Local forms for the double A_n quiver and Gopakuma–Vafa invariants

by

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A thesis submitted in fulfilment of the requirements
for the degree of

Doctor of Philosophy

at the

School of Mathematics & Statistics
College of Science & Engineering
University of Glasgow

July 2025

Abstract

This thesis investigates the crepant (partial) resolutions of cA_n singularities and their associated Gopakumar–Vafa (GV) invariants via noncommutative contraction algebras.

We begin in Chapter 3 by generalising GV invariants to crepant partial resolutions of cA_n singularities and demonstrate that these generalised invariants satisfy Toda's formula. Furthermore, we prove that generalised GV invariants are determined by the isomorphism class of the contraction algebra.

In Chapter 4 we focus on crepant resolutions of cA_n singularities, and introduce several intrinsic definitions of a Type A potential on the doubled A_n quiver Q_n, which includes a single loop at each vertex. Through applying coordinate changes, we then:

(1) Via monomialization, expresses these potentials in a particularly nice form;

(2) Show that Type A potentials classify crepant resolutions of cA_n singularities;

(3) Confirm the Realisation Conjecture of Brown–Wemyss within this context.

We also provide an example of a non-isolated cA_2 singularity which illustrates that the Donovan–Wemyss Conjecture fails for non-isolated cDV singularities.

Building upon the correspondence between crepant resolutions of cA_n singularities and monomialized Type A potentials, in Chapter 5 we:

(1) Introduce a filtration structure on the parameter space of monomialized Type A potentials with respect to the generalised GV invariants;

(2) Derive numerical constraints on the possible tuples of GV invariants, and explicitly classify all tuples arising from crepant resolutions of cA_2 singularities.

For $n \leq 3$, in Chapter 6 we further provide a complete classification of Type A potentials (without loops) up to isomorphism, as well as a classification of those with finite-dimensional Jacobi algebras up to derived equivalence. These results yield various algebraic consequences, including applications to certain tame algebras of quaternion type studied by Erdmann, for which we describe all basic algebras within the derived equivalence class.

Contents

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my supervisor, Michael Wemyss. I truly cannot imagine how I could have completed this thesis without his unwavering support and generous investment of time and energy. His guidance extended far beyond mathematics—he patiently helped me understand how to approach academic work, write research papers, and improve the structure and clarity of my English writing, which has been especially important as a non-native speaker. I am also very grateful to my second advisor, Gavin Brown, and to Dinakar Muthiah, both of whom have supported me throughout my studies, particularly in writing references.

I would like to thank the University of Glasgow and the School of Mathematics and Statistics for providing a stimulating academic environment, including many valuable seminars, discussions, and learning opportunities. I am also sincerely thankful to the China Scholarship Council for funding my studies and making it possible for me to pursue my PhD abroad.

I am fortunate to have had the companionship and intellectual camaraderie of my fellow PhD students and postdocs: Sam, Marina, Parth, Charlotte, Caroline, Matthew, Wahei, Franco, Timothy, Emma, and Isambard. Our discussions—both formal and informal—helped me grow as a mathematical researcher and deepened my passion for the subject.

Finally, I would like to thank my parents, Siyuan Zhang and Chunping Zhu, for their unconditional love, constant support, and care. Their encouragement has sustained me throughout this journey.

Author's declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

We begin with a broad introduction to several key concepts central to this thesis: the Minimal Model Program, noncommutative crepant resolutions, contraction algebras, and Gopakumar–Vafa invariants.

Following this, we summarise our main results, outline the structure of the thesis, and establish the notation and conventions used throughout.

Readers familiar with the background who wish to proceed directly to the new contributions may skip ahead to §1.5.

§ 1.1 | Minimal model program

In algebraic geometry, smooth varieties are generally better behaved than singular ones. Given a singular variety $\mathcal{X}$, a natural goal is to construct a proper birational map $\widetilde{\mathcal{X}} \to \mathcal{X}$ such that $\widetilde{\mathcal{X}}$ is smooth. Such a $\widetilde{\mathcal{X}}$ is called a *resolution* of $\mathcal{X}$, and the landmark result of [H1] guarantees the existence of resolutions in all dimensions when the base field has characteristic zero.

It is then natural to ask for the "best" resolution of $\mathcal{X}$. More precisely, one seeks a resolution $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ such that every other resolution of $\mathcal{X}$ factors through π; such a resolution is known as a *minimal resolution*. For curves and surfaces, minimal resolutions always exist and are unique [C1].

Let us consider a surface example. Let G be a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, acting on the plane $\mathbb{C}^2$ via matrix multiplication. The quotient $\mathbb{C}^2/G$ is then locally isomorphic to $\operatorname{Spec} \mathbb{C}[\![x, y]\!]^G$, where $\mathbb{C}[\![x, y]\!]^G$ denotes the ring of invariants under the group action. Under the action of G, these quotients define isolated surface singularities known as *Kleinian singularities* (or *Du Val singularities*), which are classified into types A, D, and E; see e.g. [R1].

Example 1.1.1. Let G be the group generated by $\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$, where ω is a primitive n^{th}

root of unity. Then,

$$\mathbb{C}[[x,y]]^G \cong \mathbb{C}[[x^n, y^n, xy]] \cong \mathbb{C}[[a,b,c]]/\left(ab - c^n\right).$$

Each Kleinian singularity contains a unique singular point at the origin. The unique minimal resolution of a Kleinian singularity exists and is an isomorphism away from the singular point. Moreover, the preimage of this point is a finite chain of rational curves linked in a Dynkin configuration (see [D]). We refer to this preimage as the *exceptional curves* of the resolution.

Example 1.1.1 describes the Kleinian singularity of type A_{n-1}, whose minimal resolution has $n-1$ exceptional curves.

For $n=2$ in Example 1.1.1, as the following picture illustrates, $\mathfrak{X}$ is the A_1 Kleinian singularity and its minimal resolution $\tilde{\mathfrak{X}}$ has only one exceptional curve.

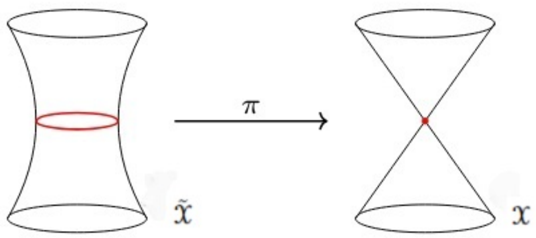


Figure 1.1: The minimal resolution $\tilde{\mathfrak{X}}$ of the A_1 Kleinian singularity $\mathfrak{X}$

However, in higher-dimensional varieties, a minimal resolution may not exist. Thus, Mori and Reid introduced the notion of a *minimal model*, whose central idea is that the crepancy property [R1]—that is, remaining close to the original space—is more important than achieving smoothness. Instead of requiring smoothness, one asks for a crepant morphism $\pi\colon \mathfrak{Y} \to \mathfrak{X}$ such that the singularities of $\mathfrak{Y}$ are "not too bad". When the minimal model $\mathfrak{Y}$ happens to be smooth, we refer to π as a *crepant resolution*.

For Kleinian singularities, the crepant resolution and minimal resolution coincide, and hence there is a unique minimal model [R1]. However, even in dimension three, the minimal model may not be unique.

Example 1.1.2. (Atiyah Flop) Let $\mathfrak{X} = \operatorname{Spec} \mathbb{C}[[u,v,x,y]]/(uv - xy)$. Blowing up the origin of $\mathfrak{X}$ yields a resolution $\pi\colon \widetilde{\mathfrak{X}} \to \mathfrak{X}$ with exceptional locus $\mathbb{P}^1 \times \mathbb{P}^1$. Although $\widetilde{\mathfrak{X}}$ is smooth, it is considered "too far away" from the original space $\mathfrak{X}$ to qualify as a minimal model.

However, we can obtain minimal models of $\mathfrak{X}$ from this resolution: contracting either copy of $\mathbb{P}^1$ gives two varieties, $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Both morphisms π_1 and π_2 are crepant resolutions (i.e., minimal models), but neither factors through the other [A2]. Thus, $\mathfrak{X}$ does not admit a unique minimal resolution, and its crepant resolutions (i.e. minimal models) are not unique.

$$
\begin{tikzcd}
 & \widetilde{\mathcal{X}} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\pi"] & \\
\mathcal{X}_1 \arrow[dr, "\pi_1"'] & & \mathcal{X}_2 \arrow[dl, "\pi_2"] \\
 & \mathcal{X} &
\end{tikzcd}
$$

Figure 1.2: Atiyah Flop

In Example 1.1.2, the minimal models $\mathcal{X}_1$ and $\mathcal{X}_2$ are smooth. However, minimal models are not always smooth (see e.g. Example 2.2.4(3) below).

The variety $\mathcal{X}$ in Example 1.1.2 belongs to the class of *compound Du Val (cDV) singularities*, which are natural three-dimensional generalisations of Du Val singularities. More precisely, they can always be locally expressed in the form $f(u,v,x)+tg(u,v,x,t)=0$, where f defines a Du Val singularity and g is any polynomial [R1].

The cDV singularities admit minimal models but do not always admit crepant resolutions, as shown in Example 2.2.4(3) (see also [K1]). Like Du Val singularities, the exceptional curves of a minimal model of a cDV singularity form a finite chain of rational curves, although they are not necessarily linked in a Dynkin configuration.

Since minimal models of cDV singularities may not be unique—as in Example 1.1.2—a natural question arises: how are different minimal models related? Kollár showed that any two minimal models of an isolated cDV singularity are connected by a finite sequence of special birational maps called *flops*, which are isomorphisms in codimension one [K1]. Roughly speaking, a flop transforms one minimal model into another by cutting some exceptional curves in the current model and re-gluing them in the opposite orientation.

Returning to Example 1.1.2, the exceptional curve in both π_1 and π_2 is $\mathbb{P}^1$, and $\mathcal{X}_2$ is a flop of $\mathcal{X}_1$. This is the simplest three-dimensional flop, known as the *Atiyah Flop* [A2].

More generally, given a minimal model $\pi\colon \mathcal{Y} \to \mathcal{X} = \operatorname{Spec}\mathcal{R}$, where $\mathcal{R}$ is a cDV singularity, by e.g. [W2, §2] we can factor π by contracting some of the exceptional curves. This yields a sequence of morphisms $\mathcal{Y} \to \mathcal{Y}_{\mathrm{con}} \xrightarrow{\pi'} \mathcal{X}$, where π' is called a *crepant partial resolution* of $\mathcal{X}$.

Thus, we have the following hierarchy:

$$\text{crepant resolutions} \subseteq \text{minimal models} \subseteq \text{crepant partial resolutions}.$$

§ 1.2 | Noncommutative crepant resolutions

There is also a ‘noncommutative geometry’ approach to the minimal model program, which takes a different tack by replacing varieties with noncommutative objects, such as noncommutative algebras and bounded derived categories of coherent sheaves.

Example 1.2.1. Consider the sheaves $\mathcal{O}$, $\mathcal{O}(1)$, ..., $\mathcal{O}(n)$ on $\mathbb{P}^n$, which generate $\mathrm{D^b}(\mathbb{P}^n)$. Set $\mathcal{V}_n := \mathcal{O} \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(n)$. By [B2] there exists a derived equivalence

$$\mathbb{R}\operatorname{Hom}(\mathcal{V}_n, -) : \mathrm{D^b}(\operatorname{coh}\mathbb{P}^n) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{mod}\operatorname{End}_{\mathbb{P}^n}(\mathcal{V}_n)),$$

where $\mathbb{R}\operatorname{Hom}(\mathcal{V}_n, -)$ denotes the derived functor of $\operatorname{Hom}(\mathcal{V}_n, -)$. The algebra $\operatorname{End}_{\mathbb{P}^n}(\mathcal{V}_n)$ is noncommutative and can be presented as a quiver with relations. Thus, one can now study the algebraic properties of this quiver to learn about the geometry of $\mathbb{P}^n$.

More generally, if a variety $\mathcal{X}$ admits a tilting bundle (essentially, a generator $\mathcal{V}$ of $\mathrm{D}^b(\operatorname{coh}\mathcal{X})$ with vanishing higher self-Ext group) then $\mathrm{D}^b(\operatorname{coh}\mathcal{X})$ is derived equivalent to $\mathrm{D}^b(\operatorname{mod}\operatorname{End}_{\mathcal{X}}(\mathcal{V}))$ [V1]. In Example 1.2.1 above, the bundle $\mathcal{O} \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(n)$ is a tilting bundle on $\mathbb{P}^n$.

This idea also applies to crepant resolutions of Kleinian singularities as follows.

Theorem 1.2.2. [KV] *Let $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ be the crepant resolution of a Kleinian singularity, and suppose $\mathcal{R}, M_1, \dots M_n$ are the indecomposable maximal Cohen–Macaulay $\mathcal{R}$-modules. Then there is a derived equivalence*

$$\mathrm{D^b}(\operatorname{coh}\mathcal{X}) \simeq \mathrm{D^b}\left(\operatorname{mod}\operatorname{End}_{\mathcal{R}}(\mathcal{R} \oplus \bigoplus_{i=1}^{n} M_i)\right).$$

Since the derived category captures all the homological data of an object, this theorem shows that the homological information of the variety $\mathcal{X}$ is precisely the same as that of the algebra $\operatorname{End}_{\mathcal{R}}(\mathcal{R} \oplus \bigoplus_{i=1}^{n} M_i)$.

Inspired by the above result for Kleinian singularities, Van den Bergh introduced the notion of *noncommutative crepant resolutions* (NCCRs) in dimension three [V2].

Definition 1.2.3. *A* noncommutative crepant resolution (NCCR) *of a Gorenstein ring $\mathcal{R}$ is a ring of the form $\Lambda := \operatorname{End}_{\mathcal{R}}(M)$ for some finitely generated reflexive $\mathcal{R}$-module M, such that Λ has finite global dimension and is maximal Cohen-Macaulay* (CM) *as an $\mathcal{R}$-module.*

Given a crepant resolution $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ where $\mathcal{R}$ is a cDV singularity, Van den Bergh demonstrated how such an algebra Λ can be constructed. This leads to the following theorem, which may be regarded as a three-dimensional analogue of Theorem 1.2.2.

Theorem 1.2.4. [V1] *Let $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ be a crepant resolution of a cDV singularity* $\operatorname{Spec}\mathcal{R}$. *Then there exists a* CM *$\mathcal{R}$-module M such that $\Lambda := \operatorname{End}_{\mathcal{R}}(M)$ is an NCCR of $\mathcal{R}$ and further, there is a derived equivalence*

$$\mathrm{D^b}(\operatorname{coh}\mathcal{X}) \simeq \mathrm{D^b}(\operatorname{mod}\Lambda).$$

The variety $\mathcal{X}$ can in fact be recovered from the algebra Λ via quiver GIT (Geometric Invariant Theory) [K4]. This shows that to study a crepant resolution of a cDV singularity, one can equivalently study the corresponding NCCR.

Example 1.2.5. (Pagoda Flop) Consider the cDV singularity given by

$$\mathcal{R} = \mathbb{C}[[u, v, x, y]]/(uv - x(x + y^n)),$$

where $n \geq 1$. Note that when $n = 1$, $\operatorname{Spec} \mathcal{R}$ is isomorphic to the $\mathcal{X}$ in Example 1.1.2 (Atiyah Flop), up to a coordinate change. Similarly, here $\operatorname{Spec} \mathcal{R}$ also admits two minimal models.

One NCCR is $\operatorname{End}_{\mathcal{R}}(\mathcal{R} \oplus (u, x))$, which can be presented as the following quiver with relations:

$$\begin{tikzcd} \mathcal{R} \arrow[loop left, "l_1"] \arrow[rr, bend left=40, "a"] \arrow[rr, bend left=10, "b"] && (u,x) \arrow[loop right, "l_2"] \arrow[ll, bend left=10, "s"] \arrow[ll, bend left=40, "t"] \end{tikzcd}
\qquad
\begin{aligned} l_1 a &= a l_2 \\ l_1 b &= b l_2 \\ l_2 s &= s l_1 \\ l_2 t &= t l_1 \\ l_1^n &= at - bs \\ l_2^n &= ta - sb \end{aligned}$$

Example 1.2.6. Consider the cDV singularity given by

$$\mathcal{R} = \mathbb{C}[[u, v, x, y]]/(uv - xy(x + y^n)),$$

where $n \geq 1$. In this example, $\operatorname{Spec} \mathcal{R}$ has six minimal models (see e.g. [SW]).

One NCCR is $\operatorname{End}_{\mathcal{R}}(\mathcal{R} \oplus (u, x) \oplus (u, xy))$, which can be presented as the following quiver with relations

$$\begin{tikzcd} (u,x) \arrow[rr, bend left=10, "a_1"] \arrow[dr, bend left=10, "b_0"] && (u,xy) \arrow[ll, bend left=10, "b_1"] \arrow[dl, bend left=10, "a_2"] \\ & \mathcal{R} \arrow[loop below, "l"] \arrow[ul, bend left=10, "a_0"] \arrow[ur, bend left=10, "b_2"] & \end{tikzcd}
\qquad
\begin{aligned} l a_0 &= a_0 a_1 b_1 \\ b_0 l &= a_1 b_1 b_0 \\ a_2 l &= b_1 a_1 a_2 \\ l b_2 &= b_2 b_1 a_1 \\ b_1 b_0 a_0 + b_1 (a_1 b_1)^n &= a_2 b_2 b_1 \\ b_0 a_0 a_1 + (a_1 b_1)^n a_1 &= a_1 a_2 b_2 \\ a_0 b_0 + l^n &= b_2 a_2 \end{aligned}$$

When π is a singular minimal model (respectively, a crepant partial resolution), Iyama–Wemyss [IW2] generalise the notion of NCCRs to *maximal modifying algebras* and *modifying algebras*, respectively. These generalizations also satisfy the derived equivalences described in Theorem 1.2.4, along with other desirable properties (see also Section 2.3).

§ 1.3 | Contraction algebras

Contraction algebras originally arose from a somewhat different motivation: the introduction of noncommutative algebras into deformation theory. The basic idea of deformation theory is to study how structures can be extended along infinitesimal directions.

More precisely, given a geometric object Y and a commutative local ring S, a deformation of Y over S is a flat family $\mathcal{Y}$ over S whose fibre over the closed point is precisely Y.

For example, given a crepant partial resolution $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ with a single exceptional curve, we can consider deformations of that curve. Donovan–Wemyss [DW1] introduced noncommutative deformations of this curve and showed that considering only commutative deformations fails to capture certain geometric features. This provides further evidence that noncommutative algebra is a powerful tool in algebraic geometry.

In this setting of a single exceptional curve, the contraction algebra can be defined as the representing object of the functor of noncommutative deformations of the curve. For crepant partial resolutions with multiple exceptional curves, the contraction algebra can be defined similarly—as the representing object of a functor of pointed noncommutative deformations.

Due to the correspondence between crepant partial resolutions of a cDV singularity and its modifying algebras (see §2.3.4), we adopt the following equivalent definition [DW1].

Definition 1.3.1. *Given a crepant partial resolution $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ of a complete local cDV singularity, let M be the corresponding* CM *$\mathcal{R}$-module in 2.3.4. Then, the contraction algebra $\Lambda_{\mathrm{con}}(\pi)$ is defined to be*

$$\underline{\operatorname{End}}_{\mathcal{R}}(M) := \operatorname{End}_{\mathcal{R}}(M)/\langle \mathcal{R} \rangle,$$

where $\langle \mathcal{R} \rangle$ denotes the two-sided ideal consisting of all morphisms which factor through $\operatorname{add} \mathcal{R}$.

Since $\mathcal{R}$ is a direct summand of the CM $\mathcal{R}$-module M, then the quiver of the contraction algebra $\underline{\operatorname{End}}_{\mathcal{R}}(M)$ can be obtained from that of the NCCR $\operatorname{End}_{\mathcal{R}}(M)$ simply by deleting the vertex corresponding to $\mathcal{R}$.

Example 1.3.2. Consider the Pagoda Flop in Example 1.2.5. The contraction algebra associated to $\operatorname{End}_{\mathcal{R}}(\mathcal{R} \oplus (u, x))$ can be presented as the following quiver with relations

$$\begin{tikzcd} (u,x) \arrow[loop right, "l_2"] \end{tikzcd} \qquad l_2^n = 0$$

Thus the contraction algebra is isomorphic to $\mathbb{C}[[l_2]]/(l_2^n)$.

Example 1.3.3. Consider Example 1.2.6. The contraction algebra associated to $\operatorname{End}_{\mathcal{R}}(\mathcal{R} \oplus (u, x) \oplus (u, xy))$ can be presented as the following quiver with relations

$$\begin{tikzcd} (u,x) \arrow[r, bend left, "a_1"] & (u,xy) \arrow[l, bend left, "b_1"] \end{tikzcd} \qquad \begin{aligned} b_1(a_1b_1)^n &= 0 \\ (a_1b_1)^n a_1 &= 0 \end{aligned}$$

Although the contraction algebra $\Lambda_{\mathrm{con}}(\pi)$ is a quotient of the modifying algebra $\mathrm{End}_{\mathcal{R}}(M)$, it still recovers all known invariants of the crepant partial resolution π of a cDV singularity $\mathcal{R}$, as follows:

(1) The quiver representation of the contraction algebra determines the dual graph, including the normal bundle of the exceptional curves [W2].

(2) The crepant partial resolution is flopping (i.e., not contracting a divisor) if and only if the dimension of its associated contraction algebra is finite [DW2] (see also §2.3.6).

If furthermore π is a crepant resolution and $\mathcal{R}$ is isolated, then

(3) The dimension of the contraction algebra is a weighted sum of the Gopakumar–Vafa (GV) invariants of the crepant resolution [T2] (see also 1.4.1).

(4) The contraction algebra determines the GV invariants [HT, T2], and moreover, it is a strictly stronger invariant than the GV invariants themselves [BW1].

(5) The contraction algebra determines the cDV singularity $\mathcal{R}$ [JKM, A.2] (see also 2.3.7).

§ 1.4 | Gopakumar–Vafa invariants

In this section, we introduce a curve invariant known as the Gopakumar–Vafa (GV) invariant, which can be thought of as a virtual count of curves in a given curve class on a variety.

Gopakumar–Vafa (GV) invariants are designed to count the number of pseudo-holomorphic curves and represent the number of BPS states on a Calabi-Yau 3-fold; it has been conjectured that this is equivalent to other curve counting Gromov-Witten invariants and Pandharipande–Thomas invariants [MT].

The general approach to calculate GV invariants is to consider the moduli space of one-dimensional stable sheaves on Calabi–Yau 3-folds satisfying some numerical conditions [K2], and as such, it is usually hard to calculate them.

Now let $\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ be a crepant resolution with exceptional curves $\bigcup_i \mathrm{C}_i$ where $\mathcal{R}$ is a cDV. Denote $A_1(\pi) := \bigoplus_i \mathbb{Z}\langle \mathrm{C}_i \rangle$ be the abelian group freely generated by C_i.

Given a curve class $\beta \in A_1(\pi)$ there is a Gopakumar–Vafa (GV) invariant $\mathrm{GV}_\beta(\pi)$ which counts the class β in $\mathcal{X}$ virtually. There are several equivalent interpretations of $\mathrm{GV}_\beta(\mathcal{X})$ (see 2.4.1).

Roughly speaking, the idea is to deform π into a disjoint union of the simplest types of exceptional curves. Then, $\mathrm{GV}_\beta(\pi)$ corresponds to the number of such curves with class β. However, the count is not naive—we refine it by using the structure of the flat family. This allows us to split the total number of curves into contributions from specific curve classes [BKL].

As noted in §1.3, if $\mathcal{R}$ is isolated, then the contraction algebra determines the GV invariants

[HT, T2]. Moreover, one can extract GV invariants from the dimension of the contraction algebra using the following result.

Theorem 1.4.1. (Toda's formula, [T2, §4.4]) *Let* $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ *be a crepant resolution of an isolated* cDV *singularity* $\mathcal{R}$ *with* m *exceptional curves. Then*

$$\dim_{\mathbb{C}} \Lambda_{\mathrm{con}}(\pi) = \sum_{\beta=(\beta_1,\dots,\beta_m)} |\beta|^2 \mathrm{GV}_\beta(\pi),$$

where $|\beta| = \beta_1 + \cdots + \beta_m$.

Example 1.4.2. Consider the Pagoda Flop from Example 1.2.5, and let π denote the crepant resolution associated to $\mathcal{R} \oplus (u, x)$. Then $\Lambda_{\mathrm{con}}(\pi) \cong \mathbb{C}[[l_2]]/(l_2^n)$, and so $\dim_{\mathbb{C}} \Lambda_{\mathrm{con}}(\pi) = n$.

Since π has a single exceptional curve C, Toda's formula (Theorem 1.4.1) implies that $\mathrm{GV}_{\mathrm{C}}(\pi) = n$.

Example 1.4.3. Consider Example 1.3.3, and let π denote the crepant resolution associated to $\mathcal{R} \oplus (u, x) \oplus (u, xy)$. There are two exceptional curves, C_1 and C_2, in π.

The $\Lambda_{\mathrm{con}}(\pi)$ has a $\mathbb{C}$-basis given by

$$\{e_1, a_1, a_1b_1, a_1b_1a_1, \dots, (a_1b_1)^n\} \cup \{e_2, b_1, b_1a_1, b_1a_1b_1, \dots, (b_1a_1)^n\},$$

where e_i denotes the trivial path at vertex i in the quiver presentation. Therefore, $\dim_{\mathbb{C}} \Lambda_{\mathrm{con}}(\pi) = 4n + 2$. It is known by e.g. [NW] that the only nonzero GV invariants are:

$$\mathrm{GV}_{\mathrm{C}_1}(\pi) = 1, \quad \mathrm{GV}_{\mathrm{C}_2}(\pi) = 1, \quad \mathrm{GV}_{\mathrm{C}_1+\mathrm{C}_2}(\pi) = n.$$

We can verify Toda's formula as follows:

$$\sum_{\beta=(\beta_1,\beta_2)} |\beta|^2 \mathrm{GV}_\beta(\pi) = 1^2 \cdot \mathrm{GV}_{\mathrm{C}_1}(\pi) + 1^2 \cdot \mathrm{GV}_{\mathrm{C}_2}(\pi) + 2^2 \cdot \mathrm{GV}_{\mathrm{C}_1+\mathrm{C}_2}(\pi) = 1 + 1 + 4n = 4n + 2.$$

§ 1.5 | Main results

In this section, we summarise the main contributions of the thesis.

§ 1.5.1 | Generalised GV invariants

Two additional tools greatly simplify the computation of GV invariants in the context of crepant partial resolutions of cA_n singularities. The first comes from Toda's formula [T2] as well as [HT, BW2], which suggests that GV invariants can be calculated by the dimension of their associated contraction algebra. The second comes from [IW3], which gives a concrete algebraic description of all crepant partial resolutions of cA_n singularities and their associated contraction algebras.

This subsection presents the consequences of these developments for curve-counting theories in algebraic geometry—specifically GV invariants—and generalises them in two important directions:

- to crepant *partial* resolutions, and
- to *non-isolated* cA_n singularities.

It is worth highlighting that similar curve-counting invariants have also been investigated in the physics literature. In particular, computations in [CSV, C3, DSV] evaluate M2-brane BPS state counts—corresponding to five-dimensional hypermultiplets—for various classes of cDV singularities. These include cases of crepant resolutions with a single exceptional curve, as well as crepant partial resolutions of quasi-homogeneous isolated cDV singularities. Although physically motivated, the resulting invariants yield precise mathematical predictions that, in the cA_n setting, coincide with our generalised GV invariants.

Thus, the framework developed in this thesis provides a natural mathematical generalisation of these physical calculations: it extends the validity of Toda's formula to crepant *partial* resolutions and to *non-isolated* cA_n singularities, while simultaneously offering an algebraic description in terms of contraction algebras. In this way, the results here not only recover the predictions from physics in special cases but also place them in a broader and more systematic mathematical setting.

Throughout, let $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ be a crepant *partial* resolution where $\mathcal{R}$ is a (not necessarily isolated) cA_n singularity. The case when $\mathcal{X}$ is smooth, equivalently when π is a crepant resolution, will recover classical invariants and results.

We first introduce our new invariants, $N_\beta(\pi)$, which does not require smoothness of $\mathcal{X}$, or $\mathcal{R}$ to be isolated. To do this, write $\mathrm{C}_1, \mathrm{C}_2, \dots, \mathrm{C}_m$ for the exceptional curves of π. For any curve class $\beta \in \bigoplus_{i=1}^m \mathbb{Z}\langle \mathrm{C}_i \rangle$, consider

$$N_\beta(\pi) := \begin{cases} \dim_{\mathbb{C}} \frac{\mathbb{C}[\![x,y]\!]}{I_\beta} & \text{if } \beta = \mathrm{C}_i + \mathrm{C}_{i+1} + \ldots + \mathrm{C}_j \\ 0 & \text{else} \end{cases}$$

where $I_\beta \in (x, y)$ is an ideal that depends on β and π (see 3.1.1).

The above generalised GV invariant is parallel to GV invariants, since when π is a crepant resolution, then $\{\mathrm{C}_i + \mathrm{C}_{i+1} + \cdots + \mathrm{C}_j \mid 1 \le i \le j \le m\}$ are the only curve classes with non-zero GV invariants [NW, V5].

We will show in 1.5.3 that in the special case when $\mathcal{X}$ is smooth, N_β is equivalent to GV_β for all curve class β, where GV_β is the integer-valued Gopakumar–Vafa (GV for short) invariant of β. This justifies us calling the N_β generalised GV invariants.

The following is our first result, which shows that Toda's formula 1.4.1 holds in this more general setting.

Proposition 1.5.1 (3.2.4, 3.3.11)**.** *Let π be a crepant partial resolution of a cA_n singularity with m exceptional curves. For any $1 \leq s \leq t \leq m$, the following equality holds.*

$$\dim_{\mathbb{C}} e_s \Lambda_{\text{con}}(\pi) e_t = \sum_{\beta=(\beta_1,\dots,\beta_m)} \beta_s \cdot \beta_t \cdot N_\beta(\pi) = \dim_{\mathbb{C}} e_t \Lambda_{\text{con}}(\pi) e_s.$$

In particular, $\dim_{\mathbb{C}} \Lambda_{\text{con}}(\pi) = \sum_\beta |\beta|^2 N_\beta(\pi)$ *where* $|\beta| = \beta_1 + \cdots + \beta_m$.

Hua–Toda [HT, T2] show that when $\mathcal{X}$ is smooth and $\mathcal{R}$ is isolated, the GV invariants are a property of the isomorphism class of the contraction algebra. The following generalises this to the crepant partial resolutions of (not necessarily isolated) cA_n singularities.

To ease notation, given a curve class $\beta = (\beta_1, \dots, \beta_m)$, denote the *reflective curve class* of β to be $\overline{\beta} := (\beta_m, \dots, \beta_1)$. This symmetry arises naturally from the involution of the doubled A_n quiver, which reverses the orientation of the chain. Since the contraction algebra is isomorphic to a quiver algebra of the doubled A_n quiver, the reflective class corresponds to this quiver involution.

Theorem 1.5.2 (3.2.7, 3.3.11)**.** *Let $\pi_k \colon \mathcal{X}_k \to \operatorname{Spec} \mathcal{R}_k$ be two crepant partial resolutions of cA_{n_k} singularities $\mathcal{R}_k$ with m_k exceptional curves for $k = 1, 2$. If $\Lambda_{\text{con}}(\pi_1) \cong \Lambda_{\text{con}}(\pi_2)$, then $m_1 = m_2$ and one of the following cases holds:*

1. $N_\beta(\pi_1) = N_\beta(\pi_2)$ *for any curve class* β,
2. $N_\beta(\pi_1) = N_{\overline{\beta}}(\pi_2)$ *for any curve class* β.

The papers [NW, V5] give a combinatorial description of the matrix which controls the transformation of the non-zero GV invariants under a flop (see §3.3.1 for cA_n cases). We show in 3.2.8 that the generalised GV invariants also satisfy this transformation.

We next restrict ourselves to cases of crepant resolutions of (not necessarily isolated) cA_n singularities and show that whilst generalised GV invariants are not always equal to the GV invariant, they are equivalent information.

Theorem 1.5.3 (3.3.8, 3.3.11)**.** *Let π be a crepant resolution of a cA_n singularity. The following holds for any curve class β.*

1. $N_\beta(\pi) = \infty \iff \mathrm{GV}_\beta(\pi) = -1$.
2. $N_\beta(\pi) < \infty \iff \mathrm{GV}_\beta(\pi) = N_\beta(\pi)$.

Together with 1.5.2, the following shows that the contraction algebra determines its associated GV invariants. This generalises the results in [HT, T2] to non-isolated cA_n cases.

Corollary 1.5.4 (3.2.7, 3.3.11)**.** *Let $\pi_k \colon \mathcal{X}_k \to \operatorname{Spec} \mathcal{R}_k$ be two crepant resolutions of cA_n singularities $\mathcal{R}_k$ for $k = 1, 2$. If $\Lambda_{\text{con}}(\pi_1) \cong \Lambda_{\text{con}}(\pi_2)$, then one of the following holds:*

1. $\mathrm{GV}_\beta(\pi_1) = \mathrm{GV}_\beta(\pi_2)$ *for any curve class* β,
2. $\mathrm{GV}_\beta(\pi_1) = \mathrm{GV}_{\overline{\beta}}(\pi_2)$ *for any curve class* β.

§ 1.5.2 | Monomialization and geometric realization

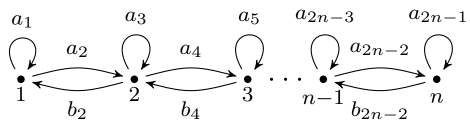
We now restrict our attention to the smooth case. Let $\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ be a crepant resolution with $\mathcal{R}$ cDV. The contraction algebra $\Lambda_{\text{con}}(\pi)$ is isomorphic to the Jacobi algebra of a quiver with some potential [V3], and it classifies $\operatorname{Spec} \mathcal{R}$ complete locally if $\mathcal{R}$ is furthermore isolated [JKM] (see also 2.3.7).

This motivates classifying Jacobi algebras (equivalently, their potentials) on various quivers, as this immediately then classifies certain crepant resolutions.

In this subsection, we introduce various intrinsic algebraic definitions of a Type A potential on the double A_n quiver Q_n (with a single loop at each vertex). Then via coordinate changes, we give a monomialization result that expresses these potentials in a particularly nice form, and show that these potentials precisely correspond to cA_n crepant resolutions, which solves the Realisation Conjecture of Brown–Wemyss in Type A cases [BW2].

Together, these results can be viewed as a noncommutative generalization of the classification of simple singularities by commutative polynomials [A1], and also a generalisation of the fact that the germ of a complex analytic hypersurface with an isolated singularity is determined by its Tjurina algebra [MY].

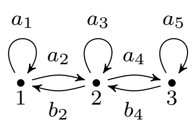
For any fixed $n \geq 1$, consider the following quiver Q_n, which is the double of the usual A_n quiver, with a single loop at each vertex. Label the arrows of Q_n left to right, as illustrated below.



Quiver Q_n which has loop a_{2i-1} at each vertex i.

From this, define elements x_i and x'_i as follows: first, set b_{2i-1} to be the trivial path e_i at vertex i, for any $1 \leq i \leq n$. Then for any $1 \leq i \leq 2n-1$, set $\mathsf{x}_i := a_i b_i$ and $\mathsf{x}'_i := b_i a_i$.

For example, in the case $n = 3$,



$$\mathsf{x}_1 = \mathsf{x}'_1 = a_1$$
$$\mathsf{x}_3 = \mathsf{x}'_3 = a_3$$
$$\mathsf{x}_5 = \mathsf{x}'_5 = a_5$$

whereas $\mathsf{x}_2 = a_2 b_2$, $\mathsf{x}'_2 = b_2 a_2$, and $\mathsf{x}_4 = a_4 b_4$, $\mathsf{x}'_4 = b_4 a_4$.

Given the above x_i and x'_i, we first define a *reduced Type A potential* on Q_n to be any reduced potential f that contains the terms $\mathsf{x}'_i \mathsf{x}_{i+1}$ for all $1 \leq i \leq 2n-2$. A *Type A potential* on Q_n is then defined in 4.1.4, but for this introduction we only require the concept of a *monomialized*

Type A potential on Q_n, which is defined to be any potential of the form

$$\sum_{i=1}^{2n-2} \mathsf{x}_i'\mathsf{x}_{i+1} + \sum_{i=1}^{2n-1}\sum_{j=2}^{\infty} k_{ij}\mathsf{x}_i^j$$

for some $k_{ij} \in \mathbb{C}$. We will show in 4.1.20 and 4.1.23 that any Type A potential is isomorphic to some monomialized Type A potential, and so the above monomialized version suffices.

The first main result is that the complete Jacobi algebra (denoted $\mathfrak{Jac}$) of any Type A potential on Q_n can be realized as the contraction algebra of a crepant resolution of some cA_n singularity.

Theorem 1.5.5 (4.2.12)**.** *For any Type A potential f on Q_n where $n \geq 1$, there exists a crepant resolution $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ where $\mathcal{R}$ is cA_n, such that $\mathfrak{Jac}(f) \cong \Lambda_{\mathrm{con}}(\pi)$.*

The Brown–Wemyss Realisation Conjecture [BW2] states that if f is any potential for which $\mathfrak{Jac}(f)$ is either finite-dimensional, or infinite-dimensional but with at most linear growth in the successive quotients by powers of its Jacobi ideal, then $\mathfrak{Jac}(f)$ is isomorphic to the contraction algebra of some crepant resolution $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$, with R cDV. The above result 1.5.5 confirms this Realisation Conjecture for any Type A potential on Q_n with $n \geq 1$.

We then obtain the converse to 1.5.5 (see 4.2.15), which shows that our definition of Type A potential is intrinsic. The definition of the quiver $Q_{n,I}$ and Type $A_{n,I}$ crepant resolutions are given in §4.1 and 4.2.11.

Corollary 1.5.6 (4.2.16)**.** *Let f be a reduced potential on $Q_{n,I}$. The following are equivalent.*

1. *f is Type A.*
2. *There exists a Type $A_{n,I}$ crepant resolution π such that $\mathfrak{Jac}(f) \cong \Lambda_{\mathrm{con}}(\pi)$.*
3. *$e_i\mathfrak{Jac}(f)e_i$ is commutative for $1 \leq i \leq n$.*

Moreover, there is a correspondence between crepant resolutions of cA_n singularities and our intrinsic noncommutative monomialized Type A potentials, as follows.

Corollary 1.5.7 (4.2.19)**.** *For any n, the set of isomorphism classes of contraction algebras associated to crepant resolutions of cA_n singularities is equal to the set of isomorphism classes of Jacobi algebras of monomialized Type A potentials on Q_n.*

Then, after restricting to those cA_n singularities which are isolated, we obtain the following consequence.

Theorem 1.5.8 (4.2.21)**.** *For any n, there exists a one-to-one correspondence*

$$\begin{array}{c} \textit{isomorphism classes of isolated } cA_n \textit{ singularities} \\ \textit{which admit smooth flopping contractions} \\ \updownarrow \\ \textit{derived equivalence classes of monomialized Type A potentials on } Q_n \\ \textit{with finite-dimensional Jacobi algebra} \end{array}$$

We establish the correspondence in 1.5.8 for isolated cA_n singularities, as our proof relies on the Donovan–Wemyss Conjecture 2.3.7, which is known to hold only for isolated cDV singularities. This naturally suggests the idea of extending the Donovan–Wemyss Conjecture to non-isolated cases.

However, in §4.2.3, we present an explicit example of a non-isolated cA_2 singularity, demonstrating in 4.2.26 that the Donovan–Wemyss Conjecture does not extend to non-isolated cDV singularities.

§ 1.5.3 | Filtrations and obstructions

In this subsection, we continue under the assumption that $\mathcal{X}$ is smooth (equivalently, that π is a crepant resolution).

The correspondence established in 1.5.7—between crepant resolutions of cA_n singularities and monomialized Type A potentials on Q_n—motivates filtration structures of the parameter space of such monomialized potentials with respect to their generalised GV invariants.

Recall that a *monomialized Type A potential* on Q_n is any potential of the form

$$\sum_{i=1}^{2n-2} \mathsf{x}_i' \mathsf{x}_{i+1} + \sum_{i=1}^{2n-1} \sum_{j=2}^{\infty} k_{ij} \mathsf{x}_i^j, \tag{1.5.A}$$

for some $k_{ij} \in \mathbb{C}$. Since contraction algebra determines its associated GV invariants in 1.5.4, the correspondence in 1.5.7 inspires us to approach GV invariants of cA_n crepant resolutions through their corresponding monomialized Type A potentials on Q_n.

So, given any n, we consider the set of all monomialized Type A potentials on Q_n (1.5.A)

$$f(\mathsf{\kappa}) = \sum_{i=1}^{2n-2} \mathsf{x}_i' \mathsf{x}_{i+1} + \sum_{i=1}^{2n-1} \sum_{j=2}^{\infty} \mathsf{\kappa}_{ij} \mathsf{x}_i^j,$$

over the parameter space

$$\mathsf{M} := \{(k_{12}, k_{13}, \dots, k_{22}, k_{23}, \dots, k_{2n-1,2}, k_{2n-1,3}, \dots) \mid \text{all } k_{ij} \in \mathbb{C}\}.$$

Based on the above correspondence between monomialized Type A potentials on Q_n and crepant resolutions of cA_n singularities, given any $f \in \mathsf{M}$ we define generalised GV invariants $N_\beta(f)$ through its associated crepant resolution (see 5.2.1).

The following gives a filtration structure on the parameter space M of monomialized Type A potentials on Q_n with respect to generalised GV invariants.

Theorem 1.5.9 (5.3.8). *Fix some s, t satisfying $1 \leq s \leq t \leq n$ and the curve class $\beta = \mathrm{C}_s + \mathrm{C}_{s+1} + \cdots + \mathrm{C}_t$. Then M has a filtration structure $\mathsf{M} = M_1 \supsetneq M_2 \supsetneq M_3 \supsetneq \cdots$ such that*

(1) *For each $i \geq 1$, $N_\beta(f(k)) = i$ for all $k \in M_i \backslash M_{i+1}$.*

(2) *Each M_i is the zero locus of some polynomial system of κ.*

(3) *If $s = t$, then for each $i \geq 2$, $M_i = \{k \in \mathsf{M} \mid k_{2s-1,j} = 0 \text{ for } 2 \leq j \leq i\}$.*

It should be emphasized that the filtration in 1.5.9 strongly depends on the curve class β; as these vary, so does the filtration.

For any curve class β and $N \in \mathbb{N}_\infty := \mathbb{N}\cup\infty$, then by 1.5.9 there exists a crepant resolution π of a cA_n singularity such that $N_\beta(\pi) = N$. However, this is no longer true when considering generalised GV invariants of different curve classes simultaneously. So we next discuss the obstructions and constructions of the generalised GV invariants that can arise from crepant resolutions of cA_n singularities.

Notation 1.5.10 (5.4.3, 5.4.4). Fix some curve class $\beta = \mathrm{C}_s + \mathrm{C}_{s+1} + \cdots + \mathrm{C}_t$, and a tuple $(q_s, q_{s+1}, \ldots, q_t) \in \mathbb{N}_\infty^{t-s+1}$. Set $\mathbf{q}_{\min} := \min\{q_i\}$, and consider the subset of crepant resolutions of cA_n singularities with respect to $(q_s, \ldots, q_t)$ defined as

$$\mathsf{CA}_{\mathbf{q}} := \{cA_n \text{ crepant resolution } \pi \mid (N_{\mathrm{C}_s}(\pi), N_{\mathrm{C}_{s+1}}(\pi), \ldots, N_{\mathrm{C}_t}(\pi)) = (q_s, q_{s+1}, \ldots, q_t)\}.$$

The following is the main obstruction result, which is new even in the case when $\mathcal{X}$ is smooth and $\mathcal{R}$ is isolated (in which case $N_\beta = \mathrm{GV}_\beta$ by 1.5.4).

Theorem 1.5.11 (5.4.7). *For any s and t with $1 \leq s \leq t \leq n$, and any tuple $(q_s, q_{s+1}, \ldots, q_t) \in \mathbb{N}_\infty^{t-s+1}$, with notation in 1.5.10 and $\beta := \mathrm{C}_s + \mathrm{C}_{s+1} + \cdots + \mathrm{C}_t$, the following statements hold.*

(1) *For any $\pi \in \mathsf{CA}_{\mathbf{q}}$ necessarily $N_\beta(\pi) \geq \mathbf{q}_{\min}$, and moreover there exists $\pi \in \mathsf{CA}_{\mathbf{q}}$ such that $N_\beta(\pi) = \mathbf{q}_{\min}$.*

(2) *When $\mathbf{q}_{\min}$ is finite, the equality $N_\beta(\pi) = \mathbf{q}_{\min}$ holds for all $\pi \in \mathsf{CA}_{\mathbf{q}}$ if and only if $\#\{i \mid q_i = \mathbf{q}_{\min}\} = 1$.*

We show in 5.4.12 that the actions on curve classes from [NW, 5.4] and [V5, 5.10], together with 1.5.11, give more obstructions and constructions of the possible tuples that can arise. One sample result is the following; many others are left to the end of §5.4.

Corollary 1.5.12 (5.4.15)**.** *The generalised GV invariants of crepant resolutions of cA_2 singularities have the following two possibilities:*

$$\begin{array}{cc} N_{\mathrm{C}_1} & N_{\mathrm{C}_2} \\ \multicolumn{2}{c}{N_{\mathrm{C}_1+\mathrm{C}_2}} \end{array} = \begin{array}{cc} p & q \\ \multicolumn{2}{c}{\min(p,q)} \end{array} \ \ \textit{or} \ \ \begin{array}{cc} p & p \\ \multicolumn{2}{c}{r} \end{array}$$

where p, q, $r \in \mathbb{N}_\infty$ with $p \neq q$ and $r \geq p$. All possible such p, q, r arise.

§ 1.5.4 | Special cases: A_3

In the case of the double A_3 quiver without loops, it is possible to describe the full isomorphism classes of Type A potentials, and the derived equivalence classes of those with finite-dimensional Jacobi algebras. This generalises [DWZ, E1, H2].

To ease notation, consider now the following labelling.

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left, "a_1"] & \underset{2}{\bullet} \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \underset{3}{\bullet} \arrow[l, bend left, "b_2"] \end{tikzcd} \qquad \mathsf{x} = b_1 a_1, \ \mathsf{y} = a_2 b_2$$

Double A_3 quiver without loops Q

Given two potentials f and g on Q, we say that f is *isomorphic* to g, written $f \cong g$, if the corresponding Jacobi algebras are isomorphic (see 2.1.8). Similarly, we say that f is *derived equivalent* to g, written $f \simeq g$, if the corresponding Jacobi algebras are derived equivalent (see 4.2.20).

Theorem 1.5.13 (6.1.17)**.** *Any Type A potential on Q must be isomorphic to one of the following isomorphism classes of potentials:*

(1) $\mathsf{x}^2 + \mathsf{xy} + \lambda\mathsf{y}^2$ *for any* $0, \frac{1}{4} \neq \lambda \in \mathbb{C}$.

(2) $\mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 + \mathsf{x}^r$ *for any* $r \geq 3$.

(3) $\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q \cong \mathsf{x}^q + \mathsf{xy} + \mathsf{y}^p$ *for any* $(p, q) \neq (2, 2)$.

(4) $\mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2$.

(5) $\mathsf{x}^p + \mathsf{xy} \cong \mathsf{xy} + \mathsf{y}^p$ *for any* $p \geq 2$.

(6) xy.

The Jacobi algebras of these potentials are all mutually non-isomorphic (except those isomorphisms stated), and in particular the Jacobi algebras with different parameters in the same item are non-isomorphic.

The Jacobi algebras in (1)*,* (2)*,* (3) *are realized by crepant resolutions of isolated cA_3 singularities, and those in* (4)*,* (5)*,* (6) *are realized by crepant resolutions of non-isolated cA_3 singularities.*

Theorem 1.5.14 (6.2.6). *The following groups the Type A potentials on Q with finite-dimensional Jacobi algebra into sets, where all the Jacobi algebras in a given set are derived equivalent.*

(1) $\{\mathsf{x}^2 + \mathsf{xy} + \lambda'\mathsf{y}^2 \mid \lambda' = \lambda, \frac{1-4\lambda}{4}, \frac{1}{4(1-4\lambda)}, \frac{\lambda}{4\lambda-1}, \frac{4\lambda-1}{16\lambda}, \frac{1}{16\lambda}\}$ *for any* $\lambda \neq 0, \frac{1}{4}$.

(2) $\{\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^2,\ \mathsf{x}^2 + \mathsf{xy} + \mathsf{y}^p,\ \mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 + \mathsf{x}^p\}$ *for* $p \geq 3$.

(3) $\{\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q,\ \mathsf{x}^q + \mathsf{xy} + \mathsf{y}^p\}$ *for* $p \geq 3$ *and* $q \geq 3$.

Moreover, the Jacobi algebras of the sets in (1)–(3) *are all mutually not derived equivalent, and in particular the Jacobi algebras of different sets in the same item are not derived equivalent. In* (1) *there are no further basic algebras in the derived equivalence class, whereas in* (2)–(3) *there are an additional finite number of basic algebras in the derived equivalence class.*

Next, recall the definition of the quaternion type quiver algebra $A_{p,q}(\mu)$ in [E1, H2], which is the completion of the path algebra of the quiver Q modulo the relations

$$a_1a_2b_2 - (a_1b_1)^{p-1}a_1, b_2b_1a_1 - \mu(b_2a_2)^{q-1}b_2, a_2b_2b_1 - (b_1a_1)^{p-1}b_1, b_1a_1a_2 - \mu(a_2b_2)^{q-1}a_2,$$

where $\mu \in \mathbb{C}$ and $p, q \geq 2$. Note we have fewer relations than in [E1, H2] since we are working with the completion. In fact $A_{p,q}(\mu) \cong \mathfrak{Jac}(Q, f)$, where

$$f = \frac{1}{p}\mathsf{x}^p - \mathsf{xy} + \frac{\mu}{q}\mathsf{y}^q \cong \mathsf{x}^p + \mathsf{xy} + (-1)^q p^{-\frac{q}{p}} q^{-1}\mu\mathsf{y}^q.$$

The following improves various results of Erdmann and Holm [E1, H2].

Corollary 1.5.15 (6.2.9). *The following groups those algebras* $A_{p,q}(\mu)$ *which are finite-dimensional into sets, where all the algebras in a given set are derived equivalent.*

(1) $\{A_{2,2}(\mu') \mid \mu' = \mu, 1-\mu, \frac{1}{1-\mu}, \frac{\mu}{\mu-1}, \frac{\mu-1}{\mu}, \frac{1}{\mu}\}$ *for* $\mu \neq 0, 1$.

(2) $\{A_{p,q}(1),\ A_{q,p}(1)\}$ *for* $(p, q) \neq (2, 2)$.

Moreover, the algebras in different sets in (1)–(2) *are all mutually not derived equivalent. In* (1) *there are no further basic algebras in the derived equivalence class, whereas in* (2) *there are an additional finite number of basic algebras in the derived equivalence class.*

§ 1.6 | Outline of the thesis

Chapter 2 provides the necessary preliminaries, particularly covering quivers with potential, contraction algebras, and Gopakumar–Vafa (GV) invariants.

Chapter 3 introduces generalised GV invariants, extending the classical GV invariants to include crepant partial resolutions of cA_n singularities. We also show that these generalised invariants satisfy Toda's formula and are determined by their associated contraction algebras.

In Chapter 4, we introduce Type A potentials on Q_n and prove that every Type A potential can be transformed into a monomialized form. We then show that each such monomialized potential can be realized by a crepant resolution of a cA_n singularity. Finally, we establish a correspondence between crepant resolutions of cA_n singularities and our intrinsic Type A potentials.

Chapter 5 constructs filtration structures on the parameter space of monomialized Type A potentials on Q_n with respect to generalised GV invariants. Based on this filtration, we describe obstructions and constructions for the possible tuples of GV invariants that can arise from crepant resolutions of cA_n singularities.

In Chapter 6, we specialize to the simplest case: the doubled A_3 quiver without loops. We classify all Type A potentials on this quiver up to isomorphism, as well as those with finite-dimensional Jacobi algebras up to derived equivalence.

The Appendix provides a quiver presentation of the NCCR of the Type A universal flop, which is used in proving the geometric realization results presented in Chapter 4.

§ 1.7 | Notation and conventions

Throughout this paper, we work over the complex number $\mathbb{C}$, which is necessary for various statements in §2.2 and §2.3.

The definitions of $Q_{n,I}$ and x_i are fundamental, and are repeated in §4.1 for reference. We adopt the following notation:

(1) The integer n always refers to the n in cA_n singularities, and also to the number of vertices in the quivers Q_n and $Q_{n,I}$. The subset $I \subseteq \{1, 2, \ldots, n\}$ denotes the vertices without loops in the quiver $Q_{n,I}$.

(2) In Chapter 3, the integer m refers to the number of exceptional curves in a crepant partial resolution of a cA_n singularity. In Chapter 4, we define $m := 2n-1-|I|$, which equals the number of variables x_i in the quiver $Q_{n,I}$.

(3) R will always denote a set of relations in a quiver, except in the Appendix, where R refers to a reduction system for a path algebra.

(4) $\mathcal{R}$ always denotes a complete local cDV singularity. Moreover, $\mathcal{R}$ refers to a complete local cA_n singularity in Chapters 3, 4, and 5.

(5) Denote $\operatorname{CM}\mathcal{R}$ for the category of maximal Cohen–Macaulay $\mathcal{R}$-modules and $\underline{\operatorname{CM}}\,\mathcal{R}$ for the stable category of $\operatorname{CM}\mathcal{R}$.

(6) Denote $\operatorname{M}\mathcal{R}$ (resp. $\operatorname{MM}\mathcal{R}$) for the category of modifying (resp. maximal modifying) $\mathcal{R}$-modules.

(7) Given a crepant partial resolution π of a cDV singularity, we write $\Lambda(\pi)$ for the modi-

fication algebra and $\Lambda_{\mathrm{con}}(\pi)$ for the contraction algebra. All modules over these noncommutative rings are taken to be right modules.

(8) e_i denotes the trivial path at vertex i in Q_n, and also the trivial path at vertex i in the quiver presentations of $\Lambda(\pi)$ or $\Lambda_{\mathrm{con}}(\pi)$.

(9) Given a crepant (resp. partial) resolution $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ of a cA_n singularity $\mathcal{R}$, we write $\mathrm{GV}_{ij}(\pi)$ (resp. $N_{ij}(\pi)$) for the classical (resp. generalised) GV invariant of the curve class $\mathrm{C}_i + \mathrm{C}_{i+1} + \cdots + \mathrm{C}_j$ in $\mathcal{X}$.

(10) In Chapter 5, when considering the parameter space of monomialized Type A potentials, κ_{ij} denotes a variable and κ represents a tuple of such variables κ_{ij} (see Definition 5.1.1). In other chapters, κ_{ij} is simply treated as a complex number.

(11) We denote isomorphisms of algebras by $\cong$, and derived equivalences of triangulated categories by $\simeq$.

(12) The dimension of a vector space V over $\mathbb{C}$ is written as $\dim_{\mathbb{C}} V$.

Chapter 2

Preliminaries

In this chapter, we provide the necessary background for the results in this thesis. This includes a brief introduction to quivers with potential, minimal models of compound Du Val (cDV) singularities, the construction of contraction algebras, and various interpretations of Gopakumar–Vafa (GV) invariants.

§ 2.1 | Quiver with potential

To set notation, consider a *quiver* $Q = (Q_0, Q_1, t, h)$ which consists of a finite set of vertices Q_0, of arrows Q_1, with two maps $h\colon Q_1 \to Q_0$ and $t\colon Q_1 \to Q_0$ called the head and tail respectively.

A *loop* a is an arrow satisfying $h(a) = t(a)$, and a *path* is a formal expression $a_1a_2 \dots a_n$ where $h(a_i) = t(a_{i+1})$ for each $1 \le i \le n-1$. For such a path $a = a_1a_2 \cdots a_n$, we extend the notation by setting $t(a) := t(a_1)$ for its starting vertex and $h(a) := h(a_n)$ for its ending vertex. A path a is *cyclic* if $h(a) = t(a)$.

Given a field k, the *complete path algebra* $k\langle\langle Q\rangle\rangle$ is defined to be the completion of the usual path algebra kQ. That is, the elements of $k\langle\langle Q\rangle\rangle$ are possibly infinite k-linear combinations of paths in Q.

Write $\mathfrak{m}_Q$, or simply $\mathfrak{m}$, for the two-sided ideal of $k\langle\langle Q\rangle\rangle$ generated by the elements of Q_1, and write A_Q, or simply A, for the k-span of the elements of Q_1.

Definition 2.1.1. *Suppose that Q is a quiver with arrow span A.*

(1) *A* relation *of Q is a k-linear combination of paths in Q, each with the same head and tail.*

(2) *Given a finite number of specified relations $R_1, \dots, R_n$, we can form the closure of the two-sided idea $R := kQR_1kQ + \dots + kQR_nkQ$ of kQ. We call (Q, R)* a quiver with relations, *and we call $k\langle\langle Q\rangle\rangle/R$* the complete path algebra of a quiver with relations.

(3) *A* quiver with potential (QP *for short*) *is a pair (Q, W) where W is a k-linear combi-*

nation of cyclic paths.

(4) *For any $n \geq 1$, set W_n to be the nth homogeneous component of W with respect to the path length.*

(5) *For each $a \in Q_1$ and cyclic path $a_1 \ldots a_d$ in Q, define the* cyclic derivative *as*

$$\partial_a (a_1 \ldots a_d) = \sum_{i=1}^{d} \delta_{a,a_i} a_{i+1} \ldots a_d a_1 \ldots a_{i-1}$$

(where δ_{a,a_i} is the Kronecker delta), and then extend ∂_a by linearity.

(6) *For every potential W, the* Jacobi ideal *$J(W)$ is defined to be the closure of the two-sided ideal in $k\langle\langle Q\rangle\rangle$ generated by $\partial_a W$ for all $a \in Q_1$.*

(7) *The* Jacobi algebra *$\mathfrak{J}\mathrm{ac}(Q, W)$ or $\mathfrak{J}\mathrm{ac}(A, W)$ is the quotient $k\langle\langle Q\rangle\rangle/J(W)$. We write $\mathfrak{J}\mathrm{ac}(W)$ when the quiver Q is obvious.*

(8) *For every potential W, write ∂W for the k-span of $\partial_a W$ for all $a \in Q_1$.*

(9) *We call a QP (Q, W)* reduced *if $W_2 = 0$. It is called* trivial *if $W_n = 0$ for all $n \geq 3$, and further $\partial W = A$.*

Example 2.1.2. Consider the one loop quiver Q with potential $W = a^2$,

The complete path algebra $k\langle\langle Q\rangle\rangle$ is $k[[a]]$. Moreover, $\mathfrak{J}\mathrm{ac}(Q, W) \cong k[[a]]/(a) \cong k$ since $\partial_a(a^2) = 2a$. Since $W_n = 0$ for all $n \geq 3$ and $\partial W = ka = A_Q$, this QP (Q, W) is trivial.

Notation 2.1.3. For $\mathcal{A} := k\langle\langle Q\rangle\rangle$, consider $\{\mathcal{A}, \mathcal{A}\}$, the commutator vector space of $k\langle\langle Q\rangle\rangle$. That is, elements of $\{\mathcal{A}, \mathcal{A}\}$ are finite sums

$$\sum_{i=1}^{n} k_i (p_i q_i - q_i p_i)$$

for elements $p_i, q_i \in k\langle\langle Q\rangle\rangle$ and $k_i \in \mathbb{C}$. Write $\{\{\mathcal{A}, \mathcal{A}\}\}$ for the closure of the commutator vector space $\{\mathcal{A}, \mathcal{A}\}$.

Definition 2.1.4. *Two potentials W and W' are* cyclically equivalent *(written $W \sim W'$) if $W - W' \in \{\{\mathcal{A}, \mathcal{A}\}\}$. We write $W \overset{i}{\sim} W'$ if $W \sim W'$ and $W - W' \in \mathfrak{m}^i$.*

Remark 2.1.5. Note that if two potentials W and W' are cyclically equivalent, then $\partial_a W = \partial_a W'$ for all $a \in Q_1$, and hence $\mathfrak{J}\mathrm{ac}(Q, W) = \mathfrak{J}\mathrm{ac}(Q, W')$ [DWZ, 3.3]. Since we aim to classify the Jacobi algebras up to isomorphism, we always consider the potentials up to cyclic equivalence.

Given an algebra homomorphism $\varphi\colon k\langle\langle Q\rangle\rangle \to k\langle\langle Q'\rangle\rangle$ such that $\varphi|_k = id$ which sends $\mathfrak{m}_Q$ to $\mathfrak{m}_{Q'}$, write $\varphi|_{A_Q} = (\varphi_1, \varphi_2)$ where $\varphi_1\colon A_Q \to A_{Q'}$ and $\varphi_2\colon A_Q \to \mathfrak{m}_{Q'}^2$ are k-module homomorphisms.

Proposition 2.1.6. [DWZ, 2.4] *Given two quivers Q and Q', any pair (φ_1, φ_2) of k-module homomorphisms $\varphi_1\colon A_Q \to A_{Q'}$ and $\varphi_2\colon A_Q \to \mathfrak{m}_{Q'}^2$ gives rise to a unique homomorphism of algebras $\varphi\colon k\langle\langle Q\rangle\rangle \to k\langle\langle Q'\rangle\rangle$ such that $\varphi|_k = id$ and $\varphi|_{A_Q} = (\varphi_1, \varphi_2)$. Furthermore, φ is an isomorphism if and only if φ_1 is a k-module isomorphism $A_Q \to A_{Q'}$.*

From the above result, whenever we construct an automorphism $\varphi\colon k\langle\langle Q\rangle\rangle \to k\langle\langle Q\rangle\rangle$ in §4.1 and §6, it will always be the case that $\varphi|_k = id$, so we will only describe $\varphi|_{A_Q}$.

Definition 2.1.7. *An algebra homomorphism $\varphi\colon k\langle\langle Q\rangle\rangle \to k\langle\langle Q\rangle\rangle$ is called a* unitriangular automorphism *if $\varphi|_k = id$ and $\varphi_1 = id$. For $i \geq 1$, we say that φ has* depth i *provided that $\varphi_2(a) \in \mathfrak{m}_Q^{i+1}$ for all $a \in Q_1$.*

Definition 2.1.8. *Let f and g be potentials on a quiver Q.*

(1) *We say that f is* isomorphic *to g (written $f \cong g$) if $\mathfrak{J}\mathrm{ac}(f) \cong \mathfrak{J}\mathrm{ac}(g)$ as algebras.*

(2) *If there exists an algebra isomorphism $\varphi\colon k\langle\langle Q\rangle\rangle \to k\langle\langle Q\rangle\rangle$ such that $\varphi|_k = id$ and $\varphi(f) = g$, then we write $\varphi\colon f \mapsto g$ and say that f is* equivalent *to g.*

(3) *If there exists an algebra isomorphism $\varphi\colon k\langle\langle Q\rangle\rangle \to k\langle\langle Q\rangle\rangle$ such that $\varphi|_k = id$ and $\varphi(f) \sim g$, then we write $\varphi\colon f \rightsquigarrow g$ and say that f is* right-equivalent *to g.*

(4) *For $i \geq 1$, if there exists a unitriangular $\varphi\colon k\langle\langle Q\rangle\rangle \to k\langle\langle Q\rangle\rangle$ such that φ has depth greater than or equal to i, and further $\varphi(f) \overset{i+1}{\sim} g$, then we write $\varphi\colon f \overset{i}{\rightsquigarrow} g$ and say that f is* path degree i right-equivalent *to g.*

We follow the definition of right-equivalence in [DWZ, 4.2]. Moreover, from [DWZ, p12], $f \rightsquigarrow g$ induces $f \cong g$, and further a finite sequence of right-equivalences is still a right-equivalence. By 2.1.6, $f \overset{i}{\rightsquigarrow} g$ induces $f \rightsquigarrow g$. Thus, together with the above definition, we obtain

$$f \sim g \quad \text{or} \quad f \mapsto g \quad \text{or} \quad f \overset{i}{\rightsquigarrow} g \implies f \rightsquigarrow g \implies f \cong g.$$

The Jacobi algebra isomorphism $\cong$ is the equivalence relation that we aim to classify the potentials up to. The main idea is to start with a potential f, then transform it by a sequence of automorphisms which chases terms into higher and higher degrees. Composing this sequence of automorphisms then gives a single automorphism which takes f to the desired form (see §4.1 and §6.1).

The subtle point is that at each stage, the automorphism only gives the desired potential up to cyclic equivalence (e.g. $\rightsquigarrow$, $\overset{i}{\rightsquigarrow}$). Given an infinite sequence of path degree i right-equivalences $\varphi_i\colon f_i \overset{i}{\rightsquigarrow} f_{i+1}$ for $i \geq 1$, the following asserts that $\lim f_i$ exists, and further there exists a right-equivalence $F\colon f_1 \rightsquigarrow \lim f_i$.

Theorem 2.1.9. [BW2, 2.9] *Let f be a potential, and set $f_1 = f$. Suppose that there exist elements f_2, f_3, … and automorphisms φ_1, φ_2, …, such that*

(1) *Every φ_i is unitriangular of depth of $\geq i$, and*

(2) *$\varphi_i(f_i) \overset{i+1}{\sim} f_{i+1}$, for all $i \geq 1$.*

Then $\lim f_i$ exists, and there exists an automorphism F such that $F(f) \sim \lim f_i$.

