Three Essays in Microeconomic Theory

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To my parents

Abstract

This thesis explores foundational problems in microeconomic theory and advances understanding of complex decision-making processes in different environments with different parties through mathematical modelling.

Chapter 2 proves a new representation theorem for continuous-time preference over consumption or portfolio streams that take values in the probability simplex, allowing for both fully continuous and piecewise-continuous paths. Unlike existing results, our construction only relies on standard axioms. This framework is applicable to problems like portfolio allocation under a fixed budget or consumption under liquidity shocks—where payoffs may jump at discrete dates. We also show that one agent is more impatient than another if and only if their discount factor is larger.

Chapter 3 explores the asymptotic behaviour of strictly dominated strategies in random games of two players, in which the elements are i.i.d. selected from a probability distribution. As the game size increases, if the number of strategies for the two players is similar, in a sense we make precise, the probability of having strictly dominated strategies approaches 0. This is particularly evident in large nearly square games where the row size is proportionate to the column size within a specific range. Consequently, the probability of a game being dominance solvable diminishes to zero. Furthermore, we show that this result is very nearly tight: small deviations lead to a non-zero chance of strictly dominated strategies, while larger deviations make their existence nearly certain. Our findings first emphasize the significance of the parameters in the underlying probability distribution.

Chapter 3 also examines the asymptotic behaviour of the fixed proportion q of dominated strategies as $M, N \to \infty$, where M and N are the row and column sizes. Specifically, for the row player, we show that the probability of existence of q-portion-dominated strategy approaches 0 as $N \geq M/(\ln(M))^{\alpha}$ for some $\alpha > 0$ and it approaches 1 when M grow much faster than $(N/(1-\delta-q))^{N}$ for some $\delta > 0$.

Finally, Chapter 3 proposes a simple algorithm for detecting strictly dominated strategies in finite games, a topic of interest in economics and computer science since Yu and Zeleny 1975. This algorithm improves the conventional approach by utilizing minimum and maximum comparisons to reduce the expected time complexity.

Chapter 4 introduces a social planner into the random allocation framework who does not receive any object but may influence the allocation toward his preference. We present two guiding principles to clarify the conditions under which the social planner's opinion cannot be dismissed and when an agent's opinion must be respected. *Conformity-Priority Efficiency (CPF)* asserts that for any given object, we should reward the agent who conforms to social expectation more than others, which strengthens the notion of ordinal efficiency. *Indistinguishability Fairness (IF)* requires that for any object, if we can't distinguish agents due to social expectation and support of lottery, then they should be treated equally. Then, we construct a simple Flow algorithm to characterize them precisely.

Chapter 4 also examine the classic random allocation problem and, adhering to the principle of *interim favouring rank*, propose an alternative variant to the probabilistic setting to eliminate an unfair scenario where agents who rank objects higher receive more favourite objects with positive probability. We introduce the property of *interim favouring support*, which is satisfied by the fractional adaptive Boston rule. Additionally, we propose a new fairness criterion, termed *equal support equal claim*, to characterize the fractional adaptive Boston rule.

Finally, Chapter 4 introduces a new efficiency notion, *interim efficiency*, which is stronger than ex-post Pareto efficiency but weaker than ordinal efficiency. We construct the algorithm that is easy to apply in the lab, the Random Flow mechanism, to achieve interim efficiency. Numerical analysis shows that random flow results in less envy across preference profiles than the random priority mechanism.

Together, these contributions demonstrate the versatility of game theory in addressing complex decision-making, resource allocation, and strategic refinement across diverse contexts.

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Declaration

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution

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Xihao Song

Abbreviations

- `WO` - Weak Order
- `SAU` - Simplex Additive Utility
- `I` - Independence
- `C` - Continuity
- `TM` - Time Monotonicity
- `TS` - Time Stationary
- `TI` - Time Impatience
- `DC` - Dynamic Consistency
- `ND` - Non-Degeneracy
- `IESDS` - Iterated Elimination of Strictly Dominated Strategies
- `P(SD)` - The probability of the existence of a strictly dominated strategy in random game $\boldsymbol{R}_{M\times N}$
- `PNE` - Pure Nash Equilibrium
- `BRD` - Best-Response Dynamics
- `RP` - Random Priority Mechanism
- `PS` - Probabilistic serial Mechanism
- `RF` - Random Flow Mechanism
- `CFP` - Conformity-Priority Efficiency
- `IF` - Indistinguishability Fairness
- `OE` - Ordinal efficiency
- `EPPE`- Ex-post Pareto efficiency
- `IE` - Interim efficiency

Chapter 1

Introduction

In the 18th century, Daniel Bernoulli introduced the concept of ”expected utility” in the context of gambling, a concept later formalized by John von Neumann and Oskar Morgenstern in the 1940s. This foundational work inspired numerous studies in economics, particularly in decision-making under risk and uncertainty. Subsequently, Koopmans 1960 provided a rigorous axiomatic basis for intertemporal preferences. He introduced the concept of time consistency and developed a framework in which intertemporal preferences are represented as a sum of utilities, weighted by a discount factor, under specific axioms. This work extended Samuelson’s 1937 model model and addressed issues of dynamic consistency and separability (Samuelson 1937). Koopmans emphasized that resolving discontinuous cases is a simpler but critical step toward addressing continuous cases. However, many practical outcome spaces do not satisfy von Neumann-Morgenstern’s independence axiom.

Chapter 2 constructs classical models of intertemporal preferences by developing representation theorems specifically tailored to simplex spaces, where utility functions may be nonlinear. This approach models decision-making scenarios where agents evaluate the allocation of heterogeneous or homogeneous cake over time or allocate fixed capital among stocks, bonds, and real estate across different periods. Building on Qin and Rommeswinkel 2022, this work also extends the representation theorem into a dynamic framework, addressing limitations in prior research by incorporating piecewise continuous functions and accounting for utility discontinuities. This provides a robust framework for modeling preferences under risk.

In addition to individual decision-making, game theory has long served as a cornerstone for understanding strategic interactions, starting with Neumann and Morgenstern 1944. A fundamental concept in microeconomics and game theory is that of dominated strategies: a strategy is strictly dominated if it yields lower payoffs than an alternative across all possible scenarios. Players naturally avoid strictly dominated strategies if their objective is to maximize payoffs. However, identifying strictly dominated strategies becomes computationally challenging as the size of players' strategy spaces grows (Yu and Zeleny 1975).

Chapter 3 examines the existence of dominated strategies in random games with two players, where strategies are independently and identically distributed (i.i.d.) from a probability distribution. We present three main results. First, as the game size increases, if the number of strategies of two players is similar, in a sense we make precise, the probability of having strictly dominated strategies approaches 0. Small deviations in these growth rates lead to a non-zero probability of strictly dominated strategies, while larger deviations make their existence almost certain. Second, we introduce q-portion dominated strategies, where a fraction q of strategies is strictly dominated. This refined concept is particularly useful in large games. We identify thresholds for growth rates that determine the existence of q-portion dominated strategies. This result is stronger, as it yields a single strategy that strictly dominates the q-portion of strategies. Lastly, we develop an algorithm that reduces the time complexity of identifying strictly dominated strategies by approximately half, which can be integrated with other efficient algorithms.

Finally, we aim to understand the decision procedure in economics when the social planner is involved. It is common that the social planner's objective function significantly shapes economic outcomes and raises important philosophical questions about whether such interventions should be enforced when they conflict with individual preferences. In the last chapter, we take the axiomatic approach to economic design to understand this problem. The theory of economic design aims to identify and construct a decision procedure to satisfy desirable properties. And in the last twenty years, the literature on economic design, both its theoretical branch and its applied branch, has experienced spectacular growth.(Thomson 2023).

In Chapter 4, we model the social planner in the classic random allocation problem, in which every normal agent submits an ordinal preference and receives a lottery. The social planner, as the special agent, does not receive any object and holds a different preference compared to agents, depending on the nature of the social planner. We assume the social planner's preference is responsive to the order of importance, one example is lexicographical preference. To address the conflict between the social planner and normal agents, we formalize the principle of *Conformity-Priority Efficiency (CPF)*, which rewards agents who conform more closely to social expectations.

In cases where agents cannot be distinguished due to conformity, we emphasize the significance of lottery support, as every agent will receive a lottery as an outcome. We argue that it is fairer to consider lottery support rather than relative rank.[1] The second part of *Conformity-Priority Efficiency (CPF)* states that if the object a is conformed to by multiple agents at the same level regarding social expectation, priority should be given to the agent i with fewer objects assigned with positive probability—specifically, those preferred by agent i over object a, while also preferred by the object being associated with the highest violations against the social planner's preferences concerning a. To achieve this property effectively, we propose a simple Flow Algorithm.

Chapter 4 also analyzes the classic market without intervention from a social planner and formalizes the second part of *Conformity-Priority Efficiency (CPF)* in the classic model. Additionally, this chapter introduces the notion of interim efficiency, which bridges gaps in existing allocation standards. Meanwhile, our Random Flow Algorithm presents a novel mechanism enhancing fairness and efficiency compared to traditional Random Priority Mechanisms.

Together, these contributions underscore the power of game theory and economic design in addressing complex economic and public policy challenges.

1. We define an agent's most favored object as having rank 1.

Chapter 2

Intertemporal time preference on a simplex in continuous time

2.1 Introduction

Time preference and intertemporal decision making have long been central issues in economics. Koopmans' early work showed that under some conditions, an exponential discount utility model can explain how people value consumption over time (Koopmans 1960, 1966, 1972), and Bleichrodt furthered clarity on the mathematical foundations of Koopmans's theorem (Bleichrodt et al. 2008). As mentioned by Koopmans, solving the discontinuous case is simpler and is also the main step towards the continuous case.

Most traditional models assume an unconstrained outcome space and rely on the von Neumann-Morgenstern independence axiom. However, many real-life decisions, such as dividing a fixed cake over time or allocating a fixed budget among different investments, do not fit this model. In these situations, a simplex better represents outcomes (i.e., Qin and Rommeswinkel 2022) and shows how a fixed total wealth is allocated across different choices.

In this paper, we extend the static representation theorem of ibid. to a dynamic, continuous-time setting. Our approach allows for consumption streams that are not entirely smooth but can include jumps or discontinuities. This flexibility is important when modeling sudden changes in consumption or investment behavior. Meanwhile, including discontinuous functions is particularly important because solving the discontinuous case often serves as a critical step toward addressing the continuous case. For instance, Kopylov 2010 examines simple step functions as a foundation for continuous analysis. We also examine how delay affects preferences, establishing that one agent is more impatient than another precisely when their discount factor is higher.

To our understanding, there are three similar works, Harvey and Østerdal 2012, Pivato 2021, and Qin and Rommeswinkel 2022. The former considers a piece-wise continuous function with the outcome space being a product space. Moreover, they introduce a technically demanding axiom, Mid-outcome Independence, which is clearly stated in Discussion, while we use standard axioms in our representation theorem. The second one focuses on a connected space, emphasizing the importance and realism of continuous functions, while allowing for finitely many discontinuities. The last one deals with the static preference over simplex while we focus on dynamic model. Our research is necessary and complementary to this series of research and is unique in illustrating a representation theorem for preference over piecewise continuous functions within a simplex.

In the continuous-time problem, significant contributions have been made by Bell 1974 in demonstrating the exponential discount expected utility form for evaluating income time streams, Chichilnisky 1981 in identifying conditions for optimal growth in infinite continuous time, and Weibull 1985 in offering a representation theorem for single-value streams among others. However, these works often focused on spaces with either discontinuous functions in continuous time when consumption is in a product space (Harvey and Østerdal 2012, Hara 2016), or continuous functions in continuous time in a connected space (Pivato 2021), or on spaces with step functions (Kopylov 2010).

Regarding risk capture over time, one approach considers choice objects as probability measures on deterministic outcome streams in continuous time (Epstein 1983, Hayashi 2003, Hara 2016), while another views them as lottery streams or functions (Epstein and Schneider 2003, Lu and Saito 2018). These methods differ significantly in their treatment of choice objects and temporal decisions. Our study aligns with the latter approach and acknowledges the inclusion of discontinuous streams due to functions with finite jumping points. While Pivato 2021 focused on continuous trajectories, most research has concentrated on discontinuous trajectories (Kopylov 2010, Harvey and Østerdal 2012, Hara 2016).

Also, seminal works such as Fishburn 1970 and Fishburn and Rubinstein 1982, Lancaster 1963 have concentrated on the theoretical foundations of decision-making for single outcomes or multidimensional commodity bundles at discrete time points.

The article is organized as follows: Section 2.2 presents the model, Section 2.3 introduces axioms, Section 2.4 discusses the main results, Section 2.5 compares them to previous studies, and Section 2.6 concludes the study.

2.2 Model

Let $\mathscr{T} = [0,T]$ be an interval and $T < \infty$. Outcome space is a n-1 dimensional standard simplex as $\Delta = \{p \in [0,1]^n : \sum_{i \in I} p_i = 1\}$ with a natural algebraic structure, convex combination, and denote the *support* of an element $p = (p_1, p_2, ..., p_n)$ as the set $s(p) = \{i \in \{1,2,...,n\} | p_i > 0\}$. Δ is a subset of product spaces, moreover, any change in one dimension will change the value in other dimensions.

We define choice objects as outcome stream (or trajectory) denoted by the bold letter: $\mathbf{p} : \mathscr{T} \to \Delta$. In this paper, we mainly discuss piecewise continuous outcome stream. In other words, $\mathbf{p}$ is piecewise continuous if it is continuous on each piece of a finite partition of $\mathscr{T}$, and the image of each piece is contained in Δ. In the remaining part, we consider outcome stream space $\mathscr{P}$ as all piecewise continuous outcome streams $\mathbf{p}$ and the outcome stream space $\mathscr{P}_t$ that contains all piecewise continuous outcome streams $\mathbf{p}_{[t,T]}$ starting from time t for any fixed $t \in [0,T]$.

At every time t, there is an ordering $\succsim_t$ on $\mathscr{P}_t$, with the meaning that if $\mathbf{p}_{[t,T]} \succsim_t \mathbf{q}_{[t,T]}$, in words, today's preferences will only be relevant to consumption beyond today and independent from the past consumption. For simplicity, we will denote it as $\mathbf{p} \succsim_t \mathbf{q}$. Hence, $\succ_t$ and $\sim_t$ are strict preference and indifference components of $\succsim_t$.

2.3 Axioms

In the following, we define the standard axioms based on a given the outcome space Δ and outcome stream space $\mathscr{P}_t$ for any t.

First of all, we have to derive the preference over simplex (the static outcome). At each time t, we derive a preference $\succsim_t^{\Delta}$ on Δ from $\succsim_t$. For any outcome $p \in \Delta$, we denote $\mathbf{c}(p)$ as the constant outcome stream such that it generates outcome p for every time t.

Now, for all t, for all outcomes $p, p^{'} \in \Delta$, and constant outcome streams $\mathbf{c}(p), \mathbf{c}(p^{'}) \in \mathscr{P}$,we say $p \succsim_t^{\Delta} p^{'}$ if $\mathbf{c}(p) \succsim_t \mathbf{c}(p^{'})$.

In literature, it is often defined $\succsim_t$ and $\succsim_t^{\Delta}$ separately and assuming one 'agrees with' the other in some sense (i.e., Harvey and Østerdal 2012). Although there is no difference for the main result, it is useful to derive $\succsim_t^{\Delta}$ from $\succsim_t$. One advantage is that our framework can easily be adapted to situations where the utility function or preferences change over time.

Axiom 1. ***Weak order [WO]***: *$\forall t \in \mathscr{T}, \succsim_t$ is complete and transitive over $\mathscr{P}_t$.*

Axiom 1 also implies $\succsim_t^{\Delta}$ satisfies Weak Order because $\succsim_t^{\Delta}$ is derived from the orderings $\succsim_t$. Next, we state the axiom for preference $\succsim_t^{\Delta}$ only according to Qin and Rommeswinkel 2022. We need one more notation $[n] = \{1, 2, ..., n\}$. Reminder $s(p)$ is the *support* of an element $p = (p_1, p_2, ..., p_n)$.

Axiom 2. ***Simplex Additive Utility [SAU]*** *: At any time t, we state $\succsim_t^{\Delta}$ satisfies Strong Essentiality and Comeasurability, and Dimensional Independence, where*

1. ***Strong Essentiality*** *We say $\succsim_t^{\Delta}$ on set Δ fulfills strong essentiality for a dimension set $[n]$ if for all $\alpha \in (0,1)$ there exist some $p, p^{'}, p^{''}$ such that $s(p), s(p^{'}) \subseteq [n]$, and $s(p^{''}) \cap [n] = \emptyset$, then $\alpha p + (1-\alpha) p^{''} \succ_t^{\Delta} \alpha p^{'} + (1-\alpha) p^{''}$*
2. ***Comeasurability*** *For every t, $\forall p, p^{'}, p^{''}, p^{'''}, \tilde{p}, \tilde{p}^{'}, \tilde{p}^{''}, \tilde{p}^{'''} \in \Delta, a, \tilde{a} \in (0,1)$,*

if $[s(p)\cup[s(p^{'})]\cap[s(p^{''})\cup[s(p^{'''})] = \emptyset$, *and* $[s(\tilde{p})\cup[s\tilde{p}^{'})]\cap[s(\tilde{p}^{''})\cup[s(\tilde{p}^{'''})] = \emptyset$, *s.t.*

$$\begin{cases} ap+(1-a)p^{''} \sim_t^{\Delta} \tilde{a}\tilde{p}+(1-\tilde{a})\tilde{p}^{''} \\ ap+(1-a)p^{'''} \sim_t^{\Delta} \tilde{a}\tilde{p}+(1-\tilde{a})\tilde{p}^{'''} \\ ap^{'}+(1-a)p^{''} \sim_t^{\Delta} \tilde{a}\tilde{p}^{'}+(1-\tilde{a})\tilde{p}^{''} \end{cases} \Rightarrow [ap^{'}+(1-a)p^{'''} \sim_t^{\Delta} \tilde{a}\tilde{p}^{'}+(1-\tilde{a})\tilde{p}^{'''}] \tag{2.1}$$

3. ***Dimensional Independence*** *For every t,*$\forall p, p^{'}, p^{''}, p^{'''} \in \Delta, a \in (0,1)$, *if* $[s(p)\cup[s(p^{'})]\cap[s(p^{''})\cup[s(p^{'''})] = \emptyset$, *s.t.*

$$[ap+(1-a)p^{''} \succsim_t^{\Delta} ap^{'}+(1-a)p^{''}] \Rightarrow [ap+(1-a)p^{'''} \succsim_t^{\Delta} ap^{'}+(1-a)p^{'''}] \tag{2.2}$$

Axiom 2 is the same necessary and sufficient condition for additive representation on the simplex from Qin and Rommeswinkel 2022. This ensure the induced preference on Δ to admit a simplex-additive representation (Theorem 1 in Qin and Rommeswinkel 2022).

Now we state the axiom for time preference. We define $\alpha\mathbf{p}+(1-\alpha)\mathbf{q}$ as pointwise convex combination between trajectories $\mathbf{p}$ and $\mathbf{q}$. Moreover we define $(\mathbf{p}, {}_x\mathbf{q}_{[a,b)})$ as the stream on the time interval $[0,T]$ which follows $\mathbf{q}(t-x)$ from time a to b and follows $\mathbf{p}$ on the time interval $\mathscr{T}\setminus[a,b)$. Particularly, we denote $(\mathbf{p}, {}_0\mathbf{q}_{[a,b)})$ as $(\mathbf{p}, \mathbf{q}_{[a,b)})$ (see Figure 2.1).

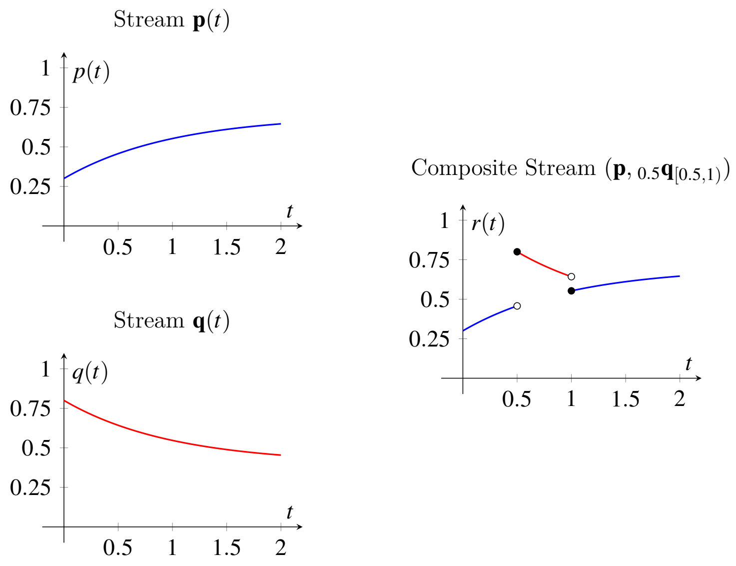

Figure 2.1: Left: Original streams $\mathbf{p}(t)$ (top) and $\mathbf{q}(t)$ (bottom).
Right: Composite stream $(\mathbf{p}, {}_{0.5}\mathbf{q}_{[0.5,1)})$.

Axiom 3. ***Independence [I]***: *For all t, for all sub half closed interval V of $[t,T]$, for all $\mathbf{p},\mathbf{p}^{'},\mathbf{q},\mathbf{q}^{'} \in \mathscr{P}_t$ we have*

$$(\mathbf{p},\mathbf{q}_V) \gtrsim_t (\mathbf{p}^{'},\mathbf{q}_V) \Rightarrow (\mathbf{p},\mathbf{q}_V^{'}) \gtrsim_t (\mathbf{p}^{'},\mathbf{q}_V^{'}) \tag{2.3}$$

Axiom 3 is the traditional separability: one may wish to assume that preferences between money and gold are separable for any period.

Axiom 4. ***Continuity [C]***:

For all t, for any $\mathbf{p} \in \mathscr{P}_t$ and any $\mathbf{q} \in \mathscr{P}_t$

1. *If $\mathbf{p} \succ_t \mathbf{q}$, then there exists $\delta > 0$ and $\mathbf{h}$ such that $d(\mathbf{q}(t^{'}),\mathbf{h}(t^{'})) < \delta$ for all $t^{'} > t$ implies $\mathbf{p} \succ_t \mathbf{h}$ for all $\mathbf{h} \in \mathscr{P}_t$.*
2. *If $\mathbf{q} \succ_t \mathbf{p}$, then there exists $\delta > 0$ and $\mathbf{h}$ such that $d(\mathbf{q}(t^{'}),\mathbf{h}(t^{'})) < \delta$ for all $t^{'} > t$ implies $\mathbf{h} \succ_t \mathbf{p}$ for all $\mathbf{h} \in \mathscr{P}_t$.*

Debreu 1959, Gorman 1968, and Harvey and Østerdal 2012 employed conditions that are analogous to Axioms 1, 3, and 4 in order to formulate additive-utility models. These models are specifically tailored for the scenario involving multivariable consequences and employ step functions that are contingent upon a designated partition of the interval $[0,T]$.

Axiom 5. ***Time Monotonicity [TM]***: $\forall t \in \mathscr{T}$.

1. $\forall t \in \mathscr{T}, \forall \mathbf{p},\mathbf{p}^{'} \in \mathscr{P}_t$

   $$\forall \tau \geq t, \mathbf{p}(\tau) \gtrsim_t^{\Delta} \mathbf{p}^{'}(\tau) \Rightarrow \mathbf{p} \gtrsim_t \mathbf{p}^{'}$$

   Moreover, if $\mathbf{p}(\tau) \gtrsim_t^{\Delta} \mathbf{q}(\tau)$ almost everywhere except a non-point interval, and $\mathbf{p}(\tau) \succ_t^{\Delta} \mathbf{q}(\tau)$ on that non-point interval, then $\mathbf{p} \succ_t \mathbf{q}$.
2. *For all t, for all constant outcome stream $\mathbf{c}(p)$ and $\mathbf{c}(p')$, we have*

   $$\textit{if } p \gtrsim_\tau^{\Delta} p' \textit{ for all } \tau > t \textit{ except a finite numbers, then } \mathbf{c}(p) \gtrsim_t \mathbf{c}(p')$$

Axiom 5 suggests that if one stream is always superior to the other for any single time period based on the current period preference, then the agent should hold the same preference. If there exist a non-point interval, i.e, $[\tau, \tau')$ with $\tau < \tau'$, we have $\mathbf{p}(s) \succ_s^\Delta \mathbf{q}(s)$, then $\mathbf{p} \succ_t \mathbf{q}$. It's worth clarifying that the second part differs from the definition of $\gtrsim^\Delta$.

Axiom 6. ***Time Stationary [TS]:*** *$\forall t \in \mathscr{T}, \mathbf{p} \in \mathscr{P}_t, \gamma > 0$, and $\mathbf{q}, \mathbf{q}^{'} : [0, \gamma] \to \Delta$, if for some $\tau \geq t$, we have*

$$(\mathbf{p}, {}_\tau\mathbf{q}_{[\tau,\tau+\gamma)}) \gtrsim_t (\mathbf{p}, {}_\tau\mathbf{q}^{'}_{[\tau,\tau+\gamma)}) \tag{2.4}$$

then it's true for any τ with $\tau \geq t$ and $\tau + \gamma \leq T$.

Axiom 6 is a reduced continuous-time form of risk preference introduced by Epstein and Schneider 2003. This imposes a similar time-stationary property on the ranking of probability measure streams. In continuous time, this axiom applies not to two single time periods but to a small time interval. Consequently, a specific component of the probability measure stream, $\mathbf{p}, \mathbf{q}$, remains invariant starting time τ. Axiom 6 necessitates invariance with time τ and γ, such that $\tau + \gamma \leq T$, implying that if a discounted utility function exists at time $\tau = t$ for some time periods $[\tau, \tau + \gamma]$, then a discounted utility function exists at any time t.

Axiom 7. ***Time Impatience [TI]:*** *$\forall t \in \mathscr{T}, \forall \gamma > 0$ and $\gamma + t \leq T$, $\forall \mathbf{p}, \mathbf{q} \in \mathscr{P}_t$ with $\mathbf{p}(s) \gtrsim_s^\Delta \mathbf{q}(s), \forall s \in [t, T]$, if for some $\tau \geq t$ we have*

$$(\mathbf{q}, \mathbf{p}_{[t,t+\gamma)}) \gtrsim_t (\mathbf{q}, \mathbf{p}_{[\tau,\tau+\gamma)}) \tag{2.5}$$

then this is true for all $\tau \geq 0$ and $\tau + \gamma \leq T$.

Axiom 7 enforces that if a player prefers a probability measure stream now over one in the future for any delayed period $[0, \tau]$ at the outset, they need to hold the same attitude at any future time t, and vice versa.

Axiom 8. ***Dynamic Consistency [DC]:*** *$\forall t \in \mathscr{T}, \mathbf{p}, \mathbf{p}^{'} \in \mathscr{P}_t, \forall \tau \leq t$*

$$[\mathbf{p}(\tau) = \mathbf{p}^{'}(\tau)] \wedge [\mathbf{p} \gtrsim_t \mathbf{p}^{'}] \Rightarrow [\mathbf{p} \gtrsim_\tau \mathbf{p}^{'}] \tag{2.6}$$

with the strict inequality holding if $\mathbf{p}^{'} \succ_t \mathbf{p}$.

Axiom 8 posits that if an agent opts for the outcome stream $\mathbf{p}$ over $\mathbf{p}^{'}$ at time t when everything before time t is identical, it would appear irrational if they changed their preference to $\mathbf{p}^{'}$ over $\mathbf{p}$ when returning to a previous time. This axiom reflects classic dynamic consistency within the context of intertemporal choice. While Axiom 6 and Axiom 8 appear to capture similar behavior when considering a single preference relation, there are conceptual differences between them within the context of the current model. Specifically, Axiom 6 pertains to a single preference relation within a specific period, whereas Axiom 8 encompasses the entire process.

Although the relation $\succsim_t$ originates from $\succsim_0$ due to the principle of Dynamic Consistency, it is important to clarify that the focus of the model under consideration is on the whole process of preferences. For this reason, the notation $\succsim_t$ will be employed in subsequent discussions.

Axiom 9. ***Non-Degeneracy [ND]***: *For every t, there are some* $\mathbf{p}, \mathbf{p}^{'} \in \mathscr{P}_t$, *such that* $\mathbf{p}^{'} \succ_t \mathbf{p}$.

Axiom 9 stipulates that the decision-making problem and qualitative consumption are nontrivial by eliminating the possibility of the decision-maker being indifferent among all consumption at every time, implying that a strict component is non-empty.

2.4 Result

In this work, we provide two representation theorems under the restriction of $\mathscr{P}$ and present a representation of when consumption is under budget, enabling the natural construction of an expected utility version of the utility function through homogeneity, as the expected utility is a linear variant of our model. Specifically, Proposition 1 characterizes the preference $\succsim_0$ at a fixed date via an integral over an arbitrary weight function $\delta(t)$ and Proposition 2 strengthens this to the dynamic setting $\{\succsim_t\}$, deriving the unique exponential form and the discount function $\delta(t) = e^{-\beta t}$.

Proposition 1. *For a binary relation* $\succsim_0$ *on* $\mathscr{P}$ *and a derived preference* $\{\succsim_0^{\Delta}\}$ *on* $\boldsymbol{P}$, *the following statements are equivalent:*

1. $\succsim_0$ *satisfies WO, I, TM, TS, C, ND and* $\succsim_0^{\Delta}$ *satisfies SAU.*

2. *There exists a utility function $U(p)=\sum_i^n u_i(p_i)$ for all $p\in\Delta$, exists a continuous function $\delta:\mathscr{T}\to\mathbb{R}$, such that for any $\mathbf{p},\mathbf{q}\in\mathscr{P}$,*

$$\mathbf{p}\succsim_0\mathbf{q}\Leftrightarrow\int_0^T U(\mathbf{p}(t))\delta(t)dt\geq\int_0^T U(\mathbf{q}(t))\delta(t)dt \tag{2.7}$$

The proof of Proposition 1 is based on the proof of Theorem 1 in Harvey and Østerdal 2012 and the additive representation theorem in Debreu et al. 1954 and Gorman 1968, but it relies on a set of standard axioms compared to Harvey and Østerdal 2012. There is no direct logical connection between the two results due to differences in the properties of the outcome spaces. Specifically, we require the outcome space to be a simplex, whereas ibid. relies on the full product space of intervals. Moreover, their framework and axioms depend on the existence of a null outcome, which is also impossible in Δ. Instead, we incorporate the commonly used axiom of Non-degeneracy to address this limitation.

When we restrict attention to static outcomes, Weak Order, SAU and Continuity on the simplex Δ guarantee by standard results (Qin and Rommeswinkel 2022) a continuous additive utility $U(p)=\sum_{i=1}^n u_i(p_i)$. Next, any piecewise-continuous stream can be approximated by a step function on a finite partition of $[0,T]$, and Independence together with Goldman 1957 and Gorman 1968 on the product space Δ^m yields an additive representation (Lemma 5). To use that, we have to show that a point interval is inessential and non-trivial intervals are essential (Lemma 4). Then we have to show that all functions are cardinal equal by the Jensen functional equation. This is the key difference from Harvey and Østerdal 2012, and they use a new complicated axiom called 'Mid-outcome independence'. We instead use Time Stationary to show the cardinal equivalence in Lemma 6. After that, we follow the last step in Theorem A.1 (ibid.) to show it has the integral representation, which completes the proof.

Proposition 2. *For a set of preference $\{\succsim_t\}_{t\in\mathscr{T}}$ on $\mathscr{P}$ and a set of derived preference $\{\succsim_t^\Delta\}_{t\in\mathscr{T}}$ on P, following statements are equivalent:*

1. *$\forall t$, $\succsim_t$ satisfies Axiom WO, I, TM, TS, TI, DC, C, ND and $\succsim_t^\Delta$ satisfies SAU.*
2. *There exists a utility function $U=\sum_i^n u_i(p_i)$, exists a discount factor β, s.t. for any $t\in\mathscr{T}$, and $\mathbf{p},\mathbf{q}\in\mathscr{P}$*

$$\mathbf{p}\succsim_t\mathbf{q}\Leftrightarrow\int_t^T e^{-\beta(s-t)}U(\mathbf{p}(s))ds\geq\int_t^T e^{-\beta(s-t)}U(\mathbf{q}(s))ds \tag{2.8}$$

To prove Proposition 2, we use Time stationary and Time impatience and Cauchy's multiplicative functional equation to show the exponential expression of $\delta(t)$. Then we use Dynamic Consistency to show $\succsim_t$ share the same discount factor β. To do so, we will pick one utility index $(u_i)_{i\in n}$ that assigns utility 1 to the first dimension and 0 to the others. Then we will construct the two trajectories that are the same up to some time t and different in the first dimension such that $\mathbf{p} \succsim_t \mathbf{q}$. Dynamic consistency say the relation between $\mathbf{p}$ and $\mathbf{q}$ will remain for all $\tau < t$. We can show a contradiction to assumption $\mathbf{p} \succsim_t \mathbf{q}$ if $\delta_t \neq \delta_{t'}$ for some $t \neq t'$.

The results presented in this paper diverge from previous findings from two perspectives. For one thing, the outcome space considered is distinct from those in earlier studies. For instance, Harvey, Østerdal et al. 2007; Harvey and Østerdal 2012, and Hara 2016 stipulated that the outcome space must be a product set. On the opposite, this research examines a subset of the product space, leading to distinct assumptions about preferences in the outcome space. Also, we focus on an intertemporal preference structure as opposed to the temporal preference employed by Weibull 1985, Harvey, Østerdal et al. 2007; Harvey and Østerdal 2012, Hara 2016, and Kopylov 2010, among others. For another, Pivato 2021 offered a representation theorem for intertemporal preferences with any connected outcome space in 2021. Nevertheless, discontinuity is not allowed in his research.

Returning to our results, Time Monotonicity (2) implies that the utility function remains the same for all time t, and Dynamic Consistency suggests that the discount factor is constant over time t. However, it may be more practical to drop Time Monotonicity (2) in certain situations, for example, our preference over gold and currency varies. Now, we analyze how the parameter β changes influence investor behaviour.

Definition 1. *(at least as impatient as) For any time t, we say a preference* $\succsim_t^*$ *is at least as impatient as a preference* $\succsim_t$*, where* $\succsim_t^\Delta = \succsim_t^{*\Delta}\ \forall t$ *if and only if for all* $\gamma > 0$ *for all* $\mathbf{p},\mathbf{q} \in \mathscr{P}_t$ *with* $\mathbf{p}(s) \succsim_t^\Delta \mathbf{q}(s)\ \forall s$*, there exists some* $\tau \geq t$ *we have*

$$(\mathbf{q}, \mathbf{p}_{[t,t+\gamma)}) \sim_t (\mathbf{q}, \mathbf{p}_{[\tau,\tau+\gamma)}) \Rightarrow (\mathbf{q}, \mathbf{p}_{[t,t+\gamma)}) \succsim_t^* (\mathbf{q}, \mathbf{p}_{[\tau,\tau+\gamma)}) \tag{2.9}$$

We compare two people, A and B, who agree on which payoffs are better at each moment. We say "A is at least as impatient as B" if whenever B is exactly torn between getting a payoff now versus a payoff later, A would prefer getting it sooner. We say a preference $\succsim_t^*$ is strictly more impatient than a preference $\succsim_t$ if $(\mathbf{q}, \mathbf{p}_{[t,t+\gamma)}) \sim_t (\mathbf{q}, \mathbf{p}_{[\tau,\tau+\gamma)}) \Rightarrow (\mathbf{q}, \mathbf{p}_{[t,t+\gamma)}) \succ_t^* (\mathbf{q}, \mathbf{p}_{[\tau,\tau+\gamma)})$.

Proposition 3. *For two different preference relations $\succsim_0$ and $\succsim_0^*$, $\succsim_0^*$ is at least as impatient as $\succsim_0$ if and only if $\beta^* \geq \beta$.*

In the standard exponential-discounting model, there's a single number β that tells decision makers how sharply they discount the future. The proposition shows that raising β exactly results in impatience in the sense above, and vice versa.

