On the monodromies of strata of (half-)translation surfaces

by

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Abstract

The present work consists of two independent parts devoted to the study of topological monodromy maps of strata of (half-)translation surfaces. As the name suggests, topological monodromies provide some important topological data for such strata (Lemma 2.4.1), considered by many authors as the loci of fundamental dynamical actions.

Inspired by Calderon–Salter's description of the images of the topological monodromy maps of non-hyperelliptic strata of translation surfaces ([Cal20], [CS21] and [CS22]), we estimated the size of the respective kernels in low genera and in particular in the case of the non-hyperelliptic strata $\mathcal{H}^{\mathrm{odd}}(4)$ and $\mathcal{H}(3,1)$ in genus 3 and $\mathcal{H}^{\mathrm{even}}(6)$ in genus 4. We prove that the kernel contains a non-abelian free group of rank 2.

In the second part, we study the topological monodromy maps of strata of half-translation surfaces, a generalisation of the concept of translation surface. Here, we improve Walker's result [Wal09] and provide a promising candidate for the image of the topological monodromies as subgroups of the mapping class groups.

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Acknowledgements

Though my name stands on the cover, this Thesis carries the silent presence of those who walked beside me during these 4 years; I consider this work a shared achievement with all those who supported me, whose presence and encouragement have shaped my academic journey, even in indirect ways.

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Author’s declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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Chapter 1

Introduction

Monodromy kernels for strata of low genus translation surfaces

Translation surfaces and their moduli spaces naturally arise in the interplay of topology, algebraic geometry, dynamics and number theory as shown through the work of Veech [Vee78], Masur [Mas82], Thurston [Thu88] and many subsequent authors. The moduli spaces of Riemann surfaces have a contractible cover and, for this reason, the topology is understood; see, for example, the work of Harer–Zagier [HZ86] and Maclachlan [Mac71]. Much less is known about the topology of the moduli spaces of translation surfaces.

We analyze the fundamental group of some covering spaces of the stratum components $\mathcal{H}^{\text{odd}}(4)$, $\mathcal{H}(3,1)$ and $\mathcal{H}^{\text{even}}(6)$ of the moduli space of translation surfaces, the *Teichmüller strata of blown-up translation surfaces.*

Theorem A. The fundamental groups of the Teichmüller strata of blown-up translation surfaces of $\mathcal{H}^{\text{odd}}(4)$, $\mathcal{H}(3,1)$ or $\mathcal{H}^{\text{even}}(6)$ contain a rank 2 non-abelian free group.

We achieve the result by studying the (pronged) topological monodromy maps, homomorphisms from the orbifold fundamental groups onto the images in the respective mapping class groups of surfaces with non-empty boundary components. Topological monodromies have kernels isomorphic to the fundamental groups of Teichmüller-like covers. We prove that, in these cases, the monodromies are far from being isomorphisms, as the kernels contain a non-abelian free group of rank 2.

The orbifold fundamental groups of the stratum components involved in the statement of Theorem A are closely related to Artin groups. Looijenga–Mondello showed that the stratum components $\mathcal{H}^{\text{odd}}(4)$ and $\mathcal{H}(3,1)$ are orbifold classifying space [LM14, Corollary 1.2] and the groups $\pi_1^{orb}(\mathcal{H}^{\text{odd}}(4))$ and $\pi_1^{orb}(\mathcal{H}(3,1))$ are infinite-cyclic central extensions of the inner automorphism groups of some Artin groups [LM14]. We prove the same for $\mathcal{H}^{\text{even}}(6)$.

Theorem B. The stratum component $\mathcal{H}^{\text{even}}(6)$ has a contractible cover. Moreover, the orbifold fundamental group is an infinite-cyclic central extension of the inner automorphism group of the Artin group of type E_8.

Translation surfaces. Let Σ denote a closed oriented surface of genus g and let $\mathcal{Z} \subset \Sigma$ be a finite set of points. A *translation structure* on Σ is an atlas of charts with values in $\mathbb{C}$ where the transition maps of $\Sigma \setminus \mathcal{Z}$ are translation, points in $\mathcal{Z}$ are cone type singularities and the holonomy $\pi_1(\Sigma \setminus \mathcal{Z}) \to \mathrm{SO}(2)$ is trivial. In particular, the complex structure on $\Sigma \setminus \mathcal{Z}$ can be extended to Σ by Riemann's removable singularity theorem and the metric around each point $p \in \mathcal{Z}$ can be given by cyclically gluing half-planes around p. The flat metric can be represented by polygons in $\mathbb{C}$.

Equivalently, a translation structure on Σ can also be given by pairs of the form (X, ω), where X is genus g Riemann surface and ω is a non-zero holomorphic one form on X. The finite set $\mathcal{Z}$ is identified with $\mathcal{Z}_\omega = \{p \in X \mid \omega_p \equiv 0\}$. Since the holonomy around every cone singularity is trivial, the number k_p of half-planes glued around each point $p \in \mathcal{Z}$ is even and the multiplicity of ω at p is $\frac{k_p}{2} + 1$.

Strata of translation surfaces. The moduli space of genus g translation surfaces is the set of all translation structures (X, ω) of Σ up to isomorphisms. The whole moduli space can be stratified in orbifolds $\mathcal{H}(k_1, \dots, k_n)$ characterized by the combinatorial data given by the orders of ω at its zeros.

Even though the topology of the strata of translation surfaces is poorly understood, our knowledge has improved in the past years. Costantini–Möller–Zoachhuber gave a recursive computable formula for the Euler characteristic of the moduli space of translation surfaces [CMZ22]. Further, Zykoski has constructed a finite simplicial complex with the same homotopy type of the strata $\mathcal{H}(k_1, \dots, k_n)$, motivated by Harer's construction of a simplicial complex those quotient by the mapping class group is homotopic equivalent to the moduli space of Riemann surfaces [Zyk22].

Kontsevich–Zorich showed that each stratum has at most 3 connected components and in every genus some components are *hyperelliptic* [KZ03]. Namely, hyperelliptic components consist of translation surfaces (X, ω) where X is a hyperelliptic Riemann surface and $\tau^*(X, \omega) = (X, -\omega)$ where τ is the hyperelliptic involution of X. These connected components are orbifold classifying spaces for finite extensions of braid groups; see [LM14, Section 1.4] for a proof. Much less is known about the non-hyperelliptic stratum components and, in particular, whether or not there exists a contractible cover. Natural candidates to consider are Teichmüller-like parameter spaces; see Section 2.4 for details. Their fundamental groups are isomorphic to the kernels of the topological monodromy maps.

Monodromy maps. The mapping class group $\mathrm{Mod}(\Sigma, \mathcal{Z})$ is the group of all orientation-preserving self-diffeomorphisms of Σ that leave the set of marked points invariant, up to isotopies relative to the set of marked points. If $\mathcal{C}$ is a connected component of a stratum $\mathcal{H}(k_1, \ldots, k_n)$, then any (orbifold) homotopy class of loops based at (X, ω) gives rise to some self-diffeomorphism of X that preserves the zeros of ω. These data are recorded by the (labelled) topological monodromy map

$$\tilde{\rho}_{\mathcal{C}} : \pi_1^{orb}(\mathcal{C}) \to \mathrm{Mod}(\Sigma, \mathcal{Z}).$$

Calderon–Salter's work resulted in a complete description of the images of the monodromy maps associated with all non-hyperelliptic connected components of the strata $\mathcal{H}(k_1, \ldots, k_n)$ for $g \geq 5$. In other words, the orbifold fundamental groups of all non-hyperelliptic connected components are projected onto subgroups of the mapping class groups called *framed mapping class groups* [CS22].

The kernel of the punctured monodromy. If $\mathcal{C}$ is hyperelliptic then $\mathrm{Im}\,\tilde{\rho}_{\mathcal{C}}$ is isomorphic to the symmetric mapping class group $\mathrm{SMod}(\Sigma, \mathcal{Z})$, and $\ker \tilde{\rho}_{\mathcal{C}}$ is finite [Cal20, Section 2.1]. It is natural to ask whether or not the topological monodromy is injective or if the kernel is small enough so to have a description of the commensurability class of the orbifold fundamental groups. Analogously, it is natural to ask if Teichmüller-like covers of stratum components are universal covers, as the orbifold fundamental groups are orbifold deck transformation groups. For this reason, we are interested in estimating the size of the kernels of the monodromies $\tilde{\rho}_{\mathcal{C}}$ for non-hyperelliptic connected components. This thesis's first main result is that the kernel is large in some cases.

Theorem C. Let $\tilde{\rho}_{\mathcal{H}^{\mathrm{odd}}(4)}$, $\tilde{\rho}_{\mathcal{H}(3,1)}$ and $\tilde{\rho}_{\mathcal{H}^{\mathrm{even}}(6)}$ be the labelled topological monodromy maps of the non-hyperelliptic connected components $\mathcal{H}^{\mathrm{odd}}(4)$, of $\mathcal{H}(3,1)$ and of $\mathcal{H}^{\mathrm{even}}(6)$, respectively. The kernels of the monodromies into $\mathrm{Mod}(\Sigma, \mathcal{Z})$ contain a rank 2 non-abelian free group.

It turns out that Theorem C is an example of a more general phenomenon related to *geometric homomorphisms* from Artin groups to mapping class groups.

Geometric homomorphisms. If Γ is a finite, connected and undirected simple graph with $\mathcal{V}(\Gamma) = \{v_1, \ldots, v_n\}$ as its set of vertices, an *Artin group* is a group that admits a presentation of the following form

$$A_\Gamma = \left\langle a_1, \ldots, a_n \;\middle|\; \begin{array}{lll} a_i a_j a_i = a_j a_i a_j & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ a_i a_j = a_j a_i & \text{otherwise} \end{array} \right\rangle. \tag{1.1}$$

Roughly speaking, a geometric homomorphism $A_\Gamma \to \mathrm{Mod}(\Sigma, \mathcal{Z})$ arises as the correspondence between the vertices of the defining graph Γ and a family of simple closed curves on the surface Σ. The standard Artin generators in the presentation (1.1) map to Dehn twists about curves that respect the intersection pattern given by the defining graph; see Figure 1.1.

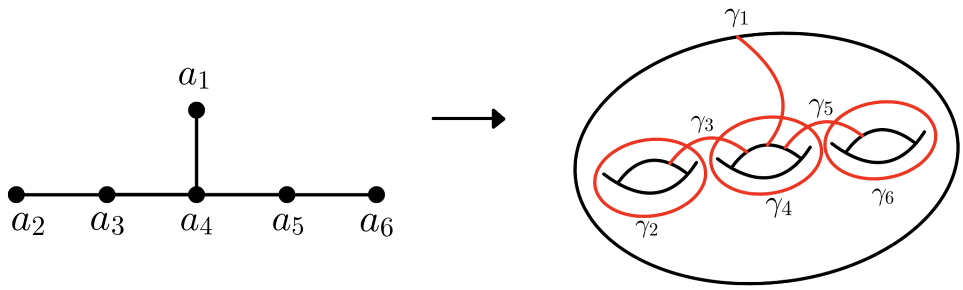


Figure 1.1: The map sending every standard generator a_i to the Dehn twist T_{γ_i} determines a geometric homomorphism for the Artin group A_{E_6}.

Possibly, there might exist relations between Dehn twists that do not hold for standard generators of Artin groups or, in other words, geometric homomorphisms might have non-trivial kernels. However, there is no known algorithm that can solve the word problem for a generic Artin group (for further details, see, for example [McC17, Conjecture 5.2]), and this is the main obstruction to characterize kernels of geometric homomorphisms.

Wajnryb proved that if the graph Γ contains E_6 as a subgraph, any geometric homomorphism cannot be an injection [Waj99]. In particular, Wajnryb found an element w given explicitly in terms of the standard generators in the presentation (1.1) and adopted the following strategy: as every inclusion of graphs induces a monomorphism of the respective Artin groups [Van83], it is enough to find a non-trivial element w in A_{E_6} which can be written in the mapping class group of genus 3 surface as a braid relator of Dehn twists. The group A_{E_6} is a spherical-type Artin group, a class of groups for which the word problem has been solved by means of their *Garside structure*. Our next result builds on Wajnryb's work and Theorem C can be thought of as a corollary of the following theorem.

Theorem D. Let Γ be any finite and undirect simple graph with E_6 as a subgraph. Any geometric homomorphism of A_Γ in the mapping class group of a surface with non-empty boundary has a large kernel that contains a non-abelian free group $\mathbb{F}_2$ of rank 2. Indeed, there is some $g \in A_\Gamma$ such that $\mathbb{F}_2$ is generated by the Wajnryb element w and its conjugate $g^{-1}wg$.

Theorem D follows from the acylindrical hyperbolicity of spherical-type Artin groups modulo their center or, equivalently, the inner automorphism group $\operatorname{Inn} A_\Gamma$. Here, the Ping-Pong strategy can be adopted to detect non-abelian free groups.

Acylindrical hyperbolicity. Let $\operatorname{Inn} A_\Gamma$ denote the inner automorphism group of the spherical-type Artin group A_Γ or, equivalently, the quotient of A_Γ by its infinite-cyclic center. Calvez–Wiest proved that the group $\operatorname{Inn} A_\Gamma$ acts *acylindrically* on a δ-hyperbolic graph, which is known in the literature as the *additional length graph* $C_{AL}(\Gamma)$ [CW16a, Theorem 1.3].

Calvez–Wiest found a group element $\kappa \in A_\Gamma$ representing a loxodromic isometry of $C_{AL}(\Gamma)$ that acts weakly properly discontinuously. By Osin's criterion [Osi15, Theorem 1.2] the existence of the *Calvez–Wiest element* κ is enough to conclude the acylindrical hyperbolicity of $\operatorname{Inn} A_\Gamma$.

We prove that the infinite order Wajnryb element acts elliptically on $C_{AL}(\Gamma)$ and a classical result shows that the image of w in $\operatorname{Inn} A_\Gamma$ cannot fix $\kappa \in \operatorname{Inn} A_\Gamma$ in the Gromov boundary of the additional length graph [AC21, Lemma 25]. The following is due to Abbott–Dahmani and is the key ingredient we need to prove Theorem D.

Proposition. [AD19, Proposition 2.1] Let G be a group acting acylindrically hyperbolic on a geodesic δ-hyperbolic space X. Suppose $\sigma \in G$ is elliptic and $\gamma \in G$ is loxodromic. If

1. the set $\mathrm{A}_{10\delta}(\gamma) = \{x \in X \mid d(x, \gamma x) \leq \inf_{y \in X} d(y, \gamma y) + 10\delta\}$ is not preserved by any non-trivial power of σ and
2. the diameter of $\mathrm{Fix}_{50\delta}(\sigma) = \{x \in X \mid d(x, \sigma^n x) \leq 50\delta \text{ for all } n \in \mathbb{Z}\}$ is finite,

then there is some $n \in \mathbb{Z}$ such that the group generated by σ and γ^n is a non-abelian free group of rank 2.

We conclude that there exists a positive integer n such that the group generated by $\kappa^{-n} w \kappa^n$ and w is a non-abelian free group of rank 2.

Projective strata. We now explain how Artin groups arise in the context of the non-hyperelliptic components of the strata mentioned in Theorem C.

The multiplicative group $\mathbb{C}^*$ acts on the cotangent bundle of each Riemann surface X by multiplication. The action preserves the multiplicity at the cone points of each holomorphic 1-form and is well-defined on each connected component $\mathcal{C}$ of a stratum $\mathcal{H}(k_1, \ldots, k_n)$. The resulting quotient is denoted by $\mathbb{P}\mathcal{C}$ and is known as a *projective stratum* of translation surfaces.

Looijenga–Mondello proved that the orbifold fundamental groups of $\mathbb{P}\mathcal{H}^{\mathrm{odd}}(4)$ and $\mathbb{P}\mathcal{H}(3,1)$ are the inner automorphism groups of the E_6-type and E_7-type spherical Artin groups, respectively [LM14]. We prove that the same holds for $\mathbb{P}\mathcal{H}^{\mathrm{even}}(6)$, which has an orbifold fundamental group isomorphic to the inner automorphism group of the E_8-type spherical Artin group. A result of Pinkham implies that the monodromy map of $\mathbb{P}\mathcal{H}^{\mathrm{odd}}(4)$ is geometric [Pin74], meaning that standard Artin generators representing classes of elements in $\operatorname{Inn}(A_{E_6})$ are mapped to some Dehn twists. The same holds for the monodromy of $\mathbb{P}\mathcal{H}(3,1)$ and of $\mathbb{P}\mathcal{H}^{\mathrm{even}}(6)$.

Theorem E. The topological monodromies $\rho_{\mathbb{P}\mathcal{H}(3,1)}$ and $\rho_{\mathbb{P}\mathcal{H}^{\mathrm{even}}(6)}$ map the classes of the standard generators to Dehn twists.

Theorem E and Pinkham's result are then enough to conclude that the kernels of the monodromies associated with the strata $\mathcal{H}^{\mathrm{odd}}(4)$, $\mathcal{H}(3,1)$ and $\mathcal{H}^{\mathrm{even}}(6)$ contain a copy of a non-abelian free group $\mathbb{F}_2$ of rank 2.

Monodromy of some strata of quadratic differentials

Abelian differentials on Riemann surfaces of genus $g \geq 1$ can be represented via families of polygons in $\mathbb{C}$ with edges paired and identified via translations. A generalisation can be made in such a way that edges are identified via half-translations $z \mapsto \pm z + c$. The resulting surfaces are *half-translation surfaces*. Meromorphic quadratic differentials with at least simple poles are to half-translation surfaces what holomorphic abelian differentials are to translation surfaces. That is, there exists a one-to-one correspondence between equivalence classes of quadratic differentials and half-translation surfaces. If we do not allow simple poles, the moduli space of genus g quadratic differentials $\mathcal{Q}_g$ has a Teichmüller-like cover $\mathcal{TQ}_g$ which is the cotangent bundle of the the classical Teichmüller space $\mathcal{T}_g$.

We define $\mathcal{TQ}_g$ to be the set of triples (X, ϕ, q) where (X, ϕ) is an element of the $\mathcal{T}_g$ and q is a non-zero holomorphic differential on X that is not a square of an abelian differential. Both $\mathcal{TQ}_g$ and $\mathcal{Q}_g$ can be stratified by the orders of the zeros of the quadratic differentials: if $\underline{k} = (k_1, \ldots, k_n)$ is a partition of $4g-4$, then $\mathcal{Q}(\underline{k})$ and $\mathcal{TQ}(\underline{k})$ are the spaces of holomorphic and marked holomorphic quadratic differentials with n zeros with multiplicities given by $\underline{k}$, respectively. The notation $(k_1^{r_1}, \ldots, k_n^{r_n})$ is used to denote multiple zeros of the same order.

Lanneau proved that each stratum of quadratic differentials $\mathcal{Q}(\underline{k})$ in genus $g \geq 3$ is either connected or has two connected components, one of which is hyperelliptic or, equivalently, consists of hyperelliptic Riemann surfaces and quadratic differentials preserved by the hyperelliptic involution [Lan08]. Later, Walker proved that the components of the Teichmüller strata $\mathcal{TQ}(2^g, k_1, \ldots, k_n)$ are exactly $2^{2g}-1$ if all the k_i are even, while the components of the Teichmüller strata $\mathcal{TQ}(r^g, k_1, \ldots, k_n)$ are at most r^{2g} if $r = 1, 3$ divides all the k_i. More generally, Walker proved that the spaces $\mathcal{TQ}(k_1, \ldots, k_n)$ with all the k_i even have at least $2^{2g}-1$ components [Wal09, Theorem 1], but the bound is not always sharp. We prove the following.

Theorem F. Let $\mathcal{L}$ be a non-hyperellipitc stratum component in $\mathcal{Q}(\underline{k})$ and $r = \gcd(\underline{k})$.

a) If r is even, the connected components of $\mathcal{TL}$ are at least $(r/2)^{2g}(2^{2g}-1)$.

b) If r is odd, the connected components of $\mathcal{TL}$ are at least $r^{2g} - ((r+1)/2)^{2g}$.

Theorem F has been proved using the (non-labelled) topological monodromy map associated to any stratum component of $\mathcal{Q}(k_1, \ldots, k_n)$, each of which is a good orbifold. Once again, the (orbifold) homotopy class of loops based at a point $Q_0 = (X, \phi, q)$ in a Teichmüller cover $\mathcal{TL}$ of $\mathcal{L}$ give rise to an element in the mapping class group $\mathrm{Mod}(\Sigma)$

of a closed and connected orientable surface of genus g. In other words, there exists a homomorphism

$$\rho_{\mathcal{L}} : \pi_1^{orb}(\mathcal{L}, Q_0) \to \mathrm{Mod}(\Sigma)$$

from the orbifold fundamental group of $\mathcal{L}$ at Q_0, and the index of its image in $\mathrm{Mod}(\Sigma)$ coincides with the number of connected components of $\mathcal{TL}$ (Lemma 2.4.1). The above theorem follows from the characterisation of a subgroup of $\mathrm{Mod}(\Sigma)$ containing the image of $\rho_{\mathcal{L}}$. The subgroup of $\mathrm{Mod}(\Sigma)$ we have considered is the stabilizer of the winding number function associated to the *horizontal line field* of Q_0. Calderon–Salter used a similar argument to compute the number of connected components of the non-hyperelliptic Teichmüller strata of abelian differential [CS21]. Hyperelliptic stratum components have not been considered in the above theorem, since the number of Teichmüller connected components is infinite.

Some filling multicurve systems of a closed surface can be upgraded into an actual marked flat structure by a well-known construction attributed to Thurston–Veech [Thu88]. Any curve in the system corresponds to a vertical or horizontal cylinder on the flat surface and can be sheared to obtain Dehn twists via the topological monodromy. Starting from a filling multicurve system containing simple curves in the Humphries generating set of $\mathrm{Mod}(\Sigma)$, we construct marked quadratic differentials in $\mathcal{TQ}(\underline{k})$, where $\underline{k} = (1^3, k_1, \ldots, k_n)$ and $(k_1, \ldots, k_n)$ is a partition of $4g-7$ that can be obtained from a labelled graph as follows.

Suppose C_g is one of the planar graphs shown in Figure 1.2 when g is odd, or in Figure 1.3 when g is even. The vertices of C_g are labelled by the integers 1 and 2, while some edges have labels $\delta_i, \ldots, \delta_d$ for $d = \frac{g-3}{2}$ if g is even or $d = \frac{g-4}{2}$ if g is odd. The labels of the vertices of C_g form a partition of $4g-7$. A new partition of $4g-7$ can be obtained from C_g by applying a finite number of the following two *elementary edge-collapses*. We either replace an unlabelled edge and its endpoints labelled by $l_1, l_2 \in \mathbb{N}$ with a new vertex having as a label $l_1 + l_2$ and edges inherited by the endpoints of the collapsed edge, or simultaneously collapse two edges with the same label.

Theorem G. Suppose $g \geq 5$ and $(k_1, \ldots, k_n)$ is a partition of $4g-7$ obtained from C_g by applying a finite number of elementary collapses. If $\underline{k} = (1^3, k_1, \ldots, k_n)$, then $\mathcal{TQ}(\underline{k})$ is connected.

We have also considered the composition of the monodromies $\rho_{\mathcal{L}}$ with the symplectic representation $\psi_g : \mathrm{Mod}(\Sigma) \to \mathrm{Sp}(2g, \mathbb{Z})$. A complete description of the topological monodromy images is not available yet and one of the intermediate steps Calderon–Salter achieved to describe the image of topological monodromies of non-hyperelliptic strata of abelian differentials is that of describing the image of the fundamental groups in the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. The composition $\psi_g \circ \rho_{\mathcal{L}}$ defines the *symplectic monodromy* of $\mathcal{L}$ based at Q_0 that we denote by $\rho_{\mathcal{L}}^{\mathbb{Z}}$. Here, the (orbifold) homotopy

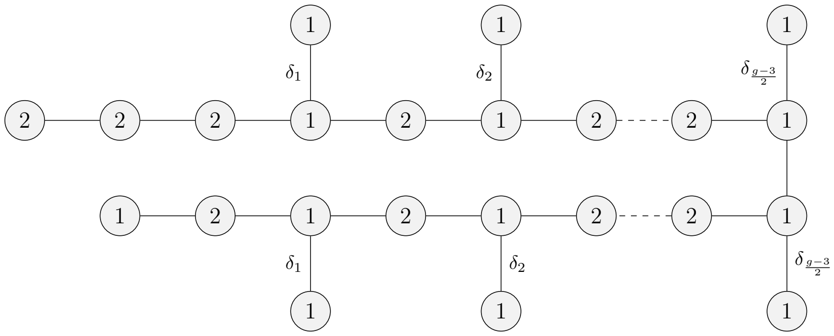


Figure 1.2: The C_g graph if g is odd.

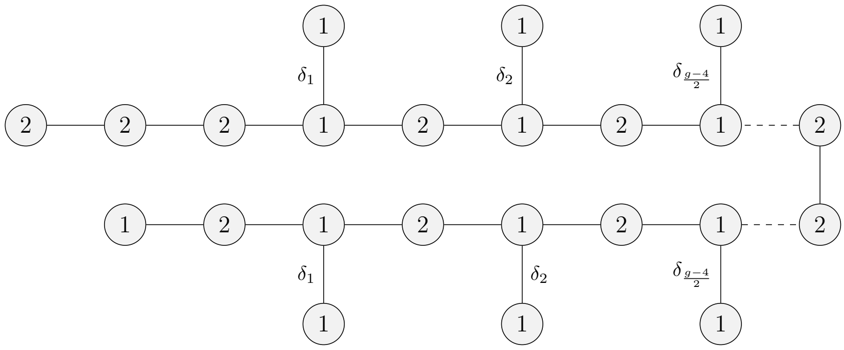


Figure 1.3: The C_g graph if g is even.

class of loops based at a point $Q_0 \in \mathcal{TL}$ give rise to automorphisms of $H_1(X, \mathbb{Z})$ that preserves the algebraic intersection number. Gutiérrez Romo's result on plus and minus Rauzy–Veech groups [Gut17] implies that, for certain strata of quadratic differentials, the symplectic image of $\rho^{\mathbb{Z}}_{\mathcal{L}}$ is the entire ambient group $\mathrm{Sp}(2g, \mathbb{Z})$. More precisely, in genus $g \geq 4$, the image of $\rho^{\mathbb{Z}}_{\mathcal{L}}$ is the full symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ if $\mathcal{L}$ is in a stratum component having at least three singularities (zeros or poles), not all of even order. However, in the case where all the k_i's are even, Walker's classification of the connected components of the Teichmüller strata $\mathcal{TQ}(2^g, k_1, \dots, k_n)$ implies that the symplectic image $\mathrm{Im}(\rho^{\mathbb{Z}}_{\mathcal{L}})$ is the full stabilizer of the *Rauzy-Veech co–cycle* $RV^*_{Q_0}$ at Q_0. The Rauzy-Veech co–cycle $RV^*_{Q_0}$ is induced by the winding number function at Q_0. Here, we improve Walker's result. Note that the strata of Corollary 1 are connected, according to Lanneau's description.

Corollary 1. Let $g \geq 6$ and $\underline{k} = (2^4, k_1, \dots, k_n)$ be a partition of $4g - 4$ such that all the k_i's are even. Then the image of the symplectic monodromy $\rho^{\mathbb{Z}}_{\mathcal{Q}(\underline{k})}$ is the stabilizer of the Rauzy–Veech co-cycle $RV^*_{Q_0}$.

Winding number function can also be used to show that, if r is even, the symplectic image $\mathrm{Im}(\rho^{\mathbb{Z}}_{\mathcal{L}})$ does not always coincide with the stabilizer of a Rauzy–Veech cycle.

Corollary 2. Let $g \geq 3$ and $\mathcal{L}$ a non-hyperelliptic connected component of $\mathcal{Q}(\underline{k})$ if $\underline{k} = (k_1, \dots, k_n)$ and all the k_i are divisible by 4, the index $[\mathrm{Sp}(2g, \mathbb{Z}) : \mathrm{Im}\, \rho^{\mathbb{Z}}_{\mathcal{L}}]$ is at least $2^{2g}(2^{2g} - 1)$.

Chapter 2

Translation surfaces

§ 2.1 | Equivalent definitions

Translation surfaces are defined via families of polygons in the complex plane or, equivalently, as abelian differentials on closed Riemann surfaces. The first section of this chapter sets the foundational knowledge required to define strata of translation surfaces and their topology. Mutatis mutandis, the same theory can be applied to define strata of half-translation surfaces.

Definition via polygons

Let $P = \{P_1, \dots, P_n\}$ be a finite collection of polygons in the complex plane $\mathbb{C}$ and let $\mathcal{S}$ be the set of sides of P, each given with an orientation. Suppose the collection $\mathcal{S}$ comes with pairings $\{s, t\}$ for $s, t \in \mathcal{S}$ with the following three conditions satisfied:

1. each side $s \in \mathcal{S}$ appears in exactly one pair;
2. two sides of each pair are parallel and of the same length;
3. for each pair, the polygon of one oriented side lies on the left of the side, and the polygon corresponding to the other oriented side lies to the right.

Then, a *translation surface* is a closed Riemann surface obtained by identifying the sides of the polygons in P by complex translations in $\mathbb{C}$ respecting the given pairing.

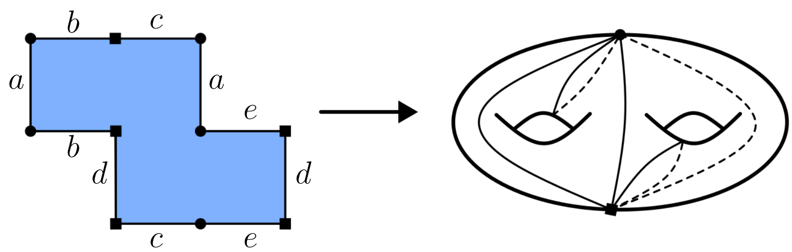


Figure 2.1: An example of a genus 2 translation surface. Opposite sides are identified via the appropriate translations. Here, the orientation of each edge is given by the natural orientation inherited by the complex plane. The thinner curves on the surface are the images of the sides of the polygon after the identification.

The above construction topologically determines a closed orientable surface. An atlas of charts is naturally defined and the changes of coordinates are complex translations. Note that a translation surface does not need to be connected. However, from now on, we will only consider connected translation surfaces.

The third property guarantees that the resulting closed Riemann surface is oriented. Figure 2.2 shows an example of a non-orientable Riemann surface obtained from a polygon. Each of the sides s and t, oriented using the orientation of $\mathbb{C}$, lies on the same side of the polygons they bound. We can find a simple arc γ connecting the middle points of s and t. The thickened arc γ is a homeomorphic image of the Möbius band. Hence, the polygon in Figure 2.2 does not define an oriented surface.

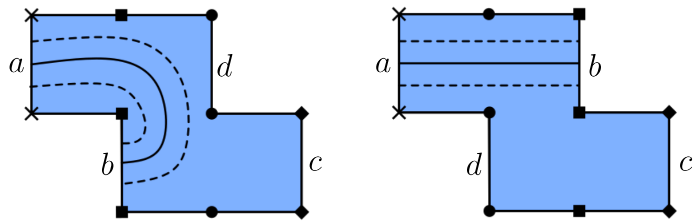


Figure 2.2: A non-orientable connected translation surface defined by two polygons. Opposite sides with no labels are identified.

Any Riemann surface X is conformally isomorphic to the quotient of a simply connected Riemann surface $\tilde{X}$, the universal cover of X, by a group of conformal automorphisms which acts freely and properly discontinuously on $\tilde{X}$. The metric of $\tilde{X}$ descends to a metric on X, that might be spherical, flat or hyperbolic depending on the genus of X [Jos13, Corollary 4.4.1].

Theorem 2.1.1 (Uniformitazion Theorem)**.** Any simply connected Riemann surface is conformally isomorphic either to the complex plane, the Riemann sphere $\mathbb{CP}^1$, or the unit open disk with the hyperbolic metric.

If $g = 1$, the Riemann surface X is a torus and its universal cover is the complex plane. In this case, X has a flat metric: the sectional curvature vanishes everywhere. If $g \geq 2$, then X has a hyperbolic metric.

Intriguingly, translation surfaces define a flat metric on the complement of finitely many points of a Riemann surface, regardless of the genus. A polygonal representation of a translation surface defines an atlas of charts in $\mathbb{C}$ on the underlying Riemann surface, such that changes of coordinates are translations. Every non-vertex point comes with a centred chart in $\mathbb{C}$, where a neighbourhood is conformally mapped to an open ball of $\mathbb{C}$. Since translations preserve the standard flat Euclidean metric on $\mathbb{C}$, the complement of the vertex-points set inherits a flat metric.

The same does not hold if the open ball is centred at one of the points corresponding to a vertex of the polygons. In this case, the metric is given by cyclically gluing circular sectors around a common point, addressed as a *cone point*. The total angle around a cone point might not be 2π. In this case, the Euclidean metric inherited by the complex plane cannot be extended to the whole Riemann surface unless $g = 1$.

Proposition 2.1.2. [AM24b, Lemma 2.1.2] The angle around a cone point of a translation surface is an even multiple of π. In particular, if $\mathcal{Z} = \{p_1, \dots, p_n\}$ is the set of cone points of a translation surface, the angle around each p_i can be written in the form $2(k_i + 1)\pi$ for some $k_i \in \mathbb{N}$.

Every translation surface comes with a partition of $2g - 2$.

Proposition 2.1.3. [AM24b, Theorem 2.1.3] Let the angles at the cone points $p_1, \dots, p_n$ of a genus g translation surface be $2(k_1 + 1)\pi, \dots, 2(k_n + 1)\pi$, respectively. Then $\sum_{i=1}^{n} k_i = 2g - 2$ holds.

We introduce a notion of isomorphism between translation surfaces in order to define their moduli spaces later on. The following equivalence relation preserves the partition $k_1, \dots, k_n$.

Let P_1 and P_2 be a pair of polygon collections defining translation surfaces. The collections P_1 and P_2 are *flat equivalent* if one can be obtained from the other by a finite sequence of *scissor moves* (Figure 2.3). A scissors move is performed by cutting one of the two polygons along a straight segment joining two vertices and gluing back the two cut pieces along identified sides via a translation.

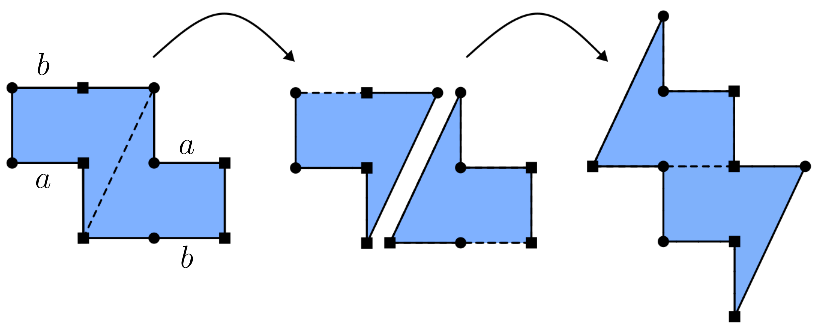


Figure 2.3: A scissor move for a translation surface of genus 2. Opposite sides with no labels are identified. The orientation of each side is naturally inherited by $\mathbb{C}$.

Scissor moves only change the underlying Riemann structure up to a bilohomorphism. If two translation surfaces are flat equivalent, they are also conformally equivalent.

Definition via abelian differentials

If $g \geq 1$, translation surfaces can equivalently be defined as *abelian differentials*. In this section, we show that the two definitions are equivalent. Once this equivalence is established, we will use the terms abelian differential and translation surface interchangeably.

An *abelian differential* is a pair (X, ω) where X is a Riemann surface and ω is a holomorphic 1-form. Two pairs (X_1, ω_1) and (X_2, ω_2) are *flat equivalent* is there exists a bilohomorphism $I : X_1 \to X_2$ such that $I^*\omega_2 = \omega_1$.

An abelian differential is a holomorphic section of the cotangent bundle of its Riemann surface X. The cotangent bundle is usually referred to as the canonical bundle K_X. In local coordinates, the 1-form ω can be written in the form $f(z)dz$ where f is a holomorphic map. The set of abelian differentials on X is a complex vector space that will be denote by $\Omega(X)$. If X has genus g, then the complex dimension of $\Omega(X)$ is g.

Before making the correspondence between translation surfaces and abelian differentials explicit, we recall some properties of abelian differentials. Firstly, a 1-form ω on a Riemann surface X vanishes with order k in a point $p \in X$ if there exists a local chart z centred in p such that ω can locally be written as $z^k f(z)dz$ for f holomorphic and $f(0) \neq 0$. Also, if ω is not identically zero, its zeros are isolated and the vanishing locus of ω is a discrete closed set of X. As X is compact, the 1-form ω has only finitely many zeros and can define a *divisor* on X. See Section 2.5 for more details on divisors. The following is a straightforward application of the Riemann–Roch Theorem, and it can also be obtained as a corollary of Proposition 2.1.5.

Proposition 2.1.4. Any non-zero $\omega \in \Omega(X)$ vanishes in finitely many points $p_1, \dots, p_n$ with orders $k_1, \dots, k_n$, such that $\sum_{i=1}^n k_i = 2g - 2$.

Any abelian differential on a Riemann surface X defines a translation surface structure on the same Riemann surface X, and viceversa. In the upcoming pages, we are going to describe explicitly the following correspondence.

Proposition 2.1.5. There is a one-to-one correspondence

$$\left\{ \begin{matrix} \text{Translation surfaces} \\ \text{modulo scissor moves} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{Abelian differentials} \\ \text{modulo flat equivalence} \end{matrix} \right\}$$

so that cone points $p_1, \dots, p_n$ with angles $2(k_1+1)\pi, \dots, 2(k_n+1)\pi$ correspond to zeros of holomorphic 1-forms with multiplicities $k_1, \dots, k_n$, respectively.

Translation surfaces can also be defined via a third equivalent definition we will only use below to prove Proposition 2.1.5. In particular, a translation surface is an oriented, compact and connected surface with a flat metric on the complement of finitely many points. The metric around the remaining points is given by cyclically glueing an even number of half-planes. The changes of coordinates are complex translations. For more details, see, for example, [Wri15].

Any pair (X, ω) defines an atlas of charts on $X \setminus \mathcal{Z}$, where $\mathcal{Z}$ is the set of the zeros of ω, such that the changes of coordinates are translations and the metric is flat. Let $p \in X$ be a non-zero point of ω and pick a simply connected open neighbourhood U_α

of p which does not intersect any zeros of ω. Let $w_\alpha : U_\alpha \to \mathbb{C}$ be a chart centered at p, such that $\omega_{|U_\alpha} = f_{w_\alpha} dw_\alpha$. Without loss of generality, we can suppose that f_{w_α} does not vanish on $w_\alpha(U_\alpha)$. Then, we define the chart

$$\begin{aligned} z_\alpha : w_\alpha(U_\alpha) &\to \mathbb{C} \\ w_\alpha(q) &\mapsto \int_0^{w_\alpha(q)} f_{w_\alpha}(\tau) d\tau. \end{aligned}$$

The chart z_α is well-defined by Cauchy's integral theorem since U_α is simply connected, and the integral is independent of the path chosen to connect 0 and $w_\alpha(q)$. Let f_{z_α} be the new coordinate chart of ω associated with the chart z_α. As the 1-form ω is well-defined, we have that

$$f_{z_\alpha}(w_\alpha(q)) \frac{dz_\alpha}{dw_\alpha}(w_\alpha(q)) = f_{w_\alpha}(w_\alpha(q)) \text{ for any } q \in U_\alpha.$$

From the definition of the chart z_α we obtain $\frac{dz_\alpha}{dw_\alpha}(w_\alpha(q)) = f_{w_\alpha}(w_\alpha(q)) \neq 0$ and therefore $f_{z_\alpha} \equiv 1$. Suppose now that U_α and U_β are two overlapping simply-connected domains. Since by the above argument both f_{z_α} and f_{z_α} are constantly equal to 1, then $\frac{dz_\alpha}{dz_\beta} \equiv 1$ and there is a constant $c \in \mathbb{C}$ such that

$$z_\alpha(q) + c = z_\beta(q) \text{ for any } q \in U_\alpha \cap U_\beta.$$

In these coordinates, the metric on the complement of the vanishing points of ω is flat and is given locally by the form $|\omega| = |dz|$

Now let $p \in X$ be a vanishing point of ω of order k and choose a chart $z : U_p \to \mathbb{C}$ centred at p. We can write ω locally around p as $f_z dz$ where $f_z(z) = z^k g(z)$, g is holomorphic and $g(0) \neq 0$. The map

$$\begin{aligned} h_\alpha : z(U_p) &\to \mathbb{C} \\ z(q) &\mapsto (k+1) \int_0^{z(q)} z^k g(\tau) d\tau, \end{aligned}$$

has a zero of order $k+1$ at 0. Consider a $(k+1)^{st}$ holomorphic root of h and denote it by w. In particular, we have that

$$\frac{dw}{dz} = w^{-k} f_z(z)$$

and the new coordinate chart f_w for ω is $f_w(w) = w^k$. Hence, the metric around p is given by $|\omega| = |w^k dw|$ and can be concretely described, as follows. Divide the w-plane in $2(k+1)$ sectors of equal angle, with the convention that four sectors have a ray along either the positive or negative real axis. The metric $|w^k dw|$ is the pullback of the flat metric of $\mathbb{C}$ by the map $w \mapsto w^{k+1}$, where $\mathbb{C}$, divide in sector, is pulled back isometrically to $k+1$ copies of the upper half plane and $(k+1)$ copies of the lower half plane, glued isometrically along the half-infinite rays.

