Local solubility of a family of quadric surfaces over a biprojective base

Cameron Wilson BSc, MSc

SUBMITTED IN FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

SCHOOL OF MATHEMATICS & STATISTICS
COLLEGE OF SCIENCE & ENGINEERING

July 2025

To Mum and Dad, for putting up with me.

Abstract

We prove an asymptotic formula for the number of everywhere locally soluble diagonal quadric surfaces

$$y_0x_0^2 + y_1x_1^2 + y_2x_2^2 + y_3x_3^2 = 0$$

parametrised by points $y \in \mathbb{P}^3(\mathbb{Q})$ lying on the split quadric surface $y_0y_1 = y_2y_3$ which do not satisfy $-y_0y_2 = \square$ nor $-y_0y_3 = \square$. Our methods involve proving asymptotic formulae for character sums with a hyperbolic height condition and proving variations of large sieve inequalities for quadratic characters.

Contents

Acknowledgements

First, last, and always: to my parents, whose support has given me strength when frustration threatened to take over; also to my brother, Connor, and my Papa, who have both been sources of inspiration for many years.

To Andrew, Jonathan and Leo, whose friendship I hold close, despite the distance; to Craig and Alec, for the games of chess and the nights playing Catan or shooting (mostly hostile) aliens; and to Ujan, whose sense of humour has been a welcome distraction, though also the cause of many, many face palms - thank you.

I am grateful for the advice and guidance of Alex Bartel and for the patience of Damián Gvirtz-Chen and Dan Loughran in answering many questions, which led to the formation of Chapter 3 of this thesis.

Thanks are also due to Tim Browning, Julian Lyczak, and Nick Rome for helpful comments on the papers comprising this thesis and for suggestions of future work.

Funding for my Ph.D has been provided by the Carnegie Trust for the Universities of Scotland, and I am thankful for the opportunities this has provided me.

Finally, I thank my supervisor, Efthymios Sofos, for his professional guidance, patience, and utterly baffling and chaotic sense of humour - all of which have been sources of motivation in the last four years.

Declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Cameron Wilson

Notation

Let f be a complex valued functions and g be a positive, real valued function. We will write both $f(x) = O(g(x))$ and $f(x) \ll g(x)$ to mean that there exists a constant $C \in \mathbb{R}_{\geqslant 0}$ such that $|f(x)| \leqslant Cg(x)$ for sufficiently large x.
We will write $f(x) \sim g(x)$ to denote that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

We will use the standard notation for projective n space, $\mathbb{P}^n$ and its set of rational points $\mathbb{P}^n(\mathbb{Q})$. Whenever we write $a \in \mathbb{P}^n(\mathbb{Q})$ we will mean the primitive integer representation $a = [a_0 : \ldots : a_n]$, where $a_0, \ldots, a_n \in \mathbb{Z}$, $\gcd(a_0, \ldots, a_n) = 1$.

Chapter 1

Introduction

1.1 Background

A Diophantine equation is an equation of the form

$$f(x_0, \ldots, x_n) = 0 \tag{1.1.1}$$

where $f \in \mathbb{Z}[x_0, \ldots, x_n]$. Given such an equation, we aim to determine whether it is soluble over the rational numbers[1], that is, do there exist $X_0, \ldots, X_n \in \mathbb{Q}$ such that

$$f(X_0, \ldots, X_n) = 0.$$

Research into this problem goes back 2000 years, drawing on many fields and techniques in mathematics and spawning new directions of investigation. Two of the most potent directions in recent decades are Manin's problem and Serre's problem. The first of these concerns counting rational points on a special class of algebraic varieties; the second aims to count the number of Diophantine equations within a family which are rationally soluble.

1.1.1 Manin's Conjecture

Let X be a smooth projective variety over $\mathbb{Q}$. (In this thesis, by variety, we mean an integral and quasi-projective algebraic variety.) Then X is said to be *Fano*, if the anticanonical line bundle of X is ample. It is conjectured that the set $X(\mathbb{Q})$ of rational points on a Fano variety X is Zariski dense as soon as it is non-empty[2]. To study the set of rational points in a quantitative way, we make use of a anticanonical height

[1] Classically, one considers *integral* solubility instead of rational solubility. The Diophantine equations studied in this thesis, however, will be homogeneous. A consequence of this is that these notions of solubility will be equivalent.

[2] This is a consequence of a more general conjecture of Colliot-Thélène [10].

function $H : X(\mathbb{Q}) \to \mathbb{R}_{\geqslant 0}$ which measures the complexity of a rational point. A key goal of rational points research is to determine the asymptotic behaviour of the counting function

$$M_U(B) = \sharp\{x \in U(\mathbb{Q}) : H(x) \leqslant B\} \tag{1.1.2}$$

as $B \to \infty$, where $U \subseteq X$. The following conjecture is due to Manin and Batyrev:

Conjecture 1.1.1 (Manin et. al. [16]). *Let X be a Fano variety. Then there exists a Zariski dense, open subset $U \subseteq X$, and constants $a \in \mathbb{R}_{\geqslant 0}$, $b \in \mathbb{Z}_{>0}$ such that*

$$M_U(B) \sim aB^b(\log B)^{\rho_X - 1},$$

where ρ_X is the rank of the Picard group of X and b has an explicit description.

To see why it is necessary to take a Zariski dense, open subset $U \subseteq X$ in the above conjecture, suppose that X is a cubic surface. It is well known that X contains 27 lines and Manin's conjecture predicts that $b = 1$. If one of these lines is rational, then the rational points on this line will dominate the counting function $M_X(B)$, leading to a lower bound $\gg B^2$. It therefore follows that for a cubic surface X to follow Conjecture 1.1.1, we must have $U \subseteq X \setminus \{\text{lines}\}$.

The removal of dominating rational subvarieties is also not enough. Following a counter-example of earlier incarnations of Conjecture 1.1.1 by Batyrev and Tschinkel [2], recent investigations into Manin's conjecture take the open set U to be the complement in X of accumulating thin sets (see, for example, [34, §8]). The following definition may be found in Chapter 3 of [39].

Definition 1.1.2. Let X be a Fano variety over $\mathbb{Q}$ and suppose T is a subset of $X(\mathbb{Q})$.

- T is said to be a type 1 thin set of X if there is a proper, Zariski closed subset W of X such that $T \subseteq W(\mathbb{Q})$.
- T is said to be a type 2 thin set of X if there exists an irreducible, quasi-projective variety X' with $\dim(X') = \dim(X)$, and a generically surjective morphism $\varphi : X' \to X$ of degree $\geqslant 2$ such that $T \subset \varphi(X'(\mathbb{Q}))$.

Lastly, T is said to be a *thin* set of X if it is contained in a finite union of type 1 and type 2 thin sets.

In Peyre's reformulation of Conjecture 1.1.1, it is stated that there exists a thin set $T \subseteq X(\mathbb{Q})$ such that

$$M_{X\setminus T}(B) = \sharp\{x \in X(\mathbb{Q}) : H(x) \leqslant B,\ x \notin T\} \sim aB^b(\log B)^{\rho_X - 1}.$$

Peyre also gives an explicit prediction for the constant a in terms of geometric invariants of the variety X [33].

Example 1.1.3. Of key importance to this thesis is the quadric surface $Y \subseteq \mathbb{P}^3$ defined by the equation $y_0y_1 = y_2y_3$. This is an example of a Fano variety of dimension 2. For now, let $H : \mathbb{P}^3(\mathbb{Q}) \to \mathbb{R}_{\geqslant 0}$ be the naive Weil height on $\mathbb{P}^3(\mathbb{Q})$ and restrict it to $Y(\mathbb{Q})$. A straightforward argument using the $\mathbb{Q}$-isomorphism of Y from the product variety $\mathbb{P}^1 \times \mathbb{P}^1$ and the hyperbola method allows us to conclude Manin's conjecture for Y, that is:

$$\sharp\{y \in Y(\mathbb{Q}) : H(y) \leqslant B\} = cB^2 \log B + O(B^2)$$

for some positive constant c. Note that the Picard rank of Y is 2. Also of note is that we have taken $U = Y$, i.e there is no need to remove any thin sets from Y to obtain Conjecture 1.1.1. To conclude this example, and this section, we will draw attention to the following sets of rational points on Y

$$T_1 = \{y = [y_0 : y_1 : y_2 : y_3] \in Y(\mathbb{Q}) : -y_0y_2 = \square\}$$
$$T_2 = \{y = [y_0 : y_1 : y_2 : y_3] \in Y(\mathbb{Q}) : -y_0y_3 = \square\}.$$

It can be proven with similar methods that

$$B^2 \ll \sharp\{y \in Ti : H(y) \leqslant B\} \ll B^2$$

for $i = 1, 2$. Although $T = T_1 \cup T_2$ does not dominate the set of rational points on Y, it is thin in the sense of Definition 1.1.2. Indeed, if we take $Y' \subseteq \mathbb{P}^3$ to be the smooth variety defined by the equation $-z_0^2z_1 = z_2^2z_3$ and $\varphi : Y' \to Y$, and the degree 2 rational map $\varphi([z_0 : z_1 : z_2 : z_3]) = \left[-\frac{z_0^2}{z_2} : z_1 : z_2 : z_3\right]$, then by blowing up Y' along the points where φ is undetermined (those points where $z_2 = 0$) to obtain a variety $\tilde{Y}$ and a degree 2 morphism $\tilde{\varphi} : \tilde{Y} \to Y$ such that $T_1 \subseteq \tilde{\varphi}(\tilde{Y}(\mathbb{Q}))$[3].

1.1.2 Serre's problem

Suppose $\mathcal{F} \subseteq \mathbb{Z}[x_0, \ldots, x_n]$ is an infinite family of polynomials. A natural question to ask is: for how many $f \in \mathcal{F}$ does the Diophantine equation (1.1.1) have a rational solution? More formally, if coef(f) is the set of coefficients of f, we want to study the asymptotic behaviour of the following counting function:

$$\sharp\left\{f \in \mathcal{F} : \begin{array}{l} f(x_0, \ldots, x_n) = 0 \text{ has a } \mathbb{Q} \text{ solution} \\ \max\{|a| : a \in \mathrm{coef(f)}\} \leqslant B \end{array}\right\}.$$

We are particularly interested in families that can be parametrised by the rational points of an algebraic variety. The earliest result in this direction is due to Serre, who used the large sieve to prove upper bounds for the families of ternary conics parametrised by projective space. He proved that

$$\sharp\left\{a \in \mathbb{P}^5(\mathbb{Q}) : \begin{array}{c} a_0x^2 + a_1y^2 + a_2z^2 + a_3xy + a_4xz + a_5yz = 0 \\ \text{has a solution over } \mathbb{Q}, \\ H(a) \leqslant B \end{array}\right\} \ll \frac{B^6}{\sqrt{(\log B)}}, \tag{1.1.3}$$

[3]This argument on the thinness of the set T was suggested by Florian Wilsch.

and

$$\sharp\left\{a\in\mathbb{P}^2(\mathbb{Q}):\begin{array}{c}a_0x^2+a_1y^2+a_2z^2=0\\ \text{has a solution over }\mathbb{Q},\\ H(a)\leqslant B\end{array}\right\}\ll\frac{B^3}{(\log B)^{3/2}},\tag{1.1.4}$$

where H is the naive Weil height on $\mathbb{P}^5(\mathbb{Q})$ and $\mathbb{P}^2(\mathbb{Q})$ respectively. This was followed by results of Hooley, who proved matching lower bounds for the family of diagonal planar conics [23], and Guo, who proved asymptotics for the problem of diagonal planar conics whose coefficients are restricted to be square-free and pairwise co-prime [19]. Subsequently, Hooley gave lower bounds for the family of general planar conics [24].

In the case of families of conics, it is known that the Hasse Principle is satisfied; this means that a variety in the family has a rational solution if and only if it has a solution over every local field, $\mathbb{Q}_p$ for $p\in\{\text{primes}\}\cup\{\infty\}$. A benefit of studying families which satisfy this principle is that the otherwise difficult problem of asking whether an equation has rational points is equivalent to the more tractable problem of asking whether or not it has a solution in each of these local fields.

Not every family satisfies the Hasse Principle — for example, it is a famous example of Selmer that the plane cubic cut out by the equation $3x^3+4y^3+5z^3=0$ has points in every local field over $\mathbb{Q}$ but fails to have a rational point [36]. In fact, Bhargava has proven that a positive proportion of ternary plane cubics fail the Hasse Principle [3]. Such failures in the Hasse Principle mean that rational solubility requires more effort to study than just considering local solubility. For this reason, we study the simpler problem of counting equations in families which are everywhere locally soluble. The general set-up in this direction is as follows: suppose that X and W are smooth, proper projective varieties over $\mathbb{Q}$ and that $\phi:W\to X$ is a dominant map with geometrically integral generic fibre. The fibre of a rational point $x\in X(\mathbb{Q})$, $\phi^{-1}(x)$, is then an algebraic variety corresponding to a for which we ask if it has points in every local field of $\mathbb{Q}$. We now aim to study the asymptotic behaviour of the counting function

$$N_U(\phi,B)=\sharp\left\{x\in U(\mathbb{Q}):\begin{array}{l}\phi^{-1}(x)(\mathbb{Q}_p)\neq\emptyset\text{ for all }p\in\{\text{primes}\}\cup\{\infty\}\\ H(x)\leqslant B\end{array}\right\}\tag{1.1.5}$$

where H is some height function on the variety X and U is some Zariski open subset of X.

To understand the results we will need the following quantity, defined for $X=\mathbb{P}^n$ by Loughran and Smeets [30] and for more general varieties X by Browning and Loughran in [7]. Set $X^{(1)}$ to be the collection of codimension 1 points of X. Then recall that for any $D\in X^{(1)}$, the absolute Galois group $\mathrm{Gal}(\overline{\kappa(D)}/\kappa(D))$ of the residue field of D acts on the irreducible components of the reduced fibres $\pi^{-1}(D)\otimes\overline{\kappa(D)}$. Choose some finite group $\Gamma_D(\phi)$ through which the action is factored and define $\Gamma_D^\circ(\phi)$ to be the collection of $\gamma\in\Gamma_D(\phi)$ which fix some multiplicity 1 irreducible component of $\phi^{-1}(D)\otimes\overline{\kappa(D)}$. We then define $\delta_D(\phi)=\sharp\Gamma_D^\circ(\phi)/\sharp\Gamma_D(\phi)$ and

$$\Delta_X(\phi)=\sum_{D\in X^{(1)}}(1-\delta_D(\phi)).\tag{1.1.6}$$

Note that the assumption that the generic fibre is geometrically integral ensures that this sum is finite.

Let us first consider (1.1.5) when $X = \mathbb{P}^n$ for $n \geqslant 1$ with the naive Weil height. Then Loughran and Smeets proved the following general upper bound:

$$N_{\mathbb{P}^n}(\phi, B) \ll \frac{B^{n+1}}{(\log B)^{\Delta_{\mathbb{P}^n}(\phi)}}. \tag{1.1.7}$$

This upper bound generalised those given by Serre for the families of conics over $\mathbb{P}^5(\mathbb{Q})$ or $\mathbb{P}^2(\mathbb{Q})$ and gave a geometric interpretation to the growth rate. Indeed the value of the invariants for the cases considered in (1.1.3) and (1.1.4) are $\frac{1}{2}$ and $\frac{3}{2}$ respectively. Loughran and Smeets further conjectured that this upper bound is optimal when at least one fibre of ϕ has solutions in every local field and that the fibre over every codimension 1 point has an irreducible component of multiplicity 1. They confirmed this prediction in the cases where $\Delta(\phi) = 0$, generalising earlier work by Poonen and Voloch on the local solubility of hypersurfaces [35]. Later, Loughran and Matthiesen proved matching lower bounds for (1.1.7) for $X = \mathbb{P}^1$ and for some special families over $\mathbb{P}^n$ [28]. Loughran, Rome and Sofos have provided a conjecture on the leading constant, as well as its geometric interpretation, for counting problems of this type [29]. Recently, Browning, Lyczak and Smeets investigated the paucity of rationally soluble fibrations where the map ϕ is allowed to have multiple fibres [9].

Now let us consider (1.1.5) for more general varieties X with an anticanonical height function. By combining the conjecture of Loughran and Smeets with Manin's conjecture, we may predict that there exists a thin set $T \subseteq X$ such that, for $U = X \setminus T$,

$$N_U(\phi, B) \sim \frac{c' M_U(B)}{(\log B)^{\Delta(\phi)}} \tag{1.1.8}$$

for some constant $c' > 0$. Of particular interest are the cases where X is a Fano variety over $\mathbb{Q}$ with no accumulating thin subsets in the sense considered in §1.1.1. In such cases, it is natural to think that we may take $U = X$, as we do in Manin's conjecture when there are no accumulating subsets. This problem has been investigated when X is a quadric of dimension $\geqslant 3$ by Browning and Loughran [7], where the upper bound of the shape (1.1.8) is proven. Furthermore, when X is a quadratic form of rank $\geqslant 5$ and all fibres over codimension 1 points of X are split (ensuring that $\Delta(\phi) = 0$), Browning and Heath-Brown proved that a positive proportion of fibres are everywhere locally soluble [5]. In these cases $\rho_X = 1$. Other examples include: fibrations over algebraic groups, which have been studied by Loughran [27], and Loughran, Takloo-Bighash, and Tanimoto [31]; and fibrations over hypersurfaces studied by Sofos and Visse-Martindale [42].

1.2 Main Result

This thesis investigates the Serre problem when the base variety is the split quadric surface $Y \subseteq \mathbb{P}^3$ defined in Example 1.1.3. Let $Z \subset \mathbb{P}^3 \times \mathbb{P}^3$ be the variety cut out by the equations

$$y_0x_0^2 + y_1x_1^2 + y_2x_2^2 + y_3x_3^2 = 0 \text{ and } y_0y_1 = y_2y_3, \tag{1.2.1}$$

and let $\pi : Z \to Y$ be the dominant morphism sending $([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3]) \in Z$ to the point $[y_0 : y_1 : y_2 : y_3] \in Y$. Write $\widetilde{Z} \to Z$ for a desingularisation of Z

and write $\widetilde{\pi} : \widetilde{Z} \to Y$ for its composition with π. Furthermore, from here on out we will set $H : \mathbb{P}^3(\mathbb{Q}) \to \mathbb{R}_{\geqslant 0}$ to be the naive Weil height on $\mathbb{P}^3(\mathbb{Q})$ and restrict it to $Y(\mathbb{Q})$. Notice that the variety Z is a subvariety of the universal family of diagonal quadric surfaces, Q, say. Furthermore, since the product of coefficients of the quadric fibres is always a square, Z is contained inside the accumulating subset for Manin's conjecture for Q [4].

This fibration problem was first considered by Browning, Lyczak and Sarapin [8]. In particular, they showed that

$$B^2 \ll N_Y(\widetilde{\pi}, B) \ll B^2. \tag{1.2.2}$$

In this case $\rho_Y = 2$ and it is demonstrated in the paper of Browning, Lyczak and Sarapin that $\Delta_Y(\widetilde{\pi}) = 2$. It follows that (1.1.8) predicts a growth rate of $\frac{B^2}{\log B}$, as Y has no accumulating thin sets. The anomaly in this particular case may be seen to arise from the presence of the thin set of points T defined in Example 1.1.3: the fibres of the points in T each have a rational point and we saw in this example that the number of rational points in T of height $\leqslant B$ is of order B^2. Following Peyre's modern reformulation of the Manin conjecture [32], it was conjectured in [8] that prediction (1.1.8) should hold for this fibration problem after the removal of the thin set T. To this purpose, we set

$$N(B) := N_{Y\setminus T}(\widetilde{\pi}, B) = \sharp \left\{ y \in Y(\mathbb{Q}) : \begin{array}{c} -y_0y_2 \neq \square, \ -y_0y_3 \neq \square \\ \widetilde{\pi}^{-1}(y) \text{ has a } \mathbb{Q}\text{-point} \\ H(y) \leqslant B \end{array} \right\}. \tag{1.2.3}$$

The main result of this thesis is the following.

Theorem 1.2.1. *As $B \to \infty$,*

$$N(B) = \frac{cB^2 \log\log B}{\log B} + O\left(\frac{B^2 \sqrt{\log\log B}}{\log B} \right)$$

where c is given by

$$\begin{aligned} &\frac{935}{36\pi^2} \prod_{p \neq 2} \left(1 + \frac{1}{p}\right)^{-2} \left(1 + \frac{2}{p} + \frac{4}{p^2} + \frac{2}{p^3} + \frac{1}{p^4}\right) \\ &+ \frac{25}{36\pi^2} \prod_{p \neq 2} \left(1 + \frac{1}{p}\right)^{-2} \left(1 + \frac{2}{p} + \frac{2\left(1 + \left(\frac{-1}{p}\right)\right)}{p^2} + \frac{2}{p^3} + \frac{1}{p^4}\right), \end{aligned}$$

which is > 0.

This result contradicts the prediction made by Browning, Lyczak and Sarapin that (1.1.8) should hold for $U = Y \setminus T$. However, the primary novelty of this result is the completely surprising appearance of the $\log\log B$ factor in the main term. Prior to this paper the only $\log\log B$ factor occurring in a fibration problem was the following example with $X = \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$ [29]: if $H' : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{R}_{\geqslant 0}$ is the naive Weil height on $\mathbb{P}^1(\mathbb{Q})$ then it may be proven that

$$\sharp \left\{ (t_1, t_2) \in \mathbb{Q}^2 : \begin{array}{l} H'(t_1)H'(t_2) \leqslant B, \\ \text{each } t_i \text{ is the sum of two squares} \end{array} \right\} \sim \frac{c' B^2 \log\log B}{\log B}$$

for some $c' > 0$[4]. In the broader topic of rational points, a double logarithmic factor has also appeared in the study of Brauer–Manin obstruction for $K3$ surfaces [20]. A geometric interpretation of either occurrence of the $\log\log B$ factor is yet to be found.

1.3 Outline

This thesis is broken into two parts: an arithmetic-geometric part consisting of Chapters 2 and 3, and an analytic part comprised of Chapters 4 and 5.
Chapter 2 covers the proof of Theorem 1.2.1, assuming the analytical results proven in Chapters 4 and 5.
In Chapter 3 we will discuss some geometric properties of the family of quadrics studied in Theorem 1.2.1. Specifically, we will tie the family of quadrics to a family of conics, and use this relationship to study the rational solubility for the conics using Theorem 1.2.1. We will also compute the subordinate Brauer group for these families, in an attempt to understand the appearance of two Euler products in Theorem 1.2.1.
As we will discuss in more detail in Chapter 2, the analytic methods required to prove Theorem 1.2.1 comprise summing quadratic characters over hyperbolic regions. To study these sums we adapt the character sum methods of Friedlander and Iwaniec [17], and Fouvry and Klüners [15]. The primary tools used in these papers are the large sieve for quadratic characters and Selberg–Delange methods. However, due to the nature of our height conditions, we require more acute versions of the large sieve. We will introduce and prove these large sieve results in Chapter 4.
We conclude this thesis by proving the required bounds for our character sums in Chapter 5.
Lastly, we include an appendix with tables that list the solutions to certain congruence equations $\bmod 8$ required in the final stages of the proof of Theorem 1.2.1.

[4]Note that the product of two naive Weil heights on $\mathbb{P}^1(\mathbb{Q})$ is the square root of the anticanonical height on $\mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$.

Chapter 2

Asymptotics for local solubility of diagonal quadrics over a biprojective base

2.1 Introduction

The purpose of this chapter is to prove the main theorem of this thesis, Theorem 1.2.1, assuming the analytical results that will be proven in Chapters 4 and 5. First we will give a brief overview of the section and outline the proof.

In §2.2 we prove some technical results that are required at various points throughout the proof to simplify expressions. We will also use this section to list the main results of Chapters 4 and 5 for reference.

The proof of Theorem 1.2.1 is contained within §2.3-§2.10. In §2.3 we apply the parameterisation of the quadric surface Y by $\mathbb{P}^1 \times \mathbb{P}^1$ and then express $N(B)$ as a counting problem over the integers. As well as transforming our height condition this will change the form of the diagonal quadric fibres to the form

$$t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 + t_0t_3x_3^2 = 0.$$

Local conditions for the solubility of general diagonal quadrics are considered throughout §2.4. Of particular note is that the real conditions result in $N(B)$ being split into two similar but separate counting problems, $N_1(B)$ and $N_2(B)$ depending on the sign of the coefficients.

The next step is to reduce to odd, square-free, and co-prime variables, apply the Hasse Principle to express the indicator function of the diagonal quadrics having a rational point as a sum over Jacobi symbols in these new variables and sum over them. This is the content of §2.5.1 and §2.5.2. With this we express each $N_1(B)$ and $N_2(B)$ as a sums over Jacobi symbols involving 40 variables. The purpose of §2.5.3 is to decompose this expression into smaller pieces, isolating the main terms and the error terms. At this stage, we will outline §2.6-§2.9, where each of these smaller pieces is dealt with using the results listed in §2.2. Finally, in §2.10, we express the constant as a sum of Euler products.

Throughout this chapter, we use the notation $\|a, b\|$ to denote the maximum of $|a|$ and $|b|$ for $a, b \in \mathbb{R}$.

2.2 Analytical pre-requisites

2.2.1 Simplification results

In this section we prove some lemmas which will help us simplify our multivariable sums. Our first lemma will allow us to limit our count over our coefficents to one where the square parts and common factors are small (see §2.5). We remark that throughout this thesis, the set of natural numbers $\mathbb{N}$ does not contain 0.

Lemma 2.2.1. *Fix $c_0, c_1, c_2, c_3 \in \mathbb{N}$. Then for all $B \geqslant 2$ we have,*

$$\sharp\{(t_0, t_1, t_2, t_3) \in \mathbb{N}^4 : \|t_0, t_1\| \cdot \|t_2, t_3\| \leqslant B; c_i | t_i\} \ll \frac{B^2(\log B)}{c_0c_1c_2c_3}$$

where the implied constant is absolute.

Proof. Let $S_{\mathbf{c}}(B)$ denote the expression on the left hand side. Then

$$S_{\mathbf{c}}(B) = \sum\sum_{nm \leqslant B} \sharp\left\{(t_0, t_1) \in \mathbb{N}^2 : \begin{array}{c} \|t_0, t_1\| = n, \\ c_0|t_0, \ c_1|t_1 \end{array}\right\} \sharp\left\{(t_2, t_3) \in \mathbb{N}^2 : \begin{array}{c} \|t_2, t_3\| = m \\ c_2|t_2, \ c_3|t_3 \end{array}\right\}.$$

Note that, if $\mathbb{1}(c|k)$ denotes the indicator function for c dividing k then

$$\sharp\left\{(t_0, t_1) \in \mathbb{N}^2 : \begin{array}{c} \|t_0, t_1\| = n, \\ c_0|t_0, \ c_1|t_1 \end{array}\right\} \ll \frac{n}{c_1}\mathbb{1}(c_0|n) + \frac{n}{c_0}\mathbb{1}(c_1|n)$$

and similarly

$$\sharp\left\{(t_2, t_3) \in \mathbb{N}^2 : \begin{array}{c} \|t_2, t_3\| = m \\ c_2|t_2, \ c_3|t_3 \end{array}\right\} \ll \frac{m}{c_3}\mathbb{1}(c_2|m) + \frac{m}{c_2}\mathbb{1}(c_3|m).$$

Therefore, $S_{\mathbf{c}}(B)$ is

$$\ll \sum\sum_{nm \leqslant B} nm \left(\frac{\mathbb{1}(c_0|n)\mathbb{1}(c_2|m)}{c_1c_3} + \frac{\mathbb{1}(c_0|n)\mathbb{1}(c_3|m)}{c_1c_2} + \frac{\mathbb{1}(c_1|n)\mathbb{1}(c_2|m)}{c_0c_3} + \frac{\mathbb{1}(c_1|n)\mathbb{1}(c_3|m)}{c_0c_2}\right).$$

Let us look at the sum over the first term:

$$\sum\sum_{nm\leqslant B} \frac{nm}{c_1c_3}\mathbb{1}(c_0|n)\mathbb{1}(c_2|m) \ll \frac{B}{c_1c_3}\sum_{m\leqslant B}\sum_{n\leqslant B/m}\mathbb{1}(c_0|n)\mathbb{1}(c_2|m)$$
$$\ll \frac{B^2}{c_0c_1c_3}\sum_{m\leqslant B}\frac{1}{m}\mathbb{1}(c_2|m).$$

This is

$$\ll \frac{B^2}{c_0c_1c_3}\sum_{k\leqslant B/c_2}\frac{1}{c_2k} \ll \frac{B^2}{c_0c_1c_2c_3}\sum_{k\leqslant B}\frac{1}{k} \ll \frac{B^2(\log B)}{c_0c_1c_2c_3}.$$

The sums over the other terms above are equivalent. □

Here and throughout, let μ denote the Möbius function. The next lemma will allow us to get rid of terms regarding μ^2, which will make analysis over four dimensions easier.

Lemma 2.2.2. *Assume that* $g_0, g_1, g_2, g_3 : \mathbb{N} \to \mathbb{C}$ *are multiplicative functions with* $|g_i(n)| \leqslant 1$ *for all* $n \in \mathbb{N}$ *and all* $0 \leqslant i \leqslant 3$*. Then for all* $X \geqslant 2$, $0 \leqslant z, w_0, w_1 \leqslant X^{1/4}$, *and* $\mathbf{q} \in \mathbb{N}^{\mathbf{4}}_{\text{odd}}$, $\mathbf{c} \in \mathbb{N}^{\mathbf{4}}$,

$$\sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4, \|n_0,n_1\|,\|n_2,n_3\|>z \\ \|n_0c_0,n_1c_1\|\cdot\|n_2c_2,n_3c_3\|\leqslant X \\ \mathbf{n}\equiv\mathbf{q} \bmod 8}} \mu^2(n_0n_1n_2n_3)\left(\prod_{i=0}^{3} g_i(n_i)\right)$$
$$= \sum_{r\leqslant w_0}\mu(r)\sum\sum\sum\sum_{\substack{\mathbf{n}'\in\prod_{i=0}^{3}(\mathbb{N}\cap[1,w_1]) \\ p|n_0'n_1'n_2'n_3'\Rightarrow p|r \\ r^2|n_0'n_1'n_2'n_3' \\ \gcd(2,n_i)=1\ \forall\ i}} \left(\prod_{i=0}^{3} g_i(n_i')\right) G(X,\mathbf{n}')$$
$$+ O\left(\frac{X^2(\log X)}{\min(w_0^{5/12}, w_1^{1/3})c_0c_1c_2c_3}\right),$$

where we have defined, for any $\mathbf{a} \in \mathbb{N}^4_{\text{odd}}$,

$$G(X,\mathbf{a}) = \sum\sum\sum\sum_{\substack{\mathbf{n}''\in\mathbb{N}^4, \|n_0''a_0,n_1''a_1\|,\|n_2''a_2,n_3''a_3\|>z \\ \|a_0n_0''c_0,a_1n_1''c_1\|\cdot\|a_2n_2''c_2,a_3n_3''c_3\|\leqslant X \\ n_i''\equiv q_i/a_i \bmod 8\ \forall\ i \\ \gcd(n_i'',r)=1}} \left(\prod_{i=0}^{3} g_i(n_i'')\right),$$

and the implied constant is absolute.

Proof. We begin by using the convolution identity $\mu^2(s) = \sum_{r^2|s}\mu(r)$ with $s = n_0n_1n_2n_3$.

Then

$$\sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4, \|n_0,n_1\|,\|n_2,n_3\|>z \\ \|n_0c_0,n_1c_1\|\cdot\|n_2c_2,n_3c_3\|\leqslant X \\ \mathbf{n}\equiv\mathbf{q} \bmod 8}} \mu^2(n_0n_1n_2n_3)\left(\prod_{i=0}^{3} g_i(n_i)\right)$$
$$= \sum_{r\leqslant X}\mu(r) \sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4, \|n_0,n_1\|,\|n_2,n_3\|>z \\ \|n_0c_0,n_1c_1\|\cdot\|n_2c_2,n_3c_3\|\leqslant X \\ \mathbf{n}\equiv\mathbf{q} \bmod 8 \\ r^2|n_0n_1n_2n_3}} \left(\prod_{i=0}^{3} g_i(n_i)\right).$$

Now write $n_i = n_i'n_i''$ for each i where the n_i'' are co-prime to r and all prime factors of each n_i' divide r. Notice that because $n_i \equiv q_i \bmod 8$ is odd, each of the n_i' will be odd. This will yield

$$\sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4, \|n_0,n_1\|,\|n_2,n_3\|>z \\ \|n_0c_0,n_1c_1\|\cdot\|n_2c_2,n_3c_3\|\leqslant X \\ \mathbf{n}\equiv\mathbf{q} \bmod 8}} \mu^2(n_0n_1n_2n_3)\left(\prod_{i=0}^{3} g_i(n_i)\right)$$
$$= \sum_{r\leqslant X}\mu(r) \sum\sum\sum\sum_{\substack{\mathbf{n}'\in\prod_{i=0}^{3}(\mathbb{N}\cap[1,X]) \\ p|n_0'n_1'n_2'n_3'\Rightarrow p|r \\ r^2|n_0'n_1'n_2'n_3' \\ \gcd(2,n_i)=1\ \forall\ i}} \left(\prod_{i=0}^{3} g_i(n_i')\right) G(X,\mathbf{n}').$$

We may bound the $G(X,\mathbf{n}')$ sum using Lemma 2.2.1:

$$G(X,\mathbf{n}') \ll \sum\sum\sum\sum_{\substack{\mathbf{n}''\in\mathbb{N}^4 \\ \|n_0'n_0''c_0,n_1'n_1''c_1\|\cdot\|n_2'n_2''c_2,n_3'n_3''c_3\|\leqslant X}} 1 \ll \frac{X^2\log X}{c_0c_1c_2c_3n_0'n_1'n_2'n_3'}.$$

Therefore, the contribution coming from terms where $n_i' > w_1$ for some i may be seen to be:

$$\begin{aligned}\sum_{r\leqslant X}\mu(r) \sum\sum\sum\sum_{\substack{\mathbf{n}'\in\prod_{i=0}^{3}(\mathbb{N}\cap[1,X]) \\ p|n_0'n_1'n_2'n_3'\Rightarrow p|r \\ r^2|n_0'n_1'n_2'n_3' \\ \gcd(2,n_i)=1\ \forall\ i \\ n_i>w_1 \text{ for some } i}} \left(\prod_{i=0}^{3} g_i(n_i')\right) G(X,\mathbf{n}') &\ll \sum_{r\leqslant X} \sum\sum\sum\sum_{\substack{\mathbf{n}'\in\prod_{i=0}^{4}(\mathbb{N}\cap[1,X]) \\ p|n_0'n_1'n_2'n_3'\Rightarrow p|r \\ r^2|n_0'n_1'n_2'n_3' \\ \gcd(2,n_i')=1\ \forall\ i \\ n_i'>w_1 \text{ for some } i}} \frac{X^2\log X}{c_0c_1c_2c_3n_0'n_1'n_2'n_3'} \\ &\ll \frac{X^2\log X}{c_0c_1c_2c_3}\sum_{r\leqslant X}\sum_{\substack{n\in\mathbb{N} \\ p|n\Rightarrow p|r \\ r^2|n \\ n>w_1}} \frac{\tau_4(n)}{n},\end{aligned} \tag{2.2.1}$$

where $\tau_4(n)$ denotes the number of ways n can be written as the product of 4 positive integers. Then, noting that $\tau_4(a) \leqslant \tau^3(a) \leqslant \tau^3(a) \ll a^{1/12}$ for all $a\in\mathbb{N}$, this expression becomes:

$$\ll \frac{X^2\log X}{w_1^{1/3}c_0c_1c_2c_3}\sum_{r\leqslant X}\sum_{\substack{n\in\mathbb{N} \\ p|n\Rightarrow p|r \\ r^2|n \\ n>w_1}} \frac{1}{n^{7/12}}.$$

We now use Lemma 5.7 from [29] with $\epsilon = 1/12$ to determine that

$$\sum_{\substack{n\in\mathbb{N}\\ p|n\Rightarrow p|r\\ r^2|n\\ n>w_1}} \frac{1}{n^{7/12}} \ll \frac{1}{r^{13/12}}.$$

Therefore:

$$\sum_{r\leqslant X}\mu(r) \sum\sum\sum\sum_{\substack{\mathbf{n}'\in\prod_{i=0}^{4}(\mathbb{N}\cap[1,X])\\ p|n_0'n_1'n_2'n_3'\Rightarrow p|r\\ r^2|n_0'n_1'n_2'n_3'\\ \gcd(2,n_i')=1\ \forall\ i\\ n_i'>w_1 \text{ for some } i}} \left(\prod_{i=0}^{3} g_i(n_i')\right) G(X,\mathbf{n}') \ll \frac{X^2\log X}{w_1^{1/3}c_0c_1c_2c_3}\sum_{r\leqslant X}\frac{1}{r^{13/12}}$$

$$\ll \frac{X^2\log X}{w_1^{1/3}c_0c_1c_2c_3}.$$

Lastly we bound the terms $r > w_0$. Again we use Lemma 2.2.1 and the bound $\tau_4(a) \ll a^{1/12}$ to obtain:

$$\sum_{r>w_0}\mu(r) \sum\sum\sum\sum_{\substack{\mathbf{n}'\in\prod_{i=0}^{4}(\mathbb{N}\cap[1,w_1])\\ p|n_0'n_1'n_2'n_3'\Rightarrow p|r\\ r^2|n_0'n_1'n_2'n_3'\\ \gcd(2,n_i')=1\ \forall\ i}} \left(\prod_{i=0}^{3} g_i(n_i')\right) G(X,\mathbf{n}') \ll \frac{X^2\log X}{c_0c_1c_2c_3}\sum_{r>w_0}\sum_{\substack{n\in\mathbb{N}\\ p|n\Rightarrow p|r\\ r^2|n}}\frac{1}{n^{11/12}}. \quad (2.2.2)$$

Then, using Lemma 5.7 of [29] with $\epsilon = 5/12$ this will be bounded by

$$\ll \frac{X^2\log X}{c_0c_1c_2c_3}\sum_{r>w_0}\frac{1}{r^{17/12}} \ll \frac{X^2\log X}{w_0^{5/12}c_0c_1c_2c_3}.$$

□

Lemma 2.2.3. *Assume that $g_0, g_1, g_2, g_3 : \mathbb{N} \to \mathbb{C}$ are multiplicative functions with $|g_i(n)| \leqslant 1$ for all $n \in \mathbb{N}$ and all $0 \leqslant i \leqslant 3$. Then for all $X \geqslant 2$, $0 \leqslant z, w_0, w_1 \leqslant X^{1/4}$, and $\mathbf{q} \in \mathbb{N}_{\mathrm{odd}}^4$, $c_{01}, c_{23}, M \in \mathbb{N}$ with $c_{01}, c_{23}, M \leqslant X^{1/4}$,*

$$\sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4\\ \|n_0n_1c_{01},n_2n_3c_{23}\|\cdot M\leqslant X\\ \mathbf{n}\equiv\mathbf{q} \bmod 8}} \mu^2(n_0n_1n_2n_3)\left(\prod_{i=0}^{3} g_i(n_i)\right)$$

$$= \sum_{r\leqslant w_0}\mu(r) \sum\sum\sum\sum_{\substack{\mathbf{n}'\in\prod_{i=0}^{3}(\mathbb{N}\cap[1,w_1])\\ p|n_0'n_1'n_2'n_3'\Rightarrow p|r\\ r^2|n_0'n_1'n_2'n_3'\\ \gcd(2,n_i')=1\ \forall\ i}} \left(\prod_{i=0}^{3} g_i(n_i')\right)\tilde{G}(X,\mathbf{n}')$$

$$+ O\left(\frac{X^2(\log X)^2}{\min(w_0^{5/12}, w_1^{1/3})c_{01}c_{23}M^2}\right)$$

where we have defined, for any $\mathbf{a} \in \mathbb{N}_{\mathrm{odd}}^4$,

$$\tilde{G}(X,\mathbf{a}) = \sum\sum\sum\sum_{\substack{\mathbf{n}''\in\mathbb{N}^4\\ \|a_0a_1n_0''n_1''c_{01},a_2a_3n_2''n_3''c_{23}\|\cdot M\leqslant X\\ n_i''\equiv q_i/a_i \bmod 8\ \forall\ i\\ \gcd(n_i'',r)=1}} \left(\prod_{i=0}^{3} g_i(n_i'')\right)$$

and the implied constant is absolute.

Proof. As in the previous proof we use the identity $\mu^2(n_0n_1n_2n_3) = \sum_{r^2|n_0n_1n_2n_3}\mu(r)$ and then write $n_i = n_1'n_i''$ for each i where each n_i'' are co-prime to r and all prime factors of each n_i' divide r. It follows that

$$\sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4\\ \|n_0n_1c_{01},n_2n_3c_{23}\|\cdot M\leqslant X\\ \mathbf{n}\equiv\mathbf{q} \bmod 8}} \mu^2(n_0n_1n_2n_3)\left(\prod_{i=0}^{3} g_i(n_i)\right)$$
$$= \sum_{r\leqslant X}\mu(r) \sum\sum\sum\sum_{\substack{\mathbf{n}'\in\prod_{i=0}^{3}(\mathbb{N}\cap[1,X])\\ p|n_0'n_1'n_2'n_3'\Rightarrow p|r\\ r^2|n_0'n_1'n_2'n_3'\\ \gcd(2,n_i')=1\ \forall\, i}} \left(\prod_{i=0}^{3} g_i(n_i')\right)\tilde{G}(X,\mathbf{n}').$$

It remains to bound the large terms. First note the following bound for $\tilde{G}(X,\mathbf{n}')$:

$$\begin{aligned}\tilde{G}(X,\mathbf{n}') &\ll \left(\sum\sum_{n_0''n_1''\leqslant X/Mn_0'n_1'c_{01}} 1\right)\left(\sum\sum_{n_2''n_3''\leqslant X/Mn_2'n_3'c_{23}} 1\right)\\ &\ll \left(\frac{X(\log X)}{n_0'n_1'c_{01}M}\right)\left(\frac{X(\log X)}{n_2'n_3'c_{23}M}\right)\\ &\ll \frac{X^2(\log X)^2}{n_0'n_1'n_2'n_3'c_{01}c_{02}M^2}.\end{aligned}$$

Now, we sum over the large n_i' and large r terms as in the previous proof, yielding the same result. □

2.2.2 Large sieve results

As has already been mentioned, the large sieve for quadratic characters will make a regular appearance throughout this thesis. For now, we will only need the following two versions.

The first is due to Friedlander and Iwaniec over rectangular regions.

Lemma 2.2.4 ([17], Lemma 2)**.** *Let $N, M \geqslant 2$ and suppose (a_n), (b_m) are any complex sequences supported on the odd integers such that $|a_n|, |b_m| \leqslant 1$. Then*

$$\sum_{n\leqslant N}\sum_{m\leqslant M} a_nb_m\left(\frac{n}{m}\right) \ll (MN^{5/6}+M^{5/6}N)(\log 3NM)^{7/6}$$

where the implied constant is absolute.

The second version is a result due to the author and will be proven in Chapter 4. The benefit of this result is that it exhibits cancellation for sums of Jacobi symbols in a hyperbolic region, provided that the variables are bounded away from the axes.

Lemma 2.2.5 ([44], Theorem 1.1). *Let $X, z \geqslant 2$ and let (a_n), (b_m) be complex sequences supported on the odd square-free integers such that $|a_n|, |b_m| \leqslant 1$. Then if there exists an $\epsilon > 0$ such that $z \leqslant X^{1/3-\epsilon}$ then*

$$\sum_{\substack{z<n,m\leqslant X\\ nm\leqslant X}} a_n b_m \left(\frac{n}{m}\right) \ll_\epsilon \frac{X(\log X)^3}{z^{1/2}},$$

where the implied constant depends at most on ϵ.

2.2.3 Hyperbolic character sums

The following six results are the main results of Chapter 5. They are the primary technical tools for sections §2.7-§2.9.

The first will be used to handle the main term. For this result, we define

$$f_0 = \frac{1}{\sqrt{\pi}} \prod_{p \text{ prime}} f_p \left(1-\frac{1}{p}\right)^{1/2} \quad \text{and} \quad f_p = 1 + \sum_{j=1}^{\infty} \frac{1}{(j+1)p^j}. \tag{2.2.3}$$

Proposition 2.2.6. *Let $X \geqslant 3$, $C_1, C_2, C_3 > 0$ and take any $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$. Then for any fixed odd integers $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ and fixed integers $1 \leqslant c_0, c_1, c_2, c_3 \leqslant (\log X)^{C_2}$, $1 \leqslant d_0, d_1, d_2, d_3 \leqslant (\log X)^{C_3/2}$ we have*

$$\sum\sum\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\cdot\|n_2c_2,n_3c_3\|\leqslant X\\ \|n_0d_0,n_1d_1\|,\|n_2d_2,n_3d_3\|>(\log X)^{C_3}\\ \gcd(n_i,r_i)=1\ \forall\ 0\leqslant i\leqslant 3\\ n_i\equiv q_i \bmod 8\ \forall\ 0\leqslant i\leqslant 3}} \frac{1}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}$$
$$= \frac{\mathfrak{S}_2(\mathbf{r})X^2\log\log X}{c_0c_1c_2c_3\log X}\left(1+O_{C_1,C_2,C_3}\left(\frac{\tau(r_0)\tau(r_1)\tau(r_2)\tau(r_3)}{\sqrt{\log\log X}}\right)\right),$$

where the implied constant depends at most on C_1, C_2, C_3 and we define

$$\mathfrak{S}_2(\mathbf{r}) = \frac{4f_0^4}{\phi(8)^4\left(\prod_{p|2r_0} f_p\right)\left(\prod_{p|2r_1} f_p\right)\left(\prod_{p|2r_2} f_p\right)\left(\prod_{p|2r_3} f_p\right)}.$$

The next five will be used at various points to bound the error terms. For a positive integer m we will henceforth denote by ψ_m the Jacobi symbol $\left(\frac{\cdot}{m}\right)$ or $\left(\frac{m}{\cdot}\right)$ generically.

Proposition 2.2.7. *Let $X \geqslant 3$, $C_1, C_2, C_3 > 0$ and fix odd integers Q_0, Q_2 and some $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$, $\tilde{\mathbf{q}} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$. Fixing some odd integers $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ such that $\gcd(r_i, Q_i) = 1$ for $i = 0, 2$ and any $1 \leqslant c_0, c_1, c_2, c_3 \leqslant (\log X)^{C_2}$, $1 \leqslant d_0, d_1, d_2, d_3 \leqslant (\log X)^{C_3/2}$ we define, for any $\mathbf{m} \in \mathbb{N}^4$,*

$$H(X,\mathbf{m}) = \sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4,\|n_0m_0c_0,n_1m_1c_1\|\cdot\|n_2m_2c_2,n_3m_3c_3\|\leqslant X\\ (5.2.3)}} \frac{\psi_{Q_0m_0m_1}(n_2n_3)\psi_{Q_2m_2m_3}(n_0n_1)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)},$$

where we use (5.2.3) with $D = C_3$. Then for any $C_4 > 0$:

$$\sum\sum\sum\sum_{\substack{\mathbf{m}\in\mathbb{N}^4, \|m_0,m_1\|,\|m_2,m_3\|\leqslant(\log X)^{C_3} \\ \gcd(m_0m_1,Q_0r_2r_3)=\gcd(m_2m_3,Q_1r_0r_1)=1 \\ \mathbf{m}\equiv\tilde{\mathbf{q}} \bmod 8 \\ Q_0m_0m_1 \text{ and } Q_2m_2m_3\neq 1}} \frac{\mu^2(2m_0m_1m_2m_3)|H(X,\mathbf{m})|}{\tau(m_0)\tau(m_1)\tau(m_2)\tau(m_3)}$$

$$\ll_{C_1,C_2,C_3,C_4} \frac{Q_0Q_2X^2}{c_0c_1c_2c_3(\log X)^{C_4}},$$

where the implied constant depends at most on C_1, C_2, C_3, C_4.

Proposition 2.2.8. *Let $X \geqslant 3$, $C_1, C_2 > 0$ be such that $(C_1 \log\log X)^{C_2} > 2$. Fix some odd square-free integers $Q_1, Q_2, Q_3 \in \mathbb{N}$ such that $Q_1 \leqslant (\log\log X)^{C_2}$, and some $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$, $\tilde{\mathbf{q}} \in (\mathbb{Z}/8\mathbb{Z})^{*2}$. Suppose χ_2 and χ_3 are characters modulo Q_2 and Q_3 respectively. Fixing any odd integers $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ such that* $\gcd(Q_1, r_0r_1r_2r_3) = \gcd(Q_2Q_3, r_2r_3) = 1$ *and fixing any $1 \leqslant c_0, c_1, c_2, c_3 \leqslant (\log X)^{C_2/32}$,* $1 \leqslant d_0, d_1, d_2, d_3 \leqslant (\log X)^{C_2/4}$ *we define, for any $\mathbf{m} \in \mathbb{N}^2$*

$$H'(X,\mathbf{m}) = \sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4, \|n_0d_0,n_1d_1\|,\|n_2d_2,n_3d_3\|>(\log X)^{C_2} \\ \|n_0c_0,n_1c_1\|\cdot\|n_2m_2c_2,n_3m_3c_3\|\leqslant X \\ \gcd(n_i,2r_i)=1\ \forall\ 0\leqslant i\leqslant 3 \\ \mathbf{n}\equiv\mathbf{q} \bmod 8}} \frac{\psi_{m_2m_3}(n_2n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}.$$

Then,

$$\sum\sum_{\substack{\mathbf{m}\in\mathbb{N}^2, \|m_2,m_3\|\leqslant(\log X)^{C_2} \\ \gcd(m_i,2Q_1Q_2Q_3r_i)=1\ \forall\ 2\leqslant i\leqslant 3 \\ \mathbf{m}\equiv\tilde{\mathbf{q}} \bmod 8 \\ Q_1m_2m_3\neq 1}} \frac{\mu^2(m_2m_3)\chi_2(m_2)\chi_3(m_3)}{\tau(m_2)\tau(m_3)} H'(X,\mathbf{m}) \ll_{C_2} \frac{\tau(r_0)\tau(r_1)X^2}{c_0c_1c_2c_3(\log X)(\log\log X)^{C_3}},$$

where $C_3 = C_2/2 - 1$ and where the implied constant depends at most on C_1 and C_2.

Lemma 2.2.9. *Let $X \geqslant 3$, $C_1, C_2 > 0$. Suppose χ_0, χ_1, χ_2 and χ_3 are Dirichlet characters modulo 8 such that χ_i and χ_j are non-principal for some pair $(i,j) \in \{0,1\} \times \{2,3\}$. Then for any odd integers $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ and any integers $1 \leqslant c_{01}, c_{23}, M \leqslant (\log X)^{C_2}$ we have,*

$$\sum\sum\sum\sum_{\substack{\|n_0n_1c_{01},n_2n_3c_{23}\|\cdot M\leqslant X \\ \gcd(n_i,2r_i)=1\ \forall\ 0\leqslant i\leqslant 3}} \frac{\chi_0(n_0)\chi_2(n_2)\chi_1(n_1)\chi_3(n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} \ll_{C_2} \frac{\tau(r_0)\tau(r_1)\tau(r_2)\tau(r_3)X^2}{c_{01}c_{23}M^2(\log X)},$$

where the implied constant depends at most on C_2.

Proposition 2.2.10. *Let $X \geqslant 3$, $C_1, C_2, C_3 > 0$, let Q_{02}, Q_{13} be odd integers and take $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$, $\tilde{\mathbf{q}} \in (\mathbb{Z}/8\mathbb{Z})^{*2}$. Let $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ be odd integers such that* $\gcd(Q_{ij}, 2r_ir_j) = 1$ *for $i \in \{(0,2),(1,3)\}$ and any $1 \leqslant c_0, c_1, c_2, c_3 \leqslant (\log X)^{C_2}$. Define, for any $\mathbf{m} \in \mathbb{N}^4$,*

$$H''(X,\mathbf{m}) = \sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4 \\ \|n_0n_1c_0,n_2n_3c_1\|\cdot\|m_0m_1c_2,m_2m_3c_3\|\leqslant X \\ \gcd(n_i,r_i)=1\ \forall\ 0\leqslant i\leqslant 3 \\ n_i\equiv q_i \bmod 8\ \forall\ 0\leqslant i\leqslant 3}} \frac{\psi_{Q_{02}m_0m_2}(n_0n_2)\psi_{Q_{13}m_1m_2}(n_1n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}.$$

Then

$$\sum\sum\sum\sum_{\substack{\mathbf{m}\in\mathbb{N}^4, \|m_0,m_1,m_2,m_3\|\leqslant(\log X)^{C_3}\\ \gcd(m_0m_2,2Q_{02}r_0r_2)=\gcd(m_1m_3,Q_{13}r_1r_3)=1\\ Q_{02}m_0m_2\neq 1 \text{ and } Q_{13}m_1m_3\neq 1\\ \mathbf{m}\equiv\tilde{\mathbf{q}} \bmod 8}} \frac{\mu^2(2m_0m_1m_2m_3)|H''(X,\mathbf{m})|}{\tau(m_0)\tau(m_1)\tau(m_2)\tau(m_3)}$$

$$\ll_{C_1,C_2,C_3,C_4} \frac{Q_{02}Q_{13}X^2}{c_0c_1c_2c_3(\log X)^{C_4}},$$

for any $C_4>0$ where the implied constant depends at most on the C_i.

Proposition 2.2.11. *Let $X\geqslant 3$, $C_1,C_2>0$. Let us fix vectors $\mathbf{q}\in(\mathbb{Z}/8\mathbb{Z})^{*4}$ and $\tilde{\mathbf{q}}\in(\mathbb{Z}/8\mathbb{Z})^{*2}$, $\tilde{\mathbf{r}}\in\mathbb{N}^2$ be vectors of odd integers. Fix odd integers $1\leqslant r_0,r_1,r_2,r_3,\tilde{r}_0,\tilde{r}_1\leqslant(\log X)^{C_1}$ and fix $1\leqslant c_{01},c_{23},\tilde{c}_0,\tilde{c}_1\leqslant(\log X)^{C_2}$. Then for any $\mathbf{m}\in\mathbb{N}^2$ we define*

$$T(X,\mathbf{m}) = \sum\sum\sum\sum_{\substack{\|n_0n_1c_{01},n_2n_3c_{23}\|\cdot\|m_0\tilde{c}_0,m_1\tilde{c}_1\|\leqslant X\\ \gcd(n_i,2r_i)=1\ \forall\ 0\leqslant i\leqslant 3\\ n_i\equiv q_i \bmod 8\ \forall\ 0\leqslant i\leqslant 3}} \frac{\psi_{m_0m_1}(n_0n_2)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}.$$

Then for any $C_3>0$,

$$\sum\sum_{\substack{\|m_0,m_1\|\leqslant(\log X)^{C_3}\\ m_i\equiv\tilde{q}_i \bmod 8\ \forall\ 0\leqslant i\leqslant 1\\ \gcd(m_i,\tilde{r})=1\ \forall\ 0\leqslant i\leqslant 1\\ m_0m_1\neq 1}} \frac{\mu^2(m_0m_1)}{\tau(m_1)\tau(m_2)}|T(X,\mathbf{m})| \ll_{C_1,C_2,C_3} \frac{\tau(r_0)\tau(r_2)X^2(\log\log X)^{1/2}}{c_{01}c_{23}\tilde{c}_0\tilde{c}_1(\log X)},$$

where the implied constant depends at most on the C_i.

2.3 Geometric Input

Recall that $Z\subset\mathbb{P}^3\times\mathbb{P}^3$ is the variety over $\mathbb{Q}$ cut out by the equations

$$y_0x_0^2+y_1x_1^2+y_2x_2^2+y_3x_3^2=0 \text{ and } y_0y_1=y_2y_3,$$

and that $\pi: Z(\mathbb{Q})\to Y(\mathbb{Q})$ be the dominant map sending $([x_0:x_1:x_2:x_3],[y_0:y_1:y_2:y_3])\in Z(\mathbb{Q})$ to the point $[y_0:y_1:y_2:y_3]\in Y(\mathbb{Q})$ where $Y\subset\mathbb{P}^3$ is the rational quadric surface cut out by the equation

$$y_0y_1=y_2y_3.$$

We then want to find asymptotics for the following quantity:

$$N(B)=\sharp\left\{y\in\pi(Z)(\mathbb{Q}):\ \begin{array}{c}-y_0y_2,-y_0y_3\neq\square;\\ \pi^{-1}(y)\text{ has a }\mathbb{Q}\text{-point};\\ H(y)\leqslant B\end{array}\right\}, \tag{2.3.1}$$

where H is the naive Weil height in $\mathbb{P}^3(\mathbb{Q})$. The variety Y is $\mathbb{Q}$-isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$ via the regular map $\phi:\mathbb{P}^1\times\mathbb{P}^1\to Y$ given by

$$y_0=t_0t_2,\ y_1=t_1t_3,\ y_2=t_1t_2,\ y_3=t_0t_3.$$

We may then reformulate our counting function in the following way:

$$N(B) = \sharp \left\{ t \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) : \begin{array}{c} -t_0t_1, -t_2t_3 \neq \square; \\ \pi^{-1}(\phi^{-1}(t)) \text{ has a } \mathbb{Q}\text{-point;} \\ H([t_0 : t_1])H([t_2 : t_3]) \leqslant B \end{array} \right\}, \tag{2.3.2}$$

where here $H([a : b])$ is the naive Weil height on $\mathbb{P}^1(\mathbb{Q})$ and the fibre $\pi^{-1}(\phi^{-1}(t))$ is given by the equation

$$t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 + t_0t_3x_3^2 = 0.$$

We now wish to write this counting problem as one over the integers. This is done by noting the correspondence between $\mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$ and $\mathbb{Z}^2_{\text{prim}} \times \mathbb{Z}^2_{\text{prim}}$ where $\mathbb{Z}^2_{\text{prim}}$ is the set of all coprime integer pairs (n, m). For each point in $\mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$ there are four points in $\mathbb{Z}^2_{\text{prim}} \times \mathbb{Z}^2_{\text{prim}}$ corresponding to it. Therefore our counting problem becomes

$$N(B) = \frac{1}{4}\sharp \left\{ \mathbf{t} \in \mathbb{Z}^2_{\text{prim}} \times \mathbb{Z}^2_{\text{prim}} : \begin{array}{c} t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 + t_0t_3x_3^2 = 0 \text{ has a } \mathbb{Q}\text{-point,} \\ -t_0t_1, -t_2t_3 \neq \square, \\ \|t_0, t_1\| \cdot \|t_2, t_3\| \leqslant B \end{array} \right\}$$

where, as before, $\|t_0, t_1\| = \max\{|t_0|, |t_1|\}$ and $\|t_2, t_3\| = \max\{|t_2|, |t_3|\}$. We will henceforth write this as

$$N(B) = \frac{1}{4}\sharp \left\{ \mathbf{t} \in \mathbb{Z}^4 : \begin{array}{c} t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 + t_0t_3x_3^2 = 0 \text{ has a } \mathbb{Q}\text{-point,} \\ \gcd(t_0, t_1) = \gcd(t_2, t_3) = 1, -t_0t_1, -t_2t_3 \neq \square, \\ \|t_0, t_1\| \cdot \|t_2, t_3\| \leqslant B \end{array} \right\}. \tag{2.3.3}$$

To conclude this section, we consider the points $\mathbf{t}$ such that one of the entries is 0. First, we assume $t_0 = 0$. Then since $\gcd(t_0, t_1) = 1$ we must then have $t_1 = \pm 1$. In this case we therefore want pairs (t_2, t_3) such that $\|t_2, t_3\| \leqslant B$, $-t_2t_3 \neq \square$ and $t_3x_1^2 + t_2x_2^2 = 0$ has a $\mathbb{Q}$-point. For the latter to be true, however, we must have $-t_2t_3$ equal to a square, which is a contradiction. Therefore there are no points with $t_0 = 0$ included in the count. A symmetric argument may be given for the cases $t_1 = 0$, $t_2 = 0$ and $t_3 = 0$. We may therefore write our counting problem as

$$N(B) = \frac{1}{4}\sharp \left\{ \mathbf{t} \in (\mathbb{Z} \setminus \{0\})^4 : \begin{array}{c} t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 + t_0t_3x_3^2 = 0 \text{ has a } \mathbb{Q}\text{-point,} \\ \gcd(t_0, t_1) = \gcd(t_2, t_3) = 1, -t_0t_1, -t_2t_3 \neq \square, \\ \|t_0, t_1\| \cdot \|t_2, t_3\| \leqslant B \end{array} \right\}. \tag{2.3.4}$$

Remark 2.3.1. Note that the point where one of the $t_i = 0$ correspond to points $[y_0 : y_1 : y_2 : y_3]$ such that two of the $y_j = 0$. These are the lines on the quadric surface which have singular fibres under the map π, and thus we may ignore the desingularisation used in the introduction.

2.4 Local Solubility

2.4.1 Real Points

Our first step is to guarantee that our quadrics have real points. This will occur whenever the coefficients are not all positive and not all negative. However, we may use the symmetry of our surface to ensure we are counting over purely positive integers and simplify our argument. First, we split $\mathbb{R}^4$ into 16 regions determined by the sign of the t_i. For $\mathbf{l} \in \{0,1\}^4$ we will write $R_{\mathbf{l}}$ to be the regions defined by the points $\mathbf{t} \in \mathbb{R}^4$ such that $t_i > 0$ if $l_i = 0$ and $t_i < 0$ if $l_i = 1$. For example, $R_{(1,0,0,0)} = \{\mathbf{t} \in \mathbb{R}^4 : t_0 < 0,\ t_1, t_2, t_3 > 0\}$. We will then write

$$N_{\mathbf{l}}(B) = \sharp \left\{ \mathbf{t} \in (\mathbb{Z}\setminus\{0\})^4 \cap R_{\mathbf{l}} : \begin{array}{c} t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 + t_0t_3x_3^2 = 0 \text{ has a } \mathbb{Q}\text{-point;} \\ \gcd(t_0,t_1) = \gcd(t_2,t_3) = 1; -t_0t_1, -t_2t_3 \neq \square; \\ \|t_0,t_1\| \cdot \|t_2,t_3\| \leqslant B \end{array} \right\}.$$

It is clear that $N_{(0,0,0,0)}(B) = N_{(1,1,1,1)}(B) = 0$ since in these cases the corresponding quadrics have no real solutions. In order to streamline our argument we will prove the following:

Lemma 2.4.1. *For notation as above we have the following,*

- *if* $\sum_{i=0}^3 l_i = 1$ *or* 3 *then,* $N_{\mathbf{l}}(B) = N_{(1,0,0,0)}(B)$,
- *if* $\sum_{i=0}^3 l_i = 2$ *and* $\mathbf{l} \notin \{(1,1,0,0),(0,0,1,1)\}$ *then,* $N_{\mathbf{l}}(B) = N_{(1,0,1,0)}(B)$,
- *if* $\mathbf{l} \in \{(1,1,0,0),(0,0,1,1)\}$ *then* $N_{\mathbf{l}}(B) = 0$.

Proof. The last of these assertions is immediate from the fact that, if $\mathbf{t} \in R_{(1,1,0,0)} \cup R_{(0,0,1,1)}$, then the coefficients of the equation

$$t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 + t_0t_3x_3^2 = 0$$

are all negative. We now look at the first assertion. The key observation is that, if $\mathbf{t} \in R_{\mathbf{l}}$ where $\sum_i l_i = 1$ or 3, then we may find a unique point $\tilde{\mathbf{t}} \in R_{(1,0,0,0)}$, of equal height which corresponds to a quadric which is equivalent – under the natural action of $\mathrm{SL}_4(\mathbb{Z})$ on the set of quadratic forms in four variables – to the quadric corresponding to $\mathbf{t}$. Then the quadric defined by $\mathbf{t}$, say $\mathcal{C}_{\mathbf{t}}$, has a rational point if and only if the quadric defined by $\tilde{\mathbf{t}}$, say $\mathcal{C}_{\tilde{\mathbf{t}}}$, has a rational point. First suppose that $\sum_i l_i = 1$. Then only one of the components is negative. If $l_0 = 1$, then the result is trivial. For a point $\mathbf{t} = (t_0,t_1,t_2,t_3) \in R_{(0,1,0,0)}$, the corresponding quadric is

$$\mathcal{C}_{\mathbf{t}} : t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 + t_0t_3x_3^2 = 0$$

where here, the coefficients of x_1^2 and x_2^2 are negative. We map $\mathbf{t}$ to the point $\tilde{\mathbf{t}} = (t_1,t_0,t_2,t_3) \in R_{(1,0,0,0)}$. Then the quadric corresponding to $\tilde{\mathbf{t}}$ is

$$\mathcal{C}_{\tilde{\mathbf{t}}} : t_1t_2x_0^2 + t_0t_3x_1^2 + t_0t_2x_2^2 + t_1t_3x_3^2 = 0.$$

It is clear that $\mathcal{C}_{\tilde{\mathbf{t}}}$ is equivalent to $\mathcal{C}_{\mathbf{t}}$ since we have only permuted the coefficients. This mapping from $R_{(0,1,0,0)}$ to $R_{(1,0,0,0)}$ is clearly a bijection and it is easy to show that it preserves height. Next we consider $l_2 = 1$. Here, $\mathbf{t} = (t_0,t_1,t_2,t_3) \in R_{(0,0,1,0)}$, with quadric

$$\mathcal{C}_{\mathbf{t}} : t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 + t_0t_3x_3^2 = 0,$$

where the coefficients of x_0^2 and x_2^2, t_0t_2 and t_1t_2, are negative. We map $\mathbf{t}$ to $\tilde{\mathbf{t}} = (t_2, t_3, t_0, t_1)$, which has the quadric

$$\mathcal{C}_{\tilde{\mathbf{t}}} : t_0t_2x_0^2 + t_1t_3x_1^2 + t_3t_0x_2^2 + t_2t_1x_3^2 = 0.$$

Again, $\mathcal{C}_{\tilde{\mathbf{t}}}$ is equivalent to $\mathcal{C}_{\mathbf{t}}$ since we have only permuted the coefficients of x_2^2 and x_3^2. This mapping is also a bijection and height preserving. Finally, if $\mathbf{t} = (t_0, t_1, t_2, t_3) \in R_{(0,0,0,1)}$, then we map to $\tilde{\mathbf{t}} = (t_3, t_2, t_1, t_0) \in R_{(1,0,0,0)}$. As before this is a height preserving, bijective map such that $\mathcal{C}_{\mathbf{t}}$ and $\mathcal{C}_{\tilde{\mathbf{t}}}$ are equivalent quadrics. If $\sum_i l_i = 3$, then we reduce to one of the above cases by sending $\mathbf{t} = (t_0, t_1, t_2, t_3) \in R_{\mathbf{l}}$ to $\tilde{\mathbf{t}} = (-t_0, -t_1, -t_2, -t_3) \in R_{\tilde{\mathbf{l}}}$ where $\sum_i \tilde{l}_i = 1$. This mapping is a height preserving bijection and furthermore $\mathcal{C}_{\mathbf{t}}$ and $\mathcal{C}_{\tilde{\mathbf{t}}}$ are the same quadric. We have now proved the first assertion. The strategy for the second assertion is the same: if $\sum_i l_i = 2$ and $\mathbf{l} \notin \{(1,1,0,0), (0,0,1,1)\}$ we find a height and quadric preserving, bijective mapping from $R_{\mathbf{l}}$ to $R_{(1,0,1,0)}$. For $R_{(1,0,1,0)}$ this is trivial. For $\mathbf{t} = (t_0, t_1, t_2, t_3) \in R_{(0,1,0,1)}$ we have

$$\mathcal{C}_{\mathbf{t}} : t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 + t_0t_3x_3^2 = 0,$$

where the coefficients of x_2^2 and x_3^2, t_1t_2 and t_0t_3 are negative. We send $\mathbf{t}$ to $\tilde{\mathbf{t}} = (t_1, t_0, t_3, t_2) \in R_{(1,0,1,0)}$. Again, this map is bijective and height preserving, and the quadric corresponding to $\tilde{\mathbf{t}}$ is

$$\mathcal{C}_{\tilde{\mathbf{t}}} : t_1t_3x_0^2 + t_0t_2x_1^2 + t_0t_3x_2^2 + t_1t_2x_3^2 = 0,$$

which is equivalent to $\mathcal{C}_{\mathbf{t}}$. For $\mathbf{t} = (t_0, t_1, t_2, t_3) \in R_{(1,0,0,1)}$ we map it to $\tilde{\mathbf{t}} = (t_0, t_1, t_3, t_2) \in R_{(1,0,1,0)}$ and for $\mathbf{t} = (t_0, t_1, t_2, t_3) \in R_{(0,1,1,0)}$ we map it to $\tilde{\mathbf{t}} = (t_1, t_0, t_2, t_3) \in R_{(1,0,1,0)}$. Both of these are easily checked to be height preserving bijections resulting in points with equivalent quadrics. This completes the proof. □

Lemma 2.4.1 allows us to rephrase our counting problem so that we only count over positive integers while ensuring all quadrics considered have real points. This is encoded in the lemma below.

Proposition 2.4.2. *We have $N(B) = 2N_1(B) + N_2(B)$, where*

$$N_1(B) = \sharp\left\{ \mathbf{t} \in \mathbb{N}^4 : \begin{array}{c} -t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 - t_0t_3x_3^2 = 0 \textit{ has a } \mathbb{Q}\textit{-point}; \\ \gcd(t_0, t_1) = \gcd(t_2, t_3) = 1, t_0t_1 \neq \square; \\ \|t_0, t_1\| \cdot \|t_2, t_3\| \leqslant B \end{array} \right\},$$

and

$$N_2(B) = \sharp\left\{ \mathbf{t} \in \mathbb{N}^4 : \begin{array}{c} t_0t_2x_0^2 + t_1t_3x_1^2 - t_1t_2x_2^2 - t_0t_3x_3^2 = 0 \textit{ has a } \mathbb{Q}\textit{-point}; \\ \gcd(t_0, t_1) = \gcd(t_2, t_3) = 1, t_0t_1, t_2t_3 \neq \square; \\ \|t_0, t_1\| \cdot \|t_2, t_3\| \leqslant B \end{array} \right\},$$

Proof. We may write $4N(B) = \sum_{\mathbf{l}\in\{0,1\}^4} N_{\mathbf{l}}(B)$. Thus, using Lemma 2.4.1,

$$N(B) = \frac{1}{4}\left(8N_{(1,0,0,0)}(B) + 4N_{(1,0,1,0)}(B)\right) = 2N_1(B) + N_2(B).$$

This last equality comes from noting that $N_{(1,0,0,0)}(B) = N_1(B)$ and $N_{(1,0,1,0)}(B) = N_2(B)$. □

2.4.2 p-adic Points

Using the Hasse principle for quadrics, we may equate the problem of detecting rational points to detecting p-adic points. We have already ensured that all quadrics we are considering have a real point and so we only need to detect $\mathbb{Q}_p$-points for every prime p. The advantage of this is that we will be able to express solubility conditions as a sum over Jacobi symbols. To do this we define, for $(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4$, the indicator function,

$$\langle a_0, a_1, a_2, a_3\rangle_p = \begin{cases} 1 \text{ if } \mathcal{D}_{\mathbf{a}} \text{ has a } \mathbb{Q}_p\text{-point} \\ 0 \text{ otherwise,} \end{cases}$$

where $\mathcal{D}_{\mathbf{a}}$ is the quadric defined by the equation

$$a_0x_0^2 + a_1x_1^2 + a_2x^2 + a_3x_3^2 = 0.$$

Then we obtain the following result:

Lemma 2.4.3. *Let p be an odd prime and suppose a_0, a_1, a_2, a_3 are square-free, non-zero integers such that $\gcd(a_0, a_1, a_2, a_3) = 1$. Then:*

(a) If $v_p(a_0a_1a_2a_3) \neq 2$ then $\langle a_0, a_1, a_2, a_3\rangle_p = 1$;

(b) Otherwise, if $p \mid a_i, a_j$ for any distinct $i, j \in \{0, 1, 2, 3\}$ and $p \nmid a_ka_l$ for the distinct $k, l \in \{0, 1, 2, 3\} \setminus \{i, j\}$ then

$$\langle a_0, a_1, a_2, a_3\rangle_p = \frac{1}{4}\left(3 + \left(\frac{-a_ka_l}{p}\right) + \left(\frac{-(a_ia_j)/p^2}{p}\right) - \left(\frac{-a_ka_l}{p}\right)\left(\frac{-(a_ia_j)/p^2}{p}\right)\right),$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the quadratic Jacobi symbol.