Corollary 1. *For two different preference relations $\succsim_0$ and $\succsim_0^*$, $\succsim_0^*$ is more impatient than $\succsim_0$ if and only if $\beta^* > \beta$.*

2.5 Discussion

One close paper is Pivato 2021 which considers all bounded continuous trajectories from (finite or infinite) time intervals to connected topological space. And the other close paper is Harvey, Østerdal et al. 2007; Harvey and Østerdal 2012 which consider piece-wise trajectories with the assumption 'Weakly Increasing Assumption'. However, this is not feasible considering that increasing the value in one dimension while keeping other dimensions constant is impossible in Δ.

Assumption 1. *Weakly Increasing Assumption*

We say $\succsim$ on outcome set $X = \prod_{i=1}^n X_i$ is general weakly increasing if $\forall j = 1,..,n, \forall x_j \in X_j, \forall (x_j, x_{-j}), (x_j^{'}, x_{-j}) \in X$:

$$x_j \geq x_j^{'} \Leftrightarrow (x_j, x_{-j}) \succsim (x_j^{'}, x_{-j}) \tag{2.10}$$

Axiom* 1. ***Agree with**: For any $\mathbf{p}, \mathbf{q} \in \mathscr{P}, \forall t$, we have*

1. $[\mathbf{p}(\tau) \succsim_t^{\Delta} \mathbf{q}(\tau)]$ *almost everywhere for $\tau \geq t$ in P and $\mathbf{p}(\tau) \succ_t^{\Delta} \mathbf{q}(\tau)$ on a non-point interval* $\Rightarrow [\mathbf{p} \succ_t \mathbf{q}]$

2. $[\mathbf{p}(\tau) \succsim_t^\Delta \mathbf{q}(\tau)]$ *almost everywhere for* $\tau \geq t$ *in* $P \Rightarrow [\mathbf{p} \succsim_t \mathbf{q}]$

People might notice a similarity between "Agree with" and "Time Monotonicity." The first part of "Agree with" is the same as the first part of "Time Monotonicity." However, the second part of "Agree with" is stronger because it applies to all outcome streams, while "Time Monotonicity" only applies to constant outcome streams.

Lastly, we state one axiom used in Harvey and Østerdal 2012: mid-outcome independent, instead we use Time monotonicity and non-degeneracy to ensure the outcome scale does not depend on the time. Let outcome space X be the product of intervals that contain the null outcome 0. Let $\mathbf{0}$ be the null outcome stream that generates 0 for all time. Then let $\succsim^X$ be the preference over outcomes and $\succsim$ be the preference over outcome streams. We use the same operation in the Previous section: $(\mathbf{x}_E, \mathbf{y}_F, \mathbf{z})$ is the outcome stream that generates $\mathbf{x}(t)$ on time period E, generates $\mathbf{y}(t)$ on time period F, and generates $\mathbf{z}(t)$ on the rest.

Now we define $x^{1/2} \in X$ as the mid-outcome of two outcomes x^0 and x^1 if 1) $x^1 \succ^X x^{1/2} \succ^X x^0$ and 2) exist $y^1 \succ^X y^0$ and $z \in \Delta$ such that $(\mathbf{c}(x^{1/2})_{<s,s'>}, \mathbf{c}(y^0)_{<t,t'>}, \mathbf{c}(z)) \sim (\mathbf{c}(x^0)_{<s,s'>}, \mathbf{c}(y^1)_{<t,t'>}, \mathbf{c}(z))$ and $(\mathbf{c}(x^{1/2})_{<s,s'>}, \mathbf{c}(y^1)_{<t,t'>}, \mathbf{c}(z)) \sim (\mathbf{c}(x^1)_{<s,s'>}, \mathbf{c}(y^0)_{<t,t'>}, \mathbf{c}(z))$ for some disjoint interval $< s,s' >$ and $< t,t' >$.

To notice, ibid. requires zero consumption instead of a constant outcome z. However, it's equivalent in their model when we impose the Axiom independence because we could replace z with any other constant outcome.

Moreover, ibid. needs one complicated axiom: Mid-outcome independence.

Axiom* 2. ***Mid-outcome independence*** *For any disjoint, non-point intervals* $< s,s' >$ *and* $< t,t' >$*, and outcome* $x^1 \succ^X x^0$*, if the pair* x^1,x^0 *has a mid-outcome with respect to outcomes in* $< t,t' >$*, and the pair* x^1,x^0 *has a mid-outcome with respect to outcomes in* $< s,s' >$*, then* x^1,x^0 *has the same mid-outcome in both cases.*

From their result, if x^1,x^0 has same mid-outcome $x^{1/2}$ in two disjoint interval $< s,s' >$ and $< t,t' >$, then $u(x^{1/2}) = 1/2u(x^0) + 1/2u(x^1)$ holds in both time intervals $< s,s' >$ and $< t,t' >$. However, without requiring this axiom, we can obtain a similar conclusion with the combination of Independence, Time monotonicity, Time stationary, and Non-Degeneracy. We mainly show if $x^{1/2}$ is mid-outcome to x^1 and x^0 with respect to outcomes in $< t, t' >$, then it's indifferent to the mid-outcome to x^1 and x^0 with respect to outcomes in $< s, s' >$. To notice, it doesn't imply equivalence or other logical relation it's weaker in the sense we can't guarantee they have the same mid-outcome, but we can conclude those two mid-outcomes are indifferent, hence equivalent in cardinal utility.

Harvey and Østerdal 2012 use this axiom and Assumption 1 to provide Riemann integral for piecewise continuous function from time interval to product space. It is worth noting that the 'Weakly Increasing Assumption' and 'Mid-outcome independence' is a pivotal in their theorem. As a consequence, the representation theorems stated in Proposition 1 and Proposition 2 cannot be directly derived from Harvey and Østerdal's contributions.

2.6 Conclusion

This paper introduces a novel representation theorem for exponential discounted additive utility within a continuous-time framework, where consumption exists in a simplex.

Our primary contribution is the formulation of a representation theorem that elucidates the decision-makers preferences over consumption streams within a budget, integrating risk into the model. A potential future extension could be to replace exponential discounting with a broader class of discount functions or test the axiom Time Monotonicy (2) instead of focusing on Dynamic consistency in literature. This would necessitate a new set of axioms, possibly leading to alternative representations of preferences. Incorporating stochastic elements into the outcome domain would offer a more comprehensive perspective on decision-making under uncertainty over time, capturing the complexities of real-world scenarios that are influenced by both temporal preferences and the uncertain nature of future events.

Our approach differs significantly from existing literature in several key aspects. First, the consumption space in our study is a subset of a product space without null consumption, which limits the direct application of theorems by Hara 2016; Harvey, Østerdal et al. 2007; Harvey and Østerdal 2012; Kopylov 2010; Weibull 1985. Second, we expand the consumption stream space to include piecewise continuous functions, differentiating our work from that of Pivato 2021 and Hara 2016.

Contrary to previous studies (Hara 2016; Epstein 1983; Hayashi 2003), we move away from the traditional risk paradigm associated with probabilistic distributions over certain outcome streams. Instead, we adopt Epstein and Schneider 2003 and Lu and Saito 2018, treating consumption as a probabilistic measure in a broader context. However, our model differs from Harvey and Østerdal 2012 by allowing Lebesgue integrals. Our approach allows for greater flexibility by considering time preferences beyond a fixed preference at t=0, as ensured by Dynamic Consistency. We also introduce a unique structure for the discount factor, focusing on Time Stationarity and Time Impatience, aspects not explored in Harvey, Østerdal et al. 2007.

This framework lays a solid groundwork for analyzing various decision problems involving intertemporal preference. By adjusting some assumptions and extending the model to more encompassing settings, we can obtain deeper insights into the mechanisms of intertemporal choice, leading to more precise and practical models for decision-making in continuous time.

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Chapter 3

Dominated Strategy in Random Game

3.1 Introduction

The concept of a dominated strategy is fundamental in microeconomics and game theory: a strategy is strictly dominated by another strategy if it yields a lower payoff than another strategy in every possible scenario. It's used in many solution concepts such as *dominance solvability* (Gale 1953, Luce and Raiffa 1957, Moulin 1979). Recently, the dominance solvability of random games has gained attention (Alon et al. 2021), although strictly dominated strategies themselves have not been the primary focus. A random game, as introduced by Goldman 1957, is a non-cooperative game where payoffs are independently and identically distributed (i.i.d.) according to a probability distribution. Identifying strictly dominated strategies or computing sets of undominated strategies becomes computationally intensive as the size of the players' strategy spaces grows (Yu and Zeleny 1975). Moreover, in large games, the existence of a strictly dominated strategy may be less significant than in smaller games. Hence, we seek the existence of $q - portion$ strictly dominated strategies.

Alon et al. 2021 indicates that the probability that the random game has a strictly dominated strategy trends towards zero when the size of the strategy space is not excessively unbalanced. In this study, we at first show a positive result and conform the conjecture in ibid.: there exists a specific threshold defined by $f(n) \in \Theta(\ln(n))$,[1] such that as m, n go to infinity with $m > f(n)$, the probability of a game lacking any

1. We write $f(n) = \Theta(g(n))$ if both $g(n) = O(f(n))$ and $f(n) = O(g(n))$. Informally, it means that f is bounded both above and below by g asymptotically.

dominated strategies approaches 1, and consequently, the probability of the game being dominance solvable tends to zero. For example, in an $m \times n$ random game with uniformly distributed payoffs, if the growth rate $m(n)$ falls within the range $[\log_2(n^{2+\delta}), 2^{\frac{n}{2+\delta}}]$ for some $\delta > 0$, our findings indicate that the probability of the existence of a strictly dominated strategy, denoted as P(SD), nears zero. This implies that the probability of a game being dominance solvable also approaches zero, as illustrated in Figure 3.1 (Yellow Area).

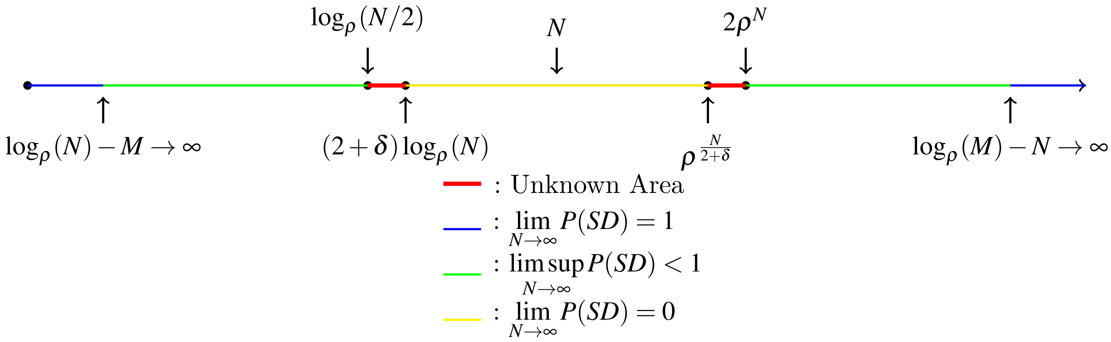

Figure 3.1: Summary of Results

Note: This figure is as M varies for fixed N

This paper additionally analyzes the 'only if' part: for the majority of cases outside the specified region, P(SD) is bounded away from 0 as N approaches infinity. Importantly, this represents an upper limit on the probability that a game is dominance solvable, as the initially mentioned relationship is not an equivalence but a strict subset. Therefore, it encompasses the growth rate $m = \log_2(n) + \omega(1)$. This study is also the first to quantify the results in terms of the size of the atoms of the underlying probability distribution, allowing the results to be applicable across various distributions. However, due to our focus on defining constraints and identifying a region for positive outcomes, we do not provide characterizations for the probabilities of 'dominance solvability' as detailed by Alon et al. 2021, rather only for the existence of a strictly dominated strategy. Nonetheless, our findings present a precise cutoff in the order of $\Theta(\ln(n))$.

For the large game, we answer another question: what is the asymptotic behavior of a fixed portion of strategies as M and N approach infinity? This question is important for three reasons. First of all, this notion is much more meaningful in the large game. Secondly, it is practical to eliminate many strategies instead of just a single strategy. Lastly, Dufwenberg and Stegeman 2002 finds that if strategy spaces are compact and payoff functions are upper semicontinuous in their own strategies, then the order of elimination does not matter, which means that if we eliminate all current strictly dominated strategies at once, it will lead to the same reduction game in the end. However, it will reduce the number of iterations and minimize redundancy in checking strategies repeatedly.

However, addressing this question is non-trivial due to the complexity of conditional events across multiple strategies. We establish the negative result using binomial coefficient approximations and the positive result using the Chernoff bound. Specifically, for the row player (with M), the probability of the existence of a q-portion of dominated strategies will converge to 0 as M and N approach infinity, provided that $N \geq M/(\ln(M))^{\alpha}$ for some fixed $\alpha > 0$. Conversely, the probability will converge to 1 as M and N approach infinity, provided that $M \gg (N/(1-\delta-q))^{N}$ for some fixed $\delta > 0$. Notice that, this is a stronger result as we require that there exists $q-$portion dominated strategies are dominated by a single strategy. Perhaps the threshold could be improved by relaxing this condition.

Finding an efficient algorithm to obtain all dominated strategies or undominated strategies has been the interest in economics and computer science since Yu and Zeleny 1975. They propose a much more efficient algorithm, but rely on the particular constraint of the problem. Here, we try to propose an efficient algorithm without constraint. Current computational complexity is $O(M^2N)$ to find strictly dominated strategies for row player in a given payoff matrix, which involves comparing every element of their payoff vectors across different strategies.

In this work, we aim to find a simple algorithm that is faster, at least by a constant factor, than the most direct brute-force algorithms. Such an algorithm could potentially be combined with existing ones to achieve even greater efficiency. From the simple observation, if the maximum of one payoff vector is smaller than the minimum of the other payoff vector, then it's strictly dominated for sure. Even though this will help, the probability of this event approaches 0 as the number of strategies tends to infinity. We have another observation for the rest of the events when the maximum of one payoff vector is larger than the minimum of another payoff vector. If a particular pattern exists—where the range of one payoff vector is contained within the range of the other—then it is certain that neither dominates the other. Importantly, the probability of this event approaches $\frac{1}{2}$ as the number of strategies tends to infinity. If neither of these cases applies, we compare every element of their payoff vectors.

In practice, this is efficient because there are two cases where checking every element is unnecessary, and they will happen about 50% of the time as M,N go to infinity. By Knuth 1997, the minimum and maximum of a set of n elements can be found with $3n/2-2$ comparisons when n is even and with $3(n-1)/2$ comparisons when n is odd. Therefore, in the first two cases, we require at most $M \times (3N/2-2)+M(M-1)$ times instead $M(M-1)/2 \times N$ times when n is even. In the remaining case, we need to compare every element of the payoff vectors.

3.2 Literature Review

To best of our knowledge, Goldman 1957 is the first study to consider random games in the context of zero-sum games. As random games became a common topic of interest, many other concepts in this framework were investigated, including: **value of a random zero-sum game** (Thrall and Falk 1965,Thomas 1965,Cover 1966), **Nash Equilibrium** (Goldberg et al. 1968, Dresher 1970, Powers 1990, Rinott and Scarsini 2000, Bárány et al. 2007, McLennan 2005), **best-response dynamics** (Galla and Farmer 2013, Sanders et al. 2018, Pangallo et al. 2019, Heinrich et al. 2021, Amiet et al. 2021), or **equilibrium properties in evolutionary games** (Gokhale and Traulsen 2010, Han et al. 2012, Galla and Farmer 2013, Gokhale and Traulsen 2014, Duong and Han 2016b, Duong and Han 2016a). Also, modeling payoffs as independent random variables enables tractable, average-case analysis across diverse settings. For example, using random games is useful to model and understand social and biological systems in which very limited information is available, or where the environment changes so rapidly and frequently that one cannot predict the payoffs of their inhabitant (Fudenberg and Harris 1992, Han et al. 2012, May 2001, Gross et al. 2009). In addition, some experimentalists randomly select games to test theories(Erev et al. 2007).

However, strictly dominated strategies and related solution concepts were not a primary focus of the random-games literature. The closest one is by Pei and Takahashi 2019. They examine point-rationalizable and rationalizable strategies in random games, showing that all strategies are rationalizable with probability one as n approaches infinity in both symmetric and asymmetric games when the two players' strategy spaces are of the same order. Specifically, they show that all strategies survive iterated elimination of strategies strictly dominated by pure strategies with probability close to one when the strategy space is large in a symmetric random game. Notably, we analyze the upper bound of the probability that a strategy is rationalizable, as a rationalizable strategy cannot be strictly dominated by any pure strategy. Our main contribution is determining the threshold of how "imbalanced" a game must be to ensure the existence of strictly dominated strategies or "q-portion" dominated strategies.

The other relevant paper is Alon et al. 2021, which demonstrated that the probability of a two-person random game being dominance-solvable decreases as the number of actions increases, aligning closely with our results.

Their research notably suggests that when there is a considerable imbalance in players' action sets, the IESDS can still effectively simplify the game. However, the precise extent of imbalance required for the probability of dominance solvability to approach zero remains an open question. This leads to further inquiry into the potential effectiveness of IESDS. Furthermore, they hypothesize that for growth rates defined by $m = o(\ln(n))$ or $m = \log_2(n) + \omega(1)$,[2] m, n are the number of strategies for row and column players, a substantial proportion of strategies will remain undominated, nearly approaching 1. The crux of these inquiries lies in the probability of encountering a strictly dominated strategy. Specifically, when the growth rate of a function is in this range, this probability goes to one as the n goes to infinity, we can straightforwardly assess the probability of a game being dominance-solvable.

The article is organized as follows: Section 3.3 presents the model, Section 3.4 introduces the result in existence of strictly dominated strategy, Section 3.5 delivers the result in existence of $q-$portion strictly dominated strategy, Section 3.6 introduce new simple algorithm to check strictly dominance, and Section 3.7 concludes the study.

3.3 Preliminary

A *two-player finite game* is $G = \{S, T, u_1, u_2\}$, where S is set of strategies for player 1 with $|S| = M$, T is set of strategies for player 2 with $|T| = N$, $u_i : S \times T \to \mathbb{R}$ is the payoff for i=1,2. G can also be represented by a pair of payoff matrix $R_{M\times N} = \{R^1_{M\times N}, R^2_{M\times N}\}$, then any pair of two elements $(u_1(s_i,t_j), u_2(s_i,t_j)) \in R_{M\times N}$ is a payoff of player 1 and 2 generated by the strategy profile $\mathbf{s} = (s_i, t_j)$.

2. We write $f(n) = \omega(g(n))$ if $g(n) = o(f(n))$. Informally, it means that f dominates g asymptotically.

$$R_{M\times N}=\{R^1_{M\times N},R^2_{M\times N}\}=\{\begin{bmatrix}u_1(s_1,t_1) & \cdots & u_1(s_1,t_N)\\ \vdots & \ddots & \vdots\\ u_1(s_M,t_1) & \cdots & u_1(s_M,t_N)\end{bmatrix},\begin{bmatrix}u_2(s_1,t_1) & \cdots & u_2(s_1,t_N)\\ \vdots & \ddots & \vdots\\ u_2(s_M,t_1) & \cdots & u_2(s_M,t_N)\end{bmatrix}\} \tag{3.1}$$

In this work, we consider a random game G, where payoff $R^1_{M\times N}$ and $R^2_{M\times N}$ is a realization of random variable in $(X^i_{\mathbf{s}})_{i=1,2,\mathbf{s}\in S\times T}$ where $(X^i_{\mathbf{s}})_{i=1,2,\mathbf{s}\in S\times T}$ are selected i.i.d from probability distribution on [0,1]. We denote $\rho=\left[P(X^i_{\mathbf{s}}>X^i_{\mathbf{s}'})\right]^{-1}$, for any two random variables $X^i_{\mathbf{s}}$ and $X^i_{\mathbf{s}'}$. For example, if the distribution is non-atomic, then $\rho=2$; otherwise, $\rho>2$.

In a game G, we say the strategy s_i (or t_i) is *strictly dominated* for player 1 (or 2) if there exists another strategy $s_i^{'}$ (or $t_i^{'}$), s.t. s_i (or t_i) generates strict worse payoffs than $s_i^{'}$ (or $t_i^{'}$) for all $t\in T$ (or $s\in S$). We denote s_i strictly dominates s_j as $s_i\succ_1 s_j$ and denote s_i doesn't strictly dominates s_j as $s_i\sim_1 s_j$ for player 1, similarly $t_i\succ_2 t_j$ and $t_i\sim_2 t_j$ for player 2.

We denote the set of strictly dominated strategies of game G by $SD(G)$ with the cardinality $\mathbf{card}(SD(G))$.[3] In the following parts, we denote the probability that there doesn't exist strictly dominated strategy as $P(\neg SD)=P(\mathbf{card}(SD(G))=0)$. Due to independence of $(X^s_i)_{i=1,2,s\in S\cup T}$, we have $P(\neg SD)=P(SD^1=0)P(SD^2=0)$.

We will focus on asymptotic behavior of strictly dominated strategies because the specific distribution of strictly dominated strategies is far too cumbersome to calculate regarding M and N (see Hammett and Pittel 2008; Gunby and Pálvölgyi 2019; Alon et al. 2021). The reason it's difficult is that the domination relationships are complex. Strategy A might dominate Strategy B. Strategy C might also dominate Strategy B. Determining the final set of undominated strategies requires considering all these relationships simultaneously. Notably, even for a relatively simple case (M=3), just finding the probability that the other player (Column) has no strictly dominated strategies is equivalent to a known difficult mathematical problem ("permutation avoidance") that doesn't have an explicit, simple formula (Alon et al. 2021 Online Appendix).

3. For short, we use SD^1 and SD^2 to represent the cardinality of the set of strictly dominated strategies for players 1 and 2 respectively.

3.4 Existence of Strictly Dominated Strategy

In this work, we will study the probability that a game doesn't have a strictly dominated strategy. We observe the fact of bound of $P(SD) = 1 - P(\neg SD)$ where $P(SD)$ is the probability that there exists a strictly dominated strategy:

1. If for some strategy s_i, exist s_j such that $s_i \succ s_j$, then it implies there exists a strictly dominated strategy, hence the $P(\neg SD) = 1 - P(SD) \leq 1 - P(\text{for some } s_i, \exists s_j, s_i \succ s_j)$.
2. If there is no strict domination relation for any pair of strategies, then there is no strictly dominated strategy. Hence $\max\{(1 - \sum_{s_i \neq s_j} P(s_i \succ s_j)), 0\} \max\{(1 - \sum_{t_i \neq t_j} P(t_i \succ t_j)), 0\} \leq P(\neg SD)$.

In this part, we mainly check the following:

1. Under which condition, do those bounds converge to 1?
2. Under which condition, does the upper bound not converge to one?
3. Under which condition, do those bounds converge to 0?

Proposition 4. *In a $M \times N$ random game,*

1. $P(SD) \to 0$ *if* $M, N \to \infty$ *with* $M = N$.
2. $P(SD) \to 0$ *if* $M, N \to \infty$ *with* $(2+\delta) log_\rho(N) \leq M \leq \rho^{\frac{N}{2+\delta}}$ *for some* $\delta > 0$.

Proposition 4 provides an approximate result for the probability of a dominated strategy in a random game with payoffs that are independently and identically distributed over any distribution. The first statement is a special case of the second statement. The proof relies on the second observation

$$\max\{(1 - \sum_{s_i \neq s_j} P(s_i \succ s_j)), 0\} \max\{(1 - \sum_{t_i \neq t_j} P(t_i \succ t_j)), 0\} \leq P(\neg SD)$$

Take the first statement as an example. In an $M \times N$ random game with uniformly distributed payoffs, for player 1, the probability that a strategy strictly dominates the other strategy is $\frac{1}{2^N}$. Then this observation tells us $P(\neg SD) \geq (\max\{1 - N^2(\frac{1}{2^N}), 0\})^2$ which goes to 1 as N goes to infinity. The second statement is less straightforward, as one needs to deduce reasonable bounds on the rate of growth; however, after some trial and error, a proof with these given bounds follows from a straightforward calculation. It indicates that the probability of there being no dominated strategy will increase to 1 as the number of strategies satisfies certain conditions and increases to infinity.

Now, we can directly analyze the probability that a game is dominance solvable, denoted by P(Dominance Solvability).

Corollary 2. *In a $M \times N$ random game, P(Dominance Solvability) $\to 0$ if $M,N \to \infty$ with $(2+\delta)log_\rho(N) \leq M \leq \rho^{\frac{N}{2+\delta}}$, for some $\delta > 0$.*

Notice that this does not contain the 'only if' part because the event 'a game doesn't have a strictly dominated strategy' is a strict subset of the event 'a game is not dominance solvable'. In the following, we can show the 'only if' part for Proposition 4 that the range of $M \in [(2+\delta)log_\rho(N), 2^{\frac{N}{2+\delta}}], \forall \delta > 0$ is very nearly tight.

Proposition 5. *In a $M \times N$ random game,*

1. $\limsup P(SD) < 1$ *if* $M,N \to \infty$ *with* $N \leq \log_\rho(M/2)$.
2. $P(SD) \to 1$ *if* $M,N \to \infty$ *with* $\log_\rho(M) - N \to \infty$.

To prove the first statement in Proposition 5, we rely on the inclusion and exclusion equation of $P(\neg SD) \leq 1 - P$ (for some s_i, s_i is dominated)$= \lim\limits_{M,N\to\infty}(1 - \binom{N-1}{1}/\rho^M + \binom{N-1}{2}/(\rho^M)^2 - \cdots - (-1)^{N-2}\binom{N-1}{N-1}/(\rho^M)^{N-1})$, where $\rho = [P(X^i_{\mathbf{s}} > X^i_{\mathbf{s}'})]^{-1}$. We focus on the function $f(k) = \binom{N-1}{k}/(\rho^M)^k$ and find the condition such that $f(k+1)/f(k) \leq \alpha < 1$ for each k and $\liminf\limits_{M,N\to\infty} f(1) > 0$. As a result, the probability will be positive and less than 1. To prove the second statement in Proposition 5, we will divide the game into Z equal-sized subgames such that all subgames will satisfy the conditions given in first statement.

By the first statement, each such subgame has at most probability $\eta < 1$ of lacking any strictly dominated strategy, Since these subgames are generated independently, the probability that all Z subgames simultaneously lack a strictly dominated strategy is at most η^Z. Hence, the probability that the original large game contains no strictly dominated strategy approaches zero as well.

Notice, there is a small uncovered area in the line as shown in Figure 3.1, which summarizes the results from Propositions 4 and 5.

Now, we simulate random normal-form games of all sizes from 2×2 up to 20×20. For each size, we generated 100,000 payoff matrices at random, and used a 'brute-force' algorithm to test whether either player has a strictly dominated strategy (the algorithm 1 in Section 6). Finally, for each game size, we recorded the proportion of games in which a strictly dominated strategy was present.

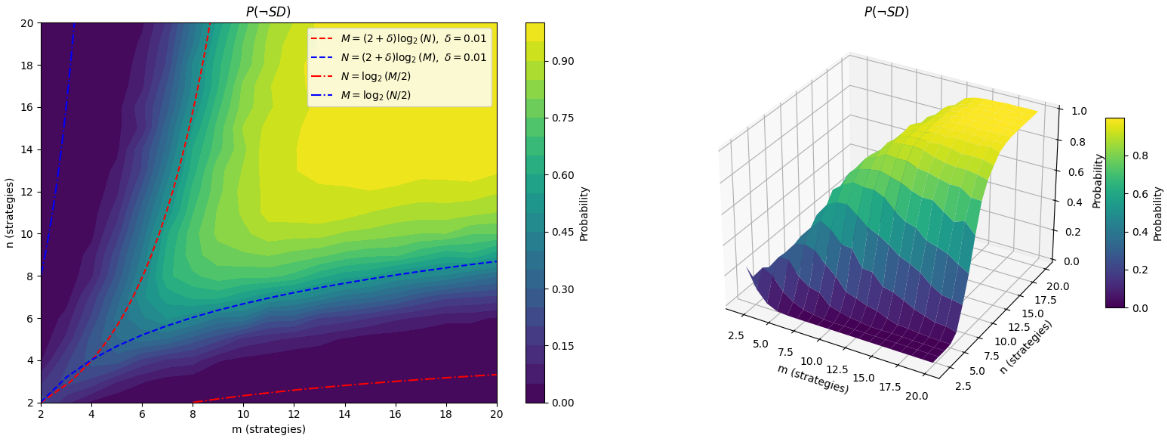

Figure 3.2: Simulation Result for Propositions 1 and 2

Figure 3.2 displays the result. In the right graph, the x-axis represents the number of strategies for player 2 (n), the y-axis represents the number of strategies for player 1 (m), and the z-axis represents the probability of strictly dominated strategies. It can be observed that the $P(\neg SD)$ approaches 1 with increasing values of n and m within the yellow region. However, $P(\neg SD)$ does not approach 1 in other areas of the graph. In the left one, we point out the specific growth rate $N = 2.01log_2(M)$ and $N = log_2(M/2)$, similar to the row player. The probability will not approach 1 in the large game if the growth rate is slower than $N = log_2(M/2)$ and will approach 1 if the growth rate is faster than $N = 2.01log_2(M)$, which is consistent with our analysis. Note that Pei and Takahashi 2019 also stimulate the symmetric (Figure 4 and Proposition 5), which is exactly the '45 degree line' of this graph.

3.5 Existence of q-portion Strictly Dominated Strategy

Proposition 4 told us if the game is 'imbalanced' then there will exist a strictly dominated strategy when the game is very large. But if the game is very large, it's practical and meaningful to eliminate a portion of strategies instead of a single strategy. Hence, we wish to determine the conditions for the existence or nonexistence of a 'q-portion' dominated strategies. Notice that, unlike the previous section relying relationship between the two variables based on their comparative probabilities, we are focusing on the ordinal comparison between elements in this section. Hence, the results of this section will hold under continuous distributions. For simplicity, the arguments and proofs depending on the uniform distribution.

Definition 2. *For any $S' \subseteq S$, we say S' is strictly dominated if $\forall s' \in S'$, $\exists s \in S$, s.t. $s \succ s'$.*

By definition, we have S' is strictly dominated if and only if $\forall s' \in S'$, $\exists s \in S \setminus S'$, s.t. $s \succ s'$.

For any $x \in \mathbb{R}^+$, $\lceil x \rceil = \min\{z \in \mathbb{Z}^+ : x \leq z\}$. Whenever $q \in (0,1)$, we define $P^1_{q,M,N}$ as the probability of there are $S_{qM} \subset S$ of size $\lceil qM \rceil$ strictly dominated strategies for player 1 in the game $R_{M,N}$. Fix S_{qM}, We denote

$$E^k(S_{qM}) = \left[\exists\ {}_kS \subseteq S \setminus S_{qM},\ s.t.\ {}_kS \text{ of size } k \text{ strictly dominates } S_{qM}\right].$$

Then, by definition, for a set S_{qM} of size $\lceil qM \rceil$, which does not depend on the particular choice of S_{qM},

$$P\left[\exists S_{qM} \subset S,\ \exists\, S' \subset S \text{ such that } S_{qM} \text{ is strictly dominated by } S'\right] = \binom{M}{qM} \bigcup_{k=1}^{qM} P\left[E^k(S_{qM})\right]$$

In particular, we denote $P^1_{q,M,N,\downarrow}$ as the probability that there is a particular (for example on the bottom) $\lceil qM \rceil$ strategies are strictly dominated for player 1 in the game $R_{M,N}$.

Proposition 6. *In a $M \times N$ random game, for any $q \in (0,1)$, for any M,*

1. *$P^1_{q,M,N,\downarrow}$ will go to 0 when N,M goes to infinity.*
2. *$P^1_{q,M,N}$ will go to 0 when N,M goes to infinity with $N \geq M/(\ln(M))^\alpha$ for some fixed $\alpha > 0$.*

Proposition 6 states that if, for any q, we fix their positions (e.g., placing them all at the bottom), the probability that all are strictly dominated strategies approaches 0, regardless of the growth rate between M and N. The proof is straightforward. We focus on the logic: in a payoff matrix, if a set of rows is strictly dominated, then for any column, the highest payoff in that column cannot come from a row in this set. Therefore, it's easy to find the upper bound of $P^1_{q,M,N,\downarrow}$ and it doesn't depend on any growth rate.

Furthermore, if no restrictions are placed on their positions (e.g., the set can be located anywhere in the matrix, not just at the bottom), the probability of the existence of q-portion dominated strategies depends on the growth rate between M and N. Specifically, if $N \geq M/(\ln(M))^\alpha$ for some fixed $\alpha > 0$, then no q-portion dominated strategies exist, which includes, but not limited to, scenarios with linear growth rates. To prove this, we characterizes the event $E^k(S_{qM})$ by partitioning the strategy set S_{qM} into subsets. For each subset, we check if the corresponding strategy in $_kS$ strictly dominates all strategies in that subset. Then we can find the upper bound of $P[E^k(S_{qM})]$. After that, we apply logarithmic transformations and use Shannon entropy to approximate $P^1_{q,M,N}$ and show it will go to infinity under the condition in the second statement.

However, we can show for any q, when M,N go to infinite when M growth much faster than $(N/(1-q-\delta))^N$, there is a block of a q-fraction of strategies which are all strictly dominated by one single strategy. Figure 3.3 illustrates the idea. We assume all elements in the payoff matrix are independently and identically distributed (i.i.d.) according to a probability distribution. Thus, the question can be reformulated to demonstrate the existence of one point in the Red area and the existence of qM points in the Purple area when M points are randomly selected. Namely, we aim to show there exists a subset of rows S such that $|S| \geq qM$ and each $x_i \in S$ is in the selected rows is in the Purple area, namely $x_{ij} \leq (\delta + q)^{1/N}, \quad \forall j = 1,\dots,N$, for some $\delta > 0$ with $\delta + q \in (0,1)$. Namely,

$$P(\exists S' \subseteq S, |S'| \geq qM, \forall x \in S', x_{ij} \leq (\delta + q)^{1/N}, \forall j = 1,..,N)$$

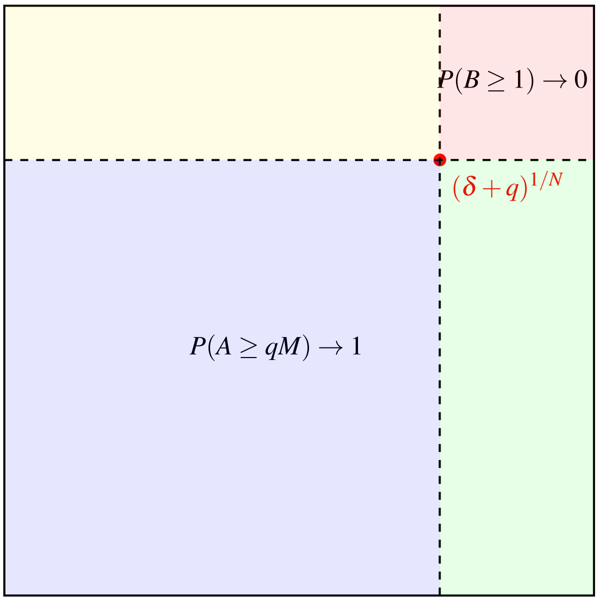

Figure 3.3: Chernoff Bound of Event A and B

We could find this probability using the Chernoff bound,[4] particularly when $M,N \to \infty$ and each element $x_{i,j}$ is uniformly distributed in $[0,1]$. As remarked, the argument works for any nonatomic distribution.

From the construction, we have $P(x_{ij} \leq (\delta+q)^{1/N}) = (\delta+q)^{1/N}$. Now we define the event that a row satisfies this condition as $A_i = \{x_{ij} \leq (\delta+q)^{1/N}, \forall j = 1,\dots,N\}$. Then

$$P(A_i) = \prod_{j=1}^{N} P(x_{ij} \leq (\delta+q)^{1/N}) = \left((\delta+q)^{1/N}\right)^N = (\delta+q).$$

Since we are interested in the probability of having at least qM rows satisfy the condition, we define the following sum: $A = \sum_{i=1}^{M} \mathbf{1}_{A_i}$, where $\mathbf{1}_A$ is the indicator random variable that the i-th row satisfies the condition A_i and $\mathbb{E}[A] = M(\delta+q)$.