§ 2.2 | Minimal models and flops

Throughout the remainder of this thesis, the notation $\mathcal{R}$ will be reserved for the singularities of the following form.

Definition 2.2.1. *A complete local $\mathbb{C}$-algebra $\mathcal{R}$ is called a* compound Du Val (cDV) singularity *if*

$$\mathcal{R} \cong \frac{\mathbb{C}[[u, v, x, t]]}{f + tg}$$

where $f \in \mathbb{C}[[u, v, x]]$ defines a Du Val, or equivalently Kleinian, surface singularity and $g \in \mathbb{C}[[u, v, x, t]]$ is arbitrary.

In other words, a cDV singularity is a threefold singularity such that any generic surface slice through it is a Kleinian singularity. These surface singularities are well understood and are classified by simply laced Dynkin diagrams.

Like the Kleinian surface singularities they generalise, cDV singularities are also categorized into types A, D, and E, corresponding to the ADE Dynkin diagrams. Type A cDV singularities take the following form:

$$\mathcal{R} \cong \frac{\mathbb{C}[[u, v, x, y]]}{uv - f_0 f_1 \dots f_t}, \tag{2.2.A}$$

where each f_i is a prime element of $\mathbb{C}[[x, y]]$. We refer to such an $\mathcal{R}$ as a cA_{n-1} singularity, where n is the order of the product $f_0 f_1 \dots f_t$ viewed as a power series.

Definition 2.2.2. *Let $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ be a proper birational morphism.*

(1) *We call π a* crepant partial resolution *if $\omega_{\mathcal{X}} \cong \pi^* \omega_{\mathcal{R}}$.*

(2) *We call $\mathcal{X}$ a* minimal model *of $\operatorname{Spec} \mathcal{R}$ if π is a crepant partial resolution and $\mathcal{X}$ has only $\mathbb{Q}$-factorial terminal singularities (see [W2, §2] for a definition). When $\mathcal{X}$ is furthermore smooth, we call π a* crepant resolution*.*

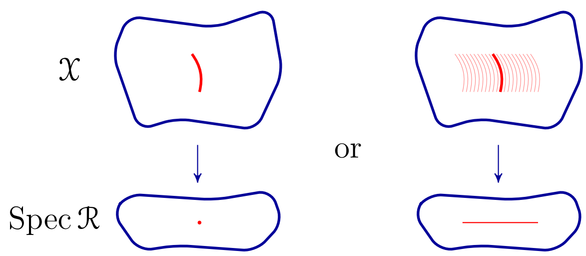
(3) *When $\mathcal{R}$ is isolated, crepant partial resolutions and crepant resolutions are equivalently called* flopping contractions *and* smooth flopping contractions*, respectively.*

Remark 2.2.3. In general, the definition of a crepant (partial) resolution only requires π to be *proper*. For crepant partial resolutions of cDV singularities, projectivity follows automatically (see e.g. [W2]), so in this case every crepant resolution is projective.

All minimal models of cDV singularities have fibres of dimension at most one. Therefore, any crepant partial resolution $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ falls into one of two types:

- A *curve-to-point contraction*, which arises when $\mathcal{R}$ is isolated, or
- A *divisor-to-curve contraction*, which arises when $\mathcal{R}$ is non-isolated.

To illustrate this distinction intuitively, consider the following sketch depiction of the simplest case—where a single curve lies above the origin. The difference between the two cases is visualised below:



We already noted in §1.1 that crepant resolutions do not always exist for cDV singularities. For the Type A cDV singularity defined by (2.2.A), we have the following facts.

- $\mathcal{R}$ is isolated. $\iff$ $(f_i) \neq (f_j)$ for all $i \neq j$ [IW3].
- $\mathcal{R}$ admits a crepant resolution. $\iff$ each f_i has a linear term [BIKR, IW3].

Example 2.2.4. We present three examples of Type A cDV singularities.

(1) Recall the *Pagoda Flop* from Example 1.1.2, given by

$$\mathcal{R} = \mathbb{C}[[u, v, x, y]]/(uv - x(x + y^n)).$$

The $\operatorname{Spec} \mathcal{R}$ is an isolated cA_1 singularity. Blowing up the ideal (u, x) yields a resolution $\mathcal{X}_1 \to \operatorname{Spec} \mathcal{R}$, and blowing up $(u, x + y^n)$ gives $\mathcal{X}_2 \to \operatorname{Spec} \mathcal{R}$. Both $\mathcal{X}_1$ and $\mathcal{X}_2$ are smooth, and the morphisms π_1 and π_2 are smooth flopping contractions.

(2) Let

$$\mathcal{R} = \mathbb{C}[[u, v, x, y]]/(uv - x^2).$$

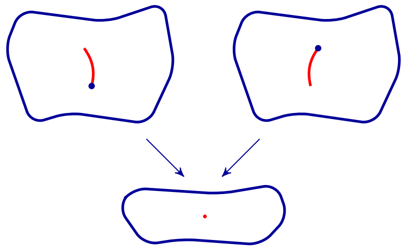
Then $\operatorname{Spec} \mathcal{R}$ is a non-isolated cA_1 singularity, with singularities along the x-axis. There is only one crepant resolution, obtained by blowing up the ideal (u, x).

(3) Let

$$\mathcal{R} = \mathbb{C}[[u, v, x, y]]/(uv - x(x^2 + y^3)).$$

The $\operatorname{Spec} \mathcal{R}$ is an isolated cA_2 singularity. Blowing up the ideal (u, x) gives a resolution $\mathcal{X}_1 \to \operatorname{Spec} \mathcal{R}$, and blowing up $(u, x^2 + y^3)$ gives $\mathcal{X}_2 \to \operatorname{Spec} \mathcal{R}$.

However, since $x^2 + y^3$ has no linear term, neither $\mathcal{X}_1$ nor $\mathcal{X}_2$ is smooth. A sketch illustrating this structure follows:



In this diagram, the dots on the exceptional curves represent the singular point $uv = x^2 + y^3$, which is factorial. Hence, both $\mathcal{X}_1$ and $\mathcal{X}_2$ are minimal models of $\operatorname{Spec}\mathcal{R}$.

As noted in §1.1, a given threefold $\operatorname{Spec}\mathcal{R}$ may admit multiple minimal models. A natural question then arises: *how many* such models exist, and *how* are they related?

It is known that there are only finitely many minimal models [KM], and they are all connected via sequences of codimension-two surgery operations known as *flops* [K5].

We now describe flops and flopping curves in detail. Let $\pi : \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ be a crepant partial resolution. The reduced fibre above the origin $\pi^{-1}(0)^{\mathrm{red}} = \bigcup_i \mathrm{C}_i$ is a union of rational curves. Choose any such C_i. Since $\mathcal{R}$ is complete local, by e.g. [W2, §2] we may factor π as

$$\mathcal{X} \xrightarrow{f} \mathcal{X}_{\mathrm{con}} \xrightarrow{g} \operatorname{Spec}\mathcal{R}$$

where f contracts C_j to a closed point if and only if $j = i$.

For any such factorisation, whenever f is a flopping contraction one can construct a birational map $f^i \colon \mathcal{X}^i \to \mathcal{X}_{\mathrm{con}}$, satisfying technical conditions described in [W2, 2.6], and fitting into the following commutative diagram:

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"] \arrow[ddr, bend right, "\pi"'] & & \mathcal{X}^i \arrow[dl, "f^i"'] \arrow[ddl, bend left, "\pi^i"] \\
& \mathcal{X}_{\mathrm{con}} \arrow[d, "g"] & \\
& \operatorname{Spec}\mathcal{R} &
\end{tikzcd}$$

where φ is a birational equivalence. The map $\pi^i \colon \mathcal{X}^i \to \operatorname{Spec}\mathcal{R}$ is called the *flop* of π at the curve C_i. It is also a crepant partial resolution, and π is a minimal model (resp. crepant resolution) if and only if π^i is [K6, 4.11]. Note that Examples (1) and (3) in Example 2.2.4 are flops.

It is well known that the number of curves in the exceptional locus of π^i matches that of π, and there is a natural correspondence between them. If we fix an ordering $\mathrm{C}_1, \ldots, \mathrm{C}_n$ on the curves in π, this ordering is preserved under the flop π^i, and we will often abuse notation by using the same symbols $\mathrm{C}_1, \ldots, \mathrm{C}_n$ for the curves in π^i.

We also emphasise that flopping is an involution: if $\pi^i\colon \mathcal{X}^i \to \operatorname{Spec} \mathcal{R}$ is the flop of π at the curve C_i, then conversely, $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ is the flop of π^i at the same curve C_i. This symmetry makes flops especially interesting objects of study in birational geometry.

§ 2.3 | Modifying modules and contraction algebras

The Homological Minimal Model Programme uses noncommutative algebra to study minimal models. In this section, we introduce the constructions of modifying modules [IW3] and contraction algebras, which are central to analysing crepant partial resolutions of cDV singularities.

In particular, one key feature of any cDV singularity $\mathcal{R}$ is that all of its birational geometry—i.e., the geometry "above" $\operatorname{Spec} \mathcal{R}$—can be recovered from the category of maximal Cohen–Macaulay modules $\operatorname{CM} \mathcal{R}$, through various different approaches.

Definition 2.3.1. *Given $\mathcal{R}$* cDV *as before, $M \in \operatorname{mod} \mathcal{R}$ is called* maximal Cohen–Macaulay (CM) *provided*

$$\operatorname{depth}_{\mathcal{R}} M := \inf\{i \geq 0 \mid \operatorname{Ext}^i_{\mathcal{R}}(\mathcal{R}/\mathfrak{m}, M) \neq 0\} = \dim \mathcal{R}.$$

We write $\operatorname{CM} \mathcal{R}$ for the category of $\operatorname{CM} \mathcal{R}$-modules, and $\underline{\operatorname{CM}}\, \mathcal{R}$ for the stable category of $\operatorname{CM} \mathcal{R}$. Further, for $(-)^* := \operatorname{Hom}_{\mathcal{R}}(-, \mathcal{R})$, $M \in \operatorname{mod} \mathcal{R}$ is called reflexive if the natural morphism $M \to M^{**}$ is an isomorphism, and we write $\operatorname{ref} \mathcal{R}$ for the category of reflexive $\mathcal{R}$-modules.

Definition 2.3.2. *We say $N \in \operatorname{ref} \mathcal{R}$ is a* modifying (M) module *if $\operatorname{End}_{\mathcal{R}}(N) \in \operatorname{CM} \mathcal{R}$, and we say that $N \in \operatorname{ref} \mathcal{R}$ is a* maximal modifying (MM) module *if it is modifying and it is maximal with respect to this property; equivalently,*

$$\operatorname{add} N = \{X \in \operatorname{ref} \mathcal{R} \mid \operatorname{End}_{\mathcal{R}}(N \oplus X) \in \operatorname{CM} \mathcal{R}\}.$$

If N is an M *module (resp.* MM *module), we call $\operatorname{End}_{\mathcal{R}}(N)$ a* modification algebra *(resp.* maximal modification algebra*).*

The concept of a smooth noncommutative minimal model—called a noncommutative crepant resolution—is due to Van den Bergh [V2].

Definition 2.3.3. *A* noncommutative crepant resolution (NCCR) *of $\mathcal{R}$ is a ring of the form $\Lambda := \operatorname{End}_{\mathcal{R}}(N)$ where $N \in \operatorname{ref} \mathcal{R}$, such that $\Lambda \in \operatorname{CM} \mathcal{R}$ and has finite global dimension.*

If an NCCR $\operatorname{End}_{\mathcal{R}}(N)$ exists, then N is automatically a maximal modifying (MM) module, and moreover every MM module gives rise to an NCCR. In other words, if one noncommutative minimal model is smooth, then they all are [IW2, 5.11].

For Kleinian singularities, the McKay correspondence [M2], as reformulated by Auslander [A4, A5], provides a bijection between indecomposable non-free CM modules and the exceptional curves in the minimal resolution. This correspondence can be extended to cDV singularities, as demonstrated in 2.3.4(1)–(3) below.

As noted earlier, crepant resolutions of a cDV singularity may not be unique, but they are all connected by flops. Inspired by 1.2.4, the NCCRs and modifying algebras should reflect a similar structure to the geometry of flops, as shown in 2.3.4(4) below.

Theorem 2.3.4. [IW2] *Let $\mathcal{R}$ be a* cDV *singularity, then there exist bijections*

$$(\mathrm{M}\,\mathcal{R}) \cap (\mathrm{CM}\,\mathcal{R}) \longleftrightarrow \{\textit{crepant partial resolutions } \pi : \mathcal{X} \to \operatorname{Spec}\mathcal{R}\}\,,$$
$$(\mathrm{MM}\,\mathcal{R}) \cap (\mathrm{CM}\,\mathcal{R}) \longleftrightarrow \{\textit{minimal models } \pi : \mathcal{X} \to \operatorname{Spec}\mathcal{R}\}\,.$$

If further $\mathcal{R}$ admits a crepant resolution, then

$$(\mathrm{MM}\,\mathcal{R}) \cap (\mathrm{CM}\,\mathcal{R}) \longleftrightarrow \{\textit{crepant resolutions } \pi : \mathcal{X} \to \operatorname{Spec}\mathcal{R}\}\,.$$

Moreover, under this bijection:

(1) *There is a one-to-one correspondence between the exceptional curves $\mathrm{C}_1, \dots, \mathrm{C}_m$ of the crepant partial resolution π and the non-free indecomposable summands of the corresponding module N.*

(2) *The quiver of $\underline{\mathrm{End}}_{\mathcal{R}}(N)$ encodes the dual graph of the corresponding crepant partial resolution π, recording how the exceptional curves $\mathrm{C}_1, \dots, \mathrm{C}_m$ intersect.*

(3) *When $\mathcal{X}$ is smooth, the number of loops at a vertex in the quiver of $\underline{\mathrm{End}}_{\mathcal{R}}(N)$ determines the normal bundle of the corresponding exceptional curve.*

(4) *The flops of the crepant partial resolution π correspond to mutations of the module N.*

The passage from left to right in the theorem sends a given $N \in (\mathrm{M}\,\mathcal{R}) \cap (\mathrm{CM}\,\mathcal{R})$ to a moduli space of representations of $\mathrm{End}_{\mathcal{R}}(N)$ [K4]. Therefore, an NCCR (or more generally, a modification algebra) encodes all geometric data of the associated crepant (partial) resolution of $\operatorname{Spec}\mathcal{R}$. In other words, passing to noncommutative algebra does not lose any geometric information.

We now explain the reverse direction of the bijection in 2.3.4.

Let $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ be a crepant partial resolution with exceptional curves $\mathrm{C}_1, \mathrm{C}_2, \dots, \mathrm{C}_m$. For each $1 \le i \le m$, there exists a vector bundle $\mathcal{N}_i$ on $\mathcal{X}$ [V2, 3.5.4], and we define:

$$\mathcal{N} := \mathcal{O}_{\mathcal{X}} \oplus \bigoplus_{i=1}^{m} \mathcal{N}_i.$$

This bundle is tilting on $\mathcal{X}$ [V2, 3.5.5]. Pushing forward via π yields:

$$\pi_*(\mathcal{O}_{\mathcal{X}}) = \mathcal{R}, \quad \pi_*(\mathcal{N}_i) = N_i \quad \text{for some } \mathcal{R}\text{-module } N_i.$$

Let $N := \mathcal{R} \oplus \bigoplus_{i=1}^{m} N_i$. Then both N and $\operatorname{End}_{\mathcal{R}}(N)$ lie in $\operatorname{CM}\mathcal{R}$ [V2, §4], so N is a modifying module. In other words, this construction produces a collection of indecomposable $\mathcal{R}$-modules—one for each exceptional curve—and their endomorphism algebra $\operatorname{End}_{\mathcal{R}}(N)$ is the modification algebra associated to the resolution π.

By [V2, 3.2.10], there is an isomorphism

$$\Lambda(\pi) := \operatorname{End}_{\mathcal{X}}(\mathcal{N}) \cong \operatorname{End}_{\mathcal{R}}(N) = \Lambda(N).$$

The contraction algebra associated to π is now defined as a certain quotient of this modification algebra.

Definition 2.3.5. *With notation above, define the* contraction algebra *associated to a crepant partial resolution π to be the stable endomorphism algebra*

$$\Lambda_{\mathrm{con}}(\pi) \text{ (equivalently, } \Lambda_{\mathrm{con}}(N)) := \underline{\operatorname{End}}_{\mathcal{R}}(N) = \operatorname{End}_{\mathcal{R}}(N)/\langle \mathcal{R} \rangle,$$

where $\langle \mathcal{R} \rangle$ denotes the two-sided ideal consisting of all morphisms which factor through $\operatorname{add}\mathcal{R}$.

The difference between flopping contractions and divisor-to-curve contractions can be detected by the finite dimensionality (or otherwise) of the contraction algebra as follows.

Theorem 2.3.6. (Contraction Theorem, [DW2, 4.8]) *Suppose that $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ is a crepant partial resolution. Then*

$$\pi \textit{ is a flopping contraction} \iff \dim_{\mathbb{C}}\Lambda_{\mathrm{con}}(\pi) < \infty.$$

If further $\mathcal{X}$ is smooth, these conditions are equivalent to $\mathcal{R}$ being an isolated singularity.

Donovan and Wemyss conjectured that the contraction algebra distinguishes the analytic type of the flop [DW1, 1.4], which was later proved as follows.

Theorem 2.3.7. [JKM, A.2] *Let $\pi_i\colon \mathcal{X}_i \to \operatorname{Spec}\mathcal{R}_i$ be crepant resolution of isolated* cDV *$\mathcal{R}_i$ for $i = 1, 2$. Then $\Lambda_{\mathrm{con}}(\pi_1)$ and $\Lambda_{\mathrm{con}}(\pi_2)$ are derived equivalent if and only if the singularities $\mathcal{R}_1$ and $\mathcal{R}_2$ are isomorphic.*

This means that the contraction algebras of an isolated cDV singularity $\mathcal{R}$ are all derived equivalent, and furthermore, $\mathcal{R}$ can be recovered from this derived equivalence class.

The following result connects derived equivalence of contraction algebras with the operation of flopping crepant partial resolutions.

Theorem 2.3.8. [A3, 5.2.2] *Given a crepant partial resolution $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ where $\mathcal{R}$ is isolated* cDV*, then the basic algebras derived equivalent to $\Lambda_{\mathrm{con}}(\pi)$ are precisely these $\Lambda_{\mathrm{con}}(\pi')$ where π' is obtained by a sequence of iterated flops from π. In particular, there are finitely many such algebras.*

§ 2.4 | Gopakumar–Vafa invariants

Let $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ be a crepant resolution. The reduced fibre above the origin,

$$\pi^{-1}(0)^{\mathrm{red}} = \bigcup_{i=1}^{n} \mathrm{C}_i,$$

is a union of rational curves.

Let $A_1(\pi) := \bigoplus_{i=1}^{n} \mathbb{Z}\langle \mathrm{C}_i \rangle$ be the abelian group freely generated by these exceptional curves.

Given a curve class $\beta = (\beta_1, \dots, \beta_n) \in A_1(\pi)$, there exists a genus zero Gopakumar–Vafa (GV) invariant $\mathrm{GV}_\beta(\mathcal{X})$ (or equivalently, $\mathrm{GV}_\beta(\pi)$), which virtually counts curves in the class β on $\mathcal{X}$.

Definition 2.4.1. *There are several equivalent interpretations of* $\mathrm{GV}_\beta(\mathcal{X})$.

(1) *Set*
$$\mathrm{GV}_\beta(\mathcal{X}) = \int_{\mathrm{Sh}_\beta(\mathcal{X})} \upsilon = \sum_{n\in\mathbb{Z}} n\chi\left(\upsilon^{-1}(n)\right) \qquad \textit{or} \qquad \mathrm{GV}_\beta(\mathcal{X}) = \int_{[\mathrm{Sh}_\beta(\mathcal{X})]^{vir}} 1$$
where υ is the Behrend's function [B1] *on the moduli scheme $\mathrm{Sh}_\beta(\mathcal{X})$ of one dimensional stable sheaves F with support β and Euler characteristic $\chi(F) = 1$. Moreover, there is a symmetric perfect obstruction theory on $\mathrm{Sh}_\beta(\mathcal{X})$ and virtual fundamental class* $[\mathrm{Sh}_\beta(\mathcal{X})]^{vir}$ [K2, MT].

(2) $\mathrm{GV}_\beta(\mathcal{X}) = \Omega^{num}_{\mathcal{X}}(1,\beta)$ *where $\Omega_{\mathcal{X}}(1,\beta)$ is a noncommutative BPS invariant* [V5].

(3) *If furthermore $\mathcal{R}$ is isolated, $\mathrm{GV}_\beta(\mathcal{X})$ equals to the number of $(-1,-1)$-curves with curve class β on a one-parameter deformation of $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$* [BKL].

If further $\mathcal{R}$ is isolated, GV invariants can be read off from the dimension of $\Lambda_{\mathrm{con}}(\pi)$ by Toda's formula.

Theorem 2.4.2. (Toda's formula, [T2, §4.4]) *Let $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ be a crepant resolution of an isolated* cDV *singularity $\mathcal{R}$ with exceptional curves $\bigcup_{i=1}^{n} \mathrm{C}_i$. For any $1 \le s \le t \le n$, the following equality holds.*

$$\dim_{\mathbb{C}} e_s\Lambda_{\mathrm{con}}(\pi)e_t = \sum_{\beta=(\beta_1,\dots,\beta_n)} \beta_s \cdot \beta_t \cdot \mathrm{GV}_\beta(\pi) = \dim_{\mathbb{C}} e_t\Lambda_{\mathrm{con}}(\pi)e_s.$$

In particular, $\dim_{\mathbb{C}} \Lambda_{\mathrm{con}}(\pi) = \sum_\beta |\beta|^2 \mathrm{GV}_\beta(\pi)$ *where* $|\beta| = \beta_1 + \cdots + \beta_n$.

Chapter 3

Generalised GV Invariants

In this chapter, we introduce and study generalised GV invariants, which extend the classical GV invariants to crepant partial resolutions of cA_n singularities.

In §3.1, we define these generalised GV invariants.

Next, in §3.2, we prove that these generalised invariants satisfy a version of Toda's formula and demonstrate that they are determined by their associated contraction algebras.

Finally, §3.3 restricts our focus to crepant resolutions of cA_n singularities, showing that in this context, the generalised invariants are equivalent to the classical GV invariants.

§ 3.1 | Definition of generalised GV invariants

Recall that every cA_{t-1} singularity $\mathcal{R}$ has the form

$$\mathcal{R} \cong \frac{\mathbb{C}[[u,v,x,y]]}{uv - f_0 f_1 \dots f_n},$$

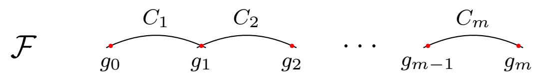
where t is the order of the polynomial $f_0 f_1 \dots f_n$ considered as a power series, and each f_i is a prime element of $\mathbb{C}[[x,y]]$. For any subset $I \subseteq \{0,1,\dots,n\}$ set $I^c = \{0,1,\dots,n\} \backslash I$ and denote

$$f_I := \prod_{i \in I} f_i \quad \text{and} \quad M_I := (u, f_I)$$

where T_I is an ideal of $\mathcal{R}$ of generated by u and f_I. For a collection of subsets $\emptyset \subsetneq I_1 \subsetneq I_2 \subsetneq \dots \subsetneq I_m \subsetneq \{0,1,\dots,n\}$, we say that $\mathcal{F} = (I_1, \dots, I_m)$ is a *flag in the set* $\{0,1,\dots,n\}$. We say that the flag $\mathcal{F}$ is *maximal* if $n = m$. Given a flag $\mathcal{F} = (I_1, \dots, I_m)$, we define

$$M^{\mathcal{F}} := \mathcal{R} \oplus \left(\bigoplus_{j=1}^{m} M_{I_j} \right).$$

To ease notation, set $I_0 := \emptyset$ and $I_{m+1} := \{0,1,\dots,n\}$, and then $g_j := f_{I_{j+1} \backslash I_j}$ for all $0 \leq j \leq m$. Thus $f_{I_j} = \prod_{i=0}^{j-1} g_i$ and $M_{I_j} = (u, \prod_{i=0}^{j-1} g_i)$. Then using [IW3, §5] $\mathcal{F}$ is given pictorially by



By [IW3, 5.1], the set $(\mathrm{M}\,\mathcal{R}) \cap (\mathrm{CM}\,\mathcal{R})$ is equal to modules $M^{\mathcal{F}}$, where $\mathcal{F}$ is a flag in $\{0, 1, \dots, n\}$. By 2.3.4, for each flag $\mathcal{F}$ there exists a crepant partial resolution $\pi^{\mathcal{F}} \colon \mathcal{X}^{\mathcal{F}} \to \operatorname{Spec} \mathcal{R}$ such that $\Lambda_{\mathrm{con}}(\pi^{\mathcal{F}}) \cong \underline{\operatorname{End}}_{\mathcal{R}}(M^{\mathcal{F}})$.

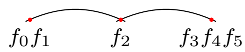
Definition 3.1.1. *With notation as above, define the* generalised GV invariant $N_\beta(\pi^{\mathcal{F}})$ *of the curve class* $\beta \in \bigoplus_{i=1}^m \mathbb{Z}\langle \mathrm{C}_i \rangle$ *to be*

$$N_\beta(\pi^{\mathcal{F}}) := \begin{cases} \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(g_{i-1}, g_j)} & \textit{if } \beta = \mathrm{C}_i + \mathrm{C}_{i+1} + \ldots + \mathrm{C}_j \\ 0 & \textit{else} \end{cases}$$

The above generalised GV invariant 3.1.1 is parallel to GV invariants, since if $\pi^{\mathcal{F}}$ is a crepant resolution, then $\{\mathrm{C}_i + \mathrm{C}_{i+1} + \cdots + \mathrm{C}_j \mid 1 \leq i \leq j \leq m\}$ are the only curve classes with non-zero GV invariants [NW, V5].

Thus throughout this paper we will often write $N_{ij}(\pi)$ (resp. $\mathrm{GV}_{ij}(\pi)$) for $N_\beta(\pi)$ (resp. $\mathrm{GV}_\beta(\pi)$) when $\beta = \mathrm{C}_i + \mathrm{C}_{i+1} + \ldots + \mathrm{C}_j$.

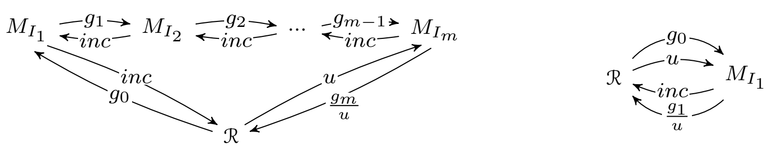
Example 3.1.2. Consider $f_0 f_1 f_2 f_3 f_4 f_5$ with a flag $\mathcal{F} = (\{0,1\} \subsetneq \{0,1,2\})$. Then $g_0 = f_0 f_1$, $g_1 = f_2$, $g_2 = f_3 f_4 f_5$, and $\mathcal{F}$ corresponds to



Then $M^{\mathcal{F}}$ is $\mathcal{R} \oplus (u, f_0 f_1) \oplus (u, f_0 f_1 f_2)$, and the generalised GV invariants are

$$N_{11}(\pi^{\mathcal{F}}) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(f_0 f_1, f_2)},\ N_{22}(\pi^{\mathcal{F}}) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(f_2, f_3 f_4 f_5)},\ N_{12}(\pi^{\mathcal{F}}) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(f_0 f_1, f_3 f_4 f_5)}.$$

Corollary 3.1.3. [IW3, 5.33] *Given a flag* $\mathcal{F} = (I_1, \dots, I_m)$*, with notation as above the quiver of* $\operatorname{End}_{\mathcal{R}}(M^{\mathcal{F}})$ *is as follows:*



$m{\geq}2$ $m{=}1$

together with the possible addition of some loops, given by the following rules:

- *Consider vertex $\mathcal{R}$. If $(g_0, g_m) = (x,y)$ in the ring $\mathbb{C}[[x,y]]$, add no loops at vertex $\mathcal{R}$. Hence suppose $(g_0, g_m) \subsetneq (x,y)$. If there exists $t \in (x,y)$ such that $(g_0, g_m, t) = (x,y)$, add a loop labelled t at vertex $\mathcal{R}$. If there exists no such t, add two loops labelled x and y at vertex $\mathcal{R}$.*

- *Consider vertex M_{I_i}. If $(g_{i-1}, g_i) = (x, y)$ in the ring $\mathbb{C}[[x, y]]$, add no loops at vertex M_{I_i}. Hence suppose $(g_{i-1}, g_i) \subsetneq (x, y)$. If there exists $t \in (x, y)$ such that $(g_{i-1}, g_i, t) = (x, y)$, add a loop labelled t at vertex M_{I_i}. If there exists no such t, add two loops labelled x and y at vertex M_{I_i}.*

§ 3.2 | Contraction algebra determines generalised GV invariants

In this subsection, we prove that the contraction algebra $\Lambda_{\mathrm{con}}(\pi^{\mathcal{F}})$ associated to a crepant partial resolution $\pi^{\mathcal{F}}$ of a cA_n singularity determines the generalised GV invariants $N_{ij}(\pi^{\mathcal{F}})$. Specifically, we express the relevant hom-spaces inside the contraction algebra in terms of power series rings, showing that their dimensions coincide with the values of the generalised invariants. This result generalises Toda's formula to the non-smooth setting and confirms that the contraction algebra encodes complete numerical curve-counting information in this context.

Through out this subsection, we follow the notation $\mathcal{R}$, $\mathcal{F}$, M_{I_j}, g_j and $\pi^{\mathcal{F}}$ in §3.1. Note in particular that the elements g_j need not be prime.

Proposition 3.2.1. *There are $\mathcal{R}$-isomorphisms*

$$\underline{\mathrm{Hom}}_{\mathcal{R}}\Big((u, g_0), (u, g_0 \dots g_{m-1})\Big) \cong \frac{\mathbb{C}[[u, v, x, y]]}{(u, v, g_0, g_m)} \cong \underline{\mathrm{Hom}}_{\mathcal{R}}\Big((u, g_0 \dots g_{m-1}), (u, g_0)\Big).$$

In particular, the dimension of each as a $\mathbb{C}$-vector space equals $\dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(g_0, g_m)$.

Proof. (1) We first prove that $\underline{\mathrm{Hom}}_{\mathcal{R}}\Big((u, g_0), (u, g_0 \dots g_{m-1})\Big) \cong \mathbb{C}[[u, v, x, y]]/(u, v, g_0, g_m)$.

We first claim that $\underline{\mathrm{Hom}}_{\mathcal{R}}\Big((u, g_0), (u, g_0 \dots g_{m-1})\Big) \cong \mathrm{Ext}^1_{\mathcal{R}}\Big((u, g_0), (u, g_m)\Big)$.

From [IW3, §5] there is an exact sequence

$$0 \to (u, g_m) \xrightarrow{\left(\frac{g_0 \cdots g_{m-1}}{u} \;\; -inc\right)} \mathcal{R}^2 \xrightarrow{\left(\substack{u \\ g_0 \cdots g_{m-1}}\right)} (u, g_0 \dots g_{m-1}) \to 0. \tag{3.2.A}$$

Thus $\Omega(u, g_0 \dots g_{m-1}) = (u, g_m)$ where Ω denotes the syzygy. Then we have

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{R}}\Big((u, g_0), (u, \prod_{i=0}^{m-1} g_i)\Big) &\cong \underline{\mathrm{Hom}}_{\mathcal{R}}\Big((u, g_0), \Omega(u, \prod_{i=0}^{m-1} g_i)[1]\Big) && (\Omega[1] = \mathrm{Id} \text{ in } \underline{\mathrm{CM}}\,\mathcal{R}) \\ &\cong \underline{\mathrm{Hom}}_{\mathcal{R}}\Big((u, g_0), (u, g_m)[1]\Big) && (\text{by above}) \\ &\cong \mathrm{Ext}^1_{\mathcal{R}}\Big((u, g_0), (u, g_m)\Big). && (\text{by e.g. [IW2]}) \end{aligned}$$

We next claim that $\mathrm{Ext}^1_{\mathcal{R}}\Big((u, g_0), (u, g_m)\Big) \cong (u, G)/(u, g_0 G, G g_m, G v)$ as $\mathcal{R}$-modules, where $G := g_1 g_2 \dots g_{m-1}$ and the right-hand side is the quotient of one ideal by another.

Applying $\mathbb{F} = \mathrm{Hom}\big((u, g_0), -\big)$ to the short exact sequence (3.2.A) gives

$$0 \to \mathbb{F}(u, g_m) \to \mathbb{F}\mathcal{R}^2 \xrightarrow{\left(\substack{u \\ \prod_{i=0}^{m-1} g_i}\right)} \mathbb{F}(u, \prod_{i=0}^{m-1} g_i) \to \mathrm{Ext}^1_{\mathcal{R}}\big((u, g_0), (u, g_m)\big) \to \mathrm{Ext}^1_{\mathcal{R}}\big((u, g_0), \mathcal{R}^2\big).$$

Since $(u, g_0) \in \mathrm{CM}\,\mathcal{R}$ by [IW3, 5.3], $\mathrm{Ext}^1_{\mathcal{R}}\big((u, g_0), \mathcal{R}^2\big) = 0$. Further, by [IW3, 5.4], there are isomorphisms

$$(u, \prod_{i=1}^{m} g_i) \cong \mathbb{F}\mathcal{R} \quad \text{via} \;\; r \mapsto (\cdot \frac{r}{u}),$$
$$(u, \prod_{i=1}^{m-1} g_i) \cong \mathbb{F}(u, \prod_{i=0}^{m-1} g_i) \quad \text{via} \;\; r \mapsto (\cdot r).$$

Combining these together gives an exact sequence

$$(u, \prod_{i=1}^{m} g_i)^{\oplus 2} \xrightarrow{d = \left(\substack{inc \\ \frac{\prod_{i=0}^{m-1} g_i}{u}}\right)} (u, \prod_{i=1}^{m-1} g_i) \to \mathrm{Ext}^1_{\mathcal{R}}\big((u, g_0), (u, g_m)\big) \to 0.$$

Thus $\mathrm{Ext}^1_{\mathcal{R}}\big((u, g_0), (u, g_m)\big) \cong (u, \prod_{i=1}^{m-1} g_i)/\operatorname{Im} d$. It is elementary to check that $\operatorname{Im} d \cong (u, g_0 G, g_m G, vG)$, proving the second claim.

Finally, we claim that $(u, G)/(u, g_0 G, g_m G, vG) \cong \mathbb{C}[[u, v, x, y]]/(u, v, g_0, g_m)$ as $\mathcal{R}$-modules.

We first define a $\mathbb{C}[[u, v, x, y]]$-homomorphism φ as follows,

$$\varphi \colon \mathbb{C}[[u, v, x, y]] \xrightarrow{\cdot G} (u, G)/(u, g_0 G, g_m G, vG).$$

Clearly, φ is well defined and $(u, v, g_0, g_m) \subseteq \ker \varphi$. We claim that $\ker \varphi \subseteq (u, v, g_0, g_m)$.

Let $r \in \mathbb{C}[[u, v, x, y]]$ be such that $\varphi(r) = 0$. Then $rG = r_1 u + r_2 g_0 G + r_3 g_m G + r_4 vG$ for some $r_i \in \mathbb{C}[[u, v, x, y]]$. Thus $r_1 u = (r - r_2 g_0 - r_3 g_m - r_4 v)G$. Since u and G have no common factors, we have $r_1 = r_5 G$ for some $r_5 \in \mathbb{C}[[u, v, x, y]]$. Thus $rG = (r_5 u + r_2 g_0 + r_3 g_m + r_4 v)G$. Since $\mathbb{C}[[u, v, x, y]]$ is domain, then $r = r_5 u + r_2 g_0 + r_3 g_m + r_4 v \in (u, v, g_0, g_m)$, and so $\ker \varphi \subseteq (u, v, g_0, g_m)$, proving the claim. Thus $\ker \varphi = (u, v, g_0, g_m)$.

Since φ is evidently surjective, it induces a $\mathbb{C}[[u, v, x, y]]$-isomorphism

$$\overline{\varphi} \colon \frac{\mathbb{C}[[u, v, x, y]]}{(u, v, g_0, g_m)} \xrightarrow{\sim} \frac{(u, G)}{(u, g_0 G, G g_m, Gv)}.$$

It is easy to check this is also an $\mathcal{R}$-module isomorphism.

(2) We next prove that $\underline{\mathrm{Hom}}_{\mathcal{R}}\big((u, g_0 \dots g_{m-1}), (u, g_0)\big) \cong \mathbb{C}[[u, v, x, y]]/(u, v, g_0, g_m)$.

We first claim that $\underline{\mathrm{Hom}}_{\mathcal{R}}\big((u, g_0 \dots g_{m-1}), (u, g_0)\big) \cong \mathrm{Ext}^1_{\mathcal{R}}\big((u, \prod_{i=0}^{m-1} g_i), (u, \prod_{i=1}^{m} g_i)\big)$.

Similar to (1), from [IW3, §5] there is an exact sequence

$$0 \to (u, g_1 \dots g_m) \xrightarrow{\left(\frac{g_0}{u}\ -inc\right)} \mathcal{R}^2 \xrightarrow{\left(\begin{smallmatrix} u \\ g_0 \end{smallmatrix}\right)} (u, g_0) \to 0. \tag{3.2.B}$$

Thus $\Omega(u, g_0) = (u, g_1 \dots g_m)$ and

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{R}}\Big((u, \prod_{i=0}^{m-1} g_i), (u, g_0)\Big) &\cong \underline{\mathrm{Hom}}_{\mathcal{R}}\Big((u, \prod_{i=0}^{m-1} g_i), \Omega(u, g_0)[1]\Big) && (\Omega[1] = \mathrm{Id} \text{ in } \underline{\mathrm{CM}}\, \mathcal{R}) \\ &\cong \underline{\mathrm{Hom}}_{\mathcal{R}}\Big((u, \prod_{i=0}^{m-1} g_i), (u, \prod_{i=1}^{m} g_i)[1]\Big) && (\text{by above}) \\ &\cong \mathrm{Ext}^1_{\mathcal{R}}\Big((u, \prod_{i=0}^{m-1} g_i), (u, \prod_{i=1}^{m} g_i)\Big). && (\text{by e.g. [IW2]}) \end{aligned}$$

We next claim that $\mathrm{Ext}^1_{\mathcal{R}}\left((u, \prod_{i=0}^{m-1} g_i), (u, \prod_{i=1}^{m} g_i)\right) \cong (u, g_0 g_m)/(u^2, ug_0, ug_m, g_0 g_m)$ as $\mathcal{R}$-modules, where the right-hand side is the quotient of two fractional ideals.

Similar to (1), applying $\mathbb{G} = \mathrm{Hom}_{\mathcal{R}}\left((u, \prod_{i=0}^{m-1} g_i), -\right)$ to the exact sequence (3.2.B) gives

$$0 \to \mathbb{G}(u, \prod_{i=1}^{m} g_i) \to \mathbb{G}\mathcal{R}^2 \xrightarrow{\left(\begin{smallmatrix} u \\ g_0 \end{smallmatrix}\right)} \mathbb{G}(u, g_0) \to \mathrm{Ext}^1_{\mathcal{R}}\Big((u, \prod_{i=0}^{m-1} g_i), (u, \prod_{i=1}^{m} g_i)\Big) \to 0.$$

By [IW3, 5.4], there are isomorphisms

$$\begin{aligned} (u, g_m) &\cong \mathbb{G}\mathcal{R} \quad \text{via} \quad r \mapsto (\cdot \frac{r}{u}), \\ (u, g_0 g_m) &\cong \mathbb{G}(u, g_0) \quad \text{via} \quad r \mapsto (\cdot \frac{r}{u}). \end{aligned}$$

Combining these together gives an exact sequence

$$(u, g_m)^{\oplus 2} \xrightarrow{d = \left(\begin{smallmatrix} u \\ g_0 \end{smallmatrix}\right)} (u, g_0 g_m) \to \mathrm{Ext}^1_{\mathcal{R}}\Big((u, \prod_{i=0}^{m-1} g_i), (u, \prod_{i=1}^{m} g_i)\Big) \to 0.$$

Thus $\mathrm{Ext}^1_{\mathcal{R}}\left((u, \prod_{i=0}^{m-1} g_i), (u, \prod_{i=1}^{m} g_i)\right) \cong (u, g_0 g_m)/\operatorname{Im} d$. It is elementary to check that $\operatorname{Im} d \cong (u^2, ug_0, ug_m, g_0 g_m)$, proving the second claim.

Finally, we claim that $(u, g_0 g_m)/(u^2, ug_0, ug_m, g_0 g_m) \cong \mathbb{C}[[u, v, x, y]]/(u, v, g_0, g_m)$ as $\mathcal{R}$-modules. Similar to (1), we first define a $\mathbb{C}[[u, v, x, y]]$-homomorphism φ as follows,

$$\varphi \colon \mathbb{C}[[u, v, x, y]] \xrightarrow{\cdot u} (u, g_0 g_m)/(u^2, ug_0, ug_m, g_0 g_m).$$

Clearly, φ is well defined and $(u, v, g_0, g_m) \subseteq \ker \varphi$. We claim that $\ker \varphi \subseteq (u, v, g_0, g_m)$.

Let $r \in \mathbb{C}[[u, v, x, y]]$ be such that $\varphi(r) = 0$. Then $ru = r_1 u^2 + r_2 g_0 u + r_3 g_m u + r_4 g_0 g_m$ for some $r_i \in \mathbb{C}[[u, v, x, y]]$. Thus $(r - r_1 u - r_2 g_0 - r_3 g_m)u = r_4 g_0 g_m$. Since u and $g_0 g_m$ have no common factors, we have $r_4 = r_5 u$ for some $r_5 \in \mathbb{C}[[u, v, x, y]]$. Thus $ru = (r_1 u + r_2 g_0 + r_3 g_m + r_5 g_0 g_m)u$. Since $\mathbb{C}[[u, v, x, y]]$ is domain, then $r = r_1 u + r_2 g_0 + r_3 g_m + r_5 g_0 g_m \in (u, v, g_0, g_m)$, and so

$\ker \varphi \subseteq (u, v, g_0, g_m)$, proving the claim.

Since φ is evidently surjective, it induces a $\mathbb{C}[[u, v, x, y]]$-isomorphism

$$\overline{\varphi}\colon \frac{\mathbb{C}[[u, v, x, y]]}{(u, v, g_0, g_m)} \xrightarrow{\sim} \frac{(u, g_0 g_m)}{(u^2, u g_0, u g_m, g_0 g_m)}.$$

It is easy to check this is also an $\mathcal{R}$-module isomorphism. □

Lemma 3.2.2. *Let $p, q \in \mathbb{C}[[x, y]]$. If the greatest common divisor $\gcd(p, q) \neq 1$, then $\dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(p, q) = \infty$.*

Proof. Write $r \in (x, y)$ for the greatest common divisor of p and q, namely $r = \gcd(p, q)$. Then $p = rp'$ and $q = rq'$ for some $p', q' \in \mathbb{C}[[x, y]]$, and so $(p, q) = (r)(p', q') \subseteq (r)$. Thus

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(r)} \leq \dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(p, q)}.$$

Since $\mathbb{C}[[x, y]]$ is a polynomial of two variables, $\dim_{\mathbb{C}} \mathbb{C}[[x, y]]/(r) = \infty$, and so the statement follows. □

Lemma 3.2.3. *Let p_i and $q_j \in \mathbb{C}[[x, y]]$ for $0 \leq i \leq s$ and $0 \leq j \leq t$. Then*

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(\prod_{i=0}^{s} p_i, \prod_{j=0}^{t} q_j)} = \sum_{i=0}^{s} \sum_{j=0}^{t} \dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(p_i, q_j)}.$$

Proof. We split the proof into two cases.

(1) There exists i' and j' such that the greatest common divisor $\gcd(p_{i'}, q_{j'}) \neq 1$.

Since $\gcd(p_{i'}, q_{j'}) \neq 1$, $\gcd(\prod_{i=0}^{s} p_i, \prod_{j=0}^{t} q_j) \neq 1$. By 3.2.2,

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(\prod_{i=0}^{s} p_i, \prod_{j=0}^{t} q_j)} = \infty = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(p_{i'}, q_{j'})}.$$

Since the dimension of a vector space can not be negative, the statement follows.

(2) The greatest common divisor $\gcd(p_i, q_j) = 1$ for each i and j.

It suffices to prove that

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(p_0, q_0 q_1)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(p_0, q_0)} + \dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(p_0, q_1)},$$

since then the statement follows by induction. We first consider the natural quotient $\mathbb{C}[[x, y]]$-homomorphism

$$\varphi\colon \frac{\mathbb{C}[[x, y]]}{(p_0, q_0 q_1)} \twoheadrightarrow \frac{\mathbb{C}[[x, y]]}{(p_0, q_0)}.$$

It is clear that $\ker \varphi = (q_0)/(p_0, q_0 q_1)$. So we only need to prove that $(q_0)/(p_0, q_0 q_1) \cong$

$\mathbb{C}[[x,y]]/(p_0,q_1)$. To see this, we define a $\mathbb{C}[[x,y]]$-homomorphism as

$$\begin{aligned}\vartheta\colon \frac{\mathbb{C}[[x,y]]}{(p_0,q_1)} &\to \frac{(q_0)}{(p_0,q_0q_1)}\\ r &\mapsto q_0r\end{aligned}$$

It is clear that ϑ is well-defined and surjective. So we only need to prove the injectivity. If $q_0r = r_1p_0 + r_2q_0q_1$ for some $r_1, r_2 \in \mathbb{C}[[x,y]]$, since $\gcd(p_0,q_0) = 1$, then $r_1 = r_3q_0$ for some $r_3 \in \mathbb{C}[[x,y]]$. Since $\mathbb{C}[[x,y]]$ is a domain, $r = r_3p_0 + r_2q_1 \in (p_0,q_1)$, and so ϑ is injective. $\square$

Recall that $\pi^{\mathcal{F}}$ is a crepant partial resolution with m excpetional curves and $\Lambda(\pi^{\mathcal{F}}) \cong \operatorname{End}_{\mathcal{R}}(M^{\mathcal{F}})$. Moreover, $\Lambda(\pi^{\mathcal{F}})$ can be presented as the quiver in 3.1.3 with the trivial path e_i at each vertex i. The following shows that generalised GV invariants also satisfy Toda's formula, which implies that these new invariants are a natural generalization.

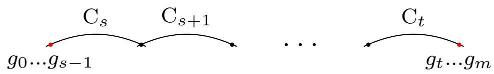
Theorem 3.2.4. *For any $1 \leq s \leq t \leq m$, the following equality holds.*

$$\dim_{\mathbb{C}} e_s\Lambda_{\mathrm{con}}(\pi^{\mathcal{F}})e_t = \sum_{i=1}^{s}\sum_{j=t}^{m} N_{ij}(\pi^{\mathcal{F}}) = \dim_{\mathbb{C}} e_t\Lambda_{\mathrm{con}}(\pi^{\mathcal{F}})e_s.$$

In particular, $\dim_{\mathbb{C}} \Lambda_{\mathrm{con}}(\pi^{\mathcal{F}}) = \sum_{i=1}^{m}\sum_{j=i}^{m}(j-i+1)^2 N_{ij}(\pi^{\mathcal{F}})$.

Proof. To ease notation, set $\pi := \pi^{\mathcal{F}}$. We first factor π as $\mathcal{X} \to \mathcal{Y} \xrightarrow{\omega} \operatorname{Spec}\mathcal{R}$ such that $A_1(\omega) = \bigcup_{k=s}^{t}\mathbb{Z}\langle \mathrm{C}_k\rangle$. By [IW3, §5], $\mathcal{Y}$ is given pictorially by

$\mathcal{Y}$



and $\Lambda_{\mathrm{con}}(\omega) \cong e_{st}\Lambda_{\mathrm{con}}(\pi)e_{st}$ where $e_{st} := e_s + \cdots + e_t$. Thus

$$\begin{aligned}
e_s\Lambda_{\mathrm{con}}(\pi)e_t &\cong e_se_{st}\Lambda_{\mathrm{con}}(\pi)e_{st}e_t && (\text{since } e_se_{st} = e_s \text{ and } e_{st}e_t = e_t)\\
&\cong e_s\Lambda_{\mathrm{con}}(\omega)e_t && (\text{since } \Lambda_{\mathrm{con}}(\omega) \cong e_{st}\Lambda_{\mathrm{con}}(\pi)e_{st})\\
&\cong \underline{\operatorname{Hom}}_{\mathcal{R}}((u, g_0\dots g_{s-1}),(u, g_0\dots g_{t-1})) && (\text{by } 3.1.3)\\
&\cong \frac{\mathbb{C}[[u,v,x,y]]}{(u,v,g_0\dots g_{s-1}, g_t\dots g_m)},
\end{aligned}$$

where in the last step uses the first isomorphism in 3.2.1, but with g_0 and g_m replaced by $g_0\dots g_{s-1}$ and $g_t\dots g_m$. Similarly,

$$\begin{aligned}
e_t\Lambda_{\mathrm{con}}(\pi)e_s &\cong e_te_{st}\Lambda_{\mathrm{con}}(\pi)e_{st}e_s && (\text{since } e_te_{st} = e_t \text{ and } e_{st}e_s = e_s)\\
&\cong e_t\Lambda_{\mathrm{con}}(\omega)e_s && (\text{since } \Lambda_{\mathrm{con}}(\omega) \cong e_{st}\Lambda_{\mathrm{con}}(\pi)e_{st})\\
&\cong \underline{\operatorname{Hom}}_{\mathcal{R}}((u, g_0\dots g_{t-1}),(u, g_0\dots g_{s-1})) && (\text{by } 3.1.3)\\
&\cong \frac{\mathbb{C}[[u,v,x,y]]}{(u,v,g_0\dots g_{s-1}, g_t\dots g_m)}
\end{aligned}$$

where again the last step uses the second isomorphism in 3.2.1, with g_0 and g_m replaced by $g_0 \dots g_{s-1}$ and $g_t \dots g_m$. Combining these together, it follows that

$$\dim_{\mathbb{C}} e_s\Lambda_{\text{con}}(\pi)e_t = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(\prod_{i=0}^{s-1} g_i, \prod_{j=t}^{m} g_j)} = \dim_{\mathbb{C}} e_t\Lambda_{\text{con}}(\pi)e_s.$$

Moreover,

$$\begin{aligned} \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(\prod_{i=0}^{s-1} g_i, \prod_{j=t}^{m} g_j)} &= \sum_{i=0}^{s-1}\sum_{j=t}^{m} \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(g_i, g_j)} && \text{(by 3.2.3)} \\ &= \sum_{i=1}^{s}\sum_{j=t}^{m} \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(g_{i-1}, g_j)} \\ &= \sum_{i=1}^{s}\sum_{j=t}^{m} N_{ij}(\pi). && \text{(by definition 3.1.1)} \end{aligned}$$

Writing $N_{ij} = N_{ij}(\pi)$ and $\Lambda_{\text{con}} = \Lambda_{\text{con}}(\pi)$ to ease notation, it follows that

$$\dim_{\mathbb{C}} e_s\Lambda_{\text{con}}e_t = \sum_{i=1}^{s}\sum_{j=t}^{m} N_{ij} = \dim_{\mathbb{C}} e_t\Lambda_{\text{con}}e_s. \tag{3.2.C}$$

Now by 3.1.3,

$$\Lambda_{\text{con}} = \begin{bmatrix} e_1\Lambda_{\text{con}}e_1 & e_1\Lambda_{\text{con}}e_2 & \cdots & e_1\Lambda_{\text{con}}e_m \\ e_2\Lambda_{\text{con}}e_1 & e_2\Lambda_{\text{con}}e_2 & \cdots & e_2\Lambda_{\text{con}}e_m \\ \vdots & \vdots & \ddots & \vdots \\ e_m\Lambda_{\text{con}}e_1 & e_m\Lambda_{\text{con}}e_2 & \cdots & e_m\Lambda_{\text{con}}e_m \end{bmatrix},$$

so using (3.2.C)

$$\dim_{\mathbb{C}} \Lambda_{\text{con}} = \begin{bmatrix} \bigoplus_{i=1}^{1}\bigoplus_{j=1}^{m} N_{ij} & \bigoplus_{i=1}^{1}\bigoplus_{j=2}^{m} N_{ij} & \cdots & \bigoplus_{i=1}^{1}\bigoplus_{j=m}^{m} N_{ij} \\ \bigoplus_{i=1}^{1}\bigoplus_{j=2}^{m} N_{ij} & \bigoplus_{i=1}^{2}\bigoplus_{j=2}^{m} N_{ij} & \cdots & \bigoplus_{i=1}^{2}\bigoplus_{j=m}^{m} N_{ij} \\ \vdots & \vdots & \ddots & \vdots \\ \bigoplus_{i=1}^{1}\bigoplus_{j=m}^{m} N_{ij} & \bigoplus_{i=1}^{2}\bigoplus_{j=m}^{m} N_{ij} & \cdots & \bigoplus_{i=1}^{m}\bigoplus_{j=m}^{m} N_{ij} \end{bmatrix}.$$

For $1 \leq i \leq j \leq m$, N_{ij} only appears in each entry of the submatrix from row i to row j and column i to column j of the above matrix, and so N_{ij} appears $(j-i+1)^2$ times in $\dim_{\mathbb{C}} \Lambda_{\text{con}}$. Thus $\dim_{\mathbb{C}} \Lambda_{\text{con}} = \sum_{i=1}^{m}\sum_{j=i}^{m}(j-i+1)^2 N_{ij}$. □

The following asserts that isomorphisms between contraction algebras of crepant partial resolutions can only map e_i to e_i or e_{m+1-i} for $1 \leq i \leq m$.

Proposition 3.2.5. *Let $\pi_k \colon \mathcal{X}_k \to \operatorname{Spec} \mathcal{R}_k$ be two crepant partial resolutions of cA_{n_k} singularities $\mathcal{R}_k$ with m_k exceptional curves for $k = 1, 2$. If there exists an algebra isomorphism $\phi \colon \Lambda_{\text{con}}(\pi_1) \cong \Lambda_{\text{con}}(\pi_2)$, then $m_1 = m_2$ and ϕ must belong to one of the following cases:*

(1) $\phi(e_i) = e_i$ *for* $1 \le i \le m$,

(2) $\phi(e_i) = e_{m+1-i}$ *for* $1 \le i \le m$,

where $m := m_1 = m_2$.

Proof. For $1 \le i \le m_1$, write $\mathcal{S}_i$ for the simple $\Lambda_{\mathrm{con}}(\pi_1)$-module corresponding to the vertex i in the quiver of $\Lambda_{\mathrm{con}}(\pi_1)$ (see [HW, §5.2]). Similarly, for $1 \le i \le m_2$, write $\mathcal{S}'_i$ for the simple $\Lambda_{\mathrm{con}}(\pi_2)$-module corresponding to the vertex i in the quiver of $\Lambda_{\mathrm{con}}(\pi_2)$. Write $\mathrm{mod}\,\Lambda_{\mathrm{con}}(\pi_k)$ for the category of finitely generated right $\Lambda_{\mathrm{con}}(\pi_k)$-modules for $k = 1, 2$.

The algebra isomorphism ϕ induces an equivalence $\varphi\colon \mathrm{mod}\,\Lambda_{\mathrm{con}}(\pi_1) \cong \mathrm{mod}\,\Lambda_{\mathrm{con}}(\pi_2)$. By Morita theory, $m_1 = m_2$, since φ maps simple modules to simple modules, and furthermore there is a σ in the symmetric group $\mathfrak{S}_m$ such that $\varphi(\mathcal{S}_i) = \mathcal{S}'_{\sigma(i)}$.

Since π_1 is a crepant partial resolution of a cA_{n_1} singularity, $\mathcal{S}_2$ is the unique simple module that satisfies $\mathrm{Ext}^1_{\Lambda_{\mathrm{con}}(\pi_1)}(\mathcal{S}_1, \mathcal{S}_2) \neq 0$ by 3.1.3 and the intersection theory of [W2, 2.15]. Since $\mathrm{mod}\,\Lambda_{\mathrm{con}}(\pi_1)$ is equivalent to $\mathrm{mod}\,\Lambda_{\mathrm{con}}(\pi_2)$, there exists unique simple module $\mathcal{T} \in \mathrm{mod}\,\Lambda_{\mathrm{con}}(\pi_2)$ such that $\mathrm{Ext}^1_{\Lambda_{\mathrm{con}}(\pi_2)}(\mathcal{S}'_{\sigma(1)}, \mathcal{T}) \neq 0$. Thus the curve $\sigma(1)$ in π_2 must be a edge curve, by 3.1.3 and the intersection theory of [W2, 2.15]. Thus $\sigma(1) = 1$ or m. We split the proof into two cases.

(1) $\sigma(1) = 1$. Since $\mathrm{Ext}^1_{\Lambda_{\mathrm{con}}(\pi_1)}(\mathcal{S}_1, \mathcal{S}_2) \neq 0$ and $\mathrm{mod}\,\Lambda_{\mathrm{con}}(\pi_1)$ is equivalent to $\mathrm{mod}\,\Lambda_{\mathrm{con}}(\pi_2)$, we have $\mathrm{Ext}^1_{\Lambda_{\mathrm{con}}(\pi_2)}(\mathcal{S}'_{\sigma(1)}, \mathcal{S}'_{\sigma(2)}) \neq 0$, and so $\mathrm{Ext}^1_{\Lambda_{\mathrm{con}}(\pi_2)}(\mathcal{S}'_1, \mathcal{S}'_{\sigma(2)}) \neq 0$. Thus the curve $\sigma(2)$ in π_2 must be connected to the curve $\sigma(1) = 1$, and so $\sigma(2) = 2$ by 3.1.3 and the intersection theory of [W2, 2.15]. Repeating the same process, we can prove $\sigma(i) = i$, and so $\varphi(\mathcal{S}_i) = \mathcal{S}'_i$, and furthermore $\phi(e_i) = e_i$ for each i.

(2) $\sigma(1) = m$. Since $\mathrm{Ext}^1_{\Lambda_{\mathrm{con}}(\pi_1)}(\mathcal{S}_1, \mathcal{S}_2) \neq 0$ and $\mathrm{mod}\,\Lambda_{\mathrm{con}}(\pi_1)$ is equivalent to $\mathrm{mod}\,\Lambda_{\mathrm{con}}(\pi_2)$, we have $\mathrm{Ext}^1_{\Lambda_{\mathrm{con}}(\pi_2)}(\mathcal{S}'_{\sigma(1)}, \mathcal{S}'_{\sigma(2)}) \neq 0$, and so $\mathrm{Ext}^1_{\Lambda_{\mathrm{con}}(\pi_2)}(\mathcal{S}'_n, \mathcal{S}'_{\sigma(2)}) \neq 0$. Thus the curve $\sigma(2)$ in π_2 must be connected to the curve $\sigma(1) = m$, and so $\sigma(2) = m - 1$ by 3.1.3 and the intersection theory of [W2, 2.15]. Repeating the same process, we can prove $\sigma(i) = m+1-i$, and so $\varphi(\mathcal{S}_i) = \mathcal{S}'_{m+1-i}$, and furthermore $\phi(e_i) = e_{m+1-i}$ for each i. □

The following strengthens 2.4.2 and 3.2.4, in that it intrinsically extracts the generalised GV invariants from the contraction algebra, and is new even in the setting of smooth flopping contractions.

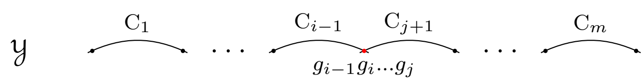
Lemma 3.2.6. *For any* $1 \le i \le j \le m$, *the following equity holds.*

$$N_{ij}(\pi^{\mathcal{F}}) = \dim_{\mathbb{C}} e_i \left(\frac{\Lambda_{\mathrm{con}}(\pi^{\mathcal{F}})}{\langle e_1, e_2, \ldots, e_{i-1}, e_{j+1}, e_{j+2}, \ldots, e_m \rangle} \right) e_j.$$

Proof. When $i = 1$ and $j = m$,

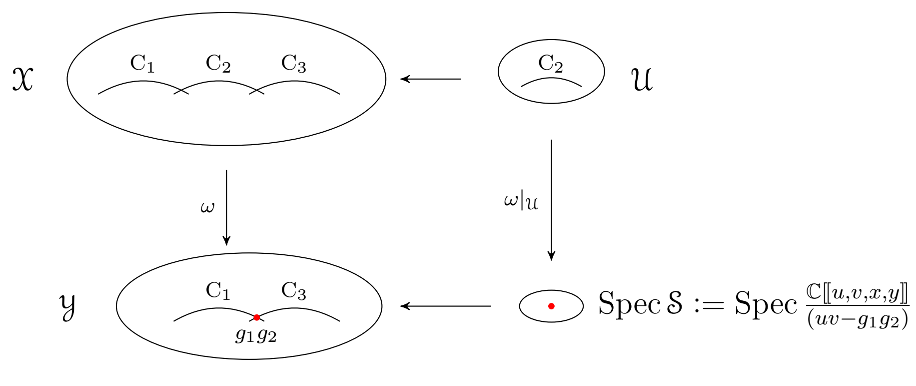
$$\begin{aligned}
N_{1m}(\pi^{\mathcal{F}}) &= \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(g_0, g_m)} && \text{(by the definition 3.1.1 of } N_{ij}(\pi^{\mathcal{F}})\text{)}\\
&= \dim_{\mathbb{C}} \underline{\mathrm{Hom}}_{\mathcal{R}}\big((u, g_0), (u, g_0 \dots g_{m-1})\big) && \text{(by 3.2.1)}\\
&= \dim_{\mathbb{C}} \underline{\mathrm{Hom}}_{\mathcal{R}}(M_{I_1}, M_{I_m}) && \text{(since } M_{I_1} = (u, g_0) \text{ and } M_{I_m} = (u, g_0 \dots g_{m-1})\text{)}\\
&= \dim_{\mathbb{C}} e_1 \Lambda_{\mathrm{con}}(\pi^{\mathcal{F}}) e_m. && \text{(by 3.1.3)}
\end{aligned}$$

Thus the statement holds. When $i \neq 1$ or $j \neq m$, we factor $\pi^{\mathcal{F}}$ as $\mathcal{X} \xrightarrow{\omega} \mathcal{Y} \to \operatorname{Spec} \mathcal{R}$ such that $A_1(\omega) = \bigcup_{k=i}^{j} \mathbb{Z}\langle \mathrm{C}_k \rangle$. By [IW3, 5.31], $\mathcal{Y}$ is given pictorially by



where the red dot labelled $g_{i-1}g_i \cdots g_j$ corresponds, complete locally, to the singularity $\mathcal{S} := \mathbb{C}[[u,v,x,y]]/(uv - g_{i-1}g_i \cdots g_j)$. Here we slightly abuse notation by again using u, v, x, y as local coordinates to define $\mathcal{S}$.

Then consider the flat morphism $\operatorname{Spec} \mathcal{S} \to \mathcal{Y}$, the fibre product $\mathcal{U} := \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} \mathcal{S}$, and the morphism $\omega|_{\mathcal{U}} \colon \mathcal{U} \to \operatorname{Spec} \mathcal{S}$. The following picture illustrates the $m = 3$, $i = j = 2$ case.



By [IW3, §5],

$$\Lambda_{\mathrm{con}}(\omega|_{\mathcal{U}}) \cong \Lambda_{\mathrm{con}}(\pi^{\mathcal{F}}) / \langle e_1, e_2, \dots, e_{i-1}, e_{j+1}, e_{j+2}, \dots, e_m \rangle.$$

Thus we have

$$\begin{aligned}
N_{ij}(\pi^{\mathcal{F}}) &= \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(g_{i-1}, g_j)} && \text{(by the definition 3.1.1 of } N_{ij}(\pi^{\mathcal{F}})\text{)}\\
&= \dim_{\mathbb{C}} \underline{\mathrm{Hom}}_{\mathcal{S}}\big((u, g_{i-1}), (u, g_{i-1} \dots g_{j-1})\big) && \text{(by 3.2.1)}\\
&= \dim_{\mathbb{C}} e_i \Lambda_{\mathrm{con}}(\omega|_{\mathcal{U}}) e_j && \text{(by 3.1.3)}\\
&= \dim_{\mathbb{C}} e_i \Big(\Lambda_{\mathrm{con}}(\pi^{\mathcal{F}}) / \langle e_1, e_2, \dots, e_{i-1}, e_{j+1}, e_{j+2}, \dots, e_m \rangle \Big) e_j. && \square
\end{aligned}$$

The following shows that the contraction algebra of a crepant partial resolution of a cA_n

singularity determines its associated generalised GV invariants.

Theorem 3.2.7. *Let* $\pi^{\mathcal{F}_k}\colon \mathfrak{X}^{\mathcal{F}_k} \to \operatorname{Spec}\mathcal{R}_k$ *be two crepant partial resolutions of* cA_{n_k} *singularities* $\mathcal{R}_k$ *with* m_k *exceptional curves for* $k=1,2$. *If* $\Lambda_{\mathrm{con}}(\pi^{\mathcal{F}_1}) \cong \Lambda_{\mathrm{con}}(\pi^{\mathcal{F}_2})$, *then* $m_1 = m_2$ *and one of the following cases holds:*

(1) $N_{ij}(\pi^{\mathcal{F}_1}) = N_{ij}(\pi^{\mathcal{F}_2})$ *for* $1 \le i \le j \le m$,

(2) $N_{ij}(\pi^{\mathcal{F}_1}) = N_{m+1-j,m+1-i}(\pi^{\mathcal{F}_2})$ *for* $1 \le i \le j \le m$,

where $m := m_1 = m_2$.