Proof. Suppose $v_p(a_0a_1a_2a_3) = 1$. We may then assume without loss in generality that $p|a_0$ and $p \nmid a_1a_2a_3$. Then $\mathcal{D}_{\mathbf{a}}$ will reduce to the smooth ternary quadratic

$$\tilde{\mathcal{D}}_{\tilde{\mathbf{a}}} : \tilde{a}_1x_1^2 + \tilde{a}_2x_2^2 + \tilde{a}_3x_3^2 = 0$$

over $\mathbb{F}_p$. By the Chevalley–Warning theorem, all smooth ternary quadratics over $\mathbb{F}_p$ have a non-zero point, $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$, which we may lift to a $\mathbb{Q}_p$-point (y_1, y_2, y_3) on the ternary quadratic $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0$ by Hensel's lifting lemma. Then $(0, y_1, y_2, y_3)$ will be a $\mathbb{Q}_p$-point on $\mathcal{D}_{\mathbf{a}}$.

If $v_p(a_0a_1a_2a_3) = 3$ then we assume without loss in generality that $p \mid a_0, a_1, a_2$. We may multiply the quadric $\mathcal{D}_{\mathbf{a}}$ by p and then apply the birational transformation $(x_0, x_1, x_2, x_3) \mapsto (x_0/p, x_1/p, x_2/p, x_3)$ to obtain an equivalent quadric $\mathcal{D}_{(a_0/p, a_1/p, a_2/p, pa_3)}$ which is of the form considered above. The proof for $v_p(a_0a_1a_2a_3) = 0$ is similar to that for $v_p(a_0a_1a_2a_3) = 1$. This completes the proof of (a) as the a_i are square-free integers with $\gcd(a_0, a_1, a_2, a_3) = 1$, so that $v_p(a_0a_1a_2a_3) \neq 2$ implies either $v_p(a_0a_1a_2a_3) = 0, 1$ or 3.

Finally we turn to case (b). Assume without loss in generality that $p \mid a_0, a_1$ and $p \nmid a_2a_3$. Then it is clear that there is a $\mathbb{Q}_p$ solution to the quadric $\mathcal{D}_{\mathbf{a}}$ if and only if there exists some $C \in \mathbb{Q}_p$ such that

$$a_0x_0^2 + a_1x_1^2 = C \text{ and } a_2x_2^2 + a_3x_3^2 = -C.$$

This is true if and only if the ternary quadratics

$$a_0x_0^2 + a_1x_1^2 = Cz^2 \text{ and } a_2x_2^2 + a_3x_3^2 = -Cw^2$$

have a solution in $\mathbb{Q}_p$. Let $(n, m)_p$ denote the Hilbert symbol for $\mathbb{Q}_p$ and write $a_0 = pu_0$, $a_1 = pu_1$ and $C = p^\alpha v$ with u_0, u_1 and v coprime to p. Then we have that $\mathcal{D}_{\mathbf{a}}$ has a solution in $\mathbb{Q}_p$ if and only if

$$\left(\frac{a_0}{C}, \frac{a_1}{C}\right)_p = \left(\frac{(-u_0u_1)^{1-\alpha}}{p}\right) = 1 \text{ and } \left(\frac{-a_2}{C}, \frac{-a_3}{C}\right)_p = \left(\frac{(-a_2a_3)^\alpha}{p}\right) = 1.$$

If $\alpha \equiv 0 \bmod 2$ then this condition simplifies to requiring that $\left(\frac{-u_0u_1}{p}\right) = 1$. If $\alpha \equiv 1 \bmod 2$, the condition requires that $\left(\frac{-a_2a_3}{p}\right) = 1$. For the backwards direction, if $\left(\frac{-u_0u_1}{p}\right) = 1$, then we may choose $C = 1$; then $\alpha = 0$ in the expressions above, and both equalities hold. Similarly, if $\left(\frac{-a_2a_3}{p}\right) = 1$ then we may choose $C = p$, in which case $\alpha = 1$ in the above expressions and both equalities above hold. Therefore, recalling that $u_0 = \frac{a_0}{p}$ and $u_1 = \frac{a_1}{p}$, we have proven that $\mathcal{D}_{\mathbf{a}}$ has a solution in $\mathbb{Q}_p$ if and only if

$$\left(\frac{-(a_0a_1)/p^2}{p}\right) = 1 \text{ or } \left(\frac{-a_2a_3}{p}\right) = 1.$$

We put these two conditions together to obtain the formula

$$\langle a_0, a_1, a_2, a_3 \rangle_p = \frac{1}{4}\left(3 + \left(\frac{-a_2a_3}{p}\right) + \left(\frac{-(a_0a_1)/p^2}{p}\right) - \left(\frac{-a_2a_3}{p}\right)\left(\frac{-(a_0a_1)/p^2}{p}\right)\right).$$

□

2.4.3 2-adic points

Our strategy will be similar as in the previous section, however we will only deal with vectors $\mathbf{a} = (a_0, a_1, a_2, a_3)$ such that the a_i are square-free, non-zero integers with $\gcd(a_0, a_1, a_2, a_3) = 1$ and $v_2(a_0a_1a_2a_3) = 0$ or 2. We begin by defining two sets:

$\mathcal{A}_1 = \{\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4} : \mathbf{q} \text{ satisfies } (2.4.1)\}$ and $\mathcal{A}_2 = \{\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4} : \mathbf{q} \text{ satisfies } (2.4.2)\}$,

where

$$\begin{cases} q_i + q_j = 0, 4 \text{ for at least one pair } (i,j) \in \{0,1\} \times \{2,3\} \text{ or,} \\ (q_0 + q_1, q_2 + q_3) \in \{(0,0),(2,0),(2,6),(0,6),(6,0),(6,2),(0,2)\}, \end{cases} \tag{2.4.1}$$

and

$$\begin{cases} \text{there is at least one choice of } i,j,k,l \text{ such that } \{(i,j),(k,l)\} = \{(0,1),(2,3)\}, \\ \text{and some } v \in (\mathbb{Z}/8\mathbb{Z})^* \text{ such that } \ (q_i + q_j = 0 \ \text{ and } \ (q_k + v)(q_l + v) = 0) \ \text{ or,} \\ (q_i + q_j = 2v \ \text{ and } \ (q_k + v)(q_l + v) = 0)\,. \end{cases} \tag{2.4.2}$$

Then we have the following.

Lemma 2.4.4. *Suppose* $a_0, a_1, a_2, a_3 \in \mathbb{N}$ *are square-free and non-zero satisfying the condition* $\gcd(a_0, a_1, a_2, a_3) = 1$. *Then:*

(a) if $2 \nmid a_0a_1a_2a_3$ *then* $\langle a_0, a_1, a_2, a_3 \rangle_2 = 1$ *if and only if* (a_0, a_1, a_2, a_3) *reduces to a vector in* $\mathcal{A}_1$ *modulo* 8*;*

(b) if $2 \mid a_i, a_j$ *for any distinct* $i, j \in \{0,1,2,3\}$ *and* $2 \nmid a_k a_l$ *for the distinct* $k, l \in \{0,1,2,3\} \backslash \{i,j\}$ *then* $\langle a_0, a_1, a_2, a_3 \rangle_2 = 1$ *if and only if* $(a_i/2, a_j/2, a_k, a_l)$ *reduces to a vector in* $\mathcal{A}_2$ *modulo 8.*

Proof. Following the same strategy used in the proof of Lemma 2.4.3(b) we have that the quadric $\mathcal{D}_{\mathbf{a}}$ has a solution in $\mathbb{Q}_2$ if and only if there exists a $C \in \mathbb{Q}_2$ such that

$$\left(\frac{a_0}{C}, \frac{a_1}{C}\right)_2 = 1 \text{ and } \left(-\frac{a_2}{C}, -\frac{a_3}{C}\right)_2 = 1 \tag{2.4.3}$$

where $(\cdot,\cdot)_2$ is the Hilbert symbol over $\mathbb{Q}_2$. Let us first consider part (a). Writing $C = 2^\alpha v$ for a unit $v \in \mathbb{Q}_2$, we use the well known formulae for these Hilbert symbols (for example see Chapter 3, Theorem 1 of [38]) to obtain the equivalent condition:

$$\frac{(a_0v^{-1} - 1)(a_1v^{-1} - 1)}{4} + \alpha\frac{(a_0^2v^{-2} + a_1^2v^{-2} - 2)}{8} \equiv 0 \bmod 2 \tag{2.4.4}$$

and

$$\frac{(a_2v^{-1} + 1)(a_3v^{-1} + 1)}{4} + \alpha\frac{(a_2^2v^{-2} + a_3^2v^{-2} - 2)}{8} \equiv 0 \bmod 2. \tag{2.4.5}$$

We have two cases: $\alpha \equiv 0 \bmod 2$ and $\alpha \equiv 1 \bmod 2$. In the former case we simplify to the condition

$$\left(\frac{(a_0v^{-1} - 1)(a_1v^{-1} - 1)}{4} \equiv 0 \bmod 2\right) \text{ and } \left(\frac{(a_2v^{-1} + 1)(a_3v^{-1} + 1)}{4} \equiv 0 \bmod 2\right),$$

which is equivalent to asking

$$(a_0 - v \equiv 0 \bmod 4 \text{ or } a_1 - v \equiv 0 \bmod 4) \ \text{ and } \ (a_2 + v \equiv 0 \bmod 4 \text{ or } a_3 + v \equiv 0 \bmod 4)\,.$$

Putting these together we obtain the first condition in (2.4.1). On the other hand, if the coefficient vector (a_0, a_1, a_2, a_3) satisfies one of these conditions then we may set

$C = a_0$ or a_1 to ensure that both Hilbert symbols are 1, giving the backwards direction. Now, if $\alpha \equiv 1 \bmod 2$ we write (2.4.4) and (2.4.5) as

$$2\left(a_0v^{-1}-1\right)\left(a_1v^{-1}-1\right)+\left(a_0^2v^{-2}+a_1^2v^{-2}-2\right) \equiv 0 \bmod 16$$

and

$$2\left(a_2v^{-1}+1\right)\left(a_3v^{-1}+1\right)+\left(a_2^2v^{-2}+a_3^2v^{-2}-2\right) \equiv 0 \bmod 16.$$

Rearranging and collecting terms we obtain

$$v^{-2}(a_0+a_1)(a_0+a_1-2v) \equiv 0 \bmod 16 \tag{2.4.6}$$

and

$$v^{-2}(a_2+a_3)(a_2+a_3+2v) \equiv 0 \bmod 16. \tag{2.4.7}$$

Now suppose $x \in \mathbb{Z}$ is any even integer and $v \in \mathbb{Z}$ any odd integer. Then

$$x(x+2v) \equiv 0 \bmod 16 \tag{2.4.8}$$

has a solution if and only if

$$\frac{x}{2}\left(\frac{x}{2}+v\right) \equiv 0 \bmod 4.$$

Since v is odd, $\frac{x}{2}$ and $\left(\frac{x}{2}+v\right)$ have opposite parity. It follows that (2.4.8) has a solution if and only if $4|\frac{x}{2}$ or $4|\left(\frac{x}{2}+v\right)$. Equivalently, (2.4.8) has a solution if and only if

$$x \equiv 0 \bmod 8 \text{ or } x+2v \equiv 0 \bmod 8.$$

Substituting in $x = a_0+a_1$ and $x = a_2+a_3$ into this tells us that (2.4.6) and (2.4.7) are equivalent to

$$(a_0+a_1) \equiv 0 \bmod 8 \text{ or } (a_0+a_1) \equiv 2v \bmod 8,$$

and

$$(a_2+a_3) \equiv 0 \bmod 8 \text{ or } (a_2+a_3) \equiv -2v \bmod 8.$$

Now $2v \equiv 2 \bmod 8$ or $2v \equiv 6 \bmod 8$ depending on whether $v \equiv 1,5 \bmod 8$ or $v \equiv 3,7 \bmod 8$. Thus we only need to consider $v \equiv 1,3 \bmod 8$. Substituting these cases into the two conditions above we obtain the second condition in (2.4.1). In the other direction, if one of these conditions is met then we may set $C = 2v$, where $v = 1$ in the first four cases and $v = 3$ in the last three cases, to ensure that the Hilbert symbols are both 1. Thus we are done with case (a).

For case (b), we may assume without loss in generality that $2|a_0, a_1$ and $2 \nmid a_2a_3$. This time we substitute $a_0 = 2u_0$, $a_1 = 2u_1$ and $C = 2^\alpha v$, where u_0, u_1, v are units in $\mathbb{Q}_2$, into the formulae for the Hilbert symbols. Then (2.4.3) becomes,

$$\frac{(u_0v^{-1}-1)(u_1v^{-1}-1)}{4}+(1-\alpha)\frac{(u_0^2v^{-2}+u_1^2v^{-2}-2)}{8} \equiv 0 \bmod 2 \tag{2.4.9}$$

and
$$\frac{(a_2v^{-1}+1)(a_3v^{-1}+1)}{4}+\alpha\frac{(a_2^2v^{-2}+a_3^2v^{-2}-2)}{8}\equiv 0 \bmod 2. \tag{2.4.10}$$

Again we consider the cases where α is even and α is odd separately. If α is even then we may simplify (2.4.9) as in the second case of (a). Doing this and then combining it with (2.4.10) we obtain the condition

$$u_0+u_1\equiv 0, 2v \bmod 8 \text{ and } (a_2+v)(a_3+v)\equiv 0 \bmod 8.$$

Since $v = 1, 3, 5$ or $7 \bmod 8$ we split into cases. Doing this, it can be seen that (u_0, u_1, a_2, a_3) must satisfy one of the conditions of (2.4.2) with $(i, j) = (0, 1)$ and $(k, l) = (2, 3)$. For the other direction, if any of these are satisfied then we may choose $v = 1, 3, 5$ or 7 appropriately and set $C = 2v$ so that each Hilbert symbol is 1. If α is odd then we simplify (2.4.10) as in case (a) and combine it with (2.4.9) to obtain the condition

$$a_2+a_3\equiv 0, -2v \bmod 8 \text{ and } (u_0-v)(u_1-v)\equiv 0 \bmod 8.$$

Once more splitting into cases for v we see that (u_0, u_1, a_2, a_3) must satisfy (2.4.2) with $(i, j) = (2, 3)$ and $(k, l) = (0, 1)$. Finally, if either of these 8 conditions are satisfied then we may choose the appropriate $v = 1, 3, 5$ or 7 such that the Hilbert symbols are 1 by choosing $C = v$. This concludes the proof. $\square$

2.5 Simplification

In this section we simplify the functions $N_r(B)$ from Lemma 2.4.2 and express them through quadratic symbols using the Hasse Principle. Let

$$\delta_r = \begin{cases} 1 & \text{if } r = 1, \\ -1 & \text{if } r = 2, \end{cases}$$

and define the quadrics

$$\mathcal{C}_{r,\mathbf{t}} : -\delta_r t_0 t_2 x_0^2 + t_1 t_3 x_1^2 + \delta_r t_1 t_2 x_2^2 - t_0 t_3 x_3^2 = 0.$$

2.5.1 Reduction to square-free and co-prime coefficients

To begin our simplification we remove the square parts of the t_i. Write $t_i = a_i b_i^2$ for a_i square-free integers. Noting that $\mathcal{C}_{r,\mathbf{t}}$ is equivalent to the quadric $\mathcal{C}_{r,\mathbf{a}}$ we have that

$$N_r(B) = \sum_{\substack{\mathbf{b}\in\mathbb{N}^4,\, b_i\leqslant B^{1/2} \\ (2.5.1)}} N_{r,\mathbf{b}}(B),$$

where

$$N_{r,\mathbf{b}}(B) = \sharp \left\{ \mathbf{a} \in \mathbb{N}^4 : \begin{array}{c} -\delta_r a_0 a_2 x_0^2 + a_1 a_3 x_1^2 + \delta_r a_1 a_2 x_2^2 - a_0 a_3 x_3^2 = 0 \text{ has a } \mathbb{Q}\text{-point;} \\ \gcd(a_0 b_0, a_1 b_1) = \gcd(a_2 b_2, a_3 b_3) = 1; \\ a_0 a_1, \frac{1-\delta_r}{2} a_2 a_3 \neq 1, \mu^2(a_i) = 1; \\ \|a_0 b_0^2, a_1 b_1^2\| \cdot \|a_2 b_2^2, a_3 b_3^2\| \leqslant B \end{array} \right\},$$

and

$$\gcd(b_0, b_1) = \gcd(b_2, b_3) = 1. \tag{2.5.1}$$

Next, we want to remove any common factors of the a_i's. Writing $a_0 = s_0 m_{02} m_{03}$, $a_1 = s_1 m_{12} m_{13}$, $a_2 = s_2 m_{02} m_{12}$ and $a_3 = s_3 m_{03} m_{13}$ where

$$m_{02} = \gcd(a_0, a_2);\; m_{03} = \gcd(a_0, a_3);\; m_{12} = \gcd(a_1, a_2);\; m_{13} = \gcd(a_1, a_3),$$

it is clear that $\mu^2(s_0 s_1 s_2 s_3 m_{02} m_{03} m_{12} m_{13}) = 1$, since $\gcd(a_0, a_1) = \gcd(a_2, a_3) = 1$. Next we note that the quadric $\mathcal{C}_{r,\mathbf{a}}$ is equivalent to the quadric

$$\mathcal{C}_{r,\mathbf{s},\mathbf{m}} : -\delta_r s_0 s_2 m_{03} m_{12} x_0^2 + s_1 s_3 m_{03} m_{12} x_1^2 + \delta_r s_1 s_2 m_{02} m_{13} x_2^2 - s_0 s_3 m_{02} m_{13} x_3^2 = 0.$$

We then write

$$N_{r,\mathbf{b}}(B) = \sum_{\substack{\mathbf{m} \in \mathbb{N}^4,\ m_{ij} \leqslant B \\ (2.5.2)}} N_{r,\mathbf{b},\mathbf{m}}(B)$$

where

$$N_{r,\mathbf{b},\mathbf{m}}(B) = \sharp \left\{ \mathbf{s} \in \mathbb{N}^4 : \mathcal{C}_{r,\mathbf{s},\mathbf{m}} \text{ has a } \mathbb{Q}\text{-point; } (2.5.3);\ (2.5.4);\ (2.5.5) \right\},$$

with

$$\begin{cases} \gcd(m_{02}, b_1 b_3) = \gcd(m_{03}, b_1 b_2) = \gcd(m_{12}, b_0 b_3) = \gcd(m_{13}, b_0 b_2) = 1, \\ \mu^2(m_{02} m_{03} m_{12} m_{13}) = 1 \end{cases} \tag{2.5.2}$$

$$\begin{cases} \gcd(s_0, m_{02} m_{03} m_{12} m_{13} b_1) = \gcd(s_1, m_{02} m_{03} m_{12} m_{13} b_0) = 1, \\ \gcd(s_2, m_{02} m_{03} m_{12} m_{13} b_3) = \gcd(s_3, m_{02} m_{03} m_{12} m_{13} b_2) = 1, \\ \mu^2(s_0 s_1 s_2 s_3) = 1, \end{cases} \tag{2.5.3}$$

$$\|s_0 m_{02} m_{03} b_0^2, s_1 m_{12} m_{13} b_1^2\| \cdot \|s_2 m_{02} m_{12} b_2^2, s_3 m_{03} m_{13} b_3^2\| \leqslant B, \tag{2.5.4}$$

$$\begin{cases} s_0 s_1 m_{02} m_{03} m_{12} m_{13} \neq 1, \\ \frac{1-\delta_r}{2} s_2 s_3 m_{02} m_{03} m_{12} m_{13} \neq 1. \end{cases} \tag{2.5.5}$$

We deal with large values of b_i and m_{ij} using Lemma 2.2.1 which shows that $N_{r,\mathbf{b},\mathbf{m}}(B)$ is

$$\ll \sharp \left\{ \mathbf{t} \in \mathbb{N}^4 : \|t_0, t_1\| \cdot \|t_2, t_3\| \leqslant B;\ b_0^2 m_{03} m_{02} | t_0;\ b_1^2 m_{13} m_{12} | t_1;\ b_2^2 m_{02} m_{12} | t_2;\ b_3^2 m_{03} m_{13} | t_3 \right\}$$
$$\ll \frac{B^2 (\log B)}{(b_0 b_1 b_2 b_3 m_{02} m_{03} m_{12} m_{13})^2}.$$

Thus summing over $\mathbf{m}$ and $\mathbf{b}$, where at least one m_{ij} or b_i is greater than $z_0 = (\log B)^A$ for some $A > 0$ we obtain

$$N_r(B) = \sum_{\substack{\mathbf{b}\in\mathbb{N}^4, \mathbf{m}\in\mathbb{N}^4, (2.5.1),(2.5.2) \\ b_i \leqslant z_0, m_{ij}\leqslant z_0}} \sum N_{r,\mathbf{b},\mathbf{m}}(B) + O\left(\frac{B^2(\log B)}{z_0}\right). \tag{2.5.6}$$

We have now reduced the counting problem $N_r(B)$ over arbitrary positive integers to the evaluation of the counting problems $N_{r,\mathbf{b},\mathbf{m}}(B)$, which count over square-free and pairwise co-prime positive integers. Next, we aim to remove factors of 2. Set $\sigma_i = v_2(s_i)$ where v_2 is the 2-adic valuation, and (relabelling s_i to henceforth be the odd part of the s_i above) and define

$$\begin{aligned}\mathcal{C}_{r,\mathbf{s},\mathbf{m},\boldsymbol{\sigma}} : &- \delta_r 2^{\sigma_0+\sigma_2} s_0 s_2 m_{03} m_{12} x_0^2 + 2^{\sigma_1+\sigma_3} s_1 s_3 m_{03} m_{12} x_1^2 \\ &+ \delta_r 2^{\sigma_1+\sigma_2} s_1 s_2 m_{02} m_{13} x_2^2 - 2^{\sigma_0+\sigma_3} s_0 s_3 m_{02} m_{13} x_3^2 = 0.\end{aligned}$$

Then we write

$$N_{r,\mathbf{b},\mathbf{m}}(B) = \sum_{\boldsymbol{\sigma}\in\{0,1\}^4, (2.5.7)} N_{r,\mathbf{b},\mathbf{m},\boldsymbol{\sigma}}(B),$$

where,

$$N_{r,\mathbf{b},\mathbf{m},\boldsymbol{\sigma}}(B) = \sharp\left\{\mathbf{s}\in\mathbb{N}^4_{\text{odd}} : \mathcal{C}_{r,\mathbf{s},\mathbf{m},\boldsymbol{\sigma}} \text{ has a } \mathbb{Q}\text{-point}, \ (2.5.8), \ (2.5.9), (2.5.10)\right\}$$

with

$$\begin{cases}\sigma_0+\sigma_1+\sigma_2+\sigma_3 \leqslant 1, \ \gcd(2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}, m_{02}m_{03}m_{12}m_{13}) = 1 \\ \gcd(2^{\sigma_0}, b_1) = \gcd(2^{\sigma_1}, b_0) = \gcd(2^{\sigma_2}, b_3) = \gcd(2^{\sigma_3}, b_2) = 1,\end{cases} \tag{2.5.7}$$

$$\begin{cases}\gcd(s_0, 2^{\sigma_1} m_{02}m_{03}m_{12}m_{13}b_1) = \gcd(s_1, 2^{\sigma_0} m_{02}m_{03}m_{12}m_{13}b_0) = 1, \\ \gcd(s_2, 2^{\sigma_3} m_{02}m_{03}m_{12}m_{13}b_3) = \gcd(s_3, 2^{\sigma_2} m_{02}m_{03}m_{12}m_{13}b_2) = 1, \\ \mu^2(2s_0s_1s_2s_3) = 1,\end{cases} \tag{2.5.8}$$

$$\|2^{\sigma_0} s_0 m_{02} m_{03} b_0^2, 2^{\sigma_1} s_1 m_{12} m_{13} b_1^2\| \cdot \|2^{\sigma_2} s_2 m_{02} m_{12} b_2^2, 2^{\sigma_3} s_3 m_{03} m_{13} b_3^2\| \leqslant B, \tag{2.5.9}$$

$$\begin{cases}2^{\sigma_0+\sigma_1} s_0 s_1 m_{02}m_{03}m_{12}m_{13} \neq 1, \\ \frac{1-\delta_r}{2} 2^{\sigma_2+\sigma_3} s_2 s_3 m_{02}m_{03}m_{12}m_{13} \neq 1.\end{cases} \tag{2.5.10}$$

We may now express our $N_{r,\mathbf{b},\mathbf{m},\boldsymbol{\sigma}}(B)$ as

$$N_{r,\mathbf{b},\mathbf{m},\boldsymbol{\sigma}}(B) = \sum_{\substack{\mathbf{s}\in\mathbb{N}^4, (2.5.8) \\ (2.5.9), (2.5.10)}} \mu^2(2s_0s_1s_2s_3)\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma}\rangle_r, \tag{2.5.11}$$

where

$$\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma}\rangle_r = \begin{cases}1 \text{ if } \mathcal{C}_{r,\mathbf{s},\mathbf{m},\boldsymbol{\sigma}} \text{ has a } \mathbb{Q}\text{-point}, \\ 0 \text{ otherwise.}\end{cases}$$

2.5.2 Application of the Hasse Principle

We now use the Hasse principle for quadrics to write our indicator function in terms of local conditions. For each prime p we define the functions

$$\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,p} = \begin{cases} 1 \text{ if } \mathcal{C}_{r,\mathbf{s},\mathbf{m},\boldsymbol{\sigma}} \text{ has a } \mathbb{Q}_p\text{-point}, \\ 0 \text{ otherwise.} \end{cases}$$

Using Lemma 2.4.3 we obtain the following:

Lemma 2.5.1. *Let p be an odd prime. Then for any $\mathbf{m} \in \mathbb{N}^4$ satisfying* (2.5.2) *and any $\boldsymbol{\sigma} \in \{0,1\}^4$ satisfying* (2.5.7),

(a) $\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,p} = 1$ *if* $p \nmid s_0 s_1 s_2 s_3 m_{02} m_{03} m_{12} m_{13}$;

(b) If $p | m_{02} m_{03} m_{12} m_{13}$ *then,*

$$\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,p} = \frac{1}{2}\left(1 + \left(\frac{\delta_r 2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3} s_0 s_1 s_2 s_3}{p}\right)\right);$$

(c) If $p | s_0 s_1$ *then,*

$$\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,p} = \frac{1}{2}\left(1 + \left(\frac{-\delta_r 2^{\sigma_2+\sigma_3} s_2 s_3 m_{02} m_{03} m_{12} m_{13}}{p}\right)\right);$$

(d) If $p | s_2 s_3$ *then,*

$$\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,p} = \frac{1}{2}\left(1 + \left(\frac{\delta_r 2^{\sigma_0+\sigma_1} s_0 s_1 m_{02} m_{03} m_{12} m_{13}}{p}\right)\right).$$

Proof. Part (a) is an immediate application of part (a) of Lemma 2.4.3. For part (b) split into two cases: $p | m_{03} m_{12}$ and $p | m_{02} m_{13}$. In the former, p divides the coefficients of x_0^2 and x_1^2 so that Lemma 2.4.3(b) will yield

$$\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,p} = \frac{1}{2}\left(1 + \left(\frac{\delta_r 2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3} s_0 s_1 s_2 s_3}{p}\right)\right)$$

since the relation on the coefficients of our quadrics ensure that the Jacobi symbols from Lemma 2.4.3 are equal, and so the indicator functions simplify to the above. The case where $p | m_{02} m_{13}$ is dealt with in the same way and yields the same result. Now consider part (c). Here $p | s_0 s_1$. Suppose first that $p | s_0$, then p divides the coefficients of x_0^2 and x_3^2. Using Lemma 2.4.3(b), again noting that the Legendre symbols simplify, we obtain:

$$\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,p} = \frac{1}{2}\left(1 + \left(\frac{-\delta_r 2^{\sigma_2+\sigma_3} s_2 s_3 m_{02} m_{03} m_{12} m_{13}}{p}\right)\right).$$

The same result will be obtained if $p | s_1$. We also note that part (d) may be obtained by the same methods, but the negative disappears since if $p | s_2$, p divides the coefficients of x_0^2 and x_2^2 and one is positive while the other is negative. The same happens if $p | s_3$. $\square$

Next we want to collect this information to obtain an expression for the indicator function $\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_r$ by applying the Hasse Principle. Henceforth, let $(n)_{\text{odd}}$ denote the odd part of n. Further, for any fixed vectors $\mathbf{m}, \mathbf{s}, \mathbf{d}, \tilde{\mathbf{d}} \in \mathbb{N}^4$ such that $d_{ij}\tilde{d}_{ij} = (m_{ij})_{\text{odd}}$, and any $\boldsymbol{\sigma} \in \{0,1\}$, we define

$$N_{r,\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma}}(\mathbf{s}, B) = \sum_{\substack{\mathbf{k},\mathbf{l} \in \mathbb{N}^4 \\ k_i l_i = s_i}}\sum \frac{(-1)^{f_r(\mathbf{d},\mathbf{k})}}{\tau\left(k_0 l_0 k_1 l_1 k_2 l_2 k_3 l_3 (m_{02}m_{03}m_{12}m_{13})_{\text{odd}}\right)} \Theta(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{k}, \mathbf{l}),$$

where

$$f_r(\mathbf{d},\mathbf{k}) = \frac{((2-\delta_r)k_0k_1k_2k_3d_{02}d_{03}d_{12}d_{13} - d_{02}d_{03}d_{12}d_{13} + k_0k_1 - k_2k_3 - (1-\delta_r))}{4}$$

and

$$\begin{aligned}\Theta(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{k}, \mathbf{l}, \boldsymbol{\sigma}) = &\left(\frac{2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3} l_0 l_1 l_2 l_3}{d_{02}d_{03}d_{12}d_{13}}\right)\left(\frac{2^{v_2(m_{02}m_{03}m_{12}m_{13})}}{k_0k_1k_2k_3}\right)\\ &\times \left(\frac{2^{\sigma_2+\sigma_3} l_2 l_3 \tilde{d}_{02}\tilde{d}_{03}\tilde{d}_{12}\tilde{d}_{13}}{k_0k_1}\right)\left(\frac{2^{\sigma_0+\sigma_1} l_0 l_1 \tilde{d}_{02}\tilde{d}_{03}\tilde{d}_{12}\tilde{d}_{13}}{k_2k_3}\right).\end{aligned}$$

We now apply the Hasse Principle and use the local conditions for odd primes given in Lemma 2.5.1 and quadratic reciprocity to express our indicator function as a sum over Jacobi symbols.

Lemma 2.5.2. *Fix some* $\mathbf{b} \in \mathbb{N}^4$. *Suppose that* $\mathbf{m} \in \mathbb{N}^4$ *satisfies* (2.5.2), *that* $\boldsymbol{\sigma} \in \{0,1\}^4$ *satisfies* (2.5.7) *and that* $\mathbf{s} \in \mathbb{N}^4$ *satisfies* (2.5.8). *Then we have that* $\mu^2(2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}s_0s_1s_2s_3m_{02}m_{03}m_{12}m_{13}) = 1$ *and*

$$\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_r = \langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,2} \sum_{\substack{\mathbf{d},\tilde{\mathbf{d}} \in \mathbb{N}^4 \\ d_{ij}\tilde{d}_{ij} = (m_{ij})_{odd}}}\sum N_{r,\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma}}(\mathbf{s}, B). \tag{2.5.12}$$

Proof. This follows from Lemma 2.5.1 and the Hasse Principle for quadrics. Indeed, using the Hasse Principle we obtain

$$\begin{aligned}\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_r &= \langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,2} \prod_{\substack{p \mid s_0s_1s_2s_3m_{02}m_{03}m_{12}m_{13} \\ p \neq 2}} \langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,p}\\ &= \langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,2} \prod_{\substack{p \mid m_{02}m_{03}m_{12}m_{13} \\ p \neq 2}} \left(\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,p}\right) \prod_{\substack{p \mid s_0s_1 \\ p \neq 2}} \left(\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,p}\right) \prod_{\substack{p \mid s_2s_3 \\ p \neq 2}} \left(\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,p}\right).\end{aligned}$$

By now applying Lemma 2.5.1, we obtain a factor of $2^{-\omega(s_0s_1s_2s_3(m_{02}m_{03}m_{12}m_{13})_{\text{odd}})}$ where $\omega(n)$ is the number of distinct primes dividing n. Since $s_0s_1s_2s_3(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}$ is square-free by assumption, this factor is exactly $\tau(s_0s_1s_2s_3(m_{02}m_{03}m_{12}m_{13})_{\text{odd}})^{-1}$.

Therefore:

$$\begin{aligned}\langle \mathbf{s},\mathbf{m},\boldsymbol{\sigma}\rangle_r &= \frac{\langle \mathbf{s},\mathbf{m},\boldsymbol{\sigma}\rangle_{r,2}}{\tau(s_0s_1s_2s_3(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}})}\\ &\quad\times \prod_{\substack{p\mid m_{02}m_{03}m_{12}m_{13}\\ p\neq 2}}\left(1+\left(\frac{\delta_r 2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}s_0s_1s_2s_3}{p}\right)\right)\\ &\quad\times \prod_{\substack{p\mid s_0s_1\\ p\neq 2}}\left(1+\left(\frac{-\delta_r 2^{\sigma_2+\sigma_3}s_2s_3m_{02}m_{03}m_{12}m_{13}}{p}\right)\right)\\ &\quad\times \prod_{\substack{p\mid s_2s_3\\ p\neq 2}}\left(1+\left(\frac{\delta_r 2^{\sigma_0+\sigma_1}s_0s_1m_{02}m_{03}m_{12}m_{13}}{p}\right)\right).\end{aligned}$$

Next we multiply out these products:

$$\langle \mathbf{s},\mathbf{m},\boldsymbol{\sigma}\rangle_r = \frac{\langle \mathbf{s},\mathbf{m},\boldsymbol{\sigma}\rangle_{r,2}}{\tau(s_0s_1s_2s_3(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}})}\sum\sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4\\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}}}\left(\sum\sum_{\substack{\mathbf{k},\mathbf{l}\in\mathbb{N}^4\\ k_il_i=s_i}}F(\mathbf{d},\tilde{\mathbf{d}},\mathbf{k},\mathbf{l},\boldsymbol{\sigma})\right),$$

where

$$\begin{aligned}F(\mathbf{d},\tilde{\mathbf{d}},\mathbf{k},\mathbf{l},\boldsymbol{\sigma}) &= \left(\frac{\delta_r 2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}k_0l_0k_1l_1k_2l_2k_3l_3}{d_{02}d_{03}d_{12}d_{13}}\right)\\ &\quad\times\left(\frac{-\delta_r 2^{\sigma_2+\sigma_3}2^{v_2(m_{02}m_{03}m_{12}m_{13})}k_2l_2k_3l_3d_{02}\tilde{d}_{02}d_{03}\tilde{d}_{03}d_{12}\tilde{d}_{12}d_{13}\tilde{d}_{13}}{k_0k_1}\right)\\ &\quad\times\left(\frac{\delta_r 2^{\sigma_0+\sigma_1}2^{v_2(m_{02}m_{03}m_{12}m_{13})}k_0l_0k_1l_1d_{02}\tilde{d}_{02}d_{03}\tilde{d}_{03}d_{12}\tilde{d}_{12}d_{13}\tilde{d}_{13}}{k_2k_3}\right).\end{aligned}$$

We are left to show that $F(\mathbf{d},\tilde{\mathbf{d}},\mathbf{k},\mathbf{l},\boldsymbol{\sigma}) = (-1)^{f_r(\mathbf{d},\mathbf{k})}\Theta(\mathbf{d},\tilde{\mathbf{d}},\mathbf{k},\mathbf{l},\boldsymbol{\sigma})$. This follows by using quadratic reciprocity for Jacobi symbols and the fact that Jacobi symbols are multiplicative in each variable. Indeed using multiplicativity:

$$\begin{aligned}F(\mathbf{d},\tilde{\mathbf{d}},\mathbf{k},\mathbf{l},\boldsymbol{\sigma}) &= \left(\frac{-1}{k_0k_1}\right)\left(\frac{k_0k_1k_2k_3}{d_{02}d_{03}d_{12}d_{13}}\right)\left(\frac{d_{02}d_{03}d_{12}d_{13}}{k_0k_1k_2k_3}\right)\left(\frac{k_2k_3}{k_0k_1}\right)\left(\frac{k_0k_1}{k_2k_3}\right)\\ &\quad\times\left(\frac{\delta_r}{d_{02}d_{03}d_{12}d_{13}k_0k_1k_2k_3}\right)\left(\frac{2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}l_0l_1l_2l_3}{d_{02}d_{03}d_{12}d_{13}}\right)\left(\frac{2^{v_2(m_{02}m_{03}m_{12}m_{13})}}{k_0k_1k_2k_3}\right)\\ &\quad\times\left(\frac{2^{\sigma_2+\sigma_3}l_2l_3\tilde{d}_{02}\tilde{d}_{03}\tilde{d}_{12}\tilde{d}_{13}}{k_0k_1}\right)\left(\frac{2^{\sigma_0+\sigma_1}l_0l_1\tilde{d}_{02}\tilde{d}_{03}\tilde{d}_{12}\tilde{d}_{13}}{k_2k_3}\right)\\ &= \left(\frac{-1}{k_0k_1}\right)\left(\frac{k_0k_1k_2k_3}{d_{02}d_{03}d_{12}d_{13}}\right)\left(\frac{d_{02}d_{03}d_{12}d_{13}}{k_0k_1k_2k_3}\right)\\ &\quad\times\left(\frac{k_2k_3}{k_0k_1}\right)\left(\frac{k_0k_1}{k_2k_3}\right)\left(\frac{\delta_r}{d_{02}d_{03}d_{12}d_{13}k_0k_1k_2k_3}\right)\Theta(\mathbf{d},\tilde{\mathbf{d}},\mathbf{k},\mathbf{l},\boldsymbol{\sigma}).\end{aligned}$$

Finally we apply quadratic reciprocity of Jacobi symbols which states that for odd integers n,m we have

$$\left(\frac{n}{m}\right)\left(\frac{m}{n}\right) = (-1)^{\frac{(n-1)(m-1)}{4}}.$$

We also note that for an odd integer n,

$$\left(\frac{-1}{n}\right) = (-1)^{\frac{(n-1)}{2}}.$$

Applying these to the remaining Jacobi symbols in the expression for $F(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{k}, \mathbf{l}, \boldsymbol{\sigma})$ and collecting the powers of -1 will yield $(-1)^{f_r(\mathbf{d},\mathbf{k})}$ as required. $\square$

Next we deal with the indicator function for 2-adic points. For this we will use the conditions given by Lemma 2.4.4 to split our sum into arithmetic progressions modulo 8. We will require some notation. Recall the set $\mathcal{A}_2$ defined in §2.4.3. We will define twists of this set. Let i, j, k, l be distinct elements of the set $\{0, 1, 2, 3\}$. Then define

$$\mathcal{A}_{i,j,k,l} = \{\mathbf{q} \in ((\mathbb{Z}/8\mathbb{Z})^*)^4 : (q_i, q_j, q_k, q_l) \in \mathcal{A}_2\}.$$

In particular, $\mathcal{A}_{0,1,2,3} = \mathcal{A}_2$. Now for $\mathbf{m} \in \mathbb{N}^4$ satisfying (2.5.2) and $\boldsymbol{\sigma} \in \{0,1\}^4$ satisfying (2.5.7) define the set function

$$\mathcal{A}(\mathbf{m}, \boldsymbol{\sigma}) = \begin{cases} \mathcal{A}_1 \text{ if } 2 \nmid m_{02}m_{03}m_{12}m_{13} \;\&\; \sigma_i = 0 \;\forall\, i \in \{0,1,2,3\}, \\ \mathcal{A}_{0,1,2,3} \text{ if } 2 \mid m_{03}m_{12},\ 2 \nmid m_{02}m_{13} \;\&\; \sigma_i = 0 \;\forall\, i \in \{0,1,2,3\}, \\ \mathcal{A}_{2,3,0,1} \text{ if } 2 \mid m_{02}m_{13},\ 2 \nmid m_{03}m_{12} \;\&\; \sigma_i = 0 \;\forall\, i \in \{0,1,2,3\}, \\ \mathcal{A}_{0,3,1,2} \text{ if } 2 \nmid m_{02}m_{03}m_{12}m_{13} \;\&\; \sigma_0 = 1 \;\&\; \sigma_i = 0 \;\forall\, i \in \{1,2,3\}, \\ \mathcal{A}_{1,2,0,3} \text{ if } 2 \nmid m_{02}m_{03}m_{12}m_{13} \;\&\; \sigma_1 = 1 \;\&\; \sigma_i = 0 \;\forall\, i \in \{0,2,3\}, \\ \mathcal{A}_{0,2,1,3} \text{ if } 2 \nmid m_{02}m_{03}m_{12}m_{13} \;\&\; \sigma_2 = 1 \;\&\; \sigma_i = 0 \;\forall\, i \in \{0,1,3\}, \\ \mathcal{A}_{1,3,0,2} \text{ if } 2 \nmid m_{02}m_{03}m_{12}m_{13} \;\&\; \sigma_3 = 1 \;\&\; \sigma_i = 0 \;\forall\, i \in \{0,1,2\}. \end{cases}$$

Remark 2.5.3. To understand this notation, notice that the vectors $\mathbf{m} \in \mathbb{N}^4$ and $\boldsymbol{\sigma} \in \{0,1\}^4$ indicate which of the coefficients of $\mathcal{C}_{r,\mathbf{s},\mathbf{m},\boldsymbol{\sigma}}$ are even. The value of $\mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$ therefore only orders these coefficients in the way required by Lemma 2.4.4

The following two lemmas regard the solubility in $\mathbb{Q}_2$.

Lemma 2.5.4. *Fix some* $\mathbf{m} = (m_{02}, m_{03}, m_{12}, m_{13}) \in \mathbb{N}^4$ *and* $\boldsymbol{\sigma} = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \in \{0,1\}^4$ *such that the conditions* (2.5.2) *and* (2.5.7) *hold. Then* $\langle \mathbf{s}, \mathbf{m}, \boldsymbol{\sigma} \rangle_{r,2} = 1$ *if and only if*

$$(-\delta_r s_0 s_2 (m_{03}m_{12})_{\text{odd}}, s_1 s_3 (m_{03}m_{12})_{\text{odd}}, \delta_r s_1 s_2 (m_{02}m_{13})_{\text{odd}}, -s_0 s_3 (m_{02}m_{13})_{\text{odd}}) \equiv \mathbf{q} \bmod 8 \tag{2.5.13}$$

for some $\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$.

Proof. By the conditions (2.5.2) and (2.5.7), the set $\mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$ is well defined. Then the result is immediate using Lemma 2.4.4. $\square$

For fixed choices of $\mathbf{b}, \mathbf{m} \in \mathbb{N}^4$ and $\boldsymbol{\sigma} \in \{0,1\}^4$ satisfying (2.5.2) and (2.5.7), any $\mathbf{d}, \tilde{\mathbf{d}} \in \mathbb{N}^4$ such that $d_{ij}\tilde{d}_{ij} = (m_{ij})_{\text{odd}}$ and any $\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$ define

$$N_{r,\mathbf{b},\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma},\mathbf{q}}(B) = \sum_{\substack{\mathbf{k},\mathbf{l} \in \mathbb{N}^4 \\ (2.5.14),\,(2.5.15) \\ (2.5.16),\,(2.5.17)}} \frac{(-1)^{f_r(\mathbf{d},\mathbf{k})} \mu^2(2k_0l_0k_1l_1k_2l_2k_3l_3)}{\tau(k_0l_0k_1l_1k_2l_2k_3l_3)} \Theta(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{k}, \mathbf{l}, \boldsymbol{\sigma})$$

where

$$\begin{cases} \gcd(k_0l_0, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_1) = \gcd(k_1l_1, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_0) = 1, \\ \gcd(k_2l_2, 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}b_3) = \gcd(k_3l_3, 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}b_2) = 1, \end{cases} \tag{2.5.14}$$

$$\|2^{\sigma_0}k_0l_0m_{02}m_{03}b_0^2, 2^{\sigma_1}k_1l_1m_{12}m_{13}b_1^2\| \cdot \|2^{\sigma_2}k_2l_2m_{02}m_{12}b_2^2, 2^{\sigma_3}k_3l_3m_{03}m_{13}b_3^2\| \leqslant B, \tag{2.5.15}$$

$$\begin{cases} 2^{\sigma_0+\sigma_1}k_0l_0k_1l_1m_{02}m_{03}m_{12}m_{13} \neq 1, \\ \frac{1-\delta_r}{2}2^{\sigma_2+\sigma_3}k_2l_2k_3l_3m_{02}m_{03}m_{12}m_{13} \neq 1, \end{cases} \tag{2.5.16}$$

$$\begin{cases} -\delta_r k_0l_0k_2l_2(m_{03}m_{12})_{\mathrm{odd}} \equiv q_0 \bmod 8;\ k_1l_1k_3l_3(m_{03}m_{12})_{\mathrm{odd}} \equiv q_1 \bmod 8; \\ \delta_r k_1l_1k_2l_2(m_{02}m_{13})_{\mathrm{odd}} \equiv q_2 \bmod 8;\ -k_0l_0k_3l_3(m_{02}m_{13})_{\mathrm{odd}}) \equiv q_3 \bmod 8. \end{cases} \tag{2.5.17}$$

Lemma 2.5.5. *For a fixed choice of* $\mathbf{b}$, $\mathbf{m}$ *and* $\boldsymbol{\sigma}$ *satisfying* (2.5.2) *and* (2.5.7) *we have:*

$$N_{r,\mathbf{b},\mathbf{m},\boldsymbol{\sigma}}(B) = \frac{1}{\tau\left((m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}\right)} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4 \\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}}} N_{r,\mathbf{b},\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma},\mathbf{q}}(B).$$

Proof. By applying Lemma 2.5.4 we may write

$$N_{r,\mathbf{b},\mathbf{m},\boldsymbol{\sigma}}(B) = \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum_{\substack{\mathbf{s}\in\mathbb{N}^4_{\mathrm{odd}},(2.5.8) \\ (2.5.9),(2.5.10),(2.5.13)}} \mu^2(2s_0s_1s_2s_3)\langle \mathbf{s},\mathbf{m},\boldsymbol{\sigma}\rangle_r.$$

Then applying Lemma 2.5.2 and swapping the summation of $\mathbf{s}\in\mathbb{N}^4$ and $\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4$ gives,

$$N_{r,\mathbf{b},\mathbf{m},\boldsymbol{\sigma}}(B) = \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4 \\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}}} \sum_{\substack{\mathbf{s}\in\mathbb{N}^4,(2.5.8) \\ (2.5.9),(2.5.10),(2.5.13)}} N_{r,\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma}}(\mathbf{s},B).$$

Now observe that,

$$\sum_{\substack{\mathbf{s}\in\mathbb{N}^4,(2.5.8) \\ (2.5.10),(2.5.9) \\ (2.5.13)}} N_{r,\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma}}(\mathbf{s},B) = \sum_{\substack{\mathbf{s}\in\mathbb{N}^4,(2.5.8) \\ (2.5.10),(2.5.9) \\ (2.5.13)}} \sum_{\substack{\mathbf{k},\mathbf{l}\in\mathbb{N}^4 \\ k_il_i=s_i}} \frac{(-1)^{f_r(\mathbf{d},\mathbf{k})}\Theta(\mathbf{d},\tilde{\mathbf{d}},\mathbf{k},\mathbf{l})}{\tau\left(k_0l_0k_1l_1k_2l_2k_3l_3(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}\right)}.$$

Swapping the order of summation of the s_i with the k_i and l_i changes (2.5.8) to (2.5.14), (2.5.9) to (2.5.15), (2.5.10) to (2.5.16) and (2.5.13) to (2.5.17). Then, using the multiplicativity of τ and condition (2.5.14) we obtain:

$$\sum_{\substack{\mathbf{s}\in\mathbb{N}^4,(2.5.8) \\ (2.5.10),(2.5.9) \\ (2.5.13)}} N_{r,\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma}}(\mathbf{s},B) = \frac{N_{r,\mathbf{b},\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma},\mathbf{q}}(B)}{\tau\left((m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}\right)}$$

which concludes the proof. □

Condition (2.5.17) is still an issue that needs to be considered as it involves products of our variables. To deal with this we note that, for a fixed $\mathbf{m}$ and $\boldsymbol{\sigma}$, this condition is solely determined on the reduction of $\mathbf{k}$ and $\mathbf{l}$ modulo 8. Therefore we will now split these vectors into appropriate arithmetic progressions modulo 8. In doing so we may also remove any even ordered characters such as $(-1)^{f_r(\mathbf{d},\mathbf{k})}$ and any Jacobi symbol involving a power of 2 from the sum over $\mathbf{k}$ and $\mathbf{l}$.

Fix some $\mathbf{b}, \mathbf{m} \in \mathbb{N}^4$ and $\boldsymbol{\sigma} \in \{0,1\}^4$ satisfying (2.5.2) and (2.5.7), some $\mathbf{d}, \tilde{\mathbf{d}} \in \mathbb{N}^4$ such that $d_{ij}\tilde{d}_{ij} = (m_{ij})_{\text{odd}}$ and some $\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$. Then for any $\mathbf{K}, \mathbf{L} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$ define

$$\Theta_{r,1}(\mathbf{d}, \mathbf{K}, \boldsymbol{\sigma}) = (-1)^{f_r(\mathbf{d},\mathbf{K})} \left(\frac{2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}}{d_{02}d_{03}d_{12}d_{13}}\right) \left(\frac{2^{\sigma_2+\sigma_3}}{K_0K_1}\right) \left(\frac{2^{\sigma_0+\sigma_1}}{K_2K_3}\right) \left(\frac{2^{v_2(m_{02}m_{03}m_{12}m_{13})}}{K_0K_1K_2K_3}\right), \tag{2.5.18}$$

$$\Theta_2(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{k}, \mathbf{l}) = \left(\frac{l_0l_1l_2l_3}{d_{02}d_{03}d_{12}d_{13}}\right) \left(\frac{\tilde{d}_{02}\tilde{d}_{03}\tilde{d}_{12}\tilde{d}_{13}}{k_0k_1k_2k_3}\right) \left(\frac{l_0l_1}{k_2k_3}\right) \left(\frac{l_2l_3}{k_0k_1}\right), \tag{2.5.19}$$

$$N_{r,\mathbf{b},\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma},\mathbf{q}}(\mathbf{K}, \mathbf{L}, B) = \sum\sum\sum\sum_{\substack{\mathbf{k},\mathbf{l}\in\mathbb{N}^4 \\ (\mathbf{k},\mathbf{l})\equiv(\mathbf{K},\mathbf{L}) \bmod 8 \\ (2.5.14),\ (2.5.15),\ (2.5.16)}} \frac{\mu^2(2k_0l_0k_1l_1k_2l_2k_3l_3)}{\tau(k_0l_0k_1l_1k_2l_2k_3l_3)} \Theta_2(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{k}, \mathbf{l}), \tag{2.5.20}$$

Then we may split $N_{r,\mathbf{b},\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma},\mathbf{q}}(B)$ into arithmetic progressions modulo 8. Doing so will allow certain conditions to be separated from our main sums since they only depend on such conditions.

Lemma 2.5.6. *For a fixed choice of* $\mathbf{b}$*,* $\mathbf{m}$ *and* $\boldsymbol{\sigma}$ *satisfying* (2.5.2) *and* (2.5.7) *we have:*

$$N_{r,\mathbf{b},\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma},\mathbf{q}}(B) = \sum\sum\sum\sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.5.17)}} \Theta_{r,1}(\mathbf{d}, \mathbf{K}, \boldsymbol{\sigma}) N_{r,\mathbf{b},\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma},\mathbf{q}}(\mathbf{K}, \mathbf{L}, B).$$

Proof. Using the multiplicity of Jacobi symbols we may write

$$(-1)^{f_r(\mathbf{d},\mathbf{K})}\Theta(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{k}, \mathbf{l}, \boldsymbol{\sigma}) = \Theta_{r,1}(\mathbf{d}, \mathbf{k}, \boldsymbol{\sigma})\Theta_2(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{k}, \mathbf{l}).$$

Then by splitting the inner sum of $N_{r,\mathbf{b},\mathbf{m},\boldsymbol{\sigma}}(B)$ into arithmetic progressions modulo 8 we obtain.

$$N_{r,\mathbf{b},\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma},\mathbf{q}}(B) = \sum\sum_{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4}} \sum\sum_{\substack{\mathbf{k},\mathbf{l}\in\mathbb{N}^4 \\ (\mathbf{k},\mathbf{l})\equiv(\mathbf{K},\mathbf{L}) \bmod 8 \\ (2.5.14),\ (2.5.15) \\ (2.5.16),(2.5.17)}} \frac{\mu^2(2k_0l_0k_1l_1k_2l_2k_3l_3)\Theta_{r,1}(\mathbf{d}, \mathbf{k}, \boldsymbol{\sigma})\Theta_2(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{k}, \mathbf{l})}{\tau(k_0l_0k_1l_1k_2l_2k_3l_3)}.$$

Now notice that, by (2.5.7), $\Theta_{r,1}$ will either be $(-1)^{f_1(\mathbf{d},\mathbf{k})}$ or $(-1)^{f_1(\mathbf{d},\mathbf{k})}$ multiplied by a Jacobi symbol of the form $\left(\frac{2}{\cdot}\right)$. For a fixed $\mathbf{d}$, both are determined completely by the the reduction modulo 8 of the $\mathbf{k}$ and $\mathbf{l}$. Thus it is enough to assert that these congruence classes satisfy (2.5.17) and bring out $\Theta_{r,1}$ from the inner sum by replacing $\mathbf{k}$ and $\mathbf{l}$ with $\mathbf{K}$ and $\mathbf{L}$ in them respectively. □

2.5.3 Isolating main terms and error terms

In this section we describe our strategy for the remainder of the proof of Theorem 1.2.1. We aim to use the character sum methods developed in §5.2 and §5.3 to handle the sums $N_{r,\mathbf{b},\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma},\mathbf{q}}(\mathbf{K},\mathbf{L},B)$. To do so we must first ensure that the size of any characters are of order $(\log B)^C$ for some $C>0$. To ensure this, we split the sum over $\mathbf{k}$ and $\mathbf{l}$ into smaller regions and manipulate the expression into a form considered in §5.1, §5.2 or §5.3. Let $z_1=(\log B)^{150A}$ for $A>0$ as given before (see the definition of z_0 before 2.5.6). The regions we will use are the following:

$$\mathcal{H}_1=\{(\mathbf{k},\mathbf{l})\in\mathbb{N}^8:\|k_0,k_1\|\leqslant z_1,\ \|k_2,k_3\|\leqslant z_1,\ \|l_0,l_1\|\leqslant z_1,\ \|l_2,l_3\|\leqslant z_1\},\quad(2.5.21)$$

$$\mathcal{H}_2=\{(\mathbf{k},\mathbf{l})\in\mathbb{N}^8:\|k_0,k_1\|\leqslant z_1,\ \|k_2,k_3\|\leqslant z_1,\ \|l_0,l_1\|> z_1,\ \|l_2,l_3\|> z_1\},\quad(2.5.22)$$

$$\mathcal{H}_3=\{(\mathbf{k},\mathbf{l})\in\mathbb{N}^8:\|k_0,k_1\|> z_1,\ \|k_2,k_3\|> z_1,\ \|l_0,l_1\|\leqslant z_1,\ \|l_2,l_3\|\leqslant z_1\},\quad(2.5.23)$$

$$\mathcal{H}_4=\{(\mathbf{k},\mathbf{l})\in\mathbb{N}^8:\|k_2,k_3\|\leqslant z_1,\ \|l_2,l_3\|\leqslant z_1\},\quad(2.5.24)$$

$$\mathcal{H}_5=\{(\mathbf{k},\mathbf{l})\in\mathbb{N}^8:\|k_0,k_1\|\leqslant z_1,\ \|l_0,l_1\|\leqslant z_1\},\quad(2.5.25)$$

$$\mathcal{H}_6=\left\{(\mathbf{k},\mathbf{l})\in\mathbb{N}^8:\ \begin{array}{c}(\|k_0,k_1\|>z_1\ \&\ \|l_2,l_3\|>z_1)\text{ or}\\(\|k_2,k_3\|>z_1\ \&\ \|l_0,l_1\|>z_1)\end{array}\right\}.\quad(2.5.26)$$

These regions cover $\mathbb{N}^8$ and the only intersections are between $\mathcal{H}_1$, $\mathcal{H}_4$ and $\mathcal{H}_5$ whose pairwise intersections are just $\mathcal{H}_1$. The following describes the contributions from the sum over each of these regions.

- $\mathcal{H}_1$ will trivially contribute an error term.
- $\mathcal{H}_2$ and $\mathcal{H}_3$ will have an oscillating part which will be shown to contribute an error term of order $O\left(\frac{B^2}{(\log B)(\log\log B)^A}\right)$ for any $A>0$ by use of Selberg–Delange methods and the neutraliser large sieve. There will also be a non-oscillating part which will contribute the main term of Theorem 1.2.1.
- $\mathcal{H}_4$ and $\mathcal{H}_5$ both contribute an error term of order $O\left(\frac{B^2\sqrt{\log\log B}}{\log B}\right)$ by the methods of Selberg–Delange and a result on averages of the L-functions $L(1,\chi)$ as χ ranges over non-principle quadratic characters. It is for the contribution from these regions that the non-square conditions of Theorem 1.2.1 play a crucial role as they force certain characters modulo 8 to be non-trivial. If this were not the case the sums over these regions would have a non-oscillating part which would contribute a term of order B^2 as in the work of Browning–Lyczak–Sarapin, [8].
- $\mathcal{H}_6$ will contribute an error term of size $O\left(\frac{B^2}{(\log B)^A}\right)$ for any $A>0$. The tools are large sieve inequalitites of Friedlander–Iwaniec, Lemma 2.2.4, and the bound for sums of Jacobi symbol over hyperbolic regions, Lemma 2.2.5.

We now express each contribution separately and bring in the other variables to express the overall main terms and error terms as sums which are malleable to the methods of §5.1, §5.2 and §5.3. Suppressing the dependence on $\mathbf{b},\boldsymbol{\sigma}$, and $\mathbf{q}$, we define

$$H_{r,i}(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L},B)=\sum_{\substack{(\mathbf{k},\mathbf{l})\in\mathcal{H}_i\\(\mathbf{k},\mathbf{l})\equiv(\mathbf{K},\mathbf{L})\bmod 8\\(2.5.14),\,(2.5.15),\,(2.5.16)}}\sum\frac{\mu^2(2k_0l_0k_1l_1k_2l_2k_3l_3)}{\tau\left(k_0l_0k_1l_1k_2l_2k_3l_3\right)}\Theta_2(\mathbf{d},\tilde{\mathbf{d}},\mathbf{k},\mathbf{l}).\quad(2.5.27)$$

Then, it is clear that

$$N_{r,\mathbf{b},\mathbf{m},\mathbf{d},\tilde{\mathbf{d}},\boldsymbol{\sigma},\mathbf{q}}(\mathbf{K},\mathbf{L},B) = \sum_{i=1}^{6} H_{r,i}(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L},B) - 2H_{r,1}(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L}). \tag{2.5.28}$$

for each $r = 1,2$. The main term will be obtained from the sums $i = 2,3$ in the cases where either $\tilde{\mathbf{d}}$ or $\mathbf{d}$ is $(1,1,1,1)$ respectively. Bringing in the other variables, our overall main terms may therefore be expressed as

$$\mathcal{M}_{r,2}(B,\mathbf{b}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^4 \\ m_{ij}\leqslant z_0 \\ (2.5.2)}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4 \\ (2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum_{\substack{\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.5.29)}} \frac{H_{r,2}(\mathbf{1},\mathbf{m}_{\mathrm{odd}},\mathbf{1},\mathbf{L},B)}{\tau\left((m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}\right)}$$

and

$$\mathcal{M}_{r,3}(B,\mathbf{b}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^4 \\ m_{ij}\leqslant z_0 \\ (2.5.2)}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4 \\ (2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum_{\substack{\mathbf{K}\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.5.29)}} \frac{\Theta_{r,1}(\mathbf{m}_{\mathrm{odd}},\mathbf{K},\boldsymbol{\sigma})H_{r,3}(\mathbf{m}_{\mathrm{odd}},\mathbf{1},\mathbf{K},\mathbf{1},B)}{\tau\left((m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}\right)}$$

where $\mathbf{m}_{\mathrm{odd}} = ((m_{02})_{\mathrm{odd}},(m_{03})_{\mathrm{odd}},(m_{12})_{\mathrm{odd}},(m_{13})_{\mathrm{odd}})$, $\mathbf{1} = (1,1,1,1)$ and

$$\begin{cases} L_0L_2(m_{03}m_{12})_{\mathrm{odd}} \equiv -\delta_r q_0 \bmod 8, \ L_1L_3(m_{03}m_{12})_{\mathrm{odd}} \equiv q_1 \bmod 8, \\ L_1L_2(m_{02}m_{13})_{\mathrm{odd}} \equiv \delta_r q_2 \bmod 8, \ L_0L_3(m_{02}m_{13})_{\mathrm{odd}} \equiv -q_3 \bmod 8. \end{cases} \tag{2.5.29}$$

The even characters from $\Theta_{r,1}$ are absent in the sums $\mathcal{M}_{r,2}(B,\mathbf{b})$ as this corresponds to the terms where $\mathbf{d} = \mathbf{1}$ and $\mathbf{K} \equiv \mathbf{1} \bmod 8$, in which case this function is trivially always 1. Notice that due to the height conditions $\mathcal{H}_2$ and $\mathcal{H}_3$ the non-square condition (2.5.16) implicitly holds and may therefore be ignored. This is not the case in $\mathcal{H}_4$ and $\mathcal{H}_5$. In these regions, we use (2.5.16) to force $\Theta_{r,1}$ to be a non-principal Dirichlet character modulo 8. By re-ordering the sums over $\mathcal{H}_4$ and $\mathcal{H}_5$, oscillation of this non-principal even character will result in an error term. We will call the contribution of the sums where this method is necessary "vanishing main terms". These are given by

$$\mathcal{V}_{r,4}(B,\mathbf{b}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^4,(2.5.2) \\ (m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}=1}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4 \\ (2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum\sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.5.30)}} \Theta_{r,1}(\mathbf{1},\mathbf{K},\boldsymbol{\sigma})H_{r,4}(\mathbf{1},\mathbf{1},\mathbf{K},\mathbf{L},B)$$

and

$$\mathcal{V}_{r,5}(B,\mathbf{b}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^4,(2.5.2) \\ (m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}=1}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4 \\ (2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum\sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.5.31)}} \Theta_{r,1}(\mathbf{1},\mathbf{K},\boldsymbol{\sigma})H_{r,5}(\mathbf{1},\mathbf{1},\mathbf{K},\mathbf{L},B)$$

where

$$\begin{cases} K_0L_0 \equiv -\delta_r q_0 \bmod 8, \ K_1L_1 \equiv q_1 \bmod 8, \\ K_1L_1 \equiv \delta_r q_2 \bmod 8, \ K_0L_0 \equiv -q_3 \bmod 8, \\ K_2, L_2, K_3, L_3 \equiv 1 \bmod 8, \end{cases} \tag{2.5.30}$$

and

$$\begin{cases} K_2L_2 \equiv -\delta_r q_0 \bmod 8, \ K_3L_3 \equiv q_1 \bmod 8, \\ K_2L_2 \equiv \delta_r q_2 \bmod 8, \ K_3L_3 \equiv -q_3 \bmod 8, \\ K_0, L_0, K_1, L_1 \equiv 1 \bmod 8. \end{cases} \tag{2.5.31}$$

The remaining terms contribute to the error term. For convenience, we split these errors into similar sections. Define $\sum^{\flat}{}_{\mathbf{m},\boldsymbol{\sigma},\mathbf{q},\mathbf{K},\mathbf{L}}$ as the sum over the conditions $\mathbf{m} \in \mathbb{N}^4, m_{ij} \leqslant z_0, (2.5.2); \boldsymbol{\sigma} \in \{0,1\}^4, (2.5.7); \mathbf{q} \in \mathcal{A}(\mathbf{m},\boldsymbol{\sigma}); \mathbf{K},\mathbf{L} \in (\mathbb{Z}/8\mathbb{Z})^{*4}, (2.5.17)$. Define also,

$$T_{r,i}(\mathbf{d},\tilde{\mathbf{d}}) = \frac{\Theta_{r,1}(\mathbf{d},\mathbf{K},\boldsymbol{\sigma})H_{r,i}(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L},B)}{\tau((m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}})}. \tag{2.5.32}$$

Then the remaining error terms are

$$\mathcal{E}_{r,1}(B,\mathbf{b}) = \sideset{}{^\flat}\sum_{\substack{\mathbf{m},\boldsymbol{\sigma},\mathbf{q}\\ \mathbf{K},\mathbf{L}}} \sum\sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4\\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}}} T_{r,1}(\mathbf{d},\tilde{\mathbf{d}}),\ \mathcal{E}_{r,2}(B,\mathbf{b}) = \sideset{}{^\flat}\sum_{\substack{\mathbf{m},\boldsymbol{\sigma},\mathbf{q}\\ \mathbf{K},\mathbf{L}}} \sum\sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4\\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}\\ \mathbf{K}\equiv\mathbf{1} \bmod 8\Rightarrow\mathbf{d}\neq\mathbf{1}}} T_{r,2}(\mathbf{d},\tilde{\mathbf{d}}), \tag{2.5.33}$$

$$\mathcal{E}_{r,3}(B,\mathbf{b}) = \sideset{}{^\flat}\sum_{\substack{\mathbf{m},\boldsymbol{\sigma},\mathbf{q}\\ \mathbf{K},\mathbf{L}}} \sum\sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4\\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}\\ \mathbf{L}\equiv\mathbf{1} \bmod 8\Rightarrow\tilde{\mathbf{d}}\neq\mathbf{1}}} T_{r,3}(\mathbf{d},\tilde{\mathbf{d}}),\ \mathcal{E}_{r,4}(B,\mathbf{b}) = \sideset{}{^\flat}\sum_{\substack{\mathbf{m},\boldsymbol{\sigma},\mathbf{q}\\ \mathbf{K},\mathbf{L}}} \sum\sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4\\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}\\ \mathbf{d}=\tilde{\mathbf{d}}=\mathbf{1}\Rightarrow \text{ at least one of}\\ K_2,L_2,K_3,L_3\not\equiv 1 \bmod 8}} T_{r,4}(\mathbf{d},\tilde{\mathbf{d}}), \tag{2.5.34}$$

$$\mathcal{E}_{r,5}(B,\mathbf{b}) = \sideset{}{^\flat}\sum_{\substack{\mathbf{m},\boldsymbol{\sigma},\mathbf{q}\\ \mathbf{K},\mathbf{L}}} \sum\sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4\\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}\\ \mathbf{d}=\tilde{\mathbf{d}}=\mathbf{1}\Rightarrow \text{ at least one of}\\ K_0,L_0,K_1,L_1\not\equiv 1 \bmod 8}} T_{r,5}(\mathbf{d},\tilde{\mathbf{d}}),\ \mathcal{E}_{r,6}(B,\mathbf{b}) = \sideset{}{^\flat}\sum_{\substack{\mathbf{m},\boldsymbol{\sigma},\mathbf{q}\\ \mathbf{K},\mathbf{L}}} \sum\sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4\\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}}} T_{r,6}(\mathbf{d},\tilde{\mathbf{d}}) \tag{2.5.35}$$

We summarise this section by bringing these contributions together and expressing each $N_r(B)$ in a concise manner. Define

$$N_{r,i}(B) = \sum_{\substack{\mathbf{b}\in\mathbb{N}^4\\ b_i\leqslant z_0\\ (2.5.1)}} (\mathcal{M}_{r,i}(B,\mathbf{b}) + \mathcal{E}_{r,i}(B,\mathbf{b})) \tag{2.5.36}$$

for $i = 2, 3$,

$$N_{r,i}(B) = \sum_{\substack{\mathbf{b}\in\mathbb{N}^4\\ b_i\leqslant z_0\\ (2.5.1)}} (\mathcal{V}_{r,i}(B,\mathbf{b}) + \mathcal{E}_{r,i}(B,\mathbf{b})) \tag{2.5.37}$$

for $i = 4, 5$ and

$$N_{r,i}(B) = \sum_{\substack{\mathbf{b}\in\mathbb{N}^4\\ b_i\leqslant z_0\\ (2.5.1)}} \mathcal{E}_{r,i}(B,\mathbf{b}) \tag{2.5.38}$$

for $i = 1, 6$. The following therefore follows from summing (2.5.28) over the remaining variables:

Proposition 2.5.7. *For $B \geqslant 3$,*

$$N_r(B) = \sum_{i=1}^{6} N_{r,i}(B) - 2N_{r,1}(B) + O\left(\frac{B^2(\log B)}{z_0}\right).$$

Moving forward we will suppress some notation by writing:

$$M_0 = 2^{\sigma_0}m_{02}m_{03}b_0^2,\ M_1 = 2^{\sigma_1}m_{12}m_{13}b_1^2,\ M_2 = 2^{\sigma_2}m_{02}m_{12}b_2^2,\ M_3 = 2^{\sigma_3}m_{03}m_{13}b_3^2.$$

2.6 Large Conductor Error Terms

In this section we bound the $\mathcal{E}_{r,6}(B, \mathbf{b})$ using Lemma 2.2.5. We begin by bounding the $H_{r,6}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B)$:

Lemma 2.6.1. *Fix some* $\mathbf{b}, \mathbf{m} \in \mathbb{N}^4$ *and* $\boldsymbol{\sigma} \in \{0,1\}^4$ *satisfying* $\mu^2(m_{02}m_{03}m_{12}m_{13}) = 1$, (2.5.2) *and* (2.5.7), *some* $\mathbf{d}, \tilde{\mathbf{d}} \in \mathbb{N}^4$ *such that* $d_{ij}\tilde{d}_{ij} = (m_{ij})_{odd}$. *Fix also some* $\mathbf{K}, \mathbf{L} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$ *satisfying* (2.5.17). *Then for* $r = 1, 2$ *we have*

$$H_{r,6}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B) \ll \frac{B^2(\log B)^6}{M_0M_1M_2M_3z_1^{1/2}}.$$

Proof. First we recall the height conditions for these expressions. The first is that given in (2.5.26) and the second is the hyperbolic height condition (2.5.15). To handle (2.5.15) we will write

$$\mathbb{1}(\|k_0l_0M_0, k_1l_1M_1\| \cdot \|k_2l_2M_2, k_3l_3M_3\| \leqslant B) = \prod_{(u,v)\in\{0,1\}\times\{2,3\}} \mathbb{1}(k_ul_uM_uk_vl_vM_v \leqslant B). \tag{2.6.1}$$

Starting from height conditions from (2.5.26), we will partition the space even further. First suppose that $\|k_0, k_1\| > z_1$ and $\|l_2, l_3\| > z_1$. Then we have 4 cases:

$\mathcal{R}_1$: $(k_1 > z_1, \quad \text{and} \quad l_3 > z_1)$;

$\mathcal{R}_2$: $(k_0 > z_1, \ k_1 \leqslant z_1 \quad \text{and} \quad l_3 > z_1)$;

$\mathcal{R}_3$: $(k_1 > z_1 \quad \text{and} \quad l_2 > z_1, \ l_3 \leqslant z_1)$;

$\mathcal{R}_4$: $(k_0 > z_1, \ k_1 \leqslant z_1 \quad \text{and} \quad l_2 > z_1, \ l_3 \leqslant z_1)$.