Similarly, we define the event when the row x_i is falling in the red area.

$$B_i = \left\{x_{ij} \in \left((\delta+q)^{1/N}, 1\right), \forall j = 1,\dots,N\right\}.$$

4. Let $X = \sum_{i=1}^{n} X_i$, let all X_i are independent and $X_i = 1$ with probability p_i. Then Chernoff bound gives us 1) $P(X \geq (1+\delta)E[X]) \leq e^{-\frac{\delta^2}{2+\delta}E[X]}$ for all $\delta > 0$; 2) $P(X \leq (1-\delta)E[X]) \leq e^{-\frac{\delta^2 E[X]}{2}}$ for all $0 < \delta < 1$.

And we define the sum $B = \sum_{i=1}^{M} \mathbf{1}_{B_i}$, where $\mathbf{1}_B$ is the indicator random variable that the i-th row satisfies the condition B_i.

Proposition 7. *In a $M \times N$ random game, for any $q \in (0,1)$, for any M,*

1. *$P(A \geq qM) \to 1$ as $M \to \infty$ and $N \to \infty$.*
2. *$P(B \geq 1) \to 1$ as $M \to \infty$ and $N \to \infty$ with $M \gg (N/(1-\delta-q))^N$ for some fixed $\delta > 0$.*

Using the Chernoff bound, we have shown that the probability of having at least qM rows in an $M \times N$ matrix where $x_{ij} \leq (\delta+q)^{1/N}$ and at least one row that $x_{ij} \in ((\delta+q)^{1/N}, 1)$ tends to 1 for each j as both M and N go to infinity with $M \gg (\frac{N}{(1-\delta-q)})^N$ for some fixed $\delta > 0$. In other words, there exist q-portion strategies that are strictly dominated by at least one strategy.

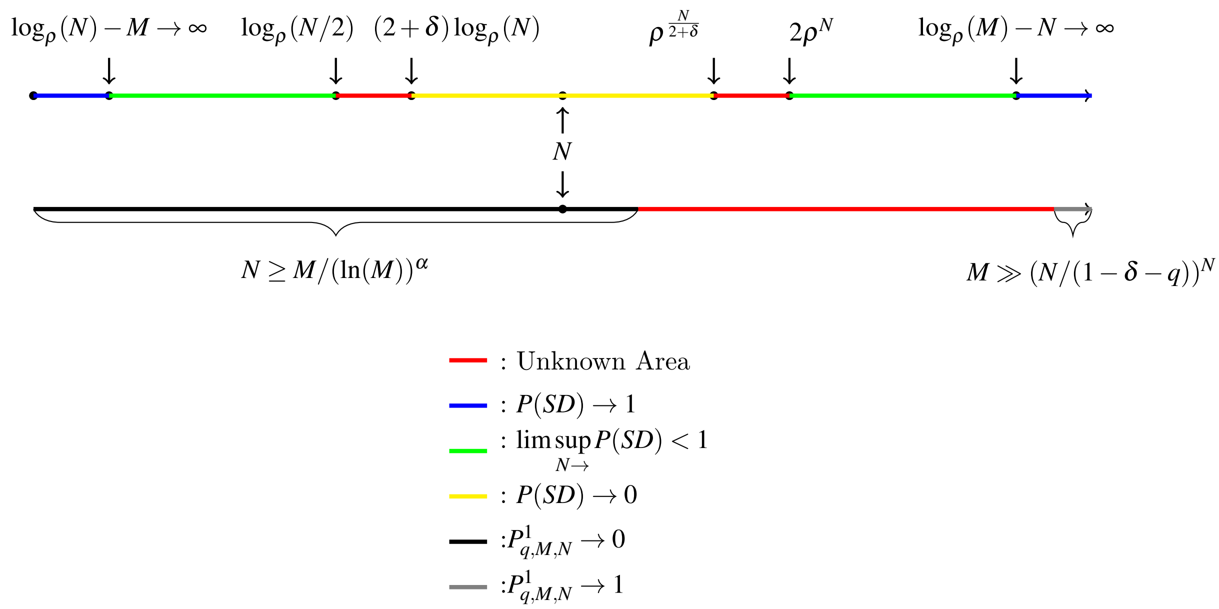

Figure 3.4: Summary of Results

Note: This figure is as M varies for fixed N

Figure 3.4 illustrates the growth rate of M with respect to N for the existence of a strictly dominated strategy (the upper line) and the existence of q-portion dominated strategies (the bottom line).

In the upper line, the yellow area represents the condition where no strictly dominated strategy exists in the game, hence Iterated Elimination of Strictly Dominated Strategies (IESDS) is not useful and the probability that the game is dominance solvable is 0. When M grows faster than N, for example, at a rate like $2\rho^N$, there may exist a strictly dominated strategy, which corresponds to the green area. When M grows significantly faster, as shown in the blue area, a strictly dominated strategy will exist in large games. The red area is currently small and unexplored, so it is unknown whether a strictly dominated strategy exists there.

The bottom line represents the partial characterization of the growth rate required for the existence of q-portion dominated strategies for player 1. In the black area, no q-portion dominated strategies exist for player 1, regardless of the value of q. It is important to note that this holds even when M grows faster than a linear growth rate with respect to N. In the grey area, q-portion dominated strategies always exist for player 1, and they are dominated by a single strategy. Similarly, the red area remains unknown. By symmetric, we can easily obtain the partial characterization of the growth rate required for the existence of q-portion dominated strategies for player 2.

3.6 Computational analysis

To determine the existence of strictly dominated strategies in a game, a comparison of each strategy's payoffs against all others for every player is required. This process factors in every strategy of the opponents as in Algorithm 1, and it's the $O(M^2 \times N)$ problem. We aim to find a simple algorithm that is faster, at least by a constant factor, than the most direct brute-force algorithms. Such an algorithm could potentially be combined with existing ones to achieve even greater efficiency.

Algorithm 1: Baseline Method

Input: Payoff matrix of a game $R_{M\times N}$

Output: A boolean value indicating if the game has a strictly dominated strategy or not

1: ▷ For player 1 (or 2), for any strategy $s_i \in S$ (or $t_i \in T$), for each other strategy $s_j \in S$ (or $t_j \in T$), compare the payoffs of strategy s_i, s_j (or t_i, t_j) for all $t \in T$ (or $s \in S$).

2: ▷ If strategy $s_i \in S$ (or $t_i \in T$) is not dominated by all the other strategies, then denote $s_i \in S$ (or $t_i \in T$) as an undominated strategy.

3: ▷ If all strategies are undominated strategies, then the game does not have a strictly dominated strategy.

return A boolean value indicating if the game has a strictly dominated strategy or not

Assume we have the matrix $R^1_{M\times N}$, and we mark the maximum as red and the minimum as green. Comparing the maximum and minimum of any two rows, i and j, we can claim Proposition 8:

$$R^1_{M\times N} = \begin{bmatrix} R^1_{M\times N}(s_1) \\ R^1_{M\times N}(s_2) \\ \vdots \\ R^1_{M\times N}(s_M) \end{bmatrix} = \begin{bmatrix} \dots & u_1(s_1,t_p) & \dots & u_1(s_1,t_q) & \dots & u_1(s_1,t_k) & \dots & u_1(s_1,t_l) & \dots \\ \dots & u_1(s_2,t_p) & \dots & u_1(s_2,t_q) & \dots & u_1(s_2,t_k) & \dots & u_1(s_2,t_l) & \dots \\ \vdots & & & & \vdots & & & & \vdots \\ \dots & u_1(s_M,t_p) & \dots & u_1(s_M,t_q) & \dots & u_1(s_M,t_k) & \dots & u_1(s_M,t_l) & \dots \end{bmatrix} \tag{3.2}$$

Proposition 8. *In above problem, for any two different $s_i, s_j \in S$, we have*

1. $P(s_i \sim_1 s_j | \min(s_i) > \max(s_j)) = P(s_i \sim_1 s_j | \min(s_j) > \max(s_i)) = 0$. *The union of those two events will happen with probability $2/\binom{2N}{N}$ and will go to 0 as N go to infinity.*
2. $P(s_i \sim_1 s_j | \max(s_i) > \max(s_j) > \min(s_j) > \min(s_i)) = P(s_i \sim_1 s_j | \max(s_j) > \max(s_i) > \min(s_i) > \min(s_j)) = 1$. *The union of two events will happen with probability $(2N-2)/(4N-2)$ and will go to $1/2$ as N go to infinty.*
3. $P(\max(s_i) > \max(s_j) > \min(s_i) > \min(s_j)) = P(\max(s_j) > \max(s_i) > \min(s_j) > \min(s_i)) = N/(4N-2) - 1/\binom{2N}{N}$. *The union of those two events will go to $1/2$ as N go to infinity.*

Proposition 8 states that if $\min(s_i) > \max(s_j)$, then s_i must strictly dominate s_j for any strategies s_i and s_j. If $\max(s_i) > \max(s_j)$ and $\min(s_j) > \min(s_i)$, then there is no domination between s_i and s_j. However, in the last situation, we cannot conclude strict domination. Since this result relies only on the ordinal comparison between the elements of s_i and s_j, regardless of the underlying probability distribution, we can show that the last case occurs with a probability of $1/2$. Hence, there is a half chance that we can tell the dominance relation between two vectors without comparing element by element. The key here is the combinatorics involved in the ordering of elements in the vectors, and how the relationships between their maximum and minimum elements affect these probabilities. The general intuition is that the probability of these specific orderings of elements in the vectors decreases as the vector size N grows. As N approaches infinity, the probabilities tend to the limits stated in this proposition.

For any comparison of two strategies, there is approximately a 50% chance of the event that we do not need to compare them element by element. For player 1, Algorithm 2 checks the minimum and maximum of each row in the payoff matrix with $3n/2-2$ comparisons when n is even and with $3(n-1)/2$ comparisons when n is odd (Knuth 1997). Therefore, in the first two cases (in 50%), we compare $M \times (3n/2-2) + M(M-1)$ times and in the last case (in 50%), we compare $M(M-1)/2 \times N$ times. Although the last case will dominate the time complexity, Algorithm 2 is more efficient in practice because of its potential for early termination via the MinMax/MaxMin checks, leading to fewer comparisons on average.

Algorithm 2: MaxMin and MinMax

Input: Payoff matrix of a game

Output: A boolean value indicating if the game has a strictly dominated strategy or not

1: ▷ For player 1 (or player 2), for any strategy $s_i \in S$ (or $t_i \in T$), for each other strategy $s_j \in S$ (or $t_j \in T$),

2:

- If $\min(s_i) > \max(s_j)$ or $\min(s_i) > \max(s_j)$, then return there exist strictly dominated strategy and the algorithm stops.
- If $\max(s_i) > \max(s_j) > \max(s_j) > \min(s_i))$ or $\max(s_j) > \max(s_i) > \max(s_i) > \min(s_j)$, then there is no dominance relation between s_i and s_j.
- Otherwise, compare the payoffs of strategy s_i, s_j (or t_i, t_j) for all $t \in T$ (or $s \in S$). If strategy $s_i \in S$ (or $t_i \in T$) is not dominated by all the other strategies, then denote $s_i \in S$ (or $t_i \in T$) as an undominated strategy.

3: ▷ If all strategies are undominated strategies, then the game does not have a strictly dominated strategy. **return** A boolean value indicating if the game has a strictly dominated strategy or not

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3.7 Conclusion

While the distribution of strictly dominated strategies cannot be explicitly characterized in general, this study verifies the absence of strictly dominated strategies and demonstrates that the probability of this occurrence approaches 1 as the number of strategies for both players tends to infinity, as shown in Proposition 4. This also indicated a clear result for the probability of a game being dominance solvable. Furthermore, we present a positive result in Proposition 5, indicating the existence of a region in which a strictly dominated strategy will be present either probably or for sure. It's important to emphasize that the above results also capture the parameter of the underlying probability distribution.

Practically, we are not only interested in the existence of dominated strategy, but we are interested in the set of dominated strategies. We then further analyze the asymptotic behaviour of the existence of a q-portion-dominated strategy in a random game. We denote $P^1_{q,M,N}$ as the probability of there are $S_{qM} \subset S$ of size $\lceil qM \rceil$ strictly dominated strategies for player 1 in the game $R_{M,N}$. In Proposition 6, we show there is no q-portion-dominated strategy when N,M goes to infinity with $N \geq \frac{M}{(\ln(M))^\alpha}$ for some $\alpha > 0$. Positively, in Proposition 7 we show there exist q-portion strategies strictly dominated by a single strategy when N,M goes to infinity with $M \gg (\frac{N}{1-\delta-q})^N$ for some fixed $\delta > 0$.

Lastly, since Yu and Zeleny 1975, the efficient algorithm to find a whole set of dominated strategies or undominated strategies has been of interest in both economics and computer science. Their result relies on the particular constraint, however, our simple algorithm doesn't depend on the extra constant and deduces the time complexity by half from the basic algorithm.

In the future, researchers can explore the unknown areas in both problems, and it would be more meaningful to improve the threshold of the growth rate to guarantee the existence of q-portion dominated strategies, i.e., when the $q-$portion dominated strategies are dominated by either 1 or 2 strategies. This is not only because this threshold is especially relevant to simplifying our understanding, but also because our result is much stronger, treated as an upper bound, where the strategy is strictly dominated by a single strategy.

Additionally, relaxing the assumption of independent and identically distributed (i.i.d.) variables could help derive more useful bounds for specific questions, as the approach in this paper can be potentially applied to such cases. Beyond directly relaxing the i.i.d. assumption, it is also meaningful to consider dominated strategies in random extensive form games, where there are n players, each choosing between two actions, and all payoffs are i.i.d. selected from a probability distribution. When transforming to a normal form game, the i.i.d. assumption no longer holds, making the problem more complex.

Furthermore, the proposed algorithm can be improved to reduce its time complexity. One potential improvement is to focus on the maximum or minimum values of the columns for player 1 (or the rows for player 2), as the row with the maximum value in the columns cannot be dominated.

Acknowledgments

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Chapter 4

Social Expectation in Random Allocation Problem

4.1 Introduction

A planner often must assign scarce slots to agents who are ranked externally by merit. This creates a unique challenge: how to allocate resources when only one side has ordinal preferences, while an exogenous priority order ranks the other side.

Consider the university admissions—such as the "consent to reassignment" mechanism[1]. Although students may privately favor certain fields, the system will treat them as having no strict preferences over majors once they consent to reassignment. However, each major should have a preference over students who are also ranked by exam grade. The university problem is to assign students to majors, respecting these one-sided preferences and external rankings to balance efficiency and fairness. Similar tensions arise in other domains. In military personnel assignments, ministries allocate officers to posts varying in strategic criticality. Officers may submit preferences, but posts themselves are ranked by criticality [2]. In development finance, public agencies allocate funding to projects ranked by social urgency, while investors have return preferences.

1. In some centralized systems (e.g., certain national entrance examinations), students submit a single broad application and indicate "consent to reassignment," meaning they agree to be placed into another major if their preferred choice is unavailable, rather than ranking multiple majors.
2. See "Determine Priority of Position Vacancies," p. 11 in U.S. Army Human Resources Command 2020.

These examples share a common structure: one side expresses preferences or priorities, while the other is guided by exogenous rankings reflecting quality, urgency, or strategic value. Traditional matching frameworks do not capture this asymmetry.

We propose a unified model that focuses on one-sided preferences, combined with external priority rankings. We analyze randomized mechanisms, explore trade-offs between fairness and efficiency, and discuss implications for education policy, military human resource management, and social finance. Our approach sheds light on a broad class of allocation problems where merit and priority override explicit agent preferences.

There are many other examples, such as dairy food for a food bank (Prendergast 2022), food rescue services (Aydin Alptekinoglu 2023), organ allocation with waiting times (Ashlagi, 2024), and status ranking (Richter and Rubinstein 2024), or a planner may rank colleges based on quality or societal benefit (Salgado-Torres 2013; Xie 2024).

On the other hand, a growing body of work shows how to design mechanism to achieve social planner optimal allocation (Abdulkadiroğlu et al. 2020, Abdulkadiroglu et al. 2021, Cowgill et al. 2024, Kang 2023, Noda 2023 etc.) or ensure truth-telling (i.e., Dworczak 2020) or ensure incentive-compatibility of social planner (Akbarpour and Li 2020). However, it's worth discussing whether a social planner's preference should be enforced if it conflicts with the individual. In this project, we will touch on the following questions: 1) When should agents follow a social planner's objective function called social expectation? 2) When should individual freedom be prioritized over social expectations? We model the social planner as an agent (Agent 0) within a random allocation framework to propose principles for balancing these two parties.

4.1.1 Model of Social planner

In our model, agent 0 plays a role in influencing the allocation but does not receive the object placement for themselves, such as a manager, government, or social planner. In this model, we must allocate n goods to n agents with the agents' ordinal preferences and agent 0's order of importance called social expectation.

Besides the previous real examples, there is another reason to model an order of importance: it's too expensive to report the information or preference over the entire allocation. Of course, some works introduce exogenous matching quality to the school choice model to measure the quality of each match, which plays a similar role (Abdulkadiroglu et al. 2021). But, as discussed by Ludwig von Mises and Friedrich Hayek, a planned economy is impossible because a central planner cannot access and collect all the necessary economic knowledge "in the absence of market prices for the factors of production" (Von Mises 2002, p.705). Hence, in our model, social planners generate a preference ordering over allocations from the whole ordinal preference profile (including both classic agents and agent 0). See Example 1.

Example 1. *We assume agent 0 has the original preference over objects. Imagine the following preference profile in Table 4.1. Agent 1's preference list* $'a_5, a_4, a_2, a_1, a_3'$ *means that the Agent 1 prefers* a_5 *over* a_4*,* a_4 *over* a_2*,* a_2 *over* a_1*, and* a_1 *over* a_3*.*

0:	a_1, a_2, a_3, a_4, a_5
1, 2:	a_5, a_4, a_2, a_1, a_3
3, 4:	a_2, a_4, a_1, a_3, a_5
5:	a_3, a_1, a_5, a_2, a_4

Table 4.1: Preference Profile

Let's consider the distribution of objects, i.e., the distribution of object a_1 *indicates how much of* a_1 *each agent receives. So, in the Table 4.2, the distribution of* a_1 *is the vector* $(\frac{1}{2}, \frac{1}{2}, 0, 0, 0)$*. Now, given agent* i*, object* a *and his preference* R_i*, we denote* $rank(a; R_i)$ *as the rank of the object* a *to agent* i *according to preference* R_i*, and* $\overline{U}(R_i; a)$ *as the agent i' weak upper counter set of object* a*. Hence, if* $rank(a; R_i) = 1$ *means the object* a *is the most favourite to agent* i*.*

For each object, Agent 0 first generates a welfare vector by reordering the distribution of a_1 *by arranging agents in ascending order of* $rank(a_1; R_i)$*. Agent 0 then evaluates the two distributions of* a_1 *based on first-order stochastic dominance. For example, the welfare vector of* a_1 *in Table 4.2 is* $[0, 0, 0, 1, 0]$ *while the the welfare vector of* a_1 *in Table 4.3 is* $[0, 1, 0, 0, 0]$*. Under this method, Agent 0 prefers the distribution of* a_1 *in Table 4.3 to the one in Table 4.2.*

After comparing the distribution of all objects between the two allocations, Agent 0 encounters a situation where they prefer the distribution of a_1 *in Table 4.3 but prefer the distribution of* a_3 *in Table 4.2. To resolve this, we employ a simple lexicographical rule: Agent 0 favours the allocation in Table 4.2 because object* a_1 *is more important. Moreover, the allocation in Table 4.3 is not dominated by the above preferences.*

	a_1	a_2	a_3	a_4	a_5
1	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
2	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
3	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
4	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
5	0	0	1	0	0

Table 4.2: Allocation 1

	a_1	a_2	a_3	a_4	a_5
1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
2	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
3	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
4	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
5	1	0	0	0	0

Table 4.3: Allocation 2

Table 4.4: Illustration for construction of agent 0's preference

This creates a potential conflict between the social planner and agents if we obey the social planner regardless of individual willingness. See Example 2

Example 2. *In Table 4.3, Agent 5 might prefer to exchange* a_3 *for a share of* a_1*, improving their welfare. Should Agent 0 reject such a trade based on social expectations? We argue that Agent 0 should not reject it on the grounds of efficiency. What we aim to evaluate is the justice of objections raised by other agents to the information provided by Agent 0. It is worth noting that the allocation in Table 4.2 is efficient, so it's not first-order-stochastic-dominated by other random allocations.*

	a_1	a_2	a_3	a_4	a_5
1	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{2}$
2	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{2}$
3	$\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{1}{4}$	0
4	$\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{1}{4}$	0
5	0	0	1	0	0

Table 4.5: Idea solution in Literature

Consider the allocation in Table 4.5, which is both efficient and envy-free (Bogomolnaia and Moulin 2001). So, all agents prefer their own allocation to that of others. Then, should Agent 0 reject to this allocation based on social expectations? We argue that this objection, while possibly in tension with fairness, reflects a broader normative question: to what extent should collective expectations shape individual entitlements? Comparing this with Table 4.2, we observe that the main tradeoff lies in the distribution of objects a_1 *and* a_4*. Agents 1 and 2 might challenge the latter allocation for violating envy-freeness; however, if Agent 0's preferences reflect shared or institutional expectations, such objections should not justify a change.*

We suggest a principle: agents who align more closely with social expectations—as encoded by order of importance—may justifiably be prioritized. While this limits purely self-interested improvements, it supports a form of fairness grounded in conformity to collective values. [3]

Now, we evaluate how much each agent's preferences align with those of Agent 0. Focusing on object a_4, note that Agents 3 and 4 prefer a_4 to a_5, as does Agent 0, whereas Agents 1, 2, and 5 do not. Hence, it is reasonable to say that Agents 3 and 4 conform more to social expectations for object a_4 compared to Agents 1, 2, and 5.

We formalize this by comparing agents' *weakly upper counter sets.* The agent whose set contains objects with lower maximum ranks in Agent 0's preference order is considered to conform more to social expectations, namely for any object a, for any two agents i, j, Agent i conform more to social expectation for object a compared to Agent j if

$$\max_{x \in \overline{U}(R_i;a)} rank(x;R_0) < \max_{x \in \overline{U}(R_j;a)} rank(x;R_0).$$

This means agent i's substitutes for object a lie closer to the top of Agent 0's preferences. Notice that $\arg\max_{x \in \overline{U}(R_i;a)} rank(x;R_0)$ could be a itself.

Example 3. *In Example 1, we suggest Agents 1, 2, and 5 should only receive a positive share of object* a_4 *if Agents 3 and 4 are fully satisfied with respect to* a_4*, i.e., they receive probability 1 from their weak upper counter set. This criterion is not met in either Table 4.3 or Table 4.5, as both allocations assign positive probability to* a_4 *for Agents 1 or 2 while Agents 3 or 4 receive less than 1 across their substitutes. Thus, these allocations violate the conformity-based fairness principle.*

This approach aligns with the idea that justice in collective outcomes may involve respecting established norms or social expectations, not just individual improvements. However, it's often to encounter a situation in which Agent i and j are not distinguished due to conformity to social expectations. For such a situation, we draw the light from the axiom called 'Interim Favoring Support'.

3. Man is by nature a social animal... Anyone who either cannot lead the common life or is so self-sufficient as not to need to, and therefore does not partake of society, is either a beast or a god. (Book 1 of the Politics, in section 1253a)

4.1.2 Interim Favoring Support

In many works, people assume that agents are justly entitled to acquire objects based on whether they prefer them more than others. Harless 2018 introduce the strong fairness notion for random assignments: *interim favoring rank*, and further explored by Ramezanian and Feizi 2021, and Chen et al. 2023.[4] *Interim favoring rank* states for any object a, for any two agents Ann and Bob, if $rank(a;R_{Ann}) < rank(a;R_{Bob})$, then Bob receives a positive share of the object a only when Ann receives 1 from Ann's weak upper counter set of object a. Now, we will see *interim favouring rank* will lead to an unfair situation in the random allocation problem.

Example 4. *Consider the preference list in the Table 4.6.*

1, 2, 3:	a_1, a_2, a_4, a_3, a_5
4,5:	a_3, a_1, a_2, a_4, a_5

Table 4.6: Preference Profile

Due to symmetry, we only analyze Agent 1 and Agent 4. Because $rank(a_1;R_1) < rank(a_1;R_4)$*, then interim favoring rank states* $P_{4,a_1} > 0$ *only if* $P_{1,a_1} = 1$*, which is impossible due to equal treatment of equals. Therefore* $P_{1,a_1} = P_{2,a_1} = P_{3,a_1} = \frac{1}{3}$*. Similar analysis holds for object* a_3*:* $P_{4,a_3} = P_{5,a_3} = \frac{1}{2}$*. For object* a_2*, because* $rank(a_2;R_1) < rank(a_2;R_4)$*, then* $P_{4,a_2} > 0$ *only if* $P_{1,a_1} + P_{1,a_2} = 1$*, which is impossible also. Therefore* $P_{1,a_2} = P_{2,a_2} = P_{3,a_2} = \frac{1}{3}$*. Similar analysis holds for object* a_4*. Lastly, equal treatment of equals implies* $P_{4,a_5} = P_{5,a_5} = \frac{1}{2}$*. Eventually, interim favouring rank support the allocation in Table 4.7.*

	a_1	a_2	a_3	a_4	a_5
1	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0
2	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0
3	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0
4	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$
5	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$

Table 4.7: Interim Favoring Rank

	a_1	a_2	a_3	a_4	a_5
1	$\frac{1}{3}$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{4}{15}$
2	$\frac{1}{3}$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{4}{15}$
3	$\frac{1}{3}$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{4}{15}$
4	0	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$
5	0	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$

Table 4.8: Interim Favoring Support

Table 4.9: Comparison

However, in Table 4.7, agents 1, 2, and 3 always have higher priority for objects a_1, a_2, a_4*, no matter what they obtain in outcome. It's unlikely that people reject agents 1, 2, and 3 to have higher priority on object* a_1 *not only because they rank object* a_1 *as top 1, but also the other agents will receive their top-choice object* a_3*. However, it's*

4. In school choice literature, Kojima and Ünver 2014 first introduce the deterministic version, called *favouring higher rank*, and their axiom support well-known *Boston mechanism.*

arguable for object a_2 and a_4. In Table 4.7, it's obvious that agents 1, 2, and 3 already have opportunities to access their top two preferred objects (a_1, a_2), while agents 4 and 5 only have a chance to secure their top-choice object (a_3). Under these circumstances, is it fair that agents 1, 2, and 3 have additional priority over object a_4?

As noted by Kojima and Ünver 2014, the deterministic version, '*favouring higher rank*', is typically viewed as a welfare criterion, yet it has also been interpreted as a fairness standard by Ramezanian and Feizi 2021. Importantly, it is not fair enough for a probabilistic setting. Specifically, this fairness concern does not arise in deterministic settings because each agent will receive only one object.

Hence, we take fairness criteria and propose the following variant to the probabilistic setting: *interim favouring support.* Now we denote the $Z(a;R_i;P_i)$ as the number of objects that agent i will receive with positive probability and that agent i prefers to a and a itself. It's different from $rank(a;R_i)$ and $rank(a;R_i) \geq Z(a;R_i;P_i)$. *Interim favouring support* states that for any object a, for any two agents Ann and Bob, if $Z(a;R_{Ann};P_{Ann}) < Z(a;R_{Bob};P_{Bob})$, no matter who prefer the object more than others, then Bob receive this object only when Ann receives 1 from Ann' weak upper counter set of object a.

Example 5. *Consider the same problem in Table 4.6, given any allocation P, if $P_{4,a_1} > 0$ then it implies $Z(a_1;R_4;P_4) \geq 1$. Hence $Z(a_1;R_1;P_1) \leq 1$, then it's only possible when $Z(a_1;R_4;P_4) = Z(a_1;R_1;P_1) = 1$. However, it implies $P_{4,a_3} = 0$ and $Z(a_3;R_4;P_4) = 0 < 1 \leq Z(a_3;R_1;P_1)$. Hence no one receives a_3, a contradiction. Therefore $P_{i,a_1} > 0$, for all $i = 1,2,3$, and $\sum_{i=1}^{3} P_{i,a_1} = 1$. Similarly, we have $P_{i,a_3} > 0$, for all $i = 4,5$, and $\sum_{i=4}^{5} P_{i,a_3} = 1$.*

However, differences from interim favouring rank arise after. For object a_2, we state that all five agents should receive a positive share of a_2. If agent i didn't receive a positive share, then $Z(a_2;R_i;P_i) < Z(a_2;R_j;P_j)$ where $P_{j,a_2} > 0$. Then interim favouring rank states agent i must receive 1 of his top favourite object, which is a contradiction to our previous analysis. Hence $P_{i,a_2} > 0$, for all i, and $\sum_{i=1}^{5} P_{i,a_2} = 1$.

The analysis for objects a_4 and a_5 is similar. It will object allocation of interim favouring rank and support the allocation in Table 4.8 is fair.

4.1.3 Main contribution: Principle and Realization

4.1.3.1 Two axioms in the random allocation problem with social planner

Based on the previous discussion, we have two criteria: conformity to social expectations and favoring support. Hence, for each object a, we can assign priority at first to agents who conform to social expectations more than others, hence the agent i with smaller $\max_{x \in \overline{U}(R_i;a)} rank(x;R_0)$. For those who are indistinguishable, we assign priority to agents based on a variant of $Z(a;R_i;P_i)$ conditional on conformity to social expectations $\max_{x \in \overline{U}(R_i;a)} rank(x;R_0)$ because of the existence of social planner.

We denote $\overline{*}(i;a) = \arg\max_{x \in \overline{U}(R_i;a)} rank(x;R_0)$. Now we denote the $\hat{Z}(a;R_i;P_i)$ as the number of objects that agent i will receive with the positive probability and that agent i prefers to a and prefered by $\overline{*}(i;a)$ including themselves.

Then, we have the first axiom mentioned before: *Conformity-Priority Efficiency*: 1) for any object, agents who conform to social expectations more than others should be rewarded; 2) for some objects, if some agents conform to the social expectation at the same level, then the one with less number $\hat{Z}(a;R_i;P_i)$ should have the high right.

The second principle is called *Indistinguishability Fairness*: for any object, when the above criteria cannot distinguish agents, they should receive equal shares of an object for fairness.

Example 6. *Continue Example 1*

Now, let's back to the Example 1, Agent 5 has the highest right to object a_1 *and* a_3*, then* $P_{5,a_3} = 1$*. Otherwise, we have contradiction to Conformity-Priority Efficiency that* $P_{i,a_3} > 0$ *and* $P_{5,a_3} < 1$ *for* $i \neq 5$*. Similarly, Agents 3 and 4 have the highest right to object* a_2 *and* $rank(a_2;R_3) = rank(a_2;R_4) = 1$*, then* $P_{3,a_2} = P_{4,a_2} = \frac{1}{2}$ *due to Indistinguishability Fairness. We have analyzed object* a_4 *and it give us* $P_{1,a_4} > 0$ *only if* $P_{i,a_2} + P_{i,a_4} = 1$ *where* $i = 3,4$*, which is impossible. Hence* $P_{3,a_4} = P_{4,a_4} = \frac{1}{2}$ *because we can't distinguish Agent*

3 and 4 due to our criteria. Lastly, we have $P_{1,a_5} = P_{2,a_5} = \frac{1}{2}$ *and* $P_{1,a_3} = P_{2,a_3} = \frac{1}{2}$ *due to Indistinguishability Fairness. Overall, it leads to the allocation in Table 4.2, the one we believe achieves a balance between conforming to social expectations and respecting individual freedom.*

Now, we propose a simple method to achieve this outcome. Following Agent 0's preference, we equally allocate to the agents who prefer the most according to their capacity. Formally, it is called Flow algorithm:

1. In round 1, we require the object in the first position of Agent 0's preference (smallest rank), denoted as $\pi(1)$, to appear in the market, then equally allocate them to agents who put them on top of the total objects and discard this object. If not, we add this object to the next position.
2. In round k, we iteratively allocate objects that have the left objects in the previous round and the objects in the $k-th$ position of Agent 0's preference, $\pi(k)$. We allocate the objects to agents who put them at the top of the remaining objects equally, considering their capacity. Whenever the capacity of an object is 0, we discard that object. We keep this process until no one puts the object in $\pi(k)$ at the top of the remaining objects or $\pi(k)$ is exhausted. Then, we add the remaining objects in the next position.

It will return the probabilistic allocation directly and use the lottery between agents during allocation. (**Proposition** 13)

4.1.3.2 Two axioms in the random allocation problem

In random allocation problem, we find unfairness resulting by *interim favoring rank* and believe *interim favoring support* will eliminate it. However, some popular mechanisms do not satisfy this axiom such as Random Priority (Abdulkadiroğlu and Sönmez 1998) and Probabilistic Serial (Bogomolnaia and Moulin 2001)(we will explain them later), while *fractional adaptive Boston rule* satisfies. *Fractional adaptive Boston rule* is the adaption of well-known *adative Boston Mechanism* and Bogomolnaia 2015 first introduce this to random allocation problem. Under this mechanism, every agent reports an ordinal preference list and will receive a lottery. Based on the preference list, each object is equally allocated to available agents who rank that object first among the rest of the objects up to their capacity, then to those who rank it second among the rest of the objects if there are any remaining capacity of the object, and so forth. Bogomolnaia 2015 shows it is *ordinally efficient*, *lexi-envy-free* and *lexi-strategy-proof*. The reader can realize the fractional adaptive Boston rule is also *interim favouring support*.

Considering the characterisation, we need one more axiom: *equal support equal claim* requires agents who are not distinguished by $Z(a;R_i;P_i)$ to have no incentive to exchange their assignment for this object. Go back to example in Table 4.8, take object a_1 for example, this axiom implies $P_{1,a_1} = P_{2,a_1} = P_{3,a_1} = \frac{1}{3}$. Together with *interim favouring support* support the allocation in Table 4.8.[5](**Proposition** 10)

4.1.3.3 New notion of efficiency in random allocation problem

In literature, there are two dominant methods to solve the general problem: Random Priority (RP) (Abdulkadiroğlu and Sönmez 1998) [6], and Probabilistic Serial (PS) (Bogomolnaia and Moulin 2001).

A sequence of agents (termed *priority*) serves as a natural tool to solve this problem. The manager sequentially asks agents, based on an exogenous priority, to choose their most preferred object from the remaining ones. This mechanism is renowned for its efficiency and incentive compatibility, but it falls short in terms of fairness. To address this, *random priority* is employed: the manager randomly determines an ordering and then queries agents to select their best object from what remains. [7]

However, ibid. shows RP lacks efficiency, namely it is not *ordinally efficient*. Therefore, they construct the *probabilistic serial*. In this method, the manager directly allocates a divisible probability weight. Agents simultaneously 'eat' the probability weight of their most preferred available object at a uniform rate. Once an object is fully 'eaten' by some agents, they move on to their next most preferred yet uneaten object. This process continues until all objects are completely allocated.

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5. To notice, Chen et al. 2023 replace 'support' with 'rank' to characterize the naive fractional Boston rule.
6. Also known as random serial dictatorship in literature.
7. Recently, RP is shown the unique rule satisfies *symmetry*,*ex-post Pareto efficient*, and *obvious strategy-proof*.

One advantage of *ordinal efficiency* is that it guarantees any decomposition of random allocation is the convex combination of efficient deterministic allocations while the efficiency of RP only guarantees there exists a convex combination of efficient deterministic allocations called *ex-post Pareto efficiency.*

We realize there is a natural extension of *ex-post Pareto efficiency* such that the random allocation can decompose into a convex combination of probabilistic ordinal efficient allocations. We call it *interim efficiency*. The *interim efficiency* is logically squeezed between ordinal efficiency and ex-post Pareto Efficiency: every ordinally efficient allocation is *'interim efficient'* and every *'interim efficient'* allocation is Ex-post Pareto efficient (the converse is not true). Unfortunately, Random Priority is not *interim efficient.* (**Proposition** 14)

We want to propose another, more efficient rule that is also easy to practice in reality. Imprecisely, we randomly generate an order of objects (permutation), $\boldsymbol{\pi}$, then run the Flow algorithm that we take $\boldsymbol{\pi}$ as Agent 0' preference. Although Random Priority and Random Flow can both be easily implemented and understood in lab settings, they exhibit distinct axiomatic properties, as shown in Table 4.10.(**Proposition** 15)

	Efficiency	Fairness	Incentive Compatibility
RF	Interim Efficient	weakly sd Envy-Free	weakly sd Strategy-Proof
RP	Ex-post Pareto Efficient	weakly sd Envy-Free	sd Strategy-Proof

Table 4.10: comparison Between RP and RF

We provide a numerical analysis among existing dominant mechanisms and RF. In 4×4 case, we observed RF generates no-envy in more preference profiles than RP: RP generates sd-envy-free allocation in 36% of preference profiles while RF generates sd-envy-free allocation in 48% of preference profiles. It suggests we can design an easy algorithm based on RF that is superior to RP with efficiency and fairness.