Proof. To ease notation, set $\pi_k := \pi^{\mathcal{F}_k}$ for $k=1,2$. Since $\Lambda_{\mathrm{con}}(\pi_1) \cong \Lambda_{\mathrm{con}}(\pi_2)$, $m_1 = m_2$ by 3.2.5. Let ϕ be the algebra isomorphism between $\Lambda_{\mathrm{con}}(\pi_1)$ and $\Lambda_{\mathrm{con}}(\pi_2)$. By 3.2.5, either $\phi(e_i) = e_i$ or $\phi(e_i) = e_{m+1-i}$ for $1 \le i \le m$. Then we split the proof into two cases.

(1) $\phi(e_i) = e_i$ for $1 \le i \le m$. In that case, for $1 \le i \le j \le m$,

$$\begin{aligned} N_{ij}(\pi_1) &\overset{3.2.6}{=} \dim_{\mathbb{C}} e_i\Big(\Lambda_{\mathrm{con}}(\pi_1)/\langle e_1, e_2, \dots, e_{i-1}, e_{j+1}, e_{j+2}, \dots, e_m\rangle\Big)e_j \\ &= \dim_{\mathbb{C}} e_i\Big(\Lambda_{\mathrm{con}}(\pi_2)/\langle e_1, e_2, \dots, e_{i-1}, e_{j+1}, e_{j+2}, \dots, e_m\rangle\Big)e_j \\ &\overset{3.2.6}{=} N_{ij}(\pi_2). \end{aligned}$$

(2) $\phi(e_i) = e_{m+1-i}$ for $1 \le i \le m$. In that case, for $1 \le i \le j \le m$,

$$\begin{aligned} N_{ij}(\pi_1) &\overset{3.2.6}{=} \dim_{\mathbb{C}} e_i\Big(\Lambda_{\mathrm{con}}(\pi_1)/\langle e_1, e_2, \dots, e_{i-1}, e_{j+1}, e_{j+2}, \dots, e_m\rangle\Big)e_j \\ &= \dim_{\mathbb{C}} e_{m+1-i}\Big(\Lambda_{\mathrm{con}}(\pi_2)/\langle e_m, e_{m-1}, \dots, e_{m-i+2}, e_{m-j}, e_{m-j+1}, \dots, e_1\rangle\Big)e_{m+1-j} \\ &\overset{3.2.4}{=} \dim_{\mathbb{C}} e_{m+1-j}\Big(\Lambda_{\mathrm{con}}(\pi_2)/\langle e_1, e_2, \dots, e_{m-j}, e_{m-i+2}, e_{m-i+3}, \dots, e_m\rangle\Big)e_{m+1-i} \\ &\overset{3.2.6}{=} N_{m+1-j,m+1-i}(\pi_2). \end{aligned}$$

□

Remark 3.2.8. The papers [NW, V5] give a combinatorial description of the matrix which controls the transformation of the non-zero GV invariants under a flop. For crepant resolutions of cA_n singularities, see §3.3.1 below.

By definition 3.1.1 and example 3.1.2, it is clear that generalised GV invariants of crepant partial resolutions of cA_n singularities also satisfy this transformation under a flop. Moreover, generalised GV invariants satisfy Toda's formula 3.2.4 and are determined by their associated contraction algebra 3.2.7. These facts give strong evidence that generalised GV invariants are a natural generalization of GV invariants.

§ 3.3 | Classical case: known facts

In this subsection, we restrict to cA_n singularities that admit a crepant resolution, and summarise several facts about their noncommutative crepant resolutions (NCCRs), as developed in [IW3]. These results serve as the foundation for comparing generalised GV invariants with classical GV invariants in the smooth case.

Recall that in §3.1, every cA_{t-1} singularity $\mathcal{R}$ has the following form

$$\mathcal{R} \cong \frac{\mathbb{C}[[u,v,x,y]]}{uv - f_0 f_1 \dots f_n},$$

where t is the order of the polynomial $f_0 f_1 \dots f_n$ considered as a power series and each f_i is a prime element of $\mathbb{C}[[x,y]]$. Moreover, $\mathcal{R}$ admits a crepant resolution if and only each f_i has a linear term by e.g. [BIKR, IW3].

In the subsection, we will only consider those $\mathcal{R}$ that admit a crepant resolution. Thus $t = n+1$, and so $\mathcal{R}$ is a cA_n singularity.

Recall in §3.1 the maximal flag $\mathcal{F}$ in the set $\{0,1,\dots,n\}$, and CM $\mathcal{R}$-module $M^{\mathcal{F}}$. Following the notation in [IW3, §5], we identify maximal flags with elements of the symmetric group $\mathfrak{S}_{n+1}$. Hence we regard each $\sigma \in \mathfrak{S}_{n+1}$ as the maximal flag

$$\{\sigma(0)\} \subset \{\sigma(0),\sigma(1)\} \subset \dots \subset \{\sigma(0),\dots,\sigma(n-1)\}.$$

Notation 3.3.1. We adopt the following notation.

(1) Consider the symmetric group $\mathfrak{S}_{n+1}$. For any $\sigma \in \mathfrak{S}_{n+1}$, set

$$M^\sigma := \mathcal{R} \oplus (u, g_{\sigma(0)}) \oplus (u, g_{\sigma(0)}g_{\sigma(1)}) \oplus \dots \oplus (u, \prod_{i=0}^{n-1} g_{\sigma(i)}) \in (\mathrm{CM}\,\mathcal{R}) \cap (\mathrm{MM}\,\mathcal{R}).$$

(2) Write $\pi^\sigma \colon \mathcal{X}^\sigma \to \operatorname{Spec}\mathcal{R}$ for the associated crepant resolution of M^σ in 2.3.4 below.

(3) Now let $k \geq 1$ and consider the k-tuple $\mathbf{r} = (r_1,\dots,r_k)$ with each $1 \leq r_i \leq n$. Set

$$\sigma(\mathbf{r}) := (r_k, r_k+1)\cdots(r_2,r_2+1)(r_1,r_1+1) \in \mathfrak{S}_{n+1},$$

and $M^{\mathbf{r}} := M^{\sigma(\mathbf{r})}$. Write $\pi^{\mathbf{r}} \colon \mathcal{X}^{\mathbf{r}} \to \operatorname{Spec}\mathcal{R}$ for $\pi^{\sigma(\mathbf{r})} \colon \mathcal{X}^{\sigma(\mathbf{r})} \to \operatorname{Spec}\mathcal{R}$.

(4) For $1 \leq i \leq n$, write π^i, $\mathcal{X}^i$ and M^i for $\pi^{(i)}$, $\mathcal{X}^{(i)}$ and $M^{(i)}$ respectively.

The following two results are the special cases of 2.3.4 and 3.1.3, when we restrict to the crepant resolutions of cA_n singularities.

Proposition 3.3.2. [IW3, 5.1, 5.27] *The modules* $(\mathrm{MM}\,\mathcal{R}) \cap (\mathrm{CM}\,\mathcal{R})$ *in* 2.3.4 *are precisely* M^σ *where* $\sigma \in \mathfrak{S}_{n+1}$. *Moreover, there is a bijection satisfying* $\Lambda(\pi^\sigma) \cong \operatorname{End}_{\mathcal{R}}(M^\sigma)$,

$$\{M^\sigma \mid \sigma \in \mathfrak{S}_{n+1}\} \longleftrightarrow \{\textit{ crepant resolutions of } \mathcal{R}\},$$
$$M^\sigma \longleftrightarrow \pi^\sigma : \mathcal{X}^\sigma \to \operatorname{Spec}\mathcal{R}.$$

Proposition 3.3.3. [IW3, W2] *Given any* $\sigma \in \mathfrak{S}_{n+1}$, *let* $\pi^\sigma \colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ *be the associated* *crepant resolution. Then the* NCCR $\Lambda(\pi^\sigma)$ *can be presented as the following quiver (with possible loops):*

$$
\begin{tikzcd}
1 \arrow[r, shift left, "g_{\sigma(1)}"] \arrow[drrr, shift left, "inc"] & 2 \arrow[l, shift left, "inc"] \arrow[r, shift left, "g_{\sigma(2)}"] & 3 \arrow[l, shift left, "inc"] & \cdots & n-1 \arrow[r, shift left, "g_{\sigma(n-1)}"] & n \arrow[l, shift left, "inc"] \arrow[dll, shift left, "\frac{g_{\sigma(n)}}{u}"] \\
 & & & 0 \arrow[ulll, shift left, "g_{\sigma(0)}"] \arrow[urr, shift left, "u"] & &
\end{tikzcd}
$$

where the vertex 0 *represents* $\mathcal{R}$ *and the vertex* i *represents* $(u, \prod_{j=0}^{i-1} g_{\sigma(j)})$ *for* $1 \leq i \leq n$.

There is a loop labelled t *at vertex* 0 *if and only if* $(g_{\sigma(0)}, g_{\sigma(n)}) \subsetneq (x, y)$ *and* $(g_{\sigma(0)}, g_{\sigma(n)}, t) = (x, y)$ *in the ring* $\mathbb{C}[[x, y]]$. *Further, for any* $1 \leq i \leq n$, *the possible loops at vertex* i *are given by the following rules:*

(1) *the normal bundle of curve* C_i *is* $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \iff (g_{\sigma(i-1)}, g_{\sigma(i)}) = (x, y)$ *in* $\mathbb{C}[[x, y]] \iff$ *add no loop at vertex* i.

(2) *the normal bundle of curve* C_i *is* $\mathcal{O}(-2) \oplus \mathcal{O} \iff (g_{\sigma(i-1)}, g_{\sigma(i)}) \subsetneq (x, y)$ *and there exists* $t \in (x, y)$ *such that* $(g_{\sigma(i-1)}, g_{\sigma(i)}, t) = (x, y)$ *in* $\mathbb{C}[[x, y]] \iff$ *add a loop labelled* t *at vertex* i.

Proof. In general, [IW3, W2] shows that either (1), (2) or the following third case holds.

(3) $(g_{\sigma(i-1)}, g_{\sigma(i)}) \subsetneq (x, y)$ and there is no t such that (2) $\iff$ add two loops labelled x and y at vertex i.

We now prove that (3) is impossible when $\mathcal{R}$ is cA_n and admits a crepant resolution. If there exist two loops at some vertex i, then $(g_{\sigma(i-1)}, g_{\sigma(i)}) \subsetneq (\!(x, y)\!)$ and there exists no $t \in (x, y)$ that satisfies $(g_{\sigma(i-1)}, g_{\sigma(i)}, t) = (x, y)$. Hence both $g_{\sigma(i-1)}$ and $g_{\sigma(i)}$ must belong to $(x, y)^2$. But this contradicts the fact that $\mathcal{R}$ admits a crepant resolution $\mathcal{X}$. □

§ 3.3.1 | Reduction steps for GV invariants

This subsection recalls various permutation results from [NW, V5], then shows that GV invariants are suitably local.

The first reduction step we will use below is to permutate the GV invariant of an arbitrary curve class into that of a particular curve class. From [NW, 5.4] and [V5, 5.10], for any cA_n crepant resolution π and $1 \leq i \leq n$, there is a linear isomorphism

$$F_i \colon A_1(\pi) \to A_1(\pi^i),$$

such that $\mathrm{GV}_\beta(\pi) = \mathrm{GV}_{|F_i(\beta)|}(\pi^i)$ for any $\beta \in A_1(\pi)$. Here we consider $A_1(\pi) \cong \mathbb{Z}^n \cong A_1(\pi^i)$, and so F_i is a elements of $\mathbf{M}_n(\mathbb{Z})$. Moreover,

$$F_i = \begin{cases} \mathbf{I}_n - 2E_{11} + E_{12}, & \text{if } i = 1 \\ \mathbf{I}_n - 2E_{nn} + E_{n,n-1}, & \text{if } i = n \\ \mathbf{I}_n - 2E_{ii} + E_{i,i-1} + E_{i,i+1}, & \text{else} \end{cases}$$

where $E_{ij} \in \mathbf{M}_n(\mathbb{Z})$ is the standard basis matrix with a one in the j-th column of the i-th row, and zeros everywhere else. Inspired by the above $\mathrm{GV}_{\beta}(\pi) = \mathrm{GV}_{|F_i(\beta)|}(\pi^i)$, we adopt the following notation.

Notation 3.3.4. For any $1 \leq i \leq n$ and $\mathbf{r}$ in 3.3.1, denote $|F_i| := |-| \circ F_i$ and $|F_{\mathbf{r}}| := |F_{r_k}| \circ \cdots \circ |F_{r_2}| \circ |F_{r_1}|$. Thus $\mathrm{GV}_{\beta}(\pi) = \mathrm{GV}_{|F_i|(\beta)}(\pi^i)$ and $\mathrm{GV}_{\beta}(\pi) = \mathrm{GV}_{|F_{\mathbf{r}}|(\beta)}(\pi^{\mathbf{r}})$.

For $1 \leq i \leq j \leq n$, write v_{ij} for the vector in $\mathbb{Z}^n$ which corresponds to the curve class $\mathrm{C}_i + \mathrm{C}_{i+1} + \cdots + \mathrm{C}_j$. Thus $v_{ij} = \sum_{k=i}^{j} \mathsf{e}_k$ where e_k is the k-th standard basis vector.

Lemma 3.3.5. *With the notation as above, the following holds.*

(1) *For $2 \leq i \leq j \leq n$, $F_{i-1}v_{ij} = v_{i-1,j}$.*

(2) *For $1 \leq i < j \leq n$, $F_j v_{ij} = v_{i,j-1}$.*

(3) *For $1 \leq i \leq j \leq n$, set*

$$\mathbf{r} = \begin{cases} \emptyset \textit{ and } F_{\mathbf{r}} = \mathrm{Id}, & \textit{if } i = j = 1 \\ (j, j-1, \ldots, 3, 2), & \textit{if } i = 1 \textit{ and } 2 \leq j \leq n \\ (i-1, i-2, \ldots, 2, 1, j, j-1, \ldots, 3, 2), & \textit{if } 2 \leq i \leq j \leq n \end{cases}$$

then $|F_{\mathbf{r}}| v_{ij} = v_{11}$.

Proof. From the basic facts of linear algebra, we have $E_{ij}\mathsf{e}_t = \begin{cases} \mathsf{e}_i, & \text{if } t = j \\ \mathbf{0}, & \text{else} \end{cases}$.

(1) When $3 \leq i \leq j \leq n$, then $F_{i-1} = \mathbf{I}_n - 2E_{i-1,i-1} + E_{i-1,i-2} + E_{i-1,i}$, and so

$$F_{i-1}v_{ij} = (\mathbf{I}_n - 2E_{i-1,i-1} + E_{i-1,i-2} + E_{i-1,i})(\sum_{k=i}^{j} \mathsf{e}_k) = v_{ij} + e_{i-1} = v_{i-1,j}.$$

When $2 = i \leq j \leq n$, then $F_{i-1} = F_1 = \mathbf{I}_n - 2E_{11} + E_{12}$, and so

$$F_{i-1}v_{ij} = F_1 v_{2j} = (\mathbf{I}_n - 2E_{11} + E_{12})(\sum_{k=2}^{j} \mathsf{e}_k) = v_{2j} + e_1 = v_{1j} = v_{i-1,j}.$$

(2) When $1 \leq i < j \leq n-1$, then $F_j = \mathbf{I}_n - 2E_{jj} + E_{j,j-1} + E_{j,j+1}$, and so

$$F_j v_{ij} = (\mathbf{I}_n - 2E_{jj} + E_{j,j-1} + E_{j,j+1})(\sum_{k=i}^{j} \mathsf{e}_k) = v_{ij} - 2e_j + e_j = v_{i,j-1}.$$

When $1 \leq i < j = n$, then $F_j = F_n = \mathbf{I}_n - 2E_{nn} + E_{n,n-1}$, and so

$$F_j v_{ij} = F_n v_{in} = (\mathbf{I}_n - 2E_{nn} + E_{n,n-1})(\sum_{k=i}^{n} \mathsf{e}_k) = v_{in} - 2e_n + e_n = v_{i,n-1} = v_{i,j-1}.$$

(3) We only prove the case of $2 \leq i \leq j \leq n$. The other two cases are similar.

By (1), $|F_1| \circ |F_2| \circ \cdots \circ |F_{i-2}| \circ |F_{i-1}| v_{ij} = v_{1j}$. By (2), $|F_2| \circ |F_3| \circ \cdots \circ |F_{j-1}| \circ |F_j| v_{1j} = v_{11}$. Thus $|F_{\mathbf{r}}| v_{ij} = |F_2| \circ |F_3| \circ \cdots \circ |F_{j-1}| \circ |F_j| \circ |F_1| \circ |F_2| \circ \cdots \circ |F_{i-2}| \circ |F_{i-1}| v_{ij} = v_{11}$. $\square$

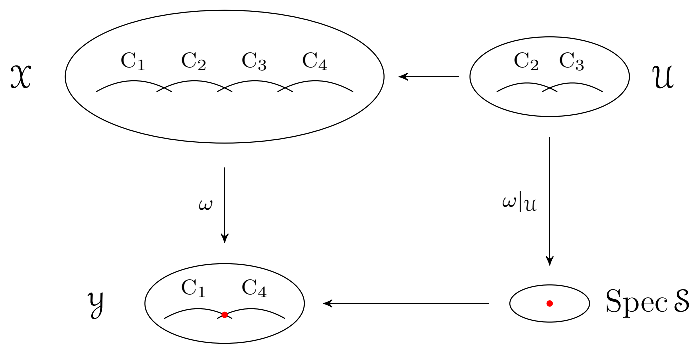
The second reduction step will show that the GV invariants are suitably local and flopping a curve only affects the neighbourhood of that curve.

Fix some integers s and t satisfying $1 \leq s \leq t \leq n$. Then we factor π as

$$\pi \colon \mathcal{X} \xrightarrow{\omega} \mathcal{Y} \to \operatorname{Spec} \mathcal{R}$$

such that $A_1(\omega) = \oplus_{k=s}^{t} \mathbb{Z}\langle \mathrm{C}_k \rangle$. Write $\operatorname{Spec} S$ for the affine patch of $\mathcal{Y}$ containing the singular point and $\mathcal{S}$ for the completion of S at the singular point. Then we consider the flat morphism $\operatorname{Spec} \mathcal{S} \to \mathcal{Y}$, the fibre product $\mathcal{U} := \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} \mathcal{S}$ and the morphism $\omega|_{\mathcal{U}} \colon \mathcal{U} \to \operatorname{Spec} \mathcal{S}$.

We abuse the notation to write the exceptional curves of $\omega|_{\mathcal{U}}$ also as $A_1(\omega|_{\mathcal{U}}) = \oplus_{k=s}^{t} \mathbb{Z}\langle \mathrm{C}_k \rangle$. The following picture illustrates the $n = 4,\ s = 2,\ t = 3$ case where the red dots represent the singular point of $\mathcal{Y}$ and $\operatorname{Spec} \mathcal{S}$.



We now prove that flopping a curve only affects the neighbourhood of that curve. Recall from notation 3.3.1 that $\mathcal{X}^i$ denotes the variety of flopping the exceptional curve C_i in $\mathcal{X}$. For $s \leq i \leq t$, we consider the diagram below: we flop C_i in $\mathcal{X}$ to obtain $\omega^i \colon \mathcal{X}^i \to \mathcal{Y}$, and denote the birational map as $\varphi \colon \mathcal{X}^i \dashrightarrow \mathcal{X}$. Pulling back ω^i along $\operatorname{Spec} \mathcal{S} \to \mathcal{Y}$, we obtain the morphism $(\omega|_{\mathcal{U}})^i$, and define the birational map $\varphi^i := (\omega|_{\mathcal{U}})^i \circ (\omega|_{\mathcal{U}})^{-1}$.

$$\begin{tikzcd}
& \mathcal{X}^i \arrow[dl, dotted, "\varphi"'] \arrow[dd, "\omega^i"] & & \mathcal{U}^+ \arrow[ll] \arrow[dl, dotted, "\varphi^i"'] \arrow[dd, "(\omega|_{\mathcal{U}})^i"] \\
\mathcal{X} \arrow[dr, "\omega"'] & & \mathcal{U} \arrow[ll] \arrow[dr, "\omega|_{\mathcal{U}}"'] & \\
& \mathcal{Y} & & \operatorname{Spec} \mathcal{S} \arrow[ll]
\end{tikzcd}$$

Lemma 3.3.6. *With the diagram above, $(\omega|_{\mathcal{U}})^i$ is the flop of $\omega|_{\mathcal{U}}$ by flopping the exceptional curve C_i in $\mathcal{U}$; that is $\mathcal{U}^i \cong \mathcal{U}^+ \cong \mathcal{X}^i \times_{\mathcal{Y}} \operatorname{Spec} \mathcal{S}$.*

Proof. Since $\mathcal{R}$ is complete local, there exists Cartier divisor D_i on $\mathcal{X}$ such that $D_i \cdot \mathrm{C}_j = \delta_{ij}$ for all $s \leq j \leq t$. Let $\widetilde{D}_i$ denote the proper transform of D_i to $\mathcal{X}^i$. Then $\widetilde{D}_i \cdot \mathrm{C}_j = -\delta_{ij}$ for all $s \leq j \leq t$. Let $D_i|_{\mathcal{U}}$ denote be the pull back of D_i to $\mathcal{U}$, and likewise $\widetilde{D}_i|_{\mathcal{U}^+}$. Then for all $s \leq j \leq t$

$$D_i|_{\mathcal{U}} \cdot \mathrm{C}_j = \delta_{ij}, \quad \widetilde{D}_i|_{\mathcal{U}^+} \cdot \mathrm{C}_j = -\delta_{ij}, \text{ and } (\varphi^i)^* D_i|_{\mathcal{U}} \cong \widetilde{D}_i|_{\mathcal{U}^+}.$$

Hence by e.g. [W2, 2.7] $(\omega|_{\mathcal{U}})^i$ is the flop of $\omega|_{\mathcal{U}}$ by flopping the exceptional curve C_i in $\mathcal{U}$. □

Proposition 3.3.7. (GV invariants are local) *With notation as above, we have $\mathrm{GV}_{ij}(\mathcal{X}) = \mathrm{GV}_{ij}(\mathcal{U})$ for any $s \leq i \leq j \leq t$.*

Proof. (1) We first prove that $\mathrm{GV}_{kk}(\mathcal{X}) = \mathrm{GV}_{kk}(\mathcal{U})$ for any $s \leq k \leq t$.

Fix k satisfying $s \leq k \leq t$. Consider the following derived equivalences from [V2, 3.5.8],

$$\begin{array}{cc}
\mathrm{D}^b(\operatorname{coh} \mathcal{X}) \xrightarrow{\sim} \mathrm{D}^b(\operatorname{mod} \Lambda(\omega)), & \mathrm{D}^b(\operatorname{coh} \mathcal{U}) \xrightarrow{\sim} \mathrm{D}^b(\operatorname{mod} \Lambda(\omega|_{\mathcal{U}})) \\
\mathcal{O}_{\mathrm{C}_k}(-1) \leftrightarrow S_k & \mathcal{O}_{\mathrm{C}_k}(-1) \leftrightarrow S'_k
\end{array}$$

where S_k denotes the simple-$\Lambda(\omega)$ module that corresponds to $\mathcal{O}_{\mathrm{C}_k}(-1)$ (see [HW, §5.2]). The S'_k is similar.

From [V5, 5.3], S_k (respectively S'_k) is the only nilpotent point in the moduli space of semisimple $\Lambda(\omega)$ (resp. $\Lambda(\omega|_{\mathcal{U}})$)-modules of its dimension vector. So, to compare $\mathrm{GV}_{kk}(\mathcal{X})$ and $\mathrm{GV}_{kk}(\mathcal{U})$, it suffices to compare the value of the Behrend functions at these two points.

From [J], these values only depend on the formal neighbourhood, which can be presented as the Maurer-Cartan locus of their enhancement algebras $\mathrm{End}^{\mathrm{DG}}_{\Lambda(\omega)}(S_k)$ and $\mathrm{End}^{\mathrm{DG}}_{\Lambda(\omega|_{\mathcal{U}})}(S'_k)$ respectively. From [DW2], these two DG-algebras are DG equivalent, via

$$\mathrm{End}^{\mathrm{DG}}_{\Lambda(\omega)}(S_k) \cong \mathrm{End}^{\mathrm{DG}}_{\mathcal{X}}(\mathcal{O}_{\mathrm{C}_k}(-1)) \cong \mathrm{End}^{\mathrm{DG}}_{\mathcal{U}}(\mathcal{O}_{\mathrm{C}_k}(-1)) \cong \mathrm{End}^{\mathrm{DG}}_{\Lambda(\omega|_{\mathcal{U}})}(S'_k).$$

Thus, these two values are the same. So, exactly as in [V5, 5.3], $\mathrm{GV}_{kk}(\mathcal{X}) = \mathrm{GV}_{kk}(\mathcal{U})$.

(2) We then prove that $\mathrm{GV}_{ij}(\mathcal{X}) = \mathrm{GV}_{ij}(\mathcal{U})$ for any $s \leq i \leq j \leq t$.

When $s \le i = j \le t$, the statement holds by (1). So we only need to prove the statement for $s \le i < j \le t$. Set $\mathbf{r} = (j, j-1, \ldots, i+1)$. Then $\mathrm{GV}_{ij}(\mathfrak{X}) = \mathrm{GV}_{ii}(\mathfrak{X}^{\mathbf{r}})$ and $\mathrm{GV}_{ij}(\mathfrak{U}) = \mathrm{GV}_{ii}(\mathfrak{U}^{\mathbf{r}})$ by 3.3.5(2). Since by 3.3.6 $\mathfrak{U}^{\mathbf{r}} \cong \mathfrak{X}^{\mathbf{r}} \times_{\mathcal{Y}} \operatorname{Spec} \mathcal{S}$, $\mathrm{GV}_{ii}(\mathfrak{X}^{\mathbf{r}}) = \mathrm{GV}_{ii}(\mathfrak{U}^{\mathbf{r}})$ by (1). So $\mathrm{GV}_{ij}(\mathfrak{X}) = \mathrm{GV}_{ij}(\mathfrak{U})$. $\square$

§ 3.3.2 | Classical case: new results

This subsection first shows in 3.3.8 that generalised GV invariants are equivalent to GV invariants. Together with 3.2.7, 3.3.10 asserts that the contraction algebra of a crepant resolution of a cA_n singularity determines its associated GV invariants. For the isolated cA_n, this result is from Toda's formula 2.4.2 and [HT]. Our result generalises this to non-isolated cA_n.

Theorem 3.3.8. *Given a crepant resolution $\pi\colon \mathfrak{X} \to \operatorname{Spec} \mathcal{R}$ where $\mathcal{R}$ is cA_n, for any $1 \le i \le j \le n$ the following holds.*

(1) $N_{ij}(\pi) = \infty \iff \mathrm{GV}_{ij}(\pi) = -1$.

(2) $N_{ij}(\pi) < \infty \iff \mathrm{GV}_{ij}(\pi) = N_{ij}(\pi)$.

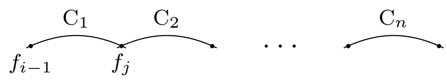
Proof. Without loss of generality, we assume

$$\mathcal{R} \cong \frac{\mathbb{C}[[u, v, x, y]]}{uv - f_0 f_1 \ldots f_n},$$

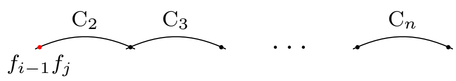
and $M = (u, f_0) \oplus (u, f_0 f_1) \oplus \ldots \oplus (u, \prod_{i=0}^{n-1} f_i)$ such that π is the associated crepant resolution with $\Lambda(\pi) \cong \operatorname{End}_{\mathcal{R}}(M)$ in 3.3.2.

Let $\mathbf{r}$ be the tuple in 3.3.5. We have $\mathrm{GV}_{ij}(\pi) = \mathrm{GV}_{11}(\pi^{\mathbf{r}})$ by 3.3.5. Then we factor $\pi^{\mathbf{r}}$ as $\mathfrak{X}^{\mathbf{r}} \xrightarrow{\omega} \mathcal{Y} \to \operatorname{Spec} \mathcal{R}$ such that $A_1(\omega) = \mathbb{Z}\langle \mathrm{C}_1 \rangle$. Since $\mathrm{GV}_{11}(\pi^{\mathbf{r}})$ only depends on $\mathfrak{X}^{\mathbf{r}}$ and the curve class C_1 by 2.4.1, then $\mathrm{GV}_{11}(\pi^{\mathbf{r}}) = \mathrm{GV}_{11}(\omega)$, and so $\mathrm{GV}_{ij}(\pi) = \mathrm{GV}_{11}(\omega)$.

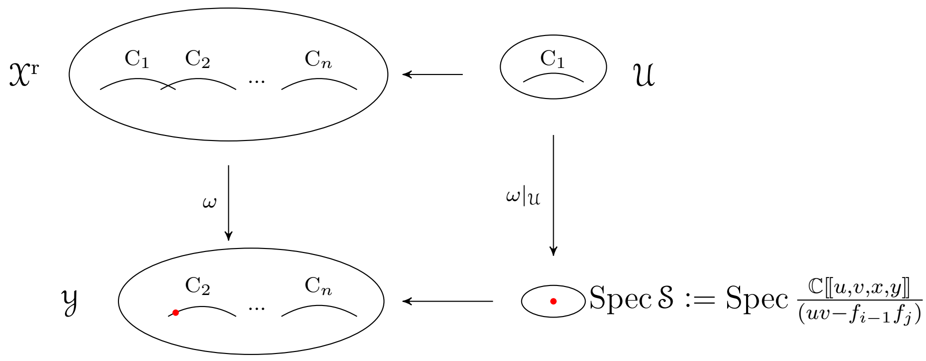
By 3.3.2, $\Lambda(\pi^{\mathbf{r}}) \cong \operatorname{End}_{\mathcal{R}}(M^{\mathbf{r}})$ where $M^{\mathbf{r}} = \mathcal{R} \oplus (u, f_{i-1}) \oplus (u, f_{i-1} f_j) \oplus \ldots \oplus (u, \prod_{i=0}^{n-1} f_i)$, then using [IW3, §5] $\mathfrak{X}^{\mathbf{r}}$ is given pictorially by

$\mathfrak{X}^{\mathbf{r}}$ 

Since $\pi^{\mathbf{r}}\colon \mathfrak{X}^{\mathbf{r}} \xrightarrow{\omega} \mathcal{Y} \to \operatorname{Spec} \mathcal{R}$ where $A_1(\omega) = \mathbb{Z}\langle \mathrm{C}_1 \rangle$, then again by [IW3, §5] $\mathcal{Y}$ is given pictorially by

$\mathcal{Y}$ 

where the singular point of $\mathcal{Y}$ is locally $S := \mathbb{C}[u, v, x, y]/(uv - f_{i-1} f_j)$. Write $\mathcal{S}$ for the completion of S at the singular point. Then consider the flat morphism $\operatorname{Spec} \mathcal{S} \to \mathcal{Y}$, the fibre product $\mathfrak{U} := \mathfrak{X}^{\mathbf{r}} \times_{\mathcal{Y}} \operatorname{Spec} \mathcal{S}$, and the morphism $\omega|_{\mathfrak{U}}\colon \mathfrak{U} \to \operatorname{Spec} \mathcal{S}$. Since GV invariants are local by 3.3.7, then $\mathrm{GV}_{11}(\omega) = \mathrm{GV}_{11}(\omega|_{\mathfrak{U}})$, and so $\mathrm{GV}_{ij}(\pi) = \mathrm{GV}_{11}(\omega|_{\mathfrak{U}})$.



Consider the $\mathcal{S}$-module $N := \mathcal{U} \oplus (u, f_{i-1})$. In 3.3.2, $\omega|_{\mathcal{U}}$ is the crepant resolution of $\operatorname{Spec}\mathcal{S}$ with respect to N. Since $\operatorname{Spec}\mathcal{S}$ is a cA_1 singularity and admits a crepant resolution, then by [R1] there exists a change of coordinates φ (possibly different in the two cases below) such that

(1) $\omega|_{\mathcal{U}}$ is a divisor-to-curve contraction. $\iff$ $\varphi(f_{i-1}) = x = \varphi(f_j)$.

(2) $\omega|_{\mathcal{U}}$ is a flop. $\iff$ $\varphi(f_{i-1}) = x + y^n$ and $\varphi(f_j) = x - y^n$ for some $n \geq 1$.

In case (1), we have $\Lambda_{\mathrm{con}}(\omega|_{\mathcal{U}}) \cong \mathbb{C}[[y]]$ from [DW1] and $\mathrm{GV}_{11}(\omega|_{\mathcal{U}}) = -1$ from [V5], and so $\mathrm{GV}_{ij}(\pi) = -1$. Moreover,

$$N_{ij}(\pi) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(f_{i-1}, f_j)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(\varphi(f_{i-1}), \varphi(f_j))} = \dim_{\mathbb{C}} \mathbb{C}[[y]] = \infty.$$

In case (2), we have $\Lambda_{\mathrm{con}}(\omega|_{\mathcal{U}}) \cong \mathbb{C}[[y]]/(y^n)$ from [DW1], and so $\mathrm{GV}_{11}(\omega|_{\mathcal{U}}) = n$ by 2.4.2, thus $\mathrm{GV}_{ij}(\pi) = n$. It follows that,

$$N_{ij}(\pi) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(f_{i-1}, f_j)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(\varphi(f_{i-1}), \varphi(f_j))} = \dim_{\mathbb{C}} \frac{\mathbb{C}[[y]]}{(y^n)} = n,$$

and so $N_{ij}(\pi) = \mathrm{GV}_{ij}(\pi)$. $\square$

Remark 3.3.9. Given a crepant resolution π of a cA_n singularity, by 3.3.8 the data of N_{ij} is equivalent to the data of GV_{ij}. We go between them freely by replacing all -1s in GV's by ∞s in N's. For example,

$$\begin{matrix} \mathrm{GV}_{11} & & \mathrm{GV}_{22} \\ & \mathrm{GV}_{12} & \end{matrix} = \begin{matrix} 1 & & 3 \\ & -1 & \end{matrix} \iff \begin{matrix} N_{11} & & N_{22} \\ & N_{12} & \end{matrix} = \begin{matrix} 1 & & 3 \\ & \infty & \end{matrix}$$

Below, the N_{ij} are mildly easier to control, and they unify statements about the filtration structure in 5.3.7 and 5.3.8.

Corollary 3.3.10. *Let $\pi_k \colon \mathcal{X}_k \to \operatorname{Spec}\mathcal{R}_k$ be two crepant resolutions of cA_n singularities $\mathcal{R}_k$ for $k = 1, 2$. If $\Lambda_{\mathrm{con}}(\pi_1) \cong \Lambda_{\mathrm{con}}(\pi_2)$, then one of the following cases holds:*

(1) $\mathrm{GV}_{ij}(\pi_1) = \mathrm{GV}_{ij}(\pi_2)$ *for* $1 \leq i \leq j \leq n$,

(2) $\mathrm{GV}_{ij}(\pi_1) = \mathrm{GV}_{n+1-j,n+1-i}(\pi_2)$ *for* $1 \leq i \leq j \leq n$.

Proof. This is immediate from 3.2.7 and 3.3.8. □

Remark 3.3.11. This section has stated various results using the indexing N_{ij} and GV_{ij}. Based on the following facts, we can rephrase these results to use the indexing N_β and GV_β as in the introduction.

Given a crepant partial resolution π of a cA_n singularity with m exceptional curves $\mathrm{C}_1, \ldots, \mathrm{C}_m$, consider the following set of exceptional curve classes

$$S := \{\mathrm{C}_i + \mathrm{C}_{i+1} + \cdots + \mathrm{C}_j \mid 1 \leq i \leq j \leq m\}.$$

Recall that given a curve class $\beta = (\beta_1, \ldots, \beta_m)$, its reflective curve class $\overline{\beta} = (\beta_m, \ldots, \beta_1)$.

(1) By [NW, V5], $\mathrm{GV}_\beta(\pi) \neq 0 \iff \beta \in S$.

(2) By the definition 3.1.1, $N_\beta(\pi) \neq 0 \iff \beta \in S$.

(3) By the definition of reflective curve class, $\beta \in S \iff \overline{\beta} \in S$.

(4) If $\beta = \mathrm{C}_i + \mathrm{C}_{i+1} + \cdots + \mathrm{C}_j$, with notation in 1.5.1, then $|\beta| = j - i + 1$.

(5) If $\beta = \mathrm{C}_i + \mathrm{C}_{i+1} + \cdots + \mathrm{C}_j$, then its reflective curve class $\overline{\beta} = \mathrm{C}_{m+1-j} + +\mathrm{C}_{m+2-j} \cdots + \mathrm{C}_{m+1-i}$.

Based on the above facts, we rephrase the results in the section to those in the introduction.

- By (2) and (4), 3.2.4 induces 1.5.1.
- By (2), (3) and (5), 3.2.7 induces 1.5.2.
- By (1) and (2), 3.3.8 induces 1.5.3.
- By (1), (3) and (5), 3.2.7 induces 1.5.4.

Chapter 4

Monomialization and Geometric Realisation

In this chapter, we focus on the smooth cases, specifically the crepant resolutions of cA_n singularities.

In §4.1, we introduce various intrinsic algebraic definitions of a Type A potential on the double A_n quiver (with a possible single loop at each vertex). Via coordinate changes, we next establish a monomialization result that expresses these potentials in a particularly nice form.

Building on this monomialization, §4.2 shows that any Type A potential on Q_n can be realised by a crepant resolution of a cA_n singularity, and thereby proving the Realisation Conjecture of Brown–Wemyss [BW2] in the setting of Type A potentials.

We further establish a correspondence between the crepant resolutions of cA_n singularities and these intrinsic Type A potentials on Q_n.

Finally, we provide an example of a non-isolated cA_2 singularity which illustrates that the Donovan–Wemyss Conjecture does not extend to non-isolated cDV singularities.

§ 4.1 | Monomialization

This section introduces the quiver $Q_{n,I}$ and Type A potentials.

The main result in §4.1.1 is that any reduced Type A potential on $Q_{n,I}$ is right-equivalent to some reduced monomialized Type A potential in 4.1.20. This is the starting point of the geometric realization in §4.2.

Then §4.1.2 shows that any monomialized Type A potential on $Q_{n,I}$ is isomorphic to a (possibly non-reduced) monomialized Type A potential on Q_n in 4.1.23, which shows that considering the monomialized Type A potentials on Q_n suffices.

Definition 4.1.1. *Given a quiver Q, let f, g and h be potentials on Q. Write $f = \sum_i \lambda_i c_i$*

as a linear combination of cycles where each $0 \neq \lambda_i \in \mathbb{C}$.

(1) *We write $V(f)$ for the $\mathbb{C}$-span of $\{\text{cycle } c \mid c \sim c_i \text{ for some } i\}$.*

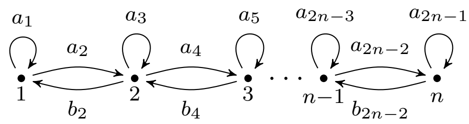
(2) *We say f is* orthogonal *to g if $V(f) \cap V(g) = \{0\}$.*

(3) *We write $f = g \oplus h$ if $f = g + h$ and g is orthogonal to h.*

(4) *We say f* contains *g if $f \sim \lambda g \oplus h'$ for some $0 \neq \lambda \in \mathbb{C}$ and potential h'.*

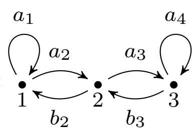
Recall the definition of the quiver Q_n in §1.5.2, which is the double of the usual A_n quiver, with a single loop at each vertex as follows.

$Q_n =$



For any $I \subseteq \{1, 2, \dots, n\}$, define the quiver $Q_{n,I}$ by removing the loop in Q_n at each vertex $i \in I$, and then relabel a_i and b_i from left to right. Similarly to before, we now set $b_i := e_i$ whenever a_i is a loop in $Q_{n,I}$, and set $\mathsf{x}_i := a_i b_i$ and $\mathsf{x}'_i := b_i a_i$ for each i. For example,

$Q_{3,\{2\}} =$



$$b_1 = e_1,\ \mathsf{x}_1 = \mathsf{x}'_1 = a_1$$
$$b_4 = e_4,\ \mathsf{x}_4 = \mathsf{x}'_4 = a_4$$

whereas $\mathsf{x}_2 = a_2 b_2$, $\mathsf{x}'_2 = b_2 a_2$, and $\mathsf{x}_3 = a_3 b_3$, $\mathsf{x}'_3 = b_3 a_3$.

Notation 4.1.2. Through this chapter, n is the number of vertices in the quiver $Q_{n,I}$, and $I \subseteq \{1, 2, \dots, n\}$ is the set of vertices *without* loop in $Q_{n,I}$. Note that $Q_{n,\emptyset}$ is just Q_n. Furthermore, set $m := 2n - 1 - |I|$, which equals the number of x_i in $Q_{n,I}$

We now give several definitions and notations with respect to $Q_{n,I}$.

Definition 4.1.3. *Given any cycle c on $Q_{n,I}$, write c as a composition of arrows. For i such that $1 \leq i \leq m$, let q_i be the number of times a_i appears in this composition. Then set $\mathbf{T}(c) := (q_1, q_2, \dots, q_m)$, and define the* degree *of c to be $\deg(c) := \sum_{i=1}^{m} q_i$.*

Definition 4.1.4. *We say that a potential f on $Q_{n,I}$ is* reduced Type A *if f is reduced in the sense of 2.1.1 and f contains $\mathsf{x}'_i \mathsf{x}_{i+1}$ in the sense of 4.1.1 for each $1 \leq i \leq m-1$. Further, we say that a (possibly non-reduced) potential f on $Q_{n,I}$ is* Type A *if*

(1) *All terms of f have degrees greater than or equal to two in the sense of 4.1.3.*

(2) *The reduced part f_{red} is Type A on $Q_{n,I'}$ for some $I \subseteq I' \subseteq \{1, 2, \dots, n\}$.*

The Splitting Theorem [DWZ, 4.6] gives the existence and uniqueness of f_{red}, so 4.1.4 is well defined.

Lemma 4.1.5. *Given any potential $f \sim \sum_{i=1}^{m-1} \lambda_i x_i' x_{i+1} + h$ where each $0 \neq \lambda_i \in \mathbb{C}$ on $Q_{n,I}$, there exists $f \rightsquigarrow f'$ such that $f' = \sum_{i=1}^{m-1} x_i' x_{i+1} + g$ for some potential g.*

Proof. Applying $a_i \mapsto k_i a_i$ where $k_i \in \mathbb{C}$ for each $1 \leq i \leq m$ gives

$$f \rightsquigarrow \sum_{i=1}^{m-1} k_i k_{i+1} \lambda_i x_i' x_{i+1} + g,$$

for some potential g. Since each $\lambda_i \neq 0$, we can always find some $(k_1, k_2, \ldots, k_m)$ that ensures $k_i k_{i+1} \lambda_i = 1$ holds for $1 \leq i \leq m-1$. $\square$

Remark 4.1.6. The above lemma shows that any reduced Type A potential f can be transformed to the form of $\sum_{i=1}^{m-1} x_i' x_{i+1} \oplus g$ for some potential g. Thus, in this paper, for any reduced Type A potential f on $Q_{n,I}$, we always assume that $f = \sum_{i=1}^{m-1} x_i' x_{i+1} \oplus g$.

Definition 4.1.7. *We call a quiver f on $Q_{n,I}$* monomialized Type A *if $f \sim \sum_{i=1}^{m-1} x_i' x_{i+1} + \sum_{i=1}^{m} \sum_{j=2}^{\infty} \kappa_{ij} x_i^j$ for some $\kappa_{ij} \in \mathbb{C}$.*

Given any monomialized Type A potential f, it is clear that f is Type A. Moreover, f is reduced if and only if $\kappa_{s2} = 0$ whenever x_s is a loop.

Definition 4.1.8. *Given a cycle c on $Q_{n,I}$, consider $\mathbf{T}(c)$ from 4.1.3. Define* left(c) *to be the smallest i such that $q_i > 0$, and* right(c) *to be the largest i such that $q_i > 0$. Then define the* length *of c to be* $\text{len}(c) := \text{right}(c) - \text{left}(c) + 1$.

From the above definition, if $\text{len}(c) = 1$ then $c \sim x_i^j$ for some $1 \leq i \leq m$ and $j \geq 1$.

Notation 4.1.9. We adopt the following notation regarding cycles on $Q_{n,I}$.

(1) Write F for the $\mathbb{C}$-span of $\{c \mid c \text{ is a cycle with } \deg(c) \geq 1\}$ where the degree is defined in 4.1.3.

(2) For any $i \in \mathbb{N}$, write D_i for the $\mathbb{C}$-span of $\{c \mid c \text{ is a cycle with } \deg(c) = i\}$.

(3) For any $i \in \mathbb{N}$, write L_i for the $\mathbb{C}$-span of $\{c \mid c \text{ is a cycle with } \text{len}(c) = i\}$ where the length is defined in 4.1.8.

(4) For any i and $j \in \mathbb{N}$ satisfying $1 \leq i \leq j \leq m$, write V_{ij} for the $\mathbb{C}$-span of $\{c \mid c \text{ is a cycle with left}(c) = i \text{ and right}(c) = j\}$.

It is clear that $F = \bigoplus_i D_i$, $F = \bigoplus_i L_i$ and $F = \bigoplus_{i \leq j} V_{ij}$.

Notation 4.1.10. Let f be a potential on $Q_{n,I}$.

(1) Write $\deg(f) = i$ if $f \in D_i$. Similarly write $\deg(f) \geq i$ if $f \in \oplus_{j \geq i} D_j$, with natural self-documenting variations such as $\deg(f) \leq i$.

(2) Write $\text{len}(f) = i$ if $f \in L_i$. Similarly write $\text{len}(f) \geq i$ if $f \in \oplus_{j \geq i} L_j$, with natural self-documenting variations such as $\text{len}(f) \leq i$.

The above degree and length notations will be important, and they will replace the common notations such as path length.

Notation 4.1.11. Let f and g be potentials on $Q_{n,I}$. With the notation in 4.1.9, since $f, g \in F$, $F = \bigoplus_i D_i$ and $F = \bigoplus_{i \leq j} V_{ij}$, we will adopt the following notation.

(1) Define f_d by decomposing $f = \sum_d f_d$ where each $f_d \in D_d$.

(2) Define $f_{<d} = \sum_{i<d} f_i$ and $f_{>d} = \sum_{i>d} f_i$, with natural self-documenting variations such as $f_{\leq d}$ and $f_{\geq d}$. Thus, if $\deg(f) \geq 2$ then $f = f_2 + f_3 + f_{>3}$.

(3) Define f_{ij} by decomposing

$$f = \sum_{i,j:1 \leq i \leq j \leq m} f_{ij},$$

where each $f_{ij} \in V_{ij}$. Variations such as $f_{ij,d}$, $f_{ij,<d}$, $f_{ij,\leq d}$, $f_{ij,>d}$ and $f_{ij,\geq d}$ are obtained by applying (1) and (2) to f_{ij}.

(4) Given s such that $1 \leq s \leq m$, set

$$f_{[s]} := \sum_{i,j:1 \leq i \leq s \leq j \leq m} f_{ij}.$$

Variations such as $f_{[s],d}$, $f_{[s],<d}$, $f_{[s],\leq d}$, $f_{[s],>d}$ and $f_{[s],\geq d}$ are obtained by applying (1) and (2) to $f_{[s]}$.

(5) Write $f = g + \mathcal{O}_d$ if $f - g \in \bigoplus_{k \geq d} D_k$, and $f = g + \mathcal{O}_{ij,d}$ if $f - g \in V_{ij} \bigcap \bigoplus_{k \geq d} D_k$.

Remark 4.1.12. We will frequently work with sequences of potentials $(\mathsf{f}_d)_{d \geq 1}$ on $Q_{n,I}$, and write f_d for the degree d pieces of f (see 4.1.11). To avoid confusion, we will systematically use Greek font f_d to denote the d-th elements in a sequence, and not the d-th degree piece.

§ 4.1.1 | Monomialization

This subsection will prove that any reduced Type A potential on $Q_{n,I}$ is right-equivalent to some reduced monomialized Type A potential (see 4.1.20).

Notation 4.1.13. To ease notation, in this subsection f will always refer to a reduced Type A potential on $Q_{n,I}$ of the form $\sum_{i=1}^{m-1} \mathsf{x}_i' \mathsf{x}_{i+1} \oplus g$ (see 4.1.6). In the statements below, to ease notation, the c and c_k will refer to a cycle on $Q_{n,I}$, possibly with a coefficient.

The following lemma allows us to monomialize the degree 2 terms in f.

Lemma 4.1.14. *Suppose that $g = h + c$ where $\operatorname{len}(c) \geq 3$ and $\deg(c) = 2$. Then there exists a path degree one right-equivalence (in the sense of 2.1.8),*

$$\rho_c \colon f \overset{1}{\rightsquigarrow} \sum_{i=1}^{m-1} \mathsf{x}_i' \mathsf{x}_{i+1} \oplus (h + c_1) + \mathcal{O}_3,$$

such that $\operatorname{len}(c_1) = 1$ and $\deg(c_1) = 2$.

Proof. Since $\deg(c) = 2$ and $\operatorname{len}(c) \geq 3$, c must have the form of $c \sim \lambda \mathsf{x}'_{s-1}\mathsf{x}_{s+1}$ for some $0 \neq \lambda \in \mathbb{C}$, where s is such that x_s is a loop. Since x_s is a loop and f is reduced, f does not contain x_s^2, and so $f_{[s],2} = \mathsf{x}'_{s-1}\mathsf{x}_s + \mathsf{x}'_s\mathsf{x}_{s+1}$.

Rewrite $f = f_{[s],2} \oplus f_{[s],\geq 3} \oplus r$. Being a loop, $\mathsf{x}_s = a_s$, so applying the depth one unitriangular automorphism $\rho_c\colon a_s \mapsto a_s - \lambda b_{s-1}a_{s-1}$ (in other words, $\mathsf{x}_s \mapsto \mathsf{x}_s - \lambda\mathsf{x}'_{s-1}$) gives

$$\begin{aligned}
\rho_c(f) &= \mathsf{x}'_{s-1}(\mathsf{x}_s - \lambda\mathsf{x}'_{s-1}) + (\mathsf{x}_s - \lambda\mathsf{x}'_{s-1})\mathsf{x}_{s+1} + f_{[s],\geq 3} + r + \mathcal{O}_3 \\
&= f - \lambda(\mathsf{x}'_{s-1})^2 - \lambda\mathsf{x}'_{s-1}\mathsf{x}_{s+1} + \mathcal{O}_3 && (f = f_{[s],2} + f_{[s],\geq 3} + r) \\
&= \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} + h + c - \lambda(\mathsf{x}'_{s-1})^2 - \lambda\mathsf{x}'_{s-1}\mathsf{x}_{s+1} + \mathcal{O}_3 && (f = \textstyle\sum_{i=1}^{m-1}\mathsf{x}'_i\mathsf{x}_{i+1} + h + c) \\
&\overset{2}{\sim} \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} + h - \lambda\mathsf{x}_{s-1}^2 + \mathcal{O}_3 && (c \sim \lambda\mathsf{x}'_{s-1}\mathsf{x}_{s+1},\ (\mathsf{x}'_{s-1})^2 \sim \mathsf{x}_{s-1}^2) \\
&= \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} \oplus (h - \lambda\mathsf{x}_{s-1}^2) + \mathcal{O}_3. && (f = \textstyle\sum_{i=1}^{m-1}\mathsf{x}'_i\mathsf{x}_{i+1} \oplus (h + c),\ \operatorname{len}(c) \geq 3)
\end{aligned}$$

Set $c_1 = -\lambda\mathsf{x}_{s-1}^2$, which satisfies $\operatorname{len}(c_1) = 1$ and $\deg(c_1) = 2$, and we are done. $\square$

The following lemmas allow us to monomialize the terms with degrees greater than two in f. More precisely, given a cycle c with $\operatorname{len}(c) \geq 2$ in f, the basic idea is to decrease $\operatorname{right}(c)$ (see 4.1.16) repeatedly through some right-equivalences until the terms that replace c have length one (see 4.1.17).

Lemma 4.1.15. *Suppose that* $\operatorname{len}(f_2) \leq 2$ *and* $g = h + c$ *where* $\operatorname{len}(c) \geq 2$, $d := \deg(c) \geq 3$. *Then there exists a path degree* $d-1$ *right-equivalence,*

$$\vartheta\colon f \overset{d-1}{\rightsquigarrow} \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} \oplus (h + c_1 + c_2) + \mathcal{O}_{d+1},$$

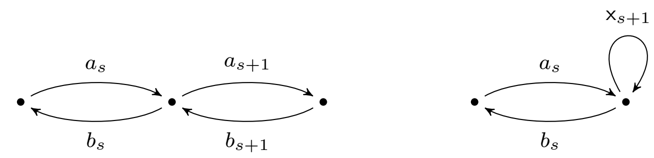
such that each c_k *is either zero or satisfies* $\operatorname{right}(c_k) \leq \operatorname{right}(c)$, $\deg(c_k) = \deg(c)$ *and* $\mathbf{T}(c_k)_{\operatorname{right}(c)} = \mathbf{T}(c)_{\operatorname{right}(c)} - 1$.

Proof. Set $s = \operatorname{right}(c) - 1$. The assumption $\operatorname{len}(f_2) \leq 2$ says that the degree two part of f (wrt. x_i, as in 4.1.10) must be spread over at most two variables. Thus the only degree two cycles in f containing x_s are $\mathsf{x}'_{s-1}\mathsf{x}_s$, x_s^2 and $\mathsf{x}'_s\mathsf{x}_{s+1}$. So, in the notation of 4.1.11, $f_{[s],2} = \mathsf{x}'_{s-1}\mathsf{x}_s + \kappa\mathsf{x}_s^2 + \mathsf{x}'_s\mathsf{x}_{s+1}$ for some $\kappa \in \mathbb{C}$.

Then separating the terms of f that contain or do not contain x_s, we may write $f = f_{[s],2} \oplus f_{[s],\geq 3} \oplus r$. The proof splits into cases.

(1) x_s is not a loop.

The assumptions that $\operatorname{len}(c) \geq 2$ and $\operatorname{right}(c) = s + 1$ imply that a_s, b_s, a_{s+1} and b_{s+1} both appear in c. Note that x_s is not a loop, thus $\mathsf{x}_s = a_s b_s$. Locally $Q_{n,I}$ looks like the following.



x_{s+1} is not a loop $\qquad$ x_{s+1} is a loop

Since right$(c) = s+1$, we can assume cycle c starts with x_{s+1} up to cyclic equivalence. In this order, c starts with some number of x_{s+1} and the next path must be b_s. Thus we may write

$$c \sim \lambda \mathsf{x}_{s+1}^N b_s p a_s r \sim \lambda b_s p a_s r \mathsf{x}_{s+1}^N$$

for some $0 \neq \lambda \in \mathbb{C}$, integer N, and paths p, r. Consider the path $q := r\mathsf{x}_{s+1}^{N-1}$, and rewrite $c \sim \lambda b_s p a_s q \mathsf{x}_{s+1}$. Since $\deg(c) \geq 3$ and $\deg(\mathsf{x}_s) = 1 = \deg(\mathsf{x}_{s+1})$, $\deg(p) + \deg(q) \geq 1$.

Then applying the depth $d-1$ unitriangular automorphism $\vartheta\colon a_s \mapsto a_s - \lambda p a_s q$ gives

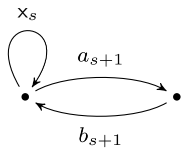
$$\begin{aligned}
\vartheta(f) &= \mathsf{x}'_{s-1}(a_s - \lambda p a_s q) b_s + \kappa[(a_s - \lambda p a_s q) b_s]^2 + b_s(a_s - \lambda p a_s q)\mathsf{x}_{s+1} + f_{[s],\geq 3} + r + \mathcal{O}_{d+1} \\
&\overset{d}{\sim} f - \lambda \mathsf{x}'_{s-1} p a_s q b_s - 2\lambda\kappa \mathsf{x}_s p a_s q b_s - \lambda b_s p a_s q \mathsf{x}_{s+1} + \mathcal{O}_{d+1} \qquad (f = f_{[s],2} + f_{[s],\geq 3} + r) \\
&= \sum_{i=1}^{m-1} \mathsf{x}'_i \mathsf{x}_{i+1} - \lambda \mathsf{x}'_{s-1} p a_s q b_s - 2\lambda\kappa \mathsf{x}_s p a_s q b_s - \lambda b_s p a_s q \mathsf{x}_{s+1} + c + h + \mathcal{O}_{d+1} \\
&\qquad\qquad (f = \textstyle\sum_{i=1}^{m-1} \mathsf{x}'_i \mathsf{x}_{i+1} + h + c) \\
&\overset{d}{\sim} \sum_{i=1}^{m-1} \mathsf{x}'_i \mathsf{x}_{i+1} - \lambda \mathsf{x}'_{s-1} p a_s q b_s - 2\lambda\kappa \mathsf{x}_s p a_s q b_s + h + \mathcal{O}_{d+1} \qquad (c \sim \lambda b_s p a_s q \mathsf{x}_{s+1}) \\
&= \sum_{i=1}^{m-1} \mathsf{x}'_i \mathsf{x}_{i+1} \oplus (-\lambda \mathsf{x}'_{s-1} p a_s q b_s - 2\lambda\kappa \mathsf{x}_s p a_s q b_s + h) + \mathcal{O}_{d+1}.
\end{aligned}$$

Set $c_1 = -\lambda \mathsf{x}'_{s-1} p a_s q b_s$ and $c_2 = -2\lambda\kappa \mathsf{x}_s p a_s q b_s$. The conclusions for c_1 are clear. Either c_2 is zero or $\kappa \neq 0$. In that case, the conclusions for c_2 are also clear.

(2) x_s is a loop.

Since x_s is a loop, from the shape of the quiver $Q_{n,I}$, x_{s+1} is not a loop. Since right$(c) = s+1$, we can assume that the cycle c ends with x_{s+1}, up to cyclic equivalence. Thus $c \sim \lambda p \mathsf{x}_{s+1}$ for some path p and $0 \neq \lambda \in \mathbb{C}$.

Since $\deg(c) \geq 3$ and $\deg(\mathsf{x}_{s+1}) = 1$, $\deg(p) \geq 2$. Moreover, since x_s is a loop and f is reduced, the κ in $f_{[s],2} = \mathsf{x}'_{s-1}\mathsf{x}_s + \kappa \mathsf{x}_s^2 + \mathsf{x}'_s \mathsf{x}_{s+1}$ equals to zero. Locally $Q_{n,I}$ looks like the following.



Being a loop, $\mathsf{x}_s = a_s$, so applying the depth $d-1$ unitriangular automorphism $\vartheta\colon a_s \mapsto a_s - \lambda p$ (in other words, $\mathsf{x}_s \mapsto \mathsf{x}_s - \lambda p$) gives

$$\begin{aligned}
\vartheta(f) &= \mathsf{x}'_{s-1}(\mathsf{x}_s - \lambda p) + (\mathsf{x}_s - \lambda p)\mathsf{x}_{s+1} + f_{[s],\geq 3} + r + \mathcal{O}_{d+1} \\
&\overset{d}{\sim} f - \lambda \mathsf{x}'_{s-1}p - \lambda p \mathsf{x}_{s+1} + \mathcal{O}_{d+1} && (f = f_{[s],2} + f_{[s],\geq 3} + r) \\
&= \sum_{i=1}^{m-1} \mathsf{x}'_i \mathsf{x}_{i+1} - \lambda \mathsf{x}'_{s-1}p - \lambda p \mathsf{x}_{s+1} + c + h + \mathcal{O}_{d+1} && (f = \textstyle\sum_{i=1}^{m-1} \mathsf{x}'_i \mathsf{x}_{i+1} + h + c) \\
&\overset{d}{\sim} \sum_{i=1}^{m-1} \mathsf{x}'_i \mathsf{x}_{i+1} - \lambda \mathsf{x}'_{s-1}p + h + \mathcal{O}_{d+1} && (c \sim \lambda p \mathsf{x}_{s+1}) \\
&= \sum_{i=1}^{m-1} \mathsf{x}'_i \mathsf{x}_{i+1} \oplus (-\lambda \mathsf{x}'_{s-1}p + h) + \mathcal{O}_{d+1}.
\end{aligned}$$

Set $c_1 = -\lambda \mathsf{x}'_{s-1}p$ and $c_2 = 0$. The conclusions for c_1 and c_2 are clear. □

We next apply the previous lemma multiple times to decrease $\mathrm{right}(c)$.

Corollary 4.1.16. *Suppose that* $\mathrm{len}(f_2) \leq 2$ *and* $g = h + c$ *where* $\mathrm{len}(c) \geq 2$, $d := \deg(c) \geq 3$. *Then there exists a path degree* $d-1$ *right-equivalence*

$$\vartheta\colon f \overset{d-1}{\rightsquigarrow} \sum_{i=1}^{m-1} \mathsf{x}'_i \mathsf{x}_{i+1} \oplus (h + \sum_k c_k) + \mathcal{O}_{d+1},$$

such that $\mathrm{right}(c_k) \leq \mathrm{right}(c) - 1$ *and* $\deg(c_k) = \deg(c)$ *for each* k.

Proof. Set $\mathbf{q} = \mathbf{T}(c)$ and $j = \mathrm{right}(c)$. By 4.1.15, there exists a path degree $d-1$ right-equivalence,

$$\vartheta_1\colon f \overset{d-1}{\rightsquigarrow} \mathsf{f}_1 := \sum_{i=1}^{m-1} \mathsf{x}'_i \mathsf{x}_{i+1} \oplus (h + \sum_{s=1}^{2} w_s) + \mathcal{O}_{d+1},$$

such that w_s is either zero, or satisfies $\mathrm{right}(w_s) \leq \mathrm{right}(c)$, $\mathbf{T}(w_s)_j = q_j - 1$ and $\deg(w_s) = \deg(c)$ for each s.

If both w_s equal zero, or both satisfy $\mathbf{T}(w_s)_j = 0$, we are done. Otherwise, we continue to apply 4.1.15 to decrease $\mathbf{T}(w_s)_j$, as follows.

$$\vartheta_2\colon \mathsf{f}_1 \overset{d-1}{\rightsquigarrow} \mathsf{f}_2 := \sum_{i=1}^{m-1} \mathsf{x}'_i \mathsf{x}_{i+1} \oplus (h + \sum_{s=1}^{2}\sum_{t=1}^{2} w_{st}) + \mathcal{O}_{d+1},$$

such that each w_{st} is either zero, or $\mathrm{right}(w_{st}) \leq \mathrm{right}(w_s) \leq \mathrm{right}(c)$, $\mathbf{T}(w_{st})_j = q_j - 2$ and $\deg(w_{st}) = \deg(c)$. The proof follows by induction. □

We next apply the previous 4.1.16 multiple times to achieve the situation where length equals one, namely a monomial type potential.

Corollary 4.1.17. *Suppose that* $\operatorname{len}(f_2) \leq 2$ *and* $g = h + c$ *where* $\operatorname{len}(c) \geq 2$, $d := \deg(c) \geq 3$. *Then there exists a path degree* $d-1$ *right-equivalence*

$$\rho_c\colon f \overset{d-1}{\rightsquigarrow} \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus (h + \sum_k c_k) + \mathcal{O}_{d+1},$$

such that $\operatorname{len}(c_k) = 1$ *and* $\deg(c_k) = \deg(c)$ *for each* k.

Proof. Set $j = \operatorname{right}(c)$. By 4.1.16, there exists a path degree $d-1$ right-equivalence

$$\vartheta_1\colon f \overset{d-1}{\rightsquigarrow} \mathsf{f}_1 := \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus (h + \sum_s w_s) + \mathcal{O}_{d+1},$$

such that $\deg(w_s) = d$ and $\operatorname{right}(w_s) \leq j-1$ for each s.

If all $\operatorname{len}(w_s) = 1$, we are done. Otherwise, we continue to apply 4.1.16 to those $\operatorname{len}(w_s) > 1$ to decrease $\operatorname{right}(w_s)$, as follows.

$$\vartheta_2\colon \mathsf{f}_1 \overset{d-1}{\rightsquigarrow} \mathsf{f}_2 := \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus (h + \sum_{s,t} w_{st}) + \mathcal{O}_{d+1},$$

such that $\deg(w_{st}) = \deg(c)$ and the w_{st} satisfies $\operatorname{right}(w_{st}) \leq j-2$.

If all $\operatorname{len}(w_{st}) = 1$, we are done. Otherwise, we can repeat this process at most $j-1$ times, as follows.

$$\rho_c\colon f \overset{d-1}{\rightsquigarrow} \mathsf{f}_{j-1} := \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus (h + \sum_k c_k) + \mathcal{O}_{d+1},$$

such that $\deg(c_k) = \deg(c)$, and either each $\operatorname{len}(c_k) = 1$ or $\operatorname{right}(c_k) = 1$. However if $\operatorname{right}(c_k) = 1$, then $\operatorname{len}(c_k) = 1$, we are done. □

Using the previous results, we next monomialize the potential f degree by degree. The following deals with degree two.

Proposition 4.1.18. *There exists a path degree one right-equivalence,*

$$\rho_2\colon f \overset{1}{\rightsquigarrow} \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus h + \mathcal{O}_3,$$

such that $\operatorname{len}(h) = 1$ *and* $\deg(h) = 2$.

Proof. We first decompose g in 4.1.13 by degree (wrt. x_i, as in 4.1.10) as $g = g_2 \oplus g_{\geq 3}$, then express g_2 as a linear combination of cycles $g_2 = \oplus_{k=1}^{s} c_k$.

