the image of $i_* : K^{-1}(W \setminus C_{R,F_R}) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$ plus the image of $i_* : K^{-1}(V) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$.

By Proposition 7.0.1, the image of $i_* : K^{-1}(W \setminus C_{R,F_R}) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$ is generated by $\{v_c\}_{c \in C_{R,F_R}}$.

We have a commutative diagram

$$\begin{array}{ccc} K^{-1}(V) & \xrightarrow{i_*} & K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle j_*} \\ K^{-1}(F_R) & \xrightarrow{i_*} & K^{-1}(\hat{\mathbb{C}} \setminus C_{R,J_R}) \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ C(J_R, \mathbb{Z}) & \xrightarrow{\quad r\quad} & C(C_{R,J_R}, \mathbb{Z}), \end{array}$$

where the map labelled r is the restriction map, and the maps labelled i_*, j_* are induced from the inclusions.

Since $K^{-1}(\hat{\mathbb{C}}) = 0$, exactness implies $\exp : C(J_R, \mathbb{Z}) \to K^{-1}(F_R)$ is surjective. Therefore, commutativity of the above diagram implies the image of $i_* : K^{-1}(F_R) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,J_R})$ is equal to the image of $\exp \circ r$.

Functions in $C(J_R, \mathbb{Z})$ must be constant on connected subsets of J_R, so the image of r is generated by the elements $\{1_{[c]} := \sum_{d \in [c]} 1_d\}_{[c] \in [C_{R,J_R}]}$. By definition, $v_{[c]} = \exp(1_{[c]})$. Therefore, the image of $i_* : K^{-1}(F_R) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,J_R})$ is generated by $\{v_{[c]}\}_{[c] \in C_{R,J_R}}$.

$i_* : K^{-1}(V) \to K^{-1}(F_R)$ is an isomorphism, so commutativity of the diagram implies the image of $j_* \circ i_*$ is generated by $\{v_{[c]}\}_{[c] \in C_{R,J_R}}$. By Proposition 7.0.1, the kernel of j_* is generated by $\{v_c\}_{c \in C_{R,F_R}}$, and $j_*(v_{[c]}) = v_{[c]}$ for all $[c]$ in $[C_{R,J_R}]$. Therefore, the image of $i_* : K^{-1}(W \setminus C_{R,F_R}) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$ plus the image of $i_* : K^{-1}(V) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$ is generated by $\{v_c\}_{c \in C_{R,F_R}}$ and $\{v_{[c]}\}_{[c] \in [C_{R,J_R}]}$. □

We will now compute the kernel and co-kernel of $\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]$ acting on K^{-1}.

Proposition 7.0.3. *Let R be a rational function of degree $d > 1$. Let $c_{R,J}$ be the size of C_{R,J_R}, $k_{R,J}$ the size of $[C_{R,J_R}]$, h_R the number of Herman cycles, f_R the number of Fatou cycles, and ω_R the greatest common divisor of their cycle lengths.*

- *If $J_R = \hat{\mathbb{C}}$, the mapping $\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}] : K^{-1}(J_R \setminus C_{R,J_R}) \to K^{-1}(J_R)$ has kernel isomorphic to $\mathbb{Z}^{c_{R,J}-1}$ and co-kernel equal to 0.*

- *If $J_R \neq \hat{\mathbb{C}}$, the mapping $\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}] : K^{-1}(J_R \setminus C_{R,J_R}) \to K^{-1}(J_R)$ has kernel isomorphic to $\mathbb{Z}^{f_R + c_{R,J} - k_{R,J} - 1}$ and co-kernel isomorphic to $\mathbb{Z}/\omega_R\mathbb{Z} \oplus \mathbb{Z}^{f_R - 1}$.*

Proof. First, assume $J_R = \hat{\mathbb{C}}$. Then $K^{-1}(J_R) = 0$, so $\ker(\iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}]) = K^{-1}(\hat{\mathbb{C}} \setminus C_{R,J_R}) \simeq \mathbb{Z}^{c_{R,J}-1}$ and $\text{co-ker}(\iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}]) = 0$.

Now, assume $J_R \neq \hat{\mathbb{C}}$. By Corollary 3.0.10, we have a commutative diagram

$$\begin{array}{ccc} K^{-1}(J_R \setminus C_{R,J}) & \xrightarrow{\exp} & K^0(F_R \setminus C_{R,F_R}) \\ \big\downarrow{\scriptstyle \iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]} & & \big\downarrow{\scriptstyle \iota - \hat{\otimes}_0[\mathcal{E}_{R,F_R}]} \\ K^{-1}(J_R) & \xrightarrow{\exp} & K^0(F_R). \end{array}$$

The bottom horizontal map is injective because $K^{-1}(\hat{\mathbb{C}}) = 0$.

By Corollary 3.0.10, we have a commutative diagram

$$\begin{array}{ccc} K^0(F_R \setminus C_{R,F_R}) & \xrightarrow{i_*} & K^0(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}) \\ \downarrow{\scriptstyle \iota - \hat{\otimes}_0[\mathcal{E}_{R,F_R}]} & & \downarrow{\scriptstyle \iota - \hat{\otimes}_0[\mathcal{E}_{R,\hat{\mathbb{C}}}]} \\ K^0(F_R) & \xrightarrow{i_*} & K^0(\hat{\mathbb{C}}). \end{array}$$

The top horizontal map is surjective by Corollary 4.1.12. Putting these two diagrams together yields a commutative diagram

$$\begin{array}{ccccccccc} & & K^{-1}(J_R \setminus C_{R,J_R}) & \longrightarrow & K^0(F_R \setminus C_{R,F_R}) & \xrightarrow{i_*} & K^0(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]} & & \downarrow{\scriptstyle \iota - \hat{\otimes}_0[\mathcal{E}_{R,F_R}]} & & \downarrow{\scriptstyle \iota - \hat{\otimes}_0[\mathcal{E}_{R,\hat{\mathbb{C}}}]} & & \\ 0 & \longrightarrow & K^{-1}(J_R) & \longrightarrow & K^0(F_R) & \xrightarrow{i_*} & K^0(\hat{\mathbb{C}}) & & \end{array}$$

with exact rows. Applying the Snake Lemma to this diagram yields a boundary map $\partial : \ker(\iota - \hat{\otimes}_0[\mathcal{E}_{R,\hat{\mathbb{C}}}]) \to \text{co-ker}(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}])$ and an exact sequence

$$\begin{array}{ccccc} \ker(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]) & \xrightarrow{\delta} & \ker(\iota - \hat{\otimes}_0[\mathcal{E}_{R,F_R}]) & \xrightarrow{i_*} & \ker(\iota - \hat{\otimes}_0[\mathcal{E}_{R,\hat{\mathbb{C}}}]) \\ & & & & \downarrow{\scriptstyle \partial} \\ \text{co-ker}(\iota - \hat{\otimes}_0[\mathcal{E}_{R,\hat{\mathbb{C}}}]) & \xleftarrow[\tilde{i_*}]{} & \text{co-ker}(\iota - \hat{\otimes}_0[\mathcal{E}_{R,F_R}]) & \xleftarrow[\tilde{\delta}]{} & \text{co-ker}(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]), \end{array}$$

where $\tilde{\delta}$ and $\tilde{i_*}$ are the descent maps of δ and i_*, respectively.

By Proposition 5.0.2, $\text{co-ker}(\iota - \hat{\otimes}_0[\mathcal{E}_{R,\hat{\mathbb{C}}}]) = K^0(\hat{\mathbb{C}})$ and $\ker(\iota - \hat{\otimes}_0[\mathcal{E}_{R,\hat{\mathbb{C}}}]) = K^0(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$. By Corollary 6.0.2, $\text{co-ker}(\iota - \hat{\otimes}_0[\mathcal{E}_{R,F_R}]) = \oplus_{P \in \mathcal{F}_R} \mathbb{Z}[f_{x_P}]$ and $\ker(\iota - \hat{\otimes}_0[\mathcal{E}_{R,F_R}]) = \oplus_{P \in \mathcal{F}_R} \mathbb{Z}[e_P]$, respectively. By Corollary 4.1.12, $\tilde{i}_*(f_{x_P}) = \beta_{\hat{\mathbb{C}}}$ and $i_*(e_P) = |P|\beta_{\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}}$ for every P in $\mathcal{F}_R$, so the kernel of $i_* : \ker(\iota - \hat{\otimes}_0[\mathcal{E}_{R,F_R}]) \to \ker(\iota - \hat{\otimes}_0[\mathcal{E}_{R,\hat{\mathbb{C}}}])$ is isomorphic to $\mathbb{Z}^{f_R - 1}$, with an image equal to $\sum_{P \in \mathcal{F}_R} |P|\mathbb{Z} = \omega_R \mathbb{Z}$. Therefore, the co-kernel of i_* is isomorphic to $\mathbb{Z}/\omega_R\mathbb{Z}$. The kernel of $\tilde{i_*} : \text{co-ker}(\iota - \hat{\otimes}_0[\mathcal{E}_{R,F_R}]) \to \text{co-ker}(\iota - \hat{\otimes}_0[\mathcal{E}_{R,\hat{\mathbb{C}}}])$ is also isomorphic to $\mathbb{Z}^{f_R - 1}$ by the same reasoning.

Exactness of the above diagram and the above computations imply we have exact sequences

$$0 \longrightarrow G \longrightarrow \ker(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]) \xrightarrow{\delta} \mathbb{Z}^{f_R - 1} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}/\omega_R\mathbb{Z} \longrightarrow \text{co-ker}(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]) \xrightarrow{\tilde{\delta}} \mathbb{Z}^{f_R - 1} \longrightarrow 0,$$

where G is the kernel of $\delta : \ker(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]) \to \ker(\iota - \hat{\otimes}_0[\mathcal{E}_{R,F_R}])$. Both sequences split, so $\ker(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]) \simeq G \oplus \mathbb{Z}^{f_R-1}$ and co-ker$(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]) \simeq \mathbb{Z}/\omega_R\mathbb{Z} \oplus \mathbb{Z}^{f_R-1}$.

We now compute G. By Corollary 3.0.10, the diagram

$$\begin{array}{ccc} K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}) & \xrightarrow{r_*} & K^{-1}(J_R \setminus C_{R,J_R}) \\ \downarrow{\scriptstyle \iota - \hat{\otimes}_1[\mathcal{E}_{R,\hat{\mathbb{C}}}]} & & \downarrow{\scriptstyle \iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]} \\ K^{-1}(\hat{\mathbb{C}}) & \xrightarrow{r_*} & K^{-1}(J_R) \end{array}$$

commutes. Therefore, $r_*(K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})) = r_*(\ker(\iota - \hat{\otimes}_1[\mathcal{E}_{R,\hat{\mathbb{C}}}])) \subseteq \ker(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}])$. By exactness, G is equal to $r_*(K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})) \cap \ker(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]) = r_*(K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}))$.

Exactness implies the kernel of r_* is equal the image of $i_* : K^{-1}(F_R \setminus C_{R,F_R}) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$, which, by Proposition 7.0.2, is generated by $\{v_c\}_{c \in C_{R,F_R}}$ and $\{v_{[c]}\}_{[c] \in [C_{R,J_R}]}$. Therefore, $r_*(K^{-1}(\hat{\mathbb{C}}))$ is isomorphic to the group generated by $\{v_c\}_{c \in C_{R,J_R}}$ satisfying the relations $\sum_{d \sim c} v_d = 0$, for all c in C_{R,J_R}. Hence, $G \simeq \mathbb{Z}^{c_{R,J} - k_{R,J}}$. Therefore, $\ker(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]) \simeq \mathbb{Z}^{c_{R,J} + f_R - k_{R,J} - 1}$.