4.2 Literature Review

This project contributes to different questions.

4.2.1 Conformity, Social planner, and individual happiness

Existing literature typically models the social planner's preference as an exogenous factor and many works have discussed how to ensure the incentive compatibility of social planners or agents when a social planner is incorporated into a model. For instance, some works study how to incentive social planners to follow the game or report true objective function (i.e., Akbarpour and Li 2020) and some works study the compatibility between the social planner optimal outcome and incentive-compatibility for agents (i.e., Noda 2023, Dworczak 2020). On the other hand, many works discussed what could be the optimal mechanism when social planners use different objective functions (i.e., Weitzman 1977, Abdulkadiroglu et al. 2021) or what would be distortion in equilibrium when there is the intervention of social planner (i.e., Kang 2023).

In contrast, we model the social planner through a strict ordering of importance and a constructed preference over allocations. This approach allows us to identify and analyze potential conflicts between the planner's priorities and individual agents' preferences. By doing so, we formulate a new criterion—*conformity*—which determines when an agent should follow the planner's ordering instead of solely pursuing their payoff.

There is also normative justification for conformity: if social expectations really are internalized virtues or shared norms that enhance cooperation and well-being, then it's rational (and welfare-enhancing) for agents to respect the planner's ordering. Philosophical and psychological theories offer the reason. Thinkers have long argued that social expectations are internalized norms that guide cooperative behavior: from Aristotle's view of humans as "social animals" whose virtues are realized through communal life (Politics, 1253a, Aristotle 1980), to Confucian role ethics emphasizing duties of self-cultivation, family, and state (Weiming 1985). Pragmatists like Dewey 1916 and James 1907 further stress that responsiveness to collective aims drives social progress. Self-Determination Theory (Ryan and Deci 2000, Deci and Ryan 2013) provides a psychological mechanism: when agents internalize pro-social norms, they satisfy core needs for competence and relatedness, yielding intrinsic motivation to comply.

4.2.2 Favor higher ranks and Boston Rule

In the literature, agents are typically prioritized by how much they wish to obtain an object—namely, by their rank (Kojima and Ünver 2014, Harless 2018, Ramezanian and Feizi 2021, Chen et al. 2023 etc.) However, we propose a new index: the number of objects an agent prefers to a given object with positive probability. To distinguish this "support-based" index from rank, we note that it corresponds exactly to the difference between the Naïve Boston rule and the Adaptive Boston rule (Mennle and Seuken 2014 and Mennle and Seuken 2021). Hence, in this paper, a key distinction between the fractional Boston rule and the fractional Adaptive Boston rule reflects the fundamental tension between welfare and fairness arising from a focus on support versus rank.

4.2.3 Mechanism Design for the market with the order of objects

It is worth noting two related works on the order of objects. Liu and Zeng 2019 provides the algorithm on restricted tier domain, simply the preference is consistent with public rank, i.e., if social planner prefers block of A to block of B, then agents can't prefer the object$b$ in B to the object a in A. However, the public rank is the complete rank over all objects, not blocks, in our model, and we do not impose restrictions on the preference domain. Also, Harless 2019 characterizes all *sd-efficient* algorithms using the *order-claim-algorithm*. Although efficiency is not the only focus of this paper, there is the same spirit between ibid. and this paper: generating the order of objects and allocating them efficiently. However, we are curious about how to allocate the object when there is an inconsistency between preference and order of importance.

4.2.4 Refinement of Ex-post Pareto Efficiency

Lastly, it is well-known that ordinal efficiency is stronger than ex-post efficiency and RP is not ordinally efficient. Then it is always interesting to check the boundary of efficiency for RP. Interim efficiency requires that the random allocation can be decomposed (if there is one) into a convex combination of ordinally efficient random allocations. It is exactly between ordinal efficiency and ex-post efficiency and we show RP does not satisfy this property. [8]

This paper is structured as follows: Section 4.3 presents the Preliminaries of classic random allocation problem. Section 4.4 describes the new axiom 'Interim Favoring Support'. Section 4.5 presents the 'fractional adaptive Boston rule' and it's characterization. Section 4.6 introduces the new model with social planner, two new desired properties, the new Flow Algorithm, and characterization. Section 4.7 introduces a new notion of efficiency and introduces the new Random Flow algorithm with its necessary conditions and numerical analysis.

4.3 Preliminary

Consider a classic assignment problem with indivisible goods. For any positive integer x, define $[x] = \{1, 2, ..., x\}$. Let $N = [n]$ denote a set of agents, and $A = [n]$ denote a set of goods. The capacity of each agent and each object is 1. We consider the set of strict preferences $\mathscr{R}$ on A, the representative element is R. We use R_N to represent the preference profile.

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8. There is one interesting question: Do interim efficiency and robust ex-post efficiency imply ordinal efficiency? Robust ex-post efficiency see Aziz et al. 2015 and Ramezanian and Feizi 2022. Abdulkadiroğlu and Sönmez 2003 also gives some thoughts on why ex-post efficiency is not ordinal efficiency: a random assignment is ordinally efficient if and only if for any given feasible support, each of its subsets is undominated.

A random assignment is a bistochastic $P = [p_{ia}]_{i \in N, a \in A}$.[9] The set of random assignments is denoted $\mathscr{P}$. We use P_i to represent the allocation of agent i. A random assignment rule is a mapping $f : \mathscr{R}^N \to \mathscr{P}$. We use $f_i(R_N)$ to represent the allocation/probability that agent i receives under the assignment rule f and use $f_{ia}(R_N)$ to represent the allocation/probability that agent i receives object a under rule f and preference profile R_N.

4.4 Interim Favoring Support

We define the upper contour set of R_i at an object $a \in A$ as $U(R_i;a) = \{x : xR_ia\}$ and weak upper contour set of R_i at an object $a \in A$ as $\overline{U}(R_i;a) = U(R_i;a) \cup \{a\}$.

We need one definition to introduce the property.

Definition 3 (Effective Rank). *Whenever R_N, P, i and a, we define*

$$Z(a;R_i;P_i) = \#\{b : b \in \overline{U}(R_i;a), P_{ib} > 0\}$$

to be the effective rank *of object a for agent i, measuring how many objects preferred to a are actually allocated to i with positive probability.*

The classical *rank* of an object a for agent i, denoted $rank(a;R_i)$, counts how many objects agent i strictly prefers to a, without considering the allocation. Our *Effective Rank* $Z(a;R_i;P_i)$ refines this concept by counting only those preferred objects that agent i actually receives with positive probability under the allocation P_i. Thus, $Z(a;R_i;P_i)$ captures the *effective rank* of a within the allocated bundle, reflecting both preferences and current assignments, i.e., $Z(a;R_i;P_i)$ for your second-choice is at most 2 (and may be smaller if the lottery never awards you one of the first two favorite objects).

Property 1. *Interim Favoring Support*

For all R_N, all a, all i, if $P_{ia} > 0$, then $\sum_{x \in \overline{U}(R_j;a)} P_{jx} = 1$ for all j that $Z(a;R_j;P_j) < Z(a;R_i;P_i)$.

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9. That is, $P \in [0,1]^{N \times A}$ and for each $i \in N$ and $a \in A$, $\sum_{b \in A} P_{ib} = 1$ and $\sum_{j \in N} p_{ja} = 1$.

Property 1 says: if you're getting any of a, the doughnut, under the assignment P, anyone who has a smaller $Z(a;R_j;P_j)$ – meaning they have fewer "top-or-equal" sweets in their lottery – must already have spent their entire probability on those sweets before you can keep any of a.

Different from Kojima and Ünver 2014 and Harless 2018, this property entitles the right of the object to agents based on what they will receive instead of how they value the object. In the previous example, if one agent (he) ranks one object a higher than another (she) but he already receives more objects (possibly) than another, *interim favoring support* will allocate a to her while *interim favoring rank* will allocate a to him. Hence, it will eliminate the unfairness in Example 4.

Now we show it's stronger than ordinal efficiency. For all i, given a preference R_i on A, we call a partial ordering of the set $\Delta(A)$ the *stochastic dominance* relation associated with R_i and denoted R_i^{sd} if $\forall P_i, P_i^{'} \in \Delta(A)$ we have

$$P_i R_i^{sd} P_i^{'} \Leftrightarrow \sum_{x \in \overline{U}(R;a)} P_i \geq \sum_{x \in \overline{U}(R;a)} P_i^{'}, \forall a \in A.$$

Given a preference R_i on A, $\forall P_i, P_i^{'} \in \Delta(A)$, we say $P_i^{'}$ is *stochastically dominated* by P_i for agent i if we have $P_i R_i^{sd} P_i^{'}$ and $P_i \neq P_i^{\prime}$. We define P as *ordinally efficient* if P_i is not stochastically dominated for all i. A random assignment rule f is *ordinally efficient* if, for all $R \in \mathscr{R}^n$, $f(R)$ is ordinally efficient.

Proposition 9. *Interim Faovring Support implies Ordinal efficiency.*

We can show if the allocation is not ordinally efficient for some preference profile, then there exist a probabilistic improvement circle (agent i prefer a_i to a_{i+1} but receive a positive share of a_{i+1}). Then there exist agent i who have the highest $Z(a_{i+1};R_i;P_i)$, then for object a_{i+1}, we must have $Z(a_{i+1};R_i;P_i) > Z(a_{i+1};R_{i+1};P_{i+1})$, but the agent $i+1$ are not satisfied to object a_{i+1} because $P_{i+1,a_{i+2}} > 0$, hence it will violate the Property 1.

Now we state the second fairness property: requires such agents to have no incentive to exchange their assignment for this object also. Together with Property 2, it will single out the allocation in Table 4.8 in Example 4.

Property 2. *Equal Support Equal Claim*

For all R_N, a, i, j *s.t.* $Z(a;R_i;P_i) = Z(a;R_j;P_j)$, *if* $P_{ia} > P_{ja} > 0$ *then* $\sum_{x \in \overline{U}(R_j;a)} P_{jx} = 1$.

Property 2 says: for any doughnut a if two people have the same effective rank under assignment P, you can't give one of them a larger share of doughnut unless the other has already used up their full probability on sweets they rank at least as highly as doughnut a.

4.5 Fractional Adaptive Boston Rule

Now, we start with the fractional adaptive Boston rule in a random assignment problem. To provide the formal definition, we need to define: whenever $B \subseteq A, N' \subseteq N, a \in B, M(a;B;N') \equiv \{i \in N' : aR_ib, \forall b \in B\}$ and $m(a;B;N') = \#M(a;B;N')$. In words, the $M(a;B;N')$ is the set of agents in N' who put a at the top of B and the set of $M(a;B;N')$ is a partition of N'.

Given a preference profile R_N, the fractional adaptive Boston rule proceeds sequentially. Let $A^0 = A, N^0 = N$, let $C = [1]^N$ be the capacity of agents, $Z = [1]^A$ be the capacity of objects.

1. In the first period, for all a if $M(a;A^0;N^0) \neq \emptyset$, we fully allocate a to $M(a;A^0;N^0)$ and every agent receives $sh_{ia} = \frac{1}{m(a;A^0;N^0)}$. Then we update capacity Z^1 and C^1, the remaining objects A^1, and the agents with positive capacity N^1.
2. For each k period, for all $a \in A^{k-1}$ if $M(a;A^{k-1};N^{k-1}) \neq \emptyset$, we allocate a to agents $M(a;A^{k-1};N^{k-1})$ and every agent receives, for some e:

$$sh_{ia} = c_i^{k-1} \wedge e \quad s.t. \sum_{i \in M(a;A^{k-1};N^{k-1})} sh_{ia} = z_a^{k-1} \wedge \sum_{i \in M(a;A^{k-1};N^{k-1})} c_i^{k-1}$$

   Then we update capacity Z^k and C^k, the remaining objects A^k, and the agents with positive capacity N^k.

This algorithm will finish in finite periods at most $|A|$ and produce an allocation matrix sh.

Proposition 10. *A random assignment function satisfies Interim Favoring Support and Equal Support Equal Claim if and only if it is a fractional adaptive Boston rule.*

We prove it by mathematical induction and simple logic. At first, we show if the allocation satisfies the Property 1 and Property 2, then all agents will receive the allocation of their favourite object as same as under the fractional adaptive Boston rule. Then we show it's true for all agents and all objects such that $Z(a:R_i;P_i)=k$. For this induction, we need to be careful because $Z(a:R_i;P_i)=k$ may imply $r(a:R_i;P_i)>k$ and the agent may receive 0 probability of object a.

Now, we show Interim favoring support and equal support equal claim are independent. Moreover, we show 1) equal claim equal support and ordinal efficiency are not sufficient to obtain interim favoring support; 2) Interim favoring support and Lexi-envy-free are not sufficient to obtain equal claim equal support.

Given the stick preference $R_i : a_1 R_i a_2 \ldots R_i a_n$, we define lexicographic preference R_i^{lex} over all probability distributions $\Delta(A)$: for all $p,q \in \Delta(A)$, then we say $pR_i^{lex}q$ as long as there is $j \in \{1,...,n\}$ such that $p_{a_j} > q_{a_j}$, while $p_{a_k} = q_{a_k}$ for all $k < j$. Given an allocation P and preference profile R_N, we say it is *lexi-envy-free*, if for any agents i and j, we have $P_i R_i^{lex} P_j$ and $P_j R_j^{lex} P_i$.

Proposition 11.

1. *Lexi-envy-free and interim favoring support do not imply equal support equal claim.*
2. *Ordinal efficiency and equal support equal claim do not imply Property interim favoring support.*

4.6 Social Planner's preference and principles for new problem

In this section, we provide the new problem: how to allocate the object when the social planner exists. We assume the social planner holds an order of importance over objects (see motivated examples and explanation in Introduction). This could refer to the tiered ordering of universities in school allocation problems, the strict ranking of placements in military graduate assignment, the priority ordering of projects in government-led investment allocation, or similar situations.

4.6.1 Model: Social Planner's preference and Problem

To distinguish from agents, we denote $\pi \in \Pi$ as it's strict preference from now on.

Definition 4 (Order of Importance)**.** *An* order of importance π *on the set of objects* A *is a strict total order. We write*

$$\pi_a < \pi_b \quad \iff \quad \textit{object } a \textit{ is more important than object } b.$$

Equivalently, π *is a bijection from* A *to* $\{1,2,\ldots,|A|\}$*, where a smaller numeric value indicates higher priority.*

Remark. This notion of Order of Importance refers to a ranking over *objects* (e.g., projects, schools, positions) as determined by a central planner or social value. This contrasts with the standard use of priority orders in matching theory, where each object typically ranks *agents* by priority. For concrete illustrations of our notion, see the examples of military-graduate placement and investor-fund allocation.

Now, we can define Agent 0's preference over allocation. We adopt a lexicographical approach to compare allocations. The social planner does not receive any objects but derives utility directly from the allocation itself, distinguishing their objective from that of individual agents. In our model and the motivating example, the social planner values objects differently from agents and wants

1. more of an object should go to those who value it more,
2. and ensure that more important objects are better allocated.

The first assumption is natural — it reflects a welfarist principle: for any given object, the planner prefers to allocate it to the agents who value it most or derive the highest utility from it. The second one is natural in this model because of the order of importance. For example, a planner wants to ensure more important posts are allocated to appropriate graduates before less important posts. The government favours allocations where higher-priority investment projects (e.g., renewable energy, AI) get better-targeted investment. Now, we define this π-lexicographic first-order dominance.

Hence, we define the welfare vector for every object that tells us the distribution among agents, who prefer a the most and get how much, and so forth. $wr(a;R;P)[1] = \sum_{i:rank(a;R_i)=1} P_{ia}$ is total share of object a given to agents who rank a first, $rank(a;R_i) = 1$.

Definition 5 (Welfare vector)**.** *For each object $a \in A$, the* welfare vector *of a under (R,P) is the n-dimensional vector*

$$wr(a;R;P) \;=\; \big(wr(a;R;P)[1], wr(a;R;P)[2], \dots, wr(a;R;P)[n]\big) \;\in\; \mathbb{R}^n,$$

whose jth component is

$$wr(a;R;P)[j] \;=\; \sum_{\substack{i\in N \\ \text{rank}(a;R_i)=j}} P_{ia}.$$

Then we define the First-order dominance relation over the Welfare vector. In other words, the social planner prefer one allocation P to the other allocation P' with the distribution of object a if the welfare vector of object a in allocation P first order stochastic dominates the welfare vector of object a in second allocation P'.

Definition 6 (First-order dominance relation over Welfare vector)**.** *Let $P, P' \in \mathscr{P}$ be two random assignments, and let $a \in A$. We define the* First-order dominance relation over Welfare vector

$$wr(a;R;P) \rhd wr(a;R;P') \qquad \Longleftrightarrow \qquad \sum_{j=1}^{k} wr(a;R;P)[j] \geq \sum_{j=1}^{k} wr(a;R;P')[j] \quad \forall k = 1, \dots, n.$$

The second assumption connects the social planner's preferences over allocations to their preferences over individual objects called *responsiveness*. Specifically, we assume that if two allocations P and P' differ only in the distribution of objects a and b, then the social planner prefers P over P' if and only if they prefer the distribution of the more desirable object (among a and b) in P. For example, in Example1, the social planner prefers the allocation in Table4.2 over that in Table4.5 if and only if they prefer the distribution of the more desirable object between a_1 and a_4 in the former.

Definition 7 (Responsiveness)**.** *The preference relation* $\succeq^R$ *over set of student is* responsive *to* π *if, for any* P *and* P *only differs in welfare vector of* a *and* b *with* $\pi_a < \pi_b$*, then* $P \succeq^R P'$ *if and only if* $wr(a;R;P) \triangleright wr(a;R;P')$.

One example is simple π-lexicographic first-order dominance.

Definition 8 (π-lexicographic first-order dominance)**.** *Let* $P,P' \in \mathscr{P}$ *be two random assignments, and let* $a \in A$*. We say*

$$P \succeq^{\pi\text{-}LSD} P' \quad \Longleftrightarrow \quad \begin{array}{l} \exists a \in A: \ wr(a;R;P) \triangleright wr(a;R;P') \\ \wedge \ \forall b \text{ with } \pi_b < \pi_a: \ wr(b;R;P) = wr(b;R;P'). \end{array}$$

π-lexicographic first-order dominance orders entire *allocations* by (1) fixing a priority ordering of *objects*, and then (2) comparing two assignments by the cumulative shares those assignments give—object by object in priority order—to the agents who most desire each object. Recall the motivated ministry officer assignment problem, a national ministry of defence assigning freshly commissioned officers to posts that range from frontline duty to technical support and administrative staff. A π-lexicographic first-order dominance orders elegantly reconciles the concern: it fills the highest-priority post by awarding it to the officers who value it most, then proceeds, in succession, through the full portfolio of assignments. See the following example.

$$\begin{array}{rl} 1, 2: & a \succ b \succ c \\ 3: & b \succ a \succ c \end{array}$$

Table 4.11: Preference Profile

	a	b	c
1	0.5	0	0.5
2	0.5	0	0.5
3	0	1	0

Table 4.12: Assignment P

	a	b	c
1	0.4	0	0.6
2	0.4	0	0.6
3	0.2	1	0

Table 4.13: Assignment P'

Example 7. *Both assignments P and P' fully assign objects to agents. The planner ranks objects as $a \succ b \succ c$. In P, object a is entirely given to agents 1 and 2, who rank it first. In P', a fraction of a is given to agent 3, who ranks it second. The welfare vector for a under P is lexicographically better: more of a goes to its top-ranked supporters. Since object a is the top priority and P outperforms P' at a, while all lower-ranked objects are equal or irrelevant, we conclude $P \succeq^{\pi\text{-}LSD} P'$.*

Definition 9 (π-lexicographic efficiency)**.** *Fix a priority order π. A random assignment $P \in \mathscr{P}$ is called* π-lexicographically efficient *if there exists no other assignment $P' \in \mathscr{P}$ such that*

$$P' \succeq^{\pi\text{-}LSD} P \quad \text{and} \quad P' \neq P,$$

i.e. no P' π-lexicographically first-order dominates P.

An assignment $P \in \mathscr{P}$ is called π-*lexicographically efficient* if there is no other assignment $P' \in \mathscr{P}$ that π-lexicographically first-order dominates it (i.e. no $P' \neq P$ with $P' \succeq^{\pi\text{-LSD}} P$). Intuitively, you cannot find any feasible re-allocation that—when you compare objects in priority order—improves the share given to agents who value the first differing object most, without worsening a strictly higher-priority object.

Lemma 1. *The relation $\succeq^{\pi\text{-}LSD}$ is:*

(i) Irreflexive*: $\neg(P \succeq^{\pi\text{-}LSD} P)$ for all P.*
(ii) Transitive*: If $P \succeq^{\pi\text{-}LSD} P'$ and $P' \succeq^{\pi\text{-}LSD} P''$, then $P \succeq^{\pi\text{-}LSD} P''$.*
(iii) Responsive to π.

However, $\succeq^{\pi\text{-}LSD}$ is not complete.

Proof. By definition, P cannot strictly dominate itself at any object. Thus $P \not\succeq^{\pi\text{-LSD}} P$.

Suppose $P \succeq^{\pi\text{-LSD}} P'$ via first differing object a, and $P' \succeq^{\pi\text{-LSD}} P''$ via first differing object b. If $\pi_a < \pi_b$, then a is the first object where P, P'' differ, and cumulative sums satisfy the lexicographic inequality by chaining the two comparisons at a. The argument is symmetric when $\pi_b < \pi_a$.

Failure of completeness. There exist P, P' whose welfare vectors at the object are incomparable under cumulative sums. Then neither dominates the other.

□

There is a clear difference between *Ordinal efficiency* and π*-lexicographic efficiency. Ordinal efficiency* ensures that no group of agents could all weakly prefer a different allocation without making someone worse off, given their ordinal preferences. In contrast, π-lexicographic efficiency reflects the social planner's prioritization over objects: it seeks to allocate higher-priority goods to agents who desire them most, in a lexicographically fair manner. The lemma 2 states that these two goals are generally incompatible—there exist allocations that are optimal under one criterion but suboptimal under the other. This misalignment arises because the planner may favor reallocating high-priority objects to better-aligned agents even when such reallocations violate Pareto improvements among the agents themselves.

Lemma 2. π*-lexicographic efficiency and ordinal efficiency are not compatible.*

Now, we conclude the model by defining the allocation role as a mapping $g : \mathscr{R}^N \times \Pi \rightarrow \mathscr{P}$. We use $g_i(R_N, \pi)$ to represent allocation that agent i receives under the assignment rule g and preference profile R_N and order of importance π.

4.6.2 Principles

We consider the properties mentioned in the Introduction. Conformity-Priority Efficiency (CPF) says that the object should be allocated to the agents who value them in the much more 'correct' position corresponding to Agent 0's preference. Indistinguishability Fairness (IF) says that those agents should receive an equal share for fairness.

We aim to identify which agents conform more closely to the social planner's expectations. Recall from the motivating example: if object a_4 ranks fourth in the planner's ordering, and some agents place it above a_5 while others place it below, then only the former are consistent with the planner's intended priority. To formalize this, we define a conformity measure that accounts not only for how an agent ranks a particular object a, but also for which high-priority objects (as judged by the planner) the agent already receives.

Let $\overline{*}(i;a) = \arg\max\{\pi_x : x \in \overline{U}(R_i;a)\}$ be the least socially important object among those that agent i prefers at least as much as a. This serves as a benchmark for evaluating the agent's alignment with planner priorities.

Define the conformity cluster.

$$\hat{U}(R_i;a) = \{b : \overline{*}(i;a)R_i bR_i a\} \cup \{a\} \cup \{\overline{*}(i;a)\}$$

This is the set of objects that agent i places between a and $\overline{*}(i;a)$ in their ranking. In other words, This is the "cluster" of objects agent i sees as near or better than a, bounded by planner priority and agent preference. Then we define Conditional Effective Rank.

Definition 10 (Conditional Effective Rank)**.** *Whenever R_N, π, P, i and a, we define*

$$\hat{Z}(i;a;P) = \#\{b : b \in \hat{U}(R_i;a), P_{ib} > 0\}$$

to be the conditional effective rank *of object a for agent i, measuring how many objects in this conformity cluster agent i are receiving a nonzero share under the current allocation P.*

This is subtle because it doesn't just say: How high is object a in your ranking? But says: How many goods that you consider as good as or better than a, and that the planner thinks are also important, are you already receiving?

Example 8. *Consider the same example in Introduction,*

0:	a_1, a_2, a_3, a_4, a_5
1, 2:	a_5, a_4, a_2, a_1, a_3
3, 4:	a_2, a_4, a_1, a_3, a_5
5:	a_3, a_1, a_5, a_2, a_4

Table 4.14: Preference Profile

We have $\overline{*}(3;a_4) = \overline{*}(4;a_4) = a_4$ but $\overline{*}(1;a_4) = \overline{*}(2;a_4) = \overline{*}(5;a_4) = a_5$. Now assume $P_1 = P_2 = P_6 = (\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4})$. Then we have: $\hat{Z}(5;a_4;P) = 2$ but $\hat{Z}(1;a_4;P) = \hat{Z}(2;a_4;P) = 3$.

Definition 11 (Agent Priority at Object a)**.** *Given preferences R, allocation P, and planner priority π, we define the binary relation $\prec^a$ as follows:*

1. *$i \prec^a j$ if*

- $\max\{\pi_x : x \in \overline{U}(R_i;a)\} > \max\{\pi_x : x \in \overline{U}(R_j;a)\}$, *or*
- *these values are equal and* $\hat{Z}(i;a;P) > \hat{Z}(j;a;P)$.

2. $i \sim^a j$ *if both expressions are equal.*

Now we are ready to state the following Properties.

Property* 1. *Conformity-Priority Efficiency (CPF)*

For all R_N, *all* π, *all* a, *all* i, *if* $P_{ia} > 0$, *then* $\sum_{x \in \overline{U}(R_i;a)} P_{jx} = 1$ *for all* j *that* $i \prec^a j$

Conformity-Priority Efficiency means that before the social planner gives a doughnut to someone, look up the line—make sure everyone with a higher need or claim has already been taken care of. This property respects a priority queue, where priority is not just based on personal taste, but is carefully measured using both personal preferences and social importance.

Proposition 12. *CPF implies ordinal efficiency.*

Suppose for contradiction that an allocation P satisfies Conformity-Priority Efficiency (CPF) but fails ordinal efficiency. Then there is an "improvement cycle"

$$i_1 : a^1 a^2, \quad i_2 : a^2 a^3, \; \ldots, \; i_n : a^n a^1,$$

where each agent i_k has $P_{i_k, a^{k+1}} > 0$ and strictly prefers a^{k+1} to a^k. Let a^1 be the highest-priority object in this cycle under π. Trace the cycle backwards from i_n: CPF requires that any agent with strictly lower planner-priority alignment or with "used up" fewer high-priority goods cannot hold a positive share of a^1 if another agent with a stronger claim does. Yet the cycle demands exactly such a reallocation, yielding a direct violation of CPF at a^1. Repeating the same argument at each link shows no improvement cycle can exist, so P must be ordinally efficient.

Property* 2. *Indistinguishability Fairness (IF)*

For all R_N, *all* π, *all* a, *all* i, j *s.t.* $i \sim^a j$, *if* $P_{ia} > P_{ja} > 0$ *then* $\sum_{x \in \overline{U}(R_j;a)} P_{jx} = 1$.

The second property says if for any object we can not distinguish agents due to two criteria, then they should have the same chance/right to that object. So, if social planner is giving agent i more of something than an equally deserving agent j, then make sure that j is already being well taken care of in terms of their top preferences. Together with Property 1, it will single out the allocation in Table 4.2 in Example 1.

4.6.3 Flow Algorithm

Before defining the new method, we define a component rule that allocates the subset of objects to agents with consideration of full preference. Now given R_N, P, B, A, N, whenever R_N is preference profile, P is the matrix, B is to-assign-objects, A is set of objects, N is set of agents, we define $BM(B;A;N)$ as the constraint fractional adaptive Boston rule in the following sequential procedure. Let the capacity of objects be $Z^0 = [1]^A - [\sum_{i \in N} P^0_{ia}]_{a \in A}$ the capacity of agents be $C^0 = [1]^N - [\sum_{a \in A} P_{ia}]_{i \in N}$.

1. In the first period, for all $a \in B$ if $M(a;A^0;N^0) \neq \emptyset$, we fully allocate a to $M(a;A^0;N^0)$ and every agent receives $sh_{ia} = \frac{1}{m(a;A^0;N^0)}$. Then we update capacity Z^1 and C^1, the remaining objects A^1 and B^1, and the agents with positive capacity N^1.
2. For each k period, for all $a \in B^{k-1}$ if $M(a;A^{k-1};N^{k-1}) \neq \emptyset$, we allocate a to agents $M(a;A^{k-1};N^{k-1})$ and every agent receives, for some e:

$$sh_{ia} = c_i \wedge e \quad s.t. \sum_{i \in M(a;A^{k-1};N^{k-1})} sh_{ia} = z_a \wedge \sum_{i \in M(a;A^{k-1};N^{k-1})} c_i^{k-1}$$

   Then we update capacity Z^k and C^k, the remaining objects A^k and B^{k-1}, and the agents with positive capacity N^k.

This algorithm will finish in finite periods at most $|B|$ and produce an allocation matrix sh. The Constraint fractional adaptive Boston rule is the variant fractional adaptive Boston rule when we allocate a set of objects B by considering full preference. To notice, the object in B is not necessary to be fully allocated because it may not be at the top of unassigned objects in A for all agent i.

Now, we construct an algorithm to satisfy CPF and IF. Given a preference profile R_N and π. The algorithm is defined by the sequential procedure. Let $A^0 = A, N^0 = N, \pi^0 = \pi, C^0 = [1]^N, Z^0 = [1]^A$,.

1. In first period, we run $BM(\pi(1);A^0;N^0)$, then update π^1 as the set of unassigned objects in $\pi(1)$, A^1 as the set of unassigned objects in A^0, N^1 as the set of agents who do not approach the capacity, C^1 as the capacity of agents, Z^1 as the capacity of objects.
2. For each k period, denote $\pi(k) = \pi(k) \cup \pi^{k-1}$ we run $BM(\pi(k);A^{k-1};N^{k-1})$, then update $\pi^{k'}$, A^k, N^k, C^k, Z^k.

This algorithm is denoted by F^{π} and will finish in finite periods. The method is simple, it allocates the object set by set according to the π, for each set we run the constraint fractional adaptive Boston rule until it is fully allocated or no agent prefers it the most. Now we use the example in Table 4.4 to illustrate this method.

Example 9. *We consider R_N and π in Table 4.4, then the method works sequentially:*

1. *In step 1, allocate the object in $\pi(1) = \{a_1\}$. No one prefers a_1 to the rest, then a_1 passes to the next period.*
2. *In step 2, allocate the object in $\pi(2) \cup \{a_1\}$.*
   - *Agent 3,4 prefer a_2 and receive $\frac{1}{2}$ each.*
   - *After this, no one prefers a_1 to the rest, then a_1 passes to the next period.*
3. *In step 3, allocate the object in $\pi(3) \cup \{a_1\}$.*
   - *Agent 5 prefers a_3 and receive 1 and leave the market.*
   - *After this, no one prefers a_1 to the rest, then a_1 passes to the next period.*
4. *In step 4, allocate the object in $\pi(4) \cup \{a_1\}$.*
   - *Agent 3,4 prefer a_4 and receive $\frac{1}{2}$ each and leave the market.*
   - *After this, no one prefers a_1 to the rest, then a_1 passes to the next period.*
5. *In step 5, allocate the object in $\pi(5) \cup a_1$.*
   - *Agent 1,2 prefer a_5 and receive $\frac{1}{2}$ each.*
   - *After this, Agent 1,2 prefer a_1 and receive $\frac{1}{2}$ each.*

So, it result in the allocation in Table 4.4. Now, we present the main result: this algorithm exactly characterizes CPF and IF.

Proposition 13. *A mechanism is F^{π} if and only if it satisfies CPF and IF.*

We prove this by mathematical induction, and it's similar but complicated than Proposition 10. The process that those two axioms and the flow algorithm result in the same allocation is illustrated clearly in the above example and the example in the Introduction.

4.7 Interim Efficiency

In random allocation, a randomization device is crucial. Ordinal efficiency (OE) is important because it ensures that every possible way of breaking down the allocations is Pareto efficient. In contrast, ex-post Pareto efficiency (EPPE) only requires that an efficient randomization device exists. Now, we consider an intermediate notion, interim efficiency (IE), which allows for more efficient randomization devices.

Given a preference profile R_N, let $\mathscr{D}^R$ denote the set of **deterministic efficient assignments**, and $\mathscr{P}^R$ denote the set of **probabilistic ordinally efficient assignments**. The union $\mathscr{D}^R \cup \mathscr{P}^R$ represents the entire set of efficient assignments.

A random assignment is *ex-post Pareto efficient* if it can be decomposed into a convex combination of deterministic efficient assignments, i.e., $\sum_{k=1}^{K} \lambda_k D_k^R$ with $\lambda_k > 0$ and $D_k^R \in \mathscr{D}^R$ for all k.

A random assignment is *'Interim Efficient'* if it can be decomposed into a convex combination of probabilistic ordinally efficient assignments, i.e., $\sum_{k=1}^{K} \lambda_k P_k^R$ with $\lambda_k > 0$ and $P_k^R \in \mathscr{P}^R$ for all k.

We will demonstrate that these two notions are distinct, specifically, that *'Interim Efficient'* refines *ex-post Pareto efficiency* but is less stringent than *ordinal efficiency.*

Proposition 14.

1. *Interim Efficiency implies ex-post pareto efficiency, but the converse is not true.*
2. *Ordinal Efficiency implies Interim Efficiency, but the converse is not true.*
3. *Random Priority is Ex-post Pareto Efficient but not Interim Efficient.*

Proposition 14 explains that interim efficiency is a more refined concept than ex-post Pareto efficiency but is less demanding than ordinal efficiency. IE is significant because it permits more types of efficient randomization devices, although it is less strict than OE, which demands efficiency in all possible deterministic assignments.

4.7.1 Random Flow

In this part, we consider methods that are easy to run in the lab, such as Random Priority, and examine those that are superior to Random Priority in terms of efficiency (based on axioms) and fairness (based on experiments).

We randomly arrange a series of objects and present them sequentially to the agents. We treat that order of objects as the Agent 0' preference, and run Flow algorithm. This method results in a random allocation directly. This is the opposite of Random Priority. We call it Random Flow and we show Random Flow has better performance in efficiency, called interim efficient (explained in the next part), which Random Priority does not. Moreover, Random Flow is at least neutral, weakly strategy-proof, and weakly envy-free.

We denote Π as the set of permutations of A. Given $\pi \in \Pi$, we denote F^{π} as the flow algorithm with order π. Then a mechanism is a Random Flow mechanism if for any R_N it selects the allocation:

$$\frac{1}{|\Pi|} \sum_{\pi \in \Pi} F^{\pi}(R_N)$$

Remark. In practical scenarios where many agents require a limited number of valuable items, the Random Flow (RF) mechanism demonstrates superior computational efficiency compared to the Random Priority method. This efficiency stems from randomizing the order of objects rather than the order of agents.

4.7.2 Necessary condition for Random Flow

A random assignment rule f is *sd-Envy-Free* if $\forall R_N, \forall i \in N$ we have $f_i(R) R_i^{sd} f_j(R), \forall j \neq i$. A random assignment rule is *weakly sd-Envy-Free* if no agent strictly prefers someone else's allocation to him or her, that is $f_j(R_N) R_i^{sd} f_i(R_N)$ indicates $f_j(R_N) = f_i(R_N)$, $\forall j \neq i, \forall i \in N$.

A random assignment rule is *sd-strategy-proof* if $\forall i \in N, \forall (R_i)_{i\in N}$ and $\forall R' \in \mathscr{R}, f_i(R) R_i^{sd} f_i(R_i^{'}, R_{-i})$. A random assignment rule is *weakly sd-strategy-proof* if an agent cannot obtain an allocation that strictly stochastic dominates to a true allocation by telling a lie, that is $f_j(R_i^{'}, R_{-i}) R_i^{sd} f_i(R_N)$ indicates $f_i(R_i^{'}, R_{-i}) = f_i(R_N)$, $\forall i \in N, \forall R_i^{'} \in \mathscr{R}$.