Since there are only a finite number of cycles with degree two on $Q_{n,I}$, necessarily s is finite. Since

$$f = \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus g = \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus g_2 \oplus g_{\geq 3},$$

then g_2 does not contain any length two terms, and so $\mathrm{len}(c_k) = 1$ or ≥ 3 for each k.

If $\mathrm{len}(c_1) = 1$, set $\rho_{c_1} = \mathrm{Id}$. Otherwise $\mathrm{len}(c_1) = 3$, so by 4.1.14 there exists

$$\rho_{c_1}\colon f \overset{1}{\rightsquigarrow} \mathsf{f}_1 := \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus (\sum_{k=2}^{s} c_k + \mathrm{h}_1) + \mathcal{O}_3,$$

such that $\mathrm{len}(\mathrm{h}_1) = 1$ and $\deg(\mathrm{h}_1) = 2$.

If $\mathrm{len}(c_2) = 1$, set $\rho_{c_2} = \mathrm{Id}$. Otherwise $\mathrm{len}(c_2) = 3$, so again by 4.1.14 there exists

$$\rho_{c_2}\colon \mathsf{f}_1 \overset{1}{\rightsquigarrow} \mathsf{f}_2 := \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus (\sum_{k=3}^{s} c_k + \sum_{k=1}^{2} \mathrm{h}_k) + \mathcal{O}_3,$$

such that $\mathrm{len}(\mathrm{h}_2) = 1$ and $\deg(\mathrm{h}_2) = 2$.

We repeat this process s times and set $\rho_2 := \rho_{c_s} \circ \cdots \circ \rho_{c_2} \circ \rho_{c_1}$. It follows that,

$$\rho_2\colon f \overset{1}{\rightsquigarrow} \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus \sum_{k=1}^{s} \mathrm{h}_k + \mathcal{O}_3,$$

such that $\mathrm{len}(\mathrm{h}_k) = 1$, $\deg(\mathrm{h}_k) = 2$ for each k. Set $h = \sum_{k=1}^{s} \mathrm{h}_k$, we are done. □

The following will allow us to monomialize the higher degree terms.

Proposition 4.1.19. *Suppose that* $\mathrm{len}(f_2) \leq 2$. *For any* $d \geq 3$, *there exists a path degree* $d-1$ *right-equivalence,*

$$\rho_d\colon f \overset{d-1}{\rightsquigarrow} \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus (g_{<d} + h) + \mathcal{O}_{d+1},$$

such that $\mathrm{len}(h) = 1$ *and* $\deg(h) = d$.

Proof. We first decompose g in 4.1.13 by degree (wrt. x_i, as in 4.1.10) as $g = g_{<d} \oplus g_d \oplus g_{>d}$, then express g_d as a linear combination of cycles $g_d = \oplus_{k=1}^{s} c_k$. Since there are only a finite number of cycles with degree d on $Q_{n,I}$, s is finite.

If $\mathrm{len}(c_1) = 1$, set $\rho_{c_1} = \mathrm{Id}$. Otherwise, by 4.1.17 there exists

$$\rho_{c_1}\colon f \overset{d-1}{\rightsquigarrow} \mathsf{f}_1 := \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} \oplus (g_{<d} + \sum_{k=2}^{s} c_k + \mathrm{h}_1) + \mathcal{O}_{d+1},$$

such that $\mathrm{len}(\mathrm{h}_1) = 1$ and $\deg(\mathrm{h}_1) = d$.

If $\operatorname{len}(c_2) = 1$, set $\rho_{c_2} = \mathrm{Id}$. Otherwise, again by 4.1.17 there exists

$$\rho_{c_2}\colon \mathsf{f}_1 \overset{d-1}{\rightsquigarrow} \mathsf{f}_2 := \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} \oplus (g_{<d} + \sum_{k=3}^{s} c_k + \sum_{k=1}^{2} \mathrm{h}_k) + \mathcal{O}_{d+1},$$

such that $\operatorname{len}(\mathrm{h}_k) = 1$ and $\deg(\mathrm{h}_k) = d$.

We repeat this process s times and set $\rho_d := \rho_{c_s} \circ \cdots \circ \rho_{c_2} \circ \rho_{c_1}$. It follows that,

$$\rho_d\colon f \overset{d-1}{\rightsquigarrow} \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} \oplus (g_{<d} + \sum_{k=1}^{s} \mathrm{h}_k) + \mathcal{O}_{d+1},$$

such that $\operatorname{len}(\mathrm{h}_k) = 1$, $\deg(\mathrm{h}_k) = d$ for each k. Set $h = \sum_{k=1}^{s} \mathrm{h}_k$, we are done. $\square$

The following is the main result of this subsection.

Theorem 4.1.20. *For any reduced Type A potential f on $Q_{n,I}$, there exists a right-equivalence $\rho\colon f \rightsquigarrow f'$ such that f' is a reduced monomialized Type A potential. In particular, f' is unique up to isomorphism of Jacobi algebras.*

Proof. We first apply the ρ_2 in 4.1.18,

$$\rho_2\colon f \overset{1}{\rightsquigarrow} \mathsf{f}_1 := \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} \oplus \mathrm{h}_2 + \mathcal{O}_3,$$

such that $\operatorname{len}(\mathrm{h}_2) = 1$ and $\deg(\mathrm{h}_2) = 2$.

Since $(\mathsf{f}_1)_2 = \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} \oplus \mathrm{h}_2$, it is clear that $\operatorname{len}((\mathsf{f}_1)_2) \leq 2$. Thus by 4.1.19 applied to f_1, there exists

$$\rho_3\colon \mathsf{f}_1 \overset{2}{\rightsquigarrow} \mathsf{f}_2 := \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} \oplus \sum_{j=2}^{3} \mathrm{h}_j + \mathcal{O}_4,$$

such that $\operatorname{len}(h_3) = 1$, $\deg(h_3) = 3$. Thus, repeating this process $s-1$ times gives

$$\rho_s \circ \cdots \circ \rho_3 \circ \rho_2\colon f \rightsquigarrow \mathsf{f}_s := \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} \oplus \sum_{j=2}^{s} \mathrm{h}_j + \mathcal{O}_{s+1},$$

such that $\operatorname{len}(\mathrm{h}_j) = 1$, $\deg(\mathrm{h}_j) = j$ for each j.

Since ρ_d is a path degree $d-1$ right-equivalence for each $d \geq 2$ by 4.1.18 and 4.1.19, by 2.1.9 $\rho := \lim_{s\to\infty} \rho_s \circ \cdots \circ \rho_3 \circ \rho_2$ exists, and further

$$\rho\colon f \rightsquigarrow \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} \oplus \sum_{j=2}^{\infty} \mathrm{h}_j,$$

such that $\operatorname{len}(\mathrm{h}_j) = 1$ and $\deg(\mathrm{h}_j) = j$ for each j.

Set $f' = \sum_{i=1}^{m-1} \mathsf{x}'_i\mathsf{x}_{i+1} + \sum_{j=2}^{\infty} \mathrm{h}_j$. Since $\operatorname{len}(\mathrm{h}_j) = 1$ for each j, f' is a monomialized Type A potential. Moreover, since f is reduced, f' is also reduced. Since a right-equivalence

$f \rightsquigarrow f'$ induces an isomorphism of Jacobi algebras $f \cong f'$, it follows that f' is unique up to isomorphism of Jacobi algebras. $\square$

§ 4.1.2 | Transform monomialized Type A potentials on $Q_{n,I}$ to Q_n

To state unified results later, it will be convenient to show that any monomialized Type A potential on $Q_{n,I}$ is isomorphic to a (possibly non-reduced) monomialized Type A potential on Q_n. This required the following results, which give precise construction on how to add a loop on $Q_{n,I}$.

Lemma 4.1.21. *Given any $I \neq \{1, 2, \ldots, n\}$ and $i \in I^c$, let x_t be the loop at vertex i of $Q_{n,I}$. Suppose that $h = \sum_{i=1}^{t-2} \mathsf{x}_i'\mathsf{x}_{i+1} + \mathsf{x}_{t-1}'\mathsf{x}_{t+1} + \sum_{i=t+1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} - \frac{1}{2}\mathsf{x}_t^2 + \sum_{i\neq t}\sum_{j=2}^{\infty} \kappa_{ij}\mathsf{x}_i^j$ where all $\kappa_{ij} \in \mathbb{C}$. There exists a right-equivalence*

$$h \rightsquigarrow \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} + \sum_{i=1}^{\infty}\sum_{j=2}^{\infty} \kappa_{ij}'\mathsf{x}_i^j,$$

where κ_{ij}' are some scalars, and further $\kappa_{t2}' \neq 0$.

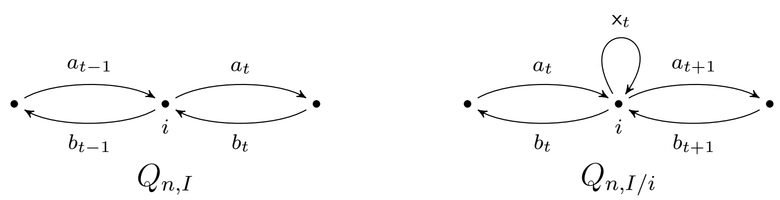
Proof. Being a loop, $\mathsf{x}_t = a_t$, so applying the automorphism $a_t \mapsto a_t - b_{t-1}a_{t-1} - a_{t+1}b_{t+1}$ (in other words, $\mathsf{x}_t \mapsto \mathsf{x}_t - \mathsf{x}_{t-1}' - \mathsf{x}_{t+1}$) gives,

$$\begin{aligned} h &\mapsto \sum_{i=1}^{t-2} \mathsf{x}_i'\mathsf{x}_{i+1} + \mathsf{x}_{t-1}'\mathsf{x}_{t+1} + \sum_{i=t+1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} - \frac{1}{2}(\mathsf{x}_t - \mathsf{x}_{t-1}' - \mathsf{x}_{t+1})^2 + \sum_{i\neq t}\sum_{j=2}^{\infty} \kappa_{ij}\mathsf{x}_i^j \\ &\sim \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} - \frac{1}{2}\mathsf{x}_{t-1}^2 - \frac{1}{2}\mathsf{x}_t^2 - \frac{1}{2}\mathsf{x}_{t+1}^2 + \sum_{i\neq t}\sum_{j=2}^{\infty} \kappa_{ij}\mathsf{x}_i^j. \end{aligned} \tag{4.1.A}$$

Then set the value of κ_{ij}' from the equation $\sum_{i=1}^{\infty}\sum_{j=2}^{\infty} \kappa_{ij}'\mathsf{x}_i^j = -\frac{1}{2}\mathsf{x}_{t-1}^2 - \frac{1}{2}\mathsf{x}_t^2 - \frac{1}{2}\mathsf{x}_{t+1}^2 + \sum_{i\neq t}\sum_{j=2}^{\infty} \kappa_{ij}\mathsf{x}_i^j$. Since κ_{t2}' is the coefficient of x_t^2 in (4.1.A), $\kappa_{t2}' = -\frac{1}{2} \neq 0$. $\square$

Corollary 4.1.22. *Given any $I \neq \emptyset$, $i \in I$ and a monomialized Type A potential f on $Q_{n,I}$, then there exists a monomialized Type A potential g on $Q_{n,I/i}$ such that $\mathfrak{Jac}(Q_{n,I}, f) \cong \mathfrak{Jac}(Q_{n,I/i}, g)$ and g contains the square of the loop at vertex i.*

Proof. Let x_t be the loop at vertex i of $Q_{n,I/i}$. Locally, $Q_{n,I}$ and $Q_{n,I/i}$ look like the following, respectively.



Relabeling the paths allows us to consider f as a potential on $Q_{n,I/i}$. More precisely, we replace the a_k and b_k in f by a_{k+1} and b_{k+1} respectively for any $k \geq t$. Then set $h := f - \frac{1}{2}\mathsf{x}_t^2$. It is clear that $\mathfrak{Jac}(Q_{n,I}, f) \cong \mathfrak{Jac}(Q_{n,I/i}, h)$.

By 4.1.21, there exists a right-equivalence $h \rightsquigarrow g$ such that g is a monomialized Type A potential on $Q_{n,I/i}$ and g contains x_t^2. Thus $\mathcal{J}\mathrm{ac}(Q_{n,I/i}, h) \cong \mathcal{J}\mathrm{ac}(Q_{n,I/i}, g)$, and so $\mathcal{J}\mathrm{ac}(Q_{n,I}, f) \cong \mathcal{J}\mathrm{ac}(Q_{n,I/i}, g)$. $\square$

Proposition 4.1.23. *Given any I and a monomialized Type A potential $f = \sum_{i=1}^{m-1} \mathsf{x}_i'\mathsf{x}_{i+1} + \sum_{i=1}^{m}\sum_{j=2}^{\infty} \kappa_{ij}'\mathsf{x}_i^j$ on $Q_{n,I}$ where all $\kappa_{ij}' \in \mathbb{C}$, then there exists a monomialized Type A potential g on Q_n, namely*

$$g = \sum_{i=1}^{2n-2} \mathsf{x}_i'\mathsf{x}_{i+1} + \sum_{i=1}^{2n-1}\sum_{j=2}^{\infty} \kappa_{ij}\mathsf{x}_i^j$$

for some $\kappa_{ij} \in \mathbb{C}$, such that $\mathcal{J}\mathrm{ac}(Q_n, g) \cong \mathcal{J}\mathrm{ac}(Q_{n,I}, f)$ and $\kappa_{2i-1,2} \neq 0$ for each $i \in I$.

Proof. If $I = \emptyset$, there is nothing to prove. Otherwise, given any $i \in I$, by 4.1.22 there exist a monomialized Type A potential g_1 on $Q_{n,I\backslash i}$ such that $\mathcal{J}\mathrm{ac}(Q_{I\backslash i}, \mathsf{g}_1) \cong \mathcal{J}\mathrm{ac}(Q_{n,I}, f)$, where g_1 contains the square of the loop at vertex i.

Similarly, by 4.1.22 we can repeat the same argument to g_1 on $Q_{n,I\backslash i}$ and any $j \in I\backslash\{i\}$ to construct a monomialized Type A potential g_2 on $Q_{n,I\backslash\{i,j\}}$ such that $\mathcal{J}\mathrm{ac}(Q_{n,I\backslash\{i,j\}}, \mathsf{g}_2) \cong \mathcal{J}\mathrm{ac}(Q_{I\backslash i}, \mathsf{g}_1)$, where g_2 contains the square of the loop at vertex i and vertex j.

Set $s = |I|$. Thus we can repeat this process s times to construct a monomialized Type A potential g_s on $Q_{n,\emptyset}$ such that $\mathcal{J}\mathrm{ac}(Q_{n,\emptyset}, \mathsf{g}_s) \cong \mathcal{J}\mathrm{ac}(Q_{n,I}, f)$, and g_s contains the square of all the loops at all vertices $i \in I$.

Set $g := \mathsf{g}_s$. Since $\kappa_{2i-1,2}$ are the coefficients of the square of the loops at the vertices $i \in I$, the statement follows. $\square$

§ 4.2 | Geometric realisation

Section §4.2.1 below shows that any Type A potential on $Q_{n,I}$ can be realised by a crepant resolution of a cA_n singularity in 4.2.12, and furthermore proves the Realisation Conjecture of Brown–Wemyss [BW2] in the setting of Type A potentials.

Section §4.2.2 gives the converse in 4.2.15, then proves a correspondence between the crepant resolutions of cA_n singularities and our intrinsic Type A potentials on Q_n in 4.2.18 and 4.2.19.

In §4.2.3, we provide an example of a non-isolated cA_2 singularity, illustrating that the Donovan–Wemyss Conjecture 2.3.7 fails for non-isolated cDV singularities in 4.2.26.

§ 4.2.1 | Geometric realisation

In this subsection, we will prove in 4.2.12 that, given any Type A potential f on $Q_{n,I}$, there is a crepant resolution π of a cA_n singularity such that $\mathcal{J}\mathrm{ac}(f) \cong \Lambda_{\mathrm{con}}(\pi)$. This proves the Realisation Conjecture of Brown–Wemyss [BW2] in the setting of Type A potentials.

Notation 4.2.1. We first fix a monomialized Type A potentials f on Q_n as follows,

$$f = \sum_{i=1}^{2n-2} \mathsf{x}_i' \mathsf{x}_{i+1} + \sum_{i=1}^{2n-1} \sum_{j=2}^{\infty} \kappa_{ij} \mathsf{x}_i^j. \tag{4.2.A}$$

Then we consider the following system of equations where each $g_i \in \mathbb{C}[[x,y]]$

$$\begin{aligned} g_0 + \sum_{j=2}^{\infty} j\kappa_{1j} g_1^{j-1} + g_2 &= 0 \\ g_1 + \sum_{j=2}^{\infty} j\kappa_{2j} g_2^{j-1} + g_3 &= 0 \\ &\vdots \\ g_{2n-2} + \sum_{j=2}^{\infty} j\kappa_{2n-1,j} g_{2n-1}^{j-1} + g_{2n} &= 0. \end{aligned} \tag{4.2.B}$$

The following lemma allows us to construct the geometric realisation of f (4.2.A) in 4.2.3(1) by the system of equations (4.2.B).

Lemma 4.2.2. *With notation in 4.2.1, fix some integer t satisfying $0 \leq t \leq 2n-1$, and set $g_t = y$, $g_{t+1} = x$. Then there exists $(g_0, g_1, \ldots, g_{2n})$ which satisfies (4.2.B) and, furthermore, each $g_s \in (\!(x,y)\!) \subseteq \mathbb{C}[[x,y]]$ is prime and has a linear term. Moreover,*

1. *For any $0 \leq s \leq 2n-1$, $(\!(g_s, g_{s+1})\!) = (\!(x,y)\!)$.*
2. *For any $1 \leq s \leq 2n-1$, $(\!(g_{s-1}, g_{s+1})\!) \subsetneq (\!(x,y)\!)$ when $\kappa_{s2} = 0$, and $(\!(g_{s-1}, g_{s+1})\!) = (\!(x,y)\!)$ when $\kappa_{s2} \neq 0$.*

Proof. We start with the equation $g_t + \sum_{j=2}^{\infty} j\kappa_{t+1,j} g_{t+1}^{j-1} + g_{t+2} = 0$ in (4.2.B) which defines $g_{t+2} = -y - \sum_{j=2}^{\infty} j\kappa_{t+1,j} x^{j-1} \in (\!(x,y)\!)$. Then we consider $g_{t+1} + \sum_{j=2}^{\infty} j\kappa_{t+2,j} g_{t+2}^{j-1} + g_{t+3} = 0$ which also defines $g_{t+3} \in (\!(x,y)\!)$. Thus we can repeat this process to construct $g_s \in (\!(x,y)\!)$ for $t+2 \leq s \leq 2n$. Similarly, the equation $g_{t-1} + \sum_{j=2}^{\infty} j\kappa_{t,j} g_t^{j-1} + g_{t+1} = 0$ defines $g_{t-1} \in (\!(x,y)\!)$. We can repeat this process to construct $g_s \in (\!(x,y)\!)$ for $0 \leq s \leq t-1$.

(1) For any $0 \leq s \leq 2n-2$, using $g_s + \sum_{j=2}^{\infty} j\kappa_{s+1,j} g_{s+1}^{j-1} + g_{s+2} = 0$ in (4.2.B), we have $(\!(g_s, g_{s+1})\!) = (\!(g_{s+1}, g_{s+2})\!)$. Moving either to the left or right until we hit t, it follows that $(\!(g_s, g_{s+1})\!) = (\!(g_t, g_{t+1})\!) = (\!(y,x)\!)$. Hence $(\!(g_s, g_{s+1})\!) = (\!(x,y)\!)$ for all $0 \leq s \leq 2n-1$, which implies that each g_s is prime and has a linear term.

(2) For any $1 \leq s \leq 2n-1$, using $g_{s-1} + \sum_{j=2}^{\infty} j\kappa_{sj} g_s^{j-1} + g_{s+1} = 0$ in (4.2.B), we have $(\!(g_{s-1}, g_{s+1})\!) = (\!(g_{s-1}, \sum_{j=2}^{\infty} j\kappa_{sj} g_s^{j-1})\!)$. Thus, if $\kappa_{s2} = 0$ then $(\!(g_{s-1}, g_{s+1})\!) \subsetneq (\!(x,y)\!)$, and if $\kappa_{s2} \neq 0$ then $(\!(g_{s-1}, g_{s+1})\!) = (\!(g_{s-1}, g_s)\!)$ which equals $(\!(x,y)\!)$ by (1). $\square$

Notation 4.2.3. For any t with $0 \leq t \leq 2n-1$, 4.2.2 calculates a solution of (4.2.B). Fix any such solution, say $(g_0, g_1, \ldots, g_{2n})$. From this, we adopt the following notation.

(1) Since each g_i is a prime element of $\mathbb{C}[[x,y]]$ with a linear term (by 4.2.2), we first define the cA_n singularity

$$\mathcal{R} := \frac{\mathbb{C}[[u,v,x,y]]}{uv - g_0g_2\dots g_{2n}},$$

and the CM $\mathcal{R}$-module

$$M := \mathcal{R} \oplus (u, g_0) \oplus (u, g_0g_2) \oplus \ldots \oplus (u, \prod_{j=0}^{n-1} g_{2j}).$$

(2) We next define

$$\mathcal{S}_1 := \frac{\mathbb{C}[[u,v,x_0,x_1,x_2,x_3\dots,x_{2n-1},x_{2n}]]}{uv - x_0x_2\dots x_{2n}}.$$

(3) Define a sequence $h_1, h_2, \dots, h_{2n-1} \in \mathcal{S}_1$ to be

$$h_i := x_{i-1} + \sum_{j=2}^{\infty} j\kappa_{ij}x_i^{j-1} + x_{i+1},$$

and set $\mathcal{S}_i := \mathcal{S}_1/(h_1, h_2, \dots, h_{i-1})$ for $2 \le i \le 2n$.

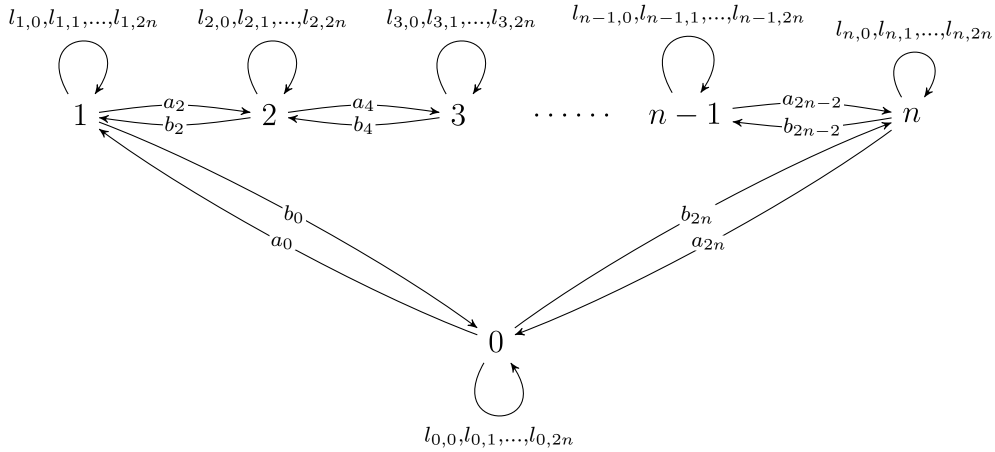
(4) For $1 \le i \le 2n$, by abuse of notation we regard (u, x_0), (u, x_0x_2), $\dots$, $\big(u, \prod_{j=0}^{n-1} x_{2j}\big)$ as $\mathcal{S}_i$-modules. Then we define the $\mathcal{S}_i$-module

$$N_i := \mathcal{S}_i \oplus (u, x_0) \oplus (u, x_0x_2) \oplus \cdots \oplus \Big(u, \prod_{j=0}^{n-1} x_{2j}\Big).$$

(5) Write π_1 for the universal flop of $\operatorname{Spec}\mathcal{S}_1$ corresponding to N_1 [IW1, §5]. For $2 \le i \le 2n$, consider the morphism $\operatorname{Spec}\mathcal{S}_i \to \operatorname{Spec}\mathcal{S}_1$, and the fiber product $\mathcal{X}_i := \mathcal{X}_1 \times_{\operatorname{Spec}\mathcal{S}_1} \operatorname{Spec}\mathcal{S}_i$. These morphisms fit into the following commutative diagram.

$$\begin{tikzcd}
\mathcal{X}_{2n} \arrow[r] \arrow[d, "\pi_{2n}"] & \cdots \arrow[r] & \mathcal{X}_2 \arrow[r] \arrow[d, "\pi_2"] & \mathcal{X}_1 \arrow[d, "\pi_1"] \\
\operatorname{Spec}\mathcal{S}_{2n} \arrow[r] & \cdots \arrow[r] & \operatorname{Spec}\mathcal{S}_2 \arrow[r] & \operatorname{Spec}\mathcal{S}_1
\end{tikzcd}$$

(6) Consider the following quiver Q.



Then define the relations R_1 of Q as follows.

$$R_1 := \begin{cases} l_{t,i}a_{2t} = a_{2t}l_{t+1,i}, \;\; l_{t+1,i}b_{2t} = b_{2t}l_{t,i}, \;\; l_{t,i}l_{t,j} = l_{t,j}l_{t,i}, \\ l_{t,2t} = a_{2t}b_{2t}, \;\; l_{t+1,2t} = b_{2t}a_{2t} \text{ for any } t \in \mathbb{Z}/(n+1) \text{ and } 0 \leq i, j \leq 2n. \end{cases} \tag{4.2.C}$$

For $2 \leq s \leq 2n$, define R_s to be R_1 with the additional relations

$$l_{t,i-1} + \sum_{j=2}^{\infty} j\mathsf{k}_{ij} l_{t,i}^{j-1} + l_{t,i+1} = 0 \text{ for any } 0 \leq t \leq n \text{ and } 1 \leq i \leq s-1. \tag{4.2.D}$$

To prepare for the main construction 4.2.9, we now establish in 4.2.4–4.2.8 a quiver presentation of the NCCR $\mathrm{End}_{\mathcal{R}}(M)$, where $\mathcal{R}$ and M are as in 4.2.3(1).

Lemma 4.2.4. *With notation in 4.2.3, for $2 \leq k \leq 2n$, $\mathcal{S}_k$ is an integral domain and normal. Furthermore, there exists a ring isomorphism $\varphi\colon \mathcal{S}_{2n} \xrightarrow{\sim} \mathcal{R}$ such that $\varphi(N_{2n}) = M$.*

Proof. Fix some k with $2 \leq k \leq 2n$. By the definition in 4.2.3(2) and 4.2.3(3),

$$\mathcal{S}_k \cong \frac{\mathbb{C}[[u, v, x_0, x_1, x_2, \ldots, x_{2n}]]}{(uv - x_0x_2 \ldots x_{2n}, h_1, h_2, \ldots, h_{k-1})},$$

where each $h_i = x_{i-1} + \sum_{j=2}^{\infty} j\mathsf{k}_{ij}x_i^{j-1} + x_{i+1}$.

Similar to 4.2.2, for each $2 \leq i \leq k$, we can express x_i as a formal power series of x_0 and x_1 using $h_1, h_2, \ldots, h_{k-1}$. Write these expressions as $x_i := H_i(x_0, x_1)$.

Thus, when k is even,

$$\mathcal{S}_k \cong \frac{\mathbb{C}[[u, v, x_0, x_1, x_{k+1}, x_{k+2}, \ldots, x_{2n}]]}{uv - x_0H_2H_4 \ldots H_kx_{k+2} \ldots x_{2n}}.$$

When k is odd,

$$\mathcal{S}_k \cong \frac{\mathbb{C}[[u, v, x_0, x_1, x_{k+1}, x_{k+2}, \dots, x_{2n}]]}{uv - x_0 H_2 H_4 \dots H_{k-1} x_{k+1} \dots x_{2n}}.$$

In both cases, $\mathcal{S}_k$ is an integral domain and normal by e.g. [S, 4.1.1].

Then we prove that $\mathcal{S}_{2n} \cong \mathcal{R}$. Recall from 4.2.2 that we start with $g_t = y$ and $g_{t+1} = x$ and then construct $(g_0, g_1, \dots, g_{2n})$ where each $g_i \in \mathbb{C}[[x, y]]$ using the equation system (4.2.B). Then, in 4.2.3(1), these g_i were used to define $\mathcal{R}$.

On the other hand,

$$\mathcal{S}_{2n} \cong \frac{\mathbb{C}[[u, v, x_0, x_1, x_2, \dots, x_{2n}]]}{(uv - x_0 x_2 \dots x_{2n}, h_1, h_2, \dots, h_{2n-1})}.$$

Similar to 4.2.2, we can express each x_s as a formal power series of x_t and x_{t+1} using $h_1, h_2, \dots, h_{2n-1}$, and indeed $x_s = g_s(x_{t+1}, x_t)$. Hence

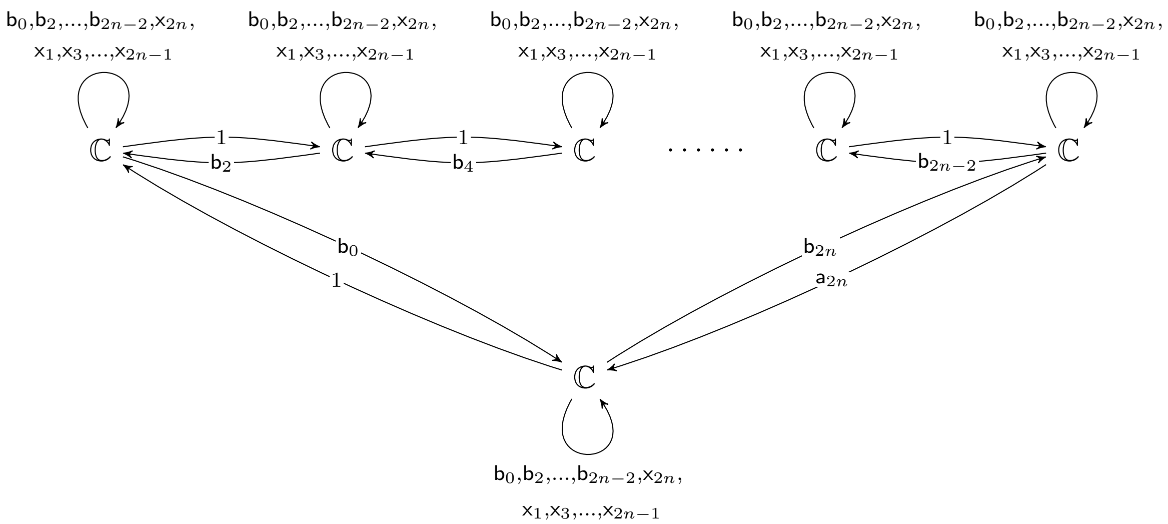
$$\mathcal{S}_{2n} \cong \frac{\mathbb{C}[[u, v, x_t, x_{t+1}]]}{uv - g_0(x_{t+1}, x_t) g_2(x_{t+1}, x_t) \dots g_{2n}(x_{t+1}, x_t)}.$$

Define a ring homomorphism $\varphi\colon \mathcal{S}_{2n} \to \mathcal{R}$ by $u \mapsto u$, $v \mapsto v$, $x_{t+1} \mapsto x$, and $x_t \mapsto y$. It is immediate that φ is an isomorphism, and moreover $\varphi(N_{2n}) = M$. $\square$

With notation in 4.2.3, consider the universal resolution $\pi_1 \colon \mathcal{X}_1 \to \operatorname{Spec} \mathcal{S}_1$ with $\Lambda(\pi_1) \cong \operatorname{End}_{\mathcal{S}_1}(N_1)$ [IW1, §5]. As shown in Appendix 7.0.18, $\operatorname{End}_{\mathcal{S}_1}(N_1) \cong \mathbb{C}\langle\langle Q \rangle\rangle / R_1$ where Q and R_1 are in (4.2.C).

To ease notation, set $\Lambda := \mathbb{C}\langle\langle Q \rangle\rangle / R_1$. By [W2, 6.2], $\mathcal{X}_1$ is isomorphic to a moduli scheme of stable representations of Λ, of dimension vector $\delta = (1, 1, \dots, 1)$ and stability $\vartheta = (-n, 1, 1, \dots, 1)$ where the $-n$ sits at vertex 0 of Q.

In notation, $\mathcal{X}_1 \cong \mathcal{M}^{\vartheta}_{\delta}(\Lambda)$, which is the moduli space of ϑ-stable representations of dimension vector δ. Moreover, exactly as in [W3, §3], $\mathcal{M}^{\vartheta}_{\delta}(\Lambda) \cong \bigcup_{i=0}^{n} \mathcal{U}_{1i}$ is a gluing of $n+1$ affine charts. Accounting for the relations R_1 (4.2.C), the first affine chart $\mathcal{U}_{10}$ is parameterised by



where $\mathsf{x}_{2n} = \mathsf{a}_{2n}\mathsf{b}_{2n}$ and (since we work on the completed path algebra) all cycles are nilpotent. We claim that $\mathcal{U}_{10} \cong \operatorname{Spec} \mathcal{A}_{10}$ where

$$\mathcal{A}_{10} := \frac{\mathbb{C}[[\mathsf{b}_0, \mathsf{b}_2, \dots, \mathsf{b}_{2n-2}, \mathsf{a}_{2n}, \mathsf{x}_1, \mathsf{x}_3, \dots, \mathsf{x}_{2n-1}, \mathsf{x}_{2n}, \mathsf{v}]][\mathsf{b}_{2n}]}{(\mathsf{x}_{2n} - \mathsf{a}_{2n}\mathsf{b}_{2n}, \mathsf{v} - \mathsf{b}_0\mathsf{b}_2 \dots \mathsf{b}_{2n})}. \tag{4.2.E}$$

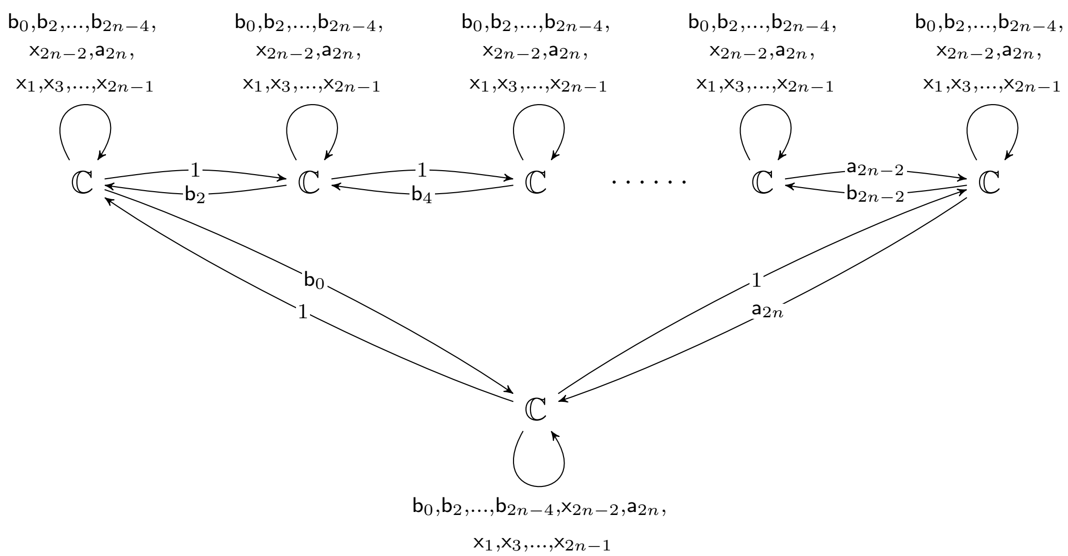
Indeed, we have used all the relations in the quiver, so the question boils down to understanding nilpotent cycles. Clearly $\mathsf{x}_1, \mathsf{x}_3, \dots, \mathsf{x}_{2n-3}, \mathsf{x}_{2n-1}, \mathsf{x}_{2n}, \mathsf{b}_0, \mathsf{b}_2, \dots, \mathsf{b}_{2n-2}$ are cycles, as is a_{2n} (once composed with all clockwise arrows marked 1), thus they are nilpotent. As is $\mathsf{b}_{2n}\mathsf{b}_{2n-2} \dots \mathsf{b}_0$.

There is no condition on b_{2n}, so it is a polynomial variable. Introducing a new completion variable v to capture the nilpotency of $\mathsf{b}_{2n}\mathsf{b}_{2n-2} \dots \mathsf{b}_0$, which has a mix of both polynomial and completion variables, the claim follows.

Moreover, $\pi_1|_{\mathcal{U}_{10}} \colon \mathcal{U}_{10} \to \operatorname{Spec} \mathcal{S}_1$ is induced by the ring homomorphism $\varphi_{10} \colon \mathcal{S}_1 \to \mathcal{A}_{10}$

$$\begin{aligned} &x_0 \mapsto \mathsf{b}_0, \quad x_2 \mapsto \mathsf{b}_2, \quad \dots, \quad x_{2n-2} \mapsto \mathsf{b}_{2n-2}, \quad x_{2n} \mapsto \mathsf{x}_{2n}, \\ &x_1 \mapsto \mathsf{x}_1, \quad x_3 \mapsto \mathsf{x}_3, \quad \dots, \quad x_{2n-1} \mapsto \mathsf{x}_{2n-1}, \quad u \mapsto \mathsf{a}_{2n}, \quad v \mapsto \mathsf{v}. \end{aligned} \tag{4.2.F}$$

Similarly, the second affine chart $\mathcal{U}_{11}$ is parameterised by



where $\mathsf{x}_{2n-2} = \mathsf{a}_{2n-2}\mathsf{b}_{2n-2}$ and (since we work on the completed path algebra) all cycles are nilpotent. We claim that $\mathcal{U}_{11} \cong \operatorname{Spec} \mathcal{A}_{11}$ where

$$\mathcal{A}_{11} := \frac{\mathbb{C}[[\mathsf{b}_0, \mathsf{b}_2, \dots, \mathsf{b}_{2n-4}, \mathsf{a}_{2n}, \mathsf{x}_1, \mathsf{x}_3, \dots, \mathsf{x}_{2n-1}, \mathsf{x}_{2n-2}, \mathsf{u}, \mathsf{v}]][\mathsf{a}_{2n-2}, \mathsf{b}_{2n-2}]}{(\mathsf{x}_{2n-2} - \mathsf{a}_{2n-2}\mathsf{b}_{2n-2}, \mathsf{u} - \mathsf{a}_{2n-2}\mathsf{a}_{2n}, \mathsf{v} - \mathsf{b}_0\mathsf{b}_2 \dots \mathsf{b}_{2n-2})}. \tag{4.2.G}$$

Similarly, we have also used all the relations in the quiver, so the question boils down to understanding nilpotent cycles. Clearly $\mathsf{x}_1, \mathsf{x}_3, \dots, \mathsf{x}_{2n-3}, \mathsf{x}_{2n-1}, \mathsf{x}_{2n-2}, \mathsf{b}_0, \mathsf{b}_2, \dots, \mathsf{b}_{2n-4}, \mathsf{a}_{2n}$ are cycles, as is $\mathsf{a}_{2n-2}\mathsf{a}_{2n}$ (once composed with all clockwise arrows marked 1), thus they are nilpotent. As is $\mathsf{b}_{2n-2}\mathsf{b}_{2n-4}\dots\mathsf{b}_0$.

There is no condition on a_{2n-2} and b_{2n-2}, so they are polynomial variables. Introducing new completion variables u and v to capture the nilpotency of $\mathsf{a}_{2n-2}\mathsf{a}_{2n}$ and $\mathsf{b}_{2n-2}\mathsf{b}_{2n-4}\dots\mathsf{b}_0$ respectively, which have a mix of both polynomial and completion variables, the claim follows.

Moreover, $\pi_1|_{\mathcal{U}_{11}}\colon \mathcal{U}_{11} \to \operatorname{Spec}\mathcal{S}_1$ is induced by the ring homomorphism $\varphi_{11}\colon \mathcal{S}_1 \to \mathcal{A}_{11}$

$$\begin{aligned} &x_0 \mapsto \mathsf{b}_0, \quad x_2 \mapsto \mathsf{b}_2, \quad \dots, \quad x_{2n-4} \mapsto \mathsf{b}_{2n-4}, \quad x_{2n-2} \mapsto \mathsf{x}_{2n-2}, \quad x_{2n} \mapsto \mathsf{a}_{2n}, \\ &x_1 \mapsto \mathsf{x}_1, \quad x_3 \mapsto \mathsf{x}_3, \quad \dots, \quad x_{2n-1} \mapsto \mathsf{x}_{2n-1}, \quad u \mapsto \mathsf{u}, \quad v \mapsto \mathsf{v}. \end{aligned} \tag{4.2.H}$$

Each of the remaining affine charts $\mathcal{U}_{1j}$ of $\mathcal{X}_1$ admits a similar parametrisation, and the corresponding morphism $\pi_1|_{\mathcal{U}_{1j}}\colon \mathcal{U}_{1j} \to \operatorname{Spec}\mathcal{S}_1$ is defined in the same way as above.

Notation 4.2.5. With the notation $\mathcal{U}_{1j}$ above and in 4.2.3, for each $2 \le i \le 2n$ and $0 \le j \le n$, we set

(1) $\mathcal{U}_{ij} := \mathcal{U}_{1j} \times_{\operatorname{Spec}\mathcal{S}_1} \operatorname{Spec}\mathcal{S}_i$, the base change of $\mathcal{U}_{1j}$ along $\operatorname{Spec}\mathcal{S}_i \to \operatorname{Spec}\mathcal{S}_1$;

(2) $\mathcal{A}_{ij} := \Gamma(\mathcal{U}_{ij}, \mathcal{O}_{\mathcal{U}_{ij}})$, the coordinate ring of $\mathcal{U}_{ij}$;

(3) $\varphi_{ij}\colon \mathcal{S}_i \to \mathcal{A}_{ij}$, the ring homomorphism associated to $\pi_i|_{\mathcal{U}_{ij}}\colon \mathcal{U}_{ij} \to \operatorname{Spec}\mathcal{S}_i$.

By definition 4.2.3(5) $\mathcal{X}_i := \mathcal{X}_1 \times_{\operatorname{Spec}\mathcal{S}_1} \operatorname{Spec}\mathcal{S}_i$, hence $\mathcal{X}_i \cong \bigcup_{j=0}^n \mathcal{U}_{ij}$.

Since $\mathcal{X}_1$ is the universal resolution of $\operatorname{Spec}\mathcal{S}_1$, it is connected and smooth. We now show that, for $2 \le i \le 2n$, the base change $\mathcal{X}_i$ is likewise connected and smooth.

The next result shows that the connectivity of $\mathcal{X}_i$ comes from the overlap of adjacent affine charts along the exceptional curves.

Proposition 4.2.6. *With notation in 4.2.3, $\mathcal{X}_i$ is connected for all $2 \le i \le 2n$.*

Proof. For $1 \le j \le n$, write C_j for the j-th exceptional curve of the universal resolution $\pi_1\colon \mathcal{X}_1 \to \operatorname{Spec}\mathcal{S}_1$ over the origin. By definition 4.2.3(3), for $2 \le i \le 2n$

$$\mathcal{S}_i := \mathcal{S}_1/(h_1, h_2, \dots, h_{i-1}),$$

where each h_i is a power series without a constant term. Thus $\operatorname{Spec}\mathcal{S}_i$ contains the origin of $\operatorname{Spec}\mathcal{S}_1$. Consequently, for $2 \le i \le 2n$, $\mathcal{X}_i$ contains the $\bigcup_{j=1}^n \mathrm{C}_j \subset \mathcal{X}_1$ supported over the origin. Moreover, for each $1 \le j \le n$ the affine charts of $\mathcal{X}_i$ satisfy

$$\mathrm{C}_j \subset \mathcal{U}_{i,j-1} \cup \mathcal{U}_{ij} \qquad \Rightarrow \qquad \mathcal{U}_{i,j-1} \cap \mathcal{U}_{ij} \neq \emptyset,$$

and so the affine charts of $\mathcal{X}_i$ pairwise overlap along the exceptional curves. Hence $\mathcal{X}_i$ is connected. $\square$

We now prove that $\mathcal{X}_i$ is smooth for $2 \le i \le 2n$ by analysing each affine chart $\mathcal{U}_{ij}$ of $\mathcal{X}_i$.

Proposition 4.2.7. *With notation in 4.2.3, for $2 \leq i \leq 2n$, $\mathcal{X}_i$ is smooth.*

Proof. Since by definition 4.2.3(5) $\mathcal{X}_i := \mathcal{X}_1 \times_{\operatorname{Spec} \mathcal{S}_1} \operatorname{Spec} \mathcal{S}_i$ for $2 \leq i \leq 2n$, we have the following pullback squares for the j-th affine chart $\mathcal{U}_{ij}$ of $\mathcal{X}_i$:

$$\begin{tikzcd} \mathcal{U}_{ij} \arrow[r] \arrow[d, "\pi_i|_{\mathcal{U}_{ij}}"] & \mathcal{U}_{1j} \arrow[d, "\pi_1|_{\mathcal{U}_{1j}}"] \\ \operatorname{Spec} \mathcal{S}_i \arrow[r] & \operatorname{Spec} \mathcal{S}_1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_{ij} & \mathcal{A}_{1j} \arrow[l] \\ \mathcal{S}_i \arrow[u, "\varphi_{ij}"] & \mathcal{S}_1 \arrow[l] \arrow[u, "\varphi_{1j}"] \end{tikzcd}$$

Recall from 4.2.3(3) that for $2 \leq i \leq 2n$

$$\mathcal{S}_i := \mathcal{S}_1/(h_1, h_2, \dots, h_{i-1}) \quad \text{with} \quad h_i := x_{i-1} + \sum_{j=2}^{\infty} j \kappa_{ij} x_i^{j-1} + x_{i+1}.$$

Therefore, for $2 \leq i \leq 2n$ and $0 \leq j \leq n$,

$$\mathcal{A}_{ij} \cong \mathcal{A}_{1j} \otimes_{\mathcal{S}_1} \mathcal{S}_i \cong \mathcal{A}_{1j} \otimes_{\mathcal{S}_1} \mathcal{S}_1/(h_1, h_2, \dots, h_{i-1}) \cong \mathcal{A}_{1j}/(\mathcal{A}_{1j}h_1, \mathcal{A}_{1j}h_2, \dots, \mathcal{A}_{1j}h_{i-1}). \quad (4.2.I)$$

First chart $(j = 0)$**.** From (4.2.E) we have

$$\mathcal{A}_{10} \cong \frac{\mathbb{C}[[\mathsf{b}_0, \mathsf{b}_2, \dots, \mathsf{b}_{2n-2}, \mathsf{a}_{2n}, \mathsf{x}_1, \mathsf{x}_3, \dots, \mathsf{x}_{2n-1}, \mathsf{x}_{2n}, \mathsf{v}]][\mathsf{b}_{2n}]}{(\mathsf{x}_{2n} - \mathsf{a}_{2n}\mathsf{b}_{2n}, \mathsf{v} - \mathsf{b}_0\mathsf{b}_2 \dots \mathsf{b}_{2n})}.$$

Moreover, by (4.2.F) $\varphi_{10} \colon \mathcal{S}_1 \to \mathcal{A}_{10}$ is given by

$$\begin{aligned} &x_0 \mapsto \mathsf{b}_0, \quad x_2 \mapsto \mathsf{b}_2, \quad \dots, \quad x_{2n-2} \mapsto \mathsf{b}_{2n-2}, \quad x_{2n} \mapsto \mathsf{x}_{2n}, \quad u \mapsto \mathsf{a}_{2n}, \quad v \mapsto \mathsf{v}, \\ &x_1 \mapsto \mathsf{x}_1, \quad x_3 \mapsto \mathsf{x}_3, \quad \dots, \quad x_{2n-1} \mapsto \mathsf{x}_{2n-1}. \end{aligned}$$

Thus, for $1 \leq i \leq 2n-1$, the images $\mathcal{A}_{10}h_i$ are

$$\mathcal{A}_{10}h_i = \begin{cases} \mathsf{b}_{i-1} + \sum_{j=2}^{\infty} j\kappa_{ij}\mathsf{x}_i^{j-1} + \mathsf{b}_{i+1}, & \text{for } i = 1, 3, \dots, 2n-3 \\ \mathsf{x}_{i-1} + \sum_{j=2}^{\infty} j\kappa_{ij}\mathsf{b}_i^{j-1} + \mathsf{x}_{i+1}, & \text{for } i = 2, 4, \dots, 2n-2 \\ \mathsf{b}_{2n-2} + \sum_{j=2}^{\infty} j\kappa_{2n-1,j}\mathsf{x}_{2n-1}^{j-1} + \mathsf{x}_{2n}, & \text{for } i = 2n-1 \end{cases}$$

Introduce the notation obtained by successive elimination:

$$\mathsf{b}_{01} := -\sum_{j=2}^{\infty} j\kappa_{1j}\, \mathsf{x}_1^{j-1} - \mathsf{b}_2 \in \mathbb{C}[[\mathsf{x}_1, \mathsf{b}_2]],$$

and, using $\mathsf{x}_1 = -\sum_{j=2}^{\infty} j\kappa_{2j}\, \mathsf{b}_2^{j-1} - \mathsf{x}_3$,

$$\mathsf{b}_{02} := -\sum_{j=2}^{\infty} j\kappa_{1j}\Big(-\sum_{r=2}^{\infty} r\kappa_{2r}\, \mathsf{b}_2^{r-1} - \mathsf{x}_3\Big)^{j-1} - \mathsf{b}_2 \in \mathbb{C}[[\mathsf{b}_2, \mathsf{x}_3]].$$

Continuing inductively, for $0 \leq t \leq n-1$ and $2t < k \leq 2n-1$, define $\mathsf{b}_{2t,k}$ with

$$\mathsf{b}_{2t,k} \in \begin{cases} \mathbb{C}[[\mathsf{x}_k, \mathsf{b}_{k+1}]], & k \text{ odd}, \ k \neq 2n-1, \\ \mathbb{C}[[\mathsf{b}_k, \mathsf{x}_{k+1}]], & k \text{ even}, \\ \mathbb{C}[[\mathsf{x}_{2n-1}, \mathsf{x}_{2n}]], & k = 2n-1. \end{cases}$$

Each $\mathcal{A}_{10}h_i$ has a linear term, hence eliminates one variable in (4.2.I). Consequently,

$$\begin{aligned}
\mathcal{A}_{20} &\cong \mathcal{A}_{10}/(\mathcal{A}_{10}h_1) \cong \mathcal{A}_{10}/(\mathsf{b}_0 + \sum_{j=2}^{\infty} j\kappa_{1j}\mathsf{x}_1^{j-1} + \mathsf{b}_2) \\
&\cong \frac{\mathbb{C}[[\mathsf{b}_2, \mathsf{b}_4, \ldots, \mathsf{b}_{2n-2}, \mathsf{a}_{2n}, \mathsf{x}_1, \mathsf{x}_3, \ldots, \mathsf{x}_{2n-1}, \mathsf{x}_{2n}, \mathsf{v}]][\mathsf{b}_{2n}]}{(\mathsf{x}_{2n} - \mathsf{a}_{2n}\mathsf{b}_{2n}, \mathsf{v} - \mathsf{b}_{01}\mathsf{b}_2 \ldots \mathsf{b}_{2n})}, \\
\mathcal{A}_{30} &\cong \mathcal{A}_{10}/(\mathcal{A}_{10}h_1, \mathcal{A}_{10}h_2) \cong \mathcal{A}_{10}/(\mathsf{b}_0 + \sum_{j=2}^{\infty} j\kappa_{1j}\mathsf{x}_1^{j-1} + \mathsf{b}_2, \mathsf{x}_1 + \sum_{j=2}^{\infty} j\kappa_{2j}\mathsf{b}_2^{j-1} + \mathsf{x}_3) \\
&\cong \frac{\mathbb{C}[[\mathsf{b}_2, \mathsf{b}_4, \ldots, \mathsf{b}_{2n-2}, \mathsf{a}_{2n}, \mathsf{x}_3, \mathsf{x}_5, \ldots, \mathsf{x}_{2n-1}, \mathsf{x}_{2n}, \mathsf{v}]][\mathsf{b}_{2n}]}{(\mathsf{x}_{2n} - \mathsf{a}_{2n}\mathsf{b}_{2n}, \mathsf{v} - \mathsf{b}_{02}\mathsf{b}_2 \ldots \mathsf{b}_{2n})}, \\
&\ \ \vdots \\
\mathcal{A}_{2n-1,0} &\cong \mathcal{A}_{10}/(\mathcal{A}_{10}h_1, \mathcal{A}_{10}h_2, \ldots, \mathcal{A}_{10}h_{2n-2}) \\
&\cong \frac{\mathbb{C}[[\mathsf{b}_{2n-2}, \mathsf{a}_{2n}, \mathsf{x}_{2n-1}, \mathsf{x}_{2n}, \mathsf{v}]][\mathsf{b}_{2n}]}{(\mathsf{x}_{2n} - \mathsf{a}_{2n}\mathsf{b}_{2n}, \mathsf{v} - \mathsf{b}_{0,2n-2}\mathsf{b}_{2,2n-2} \ldots \mathsf{b}_{2n-4,2n-2}\mathsf{b}_{2n-2}\mathsf{b}_{2n})}, \\
\mathcal{A}_{2n,0} &\cong \mathcal{A}_{10}/(\mathcal{A}_{10}h_1, \mathcal{A}_{10}h_2, \ldots, \mathcal{A}_{10}h_{2n-2}, \mathcal{A}_{10}h_{2n-1}) \\
&\cong \frac{\mathbb{C}[[\mathsf{a}_{2n}, \mathsf{x}_{2n-1}, \mathsf{x}_{2n}, \mathsf{v}]][\mathsf{b}_{2n}]}{(\mathsf{x}_{2n} - \mathsf{a}_{2n}\mathsf{b}_{2n}, \mathsf{v} - \mathsf{b}_{0,2n-1}\mathsf{b}_{2,2n-1} \ldots \mathsf{b}_{2n-4,2n-1}\mathsf{b}_{2n-2,2n-1}\mathsf{b}_{2n})}.
\end{aligned}$$

Hence, for $2 \leq i \leq 2n$, the first affine chart $\mathcal{U}_{i0} := \operatorname{Spec} \mathcal{A}_{i0}$ is smooth. Since the last affine chart $\mathcal{U}_{in}$ is analogous to $\mathcal{U}_{i0}$, it is also smooth.

Second chart $(j = 1)$**.** From (4.2.G) we have

$$\mathcal{A}_{11} \cong \frac{\mathbb{C}[[\mathsf{b}_0, \mathsf{b}_2, \ldots, \mathsf{b}_{2n-4}, \mathsf{a}_{2n}, \mathsf{x}_1, \mathsf{x}_3, \ldots, \mathsf{x}_{2n-1}, \mathsf{x}_{2n-2}, \mathsf{u}, \mathsf{v}]][\mathsf{a}_{2n-2}, \mathsf{b}_{2n-2}]}{(\mathsf{x}_{2n-2} - \mathsf{a}_{2n-2}\mathsf{b}_{2n-2}, \mathsf{u} - \mathsf{a}_{2n-2}\mathsf{a}_{2n}, \mathsf{v} - \mathsf{b}_0\mathsf{b}_2 \ldots \mathsf{b}_{2n-2})}.$$

Moreover, by (4.2.H) $\varphi_{11} \colon \mathcal{S}_1 \to \mathcal{A}_{11}$ is given by

$$x_0 \mapsto \mathsf{b}_0, \quad x_2 \mapsto \mathsf{b}_2, \quad \ldots, x_{2n-4} \mapsto \mathsf{b}_{2n-4}, \quad x_{2n-2} \mapsto \mathsf{x}_{2n-2}, \quad x_{2n} \mapsto \mathsf{a}_{2n}, \quad u \mapsto \mathsf{u}, \quad v \mapsto \mathsf{v},$$
$$x_1 \mapsto \mathsf{x}_1, \quad x_3 \mapsto \mathsf{x}_3, \quad \ldots, \quad x_{2n-1} \mapsto \mathsf{x}_{2n-1}.$$

Thus, for $1 \leq i \leq 2n-1$, the images $\mathcal{A}_{11}h_i$ are

$$\mathcal{A}_{11}h_i = \begin{cases} \mathsf{b}_{i-1} + \sum_{j=2}^{\infty} j\kappa_{ij}\mathsf{x}_i^{j-1} + \mathsf{b}_{i+1}, & \text{for } i = 1,3,\dots,2n-5 \\ \mathsf{x}_{i-1} + \sum_{j=2}^{\infty} j\kappa_{ij}\mathsf{b}_i^{j-1} + \mathsf{x}_{i+1}, & \text{for } i = 2,4,\dots,2n-4 \\ \mathsf{b}_{2n-4} + \sum_{j=2}^{\infty} j\kappa_{2n-3,j}\mathsf{x}_{2n-3}^{j-1} + \mathsf{x}_{2n-2}, & \text{for } i = 2n-3 \\ \mathsf{x}_{2n-3} + \sum_{j=2}^{\infty} j\kappa_{2n-2,j}\mathsf{x}_{2n-2}^{j-1} + \mathsf{x}_{2n-1}, & \text{for } i = 2n-2 \\ \mathsf{x}_{2n-2} + \sum_{j=2}^{\infty} j\kappa_{2n-1,j}\mathsf{x}_{2n-1}^{j-1} + \mathsf{a}_{2n}, & \text{for } i = 2n-1 \end{cases}$$

Define $\mathsf{b}_{2t,k}$ analogously for $0 \leq t \leq n-2$ and $2t < k \leq 2n-1$. From the last equation, set

$$\mathsf{x}_{2n-2,2n-1} := -\sum_{j=2}^{\infty} j\kappa_{2n-1,j}\mathsf{x}_{2n-1}^{j-1} - \mathsf{a}_{2n} \in \mathbb{C}[[\mathsf{x}_{2n-1}, \mathsf{a}_{2n}]].$$

Again, each $\mathcal{A}_{11}h_i$ has a linear term, hence eliminates one variable in (4.2.I). Consequently,

$$\begin{aligned} \mathcal{A}_{21} &\cong \mathcal{A}_{11}/(\mathcal{A}_{11}h_1) \cong \mathcal{A}_{11}/(\mathsf{b}_0 + \sum_{j=2}^{\infty} j\kappa_{1j}\mathsf{x}_1^{j-1} + \mathsf{b}_2) \\ &\cong \frac{\mathbb{C}[[\mathsf{b}_2, \mathsf{b}_4, \dots, \mathsf{b}_{2n-4}, \mathsf{a}_{2n}, \mathsf{x}_1, \mathsf{x}_3, \dots, \mathsf{x}_{2n-1}, \mathsf{x}_{2n-2}, \mathsf{u}, \mathsf{v}]][\mathsf{a}_{2n-2}, \mathsf{b}_{2n-2}]}{(\mathsf{x}_{2n-2} - \mathsf{a}_{2n-2}\mathsf{b}_{2n-2}, \mathsf{u} - \mathsf{a}_{2n-2}\mathsf{a}_{2n}, \mathsf{v} - \mathsf{b}_{01}\mathsf{b}_2 \dots \mathsf{b}_{2n-2})}, \\ \mathcal{A}_{31} &\cong \mathcal{A}_{11}/(\mathcal{A}_{11}h_1, \mathcal{A}_{11}h_2) \cong \mathcal{A}_{11}/(\mathsf{b}_0 + \sum_{j=2}^{\infty} j\kappa_{1j}\mathsf{x}_1^{j-1} + \mathsf{b}_2, \mathsf{x}_1 + \sum_{j=2}^{\infty} j\kappa_{2j}\mathsf{b}_2^{j-1} + \mathsf{x}_3) \\ &\cong \frac{\mathbb{C}[[\mathsf{b}_2, \mathsf{b}_4, \dots, \mathsf{b}_{2n-4}, \mathsf{a}_{2n}, \mathsf{x}_3, \mathsf{x}_5, \dots, \mathsf{x}_{2n-1}, \mathsf{x}_{2n-2}, \mathsf{u}, \mathsf{v}]][\mathsf{a}_{2n-2}, \mathsf{b}_{2n-2}]}{(\mathsf{x}_{2n-2} - \mathsf{a}_{2n-2}\mathsf{b}_{2n-2}, \mathsf{u} - \mathsf{a}_{2n-2}\mathsf{a}_{2n}, \mathsf{v} - \mathsf{b}_{02}\mathsf{b}_2 \dots \mathsf{b}_{2n-2})}, \\ &\vdots \\ \mathcal{A}_{2n-1,1} &\cong \mathcal{A}_{11}/(\mathcal{A}_{11}h_1, \mathcal{A}_{11}h_2, \dots, \mathcal{A}_{11}h_{2n-2}) \\ &\cong \frac{\mathbb{C}[[\mathsf{a}_{2n}, \mathsf{x}_{2n-1}, \mathsf{x}_{2n-2}, \mathsf{u}, \mathsf{v}]][\mathsf{a}_{2n-2}, \mathsf{b}_{2n-2}]}{(\mathsf{x}_{2n-2} - \mathsf{a}_{2n-2}\mathsf{b}_{2n-2}, \mathsf{u} - \mathsf{a}_{2n-2}\mathsf{a}_{2n}, \mathsf{v} - \mathsf{b}_{0,2n-2}\mathsf{b}_{2,2n-2} \dots \mathsf{b}_{2n-4,2n-2}\mathsf{b}_{2n-2})}, \\ \mathcal{A}_{2n,1} &\cong \mathcal{A}_{11}/(\mathcal{A}_{11}h_1, \mathcal{A}_{11}h_2, \dots, \mathcal{A}_{11}h_{2n-2}, \mathcal{A}_{11}h_{2n-1}) \\ &\cong \frac{\mathbb{C}[[\mathsf{a}_{2n}, \mathsf{x}_{2n-1}, \mathsf{u}, \mathsf{v}]][\mathsf{a}_{2n-2}, \mathsf{b}_{2n-2}]}{(\mathsf{x}_{2n-2,2n-1} - \mathsf{a}_{2n-2}\mathsf{b}_{2n-2}, \mathsf{u} - \mathsf{a}_{2n-2}\mathsf{a}_{2n}, \mathsf{v} - \mathsf{b}_{0,2n-1}\mathsf{b}_{2,2n-1} \dots \mathsf{b}_{2n-4,2n-1}\mathsf{b}_{2n-2})}. \end{aligned}$$

Since $\mathsf{x}_{2n-2,2n-1}$ has a linear term a_{2n}, it follows that for $2 \leq i \leq 2n$ the second affine chart $\mathcal{U}_{i1} := \operatorname{Spec} \mathcal{A}_{i1}$ is smooth. For $2 \leq j \leq n-1$, the affine charts $\mathcal{U}_{ij}$ are analogous to $\mathcal{U}_{i1}$ (for each fixed i), hence smooth as well. Therefore $\mathcal{X}_i$ is smooth for all $2 \leq i \leq 2n$. □

Corollary 4.2.8. *With notation in 4.2.3, $\operatorname{End}_{\mathcal{S}_i}(N_i) \cong \mathbb{C}\langle\langle Q \rangle\rangle / R_i$ for $1 \leq i \leq 2n$.*

Proof. Recall from 4.2.3 the commutative diagram

$$\begin{tikzcd}
\mathcal{X}_{2n} \arrow[r] \arrow[d, "\pi_{2n}"] & \cdots \arrow[r] & \mathcal{X}_2 \arrow[r] \arrow[d, "\pi_2"] & \mathcal{X}_1 \arrow[d, "\pi_1"] \\
\operatorname{Spec} \mathcal{S}_{2n} \arrow[r] & \cdots \arrow[r] & \operatorname{Spec} \mathcal{S}_2 \arrow[r] & \operatorname{Spec} \mathcal{S}_1
\end{tikzcd}$$

together with the $\mathcal{S}_i$-module N_i for $1 \le i \le 2n$.

By [IW1, §5], N_1 is the tilting bundle for π_1, and Appendix 7.0.18 shows that

$$\operatorname{End}_{\mathcal{S}_1}(N_1) \cong \mathbb{C}\langle\langle Q \rangle\rangle / R_1, \tag{4.2.J}$$

where Q and R_1 are given in (4.2.C).

Note that $\mathcal{S}_1$ is an integral domain and normal, and $\mathcal{X}_1$ is connected and smooth. By 4.2.4, $\mathcal{S}_2$ is also an integral domain and normal. By 4.2.6 and 4.2.7, $\mathcal{X}_2$ is connected and smooth. Since N_1 is the tilting bundle for π_1, we can apply [V4, 2.11] to deduce that $N_2 \cong N_1 \otimes_{\mathcal{S}_1} \mathcal{S}_2$ is the tilting bundle for π_2 and

$$\begin{aligned}
\operatorname{End}_{\mathcal{S}_2}(N_2) &\cong \operatorname{End}_{\mathcal{S}_1/h_1}(N_1 \otimes_{\mathcal{S}_1} \mathcal{S}_1/h_1) && (\text{since } \mathcal{S}_2 \cong \mathcal{S}_1/h_1,\ N_2 \cong N_1 \otimes_{\mathcal{S}_1} \mathcal{S}_2) \\
&\cong \operatorname{End}_{\mathcal{S}_1}(N_1)/(h_1) && (\text{by [V4, 2.11]}) \\
&\cong \mathbb{C}\langle\langle Q \rangle\rangle / R_2. && (\text{by (4.2.J)})
\end{aligned}$$

Here R_2 is obtained from R_1 by adding the relation (4.2.D) with $i = 1$, namely

$$l_{t,0} + \sum_{j=2}^{\infty} j \kappa_{1j} l_{t,1}^{j-1} + l_{t,2} = 0, \qquad \text{for } t \in \mathbb{Z}/(n+1),$$

which corresponds to

$$h_1 = x_0 + \sum_{j=2}^{\infty} j \kappa_{1j}\, x_1^{j-1} + x_2.$$

Iterating this argument, for any $2 \le i \le 2n$, we have $N_i \cong N_{i-1} \otimes_{\mathcal{S}_{i-1}} \mathcal{S}_i$ is the tilting bundle for π_i, and

$$\begin{aligned}
\operatorname{End}_{\mathcal{S}_i}(N_i) &\cong \operatorname{End}_{\mathcal{S}_{i-1}}(N_{i-1})/(h_{i-1}) \\
&\cong \operatorname{End}_{\mathcal{S}_{i-2}}(N_{i-2})/(h_{i-1}, h_{i-2}) \\
&\quad \vdots \\
&\cong \operatorname{End}_{\mathcal{S}_1}(N_1)/(h_{i-1}, h_{i-2}, \ldots, h_1) \\
&\cong \mathbb{C}\langle\langle Q \rangle\rangle / R_i.
\end{aligned}$$

□

The following proposition shows that any (possibly non-reduced) monomialized Type A potential on Q_n can be realised by a crepant resolution of a cA_n singularity.