We also have regions where $\|k_0, k_1\| \leqslant z_1$ or $\|l_2, l_3\| \leqslant z_1$ but $\|k_2, k_3\| > z_1$ and $\|l_0, l_1\| > z_1$. This gives 4 more regions:

$\mathcal{R}_5$: $(\|k_0, k_1\| \leqslant z_1$ or $\|l_2, l_3\| \leqslant z_1)$ and $(k_3 > z_1$ and $l_1 > z_1)$;

$\mathcal{R}_6$: $(\|k_0, k_1\| \leqslant z_1$ or $\|l_2, l_3\| \leqslant z_1)$ and $(k_2 > z_1, \ k_3 \leqslant z_1$ and $l_1 > z_1)$;

$\mathcal{R}_7$: $(\|k_0, k_1\| \leqslant z_1$ or $\|l_2, l_3\| \leqslant z_1)$ and $(k_3 > z_1$ and $l_0 > z_1, \ l_1 \leqslant z_1)$;

$\mathcal{R}_8$: $(\|k_0, k_1\| \leqslant z_1$ or $\|l_2, l_3\| \leqslant z_1)$ and $(k_2 > z_1, \ k_3 \leqslant z_1$ and $l_0 > z_1, \ l_1 \leqslant z_1)$.

Then we define

$$S_h(B) = \sum\sum\sum\sum_{\substack{(\mathbf{k},\mathbf{l})\in\mathcal{R}_{1a} \\ (\mathbf{k},\mathbf{l})\equiv(\mathbf{K},\mathbf{L}) \bmod 8 \\ (2.5.14),\,(2.5.16)}} \frac{\mu^2(2k_0l_0k_1l_1k_2l_2k_3l_3)}{\tau(k_0l_0k_1l_1k_2l_2k_3l_3)}\Theta_2(\mathbf{d},\tilde{\mathbf{d}},\mathbf{k},\mathbf{l})$$

for $1 \leqslant h \leqslant 8$. As our final notational manipulation of the section, we use the multiplicativity of the Jacobi symbol to write $\Theta_2(\mathbf{d},\tilde{\mathbf{d}},\mathbf{k},\mathbf{l})$ as

$$\Theta_2(\mathbf{d},\tilde{\mathbf{d}},\mathbf{k},\mathbf{l}) = \left(\prod_{i=0}^{3}\theta_1(\mathbf{d},l_i)\theta_2(\tilde{\mathbf{d}},k_i)\right)\left(\frac{l_0}{k_2}\right)\left(\frac{l_0}{k_3}\right)\left(\frac{l_1}{k_2}\right)\left(\frac{l_1}{k_3}\right)\left(\frac{l_2}{k_0}\right)\left(\frac{l_2}{k_1}\right)\left(\frac{l_3}{k_0}\right)\left(\frac{l_3}{k_1}\right),$$

where

$$\theta_1(\mathbf{d},l_i) = \left(\frac{l_i}{d_{02}d_{03}d_{12}d_{13}}\right) \quad\text{and}\quad \theta_2(\tilde{\mathbf{d}},k_i) = \left(\frac{\tilde{d}_{02}\tilde{d}_{03}\tilde{d}_{12}\tilde{d}_{13}}{k_i}\right).$$

Then, using (2.6.1), $\mu^2(2k_0l_0k_1l_1k_2l_2k_3l_3) = 1$ and the multiplicativity of τ we may arrange the order of summation of each $S_h(B)$ for $1 \leqslant h \leqslant 8$ and apply the triangle inequality to find that all of these satisfy an upper bound of the form

$$S_h(B) \ll \sum\sum\sum\sum_{\substack{k_ul_uM_u\leqslant B \\ k_vl_vM_v\leqslant B \\ k_ul_uk_vl_vM_uM_v\leqslant B}} \frac{1}{\tau(k_ul_u)\tau(k_vl_v)}\sum\sum_{\substack{l_i\leqslant B \\ k_j\leqslant B}}\left|\sum\sum_{\substack{z_1<k_i\leqslant B/(l_ik_jM_iM_j) \\ z_1<l_j\leqslant B/(l_ik_jM_iM_j) \\ k_il_j\leqslant B/(l_ik_jM_iM_j) \\ (2.5.14),(2.5.16)}} a_{k_i}b_{l_j}\left(\frac{l_j}{k_i}\right)\right|,$$

where the indices $i,j,u,v \in \{0,1,2,3\}$ are all distinct and depend uniquely on $1 \leqslant h \leqslant 8$ and where $|a_{k_i}|,|b_{l_j}| \leqslant 1$ are complex sequences depending independently on k_j and l_j respectively. These sequences contain the gcd conditions from the μ^2 factor, the congruence conditions on the variables k_i and l_j, $\frac{1}{\tau}$ factors, superfluous characters containing k_i or l_j and superfluous height conditions (in the form of an indicator function).

The innermost sums here are exactly of the form considered in Lemma 2.2.5. We therefore use this lemma to obtain:

$$\begin{aligned}
S_h(B) &\ll \sum\sum\sum\sum_{\substack{k_ul_uM_u\leqslant B \\ k_vl_vM_v\leqslant B \\ k_ul_uk_vl_vM_uM_v\leqslant B}} \frac{1}{\tau(k_ul_u)\tau(k_vl_v)}\sum\sum_{l_i,k_j\leqslant B}\frac{B(\log B)^3}{l_ik_jM_iM_jz_1^{1/2}} \\
&\ll \sum\sum\sum\sum_{\substack{k_ul_uM_u\leqslant B \\ k_vl_vM_v\leqslant B \\ k_ul_uk_vl_vM_uM_v\leqslant B}} \frac{1}{\tau(k_ul_u)\tau(k_vl_v)}\frac{B(\log B)^5}{M_iM_jz_1^{1/2}} \\
&\ll \frac{B(\log B)^5}{M_iM_jz_1^{1/2}}\sum\sum_{nm\leqslant B/(M_uM_v)} 1 \ll \frac{B^2(\log B)^6}{M_0M_1M_2M_3z_1^{1/2}}
\end{aligned}$$

for each $1 \leqslant h \leqslant 8$. □

Proposition 2.6.2. *Let $B \geqslant 3$. Then for $r = 1,2$,*

$$N_{r,6}(B) \ll \frac{B^2(\log B)^6}{z_1^{1/2}}.$$

Proof. Using the previous lemma we have

$$\mathcal{E}_{r,6}(B,\mathbf{b}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^4\\ m_{ij}\leqslant z_0\\ (2.5.2)}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\ (2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum\sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4}\\ (2.5.17)}} \sum\sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4\\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\text{odd}}}} \frac{\Theta_{r,1}(\mathbf{d},\mathbf{K},\boldsymbol{\sigma})H_{r,6}(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L},B)}{\tau\left((m_{02}m_{03}m_{12}m_{13})_{\text{odd}}\right)}$$

$$\ll \sum_{\substack{\mathbf{m}\in\mathbb{N}^4\\ m_{ij}\leqslant z_0\\ (2.5.2)}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\ (2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum\sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4}\\ (2.5.17)}} \frac{B^2(\log B)^6}{m_{02}^2m_{03}^2m_{12}^2m_{13}^2b_0^2b_1^2b_2^2b_3^2z_1^{1/2}}.$$

Summing this over $\mathbf{b}$ gives

$$N_{r,6}(B) \ll \sum_{\substack{\mathbf{b}\in\mathbb{N}^4\\ b_i\leqslant z_0}} \sum_{\substack{\mathbf{m}\in\mathbb{N}^4\\ m_{ij}\leqslant z_0\\ (2.5.2)}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\ (2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum\sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4}\\ (2.5.17)}} \frac{B^2(\log B)^6}{m_{02}^2m_{03}^2m_{12}^2m_{13}^2b_0^2b_1^2b_2^2b_3^2z_1^{1/2}}.$$

The result follows since there are only finitely many $\boldsymbol{\sigma},\mathbf{q},\mathbf{K}$ and $\mathbf{L}$ to consider and the sums over $\mathbf{m}$ and $\mathbf{b}$ converge. □

2.7 Small Conductor Error Term

In this section we will bound the error terms $\mathcal{E}_{r,j}(B,\mathbf{b})$ for $1\leqslant j\leqslant 5$ using the bounds from sections 5.2 and 5.3.

2.7.1 The Error Terms $\mathcal{E}_{r,1}(B,\mathbf{b})$

Here it is enough to use a trivial bound, since the variables in this region are all bounded by a power of $\log B$. We obtain:

Proposition 2.7.1. *Let $B\geqslant 3$. Then for $r=1,2$ we have*

$$N_{r,1}(B) \ll (\log B)^{1208A}.$$

Proof. Recall $\mathcal{E}_{r,1}(B,\mathbf{b})$ from (2.5.33). Now, by summing trivially over the height conditions in the region $\mathcal{H}_1$, we have

$$|T_{r,1}(\mathbf{d},\tilde{\mathbf{d}})| = \frac{H_{r,1}(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L},B)}{\tau((m_{02}m_{03}m_{12}m_{13})_{\text{odd}})} \ll z_1^8 = \frac{(\log B)^{1200A}}{\tau((m_{02}m_{03}m_{12}m_{13})_{\text{odd}})}.$$

Then,

$$\sum\sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4\\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\text{odd}}}} |T_{r,1}(\mathbf{d},\tilde{\mathbf{d}})| \ll \frac{\tau((m_{02}m_{03}m_{12}m_{13})_{\text{odd}})(\log B)^{1200A}}{\tau((m_{02}m_{03}m_{12}m_{13})_{\text{odd}})} = (\log B)^{1200A},$$

where we have implicitly used (2.5.2) to simplify the product of the $\tau(m_{ij})$. Since there are only finitely many $\mathbf{K}, \mathbf{L}, \mathbf{q}$ and $\boldsymbol{\sigma}$ to consider and the sum over $\mathbf{m}$ is trivially bounded by $z_0^4 = (\log B)^{4A}$ we obtain

$$N_{r,1}(B) \ll \sum_{\substack{\mathbf{b}\in\mathbb{N}^4 \\ b_i \leqslant z_0}} \sideset{}{^\flat}\sum_{\substack{\mathbf{m},\boldsymbol{\sigma},\mathbf{q} \\ \mathbf{K},\mathbf{L}}} (\log B)^{1200A} \ll \sum_{\substack{\mathbf{b}\in\mathbb{N}^4 \\ b_i \leqslant z_0}} (\log B)^{1204A} \ll (\log B)^{1208A}.$$

□

2.7.2 The Error Terms $\mathcal{E}_{r,2}(B,\mathbf{b})$ and $\mathcal{E}_{r,3}(B,\mathbf{b})$

These error terms are bounded using the fact that, given the conditions on the variables $\mathbf{m}, \mathbf{K}, \mathbf{L}, \mathbf{d}$ and $\tilde{\mathbf{d}}$, the sums $H_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B)$ and $H_{r,3}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B)$ are of type (b) or (c) from §5.2. We first remark that $\mathcal{E}_{r,2}(B,\mathbf{b})$ and $\mathcal{E}_{r,3}(B,\mathbf{b})$ are symmetrically equivalent, the latter being of the same form as the former with the variables $\tilde{\mathbf{d}}, \mathbf{K}$ and $\mathbf{k}$ switching roles with the variables $\mathbf{d}, \mathbf{L}$ and $\mathbf{l}$. For this reason we will restrict our focus to $\mathcal{E}_{r,2}(B,\mathbf{b})$. Our first aim is to examine the sums $H_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B)$.

Lemma 2.7.2. *Fix some* $\mathbf{b} \in \mathbb{N}^4$*, some* $\mathbf{m} \in \mathbb{N}^4$ *satisfying* (2.5.2)*, some* $\boldsymbol{\sigma} \in \{0,1\}^4$ *satisfying* (2.5.7) *and some* $\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$*. Suppose that* $\mathbf{K}, \mathbf{L} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$ *and* $\mathbf{d}, \tilde{\mathbf{d}} \in \mathbb{N}^4$ *satisfy the conditions*

$$\begin{cases} \mathbf{K}, \mathbf{L} \text{ satisfy } (2.5.17), \\ d_{ij}\tilde{d}_{ij} = (m_{ij})_{odd} \ \forall\, ij \in \{02, 03, 12, 13\}, \\ \mathbf{K} \equiv \mathbf{1} \bmod 8 \Rightarrow \mathbf{d} \neq \mathbf{1}. \end{cases} \tag{2.7.1}$$

Then

$$H_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B) \ll_A \frac{B^2 \mathcal{MAX}_1(B)}{M_0 M_1 M_2 M_3}$$

where $\mathcal{MAX}_1(B)$ *is defined as*

$$\max\left\{ \begin{array}{c} \mathbb{1}(\mathbf{d}=1)\frac{\tau(m_{02}m_{03}m_{12}m_{13})^2\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)}{(\log B)(\log\log B)^{66A}}, \\ \frac{d_{02}^2 d_{03}^2 d_{12}^2 d_{13}^2}{(\log B)^{132A}}, \frac{(\log B)(\log\log B)^4}{(\log B)^{A/3}} \end{array} \right\}.$$

Proof. Recall (2.5.19),(2.5.22) and (2.5.27). We order $H_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B)$ to sum over $\mathbf{l}$ first and write

$$\Theta_2(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{k}, \mathbf{l}) = \chi_{\tilde{\mathbf{d}}}(\mathbf{k}) \widetilde{\chi}_{\mathbf{d},k_2k_3}(l_0l_1) \widetilde{\chi}_{\mathbf{d},k_0k_1}(l_2l_3)$$

where

$$\chi_{\tilde{\mathbf{d}}}(\mathbf{k}) = \left(\frac{\tilde{d}_{02}\tilde{d}_{03}\tilde{d}_{12}\tilde{d}_{13}}{k_0k_1k_2k_3}\right) \text{ and } \widetilde{\chi}_{\mathbf{d},Q}(n) = \left(\frac{n}{d_{02}d_{03}d_{12}d_{13}Q}\right)$$

for any $Q, n \in \mathbb{N}$. Then,

$$H_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B) = \sum\sum\sum\sum_{\substack{\|k_0,k_1\|,\|k_2,k_3\|\leqslant z_1 \\ \mathbf{k}\equiv\mathbf{K} \bmod 8 \\ (2.7.2)}} \frac{\mu^2(k_0k_1k_2k_3)\chi_{\tilde{\mathbf{d}}}(\mathbf{k})}{\tau(k_0k_1k_2k_3)} H_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{k}, B)$$

where

$$H_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{k}, B) = \sum\sum\sum\sum_{\substack{\|l_0,l_1\|,\|l_2,l_3\|>z_1 \\ \mathbf{l}\equiv\mathbf{L} \bmod 8 \\ (2.5.15),(2.7.3)}} \frac{\mu^2(l_0l_1l_2l_3)}{\tau(l_0)\tau(l_1)\tau(l_2)\tau(l_3)} \widetilde{\chi}_{\mathbf{d},k_2k_3}(l_0l_1)\widetilde{\chi}_{\mathbf{d},k_0k_1}(l_2l_3),$$

$$\begin{cases} \gcd(k_0, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_1) = \gcd(k_1, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_0) = 1, \\ \gcd(k_2, 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}b_3) = \gcd(k_3, 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}b_2) = 1, \end{cases} \tag{2.7.2}$$

and

$$\begin{cases} \gcd(l_0, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}k_0k_1b_1) = \gcd(l_1, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}k_0k_1b_0) = 1, \\ \gcd(l_2, 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}k_2k_3b_3) = \gcd(l_3, 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}k_2k_3b_2) = 1. \end{cases} \tag{2.7.3}$$

Notice that these sums are now very similar to those considered in Propositions 2.2.7 and 2.2.8, except for the $\mu^2(l_0l_1l_2l_3)$ term in the inner sum. To deal with this, we apply Lemma 2.2.2 to the inner sums with: $w_0 = w_1 = z_0 = (\log B)^A$, $c_i = k_iM_i$ for $0 \leqslant i \leqslant 3$, and g_i encoding the characters, the $\frac{1}{\tau}$ factors and the gcd conditions (2.7.3). Then have that $H_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{k}, B)$ is equal to

$$\sum_{s\leqslant z_0} \mu(s) \sum\sum\sum\sum_{\substack{\|l_0',l_1'\|,\|l_2',l_3'\|\leqslant z_0 \\ p|l_0'l_1'l_2'l_3' \Rightarrow p|s \\ s^2|l_0'l_1'l_2'l_3',(2.7.4)}} \frac{\widetilde{\chi}_{\mathbf{d},k_2k_3}(l_0'l_1')\widetilde{\chi}_{\mathbf{d},k_0k_1}(l_2'l_3')}{\tau(l_0')\tau(l_1')\tau(l_2')\tau(l_3')} H'_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{k}, \mathbf{l}', B)$$
$$+ O\left(\frac{B^2(\log B)}{k_0k_1k_2k_3M_0M_1M_2M_3M_4z_0^{1/3}}\right)$$

where

$$H'_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{k}, \mathbf{l}', B) = \sum\sum\sum\sum_{\substack{\|l_0'l_0'',l_1'l_1''\|,\|l_2'l_2'',l_3'l_3''\|>z_1 \\ l_i''\equiv L_i/l_i' \bmod 8\ \forall\ 0\leqslant i\leqslant 3 \\ (2.7.5),(2.7.6)}} \frac{1}{\tau(l_0'')\tau(l_1'')\tau(l_2'')\tau(l_3'')} \widetilde{\chi}_{\mathbf{d},k_2k_3}(l_0''l_1'')\widetilde{\chi}_{\mathbf{d},k_0k_1}(l_2''l_3''),$$

$$\begin{cases} \gcd(l_0', 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}k_0k_1b_1) = \gcd(l_1', 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}k_0k_1b_0) = 1, \\ \gcd(l_2', 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}k_2k_3b_3) = \gcd(l_3', 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}k_2k_3b_2) = 1, \end{cases} \tag{2.7.4}$$

$$\begin{cases} \gcd(l_0'', 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}k_0k_1sb_1) = \gcd(l_1'', 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}k_0k_1sb_0) = 1, \\ \gcd(l_2'', 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}k_2k_3sb_3) = \gcd(l_3'', 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}k_2k_3sb_2) = 1, \end{cases} \tag{2.7.5}$$

and

$$\|2^{\sigma_0}k_0l_0'l_0''m_{02}m_{03}b_0^2, 2^{\sigma_1}k_1l_1'l_1''m_{12}m_{13}b_1^2\| \cdot \|2^{\sigma_2}k_2l_2'l_2''m_{02}m_{12}b_2^2, 2^{\sigma_3}k_3l_3'l_3''m_{03}m_{13}b_3^2\| \leqslant B. \tag{2.7.6}$$

Next, we swap the summation order of k_i and l_i'. By also summing the previous error term over k_i, we obtain:

$$H_{r,2}(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L},B) = \sum_{r\leqslant z_0}\mu(r)\sum\sum\sum\sum_{\substack{\|l_0',l_1'\|,\|l_2',l_3'\|\leqslant z_0\\ p|l_0'l_1'l_2'l_3'\Rightarrow p|r\\ r^2|l_0'l_1'l_2'l_3',(2.7.7)}}\frac{\chi_{\mathbf{d}}(\mathbf{l}')}{\tau(l_0')\tau(l_1')\tau(l_2')\tau(l_3')}H_{r,2}'(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L},\mathbf{l}',B)$$
$$+ O_A\left(\frac{B^2(\log B)(\log\log B)^4}{M_0M_1M_2M_3z_0^{1/3}}\right)$$

where $H_{r,2}'(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L},\mathbf{l}',B)$ is defined to be

$$\sum\sum\sum\sum_{\substack{\|k_0,k_1\|,\|k_2,k_3\|\leqslant z_1\\ \mathbf{k}\equiv\mathbf{K} \bmod 8\\ (2.7.8)}}\frac{\mu^2(k_0k_1k_2k_3)\widetilde{\chi}_{\tilde{\mathbf{d}},l_0'l_1'}(k_2k_3)\chi_{\tilde{\mathbf{d}},l_3'l_2'}(k_0k_1)}{\tau(k_0k_1k_2k_3)}H_{r,2}(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L},\mathbf{k},\mathbf{l}',B),$$

$$\begin{cases}\gcd(l_0', 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_1) = \gcd(l_1', 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_0) = 1,\\ \gcd(l_2', 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}b_3) = \gcd(l_3', 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}b_2) = 1,\end{cases} \tag{2.7.7}$$

$$\begin{cases}\gcd(k_0, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}l_0'l_1'b_1) = \gcd(k_1, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}l_0'l_1'b_0) = 1,\\ \gcd(k_2, 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}l_2'l_3'b_3) = \gcd(k_3, 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}l_2'l_3'b_2) = 1,\end{cases} \tag{2.7.8}$$

and for any $Q, n \in \mathbb{N}_{\mathrm{odd}}$, $\mathbf{d}, \mathbf{l} \in (\mathbb{N}\setminus\{0\})^4$ we have set

$$\widetilde{\chi}_{\mathbf{d}}(\mathbf{l}) = \left(\frac{l_0l_1l_2l_3}{d_{02}d_{03}d_{12}d_{13}}\right) \text{ and } \chi_{\mathbf{d},Q}(n) = \left(\frac{d_{02}d_{03}d_{12}d_{13}Q}{n}\right).$$

Now we claim that the sums $H_{r,2}'(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L},\mathbf{l}',B)$ are either of the form considered in Proposition 2.2.7 or of the form considered in Proposition 2.2.8. To do so we compare notation as follows:

- the n_i in §5.2 correspond to the l_i'';
- the m_i in §5.2 correspond to the k_i;
- the d_i in §5.2 correspond to l_i';
- the c_i in §5.2 correspond to the product $l_i'M_i$;
- the Q_i in §5.2 correspond to products of l_i' and d_{ij}, though we note specifically that the product corresponding to Q_0 and Q_2 in Proposition 2.2.7 are independent of the l_i', and that Q_1 in Proposition 2.2.8 is equal to 1 in all applications of this proposition (it is the product of all the d_{ij}). All characters in this application are Jacobi symbols of the corresponding modulus;

- the r_i in §5.2 are $\frac{m_{02}m_{03}m_{12}m_{13}sb_j}{d_{02}d_{03}d_{12}d_{13}}$, with $j = 1, 0, 3, 2$ for $i = 0, 1, 2, 3$ respectively.

Using this dictionary we find that conditions 2.5.2, 2.7.1 and 2.7.8 ensure that at least one of the characters $\chi_{\mathbf{d},k_0k_1}$ or $\chi_{\mathbf{d},k_2k_3}$ is non-trivial, ensuring that at least one of these results can be used. For the cases in which we use Proposition 2.2.8, which are the cases when

$$d_{02}d_{03}d_{12}d_{13} = 1,\ \|k_0, k_1\| = 1,\ \|k_2, k_3\| > 1$$
$$\text{and}$$
$$d_{02}d_{03}d_{12}d_{13} = 1,\ \|k_0, k_1\| > 1,\ \|k_2, k_3\| = 1,$$

we note also that the constants c_i, here given by $l'_i M_i$, are all $\ll (\log B)^{5A}$, the constants d_i are $l'_0, l'_1, l'_2, l'_3 \leqslant (\log B)^A$ and the lower bound in the inner most sum $H_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{k}, \mathbf{l}', B)$ is $(\log B)^{150A}$, meaning that the constants satisfy the desired bounds. Applying these propositions then give:

$$\begin{aligned} H'_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{l}', B) \ll_A & \frac{\mathbb{1}(\mathbf{d} = \mathbf{1})\tau(m_0m_1m_2m_3)^2\tau(s)^2\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2}{l'_0l'_1l'_2l'_3M_0M_1M_2M_3(\log B)(\log\log B)^{132A}} \\ & + \frac{d_{02}^2d_{03}^2d_{12}^2d_{13}^2B^2}{l'_0l'_1l'_2l'_3M_0M_1M_2M_3(\log X)^{66A}}. \end{aligned}$$

Therefore we may deduce

$$H_{r,2}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B) \ll_A \frac{\mathcal{R}B^2\mathcal{M}\mathcal{A}\mathcal{X}_1(B)}{M_0M_1M_2M_3}$$

where

$$\mathcal{R} = \sum_{s \leqslant z_0} \sum\sum\sum\sum_{\substack{\|l'_0, l'_1\|, \|l'_2, l'_3\| \leqslant z_0 \\ p | l'_0l'_1l'_2l'_3 \Rightarrow p | s \\ s^2 | l'_0l'_1l'_2l'_3}} \frac{\tau(s)^2}{l'_0l'_1l'_2l'_3\tau(l'_0)\tau(l'_1)\tau(l'_2)\tau(l'_3)}. \tag{2.7.9}$$

To conclude the proof we show that $\mathcal{R} \ll 1$. We have $\tau(l'_0)\tau(l'_1)\tau(l'_2)\tau(l'_3) \geqslant \tau(l'_0l'_1l'_2l'_3)$. Then, by writing $u = l'_0l'_1l'_2l'_3$, we have

$$\mathcal{R} \ll \sum_{s \leqslant z_0} \sum_{\substack{u \leqslant z_0^4 \\ p|u \Rightarrow p|s \\ s^2|u}} \frac{\tau(s)^2\tau_4(u)}{u\tau(u)} \ll \sum_{s \leqslant z_0} \sum_{\substack{u \leqslant z_0^4 \\ p|u \Rightarrow p|s \\ s^2|u}} \frac{\tau(s)^2}{u^{3/4}}$$

where in the last step above we have used the bound $\tau_4(u) \leqslant (\tau(u))^4 \ll \tau(u)u^{1/4}$. Now using Lemma 5.7 from [29] with $\epsilon = 1/4$ we have,

$$\sum_{\substack{u \leqslant z_0^4 \\ p|u \Rightarrow p|s \\ s^2|u}} \frac{1}{u^{3/4}} \ll \frac{1}{s^{5/4}},$$

thus $\mathcal{R} \ll \sum_{s \leqslant z_0} \frac{\tau(s)^2}{s^{5/4}} \ll 1$. □

Proposition 2.7.3. *Fix some* $\mathbf{b} \in \mathbb{N}^4$. *Then*

$$\mathcal{E}_{r,2}(B,\mathbf{b}) \ll_A \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2}{b_0^2b_1^2b_2^2b_3^2(\log B)(\log\log B)^{66A}}.$$

Proof. Recall $\mathcal{E}_{r,2}(B,\mathbf{b})$ from (2.5.33). Now we apply the Lemma 2.7.2 and use trivial bounds for the finite sums over $\boldsymbol{\sigma}, \mathbf{q}, \mathbf{K}$ and $\mathbf{L}$. This will give:

$$\mathcal{E}_{r,2}(B,\mathbf{b}) \ll_A B^2 \sum_{\substack{\mathbf{m}\in\mathbb{N}^4 \\ m_{ij}\leqslant z_0 \\ (2.5.2)}} \sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4 \\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}}}\sum \frac{\mathcal{MAX}_1(B)}{m_{02}^2m_{03}^2m_{12}^2m_{13}^2b_0^2b_1^2b_2^2b_3^2}$$

since $M_0M_1M_2M_3 = 2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}m_{02}^2m_{03}^2m_{12}^2m_{13}^2b_0^2b_1^2b_2^2b_3^2$. When $\mathbf{d}=1$ then,

$$\begin{aligned}\sum_{\substack{\mathbf{m}\in\mathbb{N}^4 \\ m_{ij}\leqslant z_0 \\ (2.5.2)}} \mathcal{MAX}_1(B) &\ll \sum_{\substack{\mathbf{m}\in\mathbb{N}^4 \\ m_{ij}\leqslant z_0 \\ (2.5.2)}} \frac{\tau(m_0m_1m_2m_3)\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)}{m_{02}^2m_{03}^2m_{12}^2m_{13}^2b_0^2b_1^2b_2^2b_3^2(\log B)(\log\log B)^{66A}} \\ &\ll \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)}{b_0^2b_1^2b_2^2b_3^2(\log B)(\log\log B)^{66A}}.\end{aligned}$$

Otherwise,

$$\begin{aligned}\sum_{\substack{\mathbf{m}\in\mathbb{N}^4 \\ m_{ij}\leqslant z_0 \\ (2.5.2)}} \sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4 \\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}}}\sum \mathcal{MAX}_1(B) &\ll \sum_{\substack{\mathbf{m}\in\mathbb{N}^4 \\ m_{ij}\leqslant z_0 \\ (2.5.2)}} \frac{1}{b_0^2b_1^2b_2^2b_3^2(\log X)^{132A}} \\ &\quad + \sum_{\substack{\mathbf{m}\in\mathbb{N}^4 \\ m_{ij}\leqslant z_0 \\ (2.5.2)}} \frac{(\log B)(\log\log B)^4}{m_{02}^2m_{03}^2m_{12}^2m_{13}^2b_0^2b_1^2b_2^2b_3^2(\log X)^{A/3}} \\ &\ll \frac{1}{b_0^2b_1^2b_2^2b_3^2(\log B)^{A/3-2}}.\end{aligned}$$

□

As alluded to above, we may use the same argument with the variables $\mathbf{d}, \mathbf{K}$ and $\mathbf{k}$ switching roles with the variables $\tilde{\mathbf{d}}, \mathbf{L}$ and $\mathbf{l}$ to obtain,

Proposition 2.7.4. *Fix some* $\mathbf{b} \in \mathbb{N}^4$. *Then*

$$\mathcal{E}_{r,3}(B,\mathbf{b}) \ll_A \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2}{b_0^2b_1^2b_2^2b_3^2(\log B)(\log\log B)^{66A}}.$$

2.7.3 The Error Terms $\mathcal{E}_{r,4}(B,\mathbf{b})$ and $\mathcal{E}_{r,5}(B,\mathbf{b})$

For these error terms we note that the conditions on the variables $\mathbf{m}, \mathbf{K}, \mathbf{L}, \mathbf{d}$ and $\tilde{\mathbf{d}}$ guarantee that the sums $H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B)$ and $H_{r,5}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B)$ are of types (b) and (c) from §5.3. Similar to the symmetry of $\mathcal{E}_{r,2}(B,\mathbf{b})$ and $\mathcal{E}_{r,3}(B,\mathbf{b})$ in the last section, $\mathcal{E}_{r,4}(B,\mathbf{b})$ and $\mathcal{E}_{r,5}(B,\mathbf{b})$ are symmetrically equivalent, the latter being of the same form of the former with the variables $k_2, l_2, k_3, l_3, K_2, L_2, K_3, L_3$ switching roles with $k_0, l_0, k_1, l_1, K_0, L_0, K_1, L_1$. We will therefore focus on $\mathcal{E}_{r,4}(B,\mathbf{b})$. We first examine $H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B)$.

Lemma 2.7.5. *Fix some* $\mathbf{b} \in \mathbb{N}^4$*, some* $\mathbf{m} \in \mathbb{N}^4$ *satisfying* (2.5.2)*, some* $\boldsymbol{\sigma} \in \{0,1\}^4$ *satisfying* (2.5.7) *and some* $\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$*. Suppose that* $\mathbf{K}, \mathbf{L} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$ *and* $\mathbf{d}, \tilde{\mathbf{d}} \in \mathbb{N}^4$ *satisfying the conditions*

$$\begin{cases} \mathbf{K}, \mathbf{L} \text{ satisfy } (2.5.17), \\ d_{ij}\tilde{d}_{ij} = (m_{ij})_{odd} \ \forall\, ij \in \{02, 03, 12, 13\}, \\ \mathbf{d} = \tilde{\mathbf{d}} = \mathbf{1} \Rightarrow \text{ one of } K_2, L_2, K_3, L_3 \not\equiv 1 \bmod 8. \end{cases} \tag{2.7.10}$$

Then

$$H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B) \ll_A \frac{B^2 \mathcal{MAX}_2(B)}{M_0M_1M_2M_3(\log B)}$$

where we define $\mathcal{MAX}_2(B)$ *as*

$$\max \left\{ \begin{array}{c} \mathbb{1}(\mathbf{d} = \tilde{\mathbf{d}} = \mathbf{1})\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)\sqrt{\log\log B}, \\ \frac{(\log B)^3(\log\log B)^4}{(\log B)^{A/3}}, \frac{d_{02}^2d_{03}^2d_{12}^2d_{13}^2(\log B)}{(\log B)^{140A}} \end{array} \right\}.$$

Proof. Recall (2.5.19),(2.5.24) and (2.5.27). We order $H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B)$ to sum over k_0, l_0, k_1 and l_1 first and therefore write

$$\Theta_2(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{k}, \mathbf{l}) = \chi_{\tilde{\mathbf{d}}}(k_2k_3)\widetilde{\chi}_{\mathbf{d}}(l_2l_3)\chi_{\tilde{\mathbf{d}},l_2l_3}(k_0k_1)\widetilde{\chi}_{\mathbf{d},k_2k_3}(l_0l_1),$$

where

$$\chi_{\tilde{\mathbf{d}}}(k_2k_3) = \left(\frac{\tilde{d}_{02}\tilde{d}_{03}\tilde{d}_{12}\tilde{d}_{13}}{k_2k_3}\right), \ \widetilde{\chi}_{\mathbf{d}}(l_2l_3) = \left(\frac{l_2l_3}{d_{02}d_{03}d_{12}d_{13}}\right),$$

and

$$\chi_{\tilde{\mathbf{d}},l_2l_3}(k_0k_1) = \left(\frac{\tilde{d}_{02}\tilde{d}_{03}\tilde{d}_{12}\tilde{d}_{13}l_2l_3}{k_0k_1}\right), \ \widetilde{\chi}_{\mathbf{d},k_2k_3}(l_0l_1) = \left(\frac{l_0l_1}{d_{02}d_{03}d_{12}d_{13}k_2k_3}\right).$$

Then

$$H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B) = \sum\sum\sum\sum_{\substack{\|k_2,k_3\|,\|l_2,l_3\| \leqslant z_1 \\ k_i \equiv K_i \bmod 8 \ \forall\, i \in \{2,3\} \\ l_i \equiv L_i \bmod 8 \ \forall\, i \in \{2,3\} \\ (2.7.11)}} \frac{\mu^2(k_2l_2k_3l_3)\chi_{\tilde{\mathbf{d}}}(k_2k_3)\widetilde{\chi}_{\mathbf{d}}(l_2l_3)}{\tau(k_2l_2k_3l_3)} H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{kl}_{23}, B)$$

where $\mathbf{kl}_{23} = (k_2, l_2, k_3, l_3)$,

$$H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{kl}_{23}, B) = \sum\sum\sum\sum_{\substack{k_0,l_0,k_1,l_1 \in \mathbb{N} \\ k_i \equiv K_i \bmod 8 \ \forall\, i \in \{0,1\} \\ l_i \equiv L_i \bmod 8 \ \forall\, i \in \{0,1\} \\ (2.5.15),(2.7.12)}} \frac{\mu^2(k_0l_0k_1l_1)}{\tau(k_0)\tau(l_0)\tau(k_1)\tau(l_1)} \chi_{\tilde{\mathbf{d}},l_2l_3}(k_0k_1)\widetilde{\chi}_{\mathbf{d},k_2k_3}(l_0l_1),$$

$$\gcd(k_2l_2, 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}b_3) = \gcd(k_3l_3, 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}b_2) = 1, \tag{2.7.11}$$

and

$$\begin{cases} \gcd(k_0, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}k_2k_3b_1) = \gcd(k_1, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}k_2k_3b_0) = 1, \\ \gcd(l_0, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}l_2l_3b_1) = \gcd(l_1, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}l_2l_3b_0) = 1. \end{cases} \tag{2.7.12}$$

Next we aim to remove the μ^2 term in these inner sums. For this we use Lemma 2.2.3 with $w_0 = w_1 = z_0$, $c_{01} = M_1$, $c_{02} = M_2$ and $M = \|k_2l_2M_2, k_3l_3M_3\|$. Then $H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{kl}_{23}, B)$ is equal to

$$\sum_{s\leqslant z_0}\mu(s)\sum\sum\sum\sum_{\substack{k_0',l_0',k_1',l_1'\leqslant z_0\\ p\mid k_0'l_0'k_1'l_1'\Rightarrow p\mid s\\ s^2\mid k_0'l_0'k_1'l_1',\ (2.7.14)}}\frac{\chi_{\tilde{\mathbf{d}},l_2l_3}(k_0'k_1')\widetilde{\chi}_{\mathbf{d},k_2k_3}(l_0'l_1')}{\tau(k_0')\tau(l_0')\tau(k_1')\tau(l_1')}H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{kl}_{23}, \mathbf{kl}_{01}', B)$$
$$+O\left(\frac{B^2(\log B)^2}{k_2l_2k_3l_3M_0M_1M_2M_3z_0^{1/3}}\right)$$

where $\mathbf{kl}_{01}' = (k_0', l_0', k_1', l_1')$ and

$$H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{kl}_{23}, \mathbf{kl}_{01}', B) = \sum\sum\sum\sum_{\substack{k_0'',l_0'',k_1'',l_1''\in\mathbb{N}\\ k_i\equiv K_i \bmod 8\ \forall\ i\in\{0,1\}\\ l_i\equiv L_i \bmod 8\ \forall\ i\in\{0,1\}\\ (2.7.13),(2.7.15)}}\frac{\chi_{\tilde{\mathbf{d}},l_2l_3}(k_0''k_1'')\widetilde{\chi}_{\mathbf{d},k_2k_3}(l_0''l_1'')}{\tau(k_0'')\tau(l_0'')\tau(k_1'')\tau(l_1'')},$$

$$\|2^{\sigma_0}k_0'k_0''l_0'l_0''m_{02}m_{03}b_0^2, 2^{\sigma_1}k_1'k_1''l_1'l_1''m_{12}m_{13}b_1^2\|\cdot\|2^{\sigma_2}k_2l_2m_{02}m_{12}b_2^2, 2^{\sigma_3}k_3l_3m_{03}m_{13}b_3^2\|\leqslant B, \tag{2.7.13}$$

$$\begin{cases}\gcd(k_0', 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}k_2k_3b_1) = \gcd(k_1', 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}k_2k_3b_0) = 1,\\ \gcd(l_0', 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}l_2l_3b_1) = \gcd(l_1', 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}l_2l_3b_0) = 1,\end{cases} \tag{2.7.14}$$

and

$$\begin{cases}\gcd(k_0'', 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}k_2k_3sb_1) = \gcd(k_1'', 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}k_2k_3sb_0) = 1,\\ \gcd(l_0'', 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}l_2l_3sb_1) = \gcd(l_1'', 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}l_2l_3sb_0) = 1.\end{cases} \tag{2.7.15}$$

Now we swap the summation of k_0', l_0', k_1', l_1' with k_2, l_2, k_3, l_3. This will yield,

$$H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B) = \sum_{s\leqslant z_0}\mu(s)\sum\sum\sum\sum_{\substack{k_0',l_0',k_1',l_1'\leqslant z_0\\ p\mid k_0'l_0'k_1'l_1'\Rightarrow p\mid s\\ s^2\mid k_0'l_0'k_1'l_1',\ (2.7.16)}}\frac{\widetilde{\chi}_{\mathbf{d}}(l_0'l_1')\chi_{\tilde{\mathbf{d}}}(k_0'k_1')}{\tau(k_0')\tau(l_0')\tau(k_1')\tau(l_1')}H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{kl}_{01}', B)$$
$$+O\left(\frac{B^2(\log B)^2(\log\log B)^4}{M_0M_1M_2M_3z_0^{1/3}}\right)$$

where $H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{kl}_{01}', B)$ is equal to

$$\sum\sum\sum\sum_{\substack{\|k_2,k_3\|,\|l_2,l_3\|\leqslant z_1\\ k_i\equiv K_i \bmod 8\ \forall\ i\in\{2,3\}\\ l_i\equiv L_i \bmod 8\ \forall\ i\in\{2,3\}\\ (2.7.17)}}\frac{\mu^2(k_2l_2k_3l_3)\chi_{\tilde{\mathbf{d}},l_0'l_1'}(k_2k_3)\chi_{\mathbf{d},k_0'k_1'}(l_2l_3)}{\tau(k_2l_2k_3l_3)}H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{kl}_{23}, \mathbf{kl}_{01}', B),$$

$$\begin{cases}\gcd(k_0', 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_1) = \gcd(k_1', 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_0) = 1,\\ \gcd(l_0', 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_1) = \gcd(l_1', 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_0) = 1,\end{cases} \tag{2.7.16}$$

and

$$\gcd(k_2l_2, 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}sb_3) = \gcd(k_3l_3, 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}sb_2) = 1. \qquad (2.7.17)$$

Now note that the condition (2.7.10) guarantees that $H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{k}\mathbf{l}'_{01}, B)$ is of the form considered in either Proposition 2.2.10 or Proposition 2.2.11 as it guarantees that either $\chi_{\tilde{\mathbf{d}},l_2l_3}$ or $\chi_{\mathbf{d},k_2k_3}$ is non-principal. The notational dictionary is as follows:

- the n_i in §5.3 correspond to $k''_0, l''_0, k''_1, l''_1$;
- the m_i in §5.3 correspond to k_2, l_2, k_3, l_3;
- the quadruple (c_0, c_1, c_2, c_3) in Proposition 2.2.10 and the quadruple $(c_{01}, c_{23}, \tilde{c}_0, \tilde{c}_1)$ in Proposition 2.2.11 both correspond to the quadruple $(k'_0l'_0M_0, k'_1l'_1M_1, M_2, M_3)$;
- the Q_{02} and Q_{13} in Proposition 2.2.10 correspond to $d_{02}d_{03}d_{12}d_{13}$ and $\tilde{d}_{02}\tilde{d}_{03}\tilde{d}_{12}\tilde{d}_{13}$ respectively;
- the r_i in §5.3 correspond to $m_{02}m_{03}m_{12}m_{13}sb_j$ divided by $d_{02}d_{03}d_{12}d_{13}$ or $\tilde{d}_{02}\tilde{d}_{03}\tilde{d}_{12}\tilde{d}_{13}$ depending on whether n_i corresponds to a k''-variable or a l''-variable and where $j = 0$ or 1 depending on whether the n_i corresponds to a $''$ variable indexed by 1 or 0 respectively;
- the $\tilde{r}_i$ in Proposition 2.2.11 are just $m_{02}m_{03}m_{12}m_{13}sb_j$ for some j.

Finally we note that the characters $\chi_{\tilde{\mathbf{d}},l'_0l'_1}(k_2k_3)$ and $\chi_{\mathbf{d},k'_0k'_1}(l_2l_3)$ in the sum over k_2, l_2, k_3, l_3 are of no import in these applications as we first use the triangle inequality to obtain the absolute value of the inner sums. Noting that we only need to apply Proposition 2.2.11 when $\mathbf{d} = \tilde{\mathbf{d}} = \mathbf{1}$ we thus have that $H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, \mathbf{k}\mathbf{l}'_{01}, B)$ is,

$$\begin{aligned} \ll_A & \frac{\mathbb{1}(\mathbf{d} = \tilde{\mathbf{d}} = \mathbf{1})\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)\tau(s)^2B^2\sqrt{\log\log B}}{k'_0l'_0k'_1l'_1M_0M_1M_2M_3(\log B)} \\ &+ \frac{d_{02}^2d_{03}^2d_{12}^2d_{13}^2B^2}{k'_0l'_0k'_1l'_1M_0M_1M_2M_3(\log B)^{140A}}. \end{aligned}$$

Now, the sum over k'_0, l'_0, k'_1, l'_1 is given by

$$\sum_{s \leqslant z_0} \sum\sum\sum\sum_{\substack{k'_0, l'_0, k'_1, l'_1 \leqslant z_0 \\ p \mid k'_0l'_0k'_1l'_1 \Rightarrow p \mid s \\ s^2 \mid k'_0l'_0k'_1l'_1, (2.7.16)}} \frac{\tau(s)^2}{\tau(k'_0)\tau(l'_0)\tau(k'_1)\tau(l'_1)k'_0l'_0k'_1l'_1} \ll \sum_{s \leqslant z_0} \sum_{\substack{u \leqslant z_0^4 \\ p \mid u \Rightarrow p \mid s \\ s^2 \mid u}} \frac{\tau(s)^2}{u^{3/4}} \ll 1.$$

The details of this bound are the same as those bounding (2.7.9) in the previous subsection. Substituting these bounds into $H_{r,4}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B)$ concludes the result. $\square$

Proposition 2.7.6. *Fix some* $\mathbf{b} \in \mathbb{N}^4$. *Then*

$$\mathcal{E}_{r,4}(B,\mathbf{b}) \ll_A \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2(\log B)}.$$

Proof. Recall $\mathcal{E}_{r,4}(B,\mathbf{b})$ from (2.5.34). Then, upon applying the Lemma 2.7.5 we are left with

$$\mathcal{E}_{r,4}(B,\mathbf{b}) \ll_A \frac{B^2}{(\log B)} \sum_{\substack{\mathbf{m}\in\mathbb{N}^4\\ m_{ij}\leqslant z_0\\ (2.5.2)}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\ (2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum\sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4}\\ (2.5.17)}} \sum\sum_{\substack{\mathbf{d},\tilde{\mathbf{d}}\in\mathbb{N}^4\\ d_{ij}\tilde{d}_{ij}=(m_{ij})_{\mathrm{odd}}}} \frac{\mathcal{MAX}_2(B)}{M_0M_1M_2M_3}.$$

Now, since $\mathbb{1}(\mathbf{d}=\tilde{\mathbf{d}}=\mathbf{1})=1$ if and only if $\mathbf{m}_{\mathrm{odd}}=\mathbf{1}$, and since (2.5.2) guarantees that $\mu^2(m_{02}m_{03}m_{12}m_{13})=1$, the only possibilities for $\mathbf{m}$ are $(1,1,1,1)$, $(2,1,1,1)$, $(1,2,1,1)$, $(1,1,2,1)$ or $(1,1,1,2)$ when this condition holds. Thus there are only finitely many $\mathbf{m}$ to consider and since there are only finitely many $\mathbf{q},\mathbf{K},\mathbf{L}$ and $\boldsymbol{\sigma}$ we have,

$$\sum_{\substack{\mathbf{m}\in\mathbb{N}^4\\ m_{ij}\leqslant z_0\\ (2.5.2),\mathbf{m}_{\mathrm{odd}}=\mathbf{1}}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\ (2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum\sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4}\\ (2.5.17)}} \frac{\mathcal{MAX}_2(B)}{M_0M_1M_2M_3} \ll \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2}.$$

Otherwise, we remark that there are only finitely many $\mathbf{q},\mathbf{K},\mathbf{L}$ and $\boldsymbol{\sigma}$ and that the sum over $\frac{1}{m_{ij}^2}$ converges so that the expression becomes bounded by

$$\ll \frac{(\log B)^3(\log\log B)^4}{b_0^2b_1^2b_2^2b_3^2(\log B)^{A/3}} + \frac{(\log B)^{4A}}{b_0^2b_1^2b_2^2b_3^2(\log B)^{140A}}$$

Noting that by choosing A to be sufficiently large and that each $b_i \leqslant (\log B)^A$, we obtain the result. □

It follows by the same argument with the variables $k_2,l_2,k_3,l_3,K_2,L_2,K_3,L_3$ switching roles with $k_0,l_0,k_1,l_1,K_0,L_0,K_1,L_1$ that:

Proposition 2.7.7. *Fix some* $\mathbf{b} \in \mathbb{N}^4$. *Then*

$$\mathcal{E}_{r,5}(B,\mathbf{b}) \ll_A \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2(\log B)}.$$

2.8 Vanishing Main Terms

In this section we handle the vanishing main terms $\mathcal{V}_{r,4}(B,\mathbf{b})$ and $\mathcal{V}_{r,5}(B,\mathbf{b})$. As in the arguments of the previous section these are similar, almost obtained from each other by switching the roles of $k_2,l_2,k_3,l_3,K_2,L_2,K_3,L_3$ with $k_0,l_0,k_1,l_1,K_0,L_0,K_1,L_1$. The key obstruction to this is condition (2.5.16), which creates an asymmetry in this "role" switching of variables when $r=1$. For this reason, and the fact that the even characters found in $\Theta_{r,1}$ will play a key role in the following arguments and this function changes with the value of r, we must separate the cases $r=1$ and $r=2$.

2.8.1 The Vanishing Main Term $\mathcal{V}_{1,4}(B,\mathbf{b})$

We begin with an examination of the inner sums $H_{1,4}(\mathbf{1},\mathbf{1},\mathbf{K},\mathbf{L},B)$. Recall that

$$H_{1,4}(\mathbf{1},\mathbf{1},\mathbf{K},\mathbf{L},B)=\sum\sum\sum\sum_{\substack{(\mathbf{k},\mathbf{l})\in\mathcal{H}_4\\(\mathbf{k},\mathbf{l})\equiv(\mathbf{K},\mathbf{L})\bmod 8\\(2.5.14),\,(2.5.15),\,(2.5.16)}}\frac{\mu^2(2k_0l_0k_1l_1k_2l_2k_3l_3)}{\tau\left(k_0l_0k_1l_1k_2l_2k_3l_3\right)}\left(\frac{l_0l_1}{k_2k_3}\right)\left(\frac{l_2l_3}{k_0k_1}\right).$$

We separate out the terms for which $k_2=l_2=k_3=l_3=1$:

$$H_{1,4}(\mathbf{1},\mathbf{1},\mathbf{K},\mathbf{L},B)=V_{1,4}(\mathbf{K},\mathbf{L},B)+EV_{1,4}(\mathbf{K},\mathbf{L},B)$$

where

$$V_{1,4}(\mathbf{K},\mathbf{L},B)=\sum\sum\sum\sum_{\substack{k_0,l_0,k_1,l_1\in\mathbb{N}^4\\(k_i,l_i)\equiv(K_i,L_i)\bmod 8\ \forall\, i\in\{0,1\}\\(2.5.14),\,(2.5.16),\,(2.8.1)}}\frac{\mu^2(2k_0l_0k_1l_1)}{\tau\left(k_0l_0k_1l_1\right)},$$

$$EV_{1,4}(\mathbf{K},\mathbf{L},B)=\sum\sum\sum\sum_{\substack{(\mathbf{k},\mathbf{l})\in\mathcal{H}_4\\(\mathbf{k},\mathbf{l})\equiv(\mathbf{K},\mathbf{L})\bmod 8\\k_2l_2k_3l_3\neq1\\(2.5.14),\,(2.5.15),\,(2.5.16)}}\frac{\mu^2(2k_0l_0k_1l_1k_2l_2k_3l_3)}{\tau\left(k_0l_0k_1l_1k_2l_2k_3l_3\right)}\left(\frac{l_0l_1}{k_2k_3}\right)\left(\frac{l_2l_3}{k_0k_1}\right),$$

and

$$\|2^{\sigma_0}k_0l_0m_{02}m_{03}b_0^2,2^{\sigma_1}k_1l_1m_{12}m_{13}b_1^2\|\cdot\|2^{\sigma_2}m_{02}m_{12}b_2^2,2^{\sigma_3}m_{03}m_{13}b_3^2\|\leqslant B. \tag{2.8.1}$$

Now $EV_{1,4}(\mathbf{K},\mathbf{L},B)$ may be treated like $H_{r,4}(\mathbf{d},\tilde{\mathbf{d}},\mathbf{K},\mathbf{L},B)$ in §2.7.3, by noting that, after breaking the μ^2 function, the condition $k_2l_2k_3l_3\neq1$ guarantees that Propositions 2.2.10 and 2.2.11 may be used. Summing over the $\mathbf{m},\boldsymbol{\sigma},\mathbf{q},\mathbf{K}$ and $\mathbf{L}$ as in Proposition 2.7.6 we are thus left with:

$$\begin{aligned}\mathcal{V}_{1,4}(B,\mathbf{b})=&\sum_{\substack{\mathbf{m}\in\mathbb{N}^4,(2.5.2)\\(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}=1}}\ \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\(2.5.7)}}\ \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})}\ \sum\sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4}\\(2.5.30)}}\Theta_{1,1}(\mathbf{1},\mathbf{K},\boldsymbol{\sigma})V_{1,4}(\mathbf{K},\mathbf{L},B)\\&+O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2(\log B)}\right).\end{aligned}$$

Now we wish to re-integrate the even characters into the sum over k_0,l_0,k_1,l_1. By doing this we obtain that $\mathcal{V}_{1,4}(B,\mathbf{b})$ is equal to

$$\sum_{\substack{\mathbf{m}\in\mathbb{N}^4,(2.5.2)\\(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}=1}}\ \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\(2.5.7)}}\ \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})}V'_{1,4}(B)+O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2(\log B)}\right) \tag{2.8.2}$$

where

$$V'_{1,4}(B)=\sum\sum\sum\sum_{\substack{k_0,l_0,k_1,l_1\in\mathbb{N}^4\\(2.5.16),\,(2.8.1),\,(2.8.3),\,(2.8.4)}}\frac{\mu^2(2k_0l_0k_1l_1)}{\tau\left(k_0l_0k_1l_1\right)}(-1)^{\frac{(k_0k_1-1)}{2}}\left(\frac{2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_0k_1}\right),$$

$$\begin{cases}\gcd(k_0l_0,2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_1)=\gcd(k_1l_1,2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_0)=1,\\\gcd(k_0l_0k_1l_1,2)=1,\end{cases} \tag{2.8.3}$$

and

$$k_0 l_0 \equiv -q_0 \bmod 8,\ k_1 l_1 \equiv q_1 \bmod 8,\ k_1 l_1 \equiv q_2 \bmod 8,\ k_0 l_0 \equiv -q_3 \bmod 8. \tag{2.8.4}$$

Notice that since $(-1)^{\frac{(k_0k_1-1)}{2}} = \left(\frac{-1}{k_0k_1}\right)$ we may write

$$(-1)^{\frac{(k_0k_1-1)}{2}} \left(\frac{2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_0k_1}\right) = \left(\frac{-2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_0k_1}\right).$$

Let us now consider (2.8.4). We first note that these conditions require that any $\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$ must satisfy

$$q_0 \equiv q_3 \bmod 8 \text{ and } q_1 \equiv q_2 \bmod 8. \tag{2.8.5}$$

We now make use of the identity

$$\mathbb{1}(a \equiv q \bmod 8) = \frac{1}{4} \sum_{\substack{\chi' \text{ char.} \\ \bmod 8}} \chi'(a)\overline{\chi'}(q) \tag{2.8.6}$$

to break to congruence conditions in (2.8.4) using (2.8.5). Putting this into $V'_{1,4}(B)$ gives:

$$V'_{1,4}(B) = \frac{1}{16} \sum\sum_{\chi,\chi' \text{ char. mod } 8} \overline{\chi}(-q_0)\overline{\chi'}(q_1)V'_{1,4}(\chi,\chi',B) \ll \sum\sum_{\chi,\chi' \text{ char. mod } 8} \left|V'_{1,4}(\chi,\chi',B)\right|.$$

where $V'_{1,4}(\chi,\chi',B)$ is defined as

$$\sum\sum\sum\sum_{\substack{k_0,l_0,k_1,l_1 \in \mathbb{N}^4 \\ (2.5.16),\ (2.8.1),\ (2.8.3)}} \frac{\mu^2(2k_0l_0k_1l_1)\chi(k_0l_0)\chi'(k_1l_1)}{\tau\left(k_0l_0k_1l_1\right)} \left(\frac{-2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_0k_1}\right).$$

The next lemma tells us that, in all of the cases we consider, we are always summing over a non-principal character.

Lemma 2.8.1. *Fix* $\mathbf{m}$ *and* $\boldsymbol{\sigma}$ *satisfying the conditions* (2.5.2), $(m_{02}m_{03}m_{12}m_{13})_{odd} = 1$ *and* (2.5.7)*. Then for any character* χ *modulo* 8 *at least one of the characters*

$$\left(\frac{-2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)\chi(\cdot) \text{ and } \chi(\cdot)$$

is not principal.

Proof. For any choice of $\mathbf{m}$ and $\boldsymbol{\sigma}$ satisfying the conditions, the quadratic character $\left(\frac{-2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)$ is a non-principal character modulo 8, since it is either equal to $\left(\frac{-1}{\cdot}\right)$ or $\left(\frac{-2}{\cdot}\right)$. It follows that, $\left(\frac{-2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)\chi(\cdot)$ is the principal character modulo 8 if and only if $\chi(\cdot) = \left(\frac{-2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)$ in which case χ is not the principal character. ☐

Therefore we should expect to see some cancellation in $V'_{1,4}(\chi,\chi',B)$ resulting from the oscillation in these non-principal characters.

Lemma 2.8.2. *Fix some* $\mathbf{b} \in \mathbb{N}^4$ *and fix* $\mathbf{m} \in \mathbb{N}^4$ *and* $\boldsymbol{\sigma} \in \{0,1\}^4$ *satisfying the conditions* (2.5.2), $(m_{02}m_{03}m_{12}m_{13})_{odd} = 1$ *and* (2.5.7). *Then*

$$V'_{1,4}(B) \ll \frac{\tau(m_0m_1m_2m_3)^4\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2}{M_0M_1M_2M_3(\log B)}.$$

Proof. We first consider $V'_{1,4}(\chi,\chi',B)$ for χ, χ' some characters modulo 8. The first thing we wish to do is remove the square-free condition. To do this we use Lemma 2.2.3 (with $w_0 = w_1 = z_0$), as well as noting that the characters of even modulus take care of the coprimality to 2. Then $V'_{1,4}(\chi,\chi',B)$ satisfies the bound

$$\ll \sum_{s \leqslant z_0} \sum\sum\sum\sum_{\substack{k'_0,l'_0,k'_1,l'_1 \leqslant z_0 \\ p|k'_0l'_0k'_1l'_1 \Rightarrow p|s \\ s^2|k'_0l'_0k'_1l'_1,(2.8.7)}} \frac{\left|V'_{1,4}(\chi,\chi',\mathbf{kl}'_{01},B)\right|}{\tau(k'_0)\tau(l'_0)\tau(k'_1)\tau(l'_1)} + O\left(\frac{B^2}{M_0M_1M_2M_3(\log B)^{A/3}}\right)$$

where,

$$V'_{1,4}(\chi,\chi',\mathbf{kl}'_{01},B) = \sum\sum\sum\sum_{\substack{k''_0,l''_0,k''_1,l''_1 \in \mathbb{N}^4 \\ (2.5.16),\ (2.8.8),\ (2.8.9)}} \frac{\chi(k''_0l''_0)\chi'(k''_1l''_1)}{\tau(k''_0)\tau(l''_0)\tau(k''_1)\tau(l''_1)} \left(\frac{-2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{k''_0k''_1}\right),$$

$$\begin{cases} \gcd(k'_0l'_0, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_1) = \gcd(k'_1l'_1, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_0) = 1, \\ \gcd(k'_0l'_0k'_1l'_1, 2) = 1, \end{cases} \tag{2.8.7}$$

$$\begin{cases} \gcd(k''_0l''_0, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}sb_1) = \gcd(k''_1l''_1, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}sb_0) = 1, \\ \gcd(k''_0l''_0k''_1l''_1, 2) = 1, \end{cases} \tag{2.8.8}$$

and

$$\|2^{\sigma_0}k'_0k''_0l'_0l''_0m_{02}m_{03}b_0^2, 2^{\sigma_1}k'_1k''_1l'_1l''_1m_{12}m_{13}b_1^2\| \cdot \|2^{\sigma_2}m_{02}m_{12}b_2^2, 2^{\sigma_3}m_{03}m_{13}b_3^2\| \leqslant B. \tag{2.8.9}$$

Now we use Lemma 2.8.1 on both χ and χ' to note that these sums satisfy the conditions of Lemma 2.2.9:

- the n_i in Lemma 2.2.9 correspond to $k''_0, l''_0, k''_1, l''_1$;
- χ_0 in Lemma 2.2.9 is thus $\chi(\cdot)\left(\frac{-2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)$; χ_1 in Lemma 2.2.9 is $\chi(\cdot)$; χ_2 in Lemma 2.2.9 is $\chi'(\cdot)\left(\frac{-2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)$ and χ_3 in Lemma 2.2.9 is $\chi'(\cdot)$;
- the remaining notation is assigned similarly to the applications of Propositions 2.2.7, 2.2.8, 2.2.10 and 2.2.11.

Upon using Lemma 2.2.9 we obtain

$$V'_{1,4}(\chi, \chi', \mathbf{kl}'_{01}, B) \ll \frac{\tau(s)^4 \tau(m_0 m_1 m_2 m_3)^4 \tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3) B^2}{k'_0 l'_0 k'_1 l'_1 M_0 M_1 M_2 M_3 (\log B)}.$$

Similar to how we dealt with (2.7.9), it can be shown that

$$\sum_{s \leqslant z_0} \sum\sum\sum\sum_{\substack{k'_0, l'_0, k'_1, k'_1 \leqslant z_0 \\ p | k'_0 l'_0 k'_1 l'_1 \Rightarrow p | s \\ s^2 | k'_0 l'_0 k'_1 l'_1, (2.8.7)}} \frac{\tau(s)^4}{k'_0 l'_0 k'_1 l'_1 \tau(k'_0)\tau(l'_0)\tau(k'_1)\tau(l'_1)} \ll 1.$$

Thus

$$V'_{1,4}(\chi, \chi', B) \ll \frac{\tau(m_0 m_1 m_2 m_3)^4 \tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3) B^2}{M_0 M_1 M_2 M_3 (\log B)} + \frac{B^2}{M_0 M_1 M_2 M_3 (\log B)^{A/3}}.$$

Now, by summing over the finitely many characters modulo 8:

$$V'_{1,4}(B) \ll \sum\sum_{\chi, \chi' \text{ char. mod } 8} \left| V'_{1,4}(\chi, \chi', B) \right| \ll \frac{\tau(m_0 m_1 m_2 m_3)^4 \tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3) B^2}{M_0 M_1 M_2 M_3 (\log B)}$$

as required. ☐

Proposition 2.8.3. *Fix some* $\mathbf{b} \in \mathbb{N}^4$*. Then*

$$\mathcal{V}_{1,4}(B, \mathbf{b}) \ll_A \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3) B^2 \sqrt{\log\log B}}{b_0^2 b_1^2 b_2^2 b_3^2 (\log B)}.$$

Proof. Recall from (2.8.2) that $\mathcal{V}_{1,4}(B, \mathbf{b})$ is equal to

$$\sum_{\substack{\mathbf{m} \in \mathbb{N}^4, (2.5.2) \\ (m_{02} m_{03} m_{12} m_{13})_{\text{odd}} = 1}} \sum_{\substack{\boldsymbol{\sigma} \in \{0,1\}^4 \\ (2.5.7)}} \sum_{\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})} V'_{1,4}(B) + O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3) B^2 \sqrt{\log\log B}}{b_0^2 b_1^2 b_2^2 b_3^2 (\log B)} \right).$$

Using Lemma 2.8.2 we therefore have:

$$\begin{aligned} \mathcal{V}_{1,4}(B, \mathbf{b}) \ll & \sum_{\substack{\mathbf{m} \in \mathbb{N}^4, (2.5.2) \\ (m_{02} m_{03} m_{12} m_{13})_{\text{odd}} = 1}} \sum_{\substack{\boldsymbol{\sigma} \in \{0,1\}^4 \\ (2.5.7)}} \sum_{\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})} \frac{\tau(m_0 m_1 m_2 m_3)^4 \tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3) B^2}{M_0 M_1 M_2 M_3 (\log B)} \\ & + O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3) B^2 \sqrt{\log\log B}}{b_0^2 b_1^2 b_2^2 b_3^2 (\log B)} \right). \end{aligned}$$

For the front term we note that in each case there are only finitely many $\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$ and $\boldsymbol{\sigma} \in \{0,1\}^4$. Also, given the conditions (2.5.2) and $(m_{02} m_{03} m_{12} m_{13})_{\text{odd}} = 1$, we must have $(m_{02}, m_{03}, m_{12}, m_{13}) \in \{(1,1,1,1), (2,1,1,1), (1,2,1,1), (1,1,2,1), (1,1,1,2)\}$. Thus there are only finitely many choices here as well. Thus after summing the first expression, the second error term will clearly dominate, giving the result. ☐

2.8.2 The Vanishing Main Term $\mathcal{V}_{1,5}(B, \mathbf{b})$

The argument in this subsection will differ to the previous one in that the condition (2.5.16) will play a key role. We begin once again by examining the inner sums $H_{1,5}(\mathbf{1}, \mathbf{1}, \mathbf{K}, \mathbf{L}, B)$, i.e (2.5.27) with $\mathbf{d} = \tilde{\mathbf{d}} = 1$. We split off the terms where $k_0 = l_0 = k_1 = l_1 = 1$, however in this case we must preserve the condition (2.5.16). Thus we write:

$$H_{1,5}(\mathbf{1}, \mathbf{1}, \mathbf{K}, \mathbf{L}, B) = \mathbb{1}(2^{\sigma_0+\sigma_1} m_{02}m_{03}m_{12}m_{13} \neq 1)V_{1,5}(\mathbf{K}, \mathbf{L}, B) + EV_{1,5}(\mathbf{K}, \mathbf{L}, B)$$

where

$$V_{1,5}(\mathbf{K}, \mathbf{L}, B) = \sum\sum\sum\sum_{\substack{k_2,l_2,k_3,l_3\in\mathbb{N}^4 \\ (k_i,l_i)\equiv(K_i,L_i) \bmod 8\ \forall\ i\in\{2,3\} \\ (2.5.14),\ (2.8.10)}} \frac{\mu^2(2k_2l_2k_3l_3)}{\tau(k_2l_2k_3l_3)},$$

$$EV_{1,5}(\mathbf{K}, \mathbf{L}, B) = \sum\sum\sum\sum_{\substack{(\mathbf{k},\mathbf{l})\in\mathcal{H}_5 \\ (\mathbf{k},\mathbf{l})\equiv(\mathbf{K},\mathbf{L}) \bmod 8 \\ k_0l_0k_1l_1\neq1 \\ (2.5.14),\ (2.5.15)}} \frac{\mu^2(2k_0l_0k_1l_1k_2l_2k_3l_3)}{\tau(k_0l_0k_1l_1k_2l_2k_3l_3)} \left(\frac{l_0l_1}{k_2k_3}\right)\left(\frac{l_2l_3}{k_0k_1}\right),$$

and

$$\|2^{\sigma_0}m_{02}m_{03}b_0^2, 2^{\sigma_1}m_{12}m_{13}b_1^2\| \cdot \|2^{\sigma_2}k_2l_2m_{02}m_{12}b_2^2, 2^{\sigma_3}k_3l_3m_{03}m_{13}b_3^2\| \leqslant B. \tag{2.8.10}$$

Similar to the previous section, we may handle the $EV_{1,5}(\mathbf{K}, \mathbf{L}, B)$ in the same way as we handled $H_{1,5}(\mathbf{d}, \tilde{\mathbf{d}}, \mathbf{K}, \mathbf{L}, B)$ in §2.7.3 this time by noting that, after breaking the μ^2 function, the condition $k_0l_0k_1l_1 \neq 1$ guarantees that the conditions of Propositions 2.2.10 or 2.2.11 are satisfied. Summing this error term over the $\mathbf{m}, \boldsymbol{\sigma}, \mathbf{q}, \mathbf{K}$ and $\mathbf{L}$ we are thus left with:

$$\mathcal{V}_{1,5}(B, \mathbf{b}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^4,(2.5.2) \\ (m_{02}m_{03}m_{12}m_{13})_{\text{odd}}=1}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4 \\ (2.8.13)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum\sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.5.31)}} \Theta_{1,1}(\mathbf{1}, \mathbf{K}, \boldsymbol{\sigma})V_{1,5}(\mathbf{K}, \mathbf{L}, B)$$
$$+ O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2(\log B)}\right).$$

Now we wish to re-integrate the even characters into the sum over k_2, l_2, k_3, l_3. By doing this we obtain:

$$\mathcal{V}_{1,5}(B, \mathbf{b}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^4,(2.5.2) \\ (m_{02}m_{03}m_{12}m_{13})_{\text{odd}}=1}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4 \\ (2.8.13)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} V'_{1,5}(B) + O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2}{b_0^2b_1^2b_2^2b_3^2(\log B)(\log\log B)^{-1/2}}\right) \tag{2.8.11}$$

where

$$V'_{1,5}(B) = \sum\sum\sum\sum_{\substack{k_2,l_2,k_3,l_3\in\mathbb{N}^4 \\ (2.5.16),\ (2.8.10),\ (2.8.12),\ (2.8.14)}} \frac{\mu^2(2k_2l_2k_3l_3)}{\tau(k_2l_2k_3l_3)} \left(\frac{2^{\sigma_0+\sigma_1+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_2k_3}\right),$$

$$\begin{cases} \gcd(k_2l_2, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_3) = \gcd(k_3l_3, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_2) = 1, \\ \gcd(k_2l_2k_3l_3, 2) = 1, \end{cases} \tag{2.8.12}$$

$$\begin{cases}\sigma_0+\sigma_1+\sigma_2+\sigma_3 \leqslant 1, \ \gcd(2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}, m_{02}m_{03}m_{12}m_{13}) = 1 \\ \gcd(2^{\sigma_0}, b_1) = \gcd(2^{\sigma_1}, b_0) = \gcd(2^{\sigma_2}, b_3) = \gcd(2^{\sigma_3}, b_2) = 1, \\ 2^{\sigma_0+\sigma_1} m_{02}m_{03}m_{12}m_{13} \neq 1,\end{cases} \tag{2.8.13}$$

and

$$k_2l_2 \equiv -q_0 \bmod 8, \ k_3l_3 \equiv q_1 \bmod 8, \ k_3l_3 \equiv q_2 \bmod 8, \ k_2l_2 \equiv -q_3 \bmod 8. \tag{2.8.14}$$

Remark 2.8.4. An important comparison to the previous case is the absence of a factor coming from $(-1)^{f_1(\mathbf{d},\mathbf{k})}$. This is because, when $d_{02}d_{03}d_{12}d_{13}k_0k_1 = 1$, as is the case here, $f_1(\mathbf{d},\mathbf{k}) = 0$ regardless of the choices of k_2 and k_3. In the previous subsection we relied on this factor to guarantee the non-principality of the characters modulo 8, see Lemma 2.8.1; in this subsection we will instead rely on the condition $2^{\sigma_0+\sigma_1} m_{02}m_{03}m_{12}m_{13} \neq 1$ from (2.8.13) to play this role, see Lemma 2.8.5 below.

Given the condition (2.8.14), any $\mathbf{q} \in \mathcal{A}(\mathbf{m},\boldsymbol{\sigma})$ we consider must satisfy

$$q_0 \equiv q_3 \bmod 8 \text{ and } q_1 \equiv q_2 \bmod 8. \tag{2.8.15}$$

Once again we make use of the identity (2.8.6), this time to break to congruence conditions (2.8.14) using (2.8.15). Putting this into $V'_{1,5}(B)$ gives:

$$V'_{1,5}(B) = \frac{1}{16} \sum_{\chi,\chi' \text{ char. mod } 8}\sum \overline{\chi}(-q_0)\overline{\chi'}(q_1) V'_{1,5}(\chi,\chi',B) \ll \sum_{\chi,\chi' \text{ char. mod } 8}\sum \left| V'_{1,5}(\chi,\chi',B)\right|.$$

where

$$V'_{1,5}(\chi,\chi',B) = \sum\sum\sum\sum_{\substack{k_2,l_2,k_3,l_3 \in \mathbb{N}^4 \\ (2.5.16),\,(2.8.10),\,(2.8.12)}} \frac{\mu^2(2k_2l_2k_3l_3)\chi(k_2l_2)\chi'(k_3l_3)}{\tau\left(k_2l_2k_3l_3\right)} \left(\frac{2^{\sigma_0+\sigma_1+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_2k_3}\right).$$

The next lemma is analogous to Lemma 2.8.1.

Lemma 2.8.5. *Fix* $\mathbf{m}$ *and* $\boldsymbol{\sigma}$ *satisfying the conditions* (2.5.2), $(m_{02}m_{03}m_{12}m_{13})_{odd} = 1$ *and* (2.8.13). *Then for any character* χ *modulo* 8 *at least one of the characters*

$$\left(\frac{2^{\sigma_0+\sigma_1+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)\chi(\cdot) \text{ and } \chi(\cdot)$$

is not principal.