Proposition 15. *RF is interim efficient, weakly envy-free, and weakly strategyproof.*

4.7.3 Numerical testing of fairness

For the random allocation problem, we consider two mechanisms: Random Priority (RP) and Random Flow (RF), and focus on the property sd-envy-free and ask the question of which algorithm will generate 'no-envy' allocation more often. We focus on the case $n = 4$ and all preference profiles. We began by selecting unique preference profiles and eliminating permutations of rows and columns. Each method was then applied to these profiles to determine allocations.

Observation: RF generates more sd envy-free allocations than RP.

Figure 4.1 shows RP generates sd-envy-free allocation in 36% of preference profiles while RF generates sd-envy-free allocation in 48% of preference profiles which is an improvement of 12% in RF. Moreover, given the nature of the Flow Algorithm, it is possible to improve this behavior by designing a more fair and easy-to-implement algorithm while keeping higher efficiency (interim) in this spirit.

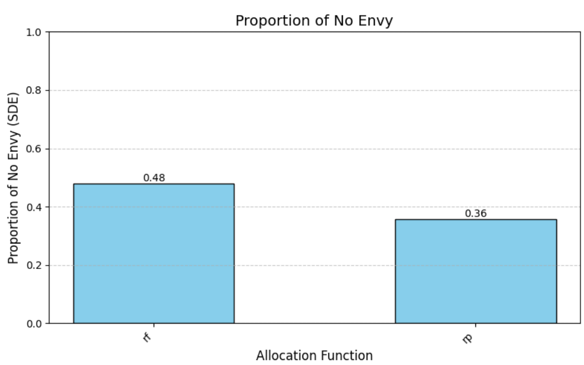

Figure 4.1: Ratio of No-envy for RP and RF

Observation: RF generates a more equalized allocation than RP and PS in about 60% of profiles.

In comparing RP, PS, and RF, Table 4.15 displays the percentage of preference profiles where the row method outperforms the column method. RF generates a more equalized allocation than RP in about 61.5% of profiles and then PS in roughly 59.7% of profiles.

	RP	PS	RF
RP		0.479	0.289
PS	0.502		0.402
RF	0.615	0.597	

Table 4.15: Better Performance for RP, PS, and RF when comparing to RP, PS, and RF

Then we analyze to what extent the RF is more equalized. In Figure 4.2, we compare RF and PS across each preference profile. The x-axis represents each preference profile, while the y-axis represents the variance difference between PS and RF, specifically calculated as the variance of PS minus the variance of RF. A positive value on the graph signifies that the allocation under PS is less equalized than that under RF, and conversely for negative values. Moreover, in those 59.7% preference profiles, the variance under PS is much larger than the variance under RF with the maximum 0.047 while in the 40.2% preference profiles, the variance under RP is much larger than variance under PS with the maximum 0.01. So we can state that for the preference profile PS is much equalized, the difference between RF and PS is not large.

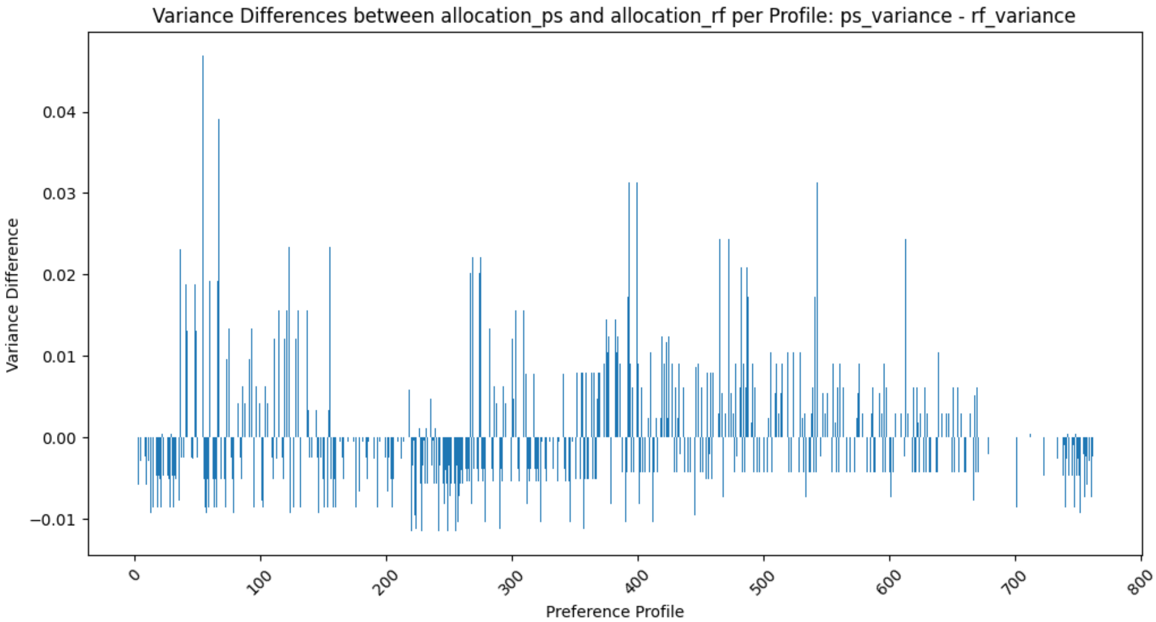

Figure 4.2: Difference between PS and RF

4.8 Conclusion

This paper investigates the classic random allocation problem with a focus on fairness and efficiency, particularly in settings involving a social planner. First, we examine the efficiency notion of *interim favouring rank*, introduced by Harless 2018, and propose a novel variant called *interim favouring support*. This suggests focusing on the support of the allocation to avoid unfair situations in which some agents have a higher chance of receiving top objects than others. We agree with the criticisms in Kojima and Ünver 2014 regarding the use of 'favouring rank' as a welfare criterion. Furthermore, we explore the applicability of the *interim favouring support* axiom and find that the popular mechanism, the fractional adaptive Boston rule, satisfies this axiom. To characterize this mechanism, we introduce a natural fairness principle: *equal support, equal claim*, which requires that agents indistinguishable by prior criteria should have equal chances of receiving an object.

Next, we introduce a new model that incorporates a social planner. We propose two main axioms based on conformity and favouring support. The first, *Conformity-Priority Efficiency*, requires that agents who conform to the social planner be rewarded. The second, *Indistinguishability Fairness*, ensures that agents indistinguishable by the previous criteria are treated equally. To characterize these axioms, we construct a simple algorithm called the *Flow Algorithm*. The algorithm allocates objects according to the social planner's preference order (from most to least preferred). If some agents conform to an object—meaning they prefer it to the remaining ones—they receive it with equal probability. Otherwise, the object is temporarily held, and the algorithm proceeds to allocate the next object in the next period. In each period, we revisit the held objects: if any agent now prefers a held object to the remaining ones, it is allocated with equal chance. This process eventually returns a probabilistic allocation that satisfies the two proposed axioms.

Finally, we consider the construction of a more efficient and practically implementable algorithm. We introduce the *Random Flow* (RF) algorithm, which is a uniform randomization over all possible Flow allocations. RF satisfies a new efficiency notion, *Interim Efficiency*, which requires that the random allocation be decomposable into a convex combination of probabilistically ordinally efficient allocations. *Interim Efficiency* lies between ordinal efficiency and ex-post Pareto efficiency: every ordinally efficient allocation is interim efficient, and every interim efficient allocation is ex-post Pareto efficient (though the converse does not hold). We show that Random Flow is interim efficient, while Random Priority is not. Although RF is not strategy-proof, it is weakly strategy-proof. Beyond our axiomatic analysis, we also provide numerical evidence. We observe that RF generates no-envy outcomes in more preference profiles than RP, with a 12% improvement. These results suggest that we can design and apply a simple, practical algorithm inspired by the Flow mechanism that outperforms RP in both efficiency and fairness.

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Chapter 5

Conclusion

This thesis explores critical aspects of intertemporal preferences, strategic behavior in random games, and fairness principles in random allocation problems, offering new theoretical insights and practical implications in these areas. Each chapter presents distinct yet interconnected contributions to the broader field of microeconomics and decision theory, advancing our understanding of individual and collective decision-making under uncertainty and competing objectives.

In Chapter 2, we introduced a novel representation theorem for exponential discounted additive utility within a continuous-time framework, where consumption exists in a simplex. This framework extends the consumption space to piecewise continuous functions and incorporates risk into decision-making. Our work diverges from traditional paradigms by addressing temporal preferences with Time Stationarity and Time Impatience, allowing for greater flexibility in modeling. By moving beyond the fixed preference at $t = 0$, this chapter lays the foundation for more robust analyses of intertemporal choice. Future directions include relaxing exponential discounting assumptions and incorporating stochastic elements into the outcome domain, enhancing the model's applicability to real-world scenarios characterized by uncertainty and evolving preferences.

Chapter 3 addresses the asymptotic behavior of strictly dominated strategies in random games, contributing both theoretical and computational advancements. We demonstrated that the probability of strictly dominated strategies vanishes as the number of strategies increases, while providing bounds on the existence of q-portion-dominated strategies. The proposed algorithm significantly reduces computational complexity compared to existing methods, paving the way for practical applications in economics and computer science. Extending these results to settings with non-i.i.d. payoffs or extensive-form games presents exciting future challenges, as does refining the algorithm to handle broader classes of strategic interactions efficiently.

In Chapter 4, we construct a new model including the social planner. The chapter explored the critical question of whether social planners' objectives should override individual freedom when conflicts arise. By modelling the social planner as an agent within the allocation framework, we assume the social planner's preference is responsive to the order of importance, one example is lexicographical preference. The analysis revealed the importance of rewarding conformity to social expectations while acknowledging the limitations of truthful reporting by both planners and agents. The chapter highlighted the interplay between individual preferences and social expectations, proposing principles like *Conformity-Priority Efficiency (CPF)* and *Indistinguishability Fairness (IF)* to balance these competing priorities. This work invites further exploration of strategy-proof mechanisms that incentivize honest behavior for both of the social planner and agents, with significant implications for tax systems, subsidy programs, centralized planning, and crisis management scenarios such as COVID-19 lockdowns.

Then, we turned to the classic random allocation problem, introducing a novel fairness criterion, *interim favouring support*, and analyzing its implications for popular mechanisms like the fractional adaptive Boston rule. Then through the Flow Algorithm and its randomized variant, Random Flow (RF), we demonstrated the feasibility of designing mechanisms that enhance fairness and efficiency while respecting practical constraints. RF's superiority over Random Priority in terms of envy-freeness and interim efficiency underscores its potential for real-world applications.

In conclusion, this thesis contributes to advancing theoretical frameworks and practical tools for understanding decision-making under risk, strategic interactions, and the trade-offs between fairness and efficiency. Each chapter offers pathways for future research, bridging gaps between theory and application, and providing a foundation for addressing pressing economic and societal challenges. By integrating innovative approaches to preferences, strategies, and allocations, this work aspires to inspire further exploration and development in microeconomics and mechanism design.

Appendices

A Proof in Chapter 2

A.1 Proof: Necessity of Axioms in Proposition 2

Now we prove the necessary part of a proposition 2. Recall the utility function:

$$DU(\mathbf{p}_{[t,b]}) = \int_t^b e^{-\beta(s-t)} \sum_{i=1}^n u_i(\mathbf{p}_i(s))ds \tag{1}$$

It is easy to check it satisfies Weak Order, Continuity, Independence, Non-degeneracy, and SAU because it is an additive utility over simplex for each time t. Now we prove it also satisfies the other time preference axioms.

Time monotonicity For all $t \in \mathscr{T}, \mathbf{p}, \mathbf{p}' \in \mathscr{P}_t, \forall \tau \geq t$, if $\mathbf{p}(\tau) \succsim_\tau^\Delta \mathbf{p}(\tau)'$: $U(\mathbf{p}(\tau)) \geq U(\mathbf{p}'(\tau)), \forall t \in \mathscr{T}$, which means

$$\int_t^T e^{-\beta(s-t)} U(\mathbf{p}(s))ds > \int_t^T e^{-\beta(s-t)} U(\mathbf{p}'(s))ds \Leftrightarrow \mathbf{p} \succsim_t \mathbf{p}' \tag{2}$$

For all $t \in \mathscr{T}$ for all constant streams, $\mathbf{c}, \mathbf{c}'$, if $\mathbf{c} \succsim_t \mathbf{c}'$, the other operation is obvious because $U(\mathbf{c}(t)) \geq U(\mathbf{c}(t))'$. □

Time stationary $\forall t, \gamma \in \mathscr{T}, \mathbf{p} \in \mathscr{P}_t$, and $\boldsymbol{f} : [0,\gamma] \to \Delta$, $\boldsymbol{g} : [0,\gamma] \to \Delta$, if for some $\tau > t$, we have

$$(\mathbf{p}, {}_\tau f_{[\tau,\tau+\gamma)}) \succsim_t (\mathbf{p}, {}_\tau \boldsymbol{g}_{[\tau,\tau+\gamma)}) \tag{3}$$

Then we have

$$
\begin{aligned}
& DU((\mathbf{p}, {}_{\tau}\boldsymbol{f}_{[\tau,\tau+\gamma)})_{[t,T]}) \\
& = \int_t^{\tau} e^{-\beta(s-t)} U(\mathbf{p}(s)) ds + \int_{\tau}^{\tau+\gamma} e^{-\beta(s-\tau)} U(\boldsymbol{f}(s)) ds + \int_{\tau+\gamma}^{T} e^{-\beta(s-\tau-\gamma)} U(\mathbf{p}(s)) ds \\
& \geq \int_t^{\tau} e^{-\beta(s-t)} U(\mathbf{p}(s)) ds + \int_{\tau}^{\tau+\gamma} e^{-\beta(s-\tau)} U(\boldsymbol{g}(s)) ds + \int_{\tau+\gamma}^{T} e^{-\beta(s-\tau-\gamma)} U(\mathbf{p}(s)) ds \\
& = DU((\mathbf{p}, {}_{\tau}\boldsymbol{g}_{[\tau,\tau+\gamma)})_{[t,T]})
\end{aligned}
\tag{4}
$$

As a result

$$
\int_{\tau}^{\tau+\gamma} e^{-\beta(s-\tau)} U(\boldsymbol{f}(s)) ds \geq \int_{\tau}^{\tau+\gamma} e^{-\beta(s-\tau)} U(\boldsymbol{g}(s)) ds \tag{5}
$$

It's true for any τ due to the construction of $\boldsymbol{g}$ and $\boldsymbol{g}$, then $\forall \delta > 0$, let $\tau' = \tau + \delta$ and $s' = s + \delta$

$$
\begin{aligned}
\int_{\tau'}^{\tau'+\gamma} e^{-\beta(s'-\tau')} U(\boldsymbol{f}(s)) ds' &= \int_{\tau'}^{\tau'+\gamma} e^{-\beta(s'-(\tau+\delta))} U(\boldsymbol{f}(s')) ds' \\
&= \int_{\tau}^{\tau+\gamma} e^{-\beta(s-\tau)} U(\boldsymbol{f}(s)) ds \\
&\geq \int_{\tau}^{\tau+\gamma} e^{-\beta(s-\tau)} U(\boldsymbol{g}(s)) ds \\
&= \int_{\tau}^{\tau+\gamma} e^{-\beta(s-(\tau'-\delta))} U(\boldsymbol{g}(s)) ds \\
&= \int_{\tau'}^{\tau'+\gamma} e^{-\beta(s'-\tau')} U(\boldsymbol{g}(s')) ds'
\end{aligned}
\tag{6}
$$

As a result, for any $\tau \geq t$, we have $(\mathbf{p}, {}_{\tau}\boldsymbol{f}_{[\tau,\tau+\gamma)})_{[t,T])} \gtrsim_t (\mathbf{p}, {}_{\tau}\boldsymbol{g}_{[\tau,\tau+\gamma)})_{[t,T])}$ is true for any $\tau \geq t$. $\square$

Time Impatience

$\forall t \in \mathscr{T}, \forall \gamma > 0, \forall \mathbf{p}, \mathbf{q} \in \mathscr{P}_t$ with $\mathbf{p}(s) \gtrsim_s^{\Delta} \mathbf{q}(s), \forall s \in [t, T]$, if for some $\tau \geq t$ we have $(\mathbf{q}, \mathbf{p}_{[t,t+\gamma)}) \gtrsim_t (\mathbf{q}, \mathbf{p}_{[\tau,\tau+\gamma)})$, then we have (as same as the Figure 6):

$$
\begin{aligned}
& \int_t^{\tau} e^{-\beta(s-t)} U(\mathbf{p}(s')) ds' + \int_{\gamma}^{\tau+\gamma} e^{-\beta(s-\gamma)} U(\mathbf{q}(s)) ds \\
& \geq \int_t^{\tau} e^{-\beta(s-t)} U(\mathbf{q}(s')) ds' + \int_{\gamma}^{\tau+\gamma} e^{-\beta(s-\gamma)} U(\mathbf{p}(s)) ds
\end{aligned}
\tag{7}
$$

Then for any $\tau > t$, the above relation doesn't change because $U(\mathbf{p}(t)) > U(\mathbf{q}(t)), \forall t$. As a result, $\forall \tau > t, (\mathbf{q}, \mathbf{p}_{[t,t+\gamma)}) \gtrsim_t (\mathbf{q}, \mathbf{p}_{[\tau,\tau+\gamma)})$. $\square$

Dynamic Consistency If $\forall t \in \mathscr{T}, \mathbf{p}, \mathbf{p}^{'} \in \mathscr{P}_t$, $\mathbf{p}^{'} \gtrsim_t \mathbf{p}$:

$$\int_t^T e^{-\beta s} U(\mathbf{p}(s))ds > \int_t^T e^{-\beta(s-\tau)} U(\mathbf{p}^{'}(s))ds \tag{8}$$

If $\forall \tau \le t, \mathbf{p}(\tau)^{'} = \mathbf{p}(\tau)$, then

$$\int_\tau^t e^{-\beta s} U(\mathbf{p}(s))ds + \int_t^T e^{-\beta s} U(\mathbf{p}(s))ds > \int_\tau^t e^{-\beta(s-\tau)} U(\mathbf{p}^{'}(s))ds + \int_t^T e^{-\beta(s-\tau)} U(\mathbf{p}^{'}(s))ds \tag{9}$$

It means $\mathbf{p} \gtrsim_\tau \mathbf{p}^{'}$.

A.2 Proof: Sufficiency in Proposition 1

Lemma 3. *If the preference $\gtrsim_t^\Delta$ satisfies conditions Weak Order, Simplex Additive Utility, Independence, Continuity, Time Monotonicity, and Non-Degeneracy, then:*

1. *There exist outcomes $x, y \in \Delta$, such that $x \succ_t^\Delta y$.*
2. *$p \succ^\Delta q$ if and only if $(\mathbf{c}(p), \mathbf{g}_{[t,t']}) \gtrsim_t (\mathbf{c}(q), \mathbf{g}_{[t,t']})$ for any outcome $p, q \in \Delta$ and any $\mathbf{g} \in \mathscr{P}$, and any $t \neq t'$.*
3. *$\gtrsim^\Delta$ is continuous in the sense that for any $p \succ^\Delta q$ there exist $\delta > 0$ such that $|g - q| < \delta$ implies $p \succ^\Delta g$ and $|g - p| < \delta$ implies $g \succ^\Delta q$.*
4. *There exists a continuous utility function for $\gtrsim^X$, specifically, $U(p) = \sum_{i=1}^n u_i(p_i)$ for all $p \in \Delta$. And it has a non-point interval range.*
5. *For any $\mathbf{p} \in \mathscr{P}$ and any time t, there exists a constant outcome stream $\mathbf{c}(x)$ such that $\mathbf{c}(x) \sim_t \mathbf{p}$.*
6. *There exist $x \in \Delta$ such that exist $p \in \Delta$, we have either $x \succ^\Delta p$ or $p \succ^\Delta x$.*

Proof. 1. By Strong Essentiality, for all $\alpha \in (0,1)$ there exist some $p, p^{'}, p^{''}$ such that $s(p), s(p^{'}) \subseteq [n]$, and $s(p^{''}) \cap [n] = \emptyset$, then $\alpha p + (1-\alpha)p^{''} \succ_t^\Delta \alpha p^{'} + (1-\alpha)p^{''}$. Then denote $x = \alpha p + (1-\alpha)p^{''}$ and $y = \alpha p^{'} + (1-\alpha)p^{''}$.

2. $\Rightarrow$ By definition of $\gtrsim^\Delta$, for any outcome $p, q \in \Delta$, we have two constant streams $\mathbf{c}(p)$ and $\mathbf{c}(q)$ such that $\mathbf{c}(p) \gtrsim_t \mathbf{c}(q)$ for every t. By Time monotonicity (1), we have for any $\mathbf{g} \in \mathscr{P}$, we have $(\mathbf{c}(p), \mathbf{g}_{[t,t']}) \gtrsim_t (\mathbf{c}(q), \mathbf{g}_{[t,t']})$ because $(\mathbf{c}(p), \mathbf{g}_{[t,t']})(\tau) \gtrsim_t^\Delta (\mathbf{c}(q), \mathbf{g}_{[t,t']})(\tau)$ for all $\tau \ge t$. $\Leftarrow$. Assume we have we have $(\mathbf{c}(p), \mathbf{g}_{[t,t']}) \gtrsim_t (\mathbf{c}(q), \mathbf{g}_{[t,t']})$ for all $\mathbf{g} \in \mathscr{P}$ and for all t. Then if $q \gtrsim^\Delta p$, we have $(\mathbf{c}(q), \mathbf{g}_{[t,t']}) \gtrsim_t (\mathbf{c}(p), \mathbf{g}_{[t,t']})$ for similar reason which is contradiction. Hence $p \gtrsim^\Delta q$.

3. By Continuity for $\gtrsim_t$ and definition of $\gtrsim^\Delta$, we have immediately the Continuity for $\gtrsim^\Delta$.

4. From Qin and Rommeswinkel 2022, the $\succsim^{\Delta}$ satisfies Weak Order, Simplex Additive Utility, and Continuity have an additive utility representation. Because Δ is connected and contains outcomes that are not indifferent, then the range of any such function is a non-point interval.
5. $\succsim^{\Delta}$ has a continuous utility function $u(p)$, and it has the maximum value u^* and minimum value u_*. By continuity, we can choose $y, z \in \Delta$ such that $u(z) = u^*$ and $u(y) = u_*$. Then by definition of $\succsim^{\Delta}$ and Time monotonicity (1), for any $\mathbf{p} \in \mathscr{P}$, it implies $\mathbf{c}(z) \succsim_t \mathbf{p} \succsim_t \mathbf{c}(y)$. By Continuity, there exist a constant stream $\mathbf{c}(x)$ such that $\mathbf{c}(x) \sim_t \mathbf{p}$.
6. By Non-Degeneracy, for every time t, there exist $\mathbf{p}, \mathbf{q} \in \mathscr{P}$ such that $\mathbf{p} \succsim_t \mathbf{q}$. Suppose $x \sim^{\Delta} y$ for all $x, y \in \Delta$, then by definition we have $\mathbf{p} \sim_t \mathbf{q}$ for all t and for all $\mathbf{p}, \mathbf{q}$ by (d), thus it's a contradiction of Non-Degeneracy.

□

Now we define π be the partition of an interval $[0, T]$ as the set of disjoint sub-intervals: $< t_0, t_1 >, \ldots, < t_{m-1}, t_m >$, where $0 = t_0 \leq t_1 \leq \cdots \leq t_{m-1} \leq t_m$ and m is an integer. A step outcome stream based on a partition π is an outcome stream that is constant on each sub-interval $< t_{i-1}, t_i >$ for all $i = 1, \ldots, m$. To abuse notation, for a step outcome stream $\mathbf{p}$, we define the value for each sub-interval $t \in < t_{i-1}, t_i >$ as the subscribe $\mathbf{p}(t) \equiv p(i)$. Hence we will use $(p(1), p(2), \ldots, p(m))$ to denote a step outcome stream based on a partition $\pi = < t_0, t_1 >, \ldots, < t_{m-1}, t_m >$.

The set of step outcome streams based on a partition π will be denoted by $S(\pi)$, and the set of step outcome streams for all partitions of $[0, T]$ will be denoted by $S(T)$. A step outcome stream $\mathbf{p}$ in $S(\pi)$ can be regarded as a vector $(p(1), p(2), \ldots, p(m))$ of outcomes, and the set $S(\pi)$ can be regarded as the product set Δ^m. For preference $\succsim$ on product space Δ^m, because Δ is connected and topological separable, Gorman 1968 and Debreu et al. 1954 states there exist a continuous function defined on $S(\pi)$ such that $U(\mathbf{p}) \geq U(\mathbf{p}')$ if and only if $\mathbf{p} \succsim \mathbf{p}'$. Then we can rely on the additive value model from Debreu 1959, Gorman 1968, or Harvey and Østerdal 2012. To do so, we need to show there are more than two *essential* factors.

Definition 12. *A factor or subinterval $< s, s' >$ is essential if there exist step outcome streams $\mathbf{p}, \mathbf{q}, \mathbf{h}$ in $S(\pi)$ such that $(\mathbf{h}, \mathbf{p}_{< s, s' >})$ and $(\mathbf{h}, \mathbf{q}_{< s, s' >})$ are not indifferent.*

Lemma 4. *If a space $(S(\pi), \succsim)$ satisfies the condition required in Lemma 3, if a subinterval $< s, s' >=< t_{i-1}, t_i >$ is non-point, then it is essential. Moreover, if subinterval $< s, s' >=< t_{i-1}, t_i >$ is a point or empty, then it is inessential.*

Proof. By Lemma 3, there exist outcomes $x, y \in \Delta$ such that $x \succ^{\Delta} y$. Then pick any $\mathbf{h} \in S(\pi) \subset \mathscr{P}$, by first statement of Time monotonicity, we have $(\mathbf{h}, \mathbf{c}(x)_{<s,s'>}) \succsim (\mathbf{h}, \mathbf{c}(y)_{<s,s'>})$. And because $< s, s' >$ is a non-point interval, then it's a strict relation, hence not indifferent.

If subinterval $< s, s' >=< t_{i-1}, t_i >$ is a point or empty, then we show $(\mathbf{h}, \mathbf{p}_{<s,s'>})$ and $(\mathbf{h}, \mathbf{q}_{<s,s'>})$ are indifferent for all $\mathbf{p}, \mathbf{q}, \mathbf{h}$ in $S(\pi)$. At first we take $\mathbf{h}$ as the constant outcome stream $\mathbf{c}(h)$, then second axiom in Time Monotonicity states $(\mathbf{h}, \mathbf{p}_{<s,s'>})$ and $(\mathbf{h}, \mathbf{q}_{<s,s'>})$ are indifferent. By Independence, we can replace $\mathbf{h}$ with any other non-constant stream, and it still holds. □

We define $(S(\pi), \succsim)$ as proper if π has at least three non-point intervals. Then we rely on the additive value model from Debreu 1959, Gorman 1968.

Lemma 5. *If a proper space $(S(\pi), \succsim)$ satisfies conditions in Lemma 3, then*

1. *$\succsim$ has an additive utility function, $U(\mathbf{p}) = \sum_{i \in E}^{m} u_i(p(i))$, where partition E is the set of all non-point intervals, and $u_i(p(i)) = \sum_{j=1}^{n} u_i(p_j(i))$.*
2. *The function $U(\mathbf{p})$ is cardinally unique upon on affine transformation.*

Proof. By definition, $S(\pi)$ is product space. The relation $\succsim$ is continuous, transitive, complete, preferential independent by Axiom Continuity, Axiom Weak Order, and Axiom Independence. Also, there are at least three non-point intervals and at least three essential factors because $(S(\pi), \succsim)$ is proper.

Then we show those m factors of $S(\pi)$ are independent. So we need to show, for any $< s, s' >=< t_{i-1}, t_i >$, we have for any $\mathbf{p}, \mathbf{q}, \mathbf{h}, \mathbf{l}$ in $S(\pi)$

$$(\mathbf{h}, \mathbf{p}_{<s,s'>}) \succsim (\mathbf{h}, \mathbf{q}_{<s,s'>}) \Leftrightarrow (\mathbf{l}, \mathbf{p}_{<s,s'>}) \succsim (\mathbf{l}, \mathbf{q}_{<s,s'>})$$

We know $\succsim$ is preferential independent. Hence we obtain the relation be same if we replace $\mathbf{h}$ with $\mathbf{l}$ in time period $<0,s>$ at first. After, we obtain $(\mathbf{l},\mathbf{p}_{<s,s'>}) \succsim (\mathbf{l},\mathbf{q}_{<s,s'>})$ by replacing the $\mathbf{h}$ with $\mathbf{l}$ in time period $<s',T>$.

Then, by Debreu 1959, the additive utility function is obtained. □

Since all functions are ordinally equal in the sense $u_j(p) = f(u_i(p))$ for all $i \neq j$ and for all $p \in \Delta$ by Lemma 3. We could prove they are also cardinal equal by the Jensen functional equation once we have the form $f(\lambda u(p^0)+(1-\lambda)u(p^1)) = \lambda f(u(p^0))+(1-\lambda)u(p^1)$.

There is a difference. We rely on the second statement of Time Monotonicity and Independence, and Time stationarity instead of the Mid-outcome independence condition in Harvey and Østerdal 2012.

Recall $x^{1/2} \in X$ as the mid-outcome of two outcomes x^0 and x^1 if

- $x^1 \succ^X x^{1/2} \succ^X x^0$
- exist $y^1 \succ^X y^0$ and $z \in \Delta$ such that

$$(\mathbf{c}(x^{1/2})_{<s,s'>}, \mathbf{c}(y^0)_{<t,t'>}, \mathbf{c}(z)) \sim (\mathbf{c}(x^0)_{<s,s'>}, \mathbf{c}(y^1)_{<t,t'>}, \mathbf{c}(z))$$

  and

$$(\mathbf{c}(x^{1/2})_{<s,s'>}, \mathbf{c}(y^1)_{<t,t'>}, \mathbf{c}(z)) \sim (\mathbf{c}(x^1)_{<s,s'>}, \mathbf{c}(y^0)_{<t,t'>}, \mathbf{c}(z))$$

  for some disjoint interval $<s,s'>$ and $<t,t'>$.

Axiom* 3. ***Mid-outcome independence*** *For any disjoint, non-point intervals* $<s,s'>$ *and* $<t,t'>$*, and outcome* $x^1 \succ^X x^0$*, if the pair* x^1,x^0 *has a mid-outcome with respect to outcomes in* $<t,t'>$*, and the pair* x^1,x^0 *has a mid-outcome with respect to outcomes in* $<s,s'>$*, then* x^1,x^0 *has the same mid-outcome in both cases.*

They use it to show for any three tuples if $u(p^1)-u(p^{1/2} = u(p^{1/2})-u(p^0))$ then $w(p^1)-w(p^{1/2} = w(p^{1/2})-w(p^0))$ which functions u,w are functions in different period. However, we do not rely on this axiom to obtain cardinal equivalence.

Lemma 6. *If* $\succsim_0$ *satisfies conditions required by Lemma 4 and Time stationary, then it has a utility function of the form,* $U(\mathbf{p}) = \sum_i \delta(i)u(p(i))$*, such that:*

1. *$u(p)$ is continuous for $p \in \Delta$ and has a non-point interval range.*
2. *A weight $\delta(i)$ is positive if the interval $< t_{i-1}, t_i >$ is non-point and is zero otherwise.*
3. *The function $u(p)$ and $\delta(i)$ are unique up to positive multiples.*

Proof. Suppose we have two different time intervals $< t_{i-1}, t_i >$ and $< t_{j-1}, t_j >$ where $i \neq j$. By Lemma 3, there is a utility function on $\succsim^{\Delta}$, $u(p)$ and $w(p)$ for each interval, so they are two different components in the additive utility function $U(\mathbf{p})$. We show they are cardinal equal.

For any tuples $p^0 \succ_i^{\Delta} p^{1/2} \succ_i^{\Delta} p^1$ such that $p^{1/2}$ is mid-outcome for p^0, p^1 with respect to outcomes in $< t_{j-1}, t_j >$. Now we show $p^{1/2}$ is indifferent to (may not be the same) the mid-outcome for p^0, p^1 with respect to outcomes in $< t_{i-1}, t_i >$.

From definition of mid-outcome, we obtain $u(p^{1/2}) = \frac{1}{2}u(p^0) + \frac{1}{2}u(p^1)$ and $p^{1/2} \sim_i^{\Delta} \frac{1}{2}p^0 + \frac{1}{2}p^1$. Therefore, $\mathbf{c}(p^{1/2}) \sim_i \mathbf{c}(\frac{1}{2}p^0 + \frac{1}{2}p^1)$. Then time stationary states

$$(\mathbf{c}(p^{1/2}), \mathbf{c}(p^{1/2})_{<t_{i-1},t_i>}) \sim_0 (\mathbf{c}(p^{1/2}), \mathbf{c}(\frac{1}{2}p^0 + \frac{1}{2}p^1)_{<t_{i-1},t_i>})$$

Then it's true for all $j - 1 > 0$,

$$(\mathbf{c}(p^{1/2}), \mathbf{c}(p^{1/2})_{<t_{j-1},t_j>}) \sim_0 (\mathbf{c}(p^{1/2}), \mathbf{c}(\frac{1}{2}p^0 + \frac{1}{2}p^1)_{<t_{j-1},t_j>})$$

By Lemma 4, we have $\sum_{k \neq j} u_k(p^{1/2}) + u_j(p^{1/2}) = \sum_{k \neq j} u_k(p^{1/2}) + u_j(\frac{1}{2}p^0 + \frac{1}{2}p^1)$. Then $w(\frac{1}{2}p^0 + \frac{1}{2}p^1) \equiv u_j(\frac{1}{2}p^0 + \frac{1}{2}p^1) = u_j(p^{1/2}) \equiv w(p^{1/2})$ and $p^{1/2} \sim_j^{\Delta} \frac{1}{2}p^0 + \frac{1}{2}p^1$.

Now assume h is the mid-outcome for p^0, p^1 with respect to outcomes in $< t_{i-1}, t_i >$. We have $w(h) = \frac{1}{2}w(p^0) + \frac{1}{2}w(p^1)$ and $h \sim_j^{\Delta} \frac{1}{2}p^0 + \frac{1}{2}p^1 \sim_j^{\Delta} p^{1/2}$. Then $w(p^1/2) = w(h) = \frac{1}{2}w(p^0) + \frac{1}{2}w(p^1)$. In other words, $f(u(p^{1/2})) = \frac{1}{2}f(u(p^0)) + f(u(p^1)))$. By Jensen functional equation, we have $f(u(p)) = au(p) + b$ for some constant a, b.

Then in conclusion, the function $U(\mathbf{p})$ can be written as $U(\mathbf{p}) = \sum_{i=1}^{m} \delta(i)u_i(p(i) = \sum_{i=1}^{m}(\delta(i)u(p(i)+b_i))$ and we can omit the constant term by cardinal uniqueness of $U(\mathbf{p})$, hence $U(\mathbf{p}) = \sum_{i=1}^{m} \delta(i)u(p(i))$. Condition 1 is implied by Lemma 4 and Lemma 3, and Condition 2 is implied by Lemma 4.

□

Then we apply the Theorem A.1 in Harvey and Østerdal 2012 if $\succsim_0$ satisfies conditions required by Lemma 5, then it has a utility function of the form, $U(\mathbf{p}) = \sum_{i=1}^{m}(\mu(s_i) - \mu(s_{i-1}))u(p(i))$. Also because $\mu(t)$ is continuous for every t, it is absolutely continuous for $t \in [0,T]$. Then $\succsim_0$ also have a utility function of the form, $U(\mathbf{p}) = \sum_{i=1}^{m}(\mu(s_i) - \mu(s_{i-1}))u(p(i)) = \sum_{i=1}^{m}(\int_{[s_{i-1},s_i]} \delta(t)dt)u(p(i)) = \sum_{i=1}^{m}(\int_{[s_{i-1},s_i]} \delta(t)u(p(i)))dt = \int_0^T \delta(t)u(\mathbf{p}(t))dt$.