Theorem 4.2.9. *Given the monomialized Type A potential f in (4.2.A) on Q_n, the cA_n singularity $\mathcal{R}$, and the* CM *$\mathcal{R}$-module M in 4.2.3, we have $\underline{\mathrm{End}}_{\mathcal{R}}(M) \cong \mathfrak{Jac}(Q_n, f)$.*

Proof. By 4.2.4, $\mathcal{R} \cong \mathcal{S}_{2n}$ and $\mathrm{End}_{\mathcal{R}}(M) \cong \mathrm{End}_{\mathcal{S}_{2n}}(N_{2n})$. By 4.2.8, $\mathrm{End}_{\mathcal{S}_{2n}}(N_{2n}) \cong \mathbb{C}\langle\langle Q\rangle\rangle/R_{2n}$. Thus $\mathrm{End}_{\mathcal{R}}(M) \cong \mathbb{C}\langle\langle Q\rangle\rangle/R_{2n}$ where Q and R_{2n} are in 4.2.3(7).

Similar to Q_n, we also define x_i and x'_i on Q as follows: for any $0 \le i \le n$, set $\mathsf{x}_{2i} := a_{2i}b_{2i}$ and $\mathsf{x}'_{2i} := b_{2i}a_{2i}$, and for any $1 \le i \le n$, set $\mathsf{x}_{2i-1} := l_{i,2i-1} =: \mathsf{x}'_{2i-1}$.

Next, we consider the following relations induced by R_{2n}. For any $1 \le t \le n-1$, left multiplying the $i = 2t$ case of (4.2.D) by b_{2t} gives

$$\begin{aligned}
& b_{2t}l_{t,2t-1} + \sum_{j=2}^{\infty} j\mathsf{k}_{2t,j}b_{2t}l_{t,2t}^{j-1} + b_{2t}l_{t,2t+1} \\
&= b_{2t}l_{t,2t-1} + \sum_{j=2}^{\infty} j\mathsf{k}_{2t,j}b_{2t}l_{t,2t}^{j-1} + l_{t+1,2t+1}b_{2t} \qquad \text{(since } b_{2t}l_{t,2t+1} = l_{t+1,2t+1}b_{2t} \text{ by (4.2.C))} \\
&= b_{2t}\mathsf{x}'_{2t-1} + \sum_{j=2}^{\infty} j\mathsf{k}_{2t,j}b_{2t}\mathsf{x}_{2t}^{j-1} + \mathsf{x}_{2t+1}b_{2t}. \\
&\qquad \text{(since } l_{t,2t} = a_{2t}b_{2t} = \mathsf{x}_{2t},\ l_{t,2t-1} = \mathsf{x}'_{2t-1} \text{ and } l_{t+1,2t+1} = \mathsf{x}_{2t+1} \text{ by (4.2.C))}
\end{aligned}$$

Similarly, for any $1 \le t \le n-1$, right multiplying the $i = 2t$ case of (4.2.D) by a_{2t} gives

$$\begin{aligned}
& l_{t,2t-1}a_{2t} + \sum_{j=2}^{\infty} j\mathsf{k}_{2t,j}l_{t,2t}^{j-1}a_{2t} + l_{t,2t+1}a_{2t} \\
&= l_{t,2t-1}a_{2t} + \sum_{j=2}^{\infty} j\mathsf{k}_{2t,j}l_{t,2t}^{j-1}a_{2t} + a_{2t}l_{t+1,2t+1} \qquad \text{(since } l_{t,2t+1}a_{2t} = a_{2t}l_{t+1,2t+1} \text{ by (4.2.C))} \\
&= \mathsf{x}'_{2t-1}a_{2t} + \sum_{j=2}^{\infty} j\mathsf{k}_{2t,j}\mathsf{x}_{2t}^{j-1}a_{2t} + a_{2t}\mathsf{x}_{2t+1}. \\
&\qquad \text{(since } l_{t,2t} = a_{2t}b_{2t} = \mathsf{x}_{2t},\ l_{t,2t-1} = \mathsf{x}'_{2t-1} \text{ and } l_{t+1,2t+1} = \mathsf{x}_{2t+1} \text{ by (4.2.C))}
\end{aligned}$$

For any $1 \le t \le n$, the $i = 2t-1$ case of (4.2.D) is

$$l_{t,2t-2} + \sum_{j=2}^{\infty} j\mathsf{k}_{ij}l_{t,2t-1}^{j-1} + l_{t,2t} = \mathsf{x}'_{2t-2} + \sum_{j=2}^{\infty} j\mathsf{k}_{2t-1,j}\mathsf{x}_{2t-1}^{j-1} + \mathsf{x}_{2t}.$$

(since $l_{t,2t-1} = \mathsf{x}_{2t-1}$, $l_{t,2t-2} = b_{2t-2}a_{2t-2} = \mathsf{x}'_{2t-2}$ and $l_{t,2t} = a_{2t}b_{2t} = \mathsf{x}_{2t}$ by notation and (4.2.C))

Combining the above three types of relations gives the following,

$$T := \begin{cases} b_i\mathsf{x}'_{i-1} + \sum_{j=2}^{\infty} j\mathsf{k}_{ij}b_i\mathsf{x}_i^{j-1} + \mathsf{x}_{i+1}b_i = 0, \text{ for } i = 2, 4, \ldots, 2n-2. \\ \mathsf{x}'_{i-1}a_i + \sum_{j=2}^{\infty} j\mathsf{k}_{ij}\mathsf{x}_i^{j-1}a_i + a_i\mathsf{x}_{i+1} = 0, \text{ for } i = 2, 4, \ldots, 2n-2. \\ \mathsf{x}'_{i-1} + \sum_{j=2}^{\infty} j\mathsf{k}_{ij}\mathsf{x}_i^{j-1} + \mathsf{x}_{i+1} = 0, \text{ for } i = 1, 3, \ldots, 2n-1. \end{cases} \tag{4.2.K}$$

Then we define the quiver $\mathcal{Q}_n$ by deleting loops on Q as follows. For each vertex t on Q with $1 \le t \le n$, we delete all loops l_{tj} except $l_{t,2t-1}$ (namely x_{2t-1}). Note that Q_n is $\mathcal{Q}_n$ by removing the vertex 0 and loops on it.

In 4.2.10 below we will show that $\mathbb{C}\langle\langle Q\rangle\rangle/\langle R_{2n}, e_0\rangle \cong \mathbb{C}\langle\langle \mathcal{Q}_n\rangle\rangle/\langle T, e_0\rangle$. Together with the isomorphism $\mathrm{End}_{\mathfrak{R}}(M) \cong \mathbb{C}\langle\langle Q\rangle\rangle/R_{2n}$ at the start of the proof, this gives

$$\underline{\mathrm{End}}_{\mathfrak{R}}(M) \cong \mathbb{C}\langle\langle Q\rangle\rangle/\langle R_{2n}, e_0\rangle \cong \mathbb{C}\langle\langle \mathcal{Q}_n\rangle\rangle/\langle T, e_0\rangle.$$

Thus $\underline{\mathrm{End}}_{\mathfrak{R}}(M)$ is isomorphic to $\mathbb{C}\langle\langle Q_n\rangle\rangle$ factored by the relations T, which after deleting paths that factor through vertex 0, become

$$b_i \mathsf{x}'_{i-1} + \sum_{j=2}^{\infty} j\kappa_{ij} b_i \mathsf{x}_i^{j-1} + \mathsf{x}_{i+1} b_i = 0, \text{ for } i = 2, 4, \dots, 2n-2.$$

$$\mathsf{x}'_{i-1} a_i + \sum_{j=2}^{\infty} j\kappa_{ij} \mathsf{x}_i^{j-1} a_i + a_i \mathsf{x}_{i+1} = 0, \text{ for } i = 2, 4, \dots, 2n-2.$$

$$\mathsf{x}'_{i-1} + \sum_{j=2}^{\infty} j\kappa_{ij} \mathsf{x}_i^{j-1} + \mathsf{x}_{i+1} = 0, \text{ for } i = 3, \dots, 2n-2.$$

$$\sum_{j=2}^{\infty} j\kappa_{1j} \mathsf{x}_1^{j-1} + \mathsf{x}_2 = 0,\ \mathsf{x}'_{2n-2} + \sum_{j=2}^{\infty} j\kappa_{2n-1,j} \mathsf{x}_{2n-1}^{j-1} = 0.$$

These are exactly the relations generated by the derivatives of f. Thus $\underline{\mathrm{End}}_{\mathfrak{R}}(M) \cong \mathfrak{Jac}(Q_n, f)$. $\square$

Lemma 4.2.10. *With notation in 4.2.3 and 4.2.9,* $\mathbb{C}\langle\langle Q\rangle\rangle/\langle R_{2n}, e_0\rangle \cong \mathbb{C}\langle\langle \mathcal{Q}_n\rangle\rangle/\langle T, e_0\rangle$.

Proof. We first divide the relations R_{2n} in 4.2.3(7) into three parts. The following are the relations in R_{2n} that factor through vertex 0.

$$T_0 := \begin{cases} l_{00} = a_0 b_0,\ l_{0,2n} = b_{2n} a_{2n}. \\ l_{0i} a_0 = a_0 l_{1i},\ l_{ni} a_{2n} = a_{2n} l_{0i},\ l_{0i} b_{2n} = b_{2n} l_{ni},\ l_{1i} b_0 = b_0 l_{0i}, \\ l_{0i} l_{0j} = l_{0j} l_{0i}, \text{ for } 0 \le i, j \le 2n. \\ l_{0,i-1} + \sum_{j=2}^{\infty} j\kappa_{ij} l_{0,i}^{j-1} + l_{0,i+1} = 0, \text{ for } 1 \le i \le 2n-1. \end{cases}$$

Then we divide the remaining relations of R_{2n} into the following two parts.

$$T_1 := \begin{cases} l_{t,2t-2} = b_{2t-2} a_{2t-2}, \text{ for } 1 \le t \le n. \\ l_{ti} l_{tj} = l_{tj} l_{ti}, \text{ for } 1 \le t \le n \text{ and } 0 \le i, j \le 2n. \\ l_{ti} a_{2t} = a_{2t} l_{t+1,i},\ l_{t+1,i} b_{2t} = b_{2t} l_{ti}, \text{ for } 1 \le t \le n-1 \text{ and } 0 \le i \le 2n. \end{cases}$$

$$T_2 := \begin{cases} l_{t,2t} = a_{2t} b_{2t}, \text{ for any } 1 \le t \le n. \\ l_{t,i-1} + \sum_{j=2}^{\infty} j\kappa_{ij} l_{t,i}^{j-1} + l_{t,i+1} = 0 \text{ for any } 1 \le t \le n \text{ and } 1 \le i \le 2n-1. \end{cases}$$

Since T in (4.2.K) is induced by R_{2n}, necessarily

$$\mathbb{C}\langle\langle Q\rangle\rangle/\langle R_{2n}\rangle \cong \mathbb{C}\langle\langle Q\rangle\rangle/\langle R_{2n}, T\rangle \cong \mathbb{C}\langle\langle Q\rangle\rangle/\langle T_0, T_1, T_2, T\rangle. \tag{4.2.L}$$

We next use T_2 to eliminate some loops at vertex $1, 2, \ldots, n$ of Q, as follows.

Fix some vertex t with $1 \leq t \leq n$ and consider the loops l_{ti} on it. Since $l_{t,2t} = a_{2t}b_{2t}$ in T_2, we can eliminate $l_{t,2t}$. From our notation, $l_{t,2t-1} := \mathsf{x}_{2t-1}$ and $\mathsf{x}_{2t} := a_{2t}b_{2t}$. Thus we can write $l_{t,2t} = \mathsf{x}_{2t}$. Since $l_{ti}l_{tj} = l_{tj}l_{ti}$ in T_1 for $0 \leq i, j \leq 2n$, we can consider $\mathbb{C}\langle\langle l_{t,2t-1}, l_{t,2t}\rangle\rangle$ as the polynomial ring $\mathbb{C}[[l_{t,2t-1}, l_{t,2t}]]$. By the relation

$$l_{t,2t-1} + \sum_{j=2}^{\infty} j\kappa_{ij} l_{t,2t}^{j-1} + l_{t,2t+1} = 0, \tag{4.2.M}$$

in T_2, we can express $l_{t,2t+1} \in \mathbb{C}[[l_{t,2t-1}, l_{t,2t}]] = \mathbb{C}[[\mathsf{x}_{2t-1}, \mathsf{x}_{2t}]]$. Thus we can eliminate $l_{t,2t+1}$. Similar to the argument in 4.2.2, for each $i \neq 2t-1$ we can express $l_{ti} := \bar{l}_{ti}(x_{2t-1}, \mathsf{x}_{2t}) \in \mathbb{C}[[\mathsf{x}_{2t-1}, \mathsf{x}_{2t}]]$ and eliminate it. So we only leave one loop $l_{t,2t-1} = \mathsf{x}_{2t-1}$ on vertex t.

Thus we can use all the relations in T_2 to eliminate all such loops at vertices $1, 2, \ldots, n$. For $0 \leq k \leq 2$, write $\overline{T}_k$ for the the relations where we have substituted l_{ti} in T_k by the polynomial $\bar{l}_{ti}$ for $1 \leq t \leq n$ and $0 \leq i \leq 2n$. So we have

$$\mathbb{C}\langle\langle Q\rangle\rangle/\langle T_0, T_1, T_2, T\rangle \cong \mathbb{C}\langle\langle \mathcal{Q}_n\rangle\rangle/\langle \overline{T}_0, \overline{T}_1, T\rangle. \tag{4.2.N}$$

Now during the above substitution process, the following expressions in $\overline{T}_2$

$$\begin{aligned} &\bar{l}_{t,2t} = \mathsf{x}_{2t} = a_{2t}b_{2t} && (\text{since } \mathsf{x}_{2t} = a_{2t}b_{2t}) \\ &\bar{l}_{t,2t-1} + \sum_{j=2}^{\infty} j\kappa_{ij}\bar{l}_{t,2t}^{j-1} + \bar{l}_{t,2t+1} = 0 && (\text{by } (4.2.M)) \end{aligned}$$

hold in $\mathbb{C}\langle\langle \mathcal{Q}_n\rangle\rangle$ tautologically. Similarly, tautologically, all the other expressions in $\overline{T}_2$ also hold in $\mathbb{C}\langle\langle \mathcal{Q}_n\rangle\rangle$.

We next prove that T in (4.2.K) induces $\overline{T}_1$.

(1) Firstly, we prove that T induces $\bar{l}_{t,2t-2} = b_{2t-2}a_{2t-2}$ for $1 \leq t \leq n$. Since

$$\begin{aligned} &\mathsf{x}'_{2t-2} + \sum_{j=2}^{\infty} j\kappa_{2t-1,j}\mathsf{x}_{2t-1}^{j-1} + \mathsf{x}_{2t} = 0, && (\text{by the } i = 2t-1 \text{ case of the third line in } (4.2.K)) \\ &\bar{l}_{t,2t-2} + \sum_{j=2}^{\infty} j\kappa_{2t-1,j}\bar{l}_{t,2t-1}^{j-1} + \bar{l}_{t,2t} = 0. && (\text{since } \overline{T}_2 \text{ holds in } \mathbb{C}\langle\langle \mathcal{Q}_n\rangle\rangle) \end{aligned}$$

and by notation $\bar{l}_{t,2t-1} = \mathsf{x}_{2t-1}$ and $\bar{l}_{t,2t} = \mathsf{x}_{2t}$, then $\bar{l}_{t,2t-2} = \mathsf{x}'_{2t-2} = b_{2t-2}a_{2t-2}$.

(2) Secondly, we prove that T induces $\bar{l}_{ti}\bar{l}_{tj} = \bar{l}_{tj}\bar{l}_{ti}$ for $1 \leq t \leq n$ and $0 \leq i, j \leq 2n$.

Left multiplying the $i = 2t$ case of the first line in (4.2.K) by a_{2t} gives

$$0 = a_{2t}(b_{2t}\mathsf{x}'_{2t-1} + \sum_{j=2}^{\infty} j\kappa_{2t,j}b_{2t}\mathsf{x}_{2t}^{j-1} + \mathsf{x}_{2t+1}b_{2t}) = \mathsf{x}_{2t}\mathsf{x}'_{2t-1} + \sum_{j=2}^{\infty} j\kappa_{2t,j}\mathsf{x}_{2t}^{j} + a_{2t}\mathsf{x}_{2t+1}b_{2t}. \quad (\text{since } \mathsf{x}_{2t} = a_{2t}b_{2t})$$

Right multiplying the $i = 2t$ case of the second line in (4.2.K) by b_{2t} gives

$$0 = (\mathsf{x}'_{2t-1}a_{2t} + \sum_{j=2}^{\infty} j\kappa_{2t,j}\mathsf{x}_{2t}^{j-1}a_{2t} + a_{2t}\mathsf{x}_{2t+1})b_{2t} = \mathsf{x}'_{2t-1}\mathsf{x}_{2t} + \sum_{j=2}^{\infty} j\kappa_{2t,j}\mathsf{x}_{2t}^{j} + a_{2t}\mathsf{x}_{2t+1}b_{2t}. \quad (\text{since } \mathsf{x}_{2t} = a_{2t}b_{2t})$$

Thus $\mathsf{x}_{2t}\mathsf{x}'_{2t-1} = \mathsf{x}'_{2t-1}\mathsf{x}_{2t}$. Since x_{2t-1} is the loop at vertex t, then by definition $\mathsf{x}'_{2t-1} = \mathsf{x}_{2t-1}$, and so $\mathsf{x}_{2t}\mathsf{x}_{2t-1} = \mathsf{x}_{2t-1}\mathsf{x}_{2t}$. Together with the fact that each $\bar{l}_{ti} \in \mathbb{C}[[\mathsf{x}_{2t-1}, \mathsf{x}_{2t}]]$ gives $\bar{l}_{ti}\bar{l}_{tj} = \bar{l}_{tj}\bar{l}_{ti}$ for $0 \le i, j \le 2n$.

(3) Finally, we prove that T induces $\bar{l}_{ti}a_{2t} = a_{2t}\bar{l}_{t+1,i}$, $\bar{l}_{t+1,i}b_{2t} = b_{2t}\bar{l}_{ti}$ for $1 \le t \le n-1$ and $0 \le i \le 2n$. For each vertex t with $1 \le t \le n-1$, we have

$$\begin{aligned} &\bar{l}_{t,2t} = a_{2t}b_{2t} = \mathsf{x}_{2t}, && (\text{since } \overline{T}_2 \text{ holds in } \mathbb{C}\langle\langle \mathcal{Q}_n \rangle\rangle) \\ &\bar{l}_{t+1,2t} = b_{2t}a_{2t} = \mathsf{x}'_{2t}, && (\text{by } (1)) \\ &\bar{l}_{t,2t-1} = \mathsf{x}_{2t-1} = \mathsf{x}'_{2t-1}, \ \bar{l}_{t+1,2t+1} = \mathsf{x}_{2t+1} = \mathsf{x}'_{2t+1}. && (\text{by the definition of } \mathsf{x}_{2t-1} \text{ and } \mathsf{x}_{2t+1}) \end{aligned}$$

Thus

$$\begin{aligned} \bar{l}_{t,2t}a_{2t} &= a_{2t}b_{2t}a_{2t} && (\text{since } \bar{l}_{t,2t} = a_{2t}b_{2t}) \\ &= a_{2t}\bar{l}_{t+1,2t}, && (\text{since } \bar{l}_{t+1,2t} = b_{2t}a_{2t}) \end{aligned}$$

and

$$\begin{aligned} \bar{l}_{t,2t-1}a_{2t} &= \mathsf{x}'_{2t-1}a_{2t} && (\text{since } \bar{l}_{t,2t-1} = \mathsf{x}'_{2t-1}) \\ &= -\sum_{j=2}^{\infty} j\kappa_{2t,j}\mathsf{x}_{2t}^{j-1}a_{2t} - a_{2t}\mathsf{x}_{2t+1} && (\text{by the } i = 2t \text{ case of the second line in (4.2.K)}) \\ &= -\sum_{j=2}^{\infty} j\kappa_{2t,j}a_{2t}\bar{l}_{t+1,2t}^{j-1} - a_{2t}\bar{l}_{t+1,2t+1} \\ & && (\text{since } \mathsf{x}_{2t} = a_{2t}b_{2t},\ \bar{l}_{t+1,2t} = b_{2t}a_{2t} \text{ and } \mathsf{x}_{2t+1} = \bar{l}_{t+1,2t+1}) \\ &= -a_{2t}(\sum_{j=2}^{\infty} j\kappa_{2t,j}\bar{l}_{t+1,2t}^{j-1} + \bar{l}_{t+1,2t+1}) \\ &= a_{2t}\bar{l}_{t+1,2t-1}. && (\text{since } \overline{T}_2 \text{ holds in } \mathbb{C}\langle\langle \mathcal{Q}_n \rangle\rangle) \end{aligned}$$

Since $\overline{T}_2$ holds in $\mathbb{C}\langle\langle \mathcal{Q}_n \rangle\rangle$, then similar to the argument in 4.2.2, each $\bar{l}_{ti} \in \mathbb{C}\langle\langle \bar{l}_{t,2t-1}, \bar{l}_{t,2t} \rangle\rangle$

and $\bar{l}_{t+1,i} \in \mathbb{C}\langle\langle \bar{l}_{t+1,2t-1}, \bar{l}_{t+1,2t} \rangle\rangle$. Furthermore,

$$\bar{l}_{ti} = H_i(\bar{l}_{t,2t-1}, \bar{l}_{t,2t}),\ \bar{l}_{t+1,i} = H_i(\bar{l}_{t+1,2t-1}, \bar{l}_{t+1,2t}).$$

for the same H_i. Together with the above $\bar{l}_{t,2t}a_{2t} = a_{2t}\bar{l}_{t+1,2t}$ and $\bar{l}_{t,2t-1}a_{2t} = a_{2t}\bar{l}_{t+1,2t-1}$, this gives $\bar{l}_{ti}a_{2t} = a_{2t}\bar{l}_{t+1,i}$ for each i.

Similarly, T (4.2.K) also induces $\bar{l}_{t+1,i}b_{2t} = b_{2t}\bar{l}_{ti}$ for each i.

Combining (1), (2) and (3), it follows that T induces $\overline{T}_1$, and so $\mathbb{C}\langle\langle \mathcal{Q}_n \rangle\rangle/\langle \overline{T}_0, \overline{T}_1, T\rangle \cong \mathbb{C}\langle\langle \mathcal{Q}_n \rangle\rangle/\langle \overline{T}_0, T\rangle$. Together with (4.2.L), this gives

$$\mathbb{C}\langle\langle Q \rangle\rangle/\langle R_{2n}\rangle \cong \mathbb{C}\langle\langle Q \rangle\rangle/\langle T_0, T_1, T_2, T\rangle \overset{(4.2.N)}{\cong} \mathbb{C}\langle\langle \mathcal{Q}_n \rangle\rangle/\langle \overline{T}_0, \overline{T}_1, T\rangle \cong \mathbb{C}\langle\langle \mathcal{Q}_n \rangle\rangle/\langle \overline{T}_0, T\rangle,$$

and so $\mathbb{C}\langle\langle Q \rangle\rangle/\langle R_{2n}, e_0\rangle \cong \mathbb{C}\langle\langle \mathcal{Q}_n \rangle\rangle/\langle \overline{T}_0, T, e_0\rangle \cong \mathbb{C}\langle\langle \mathcal{Q}_n \rangle\rangle/\langle T, e_0\rangle$. □

We now consider the quiver $Q_{n,I}$ for some $I \subseteq \{1, 2, \dots, n\}$ and prove that any Type A potential on it can be realized by a crepant resolution of a cA_n singularity as follows.

Definition 4.2.11. *We say that π is* Type A_n *if π is a crepant resolution $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ where $\mathcal{R}$ is cA_n. Moreover, we say that π is* Type $A_{n,I}$ *if the normal bundle of the exceptional curve C_i is $\mathcal{O}(-1) \bigoplus \mathcal{O}(-1)$ if and only if $i \in I$, else the normal bundle is $\mathcal{O}(-2) \bigoplus \mathcal{O}$.*

Theorem 4.2.12. *For any Type A potential f on $Q_{n,I}$, there exists a Type A_n crepant resolution $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ such that $\Lambda_{\mathrm{con}}(\pi) \cong \mathcal{J}\mathrm{ac}(Q_{n,I}, f)$. If furthermore f is reduced, then π is Type $A_{n,I}$.*

Proof. By the Splitting Theorem ([DWZ, 4.6]) and 4.1.4, there is a reduced Type A potential f_{red} on $Q_{n,I'}$ for some $I \subseteq I' \subseteq \{1, 2, \dots, n\}$ such that $\mathcal{J}\mathrm{ac}(Q_{n,I'}, f_{\mathrm{red}}) \cong \mathcal{J}\mathrm{ac}(Q_{n,I}, f)$. Then, by 4.1.20, there exists a reduced monomialized Type A potential g on $Q_{n,I'}$ such that $f_{\mathrm{red}} \cong g$. By 4.1.23, there exists a monomialized Type A potential h on Q_n such that $\mathcal{J}\mathrm{ac}(Q_{n,I'}, g) \cong \mathcal{J}\mathrm{ac}(Q_n, h)$. Thus we have

$$\mathcal{J}\mathrm{ac}(Q_{n,I}, f) \cong \mathcal{J}\mathrm{ac}(Q_{n,I'}, f_{\mathrm{red}}) \cong \mathcal{J}\mathrm{ac}(Q_{n,I'}, g) \cong \mathcal{J}\mathrm{ac}(Q_n, h).$$

By 4.2.9, there exists a cA_n singularity $\mathcal{R}$ and a maximal CM $\mathcal{R}$-module M such that $\underline{\mathrm{End}}_{\mathcal{R}}(M) \cong \mathcal{J}\mathrm{ac}(Q_n, h)$. Denote π to be the crepant resolution of $\operatorname{Spec} \mathcal{R}$, which corresponds to M in 3.3.2. Thus $\Lambda_{\mathrm{con}}(\pi) \cong \mathcal{J}\mathrm{ac}(Q_n, h)$, and so $\Lambda_{\mathrm{con}}(\pi) \cong \mathcal{J}\mathrm{ac}(Q_{n,I}, f)$.

If furthermore f is reduced, then $I' = I$, $f_{\mathrm{red}} = f$ and g is a reduced monomialized Type A potential on $Q_{n,I}$. Then write

$$h = \sum_{i=1}^{2n-2} \mathsf{x}_i'\mathsf{x}_{i+1} + \sum_{i=1}^{2n-1}\sum_{j=2}^{\infty} \kappa_{ij}\mathsf{x}_i^j,$$

for some $\kappa_{ij} \in \mathbb{C}$. Since g is reduced on $Q_{n,I}$, then by 4.1.23 $\kappa_{2i-1,2} \neq 0$ when $i \in I$, and $\kappa_{2i-1,2} = 0$ when $i \notin I$. Write $\mathcal{R}$ and M as follows,

$$\mathcal{R} = \frac{\mathbb{C}[[u,v,x,y]]}{uv - g_0 g_2 \dots g_{2n}}$$

and $M = \mathcal{R} \oplus (u, g_0) \oplus (u, g_0 g_2) \oplus \dots \oplus (u, \prod_{i=0}^{n-1} g_{2i})$. We next prove that π is Type $A_{n,I}$.

(1) For any vertex $i \in I$, since $\kappa_{2i-1,2} \neq 0$, then $(\!(g_{2i-2}, g_{2i})\!) = (\!(x, y)\!)$ by 4.2.2, and so the normal bundle of the exceptional curve C_i of π is $\mathcal{O}(-1) \bigoplus \mathcal{O}(-1)$ by 3.3.3.

(2) For any vertex $i \notin I$, since $\kappa_{2i-1,2} = 0$, then $(\!(g_{2i-2}, g_{2i})\!) \subsetneq (\!(x, y)\!)$ by 4.2.2, and so the normal bundle of the exceptional curve C_i of π is $\mathcal{O}(-2) \bigoplus \mathcal{O}$ by 3.3.3. $\square$

The Brown–Wemyss Realisation Conjecture [BW2] states that if f is any potential which satisfies $\mathrm{Jdim}(f) \leq 1$ (see [BW2, 3.4] for the definition), then $\mathcal{J}\mathrm{ac}(f)$ is isomorphic to the contraction algebra of some crepant resolution $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ with $\mathcal{R}$ cDV. The above result 4.2.12 confirms this Realisation Conjecture for Type A potentials on $Q_{n,I}$ for any $n \geq 1$ and $I \subseteq \{1, 2, \dots, n\}$.

§ 4.2.2 | Type $A_{n,I}$ crepant resolutions and potentials

In this subsection, we prove in 4.2.15 the converse of 4.2.12. More precisely, given any Type $A_{n,I}$ crepant resolution, there is a reduced Type A potential f on $Q_{n,I}$ such that $\Lambda_{\mathrm{con}}(\pi) \cong \mathcal{J}\mathrm{ac}(f)$. So, together with 4.2.12, this gives a correspondence between Type A_n crepant resolutions and monomialized Type A potentials on Q_n, in 4.2.18 and 4.2.19.

The following 4.2.13 and 4.2.14 imply that both Type A_n crepant resolutions and Type A potentials on Q_n are commutative in some sense, which is the key for proving 4.2.15.

Lemma 4.2.13. *If $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ is a Type A_n crepant resolution, then $e_i \Lambda_{\mathrm{con}}(\pi) e_i$ is commutative for any $1 \leq i \leq n$.*

Proof. Since π is a Type A_n crepant resolution, $\Lambda(\pi) \cong \operatorname{End}_{\mathcal{R}}(M)$ and $\Lambda_{\mathrm{con}}(\pi) \cong \underline{\operatorname{End}}_{\mathcal{R}}(M)$ for some maximal CM $\mathcal{R}$-module M where $M = \mathcal{R} \oplus M_1 \oplus \dots \oplus M_n$ and each M_i is an indecomposable rank one CM $\mathcal{R}$-module. Thus $\operatorname{End}_{\mathcal{R}}(M_i) \cong \mathcal{R}$ for $1 \leq i \leq n$ from e.g. [IW3, 5.4].

Denote $\mathscr{C}$ to be the stable category $\underline{\mathrm{CM}}\,\mathcal{R}$ of Cohen–Macaulay $\mathcal{R}$-modules. Then we have $\underline{\operatorname{End}}_{\mathcal{R}}(M) \cong \operatorname{End}_{\mathscr{C}}(M)$ and $\underline{\operatorname{End}}_{\mathcal{R}}(M_i) \cong \operatorname{End}_{\mathscr{C}}(M_i)$, thus for any $1 \leq i \leq n$,

$$e_i \Lambda_{\mathrm{con}}(\pi) e_i \cong e_i \underline{\operatorname{End}}_{\mathcal{R}}(M) e_i \cong e_i \operatorname{End}_{\mathscr{C}}(M) e_i \cong \operatorname{End}_{\mathscr{C}}(M_i) \cong \underline{\operatorname{End}}_{\mathcal{R}}(M_i).$$

Since $\operatorname{End}_{\mathcal{R}}(M_i) \cong \mathcal{R}$ is commutative and $\underline{\operatorname{End}}_{\mathcal{R}}(M_i)$ is a quotient of $\operatorname{End}_{\mathcal{R}}(M_i)$, then $\underline{\operatorname{End}}_{\mathcal{R}}(M_i)$ is also commutative, and so $e_i \Lambda_{\mathrm{con}}(\pi) e_i$ is commutative. $\square$

Lemma 4.2.14. *Suppose that f is a reduced potential on $Q_{n,I}$. If there is some integer j where $1 \leq j \leq m-1$ such that f does not contain $\mathsf{x}_j'\mathsf{x}_{j+1}$, then there exists some integer i (depending on j) where $1 \leq i \leq n$ such that $e_i\mathfrak{J}\mathrm{ac}(f)e_i$ is not commutative.*

Proof. (1) When x_j and x_{j+1} are not loops, then there exists a vertex $i \in I$ such that $Q_{n,I}$ at vertex i locally looks like the following.

$$Q' := \begin{tikzcd} \underset{i-1}{\bullet} \arrow[r, bend left, "a_j"] & \underset{i}{\bullet} \arrow[l, bend left, "b_j"] \arrow[r, bend left, "a_{j+1}"] & \underset{i+1}{\bullet} \arrow[l, bend left, "b_{j+1}"] \end{tikzcd}$$

We denote the above quiver with only the three vertices shown, as Q'. Then consider the noncommutative algebra J, defined as $\mathfrak{J}\mathrm{ac}(f)$ quotiented by the ideal generated by the following paths:

- $\mathfrak{m}_{Q_{n,I}}^5$ where $\mathfrak{m}_{Q_{n,I}}$ is the ideal generated by all the arrows of $Q_{n,I}$ (see 2.1.1).
- e_k for all $1 \leq k < i-1$ and $i+1 < k \leq n$.
- possible loops x_{j-1} on vertex $i-1$ and x_{j+2} on vertex $i+1$.

It is clear that $J \cong \mathfrak{J}\mathrm{ac}(Q,g)$ where $g \sim \lambda_1\mathsf{x}_j^2 + \lambda_2\mathsf{x}_j'\mathsf{x}_{j+1} + \lambda_3\mathsf{x}_{j+1}^2$ for some $\lambda_1,\ \lambda_2,\ \lambda_3 \in \mathbb{C}$. Then we suppose that $e_i\mathfrak{J}\mathrm{ac}(f)e_i$ is commutative and f does not contain $\mathsf{x}_j'\mathsf{x}_{j+1}$, and aim for a contradiction.

Since $e_i\mathfrak{J}\mathrm{ac}(f)e_i$ is commutative and e_iJe_i is a factor of $e_i\mathfrak{J}\mathrm{ac}(f)e_i$, then e_iJe_i is also commutative, and furthermore $\mathsf{x}_j'\mathsf{x}_{j+1} = \mathsf{x}_{j+1}\mathsf{x}_j'$ in e_iJe_i. Since f does not contain $\mathsf{x}_j'\mathsf{x}_{j+1}$, $g \sim \lambda_1\mathsf{x}_j^2 + \lambda_3\mathsf{x}_{j+1}^2$. It is clear that the four relations induced by differentiating $\lambda_1\mathsf{x}_j^2 + \lambda_2\mathsf{x}_{j+1}^2$ can not induce the relation $(b_ja_j)(a_{j+1}b_{j+1}) = (a_{j+1}b_{j+1})(b_ja_j)$. Thus $\mathsf{x}_j'\mathsf{x}_{j+1} \neq \mathsf{x}_{j+1}\mathsf{x}_j'$ in e_iJe_i, a contradiction.

(2) When x_j is not a loop and x_{j+1} is a loop, then there exists a vertex $i \notin I$ such that $Q_{n,I}$ at vertex i locally looks like the following.

$$Q' := \begin{tikzcd} \underset{i-1}{\bullet} \arrow[r, bend left, "a_j"] & \underset{i}{\bullet} \arrow[l, bend left, "b_j"] \arrow[loop above, "\mathsf{x}_{j+1}"] \end{tikzcd}$$

We again denote the above quiver with the only two vertices shown, as Q'. Then consider the noncommutative algebra J, defined as $\mathfrak{J}\mathrm{ac}(f)$ quotiented by the ideal generated by the following paths:

- $\mathfrak{m}_{Q_{n,I}}^4$ where $\mathfrak{m}_{Q_{n,I}}$ is the ideal generated by all the arrows of $Q_{n,I}$ (see 2.1.1).
- e_k for all $1 \leq k < i-1$ and $i < k \leq n$.
- the possible loop x_{j-1} on vertex $i-1$.

- x_{j+1}^2.

It is clear that $J \cong \mathfrak{Jac}(Q,g)/(\mathsf{x}_{j+1}^2)$ where $g \sim \lambda_1\mathsf{x}_j^2 + \lambda_2\mathsf{x}'_j\mathsf{x}_{j+1} + \lambda_3\mathsf{x}'_j\mathsf{x}_{j+1}^2 + \lambda_4\mathsf{x}_{j+1}^2 + \lambda_5\mathsf{x}_{j+1}^3 + \lambda_6\mathsf{x}_{j+1}^4$ for some $\lambda_k \in \mathbb{C}$. We suppose that $e_i\mathfrak{Jac}(f)e_i$ is commutative and f does not contain $\mathsf{x}'_j\mathsf{x}_{j+1}$, and aim for a contradiction.

Since $e_i\mathfrak{Jac}(f)e_i$ is commutative and e_iJe_i is a factor of $e_i\mathfrak{Jac}(f)e_i$, then e_iJe_i is also commutative, and furthermore $\mathsf{x}'_j\mathsf{x}_{j+1} = \mathsf{x}_{j+1}\mathsf{x}'_j$ in e_iJe_i. Since f does not contain $\mathsf{x}'_j\mathsf{x}_{j+1}$, $\lambda_2 = 0$. Since f is reduced, $\lambda_4 = 0$. Thus

$$J \cong \frac{\mathbb{C}\langle\langle Q\rangle\rangle}{(\lambda_3(b_ja_j)\mathsf{x}_{j+1} + \lambda_3\mathsf{x}_{j+1}(b_ja_j), \lambda_1 b_ja_jb_j, \lambda_1 a_jb_ja_j, \mathsf{x}_{j+1}^2)}.$$

Again, it is clear that the above relations can not induce $(b_ja_j)\mathsf{x}_{j+1} = \mathsf{x}_{j+1}(b_ja_j)$. Thus $\mathsf{x}'_j\mathsf{x}_{j+1} \neq \mathsf{x}_{j+1}\mathsf{x}'_j$ in e_iJe_i, a contradiction.

(3) When x_j is a loop and x_{j+1} is not a loop, the proof is similar to (2). □

Proposition 4.2.15. *Given any Type $A_{n,I}$ crepant resolution $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$, there exists a reduced Type A potential f on $Q_{n,I}$ such that $\Lambda_{\mathrm{con}}(\pi) \cong \mathfrak{Jac}(f)$.*

Proof. By 3.3.3, the NCCR $\Lambda(\pi)$ can be presented as the quiver in 3.3.3 with some relations. Since $\mathcal{R}$ is complete local, $\Lambda(\pi)$ is also complete local by e.g. [BW2, 8.4]. Moreover, $\Lambda(\pi)$ is 3-CY from [IW1, 2.8]. Since a complete local 3-CY algebra is a Jacobi algebra from [V3], the relations of $\Lambda(\pi)$ are generated by some reduced potential g. Since $\Lambda_{\mathrm{con}}(\pi) \cong \Lambda(\pi)/\langle e_0\rangle$, $\Lambda_{\mathrm{con}}(\pi)$ is isomorphic to $\mathfrak{Jac}(Q_{n,I}, f)$ for some reduced potential f.

Then we prove that f is Type A, namely f contains $\mathsf{x}'_i\mathsf{x}_{i+1}$ for each $1 \leq i \leq m-1$. Since $\Lambda_{\mathrm{con}}(\pi) \cong \mathfrak{Jac}(f)$ and $e_i\Lambda_{\mathrm{con}}(\pi)e_i$ is commutative for each $1 \leq i \leq n$ by 4.2.13, $e_i\mathfrak{Jac}(f)e_i$ is also commutative for each i. So f must contain $\mathsf{x}'_i\mathsf{x}_{i+1}$ for each i, by 4.2.14. □

We are now in a position to show that our definition of Type A potential 4.1.4 is intrinsic.

Corollary 4.2.16. *Let f be a reduced potential on $Q_{n,I}$. The following are equivalent.*

1. *f is Type A.*
2. *There exists a Type $A_{n,I}$ crepant resolution π such that $\mathfrak{Jac}(f) \cong \Lambda_{\mathrm{con}}(\pi)$.*
3. *$e_i\mathfrak{Jac}(f)e_i$ is commutative for $1 \leq i \leq n$.*

Proof. (1) ⇒ (2): Since f is a reduced Type A potential on $Q_{n,I}$, it is immediate by 4.2.12.

(2) ⇒ (3): Since π is a Type A_n crepant resolution, then $e_i\Lambda_{\mathrm{con}}(\pi)e_i$ is commutative by 4.2.13, and so $e_i\mathfrak{Jac}(f)e_i$ is commutative for any $1 \leq i \leq n$.

(3) ⇒ (1): Since f is a reduced potential on $Q_{n,I}$ and $e_i\mathfrak{Jac}(f)e_i$ is commutative for any $1 \leq i \leq n$, then f contains $\mathsf{x}'_i\mathsf{x}_{i+1}$ for any $1 \leq i \leq m-1$ by 4.2.14, and so f is Type A. □

Definition 4.2.17. *We say two crepant resolutions $\pi_i \colon \mathcal{X}_i \to \operatorname{Spec} \mathcal{R}_i$ for $i = 1, 2$ have the same* noncommutative deformation type (NC deformation type) *if $\Lambda_{\text{con}}(\pi_1) \cong \Lambda_{\text{con}}(\pi_2)$.*

The name NC deformation type comes from the fact that the contraction algebra represents the noncommutative deformation functor of the exceptional curves [DW1].

Together with 4.1.20, the above 4.2.15 induces a map φ from Type $A_{n,I}$ crepant resolutions to the isomorphism classes of reduced monomialized Type A potentials on $Q_{n,I}$. More precisely, for any Type $A_{n,I}$ crepant resolution π, we define $\varphi(\pi)$ to be the reduced monomialized Type A potential f on $Q_{n,I}$ that satisfies $\Lambda_{\text{con}}(\pi) \cong \mathcal{J}\mathrm{ac}(f)$ by 4.2.15 and 4.1.20. Moreover, φ is well-defined since if there are two such f_1 and f_2, then $\mathcal{J}\mathrm{ac}(f_1) \cong \mathcal{J}\mathrm{ac}(f_2)$.

Theorem 4.2.18. *The above φ induces a one-to-one correspondence as follows.*

$$\begin{gathered}\textit{Type } A_{n,I} \textit{ crepant resolutions up to NC deformation type}\\ \updownarrow \\ \textit{isomorphism classes of reduced monomialized Type } A \textit{ potentials on } Q_{n,I}\end{gathered}$$

Proof. Firstly, we prove the map from top to bottom is surjective, namely that for any reduced monomialized Type $A_{n,I}$ potential f, there is a Type $A_{n,I}$ crepant resolution $\pi \colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ such that $\mathcal{J}\mathrm{ac}(f) \cong \Lambda_{\text{con}}(\pi)$. This is immediate from 4.2.12.

Then we prove that the map from top to bottom is injective. Let $\pi \colon \mathcal{X}_k \to \operatorname{Spec} \mathcal{R}_k$ be two Type $A_{n,I}$ crepant resolutions for $k = 1, 2$. If $\Lambda_{\text{con}}(\pi_1) \cong \mathcal{J}\mathrm{ac}(f) \cong \Lambda_{\text{con}}(\pi_2)$ for some reduced monomialized Type A potential f on $Q_{n,I}$, then π_1 and π_2 have the same NC deformation type. $\square$

The following asserts that Type A potentials on Q_n describe the contraction algebra of all Type A_n crepant resolutions.

Corollary 4.2.19. *The set of isomorphism classes of contraction algebras associated to Type A_n crepant resolutions is equal to the set of isomorphism classes of Jacobi algebras of monomialized Type A potentials on Q_n.*

Proof. We first define a map ϕ from the isomorphic classes of contraction algebra associated with Type A_n crepant resolutions to the isomorphic classes of Jacobi algebra of monomialized Type A potentials on Q_n.

Given any contraction algebra $\Lambda_{\text{con}}(\pi)$ where π is a Type A_n crepant resolution, then π belongs to Type $A_{n,I}$ crepant resolution for some I. Thus $\Lambda_{\text{con}}(\pi) \cong \mathcal{J}\mathrm{ac}(Q_{n,I}, f')$ for some reduced monomialized Type A potential f' on $Q_{n,I}$ by 4.2.18. Moreover, f' is isormphic to some monomialized Type A potential f on Q_n by 4.1.23. We define $\phi(\Lambda_{\text{con}}(\pi)) := \mathcal{J}\mathrm{ac}(f)$.

Secondly, we prove that ϕ is well-defined. If there are two Type A_n crepant resolutions π_1 and π_2 such that $\Lambda_{\text{con}}(\pi_1) \cong \Lambda_{\text{con}}(\pi_2)$, then $\phi(\Lambda_{\text{con}}(\pi_1)) = \mathfrak{J}\text{ac}(f_1)$ and $\phi(\Lambda_{\text{con}}(\pi_2)) = \mathfrak{J}\text{ac}(f_2)$, so $\mathfrak{J}\text{ac}(f_1) \cong \mathfrak{J}\text{ac}(f_2)$ from the above definition of ϕ.

Thirdly, we prove that ϕ is injective. If there are two Type A_n crepant resolutions π_1 and π_2 such that $\phi(\Lambda_{\text{con}}(\pi_1)) \cong \mathfrak{J}\text{ac}(f) \cong \phi(\Lambda_{\text{con}}(\pi_2))$ for some monomialized Type A potential f on Q_n, then $\Lambda_{\text{con}}(\pi_1) \cong \Lambda_{\text{con}}(\pi_2)$ from the above definition of ϕ.

Finally, by 4.2.12 ϕ is surjective. $\square$

Notation 4.2.20. Let f and g be potentials on a quiver Q. We say that f is *derived equivalent to* to g (written $f \simeq g$) if the derived categories $\mathrm{D}^{\mathrm{b}}(\mathfrak{J}\text{ac}(f))$ and $\mathrm{D}^{\mathrm{b}}(\mathfrak{J}\text{ac}(g))$ are triangle equivalent.

Given any isolated cA_n singularity $\mathcal{R}$ which admits a crepant resolution, let $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ be one of the crepant resolutions. Then, by 4.2.19, there exists some monomialized Type A potential f on Q_n such that $\Lambda_{\text{con}}(\pi) \cong \mathfrak{J}\text{ac}(f)$, so it induces a map Φ from isolated cA_n singularities, which admit a crepant resolution to monomialized Type A potentials on Q_n.

Theorem 4.2.21. *The above Φ induces a one-to-one correspondence as follows.*

isomorphism classes of isolated cA_n singularities
which admit a crepant resolution

$$\updownarrow$$

derived equivalence classes of monomialized Type A potentials on Q_n
with finite-dimensional Jacobi algebra

Proof. Firstly, we prove that the map from top to bottom is well-defined. Given any isolated cA_n $\mathcal{R}$ which admits a crepant resolution, let $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ be one of the crepant resolutions. Then there exists some monomialized Type A potential f on Q_n such that $\Lambda_{\text{con}}(\pi) \cong \mathfrak{J}\text{ac}(f)$ by 4.2.19. Moreover, since $\mathcal{R}$ is isolated, $\mathfrak{J}\text{ac}(f)$ is finite-dimensional by 2.3.6. Let $\pi'\colon \mathcal{X}' \to \operatorname{Spec}\mathcal{R}$ be another crepant resolution such that $\Lambda_{\text{con}}(\pi') \cong \mathfrak{J}\text{ac}(f')$ for some monomialized Type A potential f' on Q_n. Since π' is a flop of π and $\mathcal{R}$ is isolated, f is derived equivalent to f' by 2.3.8.

Secondly, we prove that the map from top to bottom is surjective. Given any monomialized Type A potential f on Q_n with finite-dimensional Jacobi algebra, there exists a Type A_n crepant resolution $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ such that $\mathfrak{J}\text{ac}(f) \cong \Lambda_{\text{con}}(\pi)$ by 4.2.9. Moreover, since $\mathfrak{J}\text{ac}(f)$ is finite-dimensional, $\mathcal{R}$ is isolated by 2.3.6.

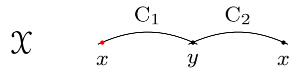
Finally, we prove the map from top to bottom is injective. This uses the proof of the Donovan-Wemyss conjecture in 2.3.7. Let $\pi_i\colon \mathcal{X}_i \to \operatorname{Spec}\mathcal{R}_i$ be two crepant resolutions of isolated cA_n $\mathcal{R}_i$ with $\Lambda_{\text{con}}(\pi_i) \cong \mathfrak{J}\text{ac}(f_i)$ for $i = 1, 2$. If f_1 is derived equivalent to f_2, together with R_1 and R_2 isolated, then $R_1 \cong R_2$ by 2.3.7. $\square$

§ 4.2.3 | Derived equivalences associated to non-isolated cases

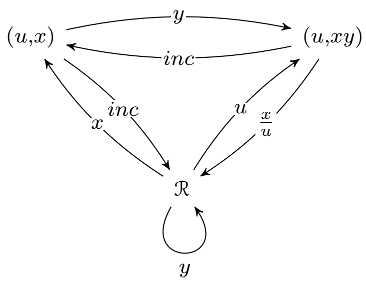
Since both 2.3.7 and 2.3.8 need the assumption of isolated cDVs, we restrict our proof of the correspondence in 4.2.21 to isolated cA_n singularities. This naturally suggests the idea of extending 2.3.7 and 2.3.8 to non-isolated cDVs.

However, by testing contraction algebras associated to crepant resolutions of the non-isolated cA_2 singularity $\mathbb{C}[[u, v, x, y]]/(uv - x^2y)$, we find in 4.2.26 and 4.2.29 that 2.3.7 and 2.3.8 do not extend directly to non-isolated cDVs.

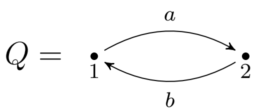
Let $\mathcal{R} := \mathbb{C}[[u, v, x, y]]/(uv - x^2y)$ and consider the $\mathcal{R}$-module $M := \mathcal{R} \oplus (u, x) \oplus (u, xy)$, and the corresponding crepant resolution $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$ with $\Lambda(\pi) \cong \operatorname{End}_{\mathcal{R}}(M)$ in 3.3.2. By [IW3, §5], $\mathcal{X}$ is given pictorially by



By 3.3.3 $\operatorname{End}_{\mathcal{R}}(M)$ can be presented as the following quiver

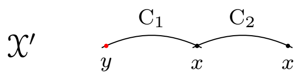


Thus $\Lambda_{\mathrm{con}}(\pi) \cong \underline{\operatorname{End}}_{\mathcal{R}}(M) \cong \mathcal{J}\mathrm{ac}(Q, f)$ for some potential f on the quiver Q where



It is easy to check that $f = 0$ by the quiver presentation of $\operatorname{End}_{\mathcal{R}}(M)$ above.

We next perform a flop of the exceptional curve C_1 in $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$, obtaining a new crepant resolution $\pi'\colon \mathcal{X}' \to \operatorname{Spec}\mathcal{R}$. Again by [IW3, §5], $\mathcal{X}'$ is given pictorially by



and $\Lambda(\pi') \cong \operatorname{End}_{\mathcal{R}}(M')$ where $M' = \mathcal{R} \oplus (u, y) \oplus (u, yx)$. Again by 3.3.3 $\operatorname{End}_{\mathcal{R}}(M')$ can be presented as the following quiver

$$
\begin{tikzcd}
(u,y) \arrow[rr, bend left, "x"] \arrow[dr, bend left, "inc"] & & (u,yx) \arrow[loop above, "y"] \arrow[ll, bend left, "inc"] \arrow[dl, bend left, "\frac{x}{u}"] \\
& \mathfrak{R} \arrow[ul, bend left, "y"] \arrow[ur, bend left, "u"] &
\end{tikzcd}
$$

Thus $\Lambda_{\text{con}}(\pi') \cong \underline{\text{End}}_{\mathfrak{R}}(M') \cong \mathfrak{J}\text{ac}(Q', f')$ for some potential f' on the quiver Q' where

$$
Q' = \begin{tikzcd}
\underset{1}{\bullet} \arrow[r, bend left, "a"] & \underset{2}{\bullet} \arrow[l, bend left, "b"] \arrow[loop above, "l"]
\end{tikzcd}
$$

It is easy to check that $f' = lba$ by the quiver presentation of $\text{End}_{\mathfrak{R}}(M')$ above.

Thus if 2.3.7 and 2.3.8 hold for non-isolated cDVs, then $\mathfrak{J}\text{ac}(Q, f)$ would need to be derived equivalent to $\mathfrak{J}\text{ac}(Q', f')$. However, we will show that this is not the case by comparing their global dimensions.

Notation 4.2.22. To ease notation, we adopt the following notation.

(1) Set $\Lambda_{\text{con}} := \mathfrak{J}\text{ac}(Q, f)$ and $\Gamma_{\text{con}} := \mathfrak{J}\text{ac}(Q', f')$.

(2) Then set $P_1 := \Lambda_{\text{con}} e_1 = \underline{\text{End}}_{\mathfrak{R}}(M, (u, x))$, $P_2 := \Lambda_{\text{con}} e_2 = \underline{\text{End}}_{\mathfrak{R}}(M, (u, xy))$ and $Q_1 := \Gamma_{\text{con}} e_1 = \underline{\text{End}}_{\mathfrak{R}}(M', (u, y))$, $Q_2 := \Gamma_{\text{con}} e_2 = \underline{\text{End}}_{\mathfrak{R}}(M', (u, yx))$.

(3) Write S_1 (resp. S_2) for the simple left Γ_{con}-module which corresponds to the following quiver representation of Γ_{con}.

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, bend left, "0"] & 0 \arrow[l, bend left, "0"] \arrow[loop above, "0"]
\end{tikzcd}
\quad \text{resp.} \quad
\begin{tikzcd}
0 \arrow[r, bend left, "0"] & \mathbb{C} \arrow[l, bend left, "0"] \arrow[loop above, "0"]
\end{tikzcd}
$$

Lemma 4.2.23. *With notation in 4.2.22, the global dimension* $\text{gl.dim}(\Lambda_{\text{con}}) \leq 1$.

Proof. Since $\Lambda_{\text{con}} \cong \mathbb{C}\langle\langle Q \rangle\rangle$ which is a complete quiver algebra with no relations, by [C3, §1] $\text{pd}_{\Lambda_{\text{con}}}(N) \leq 1$ for any left Λ_{con}-module N. Thus $\text{gl.dim}(\Lambda_{\text{con}}) \leq 1$. □

Lemma 4.2.24. *With notation in 4.2.22, there exists an exact sequence of left* Γ_{con}*-modules*

$$0 \to S_1 \to Q_2 \xrightarrow{\cdot l} Q_2 \xrightarrow{\cdot b} Q_1 \to S_1 \to 0.$$

Proof. Recall that $\Gamma_{\mathrm{con}} = \mathfrak{Jac}(Q', f')$ where $f' = lba$. Given that the relations generated by f' are $lb = 0$, $ba = 0$, and $al = 0$, we consider the following left Γ_{con}-modules as $\mathbb{C}$-vector spaces:

- $Q_1 = \Gamma_{\mathrm{con}} e_1$ is the $\mathbb{C}$-vector space generated by $\{e_1, b, ab\}$,
- $Q_2 = \Gamma_{\mathrm{con}} e_2$ is the $\mathbb{C}$-vector space generated by $\{e_2, a, l, l^2, \dots\}$,
- $\Gamma_{\mathrm{con}} b$ is the $\mathbb{C}$-vector space generated by $\{b, ab\}$,
- $\Gamma_{\mathrm{con}} l$ is the $\mathbb{C}$-vector space generated by $\{l, l^2, \dots\}$,
- $\Gamma_{\mathrm{con}} a$ is the $\mathbb{C}$-vector space generated by $\{a\}$.

Thus we have the short exact sequences

$$0 \to \Gamma_{\mathrm{con}} b \to Q_1 \to S_1 \to 0,$$
$$0 \to \Gamma_{\mathrm{con}} l \to Q_2 \xrightarrow{\cdot b} \Gamma_{\mathrm{con}} b \to 0,$$
$$0 \to \Gamma_{\mathrm{con}} a \to Q_2 \xrightarrow{\cdot l} \Gamma_{\mathrm{con}} l \to 0.$$

Combining the above three short exact sequences gives the long exact sequence

$$0 \to \Gamma_{\mathrm{con}} a \to Q_2 \xrightarrow{\cdot l} Q_2 \xrightarrow{\cdot b} Q_1 \to S_1 \to 0.$$

So we only need to prove that $\Gamma_{\mathrm{con}} a \cong S_1$ as left Γ_{con}-modules. By the one-to-one correspondence between the quiver representations of Γ_{con} and the left Γ_{con}-modules in [W3, 6.14], $S_1 = \mathbb{C}$ with the left Γ_{con}-module structure $ac = 0$, $bc = 0$, $lc = 0$, $e_2 c = 0$ and $e_1 c = c$ for any $c \in \mathbb{C}$. Thus the map $\varphi \colon \Gamma_{\mathrm{con}} a \to S_1$ defined by $\varphi(ca) = c$ for any $c \in \mathbb{C}$ is a surjective left-Γ_{con} homomorphism. Since $\dim_{\mathbb{C}} \Gamma_{\mathrm{con}} a = 1 = \dim_{\mathbb{C}} S_1$, φ is a left-Γ_{con} isomorphism. $\square$

Lemma 4.2.25. *With notation in 4.2.22, the global dimension* $\mathrm{gl.dim}(\Gamma_{\mathrm{con}}) = \infty$.

Proof. By the dimension shifting theorem of the Ext groups and the exact sequence in 4.2.24, for any $i \geq 0$ we have

$$\mathrm{Ext}^i_{\Gamma_{\mathrm{con}}}(S_1, S_2) = \mathrm{Ext}^{i+3}_{\Gamma_{\mathrm{con}}}(S_1, S_2).$$

By the shape of the quiver Q' and the intersection theory of [W2, 2.15], $\mathrm{Ext}^1_{\Gamma_{\mathrm{con}}}(S_1, S_2) = \mathbb{C}$, and so $\mathrm{Ext}^i_{\Gamma_{\mathrm{con}}}(S_1, S_2) = \mathbb{C}$ for any $i = 1, 4, 7, \cdots$. Thus $\mathrm{pd}_{\Gamma_{\mathrm{con}}}(S_1) = \infty$, and so $\mathrm{gl.dim}(\Gamma_{\mathrm{con}}) = \infty$. $\square$

Proposition 4.2.26. *With notation in 4.2.22,* Λ_{con} *is not derived equivalent to* Γ_{con}.

Proof. If Λ_{con} is derived equivalent to Γ_{con}, then by [R3] there exists a tilting complex T of Λ_{con} such that $\Gamma_{\mathrm{con}} \cong \mathrm{End}_{\mathrm{K}^b(\mathrm{proj}\,\Lambda_{\mathrm{con}})}(T)$. Since by 4.2.23 $\mathrm{gl.dim}(\Lambda_{\mathrm{con}})$ is finite, by [KK, Theorem 1] $\mathrm{gl.dim}(\mathrm{End}_{\mathrm{K}^b(\mathrm{proj}\,\Lambda_{\mathrm{con}})}(T))$ is also finite. But this contradicts with the fact that $\mathrm{gl.dim}(\Gamma_{\mathrm{con}}) = \infty$ in 4.2.25. $\square$

Although Λ_{con} is not derived equivalent to Γ_{con}, we will show that Λ_{con} is derived equivalent to $\mathbb{C}\langle\langle Q''\rangle\rangle/(l\mathsf{x})$, where

$$Q'' = \begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\mathsf{x}"] \arrow[loop above, "l"] \end{tikzcd}$$

Note that $\mathbb{C}\langle\langle Q''\rangle\rangle/(l\mathsf{x})$ is a quotient of Γ_{con} defined via the map $\vartheta\colon \Gamma_{\mathrm{con}} \to \mathbb{C}\langle\langle Q''\rangle\rangle/(l\mathsf{x})$, given by

$$\vartheta(l) = l, \quad \vartheta(b) = x, \quad \vartheta(a) = 0.$$

Thus $\mathbb{C}\langle\langle Q''\rangle\rangle/(l\mathsf{x}) \cong \Gamma_{\mathrm{con}}/\langle a\rangle$. This is a new phenomenon specific to the non-isolated cDVs.

Let $\mathrm{C}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_{\mathrm{con}})$ denote the category of bounded cochain complexes of finitely generated Λ_{con}-modules, and let $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda_{\mathrm{con}})$ denote the bounded homotopy category of finitely generated projective Λ_{con}-modules.

By [A3, §4.1], Λ_{con} has a two-term complex

$$\mathcal{P} := \Big(\underline{\mathrm{Hom}}_{\mathcal{R}}\,(M,(u,x)) \xrightarrow{\cdot y} \underline{\mathrm{Hom}}_{\mathcal{R}}\,(M,(u,xy))\Big) \oplus \Big(0 \to \underline{\mathrm{Hom}}_{\mathcal{R}}\,(M,(u,xy))\Big).$$

Since $\mathcal{R}$ is non-isolated, we can not use [A3, §4] to deduce that $\mathcal{P}$ is a titling complex. However, we next prove that $\mathcal{P}$ is still a tilting complex by checking $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda_{\mathrm{con}})}(\mathcal{P},\mathcal{P}[n]) = 0$ for $n \neq 0$. With notation in 4.2.22,

$$\mathcal{P} = (P_1 \xrightarrow{\cdot a} P_2) \oplus (0 \to P_2). \tag{4.2.O}$$

To ease notation, we write C^{b} (resp. K^{b}) for $\mathrm{C}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_{\mathrm{con}})$ (resp. $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda_{\mathrm{con}})$) throughout this subsection.

Lemma 4.2.27. *With notation in 4.2.22, $\mathcal{P}$ is a tilting complex of Λ_{con}.*

Proof. Since $\mathcal{P}$ is a two-term complex, $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(\mathcal{P},\mathcal{P}[n]) = 0$ for $n \geq 2$ or $n \leq -2$. Thus we only need to check that $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(\mathcal{P},\mathcal{P}[n]) = 0$ for $n = 1, -1$.

(1) We first check that $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(\mathcal{P},\mathcal{P}[1]) = 0$. By the construction of $\mathcal{P}$,

$$\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(\mathcal{P},\mathcal{P}[1]) = \begin{bmatrix} \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(P_1 \to P_2, (P_1 \to P_2)[1]) & \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(P_1 \to P_2, (0 \to P_2)[1]) \\ \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(0 \to P_2, (P_1 \to P_2)[1]) & \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(0 \to P_2, (0 \to P_2)[1]) \end{bmatrix}.$$

Any cochain map g in $\mathrm{Hom}_{\mathrm{C}^{\mathrm{b}}}(0 \to P_2, (P_1 \to P_2)[1])$ has the form

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d, "g_{-1}"] & 0 \arrow[r] \arrow[d, "g_0"] & P_2 \arrow[r] \arrow[d, "g_1"] & \cdots \\ \cdots \arrow[r] & P_1 \arrow[r, "\cdot(-a)"] & P_2 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

Thus $g_i = 0$ for each i, and so $\mathrm{Hom}_{\mathrm{K^b}}(0 \to P_2, (P_1 \to P_2)[1]) = 0$. Similarly, we have $\mathrm{Hom}_{\mathrm{K^b}}(0 \to P_2, (0 \to P_2)[1]) = 0$.

Any cochain map g in $\mathrm{Hom}_{\mathrm{C^b}}(P_1 \to P_2, (0 \to P_2)[1])$ has the form

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "g_{-1}"'] & P_1 \arrow[r, "\cdot a"] \arrow[d, "g_0"'] & P_2 \arrow[r] \arrow[d, "g_1"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Thus $g_i = 0$ for each $i \neq 0$. By the shape of the quiver Q, we have $g_0 = ah'$ for some $h' \in \mathrm{Hom}_{\Lambda_{\mathrm{con}}}(P_2, P_2)$. Then we define a collection of maps h_i in the diagram below as $h_0 = h'$ and $h_i = 0$ for each $i \neq 0$.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "g_{-1}"'] \arrow[dl, "0"'] & P_1 \arrow[r, "\cdot a"] \arrow[d, "g_0"'] \arrow[dl, "h_{-1}"'] & P_2 \arrow[r] \arrow[d, "g_1"'] \arrow[dl, "h_0"'] & \cdots \arrow[dl, "0"'] \\
\cdots \arrow[r] & 0 \arrow[r] & P_2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

It is clear that $g_0 = ah_0 + h_{-1} \cdot 0$. Thus h is a cochain homotopy between g and the zero cochain map, and so $\mathrm{Hom}_{\mathrm{K^b}}(P_1 \to P_2, (0 \to P_2)[1]) = 0$. Similarly, we have $\mathrm{Hom}_{\mathrm{K^b}}(P_1 \to P_2, (P_1 \to P_2)[1]) = 0$.