Proof. For any $\mathbf{m}$ and $\boldsymbol{\sigma}$ satisfying (2.8.13) we have $\left(\frac{2^{\sigma_0+\sigma_1+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right) = \left(\frac{2}{\cdot}\right)$. It follows that, in order for $\left(\frac{2^{\sigma_0+\sigma_1+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)\chi(\cdot)$ to be non-principal, we must have $\chi(\cdot) = \left(\frac{2}{\cdot}\right)$, in which case χ is non-principal. □

From here we may now follow directly the argument of the previous subsection, applying Proposition 2.2.9 to the $V'_{1,5}(\chi,\chi',B)$ and summing over the remaining variables to obtain the following:

Proposition 2.8.6. *Fix some* $\mathbf{b} \in \mathbb{N}^4$. *Then*

$$\mathcal{V}_{1,5}(B,\mathbf{b}) \ll_A \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2(\log B)}.$$

2.8.3 The Vanishing Main Term $\mathcal{V}_{2,4}(B,\mathbf{b})$

Following the same procedure as in the previous subsections, we may write

$$\mathcal{V}_{2,4}(B,\mathbf{b}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^4,(2.5.2)\\(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}=1}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\(2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4}\\(2.5.30)}}\sum \Theta_{2,1}(\mathbf{1},\mathbf{K},\boldsymbol{\sigma})V_{2,4}(\mathbf{K},\mathbf{L},B)$$
$$+O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2(\log B)}\right)$$

Returning the even characters into the sum over k_0,l_0,k_1,l_1 we obtain:

$$\mathcal{V}_{2,4}(B,\mathbf{b}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^4,(2.5.2)\\(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}=1}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\(2.8.18)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} V'_{2,4}(B) + O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2}{b_0^2b_1^2b_2^2b_3^2(\log B)(\log\log B)^{-1/2}}\right) \tag{2.8.16}$$

where

$$V'_{2,4}(B) = \sum_{\substack{k_0,l_0,k_1,l_1\in\mathbb{N}^4\\(2.5.16),\,(2.8.17),\,(2.8.19),\,(2.8.20)}}\sum\sum\sum \frac{\mu^2(2k_0l_0k_1l_1)}{\tau\left(k_0l_0k_1l_1\right)}\left(\frac{2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_0k_1}\right),$$

$$\|2^{\sigma_0}k_0l_0m_{02}m_{03}b_0^2, 2^{\sigma_1}k_1l_1m_{12}m_{13}b_1^2\|\cdot\|2^{\sigma_2}m_{02}m_{12}b_2^2, 2^{\sigma_3}m_{03}m_{13}b_3^2\|\leqslant B, \tag{2.8.17}$$

$$\begin{cases}\sigma_0+\sigma_1+\sigma_2+\sigma_3\leqslant 1,\ \gcd(2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3},m_{02}m_{03}m_{12}m_{13})=1\\ \gcd(2^{\sigma_0},b_1)=\gcd(2^{\sigma_1},b_0)=\gcd(2^{\sigma_2},b_3)=\gcd(2^{\sigma_3},b_2)=1,\\ 2^{\sigma_2+\sigma_3}m_{02}m_{03}m_{12}m_{13}\neq 1,\end{cases} \tag{2.8.18}$$

$$\begin{cases}\gcd(k_0l_0,2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_1)=\gcd(k_1l_1,2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_0)=1,\\ \gcd(k_0l_0k_1l_1,2)=1,\end{cases} \tag{2.8.19}$$

and

$$k_0l_0\equiv -q_0 \bmod 8,\ k_1l_1\equiv q_1 \bmod 8,\ k_1l_1\equiv -q_2 \bmod 8,\ k_0l_0\equiv q_3 \bmod 8. \tag{2.8.20}$$

Again, we do not have any factor coming from $(-1)^{f_2(\mathbf{d},\mathbf{k})}$. This is because

$$f_2(\mathbf{d},\mathbf{k})=k_0k_1-1$$

when $d_{02}d_{03}d_{12}d_{13}k_2k_3=1$ as is the case here. Since k_0k_1 is always odd in our sums, k_0k_1-1 is always even, and so this factor is just 1. Considering (2.8.20), we note that any $\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})$ we consider must satisfy

$$q_0\equiv -q_3 \bmod 8 \text{ and } q_1\equiv -q_2 \bmod 8. \tag{2.8.21}$$

Using this and (2.8.6) to break to congruence conditions in (2.8.20) and putting this into $V'_{2,4}(B)$ gives:

$$V'_{2,4}(B)=\frac{1}{16}\sum_{\chi,\chi'\text{ char. mod }8}\sum \overline{\chi}(-q_0)\overline{\chi'}(q_1)V'_{2,4}(\chi,\chi',B)\ll \sum_{\chi,\chi'\text{ char. mod }8}\sum \left|V'_{2,4}(\chi,\chi',B)\right|.$$

where

$$V'_{2,4}(\chi,\chi',B)=\sum_{\substack{k_0,l_0,k_1,l_1\in\mathbb{N}^4\\(2.5.16),\,(2.8.17),\,(2.8.19)}}\sum\sum\sum \frac{\mu^2(2k_0l_0k_1l_1)\chi(k_0l_0)\chi'(k_1l_1)}{\tau\left(k_0l_0k_1l_1\right)}\left(\frac{2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_0k_1}\right).$$

Lemma 2.8.7. *Fix* $\mathbf{m}$ *and* $\boldsymbol{\sigma}$ *satisfying the conditions* (2.5.2), $(m_{02}m_{03}m_{12}m_{13})_{odd} = 1$ *and* (2.8.18). *Then for any character* χ *modulo* 8 *at least one of the characters*

$$\left(\frac{2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)\chi(\cdot) \text{ and } \chi(\cdot)$$

is not principal.

Proof. For any $\mathbf{m}$ and $\boldsymbol{\sigma}$ satisfying (2.8.18) we have $\left(\frac{2^{\sigma_2+\sigma_3+v_3(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right) = \left(\frac{2}{\cdot}\right)$. It follows that, in order for $\left(\frac{2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)\chi(\cdot)$ to be non-principal, we must have $\chi(\cdot) = \left(\frac{2}{\cdot}\right)$, in which case χ is non-principal. ☐

Now we may repeat the argument of §2.8.1, applying Proposition 2.2.9 and summing over the remaining variables to obtain:

Proposition 2.8.8. *Fix some* $\mathbf{b} \in \mathbb{N}^4$. *Then*

$$\mathcal{V}_{2,4}(B,\mathbf{b}) \ll_A \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2(\log B)}.$$

2.8.4 The Vanishing Main Term $\mathcal{V}_{2,5}(B,\mathbf{b})$

We are left only to bound $\mathcal{V}_{2,5}(B,\mathbf{b})$. We follow the same procedure as in the previous subsections to obtain

$$\begin{aligned}\mathcal{V}_{2,5}(B,\mathbf{b}) = &\sum_{\substack{\mathbf{m}\in\mathbb{N}^4,(2.5.2)\\(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}=1}}\ \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\(2.5.7)}}\ \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})}\ \sum\sum_{\substack{\mathbf{K},\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4}\\(2.5.31)}} \Theta_{2,1}(\mathbf{1},\mathbf{K},\boldsymbol{\sigma})V_{2,5}(\mathbf{K},\mathbf{L},B)\\ &+ O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2(\log B)}\right).\end{aligned}$$

Returning the even characters into the sum over k_2, l_2, k_3, l_3 we obtain:

$$\mathcal{V}_{2,5}(B,\mathbf{b}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^4,(2.5.2)\\(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}=1}}\ \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\(2.8.24)}}\ \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} V'_{2,5}(B) + O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2}{b_0^2b_1^2b_2^2b_3^2(\log B)(\log\log B)^{-1/2}}\right) \tag{2.8.22}$$

where

$$V'_{2,5}(B) = \sum\sum\sum\sum_{\substack{k_2,l_2,k_3,l_3\in\mathbb{N}^4\\(2.5.16),\,(2.8.23),\,(2.8.25),\,(2.8.26)}} \frac{\mu^2(2k_2l_2k_3l_3)}{\tau(k_2l_2k_3l_3)}(-1)^{\frac{k_2k_3-1}{2}}\left(\frac{2^{\sigma_0+\sigma_1+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_2k_3}\right),$$

$$\|2^{\sigma_0}m_{02}m_{03}b_0^2, 2^{\sigma_1}m_{12}m_{13}b_1^2\|\cdot\|2^{\sigma_2}k_2l_2m_{02}m_{12}b_2^2, 2^{\sigma_3}k_3l_3m_{03}m_{13}b_3^2\| \leqslant B, \tag{2.8.23}$$

$$\begin{cases}\sigma_0+\sigma_1+\sigma_2+\sigma_3 \leqslant 1,\ \gcd(2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}, m_{02}m_{03}m_{12}m_{13}) = 1\\ \gcd(2^{\sigma_0}, b_1) = \gcd(2^{\sigma_1}, b_0) = \gcd(2^{\sigma_2}, b_3) = \gcd(2^{\sigma_3}, b_2) = 1,\\ 2^{\sigma_0+\sigma_1}m_{02}m_{03}m_{12}m_{13} \neq 1,\end{cases} \tag{2.8.24}$$

$$\begin{cases} \gcd(k_2l_2, 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}b_3) = \gcd(k_3l_3, 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}b_2) = 1, \\ \gcd(k_2l_2k_3l_3, 2) = 1, \end{cases} \tag{2.8.25}$$

and

$$k_2l_2 \equiv -q_0 \bmod 8,\ k_3l_3 \equiv q_1 \bmod 8,\ k_3l_3 \equiv -q_2 \bmod 8,\ k_2l_2 \equiv q_3 \bmod 8. \tag{2.8.26}$$

We note here that

$$(-1)^{\frac{k_2k_3-1}{2}}\left(\frac{2^{\sigma_0+\sigma_1+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_2k_3}\right) = \left(\frac{-2^{\sigma_0+\sigma_1+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_2k_3}\right).$$

Considering (2.8.26), we again note that any $\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$ we consider must satisfy

$$q_0 \equiv -q_3 \bmod 8 \text{ and } q_1 \equiv -q_2 \bmod 8, \tag{2.8.27}$$

and apply (2.8.6) to break to congruence conditions in (2.8.26) using (2.8.27). Putting this into $V'_{1,5}(B)$ gives:

$$V'_{2,5}(B) = \frac{1}{16}\sum_{\chi,\chi' \text{ char. mod } 8}\sum \overline{\chi}(-q_0)\overline{\chi'}(q_1)V'_{2,5}(\chi,\chi',B) \ll \sum_{\chi,\chi' \text{ char. mod } 8}\sum \left|V'_{2,5}(\chi,\chi',B)\right|,$$

where $V'_{2,5}(\chi,\chi',B)$ is defined as

$$\sum\sum\sum\sum_{\substack{k_2,l_2,k_3,l_3\in\mathbb{N}^4 \\ (2.5.16),\ (2.8.23),\ (2.8.25)}} \frac{\mu^2(2k_2l_2k_3l_3)\chi(k_2l_2)\chi'(k_3l_3)}{\tau\left(k_2l_2k_3l_3\right)}\left(\frac{-2^{\sigma_0+\sigma_1+v_2(m_{02}m_{03}m_{12}m_{13})}}{k_2k_3}\right).$$

Lemma 2.8.9. *Fix* $\mathbf{m}$ *and* $\boldsymbol{\sigma}$ *satisfying the conditions* (2.5.2), $(m_{02}m_{03}m_{12}m_{13})_{odd} = 1$ *and* (2.8.24). *Then for any character* χ *modulo* 8 *at least one of the characters*

$$\left(\frac{-2^{\sigma_0+\sigma_1+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)\chi(\cdot) \text{ and } \chi(\cdot)$$

is not principal.

Proof. For any $\mathbf{m}$ and $\boldsymbol{\sigma}$ satisfying (2.8.24) we have $\left(\frac{-2^{\sigma_2+\sigma_3+v_3(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right) = \left(\frac{-2}{\cdot}\right)$. It follows that, in order for $\left(\frac{-2^{\sigma_2+\sigma_3+v_2(m_{02}m_{03}m_{12}m_{13})}}{\cdot}\right)\chi(\cdot)$ to be non-principal, we must have $\chi(\cdot) = \left(\frac{-2}{\cdot}\right)$, in which case χ is non-principal. □

Now we may repeat the argument of §2.8.1, applying Proposition 2.2.9 and summing over the remaining variables to obtain the following.

Proposition 2.8.10. *Fix some* $\mathbf{b} \in \mathbb{N}^4$. *Then*

$$\mathcal{V}_{2,5}(B,\mathbf{b}) \ll_A \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2(\log B)}.$$

2.8.5 Bounding of $N_{r,4}(B)$ and $N_{r,5}(B)$

We may now prove the following:

Proposition 2.8.11. *Let $B \geqslant 3$. Then*

$$N_{r,4}(B), N_{r,5}(B) \ll_A \frac{B^2\sqrt{\log\log B}}{(\log B)}.$$

Proof. By Propositions 2.7.6,2.7.7,2.8.3,2.8.6,2.8.8 and 2.8.10 we have

$$N_{r,i}(B) \ll_A \sum_{\mathbf{b}\in\mathbb{N}^4} \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2b_1^2b_2^2b_3^2(\log B)} \ll_A \frac{B^2\sqrt{\log\log B}}{(\log B)}$$

for $r = 1, 2$ and $i = 4, 5$. □

2.9 Main Terms

In this section we will finally isolate the true main terms from $\mathcal{M}_{r,i}(B, \mathbf{b})$ where $r \in \{1, 2\}$ and $i \in \{2, 3\}$. This is achieved by trimming the remaining contributions from summing over oscillating characters in these regions. Recall that the inner sums of $\mathcal{M}_{r,2}(B, \mathbf{b})$ and $\mathcal{M}_{r,3}(B, \mathbf{b})$ are of the form (2.5.27) within the regions (2.5.22) and (2.5.23) and such that $\mathbf{d}, \tilde{\mathbf{d}} \in \{\mathbf{1}, \mathbf{m}_{\text{odd}}\}$ and are not equal. We split these inner sums into two. In $H_{r,2}(\mathbf{1}, \mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{L}, B)$ we separate the parts where $k_0k_1k_2k_3 = 1$ and in $H_{r,3}(\mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{K}, \mathbf{1}, B)$ we separate the parts where $l_0l_1l_2l_3 = 1$:

$$H_{r,2}(\mathbf{1}, \mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{L}, B) = M_{r,2}(\mathbf{1}, \mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{L}, B) + EM_{r,2}(\mathbf{1}, \mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{L}, B),$$

and

$$H_{r,3}(\mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{K}, \mathbf{1}, B) = M_{r,3}(\mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{K}, \mathbf{1}, B) + EM_{r,3}(\mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{K}, \mathbf{1}, B),$$

where

$$M_{r,2}(\mathbf{1}, \mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{L}, B) = \sum\sum\sum\sum_{\substack{\|l_0,l_1\|,\|l_2,l_3\|>z_1 \\ \mathbf{l}\equiv\mathbf{L} \bmod 8 \\ (2.9.5),\ (2.9.6)}} \frac{\mu^2(2l_0l_1l_2l_3)}{\tau\left(l_0l_1l_2l_3\right)}, \tag{2.9.1}$$

$$M_{r,3}(\mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{K}, \mathbf{1}, B) = \sum\sum\sum\sum_{\substack{\|k_0,k_1\|,\|k_2,k_3\|>z_1 \\ \mathbf{k}\equiv\mathbf{K} \bmod 8 \\ (2.9.7),\ (2.9.8)}} \frac{\mu^2(2k_0k_1k_2k_3)}{\tau\left(k_0k_1k_2k_3\right)}, \tag{2.9.2}$$

$$EM_{r,2}(\mathbf{1}, \mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{L}, B) = \sum\sum\sum\sum_{\substack{(\mathbf{k},\mathbf{l})\in\mathcal{H}_2 \\ (\mathbf{k},\mathbf{l})\equiv(\mathbf{1},\mathbf{L}) \bmod 8 \\ k_0k_1k_2k_3\neq 1 \\ (2.5.14),\ (2.5.15)}} \frac{\mu^2(2k_0l_0k_1l_1k_2l_2k_3l_3)}{\tau\left(k_0l_0k_1l_1k_2l_2k_3l_3\right)}\Theta_2(\mathbf{1}, \mathbf{m}_{\text{odd}}, \mathbf{k}, \mathbf{l}), \tag{2.9.3}$$

$$EM_{r,3}(\mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{K}, \mathbf{1}, B) = \sum\sum\sum\sum_{\substack{(\mathbf{k},\mathbf{l})\in\mathcal{H}_3 \\ (\mathbf{k},\mathbf{l})\equiv(\mathbf{K},\mathbf{1}) \bmod 8 \\ l_0l_1l_2l_3\neq 1 \\ (2.5.14),\ (2.5.15)}} \frac{\mu^2(2k_0l_0k_1l_1k_2l_2k_3l_3)}{\tau\left(k_0l_0k_1l_1k_2l_2k_3l_3\right)}\Theta_2(\mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{k}, \mathbf{l}), \tag{2.9.4}$$

$$\begin{cases} \gcd(l_0, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_1) = \gcd(l_1, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_0) = 1, \\ \gcd(l_2, 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}b_3) = \gcd(l_3, 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}b_2) = 1, \end{cases} \tag{2.9.5}$$

$$\|2^{\sigma_0}l_0m_{02}m_{03}b_0^2, 2^{\sigma_1}l_1m_{12}m_{13}b_1^2\| \cdot \|2^{\sigma_2}l_2m_{02}m_{12}b_2^2, 2^{\sigma_3}l_3m_{03}m_{13}b_3^2\| \leqslant B, \tag{2.9.6}$$

$$\begin{cases} \gcd(k_0, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_1) = \gcd(k_1, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_0) = 1, \\ \gcd(k_2, 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}b_3) = \gcd(k_3, 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}b_2) = 1, \end{cases} \tag{2.9.7}$$

$$\|2^{\sigma_0}k_0m_{02}m_{03}b_0^2, 2^{\sigma_1}k_1m_{12}m_{13}b_1^2\| \cdot \|2^{\sigma_2}k_2m_{02}m_{12}b_2^2, 2^{\sigma_3}k_3m_{03}m_{13}b_3^2\| \leqslant B. \tag{2.9.8}$$

Remark 2.9.1. Note that the we have dropped non-square condition (2.5.16). This is because the lower bounds $\|l_0, l_1\|, \|l_2, l_3\| > z_1$ and $\|k_0, k_1\|, \|k_2, k_3\| > z_1$ from $\mathcal{H}_2$ and $\mathcal{H}_3$ automatically ensure that it is satisfied.

2.9.1 The Error Terms $EM_{r,2}$ and $EM_{r,3}$

To deal with $EM_{r,2}(\mathbf{1}, \mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{L}, B)$ we note that

$$\Theta_2(\mathbf{1}, \mathbf{m}_{\text{odd}}, \mathbf{k}, \mathbf{l}) = \left(\frac{(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}{k_0k_1k_2k_3}\right)\left(\frac{l_0l_1}{k_2k_3}\right)\left(\frac{l_2l_3}{k_0k_1}\right).$$

From this we can see that the conditions $k_0k_1k_2k_3 \neq 1$ and $\mu^2(k_0k_1k_2k_3) = 1$ dictate that, by summing first over the l_i, we are summing over non-trivial characters. Similarly, for the error term $EM_{r,3}(\mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{K}, \mathbf{1}, B)$ we note that

$$\Theta_2(\mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{k}, \mathbf{l}) = \left(\frac{l_0l_1l_2l_3}{(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}\right)\left(\frac{l_0l_1}{k_2k_3}\right)\left(\frac{l_2l_3}{k_0k_1}\right),$$

and so the same observation holds here with the k_i and l_i switched. Thus we may repeat the arguments used in §2.7.2 to obtain the following:

Lemma 2.9.2. *Fix some $\mathbf{b} \in \mathbb{N}^4$, some $\mathbf{m} \in \mathbb{N}^4$ satisfying (2.5.2), some $\boldsymbol{\sigma} \in \{0,1\}^4$ satisfying (2.5.7) and some $\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$. Then for $\mathbf{L} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$ satisfying (2.5.29) we have*

$$EM_{r,2}(\mathbf{1}, \mathbf{m}_{odd}, \mathbf{1}, \mathbf{L}, B) \ll_A \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2}{M_0M_1M_2M_3(\log B)(\log\log B)^{66A}}.$$

Similarly, for $\mathbf{K} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$ *satisfying* (2.5.29) *we have*

$$EM_{r,3}(\mathbf{m}_{odd}, \mathbf{1}, \mathbf{K}, \mathbf{1}, B) \ll_A \frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2}{M_0M_1M_2M_3(\log B)(\log\log B)^{66A}}.$$

Then by summing over $H_{r,2}(\mathbf{1}, \mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{L}, B)$ and $H_{r,3}(\mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{K}, \mathbf{1}, B)$, using the same methods used to prove Propositions 2.7.3 and 2.7.4 to sum over these error parts above, we obtain:

$$\begin{aligned}\mathcal{M}_{r,2}(B, \mathbf{b}) = &\sum_{\substack{\mathbf{m}\in\mathbb{N}^4 \\ m_{ij}\leqslant z_0 \\ (2.5.2)}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4 \\ (2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum_{\substack{\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.5.29)}} \frac{M_{r,2}(\mathbf{1}, \mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{L}, B)}{\tau\left((m_{02}m_{03}m_{12}m_{13})_{\text{odd}}\right)} \\ &+ O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2}{b_0^2b_1^2b_2^2b_3^2(\log B)(\log\log B)^{66A}}\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_{r,3}(B, \mathbf{b}) = &\sum_{\substack{\mathbf{m}\in\mathbb{N}^4 \\ m_{ij}\leqslant z_0 \\ (2.5.2)}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4 \\ (2.5.7)}} \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum_{\substack{\mathbf{K}\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.5.29)}} \frac{\Theta_{r,1}(\mathbf{m}_{\text{odd}}, \mathbf{K}, \boldsymbol{\sigma})M_{r,3}(\mathbf{m}_{\text{odd}}, \mathbf{1}, \mathbf{K}, \mathbf{1}, B)}{\tau\left((m_{02}m_{03}m_{12}m_{13})_{\text{odd}}\right)} \\ &+ O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2}{b_0^2b_1^2b_2^2b_3^2(\log B)(\log\log B)^{66A}}\right).\end{aligned}$$

2.9.2 The Inner Main Terms $M_{r,2}$ and $M_{r,3}$

In this subsection, we deal with the inner sums. Once this is done, we will only be left to compute the constants. Define

$$\overline{\mathfrak{S}}(\mathbf{b}, \mathbf{m}, v) = \frac{4f_0^4}{\phi(8)^4 f_2^4} \prod_{i=0}^{3}\left(\prod_{\substack{p\mid v \\ p\nmid m_{02}m_{12}m_{03}m_{13}b_i \\ p \text{ odd}}} f_p^{-1}\right)\prod_{i=0}^{3}\left(\prod_{\substack{p\mid m_{02}m_{12}m_{03}m_{13}b_i \\ p \text{ odd}}} f_p^{-1}\right),$$

and

$$\mathfrak{C}(\mathbf{b}, \mathbf{m}, \boldsymbol{\sigma}) = \sum_{v\in\mathbb{N}} \mu(v) \sum\sum\sum\sum_{\substack{a_0,a_1,a_2,a_3\in\mathbb{N} \\ p\mid a_0a_1a_2a_3\Rightarrow p\mid v \\ v^2\mid a_0a_1a_2a_3 \\ (2.9.9)}} \frac{\overline{\mathfrak{S}}(\mathbf{b}, \mathbf{m}, v)}{a_0a_1a_2a_3\tau(a_0)\tau(a_1)\tau(a_2)\tau(a_3)}$$

where

$$\begin{cases}\gcd(a_0, 2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}b_1) = \gcd(a_1, 2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}b_0) = 1, \\ \gcd(a_2, 2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}b_3) = \gcd(a_3, 2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}b_2) = 1, \\ \gcd(a_0a_1a_2a_3, 2) = 1.\end{cases} \tag{2.9.9}$$

Remark 2.9.3. We note here that $\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m},v)$ is obtained by re-arranging $\mathfrak{S}_2(\mathbf{r})$ from Proposition 2.2.6 with $\mathbf{r}=(r_0,r_1,r_2,r_3)$ being equal to

$$\begin{aligned}&(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}vb_1,\\&(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}vb_0,\\&(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}vb_3,\\&(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}vb_2.\end{aligned}$$

respectively, using the conditions (2.5.2), (2.5.7) and (2.9.9).

Lemma 2.9.4. *Let $B\geqslant 3$. Fix $\mathbf{b}\in\mathbb{N}^4$, fix $\mathbf{m}\in\mathbb{N}^4$ and $\boldsymbol{\sigma}\in\{0,1\}^4$ satisfying* (2.5.2) *and* (2.5.7)*. Then*

$$\begin{aligned}M_{r,i}(\mathbf{1},\mathbf{m}_{odd},\mathbf{1},\mathbf{L},B)=&\frac{\mathfrak{C}(\mathbf{b},\mathbf{m},\boldsymbol{\sigma})B^2\log\log B}{M_0M_1M_2M_3\log B}\\&+O_A\left(\frac{\tau(m_{02}m_{03}m_{12}m_{13})^4\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{M_0M_1M_2M_3\log B}\right)\end{aligned}$$

for $i=2,3$.

Proof. By switching l_i with k_i the proofs for $i=2$ and $i=3$ may be seen to be identical. We therefore only deal with $M_{r,2}(\mathbf{1},\mathbf{m}_{\text{odd}},\mathbf{1},\mathbf{L},B)$. Recall (2.9.1) and Remark 2.9.1. By applying (2.2.2) with $w_0=w_1=z_0$ as we have done previously $M_{r,2}(\mathbf{1},\mathbf{m}_{\text{odd}},\mathbf{1},\mathbf{L},B)$ becomes equal to

$$\sum_{v\leqslant z_0}\mu(v)\sum\sum\sum\sum_{\substack{l_0',l_1',l_2',l_3'\leqslant z_0\\ p|l_0'l_1'l_2'l_3'\Rightarrow p|v\\ v^2|l_0'l_1'l_2'l_3',(2.9.9)}}\frac{M'_{r,2}(\mathbf{1},\mathbf{m}_{\text{odd}},\mathbf{1},\mathbf{L},\mathbf{l}',B)}{\tau(l_0')\tau(l_1')\tau(l_2')\tau(l_3')}+O\left(\frac{B^2(\log B)}{M_0M_1M_2M_3(\log B)^{A/3}}\right) \tag{2.9.10}$$

where

$$M'_{r,2}(\mathbf{1},\mathbf{m}_{\text{odd}},\mathbf{1},\mathbf{L},\mathbf{l}',B)=\sum\sum\sum\sum_{\substack{\|l_0'l_0'',l_1'l_1''\|,\|l_2'l_2'',l_3'l_3''\|>z_1\\ l_i''\equiv L_i/l_i' \bmod 8\\ (2.9.11),\ (2.9.12)}}\frac{1}{\tau(l_0'')\tau(l_1'')\tau(l_2'')\tau(l_3'')},$$

$$\begin{cases}\gcd(l_0'',2^{\sigma_1}m_{02}m_{03}m_{12}m_{13}vb_1)=\gcd(l_1'',2^{\sigma_0}m_{02}m_{03}m_{12}m_{13}vb_0)=1,\\ \gcd(l_2'',2^{\sigma_3}m_{02}m_{03}m_{12}m_{13}vb_3)=\gcd(l_3'',2^{\sigma_2}m_{02}m_{03}m_{12}m_{13}vb_2)=1,\end{cases} \tag{2.9.11}$$

$$\|2^{\sigma_0}l_0'l_0''m_{02}m_{03}b_0^2,2^{\sigma_1}l_1'l_1''m_{12}m_{13}b_1^2\|\cdot\|2^{\sigma_2}l_2'l_2''m_{02}m_{12}b_2^2,2^{\sigma_3}l_3'l_3''m_{03}m_{13}b_3^2\|\leqslant B. \tag{2.9.12}$$

Now we apply Proposition 2.2.6 with the previous remark to these sums yielding that $M'_{r,2}(\mathbf{1},\mathbf{m}_{\text{odd}},\mathbf{1},\mathbf{L},\mathbf{l}',B)$ is:

$$\begin{aligned}&=\frac{\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m},v)B^2\log\log B}{l_0'l_1'l_2'l_3'M_0M_1M_2M_3\log B}\\&+O_A\left(\frac{\tau(m_{02}m_{03}m_{12}m_{13})^4\tau(v)^4\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{l_0'l_1'l_2'l_3'M_0M_1M_2M_3\log B}\right).\end{aligned}$$

Now substitute this into (2.9.10). Then we have found $M_{r,2}(\mathbf{1}, \mathbf{m}_{\mathrm{odd}}, \mathbf{1}, \mathbf{L}, B)$ to be

$$=\frac{B^2 \log\log B}{M_0M_1M_2M_3 \log B} \sum_{v \leqslant z_0} \mu(v) \sum\sum\sum\sum_{\substack{l_0',l_1',l_2',l_3' \leqslant z_0 \\ p \mid l_0'l_1'l_2'l_3' \Rightarrow p \mid v \\ v^2 \mid l_0'l_1'l_2'l_3' \\ (2.9.9)}} \frac{\overline{\mathfrak{S}}(\mathbf{b}, \mathbf{m}, v)}{l_0'l_1'l_2'l_3'\tau(l_0')\tau(l_1')\tau(l_2')\tau(l_3')}$$
$$+ O_A\left(\frac{\tau(m_{02}m_{03}m_{12}m_{13})^4\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)\mathcal{R}'B^2\sqrt{\log\log B}}{M_0M_1M_2M_3\log B}\right)$$

To deal with the error term it is enough to show that

$$\mathcal{R}' = \sum_{v \leqslant z_0} \sum\sum\sum\sum_{\substack{l_0',l_1',l_2',l_3' \leqslant z_0 \\ p \mid l_0'l_1'l_2'l_3' \Rightarrow p \mid v \\ v^2 \mid l_0'l_1'l_2'l_3', (2.9.9)}} \frac{\tau(v)^4}{l_0'l_1'l_2'l_3'\tau(l_0')\tau(l_1')\tau(l_2')\tau(l_3')} \ll 1,$$

which is done using an identical method used to deal with (2.7.9). Finally, we need to show that

$$\sum_{v \leqslant z_0} \mu(v) \sum\sum\sum\sum_{\substack{l_0',l_1',l_2',l_3' \leqslant z_0 \\ p \mid l_0'l_1'l_2'l_3' \Rightarrow p \mid v \\ v^2 \mid l_0'l_1'l_2'l_3', (2.9.9)}} \frac{\overline{\mathfrak{S}}(\mathbf{b}, \mathbf{m}, v)}{l_0'l_1'l_2'l_3'\tau(l_0')\tau(l_1')\tau(l_2')\tau(l_3')} = \mathfrak{C}(\mathbf{b}, \mathbf{m}, \boldsymbol{\sigma}) + O\left(\frac{1}{z_0^{1/3}}\right).$$

This may be seen by noting that the following sums are $O(z_0^{-1/3})$:

$$\sum_{v > z_0} \sum\sum\sum\sum_{\substack{l_0',l_1',l_2',l_3' \in \mathbb{N} \\ p \mid l_0'l_1'l_2'l_3' \Rightarrow p \mid v \\ v^2 \mid l_0'l_1'l_2'l_3', (2.9.9)}} \frac{1}{l_0'l_1'l_2'l_3'\tau(l_0')\tau(l_1')\tau(l_2')\tau(l_3')}, \sum_{v \leqslant z_0} \sum\sum\sum\sum_{\substack{l_0',l_1',l_2',l_3' \in \mathbb{N} \\ p \mid l_0'l_1'l_2'l_3' \Rightarrow p \mid v \\ v^2 \mid l_0'l_1'l_2'l_3' \\ l_i > z_0 \text{ for some } i}} \frac{1}{l_0'l_1'l_2'l_3'\tau(l_0')\tau(l_1')\tau(l_2')\tau(l_3')}$$

which may be shown by adhering to the bounding of the error terms (2.2.1) and (2.2.2) of Lemma 2.2.2. The error term of the $M_{r,i}$ resulting from the error term $z_0^{-1/3}$ above will be absorbed into the error terms already present. $\square$

2.9.3 The Main Terms $\mathcal{M}_{r,2}(\mathbf{b}, B)$ and $\mathcal{M}_{r,3}(\mathbf{b}, B)$

Define

$$\Sigma_{r,2}(\mathbf{m}, \boldsymbol{\sigma}) = \sum_{\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})} \sum_{\substack{\mathbf{L} \in (\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.5.29)}} 1,$$

$$\Sigma_{r,3}(\mathbf{m}, \boldsymbol{\sigma}) = \sum_{\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})} \sum_{\substack{\mathbf{K} \in (\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.5.29)}} \Theta_{r,1}(\mathbf{m}_{\mathrm{odd}}, \mathbf{K}, \boldsymbol{\sigma}),$$

and

$$\mathfrak{C}_{r,i}(\mathbf{b}) = \sum_{\substack{\mathbf{m} \in \mathbb{N}^4 \\ (2.5.2)}} \sum_{\substack{\boldsymbol{\sigma} \in \{0,1\}^4 \\ (2.5.7)}} \frac{\mathfrak{C}(\mathbf{b}, \mathbf{m}, \boldsymbol{\sigma})\Sigma_{r,i}(\mathbf{m}, \boldsymbol{\sigma})}{2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}m_{02}^2m_{03}^2m_{12}^2m_{13}^2\tau((m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}})}$$

for $i = 2, 3$.

Proposition 2.9.5. *Let $B \geqslant 3$ and fix some $\mathbf{b} \in \mathbb{N}^4$. Then,*

$$\mathcal{M}_{r,i}(\mathbf{b}, B) = \frac{\mathfrak{C}_{r,i}(\mathbf{b})B^2 \log\log B}{b_0^2 b_1^2 b_2^2 b_3^2 \log B} + O_A\left(\frac{\tau(b_0)\tau(b_1)\tau(b_2)\tau(b_3)B^2\sqrt{\log\log B}}{b_0^2 b_1^2 b_2^2 b_3^2 \log B}\right),$$

for $i = 2, 3$.

Proof. The error terms above are obtained by summing the error terms of Lemma 2.9.4 over the finitely many $\mathbf{L}$ (or $\mathbf{K}$) in $(\mathbb{Z}/8\mathbb{Z})^{*4}$, finitely many $\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})$, finitely many $\boldsymbol{\sigma} \in \{0, 1\}^4$ and finally summing the $\frac{\tau(m_{ij})^4}{m_{ij}^2}$ over $\mathbf{m} \in \mathbb{N}^4$ which converges. For the main term we note that $\mathfrak{C}(\mathbf{b}, \mathbf{m}, \boldsymbol{\sigma})$ is independent of $\mathbf{q}$ and $\mathbf{L}$ (or $\mathbf{K}$) and that $\mathfrak{C}(\mathbf{b}, \mathbf{m}, \boldsymbol{\sigma}), \Sigma_{r,i}(\mathbf{m}, \boldsymbol{\sigma}) \ll 1$ independently of $\mathbf{b}, \mathbf{m}$ and $\boldsymbol{\sigma}$ so that we may extend the sum over the m_{ij} to all of $\mathbb{N}^4$ at the cost of a sufficient error term. It is then clear that we may re-arrange the sums in the remaining variables to obtain $\mathfrak{C}_{r,2}(\mathbf{b})$ (or $\mathfrak{C}_{r,3}(\mathbf{b})$). This concludes the result. □

Finally, by setting

$$\mathfrak{C}_{r,i} = \sum_{\substack{\mathbf{b} \in \mathbb{N}^4 \\ (2.5.1)}} \frac{\mathfrak{C}_{r,i}(\mathbf{b})}{b_0^2 b_1^2 b_2^2 b_3^2}$$

for $i = 2, 3$ and summing the results of Propositions 2.7.3, 2.7.4 and 2.9.5 over $b_i \leqslant z_0$ and then extending to $\mathbb{N}^4$ at the cost of another error term, we obtain the following:

Proposition 2.9.6. *For $B \geqslant 3$ we have*

$$N_{r,i}(B) = \frac{\mathfrak{C}_{r,i} B^2 \log\log B}{\log B} + O_A\left(\frac{B^2\sqrt{\log\log B}}{\log B}\right)$$

when $i = 2, 3$.

2.10 The Constants

In this section we would like to express $\mathfrak{C}_{r,i}$ in a more concise manner. First we will compute the $\Sigma_{r,i}(\mathbf{m}, \boldsymbol{\sigma})$:

2.10.1 Computation of $\Sigma_{r,2}$

Recall that

$$\Sigma_{r,2}(\mathbf{m}, \boldsymbol{\sigma}) = \sum_{\mathbf{q} \in \mathcal{A}(\mathbf{m}, \boldsymbol{\sigma})} \sum_{\substack{\mathbf{L} \in (\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.5.29)}} 1.$$

We first deal with the inner sum. We break the conditions (2.5.29) using the orthogonality relation (2.8.6). This time however, there is no easy relation between the q_i and so we use this relation to replace all four of the congruence relations above. It follows that

$$\Sigma_{r,2}(\mathbf{m},\boldsymbol{\sigma}) = \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum_{\substack{\chi,\chi',\chi'',\chi''' \\ \text{char. mod } 8}} \overline{\chi}(-\delta_r q_0 (m_{03}m_{12})_{\text{odd}}^{-1})\overline{\chi'}(q_1(m_{03}m_{12})_{\text{odd}}^{-1}) \\ \times \overline{\chi''}(\delta_r q_2(m_{02}m_{13})_{\text{odd}}^{-1})\overline{\chi'''}(-q_3(m_{02}m_{13})_{\text{odd}}^{-1})\Sigma_{r,2}(\mathbf{m},\boldsymbol{\sigma},\boldsymbol{\chi})$$

where, $\Sigma_{r,2}(\mathbf{m},\boldsymbol{\sigma},\boldsymbol{\chi})$ is given by

$$\begin{aligned}
&= \frac{1}{4^4} \sum_{\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4}} \chi(L_0L_2)\chi'(L_1L_3)\chi''(L_1L_2)\chi'''(L_0L_3),\\
&= \frac{1}{4^4} \sum_{\mathbf{L}\in(\mathbb{Z}/8\mathbb{Z})^{*4}} \chi\chi'''(L_0)\chi'\chi''(L_1)\chi\chi''(L_2)\chi'\chi'''(L_3),\\
&=\left(\frac{1}{4}\sum_{L_0\in(\mathbb{Z}/8\mathbb{Z})^*}\chi\chi'''(L_0)\right)\left(\frac{1}{4}\sum_{L_1\in(\mathbb{Z}/8\mathbb{Z})^*}\chi'\chi''(L_1)\right)\left(\frac{1}{4}\sum_{L_2\in(\mathbb{Z}/8\mathbb{Z})^*}\chi\chi''(L_2)\right)\left(\frac{1}{4}\sum_{L_3\in(\mathbb{Z}/8\mathbb{Z})^*}\chi'\chi'''(L_3)\right),\\
&= \mathbb{1}(\chi\chi'''=\chi_0)\mathbb{1}(\chi'\chi''=\chi_0)\mathbb{1}(\chi\chi''=\chi_0)\mathbb{1}(\chi'\chi'''=\chi_0),
\end{aligned}$$

and χ_0 denotes the principal character modulo 8. Since the group of principal characters modulo 8 is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, every element is equal to its own inverse. This leads us to deduce that

$$\mathbb{1}(\chi\chi'''=\chi_0)\mathbb{1}(\chi'\chi''=\chi_0)\mathbb{1}(\chi\chi''=\chi_0)\mathbb{1}(\chi'\chi'''=\chi_0) = \mathbb{1}(\chi=\chi'=\chi''=\chi''') \quad (2.10.1)$$

Thus, $\Sigma_{r,2}(\mathbf{m},\boldsymbol{\sigma},\boldsymbol{\chi}) = \mathbb{1}(\chi=\chi'=\chi''=\chi''')$. Substituting this into the expression for $\Sigma_{r,2}(\mathbf{m},\boldsymbol{\sigma})$ we obtain

$$\begin{aligned}
\Sigma_{r,2}(\mathbf{m},\boldsymbol{\sigma}) &= \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})}\sum_{\chi \text{ char. mod } 8} \overline{\chi}(q_0q_1q_2q_3(-\delta_r(m_{02}m_{03}m_{12}m_{13})_{\text{odd}})^{-2})\\
&= \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})}\sum_{\chi \text{ char. mod } 8} \overline{\chi}(q_0q_1q_2q_3)\\
&= 4 \sum_{\substack{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})\\ q_0q_1q_2q_3\equiv 1 \bmod 8}} 1 = 4\sharp\{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma}) : q_0q_1q_2q_3\equiv 1 \bmod 8\}.
\end{aligned}$$

Now, recalling the definition of $\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})$, it follows that $\Sigma_{r,2}(\mathbf{m},\boldsymbol{\sigma})$ is exactly equal to

$$\begin{cases} 4\sharp\{\mathbf{q}\in\mathcal{A}_1 : q_0q_1q_2q_3\equiv 1 \bmod 8\} \text{ if } 2\nmid m_{02}m_{03}m_{12}m_{13} \;\&\; \sigma_i=0\;\forall\, i\in\{0,1,2,3\},\\ 4\sharp\{\mathbf{q}\in\mathcal{A}_2 : q_0q_1q_2q_3\equiv 1 \bmod 8\} \text{ otherwise.}\end{cases}$$

This is because $\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})$ is equal to $\mathcal{A}_1$ in the first case and is easily seen to be bijective to $\mathcal{A}_2$ by applying a single permutation to the components of each of its elements otherwise, an operation which does not effect the condition $q_0q_1q_2q_3\equiv 1 \bmod 8$. This extra condition significantly simplifies the counting process. This is done in the following lemma:

Lemma 2.10.1. *We have the following*

$$\Sigma_{r,2}(\mathbf{m},\boldsymbol{\sigma}) = \begin{cases} 192 \textit{ if } 2\nmid m_{02}m_{03}m_{12}m_{13} \;\&\; \sigma_i=0\;\forall\, i\in\{0,1,2,3\},\\ 128 \textit{ otherwise.}\end{cases}$$

Proof. First assume that $2 \nmid m_{02}m_{03}m_{12}m_{13}$ & $\sigma_i = 0\ \forall\ i \in \{0,1,2,3\}$. Then we want compute $\sharp\{\mathbf{q} \in \mathcal{A}_1 : q_0q_1q_2q_3 \equiv 1 \bmod 8\}$. Substituting in the definition of $\mathcal{A}_1$ this is then equal to $\sharp\{\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4} : \mathbf{q} \text{ satisfies } (2.10.2)\}$ where

$$\begin{cases}
(A.1.1) : q_0 + q_2 \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.2) : q_0 + q_3 \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.3) : q_1 + q_2 \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.4) : q_1 + q_3 \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.5) : q_0 + q_2 \equiv 4 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.6) : q_0 + q_3 \equiv 4 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.7) : q_1 + q_2 \equiv 4 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.8) : q_1 + q_3 \equiv 4 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.9) : q_0 + q_1 \equiv 0 \bmod 8 \text{ and } q_2 + q_3 \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.10) : q_0 + q_1 \equiv 2 \bmod 8 \text{ and } q_2 + q_3 \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.11) : q_0 + q_1 \equiv 2 \bmod 8 \text{ and } q_2 + q_3 \equiv 6 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.12) : q_0 + q_1 \equiv 0 \bmod 8 \text{ and } q_2 + q_3 \equiv 6 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.13) : q_0 + q_1 \equiv 6 \bmod 8 \text{ and } q_2 + q_3 \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.14) : q_0 + q_1 \equiv 6 \bmod 8 \text{ and } q_2 + q_3 \equiv 2 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,}\\
(A.1.15) : q_0 + q_1 \equiv 0 \bmod 8 \text{ and } q_2 + q_3 \equiv 2 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8.
\end{cases} \tag{2.10.2}$$

Some elementary congruence computations then give

$$\begin{cases}
(A.1.1), (A.1.4) : \text{solutions are exactly those vectors } (a, b, -a, -b) \text{ for any } a, b \in (\mathbb{Z}/8\mathbb{Z})^*;\\
(A.1.2), (A.1.3) : \text{solutions are exactly those vectors } (a, b, -b, -a) \text{ for any } a, b \in (\mathbb{Z}/8\mathbb{Z})^*;\\
(A.1.5), (A.1.8) : \text{solutions are exactly those vectors}(3a, b, a, 3b) \text{ for any } a, b \in (\mathbb{Z}/8\mathbb{Z})^*;\\
(A.1.6), (A.1.7) : \text{solutions are exactly those vectors}(3a, b, 3b, a) \text{ for any } a, b \in (\mathbb{Z}/8\mathbb{Z})^*;\\
(A.1.9) : \text{solutions are exactly those vectors } (a, -a, b, -b) \text{ for any } a, b \in (\mathbb{Z}/8\mathbb{Z})^*;\\
(A.1.11), (A.1.14) : \text{solutions are exactly those vectors}(a, 2 + 7a, 7a, 6 + a),\\
(a, 6 + 7a, 7a, 2 + a) \text{ respectively for any } a, b \in (\mathbb{Z}/8\mathbb{Z})^*;\\
(A.1.10), (A.1.12), (A.1.13), (A.1.15) \text{have no solutions in } (\mathbb{Z}/8\mathbb{Z})^{*4}.
\end{cases} \tag{2.10.3}$$

From here it is easy to check directly that there are exactly 48 distinct points in $(\mathbb{Z}/8\mathbb{Z})^{*4}$ of at least one of the forms given above (see Tables A.1-A.6 in the appendix). Multiplying this by 4 gives the first case of the result. For the second case, if $2|m_{02}m_{03}m_{12}m_{13}$ or $\sigma_i = 1$ for some $i \in \{0,1,2,3\}$, then we want to compute $\sharp\{\mathbf{q} \in \mathcal{A}_2 : q_0q_1q_2q_3 \equiv 1 \bmod 8\}$, which, by substituting in the definition of $\mathcal{A}_2$, is the same as computing $\sharp\{\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4} : \mathbf{q} \text{ satisfies (2.10.4)}\}$ where

$$\begin{cases} (A.2.1): \ q_0+q_1 \equiv 0 \bmod 8, \ (q_2+1)(q_3+1) \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,} \\ (A.2.2): \ q_0+q_1 \equiv 0 \bmod 8, \ (q_2+3)(q_3+3) \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,} \\ (A.2.3): \ q_0+q_1 \equiv 0 \bmod 8, \ (q_2+5)(q_3+5) \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,} \\ (A.2.4): \ q_0+q_1 \equiv 0 \bmod 8, \ (q_2+7)(q_3+7) \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,} \\ (A.2.5): \ q_0+q_1 \equiv 2 \bmod 8, \ (q_2+1)(q_3+1) \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,} \\ (A.2.6): \ q_0+q_1 \equiv 2 \bmod 8, \ (q_2+5)(q_3+5) \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,} \\ (A.2.7): \ q_0+q_1 \equiv 6 \bmod 8, \ (q_2+3)(q_3+3) \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,} \\ (A.2.8): \ q_0+q_1 \equiv 6 \bmod 8, \ (q_2+7)(q_3+7) \equiv 0 \bmod 8 \text{ and } q_0q_1q_2q_3 \equiv 1 \bmod 8 \text{ or,} \\ (A.2.9)-(A.2.16): \ \text{same as } (A.2.1)-(A.2.8) \text{ with } q_0 \text{ and } q_1 \text{ switched} \\ \qquad\qquad\qquad \text{with } q_2 \text{ and } q_3 \text{ respectively.} \end{cases} \tag{2.10.4}$$

Congruence calculations then show:

$$\begin{cases} (A.2.1),(A.2.2),(A.2.3),(A.2.4): \text{solutions are exactly those vectors } (a,7a,1,7), \\ (a,7a,7,1),(a,7a,3,5),(a,7a,5,3) \text{ for any } a \in (\mathbb{Z}/8\mathbb{Z})^*; \\ (A.2.5),(A.2.6): \text{solutions are exactly those vectors } (a,2+7a,6a+1,7), \\ (a,2+7a,7,6a+1),(a,2+7a,6a+5,3),(a,2+7a,3,6a+5) \text{ for any } a \in (\mathbb{Z}/8\mathbb{Z})^*; \\ (A.2.7),(A.2.8): \text{solutions are exactly those vectors } (a,6+7a,6a+3,5), \\ (a,6+7a,5,6a+3),(a,6+7a,6a+7,1),(a,6+7a,1,6a+7) \text{ for any } a \in (\mathbb{Z}/8\mathbb{Z})^*; \\ (A.2.9)-(A.2.16): \text{solutions are solutions to } (A.2.1)-(A.2.8) \text{ with } q_0,q_1 \text{ swapped} \\ \text{with } q_2,q_3 \text{ respectively.} \end{cases} \tag{2.10.5}$$

As before it is now easy to check directly that there are exactly 32 distinct points in $(\mathbb{Z}/8\mathbb{Z})^{*4}$ of at least one of the forms given above (see Table A.7 in the appendix). Multiplying this by 4 gives the second part of the result. □

2.10.2 Computation of $\Sigma_{r,3}(\mathbf{m},\boldsymbol{\sigma})$

Recall that

$$\Sigma_{r,3}(\mathbf{m},\boldsymbol{\sigma}) = \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \sum_{\substack{\mathbf{K}\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.10.6)}} \Theta_{r,1}(\mathbf{m}_{\mathrm{odd}},\mathbf{K},\boldsymbol{\sigma}),$$

where $\Theta_{r,1}(\mathbf{m}_{\mathrm{odd}},\mathbf{K},\boldsymbol{\sigma})$ is defined as

$$(-1)^{f_r(\mathbf{m}_{\mathrm{odd}},\mathbf{K})} \left(\frac{2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}}{(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}}\right)\left(\frac{2^{\sigma_2+\sigma_3}}{K_0K_1}\right)\left(\frac{2^{\sigma_0+\sigma_1}}{K_2K_3}\right)\left(\frac{2^{v_2(m_{02}m_{03}m_{12}m_{13})}}{K_0K_1K_2K_3}\right)$$

and

$$\begin{cases} K_0K_2(m_{03}m_{12})_{\mathrm{odd}} \equiv -\delta_r q_0 \bmod 8, \ K_1K_3(m_{03}m_{12})_{\mathrm{odd}} \equiv q_1 \bmod 8, \\ K_1K_2(m_{02}m_{13})_{\mathrm{odd}} \equiv \delta_r q_2 \bmod 8, \ K_0K_3(m_{02}m_{13})_{\mathrm{odd}} \equiv -q_3 \bmod 8. \end{cases} \tag{2.10.6}$$

We may then use (2.10.6) to write $f_r(\mathbf{m}_{\mathrm{odd}}, \mathbf{K})$ as

$$\equiv \frac{(K_0K_1 + (\delta_r - 2)K_2K_3)(q_0q_2 + 1) + (1-\delta_r)(K_2K_3 - 1)}{4} \bmod 2$$

to remove the dependence on $\mathbf{m}_{\mathrm{odd}}$. Now notice that

$$4 \mid (K_0K_1 + (\delta_r - 2)K_2K_3)(q_0q_2 + 1) \text{ and } 4 \mid (1-\delta_r)(K_2K_3 - 1).$$

Thus we write

$$(-1)^{f_r(\mathbf{m},\mathbf{K})} = (-1)^{\tilde{f}_r(\mathbf{q},\mathbf{K})} \left(\frac{-1}{K_2K_3}\right)^{\frac{(1-\delta_r)}{2}},$$

where

$$\tilde{f}_r(\mathbf{q},\mathbf{K}) = \frac{(K_0K_1 + (\delta_r - 2)K_2K_3)(q_0q_2 + 1)}{4}.$$

Once more appealing to (2.10.6) we may see that

$$\begin{aligned} \tilde{f}_r(\mathbf{q},\mathbf{K}) &\equiv \frac{(-q_0q_2 + (1-2\delta_r)q_0q_3)(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}(q_0q_2+1)}{4} \bmod 2 \\ &\equiv \frac{((2\delta_r - 1)q_0q_3 + q_0q_2)(q_0q_2+1)}{4} \bmod 2 \end{aligned}$$

since $(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}$ is odd. Now by expanding out the numerator and recalling that (2.10.6) asserts that $q_0q_1q_2q_3 \equiv 1 \bmod 8$,

$$\begin{aligned} \tilde{f}_r(\mathbf{q},\mathbf{K}) &\equiv \frac{((2\delta_r - 1)q_0q_3 + q_0q_2)(q_0q_2+1)}{4} \bmod 2 \\ &\equiv \frac{q_0(q_0 + q_2 + (2\delta_r - 1)(q_1 + q_3))}{4} \bmod 2 \\ &\equiv \frac{(q_0 + q_2 + (2\delta_r - 1)(q_1 + q_3))}{4} \bmod 2. \end{aligned}$$

Setting the final ratio above to be $\tilde{f}_r(\mathbf{q})$, we therefore write

$$\tilde{\Theta}_{r,1}(\mathbf{m},\mathbf{q},\boldsymbol{\sigma}) = (-1)^{\tilde{f}_r(\mathbf{q})} \left(\frac{2^{\sigma_0+\sigma_1}}{\delta_r q_0 q_3}\right) \left(\frac{2^{\sigma_2+\sigma_3}}{7q_0q_2}\right) \left(\frac{2^{v_2(m_{02}m_{03}m_{12}m_{13})}}{7\delta_r q_0 q_1}\right),$$

and thus obtain

$$\Sigma_{r,3}(\mathbf{m},\boldsymbol{\sigma}) = 4 \sum_{\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})} \tilde{\Theta}_{r,1}(\mathbf{m},\mathbf{q},\boldsymbol{\sigma}) \sum_{\substack{\mathbf{K}\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ (2.10.6)}} \left(\frac{-1}{K_2K_3}\right)^{\frac{(1-\delta_r)}{2}}.$$

Using the orthogonality relation (2.8.6) to break the condition (2.10.6) the inner most sum over $\mathbf{K}$ becomes

$$\begin{aligned} &\sum_{\substack{\chi,\chi',\chi'',\chi''' \\ \text{char. mod} 8}} \overline{\chi}(-\delta_r q_0 (m_{03}m_{12})_{\mathrm{odd}}^{-1}) \overline{\chi'}(q_1 (m_{03}m_{12})_{\mathrm{odd}}^{-1}) \\ &\qquad \times \overline{\chi''}(\delta_r q_2 (m_{02}m_{13})_{\mathrm{odd}}^{-1}) \overline{\chi'''}(-q_3 (m_{02}m_{13})_{\mathrm{odd}}^{-1}) \Sigma_{r,3}(\mathbf{m},\boldsymbol{\sigma},\boldsymbol{\chi}) \end{aligned}$$

where

$$\begin{aligned}\Sigma_{r,3}(\mathbf{m},\boldsymbol{\sigma},\boldsymbol{\chi}) &= \frac{1}{4^4}\sum_{\mathbf{K}\in(\mathbb{Z}/8\mathbb{Z})^{*4}} \chi(K_0K_2)\chi'(K_1K_3)\chi''(K_1K_2)\chi'''(K_0K_3)\left(\frac{-1}{K_2K_3}\right)^{\frac{(1-\delta_r)}{2}} \\ &= \mathbb{1}\left(\chi=\chi'\left(\frac{-1}{\cdot}\right)^{\frac{(1-\delta_r)}{2}}=\chi''\left(\frac{-1}{\cdot}\right)^{\frac{(1-\delta_r)}{2}}=\chi'''\right)\end{aligned}$$

using the same method as for $\Sigma_{r,2}(\mathbf{m},\boldsymbol{\sigma},\boldsymbol{\chi})$ previously, noting that the reasoning behind (2.10.1) also applies here. Then

$$\begin{aligned}\sum_{\substack{\mathbf{K}\in(\mathbb{Z}/8\mathbb{Z})^{*4}\\ (2.10.6)}}\left(\frac{-1}{K_2K_3}\right)^{\frac{(1-\delta_r)}{2}} &= \sum_{\substack{\chi \text{ char.}\\ \text{mod } 8}} \overline{\chi}(q_0q_1q_2q_3)\left(\frac{-1}{\delta_r q_1q_2(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}^{-1}}\right)^{\frac{(1-\delta_r)}{2}} \\ &= \mathbb{1}(q_0q_1q_2q_2\equiv 1 \bmod 8)\left(\frac{-1}{\delta_r q_1q_2(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}\right)^{\frac{(1-\delta_r)}{2}}.\end{aligned}$$

Collecting this information gives

$$\Sigma_{r,3}(\mathbf{m},\boldsymbol{\sigma}) = 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}\right)^{\frac{(1-\delta_r)}{2}} \sum_{\substack{\mathbf{q}\in\mathcal{A}\\ q_0q_1q_2q_3\equiv 1 \bmod 8}} \Theta'_{r,1}(\mathbf{m},\boldsymbol{\sigma},\mathbf{q})$$

where

$$\begin{aligned}\Theta'_{r,1}(\mathbf{m},\boldsymbol{\sigma},\mathbf{q}) &= (-1)^{\tilde{f}_r(\mathbf{q})}\left(\frac{2^{\sigma_0+\sigma_1}}{\delta_r q_0q_3}\right)\left(\frac{2^{\sigma_2+\sigma_3}}{7q_0q_2}\right)\left(\frac{2^{v_2(m_{02}m_{03}m_{12}m_{13})}}{7\delta_r q_0q_1}\right)\left(\frac{-1}{\delta_r q_1q_2}\right)^{\frac{(1-\delta_r)}{2}} \\ &= (-1)^{\tilde{f}_r(\mathbf{q})}\left(\frac{2^{\sigma_0+\sigma_1}}{q_0q_3}\right)\left(\frac{2^{\sigma_2+\sigma_3}}{q_0q_2}\right)\left(\frac{2^{v_2(m_{02}m_{03}m_{12}m_{13})}}{q_0q_1}\right)\left(\frac{-1}{\delta_r q_1q_2}\right)^{\frac{(1-\delta_r)}{2}}\end{aligned}$$

since $\left(\frac{2}{7}\right)=\left(\frac{2}{\delta_r}\right)=1$. We now split into cases $r=1$ and $r=2$.

The Case $r=1$

In this case,

$$\Theta'_{1,1}(\mathbf{m},\mathbf{q},\boldsymbol{\sigma}) = (-1)^{\tilde{f}_1(\mathbf{q})}\left(\frac{2^{\sigma_0+\sigma_1}}{q_0q_3}\right)\left(\frac{2^{\sigma_2+\sigma_3}}{q_0q_2}\right)\left(\frac{2^{v_2(m_{02}m_{03}m_{12}m_{13})}}{q_0q_1}\right)$$

and

$$\tilde{f}_1(\mathbf{q}) = \frac{(q_1+q_2+q_3+q_0)}{4}.$$

We note that, since $\mathbf{q}\in\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})$ and $q_0q_1q_2q_3 \equiv 1 \bmod 8$, this exponent is always an integer. This can be seen by noting that such $\mathbf{q}\in(\mathbb{Z}/8\mathbb{Z})^{*4}$ are component-wise permutations of points which are at least one of the forms given in (2.10.3) or (2.10.5), and it is easy to check that such points have a component sum which is 0 or 4 modulo 8. Now we split into cases determined by the values of $v_2(m_{02}m_{03}m_{12}m_{13})$ and $\boldsymbol{\sigma}$. Here the precise definition of $\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})$ is required since the relative positions of the q_i will effect the value of the Jacobi symbol. We have the following cases:

(a) if $2 \nmid m_{02}m_{03}m_{12}m_{13}$, and $\sigma_i = 0 \ \forall\ i \in \{0,1,2,3\}$,

$$\Sigma_{1,3}(\mathbf{m},\boldsymbol{\sigma}) = 4 \sum_{\substack{\mathbf{q}\in\mathcal{A}_1 \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{q_0+q_1+q_2+q_3}{4}},$$

(b) if $2 \mid m_{03}m_{12},\ 2 \nmid m_{02}m_{13}$ and $\sigma_i = 0 \ \forall\ i \in \{0,1,2,3\}$,

$$\Sigma_{1,3}(\mathbf{m},\boldsymbol{\sigma}) = 4 \sum_{\substack{\mathbf{q}\in\mathcal{A}_{0,1,2,3} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{q_0+q_1+q_2+q_3}{4}} \left(\frac{2}{q_0q_1}\right),$$

(c) if $2 \mid m_{02}m_{13},\ 2 \nmid m_{02}m_{13}$ and $\sigma_i = 0 \ \forall\ i \in \{0,1,2,3\}$,

$$\Sigma_{1,3}(\mathbf{m},\boldsymbol{\sigma}) = 4 \sum_{\substack{\mathbf{q}\in\mathcal{A}_{2,3,0,1} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{q_0+q_1+q_2+q_3}{4}} \left(\frac{2}{q_0q_1}\right),$$

(d) if $2 \nmid m_{03}m_{12}m_{02}m_{13},\ \sigma_0 = 1$ and $\sigma_i = 0 \ \forall\ i \in \{0,1,2,3\} \setminus \{0\}$,

$$\Sigma_{1,3}(\mathbf{m},\boldsymbol{\sigma}) = 4 \sum_{\substack{\mathbf{q}\in\mathcal{A}_{0,3,1,2} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{q_0+q_1+q_2+q_3}{4}} \left(\frac{2}{q_0q_3}\right),$$

(e) if $2 \nmid m_{03}m_{12}m_{02}m_{13}, \sigma_1 = 1$ and $\sigma_i = 0 \ \forall\ i \in \{0,1,2,3\} \setminus \{1\}$,

$$\Sigma_{1,3}(\mathbf{m},\boldsymbol{\sigma}) = 4 \sum_{\substack{\mathbf{q}\in\mathcal{A}_{1,2,0,3} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{q_0+q_1+q_2+q_3}{4}} \left(\frac{2}{q_0q_3}\right),$$

(f) if $2 \nmid m_{03}m_{12}m_{02}m_{13}, \sigma_2 = 1$ and $\sigma_i = 0 \ \forall\ i \in \{0,1,2,3\} \setminus \{2\}$,

$$\Sigma_{1,3}(\mathbf{m},\boldsymbol{\sigma}) = 4 \sum_{\substack{\mathbf{q}\in\mathcal{A}_{0,2,1,3} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{q_0+q_1+q_2+q_3}{4}} \left(\frac{2}{q_0q_2}\right),$$

(g) if $2 \nmid m_{03}m_{12}m_{02}m_{13}, \sigma_3 = 1$ and $\sigma_i = 0 \ \forall\ i \in \{0,1,2,3\} \setminus \{3\}$,

$$\Sigma_{1,3}(\mathbf{m},\boldsymbol{\sigma}) = 4 \sum_{\substack{\mathbf{q}\in\mathcal{A}_{1,3,0,2} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{q_0+q_1+q_2+q_3}{4}} \left(\frac{2}{q_0q_2}\right).$$

Case (a): Looking at the conditions given in (2.10.3) we can see that $q_0 + q_1 + q_2 + q_3 \equiv 0 \bmod 8$. Thus, we are once more just counting elements in $\mathcal{A}_1$ satisfying $q_0q_1q_2q_3 \equiv 1 \bmod 8$, giving

$$\Sigma_{1,3}(\mathbf{m},\boldsymbol{\sigma}) = 4 \sum_{\substack{\mathbf{q}\in\mathcal{A}_1 \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{q_0+q_1+q_2+q_3}{4}} = 4 \sum_{\substack{\mathbf{q}\in\mathcal{A}_1 \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} 1 = 192.$$

Case (b): Here we note that $\mathcal{A}_{0,1,2,3} = \mathcal{A}_2$. Thus $\mathbf{q} \in \mathcal{A}_{0,1,2,3}$ must be of at least one of the forms considered in (2.10.5). For $\mathbf{q}$ of the forms solving $(A.2.1) - (A.2.4)$ or $(A.2.9) - (A.2.12)$, $q_0 + q_1 + q_2 + q_3 \equiv 0 \bmod 8$ and $q_0q_1 \equiv 7 \bmod 8$ and thus

$$(-1)^{\frac{q_0+q_1+q_2+q_3}{4}} \left(\frac{2}{q_0q_1}\right) = 1.$$

For $\mathbf{q}$ of the forms solving $(A.2.5), (A.2.6), (A.2.13), (A.2.14)$, $q_0 + q_1 + q_2 + q_3 \equiv 6a + 2 \bmod 8$ and $q_0 q_1 \equiv 2a + 7 \bmod 8$ for some $a \in (\mathbb{Z}/8\mathbb{Z})^*$ and so it follows that

$$(-1)^{\frac{q_0+q_1+q_2+q_3}{4}} \left(\frac{2}{q_0 q_1}\right) = (-1)^{\frac{3a+1}{2}} \left(\frac{2}{2a+7}\right) = 1.$$

Similarly, for $\mathbf{q}$ of the forms solving $(A.2.7), (A.2.8), (A.2.15), (A.2.16)$, $q_0+q_1+q_2+q_3 \equiv 6a + 6 \bmod 8$ and $q_0 q_1 \equiv 6a + 7 \bmod 8$ for some $a \in (\mathbb{Z}/8\mathbb{Z})^*$ and so again it follows that

$$(-1)^{\frac{q_0+q_1+q_2+q_3}{4}} \left(\frac{2}{7 q_0 q_1}\right) = (-1)^{\frac{3a+3}{2}} \left(\frac{2}{6a+7}\right) = 1.$$

Thus in this case,

$$\Sigma_{1,3}(\mathbf{m}, \boldsymbol{\sigma}) = 4 \sum_{\mathbf{q} \in \mathcal{A}_{0,1,2,3}} 1 = 128.$$

Case (c): Since we also have $\mathcal{A}_{2,3,0,1} = \mathcal{A}_2$, the only difference to the previous case is that the product $q_0 q_1$ now considers the last two components instead of the first two. However, since the set $\mathcal{A}_2$ is closed under the operation of swapping the 0th and 2nd coordinates and 1st and 3rd coordinates, this does not change anything from the previous argument. Thus we also have

$$\Sigma_{1,3}(\mathbf{m}, \boldsymbol{\sigma}) = 128$$

in this case.