A.3 Proof: Sufficiency in Proposition 2

Lemma 7. *If $\succsim_0$ and $\{\succsim_t^{\Delta}\}_{t\in\mathscr{T}}$ satisfy conditions required by Proposition 1, moreover if $\succsim_t$ satisfies Time impatience, then $\succsim_0$ is represented by an exponential discount function:*

$$DU(\mathbf{p}) = \int_{\mathscr{T}} e^{-\beta t} U(\mathbf{p}(t))dt \tag{10}$$

Proof. From Proposition 1, We know $\exists \delta(t)$, s.t.

$$DU(\mathbf{p}) = \int_{\mathscr{T}} U(\mathbf{p}(t))\delta(t)dt = \int_{\mathscr{T}} U(\mathbf{p}(t))d\mu(t), \forall \mathbf{p} \in \mathscr{P} \tag{11}$$

Now we show $\delta(t)$ has the exponential expression.

Claim 1: If μ is monotone, then $\delta(s)$ is a strictly decreasing function.

Let's set $\delta(t)=\mu([t,\infty))$, which is indeed strictly decreasing. We can show: for any $t_1>t_2$, we construct two subset $E_1=[t_1,\infty),E_2=[t_2,\infty)$, then $E_1\subset E_2$, as a result of monotonicity of measure μ, we have $\delta(t_1)=\mu(E_1)<\mu(E_2)=\delta(t_2)$. So, $\delta(t)$ is strictly decreasing and positive, then for simplicity, we assume $\delta(0)=1$.

Claim 2: $\delta(t+s)=\delta(t)\delta(s)$ if $\succeq_0$ satisfies time stationary and time impatience.

1. Assume we have a function $\delta^{'}:\mathscr{T}\to R$, with the form $\delta^{'}(t+s)=\delta(t)\delta(s)$. From construction, we know it's decreasing with $\delta^{'}(0)=1$ and $\lim_{(t+s)\to\infty}\delta^{'}(t+s)=0$. Since $\delta(t)$ is decreasing, then $\delta(t)>\delta(t+\gamma),\forall t,\gamma\in\mathscr{T}$, then assume $\delta(s)=\frac{\delta(t+\gamma)}{\delta(t)}$, then $\delta(s)\delta(t)=\delta(t+\gamma)$. So this question is equivalent to show if $\forall\gamma,t\in\mathscr{T},\delta^{'}(t+s)=\delta(s)\delta(t)=\mu([s,\infty))\mu([t,\gamma))=\mu([t+\gamma,\infty))=\delta(t+\gamma)$, which means $s=\gamma$ under specific axiom.
2. If $\gamma=0$, then this is obvious. Now assume $\forall t,\gamma\in\mathscr{T},\gamma\neq 0$ then $\exists s$, s.t. $\delta(s)=\frac{\delta(t+\gamma)}{\delta(t)}$. Now assume $\gamma>s$ Because of decreasing function $\delta(t)$, then $\delta(s)>\delta(\gamma)$, means that $\mu([0,s))<\mu([0,\gamma))$. By time impatience, if $(\boldsymbol{l}_{[0,\gamma)},\boldsymbol{l}^{'})\sim_0(\boldsymbol{l}_{[t,t+\gamma)},\boldsymbol{l}^{'})$ with appropriate choice of $\boldsymbol{l}^{'}$ s.t. $U(\boldsymbol{l}^{'}(t))=0$ and $\boldsymbol{l}(t)\gtrsim\boldsymbol{l}^{'},\forall t\in[0,\gamma)$, then it's true for all $t>0$ as well as $(\boldsymbol{l}_{[0,s)},\boldsymbol{l}^{'})\sim_0(\boldsymbol{l}_{[t,t+s)},\boldsymbol{l}^{'})$, which means that $\mu([0,s))=\mu([t,t+s))$, and $\mu([0,\gamma))=\mu([t,t+\gamma))$. From construction of $\delta(s)$, we have
$$\delta(s)=1-\mu([0,s))=\frac{1-\mu([0,t))-\mu([0,\gamma))}{1-\mu([0,t))} \tag{12}$$
thus
$$\mu([0,\gamma))=\mu([0,s))(1-\mu([0,t))) \tag{13}$$
As a result $\mu([0,s))>\mu([0,\gamma))$ and $\delta(s)<\delta(\gamma)$, which is a contradiction to $s<\gamma$.
3. Now assume $s>\gamma$, then from time impatience, we know $(\boldsymbol{l}_{[0,\frac{s-\gamma}{2})},\boldsymbol{l}^{'})\sim_0(\boldsymbol{l}_{[t,t+\frac{s-\gamma}{2})},\boldsymbol{l}^{'})$ is true for all $t>0$, now assume $t=\frac{s-\gamma}{2}>0$, then $\mu([0,\frac{s-\gamma}{2}))=\mu([\frac{s-\gamma}{2},s-\gamma))$, thus
$$\begin{aligned}\mu([0,\gamma))&=\mu([0,s))-\mu([\gamma,s))\\&=\mu([0,s))-2\mu([0,t))\\&=\mu([0,s))(1-\mu([0,t)))\end{aligned} \tag{14}$$
As a result $\mu([0,s))=2$ and $\delta(s)=1-2=-1$ which is a contradiction of $\delta(t)>0$.

To sum up, $\delta(t+s)=\delta(t)\delta(s)$, and implied by Cauchy's multiplicative functional equation[1], this means that $\delta(t)=e^{-\beta t},\exists\beta\in[0,1]$. □

1. Cauthy Equation: Let $f:R\to(0,\infty)$ be a continuous function. if f(x + y) = f(x)f(y) for all x, y, there exists c∈R such that f(x) = e^{cx} for all x.

Then for each preference relation $\succsim_t$ on subset $\mathscr{P}_t$, it can be represented by the discounted utility form:

$$DU(\mathbf{p},t) = \int_t^T e^{-\beta_t(s-t)} \sum_i^n u_i(\mathbf{p}(s))ds \tag{15}$$

Now we finish the proof by showing at any time t, $\succsim_t$ shares the same discount factor β.

Lemma 8. *If $\{\succsim_t\}_{t\in\mathscr{T}}$ satisfies Dynamic Consistency, then $\beta_t = \beta_{t'}$ for all $t \neq t'$.*

Proof. We show it by contradiction. Assume there exists τ, t, where $t \neq \tau$, s.t. $\beta_t \neq \beta_\tau$, w.o.l.g, let $t > \tau$ and $\beta_t > \beta_\tau$ (similar for $\beta_\tau > \beta_t$). Let's construct $\mathbf{p}, \mathbf{q}$ are same up to time t, and utility function u, $\exists t_\tau > t$ such that:

$$u_i(p_i) \begin{cases} p_i, & i = 1 \\ 0, & i \neq 1 \end{cases} \tag{16}$$

Then utility function $U(\mathbf{p}(t)) = \mathbf{p}_1(t)$. Now we construct two trajectories:

$$\mathbf{p} = \begin{cases} p_1(s) = 1, & s \in [t, t_\tau] \\ p_1(s) = x, & s \in [t_\tau, T] \end{cases} \tag{17}$$

$$\mathbf{q} = \begin{cases} q_1(s) = 0, & s \in [t, t_\tau] \\ q_1(s) = y, & s \in [t_\tau, T] \end{cases} \tag{18}$$

Therefore the utility for $\mathbf{p}, \mathbf{q}$ is

$$u(\mathbf{p}) = \begin{cases} 1, & s \in [t, t_\tau] \\ x, & s \in [t_\tau, T] \end{cases} \tag{19}$$

$$u(\mathbf{q}) = \begin{cases} 0, & s \in [t, t_\tau] \\ y, & s \in [t_\tau, T] \end{cases} \tag{20}$$

And they are same up to time t s.t. $\mathbf{p} \gtrsim_t \mathbf{q}$. By Dynamic Consistency, we have $\mathbf{p} \gtrsim_\tau \mathbf{q}$, which means:

1. $\mathbf{p}(\gamma) = \mathbf{q}(\gamma), \forall \gamma \le t$
2. $\int_t^T e^{-\beta_t(s-t)} \sum_i^n u_i(\mathbf{p}(s))ds > \int_t^T e^{-\beta_t(s-t)} \sum_i^n u_i(\mathbf{q}(s))ds$

Condition 2 implies

$$(y-x) < \frac{\int_{t_\tau}^T e^{-\beta_t(s-t_\tau)}ds}{\int_t^{t_\tau} e^{-\beta_t(s-t)}ds} \tag{21}$$

Now we will show if we choose $t_\tau > t$, and $\beta_t - \beta_\tau = a > 0$, s.t.

$$\int_{t_\tau}^T e^{-a(s-t_\tau)}ds > \int_t^{t_\tau} e^{-a(s-t)}ds \tag{22}$$

It means that:

$$\frac{\int_{t_\tau}^T e^{-\beta_t(s-t_\tau)}ds}{\int_t^{t_\tau} e^{-\beta_t(s-t)}ds} > \frac{\int_{t_\tau}^T e^{-\beta_\tau(s-t_\tau)}ds}{\int_t^{t_\tau} e^{-\beta_\tau(s-t)}ds} \tag{23}$$

Therefore, we can choose (y-x) between those two value, such that

$$(y-x) > \frac{\int_{t_\tau}^T e^{-\beta_\tau(s-t_\tau)}ds}{\int_t^{t_\tau} e^{-\beta_\tau(s-t)}ds} \tag{24}$$

And it equals

$$\int_t^{t_\tau} e^{-\beta_\tau(s-t)}ds < \int_{t_\tau}^T e^{-\beta_\tau(s-t_\tau)}(y-x)ds \tag{25}$$

It means $\int_t^{t_\tau} e^{-\beta_\tau(s-t)}[\sum_i^n u_i(\mathbf{p}(s)) - \sum_i^n u_i(\mathbf{q}(s))]ds < \int_{t_\tau}^T e^{-\beta_\tau(s-t_\tau)}[\sum_i^n u_i(\mathbf{q}(s)) - \sum_i^n u_i(\mathbf{p}(s))]ds$. Therefore $\int_\tau^T e^{-\beta_\tau(s-\tau)}[\sum_i^n u_i(\mathbf{p}(s)) - \sum_i^n u_i(\mathbf{q}(s))]ds < 0 \Leftrightarrow q \gtrsim_\tau^P p$, a contradiction with DC.

□

A.4 Proof for Proposition 3

Proof. By Proposition 1 and 2, we know that $\gtrsim_0^*$ and $\gtrsim_0$ can be presented by the additive exponential discounted utility function DU^* and DU respectively while $DU(\mathbf{p}) = \int_0^T e^{-\beta s}U(\mathbf{p}(s))ds$ and $DU^*(\mathbf{p}) = \int_0^T e^{-\beta^* s}U^*(\mathbf{p}(s))ds$.

Now assume there are two streams $\mathbf{p},\mathbf{q} \in \mathscr{P}$ with $\mathbf{p}(s) \gtrsim_s^\Delta \mathbf{q}(s), \forall s \in \mathscr{T}$ and $\forall \gamma > 0$, for some $\tau \geq t$ we have $\mathbf{h}^* = (\mathbf{q}, \mathbf{p}_{[0,\gamma)}) \sim_0 (\mathbf{q}, \mathbf{p}_{[\tau,\tau+\gamma)}) = \mathbf{h}$, then those two streams will share the same parts of $[\tau,\gamma)$ and $[\tau+\gamma,T)$ if $\gamma > \tau$ or the same parts of $[\gamma,\tau)$ and $[\tau+\gamma,T)$ if $\tau > \gamma$. We take $\gamma > \tau$ as an example because it will be the same proof for the other case.

By time stationary, by moving the common part of $[\tau,\gamma)$ to $[2\tau,\gamma+\tau)$, then we have $\bar{\mathbf{h}}^* = (\mathbf{p}_{[0,\tau)}, {}_{(\tau-\gamma)}\mathbf{q}_{[\tau,2\tau)}) \sim_0 (\mathbf{q}_{[0,\tau)}, {}_{(\tau-\gamma)}\mathbf{p}_{[\tau,2\tau)}) = \bar{\mathbf{h}} \Leftrightarrow DU(\bar{\mathbf{h}}^*) = DU(\bar{\mathbf{h}})$ as illustrated in Figure 6. Then there is either a single-crossing at time point τ as illustrated in Figure 6 or no crossing at all if it is a jump point in τ.

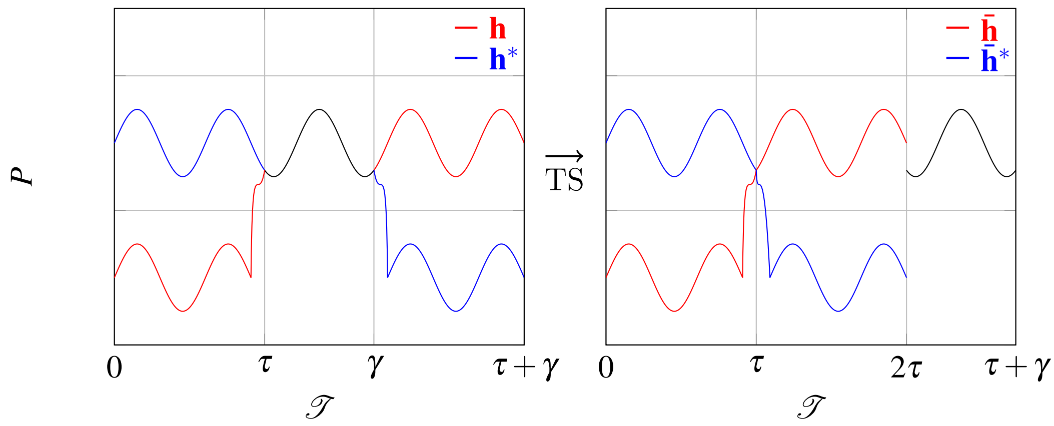

Figure 6: Time Stationary

For 'if' part. Because that $U^*(\mathbf{p}(t))$ and $U(\mathbf{p}(t))$ are identical up to an affine transformation and e^{-x} is decreasing, we assume $\beta^* = \beta + a$ and $a \geq 0$, we have

$$
\begin{aligned}
& DU(\bar{\mathbf{h}}^*) = DU(\bar{\mathbf{h}}) \\
\Leftrightarrow & \int_0^{\tau} e^{-\beta s}U(\mathbf{p}(s))ds + \int_{\tau}^{2\tau} e^{-\beta s}U(\mathbf{q}(s))ds = \int_0^{\tau} e^{-\beta s}U(\mathbf{q}(s))ds + \int_{\tau}^{2\tau} e^{-\beta s}U(\mathbf{p}(s))ds \\
\Leftrightarrow & \int_0^{\tau} e^{-\beta s}U^*(\mathbf{p}(s))ds + \int_{\tau}^{2\tau} e^{-\beta s}U^*(\mathbf{q}(s))ds = \int_0^{\tau} e^{-\beta s}U^*(\mathbf{q}(s))ds + \int_{\tau}^{2\tau} e^{-\beta s}U^*(\mathbf{p}(s))ds \\
\Leftrightarrow & \int_0^{\tau} [e^{-\beta s}(U^*(\mathbf{p}(s)) - U^*(\mathbf{q}(s)))]ds = \int_{\tau}^{2\tau} [e^{-\beta s}(U^*(\mathbf{p}(s)) - U^*(\mathbf{q}(s)))]ds \\
\Leftrightarrow & \int_0^{\tau} [e^{-\beta s}(U^*(\mathbf{p}(s)) - U^*(\mathbf{q}(s)))]ds \int_0^{\tau} e^{-ax}dx \geq \int_{\tau}^{2\tau} [e^{-\beta s}(U^*(\mathbf{p}(s)) - U^*(\mathbf{q}(s)))]ds \int_{\tau}^{2\tau} e^{-ax}dx \\
\Leftrightarrow & \int_0^{\tau} \int_0^{\tau} [e^{-\beta^* s}(U^*(\mathbf{p}(s)) - U^*(\mathbf{q}(s)))]dsds \geq \int_{\tau}^{2\tau} \int_{\tau}^{2\tau} [e^{-\beta^* s}(U^*(\mathbf{p}(s)) - U^*(\mathbf{q}(s)))]dsds \\
\Leftrightarrow & \int_0^{\tau} DU^*(p_{[0,\tau]})ds + \int_{\tau}^{2\tau} DU^*(q_{[\tau,2\tau]})ds \geq \int_0^{\tau} DU^*(q_{[0,\tau]})ds + \int_{\tau}^{2\tau} DU^*(p_{[\tau,2\tau]})ds \\
\Leftrightarrow & DU^*(\bar{\mathbf{h}}^*) \geq DU^*(\bar{\mathbf{h}})
\end{aligned}
\tag{26}
$$

The Therefore, we have $\mathbf{h}^* = (\mathbf{q}, \mathbf{p}_{[0,\gamma)}) \gtrsim_0^* (\mathbf{q}, \mathbf{p}_{[\tau,\tau+\gamma)}) = \mathbf{h}$

For 'only if' part.

$$
\begin{aligned}
& DU^*(\bar{\mathbf{h}}^*) \geq DU^*(\bar{\mathbf{h}}) \\
\Leftrightarrow & \int_0^{\tau} DU^*(p_{[0,\tau]})ds + \int_{\tau}^{2\tau} DU^*(q_{[\tau,2\tau]})ds \geq \int_0^{\tau} DU^*(q_{[0,\tau]})ds + \int_{\tau}^{2\tau} DU^*(p_{[\tau,2\tau]})ds \\
\Leftrightarrow & \int_0^{\tau} \int_0^{\tau} [e^{-\beta^* s}(U^*(\mathbf{p}(s)) - U^*(\mathbf{q}(s)))]dsds \geq \int_{\tau}^{2\tau} \int_{\tau}^{2\tau} [e^{-\beta^* s}(U^*(\mathbf{p}(s)) - U^*(\mathbf{q}(s)))]dsds \\
\Leftrightarrow & \int_0^{\tau} [e^{-\beta s}(U^*(\mathbf{p}(s)) - U^*(\mathbf{q}(s)))]ds \int_0^{\tau} e^{-ax}dx \geq \int_{\tau}^{2\tau} [e^{-\beta s}(U^*(\mathbf{p}(s)) - U^*(\mathbf{q}(s)))]ds \int_{\tau}^{2\tau} e^{-ax}dx \\
\Leftrightarrow & \int_0^{\tau} [e^{-\beta s}(U(\mathbf{p}(s)) - U(\mathbf{q}(s)))]ds \int_0^{\tau} e^{-ax}dx \geq \int_{\tau}^{2\tau} [e^{-\beta s}(U(\mathbf{p}(s)) - U(\mathbf{q}(s)))]ds \int_{\tau}^{2\tau} e^{-ax}dx
\end{aligned}
\tag{27}
$$

By assumption, we have $DU(\bar{\mathbf{h}}^*) = DU(\bar{\mathbf{h}})$, then from previous discussion, we have

$$
\int_0^{\tau} [e^{-\beta s}(U(\mathbf{p}(s)) - U(\mathbf{q}(s)))]ds = \int_{\tau}^{2\tau} [e^{-\beta s}(U(\mathbf{p}(s)) - U(\mathbf{q}(s)))]ds.
$$

Then we have $\int_0^{\tau} e^{-ax}dx \geq \int_{\tau}^{2\tau} e^{-ax}dx$ which means $a \geq 0$, otherwise it is a contradiction.

□

B Proof in Chapter 3

B.1 Proof for Proposition 4

Proof. By definition of strictly dominated strategy, we have:

$$\begin{aligned}
&\lim_{N,M\to\infty} P(\neg SD) \\
&= \lim_{N,M\to\infty} P(SD^1=0) \lim_{N,M\to\infty} P(SD^2=0) \\
&\geq \lim_{N,M\to\infty} \max\{1-\sum_{s_i\neq s_j} P(s_i\succ s_j),0\} \lim_{N,M\to\infty} \max\{1-\sum_{t_i\neq t_j} P(t_i\succ t_j),0\} \\
&= \lim_{M,N\to\infty} \max\{1-M(M-1)(\frac{1}{\rho})^N,0\} \lim_{M,N\to\infty} \max\{1-N(N-1)(\frac{1}{\rho})^M,0\}
\end{aligned} \tag{28}$$

1. If $M=N$,

$$\lim_{N,M\to\infty} P(\neg SD) \geq \lim_{N\to\infty} ((1-N^2(\frac{1}{\rho})^N)^2 = 1 \tag{29}$$

2. Simply, let start with $M=(2+\delta)log_\rho(N)$, for some $\delta>0$,

$$\lim_{N,M\to\infty} P(\neg SD) \geq \lim_{N\to\infty} (1-\frac{N(N-1)}{\rho^N})(1-\frac{N(N-1)}{N^{2+\delta}}) = 1 \tag{30}$$

Then, if $(2+\delta)log_\rho(N) \leq M \leq N$, for some $\delta>0$,

$$\lim_{N,M\to\infty} P(\neg SD) \geq \lim_{N\to\infty} ((1-N^2(\frac{1}{\rho})^N) \lim_{M\to\infty} (1-\rho^{\frac{2M}{2+\delta}-M}) = 1 \tag{31}$$

The case $M \leq N \leq \rho^{\frac{M}{2+\delta}}$ for some $\delta>0$ follows similarly because of symmetry.

□

B.2 Proof for Proposition 5

Lemma 9. *In a* $M\times N$ *random game,* $\limsup_{M,N\to\infty} P(\neg SD) \leq 1-(1-\alpha)2\alpha$ *if* $N=\log_\rho(M)-\log_\rho(2\alpha)$ *for some* $\alpha\in(0,1]$.

Proof. By definition of strictly dominated strategy, we have:

$$\begin{aligned}
&\lim_{M,N\to\infty} P(SD^1=0)) \\
&\le \lim_{M,N\to\infty}(1-P(\exists j, s_1 \succ s_j))) \\
&= \lim_{M,N\to\infty}(1-\sum_j P(s_1\succ s_j)+\sum_{1<j<k}P([s_1\succ s_j]\cap[s_1\succ s_k])-\cdots-(-1)^{M-2}P(\sum_{j\neq 1} s_1\succ s_j)) \\
&= \lim_{M,N\to\infty}(1-\frac{\binom{M-1}{1}}{\rho^N}+\frac{\binom{M-1}{2}}{(\rho^N)^2}-\cdots-(-1)^{M-2}\frac{\binom{M-1}{M-1}}{(\rho^N)^{M-1}})
\end{aligned} \tag{32}$$

Similarly.

$$\lim_{M,N\to\infty} P(SD^2=0)) \le \lim_{M,N\to\infty}(1-\frac{\binom{N-1}{1}}{\rho^M}+\frac{\binom{N-1}{2}}{(\rho^M)^2}-\cdots-(-1)^{N-2}\frac{\binom{N-1}{N-1}}{(\rho^M)^{N-1}}) \tag{33}$$

Now focus on the function $f(k)=\frac{\binom{M-1}{k}}{(\rho^N)^k}$, we will show at first $\frac{f(k+1)}{f(k)}\le\alpha<1, \forall k\in\mathbb{N}^+$ if $N=\log_\rho(M)-\log_\rho(2\alpha)$ for some $\alpha\in(0,1]$. We observe $\frac{f(k+1)}{f(k)}=\frac{M-k-1}{(k+1)\rho^N}, \forall k$ and $\frac{M-k-1}{k+1}$ is decreasing on k, so we have $\frac{f(2)}{f(1)}=\frac{M-2}{2\rho^N}<\frac{M}{2\rho^N}\le\alpha\Rightarrow\frac{f(k+1)}{f(k)}\le\frac{f(2)}{f(1)}=\alpha<1$.

Then if $M=2\alpha\rho^N\Leftrightarrow N=\log_\rho(M)-\log_\rho(2\alpha)$, we have

$$\begin{aligned}
&\limsup_{N\to\infty} P(SD^1=0)) \\
&\le \limsup_{N\to\infty}(1-f(1)+f(2)-\cdots-(-1)^{M-2}f(M-1)) \\
&\le \limsup_{N\to\infty}(1-f(1)+f(2)) \\
&= 1-\limsup_{N\to\infty}((1-\alpha)f(1)) \\
&= 1-\limsup_{N\to\infty}((1-\alpha)\frac{M-1}{\rho^N})) \\
&\le 1-(1-\alpha)2\alpha<1
\end{aligned} \tag{34}$$

To sum up, we have $\limsup_{M,N\to\infty} P(\neg SD)\le\limsup_{M,N\to\infty} P(\neg SD^1)<1$ when $N=\log_\rho(M)-\log_\rho(2\alpha)$.

□

Now we prove the Proposition 5.

Proof. 1. Fix some $0 < \alpha < 1$. Given M, N, with $N \le log_\rho(\frac{M}{2})$, then if we set $N' = log_\rho(\frac{M}{2\alpha})$, then $N' \ge N$. The probability of no strictly dominated strategy for Player 1 is greater in the $M \times N'$ game than in the $M \times N$ game, and in the game $M \times N'$, we have $\limsup_{M,N' \to \infty} P(\neg SD)$ as $N' = log_\rho(\frac{M}{2\alpha})$ is bounded from above by $1-(1-\alpha)2\alpha$; hence, $\lim_{M,N\to\infty} P(\neg SD)$ as $N \le log_\rho(\frac{M}{2})$ is bounded from above by $1-(1-\alpha)2\alpha$.

2. Fix Z arbitrarily large, we may assume $\log_\rho(M) \ge N + Z$. Then we divid the rows into $\frac{Z}{2}$ groups, each forming a matrix of $M' = \frac{2M}{Z}$ rows and N columns. In the below, we will denote G_z as a subgame for all positive integer $z \le Z$. If there is no ambiguity, we denote $G[s_i]$ as the subgame that strategy s_i belongs to.

$$G = \begin{bmatrix} G1 \\ \hline \vdots \\ \hline G_z \\ \hline \vdots \\ \hline G_Z \end{bmatrix}$$

Now, let's denote two events:

SD_z: The subgame G_z has a strategy that is strictly dominated by a strategy in G_z.

$SD_z^{'}$: The subgame G_z has a strategy that is strictly dominated by a strategy in G.

Therefore, $SD_z \subset SD_z^{'}$ for all z and $SD_1, SD_2, ..., SD_Z$ are independent.

Claim: $P(\neg SD) \le \prod_s (\neg SD_z)$**.**

$$P(\neg SD) = 1 - P(SD) = 1 - P(\cup_z SD_z^{'}) \le 1 - P(\cup_z SD_z) = P(\cap_z \neg SD_z) = \prod_z P(\neg SD_z)$$

From Proposition 5 (1), there exist $\eta \in (0,1)$ such that in each subgame G_z, when $N \le \log_\rho(M) - Z \le \log_\rho(\frac{M}{Z}) = \log_\rho(\frac{M'}{2})$, then $\lim_{N,M\to\infty} P(\neg SD_z) \le \eta$ for all z. Then we have

$$\lim_{N,M\to\infty} P(\neg SD) \le \lim_{N,M\to\infty} \prod_z P(\neg SD_z) = \lim_{N,M\to\infty} P(\neg SD_1)^Z \le \lim_{N,M\to\infty} \eta^Z = 0$$

□

B.3 Proof for Proposition 6

Proof. 1.

Lemma 10. *For any $S' \subset S$, if S' is strictly dominated, then for any $j \in N$, $\arg\max_{i \in M} a_{ij} \notin S'$.*

Proof. We show if there exist some j that $\max_{i \in M} a_{ij} \in S'$, then S' is not strictly dominated. Denote $\hat{s}' \in S'$ be the strategy contains at least one maximum, then it can not be strictly dominated by any strategy in S, thus S' is not strictly dominated.

□

Corollary 3. *$P^1_{q,M,N,\downarrow}$ is bounded by $(1-q)^N$ and it will go to 0 when $N.M$ go to infinity.*

Proof. Given M,N, the probability that the maximum in the random matrix R^1_{MxN} in column j is in S/S_{qM} is $1-q$. Then by i.i.d., we have $P^1_{q,M,N,\downarrow} \leq (1-q)^N$. Hence, $P^1_{q,M,N,\downarrow}$ will go to 0 when $N.M$ go to infinity. □

2.

Lemma 11. *For all $1 \leq k \leq qM$, $P\left[E^k(S_{qM})\right] \leq \sum_{\boldsymbol{a}} \frac{(M-qM)!(qM)!}{(M-qM-k)!k!} \frac{1}{\prod_{i=1} \boldsymbol{a}_i!(\boldsymbol{a}_i+1)^N}$, where $\boldsymbol{a} \in \mathbb{Z}^k_+$ s.t. $\sum_{i=1}^{k} \boldsymbol{a}_i = qM$.*

Proof. In the following, without misunderstanding, we refer 'doms' or 'dominates' as 'strictly dominates'. For each event $E^k(S_{qM})$, it equals to

$$\bigcup_{\mathbf{a}} \left\{ \begin{array}{l} \{\text{pick } T_1 \text{ of size } \mathbf{a}_1 \text{ from } S_{qM}, \text{ s.t. } 1_{st} \text{ strategy in } {}_kS \text{ doms } T_1\} \\ \bigcap \{\text{pick } T_2 \text{ of size } \mathbf{a}_2 \text{ from } S_{qM} \setminus T_1, \text{ s.t. } 2_{nd} \text{ strategy in } {}_kS \text{ doms } T_2\} \\ \vdots \\ \bigcap \{\text{k-th strategy in } {}_kS \text{ doms remaining } T_k\} \end{array} \right\}$$

We obtain

$$\begin{aligned} P\left[E^k(S_{qM})\right] &\leq \binom{M-qM}{k} \sum_{\mathbf{a}} \binom{qM}{\mathbf{a}_1} \frac{1}{(\mathbf{a}_1+1)^N} \cdot \binom{qM-\mathbf{a}_1}{\mathbf{a}_2} \frac{1}{(\mathbf{a}_2+1)^N} \cdots \frac{1}{(\mathbf{a}_k+1)^N} \\ &= \binom{M-qM}{k} \sum_{\mathbf{a}} \binom{qM}{\mathbf{a}_1, \ldots, \mathbf{a}_k} \cdot (\prod_i (\mathbf{a}_i+1))^{-N} \end{aligned}$$

After simplification, we have

$$P\left[E^k(S_{qM})\right] \leq \sum_{\mathbf{a}} \frac{(M-qM)!(qM)!}{(M-qM-k)!k!} \frac{1}{\prod_{i=1}^k \mathbf{a}_i!(\mathbf{a}_i+1)^N}.$$

□

For example, if $k=1$ then $P\left[E^1(S_{qM})\right] = \frac{M-qM}{(qM+1)^N}$ which is the probability that pick one strategy, $s_i^* \in_1 S$ from $S \setminus S_{qM}$ such that $s_i^* \succ s_i$ for all $s_i \in S_{qM}$. If $k = qM$, then $P\left[E^{qM}(S_{qM})\right] = \frac{(M-qM)!}{(M-qM-qM)!2^{qMN}} = \binom{M-qM}{qM}\frac{(qM)!}{2^{qMN}}$ which is the probability that pick qM strategies, S^*, from $S \setminus S_{qM}$ and pick any possible (one to one) matches with ${}_{qM}S$ and S_{qM}, we have $s^{'} \succ s$ where $s' \in {}_{qM}S$ and $s \in S_{qM}$.

Lemma 12. *For any* $q \leq \frac{1}{3}$, $\sum\limits_k \binom{M}{qM} P\left[E^k(S_{qM})\right] \to 0$ *for any when* N,M *goes to infinity with* $N \geq \frac{M}{(\ln(M))^\alpha}$ *for some* $0 < \alpha < 1$.

Proof.

$$\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right] \leq \sum_k \binom{M-qM}{k} \sum_{\mathbf{a}} \binom{qM}{\mathbf{a}_1,\ldots,\mathbf{a}_k} \cdot (\prod_i (\mathbf{a}_i+1))^{-N}$$

Also, given k and q, there are $\binom{qM-1}{k-1}$ distinct $\mathbf{a}$ which are positive integer solutions to $\sum_{i=1}^k \mathbf{a}_i = qM$. Now, we have

$$\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right] \leq qM \binom{M-qM}{k}\binom{qM-1}{k-1}\binom{qM}{\mathbf{a}_1,\ldots,\mathbf{a}_k}(\prod_i (\mathbf{a}_i+1))^{-N}$$

For binomial coefficients, after logarithmic transformations and approximations,

$$\begin{aligned}\ln(\binom{n}{m}) &= n\ln(n) - m\ln(m) - (n-m)ln(n-m) \approx n[-(\frac{m}{n})\ln(\frac{m}{n}) - (1-\frac{m}{n})\ln(1-\frac{m}{n})] \\ &= n\mathscr{H}(\frac{m}{n})\end{aligned}$$

where $\mathscr{H}(\frac{m}{n}) = -\frac{m}{n}\ln(\frac{m}{n}) - (1-\frac{m}{n})\ln(1-\frac{m}{n})$ is the Shannon entropy.

For multinomial coefficient, we make logarithm and according to Lemma 2.2 in Csiszár, Shields et al. 2004, we have $\ln(\binom{qM}{\mathbf{a}_1,\ldots,\mathbf{a}_k}) \approx qM\mathscr{H}(\frac{\mathbf{a}_1}{qM},\ldots,\frac{\mathbf{a}_k}{qM})$ where $\mathscr{H}$ is also the Shannon entropy $\mathscr{H}$ for a distribution of values $\left(\frac{\mathbf{a}_1}{qM},\ldots,\frac{\mathbf{a}_k}{qM}\right)$ as follows:

$$\mathscr{H}\left(\frac{\mathbf{a}_1}{qM},\ldots,\frac{\mathbf{a}_k}{qM}\right) = -\sum_{i=1}^k \frac{\mathbf{a}_i}{qM}\ln\left(\frac{\mathbf{a}_i}{qM}\right).$$

By Shannon 1948, $H(p)$ will be maximized when all observations have the same probability of occurrence $p_1 = p_2 = \cdots = p_n$. Hence, we have

$$\ln(\binom{qM}{\mathbf{a}_1,\ldots,\mathbf{a}_k}) \approx qM\mathscr{H}\left(\frac{\mathbf{a}_1}{qM},\ldots,\frac{\mathbf{a}_k}{qM}\right) \leq qM\ln(k)$$

and

$$\ln(\binom{n}{m}) \approx n\mathscr{H}(\frac{m}{n}) \leq n\ln(2)$$

Lastly, for term $\frac{1}{\prod_i (\mathbf{a}_i+1))^N}$, we have

$$\frac{1}{\prod_i(\mathbf{a}_i+1))^N} \leq \frac{1}{(qM+1)^N 2^{k(N-1)}}$$

Hence, we obtain

$$\begin{aligned}
&\ln(\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right]) \\
&\leq \ln(k) + \ln(\binom{M-qM}{k}) + \ln(\binom{qM-1}{k-1}) + \ln(\binom{qM}{\mathbf{a}_1,\ldots,\mathbf{a}_k}) + \ln((\prod_i(\mathbf{a}_i+1))^{-N}) \\
&\leq \ln(k) + (M-qM)\ln(2) + (qM-1)\ln(2) + qM\ln(k) - N\ln(qM+1) - k(N-1)\ln(2) \\
&\approx M\ln(2) - N\ln(qM) + \ln(\frac{k^{qM}}{2^{kN}})
\end{aligned}$$

For function $\ln(\frac{k^{qM}}{2^{kN}})$, by F.O.C, it will be maximized when $k = \frac{qM}{N\ln(2)}$. Hence

$$\begin{aligned}
\ln(\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right]) &\leq M\ln(2) - N\ln(qM) + qM\ln(\frac{qM}{N\ln(2)}) - qM \\
&= M(\ln(2)-q) + qM\ln(\frac{qM}{N\ln(2)}) - N\ln(qM)
\end{aligned}$$

If $N \geq \frac{M}{(\ln(M))^\alpha}$ for some $0 < \alpha < 1$, then
we have $N\ln((qM)) \geq M(\ln(M))^{1-\alpha}$ and $qM\ln(\frac{qM}{N\ln(2)}) \leq qM\alpha\ln(\ln(M)) - qM\ln(\frac{q}{\ln(2)})$.
Hence

$$\ln(\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right]) \leq M(\ln(2) - q - q\ln(\frac{q}{\ln(2)}) + q\alpha\ln(\ln(M))) - M(\ln(M))^{1-\alpha} \tag{35}$$

Since $\ln(M)$ goes faster than $\ln(\ln(M))$ and $\ln(2) - q - q\ln(\frac{q}{\ln(2)})$ is a constant number between 0 and 1 given $0 < q < \frac{1}{3}$. We conclude $\ln(\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right])$ goes to negative infinity when N,M goes to infinity with $N \geq \frac{M}{(\ln(M))^\alpha}$ for some $0 < \alpha < 1$. It implies $\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right] \to 0$.