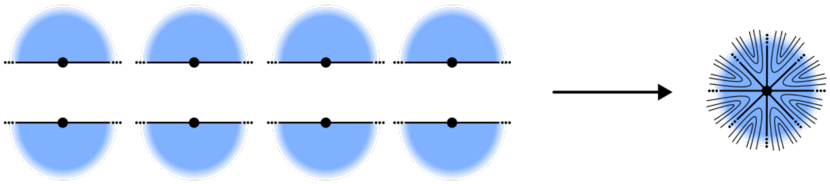


Figure 2.4: Two pairs of four half-planes representing a neighbourhood of a vanishing point.

The following is a result about triangulating a Riemann surface X with the metric induced by a holomorphic 1-form $\omega \in \Omega(X)$. The triangulation is defined using geodesics with endpoints in the zeros of ω, that are called *saddle connections*. A saddle connection on an abelian differential (X, ω) is a geodesic in the metric induced by ω joining (non necessarily distinct) zeros of ω with no zeros in the interior.

Proposition 2.1.6. [AM24b, Proposition 2.5.1] Let γ be an arc in X joining two zeros p_1 and p_2 of $\omega \in \Omega(X)$. There exists a unique saddle connection in the homotopy class of γ relative to the endpoints. Moreover, any abelian differential (X, ω) admits a triangulation in saddle connections.

The following tool will be used to prove Proposition 2.1.5. Let Σ be a connected closed oriented surface with finitely many marked points. The *flip graph* is a graph with vertices in one-to-one correspondence with the triangulations of Σ by triangles having vertices at the marked point. Two vertices share an edge if the respective triangulations differ by a flip move. A flip move is performed replacing the shared side of two adjacent triangles with the other diagonal of the quadrilateral they define.

Theorem 2.1.7 ([Mos88])**.** The flip graph of a connected closed oriented surface is connected.

We are now ready to prove the main proposition of this section.

Proof of Proposition 2.1.5. Any abelian differential can be triangulated by saddle connections, which means that it can be represented by a collection of polygon. More precisely, if γ is a saddle connection that bounds a triangle, the corresponding side of the polygonal representation is given by the complex number $\int_\gamma \omega$. Conversely, any polygonal representation of a translation surface defines a Riemann surface structure X on a connected closed oriented surface, and an holomorphic 1-form by locally pulling back the Euclidean form dz from $\mathbb{C}$ to X via the local coordinates. The resulting 1-form is well defined as the coordinates changes are complex translations.

Suppose two abelian differentials (X_1, ω_1) and (X_2, ω_2) are flat equivalent via a bilomorphism $I : X_1 \to X_2$. Consider triangulations τ_1 and τ_2 in saddle connections for both (X_1, ω_1) and (X_2, ω_2), respectively, so that every triangle is realised by geodesics.

Then, the pullback via I of τ_2 can be obtained by a finite sequence of flip moves from τ_1, by Theorem 2.1.7, so that all intermediate triangulations are realised by saddle connections. In particular, any flip move relates polygonal representations by a scissor move. Conversely, if two translation surfaces are equivalent via a scissor move, then they are also flat equivalent, since the cutting and pasting operations do not change the underlying holomorphic structure. □

§ 2.2 | Strata of translation surfaces

In this section, we define the strata of translation surfaces and their topology. Recall that any genus g translation surface comes with a partition of $2g-2$ by Proposition 2.1.3 or, equivalently, by Proposition 2.1.4.

Let $k_1, \ldots, k_n$ be a partition of $2g-2$ for some integer $g \geq 1$. The stratum $\mathcal{H}(k_1, \ldots, k_n)$ is the set of all translation surfaces with exactly n cone points with total angles $2(k_1 + 1)\pi, \ldots, 2(k_n+1)$, modulo scissor moves. Equivalently, by Proposition 2.1.5 the stratum $\mathcal{H}(k_1, \ldots, k_n)$ is the set of all genus g abelian differentials (X, ω) such that $\omega \in \Omega(X)$ has exactly n zeros of order $k_1, \ldots, k_n$, modulo flat equivalence.

Before defining the topology on the strata $\mathcal{H}(k_1, \ldots, k_n)$, we recall the definition of the moduli spaces $\mathcal{M}_g^n$ of genus g Riemann surfaces with n marked points. The two families of moduli spaces given by $\mathcal{M}_g^n$ for $g \geq 1, n \geq 0$ and by $\mathcal{H}(\underline{k})$ for $k_1 \ldots, k_n$ partition of $2g-2$, are *good orbifolds*.

If X is an orientable smooth manifold with empty boundary and G a subgroup of the self-diffeomorphism group $\mathrm{Diff}(X)$ of X, the quotient X/G is a good orbifold if G acts *properly*. That is, for any compact set $K \subset X$, we have that $\{g \in G \mid gK \cap K \neq \emptyset\}$ is compact in G with the compact-open topology inherited by $\mathrm{Diff}(X)$.

Note that, by the Quotient Manifold Theorem, if G is a Lie group then the quotient X/G is a manifold provided the G-action on X has no non-trivial element of G fixing a point of X. In other words, the space X/G is again a manifold if the action is free. The orbit space X/G is also a smooth manifold of dimension $\dim X - \dim G$ and the quotient map $p : X \to X/G$ is a smooth submersion and a principal G-bundle.

Let now Σ be a closed oriented surface of genus $g \geq 1$ with a finite set of marked points $\mathcal{Z} \subset \Sigma$ of cardinality $n \geq 0$. The *mapping class group* $\mathrm{Mod}(\Sigma, \mathcal{Z})$ of the pair $(\Sigma, \mathcal{Z})$ is the group $\mathrm{Diff}^+(\Sigma, \mathcal{Z})$ of orientation preserving self-diffeomorphisms of Σ that set-wise preserve $\mathcal{Z}$, up to isotopes of Σ leaving $\mathcal{Z}$ invariant. The easiest infinite order mapping classes are the *Dehn twists*, represented by diffeomorphisms of Σ supported on the tubular neighborhood of some simple closed curve, as in Figure 3. Two Dehn twists T_{γ_1} and T_{γ_2} about the simple closed curves γ_1 and γ_2 commute if and only if γ_1 and γ_2 are disjoint, and satisfy the braid relation $T_{\gamma_1} T_{\gamma_2} T_{\gamma_1} = T_{\gamma_2} T_{\gamma_1} T_{\gamma_2}$ if and only if the geometric intersection number of γ_1 and γ_2 is 1.

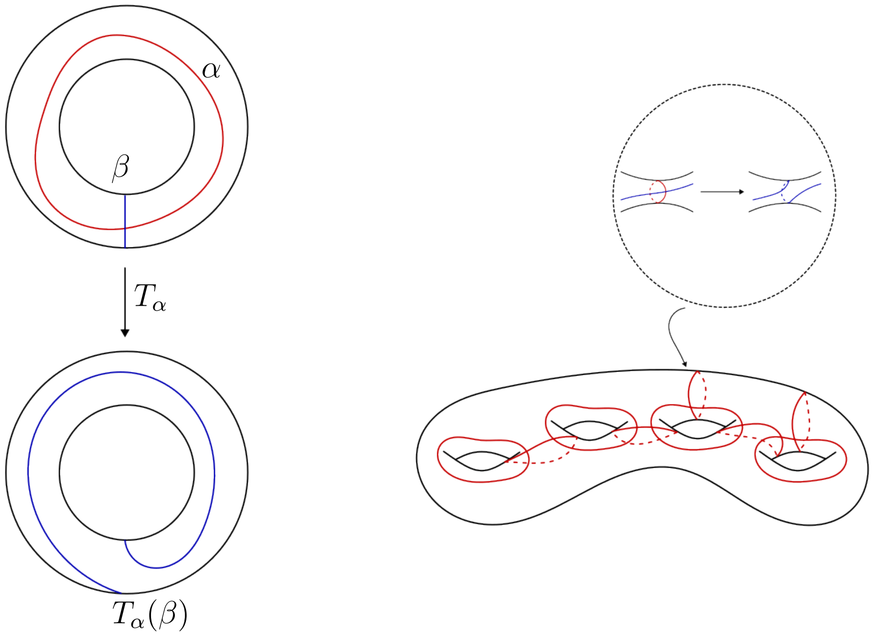


Figure 2.5: The picture on the left represents the action of a Dehn twist about the curve α on the arc β supported on an annulus. On the bottom right-hand side, the family of red simple closed curves represents the Humphries generating set, consisting of Dehn twists about the 9 curves shown. The top right-hand side of the picture shows the action of the Dehn twists about the red curve α on the blue curve β that intersects α once.

Let us fix a pair $(\Sigma, \mathcal{Z})$ of a closed oriented surface of genus g and a finite set of points $\mathcal{Z} \subset \Sigma$ of cardinality $n \geq 0$. A *marking* of a genus g Riemann surface X is an orientation-preserving diffeomorphism $\phi : \Sigma \to X$. The *Teichmüller space* $\mathcal{T}(\Sigma, \mathcal{Z})$ is the set of equivalence classes of pairs (X, ϕ) where X is a closed genus g Riemann surface and $\phi : \Sigma \to X$ is marking, where (X_1, ϕ_1) and (X_2, ϕ_2) are equivalent if there exists a bilohomorphism $I : X_1 \to X_2$ so that

- $\phi_2^{-1} \circ I \circ \phi_1|_{\mathcal{Z}} = id_\Sigma|_{\mathcal{Z}}$ and
- $\phi_2^{-1} \circ I \circ \phi_1 : \Sigma \to \Sigma$ is isotopic to id_Σ relative $\mathcal{Z}$.

Teichmüller spaces can be given a natural topology in terms of *Fenchel–Nielsen coordinates*. We briefly describe the topology of $\mathcal{T}(\Sigma, \mathcal{Z})$ and refer the reader to [FM12, Part 2] for further details. There is a maximum total of $3g - 3 + n$ simple closed curves cutting a closed genus g surface in subsurfaces with the diffeomorphic type of (possibly degenerate) pair of pants or, in other words, spheres with at least one boundary component and exactly three between marked points and boundary components. Let us fix $[(X, \phi)] \in \mathcal{T}(\Sigma, \mathcal{Z})$ and a collection $\{\gamma_1, \dots, \gamma_{3g-3+n}\}$ of simple closed curves cutting Σ in (possibly degenerate) pair of pants. The length parameters $(l_{\gamma_1}([(X, \phi)]), \dots, l_{\gamma_{3g-3+n}}([(X, \phi)]) \in \mathbb{R}_{\geq 0}^{3g-3+n}$, given by the lengths $l_{\gamma_i}([(X, \phi)]) = l_X(\phi(\gamma_i))$ on X for $i = 1, \dots, 3g - 3 + n$, are well-defined on $\mathcal{T}(\Sigma, \mathcal{Z})$. Moreover, every curve γ_i defines a twisting parameter $\theta_{\gamma_i}([X,\phi]) \in \mathbb{R}$, recording how the pair of pants glue together in X.

Theorem 2.2.1 (Fenchel–Nielsen coordinates)**.** Let Σ be a closed, oriented Riemann surface of genus g and $\mathcal{Z} \subset X$ a finite set of n points. The map

$$\begin{aligned}\mathcal{FN} : \mathcal{T}(\Sigma,\mathcal{Z}) &\to \mathbb{R}^{3g-3+n}_{\geq 0} \times \mathbb{R}^{3g-3+n} \\ [(X,\phi)] &\mapsto (l_{\gamma_i}([(X,\phi)]), \theta_{\gamma_i}([X,\phi]))_{i=1}^{3g-3+n}\end{aligned}$$

is a bijection.

We endow $\mathcal{T}(\Sigma,\mathcal{Z})$ with the $\mathbb{R}^{3g-3+n}_{\geq 0} \times \mathbb{R}^{3g-3+n}$ topology given by the Fenchel–Nielsen coordinates. In particular, each Teichmüller space is contractible.

Relative diffeomorphisms between pairs of closed genus g surfaces and finite subsets of points $(\Sigma,\mathcal{Z})$ and $(\Sigma',\mathcal{Z}')$ induce isomorphism of groups $\mathrm{Mod}(\Sigma,\mathcal{Z}) \cong \mathrm{Mod}(\Sigma',\mathcal{Z}')$ and homeomorphisms $\mathcal{T}(\Sigma,\mathcal{Z}) \cong \mathcal{T}(\Sigma',\mathcal{Z}')$. Without loss of generality, we will denote by Mod_g^n and $\mathcal{T}_g^n$ the mapping class group and the Teichmüller space of a genus g closed surface with a finite set of marked points of cardinality n. If $n = 0$, then we will suppress the superscript n and write Mod_g and $\mathcal{T}_g$ instead.

The mapping class groups Mod_g^n act on $\mathcal{T}_g^n$ by precomposiiton on the marking. In particular, the mapping class groups Mod_g^n are discrete groups acting properly on $\mathcal{T}_g^n$ [FM12, Theorem 12.2]. The orbit spaces $\mathcal{M}_g^n = \mathcal{T}_g^n / \mathrm{Mod}_g^n$ are called *moduli spaces of Riemann surfaces* and are good orbifolds parametrizing isomorphism classes of genus g Riemann surfaces with n marked points.

Analogously, we define *Teichmüller strata of translation surfaces*, so to give each stratum $\mathcal{H}(k_1,\ldots,k_n)$ a well-defined topology and good orbifold structure. Let Σ be a closed, oriented surface of genus g. If $k_1,\ldots,k_n \in \mathbb{Z}^+$ is a partition of $2g-2$, a marked translation surface (X,ϕ,ω) consists of a genus g abelian differential (X,ω) with prescribed order of the zeros given by the partition $k_1,\ldots,k_n$[1], and a marking $\phi : \Sigma \to X$. The Teichmüller stratum of translation surfaces $\mathcal{T}(k_1,\ldots,k_n)$ is the set of marked translation surfaces, up to homotopy. In particular, the marked translation surfaces (X_1,ω_1,ϕ_1) and (X_2,ω_2,ϕ_2) are equivalent if there exists a bilomorphism $I : X_1 \to X_2$ so that

- $I^*\omega_2 = \omega_1$;
- $\phi_2^{-1} \circ I \circ \phi_1 : \Sigma \to \Sigma$ is isotopic to id_Σ.

As for the definition of mapping class groups and Teichmüller spaces, we will omit the dependence of the definition of $\mathcal{T}(k_1,\ldots,k_n)$ from the choice of Σ, since diffeomorphic surfaces induce bijections and, ultimately, homeomorphisms with respect to their

[1]From now on, we will write that such an abelian differential is of type $k_1,\ldots,k_n$.

topology defined in the following.

Let $\mathcal{Z} \subset \Sigma$ be a finite set of cardinality n. Suppose the relative homology group $H_1(\Sigma, \mathcal{Z}; \mathbb{Z})$ has a *relative symplectic basis* given by simple closed curves $\gamma_1, \ldots, \gamma_{2g}$ and arcs $\gamma_{2g+i}, \ldots, \gamma_{2g+n-1}$ such that each γ_{2g+i} joins p_i and p_n. In particular, the arcs $\gamma_{2g+i}, \ldots, \gamma_{2g+n-1}$ can be homotoped to be disjoint except at the endpoints, while the simple closed curves $\gamma_1, \ldots, \gamma_{2g}$ have intersection numbers given by $i(\gamma_i, \gamma_{i+1}) = 1$ if i is odd and zero otherwise. Let us fix a triangulation τ of Σ with n vertices corresponding to the set $\mathcal{Z}$. The subset $U_\tau \subset \mathcal{T}(k_1, \ldots, k_n)$ of all equivalence classes of marked translation surfaces (X, ω, ϕ) with marking ϕ taking the edges of τ to saddle connections of (X, ω), is going to be a local patch for $\mathcal{T}(k_1, \ldots, k_n)$.

Proposition 2.2.2. The map

$$\begin{aligned} \mathrm{hol}_\tau : U_\tau &\to \mathbb{C}^{2g+n-1} \\ [(X, \omega, \phi)] &\mapsto \left\{ \int_{\phi(\gamma_i)} \omega \right\}_{i=1}^{2g+n-1} \end{aligned}$$

is injective and has an open image.

Proof. Suppose the marked translation surfaces (X_1, ω_1, ϕ_1) and (X_2, ω_2, ϕ_2) are equivalent. Let Φ be the diffeomorphism $\phi_2 \circ \phi_1^{-1} : X_1 \to X_2$. If T is a triangle of the triangulation $\phi_1(\tau)$ bounded by saddle connections in (X_1, ω_1), then $\Phi(T)$ is a triangle of the triangulation $\phi_2(\tau)$ bounded by saddle connections in (X_2, ω_2), up to isotopy relative endpoints. In particular, we can suppose the triangles T and $\Phi(T)$ to be affine, as the holonomy of the respective edges is the same by hypothesis. Hence, by Proposition 2.1.5 there exists a bilohomorphism $I : X_1 \to X_2$ isotopic to Φ such that $I^*\omega_2 = \omega_1$. In particular, the marked translation surfaces (X_1, ω_1, ϕ_1) and (X_2, ω_2, ϕ_2) are equivalent. To see that $\mathrm{hol}_\tau(U_\tau)$ is open in $\mathbb{C}^{2g+n-1}$, we notice that a complex number z is in $\mathrm{hol}_\tau(U_\tau)$ if and only if the coordinates of z corresponds to the sides of non-degenerate triangles in τ, which is an open condition in $\mathbb{C}^{2g+n-1}$. □

Theorem 2.2.3. Let $k_1, \ldots, k_n$ be a partition of $2g-2$ for $g \geq 1$. The Teichmüller stratum $\mathcal{T}(k_1, \ldots, k_n)$ is a complex manifold of dimension $2g+n-1$.

Proof. Let T be the set of all triangulations of Σ with vertices n vertices corresponding to the set $\mathcal{Z}$. By Proposition 2.1.6, the family of sets $\{U_\tau\}_{\tau \in T}$ covers $\mathcal{T}(k_1, \ldots, k_n)$. We will show that the family $\{\mathrm{hol}_\tau : U_\tau \to \mathbb{C}^{2g+n-1}\}_{\tau \in T}$ defines an atlas of charts of $\mathcal{T}(k_1, \ldots, k_n)$. Suppose two triangulations τ_1 and τ_2 with triangles bounded by curves $\{\gamma_1, \ldots, \gamma_{2g+n-1}\}$ and $\{\delta_1, \ldots, \delta_{2g+n-1}\}$ are such that the corresponding sets U_{τ_1} and U_{τ_2} have non-trivial intersection. The complex coordinates $\{\int_{\phi(\gamma_i)} \omega\}_{i=1}^{2g+n-1}$ and $\{\int_{\phi(\delta_i)} \omega\}_{i=1}^{2g+n-1}$ differ by the action of a matrix in $\mathrm{GL}_{2g+n-1}(\mathbb{C})$. Indeed, the triangulations τ_1 and τ_2 are connected in the flip graph of $(\Sigma, \mathcal{Z})$ and any flip move acts linearly on some saddle connection. □

Theorem 2.2.3 proves that the changes of coordinates are linear maps, as any flip move of a triangulation induces a reflection in $\mathbb{C}^{2g+n-1}$. In particular, each $\mathcal{T}(k_1, \ldots, k_n)$ is an (X, G)-manifold, where $X = \mathbb{C}^{2g+n-1}$ and $G = \mathrm{GL}_{2g+n-1}(\mathbb{C})$.

Lemma 2.2.4. Let X and Y be smooth manifolds, F a discrete subgroup of $\mathrm{Diff}(Y)$ and $G < F$ a subgroup of $\mathrm{Diff}(X)$. If $f : X \to Y$ is continuous and G-equivariant, then X/G is a good orbifold if Y/G is.

Proof. Let K be compact in X. Since Y/G is a good orbifold and $f(K)$ is compact in Y, the inclusions

$$\{g \in G \mid gK \cap K \neq \emptyset\} \subset \{g \in G \mid f(gK \cap K) \neq \emptyset\} \subset \{g \in F \mid gf(K) \cap f(K) \neq \emptyset\} < \infty$$

imply that X/G is a good orbifold, since the set $\{g \in G \mid gf(K) \cap f(K) \neq \emptyset\}$ is finite by hypothesis. □

Proposition 2.2.5. The mapping class group Mod_g acts properly on the Teichmüller strata $\mathcal{T}(k_1, \ldots, k_n)$.

Proof. We only need to show that the Mod_g-equivariant function $f : \mathcal{T}(k_1, \ldots, k_n) \to \mathcal{T}_g$ mapping the equivalence class of triples $[(X, \phi, \omega)]$ in the equivalence class of pairs $[(X, \phi)]$ is well-defined and continuous. Equivalent markings in $\mathcal{T}(k_1, \ldots, k_n)$ respect the same condition of equivalence of markings stated for $\mathcal{T}_g$. Hence, the continuous function f is well-defined and Mod_g-equivariant [Vee90]. □

Corollary 2.2.6. The strata of translation surfaces $\mathcal{H}(k_1, \ldots, k_n)$ are good orbifolds.

We might expect that, as for $\mathcal{M}_g$, the strata of translation surfaces are connected. Surprisingly, this is not always the case. The following two definitions introduce the classification of connected components of strata of translation surfaces, also called the *stratum components*.

An abelian differential (X, ω) is hyperelliptic if there exists a self-bilohomorphism τ of X such that $\tau^2 = id_X$, the quotient $X/\langle\tau\rangle$ is bilohomorphic to the Riemann sphere $\mathbb{CP}^1$ and $\tau^*\omega = -\omega$ holds.

A translation surface is hyperelliptic if it admits a polygonal representation that is symmetric with respect to the π-rotation of the complex plane. More precisely, this is the case if the π-rotation preserves the shape of the polygons, it reverses the orientation of the sides and the quotient by the π-rotation provides a polygonal representation of $\mathbb{CP}^1$. In this case the π-rotation is a bilohomorphic involution and, as the sides of the polygonal representation change orientation, the holonomy vectors $\int_{\gamma_i} \omega$ are mapped to their inverses $-\int_{\gamma_i} \omega$. Any holomorphic 1-form is closed but not exact and $\Omega(X)$ is naturally a subspace of the first cohomology $H^1(X, \mathbb{C})$, which is isomorphic to $\mathrm{Hom}(\pi_1(X), \mathbb{C})$ via the map $\omega \mapsto (\gamma \mapsto \int_\gamma \omega)$. We can then conclude that the action by the pullback of the involution defined by the π-rotation maps ω to $-\omega$.

If (X, ω) is a hyperelliptic abelian differential, then X is a hyperelliptic Riemann surface and the hyperelliptic involution τ acts on any other holomorphic 1-form $\omega \in \Omega(X)$ by pullback as $-\mathrm{Id}$. To see this, note that τ acts linearly on $\Omega(X)$ and can only have ± 1 as eigenvalues since it is an involution. However, any involution acting on a $\mathbb{C}$-vector space is diagonalizable and the dimension d_1 of the eigenspace relative to the eigenvalue 1 and the dimension d_{-1} of the eigenspace relative to the eigenvalue -1 add up to g. One can use a version of the Lefschetz fixed point theorem for differential sheaf cohomology to show that the number of fixed points of τ, that is $2g+2$, is the same as $1 - \mathrm{Trace}(\tau^* : \Omega(X) \to \Omega(X)) + 1 = 2 - (d_1 - d_{-1})$. Hence, we obtain that g is equal to both $d_{-1} - d_1$ and $d_1 + d_{-1}$ so that $g = d_1$.

The classification of stratum components also relies on spin structures associated with holomorphic 1-forms whose zeros have even order. Recall that a divisor D on a Riemann surface X is a finite formal sum

$$D = \sum_{i=1}^{n} k_i p_i, \text{where } p_i \in X \text{ and } k_i \in \mathbb{Z} \text{ for any } i = 1 \ldots, n.$$

The degree $\deg(D)$ of the divisor D is the sum $\sum_{i=1}^{n} k_i$. The divisor D is *effective* if each of the k_i's are non-negative. The divisor D is *canonical* if there exists $\omega \in \Omega(X)$ with zeros $p_1, \ldots, p_n$ of order $k_1, \ldots, k_n$, respectively. In this case, the canonical divisor D will be denoted by $\mathrm{div}(\omega)$.

If X is a closed Riemann surface, there is a one-to-one correspondence

$$\left\{ \begin{matrix} \text{Divisors on X} \\ \text{modulo linear equivalence} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{Meromorphic sections of line bundles on X} \\ \text{modulo isomorphism} \end{matrix} \right\}$$

so that effective divisors correspond to holomorphic sections and the sum of two divisors corresponds to the tensor product of the respective sections [Jos13, Chapter 5].

If (X, ω) is an abelian differential in a stratum of the form $\mathcal{H}(2k_1, \ldots, 2k_n)$, the divisor $\mathrm{div}(\omega)/2$ corresponds to a section of some line bundle $\mathcal{L}$ on X such that $\mathcal{L}^{\otimes 2}$ is the canonical cotangent bundle K_X of X. The line bundle $\mathcal{L}$ is a *spin structure* of (X, ω) and its *parity* is the complex dimension $h^0(X, \mathcal{L}) \mod 2$ of the space of holomorphic sections $X \to \mathcal{L}$.

The following theorem is due to Kontsevich–Zorich [KZ03]. They proved that any stratum $\mathcal{H}(k_1, \ldots, k_n)$ has at most three connected components that are denoted by

$\mathcal{H}^{\text{hyp}}(k_1, \ldots, k_n)$, $\mathcal{H}^{\text{odd}}(k_1, \ldots, k_n)$ and $\mathcal{H}^{\text{even}}(k_1, \ldots, k_n)$, whenever defined. The stratum components $\mathcal{H}^{\text{hyp}}(k_1, \ldots, k_n)$ consists of hyperelliptic translation surfaces and are only defined if the partition $(k_1, \ldots, k_n)$ is of the form $(2g-2)$ or $(g-1, g-1)$. The other stratum components $\mathcal{H}^{\text{odd}}(k_1, \ldots, k_n)$ and $\mathcal{H}^{\text{even}}(k_1, \ldots, k_n)$ are called non-hyperelliptic components and are distinguished by the parity of the spin structures. These are defined only if the terms of the partition $(k_1, \ldots, k_n)$ are all even. A non-hyperelliptic stratum component might still contain hyperelliptic translation surfaces and a stratum component is said to be *totally non-hyperelliptic* if it does not contain hyperelliptic translation surfaces.

Theorem 2.2.7 (Classification of stratum components)**.** The strata of translation surfaces in genus $g = 2$ are connected. The only non-connected strata in genus $g = 3$ are $\mathcal{H}(4)$ and $\mathcal{H}(2, 2)$, which have exactly a hyperelliptic a totally non-hyperelliptic odd spin structure component. If $g \geq 4$, then

- the strata $\mathcal{H}(2g-2)$ have three connected components, namely $\mathcal{H}^{\text{hyp}}(2g-2)$, $\mathcal{H}^{\text{odd}}(2g-2)$ and $\mathcal{H}^{\text{even}}(2g-2)$. The non-hyperelliptic strata are totally non-hyperelliptic;
- if $g-1$ is even, the strata $\mathcal{H}(g-1, g-1)$, have three connected components, namely $\mathcal{H}^{\text{hyp}}(g-1, g-1)$, $\mathcal{H}^{\text{odd}}(g-1, g-1)$ and $\mathcal{H}^{\text{even}}(g-1, g-1)$;
- if $g-1$ is odd, the strata $\mathcal{H}(g-1, g-1)$, have two connected components, namely $\mathcal{H}^{\text{hyp}}(g-1, g-1)$ and a non-hyperelliptic component $\mathcal{H}^{\text{nonhyp}}(g-1, g-1)$;
- all the other strata of the form $\mathcal{H}(2k_1, \ldots, 2k_n)$ have two connected components, distinguished my the parity of the spin structure.

The remaining strata are non-empty and connected.

§ 2.3 | Orbifold fundamental groups and monodromy maps of the strata

Kontsevich–Zorich have classified the connected components of the strata of abelian differentials. The next natural question we might want to answer is how to describe the fundamental groups. However, when studying good orbifolds, orbifold fundamental groups retain many more symmetries than the classical fundamental groups. In the first subsection, we define orbifold fundamental groups for good orbifolds and list the properties we will use in the upcoming chapters. The second subsection will be devoted to the description of the monodromy maps associated to strata of translation surfaces: the topological monodromy, the labelled topological monodromy and the pronged topological monodromy.

Orbifold fundamental groups

Let X/G be a good orbifold and $x_0 \in X$. The *orbifold fundamental group* of X/G at x_0 is the group $\pi_1^{orb}(X/G, x_0)$ of pairs (g, γ), where $g \in G$ and γ is a homotopy class of arcs in X relative the endpoints, connecting x_0 with $g \cdot x_0$. The group operation is given by the composition law

$$(\gamma_1, g_1)(\gamma_2, g_2) = (\gamma_1 * (g_1 \cdot \gamma_2), g_1 g_2),$$

where the operation $*$ stands for the usual concatenation of paths. As for the classic fundamental groups, orbifold fundamental groups with different base points are isomorphic.

Suppose that $\tilde{X}$ is the universal cover of X. Then X/G is an *orbifold classifying space* for a group π if $\tilde{X}$ is contractible and $\pi_1^{orb}(X/G, x_0)$ is isomorphic to π for some (or equivalently for any) $x_0 \in X$. Alternatively, we say that X/G is $K(\pi, 1)$. Such property holds for some stratum components of translation surfaces, namely for the hyperelliptic stratum components, for $\mathcal{H}^{\text{odd}}(4)$, $\mathcal{H}(3,1)$ [LM14, Theorem 1.1] and for $\mathcal{H}^{\text{even}}(6)$ (Theorem B).

If G is a discrete group acting properly and freely on a manifold X by diffeomorphisms, then the quotient map $p : X \to X/G$ is a covering map, the orbifold fundamental group of X/G at $x_0 \in X$ is isomorphic to $\pi_1(X/G, p(x_0))$ and to the deck transformation group of the cover $\tilde{X} \to X/G$. The map $\tilde{X} \to X/G$ is not a covering map if G does not act freely on X, but the deck transformation group $\mathrm{Deck}(\tilde{X} \to X/G)$ can still be defined.

Proposition 2.3.1. Let X/G be a good orbifold and $x_0 \in X$. Then $\pi_1^{orb}(X/G, x_0)$ is isomorphic to $\mathrm{Deck}(\tilde{X} \to X/G)$.

Proof. Let $\tilde{x_0} \in \tilde{X}$ be a lift of x_0. If γ is an homotopy class of arcs from x_0 to $g \cdot x_0$ for some $g \in G$, then there exists a unique lift $\tilde{\gamma}$ of γ such that $\tilde{\gamma}(0) = \tilde{x_0}$. Similarly, there exists a unique lift $d_\gamma : \tilde{X} \to \tilde{X}$ of the universal covering map $p : \tilde{X} \to X$ such that $d_\gamma(\tilde{x_0}) = \tilde{\gamma}(1)$. The map d_γ commutes with $\tilde{X} \to X/G$ and is an element of $\mathrm{Deck}(\tilde{X} \to X/G)$. We want to show that

$$\pi_1^{orb}(X/G, x_0) \to \mathrm{Deck}(\tilde{X} \to X/G)$$
$$(\gamma, g) \mapsto d_\gamma$$

is an isomorphism of groups. It is easy to see that for any two pairs $(\gamma_1, g_1), (\gamma_2, g_2) \in \pi_1^{orb}(X/G, x_0)$, the deck transformation $d_{\gamma_1 * (g_1 \cdot \gamma_2)}$ is the composition $d_{\gamma_1} \circ d_{\gamma_2}$, since the unique lift associated to $\gamma_1 * (g_1 \cdot \gamma_2)$ is $\tilde{\gamma_1} * (d_{\gamma_1} \cdot \tilde{\gamma_2})$. If $d_\gamma = id$ then $\tilde{\gamma}(1) = x_0$ and $\tilde{\gamma}$ is a loop in $\tilde{X}$, which is simply connected and therefore γ is homotopic to the constant path. Surjectivity can be proved as follows. If d is a deck transformation of $\tilde{X} \to X/G$, then consider the homotopy class $\tilde{\gamma}$ of arcs from $\tilde{x_0}$ to $d(\tilde{x_0})$. Since d is a deck transformation, there is $g \in G$ such that $p(\tilde{\gamma})(1) = g \cdot x_0$. □

Suppose X is both path-connected and simply connected. In this case, the orbifold fundamental group $\pi_1^{orb}(X/G, x_0)$ is isomorphic to G via the map projecting any pair (γ, g) to g. Indeed, the kernel of the projection $(\gamma, g) \mapsto g$ is the fundamental group of X based at x_0, while a group element g is in the image if there exists a path from x_0 to $g \cdot x_0$.

The orbifold fundamental group of a good orbifold, in general, retains more information about the group action than the fundamental group. For example, if τ is a reflection in $\mathbb{C}$, the quotient group $\mathbb{C}/\langle\tau\rangle$ is a proper good orbifold since τ has a one-dimensional fixed locus. The orbifold $\mathbb{C}/\langle\tau\rangle$ has an orbifold fundamental group isomorphic to $\mathbb{Z}_2$ since $\mathbb{C}$ is connected and simply connected. However, the quotient $\mathbb{C}/\langle\tau\rangle$ is also simply connected: if, for example, $\tau(z) = \overline{z}$, then $\mathbb{C}/\langle\tau\rangle$ is homeomorphic to the closure of the upper half-plane in $\mathbb{C}$.

Another example comes from the moduli space of Riemann surfaces $\mathcal{M}_g$, that has a contractible universal cover and the orbifold fundamental group $\pi_1^{orb}(\mathcal{M}_g, (X, \phi))$ is isomorphic to Mod_g. However, the moduli space $\mathcal{M}_g$ is simply connected [Mac71].

The orbifold fundamental group of a good orbifold can be expressed as the classical fundamental group of a specific topological space. We first recall the notion of *semi-simplicial* set. Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of sets. Each set S_n is given with the discrete topology and called the *set of n-simplices*. If there are functions $d_n^i : S_{n+1} \to S_n$ for any $n \in \mathbb{N}$ and $i = 0, \dots, n+1$ such that $d_n^i \circ d_{n+1}^j = d_n^j \circ d_{n+1}^i$ for $i < j$, then the semi-simplicial set of the collection $\{S_n\}_{n\in\mathbb{N}}$ is the geometric realization

$$\Big(\coprod_{n=0}^{\infty} S_n \times \Delta^n\Big)/\sim,$$

where Δ^n is the standard n-simplex and $(\sigma, \iota^i t) \sim (d_i\sigma, t)$ for all $\sigma \in S_n, t \in \Delta^{n-1}$ and $\iota^i \colon \Delta^{n-1} \to \Delta^n$ inclusion of the i^{th} face.

Let now G be a topological group and EG the semi-simplicial set obtained as follows. The n-simplices are the ordered $(n+1)$ tuples of elements of G of the form $[g_0, \dots, g_n]$ and such n-simplices are naturally attached to the $(n-1)$-simplices $[g_0, \dots, \hat{g}_i, \dots, g_n]$, where $\hat{g}_i$ denotes that this vertex has been deleted. The group G acts on the simplices by left multiplication on the vertices

$$g \cdot [g_0, \dots, g_n] = [gg_0, \dots, gg_n].$$

The semi-simplicial complex EG is contractible. Indeed, there is a homotopy dragging every point $x \in [g_0, \dots, g_n]$ to the vertex $[e]$ of the identity element, along the straight segment in $[e, g_0, \dots, g_n]$ from x to $[e]$. Moreover, the group G acts freely and simplicially on EG. Hence, if BG denotes the orbit space EG/G, the quotient map $EG \to BG$ is a fiber bundle with fibers diffeomorphic to G. The associated long exact sequence can be used to show that $\pi_n(BG, *) \cong \pi_{n-1}(G, e)$ for every point $* \in BG$. If now X/G is a good orbifold, the *Borel construction* X_G is the orbit space obtained form the diagonal G-action on $EG \times X$.

Proposition 2.3.2. Let X/G be a good orbifold and $p \in X$. If X is connected, the orbifold fundamental group $\pi_1^{orb}(X/G, p)$ is isomorphic to $\pi_1(X_G)$.

Proof. The universal cover of X_G is homeomorphic to the product $EG \times \tilde{X}$ since EG is contractible. Moreover, if $g \in G$ and $d \in \mathrm{Deck}(\tilde{X} \to X)$, then (g, d) is deck transformation of $EG \times \tilde{X} \to X_G$. Let x be in EG. The homomorphism

$$\pi_1^{orb}(X/G, p) \to \pi_1(X_G, [(x, p)])$$
$$(g, \gamma) \mapsto (g, d_\gamma)$$

is an isomorphism. While injectivity is immediate, surjectivity follows from the existence and uniqueness of lifts of $EG \times \tilde{X} \to X_G$. □

Corollary 2.3.3. Let X/G be a good orbifold and $p \in X$. If X is connected, there is an exact sequence

$$\pi_2(X_G) \to \pi_1(G, e) \to \pi_1(X, p) \to \pi_1^{orb}(X/G, p) \to \pi_0(G) \to 1.$$

Proof. The claim follows from Proposition 2.3.2, since the map $X_G \to BG$ is a fiber bundle with fibers diffeomorphic to X. □

Monodromy maps

Hyperelliptic connected components are topologically well understood.

Theorem 2.3.4. [LM14, Introduction] The strata $\mathcal{H}^{\mathrm{hyp}}(2g-2)$ and $\mathcal{H}^{\mathrm{hyp}}(g-1, g-1)$ are isomorphic, as good orbifolds, to the quotient of configuration spaces of points on $\mathbb{CP}^1$ by the action of the group of $(2g+1)^{st}$ and $(2g+2)^{nd}$ roots of unity, respectively.

The topology of the non-hyperelliptic stratum components proves to be more intricate to study, and few results are available in the literature. However, *topological monodromy maps* can help describe the isomorphism type of orbifold fundamental groups.

Morphisms of good orbifolds induce group homomorphisms between orbifold fundamental groups. Let X/G and Y/F be good orbifolds and let us fix the points $x_0 \in X$ and $y_0 \in Y$. Suppose there exists a group homomorphism $\varphi : G \to F$ and a $\varphi-$equivariant continuous map $f : X \to Y$. Then, the map f induces a group homomorphism

$$\begin{aligned}\pi_1^{orb}(f) : \pi_1^{orb}(X/G, x_0) &\to \pi_1^{orb}(Y/F, y_0)\\ (\gamma, g) &\mapsto (f(\gamma), \phi(g)).\end{aligned}$$

Let now T_0 be a marked translation surface in some Teichmüller stratum $\mathcal{T}(k_1, \ldots, k_n)$ and let $\mathcal{C}$ be the unique connected component of $\mathcal{H}(k_1, \ldots, k_n)$ containing (X, ω). Since the forgetful map $\mathcal{T}(k_1, \ldots, k_n) \to \mathcal{T}_g$ is continuous [Vee90] and Mod_g-equivariant, there exists an induced π_1^{orb}-homomorphism

$$\rho_{\mathcal{C}} : \pi_1^{orb}(\mathcal{C}, T_0) \to \pi_1^{orb}(\mathcal{M}_g, (X, \phi)) \cong \mathrm{Mod}_g,$$

the *topological monodromy.* The image of the topological monodromy map of a stratum component $\mathcal{C}$ describe the maximal subgroup of Mod_g acting on $\mathcal{TC}$ so that its quotient is $\mathcal{C}$ and Calderon–Salter described $\mathrm{Im}(\rho_{\mathcal{C}})$ for non-hyperelliptic connected components in genus $g \geq 3$ as *higher-spin mapping class groups* [CS22].

If ξ is a vector field on a Riemann surface X vanishing on a set of finitely many points $\mathcal{Z}$, for any homotopy class γ of oriented smooth loops in $X \setminus \mathcal{Z}$, we denote by $\phi_\xi(\gamma)$ the winding number of γ with respect to the vector field ξ. The integer $\phi_\xi(\gamma)$ counts the number of times the tangent vector $d\gamma$ turns about ξ.

Let Σ be a genus g Riemann surface and $\mathcal{Z}$ a finite set of n marked points on X. If $\phi : \Sigma \to X$ is a marking for X, the mapping class group $\mathrm{Mod}(\Sigma, \mathcal{Z})$ is isomorphic to $\mathrm{Mod}(X, \phi(\mathcal{Z}))$ through the isomorphism induced by ϕ. If ξ is a vector field on X vanishing on $\phi(\mathcal{Z})$, then the mapping class group Mod_g^n acts of the winding number function ϕ_ξ as

$$f \cdot \phi_\xi(\gamma) = \phi_\xi(f^{-1}(\gamma))$$

for any homotopy class of smooth curves in $X \setminus \phi(\mathcal{Z})$ and $f \in \mathrm{Mod}_g^n$. The framed mapping class group $\mathrm{Mod}_g^n(\phi_\xi)$ is the stabilizer of ϕ_ξ in Mod_g^n.