□

Recall from Chapter 6 the notion of an orientation for a Herman cycle Q. We now compute, for any x in Q, $i_* : K^{-1}(U_x) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$ using the descriptions of the domains and co-domains provided by their orientations.

Lemma 7.0.4. *Let R be a rational function with an oriented Herman cycle Q, and let x be in Q. Let D be a finite set not intersecting U_x, and let D_x^+ be the points in D contained in the connected component of $\hat{\mathbb{C}} \setminus U_x$ containing $\partial^+ U_x$. Then, $i_* : K^{-1}(U_x) \to K^{-1}(\hat{\mathbb{C}} \setminus D)$ sends u_x to $\sum_{d \in D_x^+} v_d$.*

Proof. Let U_x^+ be the union of U_x with the connected component of $\hat{\mathbb{C}} \setminus U_x$ containing $\partial^+ U_x$. Since the complement of U_x^+ is connected and closed, U_x^+ is a simply connected open set such that $U_x^+ \cap D = D_x^+$.

The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(U_x^+ \setminus D_x^+) & \longrightarrow & C_0(U_x^+) & \longrightarrow & C(D_x^+) & \longrightarrow & 0 \\ & & \uparrow & & \| & & \uparrow & & \\ 0 & \longrightarrow & C_0(U_x) & \longrightarrow & C_0(U_x^+) & \longrightarrow & C(U_x^+ \setminus U_x) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_0(U_x) & \longrightarrow & C_0(U_x \bigsqcup \partial^+ U_x) & \longrightarrow & C(\partial^+ U_x) & \longrightarrow & 0 \end{array}$$

commutes, and has exact rows. Therefore, by naturality of exp, we have a commutative diagram

$$\begin{array}{ccccc} K^0(\partial^+ U_x) & \longleftarrow & K^0(U_x^+ \setminus U_x) & \longrightarrow & K^0(D_x^+) \\ \downarrow \exp & & \downarrow \exp & & \downarrow \exp \\ K^{-1}(U_x) & = & K^{-1}(U_x) & \xrightarrow{i_*^+} & K^{-1}(U_x^+ \setminus D_x^+), \end{array}$$

where $i^+ : C_0(U_x) \to C_0(U_x^+ \setminus D_x^+)$ is the inclusion. Hence, $i_*^+(u_x) = i_*^+(\exp([1_{U_x^+ \setminus U_x}])) = \sum_{d \in D_x^+} \exp([1_d]) = \sum_{d \in D_x^+} v_d$.

Let $j_* : K^{-1}(U_x^+ \setminus D_x^+) \to K^{-1}(\hat{\mathbb{C}} \setminus D)$ be the map induced from the inclusion. From the above calculation and Proposition 7.0.1, we have $i_*(u_x) = j_*(i_*^+(u_x)) = \sum_{d \in D_x^+} v_d$. □

For Q in $\mathcal{H}_R$ and x in Q, let $(\hat{\mathbb{C}} \setminus U_x)^+$, $(\hat{\mathbb{C}} \setminus U_x)^-$ denote the connected component of $\hat{\mathbb{C}} \setminus U_x$ containing $\partial^+ U_x$, $\partial^- U_x$, respectively. Denote $J_x^\pm := (\hat{\mathbb{C}} \setminus U_x)^\pm \cap J_R$ and $\overline{U_x}^\pm := U_x \cup \partial^\pm U_x$.

Lemma 7.0.5. *For Q in $\mathcal{H}_R$ and x in Q, the homomorphism exp : $C(J_R, \mathbb{Z}) \to K^{-1}(F_R)$ satisfies exp$(1_{J_x^+}) = u_x$.*

Proof. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(F_R \setminus C_{R,F_R}) & \longrightarrow & C_0(\hat{\mathbb{C}} \setminus C_{R,F_R}) & \longrightarrow & C(J_R) & \longrightarrow & 0 \\ & & \downarrow r & & \downarrow r & & \downarrow r & & \\ 0 & \longrightarrow & C_0(U_x) & \longrightarrow & C_0(\overline{U_x}^+) & \longrightarrow & C(\partial^+ U_x) & \longrightarrow & 0 \end{array}$$

commutes and has exact rows, where the vertical maps are the restrictions. Naturality of exp then implies the diagram

$$\begin{array}{ccc} C(J_R, \mathbb{Z}) & \xrightarrow{\exp} & K^{-1}(F_R \setminus C_{R,F_R}) \\ \downarrow r & & \downarrow r_* \\ C(\partial^+ U_x, \mathbb{Z}) & \xrightarrow{\exp} & K^{-1}(U_x) \end{array}$$

commutes. We have, by definition, $\exp(1_{\partial^\pm U_x}) = \pm u_x$, so commutativity of the diagram implies $\exp(1_{J_x^\pm})$ satisfies $r_*(\exp(1_{J_x^\pm})) = \pm u_x$. Since U_x is a connected component of $F_R \setminus C_{R,F_R}$, r_* is the projection onto the direct summand $K^{-1}(U_x)$ of $K^{-1}(F_R \setminus C_{R,F_R})$; let's denote this projection q_{U_x}.

Now, let $U \neq U_x$ be a Fatou component, and q_U denote the projection of $K^{-1}(F_R \setminus C_{R,F_R})$ onto the direct summand $K^{-1}(U \setminus C_{R,F_R})$. To prove the lemma, it remains to show

$q_U(\exp(u_x)) = 0$. Denote $J_U = J_R \cup \{U \setminus C_{R,F_R}\}$ The diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(F_R \setminus C_{R,F_R}) & \longrightarrow & C_0(\hat{\mathbb{C}} \setminus C_{R,F_R}) & \longrightarrow & C(J_R) & \longrightarrow & 0 \\
 & & \downarrow r & & \downarrow r & & \| & & \\
0 & \longrightarrow & C_0(U \setminus C_{R,F_R}) & \longrightarrow & C_0(J_U) & \longrightarrow & C(J_R) & \longrightarrow & 0 \\
 & & \| & & \downarrow r & & \downarrow r & & \\
0 & \longrightarrow & C_0(U \setminus C_{R,F_R}) & \longrightarrow & C_0(\overline{U} \setminus C_{R,F_R}) & \longrightarrow & C(\partial U) & \longrightarrow & 0
\end{array}$$

commutes, and has exact rows, where the vertical maps labelled r are the restrictions. If we denote the exponential map of the middle and bottom row by $\exp_U$ and $\exp_{\partial U}$, respectively, then naturality of exp implies $q_U \circ \exp = \exp_U = \exp_{\partial U} \circ r$.

Either $\overline{U} \subseteq (\hat{\mathbb{C}} \setminus U_x)^+$ or $\overline{U} \subseteq (\hat{\mathbb{C}} \setminus U_x)^-$. In the first case, $\partial U \cap J_x^- = \emptyset$. Therefore, $r(1_{J_x^-}) = 0$ and hence $-q_U(\exp(1_{J_x^+})) = q_U(\exp(1_{J_x^-})) = \exp_{\partial U}(r(1_{J_x^-})) = 0$. In the second case, $r(1_{J_x^+}) = 0$, so that $q_U(\exp(1_{J_x^+})) = \exp_{\partial U}(r(1_{J_x^+})) = 0$. □

If Q is a oriented Herman cycle and d is a point not in U_Q, then we let $H_Q(d)$ be the number of x in Q for which d is in the connected component of $\hat{\mathbb{C}} \setminus U_x$ containing $\partial^+ U_x$. Lemma 7.0.4 implies that if D is a finite set not intersecting U_Q, then $i_* : K^{-1}(U_Q) \to K^{-1}(\hat{\mathbb{C}} \setminus D)$ sends u_Q to $\sum_{d \in D} H_Q(d) v_d$. The functions $\{H_Q\}_{Q \in \mathcal{H}_R}$ will also play a role in describing $\ker(\mathrm{id} - \Phi)$, as the following Corollary to the above lemma may suggest.

Corollary 7.0.6. *H_Q is locally constant on $\hat{\mathbb{C}} \setminus U_Q$, and $H_Q - \Phi(H_Q)$ is constant on J_R.*

Proof. $H_Q = \sum_{x \in Q} 1_{(\hat{\mathbb{C}} \setminus U_x)^+}$ and is therefore locally constant. We have $H_Q|_{J_R} = \sum_{x \in Q} 1_{J_x^+}$, so, by Lemma 7.0.5, we have $\exp(H_Q) = u_Q$, where $\exp = \exp : C(J_R, \mathbb{Z}) \to K^{-1}(F_R \setminus C_{R,F_R})$.

Therefore, Proposition 4.1.8 and 6.0.4 (respectively) imply $\exp(H_Q - \Phi(H_Q)) = \iota(u_Q) - u_Q \hat{\otimes}_1 [\mathcal{E}_{R,F_R}]$ and $\iota(u_Q) - u_Q \hat{\otimes}_1 [\mathcal{E}_{R,F_R}] = 0$. Since the kernel of $\exp : C(J_R, \mathbb{Z}) \to K^{-1}(F_R)$ equals $\mathbb{Z}[1_{J_R}]$, it follows that $H_Q - \Phi(H_Q)$ must be constant on J_R. □

Let $\alpha_R : \mathbb{Z}[\mathcal{H}_Q] \oplus \mathbb{Z} \to \mathbb{Z}$ be the homomorphism sending 1 to $1 - d$ and Q to $H_Q - \Phi(H_Q)$, for all Q in $\mathcal{H}_Q$. Denote the greatest common divisor of $\{H_Q - \Phi(H_Q)\}_{Q \in \mathcal{H}_R} \cup \{1 - d\}$ by a_R. We now compute the kernel and co-kernel of $\mathrm{id} - \Phi : C(J_R, \mathbb{Z}) \to C(J_R, \mathbb{Z})$

Proposition 7.0.7. *Let R be a rational function of degree $d > 1$. Then,*

(1) The homomorphism $\varphi_R : \mathbb{Z}^{\mathcal{H}_R} \oplus \mathbb{Z} \to C(J_R, \mathbb{Z})$ sending 1 to 1_{J_R} and Q to H_Q, for all Q in $\mathcal{H}_\mathcal{R}$, is an isomorphism onto $(id - \Phi)^{-1}(\mathbb{Z}[1_{J_R}])$. Hence, φ_R maps ker(α_R) isomorphically onto ker$(id - \Phi) \simeq \mathbb{Z}^{h_R}$.

(2) *The torsion degree of 1_{J_R} in co-ker($id-\Phi$) is a_R, and for any choice of x_Q in Q, for Q in $\mathcal{H}_R$, the homomorphism $\phi_R : \mathbb{Z}^{\mathcal{H}_R} \oplus \mathbb{Z}/a_R\mathbb{Z} \to$ co-ker($id-\Phi$) sending 1 to the image of 1_{J_R} in the co-kernel and Q to the image of $1_{J^+_{x_Q}}$, for all Q in $\mathcal{H}_R$ in the co-kernel is an isomorphism onto co-ker($id-\Phi$).*

Proof. First, assume $J_R = \hat{\mathbb{C}}$. Then, $C(J_R, \mathbb{Z}) = \mathbb{Z}1_{J_R}$ and $\mathcal{H}_R = \emptyset$. Hence, $(\mathrm{id} - \Phi)^{-1}(\mathbb{Z}[1_{J_R}]) = \mathbb{Z}[1_{J_R}]$, $a_R = 1-d$, and $\text{co-ker}(\mathrm{id}-\Phi) = \mathbb{Z}[1_{J_R}]/(1-d)\mathbb{Z}[1_{J_R}]$.