Case (d): Here we begin by noting that $\mathbf{q} \in \mathcal{A}_{0,3,1,2}$ is equivalent to $(q_0, q_3, q_1, q_2) \in \mathcal{A}_2$. Thus by taking the product $q_0 q_3$ we are once again just taking the product of the first two components of the solutions in (2.10.5). Following the same procedure as in case (b) it may therefore be seen that

$$(-1)^{\frac{q_0+q_1+q_2+q_3}{4}} \left(\frac{2}{q_0 q_3}\right) = 1,$$

for any $\mathbf{q} \in \mathcal{A}_{0,3,1,2}$. Thus

$$\Sigma_{1,3}(\mathbf{m}, \boldsymbol{\sigma}) = 128.$$

Case (e): This case is symmetric to case (d) and so we will also obtain

$$\Sigma_{1,3}(\mathbf{m}, \boldsymbol{\sigma}) = 128.$$

Case (f): Noting that $\mathbf{q} \in \mathcal{A}_{0,2,1,3}$ is equivalent to $(q_0, q_2, q_1, q_3) \in \mathcal{A}_2$, it is easy to see that the same arguments as before give

$$\Sigma_{1,3}(\mathbf{m}, \boldsymbol{\sigma}) = 128.$$

Case (g): Being symmetric to case (f) this case will also give

$$\Sigma_{1,3}(\mathbf{m}, \boldsymbol{\sigma}) = 128.$$

Thus we have shown the following;

Lemma 2.10.2. *We have the following*

$$\Sigma_{1,3}(\mathbf{m},\boldsymbol{\sigma}) = \begin{cases} 192 \textit{ if } 2 \nmid m_{02}m_{03}m_{12}m_{13} \ \& \ \sigma_i = 0 \ \forall \ i \in \{0,1,2,3\}, \\ 128 \textit{ otherwise.} \end{cases}$$

The Case $r = 2$

In this case

$$\Theta'_{2,1}(\mathbf{m},\mathbf{q},\boldsymbol{\sigma}) = (-1)^{\tilde{f}_2(\mathbf{q})} \left(\frac{2^{\sigma_0+\sigma_1}}{q_0q_3}\right)\left(\frac{2^{\sigma_2+\sigma_3}}{q_0q_2}\right)\left(\frac{2^{v_2(m_{02}m_{03}m_{12}m_{13})}}{q_0q_1}\right)\left(\frac{-1}{7q_1q_2}\right)$$

and

$$\tilde{f}_2(\mathbf{m}_{\text{odd}},\mathbf{q}) = \frac{(q_0+5q_1+q_2+5q_3)}{4}.$$

We split up into similar cases as before, again noting that the precise definition of $\mathcal{A}(\mathbf{m},\boldsymbol{\sigma})$ is once more required since the relative positions of the q_i will effect the value of the Jacobi symbol and the reciprocity factor. We have the following cases:

(*a*) if $2 \nmid m_{02}m_{03}m_{12}m_{13}$, and $\sigma_i = 0 \ \forall \ i \in \{0,1,2,3\}$,

$$\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma}) = 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}\right) \sum_{\substack{\mathbf{q}\in\mathcal{A}_1 \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}} \left(\frac{-1}{7q_1q_2}\right),$$

(*b*) if $2 \mid m_{03}m_{12},\ 2 \nmid m_{02}m_{13}$ and $\sigma_i = 0 \ \forall \ i \in \{0,1,2,3\}$,

$$\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma}) = 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}\right) \sum_{\substack{\mathbf{q}\in\mathcal{A}_{0,1,2,3} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}} \left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_2}\right),$$

(*c*) if $2 \mid m_{02}m_{13},\ 2 \nmid m_{02}m_{13}$ and $\sigma_i = 0 \ \forall \ i \in \{0,1,2,3\}$,

$$\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma}) = 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}\right) \sum_{\substack{\mathbf{q}\in\mathcal{A}_{2,3,0,1} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}} \left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_2}\right),$$

(*d*) if $2 \nmid m_{03}m_{12}m_{02}m_{13},\ \sigma_0 = 1$ and $\sigma_i = 0 \ \forall \ i \in \{0,1,2,3\} \setminus \{0\}$,

$$\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma}) = 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}\right) \sum_{\substack{\mathbf{q}\in\mathcal{A}_{0,3,1,2} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}} \left(\frac{2}{q_0q_3}\right)\left(\frac{-1}{7q_1q_2}\right),$$

(*e*) if $2 \nmid m_{03}m_{12}m_{02}m_{13}, \sigma_1 = 1$ and $\sigma_i = 0 \ \forall \ i \in \{0,1,2,3\} \setminus \{1\}$,

$$\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma}) = 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}\right) \sum_{\substack{\mathbf{q}\in\mathcal{A}_{1,2,0,3} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}} \left(\frac{2}{q_0q_3}\right)\left(\frac{-1}{7q_1q_2}\right),$$

(f) if $2 \nmid m_{03}m_{12}m_{02}m_{13}, \sigma_2 = 1$ and $\sigma_i = 0 \;\forall\; i \in \{0,1,2,3\} \setminus \{2\}$,

$$\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma}) = 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}\right) \sum_{\substack{\mathbf{q}\in\mathcal{A}_{0,2,1,3} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}} \left(\frac{2}{q_0q_2}\right)\left(\frac{-1}{7q_1q_2}\right),$$

(g) if $2 \nmid m_{03}m_{12}m_{02}m_{13}, \sigma_3 = 1$ and $\sigma_i = 0 \;\forall\; i \in \{0,1,2,3\} \setminus \{3\}$,

$$\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma}) = 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}\right) \sum_{\substack{\mathbf{q}\in\mathcal{A}_{1,3,0,2} \\ q_0q_1q_2q_3\equiv 1 \bmod 8}} (-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}} \left(\frac{2}{q_0q_2}\right)\left(\frac{-1}{7q_1q_2}\right).$$

Case (a): Denote by $\mathcal{A}_1(j)$ the solutions to $(A.1.j)$ for $1 \leqslant j \leqslant 15$. Noting that by adhering to (2.10.3), $(q_0 + 5q_1 + q_2 + 5q_3) \equiv 0 \bmod 8$ for any choice of $\mathbf{q} \in \mathcal{A}_1$. We may therefore write this sum as follows:

$$\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma}) = 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\text{odd}}}\right)\left(\sum_{j=1}^{15} \sum_{\substack{\mathbf{q}\in\mathcal{A}_1(j) \\ \mathbf{q}\notin\cup_{i<j}\mathcal{A}_1(i)}} \left(\frac{-1}{7q_1q_2}\right)\right).$$

The distinct points of each set $\mathbf{q} \in \mathcal{A}_1(j) \setminus \cup_{i<j}\mathcal{A}_1(i)$ are displayed in Tables A.1-A.4 of §A. Note also that the inner sums above are empty for $j = 3,4,7,8$ as $\mathcal{A}_1(j) = \mathcal{A}_1(i)$ for $i = 2,1,6,5$ respectively. More generally, $\mathcal{A}_1(j) \setminus \cup_{i<j}\mathcal{A}_1(i) = \emptyset$ for $j \in \{3,4,7,8,10,11,12,13,14,15\}$. For $\mathbf{q} \in \mathcal{A}_1(1)$, $q_1q_2 \equiv 7ab \bmod 8$ for any $a,b \in (\mathbb{Z}/8\mathbb{Z})^{*4}$, so

$$\sum_{\mathbf{q}\in\mathcal{A}_1(1)} \left(\frac{-1}{7q_1q_2}\right) = \sum_{a,b\in(\mathbb{Z}/8\mathbb{Z})^{*4}} \left(\frac{-1}{ab}\right) = 0.$$

For $\mathbf{q} \in \mathcal{A}_1(2) \setminus \mathcal{A}_1(1)$, $q_1q_2 \equiv 7 \bmod 8$, and so by looking at Table A.2,

$$\sum_{\substack{\mathbf{q}\in\mathcal{A}_1(2) \\ \mathbf{q}\notin\mathcal{A}_1(i)}} \left(\frac{-1}{7q_1q_2}\right) = \sum_{\substack{a,b\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ a\neq b}} 1 = 12.$$

For $\mathbf{q} \in \mathcal{A}_1(5) \setminus \cup_{i<5}\mathcal{A}_1(i)$, $q_1q_2 \equiv ab \bmod 8$ for $a,b \in (\mathbb{Z}/8\mathbb{Z})^{*4}$, $ab \equiv 1,3 \bmod 8$ so,

$$\sum_{\substack{\mathbf{q}\in\mathcal{A}_1(5) \\ \mathbf{q}\notin\cup_{i<5}\mathcal{A}_1(i)}} \left(\frac{-1}{7q_1q_2}\right) = \sum_{\substack{a,b\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ ab\equiv 1,3 \bmod 8}} \left(\frac{-1}{7ab}\right) = 0.$$

For $\mathbf{q} \in \mathcal{A}_1(6) \setminus \cup_{i<6}\mathcal{A}_1(i)$, $q_1q_2 \equiv 3 \bmod 8$, and looking at Table A.4 we have

$$\sum_{\substack{\mathbf{q}\in\mathcal{A}_1(6) \\ \mathbf{q}\notin\cup_{i<6}\mathcal{A}_1(i)}} \left(\frac{-1}{7q_1q_2}\right) = \sum_{\substack{a,b\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ a\equiv b \bmod 8}} 1 = 4.$$

Finally, for $\mathbf{q} \in \mathcal{A}_1(9) \setminus \cup_{i<9}\mathcal{A}_1(i)$, $q_1q_2 \equiv -ab \bmod 8$ for $a,b \in (\mathbb{Z}/8\mathbb{Z})^{*4}$ such that $ab \equiv 3,5 \bmod 8$ giving

$$\sum_{\substack{\mathbf{q}\in\mathcal{A}_1(9) \\ \mathbf{q}\notin\cup_{i<9}\mathcal{A}_1(i)}} \left(\frac{-1}{7q_1q_2}\right) = \sum_{\substack{a,b\in(\mathbb{Z}/8\mathbb{Z})^{*4} \\ ab\equiv 3,5 \bmod 8}} \left(\frac{-1}{7ab}\right) = 0.$$

Therefore, in case (a),

$$\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma}) = 64\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}}\right).$$

Case (b): Following the same method of case (a) we write $\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma})$ as,

$$= 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}}\right)\left(\sum_{j=1}^{8}\sum_{\substack{\mathbf{q}\in\mathcal{A}_{0,1,2,3}(j)\\ \mathbf{q}\notin\cup_{i<j}\mathcal{A}_{0,1,2,3}(i)}}(-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}}\left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_2}\right)\right),$$

where here $\mathcal{A}_{0,1,2,3}(j)$ is set of solutions to $(A.2.j)$ for $1\leqslant j\leqslant 16$. Looking at Table A.7 it may be seen that we can ignore the inner sums for $j=2,3,4,6,8$ and $j\geqslant 9$. When $j=1$ there are 16 elements of $\mathcal{A}_{0,1,2,3}(j)$, all of the form $(a,7a,b,7b)$ for $a,b\in(\mathbb{Z}/8\mathbb{Z})^*$. Then $(q_0+5q_1+q_2+5q_3)\equiv(a+3a+b+3b)\equiv 0 \bmod 8$ for all possible solutions. Also $q_0q_1\equiv 7 \bmod 8$, while $7q_1q_2\equiv 7ab \bmod 8$ for some $a,b\in(\mathbb{Z}/8\mathbb{Z})^*$. Thus, we may write

$$\sum_{\mathbf{q}\in\mathcal{A}_{0,1,2,3}(1)}(-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}}\left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_2}\right) = \sum_{a,b\in(\mathbb{Z}/8\mathbb{Z})^*}\left(\frac{-1}{7ab}\right) = 0.$$

Since the sums over $\mathbf{q}\in\mathcal{A}_{0,1,2,3}(j)\setminus\cup_{i<j}\mathcal{A}_{0,1,2,3}(j)$ for $j=2,3,4$ is empty we turn to $j=5$. Here, every solution has $(q_0+5q_1+q_2+5q_3)\equiv 6a+2 \bmod 8$ and $q_0q_1\equiv 2a+7$ for some $a\in(\mathbb{Z}/8\mathbb{Z})^*$. There are 8 elements of $\mathcal{A}_{0,1,2,3}(5)\setminus\cup_{i<5}\mathcal{A}_{0,1,2,3}(i)$: the first 4 solutions have $7q_1q_2\equiv a \bmod 8$ for $a\in(\mathbb{Z}/8\mathbb{Z})^*$; the next 2 solutions correspond to $a\equiv 3,7 \bmod 8$ and have $7q_1q_2\equiv 3 \bmod 4$ and the last 2 solutions correspond to $a\equiv 1,5 \bmod 8$ have $7q_1q_2\equiv 1 \bmod 4$. We obtain

$$\begin{aligned}\sum_{\substack{\mathbf{q}\in\mathcal{A}_{0,1,2,3}(5)\\ \mathbf{q}\notin\cup_{i<5}\mathcal{A}_{0,1,2,3}(i)}}(-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}}\left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_2}\right) &= \sum_{a\in(\mathbb{Z}/8\mathbb{Z})^*}\left(\frac{2}{2a+7}\right) - \sum_{a\in(\mathbb{Z}/8\mathbb{Z})^*}\left(\frac{2}{2a+7}\right)\\ &= 0.\end{aligned}$$

For $j=6$, the sum is once again empty. For $j=7$, every solution has $(q_0+5q_1+q_2+5q_3)\equiv 6a+6 \bmod 8$ and $q_0q_1\equiv 6a+7$ for some $a\in(\mathbb{Z}/8\mathbb{Z})^*$. There are 8 elements of $\mathcal{A}_{0,1,2,3}(7)\setminus\cup_{i<7}\mathcal{A}_{0,1,2,3}(i)$: the first 4 solutions have $7q_1q_2\equiv 3a \bmod 8$; the next 2 solutions correspond to $a\equiv 1,5 \bmod 8$ and have $7q_1q_2\equiv 3 \bmod 4$ and the last 2 solutions correspond to $a\equiv 3,7 \bmod 8$ where $7q_1q_2\equiv 1 \bmod 4$ and so

$$\begin{aligned}\sum_{\substack{\mathbf{q}\in\mathcal{A}_{0,1,2,3}(7)\\ \mathbf{q}\notin\cup_{i<7}\mathcal{A}_{0,1,2,3}(i)}}(-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}}\left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_2}\right) &= 2\sum_{a\in(\mathbb{Z}/8\mathbb{Z})^*}\left(\frac{2}{6a+7}\right)\\ &= 0.\end{aligned}$$

Finally, since the inner sums are once again empty for $j\geqslant 8$, we have $\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma})=0$ for case (b).

Case (c): Recall that $\mathbf{q}\in\mathcal{A}_{2,3,0,1}$ is equivalent to $(q_2,q_3,q_0,q_1)\in\mathcal{A}_2$. Since $\mathcal{A}_2$ is invariant under this permutation, and for each $\mathbf{q}\in\mathcal{A}_2$, $q_0q_1\equiv q_2q_3 \bmod 8$, we may re-order the sum:

$$\begin{aligned}\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma}) &= 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}}\right)\sum_{\substack{\mathbf{q}\in\mathcal{A}_{2,3,0,1}\\ q_0q_1q_2q_3\equiv 1 \bmod 8}}(-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}}\left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_2}\right)\\ &= 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}}\right)\sum_{\substack{\mathbf{q}\in\mathcal{A}_{2}\\ q_0q_1q_2q_3\equiv 1 \bmod 8}}(-1)^{\frac{(q_0+5q_1+q_2+5q_3)}{4}}\left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_2}\right).\end{aligned}$$

This is now just case (b), and thus is equal to 0.

Case (d) and Case (e): Note that we may compare cases (d) and (e) in the same way as we compared (b) and (c). Thus we only need to handle (d). Noting that $\mathbf{q}\in\mathcal{A}_{0,3,1,2}$ implies $(q_0,q_3,q_1,q_2)\in\mathcal{A}_2$, we may write $\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma})$ in this case as follows:

$$\begin{aligned}&= 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}}\right)\sum_{\substack{\mathbf{q}\in\mathcal{A}_{2}\\ q_0q_1q_2q_3\equiv 1 \bmod 8}}(-1)^{\frac{(q_0+q_1+5q_2+5q_3)}{4}}\left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_3}\right)\\ &= 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}}\right)\left(\sum_{j=1}^{8}\sum_{\substack{\mathbf{q}\in\mathcal{A}_{2}(j)\\ \mathbf{q}\notin\cup_{i<j}\mathcal{A}_2(j)}}(-1)^{\frac{(q_0+q_1+5q_2+5q_3)}{4}}\left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_3}\right)\right).\end{aligned}$$

Examining each of the inner sums individually, looking at each the forms of solutions for each $(A.2.j)$, and their solutions in Table A.7 as with case (b), we will obtain:

$$\sum_{\substack{\mathbf{q}\in\mathcal{A}_{2}(j)\\ \mathbf{q}\notin\cup_{i<j}\mathcal{A}_2(j)}}(-1)^{\frac{(q_0+q_1+5q_2+5q_3)}{4}}\left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_3}\right)=0$$

for all $1\leqslant j\leqslant 16$. Thus $\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma})=0$ in these cases as well.

Cases (f) and (g): As with (b) and (c), cases (f) and (g) are symmetric. Dealing with (f) we write

$$\begin{aligned}\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma}) &= 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}}\right)\sum_{\substack{\mathbf{q}\in\mathcal{A}_{2}\\ q_0q_1q_2q_3\equiv 1 \bmod 8}}(-1)^{\frac{(q_0+q_1+5q_2+5q_3)}{4}}\left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_2}\right)\\ &= 4\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}}\right)\left(\sum_{j=1}^{8}\sum_{\substack{\mathbf{q}\in\mathcal{A}_{2}(j)\\ \mathbf{q}\notin\cup_{i<j}\mathcal{A}_2(j)}}(-1)^{\frac{(q_0+q_1+5q_2+5q_3)}{4}}\left(\frac{2}{q_0q_1}\right)\left(\frac{-1}{7q_1q_2}\right)\right).\end{aligned}$$

Similar calculations to before give $\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma})=0$ here as well.

Overall, when $r=2$ we have proven the following:

Lemma 2.10.3. *We have the following*

$$\Sigma_{2,3}(\mathbf{m},\boldsymbol{\sigma})=\begin{cases}64\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{odd}}\right) & \textit{if } 2\nmid m_{02}m_{03}m_{12}m_{13}\ \&\ \sigma_i=0\ \forall\, i\in\{0,1,2,3\},\\ 0 & \textit{otherwise.}\end{cases}$$

2.10.3 Removing Dependency on $\boldsymbol{\sigma}$

In this section we deal with the condition (2.5.7). In doing so we will simplify our expression for $\mathfrak{C}$ into sums over $\mathbf{b}$ and $\mathbf{m}$ whose components are all odd. To start we note that a close examination of $\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m},v)$ and (2.9.9) tells us that $\mathfrak{C}(\mathbf{b},\mathbf{m},\boldsymbol{\sigma})$ is independent of both $\boldsymbol{\sigma}$ and $v_2(m_{02}m_{03}m_{12}m_{13})$. Write

$$\mathfrak{C}_{r,i}(\mathbf{b}) = \frac{4f_0^4}{\phi(8)^4 f_2^4} \sum_{\substack{\mathbf{m}\in\mathbb{N}^4\\(2.5.2)}} \sum_{\substack{\boldsymbol{\sigma}\in\{0,1\}^4\\(2.5.7)}} \frac{\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m})\mathfrak{C}(\mathbf{b},\mathbf{m})\Sigma_{r,i}(\mathbf{m},\boldsymbol{\sigma})}{2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}m_{02}^2m_{03}^2m_{12}^2m_{13}^2\tau((m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}})}$$

for $i=2,3$ where

$$\mathfrak{C}(\mathbf{b},\mathbf{m}) = \sum_{v\in\mathbb{N}}\mu(v) \sum\sum\sum\sum_{\substack{a_0,a_1,a_2,a_3\in\mathbb{N}\\ p\mid a_0a_1a_2a_3\Rightarrow p\mid v\\ v^2\mid a_0a_1a_2a_3,(2.9.9)}} \frac{\overline{\mathfrak{S}'}(v)}{a_0a_1a_2a_3\tau(a_0)\tau(a_1)\tau(a_2)\tau(a_3)}$$

with

$$\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m}) = \prod_{i=0}^{3}\left(\prod_{\substack{p\mid m_{02}m_{12}m_{03}m_{13}b_i\\ p \text{ odd}}} f_p^{-1}\right) \text{ and } \overline{\mathfrak{S}'}(v) = \prod_{i=0}^{3}\left(\prod_{\substack{p\mid v\\ p\nmid m_{02}m_{12}m_{03}m_{13}b_i\\ p\text{ odd}}} f_p^{-1}\right),$$

noting that the dependency of $\mathfrak{C}(\mathbf{b},\mathbf{m})$ on $\mathbf{b}$ and $\mathbf{m}$ is contained in the condition (2.9.9). Now we observe that only the $\Sigma_{r,i}(\mathbf{m},\boldsymbol{\sigma})$ depend on $\boldsymbol{\sigma}$ and $v_2(m_{02}m_{03}m_{12}m_{13})\in\{0,1\}$, allowing us to write $\mathfrak{C}_{r,i}(\mathbf{b})$ as

$$\frac{4f_0^4}{\phi(8)^4f_2^4}\sum_{\substack{\mathbf{m}\in\mathbb{N}^4_{\mathrm{odd}}\\(2.5.2)}}\frac{\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m})\mathfrak{C}(\mathbf{b},\mathbf{m})}{m_{02}^2m_{03}^2m_{12}^2m_{13}^2\tau(m_{02}m_{03}m_{12}m_{13})}\sum_{\substack{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}\in\{0,1\}^4\\(2.10.7)}}\frac{\Sigma_{r,i}(\mathbf{m},\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})}{2^{\sigma_0+\sigma_1+\sigma_2+\sigma_3}4^{\tilde{\sigma}_{02}+\tilde{\sigma}_{03}+\tilde{\sigma}_{12}+\tilde{\sigma}_{13}}}$$

where

$$\begin{cases}\sigma_0+\sigma_1+\sigma_2+\sigma_3+\tilde{\sigma}_{02}+\tilde{\sigma}_{03}+\tilde{\sigma}_{12}+\tilde{\sigma}_{13}\leqslant 1,\\ \gcd(2^{\sigma_0},b_1)=\gcd(2^{\sigma_1},b_0)=\gcd(2^{\sigma_2},b_3)=\gcd(2^{\sigma_3},b_2)=1,\\ \gcd(2^{\tilde{\sigma}_{02}},b_1b_3)=\gcd(2^{\tilde{\sigma}_{03}},b_1b_2)=\gcd(2^{\tilde{\sigma}_{12}},b_0b_3)=\gcd(2^{\tilde{\sigma}_{13}},b_0b_2)=1.\end{cases} \tag{2.10.7}$$

and

$$\Sigma_{r,i}(\mathbf{m},\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})=\Sigma_{r,i}(\tilde{\mathbf{m}},\boldsymbol{\sigma}),$$

where here $\tilde{\mathbf{m}}=(2^{\tilde{\sigma}_{02}}m_{02},2^{\tilde{\sigma}_{03}}m_{03},2^{\tilde{\sigma}_{12}}m_{12},2^{\tilde{\sigma}_{13}}m_{13})$. Now using the previous subsection, we may compute the sum over $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$, which we will call $\Delta_{r,i}(\mathbf{b},\mathbf{m})$. We have the following:

$$\Delta_{1,i}(\mathbf{b},\mathbf{m})=192+\sharp\{0\leqslant j\leqslant 3: 2\nmid b_j\}\frac{128}{2}+\sharp\{0\leqslant j\leqslant 1:2\nmid b_j\}\cdot\sharp\{2\leqslant j\leqslant 3:2\nmid b_j\}\frac{128}{4}$$

for $i=2,3$ and

$$\Delta_{2,2}(\mathbf{b},\mathbf{m})=192+\sharp\{0\leqslant j\leqslant 3: 2\nmid b_j\}\frac{128}{2}+\sharp\{0\leqslant j\leqslant 1:2\nmid b_j\}\cdot\sharp\{2\leqslant j\leqslant 3:2\nmid b_j\}\frac{128}{4},$$

$$\Delta_{2,3}(\mathbf{b},\mathbf{m}) = 64\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}}\right).$$

It follows that we may write

$$\mathfrak{C}_{r,i}(\mathbf{b}) = \frac{4f_0^4}{\phi(8)^4 f_2^4}\sum_{\substack{\mathbf{m}\in\mathbb{N}_{\mathrm{odd}}^4\\ (2.5.2)}} \frac{\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m})\mathfrak{C}(\mathbf{b},\mathbf{m})\Delta_{r,i}(\mathbf{b},\mathbf{m})}{m_{02}^2m_{03}^2m_{12}^2m_{13}^2\tau(m_{02}m_{03}m_{12}m_{13})}.$$

Next, we remove the even parts in the sum over b_i. Noting that only the $\Delta(\mathbf{b},\mathbf{m})$ depend on the even part of the b_i, we write $b_i = (b_i)_{\mathrm{odd}}2^{\mu_i}$ for $\mu_i = v_2(b_i)$. Then

$$\mathfrak{C}_{r,i} = \frac{4f_0^4}{\phi(8)^4 f_2^4}\sum_{\substack{\mathbf{b},\mathbf{m}\in\mathbb{N}_{\mathrm{odd}}^4\\ (2.5.1),(2.5.2)}} \frac{\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m})\mathfrak{C}(\mathbf{b},\mathbf{m})}{b_0^2b_1^2b_2^2b_3^2m_{02}^2m_{03}^2m_{12}^2m_{13}^2\tau(m_{02}m_{03}m_{12}m_{13})}\sum_{\substack{\boldsymbol{\mu}\in(\mathbb{N}\cup\{0\})^4\\ (2.10.8)}}\frac{\Delta_{r,i}(\mathbf{b},\mathbf{m})}{4^{\mu_0+\mu_1+\mu_2+\mu_3}}$$

where

$$\gcd(2^{\mu_0},2^{\mu_1}) = \gcd(2^{\mu_2},2^{\mu_3}) = 1. \tag{2.10.8}$$

Now,

$$\sum_{\substack{\boldsymbol{\mu}\in(\mathbb{N}\cup\{0\})^4\\ (2.10.8)}}\frac{1}{4^{\mu_0+\mu_1+\mu_2+\mu_3}} = \frac{25}{9},$$

$$\sum_{\substack{\boldsymbol{\mu}\in(\mathbb{N}\cup\{0\})^4\\ (2.10.8)}}\frac{\sharp\{0\leqslant j\leqslant 3:\mu_j=0\}}{4^{\mu_0+\mu_1+\mu_2+\mu_3}} = 4+12\sum_{\substack{\tilde\mu\in\mathbb{N}\\ \tilde\mu>0}}\frac{1}{4^{\tilde\mu}}+8\sum_{\substack{\tilde\mu_0,\tilde\mu_1\in\mathbb{N}\\ \tilde\mu_0,\tilde\mu_1>0}}\sum\frac{1}{4^{\tilde\mu_0+\tilde\mu_1}} = \frac{80}{9},$$

and

$$\sum_{\substack{\boldsymbol{\mu}\in(\mathbb{N}\cup\{0\})^4\\ (2.10.8)}}\frac{\sharp\{0\leqslant j\leqslant 1:\mu_j=0\}\cdot\sharp\{2\leqslant j\leqslant 3:\mu_j=0\}}{4^{\mu_0+\mu_1+\mu_2+\mu_3}} = 4+8\sum_{\substack{\tilde\mu\in\mathbb{N}\\ \tilde\mu>0}}\frac{1}{4^{\tilde\mu}}+4\sum_{\substack{\tilde\mu_0,\tilde\mu_1\in\mathbb{N}\\ \tilde\mu_0,\tilde\mu_1>0}}\sum\frac{1}{4^{\tilde\mu_0+\tilde\mu_1}} = \frac{64}{9}.$$

It follows that

$$\sum_{\substack{\boldsymbol{\mu}\in(\mathbb{N}\cup\{0\})^4\\ (2.10.8)}}\frac{\Delta_{1,i}(\mathbf{b},\mathbf{m})}{4^{\mu_0+\mu_1+\mu_2+\mu_3}} = \frac{4800}{9}+\frac{5120}{9}+\frac{2048}{9} = \frac{11968}{9}$$

for $(r,i) = (1,2),(1,3),(2,2)$, and

$$\sum_{\substack{\boldsymbol{\mu}\in(\mathbb{N}\cup\{0\})^4\\ (2.10.8)}}\frac{\Delta_{2,3}(\mathbf{b},\mathbf{m})}{4^{\mu_0+\mu_1+\mu_2+\mu_3}} = \frac{1600}{9}\left(\frac{-1}{(m_{02}m_{03}m_{12}m_{13})_{\mathrm{odd}}}\right).$$

Now for $m\in\mathbb{N}$ and $(r,i)\in\{(1,2),(1,3),(2,2),(2,3)\}$ define

$$\rho_{(r,i)} = \begin{cases}\frac{11968}{9} & \text{if } (r,i)\in\{(1,2),(1,3),(2,2)\},\\ \frac{1600}{9} & \text{if } (r,i)=(2,3),\end{cases}$$

and

$$\rho'_{(r,i)}(m) = \begin{cases}1 & \text{if } (r,i)\in\{(1,2),(1,3),(2,2)\},\\ \left(\frac{-1}{m}\right) & \text{if } (r,i)=(2,3).\end{cases}$$

Then

$$\mathfrak{C}_{r,i} = \frac{f_0^4\rho_{(r,i)}}{64f_2^4}\sum_{\substack{\mathbf{b},\mathbf{m}\in\mathbb{N}_{\mathrm{odd}}^4\\ (2.5.1),(2.5.2)}}\frac{\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m})\mathfrak{C}(\mathbf{b},\mathbf{m})\rho'_{(r,i)}(m_{02}m_{03}m_{12}m_{13})}{b_0^2b_1^2b_2^2b_3^2m_{02}^2m_{03}^2m_{12}^2m_{13}^2\tau(m_{02}m_{03}m_{12}m_{13})}$$

for all $(r,i)\in\{(1,2),(1,3),(2,2),(2,3)\}$.

2.10.4 Simplification of $\mathfrak{C}(\mathbf{b},\mathbf{m})$

Let $\mathbf{x}\in\mathbb{N}^4$ and define

$$\mathfrak{C}(\mathbf{x})=\sum_{v\in\mathbb{N}}\mu(v)\sum\sum\sum\sum_{\substack{a_0,a_1,a_2,a_3\in\mathbb{N}\\ p\mid a_0a_1a_2a_3\Rightarrow p\mid v\\ v^2\mid a_0a_1a_2a_3\\ \gcd(a_i,2x_i)=1\ \forall\ i}}\frac{\overline{\mathfrak{S}'}(v,\mathbf{x})}{a_0a_1a_2a_3\tau(a_0)\tau(a_1)\tau(a_2)\tau(a_3)},$$

where

$$\mathfrak{S}'(v,\mathbf{x})=\prod_{i=0}^{3}\left(\frac{1}{\prod_{\substack{p\mid v\\ p\nmid 2x_i}}}f_p\right).$$

Then we have the following,

Lemma 2.10.4. *For any* $\mathbf{x}\in\mathbb{N}^4$,

$$\mathfrak{C}(\mathbf{x})=\prod_{p\neq 2}\frac{1}{f_p^{\sharp\{0\leqslant i\leqslant 3:p\nmid x_i\}}}\left(1+\frac{\sharp\{0\leqslant i\leqslant 3:p\nmid x_i\}}{2p}\right).$$

Proof. We write,

$$\mathfrak{C}(\mathbf{x})=\sum_{v\in\mathbb{N}_{\mathrm{odd}}}\mu(v)\overline{\mathfrak{S}'}(v,\mathbf{x})\sum_{\substack{w\in\mathbb{N}^4_{\mathrm{odd}}\\ v^2\mid w\\ p\mid w\Rightarrow p\mid v}}\frac{1}{w}\sum_{\substack{\mathbf{a}\in\mathbb{N}^4\\ a_0a_1a_2a_3=w\\ \gcd(a_i,2x_i)=1\forall i}}\frac{1}{\tau(a_0)\tau(a_1)\tau(a_2)\tau(a_3)}.$$

Then the sum over w is

$$\prod_{p\mid v}\left(\sum_{j=2}^{\infty}\frac{1}{p^j}\sum_{\substack{\mathbf{e}\in\mathbb{Z}^4_{\geqslant 0}\\ e_0+e_1+e_2+e_3=j\\ \gcd(p^{e_i},x_i)=1\ \forall\ i}}\prod_{i=0}^{3}\left(\frac{1}{e_i+1}\right)\right)=\prod_{p\mid v}\left(f_p^{\sharp\{0\leqslant i\leqslant 3:p\nmid x_i\}}-1-\frac{\sharp\{0\leqslant i\leqslant 3:p\nmid x_i\}}{2p}\right)$$

the equality coming from adding in the terms for which $j=0$ and $j=1$. Call the term inside the product c_p for each prime p then by summing over v we conclude that

$$\mathfrak{C}(\mathbf{x})=\prod_{p\neq 2}\left(1-\frac{c_p}{f_p^{\sharp\{0\leqslant i\leqslant 3:p\nmid x_i\}}}\right)=\prod_{p\neq 2}\frac{1}{f_p^{\sharp\{0\leqslant i\leqslant 3:p\nmid x_i\}}}\left(1+\frac{\sharp\{0\leqslant i\leqslant 3:p\nmid x_i\}}{2p}\right).$$

□

From this we may prove the following:

Lemma 2.10.5. *For* $\mathbf{b},\mathbf{m}\in\mathbb{N}^4_{odd}$ *satisfying* (2.5.1) *and* (2.5.2),

$$\frac{f_0^4}{f_2^4}\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m})\mathfrak{C}(\mathbf{b},\mathbf{m})=\frac{1}{(2\pi)^2}\prod_{p\neq 2}\left(1-\frac{1}{p}\right)^2\left(1+\frac{\sharp\{0\leqslant i\leqslant 3:p\nmid m_{02}m_{03}m_{12}m_{13}b_i\}}{2p}\right).$$

Proof. From Lemma 2.10.4 we have

$$\mathfrak{C}(\mathbf{b},\mathbf{m}) = \prod_{p\neq 2} \frac{1}{f_p^{\sharp\{0\leqslant i\leqslant 3: p\nmid m_{02}m_{03}m_{12}m_{13}b_i\}}} \left(1+\frac{\sharp\{0\leqslant i\leqslant 3: p\nmid m_{02}m_{03}m_{12}m_{13}b_i\}}{2p}\right),$$

and by re-arranging we have

$$\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m}) = \prod_{i=0}^{3}\left(\frac{1}{\prod_{\substack{p\mid m_{02}m_{12}m_{03}m_{13}b_i\\ p \text{ odd}}} f_p}\right) = \prod_{p\neq 2}\left(\frac{1}{f_p^{\sharp\{0\leqslant i\leqslant 3: p\mid m_{02}m_{03}m_{12}m_{13}b_i\}}}\right).$$

Thus

$$\mathfrak{C}(\mathbf{b},\mathbf{m})\overline{\mathfrak{S}}(\mathbf{b},\mathbf{m}) = \prod_{p\neq 2}\frac{1}{f_p^4}\left(1+\frac{\sharp\{0\leqslant i\leqslant 3: p\nmid m_{02}m_{03}m_{12}m_{13}b_i\}}{2p}\right),$$

and by recalling the definition of f_0, the result follows. □

Define the function $\gamma:\mathbb{N}_{\text{odd}}^4\to\mathbb{R}$ by

$$\gamma(\mathbf{x}) = \prod_{p\neq 2}\left(1-\frac{1}{p}\right)^2\left(1+\frac{\sharp\{0\leqslant i\leqslant 3: p\nmid x_i\}}{2p}\right).$$

Then what we have now shown is that

$$\mathfrak{C}_{r,i} = \frac{\rho_{(r,i)}}{64(2\pi)^2}\sum_{\substack{\mathbf{b},\mathbf{m}\in\mathbb{N}_{\text{odd}}^4\\ (2.5.1),(2.5.2)}}\frac{\gamma(m_{02}m_{03}m_{12}m_{13}\mathbf{b})\rho'_{(r,i)}(m_{02}m_{03}m_{12}m_{13})}{b_0^2b_1^2b_2^2b_3^2m_{02}^2m_{03}^2m_{12}^2m_{13}^2\tau(m_{02}m_{03}m_{12}m_{13})}.$$

2.10.5 Sum over m

Noting that the summand only depends on the product of the components of **m** we collect terms to write

$$\mathfrak{C}_{r,i} = \frac{\rho_{(r,i)}}{64(2\pi)^2}\sum_{\substack{\mathbf{b}\in\mathbb{N}_{\text{odd}}^4\\ (2.5.1)}}\sum_{m\in\mathbb{N}_{\text{odd}}}\frac{\mu^2(m)\gamma(m\mathbf{b})\rho'_{(r,i)}(m)}{b_0^2b_1^2b_2^2b_3^2m^2\tau(m)}\left(\sum_{\substack{m_{02}m_{03}m_{12}m_{13}=m\\ (2.5.2)}}1\right).$$

This inner sum is a four-fold Dirichlet convolution of multiplicative functions (indicator functions of the gcd conditions in (2.5.2)) applied to a square-free integer m. By considering its behaviour for m prime, we may therefore deduce that this inner sum can be written as

$$\mu^2(m)\beta(m,\mathbf{b}) = \mu^2(m)\prod_{\substack{p\mid m\\ p\neq 2}}\left(\sharp\{0\leqslant i\leqslant 1: p\nmid b_i\}\cdot\sharp\{2\leqslant i\leqslant 3: p\nmid b_i\}\right),$$

which is multiplicative in m. Note also that we may untangle the dependence of m from $\gamma(m\mathbf{b})$ since

$$\gamma(m\mathbf{b}) = \gamma(\mathbf{b})\gamma_0(m,\mathbf{b})$$

where

$$\gamma_0(m,\mathbf{b}) = \prod_{\substack{p|m\\ p\neq 2}} \left(1+\frac{\sharp\{0\leqslant i\leqslant 3 : p\nmid b_i\}}{2p}\right)^{-1},$$

which is also multiplicative in m. Therefore the sum over m in $\mathfrak{C}_{r,i}$ becomes

$$\sum_{m\in\mathbb{N}_{\mathrm{odd}}} \frac{\mu^2(m)\gamma_0(m,\mathbf{b})\beta(m,\mathbf{b})\rho'_{(r,i)}(m)}{m^2\tau(m)} = \prod_{p\neq 2}\left(1+\frac{\rho'_{(r,i)}(p)\gamma_0(p,\mathbf{b})\beta(p,\mathbf{b})}{2p^2}\right).$$

Writing $\gamma_0(p)=\gamma_0(p,\mathbf{1})$ and $\beta(p)=\beta(p,\mathbf{1})$ we define $\kappa_p^{(1)}$ and $\kappa_p^{(2)}$ as

$$\left(1-\frac{1}{p}\right)^2\gamma_0(p)^{-1} \text{ and } \gamma_0(p)^{-1}\left(1+\frac{\rho'_{(r,i)}(p)\gamma_0(p)\beta(p)}{2p^2}\right) = \left(1+\frac{2}{p}+\frac{2\rho'_{(r,i)}(p)}{p^2}\right),$$

respectively. Then

$$\gamma(\mathbf{b}) = \left(\prod_{p\neq 2}\kappa_p^{(1)}\right) g^{(1)}(\mathbf{b}),$$

and

$$\prod_{p\neq 2}\left(1+\frac{\rho'_{(r,i)}(p)\gamma_0(p,\mathbf{b})\beta(p,\mathbf{b})}{2p^2}\right) = \left(\prod_{p\neq 2}\gamma_0(p)\kappa_p^{(2)}\right) g^{(2)}(\mathbf{b}),$$

where $g^{(1)}(\mathbf{b})$ and $g^{(2)}(\mathbf{b})$ are defined by

$$\prod_{\substack{p|b_0b_1b_2b_3\\ p\neq 2}}\gamma_0(p)\gamma_0(p,\mathbf{b})^{-1} \text{ and } \prod_{\substack{p|b_0b_1b_2b_3\\ p\neq 2}}(\gamma_0(p)\kappa_p^{(2)})^{-1}\left(1+\frac{\rho'_{(r,i)}(p)\gamma_0(p,\mathbf{b})\beta(p,\mathbf{b})}{2p^2}\right)$$

respectively. We are left with

$$\mathfrak{C}_{r,i} = \frac{\rho_{(r,i)}}{256\pi^2}\prod_{p\neq 2}\left(\left(1-\frac{1}{p}\right)^2\kappa_p^{(2)}\right)\sum_{\substack{\mathbf{b}\in\mathbb{N}_{\mathrm{odd}}^4\\ (2.5.1)}}\frac{g^{(1)}(\mathbf{b})g^{(2)}(\mathbf{b})}{b_0^2b_1^2b_2^2b_3^2}. \tag{2.10.9}$$

2.10.6 Sum over b

Let $g(\mathbf{b}) = g^{(1)}(\mathbf{b})g^{(2)}(\mathbf{b})$. Then $g(\mathbf{b})$ may be seen to be

$$\prod_{\substack{p|b_0b_1b_2b_3\\ p\neq 2}}(\kappa_p^{(2)})^{-1}\left(1+\frac{\sharp\{0\leqslant i\leqslant 3 : p\nmid b_i\}}{2p}+\frac{\rho'_{(r,i)}(p)\sharp\{0\leqslant i\leqslant 1 : p\nmid b_i\}\cdot\sharp\{2\leqslant i\leqslant 3 : p\nmid b_i\}}{2p^2}\right)$$

which is clearly multiplicative in the sense that, if the products $b_0b_1b_2b_3$ and $\tilde{b}_0\tilde{b}_1\tilde{b}_2\tilde{b}_3$ are coprime, then $g(b_0\tilde{b}_0, b_1\tilde{b}_1, b_2\tilde{b}_2, b_3\tilde{b}_3) = g(\mathbf{b})g(\tilde{\mathbf{b}})$. Therefore,

$$\sum_{\substack{\mathbf{b}\in\mathbb{N}_{\mathrm{odd}}^4\\ (2.5.1)}}\frac{g(\mathbf{b})}{b_0^2b_1^2b_2^2b_3^2} = \prod_{p\neq 2}\left(\sum_{\substack{\mathbf{e}\in(\mathbb{N}\cup\{0\})^4\\ \min(e_0,e_1)=0\\ \min(e_2,e_3)=0}}\frac{g(p^{e_0},p^{e_1},p^{e_2},p^{e_3})}{p^{2e_0+2e_1+2e_2+2e_3}}\right) \tag{2.10.10}$$

We now consider the sums inside the product. There is a single term for which $e_i = 0$ for all $i \in \{0,1,2,3\}$ given by

$$g(1,1,1,1) = (\kappa_p^{(2)})^{-1}(\kappa_p^{(2)}) = (\kappa_p^{(2)})^{-1}\left(\frac{p^2+2p+2\rho'_{(r,i)}(p)}{p^2}\right). \tag{2.10.11}$$

When $e_i \geqslant 1$ for a single $i \in \{0,1,2,3\}$,

$$g(p^{e_0},p^{e_1},p^{e_2},p^{e_3}) = (\kappa_p^{(2)})^{-1}\left(1+\frac{3}{2p}+\frac{\rho'_{(r,i)}(p)}{p^2}\right) \text{ and } \sum_{e\geqslant 1}\frac{1}{p^{2e}} = \frac{1/p^2}{(1-1/p^2)}.$$

There are four such terms, together giving a contribution of

$$(\kappa_p^{(2)})^{-1}\left(\frac{4p^2+6p+4\rho'_{(r,i)}(p)}{p^2(p^2-1)}\right). \tag{2.10.12}$$

When $e_i \geqslant 1$ for exactly two $i \in \{0,1,2,3\}$, the minimum conditions on the e_i dictate that

$$g(p^{e_0},p^{e_1},p^{e_2},p^{e_3}) = (\kappa_p^{(2)})^{-1}\left(1+\frac{1}{p}+\frac{\rho'_{(r,i)}(p)}{2p^2}\right) \text{ and } \sum_{e_i,e_j\geqslant 1}\frac{1}{p^{2e_i+2e_j}} = \left(\frac{1/p^2}{(1-1/p^2)}\right)^2.$$

There are four possible pairs $(i,j) \in \{0,1,2,3\}^2$ $(i<j)$ in which this can occur, together giving a contribution of

$$(\kappa_p^{(2)})^{-1}\left(\frac{4p^2+4p+2\rho'_{(r,i)}(p)}{p^2(p^2-1)^2}\right). \tag{2.10.13}$$

The condition $\min(e_0,e_1) = \min(e_2,e_3) = 0$ does not allow any contribution from $\mathbf{e} \in (\mathbb{N}\cup\{0\})^4$ where three or four $e_i \geqslant 1$. Therefore, inputting (2.10.11),(2.10.12) and (2.10.13) into the right hand side of (2.10.10) for each prime $p \neq 2$ tells us that (2.10.10) is equal to

$$\begin{aligned}&\prod_{p\neq 2}(\kappa_p^{(2)})^{-1}\left(\frac{p^2+2p+2\rho'_{(r,i)}(p)}{p^2}+\frac{4p^2+6p+4\rho'_{(r,i)}(p)}{p^2(p^2-1)}+\frac{4p^2+4p+2\rho'_{(r,i)}(p)}{p^2(p^2-1)^2}\right)\\ &=\prod_{p\neq 2}(\kappa_p^{(2)})^{-1}\left(\frac{p^6+2p^5+2(\rho'_{(r,i)}(p)+1)p^4+2p^3+2p^2}{p^2(p^2-1)^2}\right).\end{aligned}$$

Inputting this into (2.10.9) and using $(p^2-1) = p^2(1-1/p)(1+1/p)$, we have now proved the following:

Proposition 2.10.6. *For each $(r,i) \in \{(1,2),(1,3),(2,2),(2,3)\}$, the constant $\mathfrak{C}_{r,i}$ is equal to*

$$\frac{\rho_{(r,i)}}{256\pi^2}\prod_{p\neq 2}\left(1+\frac{1}{p}\right)^{-2}\left(1+\frac{2}{p}+\frac{2(\rho'_{(r,i)}(p)+1)}{p^2}+\frac{2}{p^3}+\frac{1}{p^4}\right).$$

2.10.7 Conclusion of the proof of Theorem 1.2.1

We combine Propositions 2.5.7, 2.8.11 and 2.9.6 to obtain

$$N_r(B) = \frac{(\mathfrak{C}_{r,2}+\mathfrak{C}_{r,3})B^2\log\log B}{\log B} + O_A\left(\frac{B^2\sqrt{\log\log B}}{\log B}\right)$$

for sufficiently large $A > 0$. Then, by Proposition 2.4.2,

$$N(B) = \frac{(2\mathfrak{C}_{1,2} + 2\mathfrak{C}_{1,3} + \mathfrak{C}_{2,2} + \mathfrak{C}_{2,3})B^2 \log\log B}{\log B} + O_A\left(\frac{B^2\sqrt{\log\log B}}{\log B}\right).$$

Finally, using Proposition 2.10.6, $(2\mathfrak{C}_{1,2} + 2\mathfrak{C}_{1,3} + \mathfrak{C}_{2,2} + \mathfrak{C}_{2,3})$ is then

$$\frac{935}{36\pi^2}\prod_{p\neq 2}\left(1+\frac{1}{p}\right)^{-2}\left(1+\frac{2}{p}+\frac{4}{p^2}+\frac{2}{p^3}+\frac{1}{p^4}\right)$$
$$+\frac{25}{36\pi^2}\prod_{p\neq 2}\left(1+\frac{1}{p}\right)^{-2}\left(1+\frac{2}{p}+\frac{2\left(1+\left(\frac{-1}{p}\right)\right)}{p^2}+\frac{2}{p^3}+\frac{1}{p^4}\right).$$

Chapter 3

Local solubility for a family of conics

3.1 Introduction

In this chapter, we will relate the family of quadric surfaces covered in Theorem 1.2.1 with a family of planar conics. As a result of this comparison, we will obtain the asymptotic for the rational solubility of this family of conics for free. We will also compute the Subordinate Brauer group for these families.

3.2 Set-up

To begin with, we make some key remarks about our family of quadric surfaces that will allow us to understand the geometric invariants of our family. Recall that $Z \subset \mathbb{P}^3 \times \mathbb{P}^3$ is the variety cut out by the equations

$$y_0x_0^2 + y_1x_1^2 + y_2x_2^2 + y_3x_3^2 = 0 \text{ and } y_0y_1 = y_2y_3,$$

that $Y \subset \mathbb{P}^3$ is the quadric surface cut out by the equation

$$y_0y_1 = y_2y_3$$

and that $\pi : Z \to Y$ is the dominant map sending $([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3]) \in Z$ to $[y_0 : y_1 : y_2 : y_3] \in Y$. It was presented in §2.3 that by applying the $\mathbb{Q}$-isomorphism from $\mathbb{P}^1 \times \mathbb{P}^1$ to Y given by,

$$y_0 = t_0t_2,\ y_1 = t_1t_3,\ y_2 = t_1t_2,\ y_3 = t_0t_3,$$

that Z is $\mathbb{Q}$-isomorphic to the variety $Z' \subset \mathbb{P}^3 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ given by

$$t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 + t_0t_3x_3^2 = 0 \tag{3.2.1}$$

where $[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3$ and $([t_0 : t_1], [t_2 : t_3]) \in \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, setting $\phi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to Y$ and $\phi_2 : Z \to Z'$ to be the isomorphisms described above, we obtain the dominant map $\varphi : Z' \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by $\varphi = \phi_1^{-1} \circ \pi \circ \phi_2^{-1}$. It is in this form that we will consider our problem in this section. We will denote by Q_t the quadric fibre $\varphi^{-1}(t)$ associated to the point $t \in \mathbb{P}^1 \times \mathbb{P}^1$.

We now remark that the variety Z' (and by consequence Z) is singular. Indeed the Jacobian of this variety is given by

$$\Big(2t_0t_2x_0, 2t_1t_3x_1, 2t_1t_2x_2, 2t_0t_3x_3, (t_2x_0^2 + t_3x_3^2), (t_3x_1^2 + t_2x_2^2), (t_0x_0^2 + t_1x_2^2), (t_1x_1^2 + t_0x_3^2)\Big)$$

and so it has a singular locus contained in the union

$$S = \bigcup_{i=0}^{3} \{(x, t) \in \mathbb{P}^3 \times (\mathbb{P}^1 \times \mathbb{P}^1) : t_i = 0,\ x_{i'} = x_{j'} = 0,\ t_{i'}x_i^2 + t_jx_j^2 = 0\} \tag{3.2.2}$$

where we have:

$$(j, i', j') = \begin{cases} (3, 2, 1) \text{ if } i = 0 \\ (2, 3, 0) \text{ if } i = 1 \\ (0, 1, 3) \text{ if } i = 2 \\ (1, 0, 2) \text{ if } i = 3. \end{cases}$$

Note that this singular locus is the image of the singular locus for Z defined in section 2 of [8] under the isomorphism ϕ_2. Now let L_i denote the line defined by $t_i = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{L} = \cup_{i=0}^3 L_i$. Then we find that S lies above the union of lines $\mathcal{L}$. Henceforth we will use the notation $\mathcal{V} := (\mathbb{P}^1 \times \mathbb{P}^1) \backslash \mathcal{L}$ and remark that this is an open subset of $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark 3.2.1. Recall the thin set $T = T_1 \cup T_2$ of Y from Example 1.1.3, which gave the abundance of rational points in [8]. Let $\mathcal{T} \subset \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$ be the image of this thin set under the isomorphism ϕ_1. Then

$$\mathcal{T} = \{([t_0 : t_1], [t_2 : t_3]) \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) : -t_0t_1 = \square, -t_2t_3 = \square\}.$$

Considering the conditions of the form $t_{i'}x_i^2 + t_jx_j^2 = 0$ in the definition of S, we see that $\mathcal{L}(\mathbb{Q}) \subset \mathcal{T}$. This observation was used in §2.3 to remove the points $t_i = 0$ from the counting problem.

Finally, we pre-compose φ with a desingularisation $\widetilde{Z} \to Z'$ of Z' to obtain a dominant map $\widetilde{\varphi} : \widetilde{Z} \to \mathbb{P}^1 \times \mathbb{P}^1$. The precise form of the fibres along $\mathcal{L}$ under $\widetilde{\varphi}$ will not affect our results; the fibres of points outside $\mathcal{L}$ are unchanged.

Proposition 3.2.2. *For every quadratic fibre, Q_t, over a point $t \in \mathcal{V}$, there exists a conic C_t such that $Q_t(\mathbb{Q}) \neq \emptyset$ if and only if $C_t(\mathbb{Q}) \neq \emptyset$. Furthermore, $Q_t \cong C_t \times C_t$, and we may choose the desingularisation $\widetilde{\varphi} : \widetilde{Z} \to \mathbb{P}^1 \times \mathbb{P}^1$ to be a smooth proper model of the fibre product $C_t \times C_t \to \mathcal{V}$.*

Proof. We note that for $t \in \mathcal{V}$, the quadric surface Q_t is smooth and has square determinant $(t_0t_1t_2t_3)^2$. It follows from [11, Théorème 2.5] that $Q_t \cong C_t \times C_t$ over $\mathbb{Q}$ and that $Q_t(\mathbb{Q}) \neq \emptyset$ if and only if $C_t(\mathbb{Q}) \neq \emptyset$. That we may choose $\widetilde{\varphi} : \widetilde{Z} \to \mathbb{P}^1 \times \mathbb{P}^1$ to be a smooth model of the fibre product follows from Hironaka's theorem for the resolution of singularities. $\square$

Before we continue, it is worth examining the form of the conics C_t for a fixed $t \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$. Upon remarking that Q_t is birational to the quadric

$$x_0^2 + t_0t_1x_1^2 + t_2t_3x_2^2 + t_0t_1t_2t_3x_3^2 = 0$$

we may take [11, Théorème 2.5] and [12, Proposition 1.1.8] to see that C_t can be written as

$$C_t : x_0^2 + t_0t_1x_1^2 + t_2t_3x_2^2 = 0. \tag{3.2.3}$$

Note that we are working with $t \in \mathcal{V}$ here so that none of the t_i are 0. From [12, Proposition 1.1.8] we also obtain a quaternion algebra $q_{C_t} = (-t_0t_1, -t_2t_3)$ over $\mathbb{Q}$.

3.3 Solubility for a family of conics

We can now prove the main theorem for this chapter. Keeping the notation of the previous section, we will now consider the variety $\mathcal{C} \subset \mathbb{P}^3 \times \mathbb{P}^3$ defined by the equations

$$y_0x_0^2 + y_1x_1^2 + y_2x_2^2 = 0 \text{ and } y_0y_1 = y_2y_3$$

and the obvious dominant map $\nu : \mathcal{C} \to Y$. As with the quadric fibre bundle, this conic fibre bundle has an isomorphism $\iota : \mathcal{C} \to \mathcal{C}'$ where $\mathcal{C}' \subseteq \mathbb{P}^3 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ is defined by the equation

$$t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^2 = 0$$

with $t = ([t_0 : t_1], [t_2 : t_3]) \in \mathbb{P}^1 \times \mathbb{P}^1$. Following [8], we consider a desinguarisation of this problem, namely the maps $\widetilde{\nu} : \widetilde{\mathcal{C}} \to Y$ and $\widetilde{\iota} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}'$. Using the notation of the introduction, we aim to consider the counting problem

$$N_{Y \setminus T}(\widetilde{\nu}, B) = \sharp \left\{ y \in Y(\mathbb{Q}) : \begin{array}{c} -y_0y_2 \neq \square, \ -y_0y_3 \neq \square \\ \widetilde{\nu}^{-1}(y) \text{ has a } \mathbb{Q}\text{-point} \\ H(y) \leqslant B \end{array} \right\}. \tag{3.3.1}$$

In [8], they note that using their methods, one may prove

$$B^2 \ll N_Y(\widetilde{\nu}, B) \ll B^2.$$

In other words, including the thin set T, we have the same upper and lower bounds for the count over the family of quadrics. The next theorem shows that the asymptotic formula for $N_{Y\setminus T}(\widetilde{\nu}, B)$ is the same as that for $N(B) = N_{Y\setminus T}(\widetilde{\pi}, B)$.

Theorem 3.3.1. *As $B \to \infty$,*

$$N_{Y\setminus T}(\widetilde{\nu}, B) = \frac{cB^2 \log\log B}{\log B} + O\left(\frac{B^2\sqrt{\log\log B}}{\log B}\right)$$

where $c > 0$ is the same constant given in the statement of Theorem 1.2.1.

Proof. We begin, as usual, with the parameterisation of Y by $\mathbb{P}^1 \times \mathbb{P}^1$ to transform the counting problem to

$$N_{Y\setminus T}(\widetilde{\nu}, B) = N_{(\mathbb{P}^1\times\mathbb{P}^1)\setminus\mathcal{T}}(\widetilde{w}, B) = \sharp \left\{ t \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) : \begin{array}{c} -t_0t_1 \neq \square, \ -t_2t_3 \neq \square \\ \widetilde{w}^{-1}(t) \text{ has a } \mathbb{Q}\text{-point} \\ H([t_0:t_1])H([t_2:t_3]) \leqslant B \end{array} \right\},$$

where $\widetilde{w} = \phi_1 \circ \widetilde{\nu} \circ \iota^{-1}$. For the set of points $t \notin \mathcal{T}$, the fibres $\widetilde{w}^{-1}(t)$ are given by the conic

$$C_t' : t_0t_2x_0^2 + t_1t_3x_1^2 + t_1t_2x_2^3 = 0. \tag{3.3.2}$$

We see from Remark 3.2.1 that points $([t_0 : t_1], [t_2 : t_3]) \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$ with $t_i = 0$ for some $0 \leqslant i \leqslant 3$ lie inside $\mathcal{T}$. Therefore, by multiplying the equation (3.3.2) by t_1t_2 and permuting the variables we find that the fibres C_t' for $t \notin \mathcal{T}$ are equivalent to the conic C_t as defined in (3.2.3). Thus, by Proposition 3.2.2, we have

$$C_t'(\mathbb{Q}) \neq \emptyset \iff C_t(\mathbb{Q}) \neq \emptyset \iff Q_t(\mathbb{Q}) \neq \emptyset$$

for $t \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \setminus \mathcal{T}$. In particular, we have the equality,

$$N_{Y\setminus T}(\widetilde{\nu}, B) = N(B)$$

where we recall that $N(B)$, as defined in (1.2.3), is the counting problem determined by Theorem 1.2.1. An application of this theorem, therefore, concludes this proof. □

3.4 The Subordinate Brauer Group

In this section, we will recall the subordinate Brauer group of a proper morphism $f : X \to Y$ of integral Noetherian schemes over a field, and compute this group for our map $\varphi : Z' \to \mathbb{P}^1 \times \mathbb{P}^1$.

3.4.1 The Brauer Group

For a scheme X over a field k, we will define its Brauer group by $\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$ where $\mathbb{G}_m$ denotes the multiplicative algebraic torus. Furthermore, we will write $k(X)$ for the function field of X and $X^{(1)}$ to be the set of codimension 1 points of X. The following is Grothendieck's Purity Theorem (see [12, Theorem 3.7.2]):

Theorem 3.4.1. *Suppose X is a regular, integral scheme over a field k of characteristic 0. Then we have the following exact sequence:*

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(k(X)) \longrightarrow \bigoplus_{D\in X^{(1)}} H^1(k(D), \mathbb{Q}/\mathbb{Z})$$

where the last map is the direct sum of the residue maps $\partial_D : \mathrm{Br}(k(X)) \to H^1(k(D), \mathbb{Q}/\mathbb{Z})$ *at the codimension* 1 *points* D*.*

An element of the Brauer group $b \in \mathrm{Br}(X)$ is called unramified at a point D if $\partial_D(b) = 0$.

Now suppose that k is a number field. The following exact sequence is a well-known result of class field theory:

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \bigoplus_{v \in \mathrm{Val}(k)} \mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \tag{3.4.1}$$

where the last non-trivial map is the sum of local invariant maps $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$. Let $\mathbf{A}_k$ denote the adelic numbers over k. Then $X(\mathbf{A}_k)$ denotes the adelic points of X. We will also make use of the Brauer-Manin pairing

$$X(\mathbf{A}_k) \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z},\ ((x_v)_{v \in \mathrm{Val(k)}}, b) \longmapsto \sum_v \mathrm{inv}_v(b(x_v)).$$

We remark here that $\mathrm{Br}(\mathbb{P}^n_k) \cong \mathrm{Br}(\mathbb{A}^n_k) \cong \mathrm{Br}(k)$.

3.4.2 The Subordinate Brauer Group

Let $f : W \to X$ be a proper morphism of regular, integral Noetherian schemes over a field k with geometrically irreducible fibre. Suppose throughout that k has characteristic 0.

Definition 3.4.2 ([29], Definition 2.1)**.** For $f : W \to X$ above, we define the subordinate Brauer group as

$$\mathrm{Br}_{\mathrm{sub}}(X, f) = \bigcap_{D \in X^{(1)}} \left\{ \alpha \in \mathrm{Br}(k(X)) : \begin{array}{l} \partial_E f^*(\alpha) = 0 \text{ for all irreducible components} \\ E \subseteq f^{-1}(D) \text{ of multiplicity } 1 \end{array} \right\}.$$

Now suppose that X admits an ample line bundle. Let $U \subset X$ be an open subset and suppose that $\mathscr{B}$ is a finite multi-set of elements in $\mathrm{Br}(U)$. Denote by $\langle \mathscr{B} \rangle$ the subgroup of $\mathrm{Br}(U)$ generated by $\mathscr{B}$. Then we may also define the following.

Definition 3.4.3 ([29], Definition 2.8)**.** We say that an element $b \in \mathrm{Br}(k(X))$ is *subordinate* to $\mathscr{B}$ if for every $D \in X^{(1)}$, $\partial_D(b)$ lies in $\partial_D(\langle \mathscr{B} \rangle)$. We may also define

$$\mathrm{Br}_{\mathrm{sub}}(X, \mathscr{B}) = \left\{ \alpha \in \mathrm{Br}(k(X)) :\ \partial_D(b) \in \partial_D(\langle \mathscr{B} \rangle) \text{ for all } D \in X^{(1)} \right\}.$$

Since X admits an ample line bundle, every element $b \in \mathrm{Br}(U)$ is the Brauer class of a *Severi-Brauer* scheme V_b. This follows from a theorem of Gabber. In our setting, $U = \mathcal{V}$ and $\mathscr{B} = \{q_{C_t}, q_{C_t}\}$ where q_{C_t} is the quaternion algebra associated to a given conic C_t from Proposition 3.2.2.

Lemma 3.4.4 ([29], Lemma 2.9)**.** *Suppose that X and W, $U \subseteq X$, $\mathscr{B}$ are as defined above. Suppose that $f : W \to X$ is a smooth proper model of the fibre product $F : \bigtimes_{b\in\mathscr{B}} V_b \to U$. Then*

$$\mathrm{Br}_{\mathrm{sub}}(X, f) = \mathrm{Br}_{\mathrm{sub}}(X, \mathscr{B}).$$

Remark 3.4.5. The exposition in §2.4 of [29] does not explicitly define $\mathscr{B}$ to be a multi-set, instead defining it to be a finite subset of $\mathrm{Br}(U)$. The inclusion of multi-sets does not change the definition of $\mathrm{Br}_{\mathrm{sub}}(X, \mathscr{B})$ as this depends only on the group $\langle\mathscr{B}\rangle$, which remains unchanged. Allowing multi-sets also does not change the proof of the above lemma.

We are now ready to compute $\mathrm{Br}_{\mathrm{sub}}(\widetilde{Z}, \widetilde{\varphi})$.

Lemma 3.4.6. *We have*

$$\mathrm{Br}_{\mathrm{sub}}(\widetilde{Z}, \widetilde{\varphi})/\mathrm{Br}(\mathbb{Q}) = \langle q_{C_t}\rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. By taking $\mathscr{B} = \{q_{C_t}, q_{C_t}\}$, then we may use Proposition 3.2.2 and Lemma 3.4.4; therefore we aim to compute $\mathrm{Br}_{\mathrm{sub}}(\widetilde{Z}, \mathscr{B})$. Recall that $q_{C_t} = (-t_0t_1, -t_2t_3)$, which ramifies precisely at the lines $\mathcal{L}_i = \{t_i = 0\}$. Using [18, Example 7.1.5] the residues of q_{C_t} at $\mathcal{L}_i$ are

$$\partial_{\mathcal{L}_i}(q_{C_t}) = \begin{cases} -(t_2t_3)^{-1} = -t_2t_3 \in \mathbb{Q}(\mathcal{L}_i)^\times/\mathbb{Q}(\mathcal{L}_i)^{\times 2} & \text{if } i \in \{0,1\} \\ -t_0t_1 \in \mathbb{Q}(\mathcal{L}_i)^\times/\mathbb{Q}(\mathcal{L}_i)^{\times 2} & \text{if } i \in \{2,3\}. \end{cases}$$

Now suppose $b \in \mathrm{Br}_{\mathrm{sub}}(\widetilde{Z}, \mathscr{B})$. Then by definition, b can only ramify along a subset of the $\mathcal{L}_i$ with the prescribed residues above. If b is unramified, $b \in \mathrm{Br}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathrm{Br}(\mathbb{Q})$ and so is constant. If b is ramified only along one of the $\mathcal{L}_i$ then $b \in \mathrm{Br}(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathcal{L}_i) \cong \mathrm{Br}(\mathbb{A}^1 \times \mathbb{P}^1) \cong \mathrm{Br}(\mathbb{Q})$ and so is also constant. Similarly, if b ramifies at exactly three of the lines $\mathcal{L}_i$ then $b - q_{C_t}$ is ramified at only one line and so is constant. We therefore suppose that b ramifies at precisely 2 of the lines $\mathcal{L}_i$. Since

$$\mathbb{P}^1 \times \mathbb{P}^1 \setminus (\mathcal{L}_i \cup \mathcal{L}_j) \cong \mathbb{A}^2$$

when $(i,j) \in \{0,1\} \times \{2,3\}$, we therefore have that b will be constant or equivalent to q_{C_t} unless it ramifies along $\mathcal{L}_0 \cup \mathcal{L}_1$ or $\mathcal{L}_2 \cup \mathcal{L}_3$. We may deal with each case similarly, so suppose that b ramifies along $\mathcal{L}_0 \cup \mathcal{L}_1$. Then it must ramify along each line with residue $-t_2t_3$. Furthermore, since the residue maps are homomorphisms, the residue of $2b$ along these lines is $(t_2t_3)^2 = 1 \in \mathbb{Q}(\mathcal{L}_i)^\times/\mathbb{Q}(\mathcal{L}_i)^{\times 2}$, $i \in \{0,1\}$. It follows that $2b$ is unramified everywhere, and thus constant. This implies that b is of order at most 2 in $\mathrm{Br}_{\mathrm{sub}}(\widetilde{Z}, \mathscr{B})/\mathrm{Br}(\mathbb{Q})$. Assume it is of order exactly 2; otherwise, it is constant. By the Merkuyev-Susin Theorem, [18, Theorem 2.5.7], this implies that up to addition by $\mathrm{Br}(\mathbb{Q})$,

$$b = \sum_{l=0}^{L} q_l$$

for q_l quaternion algebras over $\mathbb{Q}(\mathbb{P}^1 \times \mathbb{P}^1)$. Write $q_l = (\alpha_l, \beta_l)$ for $\alpha_l, \beta_l \in \mathbb{Q}(\mathbb{P}^1 \times \mathbb{P}^1)$. Writing v_j for the valuation map at the place t_j, we may decompose the α_l and β_l as

$$\alpha_l = t_0^{v_0(\alpha_l)} t_1^{v_1(\alpha_l)} t_2^{v_2(\alpha_l)} t_3^{v_3(\alpha_l)} \widetilde{\alpha}_l \text{ and } \beta_l = t_0^{v_0(\beta_l)} t_1^{v_1(\beta_l)} t_2^{v_2(\beta_l)} t_3^{v_3(\beta_l)} \widetilde{\beta}_l$$

where $\widetilde{\alpha}_l, \widetilde{\beta}_l \in \mathbb{Q}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $v_j(\widetilde{\alpha}_l) = v_j(\widetilde{\beta}_l) = 0$ for all j and all l. Then, upon using [18, Example 7.1.5] we have,

$$\partial_{\mathcal{L}_k}(b) = (-1)^{\sigma_k(\boldsymbol{\alpha},\boldsymbol{\beta})} \left[\prod_{\substack{j=0\\ j\neq k}}^{3} t_j^{\Sigma_j^k}\right] \left[\prod_l \widetilde{\alpha}_l^{v_{t_k}(\beta_l)}\right] \left[\prod_l \widetilde{\beta}_l^{-v_{t_k}(\alpha_l)}\right] \in \mathbb{Q}(\mathcal{L}_i)^\times / \mathbb{Q}(\mathcal{L}_i)^{2\times}$$

where

$$\Sigma_j^k = \sum_l (v_j(\alpha_l) v_k(\beta_l) - v_j(\beta_l) v_k(\alpha_l)) = -\Sigma_k^l.$$

Since b ramifies precisely on $\mathcal{L}_0 \cup \mathcal{L}_1$ and is subordinate to q_{C_t}, we know that $\partial_{\mathcal{L}_0}(b) = -t_2 t_3 \in \mathbb{Q}(\mathcal{L}_0)^\times / \mathbb{Q}(\mathcal{L}_0)^{2\times}$. From this, and the general formula above, we can see that $\Sigma_2^0 \equiv 1 \bmod 2$; therefore we must also have $\Sigma_0^2 \equiv 1 \bmod 2$. Then,

$$\partial_{\mathcal{L}_2}(b) = t_0 \left[(-1)^{\sigma_2(\boldsymbol{\alpha},\boldsymbol{\beta})} \left[\prod_{\substack{j=1\\ j\neq 2}}^{3} t_j^{\Sigma_j^2}\right] \left[\prod_l \widetilde{\alpha}_l^{v_{t_2}(\beta_l)}\right] \left[\prod_l \widetilde{\beta}_l^{-v_{t_2}(\alpha_l)}\right]\right] \in \mathbb{Q}(\mathcal{L}_2)^\times / \mathbb{Q}(\mathcal{L}_2)^{2\times}.$$

But since $t_1, t_3, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_L, \widetilde{\beta}_1, \ldots, \widetilde{\beta}_L$ all have valuation 0 at t_0, this implies that $\partial_{\mathcal{L}_2}(b) \neq 1 \in \mathbb{Q}(\mathcal{L}_2)^\times / \mathbb{Q}(\mathcal{L}_2)^{2\times}$ - i.e that b is ramified at $\mathcal{L}_2$. This is a contradiction. Therefore, b cannot be subordinate to q_{C_t} and ramify at precisely $\mathcal{L}_0 \cup \mathcal{L}_1$. The result follows. □

Chapter 4

Variations on the Large Sieve

4.1 Introduction

In this chapter, we prove the bounds for bilinear sums over the Jacobi symbol $\left(\frac{n}{m}\right)$ which are necessary for the conclusion of Theorem 1.2.1. First, let us recall the previous results in this direction. Suppose a_n and b_m are arbitrary complex sequences bounded in magnitude by 1 and supported on the odd integers. Let $N, M \geqslant 2$. Then we have the following well-known bounds:

- Elliott [14, 22]:

$$\sum_{n \leqslant N,} \sum_{m \leqslant M} a_n b_m \left(\frac{n}{m}\right) \ll NM \left(N^{-1/2} + N^{1/2} M^{-1/2} \log N\right); \tag{4.1.1}$$

- Heath-Brown [22]: for any $\epsilon > 0$,

$$\sum_{n \leqslant N,} \sum_{m \leqslant M} \mu^2(2n)\mu^2(2m) a_n b_m \left(\frac{n}{m}\right) \ll_\epsilon (NM)^{1+\epsilon} \left(N^{-1/2} + M^{-1/2}\right); \tag{4.1.2}$$

- Friedlander–Iwaniec [17]:

$$\sum_{n \leqslant N,} \sum_{m \leqslant M} a_n b_m \left(\frac{n}{m}\right) \ll NM \left(N^{-1/6} + M^{-1/6}\right) (\log 3NM)^{7/6}. \tag{4.1.3}$$

These bounds have been used in many problems, for example:

- Values of L-functions (Soundararajan [43]);
- 4-ranks of class groups (Fouvry–Klüners [15]);
- Manin's conjecture (Browning–Heath-Brown [4]);
- Bateman–Horn's conjecture on average (Baier–Zhao [1]).

When $N^2 \ll M$ the Elliott bound is the most effective bound; when N and M are of comparable size the Heath-Brown result is more effective. In applications to rational points problems the Friedlander–Iwaniec bound is more versatile as the presence of $(NM)^\epsilon$ in the Heath-Brown bound may lead to problems when we require no loss of logarithms. One may also combine the Elliott and Heath-Brown results into one as in the work of Fouvry and Klüners on the 4-rank of class groups [15, Lemma 15].

For the hyper-skewed regions that occur in our problem, each of the above results will fail to give bounds that are smaller than the main term. To solve this problem we provide two variations of the above bounds which suffice for our purposes. The first is an improved version of Elliott's bound for when the complex sequence b_m has some multiplicative structure and will play a crucial role in Chapter 5. The second is Theorem 2.2.5 for sums of the Jacobi symbol in hyperbolic regions, which was used in Chapter 2, §2.6 to bound the regions of our character sum where the Jacobi symbols have large conductions.

4.1.1 Hooley neutralisers and the large sieve

To prove Propositions 2.2.8 and 2.2.11 we will require improvements over (4.1.1) in regions that are hyper-skewed. The following result states that this may be achieved when the complex sequence b_m has some multiplicative structure.

Theorem 4.1.1. *Let $M, N \geqslant 2$, and fix some $\epsilon > 0$. Let f be any multiplicative function such that $0 \leqslant f(p) \leqslant 1$ and $f(p^m) \leqslant f(p)$ for all primes p and all $m \geqslant 2$. Suppose also that there exists an $0 < \alpha \leqslant 1$ such that for all $X \geqslant 2$ we have,*

$$\sum_{p \leqslant X} \frac{f(p)}{p} = \alpha \log\log X + O(1). \tag{4.1.4}$$

Then for any complex sequences a_n,b_m which are supported on the odd integers such that $|a_n| \leqslant 1$ and $|b_m| \leqslant 1$ we have:

$$\sum_{n \leqslant N,}\sum_{m \leqslant M} a_n b_m f(m) \left(\frac{n}{m}\right) \ll_\epsilon \frac{MN^{1/2}(\log N)}{(\log M)^{(1-\alpha)}} + \frac{M^{1/2+\epsilon}N^{3/2}(\log N)^{1/2}}{(\log M)^{(1-\alpha)/2}},$$

where the implied constant depends at most on ϵ.

This theorem is most effective when $N^2 \ll M$. The main benefit here is that we have maintained saving from summing over the multiplicative function. Indeed, it follows from a result of Shiu [40, Theorem 1] that for a multiplicative function satisfying the conditions of Theorem 4.1.1 that we have

$$\sum_{m \leqslant M} f(m) \ll \frac{M}{(\log M)^{1-\alpha}}.$$

We will prove this in §4.2 by using Hooley Neutralisers to insert the Brun Sieve into standard large sieve methods. Using partial summation we obtain further improvement when a_n contains a harmonic factor:

Corollary 4.1.2. *Let $M \geqslant N \geqslant W \geqslant 2$, and fix some $\epsilon > 0$. Let f be any multiplicative function such that $0 \leqslant f(p) \leqslant 1$ and $f(p^m) \leqslant f(p)$ for all primes p and all $m \geqslant 2$. Suppose also that for all $X \geqslant 2$ we have,*

$$\sum_{p \leqslant X} \frac{f(p)}{p} = \alpha \log\log X + O(1) \tag{4.1.5}$$

for some $0 < \alpha < 1$. Then for any complex sequences a_n,b_m which are supported on the odd integers such that $|a_n| \leqslant 1$ and $|b_m| \leqslant 1$ we have:

$$\sum\sum_{W<n\leqslant N,\, m\leqslant M} \frac{a_n}{n} b_m f(m) \left(\frac{n}{m}\right) \ll_\epsilon \frac{M(\log N)}{W^{1/2}(\log M)^{(1-\alpha)}} + \frac{M^{1/2+\epsilon}N^{1/2}(\log N)^{1/2}}{(\log M)^{(1-\alpha)/2}},$$

where the implied constant depends at most on ϵ.

The benefit of this result is that it encodes not only the saving from the multiplicative function but also the convergence from the sum $\sum_{W<n\leqslant N} \frac{1}{n}\left(\frac{n}{m}\right)$ when m is non-square. Again, this result is most effective when $N^2 \ll M$. This, however, will be sufficient for our purposes provided that they are aptly applied alongside the Friedlander–Iwaniec bound (4.1.3). For the proof of Theorem 1.2.1 the corollary will be applied with the multiplicative function $\frac{1}{\tau(m)}$.

As an example of how these results may be applied elsewhere, we remark that in the proof of Proposition 2.2.11, we encounter the simultaneous average of $\frac{1}{\tau(m)}$ and special values of L-functions, $L\left(1, \left(\frac{\cdot}{m}\right)\right)$. An application of Corollary 4.1.2 will yield the following bound:

Corollary 4.1.3. *For all $X \geqslant 3$ we have*

$$\sum_{1<m\leqslant X} \frac{\mu^2(2m)}{\tau(m)} L\left(1, \left(\frac{\cdot}{m}\right)\right) \ll \frac{X}{\sqrt{\log X}}.$$

Since the average of $\frac{1}{\tau(m)}$ is $\frac{c_1}{\sqrt{\log M}}$ for some constant c_1 and the average of $L\left(1, \left(\frac{\cdot}{m}\right)\right)$ is a constant, this bound suggests that the distributions of these two functions are independent. This result is a consequence of Lemma 5.3.5, which will be proven in Chapter 5.