Proposition 2.3.5. Let ξ be a non-vanishing vector field on $\Sigma \setminus \mathcal{Z}$. Then, the following properties hold for the winding number function ϕ_ξ.

(1) (Normalization) If z a non-essential curve on $\Sigma \setminus \mathcal{Z}$, then $\phi_s(z) = 1$;

(2) (Twist-linearity) if δ and γ are oriented simple closed curves and $T_\delta(\gamma)$ is the Dehn twist of γ about δ, then $\phi_\xi(T_\delta(\gamma)) = \phi_\xi(\gamma) + \phi_\xi(\gamma) i(\gamma, \delta)$, where $i(\gamma, \delta)$ denotes the algebraic intersection pairing;

(3) (Homological Coherence) if the curves $\gamma_1, \ldots, \gamma_m$ bound a subsurface $S \subset \Sigma \setminus \mathcal{Z}$ such that S lies to the left of each curve, then

$$\sum_{i=1}^{n} \phi_\xi(\gamma_i) = \chi(S).$$

Any translation surface (X, ω) can be associated with a *non-vanishing horizontal unit vector field* ξ_ω on the complement of the zeros of ω. Indeed, for any $x \in \Sigma \setminus \mathcal{Z}$, one can define $\xi_\omega(x)$ to be the unique unit tangent vector such that $\omega(\xi_\omega(x))$ is a positive real number. Moreover, if $\phi : \Sigma \to X$ is a marking, then the vector field ξ can be pulled back by ϕ to $\Sigma \setminus \mathcal{Z}$.

Note that the horizontal foliation of a translation surface inherited by its polygonal representations consists of the integral curves of its non-vanishing horizontal unit vector field. At each cone point p_i with multiplicity k_i, the winding number of a small loop γ_i oriented counterclockwise around p_i has $\phi_\xi(\gamma_i) = k_i + 1$, that is the number of half-planes glued around p_i or, equivalently, the number of tangency points between the integral curves of ξ and $d\gamma_i$. In particular, the function ϕ_ξ is not well-defined for homotopy classes of oriented curves on Σ, as the value of a small essential loop in $\Sigma \setminus \mathcal{Z}$ is 1, while the value of any γ_i is $k_i + 1$. However, if $r = \gcd(k_1, \ldots, k_n)$, then the *r-spin structure* $\phi_\xi \mod r$ is well-defined on the set of homotopy classes of oriented curves on Σ [Cal20, Section 4.1].

Consider the forgetful map $F : \mathrm{Mod}_g^n \to \mathrm{Mod}_g$ realized by forgetting the marked points. If ξ is a vector field on Σ as above, then the *r-spin mapping class group* $\mathrm{Mod}_g(\phi_\xi)$ is the image through F of the associated framed mapping class group $\mathrm{Mod}_g^n(\phi_\xi)$.

Theorem 2.3.6. Let $\mathcal{C}$ be a non-hyperelliptic connected component of a stratum $\mathcal{H}(k_1, \ldots, k_n)$ in genus $g \geq 3$ and $\mathcal{TC}$ its Teichmüller cover in $\mathcal{T}(k_1, \ldots, k_n)$. Let $T_0 = (X, \omega, \phi)$ be a marked translation surface in $\mathcal{TC}$. The image of the topological monodromy

$$\rho_{\mathcal{C}} : \pi_1^{orb}(\mathcal{C}, T_0) \to \mathrm{Mod}_g$$

is the r-spin mapping class group $\mathrm{Mod}_g(\phi_\xi)$ associated to the vector field ξ on (X, ω, ϕ). The index of $\mathrm{Mod}_g(\phi_\xi)$ is finite.

§ 2.4 | Finer Teichmüller covers

The topology of the non-hyperelliptic stratum components can be understood from the orbifold structure. Generally, the monodromy $\rho : \pi_1^{orb}(X/G, x_0) \to G$, given by $(\gamma, g) \mapsto g$, provides some useful topological data.

Lemma 2.4.1. Let X/G be a good orbifold and $x_0 \in X$. Let $\rho : \pi_1^{orb}(X/G, x_0) \to G$ be the associated monodromy in G. Then:

- X has $[G : \mathrm{Im}\,\rho]$ many connected components;
- the kernel of ρ is isomorphic to the fundamental group of the unique connected component of X containing the base point x_0.

Proof. If Ω is the connected component containing x_0, the cardinality of the orbit set

$\{f \cdot \Omega \mid f \in G\}$ coincides with the number of connected components of X and by the Orbit-Stabilizer theorem this is the index of the stabilizer of Ω in G. If f acts trivially on Ω there exists a path connecting x_0 to $f \cdot x_0$. Hence, the group element f is in the image of the monodromy ρ. Viceversa, if $f \in \operatorname{Im} \rho$ then there is a path in Ω connecting x_0 to $f \cdot x_0$ and for any $x \in \Omega$ there exists a path δ with endpoints x and x_0. The concatenation $\delta * \gamma * (f \cdot \delta)$ connects x with $f \cdot x$ and we can conclude that f stabilizes Ω. To prove the second claim, suppose $(\gamma, f) \in \pi_1^{orb}(X/G, x_0)$ and $\rho(\gamma, f) = id_G$, then $f = id_G$ and γ is the homotopy class of a loop based at x_0; vice-versa, any loop γ in X based at x_0 can be seen as an element in $\ker \rho$ as (γ, id_G). □

By Theorem 2.3.6, the images of the topological monodromy $\rho_{\mathcal{C}}$ are higher-spin mapping class groups and have finite index. The Calderon–Salter result turns out to be a topological statement on the connected components of the Teichmüller non-hyperelliptic strata in genus $g \geq 3$. Then, by the previous lemma, it is natural to ask if the topological monodromies $\rho_{\mathcal{C}}$ of non-hyperelliptic stratum components are injective or, equivalently, the connected components of $\mathcal{TC}$ are simply connected.

In the first subsection, we prove that any connected component of the Teichmüller lifts $\mathcal{TC}$ of non-hyperelliptic strata in genus $g \geq 3$ is far from being universal, as the fundamental groups contain a non-abelian free group of rank 2. We will define two additional covering maps, the Teichmüller strata of labelled translation surfaces $\mathcal{T}\tilde{\mathcal{C}}$ and the Teichmüller strata of blown-up translation surfaces $\mathcal{T}\hat{\mathcal{C}}$. Both $\mathcal{T}\tilde{\mathcal{C}}$ and $\mathcal{T}\hat{\mathcal{C}}$ fit in a sequence of covers

$$\mathcal{T}\hat{\mathcal{C}} \to \mathcal{T}\tilde{\mathcal{C}} \to \mathcal{TC} \to \mathcal{C}$$

and both come with their own monodromy maps. The kernel of the monodromy associated with $\mathcal{T}\tilde{\mathcal{C}}$ is known to have a non-trivial kernel, as it contains a copy of $\mathbb{Z}$. However, we don't know if the Large Kernel Theorem holds in general or, in other words, if the kernel of the monodromy associated with $\mathcal{T}\tilde{\mathcal{C}}$ contain a non-abelian free group of rank 2. Nothing about the kernels of the monodromy of $\mathcal{T}\hat{\mathcal{C}}$ is known.

Teichmüller strata of labelled translation surfaces

Even if $\mathcal{TC}$ is not connected, one might restrict to a connected component and hope that, as in the case of the Teichmüller cover $\mathcal{T}_g$ for the moduli space of Riemann surfaces $\mathcal{M}_g$, the component is simply-connected and therefore a universal cover for $\mathcal{C}$. This is never the case if $g \geq 3$.

Theorem 2.4.2 (Large Kernel Theorem for $\rho_{\mathcal{C}}$)**.** If $g \geq 3$ and $\mathcal{C}$ is a non-hyperelliptic stratum component, than $\pi_1(\mathcal{TC}, T_0)$ contains $\mathbb{F}_2$, a non-abelian free group of rank 2. In particular, the kernel of the monodromy $\rho_{\mathcal{C}}$ contains a copy of $\mathbb{F}_2$.

To prove Theorem 2.4.2, we will define a new orbifold structure. Let Σ be a closed, oriented Riemann surface of genus g and $\mathcal{Z} = \{p_1, \ldots, p_n\} \subset X$ a finite set of points. If

$k_1, \dots, k_n \in \mathbb{Z}^+$ is a partition of $2g-2$, a labelled marked translation surface (X, ω, ϕ) consists of a genus g abelian differential (X, ω) with prescribed order of the zeros given by the partition $k_1, \dots, k_n$, and a marking $\phi : \Sigma \to X$ such that $\operatorname{ord}_{\phi(p_i)} \omega = k_i$ for $i = 1, \dots, n$. The *Teichmüller stratum of labelled translation surfaces* $\mathcal{T}^{lab}(k_1, \dots, k_n)$ is the set of marked translation surfaces up to homotopy fixing $\mathcal{Z}$ point-wise. That is, two labelled marked translation surfaces (X_1, ω_1, ϕ_1) and (X_2, ω_2, ϕ_2) are equivalent if there exists a bilomorphism $I : X_1 \to X_2$ so that

- $I^*\omega_2 = \omega_1$;
- $\phi_2^{-1} \circ I \circ \phi_1|_{\mathcal{Z}} = id_\Sigma|_{\mathcal{Z}}$ and
- $\phi_2^{-1} \circ I \circ \phi_1 : \Sigma \to \Sigma$ is isotopic to id_Σ relative $\mathcal{Z}$.

The topology of each of the labelled Teichmüller strata $\mathcal{T}^{lab}(k_1, \dots, k_n)$ can be defined analogously as the topology of the Teichmüller strata $\mathcal{T}(k_1, \dots, k_n)$ via period coordinates. The action of the mapping class group Mod_g^n on a labelled Teichmüller stratum $\mathcal{T}^{lab}(k_1, \dots, k_n)$ does not necessarily fix points-wise the labelled points $\mathcal{Z} = \{p_1, \dots, p_n\}$. However, the *pure mapping class group* $\operatorname{PMod}(\Sigma, \mathcal{Z})$, the set of orientation-preserving self-diffeomorphisms of Σ that pointwise preserve $\mathcal{Z}$ up to isotopy relative $\mathcal{Z}$, fixes $\mathcal{Z}$ pointwise. Similarly as in the previous cases, we will denote $\operatorname{PMod}(\Sigma, \mathcal{Z})$ by PMod_g^n. The pure mapping class group PMod_g^n acts properly on $\mathcal{T}^{lab}(k_1, \dots, k_n)$ and the orbit space $\mathcal{T}^{lab}(k_1, \dots, k_n)/\operatorname{PMod}_g^n$ coincide with a finite cover of the stratum $\mathcal{H}(k_1, \dots, k_n)$.

We denote by $\mathcal{H}^{lab}(k_1, \dots, k_n)$ the space of translation surfaces (X, ω) of type $(k_1, \dots, k_n)$, such that (X_1, ω_1) is equivalent to (X_2, ω_2) if and only if there exists a bilohomorphism $I : X_1 \to X_2$ that preserves point-wise the set of singularities and such that $I^*\omega_2 = \omega_1$. The space $\mathcal{H}^{lab}(k_1, \dots, k_n)$ is the quotient of $\mathcal{T}^{lab}(k_1, \dots, k_n)$ by the action of PMod_g^n.

Suppose $\underline{k} = (k_1, \dots, k_m)$ is a partition of $2g-2$, such that every k_i appears exactly r_i times. The notation $(k_1^{r_1}, \dots, k_m^{r_m})$ stands for

$$(\underbrace{k_1, \dots, k_1}_{r_1 \text{ times}}, \dots, \underbrace{k_m, \dots, k_m}_{r_m \text{ times}}),$$

and $\sum_{i=1}^m r_i = n$. The product of the symmetric groups $\prod_{i=1}^m \operatorname{Sym}(r_i)$, denoted by $\operatorname{Sym}(\underline{k})$, acts on the Teichmüller space $\mathcal{T}^{lab}(\underline{k})$ permuting the labels on the set $\mathcal{Z}$. The shot exact sequence

$$1 \to \operatorname{PMod}_g^n \to \operatorname{Mod}_g^n \to \operatorname{Sym}(n) \to 1$$

is split and we can consider $\operatorname{PMod}_g^n \rtimes \operatorname{Sym}(\underline{k})$ as a subgroup of Mod_g^n. Then, the quotient of the labelled Teichmüller space $\mathcal{T}^{lab}(k_1, \dots, k_m)$ by the action of

$\mathrm{PMod}_g^n \rtimes \mathrm{Sym}(\underline{k})$ corresponds to the stratum $\mathcal{H}(\underline{k})$. The symmetric group $\mathrm{Sym}(\underline{k})$ acts freely on $\mathcal{T}^{lab}(\underline{k})$ and therefore the quotient map $\mathcal{T}^{lab}(\underline{k}) \to \mathcal{T}^{lab}(\underline{k})/\mathrm{Sym}(\underline{k})$ is a finite covering map. Hence, the forgetful functions $\mathcal{T}^{lab}(\underline{k}) \to \mathcal{T}(\underline{k})$ are covering maps with deck transformation groups $\mathrm{Sym}(\underline{k})$.

The finer orbifold structure of $\mathcal{H}(\underline{k})$ given by the Teichmüller spaces $\mathcal{T}^{lab}(\underline{k})$ define new topological monodromies. Let $\tilde{T}_0 \in \mathcal{T}^{lab}(\underline{k})$ be a marked translation surface. The forgetful map $\mathcal{T}^{lab}(\underline{k}) \to \mathcal{T}_g^n$ is continuous and equivariant with respect to the inclusion of $\mathrm{PMod}_g^n \rtimes \mathrm{Sym}(\underline{\lambda})$ in Mod_g^n. Then, there exists an induced π_1^{orb}-homomorphism

$$\tilde{\rho}_{\mathcal{C}} : \pi_1^{orb}(\mathcal{C}, \tilde{T}_0) \to \mathrm{Mod}_g^n,$$

the *labelled topological monodromy.*

The image of the labelled topological monodromy $\tilde{\rho}_{\mathcal{C}}$ is a subgroup of the semidirect product $\mathrm{PMod}_g^n \rtimes \mathrm{Sym}(\underline{k})$. Let $\tilde{\mathcal{C}}$ be a connected component of $\mathcal{H}^{lab}(k_1, \dots, k_n)$ lying over some stratum components $\mathcal{C}$. If $\tilde{T}_0 = (X, \phi, \omega) \in \mathcal{T}^{lab}(\underline{k})$ is a labelled marked translation surface with image $T_0 \in \mathcal{T}(\underline{k})$ and such that $(X, \omega) \in \tilde{\mathcal{C}}$, there is a short exact sequence

$$1 \to \pi_1^{orb}(\tilde{\mathcal{C}}, \tilde{T}_0) \to \pi_1^{orb}(\mathcal{C}, T_0) \to \mathrm{Sym}(\underline{k}) \to 1$$

induced by the homomorphism $\pi_1^{orb}(\mathcal{C}, T_0) \to \mathrm{Sym}(\underline{k})$ mapping each $(\gamma, f) \in \pi_1^{orb}(\mathcal{C}, T_0)$ to the respective permutation of $f \in \mathrm{Mod}_g^n$ on the labels of $\mathcal{Z}$ [Boi13]. The orbifold fundamental group $\pi_1^{orb}(\tilde{\mathcal{C}}, \tilde{T}_0)$ can be then thought of as a finite index subgroup of $\pi_1^{orb}(\mathcal{C}, T_0)$. The restriction of the labelled topological monodromy $\tilde{\rho}_{\mathcal{C}}$ to the copy of $\pi_1^{orb}(\tilde{\mathcal{C}}, T_0^{lab})$ has its image in the pure mapping class group $\mathrm{PMod}_g^n \triangleleft \mathrm{Mod}_g^n$, as any two marked translation surfaces (X, ϕ_1, ω) and (X, ϕ_2, ω) in $\mathcal{T}^{lab}(\underline{k})$ that are connected by an arc differ by the action of the mapping class $\phi_1^{-1}\phi_2 \in \mathrm{PMod}_g^n$.

The kernel of topological monodromy $\rho_{\mathcal{C}}$ is isomorphic to the fundamental group of $\mathcal{TC}$, Similarly, the kernel of the labelled topological monodromy $\tilde{\rho}_{\mathcal{C}}$ is isomorphic to $\pi_1(\mathcal{T}\tilde{\mathcal{C}}, \tilde{T}_0)$, where $\mathcal{T}\tilde{\mathcal{C}}$ denotes the set of labelled marked translation surfaces $(X, \omega, \phi) \in \mathcal{T}^{lab}(\underline{k})$ such that $(X, \omega) \in \tilde{\mathcal{C}}$.

Calderon–Salter have described the images of the labelled topological monodromies and proved that these have infinite index if $g \geq 5$ [CS22, Theorem A].

Theorem 2.4.3. Let $\mathcal{C}$ be a non-hyperelliptic connected component of a stratum $\mathcal{H}(k_1, \dots, k_n)$ in genus $g \geq 5$ and $\tilde{\mathcal{C}}$ a connected component lying over $\mathcal{C}$ in $\mathcal{H}^{lab}(k_1, \dots, k_n)$. Let $\tilde{T}_0 \in \mathcal{T}\tilde{\mathcal{C}}$ be a labelled marked translation surface and ξ the associated horizontal vector field. The image of the labelled topological monodromy

$$\tilde{\rho}_{\mathcal{C}} : \pi_1^{orb}(\mathcal{C}, \tilde{T}_0) \to \mathrm{Mod}_g^n$$

and its restiction

$$\tilde{\rho}_{\mathcal{C}} : \pi_1^{orb}(\tilde{\mathcal{C}}, \tilde{T}_0) \to \mathrm{PMod}_g^n$$

are, respectively, the framed mapping class group $\mathrm{Mod}_g^n(\phi_\xi)$ and the stabilizer of the absolute framing ϕ_ξ in PMod_g, denoted by $\mathrm{PMod}_g(\phi_\xi)$. The index of both $\mathrm{Mod}_g(\phi_\xi)$ and $\mathrm{PMod}_g(\phi_\xi)$ is infinite.

We will prove that the fundamental groups of each connected component of $\mathcal{TC}$, when $\mathcal{C}$ is a non-hyperelliptic stratum component in genus $g \geq 3$, are infinite and contain a non-abelian free group of rank 2. Let us consider the following commutative diagram.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathcal{TC}, T_0) & \longrightarrow & \pi_1^{orb}(\mathcal{C}, T_0) & \xrightarrow{\rho_{\mathcal{C}}} & \mathrm{Mod}_g(\phi_\xi) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \tilde{\rho}_{\mathcal{C}}} & & \downarrow & & \\ 1 & \longrightarrow & \ker(F) & \longrightarrow & \mathrm{Mod}_g^n & \xrightarrow{F} & \mathrm{Mod}_g & \longrightarrow & 1. \end{array}$$

The vertical map on the left is the restriction of $\tilde{\rho}_{\mathcal{C}}$ to the kernel of $\rho_{\mathcal{C}}$ and detects loops in the connected component of the basepoint $T_0 \in \mathcal{TC}$ that drag the set of marked points $\mathcal{Z}$ back on the reference topological surface Σ. Indeed, the kernel of the forgetful map $F : \mathrm{Mod}_g^n \to \mathrm{Mod}_g$ restricted to PMod_g^n, is the fundamental group of the configuration space of $n = |\mathcal{Z}|$ un-ordered points of Σ.

Let Σ be a closed and oriented surface of genus g. Denote by $\mathrm{Conf}(\Sigma, n)$ the configuration set of n ordered points on Σ, that is the set $\{(p_1, \dots, p_n) \in \Sigma^n \mid p_i \neq p_j \text{ for } i \neq j\}$. If $\mathcal{Z}$ is a finite set of n points on Σ, the *pure surface braid group* of the pair $(\Sigma, \mathcal{Z})$ is the group $\pi_1(\mathrm{Conf}(\Sigma, n), \mathcal{Z})$ denoted by $\mathcal{PB}_g^n$. The *surface braid group* $\mathcal{SB}_g^n$ of the pair $(\Sigma, \mathcal{Z})$ is the fundamental group $\pi_1(\mathrm{Conf}(\Sigma, n)/\mathrm{Sym}(n), [\mathcal{Z}])$. The pure surface braid group $\mathcal{PB}_g^n$ is the fundamental group of the configuration set of n ordered points on Σ, based at $\mathcal{Z}$, while the surface braid group $\mathcal{SB}_g^n$ is the fundamental group of the configuration set of n un-ordered points on Σ.

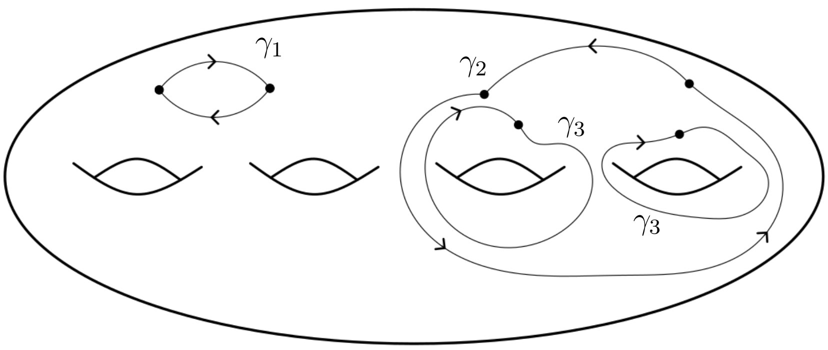


Figure 2.6: A surface braid on a genus 4 surface. The sum of the isotopy classes of the curves $\gamma_1, \gamma_2, \gamma_3$ and γ_4 is trivial in homology.

Theorem 2.4.4. [FM12, Theorem 9.1] Let Σ be an oriented closed surface of genus

$g \geq 2$. Then, the following sequence is exact

$$1 \to \mathcal{SB}_g^n \to \mathrm{Mod}_g^n \xrightarrow{F} \mathrm{Mod}_g \to 1$$

and the restriction of F to PMod_g^n is $\mathcal{PB}_g^n$.

Any loop in a configuration space $\mathrm{Conf}(\Sigma, n)$ identifies a collection of simple closed loops on Σ. The inclusion of $\mathcal{PB}_g^n$ in PMod_g^n can be described as follows. Let $\sigma \in \mathcal{PB}_g^n$ be a pure surface braid and $\gamma_1, \ldots, \gamma_n$ the loops traced by σ, such that each γ_i is based at a distinct point $p_i \in \mathcal{Z}$. For every $i = 1, \ldots, n$, let $\gamma_{L,i}$ and $\gamma_{R,i}$ be the curves obtained from γ_i by pushing γ off itself to the left and the right, respectively. Then, the *push map* $\mathcal{P} : \mathcal{PB}_g^n \to \mathrm{PMod}_g^n$ given by $\mathcal{P}(\sigma) = \prod_{i=1}^n T_{\gamma_{L,i}} T_{\gamma_{R,i}}^{-1}$ is injective and fits inside the Generalized Birman exact sequence.

The restriction of the labelled topological monodromy $\tilde{\rho}_{\mathcal{C}}$ to the normal subgroup $\pi_1(\mathcal{TC}, T_0)$ defines a group homomorphism $\pi_1(\mathcal{TC}, T_0) \to \mathcal{SB}_g^n$ as in the commutative diagram above. A priori, the homomorphism might be trivial, and no loop in $\mathcal{TC}$ might drag non-trivially the set of points $\mathcal{Z}$ back. Our goal is to show that $\pi_1(\mathcal{TC}, T_0)$ is non-trivial, so to conclude that the connected components of $\mathcal{TC}$ are not universal. We will prove more: the image of $\pi_1(\mathcal{TC}, T_0)$ in $\mathcal{SB}_g^n$ contains a non-abelian free group of rank 2 and the monodromies $\rho_{\mathcal{C}}$ have the so called *Large Kernel Property*. Without loss of generality, since $\pi_1^{orb}(\tilde{\mathcal{C}}, T_0)$ is a finite index subgroup of $\pi_1^{orb}(\mathcal{C}, T_0)$, we will consider the restriction of the monodromy $\tilde{\rho}_{\mathcal{C}}$ to the subgroup of $\pi_1(\mathcal{TC}, T_0) \cap \pi_1^{orb}(\tilde{\mathcal{C}}, \tilde{T}_0)$, which can be described as the fundamental group $\pi_1(\mathcal{T}\tilde{\mathcal{C}}, \tilde{T}_0)$ of the Teichmüller cover of $\tilde{\mathcal{C}}$ based at some point $\tilde{T}_0$.

The image in $\mathcal{PB}_g^n$ of the fundamental group $\pi_1(\mathcal{T}\tilde{\mathcal{C}}, \tilde{T}_0)$ can sometimes be explicitly described. Let Σ be an oriented closed surface of genus g and consider the first homology group $H_1(\Sigma, \mathbb{Z})$. The Abel–Jacobi map is the group homomorphism $AJ_* : \mathcal{PB}_g^n \to H_1(\Sigma, \mathbb{Z})$ given by $AJ_*(\sigma) = \sum_{i=1}^n [\gamma_i]$ mapping any homotopy class of loops σ in $\mathrm{Conf}(\Sigma, n)$ to the sum of the homology classes of the simple closed loops $\gamma_1, \ldots, \gamma_n$ on Σ that are defined by $\sigma \in \mathcal{PB}_g^n$.

The kernel of the Abel–Jacobi map $AJ_* : \mathcal{PB}_g^n \to H_1(\Sigma, \mathbb{Z})$ consists of all the surface braids in $\mathcal{PB}_g^n$ that define simple closed loops on Σ so that the sum of their homology classes is trivial or, in other words, simple closed loops that bound an immersed subsurface in Σ. Note that, if $n = 1$ the configuration space is Σ. Hence, the pure surface braid group $\mathcal{PB}_g^1$ is the fundamental group $\pi_1(\Sigma)$ and the Abel–Jacobi map is the abelianization map of $\pi_1(\Sigma)$. In particular, the kernel of AJ_* is the commutator group $[\pi_1(\Sigma), \pi_1(\Sigma)]$.

Proposition 2.4.5. Let $\mathcal{C}$ be a non-hyperelliptic connected component of a stratum of the form $\mathcal{H}(2g-2)$ or $\mathcal{H}(k, \ldots, k)$ for some $k \in \mathbb{Z}^+$ in genus $g \geq 5$. If $\tilde{T}_0 = (X, \phi, \omega)$ is marked abelian differential in $\mathcal{T}\tilde{\mathcal{C}}$, the image of $\pi_1(\mathcal{T}\tilde{\mathcal{C}}, \tilde{T}_0)$ via $\tilde{\rho}_{\mathcal{C}}$ in $\mathcal{PB}_g^n$ is the kernel

AJ_*.

Proof. Let $\sigma \in \mathcal{PB}_g^n$ be a pure surface braid. We will prove that $\mathcal{P}(\sigma) \in \mathrm{PMod}_g^n$ if and only if $\sum_{i=1}^n [\gamma_i]$ is trivial in $H_1(\Sigma, \mathbb{Z})$. Note that this is enough to prove the claim, as the image of $\tilde{\rho}_{\mathcal{C}}$ restricted to $\pi_1(\mathcal{T}\tilde{\mathcal{C}}, \tilde{T}_0)$ is the intersection $\mathcal{PB}_g^n \cap \mathrm{PMod}_g^n(\phi_\xi)$ by Theorem 2.4.3. By successive use of twist-linearity, it can be shown that

$$\phi_\xi(\mathcal{P}(\sigma)(\alpha)) = \sum_{i=1}^n (\phi_\xi(\gamma_{L,i}) - \phi_\xi(\gamma_{R,i}))i(\gamma_i, \alpha) + \phi_\xi(\alpha) = -i(\sum_{i=1}^n k_i\gamma_i, \alpha) + \phi_\xi(\alpha)$$

for any simple closed loop α on $\Sigma \setminus \mathcal{Z}$, where the last equality follows from homological coherence (Proposition 2.3.5). Indeed, the curves $\gamma_{L,i}$ and $\gamma_{R,i}$ bound a pair of pants, together with a small loop around the cone point p_i. If $n = 1$, then $\phi_\xi(\mathcal{P}(\sigma)(\alpha)) = \phi_\xi(\alpha)$ for any simple closed loop α if and only if $i(\gamma_1, \alpha) = 0$ for any α, that is if $\gamma_1 \in [\pi_1(\Sigma), \pi_1(\Sigma)]$. If $k_i = k$ for all $i = 1, \ldots, n$, then $i(\sum_{i=1}^n k_i\gamma_i, \alpha) = k \cdot i(\sum_{i=1}^n \gamma_i, \alpha)$ and $\phi_\xi(\mathcal{P}(\sigma)(\alpha)) = \phi_\xi(\alpha)$ for any simple closed loop α if and only if $i(\sum_{i=1}^n \gamma_i, \alpha) = 0$ for any α, that is $\sum_{i=1}^n \gamma_i$ is homologically trivial and $\sigma \in \ker AJ_*$. □

Proof of Theorem 2.4.2. The commutator $[\pi_1(\Sigma), \pi_1(\Sigma)]$ is a non-abelian free group of infinite rank [Put22, Theorem B]. Hence, the homomorphism $\tilde{\rho}_{\mathcal{C}} : \pi_1(\mathcal{TC}, T_0) \to \mathcal{PB}_g^n$ maps isomorphically a subgroup of $\pi_1(\mathcal{TC}, T_0)$ to $[\pi_1(\Sigma), \pi_1(\Sigma)]$. We only need to show that there exists a copy of $[\pi_1(\Sigma), \pi_1(\Sigma)]$ in the image of $\tilde{\rho}_{\mathcal{C}}|_{\pi_1(\mathcal{TC}, T_0)}$. Let γ be the homotopy class of a loop in Σ based at a point $p_i \in \mathcal{Z}$ such that $\gamma \in [\pi_1(\Sigma), \pi_1(\Sigma)]$. Then, by the same argument in the above Proposition 2.4.5, the respective $\sigma \in \mathcal{PB}_g^n$ is in $\mathrm{PMod}_g^n(\phi_\xi)$, and therefore in the image of $\tilde{\rho}_{\mathcal{C}}|_{\pi_1(\mathcal{TC}, T_0)}$. □

Teichmüller strata of blown-up translation surfaces

The labelled Teichmüller space $\mathcal{T}\tilde{\mathcal{C}}$ has infinitely many connected components by Theorem 2.4.3, which are counted by the index of the monodromy $\tilde{\rho}_{\mathcal{C}}$ (Lemma 2.4.1). The connected components of the labelled Teichmüller space $\mathcal{T}^{lab}(\underline{k})$ are not universal: the fundamental groups $\pi_1(\mathcal{T}\tilde{\mathcal{C}}, \tilde{T}_0)$ are infinite [BSW22, Corollary 3.10].

Theorem 2.4.6 (Weak Large Kernel Theorem for $\tilde{\rho}_{\mathcal{C}}$)**.** Every connected component of $\mathcal{T}^{lab}(\underline{k})$ in genus $g \geq 3$ that covers a non-hyperelliptic connected component has an infinite fundamental group.

We define a covering map for the Teichmüller spaces $\mathcal{T}^{lab}(\underline{k})$ that has *blown-up translation surfaces* as fibers. Let us describe the blow-up process locally first. For more details, see [BSW22, Section 3]

An *oriented real blow up* of a point $p \in \mathbb{C}$ is new topological space B_p together with a continuous map $c_p : B_p \to \mathbb{C}$, called collapsing map, so that the preimage of p is homeomorphic to the unit circle while the inverse image of any other point is a single point. *The blow-up of a triangle* T in $\mathbb{C}$ is obtained by replacing every vertex with an edge, in such a way that the resulting space is a hexagon with two types of edges, those corresponding to edges of triangles and those corresponding to vertices of the triangle. There are collapsing maps identifying the new edges with the vertices blown up.

Let (X, ω) be a translation surface triangulated in saddle connections. Let $B(X, \omega)$ be the surface obtained by blowing up each vertex of each triangle of the triangulation, so to have a decomposition of $B(X, \omega)$ in hexagons. The boundary components of the blown-up surface are denoted by $\partial_i Bl(X, \omega)$. Note that this is equivalent to locally blow up the singularities of (X, ω). A *blow-up* of (X, ω) is the surface $B(X, \omega)$ together with a choice of angular coordinates given by maps $\alpha_i : \partial_i Bl(X, \omega) \to S^1$ for $i = 1, \ldots, n$. We say that $B(X, \omega)$ is of type $(k_1, \ldots, k_n)$ if (X, ω) is.

In the new surface $B(X, \omega)$, every cone point p_i of (X, ω) is being replaced with a circle given by gluing together the edges of the blown-up triangles corresponding to the vertices of the triangle. The total angle around p_i measures $2\pi(k_i + 1)$. The metric of $B(X, \omega)$ does not depend on the choice of the triangulation. There exists a collapsing map $c : B(X, \omega) \to X$ given by identifying the boundary components $\partial_i Bl(X, \omega)$ with the respective singularity p_i.

Suppose $g \geq 2$ and $k_1, \ldots, k_n$ is a partition of $2g - 2$ and let (X, ω) and (X', ω') be translation surfaces of type $(k_1, \ldots, k_n)$. The blow-ups $B(X, \omega)$ and $B(X', \omega')$ are said to be equivalent if there exists a bilohomorphism $I : X \to X'$ such that $I^*\omega' = \omega$, together with a diffeomorphism $\tilde{I} : B(X, \omega) \to B(X', \omega')$ fitting in the following commutative diagram

$$\begin{array}{ccc} B(X,\omega) & \xrightarrow{\;c\;} & X \\ \big\downarrow{\scriptstyle \tilde{I}} & & \big\downarrow{\scriptstyle I} \\ B(X',\omega') & \xrightarrow{\;c'\;} & X', \end{array}$$

that moreover preserves the angular coordinates. In other words, if $\alpha_i : \partial_i B(X, \omega) \to S^1$ and $\alpha_i' : \partial_i B(X', \omega') \to S^1$ are the angular coordinates of $B(X, \omega)$ and $B(X', \omega')$ for $i = 1, \ldots, n$, then $\alpha_i' \circ \tilde{I}|_{\partial_i Bl(X,\omega)} = \alpha_i$ for every $i = 1, \ldots, n$. we denote by $\mathcal{H}^{pr}(k_1, \ldots, k_n)$ the set of equivalence classes of blown-up translation surfaces of type $(k_1, \ldots, k_n)$[2].

Let us denote by $\tilde{\Sigma}$ a surface of genus g and n boundary components denoted by $\partial_1 \tilde{\Sigma}, \ldots, \partial_n \tilde{\Sigma}$. We equip every boundary component with angular coordinates given by a orientation-reversing maps $\beta_i : \partial_1 \tilde{\Sigma} \to \mathbb{R}/(2\pi(k_i + 1)\mathbb{Z})$. Let $(\Sigma, \mathcal{Z})$ be the pair consisting of a surface Σ of genus g obtained from $\tilde{\Sigma}$ by capping the boundary components by open disks, and a finite set $\mathcal{Z}$ given by marking every added disk with a unique point p_i. If (X, ϕ, ω) is a marked translation surface of type $(k_1, \ldots, k_n)$ on

[2]The superscript "pr" stands for "pronged". Indeed, fixing angular coordinates is the same as fixing a prong for every cone point.

$(\Sigma, \mathcal{Z})$, a *marked blown-up translation surface* of (X, ϕ, ω) is a pair $(Bl(X,\omega), \tilde{\phi})$ where $\tilde{\phi} : \tilde{\Sigma} \to Bl(X,\omega)$ is an orientation-preserving diffeomorphism mapping every $\partial_1 \tilde{\Sigma}$ to $\partial_i Bl(X,\omega)$, such that $\alpha_i \circ \tilde{\phi} \equiv \beta_i \mod 2\pi$ for every $i = 1, \dots, n$ and the following diagram commutes

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\tilde{\phi}} & Bl(X,\omega) \\ \downarrow \text{Cap} & & \downarrow c \\ \Sigma & \xrightarrow{\phi} & X, \end{array}$$

where the map $\text{Cap} : \tilde{\Sigma} \to \Sigma$ is the capping map.

Let $\tilde{\Sigma}$ and $(\Sigma, \mathcal{Z})$ be as above. If $(Bl(X_1,\omega_1), \phi_1)$ and $(Bl(X_2,\omega_2), \phi_2)$ are the blow-up of the marked translation surfaces (X_1, ϕ_1, ω_1) and (X_2, ϕ_2, ω_2), respectively, then $(Bl(X_1,\omega_1), \tilde{\phi_1})$ and $(Bl(X_2,\omega_2), \tilde{\phi_2})$ are equivalent if (X_1, ϕ_1, ω_1) and (X_2, ϕ_2, ω_2) are equivalent via a bilohomorphism $I : X_1 \to X_2$ and there exists a diffeomorphism $\tilde{I} : Bl(X_1,\omega_1) \to Bl(X_2,\omega_2)$ preserving the angular coordinates such that

- $\tilde{\phi_2}^{-1} \circ \tilde{I} \circ \tilde{\phi_1}|_{\partial\tilde{\Sigma}} = id_{\partial\tilde{\Sigma}}$ and
- $\tilde{\phi_2}^{-1} \circ \tilde{I} \circ \tilde{\phi_1}|_{\partial\tilde{\Sigma}}$ is isotopic to $id_{\tilde{\Sigma}}$ relative the boundary.

The Teichmüller strata $\mathcal{T}^{pr}(\underline{k})$ is the set of equivalence classes of blown-up marked translation surfaces of type $\underline{k} = (k_1, \dots, k_n)$.

The topology of $\mathcal{T}^{pr}(\underline{k})$ can be defined as follows [BSW22, Section 3]. Let us fix $(X, \phi, \omega) \in \mathcal{T}^{lab}(\underline{k})$ and a marked blow-up $(B(X,\omega), \tilde{\phi}) \in \mathcal{T}^{pr}(\underline{k})$. If τ is a triangulation in saddle connections of (X,ω), we denote by $\tilde{\tau}$ the respective decomposition in hexagons of $B(X,\omega)$, pulled back by the marking $\tilde{\phi} : \tilde{\Sigma} \to B(X,\omega)$. The edges of τ lift to arcs in $\tilde{\tau}$ with endpoints lying on $\partial\tilde{\Sigma}$ that decompose $\tilde{\Sigma}$ in hexagons. If $\varepsilon > 0$, we define $U_{\tilde{\tau},\varepsilon}$ to be the set of marked blown-up translation surfaces $(B(X',\omega'), \tilde{\phi}') \in \mathcal{T}^{pr}(k_1, \dots, k_n)$ on which there is a triangulation τ' in saddle connections of (X', ω') lifting to an hexagonal decomposition $\tilde{\tau}'$ of $\tilde{\Sigma}$ such that the following two properties are satisfied:

- the arcs in the two hexagonal decompositions are pairwise homotopic;
- if γ and γ' are homotopic arcs from $\partial_i\tilde{\Sigma}$ to $\partial_j\tilde{\Sigma}$ that belong to the decompositions $\tilde{\tau}$ and $\tilde{\tau}'$, respectively, there exist intervals $I_i \subset \partial_i\tilde{\Sigma}$ and $I_j \subset \partial_j\tilde{\Sigma}$ of length ε containing the endpoints of γ and γ' and such that γ is homotopic to γ' through a family of paths each of which has one endpoint in I_i and one endpoint in I_j.

The subsets $U_{\tilde{\tau},\varepsilon}$ come with injective holonomy maps $\text{hol}_{\tilde{\tau}} : U_{\tilde{\tau},\varepsilon} \to \mathbb{C}^{2g+n-1}$ as for the Teichmüller manifolds $\mathcal{T}(\underline{k})$ and $\mathcal{T}^{lab}(\underline{k})$.

There exists a well-defined map $\mathcal{T}^{pr}(\underline{k}) \to \mathcal{T}^{lab}(\underline{k})$ which takes any $(Bl(X,\omega), \tilde{\phi})$ and returns (X, ϕ, ω). Continuity can be checked on local charts, where the restriction

$U_{\tilde{\tau},\varepsilon} \to U_\tau$ is local diffeomorphism given by the capping map $\mathrm{Cap} : \tilde{\Sigma} \to \Sigma$ performed by glueing open punctures disks to the boundary components, that induces an isomorphism $H^1(\tilde{\Sigma}, \partial\tilde{\Sigma}; \mathbb{Z}) \cong H^1(\Sigma, \mathcal{Z}; \mathbb{Z})$ in cohomology.

Let $\mathrm{Diff}^+_{pr}(\tilde{\Sigma}, \partial\tilde{\Sigma})$ be the group of orientation-preserving self-diffeomorphisms $f : \tilde{\Sigma} \to \tilde{\Sigma}$ fixing set-wise the boundary components and such that the restriction to each $\partial_i\tilde{\Sigma}$ is a rotation by a multiple of 2π with respect to the angular coorinates given by the maps $\beta_i : \partial_1\tilde{\Sigma} \to \mathbb{R}/(2\pi(k_i + 1)\mathbb{Z})$. The group $\mathrm{Mod}_{pr}(\tilde{\Sigma})$ is the group of diffeomorphisms in $\mathrm{Diff}^+_{pr}(\tilde{\Sigma}, \partial\tilde{\Sigma})$ up to isotopy relative the boundary.