Now, assume $J_R \neq \hat{\mathbb{C}}$. $K^{-1}(\hat{\mathbb{C}}) = 0$, so the diagram

$$\begin{array}{ccc} K^{-1}(F_R \setminus C_{R,F_R}) & \xrightarrow{i_*} & K^{-1}(\hat{\mathbb{C}} \setminus C_{R,F_R}) \\ \big\downarrow{\scriptstyle \iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}]} & & \big\downarrow \\ K^{-1}(F_R) & \xrightarrow{i_*} & 0 \end{array}$$

automatically commutes. By Proposition 7.0.2, the subgroup $G(F_R, C_R)$ of $K^{-1}(F_R \setminus C_{R,F_R})$ has $\mathbb{Z}$-linear independent generators $\{v_c\}_{c \in C_{R,F_R}}$ such that $i_*(v_c) = v_c$, for all c in C_{R,F_R}. Therefore, $i_* : K^{-1}(F_R \setminus C_{R,F_R}) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,F_R})$ is surjective.

By Proposition 4.1.8, the diagram

$$\begin{array}{ccc} C(J_R, \mathbb{Z}) & \xrightarrow{\exp} & K^{-1}(F_R \setminus C_{R,F_R}) \\ \big\downarrow{\scriptstyle \mathrm{id}-\Phi} & & \big\downarrow{\scriptstyle \iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}]} \\ C(J_R, \mathbb{Z}) & \xrightarrow{\exp} & K^{-1}(F_R) \end{array}$$

commutes. The bottom map has kernel equal to $\mathbb{Z}[1_{J_R}]$. Hence, the diagram

$$\begin{array}{ccccccccc} & & C(J_R, \mathbb{Z}) & \xrightarrow{\exp} & K^{-1}(F_R \setminus C_{R,F_R}) & \xrightarrow{i_*} & K^{-1}(\hat{\mathbb{C}} \setminus C_{R,F_R}) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \widetilde{\mathrm{id}-\Phi}} & & \big\downarrow{\scriptstyle \iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}]} & & \big\downarrow & & \\ 0 & \longrightarrow & C(J_R, \mathbb{Z})/\mathbb{Z}[1_{J_R}] & \xrightarrow{\widetilde{\exp}} & K^{-1}(F_R) & \longrightarrow & 0 & & \end{array}$$

commutes, and has exact rows, where $\widetilde{\mathrm{id}-\Phi}$ and $\widetilde{\exp}$ denote the descent maps of $\mathrm{id}-\Phi$, exp, respectively.

By the Snake Lemma, there is a boundary map $\partial : K^{-1}(\hat{\mathbb{C}} \setminus C_{R,F_R}) \to \text{co-ker}(\mathrm{id}-\Phi)/\mathbb{Z}[1_{J_R}]$

making the sequence

$$\begin{array}{ccccc}(\mathrm{id}-\Phi)^{-1}(\mathbb{Z}[1_{J_R}]) & \xrightarrow{\exp} & \ker(\iota-\hat{\otimes}_1[\mathcal{E}_{R,F_R}]) & \xrightarrow{i_*} & K^{-1}(\hat{\mathbb{C}}\setminus C_{R,F_R}) \\ & & & & \downarrow \partial \\ 0 & \longleftarrow & \text{co-ker}(\iota-\hat{\otimes}_1[\mathcal{E}_{R,F_R}]) & \xleftarrow[\widetilde{\exp}]{} & \text{co-ker}(\mathrm{id}-\Phi)/\mathbb{Z}[1_{J_R}]\end{array}$$

exact.

By Proposition 6.0.4, $\ker(\iota-\hat{\otimes}_1[\mathcal{E}_{R,F_R}]) = G(F_R, C_{R,F_R})\oplus(\bigoplus_{Q\in\mathcal{H}_R}\mathbb{Z}[u_Q])$. By Proposition 7.0.4, $i_*(u_Q) = \sum_{c\in C_{R,F_R}} H_Q(c)v_c$, for all Q in $\mathcal{H}_R$. So, i_* is surjective with kernel freely generated by the elements $w_Q := u_Q - \sum_{c\in C_{R,F_R}} H_Q(c)v_c$, for all Q in $\mathcal{H}_R$, and $\sum_{c\in C_{R,F_R}} v_c$. Surjectivity of i_* implies, by exactness of the above diagram, the sequences

$$(\mathrm{id}-\Phi)^{-1}(\mathbb{Z}[1_{J_R}]) \xrightarrow{\exp} \mathbb{Z}[\textstyle\sum_{c\in C_{R,F_R}} v_c]\bigoplus_{Q\in\mathcal{H}_\mathcal{R}}\mathbb{Z}[w_Q] \longrightarrow 0$$

$$0 \longleftarrow \text{co-ker}(\iota-\hat{\otimes}_1[\mathcal{E}_{R,F_R}]) \xleftarrow[\tilde{\exp}]{} \text{co-ker}(\mathrm{id}-\Phi) \longleftarrow \mathbb{Z}[1_{J_R}] \qquad (*)$$

are exact.

We now prove (1). It suffices to show $\{H_Q\}_{Q\in\mathcal{H}_R}\cup\{1_{J_R}\}$ is a $\mathbb{Z}$-linear independent generating set for $(\mathrm{id}-\Phi)^{-1}(\mathbb{Z}[1_{J_R}])$.

First, assume $C_{R,F_R}=\emptyset$. In this case, $w_Q = u_Q$, for all Q in $\mathcal{H}_R$, and exactness implies exp has kernel equal to $\mathbb{Z}[1_{J_R}]$. From above, $\exp : (\mathrm{id}-\Phi)^{-1}(\mathbb{Z}[1_{J_R}]) \to \bigoplus_{Q\in\mathcal{H}_R}\mathbb{Z}[w_Q]$ is a surjection, and, by Lemma 7.0.5, $H_Q = \sum_{x\in Q} 1_{J_x^+}$ satisfies $\exp(H_Q) = \sum_{x\in Q} u_x = u_Q = w_Q$. Therefore, the functions $\{H_Q\}_{Q\in\mathcal{H}_R}\cup\{1_{J_R}\}$ generate $(\mathrm{id}-\Phi)^{-1}(\mathbb{Z}[1_{J_R}])$ and are $\mathbb{Z}$-linearly independent (since $\{u_Q\}_{Q\in\mathcal{H}_R}$ are).

Assume $C_{R,F_R}\neq\emptyset$. Exactness then implies exp is injective, so that $\exp : (\mathrm{id}-\Phi)^{-1}(\mathbb{Z}[1_{J_R}]) \to \bigoplus_{Q\in\mathcal{H}_R}\mathbb{Z}[w_Q]\oplus\mathbb{Z}[\sum_{c\in C_{R,F_R}} v_c]$ is an isomorphism.

Naturality implies the diagram

$$\begin{array}{ccc} C(J_R,\mathbb{Z}) & \xrightarrow{\exp} & K^{-1}(F_R\setminus C_{R,F_R}) \\ & \searrow^{\exp} & \downarrow i_* \\ & & K^{-1}(F_R)\end{array}$$

where $i : C_0(F_R\setminus C_{R,F_R}) \to C_0(F_R)$ is the inclusion. Lemma 6.0.3 implies $i_*(w_Q) = u_Q$ for all Q in $\mathcal{H}_\mathcal{R}$ and $i_*(\sum_{c\in C_{R,F_R}} v_c) = 0$. Therefore, $i_*\exp(H_Q) = u_Q$ implies $\exp(H_Q) = w_Q + a_Q(\sum_{c\in C_{R,F_R}} v_c)$, for some a_Q in $\mathbb{Z}$, for all Q in $\mathcal{H}_R$. It remains to show $\exp(1_{J_R}) =$

$-\sum_{c\in C_{R,F_R}} v_c$.

The diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(F_R\setminus C_{R,F_R}) & \longrightarrow & C_0(\hat{\mathbb{C}}\setminus C_{R,F_R}) & \longrightarrow & C(J_R) & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0(F_R\setminus C_{R,F_R}) & \longrightarrow & C(\hat{\mathbb{C}}) & \longrightarrow & C(J_R\cup C_{R,F_R}) & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C_0(F_R\setminus C_{R,F_R}) & \longrightarrow & C_0(F_R) & \longrightarrow & C(C_{R,F_R}) & \longrightarrow & 0
\end{array}$$

commutes, and has exact rows. Naturality of exp then implies the diagram

$$\begin{array}{ccccc}
C(J_R,\mathbb{Z}) & \longrightarrow & C(J_R\cup C_{R,F_R},\mathbb{Z}) & \longleftarrow & C(C_{R,F_R},\mathbb{Z})\\
 & \searrow^{\exp} & \downarrow{\scriptstyle \exp} & \swarrow_{\exp} & \\
 & & K^{-1}(F_R\setminus C_{R,F_R}) & &
\end{array}$$

commutes. By exactness, $\exp(1_{J_R}+1_{C_{R,F_R}})=0$, so that $\exp(1_{J_R})=-\exp(1_{C_{R,F_R}})=-\sum_{c\in C_{R,F_R}} v_c$. This finishes the proof of (1).

Consequently, the subgroup of $C(J_R,\mathbb{Z})$ generated by $\{H_Q\}_{Q\in\mathcal{H}_\mathcal{R}}\cup\{1_{J_R}\}$ surjects onto $\mathrm{im}(\mathrm{id}-\Phi)\cap\mathbb{Z}[1_{J_R}]$ via $\mathrm{id}-\Phi$. Therefore, the torsion degree of 1_{J_R} in co-ker$(\mathrm{id}-\Phi)$ is equal to a_R, the greatest common divisor of $\{H_Q-\Phi(H_Q)\}_{Q\in\mathcal{H}_\mathcal{R}}\cup\{1-d\}$.

We now prove (2). By Proposition 6.0.4, co-ker$(\iota-\hat{\otimes}_1[\mathcal{E}_{R,F_R}])\simeq\mathbb{Z}^{h_R}$, and the elements $\{u_{x_Q}\}_{Q\in\mathcal{H}_R}$ under the quotient map form a $\mathbb{Z}$-linear independent generating set. exactness of the bottom sequence in diagram (*) then implies we have a short exact sequence

$$0\longleftarrow \sum_{Q\in\mathcal{H}_R}\mathbb{Z}[u_{x_Q}]\xleftarrow{\exp}\text{co-ker}(\mathrm{id}-\Phi)\longleftarrow\mathbb{Z}[1_{J_R}]/a_R\mathbb{Z}[1_{J_R}]\longleftarrow 0.$$

This short exact sequence splits and, by Lemma 7.0.5, $\exp(1_{J^+_{x_Q}})=u_{x_Q}$ for all Q in $\mathcal{H}_R$. Therefore, ϕ_R is equal to the direct sum of the inclusion of the kernel of $\widetilde{\exp}$ and a splitting, and is hence an isomorphism. □

For every Q in $\mathcal{H}_R$, Corollary 7.0.6 implies $H_Q(c)=H_Q(d)$ for any c,d in C_{R,J_R} such that $c\sim d$. Denote $H_Q([c]):=H_Q(c)$, for all $[c]$ in $[C_{R,J_R}]$.