(2) We next check that $\mathrm{Hom}_{\mathrm{K^b}}(\mathcal{P}, \mathcal{P}[-1]) = 0$. By the construction of $\mathcal{P}$,

$$\mathrm{Hom}_{\mathrm{K^b}}(\mathcal{P}, \mathcal{P}[-1]) = \begin{bmatrix} \mathrm{Hom}_{\mathrm{K^b}}(P_1 \to P_2, (P_1 \to P_2)[-1]) & \mathrm{Hom}_{\mathrm{K^b}}(P_1 \to P_2, (0 \to P_2)[-1]) \\ \mathrm{Hom}_{\mathrm{K^b}}(0 \to P_2, (P_1 \to P_2)[-1]) & \mathrm{Hom}_{\mathrm{K^b}}(0 \to P_2, (0 \to P_2)[-1]) \end{bmatrix}.$$

Any cochain map g in $\mathrm{Hom}_{\mathrm{C^b}}(P_1 \to P_2, (0 \to P_2)[-1])$ has the form

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "\cdot a"] \arrow[d, "g_{-1}"] & P_2 \arrow[r] \arrow[d, "g_0"] & 0 \arrow[r] \arrow[d, "g_1"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & P_2 \arrow[r] & \cdots
\end{tikzcd}$$

Thus $g_i = 0$ for each i, and so $\mathrm{Hom}_{\mathrm{K^b}}(P_1 \to P_2, (0 \to P_2)[-1]) = 0$. Similarly, we have $\mathrm{Hom}_{\mathrm{K^b}}(0 \to P_2, (0 \to P_2)[-1]) = 0$.

Any cochain map g in $\mathrm{Hom}_{\mathrm{C^b}}(P_1 \to P_2, (P_1 \to P_2)[-1])$ has the form

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "\cdot a"] \arrow[d, "g_{-1}"] & P_2 \arrow[r] \arrow[d, "g_0"] & 0 \arrow[r] \arrow[d, "g_1"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_1 \arrow[r, "\cdot(-a)"] & P_2 \arrow[r] & \cdots
\end{tikzcd}$$

Thus $g_i = 0$ for each $i \neq 0$. Since g commutes with the boundary operator, $g_0 \cdot (-a) =$

$0 \cdot g_1 = 0$. Since $\Lambda_{\mathrm{con}} \cong \mathbb{C}\langle\langle Q \rangle\rangle$ with no relations, $g_0 = 0$. Thus $g = 0$, and so we have $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(P_1 \to P_2, (P_1 \to P_2)[-1]) = 0$. Similarly, $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(0 \to P_2, (P_1 \to P_2)[-1]) = 0$. $\square$

Lemma 4.2.28. *With notation as above and 4.2.22,* $\mathrm{End}_{\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda_{\mathrm{con}})}(\mathcal{P}) \cong \mathbb{C}\langle\langle Q'' \rangle\rangle/(l\mathsf{x})$.

Proof. Recall that $\mathcal{P} = (P_1 \xrightarrow{\cdot a} P_2) \oplus (0 \to P_2)$ (4.2.O). Thus

$$\mathrm{End}_{\mathrm{K}^{\mathrm{b}}}(\mathcal{P}) = \begin{bmatrix} \mathrm{End}_{\mathrm{K}^{\mathrm{b}}}(P_1 \to P_2) & \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(P_1 \to P_2, 0 \to P_2) \\ \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(0 \to P_2, P_1 \to P_2) & \mathrm{End}_{\mathrm{K}^{\mathrm{b}}}(0 \to P_2) \end{bmatrix}.$$

(1) Any cochain map g in $\mathrm{End}_{\mathrm{C}^{\mathrm{b}}}(P_1 \to P_2)$ has the form

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "g_{-2}"'] & P_1 \arrow[r, "\cdot a"] \arrow[d, "g_{-1}"'] & P_2 \arrow[r] \arrow[d, "g_0"'] & 0 \arrow[r] \arrow[d, "g_1"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_1 \arrow[r, "\cdot a"] & P_2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Thus $g_i = 0$ for each $i \neq -1, 0$. Since g commutes with the boundary operator, $g_{-1}a = ag_0$. Recall that $P_1 = \Lambda_{\mathrm{con}}e_1$ and $P_2 = \Lambda_{\mathrm{con}}e_2$ in 4.2.22 where $\Lambda_{\mathrm{con}} \cong \mathbb{C}\langle\langle Q \rangle\rangle$ and

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left, "a"] & \underset{2}{\bullet} \arrow[l, bend left, "b"] \end{tikzcd}$$

Thus $g_{-1}a = ag_0$ induces

$$\begin{aligned} g_{-1} &= c_n(ab)^n + c_{n-1}(ab)^{n-1} + \cdots + c_1 ab + c_0 e_1, \\ g_0 &= c_n(ba)^n + c_{n-1}(ba)^{n-1} + \cdots + c_1 ba + c_0 e_2, \end{aligned}$$

for some $n \geq 0$ and each $c_i \in \mathbb{C}$.

We next define a cochain map $E_1 \in \mathrm{End}_{\mathrm{C}^{\mathrm{b}}}(P_1 \to P_2)$ as $(E_1)_{-1} = e_1$, $(E_1)_0 = e_2$ and $(E_1)_i = 0$ for each $i \neq -1, 0$.

Then we define a collection of maps h_i in the diagram below as $h_i = 0$ for each $i \neq 0$ and

$$h_0 = c_n b(ab)^{n-1} + c_{n-1}b(ab)^{n-2} + \cdots + c_1 b.$$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "g_{-2}"] \arrow[dl, "0"] & P_1 \arrow[r, "\cdot a"] \arrow[d, "g_{-1}"] \arrow[dl, "h_{-1}"'] & P_2 \arrow[r] \arrow[d, "g_0"] \arrow[dl, "h_0"'] & 0 \arrow[r] \arrow[d, "g_1"] \arrow[dl, "0"'] & \cdots \arrow[dl, "0"'] \\
\cdots \arrow[r] & 0 \arrow[r] & P_1 \arrow[r, "\cdot a"] & P_2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

It is clear that

$$g_{-1} - c_0(E_1)_{-1} = ah_0 + h_{-1} \cdot 0 \text{ and } g_0 - c_0(E_1)_0 = h_0 a + 0 \cdot h_1.$$

Thus h is a cochain homotopy between g and c_0E_1, and so $g = c_0E_1$ in $\mathrm{End}_{\mathrm{K}^{\mathrm{b}}}(P_1 \to P_2)$. Thus $\mathrm{End}_{\mathrm{K}^{\mathrm{b}}}(P_1 \to P_2) \cong \mathbb{C}E_1$

(2) Any cochain map g in $\mathrm{Hom}_{\mathrm{C}^{\mathrm{b}}}(0 \to P_2, P_1 \to P_2)$ has the form

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "g_{-2}"'] & 0 \arrow[r] \arrow[d, "g_{-1}"'] & P_2 \arrow[r] \arrow[d, "g_0"'] & 0 \arrow[r] \arrow[d, "g_1"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_1 \arrow[r, "\cdot a"] & P_2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Thus $g_i = 0$ for each $i \neq 0$, and

$$g_0 = c_n(ba)^n + c_{n-1}(ba)^{n-1} + \cdots + c_1ba + c_0e_2$$

for some $n \geq 0$ and each $c_i \in \mathbb{C}$.

We next define a cochain map $\mathsf{X} \in \mathrm{Hom}_{\mathrm{C}^{\mathrm{b}}}(0 \to P_2, P_1 \to P_2)$ as $\mathsf{X}_0 = e_2$ and $\mathsf{X}_i = 0$ for each $i \neq 0$. Similar to (3), we can also construct a cochain homotopy h between g and $c_0\mathsf{X}$. So $g = c_0\mathsf{X}$ in $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(0 \to P_2, P_1 \to P_2)$. Thus $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(0 \to P_2, P_1 \to P_2) \cong \mathbb{C}\mathsf{X}$.

(3) Any cochain map g in $\mathrm{Hom}_{\mathrm{C}^{\mathrm{b}}}(P_1 \to P_2, 0 \to P_2)$ has the form

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "g_{-2}"'] & P_1 \arrow[r, "\cdot a"] \arrow[d, "g_{-1}"'] & P_2 \arrow[r] \arrow[d, "g_0"'] & 0 \arrow[r] \arrow[d, "g_1"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & P_2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Thus $g_i = 0$ for each $i \neq 0$. Since g commutes with the boundary operator, $a \cdot g_0 = g_{-1} \cdot 0 = 0$. Since $\Lambda_{\mathrm{con}} \cong \mathbb{C}\langle\langle Q \rangle\rangle$ with no relations, $g_0 = 0$. Thus $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}}(P_1 \to P_2, 0 \to P_2) = 0$.

(4) Any cochain map g in $\mathrm{End}_{\mathrm{C}^{\mathrm{b}}}(0 \to P_2)$ has the form

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "g_{-2}"'] & 0 \arrow[r] \arrow[d, "g_{-1}"'] & P_2 \arrow[r] \arrow[d, "g_0"'] & 0 \arrow[r] \arrow[d, "g_1"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & P_2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Thus $g_i = 0$ for each $i \neq 0$ and

$$g_0 = c_n(ba)^n + c_{n-1}(ba)^{n-1} + \cdots + c_1ba + c_0e_2$$

for some $n \geq 0$ and each $c_i \in \mathbb{C}$.

We next define the cochain maps $E_2, L \in \mathrm{End}_{\mathrm{C}^{\mathrm{b}}}(0 \to P_2)$ as

- $(E_2)_0 = e_2$ and $(E_2)_i = 0$ for each $i \neq 0$,
- $L_0 = ba$ and $L_i = 0$ for each $i \neq 0$.

It is clear that $\mathrm{End}_{\mathrm{C^b}}(0 \to P_2)$ is a $\mathbb{C}$-algebra generated by E_2 and L with relations $E_2^2 = E_2$, $E_2 L = L = LE_2$. Since any cochain homotopy h in the diagram above must be zero, $\mathrm{End}_{\mathrm{K^b}}(0 \to P_2)$ has no more relations, and so $\mathrm{End}_{\mathrm{K^b}}(0 \to P_2) \cong \mathbb{C}[[L]]$, and as a $\mathbb{C}$-vector space is spanned by $\{E_2, L, L^2, \cdots\}$.

Combining (1), (2), (3) and (4), it follows that

$$\mathrm{End}_{\mathrm{K^b}}(\mathcal{P}) = \begin{bmatrix} \mathrm{End}_{\mathrm{K^b}}(P_1 \to P_2) & \mathrm{Hom}_{\mathrm{K^b}}(P_1 \to P_2, 0 \to P_2) \\ \mathrm{Hom}_{\mathrm{K^b}}(0 \to P_2, P_1 \to P_2) & \mathrm{End}_{\mathrm{K^b}}(0 \to P_2) \end{bmatrix} \cong \begin{bmatrix} \mathbb{C}E_1 & 0 \\ \mathbb{C}\mathsf{X} & \mathbb{C}[[L]] \end{bmatrix},$$

which is the $\mathbb{C}$-algebra generated by $\mathbb{E}_1$, $\mathbb{X}$, $\mathbb{E}_2$ and $\mathbb{L}$ where

$$\mathbb{E}_1 := \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbb{X} := \begin{bmatrix} 0 & 0 \\ \mathsf{X} & 0 \end{bmatrix}, \quad \mathbb{E}_2 := \begin{bmatrix} 0 & 0 \\ 0 & E_2 \end{bmatrix}, \quad \mathbb{L} := \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix}.$$

Moreover, $\mathrm{End}_{\mathrm{K^b}}(\mathcal{P})$ is the $\mathbb{C}$-vector space spanned by $\{\mathbb{E}_1, \mathbb{X}, \mathbb{E}_2, \mathbb{L}, \mathbb{L}^2, \cdots\}$.

Recall that

$$Q'' = \begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\mathsf{x}"'] \arrow[loop above, "l"] \end{tikzcd}$$

We define an algebra homomorphism $\varphi\colon \mathbb{C}\langle\langle Q''\rangle\rangle/(l\mathsf{x}) \to \mathrm{End}_{\mathrm{K^b}}(\mathcal{P})$ by $\varphi(e_1) = \mathbb{E}_1$, $\varphi(e_2) = \mathbb{E}_2$, $\varphi(X) = \mathbb{X}$ and $\varphi(l) = \mathbb{L}$. We next check that φ is well-defined, which only requires verifying that $\mathrm{End}_{\mathrm{K^b}}(\mathcal{P})$ satisfies the relations of $\mathbb{C}\langle\langle Q''\rangle\rangle/(l\mathsf{x})$.

The relations of $\mathbb{C}\langle\langle Q''\rangle\rangle/(l\mathsf{x})$ are

$$\begin{aligned} &e_1^2 = e_1,\ e_1 e_2 = 0,\ e_1 x = 0,\ e_1 l = 0, \\ &e_2 e_1 = 0,\ e_2^2 = e_2,\ e_2 x = x,\ e_2 l = l, \\ &x e_1 = x,\ x e_2 = 0,\ x^2 = 0,\ xl = 0, \\ &l e_1 = 0,\ l e_2 = l,\ lx = 0. \end{aligned}$$

Recall that

(1) $E_1 \in \mathrm{End}_{\mathrm{K^b}}(P_1 \to P_2)$ with $(E_1)_{-1} = e_1$, $(E_1)_0 = e_2$ and $(E_1)_i = 0$ for each $i \neq -1, 0$,

(2) $\mathsf{X} \in \mathrm{Hom}_{\mathrm{K^b}}(0 \to P_2, P_1 \to P_2)$ with $\mathsf{X}_0 = e_2$ and $\mathsf{X}_i = 0$ for each $i \neq 0$,

(3) $E_2 \in \mathrm{End}_{\mathrm{K^b}}(0 \to P_2)$ with $(E_2)_0 = e_2$ and $(E_2)_i = 0$ for each $i \neq 0$,

(4) $L \in \mathrm{End}_{\mathrm{K^b}}(0 \to P_2)$ with $L_0 = ba$ and $L_i = 0$ for each $i \neq 0$.

Thus $\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{X}$ and $\mathbb{L}$ clearly satisfy all the relations above, except for $\mathbb{L}\mathbb{X} = 0$. So it remains to check that $L\mathsf{X} = 0$, which is illustrated as follows,

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] \arrow[ddl, "h_{-2}"'] & 0 \arrow[r] \arrow[d] \arrow[ddl, "h_{-1}"'] & P_2 \arrow[r] \arrow[d, "\cdot ba"] \arrow[ddl, "h_0"'] & 0 \arrow[r] \arrow[d] \arrow[ddl, "h_1"'] & \cdots \arrow[ddl, "h_2"'] \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P_2 \arrow[r] \arrow[d, "\cdot e_2"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_1 \arrow[r, "\cdot a"] & P_2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

We define a collection of maps h_i in the diagram above as $h_0 = b$ and $h_i = 0$ for each $i \neq 0$. It is clear that h is a cochain homotopy between $L\mathsf{X}$ and 0. So we have $L\mathsf{X} = 0$ in $\mathrm{Hom}_{\mathrm{K^b}}(0 \to P_2, P_1 \to P_2)$. So $\varphi\colon \mathbb{C}\langle\!\langle Q'' \rangle\!\rangle/(l\mathsf{x}) \to \mathrm{End}_{\mathrm{K}^b}(\mathcal{P})$ is an algebra homomorphism.

Since $\mathrm{End}_{\mathrm{K^b}}(\mathcal{P})$ is the $\mathbb{C}$-algebra generated by $\mathbb{E}_1$, $\mathbb{X}$, $\mathbb{E}_2$ and $\mathbb{L}$, and $\varphi(e_1) = \mathbb{E}_1$, $\varphi(e_2) = \mathbb{E}_2$, $\varphi(X) = \mathbb{X}$ and $\varphi(l) = \mathbb{L}$, it follows that φ is surjective. By the relations of $\mathbb{C}\langle\!\langle Q'' \rangle\!\rangle/(l\mathsf{x})$, it is a $\mathbb{C}$-vector space spanned by $\{e_1, x, e_2, l, l^2, \cdots\}$. Since $\mathrm{End}_{\mathrm{K^b}}(\mathcal{P})$ is the $\mathbb{C}$-vector space spanned by $\{\mathbb{E}_1, \mathbb{X}, \mathbb{E}_2, \mathbb{L}, \mathbb{L}^2, \cdots\}$, φ is injective. So φ is an algebra isomorphism. □

Proposition 4.2.29. *With notation as above and 4.2.22, Λ_{con} is derived equivalent to $\mathbb{C}\langle\!\langle Q'' \rangle\!\rangle/(l\mathsf{x})$.*

Proof. By 4.2.27, $\mathcal{P}$ in (4.2.O) is a tilting complex of Λ_{con}. Thus by [R3] Λ_{con} is derived equivalent to $\mathrm{End}_{\mathrm{K^b}(\mathrm{proj}\,\Lambda_{\mathrm{con}})}(\mathcal{P})$. Since $\mathrm{End}_{\mathrm{K^b}(\mathrm{proj}\,\Lambda_{\mathrm{con}})}(\mathcal{P}) \cong \mathbb{C}\langle\!\langle Q'' \rangle\!\rangle/(l\mathsf{x})$ by 4.2.28, the statement follows. □

Chapter 5

Filtrations and Obstructions

In §5.1 and §5.2, we first define some matrices and generalised GV invariants associated to a monomialized Type A potential.

Using these matrices, §5.3 gives filtration structures of the parameter space of monomialized Type A potentials on Q_n with respect to generalised GV invariants.

Finally, §5.4 uses these filtration structures to give the obstructions of generalised GV invariants that can arise from crepant resolutions of cA_n singularities.

§ 5.1 | Matrices from potentials

This section introduces some matrices associated with monomialized Type A potentials. With these matrices, §5.3 gives a filtration structure of the parameter space of monomialized Type A potentials on Q_n with respect to generalised GV invariants.

Throughout this section, we fix some $n \geq 1$ and consider monomialized Type A potentials on the quiver Q_n (1.5.A).

Notation 5.1.1. Since §5.3 and §5.4 will consider the parameter space of monomialized Type A potentials on Q_n, we introduce the following notation.

(1) Define the set of monomialized Type A potentials on Q_n

$$\mathsf{MA} := \{\sum_{i=1}^{2n-2} \mathsf{x}_i'\mathsf{x}_{i+1} + \sum_{i=1}^{2n-1}\sum_{j=2}^{\infty} k_{ij}\mathsf{x}_i^j \mid k_{ij} \in \mathbb{C} \text{ for } 1 \leq i \leq 2n-1 \text{ and } 2 \leq j \leq \infty\}.$$

(2) Then set the parameter space M associated to MA to be

$$\mathsf{M} := \{(k_{12}, k_{13}, \dots, k_{2n-1,2}, k_{2n-1,3}, \dots) \mid k_{ij} \in \mathbb{C} \text{ for } 1 \leq i \leq 2n-1 \text{ and } 2 \leq j \leq \infty\}.$$

(3) Write κ for the tuple of *variables* κ_{ij} for $1 \leq i \leq 2n-1$ and $2 \leq j \leq \infty$, inside the infinite polynomial ring $\mathbb{C}[[\kappa_{12}, \kappa_{13}, \dots \kappa_{2n-1,2}, \kappa_{2n-1,3}, \dots]] := \mathbb{C}[[\kappa]]$.

(4) For each i and j, define the map $\varepsilon_{ij}\colon \mathsf{MA} \to \mathbb{C}$ to be $\varepsilon_{ij}(f) := jk_{ij}$. By the obvious

bijection map $\mathsf{M} \to \mathsf{MA}$, sometimes we abuse the notation to consider $\varepsilon_{ij} : \mathsf{M} \to \mathsf{MA} \to \mathbb{C}$ and so $\varepsilon_{ij}(\kappa) = j\kappa_{ij}$.

Given two matrices $A = (a_{ij})_{p\times q}$ and $B = (b_{ij})_{s\times t}$ with $a_{pq} = b_{11}$, define $A\square B \in \mathbf{M}_{(p+s-1)\times(q+t-1)}$ to be

$$A \square B := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p-1,1} & a_{p-1,2} & \cdots & a_{p-1,q-1} & a_{p-1,q} & 0 & \cdots & 0 \\ a_{p1} & a_{p2} & \cdots & a_{p-1,q} & a_{pq} & b_{12} & \cdots & b_{1t} \\ 0 & 0 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2t} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & b_{s1} & b_{s2} & \cdots & b_{st} \end{bmatrix}.$$

Definition 5.1.2. *With the ε_{ij} in 5.1.1(4), we next define a set of matrices A^d_{ij} for*

(1) $1 \le i \le j \le 2n-1$, *$j-i$ is odd, and $d = 2$,*

(2) $1 \le i \le j \le 2n-1$, *$j-i$ is even, and $d \ge 2$.*

For any $1 \le i \le 2n-1$ and $d \ge 2$, define $A^d_{i,i} := \begin{bmatrix}\varepsilon_{i,d}\end{bmatrix}$.

For any $1 \le i \le 2n-2$, define $A^2_{i,i+1} := \begin{bmatrix} \varepsilon_{i,2} & 1 \\ 1 & \varepsilon_{i+1,2} \end{bmatrix}$.

For any $1 \le i \le 2n-3$ and $d > 2$, define $A^d_{i,i+2} \in \mathbf{M}_{(d+1)\times(d+1)}$ to be

$$A^d_{i,i+2} := \begin{bmatrix} \varepsilon_{i,d} & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \varepsilon_{i+2,d} \end{bmatrix}. \tag{5.1.A}$$

The other A^d_{ij} are defined inductively. For any i, j satisfying $j-i \ge 2$, define

$$A^2_{i,j} := A^2_{i,i+1} \square A^2_{i+1,i+2} \square \cdots \square A^2_{j-1,j}. \tag{5.1.B}$$

For any $d > 2$, and i, j satisfying $j-i \ge 4$ and even, define

$$A^d_{i,j} := A^d_{i,i+2} \square A^d_{i+2,i+4} \square \cdots \square A^d_{j-2,j}. \tag{5.1.C}$$

Given any $f \in \mathsf{MA}$, define $A^d_{ij}(f)$ as replacing all $\varepsilon_{*,d}$ in A^d_{ij} with $\varepsilon_{*,d}(f)$.

Remark 5.1.3. Since $\varepsilon_{ij}\colon \mathsf{MA} \to \mathbb{C}$ in 5.1.1(4), for any i, j, d in 5.1.2, we have

$$\begin{aligned} A^d_{ij}\colon \mathsf{MA} &\to \mathbf{M}(\mathbb{C}), \\ f &\mapsto A^d_{ij}(f) \end{aligned}$$

where $\mathbf{M}(\mathbb{C})$ is the set of matrices over the complex numbers. By the obvious bijection map $\mathsf{M} \to \mathsf{MA}$, sometimes we abuse the notation and consider $A^d_{ij}\colon \mathsf{M} \to \mathsf{MA} \to \mathbf{M}(\mathbb{C})$, and so $A^d_{ij}(\kappa) \in \mathbf{M}(\mathbb{C}[[\kappa]])$ and $\det A^d_{ij}(\kappa) \in \mathbb{C}[[\kappa]]$.

Example 5.1.4. $A^d_{i,i}(\kappa) = \begin{bmatrix} d\kappa_{id} \end{bmatrix}$, $A^2_{i,i+1}(\kappa) = \begin{bmatrix} 2\kappa_{i,2} & 1 \\ 1 & 2\kappa_{i+1,2} \end{bmatrix}$, and for $d > 2$

$$A^d_{i,i+2}(\kappa) = \begin{bmatrix} d\kappa_{id} & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & d\kappa_{i+2,d} \end{bmatrix}.$$

Then we consider some subsets of the monomialized Type A potentials MA on Q_n.

Notation 5.1.5. Fix a tuple $\mathbf{p} = (p_1, p_2, \ldots, p_{2n-1})$ where each $2 \le p_i \in \mathbb{N}_\infty$, we adopt the following notation, which is parallel to that in 5.1.1.

(1) Define the following subset of monomialized Type A potentials on Q_n

$$\mathsf{MA}_{\mathbf{p}} := \{ \sum_{i=1}^{2n-2} \mathsf{x}'_i \mathsf{x}_{i+1} + \sum_{i=1}^{2n-1} \sum_{j=2}^{\infty} k_{ij} \mathsf{x}^j_i \mid k_{i,j_i} = 0 \text{ for } 1 \le i \le 2n-1, 2 \le j_i < p_i \}. \tag{5.1.D}$$

(2) Then set the parameter space $\mathsf{M}_{\mathbf{p}}$ associated to $\mathsf{MA}_{\mathbf{p}}$ to be

$$\mathsf{M}_{\mathbf{p}} := \{(k_{12}, k_{13}, \ldots, k_{2n-1,2}, k_{2n-1,3}, \ldots) \mid k_{i,j_i} = 0 \text{ for } 1 \le i \le 2n-1, 2 \le j_i < p_i\}. \tag{5.1.E}$$

(3) Write $\kappa_{\mathbf{p}}$ for the tuple of *variables* κ_{ij_i}, for $1 \le i \le 2n-1$ and $p_i \le j_i \le \infty$.

(4) For any i, j satisfying $1 \le i \le j \le 2n-1$, define $d_{ij}(\mathbf{p})$ to be

$$d_{ij}(\mathbf{p}) := \begin{cases} 2 & \text{if } j-i \text{ is odd} \\ \min(p_i, p_{i+2}, \ldots, p_j) & \text{if } j-i \text{ is even} \end{cases} \tag{5.1.F}$$

(5) Given another tuple $\mathbf{p}' = (p'_1, p'_2, \ldots, p'_{2n-1})$, write $\mathbf{p}' \ge \mathbf{p}$ if $p'_i \ge p_i$ for each i.

Remark 5.1.6. We next make some remarks about the above notations.

(1) If $\mathbf{p} = (2, 2, \ldots, 2)$, then $\kappa_{\mathbf{p}}$, $\mathsf{M}_{\mathbf{p}}$ and $\mathsf{MA}_{\mathbf{p}}$ coincide with κ, M and MA respectively.

(2) By the inclusion map $\mathsf{MA}_{\mathbf{p}} \hookrightarrow \mathsf{MA}$, for any $f \in \mathsf{MA}_{\mathbf{p}}$ and i, j, d in 5.1.2, $f \in \mathsf{MA}$ and so $\varepsilon_{ij}(f)$, $A^d_{ij}(f)$ have been defined.

(3) By the inclusion map $\mathsf{M}_{\mathbf{p}} \hookrightarrow \mathsf{M}$, for any i, j, d in 5.1.2 sometimes we abuse the notation to consider ε_{ij} and A^d_{ij} are defined on the subspace $\mathsf{M}_{\mathbf{p}}$, and so $A^d_{ij}(\kappa_{\mathbf{p}}) \in \mathbf{M}(\mathbb{C}[[\kappa_{\mathbf{p}}]])$ and $\varepsilon_{ij}(\kappa_{\mathbf{p}}), \det A^d_{ij}(\kappa_{\mathbf{p}}) \in \mathbb{C}[[\kappa_{\mathbf{p}}]]$.

(4) Let $f \in \mathsf{MA}_{\mathbf{p}}$ and write
$$f = \sum_{i=1}^{2n-2} \mathsf{x}'_i \mathsf{x}_{i+1} + \sum_{i=1}^{2n-1} \sum_{j=2}^{\infty} k_{ij} \mathsf{x}^j_i .$$
For $1 \leq i \leq 2n-1$, if $d < p_i$, then $k_{id} = 0$, and so $\varepsilon_{id}(f) = dk_{id} = 0$. Thus ε_{id} is a zero function over the domain $\mathsf{MA}_{\mathbf{p}}$, and so $\varepsilon_{id}(\kappa_{\mathbf{p}}) = 0$.

(5) If $\mathbf{p}' \geq \mathbf{p}$, then $\mathsf{MA}_{\mathbf{p}'} \subseteq \mathsf{MA}_{\mathbf{p}}$ and $\mathsf{M}_{\mathbf{p}'} \subseteq \mathsf{M}_{\mathbf{p}}$.

The following results of this subsection come from the inductive definition of A^d_{ij}. They will be used in §5.3 to give the general position of the parameter space $\mathsf{M}_{\mathbf{p}}$ with respect to generalised GV invariants.

Lemma 5.1.7. *Given any i and j satisfying $j - i \geq 2$, the following holds.*

(1) $\det A^2_{ij} = \varepsilon_{j2} \det A^2_{i,j-1} - \det A^2_{i,j-2}$,

(2) $\det A^2_{ij} = \varepsilon_{i2} \det A^2_{i-1,j} - \det A^2_{i-2,j}$.

When furthermore $j - i$ is even, for any $d > 2$, the following holds.

(3) $\det A^d_{ij} = -\det A^d_{i,j-2} + (-1)^{(j-i)(d-1)/2} \varepsilon_{jd}$,

(4) $\det A^d_{ij} = (-1)^{d-1} \det A^d_{i+2,j} + (-1)^{(j-i)/2} \varepsilon_{id}$.

Proof. (1) By the inductive definition of A^2_{ij} and $A^2_{i,j-1}$ (5.1.B),
$$A^2_{ij} = A^2_{i,j-1} \square A^2_{j-1,j} \text{ and } A^2_{i,j-1} = A^2_{i,j-2} \square A^2_{j-2,j-1}.$$
Set v_n to be the $1 \times n$ matrix $[0, 0, \ldots, 0, 1]$. Thus
$$A^2_{ij} = \left[\begin{array}{c|c} A^2_{i,j-1} & v^T_{j-i} \\ \hline v_{j-i} & \varepsilon_{j2} \end{array}\right], \qquad A^2_{i,j-1} = \left[\begin{array}{c|c} A^2_{i,j-2} & v^T_{j-i-1} \\ \hline v_{j-i-1} & \varepsilon_{j-1,2} \end{array}\right].$$
Write B for the matrix by removing the last row and the second to last column of A^2_{ij}. By expanding along the last row of A^2_{ij}, $\det A^2_{ij} = \varepsilon_{j2} \det A^2_{i,j-1} - \det B$. Moreover, by the forms

of A^2_{ij} and $A^2_{i,j-1}$ as above,

$$B = \left[\begin{array}{c|c} A^2_{i,j-2} & 0 \\ \hline v_{j-i-1} & 1 \end{array}\right].$$

Thus by expanding along the last column of B, $\det B = \det A^2_{i,j-2}$, and so $\det A^2_{ij} = \varepsilon_{j2} \det A^2_{i,j-1} - \det A^2_{i,j-2}$.

(2) This is similar, by expanding along the first row of A^2_{ij}.

(3) By the inductive definition of A^d_{ij} (5.1.C), $A^d_{ij} = A^d_{i,j-2} \,\square\, A^d_{j-2,j}$. Together with (5.1.A), A^d_{ij} has the following form

$$A^d_{ij} = \left[\begin{array}{cccc|cccccc}
 & & & & 0 & 0 & \cdots & 0 & 0 & 0 \\
 & & & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & \multicolumn{2}{c}{A^d_{i,j-2}} & & 0 & 0 & \cdots & 0 & 0 & 0 \\
 & & & & 0 & 0 & \cdots & 0 & 1 & 0 \\
\hline
0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \varepsilon_{jd}
\end{array}\right].$$

Write C^d_{ij} for the matrix by removing the last row and the last column of A^d_{ij}, D for the matrix by removing the last row and the second to last column of A^d_{ij}. By expanding along the last row of A^d_{ij}, $\det A^d_{ij} = \varepsilon_{jd} \det C^d_{ij} - \det D$. We claim that $\det D = \det A^d_{i,j-2}$ and $\det C^d_{ij} = (-1)^{(j-i)(d-1)/2}$. So the statement follows.

To see this, by the form of A^d_{ij},

$$D = \left[\begin{array}{cccc|ccccc} \multicolumn{4}{c|}{\multirow{4}{*}{$A^d_{i,j-2}$}} & 0 & 0 & \cdots & 0 & 0 \\ &&&& \vdots & \vdots & \ddots & \vdots & \vdots \\ &&&& 0 & 0 & \cdots & 0 & 0 \\ &&&& 0 & 0 & \cdots & 1 & 0 \\ \hline 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 1 \end{array}\right].$$

By expanding along the last column repeatedly, $\det D = \det A^d_{i,j-2}$.

By the definition of C^d_{ij}, $C^d_{i,j-2}$ and the form of A^d_{ij},

$$C^d_{ij} = \left[\begin{array}{cccc|cccccc} \multicolumn{4}{c|}{\multirow{4}{*}{$C^d_{i,j-2}$}} & 0 & 0 & 0 & \cdots & 0 & 0 \\ &&&& \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ &&&& 0 & 0 & 0 & \cdots & 0 & 0 \\ &&&& 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & \cdots & 1 & \varepsilon_{j-2,d} & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right],$$

where the lower right corner block is a d by d matrix. Since C^d_{ij} has the above form, by expanding along the last row $d-1$ times, it follows that $\det C^d_{ij} = (-1)^{d-1} \det C$ where

$$C := \left[\begin{array}{cccc|c} \multicolumn{4}{c|}{\multirow{4}{*}{$C^d_{i,j-2}$}} & 0 \\ &&&& \vdots \\ &&&& 0 \\ &&&& 0 \\ \hline 0 & 0 & \cdots & 1 & 1 \end{array}\right].$$

Thus $\det C = \det C^d_{i,j-2}$, and so $\det C^d_{ij} = (-1)^{d-1} \det C^d_{i,j-2}$. Since $C^d_{i,i+2}$ is obtained by removing the last row and the last column of $A^d_{i,i+2}$ (5.1.A), $\det C^d_{i,i+2} = (-1)^{d-1}$. So

$$\det C^d_{ij} = (-1)^{d-1} \det C^d_{i,j-2} = (-1)^{(j-i-2)(d-1)/2} \det C^d_{i,i+2} = (-1)^{(j-i)(d-1)/2}.$$

(4) This is similar, by expanding along the first row of A^d_{ij}. $\square$

Notation 5.1.8. For any i and j satisfying $i \le j$ and $j-i$ is even, we adopt the following notation for the ideals in $\mathbb{C}[\varepsilon_{i-1,2}, \varepsilon_{i,2}, \ldots, \varepsilon_{j+1,2}]$.

(1) Write m_{ij} for the ideal $(\varepsilon_{i,2}, \varepsilon_{i+2,2}, \ldots, \varepsilon_{j,2})$.

(2) Write E_{ij} for the ideal generated by all the degree two terms of $\varepsilon_{i,2}, \varepsilon_{i+2,2}, \ldots, \varepsilon_{j,2}$ except $\varepsilon^2_{i,2}, \varepsilon^2_{i+2,2}, \ldots, \varepsilon^2_{j,2}$.

Lemma 5.1.9. *Given any i, j satisfying $i \le j$, the following holds.*

(1) *If $j-i$ is odd, then* $\det A^2_{ij} = (-1)^{(j-i+1)/2} + \epsilon$, *where* $\epsilon \in m_{i,j-1} \cap m_{i+1,j}$.

(2) *If $j-i$ is even, then* $\det A^2_{ij} = (-1)^{(j-i)/2}(\varepsilon_{i2} + \varepsilon_{i+2,2} + \cdots + \varepsilon_{j2}) + \epsilon$ *where* $\epsilon \in E_{ij}$.

(3) *If $j-i$ is even and $d > 2$, then* $\det A^d_{ij} = (-1)^{(j-i)/2}(\varepsilon_{id} + (-1)^d\varepsilon_{i+2,d} + \cdots + (-1)^{(j-i)d/2}\varepsilon_{jd})$.

Proof. (1) If $j-i=1$, then by definition $\det A^2_{ij} = -1 + \varepsilon_{i,2}\varepsilon_{i+1,2}$. Since $\varepsilon_{i,2}\varepsilon_{i+1,2} \in (\varepsilon_{i,2}) \cap (\varepsilon_{i+1,2}) = m_{i,j-1} \cap m_{i+1,j}$, the statement follows.

We next prove this statement by induction. Fix some i, j satisfying $j-i \ge 3$ and odd. Assume that $\det A^2_{i,j-2} = (-1)^{(j-i-1)/2} + \epsilon'$ where $\epsilon' \in m_{i,j-3} \cap m_{i+1,j-2}$. So we have

$$\begin{aligned} \det A^2_{ij} &= \varepsilon_{j2} \det A^2_{i,j-1} - \det A^2_{i,j-2} && \text{(by 5.1.7(1))} \\ &= \varepsilon_{j2} \det A^2_{i,j-1} - (-1)^{(j-i-1)/2} - \epsilon' && \text{(by assumption)} \\ &= (-1)^{(j-i+1)/2} + \varepsilon_{j2} \det A^2_{i,j-1} - \epsilon'. \end{aligned}$$

Set $\epsilon := \varepsilon_{j2} \det A^2_{i,j-1} - \epsilon'$. So it suffices to prove that $\epsilon \in m_{i,j-1} \cap m_{i+1,j}$.

Since by definition (5.1.B) $\det A^2_{i,j-1} \in \mathbb{C}[\varepsilon_{i,2}, \varepsilon_{i+1,2}, \ldots, \varepsilon_{j-1,2}]$, $\varepsilon_{j2} \det A^2_{i,j-1} \in m_{i+1,j}$. Together with $\epsilon' \in m_{i+1,j-2} \subseteq m_{i+1,j}$, it follows that $\epsilon \in m_{i+1,j}$. Similarly, we can prove $\epsilon \in m_{i,j-1}$ by $\det A^2_{ij} = \varepsilon_{i2} \det A^2_{i-1,j} - \det A^2_{i-2,j}$ in 5.1.7(2). So $\epsilon \in m_{i,j-1} \cap m_{i+1,j}$.

(2) If $j-i=0$, then by definition $\det A^2_{ij} = \varepsilon_{i,2}$. Thus the statement follows.

We next prove this statement by induction. Fix some i, j satisfying $j-i \ge 2$ and even. Assume that $\det A^2_{i,j-2} = (-1)^{(j-2-i)/2}(\varepsilon_{i,2} + \varepsilon_{i+2,2} + \cdots + \varepsilon_{j-2,2}) + \epsilon_1$ where $\epsilon_1 \in E_{i,j-2}$. Then by (1) $\det A^2_{i,j-1} = (-1)^{(j-i)/2} + \epsilon_2$ where $\epsilon_2 \in m_{i,j-2}$. So we have

$$\begin{aligned} \det A^2_{ij} &= \varepsilon_{j2} \det A^2_{i,j-1} - \det A^2_{i,j-2} && \text{(by 5.1.7(1))} \\ &= \varepsilon_{j2}((-1)^{(j-i)/2} + \epsilon_2) - (-1)^{(j-2-i)/2}(\varepsilon_{i,2} + \varepsilon_{i+2,2} + \cdots + \varepsilon_{j-2,2}) - \epsilon_1 \\ & && \text{(by (1) and assumption)} \\ &= (-1)^{(j-i)/2}(\varepsilon_{i,2} + \varepsilon_{i+2,2} + \cdots + \varepsilon_{j,2}) + \varepsilon_{j2}\epsilon_2 - \epsilon_1. \end{aligned}$$

Set $\epsilon := \varepsilon_{j2}\epsilon_2 - \epsilon_1$. Thus it suffices to prove that $\epsilon \in E_{ij}$. Since $\epsilon_2 \in m_{i,j-2} =$

$(\varepsilon_{i2}, \varepsilon_{i+2,2}, \ldots, \varepsilon_{j-2,2})$, $\varepsilon_{j2}\epsilon_2 \in (\varepsilon_{j2}\varepsilon_{i2}, \varepsilon_{j2}\varepsilon_{i+2,2}, \ldots, \varepsilon_{j2}\varepsilon_{j-2,2}) \in E_{ij}$. Together with $\epsilon_1 \in E_{i,j-2} \subseteq E_{ij}$, it follows that $\epsilon \in E_{ij}$.

(3) If $j - i = 0$ and $d > 2$, then by definition $\det A^d_{ij} = \varepsilon_{i,d}$. Thus the statement follows.

We next prove this statement by induction. Fix some i, j and d satisfying $d > 2$, and $j-i \geq 2$ and even. Assume that $\det A^d_{i,j-2} = (-1)^{(j-2-i)/2}(\varepsilon_{id}+(-1)^d\varepsilon_{i+2,d}+\cdots+(-1)^{(j-2-i)d/2}\varepsilon_{j-2,d})$. So we have

$$\begin{aligned}\det A^d_{ij} &= -\det A^d_{i,j-2} + (-1)^{(j-i)(d-1)/2}\varepsilon_{jd} && \text{(by 5.1.7(3))}\\ &= (-1)^{(j-i)/2}(\varepsilon_{id} + (-1)^d\varepsilon_{i+2,d} + \cdots + (-1)^{(j-i)d/2}\varepsilon_{jd}). && \text{(by assumption)}\end{aligned}$$

Thus the statement follows. □

Proposition 5.1.10. *Let $f \in \mathsf{MA}$ and write*

$$f = \sum_{i=1}^{2n-2} \mathsf{x}'_i\mathsf{x}_{i+1} + \sum_{i=1}^{2n-1}\sum_{j=2}^{\infty} k_{ij}\mathsf{x}_i^j.$$

For any $1 \leq i \leq j \leq 2n-1$ such that $j - i$ is odd, the following holds.

(1) *If $k_{t2} = 0$ for $t = i, i+2, \ldots, j-1$, then* $\det A^2_{ij}(f) = (-1)^{(j-i+1)/2}$.

(2) *If $k_{t2} = 0$ for $t = i+1, i+3, \ldots, j$, then* $\det A^2_{ij}(f) = (-1)^{(j-i+1)/2}$.

In particular, given some $\mathbf{p}$ satisfying $d_{i,j-1}(\mathbf{p}) > 2$ or $d_{i+1,j}(\mathbf{p}) > 2$, then we have $\det A^2_{ij}(\kappa_{\mathbf{p}}) = (-1)^{(j-i+1)/2}$.

Proof. (1) For $t = i, i+2, \ldots, j-1$, since $k_{t2} = 0$, then $\varepsilon_{t2}(f) = 2k_{t2} = 0$. By 5.1.9(1), $\det A^2_{ij}(f) = (-1)^{(j-i+1)/2} + \epsilon(f)$ where $\epsilon \in m_{i,j-1} \cap m_{i+1,j}$. In particular ϵ belongs to the ideal generated by the functions $\varepsilon_{i2}, \varepsilon_{i+2,2}, \ldots, \varepsilon_{j-1,2}$, all of which evaluate at f to be zero. Thus $\epsilon(f) = 0$, and so $\det A^2_{ij}(f) = (-1)^{(j-i+1)/2}$.

(2) This is similar.

If $d_{i,j-1}(\mathbf{p}) > 2$, then by (5.1.F) $p_i, p_{i+2}, \ldots, p_{j-1} > 2$. If further $f \in \mathsf{MA}_{\mathbf{p}}$, then $k_{t2} = 0$ for $t = i, i+2, \ldots, j-1$ by (5.1.D), and so by (1) $\det A^2_{ij}(f) = (-1)^{(j-i+1)/2}$. Since f is an arbitrary potential in $\mathsf{MA}_{\mathbf{p}}$, $\det A^2_{ij}(\kappa_{\mathbf{p}}) = (-1)^{(j-i+1)/2}$. Similarly, if $d_{i+1,j}(\mathbf{p}) > 2$, then by (2) $\det A^2_{ij}(\kappa_{\mathbf{p}}) = (-1)^{(j-i+1)/2}$. □

Recall the notation $\kappa_{\mathbf{p}}$, $d_{ij}(\mathbf{p})$ in 5.1.5, and $\det A^d_{ij}(\kappa_{\mathbf{p}})$ in 5.1.6. The following is the main technical result of this subsection. It will be used in §5.3 below to construct a filtration structure on $\mathsf{M}_{\mathbf{p}}$ (for some fixed $\mathbf{p}$) with respect to the generalised GV invariant of some chosen curve class $\mathrm{C}_i+\ldots+\mathrm{C}_j$. The zero locus of the polynomial $\det A^{d_{ij}(\mathbf{p})}_{ij}(\kappa_{\mathbf{p}}) \in \mathbb{C}[[\kappa_{\mathbf{p}}]]$ will turn out to be the first strata in the filtration, which motivates proving that this polynomial is nonzero in part (2) below. Part (1) is more technical, but will be needed for inductive proof in 5.3.2.

Proposition 5.1.11. *Given some* $\mathbf{p}$*, and any* i, j, d *in* 5.1.2*, then the following holds.*

(1) *If* $d < d_{ij}(\mathbf{p})$*, then* $\det A^d_{ij}(\kappa_{\mathbf{p}}) = 0 \in \mathbb{C}[[\kappa_{\mathbf{p}}]]$.

(2) *If* $d = d_{ij}(\mathbf{p})$ *and* d *is finite, then* $\det A^d_{ij}(\kappa_{\mathbf{p}}) \neq 0$ *in* $\mathbb{C}[[\kappa_{\mathbf{p}}]]$.

Proof. For any $d \geq 2$, consider two complementary subsets of $S := \{i, i+2, \dots, j\}$

$$S_d := \{t \in S \mid p_t \leq d\}, \quad \overline{S_d} := \{t \in S \mid p_t > d\}.$$

Then by 5.1.6(4),

$$t \in \overline{S_d} \iff \varepsilon_{td}(f) = 0 \text{ for all } f \in \mathsf{MA}_{\mathbf{p}} \iff \varepsilon_{td}(\kappa_{\mathbf{p}}) \text{ is the zero function over } \mathsf{M}_{\mathbf{p}}. \tag{5.1.G}$$

If $j - i$ is even and $d < d_{ij}(\mathbf{p})$, then by (5.1.F) $d < \min(p_i, p_{i+2}, \dots, p_j)$, and so $S_d = \emptyset$, $\overline{S_d} = S$. If $j - i$ is even and $d = d_{ij}(\mathbf{p})$, then by (5.1.F) $d = \min(p_i, p_{i+2}, \dots, p_j)$, and so $S_d \neq \emptyset$, $\overline{S_d} \neq S$.

(1) Since $d \geq 2$, the case $d_{ij}(\mathbf{p}) = 2$ cannot occur. Consequently $d_{ij}(\mathbf{p}) > 2$, and thus $j - i$ must be even by (5.1.F). Since $d < d_{ij}(\mathbf{p})$, $\overline{S_d} = S$, and so by (5.1.G) $\varepsilon_{td}(\kappa_{\mathbf{p}})$ is a zero function for each $t \in S = \{i, i+2, \dots, j\}$.

If furthermore $d > 2$, then

$$\begin{aligned} \det A^d_{ij}(\kappa_{\mathbf{p}}) &= (-1)^{(j-i)/2}(\varepsilon_{id}(\kappa_{\mathbf{p}}) + (-1)^d \varepsilon_{i+2,d}(\kappa_{\mathbf{p}}) + \cdots + (-1)^{(j-i)d/2}\varepsilon_{jd}(\kappa_{\mathbf{p}})) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by 5.1.9(3))} \\ &= 0. \qquad\qquad\qquad (\text{since } \varepsilon_{td}(\kappa_{\mathbf{p}}) = 0 \text{ for } t = i, i+2, \dots, j) \end{aligned}$$

Otherwise, if $d = 2$, then

$$\begin{aligned} \det A^d_{ij}(\kappa_{\mathbf{p}}) &= \det A^2_{ij}(\kappa_{\mathbf{p}}) \\ &= (-1)^{(j-i)/2}(\varepsilon_{i2}(\kappa_{\mathbf{p}}) + \varepsilon_{i+2,2}(\kappa_{\mathbf{p}}) + \cdots + \varepsilon_{j2}(\kappa_{\mathbf{p}})) + \epsilon(\kappa_{\mathbf{p}}) \qquad \text{(by 5.1.9(2))} \\ &= \epsilon(\kappa_{\mathbf{p}}), \qquad\qquad (\text{since } \varepsilon_{t2}(\kappa_{\mathbf{p}}) = 0 \text{ for } t = i, i+2, \dots, j) \end{aligned}$$

where $\epsilon \in E_{ij}$ and E_{ij} is the ideal generated by some degree two terms of $\varepsilon_{i2}, \varepsilon_{i+2,2}, \dots, \varepsilon_{j2}$. Since $\varepsilon_{t2}(\kappa_{\mathbf{p}}) = 0$ for $t = i, i+2, \dots, j$, $\epsilon(\kappa_{\mathbf{p}}) = 0$, and so $\det A^d_{ij}(\kappa_{\mathbf{p}}) = 0$.

(2) We split the proof into cases.

(i) $j - i$ is odd, $d = d_{ij}(\mathbf{p})$ and finite.

Since $j - i$ is odd, $d = d_{ij}(\mathbf{p}) = 2$ by (5.1.F). Thus by 5.1.9(1),

$$\det A^d_{ij}(\kappa_{\mathbf{p}}) = \det A^2_{ij}(\kappa_{\mathbf{p}}) = (-1)^{(j-i+1)/2} + \epsilon(\kappa_{\mathbf{p}}),$$

where $\epsilon \in m_{i,j-1}$ and $m_{i,j-1}$ is the ideal generated by $\varepsilon_{i,2}, \varepsilon_{i+2,2} \ldots, \varepsilon_{j-1,2}$. Since by 5.1.6(4) $\varepsilon_{t2}(\kappa_{\mathbf{p}})$ is either $2\kappa_{t2}$ or zero for any t, $\epsilon(\kappa_{\mathbf{p}}) \in (\kappa_{\mathbf{p}})$, and so $\det A_{ij}^d(\kappa_{\mathbf{p}})$ is a non-zero polynomial.

(ii) $j-i$ is even, $d = d_{ij}(\mathbf{p}) > 2$ and finite.

Since $j-i$ is even and $d > 2$,

$$\begin{aligned}\det A_{ij}^d(\kappa_{\mathbf{p}}) &= (-1)^{(j-i)/2}(\varepsilon_{id}(\kappa_{\mathbf{p}}) + (-1)^d \varepsilon_{i+2,d}(\kappa_{\mathbf{p}}) + \cdots + (-1)^{(j-i)d/2}\varepsilon_{jd}(\kappa_{\mathbf{p}})) \\ &\qquad\qquad \text{(by 5.1.9(3))} \\ &= (-1)^{(j-i)/2}\sum_{t\in S_d}(-1)^{(t-i)d/2} d\kappa_{td}. \qquad \text{(by (5.1.G))}\end{aligned}$$

Since $j-i$ is even and $d = d_{ij}(\mathbf{p})$, $S_d \neq \emptyset$, and so $\det A_{ij}^d(\kappa_{\mathbf{p}})$ is a non-zero polynomial.

(iii) $j-i$ is even and $d = d_{ij}(\mathbf{p}) = 2$.

Since $j-i$ is even and $d = 2$,

$$\begin{aligned}\det A_{ij}^d(\kappa_{\mathbf{p}}) &= \det A_{ij}^2(\kappa_{\mathbf{p}}) \\ &= (-1)^{(j-i)/2}(\varepsilon_{i2}(\kappa_{\mathbf{p}}) + \varepsilon_{i+2,2}(\kappa_{\mathbf{p}}) + \cdots + \varepsilon_{j2}(\kappa_{\mathbf{p}})) + \epsilon(\kappa_{\mathbf{p}}) \qquad \text{(by 5.1.9(2))} \\ &= (-1)^{(j-i)/2}\Big(\sum_{t\in S_d} 2\kappa_{t2}\Big) + \epsilon(\kappa_{\mathbf{p}}), \qquad \text{(by (5.1.G))}\end{aligned}$$

where $\epsilon \in E_{ij}$ and E_{ij} is the ideal generated by some degree two terms of $\varepsilon_{i2}, \varepsilon_{i+2,2}, \ldots, \varepsilon_{j2}$. Since by 5.1.6(4) $\varepsilon_{t2}(\kappa_{\mathbf{p}})$ is either $2\kappa_{t2}$ or zero for any t, $\epsilon(\kappa_{\mathbf{p}})$ is a degree two term in $\mathbb{C}[[\kappa_{\mathbf{p}}]]$. Since $j-i$ is even and $d = d_{ij}(\mathbf{p})$, $S_d \neq \emptyset$, and so $\sum_{t\in S_d} 2\kappa_{t2}$ is a non-zero degree one term in $\mathbb{C}[[\kappa_{\mathbf{p}}]]$. Combining these facts together, it follows that $\det A_{ij}^d(\kappa_{\mathbf{p}})$ is a non-zero polynomial. □

§ 5.2 | Generalised GV invariants of potentials

This section introduces generalised GV invariants of a monomialized Type A potential on Q_n, which parallels those of a crepant resolution of a cA_n singularity in 3.1.1.

Inspired by the correspondence between monomialized Type A potentials on Q_n and crepant resolutions of cA_n singularities in 4.2.19, we define generalised GV invariants of a monomialized Type A potential by its associated crepant resolution as follows.

We first recap the geometric realization in §4.2.1. Fix a monomialized Type A potentials f on Q_n

$$f = \sum_{i=1}^{2n-2} \mathsf{x}_i'\mathsf{x}_{i+1} + \sum_{i=1}^{2n-1}\sum_{j=2}^{\infty} k_{ij}\mathsf{x}_i^j,$$

where each $k_{ij} \in \mathbb{C}$. Then we consider the following system of equations where each $g_i \in$

$\mathbb{C}[[x,y]]$

$$
\begin{aligned}
g_0 + \sum_{j=2}^{\infty} j k_{1j} g_1^{j-1} + g_2 &= 0 \\
g_1 + \sum_{j=2}^{\infty} j k_{2j} g_2^{j-1} + g_3 &= 0 \\
&\vdots \\
g_{2n-2} + \sum_{j=2}^{\infty} j k_{2n-1,j} g_{2n-1}^{j-1} + g_{2n} &= 0.
\end{aligned} \tag{5.2.A}
$$

Fix some integer s satisfying $0 \leq s \leq 2n-1$, and set $g_s = y$, $g_{s+1} = x$. Then there exists $g_0, g_1, \ldots, g_{2n}$ which satisfies (5.2.A) and each $g_i \in (x,y) \subseteq \mathbb{C}[[x,y]]$. Furthermore, for any $0 \leq i \leq 2n-1$, $(g_i, g_{i+1}) = (x,y)$.

Definition 5.2.1. *With notation as above, for any $1 \leq i \leq j \leq n$, define the* generalised GV invariant *$N_{ij}(f)$ associated to f to be*

$$N_{ij}(f) := \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(g_{2i-2}, g_{2j})}.$$

We then consider the cA_n singularity

$$\mathcal{R} := \frac{\mathbb{C}[[u,v,x,y]]}{uv - g_0 g_2 \ldots g_{2n}},$$

and consider the $\mathcal{R}$-module

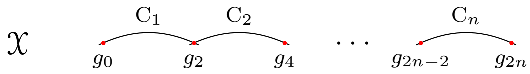
$$M := \mathcal{R} \oplus (u, g_0) \oplus (u, g_0 g_2) \oplus \ldots \oplus (u, \prod_{i=0}^{n-1} g_{2i}) \in (\mathrm{MM}\,\mathcal{R}) \cap (\mathrm{CM}\,\mathcal{R}).$$

In view of the above results 3.2.5 and 3.2.7, we introduce the following notation.

Notation 5.2.2. Suppose that Λ_1, Λ_2 are complete quiver algebras of Q_n subject to some relations. Write e_i for the trivial path at vertex i of Q_n, and write $\varphi\colon \Lambda_1 \xrightarrow{\sim} \Lambda_2$ if φ is an algebra isomorphism satisfying $\varphi(e_i) = e_i$ for each i.

By 4.2.9, $\underline{\mathrm{End}}_{\mathcal{R}}(M) \cong \mathcal{J}\mathrm{ac}(f)$. Since $(g_i, g_{i+1}) = (x,y)$ for $0 \leq i \leq 2n-1$, each g_i has a linear term, and so $\mathcal{R}$ admits a crepant resolution by e.g. [IW3, 5.1]. Together with $M \in (\mathrm{MM}\,\mathcal{R}) \cap (\mathrm{CM}\,\mathcal{R})$, by 3.3.2 there exists a crepant resolution $\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ such that $\Lambda_{\mathrm{con}}(\pi) \cong \underline{\mathrm{End}}_{\mathcal{R}}(M)$.

By 3.3.3, $\underline{\mathrm{End}}_{\mathcal{R}}(M)$ and $\Lambda_{\mathrm{con}}(\pi)$ can be presented as a complete quiver algebra of Q_n with some relations. In this chapter, we declare that the ith vertex of $\underline{\mathrm{End}}_{\mathcal{R}}(M) \cong \Lambda_{\mathrm{con}}(\pi)$ is the vertex corresponding to the summand $(u, \prod_{i=0}^{i-1} g_{2i})$. Using [IW3, §5] $\mathcal{X}$ is given pictorially by



and under this convention, the curve C_i corresponds to the summand $(u, \prod_{i=0}^{i-1} g_{2i})$, and thus the vertex i of $\Lambda_{\mathrm{con}}(\pi)$. Moreover, $\mathcal{J}\mathrm{ac}(f) \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathcal{R}}(M) \xrightarrow{\sim} \Lambda_{\mathrm{con}}(\pi)$.

Thus the generalised GV invariant $N_{ij}(f)$ of a monomialized Type A potential f is equal to $N_{ij}(\pi)$ (see 3.1.1), where π is its associated crepant resolution. Namely,

$$N_{ij}(\pi) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(g_{2i-2}, g_{2j})} = N_{ij}(f). \tag{5.2.B}$$

Thus the data of $N_{ij}(f)$ is equivalent to the data of $\mathrm{GV}_{ij}(\pi)$ in the sense of 3.3.8 and 3.3.9.

So in §5.3 and §5.4, we discuss generalised GV invariants of monomialized Type A potentials to reach conclusions about GV invariants of crepant resolutions of cA_n singularities.

Recall that, in order to define $N_{ij}(f)$ in 5.2.1, we first fix some integer s and set $g_s = y$, $g_{s+1} = x$, then solve to give $g_0, g_1, \dots, g_{2n}$ that satisfy (5.2.A). From this, $N_{ij}(f) = \dim_{\mathbb{C}} \mathbb{C}[[x,y]]/(g_{2i-2}, g_{2j})$.

Lemma 5.2.3. *The generalised GV invariant $N_{ij}(f)$ in 5.2.1 does not depend on s.*

Proof. We start with s, set $g_s = y$, $g_{s+1} = x$, then solve to obtain $g_0, g_1, \dots, g_{2n}$. From this, the above constructs $\mathcal{R}$, π such that $\Lambda_{\mathrm{con}}(\pi) \xrightarrow{\sim} \mathcal{J}\mathrm{ac}(f)$.

We next start with another integer t and set $g'_t = y$, $g'_{t+1} = x$, then solve to obtain $g'_0, g'_1, \dots, g'_{2n}$. Similarly, the above constructs $\mathcal{R}'$, π' such that $\Lambda_{\mathrm{con}}(\pi') \xrightarrow{\sim} \mathcal{J}\mathrm{ac}(f)$. Thus $\Lambda_{\mathrm{con}}(\pi) \xrightarrow{\sim} \Lambda_{\mathrm{con}}(\pi')$, and so $N_{ij}(\pi) = N_{ij}(\pi')$ by 3.2.7. In particular

$$\dim_{\mathbb{C}} \mathbb{C}[[x,y]]/(g_{2i-2}, g_{2j}) = N_{ij}(\pi) = N_{ij}(\pi') = \dim_{\mathbb{C}} \mathbb{C}[[x,y]]/(g'_{2i-2}, g'_{2j}),$$

and so $N_{ij}(f)$ does not depend on s. □

§ 5.3 | Filtrations

In this section, we give filtration structures of the parameter space of monomialized Type A potentials on Q_n with respect to generalised GV invariants.

§ 5.3.1 | Filtration sturctures

Fix some $\mathbf{p}$ and consider the obvious bijection map $f\colon \mathsf{M}_{\mathbf{p}} \to \mathsf{MA}_{\mathbf{p}}$ under which

$$f(\kappa_{\mathbf{p}}) = \sum_{i=1}^{2n-2} \mathsf{x}'_i \mathsf{x}_{i+1} + \sum_{i=1}^{2n-1} \sum_{j=2}^{\infty} \kappa_{ij} \mathsf{x}_i^j, \tag{5.3.A}$$

where $\kappa_{i,j_i} = 0$ for $1 \leq i \leq 2n-1$ and $2 \leq j_i < p_i$.

By considering κ_{ij} as variables and solving the system of equations (5.2.A), we can also realize the family of monomialized Type A potentials $f(\kappa_{\mathbf{p}})$ over $\mathsf{M}_{\mathbf{p}}$ (5.1.E) by a family of crepant resolutions of cA_n singularities over $\mathsf{M}_{\mathbf{p}}$. More precisely, fix some s satisfying $0 \leq s \leq 2n-1$, and set $g_s = y$, $g_{s+1} = x$, then solve $g_0, g_1, \dots, g_{2n}$ by (5.2.A) where each $g_t \in (\kappa_{\mathbf{p}}, x, y) \subseteq \mathbb{C}[[\kappa_{\mathbf{p}}, x, y]]$.

For any $k \in \mathsf{M}_{\mathbf{p}}$, write $g_t(k) \in \mathbb{C}[[x, y]]$ for g_t evaluated at k, and consider the cA_n singularity

$$\mathcal{R}_k := \frac{\mathbb{C}[[u, v, x, y]]}{uv - g_0(k)g_2(k)\dots g_{2n}(k)},$$

and the $\mathcal{R}_k$-module

$$M_k := \mathcal{R} \oplus (u, g_0(k)) \oplus (u, g_0(k)g_2(k)) \oplus \dots \oplus (u, \prod_{i=0}^{n-1} g_{2i}(k)) \in (\mathrm{MM}\,\mathcal{R}_k) \cap (\mathrm{CM}\,\mathcal{R}_k).$$

Similar to §5.2, $\mathcal{J}\mathrm{ac}(f(k)) \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathcal{R}_k}(M_k) \xrightarrow{\sim} \Lambda_{\mathrm{con}}(\pi_k)$. Thus if we vary k over the parameter space $\mathsf{M}_{\mathbf{p}}$, the family of crepant resolutions π_k realizes $f(\kappa_{\mathbf{p}})$.

Recall that in the above construction, we first fix some integer s satisfying $0 \leq s \leq 2n-1$, then construct $g_0, g_1, \dots, g_{2n}$ with $g_s = y$ and $g_{s+1} = x$ to realize $f(\kappa_{\mathbf{p}})$.

Notation 5.3.1. With the fixed s as above, we adopt the following notation in 5.3.2.

(1) Set $(g_{s0}, g_{s1}, \dots, g_{s,2n}) := (g_0, g_1, \dots, g_{2n})$.

(2) For $0 \leq t \leq 2n$, set $h_{st} := g_{st}(\kappa_{\mathbf{p}}, x, 0) \in \mathbb{C}[[\kappa_{\mathbf{p}}, x]]$.

(3) Give any $h \in \mathbb{C}[[\kappa_{\mathbf{p}}, x]]$, write $[h]_i$ for the degree i graded piece with respect to x.

(4) Write $\mathcal{O}_d$ for a element in $\mathbb{C}[[\kappa_{\mathbf{p}}, x]]$ that satisfies $[\mathcal{O}_d]_i = 0$ for each $i < d$.

(5) For $1 \leq t \leq 2n-1$, write $\kappa_{t,\mathbf{p}}$ for the tuple of *variables* κ_{ij_i} for $1 \leq i \leq t$ and $p_i \leq j_i \leq \infty$.

For $0 \leq s \leq 2n-1$, since $g_{ss} = y$, for any t we have $(g_{ss}, g_{st}) = (y, g_{st}) = (h_{st})$. Thus

$$\begin{aligned} N_{ij}(f(\kappa_{\mathbf{p}})) &= \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(g_{2i-2,2i-2}, g_{2i-2,2j})} && \text{(by 5.2.3 with } s = 2i-2) \\ &= \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(y, g_{2i-2,2j})} \\ &= \dim_{\mathbb{C}} \frac{\mathbb{C}[[x]]}{(h_{2i-2,2j})}. \end{aligned} \tag{5.3.B}$$

So $h_{2i-2,2j}$ determines the generalised GV invariant $N_{ij}(f(\kappa_{\mathbf{p}}))$. In particular, the lowest degree term (wrt. x) of $h_{2i-2,2j}$ determines the general value and general position of $N_{ij}(f(\kappa_{\mathbf{p}}))$ over the parameter space $\mathsf{M}_{\mathbf{p}}$. The following establishes that the lowest degree term can be described by the matrix $A^{d}_{2i-1,2j-1}(\kappa_{\mathbf{p}})$ where $d = d_{2i-1,2j-1}(\mathbf{p})$ in 5.1.5.

Proposition 5.3.2. *Given the monomialized Type A potentials $f(\kappa_{\mathbf{p}})$ (5.3.A) on Q_n and with notation in 5.3.1, for any $1 \leq s \leq t \leq 2n-1$, we have*

$$h_{s-1,t+1} = \sum_{i=r}^{\infty} c_i x^i$$

for some $1 \leq r \in \mathbb{N}_\infty$ and each $c_i \in \mathbb{C}[[\kappa_{\mathbf{p}}]]$. Moreover, the following hold.