On a tubular neighbourhood of a boundary component $\partial_i\tilde{\Sigma}$, one might define a diffeomorphism f_i which rotates $\partial_i\tilde{\Sigma}$ by 2π and extends by the identity of the rest of the surface. The equivalence class of the map f_i is an element of $\mathrm{Mod}_{pr}(\tilde{\Sigma})$ such that $f_i^{k_i+1}$ is a full Dehn twist about the boundary component $\partial_i\tilde{\Sigma}$.

Let $\mathrm{Diff}^+(\tilde{\Sigma}, \partial\tilde{\Sigma})$ be the group of self-diffeomorphisms $f : \tilde{\Sigma} \to \tilde{\Sigma}$ fixing point-wise the boundary components. The group $\mathrm{Mod}(\tilde{\Sigma})$ is the group of diffeomorphisms in $\mathrm{Diff}^+(\tilde{\Sigma}, \partial\tilde{\Sigma})$ up to isotopy relative the boundary.

Proposition 2.4.7. [BSW22, Lemma 3.3 and Section 4.3]. Let FT be the free abelian group generated by f_i for $i = 1, \ldots, n$ and PR the product $\prod_{i=1}^n \mathbb{Z}/(k_i + 1)\mathbb{Z}$. The capping map $\mathrm{Cap} : \tilde{\Sigma} \to \Sigma$ induces a short exact sequence

$$1 \to FT \to \mathrm{Mod}_{pr}(\tilde{\Sigma}) \to \mathrm{PMod}(\Sigma, \mathcal{Z}) \to 1,$$

while the action of $\mathrm{Mod}_{pr}(\tilde{\Sigma})$ of the angular coordinates $\beta_1, \ldots, \beta_n$ induce the short exact sequence

$$1 \to \mathrm{Mod}(\tilde{\Sigma}) \to \mathrm{Mod}_{pr}(\tilde{\Sigma}) \to PR \to 1.$$

Theorem 2.4.8. The map $\mathcal{T}^{pr}(\underline{k}) \to \mathcal{T}^{lab}(\underline{k})$ is a covering map with a deck transformation group isomorphic to FT. In particular, the connected components of $\mathcal{T}^{pr}(\underline{k})$ are covering manifolds for the stratum $\mathcal{H}(\underline{k})$.

The spaces $\mathcal{H}^{pr}(\underline{k})$ are manifolds for the Quotient Manifold Theorem. Indeed, the spaces $\mathcal{H}^{pr}(\underline{k})$ are the quotient of $\mathcal{T}^{pr}(\underline{k})$ by $\mathrm{Mod}(\tilde{\Sigma})$. The action of $\mathrm{Mod}(\tilde{\Sigma})$ on $\mathcal{T}^{pr}(\underline{k})$ is proper since $\mathcal{T}^{pr}(\underline{k}) \to \mathcal{T}^{lab}(\underline{k})$ is continuous and equivariant with respect to the surjective homomorphism $\mathrm{Mod}(\tilde{\Sigma}) \to \mathrm{PMod}(\Sigma, \mathcal{Z})$ induced by capping $\tilde{\Sigma}$ with open punctured disks. Moreover, the action is also free as $\mathrm{Mod}(\tilde{\Sigma})$ is torsion-free [FM12, Corollary 7.3]. Hence, the orbifold fundamental group $\pi_1^{orb}(\hat{\mathcal{C}}, \hat{T}_0)$ of a connected component $\hat{\mathcal{C}}$ of $\mathcal{H}^{pr}(\underline{k})$, based at some $\hat{T}_0 \in \mathcal{T}^{pr}(\underline{k})$ and lying over a stratum component $\mathcal{C}$, is isomorphic to the fundamental group $\pi_1(\hat{\mathcal{C}}, [\hat{T}_0])$.

The new orbifold structure of $\mathcal{H}^{lab}(\underline{k})$, given by the Teichmüller spaces of blown-up marked translation surfaces $\mathcal{T}^{pr}(\underline{k})$, provides us with finer topological monodromies.

Let $\hat{T}_0 \in \mathcal{T}^{pr}(\underline{k})$ be a blow-up of a marked translation surface and let $\tilde{\mathcal{C}}$ be the connected component of its projection in $\mathcal{H}^{lab}(\underline{k})$. The orbifold structure $\mathcal{T}^{pr}(\underline{k})/\operatorname{Mod}_{pr}(\tilde{\Sigma})$ of $\mathcal{H}^{lab}(\underline{k})$ defines a homomorphism

$$\hat{\rho}_C : \pi_1^{orb}(\tilde{\mathcal{C}}, \hat{T}_0) \to \operatorname{Mod}_{pr}(\tilde{\Sigma}),$$

the *pronged topological monodromy.*

Let $\hat{\mathcal{C}}$ be a connected component of $\mathcal{H}^{pr}(\underline{k})$ lying over some connected component components $\tilde{\mathcal{C}}$. If $\hat{T}_0 \in \mathcal{T}^{pr}(\underline{k})$ is a blow-up of a marked translation surface with image $\tilde{T}_0 \in \mathcal{T}^{lab}(\underline{k})$ such that $(X, \omega) \in \hat{\mathcal{C}}$, there is an exact sequence

$$1 \to \pi_1(\hat{\mathcal{C}}, [\hat{T}_0]) \to \pi_1^{orb}(\tilde{\mathcal{C}}, \tilde{T}_0) \to PR' \to 1$$

induced by the action each $(\gamma, f) \in \pi_1^{orb}(\tilde{\mathcal{C}}, \tilde{T}_0)$ on angular coordinates of $\tilde{T}_0$, where PR' is the subgroup of PR consisting of elements $(z_1, \ldots, z_n) \in PR$ such that the sum of the z_i corresponding to even k_i is even [CS22, Corollary 7.6]. In particular, the fundamental group $\pi_1(\hat{\mathcal{C}}, \hat{T}_0)$ is a finite index subgroup of $\pi_1^{orb}(\tilde{\mathcal{C}}, \tilde{T}_0)$. The restriction of the pronged topological monodromy $\hat{\rho}_{\mathcal{C}}$ to the copy of $\pi_1(\hat{\mathcal{C}}, \hat{T}_0)$ has its image in the boundary-preserving mapping class group $\operatorname{Mod}(\tilde{\Sigma}) \triangleleft \operatorname{Mod}_{pr}(\tilde{\Sigma})$, as any two blow-up of marked translation surfaces $(B(X,\omega), \phi_1), (B(X,\omega), \phi_2) \in \mathcal{T}^{pr}(\underline{k})$ connected by an arc differ by the action of the mapping class $\phi_1^{-1}\phi_2 \in \operatorname{Mod}(\tilde{\Sigma})$.

The kernel of topological monodromy $\tilde{\rho}_{\mathcal{C}}$ is isomorphic to the fundamental group of $\mathcal{T}\tilde{\mathcal{C}}$. Similarly, the kernel of the pronged topological monodromy $\hat{\rho}_{\mathcal{C}}$ is isomorphic to $\pi_1(\mathcal{T}\hat{\mathcal{C}}, \hat{T}_0)$, where $\mathcal{T}\hat{\mathcal{C}}$ denotes the set of blown-up marked translation surfaces $(B(X,\omega), \phi_1) \in \mathcal{T}^{pr}(\underline{k})$ such that $B(X,\omega) \in \tilde{\mathcal{C}}$.

Similarly as for the labelled case, Calderon–Salter have described the images of the pronged topological monodromies if $g \geq 5$ [CS22, Theorem B]. Let $\tilde{\xi}$ be the unit horizontal non-vanishing vector field of a blown-up translation surface $B(X,\omega)$, defined as the pullback of ξ by the collapsing map $c : B(X,\omega) \to X$. If $\tilde{\phi} : \tilde{\Sigma} \to B(X,\omega)$ is a marking, then we denote by $\phi_{\tilde{\xi}}$ the winding number function on $\tilde{\Sigma}$ of the pullback of $\tilde{\xi}$ by the marking ϕ.

Theorem 2.4.9. Let $\tilde{\mathcal{C}}$ be a connected components of $\mathcal{H}^{lab}(\underline{k})$ lying over a non-hyperelliptic stratum component of $\mathcal{H}(\underline{k})$ in genus $g \geq 5$. If $\hat{\mathcal{C}}$ is a connected component lying over $\tilde{\mathcal{C}}$ in $\mathcal{H}^{pr}(\underline{k})$ and $\tilde{T}_0 \in \mathcal{T}\hat{\mathcal{C}}$, the image of the pronged topological monodromy

$$\hat{\rho}_{\mathcal{C}} : \pi_1^{orb}(\tilde{\mathcal{C}}, \hat{T}_0) \to \operatorname{Mod}_{pr}(\tilde{\Sigma})$$

and its restriction

$$\hat{\rho}_{\mathcal{C}} : \pi_1(\hat{\mathcal{C}}, [\hat{T}_0]) \to \operatorname{Mod}(\tilde{\Sigma})$$

are, respectively, the stabilizers of the relative framing $\phi_{\tilde{\xi}}$, denoted by $\operatorname{Mod}_{pr}(\tilde{\Sigma})[\phi_{\tilde{\xi}}]$

and $\mathrm{Mod}(\tilde{\Sigma})[\phi_{\tilde{\xi}}]$, respectively. In particular, the index of both the images are infinite.

Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathcal{T}\tilde{\mathcal{C}}, \tilde{T}_0) & \longrightarrow & \pi_1^{orb}(\tilde{\mathcal{C}}, \tilde{T}_0) & \xrightarrow{\tilde{\rho}_{\mathcal{C}}} & \mathrm{PMod}_g^n(\phi_\xi) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \hat{\rho}_{\mathcal{C}}} & & \downarrow & & \\ 1 & \longrightarrow & FT & \longrightarrow & \mathrm{Mod}_{pr}(\tilde{\Sigma}) & \longrightarrow & \mathrm{PMod}(\Sigma, \mathcal{Z}) & \longrightarrow & 1. \end{array}$$

The vertical map on the left detects loops in the connected component of the basepoint $\tilde{T}_0 \in \mathcal{T}\mathcal{C}$ that change the angular coordinates of a blown-up marked translation surface lying over T_0^{lab}.

Proof of Theorem 2.4.6. The image of the map $\pi_1(\mathcal{T}\tilde{C}, \tilde{T}_0) \to FT$ is cyclic and generated by a loop with image in FT given by $\prod_{i=1}^n f_i$ [CS22]. $\square$

It is not known if the Large Kernel Theorem holds for the monodromy associated to the non-hyperelliptic components $\tilde{\mathcal{C}}$ or $\hat{\mathcal{C}}$, that is, to the homomorphisms $\tilde{\rho}_{\mathcal{C}} : \pi_1^{orb}(\tilde{\mathcal{C}}, \tilde{T}_0) \to \mathrm{PMod}(\Sigma, \mathcal{Z})$ and $\hat{\rho}_{\mathcal{C}} : \pi_1^{orb}(\hat{\mathcal{C}}, [\hat{T}_0]) \to \mathrm{Mod}_{pr}(\tilde{\Sigma})$. In other words, it is not known if the connected components of the Teichmüller manifolds $\mathcal{T}^{lab}(\underline{k})$ and $\mathcal{T}^{pr}(\underline{k})$ contain a non-abelian free group of rank 2. We only know that the connected components of $\mathcal{T}^{lab}(\underline{k})$ are infinite. In the following chapters, we will prove that the connected components of $\mathcal{T}^{pr}(\underline{k})$ associated to the non-hyperelliptic components $\mathcal{H}^{\mathrm{odd}}(4)$, $\mathcal{H}(3,1)$ and $\mathcal{H}^{\mathrm{even}}(6)$ have a large fundamental group, containing a non-abelian free group of rank 2. To prove it, we will need the definition of *projective strata*.

§ 2.5 | Projective strata

Strata of translation surfaces can be projectivized: any non-zero complex number z acts on the abelian differential (X, ω) by multiplication on the 1-form, that is

$$z \cdot (X, \omega) = (X, z\omega).$$

We will denote by $\mathbb{P}(k_1, \ldots, k_n)$ the quotient of $\mathcal{H}(k_1, \ldots, k_n)$ by the above $\mathbb{C}^*$ action.

Via polar coordinates $z = \rho e^{i\theta}$, any non-zero complex number z can be seen as the composition of a rotation, given by $e^{i\theta}$, and (possibly) a dilation or contraction given by ρ. The above $\mathbb{C}^*$-action on abelian differentials corresponds, at the level of polygonal representations, to rotating and scaling the sides. To see this, we recall that the sides of the polygonal representations are given, up to complex translations, by the period coordinates $\int_\gamma \omega$ where γ belongs to a fixed relative symplectic basis. Hence, the action of a non-zero complex number z changes the period coordinates, from $\int_\gamma \omega$ to $z \int_\gamma \omega$, thus acting on the whole polygonal representation.

$$\mathbb{C}^* \overset{\text{by rotation}}{\underset{\text{and scaling}}{\curvearrowright}} \left\{ \begin{array}{c} \text{Translation surfaces} \\ \text{modulo scissor moves} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Abelian differentials} \\ \text{modulo flat equivalence} \end{array} \right\} \overset{\text{by left}}{\underset{\text{multiplication}}{\curvearrowleft}} \mathbb{C}^*$$

Proposition 2.5.1. The $\mathbb{C}^*$-action is continuous on Teichmüller strata $\mathcal{T}(\underline{k})$.

Proof. Note that the $\mathbb{C}^*$-action is well-defined on every local chart U_τ. Hence, without loss of generality, the continuity of the group action map $\Phi : \mathbb{C}^* \times \mathcal{T}(\underline{k}) \to \mathcal{T}(\underline{k})$ given by $\Phi(z, (X, \omega, \phi)) = (X, z \cdot \omega, \phi)$ can be checked on the single charts, where it restricts to a multiplication map on the period coordinates. □

Any projective stratum is also a good orbifold. Indeed, every $\mathbb{P}(\underline{k})$ is the quotient of a smooth manifold by the proper action of a discrete group. Specifically, there is a subgroup of the mapping class group Mod_g^n acting on the smooth manifold $\mathbb{P}\mathcal{T}(\underline{k})$, defined as the orbit space $\mathcal{T}(\underline{k})/\mathbb{C}^*$, so that the resulting quotient is $\mathbb{P}(\underline{k})$. Let us prove that $\mathbb{P}(\underline{k})$ is indeed a good orbifold.

Proposition 2.5.2. The spaces of the form $\mathbb{P}\mathcal{T}(\underline{k})$ are $(2g + n - 2)$-manifolds.

Proof. We claim that the action of $\mathbb{C}^*$ on each $\mathcal{T}(\underline{k})$ is free and proper. Since the action is free and well-defined on local charts U_τ, we can check properness locally. However, the map $\Phi : U_\tau \times \mathbb{C}^* \to U_\tau \times U_\tau$ given by $\Phi(x, \lambda) = (x, \lambda x)$ is proper,[3], meaning that the map is closed and has compact fibers. □

Since the punctured complex plane $\mathbb{C}^*$ acts freely and properly on each Teichmüller stratum component $\mathcal{TC}$, by the Quotient Manifold Theorem $\mathcal{TC}/\mathbb{C}^*$ is a smooth manifold of dimension $\dim \mathcal{C} - 1$. Moreover, the quotient map $q_\mathcal{C} : \mathcal{TC} \to \mathcal{TC}/\mathbb{C}^*$ is a smooth submersion and every $q_\mathcal{C}$ is a principal $\mathbb{C}^*$-bundle.

The relation between the orbifold fundamental groups of a stratum component $\mathcal{C}$ and its projectivization $\mathbb{P}\mathcal{C}$ is summarized below in Proposition 2.5.3.

Proposition 2.5.3. Let $\mathcal{C}$ be a stratum component and $\mathbb{P}\mathcal{C}$ its projectivization. The following is a short exact sequence

$$0 \to \pi_1(\mathbb{C}^*) \to \pi_1^{orb}(\mathcal{C}) \to \pi_1^{orb}(\mathbb{P}\mathcal{C}) \to 1.$$

Proof. Let us fix a base point $(X, \phi, \omega) \in \mathcal{TC}$. Suppose $G_\mathcal{C}$ is the subgroup of Mod_g^n of all the mapping classes obtained as the composition $\phi^{-1}\phi'$ where $(X, \phi', \omega) \in \mathcal{TC}$ is the endpoint of a path in $\mathcal{TC}$ starting at (X, ϕ, ω). Note that $G_\mathcal{C}$ acts on both $\mathcal{TC}/\mathbb{C}^*$ and $\mathcal{TC}$ so to obtain the orbit spaces $\mathbb{P}\mathcal{C}$ and $\mathcal{C}$, respectively. Moreover, the map $q_\mathcal{C}$ is a fiber bundle. From the long exact sequence it induces, we can conclude that the corresponding map on fundamental groups is surjective since $\mathbb{C}^*$ is connected. In

[3] This is an equivalent condition to the notion of properness previously given

particular, the map $q_{\mathcal{C}}$ is $G_{\mathcal{C}}$-equivariant and induces a surjection $\pi_1^{orb}(\mathcal{C}) \twoheadrightarrow \pi_1^{orb}(\mathbb{P}\mathcal{C})$ on the orbifolds fundamental groups given by $(\gamma, \phi) \mapsto (q_{\mathcal{C}}(\gamma), \phi)$. Indeed, we can apply a weaker version of the 5-lemma to the commutative diagram below

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(\mathcal{TC}) & \longrightarrow & \pi_1^{orb}(\mathcal{C}) & \xrightarrow{\rho_{\mathcal{C}}} & G_{\mathcal{C}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (q_{\mathcal{C}})_*} & & \downarrow{\scriptstyle \pi_1^{orb}(q_{\mathcal{C}})} & & \| & & \\ 0 & \longrightarrow & \pi_1(\mathbb{P}\mathcal{TC}) & \longrightarrow & \pi_1^{orb}(\mathbb{P}\mathcal{C}) & \xrightarrow{\rho_{\mathbb{P}\mathcal{C}}} & G_{\mathcal{C}} & \longrightarrow & 0. \end{array}$$

The kernel of $\pi_1^{orb}(q_{\mathcal{C}})$ is isomorphic to $\pi_1(\mathbb{C}^*)$ and generated by a loop in a fiber of $q_{\mathcal{C}}$, as it can be seen by the long exact sequence associated. □

The projective stratum components $\mathbb{P}\mathcal{C}$ parameterize the isomorphism classes of pairs (X, D), where X is a closed Riemann surface and D is an effective canonical divisor with prescribed multiplicities provided by the stratum component $\mathcal{C}$.

Two pairs (X_1, D_1) and (X_2, D_2) are equivalent in $\mathbb{P}\mathcal{C}$ if there exists a bilohomorphism $I : X_1 \to X_2$ such that $I^* D_2 = D_1$. If $\mathcal{C}$ is a stratum component, its projectivization $\mathbb{P}\mathcal{C}$ can be projected in $\mathcal{M}_g^n$, the moduli space of pointed Riemann surfaces. Sometimes, the projection is an isomorphism of orbifolds. We first recall the following result.

Proposition 2.5.4. Let $f : M \to N$ be a continuous bijection between manifolds without boundary. If $\dim M = \dim N$, then f is a homeomorphism.

Proof. Locally, the continuous map f is a homeomorphism by the Invariance Domain Theorem. Bijective local homeomorphisms are global homeomorphisms. □

Proposition 2.5.5. Let $\mathcal{C}$ be a connected component of the stratum $\mathcal{H}(k_1, \dots, k_n)$, such that all the k_i's are different. The forgetful map $\mathbb{P}\mathcal{C} \to \mathcal{M}_g^n$ defined by mapping $(X, \sum_{i=1}^n k_i p_i)$ to $(X, (p_1, \dots, p_n))$ is an orbifold isomorphism onto its image, provided the dimension of the image is $\dim(\mathbb{P}\mathcal{C})$.

Proof. The forgetful map is induced by the Mod_g^n-equivariant continuous map $\mathbb{P}(\mathcal{TC}) \to \mathcal{T}_g^n$ given by mapping the triple $(X, \phi, \sum_{i=1}^n k_i p_i)$ to $(X, \phi, (p_1, \dots, p_n))$. However, any bijective continuous map between manifolds of the same dimension and without boundary is a homeomorphism. □

Chapter 3

Geometric homomorphisms

The projective stratum components $\mathbb{P}\mathcal{H}^{\text{odd}}(4)$, $\mathbb{P}\mathcal{H}(3,1)$ and $\mathbb{P}\mathcal{H}^{\text{even}}(6)$ have an orbifold fundamental group that can be described using spherical-type Artin groups, defined below. Their topological monodromy maps factor through geometric homomorphism (see Subsection 3.3).

§ 3.1 | Spherical-type Artin groups

Artin groups are finitely presented groups where a finite simplicial graph, an undirected finite graph with no multiple edges nor loop-edges, gives the generators and the relations. If Γ is a finite simplicial graph and $\mathcal{V}(\Gamma) = \{v_1, \dots, v_n\}$ is its set of vertices, an Artin group is a group that admits a presentation of the following form:

$$A_\Gamma = \left\langle a_1, \dots, a_n \;\middle|\; \begin{array}{lll} a_i a_j a_i = a_j a_i a_j & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ a_i a_j = a_j a_i & \text{otherwise} \end{array} \right\rangle. \tag{3.1}$$

Any two *standard generators* a_i and a_j share a braid relation if the respective vertices are adjacent. If a pair of vertices is not connected by an edge, the corresponding standard generators must commute. An Artin group is of type Γ is the defining simplicial graph is isomorphic to Γ.

Artin groups have a more general definition given by labeling the edges of Γ and allowing relations between generators with adjacent vertices to share a braid-like relation of the form $a_i a_j a_i \cdots = a_j a_i a_j \dots$ where the lengths of both sides of the equality correspond to the label of the edge v_i and v_j share. Here, we will only consider the class of Artin groups given by the above definition or, using a more common name in literature, we will only consider *small-type Artin groups* and refer to them as Artin groups for simplicity.

The mapping class group of a disk with n marked points is an Artin group, isomorphic to the A_{n-1}-type Artin group where A_{n-1} is a simplicial graph with $n-1$ arranged in a single chain pf vertices as in Figure 3.1. The group $A_{A_{n-1}}$ is commonly known as the braid group $\mathcal{B}_n$.

The braid groups are *spherical-type* Artin groups: if we add the extra relations $a_i^2 = 1$ for every $i = 1, \ldots, n-1$ to the presentation of $\mathcal{B}_n$, the result is a group isomorphic to the symmetric group $\mathrm{Sym}(n)$ [FM12, Section 9.3]. In other words, the resulting quotient is a finite group if we impose standard generators to be involutions. Spherical-type Artin groups are those Artin groups that project to a finite group whenever we add the extra relations $a_i^2 = 1$ for every $i \in |\mathcal{V}(\Gamma)|$. The quotient of A_Γ by the subgroup normally generated by the squares of the standard generators is a *Coxeter group*, denoted by W_Γ.

Theorem 3.1.1 ([Hum90])**.** Let Γ be a connected finite simplicial graph. The Artin group A_Γ is spherical-type if and only if Γ is one of the graphs in Figure 3.1.

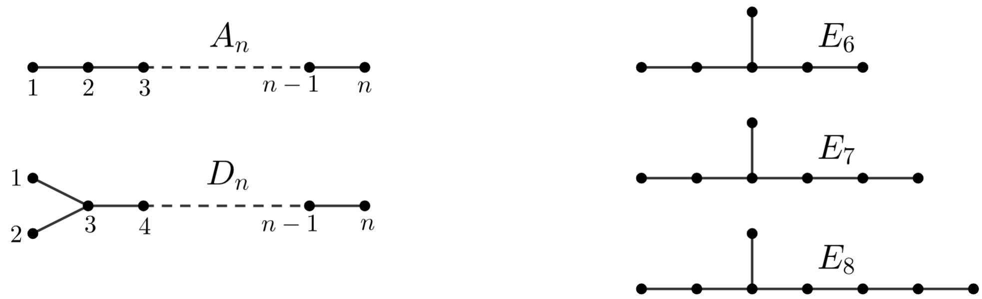


Figure 3.1: The simplicial graph of every possible spherical-type Artin group. The E_6-type, E_7-type and E_8-type describe the orbifold fundamental group of the projective components $\mathbb{P}\mathcal{H}^{\mathrm{odd}}(4)$, $\mathbb{P}\mathcal{H}(3,1)$ and $\mathbb{P}\mathcal{H}^{\mathrm{even}}(6)$.

Even though Artin groups are easy to define, they are often poorly understood. Some of the basic conjectures remain wide open about general Artin groups. There are four basic conjectures about those Artin groups that have a defining simplicial graph that is connected. It has been conjectured [GP12] that: every Artin group is torsion-free; that every non-spherical Artin group has trivial center; that every Artin group has a solvable word problem; that Artin groups satisfy the $K(\pi, 1)$ conjecture, meaning that every A_Γ is the fundamental group of an aspherical space. For spherical-type Artin groups, all the above conjectures hold.

Hyperplane arrangements

Whenever W_Γ is a finite Coxeter group, the classes of the standard generators of A_Γ in W_Γ have a geometric interpretation as reflections of a finite-dimensional real vector space, and W_Γ is isomorphic to a reflection group [Hum90]. More specifically, the simplicial graphs in Figure 3.1, called *Dynkin diagrams*, are paired with *root systems*.

A root system Φ is a configuration of vectors that span an Euclidean vector space V and have the following properties: the only scalar multiples of $\alpha \in \Phi$ are $\pm\alpha$; the set Φ is closed under reflection through the hyperplane perpendicular to any $\alpha \in \Phi$; the projection of $\beta \in \Phi$ onto the line through $\alpha \in \Phi$ is an integer or half-integer multiple of α, for any $\alpha, \beta \in \Phi$. There exists a subset of a root system that is closed under addition and, at the same time, for any multiple of any $\alpha \in \Phi$ contains exactly one vector in $\{\alpha, -\alpha\}$. Moreover, any vector in the subset cannot be written as the sum of two other elements.

The new subset of vectors in Φ is a basis for V and their configuration can be described by a Dynkin diagram: vectors are in one-to-one correspondence with vertices and edges connect vertices representing vectors that make an angle of $2\pi/3$, while no edge is drawn in case two vectors are orthogonal. There are only finitely many types of root systems, up to isomorphism, and the Dynkin diagrams in Figure 3.1 describe some of them.

Let Γ be one a of the Dynkin diagrams between A_n, D_n and E_m for $n \in \mathbb{N}$ and $m = 6, 7, 8$. The Γ-*hyperplane arrangement* $\{\mathcal{H}_\alpha\}_{\alpha \in \Phi_\Gamma}$ is the collection of hyperplanes so that each $\mathcal{H}_\alpha$ is orthogonal to the vector $\alpha \in \Phi_\Gamma$ of the root system Φ_Γ associated to Γ. The complement of the union of hyperplanes $V \setminus \cup_{\alpha \in \Phi_\Gamma} \mathcal{H}_\alpha$ is a union of contractible cones and has trivial fundamental groups. Nevertheless, the complexified complement of hyperplanes

$$M_\Gamma = (V \setminus \cup_{\alpha \in \Phi_\Gamma} \mathcal{H}_\alpha) \otimes_{\mathbb{R}} \mathbb{C}$$

have non-trivial fundamental groups, and this is because the real codimension of the hyperplanes increased by 1. The following is due to Brieskorn [Bri73, Proposition 1]. Note that any reflection group W_Γ acts faithfully on the respective M_Γ.

Proposition 3.1.2. Let Γ be a of Dynkin diagrams between A_n, D_n and E_m for $n \in \mathbb{N}$ and $m = 6, 7, 8$. If x_0 is a point in the orbit space M_Γ/W_Γ, then the fundamental group $\pi_1(M_\Gamma/W_\Gamma, x_0)$ is isomorphic to the Artin group A_Γ.

In 1972, Deligne proved that spherical-type Artin groups are $K(\pi, 1)$ [Del72].

Theorem 3.1.3. The complexified complement of hyperplanes M_Γ are aspherical if A_Γ is of spherical-type.

Garside structure

Let G be group generated by a finite set S such that $S \cap S^{-1}$ is empty. The submonoid generated by S is denoted by G^+ and its elements are called positive words. The submonoid generated by the inverse generating elements of G is denoted by $(G^+)^{-1}$. Suppose G^+ trivially intersects $(G^+)^{-1}$. The prefix order on G is the partial order $(G, \preceq)$ where $a \preceq b$ if and only if $a^{-1}b \in G^+$ or, in other words, if b can be written as a product between of a positive word in G and a.

Suppose $\mathcal{A}$ is the set of elements in G^+ that cannot be written as a product of other non-trivial elements of G^+. The monoid G^+ is *Noetherian* if for every positive word $x \in G^+$ we have that $\sup\{n \in \mathbb{N} \mid x = a_1 \dots a_n, a_i \in \mathcal{A}\}$ is finite or, in other words, if there is an upper bound to the lengths of all the possible ways x can be written as a word in $\mathcal{A}$. The group G is *Garside* if the following properties are satisfied: the monoid G^+ is Noetherian; the prefix order admits greater common divisors and lower common multiples; there exists an element $\Delta \in G^+$ that fixes G^+ by conjugation and the set $\{s \in G \mid 1 \preceq s \preceq \Delta\}$ of *simple elements* is finite and generates G.

Theorem 3.1.4 ([DDG+15])**.** Let G be a finitely generated Garside group. For every $x \in G$ there exists $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ so that x can be uniquely written as $x = \Delta^k s_1 \dots s_n$, where each $s_i \neq \Delta$ is a simple element and for every pair $\{s_i, s_{i+1}\}$ of adjacent simple elements the greater common divisor between $s_i s_{i+1}$ and Δ is exactly s_i.

For any $x \in G$, let us consider its unique representation $x = \Delta^k s_1 \dots s_n$. The power of Δ, or in other words the integer k, will be denoted by $\inf(x)$; the number of simple elements needed to write x, or in other words the integer n, by $\sup(x)$. Similarly, there exists a suffix order on a Garside group G that provides the group with right normal forms, where elements in G can be uniquely written in the form $t_1 \dots t_n \Delta^k$ with t_i simple elements different from Δ; see [AC21, Section 2] for more details. In what follows, we will only adopt the prefix order.

Theorem 3.1.5. [BS72, Théorème 7.1] Any spherical-type Artin group A_Γ is Garside. The center of A_Γ is infinite cyclic and generated by a power of Δ.

The Brieskorn–Artin algorithm has been implemented on SageMath, an open source Python mathematics software system. The Artin group's class can be called using the comand `sage.groups.artin.ArtinGroup(coxeter_matrix, names)` and the left normal form of an element by `left_normal_form(element)`. Later, we will provide an example on how to use the SageMath Artin group implementation on a specific example.

Lastly, in a Garside group G, we say that $x \in G$ absorbs $y \in G$ if either $\sup(y) = 0$ or $\inf(y) = 0$ and both the equalities $\sup(xy) = \sup(x)$ and $\inf(xy) = \inf(x)$ hold. In this case, y is absorbed by x and we say that the group element x is *absorbable*. Absorbable elements are not classified but, for example, in the braid group $\mathcal{B}_4$ any n-th powers of a generator absorb the n-th power of a non-adjacent generator. More precisely, we can write $\sigma_1^n \sigma_3^n = (\sigma_1 \sigma_3)^n$, and observe that σ_1^n absorb σ_3^n. Absorbable elements define an infinite graph we use to prove the Large Kernel Theorem E.

§ 3.2 | Plane curve singularities

Every compact Riemann surface is a complex algebraic curve that can be embedded into a projective plane as the vanishing locus of a homogeneous polynomial. For low-genus strata of translation surfaces, the underlying Riemann surfaces can be described explicitly as the zero loci of polynomials. For this reason, in this section, we introduce the definition of versal deformation spaces of plane curve singularities inspired by the work of Cuadrado–Salter [PS21]. The main reference for this section is [AGV88, Chapter 3]

A *plane curve singularity* is the germ of a complex holomorphic function $f : \mathbb{C}^2 \to \mathbb{C}$ at the origin, that has an isolated critical point in $(0,0) \in \mathbb{C}^2$. In other words, we consider an equivalence class of holomorphic functions that define the same germ in the origin and with the partial derivatives that vanish simultaneously only at $(0,0) \in \mathbb{C}^2$. Recall that a pair of functions defines the same germ in a point x if they coincide in a neighbourhood of x.

Milnor proved [Mil68] that, in the general case when the dimension of the domain of f is $n \geq 2$, there exist $\varepsilon > 0$ small enough, such that the restriction of the normalized function $f/|f|$ to the set $S^{2n-1} \setminus (f^{-1}(0) \cap S^{2n-1})$ is a locally trivial fibration. For plane curve singularities, the diffeomorphic type of the fibers is that of a connected oriented compact surface with non-empty boundary, called *Milnor fibers.* The number of boundary components is the number of branches of f, while the first Betti number coincides with the multiplicity of f at the origin.

A *deformation* of a plane curve singularity $f : \mathbb{C}^2 \to \mathbb{C}$ is the germ of a holomorphic function $F : \mathbb{C}^2 \oplus \mathbb{C}^\mu \to \mathbb{C}$ such that $F(\bullet, 0) = f$. Roughly speaking, a deformation F is *versal* if any other deformation can be obtained from F up to a change of coordinates. Formally speaking, the deformation F is versal if for any deformation $G : \mathbb{C}^2 \oplus \mathbb{C}^m \to \mathbb{C}$ there exist $\psi : \mathbb{C}^m \to \mathbb{C}^\mu$ and $g : \mathbb{C}^2 \oplus \mathbb{C}^m \to \mathbb{C}^2$ holomorphic such that $G(z, w) = F(g(z, w), \psi(w))$ for any $(z, w) \in \mathbb{C}^2 \oplus \mathbb{C}^m$.

Proposition 3.2.1. [GLS07] Up to a change of coordinates, there exists a unique versal deformation $F : \mathbb{C}^2 \oplus \mathbb{C}^\mu \to \mathbb{C}$ of a plane curve singularity $f : \mathbb{C}^2 \to \mathbb{C}$, such that the dimension μ is the multiplicity of f in the singularity at the origin.

The multiplicity of the germ f is the dimension of the algebra $\mathbb{C}\{x, y\}$ of convergent power series in two complex variables, quotient by the ideal (f_x, f_y) generated by the partial derivatives of f. A versal deformation of f can be constructed as follows. Let $g_1, \ldots, g_m \in \mathbb{C}[x, y]$ be polynomials projecting to a $\mathbb{C}$-basis of $\mathbb{C}\{x, y\}/(f_x, f_y)$. The deformation

$$F(p, s) = f(p) + \sum_{i=1}^{\mu} s_i g_i(p)$$

of f given by the parameters $s = (s_1, \ldots, s_\mu) \in \mathbb{C}^\mu$ and $p = (x, y) \in \mathbb{C}$, is versal. The restriction of the projection map $\pi_f : \mathbb{C}^2 \times \mathbb{C}^m \to \mathbb{C}^m$ to the vanishing locus of the deformation F has algebraic curves as fibers, since each $s \in \mathbb{C}^\mu$ defines the vanishing locus of $F(\bullet, s)$ in $\mathbb{C}^2$. The set of $s \in \mathbb{C}^\mu$ defining singular curves is the *discriminant locus*, denoted by $\mathcal{D}_f$. The discriminant locus is an algebraic variety and in particular, a hypersurface. We will denote by U_f the set of $s \in \mathbb{C}^m$ such that the fiber $\pi_f^{-1}(s)$ is smooth. The set U_f is the *versal deformation space* of the plane curve singularity f.

Proposition 3.2.2. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a plane curve singularity and $\pi_f : \mathbb{V}(F) \to \mathbb{C}^m$

the restriction of the projection to the vanishing locus of the versal deformation F. Then, there are $\delta, \varepsilon > 0$ such that the restriction of π_f to the set

$$V_f = \{(p,s) \in \mathbb{C}^2 \times \mathbb{C}^\mu \mid \|p\| < \delta, \|s\| < \varepsilon, s \notin \mathcal{D}_f, F(p,s) = 0\}$$

is a locally trivial fibration with fibers diffeomorphic to the Milnor fibers of f.

Any root system associated with a Dynkin diagram Γ of type A_n, D_n, E_6, E_7 or E_8 for $n \in \mathbb{N}_{\geq 3}$ comes with plane curve singularity f_Γ, as in the table below. We denote by F_Γ the versal deformation of f_Γ.

Root system	Germ f_Γ
Type A_n	$x^2 + y^{n+2}$
Type D_n	$y(x^2 + y^{n-2})$
Type E_6	$x^3 + y^4$
Type E_7	$x(x^2 + y^3)$
Type E_8	$x^3 + y^5$

The reason why we associate a Dynkin diagram as in Figure 3.1 to a plane curve singularities is explained below. The following is a Theorem of Arnol'd [Arn73, Propositions 9.1-9.3].

Theorem 3.2.3. Let Γ be one of the Dynkin digrams of type A_n, D_n, E_6, E_7 or E_8 for $n \in \mathbb{N}_{\geq 3}$. The versal deformation U_{f_Γ} is homeomorphic to M_Γ. In particular, the versal deformation space U_{f_Γ} is an Eilenberg–MacLane space $K(\pi, 1)$ for the Artin group A_Γ.

The isomorphism of Theorem 3.2.3 is given by a basis of polynomials $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$ generating the algebra of polynomials that are invariant under the action of W_Γ. In particular, the map $\tau_\Gamma : \mathbb{C}^n \to \mathbb{C}^n$ such that $\tau_\Gamma(x) = (f_1(x), \dots, f_m(x))$ induces an homeomorphism between $\mathbb{C}^n / W_\Gamma$ and $\mathbb{C}^n$ such that M_Γ is mapped homeomorphically to U_{f_Γ}.

If f is a plane singularity, the locally trivial fibration $\pi_f : V_f \to U_f$ of Theorem 3.2.2 induces a homomorphism $\rho_\Gamma : \pi_1(U_f) \to \mathrm{Mod}(\tilde{\Sigma})$ where $\tilde{\Sigma}$ has the diffeomorphic type of the smooth fibers of π_f. We will refer to such a homomorphism as the Γ-*monodromy*. Rigorously, if $\gamma : S^1 \to U_f$ is a smooth loop, then the pull-back $\gamma^*(V_f) \to S^1$ is a bundle over the circle. Any bundle over the circle is a mapping tori, and in particular there exists a mapping class $f \in \mathrm{Mod}(\tilde{\Sigma})$ such that the quotient of $\tilde{\Sigma} \times [0, 1]$ by the relation $(q, 0) \sim (f(q), 1)$ is diffeomorphic to $\gamma^*(V_f)$. The assignment $[\gamma] \mapsto f$ is a well-defined homomorphism.

A consequence of the Picard–Lefschetz formula [AGV88, Part I, page 26] is the following theorem. Recall that the fundamental groups $\pi_1(U_{f_\Gamma})$ are isomorphic to Artin groups.

Theorem 3.2.4 (Geometric monodromy of U_{f_Γ})**.** The standard generators of $\pi_1(U_{f_\Gamma})$

are mapped to Dehn twists through the Γ-monodromy ρ_Γ.

Sketch of the proof. There exists a line L in $\mathbb{C}^\mu$ that is in general position with $\mathcal{D}_f$ and intersects the discriminant locus in exactly μ points [AGV88, Part I, page 71]. The set $L \setminus \mathcal{D}_f$ coincides with the complement of the critical points of a perturbation $f_s = F(-,s)$ of f, such that $f_s^{-1}(0)$ is smooth. The restriction of $\pi_f : V_f \to U_f$ to $L \setminus \mathcal{D}_f$ coincides with the fibration of the non-singular level manifold of f_s over the set of non-critical points. Hence, the inclusion of $L \setminus \mathcal{D}_f$ in U_f induces an epimorphism of fundamental groups and the monodormy ρ_Γ can be pulled-back to the homomorphism $\pi_1(L\setminus\mathcal{D}_f) \to \mathrm{Mod}(\tilde{\Sigma})$ that is the monodromy of the fibration associated to the f_s. Note that $\pi_1(L \setminus \mathcal{D}_f)$ is a non-abelian free group. In a neighborhood of a (non-degenerate) critical point, the function f_s can be written in local coordinates as $f(x,y) = x^2 + y^2$ as a consequence of the complex Morse lemma. The fibers are open cylinders that degenerate to a cone in the origin, and a plane loop around the origin gives rise to a Dehn twist about the core curve [AGV88, Part I, Introduction]. □

§ 3.3 | Large Kernel Theorem

In this section we prove Theorem D.