We let $[H_R]$ be the following matrix with rows indexed by elements in $[C_{R,J_R}]\cup\{u\}$ and columns indexed by $\mathcal{H}_R\cup\{u\}$

- $([H_R])_{[c],Q}=H_Q(c)$ for all $[c]$ in $[C_{R,J_R}]$ and Q in $\mathcal{H}_R$,
- $([H_R])_{[c],u}=1$, for all $[c]$ in $[C_{R,J_R}]$,

- $([H_R])_{u,Q} = \Phi(H_Q) - H_Q$, for all Q in $\mathcal{H}_R$, and
- $([H_R])_{u,u} = \deg(R) - 1$.

We will write $(\mathrm{id} - \Phi)|_{C_0(J_R \setminus C_{R,J_R}, \mathbb{Z})} = \iota - \Phi_0$.

Proposition 7.0.8. *Let R be a rational function of degree $d > 1$. Then,*

(1) $(\iota - \Phi_0)^{-1}(\mathbb{Z}[1_{J_R}]) \simeq [H_R]^{-1}(\mathbb{Z}[u])$ and $ker(\iota - \Phi_0) \simeq ker([H_R])$.

(2) co-ker$(\iota - \Phi_0) \simeq$ co-ker$([H_R]) \oplus \mathbb{Z}^{h_R}$ and the isomorphism maps the class of 1_{J_R} to the class of u.

Proof. First, let's assume $J_R = \hat{\mathbb{C}}$. In this case, $\iota - \Phi_0 = 0$ and $C(J_R, \mathbb{Z}) = \mathbb{Z}[1_{J_R}]$. Therefore, $(\iota - \Phi)^{-1}(\mathbb{Z}[1_{J_R}]) = 0$ and co-ker$(\iota - \Phi_0) = \mathbb{Z}[1_{J_R}] \simeq \mathbb{Z}$.

Since $|[C_{R,J_R}]| = 1$ and $\mathcal{H}_R = \emptyset$, we have that $[H_R] : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is the map sending u to $[c] + u$. Hence, $[H_R]^{-1}(\mathbb{Z}[u]) = 0$ and co-ker$([H_R]) \simeq \mathbb{Z}$ and is generated by u.

Therefore, $\ker(\iota - \Phi_0) = (\iota - \Phi_0)^{-1}(\mathbb{Z}[1_{J_R}]) = 0 = [H_R]^{-1}(\mathbb{Z}[u]) = \ker([H_R])$ and co-ker$(\iota - \Phi_0) \simeq$ co-ker$([H_R]) \simeq \mathbb{Z}$ via the map sending the class of 1_{J_R} to the class of u.

Now, we assume throughout the proof that $J_R \neq \hat{\mathbb{C}}$. For $g = m \cdot u + \sum_{Q \in \mathcal{H}_R} a_Q \cdot Q$ in $\mathbb{Z}^{\mathcal{H}_R} \oplus \mathbb{Z}$, we have $[H_R](g) = -\alpha_R(g)u + \sum_{Q \in \mathcal{H}_R}(H_Q([c]) + m)[c]$. So, $[H_R](g)$ is in $\mathbb{Z}[u]$ if and only if $\sum_{Q \in \mathcal{H}_R} H_Q(c) + m = 0$, for all c in C_{R,J_R}.

Let φ_R be the isomorphism appearing in Proposition 7.0.7. By Proposition 7.0.7, we have $(\iota - \Phi_0)^{-1}(\mathbb{Z}[1_{J_R}]) = (\mathrm{id} - \Phi)^{-1}(\mathbb{Z}[1_{J_R}]) \cap C_0(J_R \setminus C_{R,J_R}, \mathbb{Z}) = \{m + \sum_{Q \in \mathcal{H}_R} a_Q H_Q : m + \sum_{Q \in \mathcal{H}_R} a_Q H_Q(c) = 0 \ \forall c \in C_{R,J_R}\}$. Hence, $\varphi_R((\iota - \Phi_0)^{-1}(\mathbb{Z}[1_{J_R}])) = [H_R]^{-1}(\mathbb{Z}[u])$.

Since $[H_R](g) = -\alpha_R(g)u$, for all g in $[H_R]^{-1}(\mathbb{Z}[u])$, it also follows that $\varphi_R^{-1}(\ker(H_R)) = \varphi_R^{-1}([H_R]^{-1} \cap \ker(\alpha_R))$. By Proposition 7.0.7 and the above equalities, we have $\varphi_R^{-1}([H_R]^{-1} \cap \ker(\alpha_R)) = (\iota - \Phi_0)^{-1}(\mathbb{Z}[1_{J_R}]) \cap \ker(\mathrm{id} - \Phi) = \ker(\iota - \Phi_0)$. This proves (1)

We now prove (2). We first determine the image of $i_* : K^{-1}(F_R \setminus C_{R,F_R}) \to K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}})$. Naturality of exp implies the diagram

$$\begin{array}{ccc} K^0(C_{R,\hat{\mathbb{C}}}) & \xrightarrow{\quad q \quad} & K^0(C_{R,J_R}) \\ \Big\downarrow \exp & & \Big\downarrow \exp \\ K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}) & \xrightarrow{\ r_* \ } & K^{-1}(J_R \setminus C_{R,J_R}) \end{array}$$

commutes, where the vertical maps are the restrictions. Therefore, $\mathrm{im}(i_*) = \ker(r_*) = \exp(q^{-1}(\exp^{-1}(0)))$. By exactness, the kernel of $\exp : C(C_{R,J_R}, \mathbb{Z}) \to K^{-1}(J_R \setminus C_{R,J_R})$ is the image of the restriction map $r : C(J_R, \mathbb{Z}) \to C(C_{R,J_R}, \mathbb{Z})$, which is generated by the functions $\{1_{[c]}\}_{[c] \in [C_{R,J_R}]}$. So, $\ker(r_*) = \exp(q^{-1}(\sum_{[c] \in [C_{R,J_R}]} \mathbb{Z}[1_{[c]}])) = \exp(\sum_{[c] \in [C_{R,J_R}]} \mathbb{Z}[1_{[c]}] +$

$\sum_{c\in C_{R,F_R}}\mathbb{Z}[1_c]) = \sum_{[c]\in[C_{R,J_R}]}\mathbb{Z}[v_{[c]}] + \sum_{c\in C_{R,F_R}}\mathbb{Z}[v_c] =: \tilde{W}_R$. The diagram

$$\begin{array}{ccc} K^{-1}(F_R\setminus C_{R,F_R}) & \xrightarrow{i_*} & \tilde{W}_R \\ \big\downarrow{\scriptstyle \iota-\hat{\otimes}_1[\mathcal{E}_{R,F_R}]} & & \big\downarrow \\ K^{-1}(F_R) & \longrightarrow & 0 \end{array}$$

automatically commutes and, from the above calculation, the top row map is surjective. By Corollary 3.0.10, the diagram

$$\begin{array}{ccc} C_0(J_R\setminus C_{R,J_R},\mathbb{Z}) & \xrightarrow{\exp} & K^{-1}(F_R\setminus C_{R,F_R}) \\ \big\downarrow{\scriptstyle \iota-\Phi_0} & & \big\downarrow{\scriptstyle \iota-\hat{\otimes}_1[\mathcal{E}_{R,F_R}]} \\ C(J_R,\mathbb{Z}) & \xrightarrow{\exp} & K^{-1}(F_R) \end{array}$$

commutes. The bottom map has kernel equal to $\mathbb{Z}[1_{J_R}]$. Hence, the diagram

$$\begin{array}{ccccccc} & & C(J_R\setminus C_{R,J_R},\mathbb{Z}) & \xrightarrow{\exp} & K^{-1}(F_R\setminus C_{R,F_R}) & \xrightarrow{i_*} & \tilde{W}_R & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \widetilde{\iota-\Phi_0}} & & \big\downarrow{\scriptstyle \iota-\hat{\otimes}_1[\mathcal{E}_{R,F_R}]} & & \big\downarrow & & \\ 0 & \longrightarrow & C(J_R,\mathbb{Z})/\mathbb{Z}[1_{J_R}] & \xrightarrow{\widetilde{\exp}} & K^{-1}(F_R) & \longrightarrow & 0 & & \end{array}$$

commutes, and has exact rows, where $\widetilde{\iota-\Phi}$ and $\widetilde{\exp}$ denote the descent maps of $\iota-\Phi$, exp, respectively.

By the Snake Lemma, there is a boundary map $\tilde{\partial}:\tilde{W}_R\to \text{co-ker}(\iota-\Phi)/\mathbb{Z}[1_{J_R}]$ making the sequence

$$\begin{array}{ccccc} (\iota-\Phi_0)^{-1}(\mathbb{Z}[1_{J_R}]) & \xrightarrow{\exp} & \ker(\iota-\hat{\otimes}_1[\mathcal{E}_{R,F_R}]) & \xrightarrow{i_*} & \tilde{W}_R \\ & & & & \big\downarrow{\scriptstyle \tilde{\partial}} \\ 0 & \longleftarrow & \text{co-ker}(\iota-\hat{\otimes}_1[\mathcal{E}_{R,F_R}]) & \xleftarrow[\widetilde{\exp}]{} & \text{co-ker}(\iota-\Phi_0)/\mathbb{Z}[1_{J_R}] \end{array}$$

exact. We will first determine i_* on generators. By Proposition 6.0.4, $\ker(\iota-\hat{\otimes}_1[\mathcal{E}_{R,F_R}]) = G(F_R,C_{R,F_R})\oplus(\bigoplus_{Q\in\mathcal{H}_R}\mathbb{Z}[u_Q]$. By the definition of $G(F_R,C_R)$ and Proposition 7.0.1, this group is freely generated by elements $\{v_c\}_{c\in C_{R,F_R}}$ with the property that $i_*(v_c)=v_c$, for all c in C_{R,F_R}. We also have, by Proposition 7.0.4, $i_*(u_Q)=\sum_{c\in C_{R,\hat{\mathbb{C}}}}H_Q(c)v_c$, for all Q in $\mathcal{H}_R$. So, for every Q in $\mathcal{H}_R$, $w_Q := u_Q-\sum_{c\in C_{R,F_R}}H_Q(c)v_c$ satisfies $i_*(w_Q)=\sum_{[c]\in[C_{R,J_R}]}H_Q([c])v_{[c]}$.

Therefore, the image of i_* is equal to $\text{im}(\tilde{H}_R)+\sum_{c\in C_{R,F_R}}\mathbb{Z}[v_c]$, where $\tilde{H}_R:\mathbb{Z}^{\mathcal{H}_R}\to\tilde{W}_R$ is the homomorphism such that $\tilde{H}_R(Q)=\sum_{[c]\in[C_{R,J_R}]}H_Q[c]v_{[c]}$ for all Q in $\mathcal{H}_R$.

Let V_R be the abelian group with generators $\{[c]\}_{[c]\in[C_{R,J_R}]} \cup \{u\}$ satisfying the relation $\sum_{[c]\in[C_{R,J_R}]}[c] = (1-d)u$.