1. *If $d_{st}(\mathbf{p}) = \infty$, then $h_{s-1,t+1} = 0$.*
2. *If $\mathsf{d} := d_{st}(\mathbf{p}) < \infty$, then $r = \mathsf{d} - 1$, and the lowest degree term (wrt. x) in $h_{s-1,t+1}$ has coefficient $c_r = (-1)^{t-s+1} \det A_{st}^{\mathsf{d}}(\kappa_{\mathbf{p}})$.*

Proof. Since $h_{s-1,t+1} \in \mathbb{C}[[\kappa_{\mathbf{p}}, x]]$, we first write $h_{s-1,t+1}$ as

$$h_{s-1,t+1} = \sum_{i=r_{st}}^{\infty} c_{st,i} x^i = \lambda_{st} x^{r_{st}} + \mathcal{O}_{r_{st}+1}, \tag{5.3.C}$$

for some $r_{st} \geq 0$, each $c_{st,i} \in \mathbb{C}[[\kappa_{\mathbf{p}}]]$ and $\lambda_{st} := c_{st,r_{st}}$. Now since the h's are obtained from the g's by evaluating at $y = 0$, they must satisfy the same relations as the g's. In particular, by (5.2.A),

$$h_{s-1,t-1} + \sum_{j=p_t}^{\infty} j\kappa_{tj} h_{s-1,t}^{j-1} + h_{s-1,t+1} = 0. \tag{5.3.D}$$

In the equation above, the index j starts at p_t because $\kappa_{tj} = 0$ for $j < p_t$ in $f(\kappa_{\mathbf{p}})$ (5.3.A). Rearranging (5.3.D) in the case $t = s$, then using the fact that $g_{s-1,s-1} = y$, $g_{s-1,s} = x$ (thus $h_{s-1,s-1} = 0$, $h_{s-1,s} = x$), we obtain

$$h_{s-1,s+1} = -h_{s-1,s-1} - \sum_{j=p_s}^{\infty} j\kappa_{sj} h_{s-1,s}^{j-1} = -\sum_{j=p_s}^{\infty} j\kappa_{sj} x^{j-1}. \tag{5.3.E}$$

Next, rearranging (5.3.D) in the case $t = s+1$ gives

$$\begin{aligned} h_{s-1,s+2} &= -h_{s-1,s} - \sum_{j=p_{s+1}}^{\infty} j\kappa_{s+1,j} h_{s-1,s+1}^{j-1} \\ &= -x - \sum_{j=p_{s+1}}^{\infty} j\kappa_{s+1,j} h_{s-1,s+1}^{j-1}. \end{aligned} \tag{5.3.F}$$

In the double index of $h_{s-1,*}$, we now induct on the second of the two indices to prove the result. We split the remainder of the proof into the following four lemmas (5.3.3, 5.3.4, 5.3.5 and 5.3.6). ∎

Lemma 5.3.3. *With notation in 5.3.2, if $d_{st}(\mathbf{p}) = \infty$, then $h_{s-1,t+1} = 0$.*

Proof. If $d_{st}(\mathbf{p}) = \infty$, then by (5.1.F) $t - s$ is even and $\kappa_{sj}, \kappa_{s+2,j}, \ldots, \kappa_{tj} = 0$ for all j. In particular, $h_{s-1,s+1} = 0$ via (5.3.E). Substituting this into (5.3.F), $h_{s-1,s+2} = -x$. Next, rearranging (5.3.D) in the case $t = s+2$ gives

$$h_{s-1,s+3} = -h_{s-1,s+1} - \sum_{j=p_{s+2}}^{\infty} j\kappa_{s+2,j} h_{s-1,s+2}^{j-1}.$$

Since $h_{s-1,s+1} = 0$ and $\kappa_{s+2,j} = 0$ for all j, necessarily $h_{s-1,s+3} = 0$. Repeating the same argument gives $h_{s-1,s+5}, h_{s-1,s+7}, \ldots, h_{s-1,t+1} = 0$. □

Lemma 5.3.4. *With notation in 5.3.1 and 5.3.2, for $s \le t \le 2n-1$, $h_{s-1,t+1} \in \mathbb{C}[[\kappa_{t,\mathbf{p}}, x]]$, and in particular the lowest degree (wrt. x) coefficient λ_{st} in $h_{s-1,t+1}$ (5.3.C) belongs to $\mathbb{C}[[\kappa_{t,\mathbf{p}}]]$.*

Proof. We first check that $h_{s-1,s+1}$ and $h_{s-1,s+2}$ satisfy the statement. By (5.3.E), it is straightforward that $h_{s-1,s+1} \in \mathbb{C}[[\kappa_{s,\mathbf{p}}, x]]$. Then together with (5.3.F), it follows that $h_{s-1,s+2} \in \mathbb{C}[[\kappa_{s+1,\mathbf{p}}, x]]$.

We next prove the statement by induction on the second index: we assume that $h_{s-1,t-1} \in \mathbb{C}[[\kappa_{t-2,\mathbf{p}}, x]]$ and $h_{s-1,t} \in \mathbb{C}[[\kappa_{t-1,\mathbf{p}}, x]]$ for some $t \ge s+2$, and prove that $h_{s-1,t+1} \in \mathbb{C}[[\kappa_{t,\mathbf{p}}, x]]$. This is straightforward by (5.3.D). □

Lemma 5.3.5. *With notation in 5.3.2, if $\mathsf{d} := d_{st}(\mathbf{p}) < \infty$, then $r_{st} = \mathsf{d} - 1$.*

Proof. We first check that r_{ss} and $r_{s,s+1}$ satisfy the statement. By (5.1.F), $d_{ss}(\mathbf{p}) = p_s$ and $d_{s,s+1}(\mathbf{p}) = 2$. By (5.3.E),

$$h_{s-1,s+1} = -\sum_{j=p_s}^{\infty} j\kappa_{sj} x^{j-1}$$

This has lowest degree term x^{p_s-1}, and thus by definition $r_{ss} = p_s - 1 = d_{ss}(\mathbf{p}) - 1$. Similarly, since each $j\kappa_{s+1,j} h_{s-1,s+1}^{j-1}$ in (5.3.F) contains $\kappa_{s+1,j}$, these terms can not cancel the $-x$ in (5.3.F). Thus the lowest degree of of $h_{s-1,s+2}$ is one, and so $r_{s,s+1} = 1 = d_{s,s+1}(\mathbf{p}) - 1$.

We next prove the statement by induction on the second index: we assume that $r_{s,t-2} = d_{s,t-2}(\mathbf{p}) - 1$ and $r_{s,t-1} = d_{s,t-1}(\mathbf{p}) - 1$ for some $t \ge s+2$, and prove that $r_{st} = d_{st}(\mathbf{p}) - 1$ by splitting into the following two cases.

(1) $t - s$ is odd.

Since $t-s$ is odd, $d_{s,t-2}(\mathbf{p}) = d_{st}(\mathbf{p}) = 2$ by (5.1.F). By assumption $r_{s,t-1} = d_{s,t-1}(\mathbf{p}) - 1$ and $r_{s,t-2} = d_{s,t-2}(\mathbf{p}) - 1 = 1$. Thus by (5.3.C) (applied to $t-2$ and $t-1$),

$$h_{s-1,t-1} = \lambda_{s,t-2} x + \mathcal{O}_2, \quad h_{s-1,t} = \lambda_{s,t-1} x^{d_{s,t-1}-1} + \mathcal{O}_{d_{s,t-1}},$$

where $\lambda_{s,t-2}$, $\lambda_{s,t-1} \neq 0$ by assumption. Thus by (5.3.D), in order to give the lowest degree r_{st} of $h_{s-1,t+1}$, we only need to consider the lowest degree term of $h_{s-1,t-1}$ (namely $\lambda_{s,t-2}x$) and $\sum_{j=p_t}^{\infty} j\kappa_{tj} h_{s-1,t}^{j-1}$.

Since by 5.3.4 $\lambda_{s,t-2} \in \mathbb{C}[[\kappa_{t-2,\mathbf{p}}]]$ and each $j\kappa_{tj}h_{s-1,t}^{j-1}$ contains κ_{tj}, $\lambda_{s,t-2}x$ can not be canceled by $\sum_{j=p_t}^{\infty} j\kappa_{tj}h_{s-1,t}^{j-1}$, and so the lowest degree r_{st} of $h_{s-1,t+1}$ is one. Since $d_{st}(\mathbf{p}) = 2$, $r_{st} = 1 = d_{st}(\mathbf{p}) - 1$.

(2) $t - s$ is even.

Since $t - s$ is even, $d_{s,t-1}(\mathbf{p}) = 2$ by (5.1.F). By assumption $r_{s,t-1} = d_{s,t-1}(\mathbf{p}) - 1 = 1$ and $r_{s,t-2} = d_{s,t-2}(\mathbf{p}) - 1$. Thus again by (5.3.C) (applied to $t-2$ and $t-1$),

$$h_{s-1,t-1} = \lambda_{s,t-2}x^{d_{s,t-2}-1} + \mathcal{O}_{d_{s,t-2}}, \quad h_{s-1,t} = \lambda_{s,t-1}x + \mathcal{O}_2,$$

where $\lambda_{s,t-2}$, $\lambda_{s,t-1} \neq 0$ by assumption. Thus by (5.3.D), in order to give the lowest degree r_{st} of $h_{s-1,t+1}$, we only need to consider the lowest degree term of $h_{s-1,t-1}$ (namely $\lambda_{s,t-2}x^{d_{s,t-2}-1}$) and $\sum_{j=p_t}^{\infty} j\kappa_{tj}h_{s-1,t}^{j-1}$ (namely $p_t\kappa_{t,p_t}(\lambda_{s,t-1}x)^{p_t-1}$).

Since by 5.3.4 $\lambda_{s,t-2} \in \mathbb{C}[[\kappa_{t-2,\mathbf{p}}]]$, and $p_t\kappa_{t,p_t}(\lambda_{s,t-1}x)^{p_t-1}$ contains κ_{t,p_t}, it follows that $\lambda_{s,t-2}x^{d_{s,t-2}-1}$ and $p_t\kappa_{t,p_t}(\lambda_{s,t-1}x)^{p_t-1}$ can not cancel each other. Thus the lowest degree r_{st} of $h_{s-1,t+1}$ is $\min(d_{s,t-2}(\mathbf{p}) - 1, p_t - 1)$. Since $d_{st}(\mathbf{p}) = \min(d_{s,t-2}(\mathbf{p}), p_t)$ by (5.1.F), $r_{st} = d_{st}(\mathbf{p}) - 1$. □

Lemma 5.3.6. *With notation in 5.3.2, if* $\mathsf{d} := d_{st}(\mathbf{p}) < \infty$*, then the lowest degree (wrt.* x*) coefficient in* $h_{s-1,t+1}$ *(5.3.C) is* $\lambda_{st} = (-1)^{t-s+1} \det A_{st}^{\mathsf{d}}(\kappa_{\mathbf{p}})$.

Proof. To ease notation, for any i, j, d in 5.1.2 we write d_{ij} and A_{ij}^d for $d_{ij}(\mathbf{p})$ and $A_{ij}^d(\kappa_{\mathbf{p}})$ respectively in the following proof.

We first prove that the statement holds for $t = s$. By (5.3.E), the lowest degree coefficient in $h_{s-1,s+1}$ is $-p_s\kappa_{s,p_s}$, thus

$$\begin{aligned} \lambda_{ss} &= -p_s\kappa_{s,p_s} \\ &= -d_{ss}\kappa_{s,d_{ss}} && \text{(since } p_s = d_{ss} \text{ by (5.1.F))} \\ &= -\det A_{ss}^{d_{ss}}. && \text{(since } \det A_{ss}^d = d\kappa_{sd} \text{ for any } d \text{ by 5.1.4)} \end{aligned}$$

We next prove that the statement holds for $t = s + 1$. Indeed,

$$\begin{aligned} h_{s-1,s+2} &= -h_{s-1,s} - \sum_{j=p_{s+1}}^{\infty} j\kappa_{s+1,j}h_{s-1,s+1}^{j-1} && \text{(by (5.3.F))} \\ &= -x - \sum_{j=p_{s+1}}^{\infty} j\kappa_{s+1,j}(\lambda_{ss}x^{r_{ss}} + \mathcal{O}_{r_{ss}+1})^{j-1} && \text{(since } h_{s-1,s} = x\text{, and (5.3.C))} \\ &= -x - \sum_{j=p_{s+1}}^{\infty} j\kappa_{s+1,j}(-p_s\kappa_{s,p_s}x^{r_{ss}} + \mathcal{O}_{r_{ss}+1})^{j-1} && (\lambda_{ss} = -p_s\kappa_{s,p_s}) \\ &= -x - \sum_{j=p_{s+1}}^{\infty} j\kappa_{s+1,j}(-p_s\kappa_{s,p_s}x^{p_s-1} + \mathcal{O}_{p_s})^{j-1} && (r_{ss} = d_{ss} - 1 = p_s - 1 \text{ by 5.3.5)} \\ &= -x + (-1)^{p_{s+1}}p_{s+1}\kappa_{s+1,p_{s+1}}(p_s\kappa_{s,p_s})^{p_{s+1}-1}x^{(p_s-1)(p_{s+1}-1)} + \mathcal{O}_{(p_s-1)(p_{s+1}-1)}. \end{aligned}$$

If $p_s = p_{s+1} = 2$, then $(4\kappa_{s,2}\kappa_{s+1,2} - 1)x$ is the lowest degree term in $h_{s-1,s+2}$, thus

$$\begin{aligned}
\lambda_{s,s+1} &= 4\kappa_{s,2}\kappa_{s+1,2} - 1 \\
&= \det A^2_{s,s+1} && \text{(since } \det A^2_{s,s+1} = 4\kappa_{s,2}\kappa_{s+1,2} - 1 \text{ by 5.1.4)} \\
&= \det A^{d_{s,s+1}}_{s,s+1}. && \text{(since } d_{s,s+1} = 2 \text{ by (5.1.F))}
\end{aligned}$$

Otherwise, if $p_s > 2$ or $p_{s+1} > 2$, then $-x$ is the lowest degree term in $h_{s-1,s+2}$ and by (5.3.A) $\kappa_{s,2} = 0$ or $\kappa_{s+1,2} = 0$. Thus

$$\begin{aligned}
\lambda_{s,s+1} &= -1 \\
&= 4\kappa_{s,2}\kappa_{s+1,2} - 1 && \text{(since } \kappa_{s,2} = 0 \text{ or } \kappa_{s+1,2} = 0) \\
&= \det A^2_{s,s+1} && \text{(since } \det A^2_{s,s+1} = 4\kappa_{s,2}\kappa_{s+1,2} - 1 \text{ by 5.1.4)} \\
&= \det A^{d_{s,s+1}}_{s,s+1}. && \text{(since } d_{s,s+1} = 2 \text{ by (5.1.F))}
\end{aligned}$$

We next prove the statement by induction on the second index. Fix some t satisfying $t \geq s+2$. We assume that $\lambda_{s,t-2} = (-1)^{t-s-1} \det A^{d_{s,t-2}}_{s,t-2}$ and $\lambda_{s,t-1} = (-1)^{t-s} \det A^{d_{s,t-1}}_{s,t-1}$, and prove that $\lambda_{st} = (-1)^{t-s+1} \det A^{d_{st}}_{st}$ by splitting into the following cases.

By (5.3.D), for any integer $d \geq 1$, we have

$$[h_{s-1,t-1}]_d + [\sum_{j=p_t}^{\infty} j\kappa_{tj} h^{j-1}_{s-1,t}]_d + [h_{s-1,t+1}]_d = 0, \tag{5.3.G}$$

where $[h]_d$ denotes the degree (wrt. x) d graded piece of h (see 5.3.1).

(1) $t - s$ is odd.

Since $t - s$ is odd, by (5.1.F) $d_{s,t-2} = d_{s,t} = 2$. Thus by 5.3.5, $r_{s,t-2} = r_{st} = 1$ and $r_{s,t-1} = d_{s,t-1} - 1$. So by (5.3.C),

$$\begin{aligned}
h_{s-1,t-1} &= \lambda_{s,t-2}x + \mathcal{O}_2, \\
h_{s-1,t} &= \lambda_{s,t-1}x^{r_{s,t-1}} + \mathcal{O}_{r_{s,t-1}+1} = \lambda_{s,t-1}x^{d_{s,t-1}-1} + \mathcal{O}_{d_{s,t-1}}, \\
h_{s-1,t+1} &= \lambda_{st}x + \mathcal{O}_2.
\end{aligned}$$

Thus the lowest degree of the terms in (5.3.D) is one. We then consider these lowest degree terms, thus set $d = 1$ in (5.3.G), which gives

$$\lambda_{s,t-2}x + [p_t\kappa_{t,p_t}(\lambda_{s,t-1}x^{d_{s,t-1}-1})^{p_t-1}]_1 + \lambda_{st}x = 0. \tag{5.3.H}$$

Since $t - s$ is odd, the inductive assumption becomes $\lambda_{s,t-2} = \det A^2_{s,t-2}$ and $\lambda_{s,t-1} = -\det A^{d_{s,t-1}}_{s,t-1}$. We need to prove that $\lambda_{st} = \det A^2_{st}$. We again split into subcases.

(1.1) $t - s$ is odd and $p_t > 2$.

Since $p_t > 2$, $\varepsilon_{t2}(\kappa_{\mathbf{p}}) = 2\kappa_{t2} = 0$ by 5.1.6(4) and $[p_t\kappa_{t,p_t}(\lambda_{s,t-1}x^{d_{s,t-1}-1})^{p_t-1}]_1 = 0$. To ease notation, we write ε_{t2} for $\varepsilon_{t2}(\kappa_{\mathbf{p}})$ in the following. Thus

$$\begin{aligned}
\lambda_{st} &= -\lambda_{s,t-2} && \text{(by (5.3.H) and } [p_t\kappa_{t,p_t}(\lambda_{s,t-1}x^{d_{s,t-1}-1})^{p_t-1}]_1 = 0) \\
&= -\det A^2_{s,t-2} && \text{(by assumption)} \\
&= \det A^2_{st} - \varepsilon_{t2}\det A^2_{s,t-1} && \text{(by 5.1.7)} \\
&= \det A^2_{st}. && \text{(since } \varepsilon_{t2} = 0)
\end{aligned}$$

(1.2) $t-s$ is odd, $p_t = 2$ and $d_{s,t-1} > 2$.

Since $d_{s,t-1} > 2$, $[p_t\kappa_{t,p_t}(\lambda_{s,t-1}x^{d_{s,t-1}-1})^{p_t-1}]_1 = 0$ and by 5.1.11 $\det A^2_{s,t-1} = 0$. Thus

$$\begin{aligned}
\lambda_{st} &= -\lambda_{s,t-2} && \text{(by (5.3.H) and } [p_t\kappa_{t,p_t}(\lambda_{s,t-1}x^{d_{s,t-1}-1})^{p_t-1}]_1 = 0) \\
&= -\det A^2_{s,t-2} && \text{(by assumption)} \\
&= \det A^2_{st} - \varepsilon_{t2}\det A^2_{s,t-1} && \text{(by 5.1.7)} \\
&= \det A^2_{st}. && \text{(since } \det A^2_{s,t-1} = 0)
\end{aligned}$$

(1.3) $t-s$ is odd, $p_t = 2$ and $d_{s,t-1} = 2$.

Since $p_t = 2$ and $d_{s,t-1} = 2$, $[p_t\kappa_{t,p_t}(\lambda_{s,t-1}x^{d_{s,t-1}-1})^{p_t-1}]_1 = 2\kappa_{t2}\lambda_{s,t-1}x$. Thus

$$\begin{aligned}
\lambda_{st} &= -2\kappa_{t2}\lambda_{s,t-1} - \lambda_{s,t-2} && \text{(by (5.3.H) and } [p_t\kappa_{t,p_t}(\lambda_{s,t-1}x^{d_{s,t-1}-1})^{p_t-1}]_1 = 2\kappa_{t2}\lambda_{s,t-1}x) \\
&= \varepsilon_{t2}\det A^{d_{s,t-1}}_{s,t-1} - \det A^2_{s,t-2} && \text{(by assumption and } \varepsilon_{t2} = 2\kappa_{t2}) \\
&= \varepsilon_{t2}\det A^2_{s,t-1} - \det A^2_{s,t-2} && \text{(since } d_{s,t-1} = 2) \\
&= \det A^2_{st}. && \text{(by 5.1.7)}
\end{aligned}$$

(2) $t-s$ is even.

Since $t-s$ is even, then $d_{s,t-1} = 2$ by (5.1.F). Thus by 5.3.5, $r_{s,t-1} = 1$, $r_{s,t-2} = d_{s,t-2} - 1$ and $r_{st} = d_{st} - 1$. So by (5.3.C),

$$\begin{aligned}
h_{s-1,t-1} &= \lambda_{s,t-2}x^{r_{s,t-2}} + \mathcal{O}_{r_{s,t-2}+1} = \lambda_{s,t-2}x^{d_{s,t-2}-1} + \mathcal{O}_{d_{s,t-2}}, \\
h_{s-1,t} &= \lambda_{s,t-1}x + \mathcal{O}_2, \\
h_{s-1,t+1} &= \lambda_{st}x^{r_{st}} + \mathcal{O}_{r_{st}+1} = \lambda_{st}x^{d_{st}-1} + \mathcal{O}_{d_{st}}.
\end{aligned}$$

Since by (5.1.F) $d_{s,t-2} \geq d_{st}$ and $p_t \geq d_{st}$, the lowest degree of $h_{s-1,t-1}$ and $(h_{s-1,t})^{p_t-1}$ is greater than or equal to that of $h_{s-1,t+1}$. Thus the lowest degree of the terms in (5.3.D) is $d_{st}-1$. We then consider these lowest degree terms, thus set $d = d_{st}-1$ in (5.3.G), which gives

$$[\lambda_{s,t-2}x^{d_{s,t-2}-1}]_{d_{st}-1} + [p_t\kappa_{t,p_t}(\lambda_{s,t-1}x)^{p_t-1}]_{d_{st}-1} + \lambda_{st}x^{d_{st}-1} = 0. \tag{5.3.I}$$

Since $t-s$ is even, the inductive assumption now becomes $\lambda_{s,t-2} = -\det A^{d_{s,t-2}}_{s,t-2}$ and $\lambda_{s,t-1} =$

$\det A^2_{s,t-1}$. We prove $\lambda_{st} = -\det A^{d_{st}}_{st}$ by splitting into the following subcases.

(2.1) $t-s$ is even and $p_t < d_{s,t-2}$.

Since $p_t < d_{s,t-2}$, by (5.1.F) $p_t = d_{st} < d_{s,t-2}$, and so $[\lambda_{s,t-2}x^{d_{s,t-2}-1}]_{d_{st}-1} = 0$. Thus by (5.3.I), it follows that

$$\lambda_{st} = -p_t\kappa_{t,p_t}\lambda^{p_t-1}_{s,t-1}.$$

Now, since $d_{st} < d_{s,t-2}$, by 5.1.11 $\det A^{d_{st}}_{s,t-2} = 0$. If furthermore $p_t = d_{st} = 2$, then

$$\begin{aligned}
\lambda_{st} &= -2\kappa_{t2}\lambda_{s,t-1} && \text{(since } p_t = 2)\\
&= -2\kappa_{t2}\det A^2_{s,t-1} && \text{(by assumption)}\\
&= -\varepsilon_{t2}\det A^2_{s,t-1} + \det A^2_{s,t-2} && \text{(since } \varepsilon_{t2} = 2\kappa_{t2},\ \det A^{d_{st}}_{s,t-2} = 0 \text{ and } d_{st} = 2)\\
&= -\det A^2_{st} && \text{(by 5.1.7)}\\
&= -\det A^{d_{st}}_{st}. && \text{(since } d_{st} = 2)
\end{aligned}$$

Otherwise, $p_t = d_{st} > 2$, and then by 5.1.10 $\det A^2_{s,t-1} = (-1)^{(t-s)/2}$, and so

$$\begin{aligned}
\lambda_{st} &= -p_t\kappa_{t,p_t}\lambda^{p_t-1}_{s,t-1}\\
&= -p_t\kappa_{t,p_t}(\det A^2_{s,t-1})^{p_t-1} && \text{(by assumption)}\\
&= -d_{st}\kappa_{t,d_{st}}(-1)^{(t-s)(d_{st}-1)/2} && \text{(since } \det A^2_{s,t-1} = (-1)^{(t-s)/2} \text{ and } p_t = d_{st})\\
&= -(-1)^{(t-s)(d_{st}-1)/2}\varepsilon_{t,d_{st}} + \det A^{d_{st}}_{s,t-2} && \text{(since } \varepsilon_{t,d_{st}} = d_{st}\kappa_{t,d_{st}} \text{ and } \det A^{d_{st}}_{s,t-2} = 0)\\
&= -\det A^{d_{st}}_{st}. && \text{(by 5.1.7)}
\end{aligned}$$

(2.2) $t-s$ is even and $p_t > d_{s,t-2}$.

Since $p_t > d_{s,t-2}$, by (5.1.F) $p_t > d_{s,t-2} = d_{st}$, and thus $[p_t\kappa_{t,p_t}(\lambda_{s,t-1}x)^{p_t-1}]_{d_{st}-1} = 0$. Hence by (5.3.I), it follows that

$$\lambda_{st} = -\lambda_{s,t-2}.$$

Since $p_t > d_{s,t}$, by 5.1.6(4) $\varepsilon_{t,d_{st}}(\kappa_{\mathbf{p}}) = d_{st}\kappa_{t,d_{st}} = 0$. If furthermore $d_{s,t-2} = d_{st} = 2$, then

$$\begin{aligned}
\lambda_{st} &= -\lambda_{s,t-2}\\
&= \det A^{d_{s,t-2}}_{s,t-2} && \text{(by assumption)}\\
&= \det A^2_{s,t-2} && \text{(since } d_{s,t-2} = 2)\\
&= -\varepsilon_{t2}\det A^2_{s,t-1} + \det A^2_{s,t-2} && \text{(since } \varepsilon_{t,d_{st}} = 0 \text{ and } d_{st} = 2)\\
&= -\det A^2_{st} && \text{(by 5.1.7)}\\
&= -\det A^{d_{st}}_{st}. && \text{(since } d_{st} = 2)
\end{aligned}$$

Otherwise, $d_{s,t-2} = d_{st} > 2$, and then

$$\begin{aligned}
\lambda_{st} &= -\lambda_{s,t-2} \\
&= \det A_{s,t-2}^{d_{s,t-2}} && \text{(by assumption)} \\
&= \det A_{s,t-2}^{d_{st}} && \text{(since } d_{s,t-2} = d_{st}) \\
&= -(-1)^{(t-s)(d_{st}-1)/2}\varepsilon_{t,d_{st}} + \det A_{s,t-2}^{d_{st}} && \text{(since } \varepsilon_{t,d_{st}} = 0) \\
&= -\det A_{st}^{d_{st}}. && \text{(by 5.1.7)}
\end{aligned}$$

(2.3) $t - s$ is even and $p_t = d_{s,t-2}$.

Since $p_t = d_{s,t-2}$, by (5.1.F) $p_t = d_{s,t-2} = d_{st}$. Thus by (5.3.I)

$$\lambda_{st} = -\lambda_{s,t-2} - p_t\kappa_{t,p_t}(\lambda_{s,t-1})^{p_t-1}.$$

If furthermore $p_t = d_{s,t-2} = d_{st} = 2$, then

$$\begin{aligned}
\lambda_{st} &= -\lambda_{s,t-2} - 2\kappa_{t2}\lambda_{s,t-1} && \text{(since } p_t = 2) \\
&= \det A_{s,t-2}^2 - 2\kappa_{t2}\det A_{s,t-1}^2 && \text{(by assumption and } d_{s,t-2} = 2) \\
&= \det A_{s,t-2}^2 - \varepsilon_{t2}\det A_{s,t-1}^2 && \text{(since } \varepsilon_{t2} = 2\kappa_{t2}) \\
&= -\det A_{st}^2 && \text{(by 5.1.7)} \\
&= -\det A_{st}^{d_{st}}. && \text{(since } d_{st} = 2)
\end{aligned}$$

Otherwise, $p_t = d_{s,t-2} = d_{st} > 2$. But then by 5.1.10 $\det A_{s,t-1}^2 = (-1)^{(t-s)/2}$, and so

$$\begin{aligned}
\lambda_{st} &= -\lambda_{s,t-2} - p_t\kappa_{t,p_t}(\lambda_{s,t-1})^{p_t-1} \\
&= \det A_{s,t-2}^{d_{s,t-2}} - p_t\kappa_{t,p_t}(\det A_{s,t-1}^2)^{p_t-1} && \text{(by assumption)} \\
&= \det A_{s,t-2}^{d_{st}} - d_{st}\kappa_{t,d_{st}}(-1)^{(t-s)(d_{st}-1)/2} \\
& && \text{(since } \det A_{s,t-1}^2 = (-1)^{(t-s)/2} \text{ and } p_t = d_{s,t-2} = d_{st}) \\
&= \det A_{s,t-2}^{d_{st}} - (-1)^{(t-s)(d_{st}-1)/2}\varepsilon_{t,d_{st}} && \text{(since } \varepsilon_{t,d_{st}} = d_{st}\kappa_{t,d_{st}}) \\
&= -\det A_{st}^{d_{st}}. && \text{(by 5.1.7)}
\end{aligned}$$

So by induction $\lambda_{st} = (-1)^{t-s+1}\det A_{st}^{d_{st}}$ for any $1 \le s \le t \le 2n-1$. □

We next fix $\mathbf{p}$ and curve class $\mathrm{C}_s + \mathrm{C}_{s+1} + \cdots + \mathrm{C}_t$, and from this data construct a filtration structure of $\mathsf{M}_{\mathbf{p}}$, which is the main result of this section. Recall that $\mathsf{M}_{\mathbf{p}}$ is the parameter space of monomialized Type A potentials $f(\kappa_{\mathbf{p}})$ (5.1.D), namely

$$f(\kappa_{\mathbf{p}}) = \sum_{i=1}^{2n-2} \mathsf{x}_i'\mathsf{x}_{i+1} + \sum_{i=1}^{2n-1}\sum_{j=2}^{\infty}\kappa_{ij}\mathsf{x}_i^j, \text{ where } \kappa_{i,j_i} = 0 \text{ for } 1 \le i \le 2n-1, 2 \le j_i < p_i,$$

$$\mathsf{M}_{\mathbf{p}} = \{(k_{12}, k_{13}, \ldots, k_{2n-1,2}, k_{2n-1,3}, \ldots) \mid k_{i,j_i} = 0 \text{ for } 1 \le i \le 2n-1, 2 \le j_i < p_i\}.$$

Recall the notation $d_{ij}(\mathbf{p})$ and $A^d_{ij}(\kappa_{\mathbf{p}})$ in 5.1.5.

Theorem 5.3.7. *Fix* $\mathbf{p}$*, and some* s*,* t *satisfying* $1 \le s \le t \le n$*. If* $d_{2s-1,2t-1}(\mathbf{p})$ *is finite, then* $\mathsf{M}_{\mathbf{p}}$ *has a filtration structure* $\mathsf{M}_{\mathbf{p}} = M_1 \supsetneq M_2 \supsetneq M_3 \supsetneq \cdots$ *such that*

(1) *For each* $i \ge 1$*,* $N_{st}(f(k)) = d_{2s-1,2t-1}(\mathbf{p}) + i - 2$ *for all* $k \in M_i \backslash M_{i+1}$*.*

(2) *Each* M_i *is the zero locus of some polynomial system of* $\kappa_{\mathbf{p}}$*, and moreover*

$$M_2 = \{k \in \mathsf{M}_{\mathbf{p}} \mid \det A^d_{2s-1,2t-1}(f(k)) = 0 \text{ where } d = d_{2s-1,2t-1}(\mathbf{p})\}.$$

(3) *If* $s = t$*, then for each* $i \ge 2$

$$M_i = \{k \in \mathsf{M}_{\mathbf{p}} \mid k_{2s-1,j} = 0 \text{ for } p_{2s-1} \le j \le p_{2s-1} + i - 2\}.$$

Otherwise, if $d_{2s-1,2t-1}(\mathbf{p})$ *is infinite, then* $N_{st}(f(k)) = \infty$ *for all* $k \in \mathsf{M}_{\mathbf{p}}$*.*

Proof. With notation in 5.3.1 and by (5.3.B),

$$N_{st}(f(\kappa_{\mathbf{p}})) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x]]}{(h_{2s-2,2t})}. \tag{5.3.J}$$

By 5.3.2,

$$h_{2s-2,2t} = \begin{cases} 0 & \text{if } d_{2s-1,2t-1}(\mathbf{p}) = \infty \\ \sum_{i=r}^{\infty} c_i x^i & \text{if } d_{2s-1,2t-1}(\mathbf{p}) < \infty \end{cases} \tag{5.3.K}$$

where each $c_i \in \mathbb{C}[[\kappa_{\mathbf{p}}]]$, $r = d_{2s-1,2t-1}(\mathbf{p}) - 1$ and $c_r = -\det A^{d_{2s-1,2t-1}}_{2s-1,2t-1}(\kappa_{\mathbf{p}})$.

Thus, if $d_{2s-1,2t-1}(\mathbf{p}) = \infty$, then $h_{2s-2,2t} = 0$, and so $N_{st}(f(\kappa_{\mathbf{p}})) = \infty$ by (5.3.J).

(1), (2) When $d_{2s-1,2t-1}(\mathbf{p}) < \infty$, we first define $N_1 := \mathsf{M}_{\mathbf{p}}$, and for each $i \ge 2$ define $N_i := \{k \in \mathsf{M}_{\mathbf{p}} \mid c_r = c_{r+1} = \cdots = c_{r+i-2} = 0\}$. So we have a sequence of spaces $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$. Note that there may exist some segment like $N_{i-1} \supsetneq N_i = N_{i+1} = \cdots = N_j \supsetneq N_{j+1}$. After removing the repetitive elements in all such segments, we get a sequence of filtered spaces $\mathsf{M}_{\mathbf{p}} = M_1 \supsetneq M_2 \supsetneq M_3 \cdots$. By the definition of N_i, each M_i is the zero locus of some polynomial system of $\kappa_{\mathbf{p}}$.

By (5.3.K) and (5.3.J), for each $i \ge 1$, $N_{st}(f(k))$ is constant for all $k \in M_i \backslash M_{i+1}$. Thus we can set $d_i := N_{st}(M_i \backslash M_{i+1})$, which obviously satisfies $d_1 < d_2 < \cdots$.

Since $c_r = -\det A^{d_{2s-1,2t-1}}_{2s-1,2t-1}(\kappa_{\mathbf{p}}) \neq 0$ by 5.1.11, $N_2 = \{k \in \mathsf{M}_{\mathbf{p}} \mid c_r = 0\} \subsetneq N_1$, and so $M_2 = N_2 \subsetneq N_1 = M_1$, and further $d_1 = N_{st}(M_1 \backslash M_2) = r = d_{2s-1,2t-1}(\mathbf{p}) - 1$.

We next prove that $d_i = N_{st}(M_i \backslash M_{i+1}) = d_{2s-1,2t-1}(\mathbf{p}) + i - 2$ for $i \ge 2$. Fix some i with $i \ge 2$. By (5.1.F), there exists $\mathbf{p}'$ such that $\mathbf{p}' \ge \mathbf{p}$ (see 5.1.5(5)) and $d_{2s-1,2t-1}(\mathbf{p}') = d_{2s-1,2t-1}(\mathbf{p}) + i - 1$. Since $\mathbf{p}' \ge \mathbf{p}$, $\mathsf{MA}_{\mathbf{p}'} \subseteq \mathsf{MA}_{\mathbf{p}}$ and $\mathsf{M}_{\mathbf{p}'} \subseteq \mathsf{M}_{\mathbf{p}}$ by 5.1.6(5).

Repeating the same argument as above, there is a sequence of filtered spaces $\mathsf{M}_{\mathbf{p}'} = M'_1 \supsetneq M'_2 \supsetneq \cdots$ such that $N_{st}(M'_1 \backslash M'_2) = d_{2s-1,2t-1}(\mathbf{p}') - 1 = d_{2s-1,2t-1}(\mathbf{p}) + i - 2$. Set $U_i := M'_1 \backslash M'_2$

which satisfies $U_i \subsetneq M_1' = \mathsf{M}_{\mathbf{p}'} \subseteq \mathsf{M}_{\mathbf{p}} = M_1$.

Since the above works for any $i \geq 2$, there is a sequence of spaces $U_i \subsetneq M_1$ such that $N_{st}(U_i) = d_{2s-1,2t-1}(\mathbf{p}) + i - 2$. So $U_i \subseteq M_i \backslash M_{i+1}$ and $d_i = N_{st}(M_i \backslash M_{i+1}) = d_{2s-1,2t-1}(\mathbf{p}) + i - 2$ for each $i \geq 2$.

(3) By $h_{2s-2,2s-2} = 0$, $h_{2s-2,2s-1} = x$ (see 5.3.1) and (5.3.D),

$$h_{2s-2,2s} = -\sum_{j=p_{2s-1}}^{\infty} j\kappa_{2s-1,j}x^{j-1}.$$

Then by (5.3.J),

$$N_{ss}(f(\kappa_{\mathbf{p}})) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x]]}{(h_{2s-2,2s})} = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x]]}{(\sum_{j=p_{2s-1}}^{\infty} j\kappa_{2s-1,j}x^{j-1})}.$$

Thus the statement follows immediately. $\square$

If we set $\mathbf{p} = (2, 2, \dots, 2)$ in 5.3.7, then $\mathsf{M}_{\mathbf{p}}$ coincides with M which is the parameter space of all monomialized Type A potentials $f(\kappa)$ (see 5.1.5, 5.1.6), as follows.

$$f(\kappa) = \sum_{i=1}^{2n-2} \mathsf{x}_i'\mathsf{x}_{i+1} + \sum_{i=1}^{2n-1}\sum_{j=2}^{\infty} \kappa_{ij}\mathsf{x}_i^j,$$

$$\mathsf{M} = \{(k_{12}, k_{13}, \dots, k_{22}, k_{23}, \dots, k_{2n-1,2}, k_{2n-1,3}, \dots) \mid \text{all } k_* \in \mathbb{C}\}.$$

Thus, as a special case of 5.3.7, we next give a filtration structure of M with respect to a fixed curve class.

Corollary 5.3.8. *Fix some s, t satisfying $1 \leq s \leq t \leq n$, then M has a filtration structure $\mathsf{M} = M_1 \supsetneq M_2 \supsetneq M_3 \supsetneq \cdots$ such that*

1. *For each $i \geq 1$, $N_{st}(f(k)) = i$ for all $k \in M_i \backslash M_{i+1}$.*
2. *Each M_i is the zero locus of some polynomial system of κ, and moreover*

$$M_2 = \{k \in \mathsf{M} \mid \det A^2_{2s-1,2t-1}(f(k)) = 0\}.$$

3. *If $s = t$, then for each $i \geq 2$*

$$M_i = \{k \in \mathsf{M} \mid k_{2s-1,j} = 0 \text{ for } 2 \leq j \leq i\}.$$

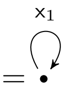
Proof. By setting $\mathbf{p} = (2, 2, \dots, 2)$ in 5.3.7, then $d_{2s-1,2t-1}(\mathbf{p}) = 2$, and so the statement follows immediately. $\square$

§ 5.3.2 | Examples

In this subsection, we will apply 5.3.7 and 5.3.8 to discuss the filtration structures of the parameter space of monomialized Type A potentials on Q_1 and Q_2.

Example 5.3.9. Consider monomialized Type A potentials $f(\mathsf{\kappa}) = \sum_{j=2}^{\infty} \mathsf{\kappa}_{1j}\mathsf{x}_1^j$ on Q_1, where

$$Q_1 =$$



The corresponding parameter space M is $\{(k_{12}, k_{13}, \dots) \mid \text{all } k_* \in \mathbb{C}\}$. Then by 5.3.8(3), for any $i \geq 1$ and $k \in \mathsf{M}$

$$N_{11}(f(k)) = i \iff k_{1,i+1} \neq 0 \text{ and } k_{1j} = 0 \text{ for } j \leq i.$$

We can also see this fact in the following way. For any $k \in \mathsf{M}$, consider the cA_1 singularity

$$\mathcal{R} := \frac{\mathbb{C}[[u,v,x,y]]}{uv - y(y + \sum_{j=2}^{\infty} jk_{1j}x^{j-1})}$$

and $\mathcal{R}$-module $M := \mathcal{R} \oplus (u, y) \oplus (u, y(y + \sum_{j=2}^{\infty} jk_{1j}x^{j-1}))$. Then $f(k)$ is realized by the crepant resolution π of $\mathcal{R}$ that corresponds to M (see §5.3). Thus by (5.2.B),

$$N_{11}(f(k)) = N_{11}(\pi) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{(y, y + \sum_{j=2}^{\infty} jk_{1j}x^{j-1})} = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x]]}{(\sum_{j=2}^{\infty} jk_{1j}x^{j-1})}.$$

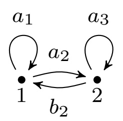
So the above fact follows immediately.

Example 5.3.10. Consider monomialized Type A potentials

$$f(\mathsf{\kappa}) = \sum_{j=2}^{\infty} \mathsf{\kappa}_{1j}\mathsf{x}_1^j + \mathsf{x}_1'\mathsf{x}_2 + \sum_{j=2}^{\infty} \mathsf{\kappa}_{2j}\mathsf{x}_2^j + \mathsf{x}_2'\mathsf{x}_3 + \sum_{j=2}^{\infty} \mathsf{\kappa}_{3j}\mathsf{x}_3^j$$

on Q_2, where

$$Q_2 =$$



$$\mathsf{x}_1 = \mathsf{x}_1' = a_1 \qquad \mathsf{x}_3 = \mathsf{x}_3' = a_3 \qquad \mathsf{x}_2 = a_2b_2, \mathsf{x}_2' = b_2a_2.$$

The parameter space M is $\{(k_{12}, k_{13}, \dots k_{22}, k_{23}, \dots k_{32}, k_{33}, \dots) \mid \text{all } \mathsf{\kappa}_* \in \mathbb{C}\}$. Recall in 5.1.2 that, for any $k \in \mathsf{M}$

$$A_{13}^2(f(k)) = \begin{bmatrix} 2k_{12} & 1 & 0 \\ 1 & 2k_{22} & 1 \\ 0 & 1 & 2k_{32} \end{bmatrix}.$$

Thus $\det A_{13}^2(f(k)) = 8k_{12}k_{22}k_{32} - 2k_{12} - 2k_{32}$. For fixed curve class $\mathrm{C}_1 + \mathrm{C}_2$, by 5.3.8(3),

$$N_{12}(f(k)) = 1 \iff \det A_{13}^2(f(k)) \neq 0 \iff 4k_{12}k_{22}k_{32} - k_{12} - k_{32} \neq 0,$$
$$N_{12}(f(k)) > 1 \iff \det A_{13}^2(f(k)) = 0 \iff 4k_{12}k_{22}k_{32} - k_{12} - k_{32} = 0.$$

Thus the generalised GV invariant N_{12} at the general position of M is one, while that at the codimension one locus defined by $4\mathsf{\kappa}_{12}\mathsf{\kappa}_{22}\mathsf{\kappa}_{32} - \mathsf{\kappa}_{12} - \mathsf{\kappa}_{32} = 0$ is greater than one.

We next choose a different $\mathbf{p}$ from the above example and consider the corresponding filtration structure, and then show that there exists a nonempty subspace of the parameter space of monomialized Type A potentials on Q_2 such that the generalised GV invariant N_{12} on this subspace is two. This also illustrates how the filtration structure in the proof of 5.3.7 was constructed.

Example 5.3.11. Set $\mathbf{p} = (3, 2, 3)$ and consider the subset $f(\kappa_{\mathbf{p}})$ of monomialized Type A potentials on Q_2, so

$$f(\kappa_{\mathbf{p}}) = \sum_{j=3}^{\infty} \kappa_{1j} x_1^j + x_1' x_2 + \sum_{j=2}^{\infty} \kappa_{2j} x_2^j + x_2' x_3 + \sum_{j=3}^{\infty} \kappa_{3j} x_3^j$$

(see 5.1.5). The parameter space $\mathsf{M}_{\mathbf{p}}$ is $\{(k_{13}, k_{14}, \dots k_{22}, k_{23}, \dots k_{33}, k_{34}, \dots) \mid \text{all } k_* \in \mathbb{C}\}$. Recall in 5.1.5 and 5.1.2 that $d_{13}(\mathbf{p}) = 3$ and for any $k \in \mathsf{M}_{\mathbf{p}}$

$$A_{13}^3(f(k)) = \begin{bmatrix} 3k_{13} & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3k_{33} \end{bmatrix}.$$

Thus $\det A_{13}^3(f(k)) = 3k_{33} - 3k_{13}$. For fixed curve class $\mathrm{C}_1 + \mathrm{C}_2$ (so, $s = 1, t = 2$), $d_{2s-1,2t-1}(\mathbf{p}) = d_{13}(\mathbf{p}) = 3$, and thus by 5.3.7 for any $k \in \mathsf{M}_{\mathbf{p}}$

$$\begin{aligned} N_{12}(f(k)) = d_{13}(\mathbf{p}) - 1 = 2 &\iff \det A_{13}^3(f(k)) \neq 0 \iff k_{33} - k_{13} \neq 0, \\ N_{12}(f(k)) > d_{13}(\mathbf{p}) - 1 = 2 &\iff \det A_{13}^3(f(k)) = 0 \iff k_{33} - k_{13} = 0. \end{aligned}$$

Thus the generalised GV invariant N_{12} at the general position of $\mathsf{M}_{\mathbf{p}}$ is two.

Since $\mathbf{p} = (3, 2, 3)$, by (5.1.E) we may view $\mathsf{M}_{\mathbf{p}} = \{k \in \mathsf{M} \mid k_{12} = 0 = k_{32}\}$. Thus,

$$U_2 := \{k \in \mathsf{M}_{\mathbf{p}} \mid k_{33} - k_{13} \neq 0\} = \{k \in \mathsf{M} \mid k_{12} = 0 = k_{32} \text{ and } k_{33} - k_{13} \neq 0\},$$

where M is the parameter space of all monomialized Type A potentials on Q_2 as in 5.3.10. Thus, by the above argument, $N_{12}(U_2) = 2$. Since $U_2 \neq \emptyset$ and $U_2 \subseteq \mathsf{M}$, U_2 is a nonempty subspace of M such that the generalised GV invariant N_{12} on this subspace is two.

Furthermore, consider

$$M_2 := \{k \in \mathsf{M} \mid 4k_{12}k_{22}k_{32} - k_{12} - k_{32} = 0\},$$

which by 5.3.10 is the first strata of M, which satisfies $N_{12}(\mathsf{M} \backslash M_2) = 1$ and $N_{12}(M_2) \geq 2$. Since $U_2 \subseteq \mathsf{M}$ and $N_{12}(U_2) = 2$, U_2 must be contained in M_2. We can also check this by some elementary calculation, namely

$$U_2 = \{k \in \mathsf{M} \mid k_{12} = 0 = k_{32}, k_{33} - k_{13} \neq 0\} \subseteq \{k \in \mathsf{M} \mid 4k_{12}k_{22}k_{32} - k_{12} - k_{32} = 0\} = M_2.$$

§ 5.4 | Obstructions

§ 5.4.1 | Obstructions

Based on the filtration structures in §5.3, this subsection in 5.4.7 gives the obstructions and constructions of generalised GV invariants that can arise from crepant resolutions of cA_n singularities.

Recall the definition of generalised GV invariants of crepant resolutions of cA_n singularities and those of monomialized Type A potentials in 3.1.1 and 5.2.1 respectively.

Definition 5.4.1. *Given a crepant resolution* π *of a* cA_n *singularity, define* generalised GV tuple *of* π *to be* $N(\pi) := (N_{st}(\pi) \mid \text{all } 1 \leq s \leq t \leq n)$.

Similarly, given a monomialized Type A potential f *on* Q_n*, define* generalised GV tuple *of* f *to be* $N(f) := (N_{st}(f) \mid \text{all } 1 \leq s \leq t \leq n)$.

Lemma 5.4.2. *Let* π *be a crepant resolution of a* cA_n *singularity and* f *be a monomialized Type A potentialon* Q_n*. If* $\Lambda_{\text{con}}(\pi) \xrightarrow{\sim} \mathcal{J}\text{ac}(f)$*, then* $N_{st}(\pi) = N_{st}(f)$ *for* $1 \leq s \leq t \leq n$*, and so* $N(\pi) = N(f)$*.*

Proof. Recall the construction of $N_{st}(f)$ in 5.2.1. There exists a crepant resolution π' such that $\Lambda_{\text{con}}(\pi') \xrightarrow{\sim} \mathcal{J}\text{ac}(f)$ and $N_{ij}(\pi') = N_{ij}(f)$ (5.2.B). Thus $\Lambda_{\text{con}}(\pi) \xrightarrow{\sim} \Lambda_{\text{con}}(\pi')$, and so $N_{st}(\pi) = N_{st}(\pi')$ by 3.2.7, and further $N_{st}(\pi) = N_{st}(f)$. □

For any s, t satisfying $1 \leq s \leq t \leq n$, and any $N \in \mathbb{N}_\infty$, by 5.3.8 there exists a crepant resolution π of a cA_n singularity such that $N_{st}(\pi) = N$. However, this is no longer true when considering generalised GV invariants of different curve classes simultaneously.

Notation 5.4.3. Fix some positive integer k, set $\mathbf{q} = \{(\beta_1, q_1), (\beta_2, q_2), \ldots, (\beta_k, q_k)\}$ where each $\beta_i \in \bigoplus_i^n \mathbb{Z}\langle \mathrm{C}_i \rangle$ and $q_i \in \mathbb{N}_\infty$. Then we denote $\mathbf{q}_{\min} := \min\{q_i\}$, and consider a subset of crepant resolutions of cA_n singularities

$$\mathsf{CA}_{\mathbf{q}} := \{cA_n \text{ crepant resolution } \pi \mid (N_{\beta_1}(\pi), N_{\beta_2}(\pi), \ldots, N_{\beta_k}(\pi)) = (q_1, q_2, \ldots, q_k)\}.$$

Notation 5.4.4. Fix some s, t with $1 \leq s \leq t \leq n$, and a tuple $(q_s, \ldots, q_t) \in \mathbb{N}_\infty^{t-s+1}$.

(1) As in 5.4.3, consider $\mathbf{q} := \{(\mathrm{C}_s, q_s), (\mathrm{C}_{s+1}, q_{s+1}), \ldots, (\mathrm{C}_t, q_t)\}$, and its associated subset of crepant resolutions of cA_n singularities $\mathsf{CA}_{\mathbf{q}}$.

(2) Furthermore, set $\mathbf{p} = (p_1, p_2, \ldots, p_{2n-1})$, where $p_{2i-1} := q_i + 1$ for $s \leq i \leq t$, else $p_i := 2$, and consider monomialized Type A potentials $\mathsf{MA}_{\mathbf{p}}$ on Q_n defined in 5.1.5.

(3) We define a nonempty subset $\mathsf{MA}^\circ_{\mathbf{p}} \subseteq \mathsf{MA}_{\mathbf{p}}$ (defined in (5.1.D)) by

$$\mathsf{MA}^\circ_{\mathbf{p}} := \{f \in \mathsf{MA}_{\mathbf{p}} \mid k_{2i-1, p_{2i-1}} \neq 0 \text{ for all } i \text{ satisfying } s \leq i \leq t \text{ and } p_{2i-1} \text{ finite}\},$$

and an open subspace $\mathsf{M}^\circ_{\mathbf{p}}$ of $\mathsf{M}_{\mathbf{p}}$ (defined in (5.1.E)) by

$$\mathsf{M}^\circ_{\mathbf{p}} := \{k \in \mathsf{M}_{\mathbf{p}} \mid k_{2i-1,p_{2i-1}} \neq 0 \text{ for all } i \text{ satisfying } s \leq i \leq t \text{ and } p_{2i-1} \text{ finite}\}.$$

We can, and will, consider $\mathsf{MA}^\circ_{\mathbf{p}}$ as a family of monomialized Type A potentials over $\mathsf{M}^\circ_{\mathbf{p}}$.

Proposition 5.4.5. *With notation in 5.4.4, the set of isomorphism classes of contraction algebras associated to $\mathsf{CA}_{\mathbf{q}}$ is equal to the set of isomorphism classes of Jacobi algebras of $\mathsf{MA}^\circ_{\mathbf{p}}$.*

Proof. For any $\pi \in \mathsf{CA}_{\mathbf{q}}$, by 4.2.19 there exists a monomialized Type A potentials f on Q_n such that $\mathcal{J}\mathrm{ac}(f) \xrightarrow{\sim} \Lambda_{\mathrm{con}}(\pi)$. We claim that $f \in \mathsf{MA}^\circ_{\mathbf{p}}$. To see this, we first fix some i satisfying $s \leq i \leq t$. Since $\mathcal{J}\mathrm{ac}(f) \xrightarrow{\sim} \Lambda_{\mathrm{con}}(\pi)$, by 5.4.2 $N_{ii}(f) = N_{ii}(\pi)$. Since $\pi \in \mathsf{CA}_{\mathbf{q}}$, $N_{ii}(\pi) = q_i$, and so $N_{ii}(f) = q_i$. Thus by 5.3.8 the following holds.

(1) If q_i is infinite, then $k_{2i-1,j} = 0$ in f for any j.

(2) If q_i is finite, then $k_{2i-1,q_i+1} \neq 0$ and $k_{2i-1,j} = 0$ in f for any $j \leq q_i$.

In either case, since $p_{2i-1} = q_i + 1$ in 5.4.4, $f \in \mathsf{MA}^\circ_{\mathbf{p}}$.

Then we prove the converse. For any $f \in \mathsf{MA}^\circ_{\mathbf{p}}$, by 1.5.5 there is a cA_n crepant resolution π such that $\Lambda_{\mathrm{con}}(\pi) \xrightarrow{\sim} \mathcal{J}\mathrm{ac}(f)$. We claim that $\pi \in \mathsf{CA}_{\mathbf{q}}$. To see this, we first fix some i satisfying $s \leq i \leq t$. Since $\Lambda_{\mathrm{con}}(\pi) \xrightarrow{\sim} \mathcal{J}\mathrm{ac}(f)$, by 5.4.2 $N_{ii}(\pi) = N_{ii}(f)$. Since $f \in \mathsf{MA}^\circ_{\mathbf{p}}$, by 5.3.8 $N_{ii}(f) = p_{2i-1} - 1 = q_i$, and so $N_{ii}(\pi) = q_i$. Thus $\pi \in \mathsf{CA}_{\mathbf{q}}$.

Together with the fact in 1.5.7 that the set of isomorphism classes of contraction algebras associated to crepant resolutions of cA_n singularities is equal to the set of isomorphism classes of Jacobi algebras of monomialized Type A potentials on Q_n, the statement follows. □

The following transfers generalised GV tuples of $\mathsf{CA}_{\mathbf{q}}$ to those of $\mathsf{MA}^\circ_{\mathbf{p}}$, which have been characterized explicitly in 5.3.7 and 5.3.8.

Corollary 5.4.6. *The set of generalised* GV *tuples of* $\mathsf{CA}_{\mathbf{q}}$ *is equal to the set of generalised* GV *tuples of* $\mathsf{MA}^\circ_{\mathbf{p}}$.

Proof. This is immediate from 5.4.5 and 5.4.2. □

Combining 5.4.6 and 5.3.7, the following gives obstructions and constructions of the possible tuples that can arise from generalised GV tuples of cA_n crepant resolutions.

Theorem 5.4.7. *For any s and t with $1 \leq s \leq t \leq n$, and any tuple $(q_s, \ldots, q_t) \in \mathbb{N}_\infty^{t-s+1}$, with notation in 5.4.4, the following statements hold.*

(1) *For any $\pi \in \mathsf{CA}_{\mathbf{q}}$ necessarily $N_{st}(\pi) \geq \mathbf{q}_{\min}$, and moreover there exists $\pi \in \mathsf{CA}_{\mathbf{q}}$ such that $N_{st}(\pi) = \mathbf{q}_{\min}$.*

(2) *When* $\mathbf{q}_{\min}$ *is finite, the equality* $N_{st}(\pi) = \mathbf{q}_{\min}$ *holds for all* $\pi \in \mathsf{CA}_{\mathbf{q}}$ *if and only if* $\#\{i \mid q_i = \mathbf{q}_{\min}\} = 1$.

Proof. By 5.4.6 it suffices to prove that the statement holds for the generalised GV invariants of $\mathsf{MA}^{\circ}_{\mathbf{p}}$. Recall in 5.4.4 that $\mathsf{MA}^{\circ}_{\mathbf{p}} \subseteq \mathsf{MA}_{\mathbf{p}}$, $\mathsf{M}^{\circ}_{\mathbf{p}} \subseteq \mathsf{M}_{\mathbf{p}}$ and $\mathbf{p} = (p_1, p_2, \dots, p_{2n-1})$ where $p_{2i-1} = q_i + 1$ for $s \le i \le t$, else $p_i = 2$. Thus

$$d_{2s-1,2t-1}(\mathbf{p}) = \min(p_{2s-1}, p_{2s+1}, \dots, p_{2t-1}) = \min(q_s + 1, q_{s+1} + 1, \dots, q_t + 1) = \mathbf{q}_{min} + 1.$$

The remainder of the proof will use the following notation and facts.

Notation 5.4.8. We list the notation and facts we will use below when $\mathbf{q}_{min}$ is finite.

(a) Set $\mathbf{I} := \{i \mid q_i \text{ is finite for } s \le i \le t\} = \{i \mid p_{2i-1} \text{ is finite for } s \le i \le t\}$. Since $\mathbf{q}_{min}$ is finite and by definition $\mathbf{q}_{min} = \min\{q_i\}$, $\mathbf{I} \neq \emptyset$.

(b) By 5.4.4, $\mathsf{M}_{\mathbf{p}} \backslash \mathsf{M}^{\circ}_{\mathbf{p}} = \{k \in \mathsf{M}_{\mathbf{p}} \mid \prod_{i \in \mathbf{I}} k_{2i-1,p_{2i-1}} = 0\}$.

(c) By 5.3.7, there exists a filtration structure $\mathsf{M}_{\mathbf{p}} = M_1 \supsetneq M_2 \supsetneq M_3 \supsetneq \cdots$ such that $N_{st}(M_1 \backslash M_2) = d_{2s-1,2t-1}(\mathbf{p}) - 1 = \mathbf{q}_{min}$, $N_{st}(M_2) > d_{2s-1,2t-1}(\mathbf{p}) - 1 = \mathbf{q}_{min}$, and $M_2 = \{k \in \mathsf{M}_{\mathbf{p}} \mid \det A^d_{2s-1,2t-1}(f(k)) = 0 \text{ where } d = d_{2s-1,2t-1}(\mathbf{p})\}$.

Notation 5.4.9. To avoid the proof difficulties encountered in infinite-dimensional vector spaces, with notation in 5.4.8, we next define some finite-dimensional linear subspaces $\mathsf{N}_{\mathbf{p}}$, $\mathsf{N}^{\circ}_{\mathbf{p}}$ and N_2 of $\mathsf{M}_{\mathbf{p}}$ to facilitate the following proof.

(a) Write $\kappa_{\mathbf{p}}$ for the tuple of variables $\mathsf{k}_{2s-1,p_{2s-1}}, \mathsf{k}_{2s,p_{2s}}, \dots, \mathsf{k}_{2t-1,p_{2t-1}}$. Note that $\kappa_{\mathbf{p}}$ only has finite variables.

(b) We next define a linear subspace $\mathsf{N}_{\mathbf{p}}$ of $\mathsf{M}_{\mathbf{p}}$ as the vector space generated by the basis corresponding to $\kappa_{\mathbf{p}}$, and a linear subspace V of $\mathsf{M}_{\mathbf{p}}$ as the vector space generated by the basis corresponding to $\mathsf{k}_{\mathbf{p}}$ except $\kappa_{\mathbf{p}}$. Thus $\mathsf{N}_{\mathbf{p}}$ is a finite dimensional vector space and $\mathsf{M}_{\mathbf{p}} = \mathsf{N}_{\mathbf{p}} \oplus V$.

(c) Parallel to $\mathsf{M}^{\circ}_{\mathbf{p}} \subseteq \mathsf{M}_{\mathbf{p}}$ in 5.4.4, define an open subspace $\mathsf{N}^{\circ}_{\mathbf{p}}$ of $\mathsf{N}_{\mathbf{p}}$ by

$$\mathsf{N}^{\circ}_{\mathbf{p}} := \{k \in \mathsf{N}_{\mathbf{p}} \mid k_{2i-1,p_{2i-1}} \neq 0 \text{ for all } i \in \mathbf{I}\}.$$

Thus $\mathsf{N}_{\mathbf{p}} \backslash \mathsf{N}^{\circ}_{\mathbf{p}} = \{k \in \mathsf{N}_{\mathbf{p}} \mid \prod_{i \in \mathbf{I}} k_{2i-1,p_{2i-1}} = 0\}$.

(d) Parallel to $M_2 \subseteq \mathsf{M}_{\mathbf{p}}$ in 5.4.8(c), define a closed subspace N_2 of $\mathsf{N}_{\mathbf{p}}$ by

$$N_2 := \{k \in \mathsf{N}_{\mathbf{p}} \mid \det A^d_{2s-1,2t-1}(f(k)) = 0 \text{ where } d = d_{2s-1,2t-1}(\mathbf{p})\}.$$

(e) By definition 5.1.2 $A^d_{2s-1,2t-1}(\mathsf{k}_{\mathbf{p}})$ only contains variables $\mathsf{k}_{2s-1,d}, \mathsf{k}_{2s,d}, \dots, \mathsf{k}_{2t-1,d}$. Thus when $d = d_{2s-1,2t-1}(\mathbf{p}) := \min(p_{2s-1}, p_{2s+1}, \dots, p_{2t-1})$, $A^d_{2s-1,2t-1}(\mathsf{k}_{\mathbf{p}})$ only contains variables in $\kappa_{\mathbf{p}}$, and so $A^d_{2s-1,2t-1}(\mathsf{k}_{\mathbf{p}}) = A^d_{2s-1,2t-1}(\kappa_{\mathbf{p}})$.

(f) Consider the natural quotient map $\varphi\colon \mathsf{M}_{\mathbf{p}} \twoheadrightarrow \mathsf{N}_{\mathbf{p}}$ with $\ker\varphi = V$. Since $\mathsf{M}^{\circ}_{\mathbf{p}}$ and $\mathsf{N}^{\circ}_{\mathbf{p}}$ are defined by the zero locus of the same polynomial, $\varphi(\mathsf{M}^{\circ}_{\mathbf{p}}) = \mathsf{N}^{\circ}_{\mathbf{p}}$, and so $\mathsf{M}^{\circ}_{\mathbf{p}} = \mathsf{N}^{\circ}_{\mathbf{p}} \oplus V$. Similarly, since by 5.4.9(e) M_2 and N_2 are also defined by the zero locus of the same polynomial, $\varphi(M_2) = N_2$, and so $M_2 = N_2 \oplus V$.

(1) If $\mathbf{q}_{min} = \infty$, then $d_{2s-1,2t-1}(\mathbf{p}) = \infty$, and so by 5.3.7 $N_{st}(\mathsf{M}_{\mathbf{p}}) = \infty$. Since $\emptyset \neq \mathsf{M}^{\circ}_{\mathbf{p}} \subseteq \mathsf{M}_{\mathbf{p}}$, there exists $f \in \mathsf{MA}^{\circ}_{\mathbf{p}}$ such that $N_{st}(f) = \infty = \mathbf{q}_{\min}$.

Otherwise, $\mathbf{q}_{min} < \infty$, and then by 5.4.8 $N_{st}(\mathsf{M}_{\mathbf{p}}) \geq \mathbf{q}_{min}$. Since $\mathsf{M}^{\circ}_{\mathbf{p}} \subseteq \mathsf{M}_{\mathbf{p}}$, $N_{st}(f) \geq \mathbf{q}_{\min}$ for any $f \in \mathsf{MA}^{\circ}_{\mathbf{p}}$. This proves the first part of the statement.

For the second part, we claim that there exists $f \in \mathsf{MA}^{\circ}_{\mathbf{p}}$ such that $N_{st}(f) = \mathbf{q}_{\min}$. Since by 5.4.8 $N_{st}(\mathsf{M}_{\mathbf{p}} \backslash M_2) = \mathbf{q}_{min}$ and $N_{st}(M_2) > \mathbf{q}_{min}$, it is equivalent to prove that $(\mathsf{M}_{\mathbf{p}} \backslash M_2) \cap \mathsf{M}^{\circ}_{\mathbf{p}} \neq \emptyset$. Since by 5.4.9(b) and 5.4.9(f), $\mathsf{M}_{\mathbf{p}} = \mathsf{N}_{\mathbf{p}} \oplus V$, $\mathsf{M}^{\circ}_{\mathbf{p}} = \mathsf{N}^{\circ}_{\mathbf{p}} \oplus V$ and $M_2 = N_2 \oplus V$, it is equivalent to prove that $(\mathsf{N}_{\mathbf{p}} \backslash N_2) \cap \mathsf{N}^{\circ}_{\mathbf{p}} \neq \emptyset$.

Since by 5.4.9(d) N_2 is the zero locus of a polynomial in $\mathbb{C}[[\kappa_{\mathbf{p}}]]$, $\mathsf{N}_{\mathbf{p}} \backslash N_2$ is an open set (wrt. Zariski topology) of the finite dimensional space $\mathsf{N}_{\mathbf{p}}$. Similarly, by 5.4.9(c) $\mathsf{N}^{\circ}_{\mathbf{p}}$ is also an open set (wrt. Zariski topology) of $\mathsf{N}_{\mathbf{p}}$. So $(\mathsf{N}_{\mathbf{p}} \backslash N_2) \cap \mathsf{N}^{\circ}_{\mathbf{p}} \neq \emptyset$.

(2) Assume that $\mathbf{q}_{\min}$ is finite.