Geometric homomorphisms

Let Ω be a finite family of isotopy classes of non-essential simple closed curves on $\tilde{\Sigma}$, such that the geometric intersection number of each pair of curves in Ω is at most 1. In other words, two isotopy classes of curves in Ω can either be represented by curves that can be homotoped to be disjoint or have geometric intersection 1. The *intersection graph* Γ_Ω of Ω is the graph with set of vertices in one-to-one correspondence with Ω and edges connecting any pair of intersecting curves. Then, any pair of Dehn twists about curves in Ω either commute or satisfy the braid relation. The subgroup of the mapping class group $\mathrm{Mod}(\tilde{\Sigma})$ generated by $\{T_\gamma \mid \gamma \in \Omega\}$ is a quotient of the Artin group A_{Γ_Ω} of type Γ_Ω. A *geometric homomorphism* is a homomorphism $A_\Gamma \to \mathrm{Mod}(\tilde{\Sigma})$ mapping standard generators to Dehn twists about isotopy classes of curves as in the family Ω. The defining graph is an intersection graph. This is the case for the Γ-monodromies ρ_Γ of Theorem 3.2.4.

The A_n and D_n type Artin groups can be embedded into the mapping class group of some surfaces via a geometric homomorphism [PV96, Théorème 1]. In particular, Perron–Vannier considered the Γ-monodromies ρ_Γ of Theorem 3.2.4. Wajnryb proved that for the E_6-type Artin group this is never the case [Waj99, Theorem 3].

Theorem 3.3.1. Any geometric homomorphism of an Artin group A_Γ is not injective if Γ contains E_6 as a subgraph.

Wajnryb used the following strategy. First, he found a non-trivial element $w \in A_{E_6}$

that maps trivially in the mapping class group of a surface of genus 3 and one boundary component via a geometric homomorphism with the set of curves Ω having intersection graph E_6. The result can be extended to Artin groups with defining graph Γ containing E_6. Indeed, in his PhD thesis, Van der Lek proved that if A_{Γ_1} and A_{Γ_2} are two Artin groups and the Dynkin diagrams Γ_1 is a full subdiagram of the Dynkin diagram Γ_2, then the inclusion of diagrams induces a monomorphism of A_{Γ_1} into A_{Γ_2} [Van83]. Hence, if $A_\Gamma \to \mathrm{Mod}(\tilde{\Sigma})$ is a geometric homomorphism and $\Sigma_3^1 \subset \tilde{\Sigma}$ is an embedded subsurface of genus 3 and one boundary component, the following diagram commutes

$$\begin{array}{ccc} A_{E_6} & \longrightarrow & \mathrm{Mod}(\Sigma_3^1) \\ \big\downarrow & & \big\downarrow \\ A_\Gamma & \longrightarrow & \mathrm{Mod}(\tilde{\Sigma}), \end{array}$$

where the vertical map on the right hand-side is the homomorphism induced by the inclusion $\Sigma_3^1 \subset \tilde{\Sigma}$, while the horizontal homomorphisms are geometric homomorphisms.

With respect to Figure 1.1, the *Wajnryb element* w can be written as a word in the alphabet $\{a_1, b\} \subset A(E_6)$, where $b = a_4a_5a_3a_4a_2a_6a_5a_3a_4$ is contained in the subgroup of A_{E_6} isomorphic to $\mathcal{B}_6$ and generated by $a_2, \ldots, a_6$. The element b has the following braid representation.

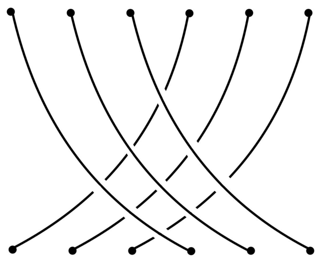


Figure 3.2: Braid representation of b.

The image of b via the geometric homomorphism ρ_{E_6} is represented by a diffeomorphism which maps the simple closed curve γ_1 to a simple closed curve β that intersects γ_1 once; see Figure 3.3.

Hence, the Dehn twists T_{γ_1} and T_β satisfy the braid relation and the Wajnryb element

$$w = a_1 a_1^b a_1 \cdot (a_1^{-1})^b a_1^{-1} (a_1^{-1})^b = [a_1, a_1^b a_1 b] \in [A_{E_6}, A_{E_6}],$$

acts trivially on Σ_3^1 as a mapping class. The notation a_1^b stands for ba_1b^{-1}. The image of w by ρ_{E_6} is precisely $T_{\gamma_1}T_\beta T_{\gamma_1} \cdot T_\beta^{-1} T_{\gamma_1}^{-1} T_\beta^{-1}$ which is the identity mapping class. Wajnryb proved that the group element w is non-trivial by applying the Garside algorithm, which can be checked on SageMath as follows. The method `.left_normal_form()` returns a tuple of simple generators in the left normal form. The first element is a power of Δ. In particular $g \in A_{E_6}$ is trivial if and only if `g.left_normal_form()` returns the tuple

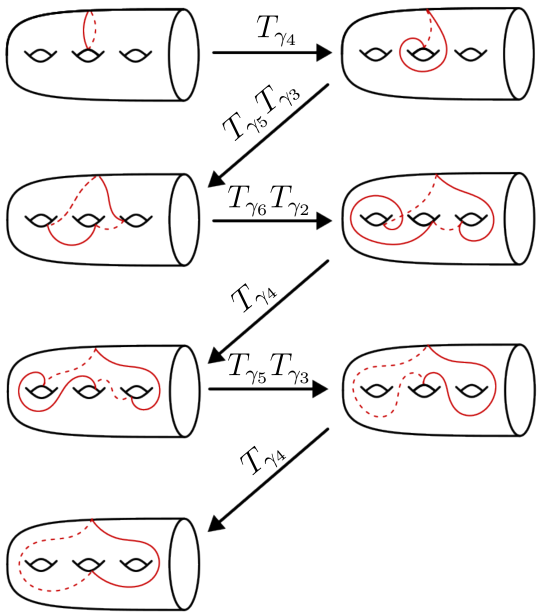


Figure 3.3: The red curve in the first surface on the left top corner is γ_1. The figure describes the diffeomorphism $\rho_{E_6}(b)$ acting on γ_1. The last curve on the bottom left side is β.

(1,). That the projection of the element $g \in A_{E_6}$ in Inn A_{E_6} is trivial if and only if $\sup(g) = 0$, or if the tuple given by the method g.left_normal_form() appears with non-trivial terms from the second onward.

```
A = ArtinGroup(['E',6])
a1 = A([2]) #In sage notation, s2
a2 = A([1]) #In sage notation, s1
a3 = A([3]) #In sage notation, s3
a4 = A([4]) #In sage notation, s4
a5 = A([5]) #In sage notation, s5
a6 = A([6]) #In sage notation, s6
b = a4*a5*a3*a4*a2*a6*a5*a3*a4
w = a1*b*a1*b^-1*a1*b*a1^-1*b^-1*a1^-1*b*a1^-1*b^-1
w.left_normal_form()

((s1^-1*(s2^-1*s3^-1*s1^-1*s4^-1)^2*s3^-1*(s5^-1*s4^-1*s2^-1*
s3^-1*s1^-1*s4^-1*s3^-1*s5^-1*s4^-1*s2^-1*s6^-1)^2*s5^-1*s4^-1*
s3^-1*s1^-1)^3, #Power of Garside element
s1*s3*s2*s4*s5*s6*s1*s3*s4*s5*s1*s3*s4*s1*s3*s2*s1, #Simple generators
s5*s6*s2*s4*s5*s3*s4*s1*s3*s2*s4*s5*s6*s5*s4*s1*s3*s2*s4*s5*s1
*s3*s4*s1*s3*s2*s1,
s1*s3*s2*s4*s5*s6*s1*s3*s4*s5*s1*s3*s4*s1*s3*s1,
s4*s5*s3*s4*s1*s3*s2*s4*s5*s6*s5*s4*s3*s2*s4*s5*s1*s3*s4,
s4*s5*s6*s3*s4*s5*s1*s3*s4,
```

```
s4*s5*s3*s4*s1*s3*s2*s4*s5*s6*s5*s4*s3*s2*s4*s5*s1*s3*s4*s2)
```

Listing 3.1: The Wajnryb element is non-trivial in both the Artin group of type E_6 and the inner automorphism group $\operatorname{Inn} A_{E_6}$.

We also remark that the Wajnryb element w is pure or, in other words, that is contained in kernel of the homomorphism $A_{E_6} \to W_{E_6}$. The triviality of w in W_{E_6} can be checked by the method `.coxeter_group_element()` on Sage.

Acylindrical hyperbolicity

Let $\mathcal{S}$ be the set of simple and absorbable elements of a spherical-type Artin group A_Γ, as defined in Section 3.1. The vertices of the *additional length graph* $\mathcal{C}_{AL}(\Gamma)$ are in one-to-one correspondence with the left cosets of the subgroup of A_Γ generated by the Garside element Δ. Two vertices, corresponding to the cosets $g_1\langle\Delta\rangle$ and $g_2\langle\Delta\rangle$, are adjacent if one can be obtained from the other by the left multiplication of some element in $\mathcal{S} \setminus \{\Delta\}$. The graph comes equipped with the metric where each edge has length one and A_Γ acts by isometries on the graph $\mathcal{C}_{AL}(\Gamma)$. Calvez–Wiest proved that $\mathcal{C}_{AL}(\Gamma)$ is a δ-hyperbolic geodesic metric space if A_Γ is spherical [CW16b, Theorem A] and $\operatorname{Inn} A_\Gamma$ is an *acylindrically hyperbolic* group for its action on $\mathcal{C}_{AL}(\Gamma)$ [CW16a, Theorem 1.3]. More precisely, they proved the following theorem.

Theorem 3.3.2. If A_Γ is a spherical-type Artin group, the isometric action of $\operatorname{Inn} A_\Gamma$ on $\mathcal{C}_{AL}(\Gamma)$ is cobounded and non-elementary. Moreover, for every $\varepsilon > 0$ there is a positive real numbers $R(\varepsilon)$ such that for each $x, y \in \mathcal{C}_{AL}(\Gamma)$ with $d(x, y) > R(\varepsilon)$, the set

$$\Gamma_\varepsilon(x, y) = \{g \in \operatorname{Inn} A_\Gamma \mid d(x, gx) < \varepsilon, d(y, gy) < \varepsilon\}$$

is finite.

The content of Theorem 3.3.2 effectively presents the definition of acylindrical hyperbolicity. Recall that an isometric action of a group G on a metric space X is cobounded if there is a ball $B \subset X$ such that $G \cdot B = X$; if X is a geodesic and hyperbolic metric space, the action of G is non-elementary if there exists a pair of independent *loxodromic* isometries $g_1, g_2 \in G$. An isometry is loxodromic if the map $n \mapsto g^n x$ is a quasi-isometry between the Cayley graph $\operatorname{Cay}(\mathbb{Z})$ and the orbit of some (equivalently any) point $x \in X$, that is, distances are only linearly distorted.

It follows from a more general theorem about acylindrical hyperbolic groups [Osi15, Theorem 1.1] that every $g \in \operatorname{Inn} A_\Gamma$ is either a loxodromic or an *elliptic* isometry of $\mathcal{C}_{AL}(\Gamma)$. If the action by an isometry g has bounded orbits, then g is elliptic. Any loxodromic element $g \in \operatorname{Inn} A_\Gamma$ is also *weakly properly discontinuous*: for every $\varepsilon > 0$ and $x \in \mathcal{C}_{AL}(\Gamma)$ there exists some $n \in \mathbb{Z}$ such that the set

$$\{g \in G \mid d(x, gx) < \varepsilon, d(\kappa^n x, g\kappa^n x) < \varepsilon\}$$

is finite. More generally, if a group G is acylindrically hyperbolic with respect to the action on a geodesic hyperbolic metric space X, then every loxodromic element also satisfies the weakly properly discontinuous condition. The existence of a loxodromic and weakly properly discontinuous group element is also a sufficient property to show that a non virtually-cyclic group is acylindrically hyperbolic [Osi15, Theorem 1.2]. Calvez–Wiest proved Theorem 3.3.2 by showing that

$$\kappa = a_4a_1a_3a_2a_4a_5a_4a_1a_3a_2a_6a_5a_5a_6a_2a_3a_1a_4a_5a_4a_2a_3a_1a_4 \tag{3.2}$$

projects to a loxodromic and weakly properly discontinuous isometry in $\operatorname{Inn} A_{E_6}$.

There is no known sufficient and necessary criterion to determine if a given isometry of an acylindrically hyperbolic group is loxodromic or elliptic. Nevertheless, Antolin–Cumplido gave a sufficient condition for an isometry of the additional length graph to have bounded orbits [AC21, Theorem 2]. We will describe this criterion below.

A *parabolic subgroup* P of an Artin group $A(\Gamma)$ is the conjugate of a subgroup generated by some strict subset of the standard generators. If P is not a direct product of non-trivial parabolic subgroups, we say that it is *irreducible*. The complex of irreducible parabolic subgroups $\mathcal{P}(\Gamma)$ is defined to have irreducible parabolic subgroups as vertices. A set of vertices $\{P_1, \dots, P_n\}$ is an n-simplex if one of the following properties is satisfied for all $i \neq j$:

- $P_i \subset P_j$ or $P_j \subset P_i$;
- $P_i \cap P_j = \{1\}$ and $[P_i, P_j] = 1$.

The complex $\mathcal{P}(\Gamma)$ can detect elliptic isometries of $\mathcal{C}_{AL}(\Gamma)$.

Theorem 3.3.3. [AC21, Theorem 2] Suppose A_Γ is an irreducible spherical-type Artin group with more than two standard generators. The elements preserving some simplex of $\mathcal{P}(\Gamma)$ act elliptically on $\mathcal{C}_{AL}(\Gamma)$. In particular, the normalizers of parabolic subgroups act elliptically on $\mathcal{C}_{AL}(A)$.

Later, we are also going to use a technical lemma borrowed from Antolin–Cumplido paper [AC21, Lemma 25]. This lemma gives the following estimate for $g \in A_{E_6}$ infinite order element in the normalizer of a proper standard parabolic subgroup and $x \in \mathcal{C}_{AL}(E_6)$:

$$d(g\kappa^n x, \kappa^n x) \geq d(x, \kappa^n x) + K, \tag{3.3}$$

for some constant $K > 0$ and $|n|$ big enough.

We now prove Theorem D and construct a non-abelian free subgroup of rank 2 in the kernel of any geometric homomorphism of Artin group with defining graph containing E_6. Recall that $\kappa \in \operatorname{Inn} A_\Gamma$ is the loxodromic isometry of $\mathcal{C}_{AL}(E_6)$ in (3.2). In view of the Abbott–Dahmani result [AD19, Proposition 2.1], we show that the Wajnryb element w is an elliptic isometry of $\mathcal{C}(E_6)$, that none of its powers preserve the quasi-axis $A_{10\delta}(\kappa)$ and that $\mathrm{Fix}_{50\delta}(w)$ is a bounded set. It will follow that there is a power $n \in \mathbb{Z}$ such that the subgroup $\langle w, \kappa^n \rangle$ is a non-abelian free group of rank 2.

Proof of Theorem D. Let Ω be a collection of isotopy classes of non-essential simple closed curves on a connected and compact orientable surface $\tilde{\Sigma}$ that pairwise intersect at most once, and suppose that the intersection graph Γ_Ω contains E_6 as a subgraph. The hypotheses of the Abbott–Dahmani result [AD19, Proposition 2.1] are satisfied for w and κ by Lemma 3.3.4, Lemma 3.3.5 and Lemma 3.3.6 below. However, the loxodromic isometry κ is not in the kernel of ρ_{E_6}. Nevertheless, if we denote by w^{κ^n} the conjugate $\kappa^{-n} w \kappa^n$, the group $\langle w, w^{\kappa^n} \rangle$ is contained in the kernel of the geometric homomprhism associated to Ω and it is also isomorphic to $\mathbb{F}_2$, as any combination of letters in $\{w, w^{\kappa^n}\}$ that represents a trivial word is also a combination of letters in $\{w, \kappa^n\}$. $\square$

Lemma 3.3.4. The projection of w in $\mathrm{Inn}\, A_{E_6}$ is an elliptic isometry of the additional length graph $\mathcal{C}_{AL}(E_6)$.

Proof. We would like to apply the Antolin–Cumplido criterion from Theorem 3.3.3. It is enough to show that the subgroup $\langle a_1, b \rangle$ normalizes the parabolic subgroup $\langle a_2, a_5 \rangle$. The action of b by conjugation on A_{E_6} permutes a_2 and a_5 (see Figure 3.4). Since a_1 is in the centralizer of both a_2 and a_5, we can conclude that the group generated by a_1 and b preserves the 2-simplex $\{\langle a_2 \rangle, \langle a_5 \rangle\}$ of the complex $\mathcal{P}(E_6)$. $\square$

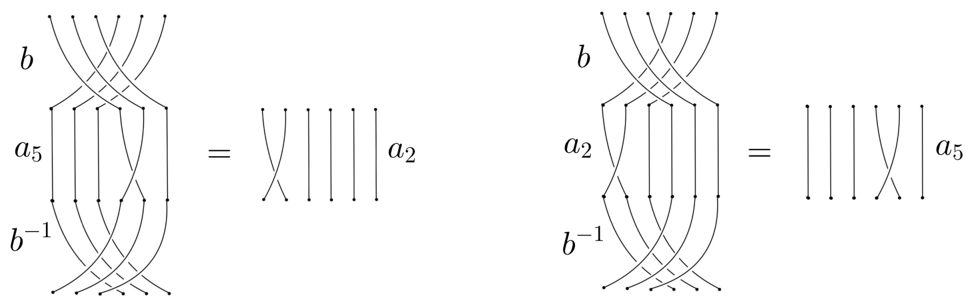


Figure 3.4: Braid representation of the conjugacy action of b on $\langle a_2, a_5 \rangle$.

Lemma 3.3.5. No non-trivial power of the Wajnryb element $w \in A(E_6)$ preserves the 10δ-quasi fixed axis $A_{10\delta}(\kappa)$ of κ.

Proof. Let $x \in \mathrm{A}_{10\delta}(\kappa)$ be a vertex of $\mathcal{C}_{AL}(E_6)$. If we suppose that w, or any of its non trivial power, preserves $\mathrm{A}_{10\delta}(\kappa)$ we would have that

$$\begin{aligned} d(w\kappa^n x, \kappa^n x) &\leq d(w\kappa^n x, \kappa^n wx) + d(\kappa^n wx, \kappa^n x) && \text{(triangular inequality)} \\ &= d(w^{\kappa^n} x, wx) + d(wx, x) && \text{(κ is an isometry)} \\ &\leq d(w^{\kappa^n} x, x) + 2d(wx, x) && \text{(triangular inequality)} \\ &\leq \inf_{y \in \mathcal{C}_{AL}(E_6)} d(y, \kappa y) + 10\delta + 2d(wx, x), && \text{(definition of $A_{10\delta}(\kappa)$)} \end{aligned}$$

for any $n \in \mathbb{Z}$, where the last inequality follows from the fact that also κ preserves $\mathrm{A}_{10\delta}(\kappa)$. However, the inequality (3.3) implies that it cannot happen, as κ is loxodromic. □

Every spherical-type Artin group has a finite-dimensional $K(\pi, 1)$ space given by the complement of a hyperplane arrangement associated with the respective Coxeter group [Del72] and a finite-dimensional CW complex that is $K(\pi, 1)$ has torsion-free fundamental groups [Hat02, Proposition 2.45]. In particular, the Artin group A_{E_6} is torsion-free. However, the group $\operatorname{Inn} A_{E_6}$ has torsion elements but the Wajnryb element w is not a periodic isometry of $\mathcal{C}_{AL}(E_6)$.

In order to prove the following lemma, we recall that standard generators $\{a_1, \dots, a_n\}$ of an Artin group A_Γ are related by length-preserving relations and in particular the map

$$\begin{aligned} \deg : A_\Gamma &\to \mathbb{Z} \\ a_{i_1}^{n_1} \dots a_{i_k}^{n_k} &\mapsto \sum_{j=1}^{k} n_k \end{aligned}$$

is a homomorphism. The commutator subgroup of A_Γ is exactly the kernel of the length homomorphism $\deg : A(\Gamma) \to \mathbb{Z}$ [MR06, Proposition 3.1].

Lemma 3.3.6. The projection of the Wajnryb element w in $\operatorname{Inn} A_{E_6}$ is not torsion.

Proof. Suppose there is some $m \in \mathbb{Z}$ such that w^m is central in A_{E_6} and can be written as Δ^k for some integer k. The degree $\deg(w)$ is zero and therefore we can write w as a commutator. However, the Garside element of A_{E_6} is $\Delta = (a_1 a_3 a_5 a_2 a_4 a_6)^6$ and has positive length. Hence, we have that

$$0 = \deg(w^m) = \deg(\Delta^k) = k \cdot \deg(\Delta)$$

and k is then forced to be equal to zero. Since A_{E_6} is torsion-free, the only possibility for the m^{th}-power of w to be trivial is that $m = 0$. □

The set $\mathrm{Fix}_{50\delta}(w)$ is then necessarily bounded.

Lemma 3.3.7. Let G be a group acting acylindrically on a δ-hyperbolic space X. If there exists $K > 0$ such that $\mathrm{Fix}_K(g)$ is unbounded, then g has finite order.

Proof. Let $x, y \in \mathrm{Fix}_K(g)$ be two points of X such that $d(x, y)$ is greater than the constant $R(K)$ from the definition of acylindrical hyperbolicity of a group (Theorem 3.3.2). Then, the set $\Gamma_K(x, y)$ is finite and contains any power of g. $\square$

Chapter 4

Monodromy kernels in low genus

The main result of this chapter is about the monodromies of the projective non-hyperelliptic stratum components $\mathbb{P}\mathcal{H}^{\text{odd}}(4)$, $\mathbb{P}\mathcal{H}(3,1)$ and $\mathbb{P}\mathcal{H}^{\text{even}}(6)$. We prove that such monodromies are geometric homomorphism in the sense of Subsection 3.3 and we observe that $\mathbb{P}\mathcal{H}^{\text{even}}(6)$ is an orbifold classifying space for the inner automorphisms group $\text{Inn}(A_{E_8})$. The same $K(\pi,1)$ result has been proved by Looijenga–Mondello for the projective stratum components $\mathbb{P}\mathcal{H}^{\text{odd}}(4)$ and $\mathbb{P}\mathcal{H}(3,1)$. By the Large Kernel Theorem for spherical-type Artin groups containing A_{E_6} (Theorem D), we will immediately deduce Theorem C.

Proof of Theorem C. The Dynkin diagrams E_Γ for $\Gamma = E_6, E_7, E_8$ contain E_6 as a subgraph and by Thereom D any geometric homomorphism $A_{E_8} \to \text{Mod}(\tilde{\Sigma})$ has the Large Kernel Property. The copy of the non-abelian free group of rank 2 of Theorem D embeds in $\text{Inn}\, A_\Gamma$ and the monodromy $\rho_{\mathbb{P}\mathcal{C}}$ of $\mathcal{C} = \mathcal{H}^{\text{odd}}(4), \mathcal{H}(3,1)$ and $\mathcal{H}^{\text{even}}(6)$ has the Large Kernel Property as $\ker \rho_{\mathbb{P}\mathcal{C}}$ contains a copy of $\mathbb{F}_2$.

The monodromy $\rho_\mathcal{C} : \pi_1^{orb}(\mathcal{C}) \to \text{Mod}(\Sigma, \mathcal{Z})$ factors through a homomorphism from $\text{Inn}(A_\Gamma)$ to the mapping class group $\text{Mod}(\Sigma, \mathcal{Z})$. Indeed, the following diagram commutes

$$\begin{array}{ccc} & & \pi_1^{orb}(\mathcal{C}) \\ & \swarrow & \downarrow \\ \pi_1^{orb}(\mathbb{P}\mathcal{C}) & \longrightarrow & \text{Mod}(\Sigma, \mathcal{Z}), \end{array}$$

where $\rho_\mathcal{C}$ is the vertical map and the oblique one is induced by the principal $\mathbb{C}^*$-bundle $q_\mathcal{C} : \mathcal{TC} \to \mathbb{P}(\mathcal{TC})$. The horizontal map is the monodromy $\rho_{\mathbb{P}\mathcal{C}}$. If the Large Kernel Property holds for the monodromy $\rho_{\mathbb{P}\mathcal{C}}$, then the Large Kernel Property also holds for the monodromy $\rho_\mathcal{C}$. □

The proof of Theorem A also follows from Theorem D. Before proceeding, however, it is useful to introduce the definition of a *cylinder shear*.

A *cylinder* ξ on a translation surface is an isometric embedding of an Euclidean cylinder whose boundary is a union of saddle connections. In particular, the interior of ξ does not contain any singular point.

If an embedded cylinder ξ is isometric to $(\mathbb{R}/a\mathbb{Z}) \times [0, b]$ for some $a, b \in \mathbb{R}^+$, the *core curve* of ξ is the isotopic class of the simple closed curve $(\mathbb{R}/a\mathbb{Z}) \times \{t\}$ for some $t \in (0, b)$.

Suppose ξ is a horizontal cylinder on a translation surface (X, ω), so that the cylinder ξ can be represented as a rectangle $[0, b] \times [0, a]$ embedded in a defining polygon of (X, ω) with a pair of sides identified. Suppose that the ratio between its height a and its weight b is R. If $t \in [0, R]$, a cylinder shear along ξ is the result of the continuous action given by the matrix

$$S_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

acting on the embedded parallelogram of the polygon representative. Analogously, by taking a suitable conjugate of S_t one can define a cylinder shear along non-horizontal cylinders.

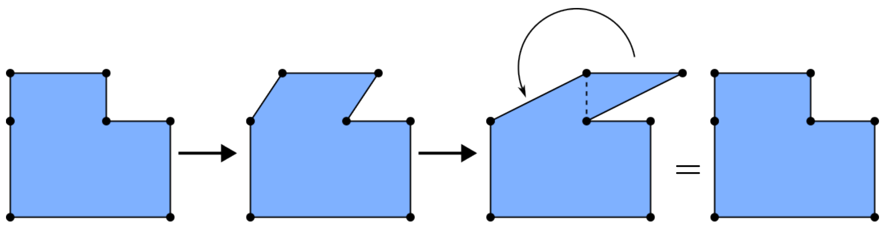
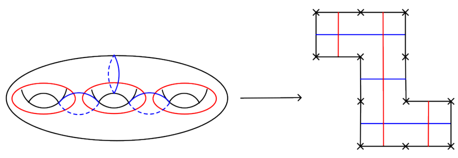


Figure 4.1: A full cylinder shear action on an L-shaped translation surface of genus 2, where opposite sides of the polygon are identified via a translation.

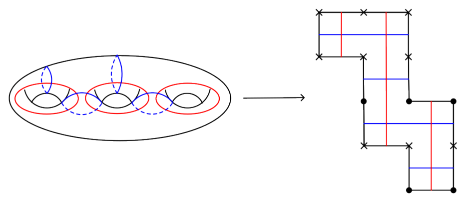
Let now $\phi : \Sigma \to X$ be a marking of (X, ω). The full shear S_R acts on (X, f, ω) preserving the translation structure of X, as the resulting polygon differs from the initial one by a scissor move, as in Figure (4.1). However, the matrix S_R changes the marking ϕ by a Dehn twist along the core curve of the cylinder ξ.

A cylinder shear is an orbifold loop induced by the continuous deformation of a conjugate of S_t on a cylinder for $t \in [0, R]$ or, in other words, it is an element $(\delta, f) \in \pi_1^{orb}(\mathcal{C})$ where $\mathcal{C}$ is the stratum component containing (X, ω) and δ is a continuous path in $\mathcal{TC}$ with endpoints (X, ϕ, ω) and $(X, \phi \circ T_\gamma^{-1}, \omega)$ for some simple closed curve γ isotopic to the core curve of cylinder sheared. The cylinder shear is mapped via the topological monodromy map of the connected component containing (X, ω) to the Dehn twist T_γ about the core curve of the sheared cylinder.

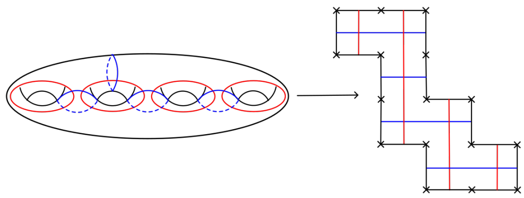
We will see in the next sections that the generators of the orbifold fundamental groups of $\mathbb{P}\mathcal{H}^{\text{odd}}(4)$, $\mathbb{P}\mathcal{H}(3, 1)$ and $\mathbb{P}\mathcal{H}^{\text{even}}(6)$ can be thought of as cylinders shears. More



(a) Marked translation surface in $\mathcal{TH}^{\mathrm{odd}}(4)$



(b) Marked translation surface in $\mathcal{T}(3,1)$



(c) Marked translation surface in $\mathcal{TH}^{\mathrm{even}}(6)$

Figure 4.2: Marked translation surfaces giving cylinder shears as generators for the orbifold fundamental group of the respective projective stratum components. The blue and red curves are horizontal and vertical cylinders, respectively.

precisely, every class of a standard generators in $\mathrm{Inn}(A_\Gamma)$ is the image of a cylinder shear in $\pi_1^{orb}(\mathcal{C})$ as in Figure 4.2.

Proof of Theorem A. The Wajnryb element w has been given in terms of standard generators of Artin groups. The class of the Wajnryb element in the kernel of $\rho_{\mathbb{P}\mathcal{C}}$ can then be lifted in $\pi_1^{orb}(\mathcal{C})$ so that each class of a standard generator in $\mathrm{Inn}(A_\Gamma)$ is lifted to a cylinder shear. The core curves of cylinders can be chosen to have an intersection graph as in Figure 4.2. In other words, the marked translation surfaces in Figure 4.2 can be chosen to be the base points of the orbifold fundamental groups. Let us denote by C_w such a lift of w in $\pi_1^{orb}(\mathcal{C})$. The $\mathbb{F}_2$ copy of Theorem D in the kernel of $\ker \rho_{\mathbb{P}\mathcal{C}}$ is generated by the class of w and a conjugate w^g for some $g \in A_\Gamma$. Hence, the $\mathbb{F}_2$ copy of Theorem C is generated by C_w and one of its conjugates. The image of C_w via the pronged monodromy map $\rho_{\hat{\mathcal{C}}}$ is trivial, since the Wajnryb element w returns a trivial mapping class, via the geometric homomorphism, even in the mapping class group of a surface with boundary components. Hence, the orbifold loop C_w is in the kernel of the pronged monodromy map $\rho_{\hat{\mathcal{C}}}$ and therefore in the fundamental group of $\mathcal{T}\hat{\mathcal{C}}$. □

§ 4.1 | Genus $g = 3$

The following result about $\mathcal{H}^{\mathrm{odd}}(4)$ is known by experts and can be also proved using the same techniques we will adopt in Section 4.2 for $\mathcal{H}^{\mathrm{even}}(6)$. It appears as a consequence of Pinkham's thesis [Pin74] and can also be found in [Ham18, Introduction].

Theorem 4.1.1. Let $\{\xi_1, \ldots, \xi_6\}$ be a collection of embedded cylinders of a translation surface $(X, \omega) \in \mathcal{H}^{\text{odd}}(4)$ such that the family of the associated core curves have an E_6-type intersection graph. Then, there exists a surjective homomorphism $A_{E_6} \to \pi_1^{orb}(\mathbb{P}\mathcal{H}^{odd}(4))$ so that each standard generator of A_{E_6} is mapped to a full cylinder shear and the kernel is the center of A_{E_6}.

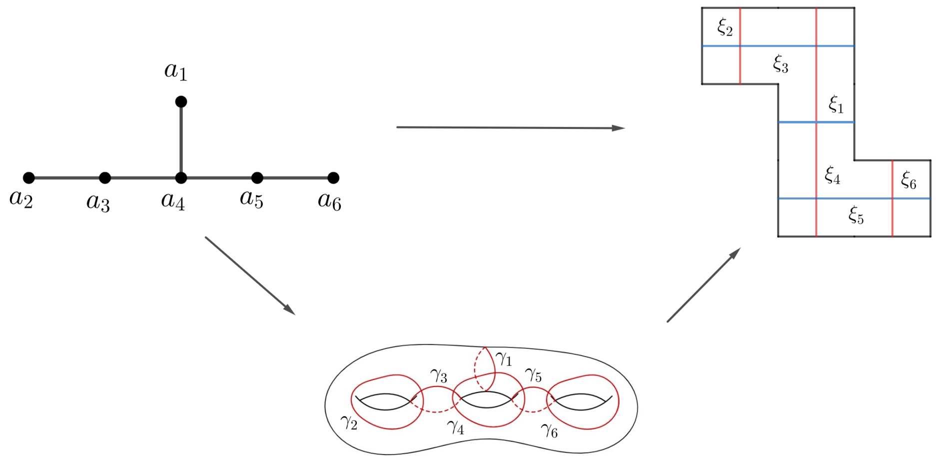


Figure 4.3: An S-shaped translation surface in $\mathcal{H}^{\text{odd}}(4)$. The red and blue segments represent the core curves of vertical and horizontal cylinders ξ_i, respectively. Their intersection graph is E_6.

The homomorphism of Theorem 4.1.1 is well-defined. Every pair of adjacent standard generators in A_{E_6} is mapped to a cylinder shears along embedded cylinders with core curves intersecting once; every pair of standard generators that commute is mapped to cylinder shears along disjoint flat cylinders. Theorem 4.1.1 shows that $\pi_1^{orb}(\mathbb{P}\mathcal{H}^{\text{odd}}(4))$ is generated by a finite family of orbifold loops coming from cylinder shears in $\pi_1^{orb}(\mathbb{P}\mathcal{H}^{odd}(4))$.

We now prove that the monodromy of $\mathbb{P}\mathcal{H}(3,1)$ is geometric. Recall that every genus 3 non-hyperelliptic Riemann surface X can be embedded in $\mathbb{CP}^2$ as the vanishing locus of a smooth plane quartic [Mir95, Chapter VII, Proposition 2.5]. The embedding of X in $\mathbb{CP}^2$ is defined as the unique projective embedding, up to linear change of coordinates, that corresponds to the linear system of positive canonical divisors on X. By abuse of notations, we identify every genus 3 non-hyperelliptic Riemann surface X with its image in $\mathbb{CP}^2$.

A *flex* of a smooth quartic X is a point $p \in X$ where the intersection multiplicity of X with its tangent space is exactly 3. A smooth and non-hyperelliptic plane quartic X with a flex point p can always be reparametrized in such a way that p is the point at infinity $[0 : 0 : 1]$ and its vanishing polynomial is of the form

$$Q_s = x^3z + y^3x + s_1xyz^2 + s_2xz^3 + s_3y^4 + s_4y^3z + s_5y^2z^2 + s_6yz^3 + s_7z^4 \in \mathbb{C}[x, y, z],$$

for some $s = (s_1, \ldots, s_7) \in \mathbb{C}^7$ [Shi93, Proposition 1].

However, there are some strata $\mathcal{H}(k_1, \ldots, k_n)$ for $n \neq 1$ where all the underlying Riemann surfaces are non-hyperelliptic. This is the case if all the odd numbers in the partition $(k_1, \ldots, k_n)$ appear an odd number of times. At its core, this can ultimately be deduced by observing that every positive canonical divisor on a hyperelliptic Riemann surface is the pullback of a divisor on the Riemann sphere $\mathbb{CP}^1$ [Har77, Chapter IV, Proposition 5.3] and apply the Riemann–Hurwitz formula [Mir95, Chapter V, Lemma 1.16]. The stratum $\mathcal{H}(3,1)$ is totally non-hyperelliptic.

Proposition 4.1.2. Let (X, ω) be a translation surface in $\mathcal{H}(3,1)$. Then X has a flex in the zero of ω of order 3. Hence, the Riemann surfaces at each point in $\mathcal{H}(3,1)$ are vanishing loci $\mathbb{V}(Q_s)$ of quartics of the form Q_s, up to isomorphism.

Proof. Since X is a genus 3 projective smooth curve, the positive canonical divisors associated with an abelian differential in $\mathcal{H}(3,1)$ coincide with divisors coming from lines in $\mathbb{CP}^2$ that intersect X in two points. One of these points, say p, has multiplicity 3; then the respective line in $\mathbb{CP}^2$ is necessarily the tangent line to X in p. In particular, X has a flex at p and is isomorphic to the vanishing locus of a quartic Q_s. □

On the other hand, every smooth vanishing locus $\mathbb{V}(Q_s)$ comes with an abelian differential in $\mathcal{H}(3,1)$ as follows. The vanishing loci $\mathbb{V}(Q_s)$ are compact Riemann surfaces and the points at infinity can be removed to get a genus 3 surface with two points removed. Equivalently, we can evaluate the homogeneous polynomial Q_s at $z = 1$ to get a polynomial $q_s \in \mathbb{C}[x,y]$ and the respective affine vanishing locus $\mathbb{V}(q_s)$ in $\mathbb{C}^2$. Since every vanishing locus $\mathbb{V}(q_s)$ is the zero level set of a holomorphic function, the two complex derivatives $\partial_x q_s$ and $\partial_y q_s$ satisfy $\partial_x q_s dx + \partial_y q_s dy = 0$. Moreover, the derivatives $\partial_x q_s$ and $\partial_y q_s$ cannot simultaneously vanish since $\mathbb{V}(q_s)$ is smooth. Hence, the abelian differential

$$\omega_s(x_0, y_0) = \begin{cases} \frac{dx}{\partial_y q_s(x_0,y_0)} & \text{if } \partial_y q_s(x_0, y_0) \neq 0 \\ -\frac{dy}{\partial_x q_s(x_0,y_0)} & \text{if } \partial_x q_s(x_0, y_0) \neq 0 \end{cases}$$

is well-defined and non-vanishing at every point $(x_0, y_0) \in \mathbb{V}(q_s)$. The volume form ω_s can be holomorphically extended to be zero on the two points at infinity $[1:0:0]$ and $[0:1:0]$, where ω_s vanishes with multiplicity 3 and 1, respectively.

Looijenga observed that a line intersecting $\mathbb{V}(Q_s)$ with multiplicity 3 is determined solely by the parameter s [Loo00, Introduction]. In particular, up to a rescaling factor, the abelian differential ω_s is the unique holomorphic 1-form on $\mathbb{V}(Q_s)$ such that the pair $(\mathbb{V}(Q_s), \omega_s)$ is a translation surface in $\mathcal{H}(3,1)$. In what follows, we are going to denote by $\mathcal{M}_{3,2}^{\text{flex}}$ the moduli space of non-hyperelliptic genus 3 Riemann surfaces with 2 marked points given by a flex $p \in X$ and the unique point of intersection between T_pX and X with multiplicity 1 (Figure 4.4). The next result is a direct consequence of the above discussion and Proposition 2.5.5.

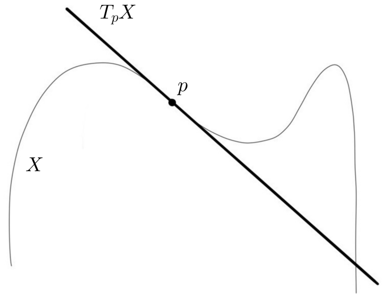


Figure 4.4: A quartic X with a flex in p.

Proposition 4.1.3. There exists an orbifold isomorphism between $\mathbb{P}\mathcal{H}(3,1)$ and $\mathcal{M}_{3,2}^{\mathrm{flex}}$. In particular, there exists an isomorphism $\theta_1 : \pi_1^{orb}(\mathbb{P}\mathcal{H}(3,1)) \to \pi_1^{orb}(\mathcal{M}_{3,2}^{\mathrm{flex}})$ that commutes with the monodromies $\rho^{\mathrm{flex}} : \pi_1^{orb}(\mathcal{M}_{3,2}^{\mathrm{flex}}) \to \mathrm{Mod}_{3,2}$ and $\rho_{\mathbb{P}\mathcal{H}(3,1)} : \pi_1^{orb}(\mathbb{P}\mathcal{H}(3,1)) \to \mathrm{Mod}_{3,2}$ of the respective moduli spaces.

The collection of parameters $s \in \mathbb{C}^7$ representing smooth quartics Q_s is an Eilenberg–Maclane space for the spherical-type Artin group A_{E_7}: the versal deformation space of the germ $f_{E_7} = x^3 + y^3x$ [Arn73, Proposition 9.3]. Indeed, suppose the $\mathbb{C}$-algebra of W_{E_7}-invariant polynomials in $\mathbb{C}[x_1, \dots, x_7]$ is generated by some homogeneous polynomials $q_1, \dots, q_7$ with degrees $d_i = \deg(q_i)$ uniquely determined by the finite group W_{E_7}. The basis $\{q_1, \dots, q_7\}$ maps (in a neighbourhood of zero) the quotient space $\mathbb{C}^7/W_{E_7}$ to $\mathbb{C}^7$ by the isomorphism $\tau_{E_7} : \mathbb{C}^7/W_{E_7} \to \mathbb{C}^7$ of complex manifolds. In particular, the image of the hyperplane arrangement $\cup_{i\in I} H_i$ modulo W_{E_7} is the hypersurface $\Pi = \{s \in \mathbb{C}^7 \mid \mathbb{V}(Q_s) \text{ is singular}\}$ defined as the vanishing locus of a weighted homogeneous polynomial with weights given by the degrees $(d_1, \dots, d_7)$ of the homogeneous polynomials $\{q_1, \dots, q_7\}$; see, for example, [OS88, Introduction] or [AGV88, Chapter 3]. The intersection of the space $\{(p, s) \in \mathbb{C}^2 \times (\mathbb{C}^7 \setminus \Pi) \mid p \in \mathbb{V}(q_s)\}$ with a sufficiently small closed polydisk $\mathbb{D}^2 \times \mathbb{D}^7$ in $\mathbb{C}^2 \times \mathbb{C}^7$ is the versal deformation space $V_{f_{E_7}}$, while its image in $\mathbb{C}^7 \setminus \Pi$ is $U_{f_{E_7}}$.