Let $\{X_{[c]}\}_{[c]\in[C_{R,J_R}]}$ be a clopen partitition of J_R such that $X_{[c]} \cap C_{R,J_R} = [c]$, for all $[c]$ in $[C_{R,J_R}]$. Define the homomorphism $\partial : V_R \to \text{co-ker}(\iota - \Phi_0)$ on generators as $\partial(u) = \overline{1_{J_R}}$ and $\partial([c]) = \overline{1_{X_{[c]}} - \Phi(X_{[c]})}$, for all $[c]$ in $[C_{R,J_R}]$. Since $\partial((1-d)u) = \overline{1_{J_R} - \Phi(1_{J_R})} = \sum_{[c]\in[C_{R,J_R}]} \overline{1_{X_{[c]}} - \Phi(1_{X_{[c]}})} = \partial(\sum_{[c]\in[C_{R,J_R}]}[c])$, this homomorphism is well-defined. Let $q : \text{co-ker}(\iota - \Phi_0) \to \text{co-ker}(\iota - \Phi_0)/\mathbb{Z}[1_{J_R}]$ be the quotient map.

We show that $q(\partial([c])) = \tilde{\partial}(v_{[c]})$ for all $[c]$ in $[C_{R,J_R}]$. By the definition of the boundary map from the Snake Lemma, $\tilde{\partial}(v_{[c]}) = \overline{a}$, for any a in $C(J_R, \mathbb{Z})$ such that $\exp(a) = \iota(w) - w\hat{\otimes}_1[\mathcal{E}_{R,F_R}]$, for any w in $K^{-1}(F_R \setminus C_{R,F_R})$ such that $i_*(w) = v_{[c]}$.

First, we choose w. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(F_R \setminus C_{R,F_R}) & \longrightarrow & C_0(\hat{\mathbb{C}} \setminus C_{R,F_R}) & \longrightarrow & C(J_R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow k & & \\ 0 & \longrightarrow & C_0(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}) & \longrightarrow & C(\hat{\mathbb{C}}) & \longrightarrow & C(C_{R,\hat{\mathbb{C}}}) & \longrightarrow & 0 \end{array}$$

commutes, where the two left-most vertical maps are inclusion and the rightmost vertical map is restriction to $C(C_{R,J_R})$, followed by the inclusion $C(C_{R,J_R}) \to C(C_{R,\hat{\mathbb{C}}})$.

Naturality of exp implies the diagram

$$\begin{array}{ccc} C(J_R, \mathbb{Z}) & \xrightarrow{k} & C(C_{R,J_R}, \mathbb{Z}) \\ \downarrow \exp & & \downarrow \exp \\ K^{-1}(F_R \setminus C_{R,F_R}) & \xrightarrow{i_*} & K^{-1}(\hat{\mathbb{C}} \setminus C_{R,\hat{\mathbb{C}}}) \end{array}$$

commutes. So, $w := \exp(1_{X_{[c]}})$ satisfies $i_*(w) = \exp(1_{[c]}) = v_{[c]}$.

Let $a = 1_{X_{[c]}} - \Phi(1_{X_{[c]}})$. Then, by Proposition 4.1.8 and naturality of exp, $\exp(a) = \iota(\exp(1_{X_{[c]}})) - \exp(1_{X_{[c]}})\hat{\otimes}_1[\mathcal{E}_{R,F_R}] = \iota(w) - w\hat{\otimes}_1[\mathcal{E}_{R,F_R}]$. Therefore, $\tilde{\partial}(v_{[c]}) = \overline{1_{X_{[c]}} - \Phi(1_{X_{[c]}})} = q(\partial([c]))$.

Since

$$\tilde{W}_R \xrightarrow{\tilde{\partial}} \text{co-ker}(\iota - \Phi_0)/\mathbb{Z}[1_{J_R}] \xrightarrow{\widetilde{\exp}} \text{co-ker}(\iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}]) \longrightarrow 0$$

is exact and $\exp(1_{J_R}) = 0$, $q^{-1}(\text{im}(\tilde{\partial}))$ equals the kernel of the surjection $\exp : \text{co-ker}(\iota - \Phi_0) \to \text{co-ker}(\iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}])$. We have $\mathbb{Z}[1_{J_R}] \subseteq \text{im}(\partial)$ and, from above, $q(\text{im}(\partial)) =$

$\tilde{\partial}(\sum_{[c]\in[C_{R,J_R}]}\mathbb{Z}[v_{[c]}]) = \mathrm{im}(\tilde{\partial})$. Therefore, $\mathrm{im}(\partial) = q^{-1}(\mathrm{im}(\tilde{\partial}))$. So,

$$V_R \xrightarrow{\ \partial\ } \text{co-ker}(\iota - \Phi_0) \xrightarrow{\ \exp\ } \text{co-ker}(\iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}]) \longrightarrow 0$$

is exact. Let $\tilde{V}_R$ be the group generated by elements $\{v_{[c]}\}_{[c]\in[C_{R,J_R}]}$ satisfying the relation $\sum_{[c]\in[C_{R,J_R}]} v_{[c]} = 0$; this group is canonically isomorphic to $\tilde{W}_R/\mathbb{Z}[\{v_c\}_{c\in C_{R,F_R}}]$ and $V_R/\mathbb{Z}[u]$. Let $\iota : \bigoplus_{Q\in\mathcal{H}_R}\mathbb{Z}[w_Q] \to V_R$ be the homomorphism sending w_Q to $\sum_{[c]\in[C_{R,J_R}]} H_Q([c])[c]$, and let $p : V_R \to \tilde{V}_R$ be the hommorphism sending u to 0, $[c]$ to $v_{[c]}$, for all $[c]$ in $[C_{R,J_R}]$. Since $i_*(G(F_R, C_R)) = \mathbb{Z}[\{v_c\}_{c\in C_{R,F_R}}] \subseteq \ker(\tilde{\partial})$ the diagram

$$\begin{array}{ccccc}
\bigoplus_{Q\in\mathcal{H}_R}\mathbb{Z}[w_Q] & \longrightarrow & \tilde{V}_R & \xrightarrow{\ \tilde{\partial}\ } & \text{co-ker}(\iota - \Phi_0)/\mathbb{Z}[1_{J_R}] \\
 & \searrow & \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle q} \\
 & & V_R & \xrightarrow{\ \partial\ } & \text{co-ker}(\iota - \Phi_0)
\end{array}$$

commutes, and the top row is exact. Hence, $\partial^{-1}(\mathbb{Z}[1_{J_R}]) = \partial^{-1}(q^{-1}(0)) = p^{-1}(\tilde{\partial}^{-1}(0)) = p^{-1}(\mathrm{im}(i_*)) = \mathrm{im}(\iota) + \mathbb{Z}[u]$.

For each $[c]$ in $[C_{R,J_R}]$, let $Y_{[c]} = X_{[c]} \cap H_Q^{-1}(H_Q([c]))$. Since $Y_{[c]}$ is a clopen set such that $[c] \subseteq Y_{[c]}$, $1_{X_{[c]}\setminus Y_{[c]}}$ is in $C_0(J_R \setminus C_{R,J_R}, \mathbb{Z})$. Hence,

$$\partial([c]) = \overline{1_{Y_{[c]}} - \Phi(1_{Y_{[c]}})} + \overline{1_{X_{[c]}\setminus Y_{[c]}} - \Phi(1_{X_{[c]}\setminus Y_{[c]}})} = \overline{1_{Y_{[c]}} - \Phi(1_{Y_{[c]}})}.$$

By construction, if we let $Y = \bigcup_{[c]\in[C_{R,J_R}]} Y_{[c]}$, then $H_Q = \sum_{[c]\in[C_{R,J_R}]} H_Q([c])1_{Y_{[c]}} + H_Q 1_{J_R\setminus Y}$, for all Q in $\mathcal{H}_R$. Since $H_Q 1_{J_R\setminus Y}$ is in $C_0(J_R \setminus C_{R,J_R}, \mathbb{Z})$, a similar calculation to that as above shows $\partial(\sum_{[c]\in[C_{R,J_R}]} H_Q([c])[c]) = \overline{H_Q - \Phi(H_Q)}$, for all Q in $\mathcal{H}_R$.

Therefore every element g in V_R of the form $g = \sum_{[c]\in[C_{R,J_R}]}\sum_{Q\in\mathcal{H}_R} a_Q H_Q([c])[c] + \sum_{Q\in\mathcal{H}_R} a_Q(\Phi(H_Q) - H_Q)$ is in $\ker(\partial)$. Let us show every element in $\ker(\partial)$ is of this form.

Let g be in $\ker(\partial)$. From above, g is in $\mathrm{im}(\iota) + \mathbb{Z}[u]$, so we can write $g = mu + \sum_{Q\in\mathcal{H}_R} a_Q \sum_{[c]\in[C_{R,J_R}]} H_Q([c])[c]$ for some $\{a_Q\}_{Q\in\mathcal{H}_R} \cup \{m\} \subseteq \mathbb{Z}$. Therefore, $0 = \partial(g) = \overline{m + \sum_{Q\in\mathcal{H}_R} H_Q - \Phi(H_Q)}$. Therefore, there are integers $\{b_Q\}_{Q\in\mathcal{H}_R} \cup \{n\} \subseteq \mathbb{Z}$ such that $n + \sum_{Q\in\mathcal{H}_R} b_Q H_Q[c] = 0$ for all c in C_{R,J_R} and $(1-d)n + \sum_{Q\in\mathcal{H}_R} b_Q(H_Q - \Phi(H_Q)) = m + \sum_{Q\in\mathcal{H}_R} H_Q - \Phi(H_Q)$.

Using the relation $(d-1)nu = -n\sum_{[c]\in[C_{R,J_R}]}[c]$, we may write $g = (m + (d-1)n)u + \sum_{[c]\in[C_{R,J_R}]}(\sum_{Q\in\mathcal{H}_R} a_Q H_Q([c]) + n)[c]$. Then, using the fact that $\sum_{Q\in\mathcal{H}_R} b_Q H_Q(c) = -n$ for all c in C_{R,J_R}, we may write

$\sum_{[c]\in[C_{R,J_R}]}(\sum_{Q\in\mathcal{H}_R} a_Q H_Q([c]) + n)[c] = \sum_{[c]\in[C_{R,J_R}]}\sum_{Q\in\mathcal{H}_R}(a_Q - b_Q)H_Q([c])[c]$. Hence, setting $a'_Q = a_Q - b_Q$, and using that $\sum_{Q\in\mathcal{H}_R} a'_Q(\Phi(H_Q) - H_Q) = m + (d-1)n$, we may write

$g = \sum_{[c]\in[C_{R,J_R}]} \sum_{Q\in\mathcal{H}_R} a'_Q H_Q([c])[c] + \sum_{Q\in\mathcal{H}_R} a'_Q(\Phi(H_Q) - H_Q)$.