4.1.2 Averages of Jacobi symbols over hyperbolic regions

The second variation of the large sieve for quadratic characters is a bound for sums over hyperbolic regions. These are sums of the general following shape:

$$\sum_{\substack{n,m\in\mathbb{N} \\ 1\leqslant nm\leqslant T}} a_n b_m \left(\frac{n}{m}\right), \tag{4.1.6}$$

where (a_n) and (b_m) are arbitrary complex sequences with $|a_n|, |b_m| \leqslant 1$. For general choices of complex sequences a_n and b_m these sums do not exhibit much cancellation - for example, we will see later that

$$\sum_{\substack{n,m\in\mathbb{N} \\ 2\nmid nm \\ 1\leqslant nm\leqslant T}} \left(\frac{n}{m}\right) \gg T,$$

which gives only logarithmic saving over the hyperbolic region of area $T(\log T)$. The main contribution of this sum will be seen to come from the points where either n or m is a square. This contribution is explained by the fact that such points have relatively large density in the hyperbolic region compared to their density in a rectangular one. For this reason we turn to study sums over pairs (n, m) where n and m are odd and square-free.

Remark 4.1.4. Throughout this chapter $\sum^*$ will denote a sum over odd, square-free integers. As usual, μ will denote the Möbius function.

In this case, however, there may still be a large contribution from points close to the axes, particularly from the lines $n = 1$ and $m = 1$. Another example which gives very small cancellation is the following: choosing $a_n = (\frac{n}{11})$ and b_m to be the characteristic function for the condition $m = 11$, one sees that

$$\sideset{}{^*}\sum_{\substack{1<n,m\leqslant T \\ 1<nm\leqslant T}} \mu^2(2nm)a_n b_m \left(\frac{n}{m}\right) = \sum_{1<n\leqslant T/11} \mu^2(22n) \gg T.$$

It is therefore clear that in order to obtain further cancellation, we must impose the extra condition that $n, m > z$ for some parameter $z = z(T)$ which tends to infinity with T. Our first bound, which is a more complete version of Lemma 2.2.5, shows that these conditions are sufficient to provide the required cancellation.

Theorem 4.1.5. *Let $T, z \geqslant 2$ and let (a_n), (b_m) be any complex sequences such that $|a_n|, |b_m| \leqslant 1$. If there exists an $\epsilon > 0$ such that $z \geqslant T^{1/3-\epsilon}$, then*

$$\sideset{}{^*}\sum_{\substack{z<n,m\leqslant T \\ nm\leqslant T}} a_n b_m \left(\frac{n}{m}\right) \ll_\epsilon \frac{T^{1+\epsilon}}{z^{1/2}},$$

where the implied constant depends at most on ϵ. If there exists an $\epsilon > 0$ such that $z \leqslant T^{1/3-\epsilon}$, then

$$\sideset{}{^*}\sum_{\substack{z<n,m\leqslant T \\ nm\leqslant T}} a_n b_m \left(\frac{n}{m}\right) \ll_\epsilon \frac{T(\log T)^3}{z^{1/2}},$$

where the implied constant depends at most on ϵ.

Theorem 4.1.5 fails to give a saving over the trivial bound of $T(\log T)$ if $z \ll (\log T)^4$. This is satisfactory for most applications, however it is also possible to obtain cancellation for any z which tends to infinity with T with the cost of a smaller exponent of z.

Theorem 4.1.6. *For all $T, z \geqslant 2$ and all complex sequences (a_n), (b_m) such that $|a_n|, |b_m| \leqslant 1$ we have,*

$$\sum_{\substack{z<n,m\leqslant T\\ nm\leqslant T}}^{*} a_n b_m \left(\frac{n}{m}\right) \ll \frac{T(\log T)}{z^{1/4}},$$

where the implied constant is absolute.

Remark 4.1.7. It is impossible to improve the exponent of z in Theorem 4.1.6 to be > 1 in general: using a similar example to before, if we take any $z \leqslant T^{1/4}$, p a prime satisfying $z < p \leqslant 2z$, $a_n = (\frac{n}{p})$ and b_m the characteristic function for the condition $m = p$, one finds that

$$\sum_{\substack{z<n,m\leqslant T\\ 1\leqslant nm\leqslant T}}^{*} a_n b_m \left(\frac{n}{m}\right) = \sum_{z<n\leqslant T/p} \mu^2(2pn) = \frac{2}{3(1+1/p)\zeta(2)}\left(\frac{T}{p} - z\right) + O\left(\frac{T^{1/2}}{p^{1/2}}\right).$$

This is $\gg \frac{T}{p} \gg \frac{T}{z}$ since $2pz < 4z^2 \leqslant T$. If this were $O(\frac{T(\log T)}{z^{\alpha}})$, then $z^{\alpha-1} = O(\log T)$, which may be contradicted by taking $z = (\log T)^A$ for $A > 0$ suitably large.

In order to prove Theorems 4.1.5 and 4.1.6 we will actually prove the following more general results.

Theorem 4.1.8. *Let $T, z \geqslant 2$, $c \geqslant 0$, and let (a_n), (b_m) be any complex sequences such that $|a_n|, |b_m| \leqslant 1$. If there exists an $\epsilon > 0$ such that $z \geqslant T^{1/3-\epsilon}$ then,*

$$\sum_{\substack{z<n,m\leqslant T\\ nm\leqslant T}}^{*} (nm)^c a_n b_m \left(\frac{n}{m}\right) \ll_{c,\epsilon} \frac{T^{1+c+\epsilon}}{z^{1/2}},$$

where the implied constant depends at most on c and ϵ. If there exists an $\epsilon > 0$ such that $z \leqslant T^{1/3-\epsilon}$, then

$$\sum_{\substack{z<n,m\leqslant T\\ nm\leqslant T}}^{*} (nm)^c a_n b_m \left(\frac{n}{m}\right) \ll_{c,\epsilon} \frac{T^{1+c}(\log T)^3}{z^{1/2}},$$

where the implied constant depends at most on c and ϵ.

Theorem 4.1.9. *For all $T, z \geqslant 2$, $c \geqslant 0$, and all complex sequences (a_n), (b_m) such that $|a_n|, |b_m| \leqslant 1$. Then*

$$\sum_{\substack{z<n,m\leqslant T\\ nm\leqslant T}}^{*} (nm)^c a_n b_m \left(\frac{n}{m}\right) \ll_{c} \frac{T^{1+c}(\log T)}{z^{1/4}},$$

where the implied constant depends at most on c.

Theorems 4.1.5 and 4.1.6 will then follow from the cases where $c = 0$. The methods in [17, 22] do not interact well with the hyperbolic region as they exploit the linear structure of the rectangular regions through the use of Hölder's inequality. We will circumvent this problem by applying the following version of Perron's formula to eliminate the hyperbolic height condition [21, Lemma 2.2]:

$$\frac{1}{\pi}\int_{-R}^{R}(nm)^{it}f_\tau(t)dt = \mathbb{1}(nm \leqslant \tau) + O(R^{-1}|\log(nm) - \log(\tau)|^{-1}), \tag{4.1.7}$$

where $f_\tau(t) = \frac{\sin(t\log(\tau))}{t}$ and

$$\mathbb{1}(\mu \leqslant \tau) = \begin{cases} 1 \text{ if } \mu \leqslant \tau \\ 0 \text{ if } \mu > \tau. \end{cases}$$

This will allow us to apply existing results. In particular, we apply Corollary 4 and Theorem 1 of [22]:

$$\sum_{\substack{m\leqslant M\\ 2\nmid m}}\sum_{n\leqslant N} a_n b_m \left(\frac{n}{m}\right) \ll_\epsilon (MN)^\epsilon (MN^{1/2} + M^{1/2}N), \tag{4.1.8}$$

and, for $I \subseteq [1, N] \cap \mathbb{N}$ of size $|I|$,

$$\sideset{}{^*}\sum_{m\leqslant M}\left|\sideset{}{^*}\sum_{n\in I} a_n\left(\frac{n}{m}\right)\right|^2 \ll_\epsilon (MN)^\epsilon(\max(M,N))|I|, \tag{4.1.9}$$

for any $\epsilon > 0$. It will also be necessary to make use of the following version of Elliott's result:

$$\sideset{}{^*}\sum_{m\leqslant M}\left|\sideset{}{^*}\sum_{n\in I} a_n\left(\frac{n}{m}\right)\right|^2 \ll (M + N^2\log(N))|I|. \tag{4.1.10}$$

This inequality goes back to Elliott [14], but was proven by Heath-Brown [22, Equation (6)]. Recall that this bound is superior to (4.1.9) if $N \leqslant M^{1/2}$. In our proof, this will be necessary when lopsided rectangles appear in our coverings of the hyperbolic region. We cannot use (4.1.9) for such lopsided rectangles, since if $z = (\log T)^A$ for some $A > 0$, we will obtain bounds of the form $\ll \frac{T^{1+c+\epsilon}}{(\log T)^{A/2}}$, which just fails to give Theorem 4.1.8.

To prove Theorem 4.1.8 we will use these results along with a dyadic covering of the hyperbolic regions. In order to obtain saving for arbitrary z in Theorem 4.1.9 it will be necessary to cover parts of the hyperbolic region with rectangles of equal width before applying Cauchy–Schwarz and (4.1.10) to each of these rectangles and summing over the results.

Lastly, we prove asymptotics for the Jacobi sums over all odd integers within the hyperbolic region, namely we have the following:

Theorem 4.1.10. *For all $T \geqslant 2$,*

$$\sum_{\substack{1\leqslant n,m\leqslant T\\ 2\nmid nm\\ nm\leqslant T}}\left(\frac{n}{m}\right) = \left(\frac{6\zeta(2)}{7\zeta(3)}\right)T + O(T^{3/4}(\log T))$$

where ζ is the Riemann-zeta function.

This result is obtained using Dirichlet's hyperbola method. A similar method will give

$$\sum_{\substack{1\leqslant n,m\leqslant T\\ nm\leqslant T}}^{*} \left(\frac{n}{m}\right) \sim c'T.$$

for some constant $c' > 0$, proving that we may not obtain saving in the square-free setting when we include points close to the axes.

4.2 Hooley Neutralisers and the Large Sieve

In this section will prove Theorem 4.1.1. This is done with the use of Hooley neutralisers. We will begin with the following lemma, which is a slight modification of [41, Proposition 4.1] and has a similar proof.

Lemma 4.2.1. *Let $P(z)$ be the product of all odd primes $\leqslant z$ and suppose (λ_d^+) is any sequence satisfying*

$$\sum_{d|n} \lambda_d^+ \geqslant \mathbb{1}(n=1). \tag{4.2.1}$$

For any function $f : \mathbb{N} \to [0,1]$ such that

(1) f is multiplicative,

(2) $f(p^m) \leqslant f(p)$ for all primes p and all $m \geqslant 1$,

define the multiplicative function $\hat{f} : \mathbb{N} \to \mathbb{R}$ by $\hat{f}(n) = \prod_{p|n}(1 - f(p))$. Then for all integers n:

$$f(n) \leqslant \sum_{\substack{d|n\\ d|P(z)}} \lambda_d^+ \hat{f}(d).$$

Proof. We let n be a square-free integer composed only of primes $p \leqslant z$, i.e. $n|P(z)$, then

$$f(n) = \sum_{m|n} \hat{f}(m) f\left(\frac{n}{m}\right) \mathbb{1}(m=1).$$

Since f and $\hat{f}$ are non-negative, we may use (4.2.1) to get the upper bound:

$$f(n) \leqslant \sum_{m|n} \hat{f}(m) f\left(\frac{n}{m}\right) \left(\sum_{d|m} \lambda_d^+\right).$$

By writing $m = dd'$ and reversing the order of summation we get

$$\begin{aligned}\sum_{m|n} \hat{f}(m) f\left(\frac{n}{m}\right)\left(\sum_{d|m}\lambda_d^+\right) &= \sum_{d|n}\lambda_d^+\left(\sum_{d'|\frac{n}{d}} \hat{f}(dd') f\left(\frac{n}{dd'}\right)\right)\\ &= \sum_{d|n}\lambda_d^+ \hat{f}(d)\left(\sum_{d'|\frac{n}{d}} \hat{f}(d') f\left(\frac{n}{dd'}\right)\right),\end{aligned}$$

this last equality being obtained by noting that since n is square-free and $dd'|n$, d and d' must be square-free and co-prime. Now

$$\sum_{d'|\frac{n}{d}} \hat{f}(d')f\left(\frac{n}{dd'}\right) = \hat{f} * f\left(\frac{n}{d}\right),$$

where $*$ denotes Dirichlet convolution. Then since f and $\hat{f}$ are multiplicative, $\hat{f} * f$ will also be multiplicative. However, $\hat{f} * f(p) = 1$ for all primes p since $\hat{f}(p) = (1 - f(p))$. Therefore $\hat{f} * f(n) = 1$ for all square-free integers n. Thus

$$\sum_{d'|\frac{n}{d}} \hat{f}(d')f\left(\frac{n}{dd'}\right) = 1.$$

It follows that

$$f(n) \leqslant \sum_{d|n} \lambda_d^+ \hat{f}(d).$$

Since we assumed that $n|P(z)$ we are done in this case. For more general integers n we write:

$$n = \left(\prod_{\substack{p|n \\ p|P(z)}} p^{v_p(n)}\right)\left(\prod_{\substack{q|n \\ q\nmid P(z)}} q^{v_q(n)}\right).$$

Then, since f is multiplicative and has image in $[0,1]$ we use assumption (2) to obtain:

$$f(n) \leqslant f\left(\prod_{\substack{p|n \\ p|P(z)}} p\right) \leqslant \sum_{\substack{d|n \\ d|P(z)}} \lambda_d^+ \hat{f}(d),$$

where we use the previous case in this last inequality. □

We will be interested in using this for upper bound sieve coefficents. Fix some $z > 0$ and let $y = z^{10}$. Then in particular we define the upper bound sieve coefficients

$$\lambda_d^+ = \mathbb{1}(d \in \mathcal{D}^+)\mu(d)$$

where

$$\mathcal{D}^+ = \{d = p_1 \dots p_k \in \mathbb{N} : z > p_1 > \dots > p_k,\ p_m < y_m \text{ for } m \text{ odd}\}$$

for $y_m = \left(\frac{y}{p_1 \dots p_m}\right)^{1/\beta}$ and some $\beta > 1$. It is well known that

$$\sum_{d|n} \lambda_d^+ \geqslant \sum_{d|n} \mu(d) = \mathbb{1}(n = 1) \tag{4.2.2}$$

and that the λ_d^+ are supported on the interval $[1, y]$. We note also that any multiplicative function f satisfying the conditions of Lemma 4.2.1 will also satisfy conditions (i) and (ii) of [40, Section 2]. We may then use [40, Theorem 1] with, $k = 1$, $Y = X$ to obtain the bound

$$\sum_{n \leqslant X} f(n) \ll \frac{X}{\log X} \exp\left(\sum_{p \leqslant X} \frac{f(p)}{p}\right). \tag{4.2.3}$$

Furthermore, we will require that our multiplicative functions satisfy,

$$\sum_{p\leqslant X}\frac{f(p)}{p}=\alpha\log\log X+O(1)$$

for some $\alpha>0$. With this in mind, (4.2.3) becomes,

$$\sum_{n\leqslant X}f(n)\ll\frac{X}{(\log X)^{1-\alpha}}. \tag{4.2.4}$$

The following lemma encodes the insertion of the Brun Sieve into the large sieve,

Lemma 4.2.2. *Let $X\geqslant 2$. Fix some $\epsilon>0$ and set $z=X^{\epsilon/10}$ and $y=X^{\epsilon}$. Let f be any function $f:\mathbb{N}\to[0,1]$ such that*

(1) f is multiplicative,

(2) $f(p^m)\leqslant f(p)$ for all primes p and all $m\geqslant 1$.

Suppose also that there exists some $0<\alpha\leqslant 1$ such that, for all $2\leqslant Y$,

$$\sum_{p\leqslant Y}\frac{f(p)}{p}=\alpha\log\log Y+O(1). \tag{4.2.5}$$

Then for the sieve coefficients (λ_d^+) defined above and any integer n,

$$\left|\sum_{\substack{d\leqslant X\\ d|P(z)\\ \gcd(d,n)=1}}\lambda_d^+f(d)\sum_{\substack{m\leqslant X\\ d|m\\ \gcd(m,n)=1}}1\right|\ll_\epsilon\frac{\phi(n)X}{n(\log X)^\alpha}\prod_{\substack{p|P(z)\\ p|n}}\left(1-\frac{f(p)}{p}\right)^{-1}+X^\epsilon\tau(n)$$

where the implied constant depends at most on ϵ.

Proof. Using the fact that λ_d^+ is supported on $[1,y]$ we have,

$$\begin{aligned}\sum_{\substack{d\leqslant X\\ d|P(z)\\ \gcd(d,n)=1}}\lambda_d^+f(d)\sum_{\substack{m\leqslant X\\ d|m\\ \gcd(m,n)=1}}1&=\frac{X\phi(n)}{n}\sum_{\substack{d\leqslant y\\ d|P(z)\\ \gcd(d,n)=1}}\lambda_d^+\frac{f(d)}{d}+O(y\tau(n))\\ &=\frac{X\phi(n)}{n}\sum_{\substack{d|P(z)\\ \gcd(d,n)=1}}\lambda_d^+\frac{f(d)}{d}+O(y\tau(n)).\end{aligned}$$

Next, we would like to apply the fundamental lemma of sieve theory to the sum over d, [25, Fundamental Lemma 6.3, pg.159]. In order to do so, we first need to satisfy the condition

$$\prod_{\substack{w\leqslant p<z\\ p\text{ prime}}}\left(1-\frac{\mathbb{1}(\gcd(n,d)=1)f(p)}{p}\right)^{-1}\leqslant K\left(\frac{\log z}{\log w}\right)$$

for any $0 < w \leqslant z$ where K is some absolute constant. For this we note that, by assumption (4.2.5) on f it follows that

$$\prod_{\substack{w<p\leqslant z\\ p\nmid n}}\left(1-\frac{f(p)}{p}\right)^{-1}=\prod_{\substack{w<p\leqslant z\\ p|n}}\left(1-\frac{f(p)}{p}\right)\prod_{w<p\leqslant z}\left(1-\frac{f(p)}{p}\right)^{-1},$$
$$\ll\left(\frac{\log z}{\log w}\right)^{\alpha}\prod_{\substack{w<p\leqslant z\\ p|n}}\left(1-\frac{f(p)}{p}\right)\ll\left(\frac{\log z}{\log w}\right)^{\alpha}$$

since $\prod_{p|n}\left(1-\frac{f(p)}{p}\right)<1$ for all n. Thus we may apply the fundamental lemma to the sum over d to obtain the upper bound:

$$\sum_{d|P(z)}\lambda_d^+\frac{\mathbb{1}(\gcd(n,d)=1)f(d)}{d}\ll\prod_{p|P(z)}\left(1-\frac{\mathbb{1}(\gcd(n,d)=1)f(p)}{p}\right)$$
$$\ll\prod_{\substack{p|P(z)\\ p|n}}\left(1-\frac{f(p)}{p}\right)^{-1}\prod_{p|P(z)}\left(1-\frac{f(p)}{p}\right)$$
$$\ll\frac{1}{(\log z)^{\alpha}}\prod_{\substack{p|P(z)\\ p|n}}\left(1-\frac{f(p)}{p}\right)^{-1}.$$

Recalling that $z=X^{\epsilon/10}$ and $y=X^{\epsilon}$ we substitute this into the our equalities above to obtain the result. $\square$

We now prove the main result of this section.

Proof of Theorem 4.1.1. Set the sum on the left to be $S(N,M)$. Then by the Cauchy–Schwarz inequality:

$$S(N,M)^2\ll\left(\sum_{m\leqslant M}|b_m|f(m)\right)\left(\sum_{m\leqslant M}|b_m|f(m)\left|\sum_{n\leqslant N}a_n\left(\frac{n}{m}\right)\right|^2\right)$$

The first sum over m is bounded using (4.2.4):

$$\sum_{m\leqslant M}|b_m|f(m)\ll\sum_{m\leqslant M}f(m)\ll\frac{M}{(\log M)^{1-\alpha}}.$$

Fix $z=M^{\epsilon/10}$ and $y=z^{10}=M^{\epsilon}$. Using Lemma 4.2.1 we have

$$|b_m|f(m)\leqslant f(m)\leqslant\sum_{\substack{d|m\\ d|P(z)}}\lambda_d^+\hat{f}(d)$$

where $\hat{f}$ is as defined in Lemma 4.2.1 and (λ_d^+) are the sieve coefficients defined previously. Therefore,

$$\sum_{m\leqslant M}|b_m|f(m)\left|\sum_{n\leqslant N}a_n\left(\frac{n}{m}\right)\right|^2\leqslant\sum_{m\leqslant M}\sum_{\substack{d|m\\d|P(z)}}\lambda_d^+\hat{f}(d)\left|\sum_{n\leqslant N}a_n\left(\frac{n}{m}\right)\right|^2$$
$$\leqslant\sum_{\substack{d\leqslant M\\d|P(z)}}\lambda_d^+\hat{f}(d)\sum_{\substack{m\leqslant M\\d|m}}\left|\sum_{n\leqslant N}a_n\left(\frac{n}{m}\right)\right|^2.$$

Now, by expanding the square this becomes:

$$\sum_{m\leqslant M}|b_m|f(m)\left|\sum_{n\leqslant N}a_n\left(\frac{n}{m}\right)\right|^2\leqslant\sum_{\substack{d\leqslant M\\d|P(z)}}\lambda_d^+\hat{f}(d)\sum_{\substack{m\leqslant M\\d|m}}\left(\sum_{n_1,n_2\leqslant N}a_{n_1}\bar{a}_{n_2}\left(\frac{n_1n_2}{m}\right)\right)$$
$$\leqslant\sum_{\substack{n_1,n_2\leqslant N\\n_1n_2=\square}}a_{n_1}\bar{a}_{n_2}\sum_{\substack{d\leqslant M\\d|P(z)}}\lambda_d^+\hat{f}(d)\sum_{\substack{m\leqslant M\\d|m\\\gcd(m,n_1n_2)=1}}1$$
$$+\sum_{\substack{n_1,n_2\leqslant N\\n_1n_2\neq\square}}a_{n_1}\bar{a}_{n_2}\sum_{\substack{d\leqslant M\\d|P(z)}}\lambda_d^+\hat{f}(d)\sum_{\substack{m\leqslant M\\d|m}}\left(\frac{n_1n_2}{m}\right).$$

We first consider the sum over $n_1n_2\neq\square$. Here we write $m=m'd$ and note that the sieve coefficents λ_d^+ are supported on the interval $[1,y]$. Then, using the Pólya–Vinogradov inequality:

$$\sum_{\substack{n_1,n_2\leqslant N\\n_1n_2\neq\square}}a_{n_1}\bar{a}_{n_2}\sum_{\substack{d\leqslant M\\d|P(z)}}\lambda_d^+\hat{f}(d)\sum_{\substack{m\leqslant M\\d|m}}\left(\frac{n_1n_2}{m}\right)\ll\sum_{\substack{n_1,n_2\leqslant N\\n_1n_2\neq\square}}\sum_{d\leqslant y}\left|\sum_{m'\leqslant M/d}\left(\frac{n_1n_2}{m'}\right)\right|$$
$$\ll y\sum_{\substack{n_1,n_2\leqslant N\\n_1n_2\neq\square}}n_1^{1/2}n_2^{1/2}(\log n_1n_2)$$
$$\ll yN^3(\log N).$$

For the sum over the squares we have

$$\sum_{\substack{n_1,n_2\leqslant N\\n_1n_2=\square}}a_{n_1}\bar{a}_{n_2}\sum_{\substack{d\leqslant M\\p|P(z)\\\gcd(d,n_1n_2)=1}}\lambda_d^+\hat{f}(d)\sum_{\substack{m\leqslant M\\d|m\\\gcd(m,n_1n_2)=1}}1\ll\sum_{n\leqslant N}\tau(n^2)\left|\sum_{\substack{d\leqslant M\\d|P(z)\\\gcd(d,n)=1}}\lambda_d^+\hat{f}(d)\sum_{\substack{m\leqslant M\\d|m\\\gcd(m,n)=1}}1\right|.$$

Note that, by assumption (4.1.4) on f, and the definition of $\hat{f}$, we may write,

$$\sum_{p\leqslant X}\frac{\hat{f}(p)}{p}=\sum_{p\leqslant X}\frac{1-f(p)}{p}=(1-\alpha)\log\log X+O(1)$$

for $2\leqslant X$. Note also that $\hat{f}(p^m)=\hat{f}(p)$ for all primes p and all $m\geqslant 1$. It follows that $\hat{f}$ satisfies the conditions of 4.2.2. Thus we may use this lemma to bound the inner sum by

$$\left| \sum_{\substack{d\leqslant M \\ d|P(z) \\ \gcd(d,n)=1}} \lambda_d^+ \hat{f}(d) \sum_{\substack{m\leqslant M \\ d|m \\ \gcd(m,n)=1}} 1 \right| \ll_\epsilon \frac{\phi(n) M g_z(n)}{n(\log M)^{1-\alpha}} + O(y\tau(n)),$$

where we have set

$$g_z(n) = \prod_{\substack{p|P(z) \\ p|n}} \left(1 - \frac{\hat{f}(p)}{p}\right)^{-1}.$$

Substituting this into the sum over $n_1 n_2 = \square$ we get,

$$\begin{aligned} \sum\sum_{\substack{n_1,n_2\leqslant N \\ n_1 n_2=\square}} a_{n_1}\bar{a}_{n_2} \sum_{\substack{d\leqslant M \\ p|P(z) \\ \gcd(d,n_1n_2)=1}} \lambda_d^+ \hat{f}(d) \sum_{\substack{m\leqslant M \\ d|m \\ \gcd(m,n_1n_2)=1}} 1 &\ll_\epsilon \left(\frac{M}{(\log M)^{1-\alpha}} \sum_{n\leqslant N} \tau(n^2) g_z(n) + y \sum_{n\leqslant N} \tau(n^2)\tau(n) \right) \\ &\ll_\epsilon \frac{MN(\log N)^2}{(\log M)^{1-\alpha}} + yN(\log N)^5 \end{aligned}$$

where we have used (4.2.3) to obtain

$$\sum_{n\leqslant N} \tau(n^2) g_z(n) \ll N(\log N)^2 \text{ and } \sum_{n\leqslant N} \tau(n^2)\tau(n) \ll N(\log N)^5,$$

since $g_z(p) \leqslant 2$ for all primes p. To conclude, we now have

$$\begin{aligned} S(N,M)^2 &\ll_\epsilon \frac{M}{(\log M)^{1-\alpha}} \left(\frac{MN(\log N)^2}{(\log M)^{1-\alpha}} + yN(\log N)^5 + yN^3(\log N) \right) \\ &\ll_\epsilon \frac{M^2 N(\log N)^2}{(\log M)^{1-\alpha}(\log M)^{1-\alpha}} + \frac{yMN^3(\log N)}{(\log M)^{1-\alpha}}. \end{aligned}$$

Taking square roots and noting that $(x+y)^{1/2} \ll (x^{1/2} + y^{1/2})$ for $x, y \geqslant 0$ gives the result, since $y = M^\epsilon$. $\square$

4.3 Proof of Theorem 4.1.8

Our strategy for this proof will be to cover the hyperbolic region in dyadic rectangles and apply (4.1.7)-(4.1.10). Set

$$S(T) = \sum_{\substack{z<n,m\leqslant T \\ nm\leqslant T}}^{*} (nm)^c a_n b_m \left(\frac{n}{m}\right).$$

If $z > T^{1/2}$ then the sum is 0, so that the bound is trivially true. If $z = T^{1/2}$ then the sum has magnitude $\leqslant 1$, so again the bound is trivial. We are now left with $z < T^{1/2}$. Our first step is to split the sum over the hyperbolic region into 4 pieces. We write

$$S(T) = S_1(T) + S_2(T) + S_3(T) - S_4(T)$$

where

$$S_1(T) = \sum_{\substack{z_1 < n,m \leqslant T \\ nm \leqslant T}}^{*} (nm)^c a_n b_m \left(\frac{n}{m}\right);$$

$$S_2(T) = \sum_{z < n \leqslant z_1}^{*} \sum_{z < m \leqslant \frac{T}{n}}^{*} (nm)^c a_n b_m \left(\frac{n}{m}\right);$$

$$S_3(T) = \sum_{z < m \leqslant z_1}^{*} \sum_{z < n \leqslant \frac{T}{m}}^{*} (nm)^c a_n b_m \left(\frac{n}{m}\right);$$

$$S_4(T) = \sum_{z < n \leqslant z_1}^{*} \sum_{z < m \leqslant z_1}^{*} (nm)^c a_n b_m \left(\frac{n}{m}\right).$$

where $z_1 = \max(z, \frac{T^{1/3}}{\log T})$. We now aim to bound each of these sums individually. First note that if $z > \frac{T^{1/3}}{\log T}$, $S_2(T) = S_3(T) = S_4(T) = 0$ and so we only need to consider them whenever $z \leqslant \frac{T^{1/3}}{\log T}$. For $S_4(T)$ we apply (4.1.8) with $N = M = \frac{T^{1/3}}{\log T}$: divide and multiply the sum by T^c so that we have $|\frac{n^c}{T^{c/2}} a_n|$, $|\frac{m^c}{T^{c/2}} b_m| \leqslant 1$. Then (4.1.8) with $\epsilon < 1/6$ gives

$$S_4(T) \ll_c \begin{cases} T^{2/3+c} & \text{if } z \leqslant \frac{T^{1/3}}{\log T}, \\ 0 & \text{if } z > \frac{T^{1/3}}{\log T}. \end{cases}$$

We now turn to the remaining 3 sums. If $z \leqslant \frac{T^{1/3}}{\log T}$, then $S_2(T)$ and $S_3(T)$ may be dealt with using symmetric arguments as a consequence of reciprocity for Jacobi symbols, $(\frac{n}{m}) = (-1)^{\frac{(n-1)(m-1)}{4}} (\frac{m}{n})$, and so we only need to deal with one: since n and m are odd and square-free, we may split $S_3(T)$ into 4 sums using the conditions $n, m \equiv 1$ or 3 (mod 4) and then reciprocity will give 4 sums in the same form as $S_2(T)$. Thus we only need to consider $S_1(T)$ and $S_2(T)$. We aim to use Perron's formula. For $S_1(T)$ we split $(z_1, T]$ into dyadic intervals to obtain $\ll (\log T)^2$ dyadic regions of the form $(N, 2N] \times (M, 2M]$ where $N, M \in (z_1, T]$. For $S_2(T)$, split the intervals $(z, z_1]$ and $(z, T]$ into dyadic intervals to obtain $\ll (\log T)^2$ dyadic regions $(N, 2N] \times (M, 2M]$ where $N \in (z, z_1]$ and $M \in (z, T]$. This will give bounds of the form

$$S_1(T) \ll (\log T)^2 \max_{\substack{z_1 < N \leqslant T \\ z_1 < M \leqslant T \\ NM \leqslant T}} |S(T; N, M)|;$$

$$S_2(T) \ll (\log T)^2 \max_{\substack{z < N \leqslant z_1 \\ z < M \leqslant T \\ NM \leqslant T}} |S(T; N, M)|;$$

where in each case

$$S(T; N, M) = \sum_{\substack{N < n \leqslant 2N \\ M < m \leqslant 2M \\ nm \leqslant T}}^{*} (nm)^c a_n b_m \left(\frac{n}{m}\right).$$

Next, we apply Perron's formula to deal with the hyperbolic conditions. Let $\theta \in [-1/2, 1/2]$ be such that $T + \theta \in \mathbb{Z} + \frac{1}{2}$ and take $\tau = T + \theta$ in (4.1.7). Then (4.1.7) becomes

$$\frac{1}{\pi} \int_{-R}^{R} (nm)^{it} f_{T+\theta}(t) dt = \mathbb{1}(nm \leqslant T) + O(R^{-1} |\log(nm) - \log(T + \theta)|^{-1})$$

for any $R > 0$ where $f_{T+\theta}(t) = \frac{\sin(t\log(T+\theta))}{t}$. Noting that here $(\log(nm) - \log(T+\theta)) \gg \frac{1}{T}$, we substitute this into $S(T; N, M)$ to obtain

$$S(T; N, M) = \frac{1}{\pi}\int_{-R}^{R} f_{T+\theta}(t) \sideset{}{^*}\sum_{\substack{N<n\leqslant 2N\\ M<m\leqslant 2M}} (nm)^{c+it} a_n b_m \left(\frac{n}{m}\right) dt + O\left(\frac{(NM)^{1+c}T}{R}\right).$$

Before we move forward, we deal with the $(nm)^{c+it}$ term: write $1 = \frac{(4NM)^c}{(4NM)^c}$, then by setting $\widetilde{a}_n = \frac{n^c}{(2N)^c} a_n$ and $\widetilde{b}_m = \frac{m^c}{(2M)^c} b_m$ we have

$$\sideset{}{^*}\sum_{\substack{N<n\leqslant 2N\\ M<m\leqslant 2M}} (nm)^{c+it} a_n b_m \left(\frac{n}{m}\right) = (4NM)^c \sideset{}{^*}\sum_{\substack{N<n\leqslant 2N\\ M<m\leqslant 2M}} n^{it}\widetilde{a}_n m^{it}\widetilde{b}_m \left(\frac{n}{m}\right)$$

where $|n^{it}\widetilde{a}_n|$, $|m^{it}\widetilde{b}_m| \leqslant 1$. By applying (4.1.8) and substituting into $S(T; N, M)$ we get

$$S(T; N, M) \ll_\epsilon 4^c(MN)^c \int_{-R}^{R} |f_{T+\theta}(t)| dt (MN)^\epsilon \left(MN^{1/2} + M^{1/2}N\right) + \frac{(NM)^{1+c}T}{R}. \tag{4.3.1}$$

By instead applying Cauchy–Schwarz and (4.1.10) we obtain

$$S(T; N, M) \ll_\epsilon 4^c(MN)^c \int_{-R}^{R} |f_{T+\theta}(t)| dt \left(MN^{1/2} + M^{1/2}N^{3/2}(\log N)^{1/2}\right) + \frac{(NM)^{1+c}T}{R}. \tag{4.3.2}$$

We will apply (4.3.1) to the dyadic regions in $S_1(T)$ to obtain

$$\begin{aligned} S_1(T) &\ll_{c,\epsilon} T^c(\log T)^2 \int_{-R}^{R} |f_{T+\theta}(t)| dt \max_{\substack{z_1<N\leqslant T\\ z_1<M\leqslant T\\ NM\leqslant T}} (MN)^\epsilon \left(MN^{1/2} + M^{1/2}N\right) + \frac{T^{2+c}(\log T)^2}{R} \\ &\ll_{c,\epsilon} T^c(\log T)^2 \int_{-R}^{R} |f_{T+\theta}(t)| dt \left(\frac{T^{1+\epsilon}}{z_1^{1/2}}\right) + \frac{T^{2+c}(\log T)^2}{R}. \end{aligned}$$

For $S_2(T)$ however, we may assume $z \leqslant \frac{T^{1/3}}{\log T}$ so that the dyadic rectangles either have the lopsided condition $N(\log N)^{1/2} \leqslant M^{1/2} \leqslant \frac{T^{1/2}}{z^{1/2}}$ or have $M^{1/2} \leqslant N(\log N)^{1/2} \ll \frac{T^{1/3}}{(\log T)^{1/2}}$. Thus we may use (4.3.2) to obtain

$$\begin{aligned} S_2(T) &\ll_c T^c(\log T)^2 \int_{-R}^{R} |f_{T+\theta}(t)| dt \max_{\substack{z<N\leqslant \frac{T^{1/3}}{\log T}\\ z<M\leqslant T\\ NM\leqslant T}} (MN^{1/2} + M^{1/2}N^{3/2}(\log N)^{1/2}) + \frac{T^{2+c}(\log T)^2}{R} \\ &\ll_c T^c(\log T)^2 \int_{-R}^{R} |f_{T+\theta}(t)| dt \left(T^{1/2}\left(\frac{T^{1/2}}{z^{1/2}} + \frac{T^{1/3}}{(\log T)^{1/2}}\right)\right) + \frac{T^{2+c}(\log T)^2}{R} \\ &\ll_c T^c(\log T)^2 \int_{-R}^{R} |f_{T+\theta}(t)| dt \left(\frac{T}{z^{1/2}}\right) + \frac{T^{2+c}(\log T)^2}{R}, \end{aligned}$$

if $z \leqslant \frac{T^{1/3}}{\log T}$ and $S_2(T) = 0$ otherwise. Choosing $R = T^2(\log T)^2$ the integral becomes bounded by $(\log T)$ therefore giving

$$S_1(T) \ll_{c,\epsilon} \frac{T^{1+c+\epsilon}(\log T)^3}{z_1^{1/2}},$$

and

$$S_2(T) \ll_c \begin{cases} \frac{T^{1+c}(\log T)^3}{z^{1/2}} & \text{if } z \leqslant \frac{T^{1/3}}{\log T}, \\ 0 & \text{if } z > \frac{T^{1/3}}{\log T}. \end{cases}$$

Finally, recalling that $S_2(T)$ and $S_3(T)$ are symmetrically equivalent and that $z_1 = \max(z, \frac{T^{1/3}}{\log T})$, we may put all of these bounds together to obtain

$$S(T) \ll_{c,\epsilon} \begin{cases} \frac{T^{1+c}(\log T)^3}{z^{1/2}} + T^{5/6+c+\epsilon}(\log T)^{7/2} + T^{2/3+c} & \text{if } z \leqslant \frac{T^{1/3}}{\log T}, \\ \frac{T^{1+c+\epsilon}(\log T)^3}{z^{1/2}} & \text{if } z > \frac{T^{1/3}}{\log T}, \end{cases}$$

for any $\epsilon > 0$. Now suppose there exists an $\epsilon > 0$ such that $z \geqslant T^{1/3-\epsilon}$. Then if $z > \frac{T^{1/3}}{\log T}$ we use the second case of the above bound with $\epsilon/2$ to obtain:

$$S(T) \ll_{c,\epsilon} \frac{T^{1+c+\epsilon/2}(\log T)^3}{z^{1/2}} \ll_{c,\epsilon} \frac{T^{1+c+\epsilon}}{z^{1/2}}.$$

If $T^{1/3-\epsilon} \leqslant z \leqslant \frac{T^{1/3}}{\log T}$ then consider the first bound with $\epsilon/2$. Then,

$$S(T) \ll_{c,\epsilon} \frac{T^{1+c}(\log T)^3}{z^{1/2}} + T^{5/6+c+\epsilon/2}(\log T)^{7/2} + T^{2/3+c} \ll_{c,\epsilon} \frac{T^{1+c+\epsilon}}{z^{1/2}}.$$

Lastly, if there exists an $\epsilon > 0$ such that $z \leqslant T^{1/3-\epsilon}$, then consider the bound for $z \leqslant \frac{T^{1/3}}{\log T}$ with $\epsilon/2$. In this case the first term dominates since

$$\frac{T^{1+c}(\log T)^3}{z^{1/2}} \geqslant T^{5/6+c+\epsilon}(\log T)^3 \gg T^{5/6+c+\epsilon/2}(\log T)^{7/2},$$

which implies the result.

4.4 Proof of Theorem 4.1.9

The key idea of this proof is to cover parts of the hyperbolic region with rectangles of equal width and apply Theorem 4.1.8 along with the Cauchy–Schwarz inequality and (4.1.10) to the sums over each of these rectangles and then sum over the results. The following lemma encodes the covering we will use:

Lemma 4.4.1. *Fix $c \geqslant 0$ and $0 < \delta \leqslant 1/2$. Then for any $T \geqslant 2$ and any $2 \leqslant z < T^\delta$ we have,*

$$\sideset{}{^*}\sum_{z<n\leqslant T^\delta} \sideset{}{^*}\sum_{z<m\leqslant \frac{T}{n}} (nm)^c a_n b_m \left(\frac{n}{m}\right) = \sum_{k\in H} \sideset{}{^*}\sum_{n\in I_k} \sideset{}{^*}\sum_{m\in J_k} (nm)^c a_n b_m \left(\frac{n}{m}\right) + O\left(\frac{T^{1+c}}{z^{1/2}}\right),$$

where

$$H = \left\{k \in \mathbb{N} : z^{1/2} \leqslant k \leqslant \frac{T^\delta}{z^{1/2}} - 1\right\};$$

$$I_k = \left\{n \in \mathbb{N} : z^{1/2}k < n \leqslant z^{1/2}(k+1)\right\};$$

$$J_k = \left\{m \in \mathbb{N} : z < m \leqslant \frac{T}{z^{1/2}(k+1)}\right\}.$$

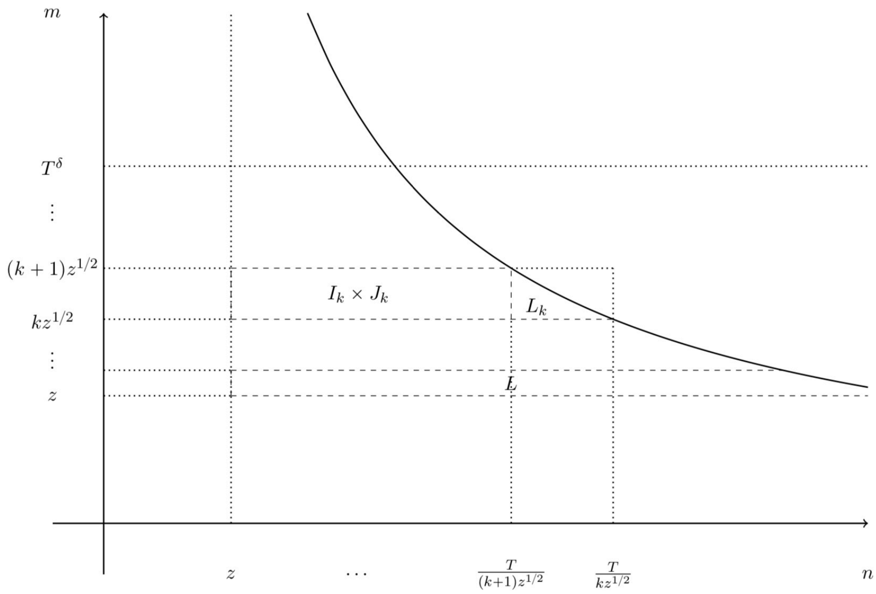

Figure 4.4.1: Lemma 4.4.1 Illustration

Remark 4.4.2. By partitioning the interval over n into intervals of equal length we are then able to make use of the fact that, close to the hyperbolic curve, the gradient $\frac{\mathrm{d}(x/n)}{\mathrm{d}n} = -\frac{x}{n^2}$, decreases rapidly in magnitude. This observation will lead to the regions leftover from the covering boxes having small volume.

Proof. We begin by partitioning the interval $(0, T^\delta]$ into $\frac{T^\delta}{z^{1/2}}$ intervals of equal length $z^{1/2}$, say $(kz^{1/2}, (k+1)z^{1/2}]$ for integers $0 \leqslant k \leqslant \frac{T^\delta}{z^{1/2}} - 1$, and intersecting this partition with $(z, T^\delta]$. Then notice that

$$\bigcup_{k \in H} I_k \subseteq (z, T^\delta] \cap \mathbb{N},$$

where the leftover part of this partition, L', satisfies

$$L' = ((z, T^\delta] \cap \mathbb{N}) \setminus \bigcup_{k \in H} I_k \subseteq ((z, z + z^{1/2}] \cup (T^\delta - z^{1/2}, T^\delta]) \cap \mathbb{N}.$$

Fix a k. Then for an $n \in I_k$, the summation index m ranges from z to $\frac{T}{n}$ where $\frac{T}{(k+1)z^{1/2}} \leqslant \frac{T}{n} < \frac{T}{kz^{1/2}}$. To create our rectangles we split all ranges over m into a range $z < m \leqslant \frac{T}{(k+1)z^{1/2}}$, giving us the intervals J_k, and $\frac{T}{(k+1)z^{1/2}} < m \leqslant \frac{T}{n}$. Notice that for $n \in I_k$, $(\frac{T}{(k+1)z^{1/2}}, \frac{T}{n}] \cap \mathbb{N} \subseteq J'_k = \{m \in \mathbb{N} : \frac{T}{z^{1/2}(k+1)} < m \leqslant \frac{T}{z^{1/2}k}\}$. Combining the ranges, for each k we have a rectangle $I_k \times J_k$ and a small section $L_k = \{(n,m) \in \mathbb{N} : n \in I_k,\ \frac{T}{(k+1)z^{1/2}} < m \leqslant \frac{T}{n}\}$ close to the hyperbolic curve which is contained in the small rectangle $I_k \times J'_k$. We also have the leftover regions coming from $n \in L'$, $L = \{(n,m) \in \mathbb{N}^2 : n \in L',\ z < m \leqslant \frac{T}{n}\}$. Then we have:

$$\{(n,m) \in \mathbb{N}^2 : nm \leqslant T,\ n \leqslant T^\delta,\ n,m > z\} = \bigcup_{k \in H} (I_k \times J_k) \cup \bigcup_{k \in H} L_k \cup L.$$

See Figure 1 for an illustration of these sets.

It follows that

$$\sum_{z<n\leqslant T^\delta}^{*}\ \sum_{z<m\leqslant \frac{T}{n}}^{*}(nm)^c a_n b_m\left(\frac{n}{m}\right) = \sum_{k\in H}\ \sum_{n\in I_k}^{*}\ \sum_{m\in J_k}^{*}(nm)^c a_n b_m\left(\frac{n}{m}\right)$$
$$+\sum_{k\in H}\ \sum_{(n,m)\in L_k}^{*}(nm)^c a_n b_m\left(\frac{n}{m}\right)$$
$$+\sum_{(n,m)\in L}^{*}(nm)^c a_n b_m\left(\frac{n}{m}\right).$$

We conclude by bounding the second and third sums trivially. For the second we use the triangle inequality and then expand the sum to $I_k \times J_k'$:

$$\sum_{k\in H}\ \sum_{(n,m)\in L_k}^{*}(nm)^c a_n b_m\left(\frac{n}{m}\right) \ll \sum_{k\in H}\ \sum_{n\in I_k}^{*}\ \sum_{m\in J_k'}^{*} T^c.$$

Note that $|I_k \times J_k'| \leqslant \frac{T}{k(k+1)}$. Summing this over $k > z^{1/2}$ gives

$$\sum_{k\in H}\ \sum_{(n,m)\in L_k}^{*}(nm)^c a_n b_m\left(\frac{n}{m}\right) \ll \frac{T^{1+c}}{z^{1/2}}.$$

For the leftovers we use the triangle inequality again and expand the double sum to the region $((z, z+z^{1/2}] \times (z, \frac{T}{z}] \cup (T^\delta - z^{1/2}, T^\delta] \times (z, \frac{T}{T^\delta - z^{1/2}}]) \cap \mathbb{N}^2$. The number of integer pairs in this region is $\ll \frac{T}{z^{1/2}} + T^{1-\delta/2} \ll \frac{T}{z^{1/2}}$ using the assumption $z < T^\delta$. Thus we obtain the bound

$$\sum_{(n,m)\in L}^{*}(nm)^c a_n b_m\left(\frac{n}{m}\right) \ll \frac{T^{1+c}}{z^{1/2}}.$$

Overall, this gives the expression

$$\sum_{z<n\leqslant T^\delta}^{*}\ \sum_{z<m\leqslant \frac{T}{n}}^{*}(nm)^c a_n b_m\left(\frac{n}{m}\right) = \sum_{k\in H}\ \sum_{n\in I_k}^{*}\ \sum_{m\in J_k}^{*}(nm)^c a_n b_m\left(\frac{n}{m}\right) + O\left(\frac{T^{1+c}}{z^{1/2}}\right).$$

□

We now complete our proof of Theorem 4.1.9. This Theorem follows directly from Theorem 4.1.8 whenever $z \geqslant (\log T)^{24}$: in these cases, $\frac{(\log T)^2}{z^{1/12}} = O(1)$, so we obtain

$$\sum_{\substack{z<n,m\leqslant T\\ nm\leqslant T}}^{*}(nm)^c a_n b_m\left(\frac{n}{m}\right) \ll_c \frac{T^{1+c}(\log T)}{z^{1/4}}.$$

We are left with the case where $z < (\log T)^{24}$, for which we aim to apply (4.1.10) and Lemma 4.4.1. However, in order for (4.1.10) to be effective, we cannot allow $N = \max(I_k)$ to exceed $M^{1/2} = (\max(J_k))^{1/2}$ in any of our covering rectangles, as then the $N^2(\log N)$ term in (4.1.10) would dominate the M term, and may lead to bounds which are too large for our purposes. To avoid this we split the hyperbolic region as before:

$$\sum_{\substack{z<n,m\leqslant T\\ nm\leqslant T}}^{*}(nm)^c a_n b_m\left(\frac{n}{m}\right) = R_1(T) + R_2(T) + R_3(T) - R_4(T)$$

where

$$\begin{aligned}
R_1(T) &= \sideset{}{^*}\sum_{\substack{T^{1/4}<n,m\leqslant T\\ nm\leqslant T}} (nm)^c a_n b_m \left(\frac{n}{m}\right);\\
R_2(T) &= \sideset{}{^*}\sum_{z<n\leqslant T^{1/4}} \sideset{}{^*}\sum_{z<m\leqslant \frac{T}{n}} (nm)^c a_n b_m \left(\frac{n}{m}\right);\\
R_3(T) &= \sideset{}{^*}\sum_{z<m\leqslant T^{1/4}} \sideset{}{^*}\sum_{z<n\leqslant \frac{T}{m}} (nm)^c a_n b_m \left(\frac{n}{m}\right);\\
R_4(T) &= \sideset{}{^*}\sum_{z<n\leqslant T^{1/4}} \sideset{}{^*}\sum_{z<m\leqslant T^{1/4}} (nm)^c a_n b_m \left(\frac{n}{m}\right).
\end{aligned}$$

This splitting allows us to apply Lemma 4.4.1 to $R_2(T)$ (and $R_3(T)$) and obtain integer intervals I_k whose maximums do not get too large, therefore allowing us to apply (4.1.10) effectively. First, we bound $R_1(T)$ and $R_4(T)$. For $R_1(T)$ we use Theorem 4.1.8 with $z_1 = T^{1/4}$:

$$R_1(T) \ll_c \frac{T^{1+c}(\log T)^3}{z_1^{1/3}} = T^{11/12+c}(\log T)^3.$$

Next, we use (4.1.8) with $N = M = T^{1/4}$ to deal with $R_4(T)$. Multiplying by $1 = \frac{T^c}{T^c}$ and setting $\widetilde{a}_n = \frac{n^c}{T^{c/2}} a_n$, $\widetilde{b}_m = \frac{m^c}{T^{c/2}} b_m$, we have $|\widetilde{a}_n|$, $|\widetilde{b}_m| \leqslant 1$. Then applying (4.1.8) gives

$$R_4(T) \ll_\epsilon T^{3/8+c+\epsilon},$$

which is sufficient by choosing $\epsilon < 13/24$, as it may then be absorbed into the bound for $R_1(T)$.

We are left with $R_2(T)$ and $R_3(T)$. Note that these sums are symmetrically equivalent using the same argument as that of $S_2(T)$ and $S_3(T)$ in Section 4.3. Thus we only need to deal with $R_2(T)$. For this we use the covering Lemma 4.4.1 with $\delta = 1/4$:

$$R_2(T) = \sum_{k\in H} \sideset{}{^*}\sum_{n\in I_k} \sideset{}{^*}\sum_{m\in J_k} (nm)^c a_n b_m \left(\frac{n}{m}\right) + O\left(\frac{T^{1+c}}{z^{1/2}}\right),$$

where

$$\begin{aligned}
H &= \left\{k \in \mathbb{N} : z^{1/2} \leqslant k \leqslant \frac{T^{1/4}}{z^{1/2}} - 1\right\};\\
I_k &= \left\{n \in \mathbb{N} : z^{1/2}k < n \leqslant z^{1/2}(k+1)\right\};\\
J_k &= \left\{m \in \mathbb{N} : z < m \leqslant \frac{T}{z^{1/2}(k+1)}\right\}.
\end{aligned}$$

To deal with this sum, we will consider the sum over n and m for a fixed k. First deal with the power term:

$$\sideset{}{^*}\sum_{n\in I_k} \sideset{}{^*}\sum_{m\in J_k} (nm)^c a_n b_m \left(\frac{n}{m}\right) = T^c \sideset{}{^*}\sum_{m\in J_k} \sideset{}{^*}\sum_{n\in I_k} \frac{n^c}{z^{c/2}(k+1)^c} \frac{m^c z^{c/2}(k+1)^c}{T^c} a_n b_m \left(\frac{n}{m}\right).$$

Now $\frac{n}{z^{1/2}(k+1)}$, $\frac{mz^{1/2}(k+1)}{T} \leqslant 1$, so that we may define the sequences $\widetilde{a}_n = \frac{n^c}{z^{c/2}(k+1)^c} a_n$ in addition to $\widetilde{b}_m = \frac{m^c z^{c/2}(k+1)^c}{T^c} b_m$ which satisfy the condition $|\widetilde{a}_n|, |\widetilde{b}_m| \leqslant 1$. Finally we apply the Cauchy–Schwarz inequality and (4.1.10) with $M = \max(J_k) \leqslant \frac{T}{z^{1/2}(k+1)}$ and $N = \max(I_k) \leqslant z^{1/2}(k+1)$:

$$\sum_{n\in I_k}^{*} \sum_{m\in J_k}^{*} (nm)^c a_n b_m \left(\frac{n}{m}\right) = T^c \sum_{m\in J_k}^{*} \sum_{n\in I_k}^{*} \widetilde{a}_n \widetilde{b}_m \left(\frac{n}{m}\right)$$

$$\begin{aligned} &\ll T^c |J_k|^{1/2} \left(\sum_{m\in J_k}^{*} \left| \sum_{n\in I_k}^{*} \widetilde{a}_n \left(\frac{n}{m}\right)\right|^2 \right)^{1/2} \\ &\ll T^c \left(\frac{T}{z^{1/2}(k+1)}\right)^{1/2} \left(\frac{T}{z^{1/2}(k+1)}\right)^{1/2} |I_k|^{1/2} \\ &\ll \frac{T^{1+c}}{z^{1/4}(k+1)} \end{aligned}$$

where we used the fact that $k^2 z(\log(kz^{1/2})) \ll T^{1/2}(\log T)$, while $\frac{T}{z^{1/2}(k+1)} \geqslant T^{3/4}$ to simplify the application of (4.1.10). Summing this bound over the given k introduces a logarithmic term, and so

$$R_2(T), R_3(T) \ll \frac{T^{1+c}(\log T)}{z^{1/4}}.$$

Combining all the bounds we get:

$$\sum_{\substack{z<n,m\leqslant T\\ nm\leqslant T}}^{*} (nm)^c a_n b_m \left(\frac{n}{m}\right) \ll_c \frac{T^{1+c}(\log T)}{z^{1/4}} + T^{11/12+c}(\log T)^3 \ll_c \frac{T^{1+c}(\log T)}{z^{1/4}}$$

(since $z < (\log T)^{24}$) as required.

4.5 Proof of Theorem 4.1.10

To begin we once more cut the hyperbolic region into regions depending on the sizes of each variable. We write

$$\sum_{\substack{1\leqslant n,m\leqslant T\\ 2\nmid nm\\ nm\leqslant T}} \left(\frac{n}{m}\right) = N_1(T) + N_2(T) - N_3(T),$$

where

$$N_1(T) = \sum_{\substack{1\leqslant n\leqslant T^{1/2}\\ 2\nmid n}} \sum_{\substack{1\leqslant m\leqslant T/n\\ 2\nmid m}} \left(\frac{n}{m}\right);$$

$$N_2(T) = \sum_{\substack{1\leqslant m\leqslant T^{1/2}\\ 2\nmid m}} \sum_{\substack{1\leqslant n\leqslant T/m\\ 2\nmid n}} \left(\frac{n}{m}\right);$$

$$N_3(T) = \sum_{\substack{1\leqslant n\leqslant T^{1/2}\\ 2\nmid n}} \sum_{\substack{1\leqslant m\leqslant T^{1/2}\\ 2\nmid m}} \left(\frac{n}{m}\right).$$

Let us first deal with $N_1(T)$. We begin by separating the square values of n:

$$N_1(T) = \sum_{\substack{1 \leqslant n \leqslant T^{1/2} \\ 2 \nmid n \\ n = \square}} \sum_{\substack{1 \leqslant m \leqslant T/n \\ 2 \nmid m}} \left(\frac{n}{m}\right) + \sum_{\substack{1 \leqslant n \leqslant T^{1/2} \\ 2 \nmid n \\ n \neq \square}} \sum_{\substack{1 \leqslant m \leqslant T/n \\ 2 \nmid m}} \left(\frac{n}{m}\right).$$

To deal with the second of these sums we use the Pólya–Vinogradov inequality for the sum over m, and then sum over $1 \leqslant n \leqslant T^{1/2}$. Thus the second sum is $O(T^{3/4}(\log T))$. For the first sum, we note that since n is a square the Jacobi symbol is the trivial character modulo n. Thus

$$\left(\frac{n}{m}\right) = \begin{cases} 1 \text{ if } \gcd(n,m) = 1, \\ 0 \text{ if } \gcd(n,m) > 1. \end{cases}$$

It is well-known that for a fixed odd n, the number of odd $1 \leqslant m \leqslant \frac{T}{n}$ co-prime to n is given by

$$\frac{T}{2n} \cdot \frac{\phi(n)}{n} + O(n^{\epsilon})$$

for any $\epsilon > 0$. Summing this error over the square values of n less than $T^{1/2}$ we will obtain an error of size $O(T^{1/4+\epsilon})$, which is satisfactory. For the main term we use the change of variables $n = k^2$:

$$\sum_{\substack{1 \leqslant n \leqslant T^{1/2} \\ 2 \nmid n \\ n = \square}} \frac{T}{2n} \cdot \frac{\phi(n)}{n} = \frac{T}{2} \sum_{\substack{1 \leqslant k \leqslant T^{1/4} \\ 2 \nmid k}} \frac{\phi(k^2)}{k^4} = \left(\sum_{\substack{k=1 \\ 2 \nmid k}}^{\infty} \frac{\phi(k^2)}{k^4} \right) \frac{T}{2} + O(T^{3/4}).$$

Noting that $N_2(T)$ may be dealt with using the same methods we obtain

$$N_1(T) + N_2(T) = \left(\sum_{\substack{k=1 \\ 2 \nmid k}}^{\infty} \frac{\phi(k^2)}{k^4} \right) T + O(T^{3/4}(\log T)).$$

Using Theorem 1 of [13] with $X = Y = T^{1/2}$ we obtain, $N_3(T) \ll T^{3/4}$. Thus we have

$$\sum_{\substack{1 \leqslant n,m \leqslant T \\ 2 \nmid nm \\ nm \leqslant T}} \left(\frac{n}{m}\right) = \left(\sum_{\substack{k=1 \\ 2 \nmid k}}^{\infty} \frac{\phi(k^2)}{k^4} \right) T + O(T^{3/4}(\log T)).$$

Lastly we evaluate the constant. To do this, let $g(n) = \mathbb{1}_{\text{odd}}(n)\mathbb{1}_{\square}(n)\phi(n)$ where $\mathbb{1}_{\text{odd}}$ and $\mathbb{1}_{\square}$ are the indicator functions for odd numbers and squares respectively. We will consider the Dirichlet series and Euler product of this multiplicative function:

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{g(n)}{n^s} &= \prod_{p}\left(1+\sum_{m=1}^{\infty}\frac{g(p)}{p^{ms}}\right)\\ &= \prod_{p\neq 2}\left(1+\sum_{m=1}^{\infty}\frac{\phi(p^{2m})}{p^{2ms}}\right)\\ &= \prod_{p\neq 2}\left(1+\frac{(p-1)}{p}\sum_{m=1}^{\infty}\frac{1}{p^{2m(s-1)}}\right)\\ &= \prod_{p\neq 2}\left(\frac{1-1/p^{(2s-1)}}{1-1/p^{(2s-2)}}\right)\\ &= \frac{1-2^{(2-2s)}}{1-2^{(1-2s)}}\frac{\zeta(2s-2)}{\zeta(2s-1)},\end{aligned}$$

where ζ is the Riemann-zeta function. By taking $s=2$ we obtain the equality

$$\sum_{\substack{k=1\\ 2\nmid k}}^{\infty}\frac{\phi(k^2)}{k^4}=\frac{6\zeta(2)}{7\zeta(3)}$$

as required.

Chapter 5

Hyperbolic Character Sums

The goal of this chapter is to prove Propostions 2.2.6-2.2.11 which we assumed in Chapter 2 to prove Theorem 1.2.1.

5.1 Technical Lemmas

In this section, we record and prove some technical lemmas that will be heavily used moving forward.

5.1.1 Large Conductor Lemmas

We first list the results on bilinear sums in the Jacobi symbol which we will need in this chapter. Recall the following result of Freidlander and Iwaniec:

Lemma 5.1.1 ([17], Lemma 2)**.** *Let $N, M \geqslant 2$ and suppose (a_n), (b_m) are any complex sequences supported on the odd integers such that $|a_n|, |b_m| \leqslant 1$. Then*

$$\sum_{n \leqslant N} \sum_{m \leqslant M} a_n b_m \left(\frac{n}{m}\right) \ll (MN^{5/6} + M^{5/6}N)(\log 3NM)^{7/6}$$

where the implied constant is absolute.

We will also need the following modification to Lemma 5.1.1:

Lemma 5.1.2. *Let $N, M \geqslant 2$ and $2 \leqslant W < N, M$. Suppose (a_n), (b_m) are any complex sequences supported on the odd integers such that $|a_n|, |b_m| \leqslant 1$.*

$$\sum_{W < n \leqslant N} \sum_{m \leqslant M} \frac{a_n}{n} b_m \left(\frac{n}{m}\right) \ll \frac{M(\log 3NM)^{7/6}}{W^{1/6}}.$$

Proof. For this proof, set

$$S(u,M)=\sum_{n\leqslant u}\sum_{m\leqslant M}a_nb_m\left(\frac{n}{m}\right).$$

Using partial summation in the n variable we have

$$\sum_{W<n\leqslant N}\sum_{m\leqslant M}\frac{a_n}{n}b_m\left(\frac{n}{m}\right)=\frac{S(N,M)}{N}-\frac{S(W,M)}{W}+\int_W^N\frac{S(u,M)}{u^2}\mathrm{d}u.$$

Then bounding $S(u,M)$ using Lemma 2.2.4, and noting that $M,N\geqslant W$, we may obtain the result by trivially bounding the logarithms and computing the integral. □

This result is particularly useful when $N\leqslant\frac{M}{W}$. Next, we will need the following special case Corollary 4.1.2:

Lemma 5.1.3. *Let $M\geqslant N\geqslant W\geqslant 2$, and fix some $\epsilon>0$. For any complex sequences a_n,b_m which are supported on the odd integers such that $|a_n|\leqslant 1$ and $|b_m|\leqslant 1$ we have:*

$$\sum_{W<n\leqslant N}\sum_{m\leqslant M}\frac{a_nb_m}{n\tau(m)}\left(\frac{n}{m}\right)\ll_\epsilon\frac{M(\log N)}{W^{1/2}(\log M)^{1/2}}+\frac{M^{1/2+\epsilon}N^{1/2}(\log N)^{1/2}}{(\log M)^{1/4}},$$

where the implied constant depends at most on ϵ.

Proof. This result is a straightforward application of Corollary 4.1.2. Indeed $\frac{1}{\tau}$ is a multiplicative function satisfying all conditions of this lemma. In particular it satisfies assumption (4.1.4) with $\alpha=1/2$, giving the result. □

This result is particularly useful in regions where $N^2\leqslant M$.

5.1.2 Small Conductor Lemmas

In order to deal with regions where our sums involve Jacobi symbols which have small conductors, we will require Siegel–Walfisz methods. Define

$$f_0=\frac{1}{\sqrt{\pi}}\prod_{p\ \text{prime}}f_p\left(1-\frac{1}{p}\right)^{1/2}\quad\text{and}\quad f_p=1+\sum_{j=1}^{\infty}\frac{1}{(j+1)p^j}. \tag{5.1.1}$$

Our key lemma for this purpose is Lemma 5.9 from [29]:

Lemma 5.1.4. *Let r and Q be integers such that $\gcd(r,Q)=1$. Let $\chi(n)$ be a character modulo Q. Fix any $C>0$. Then for all $X\geqslant 2$ we have:*

$$\sum_{\substack{n\leqslant X\\ \gcd(n,r)=1}}\frac{\chi(n)}{\tau(n)}=\frac{\mathbb{1}(\chi=\chi_0)\mathfrak{S}_0(Qr)X}{\sqrt{\log X}}\left\{1+O\left(\frac{(\log\log 3rQ)^{3/2}}{\log X}\right)\right\}+O_C\left(\frac{\tau(r)QX}{(\log X)^C}\right)$$

where χ_0 is the principal character modulo Q and

$$\mathfrak{S}_0(Qr) = \frac{f_0}{\left(\prod_{p|2rQ} f_p\right)}.$$

Furthermore, if Q and r are odd and $q \in (\mathbb{Z}/8\mathbb{Z})^*$, then we have

$$\sum_{\substack{n \leqslant X \\ \gcd(n,r)=1 \\ n \equiv q \bmod 8}} \frac{\chi(n)}{\tau(n)} = \frac{\mathbb{1}(\chi=\chi_0)\mathfrak{S}_0(Qr)X}{\phi(8)\sqrt{\log X}} \left\{1 + O\left(\frac{(\log\log 3rQ)^{3/2}}{\log X}\right)\right\} + O_C\left(\frac{\tau(r)QX}{(\log X)^C}\right).$$

Proof. We may use the same contour argument performed for Lemma 1 in [17], along with the fact that, if $\chi \neq \chi_0$, the Dirichlet series

$$\sum_{\substack{n \in \mathbb{N} \\ \gcd(n,r)=1}} \frac{\chi(n)}{\tau(n)n^s}$$

is holomorphic for $\Re(s) > 1 - \frac{c(\epsilon)}{Q^{\epsilon}(\log \mathrm{Im}(s))}$ for any $\epsilon > 0$, where $c(\epsilon)$ is some positive constant depending only on ϵ. For full details see [17, Lemma 1]. For the second part we have

$$\sum_{\substack{n \leqslant X \\ \gcd(n,r)=1 \\ n \equiv q \bmod 8}} \frac{\chi(n)}{\tau(n)} = \frac{1}{\phi(8)} \sum_{\substack{\chi' \text{char.} \\ \text{mod } 8}} \overline{\chi'}(q) \sum_{\substack{n \leqslant X \\ \gcd(n,r)=1}} \frac{\chi(n)\chi'(n)}{\tau(n)},$$

and so this follows via an application of the first part. □

This result with $\chi = \chi_0$ is used to obtain the main term in section 4 of [29]. We shall do the same, however due to difficulties arising from our height conditions, we will also require use of this result to obtain part of our error term (see Sections 5.2 and 5.3).

5.2 Character sums over hyperbolic regions I

In this section we evaluate bounds for sums over hyperbolic height conditions. In particular we will deal with sums of the form:

$$\sum\sum\sum\sum_{\substack{\|n_0c_0, n_1c_1\| \cdot \|n_2c_2, n_3c_3\| \leqslant X \\ \gcd(n_i, 2r_i)=1 \ \forall\, 0 \leqslant i \leqslant 3 \\ n_i \equiv q_i \bmod 8 \ \forall\, 0 \leqslant i \leqslant 3}} \frac{\chi_0(n_0)\chi_1(n_1)\chi_2(n_2)\chi_3(n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} \tag{5.2.1}$$

where the sum is over $\mathbf{n} \in \mathbb{N}^4$, $c_i, r_i, q_i \in \mathbb{N}$ are fixed, odd constants for each $0 \leqslant i \leqslant 3$ and χ_i are some characters. Our methods vary depending on which of the characters are principal. In particular, we will have three cases to consider:

(a) Main Term: each χ_i is principal;

(b) Small Conductor - Symmetric Hyperbola Method: χ_0 or χ_1 is non-principal and χ_2 or χ_3 is non-principal;

(c) Small Conductor - Asymmetric Hyperbola Method: χ_0 or χ_1 is non-principal but χ_2 and χ_3 are principal or vice versa.

Each case will be handled using the hyperbola method, and Lemma 5.1.4. For cases (b) and (c) we will also provide results which average over the conductors of the characters.

5.2.1 Main Term

We first provide asymptotics for (5.2.1) when all of the χ_i are principal. First we will require some preliminary lemmas:

Lemma 5.2.1. *Let $X \geqslant 3$, $C_1, C_2 > 0$ and take any $q_0, q_1 \in (\mathbb{Z}/8\mathbb{Z})^*$. Then for any odd integers $1 \leqslant r_0, r_1 \leqslant (\log X)^{C_1}$ and any fixed $1 \leqslant c_0, c_1 \leqslant (\log X)^{C_2}$:*

$$\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant X\\ \gcd(n_i,r_i)=1\ \forall i\in\{0,1\}\\ n_i\equiv q_i \bmod 8\ \forall i\in\{0,1\}}} \frac{1}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)} = \frac{\mathfrak{S}_1(r_0,r_1)\log\log X}{c_0c_1} + O_{C_1,C_2}\left(\frac{\tau(r_0)\tau(r_1)\sqrt{\log\log X}}{c_0c_1}\right),$$

where the implied constant depends only on C_1 and C_2 and for any odd r_0, r_1 we have

$$\mathfrak{S}_1(r_0,r_1) = \frac{2f_0^2}{\phi(8)^2\left(\prod_{p|2r_0} f_p\right)\left(\prod_{p|2r_1} f_p\right)}$$

and f_0, f_p are as defined in (5.1.1).

Remark 5.2.2. Note that the presence of c_0 and c_1 in the denominator of this asymptotic is due to their presence in the denominator of the summand and not due to their presence in the range of the n_i. It will be seen in the proof that they become untangled from the maximum.