The proof of Theorem C for $\mathcal{H}(3,1)$ relies on the existence of a surjective homomorphism

$$\theta : \pi_1(U_{f_{E_7}}) \to \pi_1^{orb}(\mathbb{P}\mathcal{H}(3,1))$$

such that the two monodromies $\rho_{E_7} : \pi_1(U_{f_{E_7}}) \to \mathrm{Mod}_3^2$ and $\rho_{\mathbb{P}\mathcal{H}(3,1)} : \pi_1^{orb}(\mathbb{P}\mathcal{H}(3,1)) \to \mathrm{Mod}_{3,2}$ fit inside the following commutative diagram

$$\begin{array}{ccc} \pi_1(U_{f_{E_7}}) & \xrightarrow{\ \theta\ } & \pi_1^{orb}(\mathbb{P}\mathcal{H}(3,1)) \\ \Big\downarrow \rho_{E_7} & & \Big\downarrow \rho_{\mathbb{P}\mathcal{H}(3,1)} \\ \mathrm{Mod}_3^2 & \xrightarrow{\ \mathrm{Cap}\ } & \mathrm{Mod}_{3,2}\,. \end{array} \tag{4.1}$$

Let us define $\theta : \pi_1(U_{f_{E_7}}) \to \pi_1^{orb}(\mathbb{P}\mathcal{H}(3,1))$. We do so by composing two homomorphisms, where one of them has already been given in Proposition 4.1.3. In what follows, we construct a surjective homomorphism $\theta_2 : \pi_1(U_{f_{E_7}}) \to \pi_1^{orb}(\mathcal{M}_{3,2}^{\text{flex}})$. Then, the composition $\theta_1^{-1} \circ \theta_2$ will be the homomorphism θ we need in order to prove Theorem C.

A pair of smooth quartics Q_s and Q_t might define the same isomorphism class of a Riemann surface. This is the case if and only if the parameters s and t are related by a weighted projective relation [Shi93, Proposition 1]. In particular, the vanishing loci $\mathbb{V}(Q_s)$ and $\mathbb{V}(Q_t)$ are isomorphic if and only if there exists $\lambda \in \mathbb{C}^*$ such that

$$(s_1, s_2, s_3, s_4, s_5, s_6, s_7) = (\lambda t_1, \lambda^3 t_2, \lambda^4 t_3, \lambda^5 t_4, \lambda^6 t_5, \lambda^7 t_6, \lambda^9 t_7). \tag{4.2}$$

The above relation is well-defined on Π. Indeed, the defining weighted polynomial of Π has weights compatible with the weights of the relation in (4.2); we can see it by noticing that the weights given in (4.2) coincide with half the degrees $(d_1, \dots, d_7)$ of the homogeneous polynomials $q_1, \dots, q_7$ [Hum90, Section 3.7]. In particular, the above relation is also well-defined on $\mathbb{C}^7 \setminus \Pi$ and therefore on $U_{f_{E_7}}$.

Topologically, the weighted projective space obtained from the quotient of $U_{f_{E_7}}$ by the relation in (4.2) can be realized as a subspace of the moduli space of genus 3 Riemann surfaces with 2 boundary components. We will denote it by $\mathcal{M}_{3,2}^{\text{flex},\partial}$.

Lemma 4.1.4. The quotient map $l : U_{f_{E_7}} \to \mathcal{M}_{3,2}^{\text{flex},\partial}$ induces a surjective homomorphism $l_* : \pi_1(U_{f_{E_7}}) \to \pi_1(\mathcal{M}_{3,2}^{\text{flex},\partial})$ on the respective fundamental groups.

Proof. The weighted projective relation defined in (4.2) on $\mathbb{C}^7$ pulls back to a projective relation on the quotient $\mathbb{C}^7/W_{E_7}$ via the isomorphism $\tau_{E_7} : \mathbb{C}^7/W_{E_7} \to \mathbb{C}^7$. In other words, the isomorphism τ_{E_7} induces a homeomorphism between the weighted projective space defined by (4.2) and $\mathbb{CP}^6$ modulo the induced linear action of W_{E_7}.

Then, the quotient map l can also be seen as the map

$$l : \mathbb{C}^7 \setminus \cup_{i \in I} H_i \big/ W_{E_7} \longrightarrow \mathbb{P}(\mathbb{C}^7 \setminus \cup_{i \in I} H_i) \big/ W_{E_7},$$

where $\mathbb{P}(\mathbb{C}^7 \setminus \cup_{i \in I} H_i)$ is the projectivization of the space $\mathbb{C}^7 \setminus \cup_{i \in I} H_i$.

The map l descends from the fiber bundle $\mathbb{C}^7 \setminus \cup_{i \in I} H_i \to \mathbb{P}(\mathbb{C}^7 \setminus \cup_{i \in I} H_i)$ via the free action of the finite group W_{E_7} and has connected fibers. In particular, the map l is a fiber bundle with connected fibers and the induced homomorphism on the fundamental groups is surjective by applying the long exact sequence associated with l. $\square$

The Teichmüller cover of $\mathcal{M}_{3,2}^{\text{flex},\partial}$ will be denoted by $\mathcal{T}_{3,2}^{\text{flex},\partial}$ in the following.

Proposition 4.1.5. Let $\rho^{\text{flex}} : \pi_1^{orb}(\mathcal{M}_{3,2}^{\text{flex}}) \to \text{Mod}_{3,2}$ be the monodromy of $\mathcal{M}_{3,2}^{\text{flex}}$ and let $\rho : \pi_1(\mathbb{C}^7 \setminus \Pi) \to \mathrm{Mod}_3^2$ denote the monodromy of $\mathbb{C}^7 \setminus \Pi$. There exists a surjective homomorphism $\theta_2 : \pi_1(\mathbb{C}^7 \setminus \Pi) \to \pi_1^{orb}(\mathcal{M}_{3,2}^{\text{flex}})$ that commutes with the respective monodromies. In particular, the following diagram commutes

$$\begin{array}{ccc} \pi_1(\mathbb{C}^7 \setminus \Pi) & \xrightarrow{\theta_2} & \pi_1^{orb}(\mathcal{M}_{3,2}^{\text{flex}}) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho^{\text{flex}}} \\ \mathrm{Mod}_3^2 & \xrightarrow{\text{Cap}} & \mathrm{Mod}_{3,2}\,. \end{array}$$

Proof. The surjective homomorphism $l_* : \pi_1(U_{f_{E_7}}) \to \pi_1(\mathcal{M}_{3,2}^{\text{flex},\partial})$ is induced by the quotient map, and in particular it is induced by a bundle map between surface bundles with isomorphic fibers. Therefore, the monodromies of $U_{f_{E_7}}$ and $\mathcal{M}_{3,2}^{\text{flex},\partial}$ must commute through l_*.

The group Mod_3^2 is torsion-free and therefore the orbifold structure of $\mathcal{M}_{3,2}^{\text{flex},\partial}$ is not singular. In particular, the orbifold fundamental group of $\mathcal{M}_{3,2}^{\text{flex},\partial}$ can be identified with its fundamental group $\pi_1(\mathcal{M}_{3,\partial}^{\text{flex}})$. In other words, if $\rho_\partial^{\text{flex}}$ is the monodromy of the moduli space $\mathcal{M}_{3,2}^{\text{flex},\partial}$, the diagram

$$\begin{array}{ccc} \pi_1(U_{f_{E_7}}) & \xrightarrow{l_*} & \pi_1(\mathcal{M}_{3,\partial}^{\text{flex}}) \\ & {\scriptstyle \rho} \searrow \quad \swarrow {\scriptstyle \rho_\partial^{\text{flex}}} & \\ & \mathrm{Mod}_3^2 & \end{array}$$

must commutes.

Suppose now $\mathcal{T}_{3,2}^{\text{flex},\partial}$ and $\mathcal{T}_3^{\text{flex}}$ are the Teichmüller covers of $\mathcal{M}_{3,2}^{\text{flex},\partial}$ and $\mathcal{M}_3^{\text{flex}}$, respectively. There exists a map $\mathcal{T}_{3,2}^{\text{flex},\partial} \to \mathcal{T}_3^{\text{flex}}$ given by collapsing the lengths of the boundary components to zero. In particular, this map is the restriction of the classic projection given on the respective global Teichmüller spaces where the preimage of $\mathcal{T}_3^{\text{flex}}$ is exactly $\mathcal{T}_{3,2}^{\text{flex},\partial}$. Hence, the induced map $\pi_1(\mathcal{T}_{3,2}^{\text{flex},\partial}) \to \pi_1(\mathcal{T}_3^{\text{flex}})$ on the fundamental groups is surjective.

Consider the images of $\rho_\partial^{\text{flex}}$ in Mod_3^2 and of ρ^{flex} in $\mathrm{Mod}_{3,2}$. Every marking of a Riemann surface in $\mathcal{M}_3^{\text{flex}}$ appears as the image of a marking associated with a Riemann surface in $\mathcal{M}_{3,2}^{\text{flex},\partial}$. Then, the restriction of the homomorphism $\text{Cap} : \mathrm{Mod}_3^2 \to \mathrm{Mod}_{3,2}$ on $\operatorname{im} \rho_\partial^{\text{flex}}$ is surjective onto $\operatorname{im} \rho^{\text{flex}}$ and the homomorphism $\pi_1^{orb}(\mathcal{M}_{3,2}^{\text{flex},\partial}) \to \pi_1^{orb}(\mathcal{M}_3^{\text{flex}})$ must be surjective too. □

Our final goal is to prove Theorem C. In particular, we will show that the monodromy $\rho_{\mathbb{P}\mathcal{H}(3,1)}$ is geometric. We are going to prove Theorem C using the following lemma.

Lemma 4.1.6. Every surjective endomorphism of $\mathrm{Inn}(A_{E_7})$ is an isomorphism.

Proof. Both A_{E_7} and the automorphism group $\mathrm{Aut}(A_{E_7})$ are residually finite [Bau63, Theorem 1] because A_{E_7} is linear [CW02]. Hence, the subgroup $\mathrm{Inn}(A_{E_7})$ of $\mathrm{Aut}(A_{E_7})$ is both finitely generated and residually finite. In particular, we can conclude that every surjective endomorphism of $\mathrm{Inn}(A_{E_7})$ is an isomorphism [BH99, Chapter III, Proposition 7.5]. □

Since $U_{f_{E_7}}$ is an Eilenberg–Maclane space for the Artin group A_{E_7}, the fundamental group $\pi_1(U_{f_{E_7}})$ is isomorphic to A_{E_7}. Moreover, the orbifold fundamental group $\pi_1^{orb}(\mathbb{P}\mathcal{H}(3,1))$ is isomorphic to $\mathrm{Inn}(A_{E_7})$.

Let us consider the surjective homomorphism $\theta : \pi_1(U_{f_{E_7}}) \to \pi_1^{orb}(\mathbb{P}\mathcal{H}(3,1))$ as the composition

$$\pi_1(U_{f_{E_7}}) \xrightarrow{\theta_2} \pi^{orb}(\mathcal{M}_{3,2}^{\mathrm{flex}}) \xrightarrow{\theta_1^{-1}} \pi_1^{orb}(\mathbb{P}\mathcal{H}(3,1)).$$

The group $\mathrm{Inn}(A_{E_7})$ is centerless. Therefore, the kernel of the homomorphism $\theta : A(E_7) \to \mathrm{Inn}(A_{E_7})$ contains the subgroup $\langle \Delta \rangle$. This implies that the induced map

$$\overline{\theta} : \mathrm{Inn}(A_{E_7}) \to \mathrm{Inn}(A_{E_7})$$

is a well-defined surjective endomorphism of $A(E_7)_\Delta$ and therefore an isomorphism by Lemma 4.1.6.

Since θ commutes with the monodromies $\rho_{E_7} : \pi_1(U_{f_{E_7}}) \to \mathrm{Mod}_3^2$ and $\rho_{\mathbb{P}\mathcal{H}(3,1)} : \pi_1^{orb}(\mathbb{P}\mathcal{H}(3,1)) \to \mathrm{Mod}_{3,2}$ through the capping homomorphism $\mathrm{Cap} : \mathrm{Mod}_3^2 \to \mathrm{Mod}_{3,2}$, we can conclude that $\rho_{\mathbb{P}\mathcal{H}(3,1)}$ is geometric since ρ_{E_7} is.

As in the case of the projective stratum $\mathbb{P}\mathcal{H}^{\mathrm{odd}}(4)$, the orbifold fundamental group of $\mathbb{P}\mathcal{H}(3,1)$ is generated by orbifold loops coming from cylinder shears, where the cylinders are described in Figure 4.2.

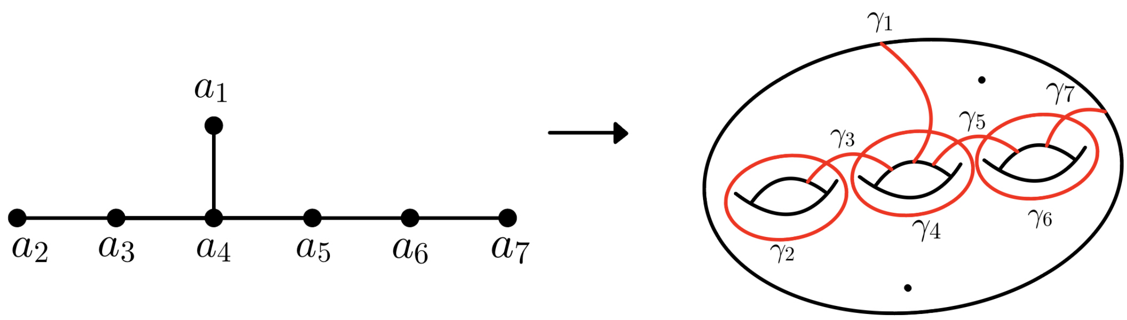


Figure 4.5: A correspondence between the E_7 Dynkin diagram on some closed curves on $\Sigma_{3,2}$. Each vertex corresponds to a simple closed curve on the punctured surface on the right-hand side. The geometric homomorphism sends each standard generator of A_{E_7} to the corresponding Dehn twist.

§ 4.2 | Genus $g = 4$

In this section, we describe the image of the minimal stratum component $\mathcal{H}^{\text{even}}(6)$ in $\mathcal{M}_{4,1}$. A general reference is [Mir95, Chapter VII, Section 4].

Each genus g pointed closed Riemann surface (X, p) comes with a sequence of g integers $\mathfrak{G}_p(X)$ called the *Weierstrass gap sequence*. A positive integer n is a Weierstrass gap number in $\mathfrak{G}_p(X)$ if and only if there is an abelian differential (X, ω) with a zero at p of order $n-1$. The complement $\mathscr{S}_p(X)$ of a gap sequence $\mathfrak{G}_p(X)$ in $\mathbb{N}$ is called a *non-gap sequence* and it is a semigroup of $\mathbb{N}$ that contains 0 and has a finite complement. In other words, the set $\mathfrak{G}_p(X)$ is a *numerical semigroup*.

Given an arbitrary numerical semigroup $\mathscr{S}$ in $\mathbb{N}$, we denote by $\mathcal{M}_{g,1}^{\mathscr{S}}$ the moduli space of pointed Riemann surfaces (X, p) such that the non-gap sequence at $p \in X$ is exactly $\mathscr{S}$. If X is hyperelliptic and p is preserved by the hyperelliptic involution of X, the Weiestrass gap sequence $\mathfrak{G}_p(X)$ is $\{1, 3, 5, \ldots, 2g-1\}$. We will show the following.

Proposition 4.2.1. Let $\mathscr{S}$ be the semigroup generated by 3 and 5. Then, a pointed Riemann surface (X, p) is in $\mathcal{M}_{4,1}^{\mathscr{S}}$ if and only if $(X, 6p) \in \mathbb{P}\mathcal{H}^{\text{even}}(6)$.

Suppose X is a non-hyperelliptic Riemann surface of genus 4. Then, the equivalence class of the canonical divisors K_X induces a holomorphic embedding $X \to \mathbb{P}^3$ of X as a smooth degree 6 curve. A consequence of Max Noether's Theorem for algebraic surfaces is that X is the complete intersection of an irreducible quadric Q and an irreducible cubic C in $\mathbb{P}^3$. Irreducible quadric on $\mathbb{P}^3$ can either be smooth or singular cones. In the first case, the Segre embedding can be use to show that Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Otherwise, the irreducible quadric Q is a cone and, up to some change of coordinates, the vanishing locus of the homogeneous polynomial $x_0^2 - x_1x_2$ in $\mathbb{P}^3$. The following can be found in [Bul13, Section 4.3].

Lemma 4.2.2. Let $(X, 6p) \in \mathbb{P}\mathcal{H}(6)$ and suppose X is a non-hyperelliptic smooth degree 6 curve in $\mathbb{P}^3$ that is a complete intersection of an irreducible quadric Q and an irreducible cubic C. Then,

- if Q is smooth, the Weierstrass gap sequence of (X, p) is $\mathfrak{G}_p(X) = \{1, 2, 3, 7\}$;
- if Q is a cone, the Weierstrass gap sequence of (X, p) is $\mathfrak{G}_p(X) = \{1, 2, 4, 7\}$.

Proof of Proposition 4.2.1. Suppose $(X, 6p) \in \mathbb{P}\mathcal{H}^{\text{even}}(6)$. By Lemma 4.2.2, there are only two possible Weierstrass gap sequences in p. However, the spin structure $\mathcal{L} = 3p$ on X is even and $h^0(X, \mathcal{L})$ is greater of equal than 2. The dimension $h^0(X, \mathcal{L})$ of the space of holomorphic differentials vanishing to order at least $g-1$ at p is the number of Weierstrass gap numbers $1 = \gamma_1 < \gamma_2 < \cdots < \gamma_g$ that are at least g [Gun66, Chapter 7, Theorem 14]. Hence, there are at least 2 Weierstrass numbers bigger than

4 if $\mathcal{L} = 3p$ is even. In particular, the Weierstrass gap sequence of (X, p) can only be $\mathfrak{G}_p(X) = \{1, 2, 4, 7\}$ and (X, p) is in $\mathcal{M}_{4,1}^{\mathscr{S}}$, where $\mathscr{S}$ is the semigroup generated by 3 and 5.

Viceversa, if a pointed Riemann surface (X, p) is in $\mathcal{M}_{4,1}^{\mathscr{S}}$ there is an abelian differential on X vanishing on p with multiplicity 6. Since the Weierstrass gap sequence of X at p is $\{1, 2, 4, 7\}$, the Riemann surface X cannot be hyperelliptic. By the above argument, the spin structure $\mathcal{L} = 3p$ is necessarily even and therefore $(X, p) \in \mathbb{P}\mathcal{H}^{\text{even}}(6)$. □

A description of the pointed Riemann surfaces in the stratum component $\mathbb{P}\mathcal{H}^{\text{even}}(6)$ is available in [Che19, Section 4.5] and [Bul13, Section 4.3]. For completeness, we briefly include such a description in the present note.

Lemma 4.2.3. Let $\mathscr{S}$ be the semigroup generated by 3 and 5. Then, the moduli space $\mathcal{M}_{4,1}^{\mathscr{S}}$ is an orbifold of dimension 7.

Sketch of the proof. Let (X, p) be a non-hyperelliptic pointed Riemann surface of genus 4 and suppose that $(X, p) \in \mathcal{M}_{4,1}^{\mathscr{S}}$. By Lemma 4.2.2, we can find an irreducible quadric cone Q and an irreducible cubic C in $\mathbb{P}^3$ such that X is the complete intersection of Q and C. Since 7 is a gap number for X in p, there exists an abelian differential ω that has order 6 in p. The complex line bundle associated with ω has intersection multiplicity 6 with X in p, and consequently the curve X is cut out by a ruling l_1 of Q in p coming from ω, with intersection multiplicity 3. There are also two rulings tangent to X in points q_1 and q_2 different from the singular point of Q.

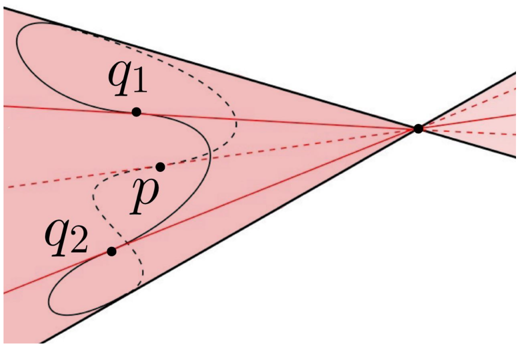


Figure 4.6: The cubic C on the cone Q with the tangent points p, q_1, q_2.

After a suitable change of coordinates, the tuple (X, p, q_1, q_2) is determined solely by the cubic equation that cuts out X from Q. After imposing the tangency requirements of the rulings, we end up with 8 free non-trivial complex parameters, where any $\lambda \in \mathbb{C}^8 \setminus \{0\}$ represents a tuple (X, p, q_1, q_2). However, any two cubic equations define the same isomorphic type of variety up to the action of a matrix in $\mathrm{GL}_4(\mathbb{C})$. The subgroup of $\mathrm{GL}_4(\mathbb{C})$ preserving Q and the three rulings is then isomorphic to $\mathbb{C}^*$. The locus in $\mathbb{C}^8 \setminus \{0\}$ parametrizing singular curves is a hypersurface and the moduli space $\mathcal{M}_{4,1}^{\mathscr{S}}$ is covered by its complement in $\mathbb{C}^8 \setminus \{0\}$ by the action of $\mathbb{C}^*$. □

The following is a consequence of Proposition 2.5.5, Proposition 4.2.1 and Lemma 4.2.3.

Corollary 4.2.4. Let $\mathscr{S}$ be the semigroup generated by 3 and 5. The orbifolds $\mathbb{P}\mathcal{H}^{\text{even}}(6)$ and $\mathcal{M}_{4,1}^{\mathscr{S}}$ are isomorphic.

We now revise some results on the moduli spaces $\mathcal{M}_{g,1}^{\mathscr{S}}$ for $\mathscr{S}$ semigroup in $\mathbb{N}$ due to Pinkham [Pin74]. In general, it is even hard to establish whether these moduli spaces are empty or not. On the other hand, we do have some results in low genera; see, for example, [MN04].

Suppose $\mathscr{S}$ is the semigroup in $\mathbb{N}$ with $\{a_1, \ldots, a_k\}$ as a minimal generating set and consider the *monomial curve* $C_{\mathscr{S}} = \{(t^{a_1}, \ldots, t^{a_k}) \in \mathbb{C}^k \mid t \in \mathbb{C}\}$. Every monomial curve $C_{\mathscr{S}}$ has an isolated singularity at the origin and the 1-dimensional algebraic torus $\mathbb{C}^*$ acts naturally on the parameter $t \in \mathbb{C}$ of $C_{\mathscr{S}}$. Pinkham proved that the moduli space $\mathcal{M}_{g,1}^{\mathscr{S}}$ is a quotient of a versal deformation space associated to the monomial curve $C_{\mathscr{S}}$ [Pin74, Proposition 13.9].

Theorem 4.2.5. Suppose f_Γ is the germ of the irreducible complex analytic map associated to a root system and such that $C_{\mathscr{S}}$ is its zero level set. The $\mathbb{C}^*$ action on $C_{\mathscr{S}}$ can be extended to U_{f_Γ}, in such a way that π_{f_Γ} is $\mathbb{C}^*$-equivariant and $U_\Gamma/\mathbb{C}^*$ is isomorphic to $\mathcal{M}_{g,1}^{\mathscr{S}}$.

The above isomorphism is given as follows. Suppose $F_\Gamma(\bullet, s) = f_\Gamma(\bullet) + \sum_{i=1}^m s_i g_i(\bullet)$ is the perturbation of f_Γ that defined the associated versal deformation space. Then, each $s \in U_{f_\Gamma}$ is mapped to the vanishing locus $\mathbb{V}(F_\Gamma(\bullet, s))$, before projectivizing the Riemann surface and marking the added point at infinity.

In case $\mathscr{S}$ is generated by $\{3, 5\}$, the monomial curve $C_{\mathscr{S}}$ is, up to change of coordinates, the vanishing locus of $f = x^3 + y^5$, germ associated to the root system E_8. Theorem B is a consequence of Theorem 3.2.3 and Corollary 4.2.4.

Proof of Theorem B. A good orbifold is $K(\pi, 1)$ if covered by a contractible manifold and $\mathcal{H}^{\text{even}}(6)$ is $K(\pi, 1)$ if $\mathbb{P}\mathcal{H}^{\text{even}}(6)$ is. The projective stratum component $\mathbb{P}\mathcal{H}^{\text{even}}(6)$ is covered by the versal deformation space associated to the root system E_8, that is a $K(\pi, 1)$ manifold and therefore covered by a contractible manifold. □

We now show that the orbifold fundamental group of $\mathbb{P}\mathcal{H}^{\text{even}}(6)$ is isomorphic to the inner automorphism group of the Artin group associated with the E_8 root system. In particular, the kernel of the monodromy is very large and contains a non-abelian free group of rank 2.

Recall that we get the short exact sequence

$$1 \to \pi_1(\mathbb{C}^*) \to \pi_1(U_{f_{E_8}}) \to \pi_1^{orb}(\mathbb{P}\mathcal{H}^{\text{even}}(6)) \to 1 \tag{4.3}$$

from the fiber bundle associated with the Borel construction. By Theorem 3.2.3, the fundamental group of $U_{f_{E_8}}$ is the Artin group A_{E_8} and its quotient by a cyclic normal subgroup is isomorphic to $\pi_1^{orb}(\mathbb{P}\mathcal{H}^{\text{even}}(6))$.

Lemma 4.2.6. Every cyclic normal subgroup of A_{E_8} is central.

Proof. Let $a \in A_{E_8}$ be the generator of an infinite cyclic normal subgroup. For every $g \in A_{E_8}$ there is an $n \in \mathbb{N}$ such that $gag^{-1} = a^n$ holds. Standard generators of A_{E_8} share only length-preserving relations. Therefore, there exists a well-defined homomorphism

$$\deg : A_{E_8} \to \mathbb{Z}$$

assigning the standard generators length 1. The following inequality shows that n must be equal to 1, provided $g \neq id$:

$$\deg(a) = \deg(gag^{-1}) = \deg(a^n) = n \deg(a).$$

Therefore, the normal subgroup $\langle a \rangle$ is central. □

Let Φ be the root system E_8 and denote by V_Φ the open complement in $\mathbb{R}^8$ of the hyperplanes family $\{H_\alpha \mid \alpha \in I_\Phi\}$ associated to Φ. The Artin group A_{E_8} has an interpretation as a fundamental group by Theorem 3.2.3. Let us pick a chamber $C \subset V_\Phi$ and a point $p \in C$. The fundamental group of the complexification of V_Φ, denoted by $\mathbb{V}_\Phi$, modulo the Coxeter group W_Φ and based at the point represented by p is isomorphic to A_{E_8}.

We now construct the Garside element Δ_Φ as the homotopy class of a loop in $\mathbb{V}_\Phi / W_\Gamma$, where Γ is the Dynkin diagram associated to Φ. The following construction is due to Brieskorn [Bri71] and can also be found in [Loo08, Section 2]. For every $x \in V_\Phi$, we define C_x to be the either V_Φ, if x is not contained in any hyperplane H_α, or the intersection of all open half-spaces H_α^+ containing the chamber C and bounded by H_α if $x \in H_\alpha$. The set

$$\mathbb{U} = \{x + iy \mid y \in C_x\}$$

is an open subset of $\mathbb{V}_\Phi$ and it is star-like with respect to any point in iC. Therefore, the set $\mathbb{U}$ is contractible. As a result, there is a unique homotopy type of arc γ_Φ between p and $-p$ entirely contained in $\mathbb{U}$. Since $-id_{V_\Phi} \in W_\Gamma$ in case $\Phi = E_8$, the arc γ_Φ projects to a loop in $\mathbb{V}_\Phi / W_\Phi$. The Garside element Δ_{E_8} can be interpreted as the homotopy class of $[\gamma_\Phi]$ in $\mathbb{V}_\Phi / W_\Phi$.

The arc γ_Φ can be taken to be the composition $\delta * \sigma$ of the following path segments

$$\sigma : [0,1] \to \mathbb{U} \qquad\qquad \delta : [0,1] \to \mathbb{U}$$
$$t \mapsto h(t)p \qquad\qquad t \mapsto ih(t)p,$$

where $h(t) = (1-t) + it$.

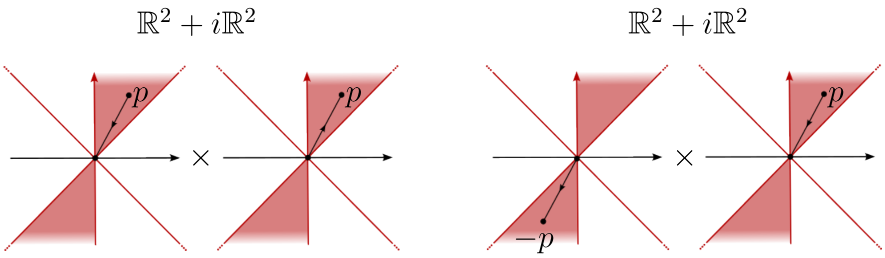


Figure 4.7: An example of the path segments σ (on the left of the picture) and of δ (on the right-hand side of the picture) in the case $\Gamma = A_3$. The colored area represents the complexified chamber $C + iC$.

Proposition 4.2.7. The image by τ_{E_8} of the homotopy class of the loop $[\gamma_{E_8}]$ in $\mathbb{V}_{E_8}/W_{E_8}$ generates the fundamental group of the $\mathbb{C}^*$-fiber associated to the quotient map $U_{f_{E_8}} \to \mathbb{P}\mathcal{H}^{\text{even}}(6)$.

Proof. Let R be the root system E_8. We want to show that the homotopy class of the loop $\tau_R(\gamma_R)$ generates the fundamental group of the $\mathbb{C}^*$-fiber associated with the quotient map $U_R \to \mathbb{P}\mathcal{H}^{\text{even}}(6)$. The punctured complex plane $\mathbb{C}^*$ acts on U_{E_8} componentwise with weights given by the degrees $d_1, \ldots, d_8$ of the homogeneous polynomials $f_1, \ldots, f_8$. In particular, the great common divisor of $d_1, \ldots, d_8$ is 2 and the the fiber $\mathcal{O}_p = \{(\lambda^{d_1} p_1, \ldots, \lambda^{d_8} p_8) \mid \lambda \in \mathbb{C}^*\}$ of $p \in U_R$ is homeomorphic to $\mathbb{C}^*/\mathbb{Z}_2$ where the underlying relation is given by $z \sim -z$. The fundamental group of $\mathcal{O}_p$ is isomorphic to $\mathbb{Z}$ and generated by the image of any arc in $\mathbb{C}^*$ tracing an angle of π. The arc γ_R traces an angle of π between the endpoints p and $-p$ and therefore the image

$$\tau_{E_8}(\gamma_R) : [0,1] \to \mathbb{U}$$
$$t \mapsto \begin{cases} (h(2t)^{d_1} f_1(p), \ldots, h(2t)^{d_8} f_8(p)) & \text{if } t \in [0, \frac{1}{2}] \\ (i^{d_1} h(2t-1)^{d_1} f_1(p), \ldots, i^{d_1} h(2t-1)^{d_8} f_8(p)) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

represents a generator of the fundamental group of the $\mathbb{C}^*$-fiber $\mathcal{O}_p$. □

We obtain the following result from the short exact sequence in (4.3).

Corollary 4.2.8. The orbifold fundamental group of $\mathbb{P}\mathcal{H}^{\text{even}}(6)$ is isomorphic to the inner automorphism group $\text{Inn}(A_{E_8})$. Then, the group $\pi_1^{orb}(\mathcal{H}^{\text{even}}(6))$ is a central extension of $\text{Inn}(A_{E_8})$.

As in the case of the projective strata $\mathbb{P}\mathcal{H}^{\text{odd}}(4)$ and $\mathbb{P}\mathcal{H}(3,1)$, the orbifold fundamental group of $\mathbb{P}\mathcal{H}^{\text{even}}(6)$ is generated by orbifold loops coming from cylinder shears, where the cylinders are described in Figure 4.2.

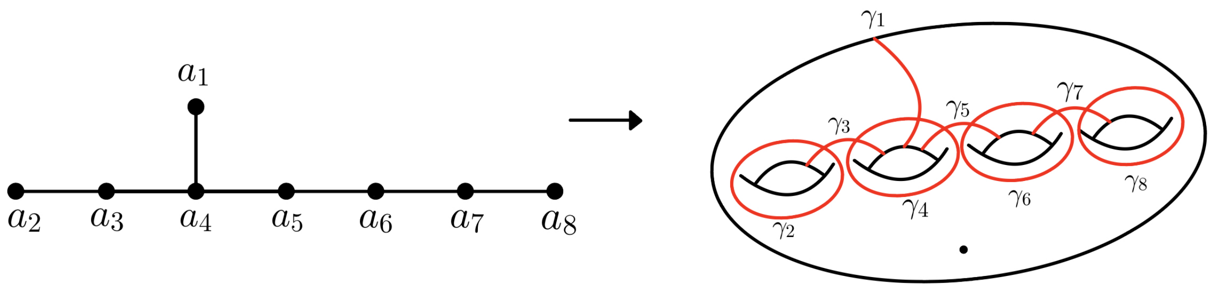


Figure 4.8: A correspondence between the E_8 Dynkin diagram and some closed curves on $\Sigma_{4,1}$. Each vertex corresponds to a simple closed curve on the punctured surface on the right-hand side. The geometric homomorphism sends each standard generator of A_{E_8} to the corresponding Dehn twist.

§ 4.3 | Final remarks

The description of the projective strata $\mathbb{P}\mathcal{H}^{\text{odd}}(4)$, $\mathbb{P}\mathcal{H}(3,1)$ and $\mathbb{P}\mathcal{H}^{\text{even}}(6)$ as quotients of versal deformation spaces of plane curve singularities allowed us to carry out the proof of the main theorems in the first part of this thesis. A partial description is also available in higher genus, at least for minimal strata thought of as subloci of the moduli spaces of pointed Riemann surfaces in the sense of Proposition 2.5.5.

Bullock proved that in genus $g \leq 5$ the projective strata can be described as the Pinkham's moduli spaces $\mathcal{M}_{g,1}^{\mathscr{S}}$ of pointed Riemann surfaces with a prescribed gap sequence [Bul13]. The moduli spaces $\mathcal{M}_{g,1}^{\mathscr{S}}$ can be thought of as quotients of versal deformation spaces [Pin74, Proposition 13.9], which might not be associated to plane curve singularities and, therefore, challenging to describe topologically. For example, we only know that $\mathbb{P}\mathcal{H}^{\text{even}}(8)$ in genus 5 is the quotient of a versal deformation space of certain complete intersection monomial curve, since the numerical semigroup $\mathscr{S}$ associated is free in the sense of [AM24a]. On top of this, in genus $g \geq 6$ there are loci within both $\mathbb{P}\mathcal{H}^{\text{odd}}(2g-2)$ and $\mathbb{P}\mathcal{H}^{\text{even}}(2g-2)$ consisting of different gap sequences.

A much more attainable problem to solve is that of a full description of the kernels of $\rho_{\mathcal{H}^{\text{odd}}(4)}$, $\rho_{\mathcal{H}(3,1)}$ and $\rho_{\mathcal{H}^{\text{even}}(6)}$. In genus 3, the Teichmüller strata $\mathcal{T}\mathcal{H}^{\text{odd}}(4)$ and $\mathcal{T}(3,1)$ project to the complement of the hyperelliptic locus $\mathscr{H}_3$ in the classical Teichmüller space $\mathcal{T}_3$, since both strata are totally non-hyperelliptic. Harris proved that $\mathcal{T}_g \setminus \mathscr{H}_g$ has the homotopy type of an infinite wedge of $2g-5$ dimensional spheres [Har21, Corollary 3.2]. In genus 3, this is an infinite wedge of circles and has and non-abelian free group of infinite rank as fundamental group. We don't know if the Wajnryb element projects to a loop wrapping around the codimension 1 connected components of the hyperelliptic locus $\mathscr{H}_3$, but we conjecture that one between $\pi_1(\mathcal{T}^{\text{odd}}(4))$ and $\pi_1(\mathcal{T}(3,1))$

is commensurable to a non-abelian free group. It might be possible to detect loops in Teichmüller strata by projecting to complements of hypersurfaces in $\mathcal{T}_g$. Chen–Möller have a list of hypersurfaces that some strata avoid [CM13].

Chapter 5

Topological monodromies of some strata of half-translation surfaces

§ 5.1 | Preliminaries on line fields

In this section, we revise the main properties of line fields and associated winding number functions. For more details, see [LP20, Section 1].

Line fields and winding number functions

Let $\tilde{\Sigma}$ be an oriented, compact, connected genus g surface with $n \neq 0$ boundary components. The projective tangent bundle $\mathbb{P}T\tilde{\Sigma}$ fibers through the fiber bundle $\pi : \mathbb{P}T\tilde{\Sigma} \to \tilde{\Sigma}$. The fiber at a point p in $\tilde{\Sigma}$ is isomorphic to the circle S^1 and consists of all the lines through the origin in $T_p\tilde{\Sigma}$.

A *line field* is a continuous section $\eta : \tilde{\Sigma} \to \mathbb{P}T\tilde{\Sigma}$ of $\pi : \mathbb{P}T\tilde{\Sigma} \to \tilde{\Sigma}$. Every line field η defines a cohomology class $[\eta] \in H^1(\mathbb{P}T\tilde{\Sigma}; \mathbb{Z})$ by taking the Poincaré–Lefschetz dual of the homology class $[\eta(\tilde{\Sigma})] \in H^1(\mathbb{P}T\tilde{\Sigma}, \partial\mathbb{P}T\tilde{\Sigma}; \mathbb{Z})$. Note that the we can use the Poincaré–Lefschetz duality since the surface $\tilde{\Sigma}$ is compact. The fiber bundle $\pi : \mathbb{P}T\tilde{\Sigma} \to \tilde{\Sigma}$ induces a short exact sequence of cohomology groups

$$0 \to H^1(\tilde{\Sigma}; \mathbb{Z}) \xrightarrow{\pi^*} H^1(\mathbb{P}T\tilde{\Sigma}; \mathbb{Z}) \xrightarrow{\iota^*} H^1(S^1; \mathbb{Z}) \to 0,$$

where the homomorphism ι^* is induced by the inclusion of the fiber in $\mathbb{P}T\tilde{\Sigma}$.

Through the assignment $\eta \mapsto [\eta]$, Lekili–Polishchuk proved that the set of homotopy classes of (unoriented) line fields on $\tilde{\Sigma}$ are in one to one correspondence with the cohomologes in the preimage $(\iota^*)^{-1}(\xi)$, where $\xi \in H^1(S^1; \mathbb{Z})$ is the cohomology class that integrates to 1 on S^1 [LP20, Lemma 1.1.2]. Hence, we can think of a (homotopy class of) line field on $\tilde{\Sigma}$ as a cohomology class in $H^1(\mathbb{P}T\tilde{\Sigma}; \mathbb{Z})$.

Suppose now $\gamma : S^1 \to \tilde{\Sigma}$ is an oriented simple closed curve on $\tilde{\Sigma}$ and consider its lift

$\overrightarrow{\gamma} : S^1 \to \mathbb{P}T\tilde{\Sigma}$. The winding number of γ with respect to the line field η is the value

$$\phi_\eta(\gamma) := \langle [\eta], [\overrightarrow{\gamma}] \rangle$$

where $\langle \cdot, \cdot \rangle : H^1(\mathbb{P}T\tilde{\Sigma};\mathbb{Z}) \times H_1(\mathbb{P}T\tilde{\Sigma};\mathbb{Z}) \to \mathbb{Z}$ is the natural pairing. Note that the value $\phi_\eta(\gamma)$ only depends on the the homotopy class of η and on the isotopy class of γ. The following proposition appears in Humphries–Johnson's work [HJ89], but can also be found in Calderon–Salter's work for winding number functions associated to non-vanishing vector fields on $\tilde{\Sigma}$ [CS22, Lemma 2.4].

Proposition 5.1.1. Let $\eta : \tilde{\Sigma} \to \mathbb{P}T\tilde{\Sigma}$ be a line field. Then, the following properties hold for the winding number function $\phi_\eta(\cdot)$.