By the above description of $\ker(\partial)$, we have that $\ker(\partial) = \mathrm{im}(\hat{H}_R)$, where $\hat{H}_R : \mathbb{Z}^{\mathcal{H}_R} \to V_R$ is the homomorphism sending Q to $(\Phi(H_Q) - H_Q)u + \sum_{[c]\in[C_{R,J_R}]} H_Q([c])[c]$. By Proposition 6.0.4, we have co-ker$(\iota - \hat{\otimes}_1[\mathcal{E}_{R,F_R}]) \simeq \mathbb{Z}^{h_R}$. Hence, we have an exact sequence

$$0 \longrightarrow \mathrm{im}(\hat{H}_R) \longrightarrow V_R \xrightarrow{\partial} \text{co-ker}(\iota - \Phi_0) \longrightarrow \mathbb{Z}^{h_R} \longrightarrow 0.$$

Therefore, co-ker$(\iota - \Phi_0) \simeq V_R/\mathrm{im}(\hat{H}_R) \oplus \mathbb{Z}^{h_R}$, via an isomorphim sending the class of u to the class of 1_{J_R}. Since $\mathrm{im}([H_R]) = q^{-1}(\mathrm{im}(\hat{H}_R))$, where $q : \mathbb{Z}^{[C_{R,J_R}]} \oplus \mathbb{Z}/ \to V_R$ is the quotient map, it follows that $V_R/\mathrm{im}(\hat{H}_R) \simeq$ co-ker$([H_R])$ via an isomorphism mapping the class of u to the class of u. □

As a Corollary to our above calculations, we can describe the K-theory of $\mathcal{O}_{R,J_R}$.

Theorem 7.0.9. *Let R be a rational function of degree $d > 1$. Then, $K_1(\mathcal{O}_{R,J_R}) \simeq \ker(H_R) \oplus \mathbb{Z}/\omega_R\mathbb{Z} \oplus \mathbb{Z}^{|f_R-1|}$ and $K_0(\mathcal{O}_{R,J_R}) \simeq$ co-ker$(H_R) \oplus \mathbb{Z}^{|f_R+h_R-1|}$, with class of the unit corresponding to the class of basis element u in co-ker(H_R).*

Proof. First, suppose $J_R = \hat{\mathbb{C}}$. By Theorem 5.0.3, we have $K_1(\mathcal{O}_{R,J_R}) \simeq \mathbb{Z}$ and $K_0(\mathcal{O}_{R,J_R}) \simeq \mathbb{Z}^{c_{R,J}+1}$, with class of the unit corresponding to a generator in a minimal generating set for $\mathbb{Z}^{c_{R,J}+1}$.

In this case, H_R is the mapping $\mathbb{Z} \to \mathbb{Z}^{C_{R,J_R}} \oplus \mathbb{Z}$ sending u to $(d-1)u + \sum_{c\in C_{R,J_R}} c$. Hence, $\ker(H_R) = 0$ and co-ker$(H_R) \simeq \mathbb{Z}^{c_{R,J_R}}$, with the class of u corresponding to a generator in a minimal generating set for $\mathbb{Z}^{c_{R,J}}$.

$\mathbb{Z}^{|f_R+h_R-1|} = \mathbb{Z}^{|f_R-1|} = \mathbb{Z}$ and $\mathbb{Z}/\omega_R\mathbb{Z} = 0$, so that $\ker(H_R) \oplus \mathbb{Z}/\omega_R\mathbb{Z} \oplus \mathbb{Z}^{|f_R-1|} \simeq \mathbb{Z}$ and co-ker$(H_R) \oplus \mathbb{Z}^{|f_R+h_R-1|} \simeq \mathbb{Z}^{c_{R,J}+1}$, with the class of u corresponding to a generator in a minimal generating set for $\mathbb{Z}^{c_{R,J}+1}$.

Now, let us assume $J_R \neq \hat{\mathbb{C}}$. By Proposition 4.1.5, we therefore have $\ker(\iota - \hat{\otimes}_0[\mathcal{E}_{R,J_R}]) \simeq \ker(\iota - \Phi_0)$ and co-ker$(\iota - \hat{\otimes}_0[\mathcal{E}_{R,J_R}]) \simeq$ co-ker$(\iota - \Phi_0)$, with the class of the unit corresponding to the class of 1_{J_R}. By the Pimsner-Voiculescu 6-term exact sequence and the above isomorphisms, we have $K_1(\mathcal{O}_{R,J_R}) \simeq \ker(\iota - \Phi_0) \oplus$ co-ker$(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}])$ and $K_0(\mathcal{O}_{R,J_R}) \simeq \ker(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]) \oplus$ co-ker$(\iota - \Phi_0)$, with the class of the unit in K_0 corresponding to the class of 1_{J_R} in co-ker$(\iota - \Phi_0)$.

By Proposition 7.0.3 and Proposition 7.0.8, we have $\ker(\iota - \Phi_0) \oplus$ co-ker$(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]) \simeq \ker([H_R]) \oplus \mathbb{Z}/\omega_R\mathbb{Z} \oplus \mathbb{Z}^{f_R-1}$ and $\ker(\iota - \hat{\otimes}_1[\mathcal{E}_{R,J_R}]) \oplus$ co-ker$(\iota - \Phi_0) \simeq \mathbb{Z}^{(f_R-1)+(c_{R,J}-k_{R,J})} \oplus \mathbb{Z}^{h_R} \oplus$ co-ker$([H_R])$, with the class of 1_{J_R} in co-ker$(\iota - \Phi_0)$ corresponding to the class of u in co-ker$([H_R])$.

Since $(H_R)_{c,Q} = (H_R)_{d,Q} = ([H_R])_{[c],Q}$ for all Q in $\mathcal{H}_R$ and $(H_R)_{c,u} = (H_R)_{d,u} = ([H_R])_{[c],u}$ for every $c \sim d$, we have that $\ker(H_R) = \ker([H_R])$ and co-$\ker(H_R) \simeq \mathbb{Z}^{c_{R,J}-k_{R,J}} \oplus$co-$\ker([H_R])$, with the class of u in co-$\ker(H_R)$ corresponding to the class of u in co-$\ker([H_R])$. Therefore, $\ker([H_R]) \oplus \mathbb{Z}/\omega_R\mathbb{Z} \oplus \mathbb{Z}^{f_R-1} \simeq \ker(H_R) \oplus \mathbb{Z}/\omega_R\mathbb{Z} \oplus \mathbb{Z}^{|f_R-1|}$ and $\mathbb{Z}^{(f_R-1)+(c_{R,J}-k_{R,J})} \oplus \mathbb{Z}^{h_R} \oplus$ co-$\ker([H_R]) \simeq$ co-$\ker(H_R) \oplus \mathbb{Z}^{|f_R+h_R-1|}$ with the class of u in co-ker$[(H_R])$ corresponding to the class of u in co-$\ker(H_R)$. $\square$

Chapter 8

Applications

8.1 A conjugacy invariant for rational functions

By the *Fatou cycle length data of* R we shall mean the tuple $L_R = (|P|)_{P \in \mathcal{F}_R}$, where the entries are ordered in non-decreasing order. Similarly, the Herman cycle length data of R shall mean the tuple $T_R = (|Q|)_{Q \in \mathcal{H}_R}$ with entries also ordered in non-decreasing order. We show that L_R and T_R are conjugacy invariants for $R : J_R \to J_R$ amongst all rational functions. This is not surprising due to the rigid nature of rational dynamics, but it is not clear how to prove it directly by a dynamical argument.

Since $J_{R^{\circ n}} = J_R$ for any n in $\mathbb{N}$ ([39, Lemma 4.4]), R and S being conjugate on their Julia sets also implies $R^{\circ n}$ and $S^{\circ n}$ are conjugate on their Julia sets, for all n in $\mathbb{N}$. Therefore, $\text{co-ker}(\iota - \hat{\otimes}_1[\mathcal{E}_{R^{\circ n}, J_R}]) \simeq \text{co-ker}(\iota - \hat{\otimes}_1[\mathcal{E}_{S^{\circ n}, J_S}])$ and $\ker(\text{id} - \Phi_{R^{\circ n}}) \simeq \ker(\text{id} - \Phi_{S^{\circ n}})$ for all n in $\mathbb{N}$. By Proposition 7.0.3 and Proposition 7.0.7, it follows that $f_{R^{\circ n}} = f_{S^{\circ n}}$ and $h_{R^{\circ n}} = h_{S^{\circ n}}$ for all n in $\mathbb{N}$.

We will now show that the sequences $\{f_{R^{\circ n}}\}_{n \in \mathbb{N}}$ and $\{h_{R^{\circ n}}\}_{n \in \mathbb{N}}$ are equivalent to the Fatou and Herman cycle length data of R. respectively.

The first observation to make is that every cycle of $R^{\circ n}$ must be contained in a cycle P for R of the same type, and that the number of distinct cycles of $R^{\circ n}$ contained in P in $\mathcal{F}_R$ is equal to the greatest common divisor between $|P|$ and n, denoted $(|P|, n)$. Moreover, each such cycle has length $\frac{|P|}{(|P|, n)}$. Hence, $f_{R^{\circ n}} = \sum_{P \in \mathcal{F}_R} (|P|, n)$ and $h_{R^{\circ n}} = \sum_{Q \in \mathcal{H}_R} (|Q|, n)$.

The rest of the argument showing equivalency is contained in a relevant lemma about elementary number theory. First, define

$$\mathcal{A} = \{(a_1, ..., a_k) \in \mathbb{N}^k : k \in \mathbb{N} \ , a_i \leq a_{i+1} \ \forall i \leq k - 1\}.$$

For $A = (a_1, ..., a_k)$ in $\mathcal{A}$ and n in $\mathbb{N}$, define $(A, n) = \sum_{i=1}^{k} (a_i, n)$.

Lemma 8.1.1. *Suppose A, B are tuples in $\mathcal{A}$. If $(A,n)=(B,n)$ for all n in $\mathbb{N}$, then $A=B$.*

Proof. Note that the tuple length of A is equal to $(A,1)$. We will prove this lemma via induction on the tuple length $k=(A,1)=(B,1)$.

If $k=1$, then $a_1=\max_{n\in\mathbb{N}}(A,n)=\max_{n\in\mathbb{N}}(B,n)=b_1$, and the lemma is proved.

Now, suppose $k>1$ and we know the lemma is true for all $m\leq k-1$. We prove for $m=k$, but first we will set some notation.

For a,b in $\mathbb{N}$, let $a|b$ mean a divides b, and $a\not| b$ mean a doesn't divide b. For C in $\mathcal{A}$ such that $C=(c_1,...,c_k)$, let $\mathcal{P}_C$ denote the set of all primes appearing in the numbers $\{c_i\}_{i=1}^k$. For each $i\leq k$ and p in $\mathcal{P}_C$, let $n_{p,C,i}=\max\{l\in\mathbb{N}_0 : p^l|c_i\}$. Hence, $c_i=\Pi_{p\in\mathcal{P}_C}p^{n_{p,C,i}}$ for all $i\leq k$.

Let $n_{p,C}=\max_{i\leq k}n_{p,C,i}$. The smallest number l for which $(C,l)=\max_{n\in\mathbb{N}}(C,n)=\sum_{i=1}^k c_i$ is denoted m_C, and it is easy to see it is equal to $\Pi_{p\in\mathcal{P}_C}p^{n_{p,C}}$.

Now, suppose A, B in $\mathcal{A}$ are such that $(A,n)=(B,n)$ for all n in $\mathbb{N}$ and $(A,1)=(B,1)=k$. Then, $\Pi_{p\in\mathcal{P}_A}p^{n_{p,A}}=m_A=m_B=\Pi_{p\in\mathcal{P}_B}p^{n_{p,B}}$, so $\mathcal{P}_A=\mathcal{P}_B=:\mathcal{P}$ and, for every p in $\mathcal{P}$, we have $n_{p,A}=n_{p,B}=:n_p$.