Proof. Set the sum to be $H(X)$. We split it into three regions depending on the value of $\|n_0c_0, n_1c_1\|$:

$$H(X) = H_0(X) + H_1(X) - H_2(X)$$

where

$$H_0(X) = \sum_{\substack{n_0c_0\leqslant X\\ \gcd(n_0,r_0)=1\\ n_0\equiv q_0 \bmod 8}} \sum_{\substack{n_1\leqslant n_0c_0/c_1\\ \gcd(n_1,r_1)=1\\ n_1\equiv q_1 \bmod 8}} \frac{1}{n_0^2c_0^2\tau(n_0)\tau(n_1)},$$

$$H_1(X) = \sum_{\substack{n_1c_1\leqslant X\\ \gcd(n_1,r_1)=1\\ n_1\equiv q_1 \bmod 8}} \sum_{\substack{n_0\leqslant n_1c_1/c_0\\ \gcd(n_0,r_0)=1\\ n_0\equiv q_0 \bmod 8}} \frac{1}{n_1^2c_1^2\tau(n_0)\tau(n_1)},$$

and

$$H_2(X) = \sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant X\\ \gcd(n_i,r_i)=1\ \forall i\in\{0,1\}\\ n_i\equiv q_i \bmod 8\ \forall i\in\{0,1\}\\ n_0c_0=n_1c_1}} \frac{1}{n_0^2c_0^2\tau(n_0)\tau(n_1)}.$$

Let us first consider $H_0(X)$. In order to use Lemma 5.1.4 on the inner sum we need to ensure that $n_0c_0/c_1 \geqslant 2$. We write:

$$H_0(X) = \sum_{\substack{2c_1\leqslant n_0c_0\leqslant X\\ \gcd(n_0,r_0)=1\\ n_0\equiv q_0 \bmod 8}} \sum_{\substack{n_1\leqslant n_0c_0/c_1\\ \gcd(n_1,r_1)=1\\ n_1\equiv q_1 \bmod 8}} \frac{1}{n_0^2c_0^2\tau(n_0)\tau(n_1)} + O\left(\sum_{n_0c_0<2c_1}\sum_{n_1\leqslant n_0c_0/c_1}\frac{1}{n_0^2c_0^2\tau(n_0)\tau(n_1)}\right),$$

To deal with this second sum we note that $n_1 \leqslant 2$ and swap the order of summation. Then it becomes

$$\ll \sum_{n_1\leqslant 2}\sum_{n_1c_1/c_0\leqslant n_0}\frac{1}{c_0^2n_0^2} \ll \frac{1}{c_0c_1}.$$

Thus we are left with

$$H_0(X) = \sum_{\substack{2c_1\leqslant n_0c_0\leqslant X\\ \gcd(n_0,r_0)=1\\ n_0\equiv q_0 \bmod 8}} \sum_{\substack{n_1\leqslant n_0c_0/c_1\\ \gcd(n_1,r_1)=1\\ n_1\equiv q_1 \bmod 8}} \frac{1}{n_0^2c_0^2\tau(n_0)\tau(n_1)} + O\left(\frac{1}{c_0c_1}\right).$$

Now applying Lemma 5.1.4 with $Q=1$ and $C=3/2$ to the sum over n_1 we obtain:

$$H_0(X) = \frac{\mathfrak{S}_0(r_1)}{\phi(8)c_0c_1}M(X) + O\left(\frac{\tau(r_1)(\log\log 3r_1)^{3/2}}{c_0c_1}E(X)\right),$$

where

$$M(X) = \sum_{\substack{2c_1\leqslant n_0c_0\leqslant X\\ \gcd(n_0,r_0)=1\\ n_0\equiv q_0 \bmod 8}} \frac{1}{n_0\tau(n_0)\sqrt{\log n_0c_0/c_1}} \text{ and } E(X) = \sum_{2c_1\leqslant n_0c_0\leqslant X}\frac{1}{n_0\tau(n_0)(\log n_0c_0/c_1)^{3/2}}.$$

Note that for small n_0 the error terms are roughly the same size as the main term; however, upon summing the n_0 over a large range, the dominance of the main term is maintained. Since $(\log\log r_1)^{3/2} \ll_{C_1} (\log\log\log X)^{3/2}$ it suffices to show that $E(X) \ll 1$. To see this, apply partial summation and Lemma 5.1.4:

$$\begin{aligned}
E(X) &\ll \frac{c_0}{X(\log X/c_1)^{3/2}}\sum_{2c_1\leqslant n_0c_0\leqslant X}\frac{1}{\tau(n_0)} + \int_{2c_1/c_0}^{X/c_0}\frac{1}{t^2(\log tc_0/c_1)^{3/2}}\sum_{2c_1/c_0\leqslant n_0\leqslant t}\frac{1}{\tau(n_0)}dt,\\
&\ll \frac{1}{(\log X)^2} + \int_{\|2,2c_1/c_0\|}^{X/c_0}\frac{1}{t(\log tc_0/c_1)^{3/2}(\log t)^{1/2}}dt\\
&\quad + \mathbb{1}(c_1/c_0<1)\int_{2c_1/c_0}^{2}\frac{1}{t^2(\log tc_0/c_1)^{3/2}}\sum_{2c_1/c_0\leqslant n_0\leqslant t}\frac{1}{\tau(n_0)}dt\\
&\ll 1 + \int_{\|2,2c_1/c_0\|}^{X/c_0}\frac{1}{t(\log tc_0/c_1)^{3/2}(\log t)^{1/2}}dt \ll 1,
\end{aligned}$$

where in the leading term of the second step we have used the fact that $c_1 \leqslant (\log X)^{C_2}$ and a Taylor expansion to assert that

$$\frac{1}{(\log X/c_1)^{3/2}} \ll \frac{1}{(\log X)^{3/2}}.$$

To see that the final integral converges, use the fact that $\sqrt{\log t} \geqslant \sqrt{\log 2}$ in this interval and we use the linear substitution $y = tc_0/c_1$. For $M(X)$ we increase the lower bound of this sums range:

$$M(X) = \sum_{\substack{(\log X)^{2C_2} \leqslant n_0 \leqslant X/c_0 \\ \gcd(n_0,r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \frac{1}{n_0 \tau(n_0)\sqrt{\log n_0 c_0/c_1}} + O\left(\sum_{n_0 \leqslant (\log X)^{2C_2}} \frac{1}{n_0 \tau(n_0)}\right).$$

A straightforward partial summation argument shows that this error term is bounded by $\ll \sqrt{\log\log X}$. For the other range we note that since $n_0 \geqslant (c_0/c_1)^2$, we may use the Taylor series expansion

$$\frac{1}{\sqrt{\log n_0 c_0/c_1}} = \frac{1}{\sqrt{\log n_0}} + O\left(\frac{(\log c_0/c_1)}{(\log n_0)^{3/2}}\right).$$

We then have

$$\begin{aligned} M(X) = &\sum_{\substack{(\log X)^{2C_2} \leqslant n_0 \leqslant X/c_0 \\ \gcd(n_0,r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \frac{1}{n_0 \tau(n_0)\sqrt{\log n_0}} + O\left(\sum_{(\log X)^{2C_2} \leqslant n_0 \leqslant X} \frac{(\log c_0/c_1)}{n_0 (\log n_0)^{3/2}}\right) \\ &+ O_{C_2}\left(\sqrt{\log\log X}\right). \end{aligned}$$

The central error term sum converges and so,

$$\sum_{(\log X)^{2C_2} \leqslant n_0 \leqslant X/c_0} \frac{(\log c_0/c_1)}{n_0 (\log n_0)^{3/2}} \ll_{C_2} \frac{(\log c_0/c_1)}{\sqrt{\log\log X}} \ll_{C_2} \sqrt{\log\log X}.$$

For the leading sum in $M(X)$ we use partial summation to obtain

$$\begin{aligned} &\int_{(\log X)^{2C_2}}^{X/c_0} \frac{1}{t^2\sqrt{\log t}} \sum_{\substack{n_0 \leqslant t \\ \gcd(n_0,r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \frac{1}{\tau(n_0)} dt + O\left(\frac{1}{X\sqrt{\log X}} \sum_{n_0 \leqslant X} \frac{1}{\tau(n_0)}\right) \\ &+ O\left(\int_2^X \frac{1}{t^2 (\log t)^{3/2}} \sum_{n_0 \leqslant t} \frac{1}{\tau(n_0)} dt\right). \end{aligned}$$

Using the trivial bound for the sums in the error terms it is clear that they are $O(1)$. Finally we may apply Lemma 5.1.4 with $Q = 1$ and $C = 3$ to the sum inside the main term, this time maintaining the constant:

$$\begin{aligned} M(X) =& \frac{\mathfrak{S}_0(r_0)}{\phi(8)} \int_{(\log X)^{2C_2}}^{X/c_0} \frac{1}{t(\log t)} dt + O\left(\int_2^X \frac{(\log\log 3r_0)^{3/2}}{t(\log t)^2} dt\right) \\ &+ O\left(\int_2^X \frac{\tau(r_0)}{t(\log t)^{7/2}} dt\right) + O_{C_2}\left(\sqrt{\log\log X}\right). \end{aligned}$$

These latter integrals will converge as X tends to infinity using $r_0 \leqslant (\log X)^{C_1}$ to deal with the presence of r_0. The integral in the main term is

$$\begin{aligned}\int_{(\log X)^{2C_2}}^{X/c_0} \frac{1}{t(\log t)} dt &= (\log\log X/c_0) + O_{C_2}(\log\log\log X)\\ &= (\log\log X) + O_{C_2}(\log\log\log X)\end{aligned}$$

so that

$$M(X) = \frac{\mathfrak{S}_0(r_0)}{\phi(8)} \log\log X + O_{C_2}\left(\sqrt{\log\log X}\right).$$

Substituting our expressions for $M(X)$ and $E(X)$ into $H_0(X)$:

$$H_0(X) = \frac{\mathfrak{S}_0(r_0)\mathfrak{S}_0(r_1)}{\phi(8)^2 c_0 c_1}(\log\log X) + O_{C_1,C_2}\left(\frac{\sqrt{\log\log X}}{c_0c_1} + \frac{\tau(r_0)\tau(r_1)(\log\log\log X)^{3/2}}{c_0c_1}\right).$$

We will similarly obtain the same expression for $H_1(X)$:

$$H_1(X) = \frac{\mathfrak{S}_0(r_0)\mathfrak{S}_0(r_1)}{\phi(8)^2 c_0 c_1}(\log\log X) + O_{C_1,C_2}\left(\frac{\sqrt{\log\log X}}{c_0c_1} + \frac{\tau(r_0)\tau(r_1)(\log\log\log X)^{3/2}}{c_0c_1}\right).$$

For $H_2(X)$, suppose without loss in generality that $c_0 \geqslant c_1$. Then

$$H_2(X) \ll \frac{1}{c_0^2} \sum_{n_0 \leqslant X/c_0} \frac{1}{n_0^2} \ll \frac{1}{c_0c_1}.$$

Since $\mathfrak{S}_1(r_0, r_1) = \frac{2\mathfrak{S}_0(r_0)\mathfrak{S}_0(r_1)}{\phi(8)^2}$, we are done. □

Using the same methods we can obtain the following variation:

Lemma 5.2.3. *Let $X \geqslant 3$, $C_1, C_2 > 0$. Then for any fixed $1 \leqslant c_0, c_1 \leqslant (\log X)^{C_1}$:*

$$\sum\sum_{\substack{\|n_0c_0,n_1c_1\| \leqslant X \\ \|n_0d_0,n_1d_1\| > (\log X)^{C_2}}} \frac{\log \|n_0c_0, n_1c_1\|}{\|n_0c_0, n_1c_1\|^2 \tau(n_0)\tau(n_1)} \ll_{C_1} \frac{\log X}{c_0c_1}$$

where the implied constant depends only on $C_1, C_2 > 0$.

Remark 5.2.4. By being more careful in the following proof we may also obtain the asymptotic

$$\sum\sum_{\substack{\|n_0c_0,n_1c_1\| \leqslant X \\ \gcd(n_i, 2r_i)=1\ \forall i \in\{0,1\} \\ n_i \equiv q_i \bmod 8}} \frac{\log \|n_0c_0, n_1c_1\|}{\|n_0c_0, n_1c_1\|^2 \tau(n_0)\tau(n_1)} \sim \frac{\mathfrak{S}_1(r_0, r_1)\log X}{\phi(8)^2 c_0c_1},$$

but this is not needed later.

Proof. Call the sum on the left-hand side $H(X)$. Then as before we may write

$$H(X) \leqslant H_0(X) + H_1(X)$$

where

$$H_0(X) = \sum_{n_0c_0\leqslant X}\sum_{n_1\leqslant n_0c_0/c_1}\frac{(\log n_0c_0)}{n_0^2c_0^2\tau(n_0)\tau(n_1)} \text{ and } H_1(X) = \sum_{n_1c_1\leqslant X}\sum_{n_0\leqslant n_1c_1/c_0}\frac{(\log n_1c_1)}{n_1^2c_1^2\tau(n_0)\tau(n_1)}.$$

Looking at $H_0(X)$ first as in the previous proof we once more ensure the range over n_1 is $\geqslant 2$ in order to apply Lemma 5.1.4: In order to use Lemma 5.1.4 on the inner sum we need to ensure that $n_0c_0/c_1 \geqslant 2$. We write:

$$\begin{aligned} H_0(X) &= \sum_{2c_1\leqslant n_0c_0\leqslant X}\sum_{n_1\leqslant n_0c_0/c_1}\frac{(\log n_0c_0)}{n_0^2c_0^2\tau(n_0)\tau(n_1)} + \sum_{n_0c_0<2c_1}\sum_{n_1\leqslant n_0c_0/c_1}\frac{(\log n_0c_0)}{n_0^2c_0^2\tau(n_0)\tau(n_1)}. \\ &= \sum_{2c_1\leqslant n_0c_0\leqslant X}\sum_{n_1\leqslant n_0c_0/c_1}\frac{(\log n_0c_0)}{n_0^2c_0^2\tau(n_0)\tau(n_1)} + O_{C_1}\left(\frac{(\log\log X)}{c_0c_1}\right). \end{aligned}$$

Now applying Lemma 5.1.4 as an upper bound to the inner sum we obtain:

$$H_0(X) \ll \frac{1}{c_0c_1}\sum_{2c_1\leqslant n_0c_0\leqslant X}\frac{(\log n_0c_0)}{n_0\tau(n_0)\sqrt{\log n_0c_0/c_1}}.$$

We split this sum into two:

$$H_0(X) \ll_{C_2} \frac{1}{c_0c_1}\sum_{(\log X)^{2C_2}\leqslant n_0\leqslant X/c_0}\frac{(\log n_0c_0)}{n_0\tau(n_0)\sqrt{\log n_0c_0/c_1}} + \frac{1}{c_0c_1}\sum_{n_0\leqslant(\log X)^{2C_2}}\frac{(\log\log X)}{n_0\tau(n_0)}.$$

The second sum here may be seen to be

$$\frac{1}{c_0c_1}\sum_{n_0\leqslant(\log X)^{2C_2}}\frac{(\log\log X)}{n_0\tau(n_0)} \ll_{C_2} \frac{(\log\log X)^{3/4}}{c_0c_1}$$

using a standard partial summation argument. For the first sum above we use a Taylor series expansion since we have $n_0 \geqslant (\log X)^{2C_2}$. Thus

$$\frac{1}{\sqrt{\log n_0c_0/c_1}} \ll_{C_2} \frac{1}{\sqrt{\log n_0}}.$$

Using the logarithmic rule we may also get rid of the c_0 in the numerator of the summand. Overall, we have,

$$\begin{aligned} H_0(X) \ll_{C_2} &\frac{1}{c_0c_1}\sum_{(\log X)^{2C_2}\leqslant n_0\leqslant X/c_0}\frac{\sqrt{\log n_0}}{n_0\tau(n_0)} + \frac{\log\log X}{c_0c_1}\sum_{(\log X)^{2C_2}\leqslant n_0\leqslant X/c_0}\frac{1}{n_0\tau(n_0)\sqrt{\log n_0}} \\ &+ \frac{(\log\log X)^{3/4}}{c_0c_1}. \end{aligned}$$

Note that the second sum above is of the same form as the main term of $M(X)$ in Lemma 5.2.1, which we evaluated to be of order $\log\log X$. For the leading term above we use partial summation and Lemma 5.1.4:

$$\begin{aligned} \frac{1}{c_0c_1}\sum_{(\log X)^{2C_2}\leqslant n_0\leqslant X/c_0}\frac{\sqrt{\log n_0}}{n_0\tau(n_0)} &\ll_{C_2} \frac{\sqrt{\log X}}{c_0c_1X}\sum_{n_0\leqslant X}\frac{1}{\tau(n_0)} + \int_2^X \frac{\sqrt{\log t}}{c_0c_1t^2}\sum_{n_0\leqslant t}\frac{1}{\tau(n_0)}dt \\ &\ll_{C_2} \frac{1}{c_0c_1} + \frac{1}{c_0c_1}\int_2^X\frac{dt}{t} \ll_{C_2} \frac{\log X}{c_0c_1}. \end{aligned}$$

Note that in the above computation, we bounded the derivative of $\frac{\sqrt{\log t}}{t}$ for brevity. Thus,

$$H_0(X) \ll_{C_2} \frac{\log X}{c_0 c_1},$$

and we may similarly obtain the same bound for $H_1(X)$. $\square$

The following result regards similar sums to the above, but over shorter ranges. We will see that they have a similar flavour to sums already seen in the above proofs:

Lemma 5.2.5. *Let $X \geqslant 3$, $C_1, C_2 > 0$. Then for any fixed $1 \leqslant c_0, c_1 \leqslant X^{C_1}$, $1 \leqslant d_0, d_1 \leqslant X^{C_2/2}$:*

$$\sum_{\|n_0 d_0, n_1 d_1\| \leqslant X^{C_2}} \sum \frac{1}{\|n_0 c_0, n_1 c_1\|^2 \tau(n_0)\tau(n_1)} \ll_{C_1, C_2} \left(\frac{\sqrt{\log X}}{c_0 c_1}\right),$$

where the implied constant only depends on C_1 and C_2.

Remark 5.2.6. The key point to note here is that the constants c_0 and c_1 are not included in the ranges for n_0 and n_1. This will cause trouble in the unwrapping argument before, especially since the constants may be larger than either variable very often. Instead it is enough just to use trivial bounds.

Proof. The sum is at most $H_0(X) + H_1(X)$ where

$$H_0(X) = \sum_{n_0 \leqslant X^{C_2}} \sum_{n_1 \leqslant n_0 c_0 / c_1} \frac{1}{n_0^2 c_0^2 \tau(n_0)\tau(n_1)} \text{ and } H_1(X) = \sum_{n_1 \leqslant X^{C_2}} \sum_{n_0 \leqslant n_1 c_1 / c_0} \frac{1}{n_1^2 c_1^2 \tau(n_0)\tau(n_1)}.$$

Here we can use a trivial bound for the inner sums, giving

$$H_0(X), H_1(X) \ll \sum_{n \leqslant X^{C_2}} \frac{1}{n c_0 c_1 \tau(n)}.$$

Upon using partial summation and Lemma 5.1.4 we obtain the desired bound. $\square$

Next we put these together to obtain Proposition 2.2.6.

Proposition 5.2.7. *Let $X \geqslant 3$, $C_1, C_2, C_3 > 0$ and take any $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$. Then for any fixed odd integers $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ and fixed integers $1 \leqslant c_0, c_1, c_2, c_3 \leqslant (\log X)^{C_2}$, $1 \leqslant d_0, d_1, d_2, d_3 \leqslant (\log X)^{C_3/2}$ we have*

$$\sum\sum\sum\sum_{\substack{\|n_0 c_0, n_1 c_1\| \cdot \|n_2 c_2, n_3 c_3\| \leqslant X \\ \|n_0 d_0, n_1 d_1\|, \|n_2 d_2, n_3 d_3\| > (\log X)^{C_3} \\ \gcd(n_i, r_i) = 1 \ \forall\ 0 \leqslant i \leqslant 3 \\ n_i \equiv q_i \bmod 8 \ \forall\ 0 \leqslant i \leqslant 3}} \frac{1}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}$$

$$= \frac{\mathfrak{S}_2(\mathbf{r}) X^2 \log\log X}{c_0 c_1 c_2 c_3 \log X}\left(1 + O_{C_1, C_2, C_3}\left(\frac{\tau(r_0)\tau(r_1)\tau(r_2)\tau(r_3)}{c_0 c_1 c_2 c_3 \sqrt{\log\log X}}\right)\right)$$

where the implied constant depends at most on C_1, C_2, C_3 and we define

$$\mathfrak{S}_2(\mathbf{r}) = \frac{4 f_0^4}{\phi(8)^4 \left(\prod_{p \mid 2r_0} f_p\right)\left(\prod_{p \mid 2r_1} f_p\right)\left(\prod_{p \mid 2r_2} f_p\right)\left(\prod_{p \mid 2r_3} f_p\right)}.$$

Proof. Call the sum on the left hand side $H(X)$. Then using the hyperbola method we may write

$$H(X) = H_0(X) + H_1(X) + O(H_2(X))$$

where

$$H_0(X) = \sum\sum\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant X^{1/2} \\ \|n_2c_2,n_3c_3\|\leqslant X/\|n_0c_0,n_1c_1\| \\ \|n_0d_0,n_1d_1\|,\|n_2d_2,n_3d_3\|>(\log X)^{C_3} \\ \gcd(n_i,2r_i)=1\ \forall\ 0\leqslant i\leqslant 3 \\ n_i\equiv q_i \bmod 8\ \forall\ 0\leqslant i\leqslant 3}} \frac{1}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)},$$

$$H_1(X) = \sum\sum\sum\sum_{\substack{\|n_2c_2,n_3c_3\|\leqslant X^{1/2} \\ \|n_0c_0,n_1c_1\|\leqslant X/\|n_2c_2,n_3c_3\| \\ \|n_0d_0,n_1d_1\|,\|n_2d_2,n_3d_3\|>(\log X)^{C_3} \\ \gcd(n_i,2r_i)=1\ \forall\ 0\leqslant i\leqslant 3 \\ n_i\equiv q_i \bmod 8\ \forall\ 0\leqslant i\leqslant 3}} \frac{1}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)},$$

and

$$H_2(X) = \sum\sum\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant X^{1/2} \\ \|n_2c_2,n_3c_3\|\leqslant X^{1/2}}} \frac{1}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}.$$

Let us first deal with $H_2(X)$. We write

$$H_2(X) \ll \prod_{i=0}^{3}\left(\sum_{n_ic_i\leqslant X^{1/2}} \frac{1}{\tau(n_i)}\right) \ll \frac{X^2}{c_0c_1c_2c_3(\log X)^2},$$

by Lemma 5.1.4. Now let us consider $H_0(X)$. Here we may add in the terms for which $\|n_2d_2, n_3d_3\| \leqslant (\log X)^{C_3}$ at the cost of a negligible error term since,

$$\sum\sum\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant X^{1/2} \\ \|n_2d_2,n_3d_3\|\leqslant(\log X)^{C_3}}} \frac{1}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} \ll X(\log X)^{2C_3}. \tag{5.2.2}$$

Then

$$H_0(X) = \sum\sum\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant X^{1/2} \\ \|n_2c_2,n_3c_3\|\leqslant X/\|n_0c_0,n_1c_1\| \\ \|n_0d_0,n_1d_1\|>(\log X)^{C_3} \\ \gcd(n_i,2r_i)=1\ \forall\ 0\leqslant i\leqslant 3 \\ n_i\equiv q_i \bmod 8\ \forall\ 0\leqslant i\leqslant 3}} \frac{1}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} + O(X(\log X)^{2C_3}).$$

Using Lemma 5.1.4 the sum over n_2 and n_3 is

$$\begin{aligned} &= \frac{\mathfrak{S}_0(r_2)\mathfrak{S}_0(r_3)X^2}{\phi(8)^2c_2c_3\|n_0c_0,n_1c_1\|^2(\log(X/\|n_0c_0,n_1c_1\|))} \\ &\quad + O\left(\frac{X^2(\log\log\|3r_2,3r_3\|)^{3/2}}{c_2c_3\|n_0c_0,n_1c_1\|^2(\log(X/\|n_0c_0,n_1c_1\|))^2}\right). \end{aligned}$$

Note that we have suppressed the arbitrary log saving error term in this calculation. This can be done by noting that $Q = 1$ and $\tau(r_i) \ll_{C_1} r_i^{1/C_1} \ll_{C_1} (\log X)$ so that

$$\frac{\tau(r_i)X^2}{(\log X/\|n_0c_0,n_1c_1\|)^{C_4}} \ll \frac{X^2(\log\log\|3r_2,3r_3\|)^{3/2}}{(\log(X/\|n_0c_0,n_1c_1\|))^2}$$

for C_4 chosen sufficiently large. Therefore we have

$$H_0(X) = \frac{\mathfrak{S}_0(r_2)\mathfrak{S}_0(r_3)X^2}{c_2c_3}M_0(X) + O\left(\frac{X^2}{c_2c_3}E_0(X)\right) + O(X(\log X)^{2C_3})$$

where

$$M_0(X) = \sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant X^{1/2}\\ \|n_0d_0,n_1d_1\|>(\log X)^{C_3}\\ \gcd(n_i,2r_i)=1\ \forall\ 0\leqslant i\leqslant 1\\ n_i\equiv q_i \bmod 8\ \forall\ 0\leqslant i\leqslant 1}} \frac{1}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)(\log X/\|n_0c_0,n_1c_1\|)}$$

and

$$E_0(X) = \sum\sum_{\|n_0c_0,n_1c_1\|\leqslant X^{1/2}} \frac{(\log\log\|3r_2,3r_3\|)^{3/2}}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)(\log X/\|n_0c_0,n_1c_1\|)^2}.$$

Using $\|n_0c_0,n_1c_1\|\leqslant X^{1/2}$, the usual Taylor series manoeuvre and Lemma 5.2.1 we have

$$\begin{aligned}E_0(X) &\ll \frac{1}{(\log X)^2}\sum\sum_{\|n_0c_0,n_1c_1\|\leqslant X^{1/2}} \frac{(\log\log\|3r_2,3r_3\|)^{3/2}}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)}\\ &\ll \frac{(\log\log X)(\log\log\|3r_2,3r_3\|)^{3/2}}{c_0c_1(\log X)^2}\end{aligned}$$

which is sufficient. Now we turn to $M_0(X)$. Since $\|n_0c_0,n_1c_1\|\leqslant\sqrt{X}$ we may use a geometric series argument to write

$$\frac{1}{\log X/\|n_0c_0,n_1c_1\|} = \frac{1}{\log X} + O\left(\frac{\log\|n_0c_0,n_1c_1\|}{(\log X)^2}\right).$$

We have from Lemma 5.2.1 (with Lemma 5.2.5 to add in the terms for which we have $\|n_0d_0,n_1d_1\|\leqslant(\log X)^{C_3}$),

$$\begin{aligned}\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant X^{1/2}\\ \|n_0d_0,n_1d_1\|>(\log X)^{C_3}\\ \gcd(n_i,2r_i)=1\ \forall\ 0\leqslant i\leqslant 1\\ n_i\equiv q_i \bmod 8\ \forall\ 0\leqslant i\leqslant 1}} \frac{1}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)} &= \frac{\mathfrak{S}_1(r_0,r_1)\log\log(X^{1/2})}{c_0c_1}\\ &\quad + O_{C_1,C_2,C_3}\left(\frac{\tau(r_0)\tau(r_1)\sqrt{\log\log(X^{1/2})}}{c_0c_1}\right)\end{aligned}$$

and from Lemma 5.2.3 that

$$\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant X^{1/2}\\ \|n_0d_0,n_1d_1\|>(\log X)^{C_3}\\ \gcd(n_i,2r_i)=1\ \forall\ 0\leqslant i\leqslant 1\\ n_i\equiv q_i \bmod 8\ \forall\ 0\leqslant i\leqslant 1}} \frac{\log\|n_0c_0,n_1c_1\|}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)} \ll_{C_1,C_2,C_3} \frac{\log X}{c_0c_1}.$$

We therefore conclude that

$$M_0(X) = \frac{\mathfrak{S}_1(r_0,r_1)\log\log(X^{1/2})}{c_0c_1} + O_{C_1,C_2,C_3}\left(\frac{\sqrt{\log\log X}}{c_0c_1}\right).$$

Note that when using the Taylor series above, the error term may in fact be of the same order as the main term. In writing it this way we are in fact splitting a constant into two parts - one independant of the n_i, which contributes to the $(\log\log X)$ by Lemma 5.2.1 and a part dependent on the n_i which contributes to an error of $O(1)$ by Lemma 5.2.3. We have now shown

$$H_0(X) = \frac{\mathfrak{S}_0(r_2)\mathfrak{S}_0(r_3)\mathfrak{S}_1(r_0,r_1)X^2\log\log X}{\phi(8)^2 c_0c_1c_2c_3(\log X)} + O_{C_1,C_2,C_3}\left(\frac{\tau(r_0)\tau(r_1)X^2\sqrt{\log\log X}}{c_0c_1c_2c_3(\log X)}\right).$$

Evaluating $H_1(X)$ in the same way we obtain the same result with r_0 and r_1 switched with r_2 and r_3. Noting that $\frac{\mathfrak{S}_0(r_0)\mathfrak{S}_0(r_1)\mathfrak{S}_1(r_2,r_3)}{\phi(8)^2} = \frac{\mathfrak{S}_0(r_2)\mathfrak{S}_0(r_3)\mathfrak{S}_1(r_0,r_1)}{\phi(8)^2} = \frac{\mathfrak{S}_2(\mathbf{r})}{2}$ we combine the expressions for $H_0(X)$, $H_1(X)$ and $H_2(X)$ to conclude the proof. □

5.2.2 Small Conductors - Symmetric Hyperbola Method

This is the easiest of the three cases: all that is required for us to do is to apply the hyperbola method and Lemma 5.1.4 appropriately. To save space we introduce the following summation conditions:

$$\begin{cases} \|n_0d_0, n_1d_1\|, \|n_2d_2, n_3d_3\| > (\log X)^D \\ \gcd(n_i, r_i) = 1 \ \forall\ 0 \leqslant i \leqslant 3 \\ n_i \equiv q_i \bmod 8 \ \forall\ 0 \leqslant i \leqslant 3, \end{cases} \tag{5.2.3}$$

where $D > 0$ and the r_i and d_i are some integers and $q_i \in (\mathbb{Z}/8\mathbb{Z})^*$ for each i.

Lemma 5.2.8. *Let $X \geqslant 3$, $C_1, C_2, C_3 > 0$ and fix odd integers Q_0, Q_2 and some $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$. Suppose χ_0 and χ_2 are non-principal Dirichlet characters modulo Q_0 and Q_2 and that $g_1, g_3 : \mathbb{N} \to \mathbb{C}$ are multiplicative functions such that $|g_1(n)|, |g_3(n)| \leqslant 1$ for all $n \in \mathbb{N}$. Then for any odd integers $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ such that $\gcd(r_i, Q_i)$ for $i = 0, 2$ and any fixed $1 \leqslant c_0, c_1, c_2, c_3 \leqslant (\log X)^{C_2}$, $1 \leqslant d_0, d_1, d_2, d_3 \leqslant (\log X)^{C_3/2}$ we have*

$$\sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4, \|n_0c_0,n_1c_1\|\cdot\|n_2c_2,n_3c_3\|\leqslant X \\ (5.2.3)}} \frac{\chi_0(n_0)g_1(n_1)\chi_2(n_2)g_3(n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} \ll_{C_1,C_2,C_3,C_4} \frac{Q_0Q_2X^2}{c_0c_1c_2c_3(\log X)^{C_4}},$$

for any $C_4 > 0$ where the implied constant depends at most on C_1, C_2, C_3 and C_4. Note we have used $D = C_3$ in (5.2.3).

Proof. Call this sum $H(X)$. Then using the hyperbola method we obtain

$$H(X) = H_0(X) + H_1(X) - H_2(X),$$

where

$$H_0(X) = \sum\sum\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant X^{1/2} \\ \|n_2c_2,n_3c_3\|\leqslant X/\|n_0c_0,n_1c_1\| \\ (5.2.3)}} \frac{\chi_0(n_0)g_1(n_1)\chi_2(n_2)g_3(n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}, \tag{5.2.4}$$

$$H_1(X) = \sum_{\substack{\|n_2c_2,n_3c_3\| \leqslant X^{1/2} \\ \|n_0c_0,n_1c_1\| \leqslant X/\|n_2c_2,n_3c_3\| \\ (5.2.3)}} \sum\sum\sum \frac{\chi_0(n_0)g_1(n_1)\chi_2(n_2)g_3(n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}, \tag{5.2.5}$$

and

$$H_2(X) = \sum_{\substack{\|n_0c_0,n_1c_1\|,\|n_2c_2,n_3c_3\| \leqslant X^{1/2} \\ (5.2.3)}} \sum\sum\sum \frac{\chi_0(n_0)g_1(n_1)\chi_2(n_2)g_3(n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}. \tag{5.2.6}$$

We deal with $H_2(X)$ first. Using the trivial bound (5.2.2), we may add in the terms for which $\|n_0d_0, n_1d_1\| > (\log X)^{C_3}$ or $\|n_2d_2, n_3d_3\| > (\log X)^{C_3}$ at the cost of a small error term:

$$H_2(X) = \sum_{\substack{\|n_0c_0,n_1c_1\|,\|n_2c_2,n_3c_3\| \leqslant X^{1/2} \\ \gcd(n_i,r_i)=1\ \forall 0\leqslant i\leqslant 3 \\ n_i\equiv q_i \bmod 8\ \forall 0\leqslant i\leqslant 3}} \sum\sum\sum \frac{\chi_0(n_0)g_1(n_1)\chi_2(n_2)g_3(n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} + O(X(\log X)^{2C_3}).$$

This sum is now separable so that, using trivial bounds for the sums over n_1 and n_3 and Lemma 5.1.4 for the sums over n_2 and n_4 we obtain:

$$\begin{aligned} H_2(X) = &\left(\sum_{\substack{n_0c_0 \leqslant X^{1/2} \\ \gcd(n_0,r_0)=1 \\ n_0\equiv q_0 \bmod 8}} \frac{\chi_0(n_0)}{\tau(n_0)}\right)\left(\sum_{\substack{n_1c_1 \leqslant X^{1/2} \\ \gcd(n_1,r_1)=1 \\ n_1\equiv q_1 \bmod 8}} \frac{g_1(n_1)}{\tau(n_1)}\right)\left(\sum_{\substack{n_2c_2 \leqslant X^{1/2} \\ \gcd(n_2,r_2)=1 \\ n_2\equiv q_2 \bmod 8}} \frac{\chi_2(n_2)}{\tau(n_2)}\right)\left(\sum_{\substack{n_3c_3 \leqslant X^{1/2} \\ \gcd(n_3,r_3)=1 \\ n_3\equiv q_3 \bmod 8}} \frac{g_3(n_3)}{\tau(n_3)}\right) \\ &+ O(X(\log X)^{2C_2}) \\ \ll_{C_1,C_2,C_3,C_4} &\left(\frac{\tau(r_0)Q_0X^{1/2}}{c_0(\log X)^{C_4}}\right)\left(\frac{X^{1/2}}{c_1}\right)\left(\frac{\tau(r_2)Q_2X^{1/2}}{c_2(\log X)^{C_4}}\right)\left(\frac{X^{1/2}}{c_3}\right) \\ \ll_{C_1,C_2,C_3,C_4} &\frac{Q_0Q_2X^2}{c_0c_1c_2c_3(\log X)^{C_4}}. \end{aligned}$$

Note also that by the assumption $r_i \leqslant (\log X)^{C_1}$ we may ignore the $\tau(r_i)$ upon choosing C_4 appropriately large, and since $c_i \leqslant (\log X)^{C_2}$ for all i we may use the bound

$$\frac{1}{(\log X/c_i)^{C_4}} \ll_{C_2,C_4} \frac{1}{(\log X)^{C_4'}}.$$

We will use these remarks again without mentioning. Next we deal with $H_0(X)$. Add in the terms where $\|n_2d_2, n_3d_3\| \leqslant (\log X)^{C_3}$ at the cost of an error term of size $O(X(\log X)^{2C_3})$ using a trivial bound. Then, performing the sum over n_2 and n_3 first we write,

$$\begin{aligned} H_0(X) \ll_{C_1,C_3} &\sum_{n_0c_0,n_1c_1 \leqslant X^{1/2}} \sum \left|\sum_{\substack{n_2c_2 \leqslant X/\|c_0n_0,c_1n_1\| \\ \gcd(n_2,r_2)=1 \\ n_2\equiv q_2 \bmod 8}} \frac{\chi_2(n_2)}{\tau(n_2)}\right| \left|\sum_{\substack{n_3c_3 \leqslant X/\|c_0n_0,c_1n_1\| \\ \gcd(n_3,r_3)=1 \\ n_3\equiv q_3 \bmod 8}} \frac{g_3(n_3)}{\tau(n_3)}\right| \\ &+ O(X(\log X)^{2C_3}). \end{aligned}$$

Using a trivial bound for the sum over n_3 and Lemma 5.1.4 for non-trivial characters (noting that the range is relatively large since that $\|n_0c_0, n_1c_1\| \leqslant X^{1/2}$) we obtain

$$\begin{aligned} H_0(X) &\ll_{C_1,C_2,C_3,C_4} \frac{\tau(r_2)Q_2X^2}{c_2c_3(\log X)^{2C_4+2}} \sum_{n_0c_0,n_1c_1\leqslant X^{1/2}} \frac{1}{\|n_0c_0, n_1c_1\|^2} \\ &\ll_{C_1,C_2,C_3,C_4} \frac{Q_2X^2}{c_0c_1c_2c_3(\log X)^{C_4}} \end{aligned}$$

where we have used the straightforward bound

$$\sum_{n_0c_0,n_1c_1\leqslant X^{1/2}} \frac{1}{\|n_0c_0, n_1c_1\|^2} \ll \frac{(\log X^{1/2}/c_0)(\log X^{1/2}/c_1)}{c_0c_1} \ll \frac{(\log X)^2}{c_0c_1}.$$

For $H_1(X)$ we use an identical argument to that of $H_0(X)$ with n_0, n_1 switching roles with n_2, n_3. This yields

$$H_1(X) \ll_{C_1,C_2,C_3,C_4} \frac{Q_0X^2}{c_0c_1c_2c_3(\log X)^{C_4}}.$$

Combining these three bounds gives the result. □

To conclude this section we want to average this result over a small range of conductors to obtain Proposition 2.2.7. For this purpose it will be necessary to specialise to the case where the characters are Jacobi symbols. For $m \in \mathbb{N}$ odd, let $\psi_m(\cdot)$ denote generically either the Jacobi symbol $\left(\frac{\cdot}{m}\right)$ or the Jacobi symbol $\left(\frac{m}{\cdot}\right)$.

Proposition 5.2.9. *Let $X \geqslant 3$, $C_1, C_2, C_3 > 0$ and fix odd integers Q_0, Q_2 and some $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$, $\tilde{\mathbf{q}} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$. Fixing some odd integers $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ such that $\gcd(r_i, Q_i) = 1$ for $i = 0, 2$ and any $1 \leqslant c_0, c_1, c_2, c_3 \leqslant (\log X)^{C_2}$, $1 \leqslant d_0, d_1, d_2, d_3 \leqslant (\log X)^{C_3/2}$ we define, for any $\mathbf{m} \in \mathbb{N}^4$,*

$$H(X, \mathbf{m}) = \sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4, \|n_0m_0c_0, n_1m_1c_1\|\cdot\|n_2m_2c_2, n_3m_3c_3\|\leqslant X \\ (5.2.3)}} \frac{\psi_{Q_0m_0m_1}(n_2n_3)\psi_{Q_2m_2m_3}(n_0n_1)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)},$$

where we use (5.2.3) with $D = C_3$. Then for any $C_4 > 0$:

$$\sum\sum\sum\sum_{\substack{\mathbf{m}\in\mathbb{N}^4, \|m_0,m_1\|,\|m_2,m_3\|\leqslant(\log X)^{C_3} \\ \gcd(m_0m_1, Q_0r_2r_3)=\gcd(m_2m_3, Q_1r_0r_1)=1 \\ \mathbf{m}\equiv\tilde{q} \bmod 8 \\ Q_0m_0m_1 \text{ and } Q_2m_2m_3\neq 1}} \frac{\mu^2(2m_0m_1m_2m_3)|H(X, \mathbf{m})|}{\tau(m_0)\tau(m_1)\tau(m_2)\tau(m_3)} \ll_{C_1,C_2,C_3,C_4} \frac{Q_0Q_2X^2}{c_0c_1c_2c_3(\log X)^{C_4}}.$$

Proof. By the gcd conditions on Q_i and the m_i, the condition that $Q_0m_0m_1$ and $Q_2m_2m_3 \neq 1$ and the term $\mu^2(m_0m_1m_2m_3) = 1$, we know that for each $\mathbf{m}$ considered in the average, the quadratic characters $\psi_{Q_0m_0m_1}$ and $\psi_{Q_1m_2m_3}$ are non-principal. Therefore, Lemma 5.2.8 tells us that, for each $\mathbf{m}$ considered in the average,

$$\begin{aligned} H(X, \mathbf{m}) &\ll_{C_1,C_2,C_3,C_4} \frac{Q_0Q_2m_0m_1m_2m_3X^2}{m_0m_1m_2m_3c_0c_1c_2c_3(\log X)^{C_4+4C_3}} \\ &\ll_{C_1,C_2,C_3,C_4} \frac{Q_0Q_2X^2}{c_0c_1c_2c_3(\log X)^{C_4+4C_3}}. \end{aligned}$$

Therefore, by summing over the given $\mathbf{m}$, the average can be seen to be bounded by

$$\ll_{C_1,C_2,C_3,C_4} \frac{Q_0 Q_2 X^2 (\log X)^{4C_3}}{c_0 c_1 c_2 c_3 (\log X)^{C_4+4C_3}} \ll_{C_1,C_2,C_3,C_4} \frac{Q_0 Q_2 X^2}{c_0 c_1 c_2 c_3 (\log X)^{C_4}}.$$

□

5.2.3 Small Conductors: Asymmetric Hyperbola Method

We begin with a technical lemma similar in form to Lemma 5.2.1.

Lemma 5.2.10. *Let $X \geqslant 3$, $C_1, C_2 > 1$, $0 < \epsilon < 1$ and define $Y = \exp((\log X)^\epsilon)$. Fix some odd integers Q_0, Q_1 and some $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*2}$ and suppose χ_0 and χ_1 are non-principal characters modulo Q_0 and Q_1 respectively. Then for any odd integers $1 \leqslant r_0, r_1 \leqslant (\log X)^{C_1}$ and any fixed integers $1 \leqslant c_0, c_1 \leqslant (\log X)^{C_2/16}$, $1 \leqslant d_0, d_1 \leqslant (\log X)^{C_2/4}$ we have*

$$\sum\sum_{\substack{n_0 c_0, n_1 c_1 \leqslant Y \\ \|n_0 d_0, n_1 d_1\| > (\log X)^{C_2} \\ \gcd(n_i, r_i)=1 \ \forall 0 \leqslant i \leqslant 1 \\ n_i \equiv q_i \bmod 8 \ \forall 0 \leqslant i \leqslant 1}} \frac{\chi_0(n_0)\chi_1(n_1)}{\|n_0 c_0, n_1 c_1\|^2 \tau(n_0)\tau(n_1)} \ll_{C_1,C_2,C_3} \frac{\tau(r_0)\tau(r_1)(Q_0+Q_1)}{c_0 c_1 (\log\log X)^{C_3}},$$

for any $C_3 > 1$ where the implied constant depends at most on C_1, C_2 and C_3.

Remark 5.2.11. The philosophy with this sum, as with many others like it that appear throughout this chapter, is that it should converge and so the lower bounds should yield some saving. To see this, note that the sums considered above are similar to

$$\sum\sum_{\substack{n_0 c_0, n_1 c_1 \leqslant Y \\ \gcd(n_i, r_i)=1 \ \forall 0 \leqslant i \leqslant 1 \\ n_i \equiv q_i \bmod 8 \ \forall 0 \leqslant i \leqslant 1}} \frac{\chi_0(n_0)\chi_1(n_1)}{n_0 n_1 c_0 c_1 \tau(n_0)\tau(n_1)},$$

which is separable in each variable and is more readily seen to converge by comparing the two sums to the Dirichlet series

$$D(1, \chi_i) = \sum_{n=1}^{\infty} \frac{\chi_i(n)}{n\tau(n)}.$$

More will be said about this idea in §5.3.

Proof. Call the sum $H(X)$. The key difference to the above remark is that we run into some difficulty when trying to untangle the constants c_0, d_0, c_1 and d_1. Indeed we have to split both maximums simultaneously in order to obtain sums of a familiar form. We have four cases:

(1) $n_0 c_0 \geqslant n_1 c_1$ and $n_0 d_0 \geqslant n_1 d_1$;

(2) $n_1c_1 > n_0c_0$ and $n_1d_1 > n_0d_0$;

(3) $n_0c_0 \geqslant n_1c_1$ and $n_1d_1 > n_0d_0$;

(4) $n_1c_1 > n_0c_0$ and $n_0d_0 \geqslant n_1d_1$.

We note immediately that the conditions of (3) imply that $\frac{d_1}{d_0} > \frac{c_1}{c_0}$ while the conditions of (4) imply $\frac{d_1}{d_0} < \frac{c_1}{c_0}$ and so only one of them will apply. Without loss in generality, we will assume that (3) case holds. Now we split $H(X)$ into regions in which n_0 and n_1 satisfy these conditions. We have

$$H(X) = H_1(X) + H_2(X) + H_3(X),$$

where

$$H_1(X) = \sum_{\substack{(\log X)^{C_2}/d_0 < n_0 \leqslant Y/c_0 \\ \gcd(n_0,r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \sum_{\substack{n_1 \leqslant \min(n_0d_0/d_1, n_0c_0/c_1) \\ \gcd(n_1,r_1)=1 \\ n_1 \equiv q_1 \bmod 8}} \frac{\chi_0(n_0)\chi_1(n_1)}{n_0^2c_0^2\tau(n_0)\tau(n_1)},$$

$$H_2(X) = \sum_{\substack{(\log X)^{C_2}/d_1 < n_1 \leqslant Y/c_1 \\ \gcd(n_1,r_1)=1 \\ n_1 \equiv q_1 \bmod 8}} \sum_{\substack{n_0 < \min(n_1d_1/d_0, n_1c_1/c_0) \\ \gcd(n_1,r_1)=1 \\ n_1 \equiv q_1 \bmod 8}} \frac{\chi_0(n_0)\chi_1(n_1)}{n_1^2c_1^2\tau(n_0)\tau(n_1)},$$

and

$$H_3(X) = \sum_{\substack{n_0 \leqslant Y/c_0 \\ \gcd(n_0,r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \sum_{\substack{\|(\log X)^{C_2}/d_1, n_0d_0/d_1\| < n_1 \leqslant n_0c_0/c_1 \\ \gcd(n_1,r_1)=1 \\ n_1 \equiv q_1 \bmod 8}} \frac{\chi_0(n_0)\chi_1(n_1)}{n_0^2c_0^2\tau(n_0)\tau(n_1)}.$$

The sums $H_1(X)$ and $H_2(X)$ may be dealt with similarly. For $H_1(X)$ we note that in these regions $\min(\frac{n_0d_0}{d_1}, \frac{n_0c_0}{c_1}) > (\log X)^{3C_2/4}$, and so the range over n_1 increases with X. We therefore apply Lemma 5.1.4 to it to obtain:

$$H_1(X) \ll_{C_3} \sum_{(\log X)^{C_2}/d_0 < n_0} \frac{\tau(r_1)Q_1 \min(d_0/d_1, c_0/c_1)}{n_0c_0^2(\log(n_0 \min(d_0/d_1, c_0/c_1)))^{C_3}},$$

for any $C_3 > 1$. Now we use $\min(d_0/d_1, c_0/c_1) \leqslant c_0/c_1$ to bound the numerator and $n_0 > (\log X)^{C_2}/d_0 \geqslant (\log X)^{3C_2/4} \geqslant \min(d_0/d_1, c_0/c_1)^2$ with a Taylor series expansion to bound the denominator. This will yield

$$H_1(X) \ll_{C_3} \sum_{(\log X)^{C_2}/d_0 < n_0} \frac{\tau(r_1)Q_1}{n_0c_0c_1(\log n_0)^{C_3}}.$$

Since $C_3 > 1$, this is the tail of a convergent series and so we obtain

$$H_1(X) \ll_{C_2,C_3} \frac{\tau(r_1)Q_1}{c_0c_1(\log\log X)^{C_3}}.$$

Similarly,

$$H_2(X)_{C_2,C_3} \ll_{C_2,C_3} \frac{\tau(r_0)Q_0}{c_0c_1(\log\log X)^{C_3}}.$$

We now turn to $H_3(X)$. First note that this sum is 0 unless

$$\frac{n_0c_0}{c_1} \geqslant \frac{(\log X)^{C_2}}{d_1}.$$

We may therefore add this condition to the sum of $H_3(X)$ with no cost. This will lead to:

$$H_3(X) = \sum_{\substack{c_1(\log X)^{C_2}/c_0d_1 < n_0 \leqslant Y/c_0 \\ \gcd(n_0,r_0)=1 \\ n_0\equiv q_0 \bmod 8}} \sum_{\substack{\|(\log X)^{C_2}/d_1, n_0d_0/d_1\| < n_1 \leqslant n_0c_0/c_1 \\ \gcd(n_1,r_1)=1 \\ n_1\equiv q_1 \bmod 8}} \frac{\chi_0(n_0)\chi_1(n_1)}{n_0^2c_0^2\tau(n_0)\tau(n_1)}.$$

In this case it is unclear whether or not the range over n_1 is guaranteed to increase with X. In order to deal with this we write the sum over n_1 as

$$\sum_{\substack{n_1 \leqslant n_0c_0/c_1 \\ \gcd(n_1,r_1)=1 \\ n_1\equiv q_1 \bmod 8}} \frac{\chi_1(n_1)}{\tau(n_1)} - \sum_{\substack{n_1 \leqslant \|(\log X)^{C_2}/d_1, n_0d_0/d_1\| \\ \gcd(n_1,r_1)=1 \\ n_1\equiv q_1 \bmod 8}} \frac{\chi_1(n_1)}{\tau(n_1)}.$$

Both of these are sums with ranges which grow with X and so we may apply Lemma 5.1.4 to them. Upon doing this to the first sum above and then summing over n_0 we obtain,

$$\ll_{C_3} \sum_{(\log X)^{11C_2/16} < n_0 \leqslant Y} \frac{\tau(r_1)Q_1}{n_0c_0c_1(\log n_0c_0/c_1)^{C_3}} \ll_{C_2,C_3} \frac{\tau(r_1)Q_1}{c_0c_1(\log\log X)^{C_3}}$$

using the usual Taylor series expansion to deal with the c_i inside of the logarithm. Note also that we have extended the range over n_0 by positivity of the summand. Now applying Lemma 5.1.4 to the second sum above and summing over n_0 we obtain (extending the range as above):

$$\ll_{C_3} \sum_{(\log X)^{11C_2/16} < n_0 \leqslant Y} \frac{\tau(r_1)Q_1\|(\log X)^{C_2}/d_1, n_0d_0/d_1\|}{n_0^2c_0^2(\log \|(\log X)^{C_2}/d_1, n_0d_0/d_1\|)^{C_3}}.$$

Here we use the upper bound

$$\|(\log X)^{C_2}/d_1, n_0d_0/d_1\| \leqslant \frac{n_0c_0}{c_1}$$

for the numerator and the lower bound

$$\|(\log X)^{C_2}/d_1, n_0d_0/d_1\| \geqslant \frac{n_0d_0}{d_1}$$

and the usual Taylor series expansion for the denominator. Then the above is

$$\sum_{(\log X)^{11C_2/16} < n_0 \leqslant Y} \frac{\tau(r_1)Q_1}{n_0c_0c_1(\log n_0)^{C_3}} \ll_{C_2,C_3} \frac{\tau(r_1)Q_1}{c_0c_1(\log\log X)^{C_3}}.$$

This concludes the proof. $\square$

Next we prove a similar result to Lemma 5.2.8:

Lemma 5.2.12. *Let $X \geqslant 3$, $C_1, C_2 > 0$ and fix some odd integers Q_0, Q_1 and some $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$. Suppose χ_0 and χ_1 are non-principal characters modulo Q_0 and Q_1 respectively. Then for any odd integers $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ and any fixed integers $1 \leqslant c_0, c_1, c_2, c_3 \leqslant (\log X)^{C_2/16}$, $1 \leqslant d_0, d_1, d_2, d_3 \leqslant (\log X)^{C_2/4}$ we have*

$$\sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4, \|n_0c_0,n_1c_1\|\cdot\|n_2c_2,n_3c_3\|\leqslant X \\ (5.2.3)}} \frac{\chi_0(n_0)\chi_1(n_1)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} \ll_{C_1,C_2,C_3} \frac{\tau(r_0)\tau(r_1)(Q_0+Q_1)X^2}{c_0c_1c_2c_3(\log X)(\log\log X)^{C_3}}$$

for any $C_3 > 0$, where the implied constant depends at most on C_1, C_2 and C_3. Here we use (5.2.3) *with $D = C_2$.*

Proof. Once more, let the sum be denoted by $H(X)$. Defining the parameter $Y = \exp((\log X)^{\epsilon})$ for some $0 < \epsilon < 1$ we use the hyperbola method to write

$$H(X) = H_0(X) + H_1(X) - H_2(X)$$

where $H_0(X), H_1(X)$ and $H_2(X)$ are

$$\sum\sum\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant Y \\ \|n_2c_2,n_3c_3\|\leqslant X/\|c_0n_0,c_1n_1\| \\ (5.2.3)}} \frac{\chi_0(n_0)\chi_1(n_1)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}, \qquad \sum\sum\sum\sum_{\substack{\|n_2c_2,n_3c_3\|\leqslant X/Y \\ \|n_0c_0,n_1c_1\|\leqslant X/\|n_2c_2,n_3c_3\| \\ (5.2.3)}} \frac{\chi_0(n_0)\chi_1(n_1)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}$$

and

$$\sum\sum\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant Y \\ \|n_2c_2,n_3c_3\|\leqslant X/Y \\ (5.2.3)}} \frac{\chi_0(n_0)\chi_1(n_1)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}$$

respectively. Following the same strategy as in the proof of Lemma 5.2.8, we may obtain

$$H_1, H_2(X) \ll_{C_1,C_2,C_3} \frac{\tau(r_0)\tau(r_1)Q_0Q_1X^2}{c_0c_1c_2c_3(\log X)^{4C_3}}.$$

Unlike in Lemma 5.2.8 however, $H_0(X)$ and $H_1(X)$ are not symmetric. Trying to use the same method as before for $H_0(X)$ will result in a bound of $X^2(\log X)^2$ which is too big. This is because we lose the information of the characters when we apply the triangle inequality and trivial bounds on the sum over n_2 and n_3. In order to maintain this information and obtain some saving over the character sum we will instead use Lemma 5.1.4 to provide an asymptotic for the inner sum. Using the trivial bound (5.2.2), we first add in the terms for which $\|n_2d_2, n_3d_3\| \leqslant (\log X)^{C_2}$ at the cost of a small error term:

$$H_0(X) = \sum\sum\sum\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant Y \\ \|n_2c_2,n_3c_3\|\leqslant X/\|c_0n_0,c_1n_1\| \\ \|n_0d_0,n_1d_1\|>(\log X)^{C_2} \\ \gcd(n_i,r_i)=1\ \forall 0\leqslant i\leqslant 3 \\ n_i\equiv q_i \bmod 8\ \forall 0\leqslant i\leqslant 3}} \frac{\chi_0(n_0)\chi_1(n_1)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} + O(X(\log X)^{2C_2}).$$

The lower bound $\|n_0d_0, n_1d_1\| > (\log X)^{C_2}$ cannot be removed as easily, and is in fact necessary for the result to hold. Using Lemma 5.1.4 on the sum over n_2 and n_3 we obtain that $H_0(X)$ is

$$= \frac{\mathfrak{S}_0(r_2)\mathfrak{S}_0(r_3)X^2}{\phi(8)^2c_2c_3} \sum\sum_{\substack{n_0c_0,n_1c_1\leqslant Y\\ \|n_0d_0,n_1d_1\|>(\log X)^{C_2}\\ \gcd(n_i,r_i)=1\ \forall 0\leqslant i\leqslant 1\\ n_i\equiv q_i \bmod 8\ \forall 0\leqslant i\leqslant 1}} \frac{\chi_0(n_0)\chi_1(n_1)}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)(\log(X/c_2c_3\|n_0c_0,n_1c_1\|))}$$
$$+ O\left(\frac{X^2(\log\log 3r_2r_3)^{3/2}}{c_2c_3(\log X)^2}\sum\sum_{n_0c_0,n_1c_1\leqslant Y}\frac{1}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)}\right).$$

Here we have to be careful with the logarithmic term in the denominator of the "main term". To deal with this we note that, since $c_2c_3\|n_0c_0,n_1c_1\| \ll Y(\log X)^{C_2/8}$, we may write:

$$\begin{aligned}\frac{1}{(\log(X/c_2c_3\|n_0c_0,n_1c_1\|))} &= \frac{1}{(\log X)} + O\left(\frac{(\log(c_2c_3\|n_0c_0,n_1c_1\|))}{(\log X)^2}\right)\\ &= \frac{1}{(\log X)} + O\left(\frac{1}{(\log X)^{2-\epsilon}}\right).\end{aligned}$$

Substituting this into the expression for $H_0(X)$ we will get

$$H_0(X) \ll \frac{X^2}{c_2c_3(\log X)}M_0(X) + O\left(\frac{X^2(\log\log 3r_2r_3)^{3/2}}{c_2c_3(\log X)^{2-\epsilon}}E_0(X)\right)$$

where $M_0(X)$ and $E_0(X)$ are

$$\sum\sum_{\substack{n_0c_0,n_1c_1\leqslant Y\\ \|n_0d_0,n_1d_1\|>(\log X)^{C_2}\\ \gcd(n_i,r_i)=1\ \forall 0\leqslant i\leqslant 1\\ n_i\equiv q_i \bmod 8\ \forall 0\leqslant i\leqslant 1}} \frac{\chi_0(n_0)\chi_1(n_1)}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)} \quad\text{and}\quad \sum\sum_{\substack{n_0c_0,n_1c_1\leqslant Y\\ \gcd(n_i,r_i)=1\ \forall 0\leqslant i\leqslant 1\\ n_i\equiv q_i \bmod 8\ \forall 0\leqslant i\leqslant 1}} \frac{1}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)}$$

respectively. For $M_0(X)$ we apply Lemma 5.2.10, giving:

$$M_0(X) \ll_{C_2,C_3} \frac{\tau(r_0)\tau(r_1)(Q_0+Q_1)}{c_0c_1(\log\log X)^{C_3}}.$$

For $E_0(X)$ we can just apply Lemma 5.2.1 to see

$$E_0(X) \ll \left(\frac{\log\log Y}{c_0c_1}\right) \ll_\epsilon \left(\frac{\log\log X}{c_0c_1}\right).$$

Substituting these bounds into our expression shows that:

$$\begin{aligned}H_0(X) &\ll_{C_2,C_3} \frac{\tau(r_0)\tau(r_1)(Q_0+Q_1)X^2}{c_0c_1c_2c_3(\log X)(\log\log X)^{C_3}} + \frac{X^2(\log\log X)(\log\log 3r_2r_3)^{3/2}}{c_0c_1c_2c_3(\log X)^{2-\epsilon}}\\ &\ll_{C_2,C_3} \frac{\tau(r_0)\tau(r_1)(Q_0+Q_1)X^2}{c_0c_1c_2c_3(\log X)(\log\log X)^{C_3}}\end{aligned}$$

concluding the proof. □

Lemma 5.2.12 is only effective when the conductors, Q_i are bounded by a power of $\log\log X$. Therefore we must turn to other methods to deal with the larger parts of such averages. In particular, we will use the large sieve results from §5.1.

Lemma 5.2.13. *Suppose a_n,b_m are any complex sequences supported on odd integers such that $|a_n| \leqslant 1$ and $|b_m| \leqslant 1$. Then for any $X \geqslant 3$, $C_1, C_2 > 1$ such that $(C_1 \log\log X)^{C_2} > 2$, and any fixed integers $1 \leqslant c_0, c_1 \leqslant (\log X)^{C_1/32}$ we have*

$$\sum_{(\log X)^{3C_1/4} < n_0 c_0 \leqslant X^{1/2}} \frac{1}{n_0^2 c_0^2 \tau(n_0)} \left| \sum_{\substack{(C_1 \log\log X)^{C_2} < m \leqslant (\log X)^{2C_1} \\ n_1 c_1 \leqslant n_0 c_0}} \sum \frac{a_m b_{n_1}}{m \tau(n_1)} \left(\frac{n_1}{m}\right) \right| \ll_{C_1, C_2} \frac{1}{c_0 c_1 (\log\log X)^{C_3}}$$

where $C_3 = C_2/2 - 1$ and the implied constant depends at most on C_1 and C_2.

Proof. For convenience we will write $Z = (\log X)^{C_1}$. Setting the sum on the left-hand side to be $T(X)$, we have

$$T(X) \leqslant T_1(X) + T_2(X) + T_3(X)$$

where

$$T_1(X) = \sum_{Z^{10} < n_0 \leqslant X^{1/2}/c_0} \frac{1}{n_0^2 c_0^2 \tau(n_0)} \left| \sum_{\substack{(\log Z)^{C_2} < m \leqslant Z^2 \\ n_1 c_1 \leqslant n_0 c_0}} \sum \frac{a_m b_{n_1}}{m \tau(n_1)} \left(\frac{n_1}{m}\right) \right|,$$

$$T_2(X) = \sum_{Z^{3/4}/c_0 < n_0 \leqslant Z^{10}} \frac{1}{n_0^2 c_0^2 \tau(n_0)} \left| \sum_{\substack{(\log Z)^{C_2} < m \leqslant Z^{1/10} \\ n_1 c_1 \leqslant n_0 c_0}} \sum \frac{a_m b_{n_1}}{m \tau(n_1)} \left(\frac{n_1}{m}\right) \right|,$$

$$T_3(X) = \sum_{Z^{3/4}/c_0 < n_0 \leqslant Z^{10}} \frac{1}{n_0^2 c_0^2 \tau(n_0)} \left| \sum_{\substack{Z^{1/10} < m \leqslant Z^2 \\ n_1 c_1 \leqslant n_0 c_0}} \sum \frac{a_m b_{n_1}}{m \tau(n_1)} \left(\frac{n_1}{m}\right) \right|.$$

For $T_1(X)$ we apply Corollary 5.1.3. Then $T_1(X)$ is

$$\ll \sum_{Z^{10} < n_0 \leqslant X^{1/2}/c_0} \frac{1}{n_0^2 c_0^2 \tau(n_0)} \left(\frac{n_0 c_0 (\log Z)}{c_1 (\log Z)^{C_2/2} (\log n_0 c_0/c_1)^{1/2}} + \frac{(n_0 c_0)^{3/5} Z (\log Z)^{1/2}}{c_1^{3/5} (\log n_0 c_0/c_1)^{1/4}} \right),$$

$$\ll \sum_{Z^{10} < n_0 \leqslant X^{1/2}/c_0} \left(\frac{(\log Z)}{n_0 c_0 c_1 (\log Z)^{C_2/2} (\log n_0 c_0/c_1)^{1/2} \tau(n_0)} \right)$$

$$+ \sum_{Z^{10} < n_0 \leqslant X^{1/2}/c_0} \left(\frac{Z (\log Z)^{1/2}}{(n_0 c_0)^{7/5} c_1^{3/5} (\log n_0 c_0/c_1)^{1/4} \tau(n_0)} \right).$$

Since $n_0 c_0/c_1 > Z^{10} c_0/c_1 > 2$ by assumption, the second sum is

$$\ll \frac{Z (\log Z)^{1/2}}{Z^4 c_0^{7/5} c_1^{3/5}} \ll \frac{(\log Z)^{1/2}}{Z^3 c_0^{7/5} c_1^{3/5}}.$$

Using similar methods used to compute $M(X)$ in Lemma 5.2.1, we can bound the first sum by

$$\ll \sum_{n_0 \leqslant X^{1/2}} \frac{1}{n_0(\log n_0 c_0/c_1)^{1/2}\tau(n_0)} \ll \log\log X.$$

Substituting the previous two bounds into $T_1(X)$ will give

$$T_1(X) \ll_{C_2} \frac{(\log\log X)}{c_0c_1(\log Z)^{C_2/2-1}} + \frac{(\log Z)^{1/2}}{Z^3 c_0^{7/5} c_1^{3/5}} \ll_{C_2} \frac{(\log\log X)}{c_0c_1(\log Z)^{C_2/2-1}},$$

using the fact that $Z > c_1^{2/5}$ to determine the dependence on c_0 and c_1. Using the same approach we can obtain the same bound for $T_2(X)$. Finally we deal with $T_3(X)$. Here it is better to apply Lemma 5.1.2 since the ranges of the inner double sum are of comparable size. Doing this we obtain,

$$\begin{aligned} T_3(X) &\ll \sum_{Z^{3/4}/c_0 < n_0 \leqslant Z^{10}} \frac{(\log Z)^{7/6}}{n_0 c_0 c_1 Z^{1/60}\tau(n_0)} \\ &\ll \frac{(\log Z)^{7/6}}{c_0c_1Z^{1/60}}. \end{aligned}$$

□

We are now ready to prove Proposition 2.2.8.

Proposition 5.2.14. *Let $X \geqslant 3$, $C_1, C_2 > 0$ be such that $(C_1\log\log X)^{C_2} > 2$. Fix some odd square-free integers $Q_1, Q_2, Q_3 \in \mathbb{N}$ such that $Q_1 \leqslant (\log\log X)^{C_2}$, and some $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$, $\tilde{\mathbf{q}} \in (\mathbb{Z}/8\mathbb{Z})^{*2}$. Suppose χ_2 and χ_3 are characters modulo Q_2 and Q_3 respectively. Fixing any odd integers $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ such that $\gcd(Q_1, r_0r_1r_2r_3) = \gcd(Q_2Q_3, r_2r_3) = 1$ and fixing any $1 \leqslant c_0, c_1, c_2, c_3 \leqslant (\log X)^{C_2/32}$, $1 \leqslant d_0, d_1, d_2, d_3 \leqslant (\log X)^{C_2/4}$ we define, for any $\mathbf{m} \in \mathbb{N}^2$*

$$H'(X,\mathbf{m}) = \sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4, \|n_0d_0,n_1d_1\|,\|n_2d_2,n_3d_3\| > (\log X)^{C_2} \\ \|n_0c_0,n_1c_1\|\cdot\|n_2m_2c_2,n_3m_3c_3\| \leqslant X \\ \gcd(n_i,2r_i)=1\ \forall\ 0\leqslant i\leqslant 3 \\ \mathbf{n}\equiv\mathbf{q} \bmod 8}} \frac{\psi_{m_2m_3}(n_2n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}.$$

Then,

$$\sum\sum_{\substack{\mathbf{m}\in\mathbb{N}^2, \|m_2,m_3\| \leqslant (\log X)^{C_2} \\ \gcd(m_i, 2Q_1Q_2Q_3r_i)=1\ \forall 2\leqslant i\leqslant 3 \\ \mathbf{m}\equiv\tilde{\mathbf{q}} \bmod 8 \\ Q_1m_2m_3\neq 1}} \frac{\mu^2(m_2m_3)\chi_2(m_2)\chi_3(m_3)}{\tau(m_2)\tau(m_3)} H'(X,\mathbf{m})$$

$$\ll_{C_2} \frac{\tau(r_0)\tau(r_1)X^2}{c_0c_1c_2c_3(\log X)(\log\log X)^{C_3}},$$

where $C_3 = C_2/2 - 1$ and where the implied constant depends at most on C_1 and C_2.