(1) (Normalization) If z not a essential curve nor a boundary components on $\tilde{\Sigma}$, then $\phi_s(z) = 2$;

(2) (Twist-linearity) if δ and γ are oriented simple closed curves and $T_\delta(\gamma)$ is the Dehn twist of γ about δ, then $\phi_\eta(T_\delta(\gamma)) = \phi_\eta(\gamma) + \phi_\eta(\gamma)i(\gamma,\delta)$, where $i(\gamma,\delta)$ denotes the algebraic intersection pairing;

(3) (Homological Coherence) if the curves $\gamma_1, \ldots, \gamma_m$ bound a subsurface $S \subset \tilde{\Sigma}$ such that S lies to the left of each curve, then

$$\sum_{i=1}^{n} \phi_\eta(\gamma_i) = 2\chi(S).$$

Proof. Suppose $v : \tilde{\Sigma} \to T\tilde{\Sigma}$ is a non-vanishing vector field on $\tilde{\Sigma}$ and ϕ_v is the winding number function of v. Then, the function ϕ_v respects twist-linearity and if z is a non-essential curve, then $\phi_v(z) = 1$. Moreover, if $\gamma_1, \ldots, \gamma_m$ are simple closed curves bounding a subsurface $S \subset \tilde{\Sigma}$ and such that S lies to the left of each curve, then $\sum_{i=1}^{n} \phi_v(\gamma_i) = \chi(S)$. If η is the line field on $\tilde{\Sigma}$ induced by the vector field v, the winding number function ϕ_η takes only even values and $\phi_\eta(\gamma) = 2\phi_v(\gamma)$ for any simple closed curve γ [LP20, Lemma 1.1.4]. The claim holds for line fields induced by vector fields.

Suppose now that η is a line field not induced by a vector field. The cohomology group $H^1(\tilde{\Sigma};\mathbb{Z})$ acts on the coset $(\iota^*)^{-1}(\xi)$ and therefore on the set of of homotopy classes of unoriented line fields. Since the action is transitive [LP20, Lemma 1.1.2], there exists $c \in H^1(\tilde{\Sigma};\mathbb{Z})$ such that $\eta + c$ is a line field induced by a vector field. Hence, the Proposition holds for $\eta + c$. For any simple closed curve γ the winding number functions ϕ_η and $\phi_{\eta+c}$ are related by the equality

$$\phi_{\eta+c}(\gamma) = \phi_\eta(\gamma) + i(c, [\gamma])$$

and the claim holds for the line field η too. □

A closed surface Σ can be obtained from $\tilde{\Sigma}$ by capping every boundary component with a disk. After capping, a closed curve on $\tilde{\Sigma}$ that is isotopic to a boundary component δ will be trivial in homology. By Proposition 5.1.1, if the winding number of δ is not equal to 2 the function ϕ_η is ill-defined on the set of isotopy classes of oriented simple closed curves of the capped surface Σ. More generally, suppose $\tilde{\Sigma}$ has n boundary components $\delta_1, \dots, \delta_n$ oriented such that $\tilde{\Sigma}$ lies on the right and $\phi_\eta(\delta_i) = k_i + 2$ for some positive integers $k_1, \dots, k_n$. Pick any pair of closed curves γ_1 and γ_2 on $\tilde{\Sigma}$ with the property that their isotopy classes coincide on Σ. Then, there exists an embedded sphere with $m + 2 \leq n + 2$ boundary components in $\tilde{\Sigma}$ given by the curves γ_1, γ_2 and some $\delta_{i_1}, \dots, \delta_{i_m}$. By homological coherence the winding numbers of γ_1 and γ_2 differ by a multiple of $r = \gcd(k_1, \dots, k_n)$, since $\phi_\eta(\gamma_1) - \phi_\eta(\gamma_2) = -\sum_{j=1}^{m} k_{i_j}$ holds. Hence, post-composing the winding number function ϕ_η by the quotient map $\mathbb{Z} \to \mathbb{Z}_r$ yields a well-defined map $\phi_\eta^r : \mathcal{S} \to \mathbb{Z}_r$ on the set $\mathcal{S}$ of isotopy classes of simple closed curves on Σ.

Any winding number function ϕ_η^r is uniquely determined by the values taken on a fixed symplectic basis $\mathcal{B} = \{\alpha_1, \beta_1 \dots, \alpha_g, \beta_g\}$ on Σ. Recall that a symplectic basis is a set of primitive vectors in $H_1(\Sigma; \mathbb{Z})$ such that $i(\alpha_i, \alpha_j) = i(\beta_i, \beta_j) = 0$ and $i(\alpha_i, \beta_j) = \delta_{ij}$ for any $i, j = 1, \dots, g$.

Lemma 5.1.2. Let $\mathcal{B}$ be a symplectic basis of vectors in $H_1(\Sigma; \mathbb{Z})$. A winding number function ϕ_η^r as above is completely determined by the values taken on $\mathcal{B}$. Moreover, given a vector $v \in \mathbb{Z}_r^{2g}$, there exists a line field η on $\tilde{\Sigma}$ such that $\phi_\eta^r(\mathcal{B}) = v$.

Proof. We can find $c \in H^1(\tilde{\Sigma}, \mathbb{Z})$ such that $\eta + c$ is a line field induced by a non-vanishing vector field on $\tilde{\Sigma}$. Then, we have that $\phi_{\eta+c}^r = \phi_\eta^r + \langle c, \cdot \rangle$ holds. The winding number function $\phi_{\eta+c}^r$ is completely determined by the values taken on the vectors in $\mathcal{B}$ by [CS21, Lemma 2.4] and so is ϕ_η^r. The second claim follows from the correspondence of homotopy class of line field on $\tilde{\Sigma}$ and the coset $(\iota^*)^{-1}(\xi) = H^1(\tilde{\Sigma}; \mathbb{Z}) + \xi$ [LP20, Lemma 1.1.2]. The freedom of the choice given by the group $H^1(\tilde{\Sigma}; \mathbb{Z})$ returns all the possible values for the vector $\phi_\eta^r(\mathcal{B})$. □

If $r = 2$, the function ϕ_η^2 can be extended to a homomorphism from $H_1(\Sigma, \mathbb{Z}_2)$ to $\mathbb{Z}_2$. Indeed, if any two sets of multicurves in Σ are homologous modulo 2, then their lifts in $\mathbb{P}T\Sigma$ differ by a $2\overrightarrow{z}$ summand, where z is a non-essential curve [Joh80, Theorem 1A].

Proposition 5.1.3. Let η be a line field on $\tilde{\Sigma}$ such that the values taken by the boundary components $\delta_i, \dots, \delta_n$ of $\tilde{\Sigma}$ via the winding number function ϕ_η are all divisible by 4. If $r = 2$, then ϕ_η^2 can be extended to a homomorphism $\phi_\eta^2 : H_1(\Sigma, \mathbb{Z}_2) \to \mathbb{Z}^2$.

Extended quadratic forms

Any line field on $\tilde{\Sigma}$ defines a map $\Omega_\eta : H_1(\tilde{\Sigma}; \mathbb{Z}_4) \to \mathbb{Z}_4$, that we will refer to as a $\mathbb{Z}_4$*-extended quadratic form* on $\tilde{\Sigma}$ [LP20, Proposition 1.2.2]. The map Ω_η is neither linear nor a quadratic form, but for any $a, b \in H_1(\tilde{\Sigma}; \mathbb{Z}_4)$ the relation

$$\Omega_\eta(a+b) = \Omega_\eta(a) + \Omega_\eta(b) + 2i(a,b)$$

holds. Indeed, for any finite collection $\gamma_i, \dots, \gamma_m$ of oriented simple closed curves on $\tilde{\Sigma}$, we can define a lift for the homology $\sum_{i=1}^m [\gamma_i] \in H_1(\tilde{\Sigma}; \mathbb{Z}_4)$ to $H_1(\mathbb{P}T\tilde{\Sigma}; \mathbb{Z}_4)$ by considering the class

$$\widetilde{\sum_{i=1}^m [\gamma_i]} := \sum_{i=1}^m [\overrightarrow{\gamma_i}] + m[z],$$

where z is as in Proposition 5.1.1. Such a lift is usually referred to as a *Johnson lift* in the context of non-vanishing vector fields, where the coefficients of the respective homologies are taken in the field $\mathbb{Z}_2$ [Joh80, Section 3]. For simplicity, we will use the same terminology here.

It is important to notice that the Johnson lift of homologies from $H_1(\tilde{\Sigma}; \mathbb{Z}_4)$ to $H_1(\mathbb{P}T\tilde{\Sigma}; \mathbb{Z}_4)$ is only well-defined up to a $2[z]$ summand. That is, changing the representatives in the homology classes of the curves $\gamma_i, \dots, \gamma_m$ will produce a lift in $H_1(\mathbb{P}T\tilde{\Sigma}; \mathbb{Z}_4)$ that can differ by a $2[z]$ summand. However, the function

$$\Omega_\eta(\sum_{i=1}^m [\gamma_i]) := \langle [\eta], \widetilde{\sum_{i=1}^m [\gamma_i]} \rangle$$

vanishes on $2[z]$ and the value $\Omega_\eta(\sum_{i=1}^m [\gamma_i])$ is well defined in $\mathbb{Z}_4$ [LP20, Proposition 1.2.2].

Under some hypothesis, a line field on $\tilde{\Sigma}$ defines a $\mathbb{Z}_4$-extended quadratic form also on the capped surface Σ. This is true is the corresponding zeros of the quadratic differential associated have order divisible by 4.

Proposition 5.1.4. Let $\delta_i, \dots, \delta_n$ be distinct simple closed curves on $\tilde{\Sigma}$ isotopic to each of the boundary components of $\tilde{\Sigma}$ and suppose that $\phi_\eta(\delta_i) \equiv 2 \mod 4$ for each $i = 1, \dots, n$. Then, the winding number function ϕ_η defines a map $\Omega_\eta : H_1(\Sigma; \mathbb{Z}_4) \to \mathbb{Z}_4$ such that

$$\Omega_\eta(\sum_{i=1}^m [\gamma_i]) = 2m + \sum_{i=1}^m \phi_\eta(\gamma_i) \mod 4$$

for any finite collection of oriented simple closed curves $\gamma_i, \dots, \gamma_m$ on Σ and

$$\Omega_\eta(a+b) = \Omega_\eta(a) + \Omega_\eta(b) + 2i(a,b) \tag{5.1}$$

for any $a, b \in H_1(\Sigma; \mathbb{Z}_4)$.

Proof. Let $\tilde{\Omega}_\eta : H_1(\tilde{\Sigma}; \mathbb{Z}_4) \to \mathbb{Z}_4$ be the $\mathbb{Z}_4$-extended quadratic form on $\tilde{\Sigma}$ defined by η via the Johnson lift. In particular, the above properties are satisfied. Recall now that the homology group $H_1(\Sigma; \mathbb{Z}_4)$ is the quotient of $H_1(\tilde{\Sigma}; \mathbb{Z}_4)$ by the boundary classes. The requirement that $\phi_\eta(\delta_i) \equiv 2 \mod 4$ for each $i = 1, \dots, n$ implies that $\tilde{\Omega}_\eta$ is trivial on the boundary classes and therefore induces a $\mathbb{Z}_4$-extended quadratic form Ω_η on the capped surface Σ. □

The following lemmas are technical results needed to proceed with the proof of Theorem 5.3.1 in the next Section. We include them here for the sake of completeness.

Lemma 5.1.5. Let $\mathcal{B} = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be a symplectic basis of vectors in $H_1(\Sigma; \mathbb{Z}_4)$. A $\mathbb{Z}_4$-extended quadratic form Ω on Σ is completely determined by the values taken on $\mathcal{B}$. Moreover, every $\mathbb{Z}_4$-extended quadratic form on Σ comes from a line field on a fixed surface with boundary components $\tilde{\Sigma}$.

Proof. If v is a vector in $H_1(\Sigma; \mathbb{Z}_4)$, it can be written as the sum $v = \sum_{i=1}^{g} a_i\alpha_i + b_i\beta_i$ for some $a_i, b_i \in \mathbb{Z}_4$. In particular, the value $\Omega(v)$ can be rewritten as the sum $\sum_{i=1}^{g} a_i\Omega(\alpha_i) + b_i\Omega(\beta_i) + 2a_ib_i$ by the property (5.1) proved in Proposition 5.1.4. The last claim follws directly form Lemma 5.1.2. □

Line fields and quadratic differentials

The winding number of a curve γ can also be given by counting with sign the number of U-turns the line field makes relatively to the tangent field of γ, as $\phi_\eta(\gamma)$ coincides with the signed intersection of $[\eta(\tilde{\Sigma})]$ with the lift of γ in $\mathbb{P}T\tilde{\Sigma}$ as defined above.

The definition of winding number has only been given for line fields of surfaces with non-empty boundary, and not for closed surfaces with points removed. However, line fields on punctured closed surfaces exist but the Poincaré–Lefschetz theorem that we used to define $[\eta]$ can only be used if the underlying manifold is compact. Hence, in what follows, we will refer to the winding number of a curve γ with respect to a line field on a punctured closed surface as the number of U-turns made by the lift $\overrightarrow{\gamma}$ with respect to the line field, similarly as defined in [Chi72, Definition 1.1].

Formally speaking, suppose $\mathcal{Z}$ is a finite set of points on Σ and let η be a line field on the complement of $\mathcal{Z}$. A C^∞-embedding $\gamma : S^1 \to \Sigma$ defines a pullback diagram

$$\begin{array}{ccc} E^\gamma & \xrightarrow{F} & \mathbb{P}T\Sigma \\ \downarrow{\scriptstyle p^\gamma} & & \downarrow{\scriptstyle \pi} \\ S^1 & \xrightarrow{\gamma} & \Sigma \end{array}$$

where $E^\gamma = \{(\theta, (l, p)) \in S^1 \times \mathbb{P}T\Sigma \mid \gamma(\theta) = p\}$. By the universal property of pullbacks, there exists a unique map $\eta^\gamma : S^1 \to E^\gamma$ such that $F \circ \eta^\gamma = \eta \circ \gamma$. Moreover, if $\overrightarrow{\gamma}$ denotes the lift of γ in $\mathbb{P}T\Sigma$, there exists a unique map $Z^\gamma : S^1 \to E^\gamma$ such that $F \circ Z^\gamma = \overrightarrow{\gamma}$ by the same principle. Both the maps η^γ and Z^γ can be seen as classes $[\eta^\gamma]$ and $[Z^\gamma]$ in $\pi_1(E^\gamma)$. In particular, it can be proved that the kernel of the homomorphism $p^\gamma_* : \pi_1(E^\gamma) \to \mathbb{Z}$ is cyclic and that $[\eta^\gamma][Z^\gamma]^{-1} \in \ker p^\gamma_*$.

The number of U-turns made by the lift $\overrightarrow{\gamma}$ with respect to the line field η is exactly $[\eta^\gamma][Z^\gamma]^{-1}$. That is, the winding number of γ for a line field on the complement of finitely many points on Σ is $[\eta^\gamma][Z^\gamma]^{-1}$. If all the above maps are pointed continuous maps between pointed topological spaces, and in particular are pointed in $p \in \Sigma \setminus \mathcal{Z}$, the winding number function we get is a map of sets $\pi_1(\Sigma \setminus \mathcal{Z}, p) \to \mathbb{Z}$ that can be extended $\phi_\eta : H_1(\Sigma \setminus \mathcal{Z}; \mathbb{Z}) \to \mathbb{Z}$.

Let (X, q) be a holomorphic quadratic differential and $\mathcal{Z}_q$ the set of zeros of $q : X \to K_X^{\otimes 2}$. Consider an isotopy class of an orientation-preserving diffeomorphism $\phi : (\Sigma, \mathcal{Z}) \to (X, \mathcal{Z}_q)$ or, in other words, a marking. The quadratic differential (X, q) comes with a line field η_q defined on the complement of $\mathcal{Z}_q$, where every point $p \in X \setminus \mathcal{Z}_q$ is mapped to the unique line l_p of the *horizontal foliation* on X defined by q as the set of smooth paths in X whose tangent vectors at each point evaluate positively under q. The differential map $d\phi^{-1}$ pushes the line field η_q forward to a line field $d(\phi^{-1})_*(\eta_q)$ on $\Sigma \setminus \mathcal{Z}$. In particular, if the quadratic differential (X, q) lies in the stratum $\mathcal{Q}(k_1, \ldots, k_n)$ and ξ_i is a free loop around the singularity p_i with order k_i, the winding number of ξ_i is $k_i + 2$ if p_i is a zero of q. If $Q_0 \in \mathcal{TQ}(\underline{k})$ is a marked quadratic differential, we will denote by $\phi^r_{Q_0}$ the winding number associated.

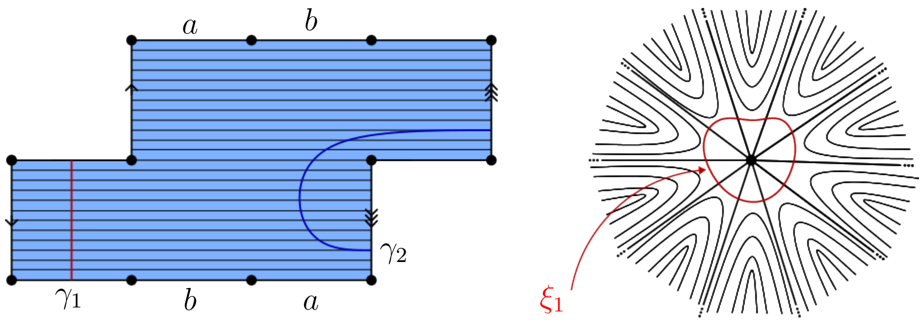


Figure 5.1: The first picture from the left is a quadratic differential in the stratum $\mathcal{Q}(8)$. Edges with no labels are identified via a translation. The thinnest horizontal lines represent some of the leaves of the horizontal foliation. The red line γ_1 has winding number $\phi_q(\gamma_1) = 0$, while the blue line γ_2 has winding number $\phi_q(\gamma_2) = 1$. The leaves of the horizontal foliation around the singularity p_1 are depicted on the picture on the right-hand side, where it can be checked that the winding number function of ξ_1 is 10. Note that it corresponds to the number of tangency points of ξ_1 with the leaves of the foliation.

We will now explain how a line field on a *marked quadratic differential* (X, ϕ, q) defines a winding number function modulo $r = \gcd(\underline{k})$ and $\mathbb{Z}_4$-extended quadratic form on Σ when 4 divides r. To do that, we will compare the induced winding number function of such a line field with the winding number function of a *blown-up quadratic differential.* Mutatis mutandis, the construction is similar to the one made for abelian differentials [BSW22, Section 3].

Let $\tilde{\Sigma}$ have $\delta_1, \ldots, \delta_n$ as boundary components. We equip every boundary component

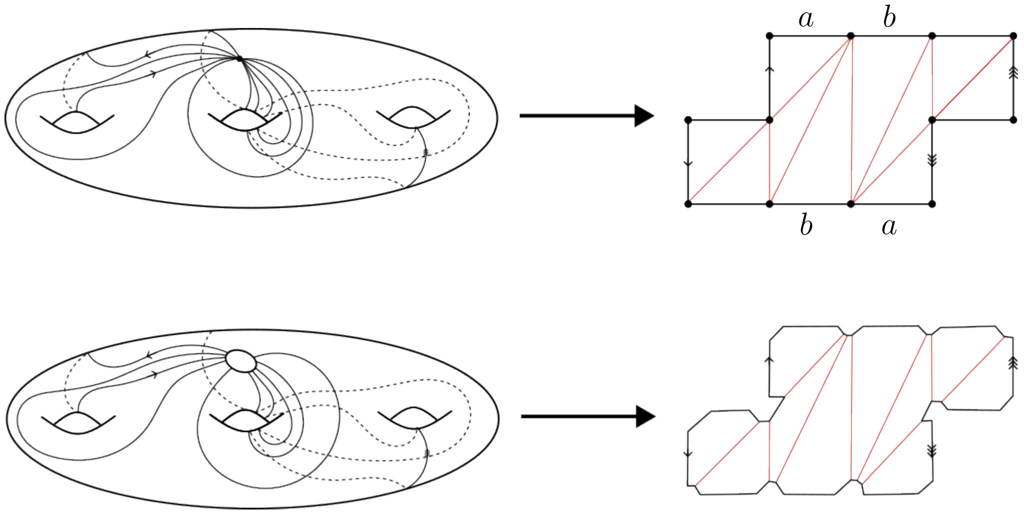


Figure 5.2: The blow-up of a quadratic differential. On the top right hand side, a polygonal representation of a quadratic differential has been cut in triangles. The picture at the bottom show how to turn the triangles in hexagon, where the new blue edges correspond to blow ups of the vertices.

δ_i with angular coordinates given by a orientation-reversing maps $\beta_i : \delta_i \to \mathbb{R}/(\pi(k_i + 2)\mathbb{Z})$. Let $(\Sigma, \mathcal{Z})$ be the pair consisting of a surface Σ of genus g obtained from $\tilde{\Sigma}$ by capping the boundary components by open disks, and a finite set $\mathcal{Z}$ given by marking every added disk with a unique point p_i. If (X, ϕ, q) is a marked translation surface of type $(k_1, \ldots, k_n)$ on $(\Sigma, \mathcal{Z})$, a *marked blown-up quadratic differential* of (X, ϕ, q) is a pair $(Bl(X, q), \tilde{\phi})$ where $\tilde{\phi} : \tilde{\Sigma} \to Bl(X, q)$ is an orientation-preserving diffeomorphism mapping every δ_i to $\partial_i Bl(X, q)$, such that $\alpha_i \circ \tilde{\phi} \equiv \beta_i \mod 2\pi$ for every $i = 1, \ldots, n$ and the following diagram commutes

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\tilde{\phi}} & Bl(X,q) \\ \Big\downarrow{\scriptstyle \mathrm{Cap}} & & \Big\downarrow{\scriptstyle c} \\ \Sigma & \xrightarrow{\phi} & X, \end{array}$$

where the map $\mathrm{Cap} : \tilde{\Sigma} \to \Sigma$ is the capping map.

Let $Q_0 = (X, \phi, q)$ be a marked quadratic differential of type $\underline{k}$ with horizontal line field η_{Q_0} and winding number ϕ_{Q_0}. The blown up quadratic differential $Bl(X, q)$ comes with a line field $\tilde{\eta}_{Q_0}$ such that $\eta_{Q_0}(c(p)) \equiv \tilde{\eta}_{Q_0}(p)$ for any $p \in Bl(X, q)$. Let us choose a marking $\tilde{\phi} : \tilde{\Sigma} \to Bl(X, q)$ lying over $\phi : \Sigma \setminus \mathcal{Z} \to X$. The line fields $d(\phi^{-1})_*(\eta_{Q_0})$ and $d(\tilde{\phi}^{-1})_*(\tilde{\eta}_{Q_0})$, respectively defined over $\Sigma \setminus \mathcal{Z}$ and $\tilde{\Sigma}$, define winding number functions where each curve around a point in $\mathcal{Z}$ or boundary curve is a multiple of r by hypothesis. Both the line fields define winding number function modulo r on the closed surface Σ that necessarily coincide. In particular, by Proposition 5.1.4, the associated $\mathbb{Z}_4$-extended quadratic form Ω_{Q_0} on Σ is well-defined if 4 divides r.

§ 5.2 | Improved lower bound on the number of connected components

For the rest of the note, we will refer to line fields induced by non-vanishing vector fields as *trivial*. A line field η is trivial if and only if its winding number function ϕ_η has only even values [LP20, Lemma 1.1.4]. Hence, a line field is non-trivial if and only if there exists at least one simple closed curve with odd winding number.

The mapping class group $\mathrm{Mod}(\tilde{\Sigma})$ acts on the set winding number functions ϕ_η of line fields. If $f \in \mathrm{Mod}(\tilde{\Sigma})$ and γ is the isotopy class of a simple closed curve on $\tilde{\Sigma}$, then

$$f \cdot \phi_\eta(\gamma) = \phi_\eta(f^{-1}(\gamma)).$$

The following results characterizes orbits of winding number functions.

Proposition 5.2.1. [LP20, Theorem 1.2.4] Suppose $g \geq 2$ and $\delta_1, \ldots, \delta_n$ are boundary components of $\tilde{\Sigma}$. Non-trivial line fields η_1 and η_2 on $\tilde{\Sigma}$ are in the same orbit of the mapping class group $\mathrm{Mod}(\tilde{\Sigma})$ if and only if $\phi_{\eta_1}(\delta_i) = \phi_{\eta_2}(\delta_i)$ for every $i = 1, \ldots, n$.

The subgroup of mapping classes in Mod_g that stabilize a winding number function ϕ_η^r modulo r on a closed and connected oriented surface Σ will be denoted by $\mathrm{Mod}_g[\phi_\eta^r]$. Here, we suppose that $r = \gcd(k_1, \ldots, k_n)$ where $\phi_\eta(\delta_i) = k_i + 2$ for every $i = 1, \ldots, n$.

Proposition 5.2.2. Let η be a non-trivial line field on $\tilde{\Sigma}$ such that $k_1+2, \ldots, k_n+2 \in \mathbb{Z}_+$ are the values of ϕ_η given by the boundary loops $\delta_1, \ldots, \delta_n$ of $\tilde{\Sigma}$, respectively. If $r = \gcd(k_1, \ldots, k_n)$ is even, the index $[\mathrm{Mod}_g : \mathrm{Mod}_g[\phi_\eta^r]]$ is $(r/2)^{2g}((r/2)^{2g} - 1)$. If r is odd, the index $[\mathrm{Mod}_g : \mathrm{Mod}_g[\phi_\eta^r]]$ is $r^{2g} - ((r+1)/2)^{2g}$.

Proof. By the Orbit-Stabilizer theorem, the index of $\mathrm{Mod}_g[\phi_\eta^r]$ is given by the cardinality of the orbit of ϕ_η^r under the action of Mod_g. By Proposition 5.2.1, the claim follows by counting the number of winding functions modulo r associated with line fields that are not trivial. Since any winding number function is uniquely determined by the values taken on a fixed symplectic basis, by Lemma 5.1.2 there is a total of r^{2g} possibilities for winding number functions modulo r associated to line fields. Trivial line fields have even winding number functions. Then, there are $(r/2)^{2g}$ winding number functions associated with trivial line fields if r is even, and $((r+1)/2)^{2g}$ if r is odd. □

Theorem F will be proved by means of the topological monodromy map. For this reason, we will now define the topology of the Teichmüller strata $\mathcal{TQ}(\underline{k})$. A holomorphic quadratic differential q on a genus g Riemann surface X defines a unique *Teichmüller mapping* $h : X \to Y$ with initial quadratic differential q and stretch factor $(1 + \|q\|)/(1 - \|q\|)$, where $\|q\| = \int_X |q|$ [FM12, Section 11.4]. The assignment

$(X, q, \phi) \mapsto (Y, h \circ \phi)$ gives the set of all genus g marked holomorphic quadratic differentials $\mathcal{TQ}_g$ the structure of cotangent bundle of the Teichmüller space $\mathcal{T}_g$. The space $\mathcal{TQ}_g$ can be then stratified in subvarieties $\mathcal{TQ}(\underline{k})$ of complex dimension $2g + n - 2$. [Vee90]. More precisely, one can assign local cohomological coordinates to $\mathcal{TQ}(\underline{k})$ starting from the following classic construction.

Proposition 5.2.3. [Lan04, Construction 1] Let (X, q) be a meromorphic quadratic differential with only simple poles, if any. If q is not the square of an abelian differential, there exists a unique minimal canonical (possibly ramified) double cover $\pi : X' \to X$ where X is endowed with an abelian differential ω and $\pi^*(q) = w^2$. The set of critical values of π is the set of singularities of Q of odd degrees.

On the complement of the set $\mathcal{Z}_q$ of singularities of (X, q), the double branched cover $\pi : X' \to X$ is regular and corresponds to the conjugacy class of a normal subgroup $N \triangleleft \pi_1(X \setminus \mathcal{Z}_q)$ of index 2. If $\phi : \Sigma \to X$ is a marking for (X, q), then the image $\phi_*^{-1}(N)$ under the homomorphism induced by ϕ^{-1} is a normal subgroup of $\pi_1(\Sigma \setminus \mathcal{Z})$ of index 2, where $\mathcal{Z} = \phi^{-1}(\mathcal{Z}_q)$. Again by the Galois correspondence, there exists a double cover $\pi_{top} : \Sigma' \setminus \mathcal{Z}' \to \Sigma \setminus \mathcal{Z}$ corresponding to the conjugacy class of the normal subgroup $\phi_*^{-1}(N)$. Moreover, the marking ϕ has two lifts $\phi_1', \phi_2' : \Sigma' \setminus \mathcal{Z}' \to X' \setminus \mathcal{Z}_\omega$, where $\mathcal{Z}_\omega$ is the set of singularities of one of the two roots of ω^2 and $\pi \circ \phi_i' = \phi \circ \pi_{top}$ holds for $i = 1, 2$. Notice that both ϕ_1' and ϕ_1' can be extended to Σ'.

The marked abelian differentials (X', ϕ_1', ω) and (X', ϕ_1', ω) are points in the Teichmüller strata of abelian differentials $\mathcal{T}(l_1, \dots, l_m)$, where $(l_1, \dots, l_m)$ is a partition of $2g' - 2$ if g' is the genus of X' and the numbers $(l_1, \dots, l_m)$ are obtained from $(k_1, \dots, k_n)$ by the following rule: each even k_i returns a pair of singularities of order $k_i/2$; each odd k_i returns a single zero of order $k_i + 1$ and associates a regular point to a pole.

If ι is the involution generating the deck transformation group of π_{top}, its induced map in homology $\iota_* : H_1(\Sigma', \mathcal{Z}'; \mathbb{C}) \to H_1(\Sigma', \mathcal{Z}'; \mathbb{C})$ splits the vector space $H_1(\Sigma', \mathcal{Z}'; \mathbb{C})$ into the direct sum $V_1 \oplus V_{-1}$ of invariant and anti-invariant vectors of ι_*, respectively. If τ is a triangulation of Σ where the vertices correspond to the points in $\mathcal{Z}$ and $\phi(\tau)$ is a collection of saddle connections, there exists a triangulation τ' of Σ' obtained by lifting the edges of τ via π_{top}. The images of the edges are still mapped to saddle connections under the markings ϕ_1' and ϕ_2' and the vertices of τ' correspond to the points in $\mathcal{Z}'$. Note that τ' is set-wise fixed by the involution ι. Then, by construction, the marked abelian differentials $T_1 = (X', \phi_1', \omega)$ and $T_2 = (X', \phi_2', \omega)$ coming from the marked quadratic differential (X, ϕ, q) are contained in the intersection $U_{\tau'} \cap \mathrm{hol}_{\tau'}^{-1}(V_{-1})$ and, since the two markings ϕ_1' and ϕ_2' differ by the action of the involution ι, we have that $\mathrm{hol}_{\tau'}(T_1) = -\mathrm{hol}_{\tau'}(T_2)$. Note that the choice of $-\omega$ instead of ω as a square root of $\pi^*(q)$ would have produced the same pair of points in cohomology. By choosing one of the two lifts, ϕ_1' or ϕ_2', we get a local mapping $\mathcal{TQ}(\underline{k}) \to \mathcal{T}(\underline{l})$.

Fix a component $\mathcal{L}$ of a stratum $\mathcal{Q}(\underline{k})$ and a marked quadratic differential $Q_0 = (X, f, q)$. Then, we can define a topological monodromy map $\rho_{\mathcal{L}} : \pi_1^{orb}(\mathcal{L}, Q_0) \to \mathrm{Mod}_g$ as induced by $\mathcal{TL} \to \mathcal{T}_g$. Recall that the group $\pi_1^{orb}(\mathcal{L}, Q_0)$ is the set of all homeomorphisms of the universal cover $\widetilde{\mathcal{TL}}$ that commute with $\widetilde{\mathcal{TL}} \to \mathcal{M}_g$. The following is an application of Lemma 2.4.1.

Lemma 5.2.4. Let $\mathcal{L}$ be a component of a stratum $\mathcal{Q}(k_1, \ldots, k_n)$ for $g \geq 2$ and Q_0 a marked quadratic differential in $\mathcal{TL}$. The image $\mathrm{Im}\,\rho_{\mathcal{L}}$ of the topological monodromy is the stabilizer in Mod_g of the connected component of $\mathcal{TL}$ containing Q_0.

The winding number ϕ_{Q_0} induced on a punctured closed surface of genus g by a marked quadratic differential (X, ϕ, q) can be extended modulo $r = \gcd(k_1, \ldots, k_n)$ to be a function on the set $\mathcal{S}$ of isotopy classes of simple closed curve of a closed genus g surface when the orders of the singularites of q are all divisible by r. More precisely, the winding numbers of cycles in $\Sigma \setminus \mathcal{Z}_q$ that encircle the singularites of q, all differ from the non-essential cycle in $\mathbb{P}T(\Sigma \setminus \mathcal{Z}_q)$ by the orders k_i, as explained in the previous section. In these cases, the winding number function ϕ_{Q_0} induces a map $\phi^r_{Q_0} : \mathcal{S} \to \mathbb{Z}_r$.

Lemma 5.2.5. Let $Q_0 = (X, f, q)$ be a marked quadratic differential in $\mathcal{TL}$ and $\mathcal{L}$ a component of $\mathcal{Q}(k_1, \ldots, k_n)$. If $r = \gcd(k_1, \ldots, k_n)$, the winding number function $\phi^r_{Q_0}$ of any smooth, oriented simple closed curve on Σ is constant on each component of $\mathcal{TL}$.

Proof. Fix a smooth, oriented simple closed curve γ on Σ. The map $\phi^r_{\bullet}(\gamma) : \mathcal{TL} \to \mathbb{Z}_r$ returning the winding number of γ modulo r with respect to the line field $d(f^{-1})_*(\eta_q)$ is a continuous map since the horizontal line field η_{Q_0} depends continuously on Q_0. However, a continuous map into a discrete space must be constant on the connected components. □

Proof of Theorem F. By Lemma 5.2.5, winding numbers modulo r are invariants for the connected components of $\mathcal{TQ}(k_1, \ldots, k_n)$. By Lemma 5.2.4, we only need to count the cardinality of the orbit of winding numbers modulo r by the action of the mapping class group Mod_g, which is given by Proposition 5.2.2. □

§ 5.3 | The case $4|r$

The symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ acts of the set of $\mathbb{Z}_4$-extended quadratic forms Ω_η. If $M \in \mathrm{Sp}(2g, \mathbb{Z})$ and $a \in H_1(\Sigma, \mathbb{Z}_4)$, then

$$M \cdot \Omega_\eta(a) = \Omega_\eta(M^{-1}(a)).$$

Let us denote by $\mathrm{Sp}(2g, \mathbb{Z})[\Omega_\eta]$ the stabilizer of a $\mathbb{Z}_4$-extended quadratic form Ω_η in the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. In this section, we prove that that the image of the symplectic monodromy $\rho_{\mathcal{L}}^{\mathbb{Z}}$ of a non-hyperelliptic connected component of a stratum of quadratic differentials in genus $g \geq 3$ is contained in some $\mathrm{Sp}(2g, \mathbb{Z})[\Omega_\eta]$ whenever 4 divides r. Once the containment has been proved, Corollary 2 will follow from Theorem 5.3.1.

Theorem 5.3.1. Let Ω_η be a $\mathbb{Z}_4$-extended quadratic form induced by a non-trivial line field on surface $\tilde{\Sigma}$ with boundary. Then, the index $[\mathrm{Sp}(2g, \mathbb{Z}) : \mathrm{Sp}(2g, \mathbb{Z})[\Omega_\eta]]$ is $2^{2g}(2^{2g} - 1)$ if $g \geq 2$.

Proof. By the Orbit-Stabilizer Theorem, the index of $\mathrm{Sp}(2g, \mathbb{Z})[\Omega_\eta]$ is the same as the cardinality of the orbit $\{M \cdot \Omega_\eta \mid M \in \mathrm{Sp}(2g, \mathbb{Z})\}$. A $\mathbb{Z}_4$-extended quadratic form is in the orbit of Ω_η if and only if is not even, that is if its image is not contained in $2\mathbb{Z}_4$.

Suppose $M \cdot \Omega_\eta$ is even. If ϕ_η is the winding number function that induces Ω_η, then $M \cdot \phi_\eta$ is even too as it differ by 2 modulo 4 from $M \cdot \Omega_\eta$. Therefore, if $f \in \mathrm{Mod}_g$ is a mapping class with image in $\mathrm{Sp}(2g, \mathbb{Z})$ the matrix M, we have that the line field $f \cdot \eta$ is trivial by Proposition 5.2.1. If Ω is a $\mathbb{Z}_4$-extended quadratic form that is not even, then there exists a line field η' on $\tilde{\Sigma}$ such that $\Omega = \Omega_{\eta'}$. In particular, its winding number function η' is non-trivial and by Proposition 5.2.1 there exists a mapping class $f \in \mathrm{Mod}_g$ such that $\eta = f \cdot \eta'$. Then, if $M \in \mathrm{Sp}(2g, \mathbb{Z})$ is the image of a mapping class f, we have $M \cdot \Omega_{\eta'} = \Omega_\eta$.

Hence, we only have to count the number of $\mathbb{Z}_4$-extended quadratic forms that are not even. By Lemma 5.1.5, any two $\mathbb{Z}_4$-extended quadratic form are equal if and only if they coincide on a fixed symplectic basis $\mathcal{B}$. Hence, there are 4^{2g} $\mathbb{Z}_4$-extended quadratic form on Σ in total. The even $\mathbb{Z}_4$-extended quadratic form are 2^{2g}, as the only values taken are even. Hence, the orbit $\{M \cdot \Omega_\eta \mid M \in \mathrm{Sp}(2g, \mathbb{Z})\}$ counts $2^{2g}(2^{2g} - 1)$ elements. □

Proof of Corollary 2. Because of Lemma 5.2.4 and Theorem 5.3.1, we only need to show that the stabilizer in Mod_g of Ω_{Q_0} contains the stabilzer of $\phi_{Q_0}^4$. However, this follows from the surjectivity of the symplectic representation $\psi_g : \mathrm{Mod}_g \to \mathrm{Sp}(2g, \mathbb{Z})$ of the mapping class group. □

§ 5.4 | The Rauzy–Veech cycle

In this section, we introduce the Rauzy–Veech cycle and prove Corollary 1.

In 2009, Walker proved that the connected components of the Teichmüller strata of the form $\mathcal{TQ}(2^g, k_1, \ldots, k_n)$ are classified by non-zero vectors in $H_1(\Sigma, \mathbb{Z}_2)$ [Wal09, Theorem 1]. When r is even, one can associate each component of the stratum $\mathcal{TQ}(k_1, \ldots, k_n)$ with a non-zero vector in $H_1(\Sigma, \mathbb{Z}_2)$ via the following assignment. First, we pick an arbitrary marked quadratic differential Q and consider the canonical double cover π :

$T \to Q$ associated to Q. Here, T is the square of an abelian differential and the canonical double cover pulls back the quadratic differential Q to T. The canonical double cover $\pi : T \to Q$ is not ramified and has a holonomy representation $\mathrm{hol}_Q : \pi_1(Q) \to \mathbb{Z}_2$. The holonomy hol_Q can be extended to the homology group $H_1(\Sigma, \mathbb{Z}_2)$ since $\mathbb{Z}_2$ is abelian and is defined as follows. If we consider the connection on the tangent space of Q induced by the flat metric of the respective quadratic differential, the holonomy $\mathrm{hol}_Q([\gamma])$ of the cycle $[\gamma]$ returns $0 \in \mathbb{Z}_2$ if the parallel transport along γ brings a vector back to itself and $1 \in \mathbb{Z}_2$ if a vector is brought back to its opposite.

The *Rauzy–Veech* cycle RV_Q associated to Q is the cycle in $H_1(\Sigma, \mathbb{Z}_2)$ that is dual to the cohomology class of hol_Q. Notice that the Rauzy–Veech cycle is the dual cycle of the winding number function ϕ_Q modulo 2. Indeed, if γ is a simple closed curve on Σ and has non-trivial holonomy, then its winding number is necessarily odd. On the other hand, if γ has trivial holonomy, its winding number is even.