Now, for $C=A$ and $C=B$, for every m in $\mathbb{N}$, and p in $\mathcal{P}$, we have

$$
\begin{aligned}
&(C,p^{n_p}m)-(C,p^{n_p-1}m)=\\
&\sum_{i=1}^k p^{n_{p,C,i}}(\frac{c_i}{p^{n_{p,C,i}}},m)-\sum_{i=1}^k p^{\min\{n_{p,C,i},n_p-1\}}(\frac{c_i}{p^{\min\{n_{p,C,i},n_p-1\}}},m)=\\
&\sum_{i:p^{n_p}|c_i} p^{n_p}(\frac{c_i}{p^{n_p}},m)-\sum_{i:p^{n_p}|c_i} p^{n_p-1}(\frac{c_i}{p^{n_p-1}},m)
\end{aligned}
$$

So, if we let C_p be the ordered tuple $(\frac{c_i}{p^{n_p}})_{i:p^{n_p}|c_i}$, and for every m in $\mathbb{N}$, let $m_p=\frac{m}{p^l}$, where $l=\max\{s\in\mathbb{N}_0 : p^s|m\}$, then

$$\frac{(C,p^{n_p}m_p)-(C,p^{n_p-1}m_p)}{p^{n_p}-p^{n_p-1}}=(C_p,m),\ \text{for every } m \text{ in } \mathbb{N}.$$

Thus, $(A_p,m)=(B_p,m)$ for all p in $\mathcal{P}$ and m in $\mathbb{N}$. There are two cases.

The first case is that $k=(A_p,1)=(B_p,1)$ for all p in $\mathcal{P}$, in which case $a_i=m_A=m_B=b_i$ for all $i\leq k$, and the lemma is proved.

The second case is that $k>(A_p,1)=(B_p,1)$ for some p in $\mathcal{P}$. By induction, it follows that $A_p=B_p$, and hence $A'_p:=(a_i)_{i:p^{n_p}|a_i}=(b_i)_{i:p^{n_p}|b_i}=:B'_p$. For $C=A$ and $C=B$, denote $C^p=(c_i)_{i:p^{n_p}\not| c_i}$. Then, for every m in $\mathbb{N}$, we have

$$(A^p,m)=(A,m)-(A'_p,m)=(B,m)-(B'_p,m)=(B^p,m).$$

By induction, it follows that $A^p = B^p$. Hence, $A = B$. □

Corollary 8.1.2. *Let R and S be rational functions. If R and S are conjugate on their Julia sets, then $L_R = L_S$ and $T_R = T_S$.*

Proof. As remarked above Lemma 8.1.1, R and S conjugate on J implies $(L_R, n) = (L_S, n)$ and $(T_R, n) = (T_S, n)$, for all n in $\mathbb{N}$. Lemma 8.1.1 then implies that $L_R = L_S$ and $T_R = T_S$. □

Corollary 8.1.3. *Let R and S be rational functions. If R and S are topologically conjugate on their Julia sets, then ker(H_R) $\simeq$ ker(H_S) and there is a bijection $b : C_{R,J_R} \cup \{u\} \to C_{S,J_S} \cup \{u\}$ mapping u to u and inducing an isomorphism b : co-ker(H_R) $\to$ co-ker(H_S).*

Proof. We will denote by $\iota - \Phi_0^R := \iota - \Phi_0 : C_0(J_R \backslash C_{R,J_R}, \mathbb{Z}) \to C(J_R, \mathbb{Z})$. Let $\varphi : J_R \to J_S$ be a homeomorphism such that $S \circ \varphi = \varphi \circ R$. Then, $(\iota - \Phi_0^R) \circ \varphi^* = \varphi^* \circ (\iota - \Phi_0^S)$ so Proposition 7.0.8 implies $\ker([H_S]) \simeq \ker(\iota - \Phi_0^S) \simeq \ker(\iota - \Phi_0^R) \simeq \ker([H_R])$. Since $\ker([H_R]) \simeq \ker(H_R)$ for any rational function R, this proves the first claim.

$\varphi : J_R \to J_S$ restricts to a bijection $b : C_{R,J_R} \to C_{S,J_S}$, so if $Y_{b(c)}$ is a connected component of J_S containing only $[b(c)]$, for c in J_R, then $Y_c := \varphi^{-1}(Y_{b(c)})$ is a connected component of J_R containing only $[c]$. It follows then by the definition of the map $\partial^R = \partial : V_R \to \text{co-ker}(\iota - \Phi_0^R)$ in the proof of Proposition 7.0.8 that $\varphi^* \circ \partial^S([b(c)]) = \varphi^*(\overline{1_{Y_{b(c)}} - \Phi(1_{Y_{b(c)}})}) = 1_{Y_c} - \Phi(Y_c) = \partial^R([c])$. Hence, $\varphi^*(\text{im}(\partial^S)) = \text{im}(\partial^R)$, and if we identify $\text{im}(\partial^S)$ and $\text{im}(\partial^R)$ with co-ker($[H_S]$) and co-ker($[H_R]$) as in the proof of Proposition 7.0.8, the isomorphism $\varphi^* : \text{im}(\partial^S) \to \text{im}(\partial^R)$ becomes b^{-1} : co-ker($[H_S]$) $\to$ co-ker($[H_R]$). It is routine to see this isomorphism lifts to an isomorphism b^{-1} : co-ker(H_S) $\to$ co-ker(H_R). □

8.2 K-theory for polynomials

When R is a polynomial, the K-theory takes on an especially simple form. We first describe some general properties of polynomial dynamics.

If R is a degree d polynomial, then there are constants $\lambda, M > 0$ such that $R(z) > \lambda |z|^d$ for $|z| > M$, so $\mathcal{A}_\infty = \{z \in \hat{\mathbb{C}} : \lim_{n\to\infty} R^{\circ n}(z) = \infty\}$ is a non-empty open set of F_R containing the critical point ∞. It is called the *attracting basin at infinity*. By [39, Lemma 9.4], $\mathcal{A}_\infty$ is connected, and it is easy to see $R^{-1}(\mathcal{A}_\infty) = \mathcal{A}_\infty$. Therefore, the attracting basin at infinity is a Fatou cycle of cycle length one. Hence, $f_R \geq 1$ and $\omega_R = 1$.

Also, $\mathcal{H}_R = \emptyset$ by the maximum modulus principle. When $\mathcal{H}_R = \emptyset$, H_R is the mapping $\mathbb{Z} \to \mathbb{Z}^{C_{R,J_R}} \oplus \mathbb{Z}$ sending u to $(d-1)u + \sum_{c \in C_{R,J_R}} c$. So, $\ker(H_R) = 0$, and co-ker(H_R) $\simeq \mathbb{Z}^{c_{R,J}}$ if $c_{R,J} \neq 0$ and co-ker(H_R) $\simeq \mathbb{Z}/(d-1)\mathbb{Z}$ otherwise, with the class of u in both cases corresponding to a generator in a minimal generating set for the group. Gathering these calculations and applying Theorem 7.0.9 yields the following Corollary.

Corollary 8.2.1. *Let P be a polynomial of degree $d > 1$. Then, $K_1(\mathcal{O}_{P,J_P}) \simeq \mathbb{Z}^{f_R-1}$. If $c_{R,J_R} = 0$, then $K_0(\mathcal{O}_{R,J_R}) \simeq \mathbb{Z}/(d-1)\mathbb{Z} \oplus \mathbb{Z}^{f_R-1}$, with the class of 1_{J_R} generating the torsion. Otherwise, $K_0(\mathcal{O}_{R,J_R}) \simeq \mathbb{Z}^{c_{R,J}+f_R-1}$, with the class of the unit corresponding to a generator in a minimal generating set for $\mathbb{Z}^{c_{R,J}+f_R-1}$.*

By the above Corollary, we can describe the isomorphism type of $\mathcal{O}_{R,J_R}$ rather easily.

Theorem 8.2.2. *Let R and S be polynomials. Then, $\mathcal{O}_{R,J_R}$ is isomorphic to $\mathcal{O}_{S,J_S}$ if and only if either*

- *$c_{R,J} = c_{S,J} = 0$, $deg(R) = deg(S)$, and $f_R = f_S$, or*
- *$c_{R,J} = c_{S,J} \neq 0$ and $f_R = f_S$.*

Consequently, for every $d > 1$ the number of isomorphism types of $\mathcal{O}_{R,J_R}$ when the degree of R is d is bounded above by $d(2d-2)$.

Proof. The characterization of the isomorphism types follows directly from the above Corollary. By [54, Corollary 2], $f_R \leq 2d-2$. Similarily, $c_{R,J} \leq d-1$. Therefore, the number of possible isomorphism types of degree d is bounded above by $(2d-2)+(d-1)(2d-2) = d(2d-2)$ □

Let us show the above inequality is sharp in the case when $d = 2$ and describe the four isomorphism types. First, we will record the following lemma, which is a collection of several results in complex dynamics.

Lemma 8.2.3. *Assume R is a rational function and P is a Fatou cycle.*

(1) If P is an attracting cycle, then it contains a critical point of R.

(2) If P is a parabolic cycle, then it contains a critical point of R.

(3) If P is a Siegel cycle, then its boundary is contained in the closure of the set of forward orbits for the critical points of R.

(4) If P is a Herman cycle, then its boundary is contained in the closure of the set of forward orbits for the critical points of R.

Proof. The proofs in the four cases can (respectively) be found in the proofs of [39, Lemma 8.5], [39, Theorem 10.15], [39, Theorem 11.17], and [39, Lemma 15.7]. □

The complement $\hat{\mathbb{C}} \setminus \mathcal{A}_\infty$ is denoted K_P and is called the *filled Julia set*. Note that K_P consists of all points in $\mathbb{C}$ with a bounded forward P-orbit. By [39, Theorem 9.5], K_P is connected if and only if $C_{P,\mathbb{C}} \subseteq K_P$, and K_P is connected if and only if J_P is connected. Also, $\partial K_P = J_P$.

It is easy to see that any quadratic map is conjugate to one of the form $f_c(z) = z^2 + c$, for some c in $\mathbb{C}$. Its critical points are thus 0 and ∞. Denote its Julia set by J_c, its filled Julia set by K_c, and the Cuntz-Pimsner algebra $\mathcal{O}_{f_c,J_c}$ by $\mathcal{O}_{c,J}$.

We show that the K-theory of a quadratic f_c depends only on the location of c in $\mathbb{C}$ relative to the filled Julia set K_c of f_c. Equivalently, the K-theory depends only on the type of bounded Fatou cycle f_c admits.