□

From Lemma 12, we have $\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right] \to 0$ when N,M goes to infinity with $N \geq \frac{M}{(\ln(M))^\alpha}$ for some $\alpha > 0$ and for any $q \leq \frac{1}{3}$, for any $k = \{1,..,qM\}$. If the probability that a small ratio of strategies is strictly dominated for player 1 will go to 0 in a large game, then it is also less likely to have the larger ratio of strategies being strictly dominated, therefore this is also true for all $q \in (\frac{1}{3}, 1)$.

□

B.4 Proof for Proposition 7

Proof. 1. We apply the Chernoff bound to estimate the probability that at most $(q+\delta+\varepsilon)M$ rows satisfy the condition. The Chernoff bound gives:

$$P(A \leq qM) = P(A \leq (1-\frac{\delta}{q+\delta})(q+\delta)M) \leq e^{-\frac{(\frac{\delta}{q+\delta})^2 E[A]}{2}} = e^{-\frac{\delta^2 M}{2(q+\delta)}}$$

where $\frac{\delta}{q+\delta} > 0$ for some $\delta > 0$.
Hence $\lim_{M,N\to\infty} P(A \geq qM) \geq \lim_{M,N\to\infty} P(A > qM) = 1 - \lim_{M,N\to\infty} P(A \leq qM) = 1$.

2. By definition,

$$P(B_i) = \prod_{j=1}^{N} P(x_{ij} \geq (\delta+q)^{1/N}) = \left(1-(\delta+q)^{1/N}\right)^N.$$

Then $P(B=0) = (1-P(B_i))^M = (1-(1-(\delta+q)^{\frac{1}{N}})^N)^M$.
Since for $0<r<1$, $\frac{1-r}{N} < 1-r^{1/N}$, then $1-(\delta+q)^{\frac{1}{N}} > \frac{1-\delta-1}{N}$. Hence

$$P(B=0) \leq (1-(\frac{1-\delta-q}{N})^N)^M$$

Also $(1-r)^N \leq e^{-Nr}$ for $0<r<1$ and all integer N. we have

$$P(B=0) \leq e^{-M(\frac{1-\delta-q}{N})^N}$$

So, if M grows much faster than $(\frac{N}{1-\delta-q})^N$, then $-M(\frac{1-\delta-q}{N})^N$ goes to negative infinity as N goes to infinity. Hence $\lim_{M,N\to\infty} P(B=0) = 0$. Immediately, we have $\lim_{M,N\to\infty} P(B \geq 1) = 1$.

□

B.5 Proof for Proposition 8

Notice that Proposition 8 depends only on the ordinal comparison between elements and the minimum/maximum values of the vectors. We can arrange the elements of two vectors into a single sequence ordered from largest to smallest. The total number of distinct arrangements of two vectors of size N each is given by $\binom{2N}{N}$.

Proof. 1. Given two payoff vectors s_i and s_j, if $\min(s_i) > \max(s_j)$, then s_j must be strictly dominated by s_i.
Moreover, it requires that all elements of s_i appear to the left of all elements of s_j in this ordered sequence. If all elements are randomly selected, there is only one arrangement that satisfies this condition.

$$P(\min(s_i) > \max(s_j)) = \frac{1}{\binom{2N}{N}}$$

and it will approach 0 as N go to infinity. Same analysis for the event $\min(s_j) > \max(s_i)$.

2. Given two payoff vectors s_i and s_j, suppose $\max(s_i) > \max(s_j) > \min(s_j) > \min(s_i)$. We denote $t^* = \arg\max(s_i)$ as the index of the column of the maximum of vector s_i. Then we have $\max(s_i) > \max(s_j) > u(s_j, t^*)$. Moreover, we denote $t_* = \arg\min(s_i)$ as the index of the column of the minimum of vector s_i. Then we have $u(s_j, t_*) > \min(s_j) > \min(s_i)$. Hence there is no dominance between s_i and s_j.
Similar to statement a), there are totally $\binom{2N-2}{N}$ distinct arrangements out of $\binom{2N}{N}$, we have

$$P(\max(s_i) > \max(s_j) > \min(s_j) > \min(s_i)) = \frac{\binom{2N-2}{N}}{\binom{2N}{N}} = \frac{N-1}{4N-2}$$

and it will approach $\frac{1}{4}$ as N go to infinity. Same analysis for the event $\max(s_j) > \max(s_i) > \min(s_i) > \min(s_j)$.

3. Given two payoff vectors s_i and s_j, suppose $\max(s_i) > \max(s_j) > \min(s_i) > \min(s_j)$. There are totally $\binom{2N-2}{N-1} - 1$ distinct arrangements out of $\binom{2N}{N}$, we have

$$P(\max(s_i) > \max(s_j) > \min(s_i) > \min(s_j)) = \frac{\binom{2N-2}{N-1}}{\binom{2N}{N}} - \frac{1}{\binom{2N}{N}} = \frac{N}{4N-2} - \frac{1}{\binom{2N}{N}}$$

and it will approach $\frac{1}{4}$ as N go to infinity. Same analysis for the event $\max(s_j) > \max(s_i) > \min(s_j) > \min(s_i)$.

□

C Proof in Chapter 4

C.1 Proof for Proposition 9

Proof. We show if the allocation is not ordinally efficient, then it violates Property 1. Fix preference profile R_N and assume the allocation P is not ordinally efficient, then there exists a (probabilistic) improvement circle τ that $a_k \tau_{i_k} a_{k+1}$ if and only if $a_k R_{i_k} a_{k+1}$ and $P_{i_k a_{k+1}} > 0$.

$$a_1 \tau_{i_1} a_2 \dots a_n \tau_{i_n} a_{n+1}$$

We denote $a_{n+1} = a_1$. Now we define $Z^* = \max_{k=1}^{n}\{Z(a_{k+1}; R_{i_k}; P_{i_k})\}$ as the highest index among those agents of the object that they wish to exchange with others. W.l.o.g, assume it's agent i_1 with the object a_2. Then in this circle, we have $Z(i_1; a_2; P) > Z(i_2; a_3; P)$. Given the construction of $Z(I; a; P)$ and construction of circle, we have $Z(i_2; a_3; P) > Z(i_2; a_2; P)$.

However, $\sum\limits_{x \in \overline{U}(R_{i_2}, a_2)} P_{i_2 x} \leq 1 - P_{i_2 a_3} < 1$ and $P_{1 a_2} > 0$ which violate Property 1. □

C.2 Proof for Proposition 11

Proof. 1. Consider the preference profile:

$$1,2 : abcd; 3,4 : bacd$$

And the allocation:

$$1,2 : (\frac{1}{2}, 0, \frac{1}{3}, \frac{1}{6})$$

$$3,4 : (0, \frac{1}{2}, \frac{1}{6}, \frac{1}{3})$$

The allocation does not violate lexi-envy-free and Property 1, but violates Property 2 because $P_{1c} > P_{3c} > 0$ but agent 3 does not satisfy with the $\overline{U}(R_3; c)$.

2. Consider the preference profile:

$$1,2 : abcd; 3,4 : bacd$$

And the allocation:

$$1,2: (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$$

$$3,4: (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2})$$

The allocation does not violate ordinally efficient, Property 2, but violates Property 1 because $P_{1b} > 0$ but agent 3,4 do not satisfy with b.

□

C.3 Proof for Proposition 12

Proof. We show if the allocation is not ordinally efficient, then it violates Property. Fix preference profile R_N and order of importance π. W.o.l.g., assume $\pi = a_1, a_2, \ldots, a_m$.

Assume the allocation P is not ordinally efficient, then there exists a (probabilistic) improvement circle τ that $a^k \tau_{i_k} a^{k+1}$ if and only if $a^k R_{i_k} a^{k+1}$ and $P_{i_k a^{k+1}} > 0$.

$$a^1 \tau_{i_1} a^2 \ldots a^n \tau_{i_n} a^{n+1}$$

We denote $a^{n+1} = a^1$. Moreover, we assume $a^1, a^2, \ldots, a^n$ are ascending in π.

Now we start with agent i_n and show the contradiction in the following.

Look agent i_1, there are only three cases: $rank(\bar{*}(i_1; a^1); \pi) > rank(\bar{*}(i_n; a^1); \pi)$ or $rank(\bar{*}(i_1; a^1); \pi)$ $rank(\bar{*}(i_n; a^1); \pi)$ or $rank(\bar{*}(i_1; a^1); \pi) < rank(\bar{*}(i_n; a^1); \pi)$.

1. If it is the last case, we obtain a contradiction directly because agent i_1 has a higher claim on object a^1 compared to agent i_n while agent i_n receives a positive share of a^1 and agent i_1 receives a positive share of a^2.
2. We show it can not be the first case. If it happens, we can look at the next agent i_2 and there are also only three cases: $rank(\bar{*}(i_2; a^2); \pi) \geq rank(\bar{*}(i_1; a^2); \pi) \geq rank(\bar{*}(i_1; a^1); \pi)$ or $rank(\bar{*}(i_2; a^2); \pi) < rank(\bar{*}(i_1; a^2); \pi)$. Due to the same reason, the second case will result in a contradiction. And we will continue this logic until we end up to the comparison between agent i_{n-1} and i_n.

As the result, we have $rank(\overline{*}(i_n;a^n);\pi) \geq rank(\overline{*}(i_{n-1};a^n);\pi) \geq rank(\overline{*}(i_1;a^1);\pi) > rank(\overline{*}(i_n;a^1);\pi)$.
However, remember for agent i_n, we have $a^n R_{i_n} a^1$, hence we have $rank(\overline{*}(i_n;a^1);\pi) \geq rank(\overline{*}(i_n;a^n)$, which is a contradiction because the week upper counter set of a^1 contains the set of a^n.

3. If it's the second case, only one possible case is possible $rank(\overline{*}(i_1;a^1);\pi) = rank(\overline{*}(i_1;a^2);\pi) = \cdots = rank(\overline{*}(i_n;a^n);\pi) = rank(\overline{*}(i_n;a^1);\pi)$. In other words, there exist a^* which is 'the object' for all agent i and two objects a^i and a^{i+1}, where 'the objects' refers $\overline{*}(i_i;a^i)$.
If it's not this case, we will result in the same contradiction $rank(\overline{*}(i_n;a^n);\pi) > rank(\overline{*}(i_n;a^1);\pi)$.
Now, we conclude for all objects a^k, i_k and i_{k+1} equally conform the social planner' preference. Then we will use the similar logic in Proposition 9 to select the agent with the $\hat{Z}(i_k;a^j;P)$ where $a^j \in \{a^k, a^{k+1}\}$. Assume it's first agent with $\hat{Z}(i_1;a^2;P)$ and it can't be $\hat{Z}(i_1;a^1;P)$. Then we obtain the contradiction: $\hat{Z}(i_2;a^2;P) < \hat{Z}(i_1;a^2;P)$, $P_{i_1a^2} > 0$, and $P_{i_2a^3} > 0$. This means agent 2 is not satisfied with the goods that are at least as good as a^2.

□

D Characterization of fractional adaptive Boston rule: Proposition 10

Whenever R_N, $a \in A$ and $P \in \mathscr{P}$, we define $N_a = \{i \in N : P_{ia} > 0\}$ be the set of agents who receive a and $N_a^{wish} = \{i \in N : P_{ia} = 0 \text{ and } \exists x, aR_i x, \text{ s.t. } P_{ix} > 0\}$ be the set of agents who wish to receive a to by replacing with other objects if it is possible.

Proposition 16. *If an allocation satisfies Property 1, then for all R_N, all a,*

1. $\max_{i \in N_a} Z(a;R_i;P_i) \leq \min_{i \in N_a^{wish}} Z(a;R_i;P_i)$.
2. *For all i, $P_{ia} = 0$ and* $\sum_{x \in U(R_i;a)} P_{ix} < 1 \Rightarrow \sum_{j: Z(a;R_j;P_j) \leq Z(a;R_i;P_i)} P_{ja} = 1$.

Proof.

Lemma 13. *Condition (a) and Condition (b) are equivalent.*

$\Rightarrow$Fix R_N and a, and assume P satisfies condition (a). Pick $i \in N_a^{wish}$, then we have $\sum_{j:Z(a;R_j;P_j)\leq Z(a;R_i;P_i)} P_{ja} = \sum_{j\in N_a} P_{ja} + 0 = 1.$

$\Leftarrow$ Pick $i^* = \arg\min_{i\in N_a^{wish}} Z(a;R_i;P_i)$, then we have $\sum_{j\in N_a} P_{ja} = 1$. Condition (b) implies $N_a \subseteq \{j : Z(a;R_j;P_j) \leq Z(a;R_{i^*};P_{i^*})\}$ and $\max_{i\in N_a} Z(a;R_i;P_i) \leq \min_{i\in N_a^{wish}} Z(a;R_i;P_i)$.

Lemma 14. *For all R_N, all a, all i, if an allocation satisfies Property 1, then* $\max_{i\in N_a} Z(a;R_i;P_i) \leq \min_{i\in N_a^{wish}} Z(a;R_i;P_i)$.

Fix R_N and P, we have N_a^{wish} and N_a. Pick $i^* = \arg\max_{i\in N_a} Z(a;R_i;P_i)$. We show $Z(a;R_i;P_i) \geq Z(a;R_{i^*};P_{i^*})$ for all $i \in N_a^{wish}$. Suppose not, then exist $i \in N_a^{wish}$ that $Z(a;R_i;P_i) < Z(a;R_{i^*};P_{i^*})$, then by Property 1, we have $\sum_{x\in\overline{U}(R_i;a)} P_{ix} = 1$ which contradict to the fact that $i \in N_a^{wish}$. Then it implies $\max_{i\in N_a} Z(a;R_i;P_i) \leq \min_{i\in N_a^{wish}} Z(a;R_i;P_i)$.

□

Proof. Now, given R_N we denote P^* as the assignment of fractional adaptive Boston rule and denote P as the assignment satisfies Property 2 and Property 1. Now we show, for all i, all a we have $P_{ia} = P^*_{ia}$. At first, we show for all i and $\forall a$ that $Z(a;R_i;P_i) = 1$, then $P_{ia} = P^*_{ia}$.

We suppose there exists agent i that $P_{ia} < P^*_{ia}$. There are two cases: $P_{ia} > 0$ or $P_{ia} = 0$.

Claim 1.1: For all i and $\forall a$ that $Z(a;R_i;P_i) = 1$ and $P_{ia} > 0$, then $P_{ia} = P^*_{ia}$.

Property 2 implies $P_{ia} = P_{ja} > 0$ for all j that $Z(a;R_j;P_j) = 1$, then it also implies $r(j;a) = 1$. The Property 2 implies $P_{ia} = P_{ja} < P^*_{ia} = P^*_{ja}, \forall i, j$ that $r(i;a) = r(j;a) = 1$, which means $\sum_{i:(r(i;a)=1} P_{ia} < \sum_{i:(r(i;a)=1} P^*_{ia} \leq 1$. Then Property 1 implies $\forall i$ that $r(i;a) = 1$, $P_{ia} = 1$, which is a contradiction. Then we conclude $P_{ia} = P^*_{ia}$. The case $P_{ia} > P^*_{ia}$ is symmetry.

Claim 1.2: For all i and $\forall a$ that $Z(a;R_i;P_i) = 1$ and $P_{ia} = 0$, then $P_{ia} = P^*_{ia}$.

If $P_{ia}=0$, Property 1 implies either $\sum_{k:Z(k;a;P)\leq 1} P_{ka}=1$ or $\sum_{x\in U(R_i;a)} P_{ix}=1$. The latter and Claim 1.1 implies $P^*_{ia}=P_{ia}=0$. Now if $\sum_{x\in U(R_i;a)} P_{ix}<1$, then a is fully allocated to agents who rank a in the top, namely for all j who $r(j;a)=1$. By Claim 1.1, we know $P_{ja}=P^*_{ja}$ for all j, $r(j;a)=1$. Then it implies $P_{ia}=1-\sum_{k:Z(k;a;P)\leq 1} P_{ka}=0=P^*_{ia}$

We conclude for all i and $\forall a$ that $Z(a;R_i;P_i)=1$, then $P_{ia}=P^*_{ia}$.

Now assume for all i and all a that $Z(a;R_i;P_i)=z$, we have $P_{ia}=P^*_{ia}$ for $z=1,..,k-1$, we show for all i and all a that $Z(a;R_i;P_i)=k$, then $P_{ia}=P^*_{ia}$. By contradiction, suppose, there exists agent i such that $P_{ia}<P^*_{ia}$. Again, there are two cases $P_{ia}>0$ or $P_{ia}=0$.

Claim 2.1: For all i and $\forall a$ that $Z(a;R_i;P_i)=k$ and $P_{ia}>0$, then $P_{ia}=P^*_{ia}$.

There are two cases:

1. If $P_{ia}=P_{ja}$ for all j that $Z(a;R_j;P_j)=k$. Then we have $P_{ia}=P_{ja}<P^*_{ia}=P^*_{ja}$ for all j that $Z(a;R_j;P_j)=k$. It implies $\sum_{i:Z(a;R_i;P_i)\leq k} P_{ia}<\sum_{i:Z(a;R_i;P_i)\leq k} P^*_{ia}\leq 1$. Then Property 1 implies $\sum_{x\in \overline{U}(R_i;a)} P_{ix}=1$ for all i that $Z(a;R_j;P_j)\leq k$. However, we have
$$\sum_{x\in \overline{U}(R_i;a)} P_{ix}=\sum_{x\in U(R_i;a)} P_{ix}+P_{ia}=\sum_{x\in U(R_i;a)} P^*_{ix}+P_{ia}<\sum_{x\in \overline{U}(R_i;a)} P^*_{ix}\leq 1$$
Which is a contradiction. Then $P^*_{ia}=P^*_{ia}$.
2. If there exists j that $P_{ia}>P_{ja}$. Property 2 implies
$$\sum_{x\in U(R_j;a)} P_{jx}+P_{ja}=\sum_{x:Z(x;R_j;P_j)<k} P^*_{jx}+P_{ja}=1.$$
Now suppose $P^*_{ja}<P_{ja}$ (it can not be $P^*_{ja}>P_{ja}$), then we have $P^*_{ia}>P_{ia}>P_{ja}>P^*_{ja}$. By Property 2, we have $\sum_{x\in U(R_j;a)} P^*_{jx}+P^*_{ja}=1$, then $P^*_{ja}=P_{ja}$, a contradiction. Therefore, we have $P^*_{ja}=P_{ja}$. Consider agent i, we have
$$\sum_{i:Z(a;R_i;P_i)\leq k} P_{ia}<\sum_{i:Z(a;R_i;P_i)\leq k} P^*_{ia}\leq 1 \text{ and } \sum_{x\in \overline{U}(R_i;a)} P_{ix}<\sum_{x\in \overline{U}(R_i;a)} P^*_{ix}\leq 1$$
which is a contradiction. Then we have $P_{ia}=P^*_{ia}$.

Then we conclude $P_{ia}=P^*_{ia}$ for all i and $\forall a$ that $Z(a;R_i;P_i)=k$ and $P_{ia}>0$.

Claim 2.2: For all i and $\forall a$ that $Z(a;R_i;P_i) = k$ and $P_{ia} = 0$, then $P_{ia} = P^*_{ia}$.

By Property 1, we have either $\sum_{x\in U(R_i;a)} P_{ix} = 1$ or $\sum_{j:Z(a;R_j;P_j)\leq Z(a;R_i;P_i)} P_{ja} = 1$.

- $\sum_{x\in U(R_i;a)} P_{ix} = 1$ implies $\sum_{x\in U(R_i;a)} P^*_{ix} = \sum_{x\in U(R_i;a)} P_{ix} = 1$, then $P^*_{ia} = P_{ia} = 0$.
- $\sum_{j:Z(a;R_j;P_j)\leq Z(a;R_i;P_i)} P_{ja} = 1$ and Claim 2.1 states for all i that $Z(a;R_i;P_i) = k$ and $0 < P_{ia}$ we have $P_{ia} = P^*_{ia}$. Therefore

$$\sum_{j:Z(a;R_j;P_j)\leq Z(a;R_i;P_i)} P^*_{ja} = \sum_{j:Z(a;R_j;P_j)\leq Z(a;R_i;P_i)} P_{ja} = 1. \Rightarrow P^*_{ia} = 0$$

Then we conclude $P_{ia} = P^*_{ia}$ for all i and $\forall a$. We complete the proof. For necessity, it exactly follows the definition.

□

E Characterization of Flow Algorithm: Proposition 13

Proof. Whenever $\succ$, P, $a \in A$, $i, j \in N$, and $P_{ia} > 0$ we define $i \preceq^a j$ if either $i \sim^a j$ or $i \prec^a j$.

Whenever R_N, $a \in A$ and $P \in \mathscr{P}$, we define $N_a = \{i \in N : P_{ia} > 0\}$ be the set of agents who receive a and $N^{wish}_a = \{i \in N : P_{ia} = 0 \text{ and } \exists x, aR_ix, \text{ s.t. } P_{ix} > 0\}$ be the set of agents who wish to receive a to by replacing with other objects if it is possible.

Lemma 15. *If Property 1 holds, then* $[P_{ia} = 0$ *and* $\sum_{x\in U(R_i;a)} P_{ix} < 1] \Rightarrow \sum_{j:i\preceq^a j} P_{ja} = 1.$

Proof. We show $i \preceq^a j$ for all $i \in N^{wish}_a$ and for all $j \in N_a$, which implies a is full distributed among agents $i \preceq^a j$ whenever $i \in N^{wish}_a$, thus complete the proof. Pick $j^* \in N_a$ with the highest $Z(a;R_i;P_i)$ among agents who have he highest $\pi_{\mp(a)}$.

Suppose not, there exist $i \in N_a^{wish}$ that $j^* \prec^a i$. By Property 1, we have $\sum_{x \in \overline{U}(R_i;a)} P_{ix} = 1$ which violate the fact that $i \in N_a^{wish}$. □

To prove the theorem, we need a few notations.

Given $\overline{*}(i;A)$, $\overline{*}(i;a)$ is called the last peak for agent i when for all $b \in \overline{L}(i;\overline{*}(i;a))$ that $\overline{*}(i;b) = \overline{*}(i;a)$. $\overline{L}(R_i;a)$ is the weak lower counter set of a for agent i under R_i. If $\overline{*}(i;a)$ is not the last peak we denote $\underline{*}(i;a)$ as the peak immediately after a, namely there exists b with smallest $r(i;b)$ that $\overline{*}(i;b) \neq \overline{*}(i;a)$. Then we define $*(i;a)$ as the set containing a as following:

1. $*(i;a) = \{b : r(i;\overline{*}(i;a)) \leq r(i;b) < r(i;\underline{*}(i;a))\}$ if $\overline{*}(i;a)$ is not last peak.
2. $*(i;a) = \{b : r(i;\overline{*}(i;a)) \leq r(i;b) \leq |A|\}$ if $\overline{*}(i;a)$ is last peak.

Whenever R_N, we define $\pi^1 = \min\{\pi_x : x \in \bigcup_{i \in N} \overline{*}(i;A)\}$ as the rank of minimal peak, and $\pi^k = \min\{\pi_x : x \in \bigcup_{i \in N} \overline{*}(i;A), \pi_x > \pi^{k-1}\}$ as the rank of $k-th$ peak. Then, given R_N, there is the largest rank π^K. For each π^k, we define $I(\pi^k) = \{i : \exists b, \overline{*}(i;b) \in \pi(\pi^k)\}$ be the set of agents who have one peak $b \in \pi(\pi^k)$. Then for all $i \in I(\pi^k)$, we denote the peak as a_i^k.

Now, given R_N we denote $P^\pi = F(R_N)$ and denote P as the assignment satisfies Property* 2 and Property* 1 that $P \neq P^\pi$.

Claim 1: For all $i \in I(\pi^1)$, $\forall a \in *(i;a_i^1)$ we have $P_{ia} = P_{ia}^\pi$.

This is easily obtained from the Proposition 10. People who are familiar with Proof of Proposition 10 can jump to Claim 2.

Claim 1.1: For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$, then $P_{ia} = P_{ia}^\pi$.

We suppose there exists agent i that $P_{ia} < P_{ia}^\pi$. There are two cases: $P_{ia} > 0$ or $P_{ia} = 0$.

Claim 1.1.1: For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$ and $P_{ia} > 0$, then $P_{ia} = P_{ia}^\pi$.

Property* 2 implies if $P_{ia} = P_{ja} > 0$ for all j that $\hat{Z}(j;a;p) = 1$, then it also implies $r(j;a) = 1$. The Property* 2 implies $P_{ia} = P_{ja} < P^{\pi}_{ia} = P^{\pi}_{ja}, \forall i, j$ that $r(i;a) = r(j;a) = 1$, which means $\sum_{i:(r(i;a)=1} P_{ia} < \sum_{i:(r(i;a)=1} P^{\pi}_{ia} \leq 1$. Then Property* 1 implies $\forall i$ that $r(i;a) = 1$, $P_{ia} = 1$, which is a contradiction. Then we conclude $P_{ia} = P^{\pi}_{ia}$. The case $P_{ia} > P^{\pi}_{ia}$ is symmetry.

Claim 1.1.2: For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$ and $P_{ia} = 0$, then $P_{ia} = P^{\pi}_{ia}$.

By Property* 1, if $P_{ia} = 0$, then either $\sum_{x \in U(R_i;a)} P_{ix} = 1$ or $\sum_{j:i \preceq^a j} P_{ja} = 1$. The former one implies $P^*_{ia} = 0 = P_{ix}$ immediately. If $\sum_{x \in U(R_i;a)} P_{ix} < 1$, then a must be fully allocated to agents who rank a in the top, namely for all j, $r(j;a) = 1$. By Claim 1.1, we know $P_{ja} = P^{\pi}_{ja}$ for all j, $r(j;a) = 1$. Then it implies $P^{\pi}_{ia} = P_{ia} = 0$.

Now assume for all For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = z$, we have $P_{ia} = P^{\pi}_{ia}$ for $z = 1, .., k-1$, we show for all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = k$, then $P_{ia} = P^{\pi}_{ia}$.

Claim 1.2: For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = k$, then $P_{ia} = P^{\pi}_{ia}$.

Claim 1.2.1: For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = k$ and $P_{ia} > 0$, then $P_{ia} = P^{\pi}_{ia}$.

By contradiction, suppose, $P_{ia} < P^{\pi}_{ia}$. Again, $P_{ia} > P^{\pi}_{ia}$ is symmetric. There are two cases:

1. If $P_{ia} = P_{ja}$ for all j that $\hat{Z}(j;a;p) = k$. Then we have $P_{ia} = P_{ja} < P^{\pi}_{ia} = P^{\pi}_{ja}$ for all j that $\hat{Z}(j;a;p) = k$. It implies $\sum_{i:\hat{Z}(i;a;p) \leq k} P_{ia} < \sum_{i:\hat{Z}(i;a;p) \leq k} P^{\pi}_{ia}$ and $\sum_{x \in \overline{U}(R_i;a)} P_{ix} = 1$ for all i that $\hat{Z}(j;a;p) \leq k$. However, we have
$$\sum_{x \in \overline{U}(R_i;a)} P_{ix} = \sum_{x \in U(R_i;a)} P_{ix} + P_{ia} = \sum_{x \in U(R_i;a)} P^{\pi}_{ix} + P_{ia} < \sum_{x \in \overline{U}(R_i;a)} P^{\pi}_{ix} \leq 1$$
Which is a contradiction. Then $P^{\pi}_{ia} = P^{\pi}_{ia}$ for all all i that $\hat{Z}(j;a;p) = k$ if $P_{ia} = P_{ja}$ for all j that $\hat{Z}(j;a;p) = k$.
2. If there exists j that $P_{ia} > P_{ja}$. Property* 2 implies
$$\sum_{x \in U(R_j;a)} P_{jx} + P_{ja} = \sum_{x:Z(x;R_j;P_j) < k} P^{\pi}_{jx} + P_{ja} = 1.$$

Now suppose $P^{\pi}_{ja} < P_{ja}$ (it can not be $P^{\pi}_{ja} > P_{ja}$), then we have $P^{\pi}_{ia} > P_{ia} > P_{ja} > P^{\pi}_{ja}$. By Property* 2, we have $\sum_{x \in U(R_j;a)} P^{\pi}_{jx} + P^{\pi}_{ja} = 1$, then $P^{\pi}_{ja} = P_{ja}$, a contradiction. Therefore, we have $P^{\pi}_{ja} = P_{ja}$. Now for i, we have $\sum_{i:\hat{Z}(i;a;p) \leq k} P_{ia} < \sum_{i:\hat{Z}(i;a;p) \leq k} P^{\pi}_{ia} \leq 1$ and $\sum_{x \in \overline{U}(R_i;a)} P_{ix} < \sum_{x \in \overline{U}(R_i;a)} P^{\pi}_{ix} \leq 1$, which is a contradiction. Then we have $P_{ia} = P^{\pi}_{ia}$.

Then we conclude $P_{ia} = P^{\pi}_{ia}$ for all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = k$ and $P_{ia} > 0$.

Claim 1.2.2: For all $i \in I(\pi^1)$ and $\forall a$ that $\hat{Z}(i;a;p) = k$ and $P_{ia} = 0$, then $P_{ia} = P^{\pi}_{ia}$.

By Property* 1, if $\sum_{x \in U(R_i;a)} < 1$ (otherwise, $P^{*}_{ia} = P_{ia} = 0$), we have $\sum_{j:i \preceq^a j} P_{ja} = 1$ and Claim 1.2.1 states $0 < P_{ia} = P^{\pi}_{ia}$ for all i that $\hat{Z}(i;a;p) = k$, also $P_{ia} = P^{\pi}_{ia}$ for all i that $\hat{Z}(i;a;p) = j$ for all $j = 1,...,k-1$. Therefore $\sum_{j:i \preceq^a j} P^{\pi}_{ja} = \sum_{j:i \preceq^a j} P_{ja} = 1$. It implies $P^{\pi}_{ia} = 0$.

Then we conclude $P_{ia} = P^{\pi}_{ia}$ for all $i \in I(\pi^1)$ and $\forall a \in *(i;a^1_i)$.

Claim 2: Assume for all $i \in I(\pi^z)$, $\forall a \in *(i;a^z_i)$ we have $P_{ia} = P^{\pi}_{ia}$ for all $z = 1,...,k-1$, then we show $i \in I(\pi^k)$, $\forall a \in *(i;a^k_i)$ we have $P_{ia} = P^{\pi}_{ia}$.

Claim 2.1: For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$, then $P_{ia} = P^{\pi}_{ia}$.

We suppose there exists agent i that $P_{ia} < P^{\pi}_{ia}$. There are two cases: $P_{ia} > 0$ or $P_{ia} = 0$.

Claim 2.1.1: For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$ and $P_{ia} > 0$, then $P_{ia} = P^{\pi}_{ia}$.

There are two cases:

1. If $P_{ia} = P_{ja}$ for all j that $\hat{Z}(j;a;p) = 1$. Then we have $P_{ia} = P_{ja} < P^{\pi}_{ia} = P^{\pi}_{ja}$ for all j that $\hat{Z}(j;a;p) = 1$. It implies $\sum_i P_{ia} < \sum_i P^{\pi}_{ia} \leq 1$ and $\sum_{x \in \overline{U}(R_i;a)} P_{ix} = 1$ for all i that $\hat{Z}(i;a;p) = 1$. However, we have

$$\sum_{x \in \overline{U}(R_i;a)} P_{ix} = \sum_{x \in \underset{z<k}{\cup} *(i;a^z_i)} P_{ix} + P_{ia} = \sum_{x \in \underset{z<k}{\cup} *(i;a^z_i)} P^{\pi}_{ix} + P_{ia} < \sum_{x \in \overline{U}(R_i;a)} P^{\pi}_{ix} \leq 1$$

Which is a contradiction. Then $P^{\pi}_{ia} = P^{\pi}_{ia}$ for all all i that $\hat{Z}(j;a;p) = 1$ if $P_{ia} = P_{ja}$ for all j that $\hat{Z}(j;a;p) = 1$.

2. If there exists j that $P_{ia} > P_{ja}$. Property* 2 implies

$$\sum_{x \in U(R_j;a)} P_{jx} + P_{ja} = \sum_{x \in \underset{z<k}{\cup} *(j;a^z_j)} P_{jx} + P_{ja} = \sum_{x \in \underset{z<k}{\cup} *(j;a^z_j)} P^{\pi}_{jx} + P_{ja} = 1$$

Now suppose $P^{\pi}_{ja} < P_{ja}$ (it can not be $P^{\pi}_{ja} > P_{ja}$), then we have $P^{\pi}_{ia} > P_{ia} > P_{ja} > P^{\pi}_{ja}$. By Property* 2, we have $\sum\limits_{x \in U(R_j;a)} P^{\pi}_{jx} + P^{\pi}_{ja} = 1$, then $P^{\pi}_{ja} = P_{ja}$, a contradiction. Therefore, we have $P^{\pi}_{ja} = P_{ja}$.

Now for i, we have

$$\sum_{i \in \underset{z<k}{\cup} I(\pi^z)} P_{ia} + \sum_{i \in I(\pi^k), \hat{Z}(i;a;p)=1} P_{ia} = \sum_{i \in \underset{z<k}{\cup} I(\pi^z)} P^{\pi}_{ia} + \sum_{i \in I(\pi^k), \hat{Z}(i;a;p)=1} P_{ia} < 1$$

which is a contradiction. Then we have $P_{ia} = P^{\pi}_{ia}$.

Then we conclude $P_{ia} = P^{\pi}_{ia}$ for all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$ and $P_{ia} > 0$.

Claim 2.1.2: For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = 1$ and $P_{ia} = 0$, then $P_{ia} = P^{\pi}_{ia}$.

By Property* 1, if $\sum\limits_{x \in U(R_i;a)} < 1$ (otherwise, $P^{*}_{ia} = P_{ia} = 0$), we have $\sum\limits_{j:i \preceq^a j} P_{ja} = 1$ and Claim 2.1.1 states $0 < P_{ia} = P^{\pi}_{ia}$ for all i that $i \in I(\pi^k)$ and $\hat{Z}(i;a;p) = 1$, also $P_{ia} = P^{\pi}_{ia}$ for all i that $i \in I(\pi^z)$ for all $z = 1, ..., k-1$. Therefore $\sum\limits_{j:i \preceq^a j} P^{\pi}_{ja} = \sum\limits_{j:i \preceq^a j} P_{ja} = 1$. It implies $P^{\pi}_{ia} = 0$.

Now assume for all For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = z$, then $P_{ia} = P^{\pi}_{ia}$ for $z = 1, .., m-1$, we show for all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = m$, then $P_{ia} = P^{\pi}_{ia}$.

Claim 2.2: For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = m$, then $P_{ia} = P^{\pi}_{ia}$.

Claim 2.2.1: For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = m$ and $P_{ia} > 0$, then $P_{ia} = P^{\pi}_{ia}$.

By contradiction, suppose, $P_{ia} < P^{\pi}_{ia}$. Then there are two cases:

1. If $P_{ia} = P_{ja}$ for all j that $\hat{Z}(j;a;p) = m$. Then we have $P_{ia} = P_{ja} < P^{\pi}_{ia} = P^{\pi}_{ja}$ for all j that $\hat{Z}(j;a;p) = m$. It implies $\sum_i P_{ia} < \sum_i P^{\pi}_{ia} \leq 1$ and $\sum_{x \in \overline{U}(R_i;a)} P_{ix} = 1$ for all i that $\hat{Z}(j;a;p) = m$. However, we have

$$\begin{aligned}\sum_{x\in\overline{U}(R_i;a)} P_{ix} &= \sum_{x\in \cup_{z<k} *(i;a_i^z)} P_{ix} + \sum_{x\in *(i;a_i^k),Z(i;x;P)<m} P_{ix} + P_{ia} \\ &= \sum_{x\in \cup_{z<k} *(i;a_i^z)} P^{\pi}_{ix} + \sum_{x\in *(i;a_i^k),Z(i;x;P)<m} P^{\pi}_{ix} + P_{ia} \\ &< \sum_{x\in\overline{U}(R_i;a)} P^{\pi}_{ix} \leq 1\end{aligned}$$

Which is a contradiction. Then $P^{\pi}_{ia} = P^{\pi}_{ia}$ for all all i that $\hat{Z}(j;a;p) = m$ if $P_{ia} = P_{ja}$ for all j that $\hat{Z}(j;a;p) = m$.