Proof. Denote $W = (\log X)^{C_2}$ and recall that $\psi_m(\cdot) = \left(\frac{\cdot}{m}\right)$. Call the average sum $S(X)$. Then we write

$$S(X) = S_1(X) + S_2(X)$$

where $S_1(X)$ and $S_2(X)$ are defined as $S(X)$ with the extra conditions $\|m_2, m_3\| \leqslant (\log W)^{C_3}$ and $(\log W)^{C_3} < \|m_2, m_3\| \leqslant W$ respectively. First we deal with $S_1(X)$. For each fixed $\mathbf{m}$ such that $\gcd(m_i, 2Q_1Q_2Q_3r_i) = 1$, $\mu^2(m_2m_3) = 1$ and $Q_1m_2m_3 \neq 1$, the sum $H'(X, \mathbf{m})$ is precisely of the form considered in Lemma 5.2.12. Furthermore, the range of m_2 and m_3 in $S_1(X)$ is small enough for this lemma to be effective. Thus,

$$\begin{aligned} S_1(X) &\ll_{C_2,C_3} \sum\sum_{\mathbf{m}\in\mathbb{N}^2, \|m_2,m_3\| \leqslant (\log W)^{C_3}} \frac{\mu^2(m_2m_3)\tau(r_0)\tau(r_1)Q_1m_2m_3X^2}{m_2m_3\tau(m_2)\tau(m_3)c_0c_1c_2c_3(\log X)(\log W)^{3C_3}} \\ &\ll_{C_2,C_3} \frac{\tau(r_0)\tau(r_1)Q_1X^2}{c_0c_1c_2c_3(\log X)(\log W)^{C_3}}. \end{aligned}$$

Next we deal with $S_2(X)$. We first use the hyperbola method on the $H'(X, \mathbf{m})$ terms again with the parameter $Y = \exp((\log X)^{\epsilon})$ to obtain

$$H'(X, \mathbf{m}) = H'_0(X, \mathbf{m}) + H'_1(X, \mathbf{m}) - H'_2(X, \mathbf{m})$$

where

$$H'_0(X, \mathbf{m}) = \sum\sum\sum\sum_{\substack{\|n_0c_0, n_1c_1\| \leqslant Y \\ \|n_2m_2c_2, n_3m_3c_3\| \leqslant X/\|n_0c_0, n_1c_1\| \\ (5.2.3)}} \frac{1}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} \left(\frac{n_0n_1}{Q_1m_2m_3}\right),$$

and $H'_1(X, \mathbf{m}), H'_2(X, \mathbf{m})$ are defined as $H'_0(X, \mathbf{m})$ with the height conditions $\{\|n_0c_0, n_1c_1\| \leqslant Y,\ \|n_2m_2c_2, n_3m_3c_3\| \leqslant X/\|n_0c_0, n_1c_1\|\}$ replaced with

$$\{\|n_2m_2c_2, n_3m_2c_3\| \leqslant X/Y,\ \|n_0c_0, n_1c_1\| \leqslant X/\|n_2m_2c_2, n_3m_2c_3\|\}$$

and

$$\{\|n_0c_0, n_1c_1\| \leqslant Y,\ \|n_2m_2c_2, n_3m_2c_3\| \leqslant X/Y\}$$

respectively. $H'_1(X, \mathbf{m})$ and $H'_2(X, \mathbf{m})$ may be dealt with using Lemma 5.1.4 since the ranges of the sums over n_0 and n_1 are guaranteed to be exponential in the size of $Q_1m_2m_3$. The conditions on Q_1, m_2 and m_3 guarantee that $\psi_{Q_1m_2m_3}$ is non-principal. Then Lemma 5.1.4 will give arbitrary logarithmic saving in the sums over n_0 and n_1 so that, upon summing over n_2 and n_3 we obtain

$$H'_1(X, \mathbf{m}), H'_2(X, \mathbf{m}) \ll_{C_1,C_2} \frac{\tau(r_0)\tau(r_1)Q_1^2m_2^2m_3^2X^2}{m_2m_3c_0c_1c_2c_3(\log X)^{6C_2}} \ll_{C_1,C_2} \frac{Q_1^2m_2m_3X^2}{c_0c_1c_2c_3(\log X)^{5C_2}}.$$

See the bounds for (5.2.4) and (5.2.6) in Lemma 5.2.8 for analogous proofs. Summing these over trivially over the m_i will then give

$$\sum\sum_{\substack{\mathbf{m}\in\mathbb{N}^2 \\ (\log W)^{C_3} < \|m_2,m_3\| \leqslant W \\ \gcd(m_i, 2Q_1Q_2Q_3r_i)=1\ \forall 2\leqslant i\leqslant 3 \\ \mathbf{m}\equiv\tilde{\mathbf{q}} \bmod 8}} \frac{\mu^2(m_2m_3)\chi_2(m_2)\chi_3(m_3)}{\tau(m_2)\tau(m_3)} H'_1(X, \mathbf{m}) \ll_{C_2,C_3} \frac{\tau(r_0)\tau(r_1)X^2}{c_0c_1c_2c_3(\log X)^{C_2}}$$

and likewise,

$$\sum_{\substack{\mathbf{m}\in\mathbb{N}^2 \\ (\log W)^{C_3}<\|m_2,m_3\|\leqslant W \\ \gcd(m_i,2Q_1Q_2Q_3r_i)=1\ \forall 2\leqslant i\leqslant 3 \\ \mathbf{m}\equiv\tilde{\mathbf{q}} \bmod 8}}\sum \frac{\mu^2(m_2m_3)\chi_2(m_2)\chi_3(m_3)}{\tau(m_2)\tau(m_3)} H_2'(X,\mathbf{m}) \ll_{C_2,C_3} \frac{\tau(r_0)\tau(r_1)X^2}{c_0c_1c_2c_3(\log X)^{C_2}}$$

since $W=(\log X)^{C_2}$. We now turn to $H_0'(X,\mathbf{m})$. We add in the terms for which $\|n_2d_2,n_3d_3\|\leqslant(\log X)^{C_2}$ at the cost of a small error term (see, for example (5.2.2)):

$$H_0'(X,\mathbf{m}) = \sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant Y \\ \|n_2m_2c_2,n_3m_3c_3\|\leqslant X/\|n_0c_0,n_1c_1\| \\ \|n_0d_0,n_1d_1\|>W \\ \gcd(n_i,r_i)=1\ \forall 0\leqslant i\leqslant 3 \\ n_i\equiv q_i \bmod 8\ \forall 0\leqslant i\leqslant 3}}\sum\sum\sum \frac{1}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}\left(\frac{n_0n_1}{Q_1m_2m_3}\right)+O(X(\log X)^{2C_2}).$$

Next we apply Lemma 5.1.4 for non-principal characters with $Q=1$ and $C=3/2$ on the sum over n_2 and n_3, allowing us to preserve the Jacobi symbol. Upon using the standard Taylor series method to the logarithmic factors (as in Lemma 5.2.12 since $m_2m_3c_2c_3\|n_0c_0,n_1c_1\|\leqslant Y(\log X)^{3C_1}=o(X)$), we obtain

$$H_0'(X,\mathbf{m}) = H_{00}'(X,\mathbf{m}) + O(H_{01}'(X,\mathbf{m}))$$

where

$$H_{00}'(X,\mathbf{m}) = \frac{\mathfrak{S}_0(r_2)\mathfrak{S}_0(r_3)X^2}{\phi(8)^2m_2m_3c_2c_3(\log X)}\left(\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant Y \\ \|n_0d_0,n_1d_1\|>W \\ \gcd(n_i,r_i)=1\ \forall 0\leqslant i\leqslant 1 \\ n_i\equiv q_i \bmod 8\ \forall 0\leqslant i\leqslant 1}}\sum \frac{\left(\frac{n_0n_1}{Q_1m_2m_3}\right)}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)}\right),$$

and

$$H_{01}'(X,\mathbf{m}) = \frac{X^2}{m_2m_3c_2c_3(\log X)^{2-\epsilon}}\sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant Y \\ \|n_0d_0,n_1d_1\|>W}}\sum \frac{1}{\|n_0c_0,n_1c_1\|^2\tau(n_0)\tau(n_1)}.$$

Note that we have used $r_i\ll(\log X)^{C_1}$ to absorb $(\log\log 3r_2r_3)^{3/2}$ and $\tau(r_2)\tau(r_3)$ into $(\log X)^{\epsilon}$. Let us first deal with the $H_{01}'(X,\mathbf{m})$. By Lemma 5.2.1 the sum is $O\left(\frac{\log\log X}{c_0c_1}\right)$, so that overall,

$$H_{01}'(X,\mathbf{m}) \ll \frac{X^2(\log\log X)}{c_0c_1c_2c_3m_2m_3(\log X)^{2-\epsilon}}.$$

Summing over $\mathbf{m}$ will give

$$\ll \sum_{\|m_2,m_3\|\leqslant W}\sum |H_{01}'(X,\mathbf{m})| \ll_{C_2,C_3} \frac{X^2(\log\log X)^4}{c_0c_1c_2c_3(\log X)^{2-\epsilon}}.$$

To deal with $H_{00}'(X,\mathbf{m})$ we use the averaging over $\mathbf{m}$. Specifically, we are left to bound

$$S_{200}(X) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^2 \\ (\log W)^{C_2}<\|m_2,m_3\|\leqslant W \\ \gcd(m_i,2Q_1Q_2Q_3r_i)=1\ \forall 2\leqslant i\leqslant 3 \\ \mathbf{m}\equiv\tilde{\mathbf{q}} \bmod 8}}\sum \frac{\mu^2(m_2m_3)\chi_2(m_2)\chi_3(m_3)}{\tau(m_2)\tau(m_3)} H_{00}'(X,\mathbf{m}).$$

This is bounded by

$$\frac{X^2}{c_2c_3(\log X)}\left|\sum_{\substack{\mathbf{m}\in\mathbb{N}^2,\mathbf{n}\in\mathbb{N}^4\\(\log W)^C<\|m_2,m_3\|\leqslant W\\\gcd(m_i,2Q_1Q_2Q_3r_i)=1\ \forall 2\leqslant i\leqslant 3\\\mathbf{m}\equiv\tilde{\mathbf{q}}\bmod 8}}\sum\ \sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant Y\\\|n_0d_0,n_1d_1\|>W\\\gcd(n_i,r_i)=1\ 0\leqslant i\leqslant 1\\n_i\equiv q_i\bmod 8\ 0\leqslant i\leqslant 1}}\sum\frac{\mu^2(m_2m_3)\chi_2(m_2)\chi_3(m_3)\left(\frac{n_0n_1}{Q_1m_2m_3}\right)}{m_2m_3\|n_0c_0,n_1c_1\|^2\tau(m_2m_3)\tau(n_0)\tau(n_1)}\right|$$

To begin we will write $m=m_2m_3$ and define

$$\bar{\tau}(m)=\sum_{\substack{m_2m_3=m,\|m_2,m_3\|>(\log W)^C\\\gcd(m_i,2Q_1Q_2Q_3r_i)=1\ \forall 2\leqslant i\leqslant 3\\m_i\equiv\tilde{q}_i\bmod 8\ \forall 2\leqslant i\leqslant 3}}\sum\chi_2(m_2)\chi_3(m_3).$$

Then, by rewriting, we see that $S_{200}(X)$ is

$$\ll\frac{X^2}{c_2c_3(\log X)}\left|\sum_{(\log W)^C<m\leqslant W^2}\ \sum_{\substack{\|n_0c_0,n_1c_1\|\leqslant Y\\\|n_0d_0,n_1d_1\|>W\\\gcd(n_i,r_i)=1\ \forall 0\leqslant i\leqslant 1\\n_i\equiv q_i\bmod 8\ \forall 0\leqslant i\leqslant 1}}\sum\frac{\mu^2(m)\bar{\tau}(m)}{m\|n_0c_0,n_1c_1\|^2\tau(m)\tau(n_0)\tau(n_1)}\left(\frac{n_0n_1}{Q_1m}\right)\right|.$$

Next we split the sum over n_0 and n_1 into a region where $n_1c_1\leqslant n_0c_0$ and a second region where $n_0c_0<n_1c_1$. The sum over each region will be of the same order, so that $S_{200}(X)$ becomes

$$\ll\frac{X^2}{c_2c_3(\log X)}\left|\sum_{\substack{W^{3/4}/c_0<n_0\leqslant Y/c_0\\\gcd(n_0,r_0)=1\\n_0\equiv q_0\bmod 8}}\frac{1}{n_0^2c_0^2\tau(n_0)}\sum_{(\log W)^C<m\leqslant W^2}\ \sum_{\substack{n_1c_1\leqslant n_0c_0\\\|n_0d_0,n_1d_1\|>W\\\gcd(n_1,r_1)=1\\n_1\equiv q_1\bmod 8}}\frac{\mu^2(m)\bar{\tau}(m)}{m\tau(m)\tau(n_1)}\left(\frac{n_0n_1}{Q_1m}\right)\right|.$$

Here we note that, although $n_0c_0\geqslant n_1c_1$, we may still have $n_0d_0<n_1d_1$; however, this can only occur when

$$\frac{c_1}{c_0}\leqslant\frac{n_0}{n_1}<\frac{d_0}{d_1}.$$

If this were the case, then we use the fact that $\|n_0d_0,n_1d_1\|>W$ to assert

$$\frac{Wc_1}{c_0}<\frac{n_1d_1c_1}{c_0}\leqslant n_0d_1,$$

from which it follows that

$$n_0c_0\geqslant\frac{Wc_1}{d_1}\geqslant W^{3/4},$$

giving the lower bound in the sum over n_0. The bound $\|n_0d_0,n_1d_1\|>W$ is maintained as a condition on n_1. Next we define

$$a_m=\frac{\mu^2(m)\bar{\tau}(m)}{\tau(m)}\left(\frac{n_0}{m}\right),$$

and

$$b_{n_1} = \mathbb{1}(\gcd(n_1, r_1) = 1)\mathbb{1}(n_1 \equiv q_1 \bmod 8)\mathbb{1}(\|n_0 d_0, n_1 d_1\| > W) \left(\frac{n_1}{Q_1}\right).$$

Then, using the triangle inequality,

$$S_{200}(X) \ll \frac{X^2}{c_2 c_3 (\log X)} \sum_{W^{3/4}/c_0 < n_0 \leqslant Y/c_0} \frac{1}{n_0^2 c_0^2 \tau(n_0)} \left| \sum\sum_{\substack{(\log W)^{C_2} < m \leqslant W^2 \\ n_1 \leqslant n_0 c_0 / c_1}} \frac{a_m b_{n_1}}{m \tau(n_1)} \left(\frac{n_1}{m}\right) \right|.$$

Finally, by applying Lemma 5.2.13, we obtain

$$S_{200}(X) \ll_{C_2, C_3} \frac{X^2}{c_0 c_1 c_2 c_3 (\log X)(\log\log X)^{9C_3/20-2}},$$

which is sufficient. □

5.3 Character sums over hyperbolic regions II

In this section we deal sums where characters are arranged in a different manner with respect to the hyperbolic height conditions. The type of sum considered is of the form

$$\sum\sum\sum\sum_{\|n_0 n_1, n_2 n_3\| \leqslant X} \frac{\chi(n_0 n_2)\psi(n_1 n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} \tag{5.3.1}$$

where χ and ψ are some Dirichlet characters. Just as in §5.2 we have three cases:

(a) Main Term: both χ and ψ are principal;

(b) Small Conductor – Symmetric Hyperbola Method: both χ and ψ are non-principal;

(c) Small Conductor – Non-symmetric Hyperbola Method: only one of χ or ψ are non-principal

The main term in this case may be seen to be of order X^2 but in our counting problem however, expressions like this come from the contribution from fibres of points $[y_0; y_1; y_2; y_3]$ where $-y_0 y_2$ or $-y_0 y_3$ are squares, which are excluded. To see this in practice see §2.5 and §2.8. We set up the preliminaries for this in the first subsection and handle the symmetric and non-symmetric cases using the results of §5.1.

5.3.1 Sums Over Fixed Conductors

First we prove Lemma 2.2.9, which is the technical result in having the X^2 term vanish is the following:

Lemma 5.3.1. *Let $X \geqslant 3$, $C_1, C_2 > 0$. Suppose χ_0, χ_1, χ_2 and χ_3 are Dirichlet characters modulo 8 such that χ_i and χ_j are non-principal for some pair $(i,j) \in \{0,1\} \times \{2,3\}$. Then for any odd integers $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ and any integers $1 \leqslant c_{01}, c_{23}, M \leqslant (\log X)^{C_2}$ we have,*

$$\sum\sum\sum\sum_{\substack{\|n_0n_1c_{01}, n_2n_3c_{23}\| \cdot M \leqslant X \\ \gcd(n_i, 2r_i)=1 \forall\ 0 \leqslant i \leqslant 3}} \frac{\chi_0(n_0)\chi_2(n_2)\chi_1(n_1)\chi_3(n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} \ll_{C_2} \frac{\tau(r_0)\tau(r_1)\tau(r_2)\tau(r_3)X^2}{c_{01}c_{23}M^2(\log X)}$$

where the implied constant depends at most on C_2.

Proof. We write the sum under consideration as $H_{01}(X)H_{23}(X)$ where,

$$H_{01}(X) = \sum\sum_{\substack{n_0n_1 \leqslant X/c_{01}M, \\ \gcd(n_i, 2r_i)=1 \forall\ i \in \{0,1\}}} \frac{\chi_0(n_0)\chi_1(n_1)}{\tau(n_0)\tau(n_1)} \quad \text{and} \quad H_{23}(X) = \sum\sum_{\substack{n_2n_3 \leqslant X/c_{23}M \\ \gcd(n_i, 2r_i)=1 \forall\ i \in \{2,3\}}} \frac{\chi_2(n_2)\chi_3(n_3)}{\tau(n_2)\tau(n_3)}.$$

These are symmetric and thus we will focus on $H_{01}(X)$ and note that any bound for this may also be obtained for $H_{23}(X)$. Further, we will assume without loss in generality that χ_0 and χ_2 are non-principal. Letting $Y = \exp((\log X)^{1/6})$ we use the classical hyperbola method to write

$$\begin{aligned} H_{01}(X) = & \sum_{\substack{n_0 \leqslant Y \\ \gcd(n_0, 2r_0)=1}} \frac{\chi_0(n_0)}{\tau(n_0)} \sum_{\substack{n_1 \leqslant X/n_0c_{01}M \\ \gcd(n_1, 2r_1)=1}} \frac{\chi_1(n_1)}{\tau(n_1)} + \sum_{\substack{n_1 \leqslant X/c_{01}MY \\ \gcd(n_1, 2r_1)=1}} \frac{\chi_1(n_1)}{\tau(n_1)} \sum_{\substack{n_0 \leqslant X/n_1c_{01}M \\ \gcd(n_0, 2r_0)=1}} \frac{\chi_0(n_0)}{\tau(n_0)} \\ & - \sum\sum_{\substack{n_0 \leqslant Y, n_1 \leqslant X/c_{01}MY \\ \gcd(n_i, 2r_i)=1}} \frac{\chi_0(n_0)}{\tau(n_0)} \frac{\chi_1(n_1)}{\tau(n_1)}. \end{aligned}$$

For the second and third sums we use Lemma 5.1.4 for the sum over $\chi_0(n_0)$ with $Q = 8$ and $C = 2025$, noting that for the second sum we use,

$$\frac{1}{(\log X/n_1c_{01}M)^{2025}} \ll \frac{1}{(\log Y)^{2025}} \ll \frac{1}{(\log X)^{2025/6}}$$

since $n_1 \leqslant X/Yc_{01}M$. In each case, upon summing over n_1, we obtain

$$\ll_C \frac{\tau(r_0)X}{c_{01}M(\log X)^{2025/6}}.$$

We are left with

$$H_{01}(X) = \sum_{\substack{n_0 \leqslant Y \\ \gcd(n_0, 2r_0)=1}} \frac{\chi_0(n_0)}{\tau(n_0)} \sum_{\substack{n_1 \leqslant X/n_0c_{01}M \\ \gcd(n_1, 2r_1)=1}} \frac{\chi_1(n_1)}{\tau(n_1)} + O_C\left(\frac{\tau(r_0)X}{c_{01}M(\log X)^{2025/6}}\right).$$

This error term is sufficient since $\tau(r_0) \ll_{C_1} (\log X)^{1/6}$. Now, if χ_1 is non-principal then this remaining sum may be handled in the same way as the second. We therefore assume that it is the principal character modulo 8. Then, given the height conditions, we may use Lemma 5.1.4 for principal characters modulo 8 on the inner sum over n_1 with $C = 2025$. This will give $H_{01}(X)$ equal to

$$\frac{\mathfrak{S}_0(r_1)X}{c_{01}M}\sum_{\substack{n_0\leqslant Y\\ \gcd(n_0,2r_0)=1}}\frac{\chi_0(n_0)}{n_0\tau(n_0)(\log X/n_0c_{01}M)^{1/2}}+O\left(\frac{X(\log\log 3r_1)^{3/2}}{c_{01}M(\log X)^{3/2}}\sum_{\substack{n_0\leqslant Y\\ \gcd(n_0,2r_0)=1}}\frac{1}{n_0\tau(n_0)}\right).$$

Using the bound

$$\sum_{\substack{n_0\leqslant Y\\ \gcd(n_0,2r_0)=1}}\frac{1}{n_0\tau(n_0)}\ll(\log Y)^{1/2}\ll(\log X)^{1/2}$$

and the typical Taylor series expansion

$$\frac{1}{(\log X/n_0c_{01}M)^{1/2}}=\frac{1}{(\log X)^{1/2}}+O\left(\frac{1}{(\log X)^{4/3}}\right)$$

(the latter a result of $n_0c_{01}M\leqslant Y(\log X)^{2C_2}$), this becomes

$$H_{01}(X)\ll\frac{X}{c_{01}M(\log X)^{1/2}}\left|\sum_{\substack{n_0\leqslant Y\\ \gcd(n_0,2r_0)=1}}\frac{\chi_0(n_0)}{n_0\tau(n_0)}\right|+O\left(\frac{X(\log\log 3r_1)^{3/2}}{c_{01}M(\log X)^{5/4}}\right).$$

Now let us consider the remaining sum over n_0. To do this first consider the Dirichlet series

$$D(s,\chi_0)=\sum_{\substack{n=1\\ \gcd(n,2r)=1}}^{\infty}\frac{\chi_0(n)}{n^s\tau(n)}.$$

Using the Euler product we may write

$$D(s,\chi_0)=P(s,r,\chi_0)R(s,\chi_0)L(s,\chi_0)^{1/2}$$

where $P(s,r,\chi_0)$ and $R(s,\chi_0)$ are

$$\prod_{\substack{p\text{ prime}\\ p\mid r}}\left(1+\sum_{j=1}^{\infty}\frac{\chi_0(p)^j}{(j+1)p^{js}}\right)^{-1}\quad\text{and}\quad\prod_{p\text{ prime}}\left(1+\sum_{j=1}^{\infty}\frac{\chi_0(p)^j}{(j+1)p^{js}}\right)\left(1-\frac{\chi_0(p)}{p^s}\right)^{1/2}$$

respectively, the second product converging absolutely when $\Re(s)>1/2$, and

$$L(s,\chi_0)=\prod_{p\text{ prime}}\left(1-\frac{\chi_0(p)}{p^s}\right)^{-1}$$

is the L-function for the character χ_0. It follows from this decomposition that the Dirichlet series converges whenever $\Re(s)>1/2$ and $L(s,\chi_0)\neq 0$. Using the zero free region for L-functions of primitive characters and Siegel's Theorem it follows, in particular, that $D(1,\chi_0)$ converges and that

$$D(1,r,\chi_0)=P(1,r,\chi_0)R(1,\chi_0)L(1,\chi_0)^{1/2}\ll\tau(r)$$

Therefore,

$$\sum_{\substack{n_0 \leqslant Y \\ \gcd(n_0, 2r_0)=1}} \frac{\chi_0(n_0)}{n_0 \tau(n_0)} \ll \tau(r_0)$$

from which it follows that

$$H_{01}(X) \ll \frac{\tau(r_0)\tau(r_1)X}{c_{01} M (\log X)^{1/2}}.$$

Similarly,

$$H_{23}(X) \ll \frac{\tau(r_2)\tau(r_3)X}{c_{23} M (\log X)^{1/2}}.$$

□

5.3.2 Small Conductor – Symmetric Hyperbola Method

As with case (b) of §5.2, we only need to apply Lemmas 5.1.4 and the hyperbola method appropriately.

Lemma 5.3.2. *Let $X \geqslant 3$, $C_1, C_2 > 0$, Q_{02}, Q_{13} be odd integers and $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$. Suppose χ_{02}, χ_{13} are non-principal Dirichlet characters modulo Q_{02}, Q_{13} respectively. Then for any odd integers $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ such that $\gcd(Q_{ij}, 2r_i r_j) = 1$ whenever $(i,j) \in \{(0,2),(1,3)\}$ and any integers $1 \leqslant c_{01}, c_{23}, M \leqslant (\log X)^{C_2}$ we have,*

$$\sum\sum\sum\sum_{\substack{\|n_0 n_1 c_{01}, n_2 n_3 c_{23}\| \cdot M \leqslant X \\ \gcd(n_i, 2r_i)=1 \forall\, 0 \leqslant i \leqslant 3 \\ n_i \equiv q_i \bmod 8 \forall\, 0 \leqslant i \leqslant 3}} \frac{\chi_{02}(n_0 n_2)\chi_{13}(n_1 n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)} \ll_{C_2, C_3} \frac{Q_{02} Q_{13} X^2}{c_{01} c_{23} M^2 (\log X)^{C_3}}.$$

for any $C_3 > 0$, where the implied constant depends at most on the C_i.

Proof. Write the sum under consideration as $H_{01}(X)H_{23}(X)$ where,

$$H_{01}(X) = \sum\sum_{\substack{n_0 n_1 \leqslant X/c_{01}M, \\ \gcd(n_i, 2r_i)=1 \forall\, i \in \{0,1\} \\ n_i \equiv q_i \bmod 8 \forall\, i \in \{0,1\}}} \frac{\chi_{02}(n_0)\chi_{13}(n_1)}{\tau(n_0)\tau(n_1)} \text{ and } H_{23}(X) = \sum\sum_{\substack{n_2 n_3 \leqslant X/c_{23}M \\ \gcd(n_i, 2r_i)=1 \forall\, i \in \{2,3\} \\ n_i \equiv q_i \bmod 8 \forall\, i \in \{2,3\}}} \frac{\chi_{02}(n_2)\chi_{13}(n_3)}{\tau(n_2)\tau(n_3)}.$$

These sums are symmetric and so we focus on $H_{01}(X)$. The hyperbola method gives

$$\begin{aligned} H_{01}(X) = &\sum_{\substack{n_0 \leqslant X^{1/2}/c_{01}^{1/2} M^{1/2} \\ \gcd(n_0, 2r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \frac{\chi_{02}(n_0)}{\tau(n_0)} \sum_{\substack{n_1 \leqslant X/n_0 c_{01} M \\ \gcd(n_1, 2r_1)=1 \\ n_1 \equiv q_1 \bmod 8}} \frac{\chi_{13}(n_1)}{\tau(n_1)} \\ &+ \sum_{\substack{n_1 \leqslant X^{1/2}/c_{01}^{1/2} M^{1/2} \\ \gcd(n_1, 2r_1)=1 \\ n_1 \equiv q_1 \bmod 8}} \frac{\chi_{13}(n_1)}{\tau(n_1)} \sum_{\substack{n_0 \leqslant X/n_1 c_{01} M \\ \gcd(n_0, 2r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \frac{\chi_{02}(n_0)}{\tau(n_0)} \\ &- \sum\sum_{\substack{n_0, n_1 \leqslant X^{1/2}/c_{01}^{1/2} M^{1/2} \\ \gcd(n_i, 2r_i)=1 \\ n_i \equiv q_i \bmod 8}} \frac{\chi_{02}(n_0)}{\tau(n_0)} \frac{\chi_{13}(n_1)}{\tau(n_1)}. \end{aligned}$$

The last of these sums may be written as the product of the sum over n_0 and the sum over n_1. Since both χ_{02} and χ_{13} are non-principal, we use Lemma 5.1.4 on each part of this product and multiply the results to show that the contribution from this sum is

$$\ll_{C_1,C_2} \frac{\tau(r_0)\tau(r_1)Q_{02}Q_{13}X}{c_{01}M(\log X/c_{01}M)^{2C_3+1}} \ll_{C_2,C_3} \frac{Q_{02}Q_{13}X}{c_{01}M(\log X)^{C_3+1}}.$$

This last bound is obtained using the assumption $c_{01}, M \leqslant (\log X)^{C_2}$. The first two sums in the expression for H_{01} are dealt with in the same way since both characters are non-principal. Looking at the first sum, we use Lemma 5.1.4 for the sum over n_1. This leads to

$$\ll_{C_3} \sum_{n_0 \leqslant X^{1/2}/c_{01}^{1/2}M^{1/2}} \frac{\tau(r_1)Q_{13}X}{n_0 c_{01}M(\log X/n_0c_{01}M)^{2C_3+2}}.$$

Now, since $n_0 \leqslant X^{1/2}/c_{01}^{1/2}M^{1/2}$ and $c_{01}, M \leqslant (\log X)^{C_2}$ it follows that the first sum is then bounded by

$$\ll_{C_2,C_3} \sum_{n_0 \leqslant X^{1/2}} \frac{\tau(r_1)Q_{13}X}{n_0 c_{01}M(\log X)^{2C_3+2}} \ll_{C_2,C_3} \frac{Q_{13}X}{c_{01}M(\log X)^{C_3+1}}.$$

Thus

$$H_{01}(X) \ll_{C_2,C_3} \frac{Q_{02}Q_{13}X}{c_{01}M(\log X)^{C_3+1}}.$$

Putting this together with the trivial bound $\frac{X(\log X)}{c_{23}M}$ for $H_{23}(X)$ gives the result. $\square$

We conclude this subsection with Proposition 2.2.10. Its proof is a direct application of Lemma 5.3.2.

Proposition 5.3.3. *Let $X \geqslant 3$, $C_1, C_2, C_3 > 0$, let Q_{02}, Q_{13} be odd integers and take $\mathbf{q} \in (\mathbb{Z}/8\mathbb{Z})^{*4}$, $\tilde{\mathbf{q}} \in (\mathbb{Z}/8\mathbb{Z})^{*2}$. Let $1 \leqslant r_0, r_1, r_2, r_3 \leqslant (\log X)^{C_1}$ be odd integers such that $\gcd(Q_{ij}, 2r_ir_j) = 1$ for $i \in \{(0,2),(1,3)\}$ and any $1 \leqslant c_0, c_1, c_2, c_3 \leqslant (\log X)^{C_2}$. Define, for any $\mathbf{m} \in \mathbb{N}^4$,*

$$H''(X,\mathbf{m}) = \sum\sum\sum\sum_{\substack{\mathbf{n}\in\mathbb{N}^4 \\ \|n_0n_1c_0, n_2n_3c_1\|\cdot\|m_0m_1c_2, m_2m_3c_3\| \leqslant X \\ \gcd(n_i,r_i)=1\ \forall 0\leqslant i\leqslant 3 \\ n_i \equiv q_i \bmod 8\ \forall 0 \leqslant i \leqslant 3}} \frac{\psi_{Q_{02}m_0m_2}(n_0n_2)\psi_{Q_{13}m_1m_2}(n_1n_3)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}.$$

Then

$$\sum\sum\sum\sum_{\substack{\mathbf{m}\in\mathbb{N}^4, \|m_0,m_1,m_2,m_3\| \leqslant (\log X)^{C_3} \\ \gcd(m_0m_2, 2Q_{02}r_0r_2) = \gcd(m_1m_3, Q_{13}r_1r_3) = 1 \\ Q_{02}m_0m_2 \neq 1 \text{ and } Q_{13}m_1m_3 \neq 1 \\ \mathbf{m} \equiv \tilde{\mathbf{q}} \bmod 8}} \frac{\mu^2(2m_0m_1m_2m_3)|H''(X,\mathbf{m})|}{\tau(m_0)\tau(m_1)\tau(m_2)\tau(m_3)}$$
$$\ll_{C_1,C_2,C_3,C_4} \frac{Q_{02}Q_{13}X^2}{c_0c_1c_2c_3(\log X)^{C_4}}.$$

for any $C_4 > 0$ where the implied constant depends at most on the C_i.

5.3.3 Small Conductor – Non-Symmetric Hyperbola Method

As in the analogous part of §5.2 the asymmetry of these sums leads to difficulty. In the previous case the lower bounds on some of the variables and averaging over the characters with the neutraliser large sieve led to saving over the desired bound. In this case we will likewise have to exploit the averaging over the conductor to obtain a valid bound, but our methods will differ as the convex factors $\frac{1}{n_0n_1}$ and $\frac{1}{\|n_0,n_1\|^2}$ switch roles from §5.2. The argument begins in a similar fashion to that of Lemma 5.3.1, but deviates in order to handle the need to average over our conductors. Assume that the character χ_{02} is non-principal with conductor Q_{02} and consider

$$A(X) = \sum\sum\sum\sum_{\substack{\|n_0n_1c_{01},n_2n_3c_{23}\|\cdot M\leqslant X\\ \gcd(n_i,r_i)=1\forall\ 0\leqslant i\leqslant 3\\ n_i\equiv q_i \bmod 8\forall\ 0\leqslant i\leqslant 3}} \frac{\chi_{02}(n_0n_2)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}. \tag{5.3.2}$$

Define also the Dirichlet series

$$\widetilde{L}_r(1,\chi) = \sum_{\substack{n=1\\ \gcd(n,r)=1\\ n\equiv q \bmod 8}}^{\infty} \frac{\chi(n)}{n\tau(n)}$$

for any odd integer r, any $q\in(\mathbb{Z}/8\mathbb{Z})^*$ and any non-principal Dirichlet character χ. Our first step is to prove the following:

Lemma 5.3.4. *Let $X\geqslant 3$, $C_1,C_2,C_3,C_4,C_5>0$ and fix some $\mathbf{q}\in(\mathbb{Z}/8\mathbb{Z})^{*4}$. Let $2<Q_{02}\leqslant(\log X)^{C_1}$ and $1\leqslant r_0,r_1,r_2,r_3\leqslant(\log X)^{C_2}$ be odd integers such that $\gcd(Q_{02},2r_0r_2)=1$. Suppose χ_{02} is a non-principal character modulo Q_{02}. Then for any integers $1\leqslant c_{01},c_{23}\leqslant(\log X)^{C_3}$, $1\leqslant M\leqslant(\log X)^{C_4}$ we have*

$$\begin{aligned}A(X) =&\frac{\mathfrak{S}_0(2r_1)\mathfrak{S}_0(2r_3)X^2}{16c_{01}c_{23}M^2\log X}\left(\sum\sum_{\chi,\chi' \bmod 8}\overline{\chi}(q_0)\overline{\chi'}(q_2)\widetilde{L}_{r_0}(1,\chi_{02}\chi)\widetilde{L}_{r_2}(1,\chi_{02}\chi')\right)\\ &+O_{C_1,C_2,C_3,C_4,C_5}\left(\frac{X^2}{c_{01}c_{23}M^2(\log X)^{3/2}}\sum_{\chi \bmod 8}(|\widetilde{L}_{r_0}(1,\chi_{02}\chi)|+|\widetilde{L}_{r_2}(1,\chi_{02}\chi)|)\right)\end{aligned}$$

where the implied constant depends at most on the C_i.

Proof. We write $A(X)$ as the product of two hyperbolic sums $H_{01}(X)H_{23}(X)$, where

$$H_{01}(X) = \sum\sum_{\substack{n_0n_1\leqslant X/c_{01}M\\ \gcd(n_i,r_i)=1\forall\ 0\leqslant i\leqslant 1\\ n_i\equiv q_i \bmod 8\forall\ 0\leqslant i\leqslant 1}} \frac{\chi_{02}(n_0)}{\tau(n_0)\tau(n_1)} \ \text{ and }\ H_{23}(X) = \sum\sum_{\substack{n_2n_3\leqslant X/c_{23}M\\ \gcd(n_i,r_i)=1\forall\ 2\leqslant i\leqslant 3\\ n_i\equiv q_i \bmod 8\forall\ 2\leqslant i\leqslant 3}} \frac{\chi_{02}(n_2)}{\tau(n_2)\tau(n_3)}.$$

We look at $H_{01}(X)$. Defining the parameter $Y=\exp((\log X)^{1/3})$, we use the standard hyperbola method we deduce that

$$H_{01}(X) = H'_{01}(X)+H''_{01}(X)-H'''_{01}(X)$$

where

$$H'_{01}(X) = \sum_{\substack{n_0 \leqslant Y \\ \gcd(n_0,r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \frac{\chi_{02}(n_0)}{\tau(n_0)} \sum_{\substack{n_1 \leqslant X/n_0 c_{01} M \\ \gcd(n_1,r_1)=1 \\ n_1 \equiv q_1 \bmod 8}} \frac{1}{\tau(n_1)},$$

$$H''_{01}(X) = \sum_{\substack{n_1 \leqslant X/c_{01} MY \\ \gcd(n_1,r_1)=1 \\ n_1 \equiv q_1 \bmod 8}} \frac{1}{\tau(n_1)} \sum_{\substack{n_0 \leqslant X/n_1 c_{01} M \\ \gcd(n_0,r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \frac{\chi_{02}(n_0)}{\tau(n_0)},$$

and

$$H'''_{01}(X) = \sum\sum_{\substack{n_0 \leqslant Y, n_1 \leqslant X/c_{01} MY \\ \gcd(n_i,r_i)=1 \forall\ 0 \leqslant i \leqslant 1 \\ n_i \equiv q_i \bmod 8 \forall\ 0 \leqslant i \leqslant 1}} \frac{\chi_{02}(n_0)}{\tau(n_0)\tau(n_1)}.$$

Using Lemma 5.1.4 for the sums over n_0 we see that

$$H'''_{01}(X) \ll_{C'_5} \frac{\tau(r_0) Q_{02} X}{c_{01} M (\log X)^{1/2} (\log X)^{C'_5/3}},$$

and

$$H''_{01}(X) \ll_{C'_5} \sum_{n_1 \leqslant X/c_{01} MY} \frac{\tau(r_0) Q_{02} X}{n_1 c_{01} M (\log X/n_1 c_{01} MY)^{(C'_5+1)/3}} \ll_{C'_5} \frac{\tau(r_0) Q_{02} X}{c_{01} M (\log X)^{C'_5/3}}.$$

In each case we have used the bound $(\log X/c_{01} MY) = (\log X)(1 + O((\log X)^{-2/3}))$ which follows from the fact that $\log c_{01} MY \ll (\log X)^{1/3}$. In the above bounds we can write $C'_5 = 3C_5$ for some $C_5 > 0$ to obtain

$$H''_{01}(X), H'''_{01}(X) \ll_{C_5} \frac{\tau(r_0) Q_{02} X}{c_{01} M (\log X)^{C_5}}.$$

For $H'_{01}(X)$ we apply Lemma 5.1.4 for non-principal characters with $C_5 > 0$ sufficiently large. This will give

$$H'_{01}(X) = \sum_{\substack{n_0 \leqslant Y \\ \gcd(n_0,2r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \frac{\chi_{02}(n_0)}{\tau(n_0)} \left(\frac{\mathfrak{S}_0(2r_1) X}{n_0 c_{01} M \sqrt{(\log X/n_0 c_{01} M)}} + O\left(\frac{X (\log\log 3r_1)^{3/2}}{n_0 c_{01} M (\log X)^{3/2}} \right) \right),$$

where we have used $\log X/c_{01} M \gg \log X$ coming from $c_{01}, M \leqslant (\log X)^{C_2}$. Using the bound

$$\sum_{n_0 \leqslant Y} \frac{1}{n_0 \tau(n_0)} \ll \sqrt{\log Y} \ll (\log X)^{1/6}.$$

we obtain

$$H'_{01}(X) = \frac{\mathfrak{S}_0(2r_1) X}{c_{01} M} \sum_{\substack{n_0 \leqslant Y \\ \gcd(n_0,2r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \frac{\chi_{02}(n_0)}{n_0 \tau(n_0) \sqrt{(\log X/n_0 c_{01} M)}} + O_{C_3}\left(\frac{X (\log\log 3r_1)^{3/2}}{c_{01} M (\log X)^{4/3}} \right).$$

For the front term we use the following:

$$\frac{1}{\sqrt{(\log X/n_0 c_{01} M)}} = \frac{1}{\sqrt{(\log X)}} \cdot \frac{1}{\left(1 - \frac{\log n_0 c_{01} M}{\log X}\right)^{1/2}} = \frac{1}{\sqrt{(\log X)}} \left(1 + O\left(\frac{1}{(\log X)^{2/3}} \right) \right).$$

It follows that

$$H'_{01}(X) = \frac{\mathfrak{S}_0(2r_1)X}{c_{01}M\sqrt{\log X}} \sum_{\substack{n_0 \leqslant Y \\ \gcd(n_0, 2r_0)=1 \\ n_0 \equiv q_0 \bmod 8}} \frac{\chi_{02}(n_0)}{n_0 \tau(n_0)} + O_{C_3}\left(\frac{X}{c_{01}M \log X}\right).$$

Next we detect the condition $8|n_0 - q_0$ using Dirichlet characters. Thus the main term sum in $H'_{01}(X)$ becomes

$$\frac{1}{4} \sum_{\chi \bmod 8} \overline{\chi}(q_0) \sum_{\substack{n_0 \leqslant Y \\ \gcd(n_0, r_0)=1}} \frac{\chi_{02}\chi(n_0)}{n_0 \tau(n_0)}, \tag{5.3.3}$$

where $\chi_{02}\chi(n_0) = \chi_{02}(n_0)\chi(n_0)$ is a non-principal character modulo $8Q_{02}$ and are non-principal since Q_{02} is odd and χ_{02} non-principal. Using a similar argument to that seen in the proof of Lemma 5.3.1, $\widetilde{L}_{r_0}(1, \chi_{02}\chi)$ converges and

$$\widetilde{L}_{r_0}(1, \chi_{02}\chi) = P(1, r_0, \chi_{02}\chi')R(1, \chi_{02}\chi)L(1, \chi_{02}\chi)^{1/2}$$

where $P(1, r_0, \chi_{02}\chi)$ and $R(1, \chi_{02}\chi)$ are

$$\prod_{\substack{p \text{ prime} \\ p|r_0}} \left(1 + \sum_{j=1}^{\infty} \frac{\chi_{02}\chi(p)^j}{(j+1)p^j}\right)^{-1} \quad \text{and} \quad \prod_{p \text{ prime}} \left(1 + \sum_{j=1}^{\infty} \frac{\chi_{02}\chi(p)^j}{(j+1)p^j}\right)\left(1 - \frac{\chi_{02}\chi(p)}{p}\right)^{1/2}$$

respectively, and

$$L(1, \chi_{02}\chi) = \prod_{p \text{ prime}} \left(1 - \frac{\chi_{02}\chi(p)}{p}\right)^{-1}$$

is the L-function for the character $\chi_{02}\chi$. Seeing this, we may extend the sum over n_0 in (5.3.3) at the cost of an error term. This equation then becomes

$$= \frac{1}{4} \sum_{\chi \bmod 8} \overline{\chi}(q_0) P(1, r_0, \chi_{02}\chi) R(1, \chi_{02}\chi) L(1, \chi_{02}\chi)^{1/2} + O_{C_5}\left(\frac{Q_{02}}{(\log X)^{C_5}}\right),$$

where we have used partial summation and Lemma 5.1.4 to bound the tail of this series. We therefore see that $H'_{01}(X)$ is equal to

$$\frac{\mathfrak{S}_0(2r_1)X}{4c_{01}M\sqrt{\log X}} \sum_{\chi \bmod 8} \overline{\chi}(q_0) P(1, r_0, \chi_{02}\chi) R(1, \chi_{02}\chi) L(1, \chi_{02}\chi)^{1/2} + O_{C_3}\left(\frac{X}{c_{01}M \log X}\right).$$

Putting this together with $H''_{01}(X)$ and $H'''_{01}(X)$ we see that $H_{01}(X)$ is then

$$\frac{\mathfrak{S}_0(2r_1)X}{4c_{01}M\sqrt{\log X}} \sum_{\chi \bmod 8} \overline{\chi}(q_0) P(1, r_0, \chi_{02}\chi) R(1, \chi_{02}\chi) L(1, \chi_{02}\chi)^{1/2} + O_{C_3}\left(\frac{X}{c_{01}M \log X}\right).$$

Similarly we may obtain that $H_{23}(X)$ is

$$\frac{\mathfrak{S}_0(2r_3)X}{4c_{23}M\sqrt{\log X}} \sum_{\substack{\chi' \text{ char.} \\ \bmod 8}} \overline{\chi'}(q_0) P(1, r_3, \chi_{02}\chi') R(1, \chi_{02}\chi') L(1, \chi_{02}\chi')^{1/2} + O_{C_3}\left(\frac{X}{c_{23}M \log X}\right).$$

Multiplying these together we obtain the result. $\square$

In order to take the desired averages over the characters, we will use the fact that $\widetilde{L}_r(1,\chi)$ looks roughly like $L(1,\chi)^{1/2}$. In fact, by noting the bounds

$$P(1,r_0,\chi) \ll \tau(r_0),\ R(1,\chi) \ll 1$$

for any non-principal character χ and absolute implied constants, we observe

$$\widetilde{L}_r(1,\chi)^2 \ll \tau(r)^2 L(1,\chi) \tag{5.3.4}$$

since $L(1,\chi) > 0$ for real non-principal characters χ.

Recall that we use the notation $\psi_m(\cdot)$ for an odd integer m to denote generically the Jacobi symbol $\left(\frac{\cdot}{m}\right)$ or $\left(\frac{m}{\cdot}\right)$. We may use quadratic reciprocity to interchange between the two if necessary (in the following proof, the characters modulo 8 and square-free functions ensure that the variables are odd).

Theorem 5.3.5. *Let χ be a character modulo 8, and m_1 be an integer in $[1,X]$. Then for all $X \geqslant 3$ we have*

$$\sum_{1<m\leqslant X} \frac{\mu^2(2mm_1)}{\tau(m)} L(1,\chi\cdot\psi_{mm_1}) \ll \frac{X}{\sqrt{\log X}}.$$

Proof. We begin by splitting the L-function in two:

$$L(1,\chi\cdot\psi_{mm_1}) = \sum_{n=1}^{\lfloor X^2\rfloor} \frac{\chi(n)\psi_{mm_1}(n)}{n} + \sum_{n>\lfloor X^2\rfloor} \frac{\chi(n)\psi_{mm_1}(n)}{n}.$$

Using partial summation and the Pólya–Vinogradov inequality [25], the tail sum may be seen to be

$$\ll \frac{(mm_1)^{1/2}\log(mm_1)}{X^2} \ll \frac{(\log X)}{X}.$$

Summing trivially over m will give $O(\log X)$ which is sufficient. The expression we have left is

$$\sum_{m\leqslant X}\sum_{n\leqslant\lfloor X^2\rfloor} \frac{\mu^2(2mm_1)}{\tau(m)}\frac{\chi(n)\psi_{m_1}(n)}{n}\left(\frac{n}{m}\right). \tag{5.3.5}$$

We will make use of the double oscillation of the character $\psi_{mm_1}(n)$ in the variables m and n. By partial summation we get

$$\sum_{m\leqslant X}\sum_{n\leqslant\lfloor X^2\rfloor} \frac{\mu^2(2mm_1)}{\tau(m)}\frac{\chi(n)\psi_{m_1}(n)}{n}\left(\frac{n}{m}\right) \ll \frac{S(X,X^2)}{X^2} + \int_2^{X^2}\frac{S(X,t)}{t^2}dt, \tag{5.3.6}$$

where

$$S(X,t) = \sum_{m\leqslant X}\sum_{n\leqslant t}\frac{\mu^2(2mm_1)}{\tau(m_1)}\chi(n)\psi_{m_1}(n)\left(\frac{n}{m}\right).$$

Using Lemma 2.2.4 we obtain

$$\frac{S(X,X^2)}{X^2} \ll \frac{X^3}{X^2}\left(X^{-1/6}+X^{-1/3}\right)(\log 3X)^{7/6} \ll X^{5/6}(\log 3X)^{7/6}.$$

The integral equals:

$$\int_2^{X^2} \frac{S(X,t)}{t^2} dt = \int_2^{X^{1/2}} \frac{S(X,t)}{t^2} dt + \int_{X^{1/2}}^{X^2} \frac{S(X,t)}{t^2} dt.$$

In the first range we apply Lemma 5.1.3 with $\epsilon = 1/6$, $N = X$ and $M = t$ as $t^2 \ll X$. In the second range we once more apply Lemma 2.2.4. The integral therefore becomes bounded by:

$$\ll \int_2^{X^{1/2}} \left(\frac{X(\log t)}{t^{3/2}\sqrt{\log X}} + \frac{X^{1/3}(\log t)^{1/2}}{t^{1/2}(\log X)^{1/4}} \right) dt + \int_{X^{1/2}}^{X^2} \left(\frac{X}{t^{7/6}} + \frac{X^{5/6}}{t} \right) (\log 3X)^{7/6} dt,$$

which is $O(X(\log X)^{-1/2})$. □

Corollary 5.3.6. *Let $X \geqslant 3$ and $C > 0$. Fix some real number $0 < c \leqslant 1$. Suppose that χ is a character modulo 8. Then*

$$\sum_{\substack{\|m_0, cm_1\| \leqslant X \\ m_0 m_1 \neq 1}}\sum \frac{\mu^2(2m_0m_1)}{\tau(m_0)\tau(m_1)} L(1, \chi \cdot \psi_{m_0m_1}) \ll \frac{X^2}{c\sqrt{\log X}},$$

where the implied constant is absolute.

Proof. When $m_1 = 1$ then we use Lemma 5.3.5 for the sum over m_0, since $m_0m_1 \neq 1$. When $m_1 > 1$ then we may use Lemma 5.3.5 to bound the sum over m_1. In this case:

$$\sum_{\substack{\|m_0, cm_1\| \leqslant X \\ m_1 > 1}}\sum \frac{\mu^2(2m_0m_1)}{\tau(m_0)\tau(m_1)} L(1, \chi \cdot \psi_{m_0m_1}) \ll \sum_{m_0 \leqslant X} \frac{X}{c\sqrt{\log X/c}} \ll \frac{X^2}{c\sqrt{\log X}}.$$

□

We conclude this chapter with the proof of Proposition 2.2.11.

Proposition 5.3.7. *Let $X \geqslant 3$, $C_1, C_2 > 0$ and fix $q \in (\mathbb{Z}/8\mathbb{Z})^{*4}$ and $\tilde{q} \in (\mathbb{Z}/8\mathbb{Z})^{*2}$. Fix odd integers $1 \leqslant r_0, r_1, r_2, r_3, \tilde{r}_0, \tilde{r}_1 \leqslant (\log X)^{C_1}$ and fix $1 \leqslant c_{01}, c_{23}, \tilde{c}_0, \tilde{c}_1 \leqslant (\log X)^{C_2}$. Then for any $\mathbf{m} \in \mathbb{N}^2$ we define*

$$T(X, \mathbf{m}) = \sum_{\substack{\|n_0n_1c_{01}, n_2n_3c_{23}\| \cdot \|m_0\tilde{c}_0, m_1\tilde{c}_1\| \leqslant X \\ \gcd(n_i, 2r_i) = 1 \forall\ 0 \leqslant i \leqslant 3 \\ n_i \equiv q_i \bmod 8 \forall\ 0 \leqslant i \leqslant 3}}\sum\sum\sum \frac{\psi_{m_0m_1}(n_0n_2)}{\tau(n_0)\tau(n_1)\tau(n_2)\tau(n_3)}.$$

Then for any $C_3 > 0$,

$$\sum_{\substack{\|m_0, m_1\| \leqslant (\log X)^{C_3} \\ m_i \equiv \tilde{q}_i \bmod 8\ \forall 0 \leqslant i \leqslant 1 \\ \gcd(m_i, Q\tilde{r}_i) = 1\ \forall 0 \leqslant i \leqslant 1 \\ m_0m_1 \neq 1}}\sum \frac{\mu^2(m_0m_1)}{\tau(m_1)\tau(m_2)} |T(X, \mathbf{m})| \ll_{C_1, C_2, C_3} \frac{\tau(r_0)\tau(r_2)X^2(\log\log X)^{1/2}}{c_{01}c_{23}\tilde{c}_0\tilde{c}_1(\log X)}.$$

where the implied constant depends at most on the C_i.

Proof. Using Lemma 5.3.4 on the $T(X, \mathbf{m})$ and using the triangle inequality we see that the sum over $\mathbf{m}$ is equal to

$$\frac{\mathfrak{S}_0(2r_1)\mathfrak{S}_0(2r_3)X^2}{c_{01}c_{23}(\log X)} \sum\sum_{\chi,\chi' \bmod 8} M(X,\chi,\chi') + O\left(\frac{X^2}{c_{01}c_{23}(\log X)^{3/2}} \sum_{\chi \bmod 8} E(X,\chi)\right),$$

where

$$M(X,\chi,\chi') = \sum\sum_{\substack{\|m_0,m_1\| \leqslant (\log X)^{C_3} \\ m_i \equiv \tilde{q}_i \bmod 8\ \forall 0\leqslant i\leqslant 1 \\ \gcd(m_i,\tilde{r}_i)=1\ \forall 0\leqslant i\leqslant 1 \\ m_0m_1\neq 1}} \frac{\mu^2(m_0m_1)}{\tau(m_1)\tau(m_2)\|m_0\tilde{c}_0, m_1\tilde{c}_1\|^2} |\widetilde{L}_{r_0}(1,\psi_{m_0m_1}\chi)||\widetilde{L}_{r_2}(1,\psi_{m_0m_1}\chi')|$$

and

$$E(X,\chi) = \sum_{j=0,2} \sum\sum_{\|m_0,m_1\| \leqslant (\log X)^{C_3}} \frac{\mu^2(m_0m_1)}{\tau(m_1)\tau(m_2)\|m_0\tilde{c}_0, m_1\tilde{c}_1\|^2} |\widetilde{L}_{r_j}(1,\psi_{m_0m_1}\chi)|.$$

We first bound $M(X,\chi,\chi')$. We may write this sum in the form

$$M(X,\chi,\chi') = \sum\sum_{\|m_0,m_1\| \leqslant (\log X)^{C_3}} a_{(m_0,m_1)} b_{(m_0,m_1)} c_{(m_0,m_1)}$$

where $b_{(m_0,m_1)} = |\widetilde{L}_{r_0}(1,\psi_{m_0m_1}\chi)|$, $c_{(m_0,m_1)} = |\widetilde{L}_{r_2}(1,\psi_{m_0m_1}\chi')|$ and $a_{(m_0,m_1)}$ represents the remaining summands and conditions. We may therefore re-index this sum as a sum over a single variable,

$$M(X,\chi,\chi') = \sum_{l \leqslant (\log X)^{2C_3}} \tilde{a}_l \tilde{b}_l \tilde{c}_l.$$

Using Cauchy's inequality, and then returning to the original double indexing, we obtain

$$M(X,\chi,\chi') = \left(\sum\sum_{\|m_0,m_1\| \leqslant (\log X)^{C_3}} a_{(m_0,m_1)} b^2_{(m_0,m_1)}\right)^{1/2} \left(\sum\sum_{\|m_0,m_1\| \leqslant (\log X)^{C_3}} a_{(m_0,m_1)} c^2_{(m_0,m_1)}\right)^{1/2}.$$

In other words, we have now obtained $M(X,\chi,\chi') \ll R_{r_0}(X,\chi)^{1/2} R_{r_2}(X,\chi')^{1/2}$ where

$$R_r(X,\chi) = \sum\sum_{\substack{\|m_0,m_1\| \leqslant (\log X)^{C_3} \\ m_0m_1\neq 1}} \frac{\mu^2(2m_0m_1)}{\tau(m_1)\tau(m_2)\|m_0\tilde{c}_0, m_1\tilde{c}_1\|^2} |\widetilde{L}_r(1,\psi_{m_0m_1}\chi)|^2.$$

By (5.3.4) we have

$$R_r(X,\chi) \ll \tau(r)^2 \sum\sum_{\substack{\|m_0,m_1\| \leqslant (\log X)^{C_3} \\ m_0m_1\neq 1}} \frac{\mu^2(2m_0m_1)}{\tau(m_1)\tau(m_2)\|m_0\tilde{c}_0, m_1\tilde{c}_1\|^2} L(1,\psi_{m_0m_1}\chi).$$

Writing

$$a(M) = \sum\sum_{\substack{\mathbf{m}\in\mathbb{N}^2, \|m_0\tilde{c}_0, m_1\tilde{c}_1\| = M \\ m_0m_1\neq 1}} \frac{\mu^2(2m_0m_1)}{\tau(m_1)\tau(m_2)} L(1,\psi_{m_0m_1}\chi)$$

we obtain

$$R_r(X,\chi) \ll \tau(r)^2 \sum_{2\leqslant M\leqslant \|\tilde{c}_0,\tilde{c}_1\|(\log X)^{C_3}} \frac{a(M)}{M^2}. \tag{5.3.7}$$

By partial summation the sum on the right hand side of (5.3.7) is then

$$\frac{1}{\|\tilde{c}_0,\tilde{c}_1\|^2(\log X)^{2C_3}} \sum_{2\leqslant M\leqslant \|\tilde{c}_0,\tilde{c}_1\|(\log X)^{C_3}} a(M) + 2\int_2^{\|\tilde{c}_0,\tilde{c}_1\|(\log X)^{C_3}} \frac{\sum_{2\leqslant M\leqslant t} a(M)}{t^3}dt.$$

Using the fact that $a(M) = 0$ unless $M \geqslant \|\tilde{c}_0,\tilde{c}_1\|$ and the change of variables $t = \|\tilde{c}_0,\tilde{c}_1\|u$ the integral becomes

$$\int_2^{\|\tilde{c}_0,\tilde{c}_1\|(\log X)^{C_3}} \frac{\sum_{2\leqslant M\leqslant t} a(M)}{t^3}dt = \int_1^{(\log X)^{C_3}} \frac{\sum_{2\leqslant M\leqslant \|\tilde{c}_0,\tilde{c}_1\|u} a(M)}{\|\tilde{c}_0,\tilde{c}_1\|^2u^3}du.$$

Thus

$$\frac{R_r(X,\chi)}{\tau(r)^2} \ll \frac{1}{\|\tilde{c}_0,\tilde{c}_1\|^2(\log X)^{2C_3}} \sum_{2\leqslant M\leqslant \|\tilde{c}_0,\tilde{c}_1\|(\log X)^{C_3}} a(M) + \int_1^{(\log X)^{C_3}} \frac{\sum_{2\leqslant M\leqslant \|\tilde{c}_0,\tilde{c}_1\|u} a(M)}{\|\tilde{c}_0,\tilde{c}_1\|^2u^3}dt. \tag{5.3.8}$$

Now, upon unwrapping $a(M)$, we may see that,

$$\sum_{2\leqslant M\leqslant \|\tilde{c}_0,\tilde{c}_1\|Y} a(M) = \sum\sum_{\substack{\|m_0\frac{\tilde{c}_0}{\|\tilde{c}_0,\tilde{c}_1\|},m_1\frac{\tilde{c}_1}{\|\tilde{c}_0,\tilde{c}_1\|}\|\leqslant Y \\ m_0m_1\neq 1}} \frac{\mu^2(m_0m_1)}{\tau(m_1)\tau(m_2)} L(1,\psi_{m_0m_1}\chi').$$

By Corollary 5.3.6 with $c = \frac{\min(\tilde{c}_0,\tilde{c}_1)}{\|\tilde{c}_0,\tilde{c}_1\|}$ we get

$$\frac{R_r(X,\chi)}{\tau(r)^2} \ll_{C_2,C_3} \frac{1}{\min(\tilde{c}_0,\tilde{c}_1)\|\tilde{c}_0,\tilde{c}_1\|\sqrt{\log\log X}} + \int_1^{(\log X)^{C_3}} \frac{1}{\min(\tilde{c}_0,\tilde{c}_1)\|\tilde{c}_0,\tilde{c}_1\|u\sqrt{\log u}}dt.$$

This is $O(\sqrt{\log\log X}/\tilde{c}_0\tilde{c}_1)$, hence

$$M(X,\chi,\chi') \ll_{C_2,C_3} \frac{\tau(r_0)\tau(r_2)\sqrt{\log\log X}}{\tilde{c}_0\tilde{c}_1}.$$

To deal with $E(X,\chi)$ we treat the sum over each $\widetilde{L}_{r_j}(1,\psi_{m_0m_1}\chi)$ separately. Calling each one $E_j(X,\chi)$ we once more use Cauchy's inequality to get $E_j(X,\chi) \ll (\mathcal{E}_j(X,\chi)\mathcal{E}_j'(X))^{1/2}$ where

$$\mathcal{E}_j(X,\chi) = \tau(r_j)^2 \sum\sum_{\substack{\|m_0,m_1\|\leqslant(\log X)^{C_3} \\ m_0m_1\neq 1}} \frac{\mu^2(m_0m_1)}{\tau(m_1)\tau(m_2)\|m_0\tilde{c}_0,m_1\tilde{c}_1\|^2} L(1,\psi_{m_0m_1}\chi)$$

and

$$\mathcal{E}_j'(X) = \sum\sum_{\substack{\|m_0,m_1\|\leqslant(\log X)^{C_3} \\ m_0m_1\neq 1}} \frac{\mu^2(m_0m_1)}{\tau(m_1)\tau(m_2)\|m_0\tilde{c}_0,m_1\tilde{c}_1\|^2}.$$

Using similar techniques to those used to bound $M(X,\chi,\chi')$ to bound $\mathcal{E}_j(X,\chi)$ and Lemma 5.2.5 to bound $\mathcal{E}_j'(X)$ we obtain

$$\frac{\mathcal{E}_j(X,\chi)}{\tau(r_0)^2}, \mathcal{E}_j'(X) \ll \frac{(\log\log X)^{1/2}}{(\tilde{c}_0\tilde{c}_1)}$$

from which it follows that

$$E(X, \chi') \ll_{C_2, C_3} \frac{\tau(r_0)\tau(r_1)(\log\log X)^{1/2}}{\tilde{c}_0 \tilde{c}_1}.$$

Finally, we inject the bounds for $M(X, \chi, \chi')$ and $E_j(X, \chi')$ into our overall expression, summing over finitely many characters χ and χ' modulo 8 and noting that $\mathfrak{S}_0(r_i) \ll 1$ for all integers, we conclude the proof.

□

Appendix A

Tables of Elements for $\mathcal{A}_1$ and $\mathcal{A}_2$

A.1 Elements of $\mathcal{A}_1$ with $q_0q_1q_2q_3 \equiv 1 \bmod 8$

Tables A.1, A.2, A.3, A.4, A.5 and A.6 display the points in $(\mathbb{Z}/8\mathbb{Z})^{*4}$ which satisfy one of the forms in (2.10.3). Note that there are 48 unique vectors displayed throughout these tables.

$b \setminus a$	1	3	5	7
1	(1,1,7,7)	(3,1,5,7)	(5,1,3,7)	(7,1,1,7)
3	(1,3,7,5)	(3,3,5,5)	(5,3,3,5)	(7,3,1,5)
5	(1,5,7,3)	(3,5,5,3)	(5,5,3,3)	(7,5,1,3)
7	(1,7,7,1)	(3,7,5,1)	(5,7,3,1)	(7,7,1,1)

Table A.1: Solutions to (A.1.1) & (A.1.4)

$b \setminus a$	1	3	5	7
1	~~(1,1,7,7)~~	(3,1,7,5)	(5,1,7,3)	(7,1,7,1)
3	(1,3,5,7)	~~(3,3,5,5)~~	(5,3,5,3)	(7,3,5,1)
5	(1,5,3,7)	(3,5,3,5)	~~(5,5,3,3)~~	(7,5,3,1)
7	(1,7,1,7)	(3,7,1,5)	(5,7,1,3)	~~(7,7,1,1)~~

Table A.2: Solutions to (A.1.2) & (A.1.3)

$b \setminus a$	1	3	5	7
1	(3,1,1,3)	(3,3,1,1)	~~(3,5,1,7)~~	~~(3,7,1,5)~~
3	(1,1,3,3)	(1,3,3,1)	~~(1,5,3,7)~~	~~(1,7,3,5)~~
5	~~(7,1,5,3)~~	~~(7,3,5,1)~~	(7,5,5,7)	(7,7,5,5)
7	~~(5,1,7,3)~~	~~(5,3,7,1)~~	(5,5,7,7)	(5,7,7,5)

Table A.3: Solutions to (A.1.5) & (A.1.8)

$b \setminus a$	1	3	5	7
1	(3,1,3,1)	~~(3,3,1,1)~~	~~(3,5,1,7)~~	~~(3,7,1,5)~~
3	~~(1,1,3,3)~~	(1,3,1,3)	~~(1,5,3,7)~~	~~(1,7,3,5)~~
5	~~(7,1,5,3)~~	~~(7,3,5,1)~~	(7,5,7,5)	~~(7,7,5,5)~~
7	~~(5,1,7,3)~~	~~(5,3,7,1)~~	~~(5,5,7,7)~~	(5,7,5,7)

Table A.4: Solutions to (A.1.6) & (A.1.7)

$b \setminus a$	1	3	5	7
1	~~(1,7,1,7)~~	(3,5,1,7)	(5,3,1,7)	~~(7,1,1,7)~~
3	(1,7,3,5)	~~(3,5,3,5)~~	~~(5,3,3,5)~~	(7,1,3,5)
5	(1,7,5,3)	~~(3,5,5,3)~~	~~(5,3,5,3)~~	(7,1,5,3)
7	~~(1,7,7,1)~~	(3,5,7,1)	(5,3,7,1)	~~(7,1,7,1)~~

Table A.5: Solutions to (A.1.9)

Equation $\setminus a$	1	3	5	7
(A.1.11)	~~(1,1,7,7)~~	~~(3,7,5,1)~~	~~(5,5,3,3)~~	~~(7,3,1,5)~~
(A.1.14)	~~(1,5,7,3)~~	~~(3,3,5,5)~~	~~(5,1,3,7)~~	~~(7,7,1,1)~~

Table A.6: Solutions to (A.1.11)&(A.1.14)

A.2 Elements of $\mathcal{A}_2$ with $q_0q_1q_2q_3 \equiv 1 \bmod 8$

Table A.7 displays the points in $(\mathbb{Z}/8\mathbb{Z})^{*4}$ which satisfy the forms in (2.10.5). We remark that here are 32 distinct points in this table.

Equation \ a	$a=1$	$a=3$	$a=5$	$a=7$
(A.2.1)-(A.2.4)				
$(a, 7a, 1, 7)$	(1,7,1,7)	(3,5,1,7)	(5,3,1,7)	(7,1,1,7)
$(a, 7a, 7, 1)$	(1,7,7,1)	(3,5,7,1)	(5,3,7,1)	(7,1,7,1)
$(a, 7a, 3, 5)$	(1,7,3,5)	(3,5,3,5)	(5,3,3,5)	(7,1,3,5)
$(a, 7a, 5, 3)$	(1,7,5,3)	(3,5,5,3)	(5,3,5,3)	(7,1,5,3)
(A.2.5)&(A.2.6)				
$(a, 2+7a, 6a+1, 7)$	(1,1,7,7)	(3,7,3,7)	(5,5,7,7)	(7,3,3,7)
$(a, 2+7a, 7, 6a+1)$	~~(1,1,7,7)~~	(3,7,7,3)	~~(5,5,7,7)~~	(7,3,7,3)
$(a, 2+7a, 6a+5, 3)$	(1,1,3,3)	~~(3,7,7,3)~~	(5,5,3,3)	~~(7,3,7,3)~~
$(a, 2+7a, 3, 6a+5)$	~~(1,1,3,3)~~	~~(3,7,3,7)~~	~~(5,5,3,3)~~	~~(7,3,3,7)~~
(A.2.7)&(A.2.8)				
$(a, 6+7a, 6a+3, 5)$	(1,5,1,5)	(3,3,5,5)	(5,1,1,5)	(7,7,5,5)
$(a, 6+7a, 5, 6a+3)$	(1,5,5,1)	~~(3,3,5,5)~~	(5,1,5,1)	~~(7,7,5,5)~~
$(a, 6+7a, 6a+7, 1)$	~~(1,5,5,1)~~	(3,3,1,1)	~~(5,1,5,1)~~	(7,7,1,1)
$(a, 6+7a, 1, 6a+7)$	~~(1,5,1,5)~~	~~(3,3,1,1)~~	~~(5,1,1,5)~~	~~(7,7,1,1)~~
(A.2.9)-(A.2.16)				
No new solutions	~~(1,7,1,7)~~	~~(1,7,3,5)~~	~~(1,7,5,3)~~	~~(1,7,7,1)~~
	~~(7,1,1,7)~~	~~(7,1,3,5)~~	~~(7,1,5,3)~~	~~(7,1,7,1)~~
	~~(3,5,1,7)~~	~~(3,5,3,5)~~	~~(3,5,5,3)~~	~~(3,5,7,1)~~
	~~(5,3,1,7)~~	~~(5,3,3,5)~~	~~(5,3,5,3)~~	~~(5,3,7,1)~~
	~~(7,7,1,1)~~	~~(3,7,3,7)~~	~~(7,7,5,5)~~	~~(3,7,7,3)~~
	~~(7,7,1,1)~~	~~(7,3,3,7)~~	~~(7,7,5,5)~~	~~(7,3,7,3)~~
	~~(3,3,1,1)~~	~~(7,3,3,7)~~	~~(3,3,5,5)~~	~~(7,3,7,3)~~
	~~(3,3,1,1)~~	~~(3,7,3,7)~~	~~(3,3,5,5)~~	~~(3,7,7,3)~~
	~~(1,5,1,5)~~	~~(5,5,3,3)~~	~~(1,5,5,1)~~	~~(5,5,7,7)~~
	~~(5,1,1,5)~~	~~(5,5,3,3)~~	~~(5,1,5,1)~~	~~(5,5,7,7)~~
	~~(5,1,1,5)~~	~~(1,1,3,3)~~	~~(5,1,5,1)~~	~~(1,1,7,7)~~
	~~(1,5,1,5)~~	~~(1,1,3,3)~~	~~(1,5,5,1)~~	~~(1,1,7,7)~~

Table A.7: Solutions to (A.2.1)-(A.2.16)

Bibliography

[1] S. Baier, L. Zhao, *Primes in quadratic progressions on average.* Mathematishe Annalen **338**(4), pp.963-982, (2007).

[2] V. V. Batyrev, Y. Tschinkel, *Rational points on some Fano cubic bundles*, C. R. Acad. Sci. Paris Sér. I Math. **323**(1), pp.41-46, (1996).

[3] M. Bhargava, *A positive proportion of plane cubics fail the Hasse Principle*, arXiv, (2014), `https://arxiv.org/abs/1402.1131`.

[4] T. D. Browning, D. R. Heath-Brown, *Density of rational points on a quadric bundle in* $\mathbb{P}^3 \times \mathbb{P}^3$. Duke Math J. **169**(16), pp.3099-3165, (2020).

[5] T. D. Browning, D. R. Heath-Brown, *The geometric sieve for quadrics*, Forum Math. **33**(1), pp.147-165, (2021).

[6] T. D. Browning, P. Le Boudec, W. Sawin, *The Hasse principle for random Fano hypersurfaces*, Annals of Math., **197**(3), pp. 1115-1203, (2023).

[7] T. D. Browning, D. Loughran, *Sieving rational points on varieties*, Trans. Amer. Math. Soc. **371**(8), pp.57575785, (2019).

[8] T. D. Browning, J. Lyczak, R. Sarapin, *Local solubility for a family of quadrics over a split quadric surface*, Involve **16**(2), pp.331-342, (2023), `https://doi.org/10.2140/involve.2023.16.331`.

[9] T. D. Browning, J. Lyczak, A. Smeets, *Paucity of rational points on fibrations with multiple fibres*, arXiv, (2023), `https://arxiv.org/abs/2310.01135`.

[10] J-L. Colliot-Thélène, *Points Rationnels sur les Fibrations*, Higher Dimensional Varieties and Rational Points, Bolyai Society Mathematical Studies **12**. Ed. K. Böröczky, J. Kollár, T. Szamuely. Springer Berlin Heidelberg, pp.171-221, (2003).

[11] J-L. Colliot-Thélène, A. N. Skorobogatov *Groupe de Chow des zéro-cycles sur les fibrés en quadriques*, K-Theory **7**(5), pp.477-500, (1993).

[12] J-L. Colliot-Thélène, A. N. Skorobogatov *The Brauer-Grothendieck Group*, Springer, **71**, (2021).

[13] J. B. Conrey, F. W. Farmer, K. Soundararajan, *Transition mean values of real characters.* Journal of Number Theory **82**(1), pp.109-120, (2000).

[14] P. D. T. A. Elliott, *On the mean value of* $f(p)$. Proc. London Math. Soc. **21**(3), pp.28-96, (1970).

[15] É. Fouvry, J. Klüners, *On the 4-rank of class groups of quadratic number fields.* Invent. Math. **167**(3), pp.455-513, (2007).

[16] J. Franke, Y.I. Manin and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. **95**, pp.421435, (1989).

[17] J. Friedlander, H. Iwaniec, *Ternary quadratic forms with rational zeros.* Journal de théorie des nombres de Bordeaux **22**(1), pp.97-113, (2010).

[18] P. Gille, T. Szamuely, *Central Simple Algebras and Galois Cohomology.* Cambridge University Press, Second Edition, (2017).

[19] C. R. Guo, *On solvability of ternary quadratic forms*, Proc. London Math. Soc. **70**(2), pp.241263, (1995).

[20] D. Gvirtz, D. Loughran, M. Nakahara, *Quantitative arithmetic of diagonal degree 2 K3 surfaces*, Mathematische Annalen, 384, no.1-2, pp.135209, (2022).

[21] G. Harman, *Prime detecting sieves.* Princeton University Press, Princeton, NJ, (2007).

[22] D. R. Heath-Brown, *A mean value estimate for real character sums.* Acta Arith. **72**(3), pp.235-275, (1995).

[23] C. Hooley, *On ternary quadratic forms that represent zero*, Glasgow Math. J. **35**(1), pp.13-23 (1993).

[24] C. Hooley, *On ternary quadratic forms that represent zero. II.*, J. reine angew. Math. **602**, pp.179-225, (2007).

[25] H. Iwaniec, E. Kowalski, *Analytic Number Theory.* Amer. Math. Soc., (2004).

[26] P. Koymans, R. Paterson, T. Santens, A. Shute, *Local solubility of generalised Fermat equations*, arXiv, (2025), `https://arxiv.org/abs/2501.17619`.

[27] D. Loughran, *The number of varieties in a family which contain a rational point*, J. Euro. Math. Soc., **20**(10), pp.25392588, (2018).

[28] D. Loughran, L. Matthiesen, *Frobenian multiplicative functions and rational points in fibrations*, J. Euro. Math. Soc., **26**(12), pp.47794830, (2024).

[29] D. Loughran, N. Rome, E. Sofos, *The leading constant for rational points in families*, arXiv, (2022), `https://arxiv.org/abs/2210.13559`.

[30] D. Loughran, A. Smeets, *Fibrations with few rational points*, Geom. Func. Anal. **26**(5), pp.14491482, (2016).

[31] D. Loughran, R. Takloo-Bighash, S. Tanimoto, *Zero-loci of Brauer group elements on semi-simple algebraic groups*, Journal de l'Institut de Mathématiques de Jussieu, **19**(5), pp.1467-1507, (2020).

[32] E. Peyre, *Liberté et accumulation*, Documenta Math. **25**, pp. 16151659, (2017).

[33] E. Peyre, *Hauteurs et mesures de Tamagawa sur les Variétés de Fano*, Duke Math. J., **79**, pp.101-218, (1995).

[34] E. Peyre, *Points de hauteur bornée, topologie adélique et mesures de Tamagawa*, Journal de Théorie des Nombres de Bordeaux, **15**(1), pp.319-349, (2003).

[35] B. Poonen, P. Voloch, *Random Diophantine equations*, Progr. Math., **226**, Arithmetic of higher-dimensional algebraic varieties, pp. 175184, Birkhäuser Boston, Boston, MA, (2004).

[36] E. S. Selmer, *The Diophantine equation* $ax^3 + by^3 + cz^3 = 0$, Acta. Math. **85**, pp.203-362, (1951).

[37] J. P. Serre, *Spécialisation des éléments de* $\mathrm{Br}_2(\mathbb{Q}(T_1, \ldots, T_n))$, C. R. Acad. Sci. Paris Sér. I Math. **311**(7), pp.397-402, (1990).

[38] J. P. Serre, *A course in arithmetic*, Springer Science & Business Media, **7**, (2012).

[39] J. P. Serre, *Topics in Galois theory*, CRC Press, (2016).

[40] P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions,* J. reine angew. Math. **313**, pp.161-170, (1980).

[41] E. Sofos, *Serre's problem on the density of isotropic fibres in conic bundles*, Proc. of the Lond. Math. Soc. **113**(2), (2016), `https://doi.org/10.1112/plms/pdw030`.

[42] E. Sofos, E. Visse-Martindale, *The density of fibres with a rational point for a fibration over hyper- surfaces of low degree*, Annales de lInstitut Fourier, **71**(2), pp.679-709, (2021).

[43] K. Soundararajan, *Non-vanishing of quadratic Dirichlet L-functions at* $s = \frac{1}{2}$. Annals of Mathematics **152**(2), pp.447-488, (2000).

[44] C. Wilson, *General bilinear forms in the Jacobi Symbol over hyperbolic regions.* Monatsh. Math., **202**, pp.435-451, (2023), `https://doi.org/10.1007/s00605-023-01848-9`.