Corollary 8.2.4. *Let $f_c(z) = z^2 + c$. Then there are four isomorphism types for $\mathcal{O}_{c,J}$, dependent on the location of* 0 *relative to the filled Julia set.*

> ***Case 0*** $(0 \notin K_c)$: *Then, $K_1(\mathcal{O}_{c,J}) = K_0(\mathcal{O}_{c,J}) = 0$.*
>
> ***Case 1*** $(0 \in int(K_c))$: *Then, $K_1(\mathcal{O}_{c,J}) \simeq K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$ and $[1_{J_c}] = 0$.*
>
> ***Case 2*** $(0 \in \partial K_c = J_c,\ int(K_c) \neq \emptyset)$: *Then $K_1(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$, $K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}^2$ and $[1_{J_c}]$ is a generator in a minimal generating set for $\mathbb{Z}^2$.*
>
> ***Case 3*** $(0 \in \partial K_c = J_c,\ int(K_c) = \emptyset)$: *Then $K_1(\mathcal{O}_{c,J}) = 0$, $K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$ and $[1_{J_c}]$ is a generator.*

Proof. **Case 0**: If 0 is not in K_c, then 0 can't be in a bounded Fatou component, and the only limit point of $\{f_c^{\circ n}(0)\}_{n\in\mathbb{N}}$ is ∞. Thus, Lemma 8.2.3 implies $K_c = J_c$. Hence $c_{f_c,J_c} = 0$, $p_{f_c} = 1$, and $\omega_{f_c} = 1$. Case 2 of Theorem 7.0.9 then implies $K_1(\mathcal{O}_{c,J}) = 0 = K_0(\mathcal{O}_{c,J})$.

Case 1: If 0 is in $\text{int}(K_c)$, then $c_{f_c,J_c} = 0$, and the Fatou set F_c contains a cycle which is distinct from the attracting basin at ∞. Hence, $p_{f_c} \geq 2$. By [54, Corollary 2], a rational function of degree d has at most $2d-2$ distinct Fatou cycles. Therefore, $p_{f_c} = 2$, and $c_{f_c,J_c} = 0$. Now, case 2 of Theorem 7.0.9 applies to show $K_1(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$, $K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}$, and $[1_{J_c}] = 0$.

Case 2: If 0 is in J_c and $\text{int}(K_c) \neq \emptyset$, then $c_{f_c,J_c} = 1$, and $p_{f_c} = 2$ by the same reasoning as before. Case 3 of Theorem 7.0.9 applies to show $K_1(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$, $K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}^2$, and $[1_{J_c}]$ is a generator in a minimal generating set for $\mathbb{Z}^2$.

Case 3: If 0 is in J_c and $\text{int}(K_c) = \emptyset$, then the only Fatou cycle is the attracting basin at infinity. Hence, $c_{f_c,J_c} = 1 = p_{f_c}$. Case 3 of Theorem 7.0.9 applies to show $K_1(\mathcal{O}_{c,J}) = 0$, $K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$, and $[1_{J_c}]$ is a generator. □

We now characterize the four isomorphism types of $\mathcal{O}_{c,J}$ above by the dynamics of f_c on its Fatou set F_c.

Corollary 8.2.5. *Let $f_c(z) = z^2 + c$. Then there are four isomorphism types for $\mathcal{O}_{c,J}$, dependent on the dynamics of f_c on the Fatou set.*

Case 0: *If $\mathcal{A}_\infty$ is not simply connected, then $K_1(\mathcal{O}_{c,J}) = K_0(\mathcal{O}_{c,J}) = 0$.*

Case 1: *If F_c contains a hyperbolic or parabolic cycle of bounded Fatou components, then $K_1(\mathcal{O}_{c,J}) \simeq K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$ and $[1_{J_c}] = 0$.*

Case 2: *If F_c contains a cycle of irrational rotations, then $K_1(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$, $K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}^2$ and $[1_{J_c}]$ is a generator.*

Case 3: *If $F_c = \mathcal{A}_\infty$ and is simply connected, then $K_1(\mathcal{O}_{c,J}) = 0$, $K_0(\mathcal{O}_{c,J}) \simeq \mathbb{Z}$ and $[1_{J_c}]$ is a generator.*

Proof. Note that $\mathcal{A}_\infty$ is simply connected if and only if $K_c = \hat{\mathbb{C}} \setminus \mathcal{A}_\infty$ is connected, which is true if and only if 0 is in K_c. So, case 0 above is equivalent to case 0 of Corollary 8.2.4.

The only other case that doesn't follow immediately from Corollary 8.2.4 and Lemma 8.2.3 is **Case 2**, which we show. By Theorem 7.0.9, it suffices to show 0 is in J_c.

Suppose the contrary. Then, Lemma 8.2.3 (3) implies 0 must be in a bounded Fatou component. If P is the Siegel cycle, then it follows that there is some k in $\mathbb{N}$ such that $\{f_c^{\circ n}(0)\}_{n\geq k} \subseteq U_P$. Since $f_c^{\circ|P|} : U_x \to U_x$ is conjugate to an irrational rotation, for all x in P, it follows that $\overline{\{f_c^{\circ n}(0)\}_{n\geq k}} \subseteq U_P$. But Lemma 8.2.3 (3) then implies $\partial U_P \subseteq \overline{\{f_c^{\circ n}(0)\}_{n\geq k}} \subseteq U_P$, a contradiction. □

Case 0,1,3 above are realized by $c = 1, 0, -2$, respectively, while case 2 is realized by the quadratic $z^2 + e^{2\pi i\varphi}z$, where φ is the golden ratio.

A complex number c is said to be *hyperbolic* (*parabolic*) if $\{f_c^n(0)\}_{n\in\mathbb{N}}$ converges to an attracting (parabolic) periodic orbit. We get the following K-theory characterization of when c is "bolic".

Corollary 8.2.6. *c in $\mathbb{C}$ is hyperbolic or parabolic if and only if $[1_{J_c}] = 0$ in $K_0(\mathcal{O}_{c,J})$.*

Proof. If $[1_{J_c}] = 0$ then f_c is a quadratic covered in Case 0 or Case 1 of Corollary 8.2.5.

In Case 0, 0 is in $\mathcal{A}_\infty$ and hence $\{f_c^n(0)\}_{n\in\mathbb{N}}$ converges to the attracting fixed point ∞.

In case 1, F_c contains either an attracting or parabolic cycle, and so Lemma 8.2.3 (1) and (2) imply $\{f_c^n(0)\}_{n\in\mathbb{N}}$ is eventually contained in a cycle of hyperbolic or parabolic Fatou components. This proves one direction of the Corollary. The converse is straightforward as well. □

Let us describe the corresponding C^*-algebra to each case. Case 0 corresponds to the Cuntz algebra $\mathcal{O}_2$ considered first in [15]. This is the universal C^*-algebra generated by two isometries s_1, s_2 such that $s_1s_1^* + s_2s_2^* = 1$. The C^*-algebra $\mathcal{O}_2$ and its generalizations for $n = 3, ...\infty$ have played a central role in the classification theory of purely infinite C^*-algebras [46] and to the general theory of C^*-algebras. For instance, a separable C^*-algebra is exact if and only if it embeds into $\mathcal{O}_2$ (see [31]). The Cuntz algebras have also found applications in wavelet theory [12] and were influential in the discovery of the Doplicher-Roberts Theorem [17], which characterizes the representations of a compact group as an abstract catgory.

Case 1 corresponds to the 2-adic ring C^*-algebra $\mathcal{Q}_2$ studied in [34], and the author thanks Chris Bruce for this identification. It also appears in other contexts; see [34, remark 3.2]. It is shown in [34] that $\mathcal{Q}_2$ is the universal C^*-algebra generated by a unitary u and isometry s satisfying $su = u^2s$ and $ss^* + uss^*u^* = 1$. Note that the isometries s and us generate $\mathcal{O}_2$ as a C^*-subalgebra of $\mathcal{Q}_2$. The representations of $\mathcal{O}_2$ that extend to $\mathcal{Q}_2$ are characterized in [34]. From this characterization, the authors of [34] motivate viewing $\mathcal{Q}_2$ as a symmetrized version of $\mathcal{O}_2$.

Case 3 corresponds to the Cuntz algebra $\mathcal{O}_\infty$, see the above remarks concerning Cuntz algebras. This is the universal C^*-algebra generated by isometries $\{s_i\}_{i=1}^{\infty}$ satisfying $\sum_{i=1}^{n} s_is_i^* \leq 1$ for every n in $\mathbb{N}$. Like $\mathcal{O}_2$, it is special amongst the Cuntz algebras. Kirchberg showed that a simple, separable, unital and nuclear C^*-algebra A is purely infinite if and only if $A \otimes \mathcal{O}_\infty \simeq A$ (see [31]).

Not much is known about the C^*-algebra representing case 2. It is isomorphic to the C^*-algebra of the partial dynamical system $z^2 : S^1 \setminus \{1\} \to S^1$. This follows by an application of the theory developed in Section 3. From this description, it follows that it is the universal C^*-algebra generated by a unitary u and isometry s satisfying $su = u^2s$ and $ss^* + uss^*u^* = \frac{u+u^*}{2}$. We shall denote it by $\mathcal{Q}_{2,\infty}$, as it shares properties of both $\mathcal{Q}_2$ and $\mathcal{O}_\infty$.

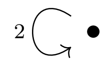
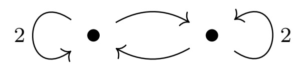
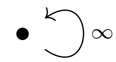
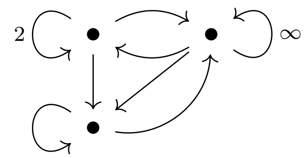
In all 4 cases, the C^*-algbera is isomorphic to a graph C^*-algebra. We follow the southern convention for graph C^*-algebras as in [48]. The below figure shows the graph corresponding to each case (the numbers in the graphs represent multiple edges):

Recall that the Mandelbrot set is $\mathcal{M} := \{c \in \mathbb{C} : 0 \in K_c\}$. Equivalently, by Corollary 8.2.4, we have that $\mathcal{M} = \{c \in \mathbb{C} : K_0(\mathcal{O}_{c,J}) \neq 0\}$.

One of the most important open problems in holomorphic dynamics is the Density of Hyperbolicity Conjecture, stated below.

Conjecture 8.2.7. $\mathcal{H} := \{c \in \mathcal{M} : f_c(z) = z^2 + c$ *is hyperbolic*$\}$ *is dense in* $\mathcal{M}$.

This conjecture can be stated as a conjecture about the K-theory for quadratics in the following way.

Parameter c	C^*-algebra	Graph
Outside $\mathcal{M}$	$\mathcal{O}_2$	
Hyperbolic or parabolic in $\mathcal{M}$	$\mathcal{Q}_2$	
In $\mathcal{M}$ and $\mathbb{C} \setminus J_c$ connected	$\mathcal{O}_\infty$	
F_c contains a Siegel cycle	$\mathcal{Q}_{2,\infty}$	

Corollary 8.2.8. *The Density of Hyperbolicity Conjecture is true if and only if* $\mathcal{H}' := \{c \in \mathcal{M} : [1_{J_c}] = 0$ *in* $K_0(\mathcal{O}_{c,J})\} = \{c \in \mathcal{M} : \mathcal{O}_{c,J} \simeq \mathcal{Q}_2\}$ *is dense in* $\mathcal{M}$.

Proof. By Corollary 8.2.6, the set $\mathcal{H}'$ is precisely the parameters c in $\mathcal{M}$ for which f_c is either parabolic or hyperbolic. By [40, Lemma 6.1] and [40, Lemma 6.2], every parabolic parameter lies on the boundary of the open set $\mathcal{H}$ of hyperbolic parameters in $\mathcal{M}$. Therefore, the set of parabolic parameters is a nowhere dense set in $\mathcal{M}$. It follows that density of $\mathcal{H}'$ is equivalent to density of $\mathcal{H}$. □

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