2. If there exists j that $P_{ia} > P_{ja}$. Property* 2 implies

$$\begin{aligned}\sum_{x\in\overline{U}(R_j;a)} P_{jx} &= \sum_{x\in \cup_{z<k} *(j;a_j^z)} P_{jx} + \sum_{x\in *(j;a_i^k),Z(x;R_j;P_j)<m} P_{jx} + P_{ja} \\ &= \sum_{x\in \cup_{z<k} *(j;a_j^z)} P^{\pi}_{jx} + \sum_{x\in *(j;a_j^k),Z(x;R_j;P_j)<m} P^{\pi}_{jx} + P_{ja} = 1\end{aligned}$$

Now suppose $P^{\pi}_{ja} < P_{ja}$, then we have $P^{\pi}_{ia} > P_{ia} > P_{ja} > P^{\pi}_{ja}$. By Property* 2, we have $\sum_{x\in \cup_{z<k} *(j;a_j^z)} P^{\pi}_{jx} + \sum_{x\in *(j;a_j^k),Z(x;R_j;P_j)<m} P^{\pi}_{jx} + P^{\pi}_{ja} = 1$, then $P^{\pi}_{ja} = P_{ja}$, a contradiction. Therefore, we have $P^{\pi}_{ja} = P_{ja}$.

Now for i, we have

$$\begin{aligned}&\sum_{i\in \cup_{z<k} I(\pi^z)} P_{ia} + \sum_{i\in I(\pi^k),Z(a;R_i;P_i)<m} P_{ia} + \sum_{i\in I(\pi^k),\hat{Z}(i;a;p)=m} P_{ia} \\ &= \sum_{i\in \cup_{z<k} I(\pi^z)} P^{\pi}_{ia} + \sum_{i\in I(\pi^k),\hat{Z}(i;a;p)<m} P^{\pi}_{ia} + \sum_{i\in I(\pi^k),\hat{Z}(i;a;p)=m} P_{ia} < 1\end{aligned}$$

which is a contradiction. Then we have $P_{ia} = P^{\pi}_{ia}$.

Then we conclude $P_{ia} = P^{\pi}_{ia}$ for all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = m$ and $P_{ia} > 0$.

Claim 2.2.2: For all $i \in I(\pi^k)$ and $\forall a$ that $\hat{Z}(i;a;p) = m$ and $P_{ia} = 0$, then $P_{ia} = P^{\pi}_{ia}$.

By Property* 1, if $\sum_{x \in U(R_i;a)} < 1$ (otherwise, $P^*_{ia} = P_{ia} = 0$), we have $\sum_{j:i \preceq^a j} P_{ja} = 1$ and Claim 2.2.1 states $P_{ia} = P^\pi_{ia}$ for all $i \in I(\pi^k)$ that $\hat{Z}(i;a;p) < m$, and $0 < P_{ia} = P^\pi_{ia}$ for all $i \in I(\pi^k)$ that $\hat{Z}(i;a;p) = m$, and for all $i \in I(\pi^z)$ for all $z = 1, ..., k-1$. Therefore $\sum_{j:i \preceq^a j} P^\pi_{ja} = \sum_{j:i \preceq^a j} P_{ja} = 1$. It implies $P^\pi_{ia} = 0$.

Then we conclude $P_{ia} = P^\pi_{ia}$ for all $i \in I(\pi^k)$ and $\forall a \in *(i; a^k_i)$. We complete the proof.

For necessity, it exactly follows the definition of F^π.

□

F Proof for Proposition 14

Proof. 1. Consider Example 10:

'Interim Efficient' requires a random assignment that can be decomposed into a convex combination of ordinally efficient random assignments, then it is ex-post Pareto efficient. Now we show the converse is not true.

Example 10. *Assume there are 4 agents and 4 objects, and the preference profile is the following:*

$$\begin{array}{l} 1\text{: } acbd \\ 2\text{: } adbc \\ 3\text{: } bcad \\ 4\text{: } bdac \end{array} \tag{36}$$

Consider a random assignment

	a	b	c	d
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
2	$\frac{1}{2}$	0	0	$\frac{1}{2}$
3	0	$\frac{1}{2}$	0	$\frac{1}{2}$
4	0	$\frac{1}{2}$	$\frac{1}{2}$	0

There are only two deterministic assignments, and it is easy to check those two are efficient because they could be represented by a priority order (1 2 3 4) and (2 1 4 3):

	a	b	c	d
1	1	0	0	0
2	0	0	0	1
3	0	1	0	0
4	0	0	1	0

	a	b	c	d
1	0	0	1	0
2	1	0	0	0
3	0	0	0	1
4	0	1	0	0

Table 6: Two deterministic assignments

Also, this random assignment can't be decomposed into the convex combination of probabilistic ordinally efficient assignments because all '0' should be unchanged after convex combination, then agent 3 will trade object 'd' with agent 4 for object 'c', therefore this random assignment is ex-post Pareto efficient, not 'Interim Efficient'.

2. Consider Example 11:

Example 11. *Assume there are 4 agents and 4 objects, and the preference profile is the following:*

$$\begin{aligned} &1,2: badc \\ &3,4: abcd \end{aligned} \tag{37}$$

Consider a random assignment

	a	b	c	d
1	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
2	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
3	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
4	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

It is the average of ordinally efficient random assignments of:

	a	b	c	d
1	$\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{1}{4}$
2	$\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{1}{4}$
3	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{1}{4}$
4	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{1}{4}$

	a	b	c	d
1	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
2	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
3	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0
4	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0

Table 7: Two Random assignments

But this random assignment is not ordinally efficient.

3. Consider the preference profile:

In this preference profile, if $\boldsymbol{P}$ is ordinally efficient, then

(a) If $P_{1c} > 0$, then $P_{2d} = P_{4d} = 0$.

(b) If $P_{4d} > 0$, then $P_{1c} = P_{3c} = 0$.

Then we try to construct the support of random assignment in Table 10 such that Table 11 must be satisfied. In the end, we will have a contradiction.

1:	*adbc*
2:	*acbd*
3:	*abdc*
4:	*abcd*

Table 8: R_N

	a	b	c	d
1	$\frac{1}{4}$	0	$\frac{1}{24}$	$\frac{17}{24}$
2	$\frac{1}{4}$	0	$\frac{17}{24}$	$\frac{1}{24}$
3	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{1}{6}$
4	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$

Table 9: Allocation of RP

Table 10: Violation of 'Interim Efficient': RP

	a	b	c	d
1		0	+	
2		0		0
3				
4				0

	a	b	c	d
1		0	0	
2		0		
3			0	
4				+

Table 11: Two possibility

Claim 1: Now, suppose $P_{1c} > 0$, then it must be Table 12 with the weight $\frac{1}{24}$:

	a	b	c	d
1	0	0	1	0
2	1	0	0	0
3	0	0	0	1
4	0	1	0	0

Table 12: Possible deterministic assignments when $P_{1c} > 0$

Claim 2: Suppose $P_{4d} > 0$, then it must be Table 13 with weight $\frac{1}{12}$.

	a	b	c	d
1	1	0	0	0
2	0	0	1	0
3	0	1	0	0
4	0	0	0	1

Table 13: Possible deterministic assignments when $P_{4d} > 0$

Now, we have:

	a	b	c	d
1	$\frac{1}{12}$	0	$\frac{1}{24}$	0
2	$\frac{1}{24}$	0	$\frac{1}{12}$	0
3	0	$\frac{1}{12}$	0	$\frac{1}{24}$
4	0	$\frac{1}{24}$	0	$\frac{1}{12}$

Table 14: $\frac{1}{12}$ Table 13+$\frac{1}{24}$ Table 12

Given Table 10 and Table 14, we know the support of Table 10 must contains following deterministic assignments:

(a) When $P_{1a} = 1$:

	a	b	c	d
1	1	0	0	0
2	0	0	0	1
3	0	1	0	0
4	0	0	1	0

	a	b	c	d
1	1	0	0	0
2	0	0	0	1
3	0	0	1	0
4	0	1	0	0

Table 15: Total Weight $\frac{1}{24}$

and

	a	b	c	d
1	1	0	0	0
2	0	0	1	0
3	0	0	0	1
4	0	1	0	0

Table 16: With Wight $\frac{1}{8}$

(b) When $P_{1d} = 1$:

	a	b	c	d
1	0	0	0	1
2	1	0	0	0
3	0	0	1	0
4	0	1	0	0

	a	b	c	d
1	0	0	0	1
2	1	0	0	0
3	0	1	0	0
4	0	0	1	0

Table 17: Total Weight $\frac{5}{24}$

and

	a	b	c	d
1	0	0	0	1
2	0	0	1	0
3	0	1	0	0
4	1	0	0	0

	a	b	c	d
1	0	0	0	1
2	0	0	1	0
3	1	0	0	0
4	0	1	0	0

Table 18: Wight with $\frac{1}{4}$ each

Now because $P_{4a} = \frac{1}{4}$ in Table 10, then we know the third deterministic assignment of Table 18 must weight $\frac{1}{4}$. And $P_{3d} = \frac{1}{6}$ implies Table 16 must be weight $\frac{1}{8} = \frac{1}{6} - \frac{1}{24}$. Then we have

	a	b	c	d
1	$\frac{5}{24}$	0	$\frac{1}{24}$	$\frac{1}{2}$
2	$\frac{1}{24}$	0	$\frac{17}{24}$	0
3	$\frac{1}{4}$	$\frac{7}{24}$	0	$\frac{1}{6}$
4	$\frac{1}{4}$	$\frac{10}{24}$	0	$\frac{1}{12}$

Table 19: $\frac{1}{12}$ Table 13+$\frac{1}{24}$ Table 12+ $\frac{1}{4}$ Table 18 each+$\frac{1}{8}$ Table 16

Given the current assignment, we have the lottery left $\frac{1}{4}$ and four deterministic assignments:

	a	b	c	d
1	1	0	0	0
2	0	0	0	1
3	0	1	0	0
4	0	0	1	0

	a	b	c	d
1	1	0	0	0
2	0	0	0	1
3	0	0	1	0
4	0	1	0	0

	a	b	c	d
1	0	0	0	1
2	1	0	0	0
3	0	0	1	0
4	0	1	0	0

	a	b	c	d
1	0	0	0	1
2	1	0	0	0
3	0	1	0	0
4	0	0	1	0

Table 20: Total weight $\frac{1}{4}$

Now denote the weight for each deterministic assignments in Table 20 from left to right as x, y, z, w. Then we have the following equations:

$$\begin{cases} x + w = \frac{1}{6} = P_{4c} \\ x + w = \frac{5}{24} = \frac{1}{2} - \frac{7}{24} = P_{3b} - \frac{7}{24} \end{cases}$$

Which is a contradiction.

4. We know F^{π} is ordinally efficient for every $\pi \in \Pi$, then RF is interim efficiency,.

□

G Proof for Proposition 15

Proof. interim efficiency is proved, now we show weakly envy-freeness and weakly strategy-proofness.

Claim 1: RF is weakly Envy-free.

Lemma 16. *RF is weakly envy-free.*

Proof. Fix a preference profile R with $R_i = a_1 R_i a_2 R \dots R_i a_n$, we show for any agent, it won't be the case $RF_j(R) R_i^{sd} RF_i(R)$.

Let's start from the a_1. Assume $R_j(1) = a_k \neq a_1$, then there is $\frac{1}{2}$ probability that a_1 proceeds a_k and $\frac{1}{2}$ probability that a_k proceeds a_1. In formal case, agent i is on the higher priority, hence $P_{ia_1} \geq P_{ja_1}$. In the later case, agent i will not be the lower priority, hence $P_{ia_1} \geq P_{ja_1}$. Overall, $P_{ia_1} > P_{ja_1}$. Then it can't be $RF_j(R) R_i^{sd} RF_i(R)$.

Then let's look at a_2. If agent i and j have a different top preference, we have a previous conclusion. Now let's assume:

$$R_i = a_1 R_i a_2 R_i \dots R_i a_k R_i \dots \tag{38}$$

$$R_j = a_1 R_j a_k R_j \dots R_i a_2 R_i \dots \tag{39}$$

Let's start to look at a_2 and a_k, there are also two possibilities, either a_k proceeds a_2 or a_2 proceeds a_k.

If a_k proceeds a_2, then when a_2 appears, similarly, agent i won't be the lower priority, hence $P_{ia_2} \geq P_{ja_2}$. If a_2 proceeds a_k, then agent i will be the higher priority, hence $P_{ia_2} > P_{ja_2}$. To sum up (by average), we get $p_{ia_2} > p_{ja_2}$. Given $P_{ia_1} = P_{ja_1}$, then it can't be $RF_j(R) R_i^{sd} RF_i(R)$.

It's enough to make induction for any a_k given $p_{ia_{k-1}} = p_{ja_{k-1}}$, by assuming $rank(R_i;k) \neq rank(R_j;k)$, then we get $P_{ia_k} \geq P_{ja_k}$ if a_k proceeds $R_j(k)$ or $P_{ia_k} > P_{ja_k}$ if $a_k$follows $R_j(k)$. and by averaging, $P_{ia_k} > P_{ja_k}$, then it can't be $RF_j(R) R_i^{sd} RF_i(R)$. □

Claim 2: RF is weakly strategy-proof.

Now we prove weakly strategy-proofness. We will show RF is upper invariance and Swap monotonicity (Mennle and Seuken 2021).

Definition 13. *Adjacent preference*

Given R_i, we say R_j is an adjacent preference of R_i if there exist $K \in [n]$, such that

1. $o(i;K+1) = o(j;K)$ *and* $o(j;K+1) = o(i;K)$
2. $o(i;k) = o(j;k)$ *for all* $k \in [n] \setminus \{K, K+1\}$

Then given a preference R, we denote the set of adjacent preferences as $\delta(R)$.

Axiom 10. *(Swap monotonicity)*

A mechanism f is swap monotonic if, for all agents $i \in N$, all R and all $R_i^{'} \in \delta(R)$ with aR_ib but $bR_i^{'}a$, one of the following holds:

1. *either:* $f_i(R_i^{'}, R_{-i}) = f_i(R)$,
2. *or:* $f_{i,b}(R_i^{'}, R_{-i}) > f_{i,b}(R)$.

In other words, swap monotonicity requires that the mechanism reacts to the swap in a direct and monotonic way: If the swap that brings b forward affects the agent's assignment at all, then at least its assignment for b must be affected directly. Moreover, this change must be monotonic in the sense that the agent's assignment for b must increase when b is reportedly more preferred.

Axiom 11. *(Upper invariance)*

A mechanism g is upper invariant if, for all agents $i \in N$, all R and all $R_i^{'} \in \delta(R)$ with aR_ib but $bR_i^{'}a$, we have $f_{i,x}(R) = f_{i,x}(R_i^{'}, R_{-i})$ for all $x \in U(R_i, a)$

From Mennel and Seuken Mennle and Seuken 2021, if one mechanism satisfies the swap monotonic and upper invariant then it is weakly strategy-proof.

Claim 2.1: RF satisfies upper invariance.

To notice F^{π} will allocate agent object a to i in round k only when every object that is better than a is allocated (according to agent **i** preference) and $\pi(k) = a$ for some k, then $R_i^{'}(j) = R_i^{'}(j)$ for all $j < k$ implies $p_{ia_j} = p_{ia_j}^{'}$ for all $j < k$ which means that RF is upper invariance because it's an average over π.

Claim 2.2: RF satisfies Swap monotonicity. For all R_N, all $\pi \in \Pi$, all i that R_i with an adjacent preference $R_i^{'}$:

$$R_i = a_1 R_i \dots a_K R_i a_{K+1} R_i a_n \tag{40}$$
$$R_i^{'} = a_1 R_i^{'} \dots a_{K+1} R_i^{'} a_K R_i^{'} a_n \tag{41}$$

We show it either be $RF_i(R) = RF_i(R_i^{'};R_{-i})$ or $RF_{ia_{K+1}}(R) < RF_{ia_{K+1}}(R_i^{'};R_{-i})$. Since it is Upper invariance, then $RF_{ia_k}(R) = RF_{ia_k}(R_i^{'};R_{-i})$ for all $k < K$.

We denote the T_K^{π} as the period when $U(R_i;a_K)$ fully allocated given the π. Notice that, agent i will receive a_K and a_{K+1} under those two situations only after T_K^{π}. If a_K is fully allocated before T_K^{π} for some $\pi \in \Pi$, then $P_{ia_K}^{\pi}(R) = P_{ia_K}^{\pi}(R_i^{'};R_{-i}) = 0$. We focus on set $\Pi_K = \{\pi_K\}$ that a_K still have capacity after period T_K^{π}. Similarly, we consider $\Pi_{K+1} = \{\pi_{K+1}\}$ that a_{K+1} still have capacity after period T_{K+1}^{π}.

At first, consider π in Π_{K+1} and not in Π_K. Then $P_{ia_K}^{\pi}(R) = P_{ia_K}^{\pi}(R_i^{'};R_{-i}) = 0$. If a_K proceeds a_{K+1} under π, then agent i will be allocated with a_{K+1} when it appears under both two preferences. Also because agent i receive the same allocation in $U(R_i;a_K)$, then due to the Flow algorithm, agent i will receive the same share of a_{K+1} up to its capacity under both two preference. If a_{K+1} proceeds a_K under π and still have capacity after $U(R_i;a_K)$ and a_K are fully allocated, then following Flow algorithm, agent i will receive the same share of a_{K+1} due to same reason. For the objects in $L(R_i;a_{K+1})$, agent i will receive the same share under those two preference because agent i will have the same capacity and have the same priority on those objects. Hence $RF_i(R) = RF_i(R')$. For π in Π_K and not in Π_{K+1}, we have similar analysis that $RF_i(R) = RF_i(R')$.

Now we consider $\pi \in \Pi_{K+1} \cap \Pi_K$. Because both a_K and a_{K+1} still have capacity after period T^* when $U(R_i;a_K)$ fully allocated. First of all, if a_K appears before time T^*, then Then $P_{ia_{K+1}}^{\pi}(R) \leq P_{ia_{K+1}}^{\pi}(R_i^{'};R_{-i})$ because agent i will receive a_K with positive probability under R_i, then his capacity when he's called to receive a_{K+1} will be smaller.

If a_{K+1} appears before time T^* but a_K doesn't, then agent i has to wait until a_K appears. If a_{K+1} is fully allocated before that period, then $0 = P_{ia_{K+1}}^{\pi}(R) < P_{ia_{K+1}}^{\pi}(R_i^{'};R_{-i})$, otherwise $P_{ia_{K+1}}^{\pi}(R) \leq P_{ia_{K+1}}^{\pi}(R_i^{'};R_{-i})$.

If both two objects appears after T^*, then there are two cases:a_{K+1} proceeds a_K or a_K proceeds a_{K+1}. If a_{K+1} appears first, then similarly there are two cases: a_{K+1} is fully allocated before period T_{a_K} or not under R. In the former case, we have $0 = P^{\pi}_{ia_{K+1}}(R) < P^{\pi}_{ia_{K+1}}(R_i^{'};R_{-i})$ and in the later case, we have $0 \leq P^{\pi}_{ia_{K+1}}(R) \leq P^{\pi}_{ia_{K+1}}(R_i^{'};R_{-i})$.

If a_K proceeds a_{K+1}, then agent i will have smaller capacity under R_i when a_{K+1} appears, hence $0 \leq P^{\pi}_{ia_{K+1}}(R) \leq P^{\pi}_{ia_{K+1}}(R_i^{'};R_{-i})$.

Overall, if $\pi \in \Pi_{K+1} \cap \Pi_K$ and it exists, then $P^{\pi}_{ia_{K+1}}(R) < P^{\pi}_{ia_{K+1}}(R_i^{'};R_{-i})$, otherwise, $RF_i(R) = RF_i(R')$. Immediate result from Mennle and Seuken 2021. RF is weakly strategy-proof.

$\square$

Bibliography

Abdulkadiroglu, Atila, Umut M Dur and Aram Grigoryan (2021). *School assignment by match quality*. Tech. rep. National Bureau of Economic Research.

Abdulkadiroğlu, Atila and Tayfun Sönmez (1998). 'Random serial dictatorship and the core from random endowments in house allocation problems'. In: *Econometrica* 66.3, pp. 689–701.

– (2003). 'Ordinal efficiency and dominated sets of assignments'. In: *Journal of Economic Theory* 112.1, pp. 157–172.

Abdulkadiroğlu, Atila et al. (2020). 'Do parents value school effectiveness?' In: *American Economic Review* 110.5, pp. 1502–1539.

Akbarpour, Mohammad and Shengwu Li (2020). 'Credible auctions: A trilemma'. In: *Econometrica* 88.2, pp. 425–467.

Alon, Noga, Kirill Rudov and Leeat Yariv (2021). 'Dominance solvability in random games'. In: *arXiv preprint arXiv:2105.10743*.

Amiet, Ben et al. (2021). 'Pure nash equilibria and best-response dynamics in random games'. In: *Mathematics of Operations Research* 46.4, pp. 1552–1572.

Aristotle (1980). *Nicomachean Ethics*. Trans. by W. D. Ross. Oxford University Press.

Aydin Alptekinoglu, Gerdus Benadè. (2023). 'Achieving Rawlsian Justice in Food Rescue'. In.

Aziz, Haris et al. (2015). 'Ex post Efficiency of Random Assignments.' In: *AAMAS*, pp. 1639–1640.

Bárány, Imre, Santosh Vempala and Adrian Vetta (2007). 'Nash equilibria in random games'. In: *Random Structures & Algorithms* 31.4, pp. 391–405.

Bell, David E (1974). 'Evaluating time streams of income'. In: *Omega* 2.5, pp. 691–699.

Bleichrodt, Han, Kirsten IM Rohde and Peter P Wakker (2008). 'Koopmans' constant discounting for intertemporal choice: A simplification and a generalization'. In: *Journal of Mathematical Psychology* 52.6, pp. 341–347.

Bogomolnaia, Anna (2015). 'Random assignment: Redefining the serial rule'. In: *Journal of Economic Theory* 158, pp. 308–318. ISSN: 0022-0531. DOI: `https://doi.org/10.1016/j.jet.2015.04.008`. URL: `https://www.sciencedirect.com/science/article/pii/S0022053115000770`.

Bogomolnaia, Anna and Hervé Moulin (2001). 'A new solution to the random assignment problem'. In: *Journal of Economic theory* 100.2, pp. 295–328.

Chen, Yajing, Patrick Harless and Zhenhua Jiao (2023). 'The fractional Boston random assignment rule and its axiomatic characterization'. In: *Review of Economic Design*, pp. 1–23.

Chichilnisky, Graciela (1981). 'Existence and characterization of optimal growth paths including models with non-convexities in utilities and technologies'. In: *The Review of Economic Studies* 48.1, pp. 51–61.

Cover, Thomas M (1966). 'The probability that a random game is unfair'. In: *The Annals of Mathematical Statistics* 37.6, pp. 1796–1799.

Cowgill, Bo et al. (2024). 'Stable Matching on the Job? Theory and Evidence on Internal Talent Markets'. In: *Management Science*.

Csiszár, Imre, Paul C Shields et al. (2004). 'Information theory and statistics: A tutorial'. In: *Foundations and Trends® in Communications and Information Theory* 1.4, pp. 417–528.

Debreu, Gerard (1959). 'Topological methods in cardinal utility theory'. In.

Debreu, Gerard et al. (1954). 'Representation of a preference ordering by a numerical function'. In: *Decision processes* 3, pp. 159–165.

Deci, Edward L and Richard M Ryan (2013). *Intrinsic motivation and self-determination in human behavior*. Springer Science & Business Media.

Dewey, John (1916). *Democracy and Education*. Macmillan.

Dresher, Melvin (1970). 'Probability of a pure equilibrium point in n-person games'. In: *Journal of Combinatorial Theory* 8.1, pp. 134–145.

Dufwenberg, Martin and Mark Stegeman (2002). 'Existence and uniqueness of maximal reductions under iterated strict dominance'. In: *Econometrica* 70.5, pp. 2007–2023.

Duong, Manh Hong and The Anh Han (2016a). 'Analysis of the expected density of internal equilibria in random evolutionary multi-player multi-strategy games'. In: *Journal of Mathematical Biology* 73, pp. 1727–1760.

– (2016b). 'On the expected number of equilibria in a multi-player multi-strategy evolutionary game'. In: *Dynamic Games and Applications* 6.3, pp. 324–346.

Dworczak, Piotr (2020). 'Mechanism Design With Aftermarkets: Cutoff Mechanisms'. In: *Econometrica* 88.6, pp. 2629–2661. DOI: `https://doi.org/10.3982/ECTA15768`. eprint: `https://onlinelibrary.wiley.com/doi/pdf/10.3982/ECTA15768`. URL: `https://onlinelibrary.wiley.com/doi/abs/10.3982/ECTA15768`.

Epstein, Larry G (1983). 'Stationary cardinal utility and optimal growth under uncertainty'. In: *Journal of Economic Theory* 31.1, pp. 133–152.

Epstein, Larry G and Martin Schneider (2003). 'Recursive multiple-priors'. In: *Journal of Economic Theory* 113.1, pp. 1–31.

Erev, Ido et al. (2007). 'Learning and equilibrium as useful approximations: Accuracy of prediction on randomly selected constant sum games'. In: *Economic Theory* 33.1, pp. 29–51.

Fishburn, Peter C (1970). *Utility theory for decision making.* Tech. rep. Research analysis corp McLean VA.

Fishburn, Peter C and Ariel Rubinstein (1982). 'Time preference'. In: *International economic review*, pp. 677–694.

Fudenberg, Drew and Christopher Harris (1992). 'Evolutionary dynamics with aggregate shocks'. In: *Journal of Economic Theory* 57.2, pp. 420–441.

Gale, David (1953). 'A theory of n-person games with perfect information'. In: *Proceedings of the National Academy of Sciences* 39.6, pp. 496–501.

Galla, Tobias and J Doyne Farmer (2013). 'Complex dynamics in learning complicated games'. In: *Proceedings of the National Academy of Sciences* 110.4, pp. 1232–1236.

Gokhale, Chaitanya S and Arne Traulsen (2010). 'Evolutionary games in the multiverse'. In: *Proceedings of the National Academy of Sciences* 107.12, pp. 5500–5504.

– (2014). 'Evolutionary multiplayer games'. In: *Dynamic Games and Applications* 4, pp. 468–488.

Goldberg, K, AJ Goldman and M Newman (1968). 'The probability of an equilibrium point'. In: *Journal of Research of the National Bureau of Standards* 72.2, pp. 93–101.

Goldman, A. J. (1957). 'The Probability of a Saddlepoint'. In: *The American Mathematical Monthly* 64.10, pp. 729–730. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2309755 (visited on 24/04/2025).

Gorman, William M (1968). 'The structure of utility functions'. In: *The Review of Economic Studies* 35.4, pp. 367–390.

Gross, Thilo et al. (2009). 'Generalized models reveal stabilizing factors in food webs'. In: *Science* 325.5941, pp. 747–750.

Gunby, Benjamin and Dömötör Pálvölgyi (2019). 'Asymptotics of pattern avoidance in the Klazar set partition and permutation-tuple settings'. In: *European Journal of Combinatorics* 82, p. 102992.

Hammett, Adam and Boris Pittel (2008). 'How often are two permutations comparable?' In: *Transactions of the American Mathematical Society* 360.9, pp. 4541–4568.

Han, The Anh, Arne Traulsen and Chaitanya S Gokhale (2012). 'On equilibrium properties of evolutionary multi-player games with random payoff matrices'. In: *Theoretical Population Biology* 81.4, pp. 264–272.

Hara, Kazuhiro (2016). 'Characterization of stationary preferences in a continuous time framework'. In: *Journal of Mathematical Economics* 63, pp. 34–43.

Harless, Patrick (2018). 'Immediate Acceptance and Weak Priorities: An Adaptation that Preserves Efficiency and Respects Rank'. In: *Working Paper*.

Harless, Patrick (2019). 'Efficient rules for probabilistic assignment'. In: *Journal of Mathematical Economics* 84, pp. 107–116. ISSN: 0304-4068. DOI: `https://doi.org/10.1016/j.jmateco.2019.07.006`. URL: `https://www.sciencedirect.com/science/article/pii/S0304406819300795`.

Harvey, Charles M and Lars Peter Østerdal (2012). 'Discounting models for outcomes over continuous time'. In: *Journal of Mathematical Economics* 48.5, pp. 284–294.

Harvey, Charles M, Lars Peter Østerdal et al. (2007). *Integral-value models for outcomes over continuous time*. Tech. rep.

Hayashi, Takashi (2003). 'Quasi-stationary cardinal utility and present bias'. In: *Journal of Economic Theory* 112.2, pp. 343–352.

Heinrich, Torsten et al. (2021). 'Best-response dynamics, playing sequences, and convergence to equilibrium in random games'. In: *arXiv preprint arXiv:2101.04222*.

James, William (1907). *Pragmatism: A New Name for Some Old Ways of Thinking*. Longmans, Green & Co.

Kang, Zi Yang (2023). *The public option and optimal redistribution*. Tech. rep. Working paper.

Knuth, Donald Ervin (1997). *The art of computer programming*. Vol. 3. Pearson Education.

Kojima, Fuhito and M Utku Ünver (2014). 'The "Boston" school-choice mechanism: an axiomatic approach'. In: *Economic Theory* 55, pp. 515–544.

Koopmans, Tjalling C (1960). 'Stationary ordinal utility and impatience'. In: *Econometrica: Journal of the Econometric Society*, pp. 287–309.

– (1966). 'Structure of preference over time'. In.

– (1972). 'Representation of preference orderings over time'. In: *Decision and organization* 57, p. 100.

Kopylov, Igor (2010). 'Simple axioms for countably additive subjective probability'. In: *Journal of Mathematical Economics* 46.5, pp. 867–876.

Lancaster, Kelvin (1963). 'An axiomatic theory of consumer time preference'. In: *International Economic Review* 4.2, pp. 221–231.

Liu, Peng and Huaxia Zeng (2019). 'Random assignments on preference domains with a tier structure'. In: *Journal of Mathematical Economics* 84, pp. 176–194.

Lu, Jay and Kota Saito (2018). 'Random intertemporal choice'. In: *Journal of Economic Theory* 177, pp. 780–815.

Luce, R Duncan and Howard Raiffa (1957). *Games and decisions: Introduction and critical survey*. Courier Corporation.

May, Robert M (2001). *Stability and complexity in model ecosystems*. Vol. 6. Princeton university press.

McLennan, Andrew (2005). 'The expected number of Nash equilibria of a normal form game'. In: *Econometrica* 73.1, pp. 141–174.

Mennle, Timo and Sven Seuken (2014). 'The Naıve versus the Adaptive Boston Mechanism'. In: *arXiv preprint arXiv:1406.3327*.

– (2021). 'Partial strategyproofness: Relaxing strategyproofness for the random assignment problem'. In: *Journal of Economic Theory* 191, p. 105144.

Moulin, Hervé (1979). 'Dominance solvable voting schemes'. In: *Econometrica: Journal of the Econometric Society*, pp. 1337–1351.

Neumann, John von and Oskar Morgenstern (1944). *Theory of Games and Economic Behavior*. 1st. Princeton, NJ: Princeton University Press.

Noda, Shunya (2023). 'A planner-optimal matching mechanism and its incentive compatibility in a restricted domain'. In: *Games and Economic Behavior* 141, pp. 364–379.

Pangallo, Marco, Torsten Heinrich and J Doyne Farmer (2019). 'Best reply structure and equilibrium convergence in generic games'. In: *Science advances* 5.2, eaat1328.

Pei, Ting and Satoru Takahashi (2019). 'Rationalizable strategies in random games'. In: *Games and Economic Behavior* 118, pp. 110–125. ISSN: 0899-8256. DOI: `https://doi.org/10.1016/j.geb.2019.08.011`. URL: `https://www.sciencedirect.com/science/article/pii/S0899825619301289`.

Pivato, Marcus (2021). 'Intertemporal choice with continuity constraints'. In: *Mathematics of Operations Research* 46.3, pp. 1203–1229.

Powers, Imelda Young (1990). 'Limiting distributions of the number of pure strategy Nash equilibria in N-person games'. In: *International Journal of Game Theory* 19, pp. 277–286.

Prendergast, Canice (2022). 'The allocation of food to food banks'. In: *Journal of Political Economy* 130.8, pp. 1993–2017.

Qin, Wei-Zhi and Hendrik Rommeswinkel (2022). 'Additive representations on a simplex'. In: *Journal of Mathematical Economics* 103, p. 102769.

Ramezanian, Rasoul and Mehdi Feizi (2021). 'Stepwise ordinal efficiency for the random assignment problem'. In: *Journal of Mathematical Economics* 92, pp. 60–65.

– (2022). 'Robust ex-post Pareto efficiency and fairness in random assignments: Two impossibility results'. In: *Games and Economic Behavior* 135, pp. 356–367.

Richter, Michael and Ariel Rubinstein (2024). 'Unilateral stability in matching problems'. In: *Journal of Economic Theory* 216, p. 105780. ISSN: 0022-0531. DOI: `https://doi.org/10.1016/j.jet.2023.105780`. URL: `https://www.sciencedirect.com/science/article/pii/S002205312300176X`.

Rinott, Yosef and Marco Scarsini (2000). 'On the number of pure strategy Nash equilibria in random games'. In: *Games and Economic Behavior* 33.2, pp. 274–293.

Ryan, Richard M. and Edward L. Deci (2000). 'Self-Determination Theory and the Facilitation of Intrinsic Motivation, Social Development, and Well-Being'. In: *American Psychologist* 55.1, pp. 68–78.

Salgado-Torres, Alfredo (2013). *Incomplete information and costly signaling in college admissions*. Tech. rep. Working paper.

Samuelson, Paul A (1937). 'A note on measurement of utility'. In: *The review of economic studies* 4.2, pp. 155–161.

Sanders, James BT, J Doyne Farmer and Tobias Galla (2018). 'The prevalence of chaotic dynamics in games with many players'. In: *Scientific reports* 8.1, p. 4902.

Shannon, Claude Elwood (1948). 'A mathematical theory of communication'. In: *The Bell system technical journal* 27.3, pp. 379–423.

Thomas, David Reginald (1965). 'Asymptotic Value Distributions for Certain 2xn Games and n-Stage Games of Perfect Information'. Mathematics. Ph.D. thesis. Ann Arbor, Michigan: Iowa State University of Science and Technology.

Thomson, William (2023). 'The Axiomatics of Economic Design, Vol. 1'. In: *Studies in Choice and Welfare. Springer, Cham*.

Thrall, R. M. and J. E. Falk (1965). 'Some Results Concerning the Kernel of a Game'. In: *SIAM Review* 7.3, pp. 359–375. ISSN: 00361445. URL: http://www.jstor.org/stable/2027846 (visited on 24/04/2025).

U.S. Army Human Resources Command (2020). *Officer's Guide to the Army Talent Alignment Process (ATAP)*. Version 5. Internal document. Retrieved from a non-public source.

Von Mises, Ludwig (2002). *Human action: a treatise on economics*. Yale University Press.

Weibull, Jörgen W (1985). 'Discounted-value representations of temporal preferences'. In: *Mathematics of Operations Research* 10.2, pp. 244–250.

Weiming, Tu (1985). *Confucian Thought: Selfhood as Creative Transformation*. SUNY Press.

Weitzman, Martin L. (1977). 'Is the Price System or Rationing More Effective in Getting a Commodity to Those Who Need it Most?' In: *The Bell Journal of Economics* 8.2, pp. 517–524. ISSN: 0361915X. URL: http://www.jstor.org/stable/3003300 (visited on 04/12/2024).

Xie, Jiarui (2024). 'Games under the Tiered Deferred Acceptance Mechanism'. In: *arXiv preprint arXiv:2406.00455*.

Yu, P.L and M Zeleny (1975). 'The set of all nondominated solutions in linear cases and a multicriteria simplex method'. In: *Journal of Mathematical Analysis and Applications* 49.2, pp. 430–468. ISSN: 0022-247X. DOI: https://doi.org/10.1016/0022-247X(75)90189-4. URL: https://www.sciencedirect.com/science/article/pii/0022247X75901894.