If Q is given together with a polygonal representation as a half-translation surface, the Rauzy–Veech cycle RV_Q can be constructed as follows. Consider the family of saddle connections of Q that bound the polygonal representation of Q. Here, we suppose that our polygonal representation consists of only one polygon and all vertices are singular. Hence, the cycles $\theta_1, \dots, \theta_{2g+n-1}$ obtained by joining the midpoints of the paired sides of Q form a natural basis for the homology of Q modulo 2, relative to the singular points. Suppose θ_i corresponds to a pair of edges a half-translation identifies. Then, the winding number of θ_i must be odd, as the holonomy is non-trivial. On the other hand, a cycle θ_i corresponding to a pair of edges a pure translation identifies has even winding number and $\phi_Q^2(\theta_i) = 0 \mod 2$. Given the epimorphism $p_* : H_1(\Sigma \setminus \mathcal{Z}; \mathbb{Z}_2) \to H_1(\Sigma; \mathbb{Z}_2)$, the Rauzy–Veech cycle RV_Q is the cycle

$$RV_Q = \sum_{i=1}^{2g+n-1} \phi_Q^2(p_*(\theta_i))p_*(\theta_i).$$

Theorem F and Corollary 2 show that, in general, the dual of the Rauzy–Veech cycle is not the correct invariant to classify the components of Teichmüller strata of quadratic differentials, as there might be roots of the dual vector RV_Q^* if r is, for example, even. However, in some cases where $r = 2$, the dual of the Rauzy–Veech cycle is the right invariant and we are able to show that the symplectic monodromy image is full inside the stabilizer in $\mathrm{Sp}(2g, \mathbb{Z})$ of the winding number function ϕ_Q^2 modulo 2, which corresponds to the stabilizer of RV_Q^*. The first step towards the proof of Corollary 1 is the following construction, known as the Thurston–Veech construction [Thu88].

Let $\alpha = \{\alpha_1, \dots, \alpha_s\}$ and $\beta = \{\beta_1, \dots, \beta_t\}$ be transverse multicurves that fill Σ into n disks $D_1, \dots, D_n$. The cellularization given by the curves in α and β defines a dual cellularization in squares. Here, each disk D_i corresponds to a point $p_i \in D_i$, and for any two adjacent disks D_i and D_j we construct an arc in $\overline{D_i} \cup \overline{D_j}$ from p_i

to q_j that passes through the common edge. If we declare each square to be a flat unit square, we get a flat metric with cone angles given by $(k_i + 2)\pi$, where $k_i + 2$ is the number of components of α (equivalently of β) that bound the disk D_i for each $i = 1, \ldots, n$. The half-translation surface structure on Σ has the curves in α and β as vertical and horizontal cylinders, respectively. Moreover, the Thurston–Veech construction returns a translation surface structure if and only if there exists a coherent way of orienting the multicurves so that their geometric intersection coincides with the algebraic intersection. We will denote by $Q(\alpha, \beta)$ the marked quadratic differential obtained from multicurves α and β via the Thurston-Veech construction.

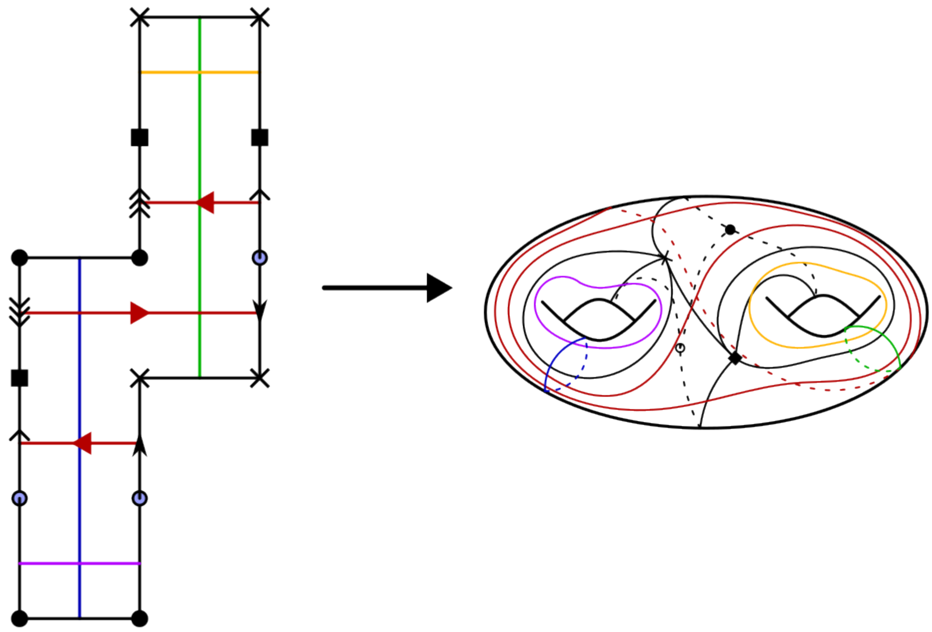


Figure 5.3: A half-translation surface in $\mathcal{Q}(1^4)$ in genus 2. Opposide sides with no arrows are identified via the appropriate complex translations.

Let us consider a genus $g \geq 5$ closed surface Σ and two sets of multicurves $\{\alpha_1, \ldots .\alpha_g\}$ and $\{\beta_1, \ldots, \beta_g\}$ that form a symplectic basis $\mathcal{B}$ on Σ. Moreover, let γ_i be a simple closed curve that intersects once β_i and β_{i+1} for every $i = 1, \ldots, g-1$ and has trivial geometric intersections with the rest of the curves in $\mathcal{B}$. If we fix $\tau \in \mathrm{Mod}_g$ to be the hyperelliptic involution that fixes the chain of curves $\{\alpha_1, \beta_1, \gamma_1, \beta_2 \ldots, \beta_g, \alpha_g\}$, the two multicurves $\{\alpha_1, \alpha_2, \tau(\alpha_2), \ldots, \alpha_{g-1}, \tau(\alpha_{g-1}), \alpha_g, \gamma_1, \ldots .\gamma_{g-1}\}$ and $\{\beta_1, \ldots, \beta_g\}$ fill the surface Σ. However, it is always possible to coherently orient the multicurves so that the geometric and algebraic intersection coincide [CS21, Lemma 6.13]. Then, we replace β_1 and β_3 by a simple closed curve b as in Figure 5.4. Note that β_1, β_3 and b bound a pair of pants and the geometric intersection of b with α_2 is 2, whereas their algebraic intersection is 0 no matter the orientation we choose.

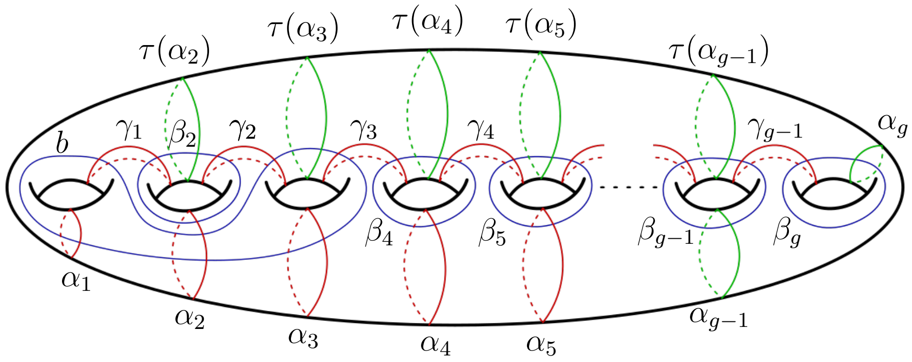


Figure 5.4: A set of multicurves on Σ, where there is no possible way to orient the multicurves so that the geometric and algebraic intersection of each pair coincide.

If we denote by α the multicurve $\{\alpha_1, \alpha_2, \tau(\alpha_2), \ldots, \alpha_{g-1}, \tau(\alpha_{g-1}), \alpha_g, , \gamma_1, \ldots . \gamma_{g-1}\}$ and by β the multicurve $\{b, \beta_2, \beta_4, \ldots, \beta_g\}$, the half-translation surface $\mathcal{Q}(\alpha, \beta)$ represents a marked quadratic differential in the Teichmüller stratum $\mathcal{TQ}(2^{2g-2})$. Indeed, the multicurves α and β fill Σ into $2g-2$ disks $D_1, \ldots, D_{2g-2}$ that are bound by exactly 4 arcs of the curves in α (or equivalently in β).

Remove from α any subcollection of curves in the set $\{\tau(\alpha_2), \ldots, \tau(\alpha_{g-1}), \alpha_6, \ldots, \alpha_g\}$. The remaining curves in α, together with the multicurve β, still fill Σ and no possible orientation can make the geometric and algebraic intersection between α_2 and b coincide, thus yielding us a proper quadratic differential. Suppose we ignore exactly one curve in α that is not one between $\alpha_1, \ldots, \alpha_5$ of any of the γ_i's. The number of disks cut is now $2g-3$, where $2g-4$ disks correspond to singularities of order 2 (or, equivalently, with cone angle 4π) as before and one of them is of order 4 (or, equivalently, with cone angle 6π).

More generally, the Thurston–Veech construction can be used to obtain a marked quadratic differential in any Teichmüller stratum $\mathcal{TQ}(2^4, \underline{k})$ where $\underline{k}$ is any partition of $4g-8$ in even natural numbers, by removing a subcollection of curves in $\{\tau(\alpha_2), \ldots, \tau(\alpha_{g-1}), \alpha_6, \ldots, \alpha_g\}$ from α. Indeed, the even partitions of $4g-8$ are in one-to-one correspondence with the partitions of $2g-4$, and the correspondence can be made explicit dividing any integer in an even partition of $4g-8$ by 2. The integers in the partition (1^{2g-4}) can be labeled by the corresponding disks of the Thurston-Veech construction obtained by filling Σ with $\alpha \cup \beta$. Starting from the partition (1^{2g-4}), it is possible to obtain any other partition of $2g-4$ by iteratively summing any two integers which corresponding disks are adjacent or, equivalently, by removing one curve between $\{\tau(\alpha_2), \ldots, \tau(\alpha_{g-1}), \alpha_6, \ldots, \alpha_g\}$ from α.

Proposition 5.4.1. Let $g \geq 5$ and $\underline{k}$ be a partition of $4g-6$. Then, there exists a multicurve $\hat{\alpha}$ of α containing $\alpha_1, \ldots \alpha_5$ and any γ_i for $i = 1, \ldots, g-1$ such that

$Q(\hat{\alpha}, \beta) \in \mathcal{TQ}(2^4, \underline{k})$. In particular, the image of the monodromy

$$\rho_{\mathcal{Q}(2^4,\underline{k})} : \pi_1^{orb}(\mathcal{Q}(2^4, \underline{k}), Q(\hat{\alpha}, \beta)) \to \mathrm{Mod}_g$$

contains the group generated by the Dehn twists about the curves in $\hat{\alpha} \cup \beta$.

Proof. The multicurve $\hat{\alpha}$ can be obtained as above by ignoring a subset of curves in $\{\tau(\alpha_2), \ldots, \tau(\alpha_{g-1}), \alpha_6, \ldots, \alpha_g\}$. The marking associated to the marked quadratic differential $Q(\hat{\alpha}, \beta)$ maps the curves in $\hat{\alpha}$ to vertical cylinders and the curves in β to horizontal cylinders. Then, the Dehn twists about any curve in $\hat{\alpha} \cup \beta$ is given by cylinder shears as in Chapter 4. □

Each of the marked quadratic differentials of the form $Q(\hat{\alpha}, \beta)$ comes with a non-trivial line field, as there is no coherent way to orient the curves in $\hat{\alpha} \cup \beta$ so to match the algebraic and geometric intersections and the flat metric of the Thurston–Veech construction is not coming from a translation surface structure. In the following, the *admissible group* will be the subgroup $T[\hat{\alpha}, \beta]$ of $\mathrm{Mod}_g[\phi^2_{Q(\hat{\alpha},\beta)}]$ generated by the Dehn twists about the curves in $\hat{\alpha} \cup \beta$.

Lemma 5.4.2. Let $Q(\hat{\alpha}, \beta)$ be the marked quadratic differential obtained from the Thurston–Veech construction via the sets of multicurves $\hat{\alpha}$ and β. Then, the Rauzy–Veech cycle $RV_{Q(\hat{\alpha},\beta)}$ associated with $Q(\hat{\alpha}, \beta)$ is the cycle $[\beta_1 + \beta_3] \in H_1(\Sigma; \mathbb{Z}_2)$.

Proof. Recall that the winding number function $\phi_{Q(\hat{\alpha},\beta)}$ modulo 2 is uniquely determined by the values taken on the symplectic basis $\mathcal{B}$ by Lemma 5.1.2. Its mod 2 version is denoted by $\phi^2_{Q(\hat{\alpha},\beta)}$. Any curve in $\hat{\alpha}$ is a cylinder and, in particular, has a trivial winding number. Also the curves $\alpha_6, \ldots, \alpha_g$ have a zero winding number, since $[\alpha_i + \alpha_{i+1}]$ and $[\gamma_i]$ represent the same cycle in $H_1(\Sigma; \mathbb{Z}_2)$ for any $i = 6, \ldots, g-1$ and by homological coherence it is possible to inductively prove that $\phi^2_{Q(\hat{\alpha},\beta)}(\alpha_i) \equiv 0 \mod 2$. Again by homological coherence, we have that $\phi^2_{Q(\hat{\alpha},\beta)}(\beta_1) \equiv \phi^2_{Q(\hat{\alpha},\beta)}(\beta_3)$ holds and must return an odd winding number. Indeed, the winding number $\phi^2_{Q(\hat{\alpha},\beta)}$ is uniquely determined by the values on $\mathcal{B}$ and cannot be trivial. Therefore, the homomorphism $\phi^2_{Q(\hat{\alpha},\beta)}$ is the dual of the cycle $[\beta_1 + \beta_3] \in H_1(\Sigma; \mathbb{Z}_2)$. □

We can now prove Corollary 1. We will prove that the stabilizer in $\mathrm{Sp}(2g, \mathbb{Z})$ of the winding number function $\phi_{Q(\hat{\alpha},\beta)}$ modulo 2 associated to the marked quadratic differential $Q(\hat{\alpha}, \beta)$ is generated by *transvections* T_v where v is a primitive vector represented by the simple closed curves in $\hat{\alpha} \cup \beta$. In other words, we will prove that $\psi_g(T[\hat{\alpha}, \beta])$ coincides with $\mathrm{Sp}(2g, \mathbb{Z})[\phi^2_{Q(\hat{\alpha},\beta)}]$. We remind the reader that for any $v, w \in \mathbb{Z}^{2g}$ represented by isotopy classes of simple closed curves on Σ we have $T_v(w) = w + i(w, v)v$, where $i(w, v)$ is the algebraic intersection between v and w. Moreover, any Dehn twist

T_γ is mapped to the transvection about the cycle of γ via the symplectic representation ψ_g [FM12, Proposition 6.3].

Before proving Corollary 1, we observe that it is possible to find a mapping class $h_i \in T[\hat{\alpha}, \beta]$ such that $h_i(\alpha_4) = \alpha_i$ for $i = 6, \dots, g$. Then, the Dehn twists T_{α_i} for $i = 6, \dots, g$ are elements of $T[\hat{\alpha}, \beta]$ since T_{α_i} can be written as the composition $hT_{\alpha_4}h^{-1}$. We first can consider h_6 to be the composition of Dehn twists $T_{\gamma_6}T_{\beta_5}T_{\beta_6}T_{\gamma_6}T_{\alpha_5}T_{\beta_5}T_{\gamma_5}T_{\beta_4}$ [FM12, Proof of Theorem 4.14] and note that by twist-linearity in Proposition 5.1.1, the Dehn twist T_{α_6} acts trivially on $\phi^2_{Q(\hat{\alpha},\beta)}$ and, more importantly, the Dehn twist T_{α_6} is contained in the admissible subgroup $T[\hat{\alpha}, \beta]$ generated by the Dehn twists about the curves in $\hat{\alpha} \cup \beta$. Iteratively, one can use the mapping class $h_i = T_{\gamma_{i+2}}T_{\beta_{i+1}}T_{\beta_{i+2}}T_{\gamma_{i+2}}T_{\alpha_{i+1}}T_{\beta_{i+1}}T_{\gamma_{i+1}}T_{\beta_1}$ in $T[\hat{\alpha}, \beta]$ to show that $\prod_{j=6}^{i} h_j(\alpha_4) = \alpha_i$ for any $i = 6, \dots, g$. Hence, we have shown that the transvections $T_{[\alpha_i]}$ for $i = 2, \dots, g$ are contained in $\psi_g(T[\hat{\alpha}, \beta])$. The transvections $T_{[\beta_j]}$ and $T_{[\gamma_k]}$ for $j = 2, 4, 5 \dots, g$ and $k = 2 \dots, g-1$ are already in $\psi_g(T[\hat{\alpha}, \beta])$ by construction. Note also that $T_{[\beta_1+\beta_3]} = T_{[b]} \in \psi_g(T[\hat{\alpha}, \beta])$.

Proof of Corollary 1. The stabilizer $\mathrm{Sp}(2g, \mathbb{Z})[\alpha_1]$ of the cycle $[\alpha_1] \in H_1(\Sigma; \mathbb{Z}_2)$ in $\mathrm{Sp}(2g, \mathbb{Z})$ is generated by finitely many transvections [DDPR25, Theorem 2]. In particular, we have

$$\mathrm{Sp}(2g, \mathbb{Z})[\alpha_1] = \langle T_{[\alpha_1]}, T_{[\alpha_i]}, T_{[\beta_i]}, T_{[\gamma_j]} \mid i = 2, \dots, g, j = 1, \dots, g-1 \rangle.$$

Since the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ acts transitively on the symplectic bases, there exists $M \in \mathrm{Sp}(2g, \mathbb{Z})$ such that $M([\alpha_1]) = [\alpha_1 + \alpha_3]$ and $M([\beta_3]) = [\beta_1 + \beta_3]$ that fixes the all the other cycles in $\mathcal{B}$ other then α_1 and β_3. Under the inner automorphism induced by M, the group $\mathrm{Sp}(2g, \mathbb{Z})[\alpha_1]$ is isomorphic to the stabilizer of the cycle $[\alpha_1 + \alpha_3] \in H_1(\Sigma; \mathbb{Z}_2)$ and

$$\mathrm{Sp}(2g, \mathbb{Z})[\alpha_1 + \alpha_3] = \left\langle T_{[\alpha_1+\alpha_3]}, T_{[\alpha_i]}, T_{[\beta_1+\beta_3]}, T_{[\beta_j]}, T_{[\gamma_1+\alpha_3]}, T_{[\gamma_k]} \;\middle|\; \begin{array}{l} \text{if } i = 2. \dots, g \\ \text{if } j = 2, 4, 5 \dots, g \\ \text{if } k = 2 \dots, g-1 \end{array} \right\rangle.$$

The vector field $\mathbb{Z}_2^{2g}$, endowed with the algebraic intersection number, is isomorphic to $\mathrm{Hom}(\mathbb{Z}_2^{2g}, \mathbb{Z}_2)$ via the symplectomorphism $v \mapsto i(v, \bullet)$, under which the cycle $[\alpha_1 + \alpha_3]$ is mapped to the dual of $[\beta_1 + \beta_3]$ or, in other words, to $\phi^2_{Q(\hat{\alpha},\beta)}$. Any symplectic matrix fixing $[\alpha_1 + \alpha_3]$ must necessarily stabilize $\phi^2_{Q(\hat{\alpha},\beta)}$, and viceversa, since

$$M \cdot i(v, w) = i(v, M^{-1}w) = i(Mv, w)$$

holds for any $v, w \in \mathbb{Z}_2^{2g}$. Then, we have that $M \in \mathrm{Sp}(2g, \mathbb{Z})[\alpha_1 + \alpha_3]$ if and only if M stabilizes the homomorphism $\phi^2_{Q(\hat{\alpha},\beta)}$. To show that the image of $T[\hat{\alpha}, \beta]$ under the symplectic representation ψ_g is the stabilizer $\mathrm{Sp}(2g, \mathbb{Z})[\phi^2_{Q(\hat{\alpha},\beta)}]$, we only need to prove that both the transvections $T_{[\gamma_1+\alpha_3]}$ and $T_{[\alpha_1+\alpha_3]}$ are in $\psi_g(T[\hat{\alpha},\beta])$.

Step 1: $T_{[\gamma_1+\alpha_3]} \in \psi_g(T[\hat{\alpha},\beta])$. We orient all the curves in $\hat{\alpha}\cup\beta$ but b so that the pairwise algebraic intersection is 1. In particular, we obtain that $[\gamma_1+\alpha_1+\alpha_2]=0$ and $[\gamma_2]=[\alpha_2+\alpha_3]$ in $H_1(\Sigma;\mathbb{Z})$ with integer coefficients[1]. The reader can check that the composition of transvections $M = T_{[\beta_2]}T_{[\gamma_1]}T^{-1}_{[\gamma_2]}T_{[\beta_2]}$ maps the cycle $[\alpha_2]$ to $-[\alpha_1+\alpha_2+\alpha_3]$. Then, the self-symplectomorphism $MT_{[\alpha_4]}M^{-1} = T_{-[\alpha_1+\alpha_2+\alpha_3]} = T_{[\alpha_1+\alpha_2+\alpha_3]}$ is in $\psi_g(T[\hat{\alpha},\beta])$.

Step 2: $T_{[\alpha_1+\alpha_3]} \in \psi_g(T[\hat{\alpha},\beta])$. Let δ be the simple closed curve as in Figure 5.5 whose homology class is $[\alpha_1+\alpha_3]$. By the lantern relation in Mod_g [FM12, Proposition 5.1], we have that

$$T_{\alpha_2}T_\delta T_c = T_{\gamma_1}T_{\gamma_2}T_{\alpha_2}T_{\alpha_3},$$

where c is a simple closed curve in Figure 5.5. The image of such a relation in $\mathrm{Sp}(2g,\mathbb{Z})$ implies that $T_{[\delta]} = T_{[\alpha_1+\alpha_3]}$ is in $\psi_g(T[\hat{\alpha},\beta])$ if and only if $T_{[c]}$ is. But $[c]=[\alpha_1+\alpha_2+\alpha_3]$ and we are done by the previous step. □

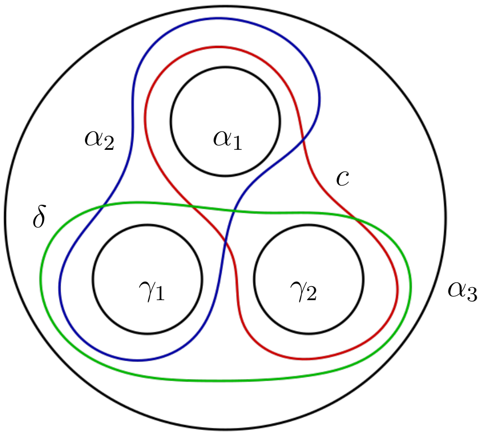


Figure 5.5: The curves involved in the lantern relation of the Step 2 in the proof of Corollary 1.

§ 5.5 | Proof of Theorem G

In this last section we show how to obtain Theorem G using the Thurston–Veech construction adopted to prove Corollary 1. The possible partitions $\underline{k}$ of $4g-4$ for which we prove $\mathcal{TQ}(\underline{k})$ to be connected are of the form $\underline{k}=(1^3,k_1,\dots,k_n)$ and can be derived recursively from the graph C_g in Figures 1.3 and 1.2 by applying a finite number of elementary edge-collapses. We recall that that an elementary edge collapse is one of the following two operations. An elementary collapse of an unlabeled edge e consists in replacing the edge e and its endpoints v_1 and v_2, labelled by l_1 and l_2, respectively, by a new vertex v with label l_1+l_2. The new vertex v is adjacent to any

[1] Here, to keep the notation as simple as possible, we did not make any distinction in notation between cycles with $\mathbb{Z}_2$ or $\mathbb{Z}$ coefficients. However, we will always specify the coefficients considered.

other vertex on the graph that previously shared an edge with one of v_1 or v_2. If e is an edge with a label δ_i for some $i = 1, \dots, d$, by an elementary collapse of e we mean the edge collapses of e together with the unique other edge labelled by δ_i.

In the appendix, we include a Python code that can be used to check whether or not a given partition can be obtained form the partition associated with C_g by a finite number of elementary edge-collapses.

Proof of Theorem G. Let $\alpha = \{\alpha_1, \alpha_2, \gamma_1, \dots, \gamma_{g-1}\}$ and $\beta = \{b, \beta_1, \dots, \beta_g\}$ be the multicurves on Σ shown in Figure 5.6. Note that the Humphries generating set of Mod_g is given by the Dehn twists about the curves in $\alpha \cup \beta \setminus \{b\}$ [Hum79]. The algebraic and geometric intersection between b and α_2 cannot coincide, whatever orientation we choose to give them. Hence, the multicurves α and β can be upgraded to a marked half-translation surface structure on Σ that is not a translation surface structure. In particular, any marked quadratic differential obtained via the Thurston–Veech construction via a family of multicurves containing $\alpha \cup \beta$ produces a surjective topological monodromy and, by Lemma 5.2.4, a connected Teichmüller stratum.

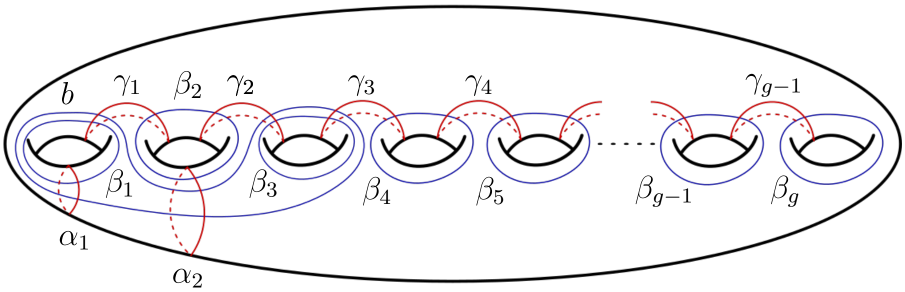


Figure 5.6: The multicurves α and β. The flat structure obtained from them produces a marked quadratic differential in $\mathcal{TQ}(1^3, 4g-7)$.

Suppose now we add to α the isotopy classes of the curves $\{\alpha_3, \dots, \alpha_g, \tau(\alpha_2), \dots, \tau(\alpha_g)\}$, where τ is the hyperelliptic involution associated with the chain $\alpha_1, \beta_1, \gamma_1, \beta_2 \dots, \beta_g, \alpha_g$, as in the Figures 5.7 and 5.8. The curves $\delta_1, \dots, \delta_d$, where $d = \frac{g-3}{2}$ if g is even and $d = \frac{g-4}{2}$ if g is odd, can be added to β so to obtain a marked quadratic differential in $\mathcal{TQ}(1^{2g-2}, 2^{g-1})$ if g is odd and $\mathcal{TQ}(1^{2g-4}, 2^g)$ if g is even.

Indeed, the disks that the curves in the Figures 5.7 and 5.8 fill Σ into are bounded by a total of 6 or 8 arcs and represent singularities of order 1 or 2, respectively, in the associated marked quadratic differentials. The graph C_g is then defined as follows. Every vertex corresponds to a disk identified by the curves in the Figures 5.7 and 5.8 that is bounded by at least one arc of a curve not in α or β. The label of a vertex is either 1 or 2 depending by the order of the singularity in the associated marked quadratic differential. Two vertices are adjacent if and only if the respective disks are bounded by the same arc. An edge has as label δ_i if the vertices it connects represent

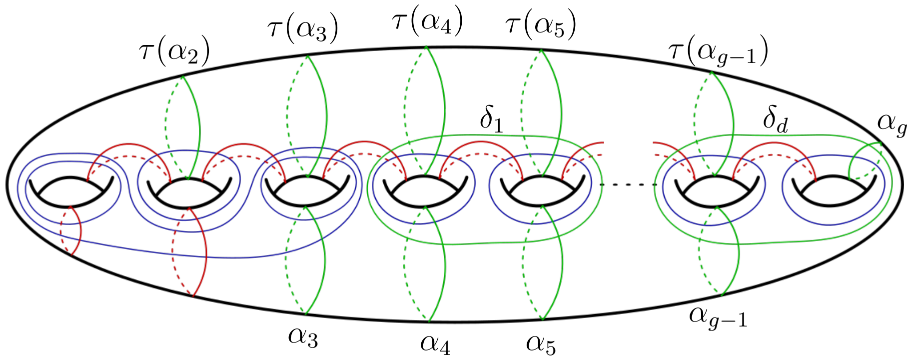


Figure 5.7: The green curves represent the curves we add to the collection $\alpha \cup \beta$, if g is odd.

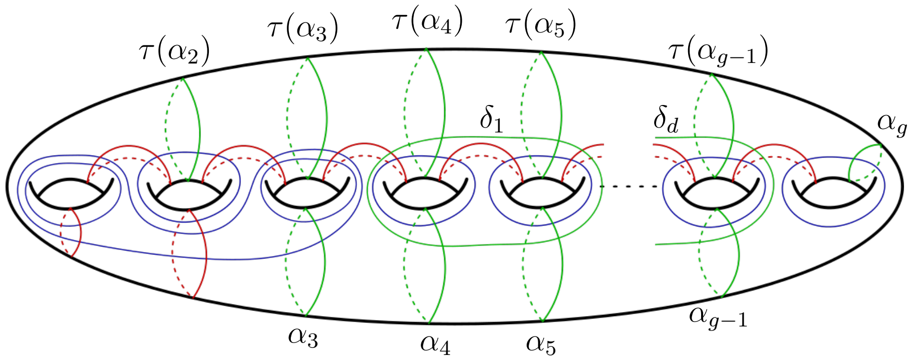


Figure 5.8: The green curves represent the curves we add to the collection $\alpha \cup \beta$, if g is even.

disks bounded by an arc of the curve δ_i. An elementary edge-collapse of C_g corresponds to the deletion of the curves associated to the collapsed edge. □

Appendix A

Python code for the partitions of C_g

Planar graphs can be implemented in Python. Here, we display our approach. The two methods `collapse_edge(self, v1, v2)`, where `v1, v2` are adjacent vertices, and `collapse_edges_with_label(self, target_label)`, where `target_label` is an integer i representing the label δ_i, correspond to the two possible elementary edge-collapses.

```python
import networkx as nx
import matplotlib.pyplot as plt
import uuid
import itertools
from itertools import combinations, product

class Vertex: #Vertex class
    def __init__(self, id, label=None):
        self.id = id
        self.label = label

    def __repr__(self):
        return f"Vertex(id={self.id}, label={self.label})"

class PlanarGraph: #Graph class
    def __init__(self):
        self.vertices = {}
        self.adjacency = {}
        self.edge_labels = {}

    def add_vertex(self, id, label=None):
        if id in self.vertices:
            raise ValueError(f"Vertex {id} already exists.")
        self.vertices[id] = Vertex(id, label)
        self.adjacency[id] = set()

    def add_edge(self, v1, v2, label=None):
        if v1 not in self.vertices or v2 not in self.vertices:
            raise ValueError("Both vertices must exist.")
        key = tuple(sorted((v1, v2)))
        if key in self.edge_labels:
            raise ValueError(f"Edge {v1}-{v2} already exists.")
        self.edge_labels[key] = label
        self.adjacency[v1].add(v2)
        self.adjacency[v2].add(v1)

    def get_vertices(self): #Returns a list of (id, label) pairs for all vertices.
        return [(v.id, v.label) for v in self.vertices.values()]

    def get_edges(self): #Returns a list of (v1, v2, label) triples for all edges.
        return [(v1, v2, lbl) for (v1, v2), lbl in self.edge_labels.items()]

    def remove_vertex(self, id):
        if id not in self.vertices:
            raise ValueError(f"Vertex {id} does not exist.")
        for neighbor in list(self.adjacency[id]):
            self.remove_edge(id, neighbor)
        del self.vertices[id]
        del self.adjacency[id]
```

```python

    def _generate_unique_id(self): #Generates a new vertex id that doesn't already exists.
        while True:
            new_id = str(uuid.uuid4())
            if new_id not in self.vertices:
                return new_id

    def remove_edge(self, v1, v2):
        key = tuple(sorted((v1, v2)))
        if key not in self.edge_labels:
            raise ValueError(f"Edge {v1}-{v2} does not exist.")
        del self.edge_labels[key]
        self.adjacency[v1].remove(v2)
        self.adjacency[v2].remove(v1)

    def update_vertex_label(self, id, new_label):
        self.vertices[id].label = new_label

    def update_edge_label(self, v1, v2, new_label):
        key = tuple(sorted((v1, v2)))
        self.edge_labels[key] = new_label

    def neighbors(self, id): #Returns adjacent edges
        return self.adjacency.get(id, set())

    def edges(self):
        return [(v1, v2, label) for (v1, v2), label in self.edge_labels.items()]

    def collapse_edge(self, v1, v2): #Elementary edge-collapse (unlabeled)
        key = tuple(sorted((v1, v2)))
        if key not in self.edge_labels:
            raise ValueError(f"No such edge: {v1}-{v2}")

        new_label = self.vertices[v1].label + self.vertices[v2].label
        neighbors = {}
        for neighbor in self.adjacency[v1]:
            if neighbor != v2:
                k = tuple(sorted((v1, neighbor)))
                neighbors[neighbor] = self.edge_labels[k]
        for neighbor in self.adjacency[v2]:
            if neighbor != v1:
                k = tuple(sorted((v2, neighbor)))
                neighbors[neighbor] = self.edge_labels[k]

        # Remove v1 and v2 and their edges
        self.remove_vertex(v1)
        self.remove_vertex(v2)

        # Create new vertex
        new_id = self._generate_unique_id()
        self.add_vertex(new_id, new_label)

        # Reconnect to former neighbors with preserved labels
        for neighbor, label in neighbors.items():
            self.add_edge(new_id, neighbor, label)

        return new_id

    def collapse_edge_multiple(self, V): #Multiple elementary edge-collapse (unlabeled)
        while len(V)!=0:
           new_vertex=self.collapse_edge(V[0][0],V[0][1])
           V=[tuple(new_vertex if x == V[0][0] or x == V[0][1] else x for x in tup) for tup in V[1:]]

    def find_edges_with_label(self, target_label):
        return [(v1, v2) for (v1, v2), lbl in self.edge_labels.items() if lbl == target_label]

    def collapse_edges_with_label(self, target_label): #Elementary edge-collapse (labeled)
        l=self.find_edges_with_label(target_label)
        self.collapse_edge(l[0][0], l[0][1])
        l=self.find_edges_with_label(target_label)
        self.collapse_edge(l[0][0], l[0][1])

    def collapse_edges_with_label_multiple(self, labels): #Multiple elementary edge-collapse (labeled)
        for l in labels:
            self.collapse_edges_with_label(str(l))
```

Listing A.1: The classes and methods used to define vertices and planar graphs.

The graphs constructed can be displayed using the method `draw`.

```python
def draw(self, layout="planar", with_labels=True):
        G_nx = nx.Graph()

        # Add nodes with labels
        for v_id, vertex in self.vertices.items():
            G_nx.add_node(v_id, label=vertex.label)

        # Add edges with labels
        for (v1, v2), label in self.edge_labels.items():
            G_nx.add_edge(v1, v2, label=label)

        # Layout options
        if layout == "spring":
            pos = nx.spring_layout(G_nx)
        elif layout == "kamada_kawai":
            pos = nx.kamada_kawai_layout(G_nx)
        else:
            try:
                pos = nx.planar_layout(G_nx)
            except nx.NetworkXException:
                print("Warning: Not planar, using spring layout.")
                pos = nx.spring_layout(G_nx)

        # Draw nodes and labels
        node_labels = {v.id: v.label for v in self.vertices.values()} if with_labels else None
        nx.draw(G_nx, pos, labels=node_labels, node_color="lightblue", node_size=100, font_size=5)

        # Draw edge labels
        edge_labels = nx.get_edge_attributes(G_nx, 'label')
        nx.draw_networkx_edge_labels(G_nx, pos, edge_labels=edge_labels, font_color="red")

        plt.title("Planar Graph")
        plt.axis("off")
        plt.show()
```

The graphs C_g can be constructed using the following two definitions, distinguished by the parity of g.

```python
def G_odd(g):
    G=PlanarGraph()
    j=3*int((g-3)/2)

    G.add_vertex('1',2)
    G.add_vertex('2',2)
    G.add_vertex('3',2)
    G.add_edge('1', '2', "")
    G.add_edge('2', '3', "")

    for i in range(1,int((g-3)/2)):
        G.add_vertex(str(3*i+1),1)
        G.add_vertex(str(3*i+2),1)
        G.add_vertex(str(3*i+3),2)
        G.add_edge(str(3*i), str(3*i+1), "")
        G.add_edge(str(3*i+1), str(3*i+2), str(i))
        G.add_edge(str(3*i+1), str(3*i+3), "")

    G.add_vertex(str(j+1),1)
    G.add_vertex(str(j+2),1)
    G.add_edge(str(j+2), str(j),'')
    G.add_edge(str(j+1), str(j+2),str(int((g-3)/2)))

    for i in range(1,int((g-3)/2)+1):
        G.add_vertex(str(j+3*i),1)
        G.add_vertex(str(j+3*i+1),1)
        G.add_vertex(str(j+3*i+2),2)
        G.add_edge(str(j+3*i), str(j+3*i-1), "")
        G.add_edge(str(j+3*i+2), str(j+3*i), "")
        G.add_edge(str(j+3*i+1), str(j+3*i), str(int((g-3)/2)-i+1))

    G.add_vertex(str(3*g-6),1)
    G.add_edge(str(3*g-6), str(3*g-7),"")
    return G

def G_even(g):
    G=PlanarGraph()
    j=3*(int((g-4)/2)+1)

    G.add_vertex('1',2)
    G.add_vertex('2',2)
    G.add_vertex('3',2)
    G.add_edge('1', '2', "")
```

```
    G.add_edge('2', '3', "")

    for i in range(1,int((g-4)/2)+1):
        G.add_vertex(str(3*i+1),1)
        G.add_vertex(str(3*i+2),1)
        G.add_vertex(str(3*i+3),2)
        G.add_edge(str(3*i), str(3*i+1), "")
        G.add_edge(str(3*i+1), str(3*i+2), str(i))
        G.add_edge(str(3*i+1), str(3*i+3), "")

    G.add_vertex(str(j+1),2)
    G.add_edge(str(j), str(j+1),"")

    for i in range(1,int((g-4)/2)+1):
        G.add_vertex(str(j+3*i-1),1)
        G.add_vertex(str(j+3*i),1)
        G.add_vertex(str(j+3*i+1),2)
        G.add_edge(str(j+3*i-1), str(j+3*i-2), "")
        G.add_edge(str(j+3*i), str(j+3*i-1), str(int((g-4)/2)-i+1))
        G.add_edge(str(j+3*i+1), str(j+3*i-1), "")

    G.add_vertex(str(3*g-7),1)
    G.add_edge(str(3*g-8), str(3*g-7),"")
    return G

    def get_G(g):
    if g%2==0:
        G=G_even(g)
    else:
        G=G_odd(g)
    return G
```

Any graph obtained from C_g by an elementary edge-collapse carries a combinatorial data given by a partition of $4g-7$. The partition can be obtained by `get. partition(G)` if G is the graph obtained from C_g.

```
def get_partition(G):
    l=G.get_vertices()
    for i in range(0,len(l)):
        l[i]=l[i][1]
    return sorted(l+[1,1,1])
```

Suppose $\underline{k}$ is a partition of $4g-7$. To test whether or not it can be obtained from C_g we should know the number of edge-collapses to perform. Let us denote by n the number of edge-collapses we need. The order in which one can apply the edge collapses does not change the final result. It only matters the number of labelled and unlabelled edge-collapses. If n_1 is the number of unlabelled edge collapses and n_2 is the number of labelled edge collapses, we have that $n = n_1 + 2n_2$.

```
def find_n1_n2_pairs(n):
    pairs = []
    for n2 in range(n // 2 + 1):
        n1 = n - 2 * n2
        pairs.append((n1, n2))
    return pairs
```

Finally, the following function checks is a given partition of $4g-7$, given as a list of positive integers, can be obtained from C_g.

```
def is_part_in_Cg_even(l,g):
    if g%2==1:
        raise ValueError(f"g is not even")
    if sum(l)!=4*g-4:
        raise ValueError(f"l non a partition of 4g-4")

    if g%2==0:
        r=int((g-3)/2)
    else:
        r=int((g-4)/2)

```

```python
    G=get_G(g)
    S=[(v1, v2) for (v1, v2), label in G.edge_labels.items() if label == ""] #Unlabelled edges
    T=list(range(1, r+1))

    for m in find_n1_n2_pairs(3*g-8-len(l)+4):
        for P in [[list(s), list(t)] for s, t in product(combinations(S, m[0]), combinations(T, m[1]))]:
            G.collapse_edge_multiple(P[0])
            G.collapse_edges_with_label_multiple(P[1])
            if get_partition(G)==sorted(l):
                G.draw(layout="planar")
                return True
            else:
                G=get_G(g)
    return False

def is_part_in_Cg_odd(l,g):
    if g%2==0:
        raise ValueError(f"g is not odd")
    if sum(l)!=4*g-4:
        raise ValueError(f"l non a partition of 4g-4")

    if g%2==0:
        r=int((g-3)/2)
    else:
        r=int((g-3)/2)

    G=get_G(g)
    S=[(v1, v2) for (v1, v2), label in G.edge_labels.items() if label == ""] #Unlabelled edges
    T=list(range(1, r+1))

    for m in find_n1_n2_pairs(3*g-7-len(l)+4):
        for P in [[list(s), list(t)] for s, t in product(combinations(S, m[0]), combinations(T, m[1]))]:
            G.collapse_edge_multiple(P[0])
            G.collapse_edges_with_label_multiple(P[1])
            if get_partition(G)==sorted(l):
                G.draw(layout="planar")
                return True
            else:
                G=get_G(g)
    return False
```

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