($\Leftarrow$) We first prove that if $\#\{i \mid q_i = \mathbf{q}_{\min}\} = 1$, then the equality $N_{st}(f) = \mathbf{q}_{\min}$ holds for all $f \in \mathsf{MA}^{\circ}_{\mathbf{p}}$. Since by 5.4.8 $N_{st}(\mathsf{M}_{\mathbf{p}} \backslash M_2) = \mathbf{q}_{min}$ and $N_{st}(M_2) > \mathbf{q}_{min}$, it is equivalent to prove that $\mathsf{M}^{\circ}_{\mathbf{p}} \cap M_2 = \emptyset$ (equivalently, $M_2 \subseteq \mathsf{M}_{\mathbf{p}} \backslash \mathsf{M}^{\circ}_{\mathbf{p}}$).

To ease notation, write m for the unique index such that $q_m = \mathbf{q}_{\min}$ and set $d := d_{2s-1,2t-1}(\mathbf{p})$. Since $p_{2i-1} = q_i + 1$ for $s \leq i \leq t$ in 5.4.4, p_{2m-1} is the unique smallest element in $\{p_{2s-1}, p_{2s+1}, \ldots, p_{2t-1}\}$, and so by (5.1.F) $d = p_{2m-1} > p_{2i-1}$ for all i satisfying $s \leq i \leq t$ and $i \neq m$. Thus by 5.1.6(4), for $s \leq i \leq t$ the following holds.

- If $i = m$, then $p_{2i-1} = d$, and so $\varepsilon_{2i-1,d}(\kappa_{\mathbf{p}}) = d\kappa_{2i-1,d}$.
- If $i \neq m$, then $p_{2i-1} > d$, and so $\varepsilon_{2i-1,d}(\kappa_{\mathbf{p}})$ is a zero function over $\mathsf{M}_{\mathbf{p}}$.

If $d > 2$, then by 5.1.9(3),

$$\begin{aligned}
\det A^d_{2s-1,2t-1}(\kappa_{\mathbf{p}}) &= (-1)^{t-s}\Big(\varepsilon_{2s-1,d}(\kappa_{\mathbf{p}}) + (-1)^d \varepsilon_{2s+1,d}(\kappa_{\mathbf{p}}) + \cdots + (-1)^{(t-s)d}\varepsilon_{2t-1,d}(\kappa_{\mathbf{p}})\Big) \\
&= (-1)^{t-s}(-1)^{(m-s)d}\varepsilon_{2m-1,d}(\kappa_{\mathbf{p}}) \\
&= (-1)^{t-s+(m-s)d} d\kappa_{2m-1,d}.
\end{aligned}$$

So by 5.4.8(c), $M_2 = \{k \in \mathsf{M}_{\mathbf{p}} \mid k_{2m-1,d} = 0\}$. Since $q_m = \mathbf{q}_{min}$ is finite, $m \in \mathbf{I}$ (see 5.4.8(a)). Together with 5.4.8(b) and $d = p_{2m-1}$, it follows that

$$\mathsf{M}_{\mathbf{p}} \backslash \mathsf{M}^{\circ}_{\mathbf{p}} = \{k \in \mathsf{M}_{\mathbf{p}} \mid k_{2m-1,d} \prod_{i \in \mathbf{I} \backslash \{m\}} k_{2i-1,p_{2i-1}} = 0\}.$$

Thus $M_2 \subseteq \mathsf{M}_{\mathbf{p}} \backslash \mathsf{M}_{\mathbf{p}}^\circ$.

Otherwise, $d = 2$, and then by 5.1.9(2),

$$\begin{aligned}\det A^2_{2s-1,2t-1}(\kappa_{\mathbf{p}}) &= (-1)^{t-s}(\varepsilon_{2s-1,2}(\kappa_{\mathbf{p}}) + \varepsilon_{2s+1,2}(\kappa_{\mathbf{p}}) + \cdots + \varepsilon_{2t-1,2}(\kappa_{\mathbf{p}})) + \epsilon(\kappa_{\mathbf{p}}) \\ &= (-1)^{t-s} 2\kappa_{2m-1,2} + \epsilon(\kappa_{\mathbf{p}}),\end{aligned}$$

where $\epsilon \in E_{2s-1,2t-1}$ and $E_{2s-1,2t-1}$ is the ideal generated by all the degree two terms of $\varepsilon_{2s-1,2}, \varepsilon_{2s+1,2}, \ldots, \varepsilon_{2t-1,2}$ except $\varepsilon^2_{2s-1,2}, \varepsilon^2_{2s+1,2}, \ldots, \varepsilon^2_{2t-1,2}$ (see 5.1.8). Together with $\varepsilon_{2m-1,2}(\kappa_{\mathbf{p}})$ is the only non-zero element in $\{\varepsilon_{2s-1,2}(\kappa_{\mathbf{p}}), \varepsilon_{2s+1,2}(\kappa_{\mathbf{p}}), \ldots, \varepsilon_{2t-1,2}(\kappa_{\mathbf{p}})\}$, thus $E_{2s-1,2t-1}(\kappa_{\mathbf{p}}) = \{0\}$, and so $\epsilon(\kappa_{\mathbf{p}}) = 0$. Thus $\det A^2_{2s-1,2t-1}(\kappa_{\mathbf{p}}) = (-1)^{t-s} 2\kappa_{2m-1,2}$. So by 5.4.8(c), $M_2 = \{k \in \mathsf{M}_{\mathbf{p}} \mid k_{2m-1,d} = 0\}$. Similarly, $M_2 \subseteq \mathsf{M}_{\mathbf{p}} \backslash \mathsf{M}_{\mathbf{p}}^\circ$.

($\Rightarrow$) We next prove the converse: if $\#\{i \mid q_i = \mathbf{q}_{\min}\} > 1$, then there exists $f \in \mathsf{MA}_{\mathbf{p}}^\circ$ such that $N_{st}(f) > \mathbf{q}_{\min}$. Since by 5.4.8(c) $N_{st}(\mathsf{M}_{\mathbf{p}} \backslash M_2) = \mathbf{q}_{min}$ and $N_{st}(M_2) > \mathbf{q}_{min}$, it is equivalent to prove $\mathsf{M}_{\mathbf{p}}^\circ \cap M_2 \neq \emptyset$ (equivalently, $M_2 \not\subseteq \mathsf{M}_{\mathbf{p}} \backslash \mathsf{M}_{\mathbf{p}}^\circ$). Since by 5.4.9(b) and 5.4.9(f), $\mathsf{M}_{\mathbf{p}} = \mathsf{N}_{\mathbf{p}} \oplus V$, $\mathsf{M}_{\mathbf{p}}^\circ = \mathsf{N}_{\mathbf{p}}^\circ \oplus V$ and $M_2 = N_2 \oplus V$, it is equivalent to prove that $N_2 \not\subseteq \mathsf{N}_{\mathbf{p}} \backslash \mathsf{N}_{\mathbf{p}}^\circ$

To ease notation, set $d := d_{2s-1,2t-1}(\mathbf{p})$ and $I := \{i \mid q_i = \mathbf{q}_{min} \text{ for } s \leq i \leq t\} = \{i \mid p_{2i-1} = d = \min(p_{2s-1}, p_{2s+1}, \ldots, p_{2t-1}) \text{ for } s \leq i \leq t\}$. Since $\#\{i \mid q_i = \mathbf{q}_{\min}\} > 1$, then the number of elements $|I| > 1$. By 5.1.6(4), for $s \leq i \leq t$ the following holds.

- If $i \in I$, then $p_{2i-1} = d$, and so $\varepsilon_{2i-1,d}(\kappa_{\mathbf{p}}) = d\kappa_{2i-1,d}$.
- If $i \notin I$, then $p_{2i-1} > d$, and so $\varepsilon_{2i-1,d}(\kappa_{\mathbf{p}})$ is a zero function over $\mathsf{M}_{\mathbf{p}}$.

If $d > 2$, then

$$\begin{aligned}\det A^d_{2s-1,2t-1}(\kappa_{\mathbf{p}}) &\overset{5.4.9(e)}{=} \det A^d_{2s-1,2t-1}(\kappa_{\mathbf{p}}) \\ &\overset{5.1.9(3)}{=} (-1)^{t-s}\Big(\varepsilon_{2s-1,d}(\kappa_{\mathbf{p}}) + (-1)^d \varepsilon_{2s+1,d}(\kappa_{\mathbf{p}}) + \cdots + (-1)^{(t-s)d}\varepsilon_{2t-1,d}(\kappa_{\mathbf{p}})\Big) \\ &= (-1)^{t-s}\Big(\sum_{i\in I}(-1)^{(i-s)d}\varepsilon_{2i-1,d}(\kappa_{\mathbf{p}})\Big) \\ &= (-1)^{t-s-sd} d \sum_{i\in I}(-1)^{id}\kappa_{2i-1,d}.\end{aligned}$$

So by 5.4.9(d), $N_2 = \{k \in \mathsf{N}_{\mathbf{p}} \mid \sum_{i\in I}(-1)^{id}k_{2i-1,d} = 0\}$. We next prove that $N_2 \not\subseteq \mathsf{N}_{\mathbf{p}} \backslash \mathsf{N}_{\mathbf{p}}^\circ$ by contradiction. Recall that $\mathsf{N}_{\mathbf{p}} \backslash \mathsf{N}_{\mathbf{p}}^\circ = \{k \in \mathsf{N}_{\mathbf{p}} \mid \prod_{i\in\mathbf{I}} k_{2i-1,p_{2i-1}} = 0\}$ in 5.4.9(c). Thus if $N_2 \subseteq \mathsf{N}_{\mathbf{p}} \backslash \mathsf{N}_{\mathbf{p}}^\circ$, then

$$\Big(\prod_{i\in\mathbf{I}}\kappa_{2i-1,p_{2i-1}}\Big) \subseteq \Big(\sum_{i\in I}(-1)^{id}\kappa_{2i-1,d}\Big)$$

in $\mathbb{C}[[\kappa_{\mathbf{p}}]]$, and so there exists $\kappa' \in \mathbb{C}[[\kappa_{\mathbf{p}}]]$ such that

$$\prod_{i\in\mathbf{I}}\kappa_{2i-1,p_{2i-1}} = \kappa'\Big(\sum_{i\in I}(-1)^{id}\kappa_{2i-1,d}\Big). \tag{5.4.A}$$

Since $\mathbb{C}[[\kappa_{\mathbf{p}}]]$ has only a finite number of variables, it is a unique factorization domain. Together with (5.4.A) and $|I| > 1$, there are two different factorizations of the same element in $\mathbb{C}[[\kappa_{\mathbf{p}}]]$, a contradiction.

Otherwise, $d = 2$, and then

$$\begin{aligned}\det A^2_{2s-1,2t-1}(\kappa_{\mathbf{p}}) &\overset{5.4.9(e)}{=} \det A^2_{2s-1,2t-1}(\kappa_{\mathbf{p}}) \\ &\overset{5.1.9(2)}{=} (-1)^{t-s}\Big(\varepsilon_{2s-1,2}(\kappa_{\mathbf{p}}) + \varepsilon_{2s+1,2}(\kappa_{\mathbf{p}}) + \cdots + \varepsilon_{2t-1,2}(\kappa_{\mathbf{p}})\Big) + \epsilon(\kappa_{\mathbf{p}}) \\ &= (-1)^{t-s}2\sum_{i\in I}\kappa_{2i-1,2} + \epsilon(\kappa_{\mathbf{p}}),\end{aligned}$$

where $\epsilon \in E_{2s-1,2t-1}$ and $E_{2s-1,2t-1}$ is the ideal generated by some degree two terms of $\varepsilon_{2s-1,2}, \varepsilon_{2s+1,2}, \ldots, \varepsilon_{2t-1,2}$. So by 5.4.9(d), $N_2 = \{k \in \mathsf{N}_{\mathbf{p}} \mid (-1)^{t-s}2\sum_{i\in I} k_{2i-1,2} + \epsilon(k) = 0\}$. Similarly, we can prove that $N_2 \not\subseteq \mathsf{N}_{\mathbf{p}}\backslash\mathsf{N}^{\circ}_{\mathbf{p}}$ by contradiction. $\square$

Example 5.4.10. Let π be a crepant resolution of a cA_3 singularity with exceptional curves C_1, C_2 and C_3. Suppose that

$$(N_{11}(\pi), N_{22}(\pi), N_{33}(\pi)) = (q_1, q_2, q_2) \text{ where } q_1 < q_2 < q_3.$$

With notation in 5.4.7, set $s = 1$, $t = 2$ and $\mathbf{q} = \{(\mathrm{C}_1, q_1), (\mathrm{C}_2, q_2)\}$. Since $N_{11}(\pi) = q_1$ and $N_{22}(\pi) = q_2$ by assumption, necessarily $\pi \in \mathsf{CA}_{\mathbf{q}}$. Since $q_1 < q_2$, $\mathbf{q}_{\min} = q_1$ is finite and $\#\{i \mid q_i = \mathbf{q}_{\min}\} = \#\{1\} = 1$. So by 5.4.7(2), $N_{12}(\pi)$ must be q_1.

Similarly, we can prove that $N_{23}(\pi) = q_2$ by setting $s = 2$, $t = 3$ and $\mathbf{q} = \{(\mathrm{C}_2, q_2), (\mathrm{C}_3, q_3)\}$, and $N_{13}(\pi) = q_1$ by setting $s = 1$, $t = 3$ and $\mathbf{q} = \{(\mathrm{C}_1, q_1), (\mathrm{C}_2, q_2), (\mathrm{C}_3, q_3)\}$.

§ 5.4.2 | Obstructions from iterated flops

Iterating flops gives more obstructions and constructions of the possible tuples that can arise from the generalised GV invariants of cA_n crepant resolutions.

Notation 5.4.11. Recall $\mathbf{r}$ and $\pi^{\mathbf{r}}$ in 3.3.1, and $|F_{\mathbf{r}}|$ in 3.3.4. There is a linear isomorphism

$$|F_{\mathbf{r}}|\colon A_1(\pi) \to A_1(\pi^{\mathbf{r}}),$$

such that $\mathrm{GV}_{\beta}(\pi) = \mathrm{GV}_{|F_{\mathbf{r}}|(\beta)}(\pi^{\mathbf{r}})$ for any $\beta \in A_1(\pi)$. By 3.3.8, $N_{\beta}(\pi) = N_{|F_{\mathbf{r}}|(\beta)}(\pi^{\mathbf{r}})$. Varying $\mathbf{r}$ over all possible flops gives the following set,

$$\mathcal{F} := \bigcup_{i=1}^{\infty}\{|F_{\mathbf{r}}| \mid \mathbf{r} = (r_1, r_2, \ldots, r_i) \text{ where each } 1 \le r_j \le n\}.$$

Given any $F \in \mathcal{F}$ and $\mathbf{q} = \{(\beta_1, q_1), (\beta_2, q_2), \ldots, (\beta_k, q_k)\}$ in 5.4.3, write

$$F(\mathbf{q}) := \{(F(\beta_1), q_1), (F(\beta_2), q_2), \ldots, (F(\beta_k), q_k)\}.$$

The flexibility of $F \in \mathcal{F}$ as above, together with 5.4.7, gives more obstructions and constructions of the possible tuples that can arise from generalised GV invariants of cA_n crepant resolutions, as follows.

Corollary 5.4.12. *For any integers s and t with $1 \leq s \leq t \leq n$, any tuple $(q_s, \ldots, q_t) \in \mathbb{N}_\infty^{t-s+1}$, and any $F \in \mathcal{F}$, with notation as in 5.4.4 and 5.4.11, the following statements hold.*

(1) *For any $\pi \in \mathsf{CA}_{F(\mathbf{q})}$ necessarily $N_{F(st)}(\pi) \geq \mathbf{q}_{\min}$, and moreover there exists $\pi \in \mathsf{CA}_{F(\mathbf{q})}$ such that $N_{F(st)}(\pi) = \mathbf{q}_{\min}$.*

(2) *When $\mathbf{q}_{\min}$ is finite, the equality $N_{F(st)}(\pi) = \mathbf{q}_{\min}$ holds for all $\pi \in \mathsf{CA}_{F(\mathbf{q})}$ if and only if $\#\{i \mid q_i = \mathbf{q}_{\min}\} = 1$.*

Proof. By the definition of $\mathcal{F}$ in 5.4.11, there exists some $\mathbf{r} = (r_1, r_2, \ldots, r_j)$ such that $F = |F_\mathbf{r}|$. Then set the reverse tuple of $\mathbf{r}$ to be $\bar{\mathbf{r}} = (r_j, r_{j-1}, \ldots, r_1)$.

Since $N_\beta(\pi) = N_{F(\beta)}(\pi^\mathbf{r})$ in 5.4.11, for any $\pi \in \mathsf{CA}_\mathbf{q}$, we have $\pi^\mathbf{r} \in \mathsf{CA}_{F(\mathbf{q})}$.

Similarly, since $N_\beta(\pi^{\bar{\mathbf{r}}}) = N_{F(\beta)}(\pi)$ in 5.4.11, for any $\pi \in \mathsf{CA}_{F(\mathbf{q})}$, we have $\pi^{\bar{\mathbf{r}}} \in \mathsf{CA}_\mathbf{q}$.

(1) If $\pi \in \mathsf{CA}_{F(\mathbf{q})}$, then $\pi^{\bar{\mathbf{r}}} \in \mathsf{CA}_\mathbf{q}$. By 5.4.7, $N_{st}(\pi^{\bar{\mathbf{r}}}) \geq \mathbf{q}_{\min}$. Since $N_{F(st)}(\pi) = N_{st}(\pi^{\bar{\mathbf{r}}})$, $N_{F(st)}(\pi) \geq \mathbf{q}_{\min}$. Again by 5.4.7, there exists $\pi_1 \in \mathsf{CA}_\mathbf{q}$ such that $N_{st}(\pi_1) = \mathbf{q}_{\min}$. Since $N_{F(st)}(\pi_1^\mathbf{r}) = N_{st}(\pi_1)$, $N_{F(st)}(\pi_1^\mathbf{r}) = \mathbf{q}_{\min}$. Since $\pi_1 \in \mathsf{CA}_\mathbf{q}$, $\pi_1^\mathbf{r} \in \mathsf{CA}_{F(\mathbf{q})}$. We are done.

(2) For any $\pi \in \mathsf{CA}_{F(\mathbf{q})}$, we have $\pi^{\bar{\mathbf{r}}} \in \mathsf{CA}_\mathbf{q}$ and $N_{F(st)}(\pi) = N_{st}(\pi^{\bar{\mathbf{r}}})$. If $\mathbf{q}_{\min}$ is finite and $\#\{i \mid q_i = \mathbf{q}_{\min}\} = 1$, then by 5.4.7 $N_{st}(\pi^{\bar{\mathbf{r}}}) = \mathbf{q}_{\min}$, and so $N_{F(st)}(\pi) = \mathbf{q}_{\min}$.

We next prove the converse. For any $\pi \in \mathsf{CA}_\mathbf{q}$, $\pi^\mathbf{r} \in \mathsf{CA}_{F(\mathbf{q})}$ and $N_{st}(\pi) = N_{F(st)}(\pi^\mathbf{r})$. Thus if $N_{F(st)}(\pi) = \mathbf{q}_{\min}$ holds for all $\pi \in \mathsf{CA}_{F(\mathbf{q})}$, then $N_{st}(\pi) = \mathbf{q}_{\min}$ holds for all $\pi \in \mathsf{CA}_\mathbf{q}$. So $\#\{i \mid q_i = \mathbf{q}_{\min}\} = 1$ by 5.4.7 and the assumption $\mathbf{q}_{\min}$ is finite. □

§ 5.4.3 | Examples

Note that 5.4.7 demonstrates that the generalised GV invariant N_{st} is constrained by properties of the tuple $(N_{ss}, \ldots, N_{tt})$, and 5.4.12 demonstrates that $N_{F(st)}$ is constrained by properties of the tuple $(N_{F(ss)}, \ldots, N_{F(tt)})$.

Example 5.4.13. Consider $n = 2$, $s = 1$ and $t = 2$, and apply different F in 5.4.12. The following table illustrates that N_β is constrained by by properties of the tuple $(N_{\beta_1}, N_{\beta_2})$ where $(\beta_1, \beta_2, \beta) := \big(F(11), F(22), F(12)\big)$.

F	β_1, β_2	β
id	$11, 22$	12
$\lvert F_{(1)}\rvert$	$11, 12$	22
$\lvert F_{(2)}\rvert$	$12, 22$	11

As an explicit example, for any cA_2 crepant resolution π the following holds. To ease notation, we write N_β for $N_\beta(\pi)$ in the following.

(1) By the first line, $N_{12} \geq \min(N_{11}, N_{22})$. Moreover, if $N_{11} \neq N_{22}$, then N_{12} must be $\min(N_{11}, N_{22})$.

(2) By the second line, $N_{22} \geq \min(N_{11}, N_{12})$. Moreover, if $N_{11} \neq N_{12}$, then N_{22} must be $\min(N_{11}, N_{12})$.

(3) By the third line, $N_{11} \geq \min(N_{12}, N_{22})$. Moreover, if $N_{12} \neq N_{22}$, then N_{11} must be $\min(N_{12}, N_{22})$.

Example 5.4.14. Consider $n = 3$, $s = 1$ and $t = 3$, and apply different F in 5.4.12. The following table illustrates that N_β is constrained by properties of the tuple $(N_{\beta_1}, N_{\beta_2}, N_{\beta_3})$ where $(\beta_1, \beta_2, \beta_3, \beta) := \big(F(11), F(22), F(33), F(13)\big)$.

F	$\beta_1, \beta_2, \beta_3$	β
id	$11, 22, 33$	13
$\lvert F_{(1)} \rvert$	$11, 12, 33$	23
$\lvert F_{(2)} \rvert$	$12, 22, 23$	13
$\lvert F_{(3)} \rvert$	$11, 23, 33$	12
$\lvert F_{(1,2)} \rvert$	$12, 11, 23$	33
$\lvert F_{(2,1)} \rvert$	$22, 12, 13$	23
$\lvert F_{(2,3)} \rvert$	$13, 23, 22$	12
$\lvert F_{(3,2)} \rvert$	$12, 33, 23$	11
$\lvert F_{(1,3)} \rvert$	$11, 13, 33$	22

With the results in 5.4.7, 5.4.12 and 5.4.13, we can give all the tuples that generalised GV tuples of cA_2 crepant resolutions can arise.

Corollary 5.4.15. *The generalised GV tuples of cA_2 crepant resolutions have the following two possibilities:*

$$\begin{array}{ccc} N_{11} & & N_{22} \\ & N_{12} & \end{array} = \begin{array}{ccc} p & & q \\ & \min(p,q) & \end{array} \quad \textit{or} \quad \begin{array}{ccc} p & & p \\ & r & \end{array}$$

where p, q, $r \in \mathbb{N}_\infty$ with $p \neq q$ and $r \geq p$. All possible such p, q, r arise.

Proof. Fix some p, $q \in \mathbb{N}_\infty$. By 5.4.7(1), for any cA_2 crepant resolution π satisfying $N_{11}(\pi) = p$ and $N_{22}(\pi) = q$, necessarily $N_{12}(\pi) \geq \min(p, q)$. Moreover, there exists such a π with $N_{12}(\pi) = \min(p, q)$. If furthermore $p \neq q$, then $N_{12}(\pi) = \min(p, q)$ by 5.4.7(2) which proves the first possibility.

Then we consider the case of $p = q$. Since by 5.4.13 N_{22} is constrained by properties of the tuple (N_{11}, N_{12}), for any $r \geq p$ by 5.4.12(1) there exists a cA_2 crepant resolution π such that $N_{11}(\pi) = p$, $N_{12}(\pi) = r$ and $N_{22}(\pi) = \min(p, r) = p$. The second possibility follows. $\square$

Chapter 6

Special Cases: A_3

This chapter considers the special case $Q_{3,\{1,2,3\}}$, namely

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left, "a_1"] & \underset{2}{\bullet} \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \underset{3}{\bullet} \arrow[l, bend left, "b_2"] \end{tikzcd} \qquad \begin{array}{l} \mathsf{x}_1 = a_1b_1,\ \mathsf{x}_1' = b_1a_1 \\ \mathsf{x}_2 = a_2b_2,\ \mathsf{x}_2' = b_2a_2 \end{array}$$

and describes the full isomorphism classes of Type A potentials, and the derived equivalence classes of those with finite-dimensional Jacobi algebras. This generalises the results in [DWZ, E1, H2].

Notation 6.0.1. In this chapter, for simplicity, we will adopt the following notation. Recall the notation f_d, $f_{\geq d}$ in 4.1.11.

(1) Write Q for $Q_{3,\{1,2,3\}}$, $\mathsf{x} := \mathsf{x}_1'$ and $\mathsf{y} := \mathsf{x}_2$, whereas $\mathsf{x}' := \mathsf{x}_1$ and $\mathsf{y}' := \mathsf{x}_2'$.

(2) Suppose that f is a Type A potential on Q. Then define the *base part* of f as $f_b := \kappa_1\mathsf{x}^p + \mathsf{x}\mathsf{y} + \kappa_2\mathsf{y}^q$ where $\kappa_1\mathsf{x}^p$, $\kappa_2\mathsf{y}^q$ is the lowest degree monomial of x, y in f respectively. If there is no monomial of x (or y) in f, we assume $\kappa_1 = 0$ (or $\kappa_2 = 0$). Then define the *redundant part* of f as $f_r := f - f_b$.

(3) Given any Type A potential f on Q with $f_b = \kappa_1\mathsf{x}^p + \mathsf{x}\mathsf{y} + \kappa_2\mathsf{y}^q$, we give a new definition of degree as follows, which differs from 4.1.3. For any $t \geq 0$, define

$$\deg(\mathsf{x}^{p+t}) := t + 2, \quad \deg(\mathsf{y}^{q+t}) := t + 2.$$

The degree of binomials in x and y is the same as 4.1.3. We also write f_d for the degree d piece of f with respect to this new definition (overwriting 4.1.11). Similar for $f_{ij,d}$, $\mathcal{O}_d$ and $\mathcal{O}_{ij,d}$. This new definition of degree is natural since now $f_b = f_2$ and $f_r = f_{\geq 3}$, which will unify the proof below.

(4) Let f be a Type A potential on Q with $f_b = \kappa_1\mathsf{x}^p + \mathsf{x}\mathsf{y} + \kappa_2\mathsf{y}^q$. Recall the definition of

$A^d_{ij}(f)$ in 5.1.2. To ease notation, write the matrices $A_{11}(f)$, $A_{22}(f)$ and $A_{12}(f)$ for

$$A_{11}(f) := A^p_{11}(f) = \begin{bmatrix} p\kappa_1 \end{bmatrix}, \qquad A_{22}(f) := A^q_{11}(f) = \begin{bmatrix} q\kappa_2 \end{bmatrix},$$

$$A_{12}(f) := A^2_{12}(f) = \begin{bmatrix} \varepsilon_{12}(f) & 1 \\ 1 & \varepsilon_{22}(f) \end{bmatrix},$$

where $\varepsilon_{12}(f) = \begin{cases} 2\kappa_1 & \text{if } p = 2 \\ 0 & \text{if } p > 2 \end{cases}$ and $\varepsilon_{22}(f) = \begin{cases} 2\kappa_2 & \text{if } q = 2 \\ 0 & \text{if } q > 2 \end{cases}$.

§ 6.1 | Normalization

The purpose of this section is to prove 6.1.17, which gives the isomorphism classes of Type A potentials on Q.

Notation 6.1.1. In this section, we assume $f = f_b + f_r$ is a Type A potential on Q with $f_b = \kappa_1 \mathsf{x}^p + \mathsf{xy} + \kappa_2 \mathsf{y}^q$, and will freely use the notations f_d, $f_{\geq d}$, $f_{ij,d}$, $\mathcal{O}_d$ and $\mathcal{O}_{ij,d}$ in 6.0.1(3).

The following results show that we can commute x and y in f.

Lemma 6.1.2. *Suppose that $f_r = g + \lambda \mathsf{yx}c$ where $0 \neq \lambda \in \mathbb{C}$, $d := \deg(\mathsf{yx}c)$ and c is a cycle with $\deg(c) \geq 1$. Then there exists a path degree $d-1$ right-equivalence,*

$$\vartheta\colon f \overset{d-1}{\rightsquigarrow} f_b + g + \lambda \mathsf{xy}c + \mathcal{O}_{d+1}.$$

Proof. Applying the depth $d-1$ unitriangular automorphism $\vartheta : a_1 \mapsto a_1 - \lambda a_1 c$, $b_1 \mapsto b_1 + \lambda c b_1$ gives,

$$\begin{aligned} \vartheta\colon f &\mapsto \kappa_1((b_1 + \lambda c b_1)(a_1 - \lambda a_1 c))^p + (b_1 + \lambda c b_1)(a_1 - \lambda a_1 c)\mathsf{y} + \kappa_2 \mathsf{y}^q + \lambda \mathsf{yx}c + g + \mathcal{O}_{d+1} \\ &= f_b - \lambda \mathsf{x}c\mathsf{y} + \lambda c\mathsf{xy} + \lambda \mathsf{yx}c + g + \mathcal{O}_{d+1} \\ &\overset{d}{\sim} f_b + g + \lambda \mathsf{xy}c + \mathcal{O}_{d+1}, \end{aligned}$$

since similar to the proof of 4.1.15, the degree of the terms generated by $f_r = g + \lambda \mathsf{yx}c$ after applying ϑ is greater than or equal to $d+1$. $\square$

Corollary 6.1.3. *Suppose that $f_r = g + \lambda c$ where $0 \neq \lambda \in \mathbb{C}$, $d := \deg(c)$ and c is a cycle with $\mathbf{T}(c)_1 = i$ and $\mathbf{T}(c)_2 = j$. Then there exists a path degree $d-1$ right-equivalence,*

$$\theta\colon f \overset{d-1}{\rightsquigarrow} f_b + g + \lambda \mathsf{x}^i \mathsf{y}^j + \mathcal{O}_{d+1}.$$

Proof. If $j = 0$, then $c \sim \mathsf{x}^i$, so there is nothing to prove. The case of $i = 0$ is similar. Thus we assume $i, j > 0$. Firstly, note that $c \sim \mathsf{x}^{i_1}\mathsf{y}^{j_1}\mathsf{x}^{i_2}\mathsf{y}^{j_2}\dots\mathsf{x}^{i_k}\mathsf{y}^{j_k}$ where $\sum_{t=1}^k i_t = i$ and

$\sum_{t=1}^{k} j_t = j$. Since 6.1.2 can commute yx contained in c to xy, we can apply the ϑ in 6.1.2 repeatedly until we commute all yx to xy, and so

$$\theta\colon f \overset{d-1}{\rightsquigarrow} f_b + g + \lambda \mathsf{x}^i \mathsf{y}^j + \mathcal{O}_{d+1}. \qquad \square$$

Remark 6.1.4. With notation in 6.1.3, we can transform λc to $\lambda \mathsf{x}^i \mathsf{y}^j$ up to higher degree $\mathcal{O}_{d+1}$ where $d = \deg(c)$. Moreover, since we will normalise the potential degree by degree in the following part of this section, we can assume $c = \mathsf{x}^i \mathsf{y}^j$.

Then, we start normalizing the potential f, and the basic idea is to use f_b to normalize f_r degree by degree. For any integer $s \geq 1$, we define the following depth $s+1$ unitriangular automorphisms.

$$\varphi_{11,s}\colon a_1 \mapsto a_1 + \lambda a_1 \mathsf{x}^s, \tag{6.1.A}$$
$$\varphi_{22,s}\colon a_2 \mapsto a_2 + \lambda \mathsf{y}^s a_2, \tag{6.1.B}$$
$$\varphi_{12,s}\colon a_1 \mapsto a_1 + \lambda_1 a_1 \mathsf{x}^{s-1}\mathsf{y},\ \ a_2 \mapsto a_2 + \lambda_2 \mathsf{x}^s a_2, \tag{6.1.C}$$

where λ, λ_1, $\lambda_2 \in \mathbb{C}$.

Lemma 6.1.5. *The $\varphi_{11,s}$ (6.1.A) induces a degree $s+1$ right-equivalence,*

$$\varphi_{11,s}\colon f \overset{s+1}{\rightsquigarrow} f + \lambda p \kappa_1 \mathsf{x}^{p+s} + \mathcal{O}_{12,s+2} + \mathcal{O}_{s+3}.$$

Proof. Applying $\varphi_{11,s}\colon a_1 \mapsto a_1 + \lambda a_1 \mathsf{x}^s$ to f gives

$$\begin{aligned} \varphi_{11,s}\colon f &\mapsto \kappa_1 (b_1(a_1 + \lambda a_1 \mathsf{x}^s))^p + b_1(a_1 + \lambda a_1 \mathsf{x}^s)\mathsf{y} + \kappa_2 \mathsf{y}^q + f_r + \mathcal{O}_{s+3} \\ &\overset{s+2}{\sim} f + \lambda p \kappa_1 \mathsf{x}^{p+s} + \lambda \mathsf{x}^{s+1}\mathsf{y} + \mathcal{O}_{s+3} \\ &= f + \lambda p \kappa_1 \mathsf{x}^{p+s} + \mathcal{O}_{12,s+2} + \mathcal{O}_{s+3}. \end{aligned} \qquad \square$$

Lemma 6.1.6. *The $\varphi_{22,s}$ (6.1.B) induces a degree $s+1$ right-equivalence,*

$$\varphi_{22,s}\colon f \overset{s+1}{\rightsquigarrow} f + \lambda q \kappa_2 \mathsf{y}^{q+s} + \mathcal{O}_{12,s+2} + \mathcal{O}_{s+3}.$$

Proof. The proof is similar to 6.1.5. $\square$

Lemma 6.1.7. *The $\varphi_{12,s}$ (6.1.C) induces a degree $s+1$ right-equivalence,*

$$\varphi_{12,s}\colon f \overset{s+1}{\rightsquigarrow} f + \begin{bmatrix} \mathsf{x}^{s+1}\mathsf{y} & \mathsf{x}^s\mathsf{y}^2 \end{bmatrix} A_{12}(f) \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \mathcal{O}_{s+3}.$$

Proof. Applying $\varphi_{12,s}$ to f gives

$$\begin{aligned}\varphi_{12,s}\colon f &\mapsto \kappa_1(\mathsf{x}+\lambda_1\mathsf{x}^s\mathsf{y})^p + (\mathsf{x}+\lambda_1\mathsf{x}^s\mathsf{y})(\mathsf{y}+\lambda_2\mathsf{x}^s\mathsf{y}) + \kappa_2(\mathsf{y}+\lambda_2\mathsf{x}^s\mathsf{y})^q + f_r + \mathcal{O}_{s+3}\\ &\overset{s+2}{\sim} f + \lambda_1 p\kappa_1\mathsf{x}^{p+s-1}\mathsf{y} + \lambda_2\mathsf{x}^{s+1}\mathsf{y} + \lambda_1\mathsf{x}^s\mathsf{y}^2 + \lambda_2 q\kappa_2\mathsf{x}^s\mathsf{y}^q + \mathcal{O}_{s+3}\\ &= f + \begin{bmatrix}\mathsf{x}^{s+1}\mathsf{y} & \mathsf{x}^s\mathsf{y}^2\end{bmatrix} A_{12}(f)\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} + \mathcal{O}_{s+3}.\end{aligned}$$

Recall the definition of $A_{12}(f)$ in 6.0.1(4). In the above last equation, we move $\lambda_1 p\kappa_1\mathsf{x}^{p+s-1}\mathsf{y}$ into $\mathcal{O}_{s+3}$ when $p>2$, and move it into $\begin{bmatrix}\mathsf{x}^{s+1}\mathsf{y} & \mathsf{x}^s\mathsf{y}^2\end{bmatrix} A_{12}(f)\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$ when $p=2$. Similar for $\lambda_2 q\kappa_2\mathsf{x}^s\mathsf{y}^q$. □

Proposition 6.1.8. *With notation in 6.1.1, for any $d\geq 3$, there exists a path degree $d-1$ right-equivalence*

$$\phi_d\colon f \overset{d-1}{\rightsquigarrow} f_{<d} + c_d + \mathcal{O}_{d+1},$$

where the c_d is defined to be

$$c_d = \begin{cases} 0 & \text{if } \det(A_{12}(f))\neq 0\\ \mu\mathsf{x}^{p+d-2} & \text{if } \det(A_{12}(f)) = 0\end{cases}$$

for some $\mu\in\mathbb{C}$.

Proof. We first rewrite $f_r = f_d + g$ and $f_d = f_{11,d} + f_{12,d} + f_{22,d}$ where $f_{11,d} = \alpha_1\mathsf{x}^{p+d-2}$ and $f_{22,d} = \alpha_2\mathsf{y}^{q+d-2}$ for some $\alpha_1,\alpha_2\in\mathbb{C}$.

Recall that $f_b = \kappa_1\mathsf{x}^p + \mathsf{x}\mathsf{y} + \kappa_2\mathsf{y}^q$ in 6.1.1. If $\kappa_2 = 0$, then there is no monomial of y in f, and so $\alpha_2 = 0$. Otherwise, $\kappa_2\neq 0$, so set $\lambda = -\alpha_2/(q\kappa_2)$ and applying 6.1.6 to obtain,

$$\begin{aligned}\varphi_{22,d-2}\colon f &\overset{d-1}{\rightsquigarrow} f + \lambda q\kappa_2\mathsf{y}^{q+d-2} + \mathcal{O}_{12,d} + \mathcal{O}_{d+1}\\ &= f_b + g + f_d + \lambda q\kappa_2\mathsf{y}^{q+d-2} + \mathcal{O}_{12,d} + \mathcal{O}_{d+1} && (f = f_b + f_r,\ f_r = f_d + g)\\ &= f_b + g + f_{11,d} + f_{12,d} + (\alpha_2 + \lambda q\kappa_2)\mathsf{y}^{q+d-2} + \mathcal{O}_{12,d} + \mathcal{O}_{d+1}\\ & && (f_d = f_{11,d} + f_{12,d} + f_{22,d},\ f_{22,d} = \alpha_2\mathsf{y}^{q+d-2})\\ &= f_b + g + f_{11,d} + f_{12,d} + \mathcal{O}_{12,d} + \mathcal{O}_{d+1} && (\text{since } \lambda = -\alpha_2/(q\kappa_2))\\ &= f_b + g + f_{11,d} + \mathcal{O}_{12,d} + \mathcal{O}_{d+1}. && (6.1.\text{D})\end{aligned}$$

Set $\mathsf{f}_1 := f_b + g + f_{11,d} + \mathcal{O}_{12,d} + \mathcal{O}_{d+1}$. The proof splits into cases.

(1) $\det(A_{12}(f)) = 0$.

By 4.1.19, we can transform the binomial terms $\mathcal{O}_{12,d}$ in f_1 to the monomials of x. More precisely, there exists a path degree $d-1$ right-equivalence,

$$\begin{aligned} \rho_d \colon \mathsf{f}_1 &\overset{d-1}{\rightsquigarrow} f_b + g + \mu \mathsf{x}^{p+d-2} + \mathcal{O}_{d+1}, && (f_{11,d} = \alpha_1 \mathsf{x}^{p+d-2}) \\ &= f_{<d} + \mu \mathsf{x}^{p+d-2} + \mathcal{O}_{d+1}. && (f_b + g = f_{<d} + f_{>d}) \end{aligned}$$

for some $\mu \in \mathbb{C}$. Set $\phi_d := \rho_d \circ \varphi_{22,d-2}$, we are done.

(2) $\det(A_{12}(f)) \neq 0$.

Similar to (6.1.D), applying 6.1.5 to f_1 gives

$$\varphi_{11,d-2} \colon \mathsf{f}_1 \overset{d-1}{\rightsquigarrow} \mathsf{f}_2 := f_b + g + \mathcal{O}_{12,d} + \mathcal{O}_{d+1}.$$

Then we continue to normalize the $\mathcal{O}_{12,d}$ in f_2. It is clear that f_2 satisfies the assumption of 4.1.15. Thus we can apply 4.1.15 repeatedly until,

$$\vartheta \colon \mathsf{f}_2 \overset{d-1}{\rightsquigarrow} \mathsf{f}_3 := f_b + g + \beta \mathsf{x}^{d-1}\mathsf{y} + \mathcal{O}_{d+1}$$

for some $\beta \in \mathbb{C}$. Then by 6.1.7,

$$\begin{aligned} \varphi_{12,d-2} \colon \mathsf{f}_3 &\overset{d-1}{\rightsquigarrow} \mathsf{f}_3 + \begin{bmatrix} \mathsf{x}^{d-1}\mathsf{y} & \mathsf{x}^{d-2}\mathsf{y}^2 \end{bmatrix} A_{12}^2(f) \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \mathcal{O}_{d+1} \\ &= f_b + g + \beta \mathsf{x}^{d-1}\mathsf{y} + \begin{bmatrix} \mathsf{x}^{d-1}\mathsf{y} & \mathsf{x}^{d-2}\mathsf{y}^2 \end{bmatrix} A_{12}^2(f) \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \mathcal{O}_{d+1}. \end{aligned}$$

Since $\det A_{12}^2(f) \neq 0$, we can solve (λ_1, λ_2) to make

$$\beta \mathsf{x}^{d-1}\mathsf{y} + \begin{bmatrix} \mathsf{x}^{d-1}\mathsf{y} & \mathsf{x}^{d-2}\mathsf{y}^2 \end{bmatrix} A_{12}^2(f) \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0.$$

Thus we have

$$\begin{aligned} \varphi_{12,d-2} \colon \mathsf{f}_3 &\overset{d-1}{\rightsquigarrow} f_b + g + \mathcal{O}_{d+1} \\ &= f_{<d} + \mathcal{O}_{d+1}. && (f_b + g = f_{<d} + f_{>d}) \end{aligned}$$

Set $\phi_d := \varphi_{12,d-2} \circ \vartheta \circ \varphi_{11,d-2} \circ \varphi_{22,d-2}$, we are done. □

Proposition 6.1.9. *With notation in 6.1.1, there exists a right-equivalence,*

$$\Phi \colon f \rightsquigarrow f_b + c$$

where the c is defined to be

$$c = \begin{cases} 0 & \textit{if } \det A_{12}(f) \neq 0 \\ \sum_{i=1}^{\infty} \mu_i \mathsf{x}^{p+i} & \textit{if } \det A_{12}(f) = 0 \end{cases}$$

for some $\mu_i \in \mathbb{C}$.

Proof. We first apply the ϕ_3 in 6.1.8,

$$\phi_3 \colon f \overset{2}{\rightsquigarrow} \mathsf{f}_1 := f_{<3} + c_3 + \mathcal{O}_4,$$

where the c_3 is the same as in 6.1.8. Then we continue to apply the ϕ_4 in 6.1.8 to f_1,

$$\phi_4 \colon \mathsf{f}_1 \overset{3}{\rightsquigarrow} \mathsf{f}_2 := (\mathsf{f}_1)_{<4} + c_4 + \mathcal{O}_5 = f_{<3} + \sum_{d=3}^{4} c_d + \mathcal{O}_5.$$

where the c_4 is the same as in 6.1.8. Thus repeating this process $s-2$ times gives

$$\phi_s \circ \cdots \circ \phi_4 \circ \phi_3 \colon f \rightsquigarrow \mathsf{f}_s := f_{<3} + \sum_{d=3}^{s} c_d + \mathcal{O}_{s+1}.$$

Since ϕ_d is a path degree $d-1$ right-equivalence for each $d \geq 3$ by 6.1.8, by 2.1.9 $\Phi := \lim_{s\to\infty} \phi_s \circ \cdots \circ \phi_4 \circ \phi_3$ exists, and further

$$\Phi \colon f \rightsquigarrow f_{<3} + \sum_{d=3}^{\infty} c_d = f_b + \sum_{d=3}^{\infty} c_d,$$

where each c_d is the same as in 6.1.8. Thus set $c := \sum_{d=3}^{\infty} c_d$, we are done. $\square$

The above 6.1.9 shows that we can eliminate all terms in f_r when $\det A_{12}(f) \neq 0$. Thus we next consider the cases of f with $\det A_{12}(f) = 0$. The following lemma holds immediately from the definition of $A_{12}(f)$ in 6.0.1(4).

Lemma 6.1.10. $\det A_{12}(f) = 0$ *if and only if* $f_b = \kappa_1 \mathsf{x}^2 + \mathsf{xy} + \kappa_2 \mathsf{y}^2$ *with* $4\kappa_1\kappa_2 = 1$.

Lemma 6.1.11. *With notation in* 6.1.1*, suppose that* f *satisfies* $\det A_{12}(f) = 0$*, and* $f_r = \mu \mathsf{x}^s + \mathcal{O}_t$ *where* $t > s \geq 3$ *and* $0 \neq \mu \in \mathbb{C}$*. Then there exists a path degree* $t-s+1$ *right-equivalence* ψ_t *such that*

$$\psi_t \colon f \overset{t-s+1}{\rightsquigarrow} f_b + \mu \mathsf{x}^s + \mathcal{O}_{t+1}.$$

Proof. Since $\det A_{12}^2(f) = 0$, by 6.1.10 $f_b = \kappa_1 \mathsf{x}^2 + \mathsf{xy} + \kappa_2 \mathsf{y}^2$ with $4\kappa_1\kappa_2 = 1$. If the degree t terms in $\mathcal{O}_t$ are zero, there is nothing to prove. Otherwise, we first apply 4.1.15 repeatedly and obtain,

$$\vartheta \colon f \overset{t-1}{\rightsquigarrow} \mathsf{f}_1 := f_b + \mu \mathsf{x}^s + \beta \mathsf{x}^{t-1}\mathsf{y} + \mathcal{O}_{t+1},$$

for some $\beta \in \mathbb{C}$. If $\beta = 0$, we are done. Otherwise, we next apply $\varphi_{12,t-s}$ in 6.1.7 which gives

$$\begin{aligned}
\varphi_{12,t-s}\colon \mathsf{f}_1 \mapsto & \kappa_1(\mathsf{x}+\lambda_1\mathsf{x}^{t-s}\mathsf{y})^2 + (\mathsf{x}+\lambda_1\mathsf{x}^{t-s}\mathsf{y})(\mathsf{y}+\lambda_2\mathsf{x}^{t-s}\mathsf{y}) + \kappa_2(\mathsf{y}+\lambda_2\mathsf{x}^{t-s}\mathsf{y})^2 \\
& + \mu(\mathsf{x}+\lambda_1\mathsf{x}^{t-s}\mathsf{y})^s + \beta\mathsf{x}^{t-1}\mathsf{y} + \mathcal{O}_{t+1} \\
\overset{t-s+2}{\sim} & f_b + \mu\mathsf{x}^s + (2\kappa_1\lambda_1+\lambda_2)\mathsf{x}^{t-s+1}\mathsf{y} + (\lambda_1+2\kappa_2\lambda_2)\mathsf{x}^{t-s}\mathsf{y}^2 + (s\mu\lambda_1+\beta)\mathsf{x}^{t-1}\mathsf{y} \\
& + (\kappa_1\lambda_1^2+\kappa_2\lambda_2^2+\lambda_1\lambda_2)\mathsf{x}^{2(t-s)}\mathsf{y}^2 + \mathcal{O}_{t+1}.
\end{aligned}$$

Since $4\kappa_1\kappa_2 = 1$, then $\frac{1}{2\kappa_1} = 2\kappa_2$, and so any λ_1 and λ_2 with $\frac{\lambda_1}{\lambda_2} = -\frac{1}{2\kappa_1} = -2\kappa_2$ satisfies the following system of equations

$$\begin{cases} 2\kappa_1\lambda_1 + \lambda_2 = 0 \\ \lambda_1 + 2\kappa_2\lambda_2 = 0 \\ \kappa_1\lambda_1^2 + \kappa_2\lambda_2^2 + \lambda_1\lambda_2 = 0. \end{cases}$$

We next choose λ_1 to satisfy $s\mu\lambda_1 + \mu_t = 0$, and set $\lambda_2 = -2\kappa_1\lambda_1$. This makes the coefficients of $\mathsf{x}^{t-s+1}\mathsf{y}$, $\mathsf{x}^{t-s}\mathsf{y}^2$, $\mathsf{x}^{t-1}\mathsf{y}$ and $\mathsf{x}^{2(t-s)}\mathsf{y}^2$ equal to zero in the above potential. Set $\psi_t := \varphi_{12,t-s}\circ\vartheta$, we are done. $\square$

The following shows that when $\det A_{12}^2(f) = 0$, the leading term of f_r can eliminate all the other terms.

Proposition 6.1.12. *With notation in 6.1.1, suppose that f satisfies $\det A_{12}(f) = 0$. Then there exists a right-equivalence Ψ such that*

$$\Psi\colon f \rightsquigarrow f_b \text{ or } f_b + \mu\mathsf{x}^s,$$

where $0 \neq \mu \in \mathbb{C}$ and $s \geq 3$.

Proof. Since $\det A_{12}(f) = 0$, then by 6.1.9

$$\Phi\colon f \rightsquigarrow \mathsf{f}_1 := f_b + \sum_{i=1}^{\infty} \mu_i\mathsf{x}^{i+2}.$$

If all $\mu_i = 0$, then $f \rightsquigarrow f_b$. Otherwise, set s to be the smallest integer satisfying $\mu_s \neq 0$. Then by 6.1.11 applied to f_1, there exists

$$\psi_{s+1}\colon \mathsf{f}_1 \overset{2}{\rightsquigarrow} \mathsf{f}_2 := f_b + \mu\mathsf{x}^s + \mathcal{O}_{s+2}.$$

Thus, repeating this process k times gives

$$\psi_{s+k}\circ\cdots\circ\psi_{s+2}\circ\psi_{s+1}\colon \mathsf{f}_1 \rightsquigarrow \mathsf{f}_{k+1} := f_b + \mu\mathsf{x}^s + \mathcal{O}_{s+k+1}.$$

Since ψ_t is a degree $t-s+1$ right-equivalence for each $t > s$ by 6.1.11, by 2.1.9 $\Psi' :=$

$\lim_{k\to\infty} \psi_{s+k} \circ \cdots \circ \psi_{s+2} \circ \psi_{s+1}$ exists, and further

$$\Psi' \colon \mathsf{f}_1 \rightsquigarrow f_b + \mu \mathsf{x}^s.$$

Set $\Psi := \Psi' \circ \Phi$, we are done. □

Combining 6.1.9, 6.1.10 and 6.1.12 gives the following result.

Proposition 6.1.13. *Any Type A potential on Q must be right-equivalent to one of the following potentials:*

(1) $\kappa_1 \mathsf{x}^2 + \mathsf{xy} + \kappa_2 \mathsf{y}^2$ *where* $\kappa_1, \kappa_2 \neq 0$ *and* $4\kappa_1\kappa_2 \neq 1$.

(2) $\kappa_1 \mathsf{x}^2 + \mathsf{xy} + \kappa_2 \mathsf{y}^2 + \mu \mathsf{x}^s$ *where* $4\kappa_1\kappa_2 = 1$, $0 \neq \mu \in \mathbb{C}$ *and* $s \geq 3$.

(3) $\kappa_1 \mathsf{x}^p + \mathsf{xy} + \kappa_2 \mathsf{y}^q$ *where* $(p,q) \neq (2,2)$ *and* $\kappa_1, \kappa_2 \neq 0$.

(4) $\kappa_1 \mathsf{x}^2 + \mathsf{xy} + \kappa_2 \mathsf{y}^2$ *where* $4\kappa_1\kappa_2 = 1$.

(5) $\kappa_1 \mathsf{x}^p + \mathsf{xy}$ *where* $p \geq 2$ *and* $\kappa_1 \neq 0$.

(6) $\mathsf{xy} + \kappa_2 \mathsf{y}^q$ *where* $q \geq 2$ *and* $\kappa_2 \neq 0$.

(7) xy.

Proof. Recall in 6.0.1 and 6.1.1, any Type A potential on Q has the form of $f = f_b + f_r$ where $f_b = \kappa_1 \mathsf{x}^p + \mathsf{xy} + \kappa_2 \mathsf{y}^q$.

When $\det A_{12}(f) = 0$, namely $p = q = 2$ and $4\kappa_1\kappa_2 = 1$ by 6.1.10, then by 6.1.9 $f \cong f_b$ or $f_b + \mu \mathsf{x}^s$ where $0 \neq \mu \in \mathbb{C}$ and $s \geq 3$. These are items (4) and (2) in the statement.

When $\det A_{12}(f) \neq 0$, by 6.1.9 $f \cong f_b$. Again by 6.1.10, we have $(p,q) \neq (2,2)$ or $4\kappa_1\kappa_2 \neq 1$, so f must belong to one of the following cases.

a) $\kappa_1 \neq 0$ and $\kappa_2 = 0$.

b) $\kappa_1 = 0$ and $\kappa_2 \neq 0$.

c) $\kappa_1 = 0$ and $\kappa_2 = 0$.

d) $\kappa_1, \kappa_2 \neq 0$, $4\kappa_1\kappa_2 \neq 1$ and $p = q = 2$.

e) $\kappa_1, \kappa_2 \neq 0$ and $(p,q) \neq (2,2)$.

The a), b), c), d) and e) are items (5), (6), (7), (1) and (3) in the statement. □

Then we continue to normalise the coefficients of the potentials in 6.1.13. In the statement of 6.1.14, case (1) is placed first since it represents the most basic family: its Jacobi algebra has the smallest dimension (namely, 20) and it exhibits moduli, in contrast to the discrete classification of Du Val singularities. By contrast, case (2) has a different structural form from all of the remaining cases, and is in fact derived equivalent to certain potentials in case (3) (see 6.2.6).

Corollary 6.1.14. *Any Type A potential on Q must be isomorphic to one of the following potentials:*

(1) $\mathsf{x}^2 + \mathsf{xy} + \lambda\mathsf{y}^2$ *where* $0, \frac{1}{4} \neq \lambda \in \mathbb{C}$.

(2) $\mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 + \mathsf{x}^s$ *where* $s \geq 3$.

(3) $\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q$ *where* $(p, q) \neq (2, 2)$.

(4) $\mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2$.

(5) $\mathsf{x}^p + \mathsf{xy}$ *where* $p \geq 2$.

(6) $\mathsf{xy} + \mathsf{y}^q$ *where* $q \geq 2$.

(7) xy.

Proof. (1) Applying $a_1 \mapsto \lambda_1 a_1,\ a_2 \mapsto \lambda_2 a_2$ where $\lambda_1, \lambda_2 \in \mathbb{C}$ to (1) gives

$$\kappa_1\mathsf{x}^2 + \mathsf{xy} + \kappa_2\mathsf{y}^2 \mapsto \lambda_1^2\kappa_1\mathsf{x}^2 + \lambda_1\lambda_2\mathsf{xy} + \lambda_2^2\kappa_2\mathsf{y}^2.$$

Since $\kappa_1 \neq 0$, we can solve (λ_1, λ_2) that ensures $\lambda_1^2\kappa_1 = \lambda_1\lambda_2 = 1$ holds. Moreover, since $\kappa_2 \neq 0$ and $4\kappa_1\kappa_2 \neq 1$, $\lambda_2^2\kappa_2 = \kappa_1\kappa_2 \neq 0, \frac{1}{4}$. Set $\lambda := \lambda_2^2\kappa_2$. Thus $\kappa_1\mathsf{x}^2+\mathsf{xy}+\kappa_2\mathsf{y}^2 \mapsto \mathsf{x}^2+\mathsf{xy}+\lambda\mathsf{y}^2$ where $\lambda \neq 0, \frac{1}{4}$.

(2) Applying $\varphi\colon a_1 \mapsto \lambda_1 a_1,\ a_2 \mapsto \lambda_2 a_2$ where $\lambda_1, \lambda_2 \in \mathbb{C}$ to (2) gives

$$\kappa_1\mathsf{x}^2 + \mathsf{xy} + \kappa_2\mathsf{y}^2 + \mu\mathsf{x}^s \mapsto \lambda_1^2\kappa_1\mathsf{x}^2 + \lambda_1\lambda_2\mathsf{xy} + \lambda_2^2\kappa_2\mathsf{y}^2 + \lambda_1^s\mu\mathsf{x}^s.$$

We next claim that we can find some (λ_1, λ_2) which satisfies

$$\lambda_1^2\kappa_1 : \lambda_1\lambda_2 : \lambda_2^2\kappa_2 : \lambda_1^s\mu = 1 : 1 : \frac{1}{4} : 1.$$

Once the claim is certified, it follows at once that $\kappa_1\mathsf{x}^2 + \mathsf{xy} + \kappa_2\mathsf{y}^2 + \mu\mathsf{x}^s \cong \mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 + \mathsf{x}^s$. To prove the claim, since $s \geq 3$ and $\mu \neq 0$, we can solve λ_1 to ensure $\lambda_1^2\kappa_1 : \lambda_1^s\mu = 1 : 1$ holds. Then we solve λ_2 from λ_1 and $\lambda_1^2\kappa_1 : \lambda_1\lambda_2 = 1 : 1$. Moreover, this choice of λ_1 and λ_2 also satisfies $\lambda_1\lambda_2 : \lambda_2^2\kappa_2 = 1 : \frac{1}{4}$ since $4\kappa_1\kappa_2 = 1$. Combining these together, (λ_1, λ_2) satisfies the claim.

(3) Applying $\varphi\colon a_1 \mapsto \lambda_1 a_1,\ a_2 \mapsto \lambda_2 a_2$ where $\lambda_1, \lambda_2 \in \mathbb{C}$ to (3) gives

$$\kappa_1\mathsf{x}^p + \mathsf{xy} + \kappa_2\mathsf{y}^q \mapsto \lambda_1^p\kappa_1\mathsf{x}^p + \lambda_1\lambda_2\mathsf{xy} + \lambda_2^q\kappa_2\mathsf{y}^q.$$

Similar to (2), the statement follows once we find some (λ_1, λ_2) which satisfies

$$\lambda_1^p\kappa_1 : \lambda_1\lambda_2 : \lambda_2^q\kappa_2 = 1 : 1 : 1.$$

The above equations induce $\lambda_2 = \lambda_1^{p-1}\kappa_1$ and $\lambda_1 = \lambda_2^{q-1}\kappa_2$, and so $\lambda_1^{(p-1)(q-1)-1}\kappa_1^{q-1}\kappa_2 = 1$.

Since κ_1, $\kappa_2 \neq 0$ and $(p,q) \neq (2,2)$, we can solve λ_1, and then λ_2 such that the above equations hold.

(4) Applying $a_1 \mapsto \lambda_1 a_1$, $a_2 \mapsto \lambda_2 a_2$ where $\lambda_1, \lambda_2 \in \mathbb{C}$ to (4) gives

$$\kappa_1 x^2 + xy + \kappa_2 y^2 \mapsto \lambda_1^2 \kappa_1 x^2 + \lambda_1\lambda_2 xy + \lambda_2^2 \kappa_2 y^2.$$

Similar to (1), we can solve (λ_1, λ_2) that ensures $\lambda_1^2\kappa_1 = \lambda_1\lambda_2 = 1$ holds, and then $\lambda_2^2\kappa_2 = \kappa_1\kappa_2 = \frac{1}{4}$ since $4\kappa_1\kappa_2 = 1$. Thus $\kappa_1 x^2 + xy + \kappa_2 y^2 \mapsto x^2 + xy + \frac{1}{4}y^2$.

(5) Applying $a_1 \mapsto \lambda_1 a_1$, $a_2 \mapsto \lambda_2 a_2$ where $\lambda_1, \lambda_2 \in \mathbb{C}$ to (5) gives

$$\kappa_1 x^p + xy \mapsto \lambda_1^p \kappa_1 x^p + \lambda_1\lambda_2 xy.$$

Since $\kappa_1 \neq 0$, we can solve (λ_1, λ_2) that ensures $\lambda_1^p\kappa_1 = \lambda_1\lambda_2 = 1$ holds. Thus $\kappa_1 x^p + xy \mapsto x^p + xy$.

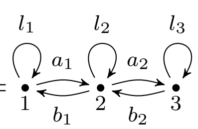
The proof of (6) and (7) is similar to (5). □

We now simplify the previous geometric realization in §4.2 for the potentials in 6.1.14.

Proposition 6.1.15. *Each Jacobi algebra of potentials in 6.1.14 is realized by a crepant resolution of a singularity of cA_3 $\mathcal{R} := \mathbb{C}[[u,v,x,y]]/(uv - h_0h_1h_2h_3)$, which corresponds to the $\mathcal{R}$-module $M := \mathcal{R} \oplus (u, h_0) \oplus (u, h_0h_1) \oplus (u, h_0h_1h_2)$ in 3.3.2 as follows.*

	h_0	h_1	h_2	h_3
(1)	$2x+y$	x	y	$x+2\lambda y$
(2)	$2x+y+sx^{s-1}$	x	y	$x+\frac{1}{2}y$
(3)	$px^{p-1}+y$	x	y	$x+qy^{q-1}$
(4)	$2x+y$	x	y	$x+\frac{1}{2}y$
(5)	$px^{p-1}+y$	x	y	x
(6)	y	x	y	$x+qy^{q-1}$
(7)	y	x	y	x

Proof. In order to construct the geometric realization by 4.2.9 and (5.2.A), we first transform the potentials in 6.1.14 to some potentials in Q_3, which has a single loop at each vertex, as illustrated below (see also 4.1.2).

$Q_3 =$ 

Consider a potential $f = \kappa_1 x^p + xy + \kappa_2 y^q + \kappa_3 x^s$ on Q where $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{C}$. By applying 4.1.22 three times, each of which adds a loop l_i at vertex i of Q for $1 \leq i \leq 3$, we have

$\mathfrak{J}\mathrm{ac}(Q,f) \cong \mathfrak{J}\mathrm{ac}(Q_3,f')$ where

$$f' = l_1\mathsf{x}' + \mathsf{x}l_2 + l_2\mathsf{y} + \mathsf{y}'l_3 - \frac{1}{2}l_1^2 - \frac{1}{2}l_2^2 - \frac{1}{2}l_3^2 - \mathsf{x}^2 - \mathsf{y}^2 + \kappa_1\mathsf{x}^p + \kappa_2\mathsf{y}^q + \kappa_3\mathsf{x}^s.$$

Then by 4.2.9, 4.2.2 and (5.2.A), we can realize f' by setting $g_2 = x$, $g_3 = x+y$ and then solving the following system of equations where each $g_i \in \mathbb{C}[[x,y]]$

$$\begin{aligned}
&g_0 - g_1 + g_2 = 0\\
&g_1 - 2g_2 + \kappa_1 p g_2^{p-1} + \kappa_3 s g_2^{s-1} + g_3 = 0\\
&g_2 - g_3 + g_4 = 0\\
&g_3 - g_4 + \kappa_2 g_4^{q-1} + g_5 = 0\\
&g_4 - g_5 + g_6 = 0.
\end{aligned}$$

Thus $(g_0,g_1,g_2,g_3,g_4,g_5,g_6) = (-\kappa_1 p x^{p-1} - \kappa_3 s x^{s-1} - y, x - \kappa_1 p x^{p-1} - \kappa_3 s x^{s-1} - y, x, x+y, y, -x+y-\kappa_2 q y^{q-1}, -x-\kappa_2 q y^{q-1})$. Set $(h_0,h_1,h_2,h_3) := (-g_0, g_2, g_4, -g_6)$ and consider

$$\mathcal{R} := \frac{\mathbb{C}[[u,v,x,y]]}{uv - h_0h_1h_2h_3} = \frac{\mathbb{C}[[u,v,x,y]]}{uv - (\kappa_1 p x^{p-1} + \kappa_3 s x^{s-1} + y)xy(x + \kappa_2 q y^{q-1})}$$

and $\mathcal{R}$-module $M = \mathcal{R} \oplus (u,h_0) \oplus (u,h_0h_1) \oplus (u,h_0h_1h_2)$. Write π for the crepant resolution of $\operatorname{Spec}\mathcal{R}$, which corresponds to M in 3.3.2. Then $\Lambda_{\mathrm{con}}(\pi) \cong \mathfrak{J}\mathrm{ac}(Q_3,f')$ by 4.2.9, and so $\Lambda_{\mathrm{con}}(\pi) \cong \mathfrak{J}\mathrm{ac}(Q,f)$. By choosing different values of κ_1, κ_2, κ_3 and p, q, s to make f become the potentials in 6.1.14, we prove the statement. $\square$

Then we classify the Type A potentials on Q up to isomorphism, which is the main result of this section.

Lemma 6.1.16. [E2] *If $\lambda_1, \lambda_2 \neq 0, \frac{1}{4}$ and $\lambda_1 \neq \lambda_2$, then $\mathsf{x}^2 + \mathsf{xy} + \lambda_1\mathsf{y}^2 \ncong \mathsf{x}^2 + \mathsf{xy} + \lambda_2\mathsf{y}^2$.*

Theorem 6.1.17. *Any Type A potential on Q must be isomorphic to one of the following isomorphism classes of potentials:*

(1) $\mathsf{x}^2 + \mathsf{xy} + \lambda\mathsf{y}^2$ *for any* $0, \frac{1}{4} \neq \lambda \in \mathbb{C}$.

(2) $\mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 + \mathsf{x}^s$ *for any* $s \geq 3$.

(3) $\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q \cong \mathsf{x}^q + \mathsf{xy} + \mathsf{y}^p$ *for any* $(p,q) \neq (2,2)$.

(4) $\mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2$.

(5) $\mathsf{x}^p + \mathsf{xy} \cong \mathsf{xy} + \mathsf{y}^p$ *for any* $p \geq 2$.

(6) xy.

The Jacobi algebras of these potentials are all mutually non-isomorphic (except those isomorphisms stated), and in particular the Jacobi algebras with different parameters in the same item are non-isomorphic.

The Jacobi algebras in (1)*,* (2)*,* (3) *are realized by crepant resolutions of isolated cA_3 singularities, and those in* (4)*,* (5)*,* (6) *are realized by crepant resolutions of non-isolated cA_3 singularities.*

Proof. We first prove the isomorphisms in the statement. Applying $a_1 \mapsto b_2, b_1 \mapsto a_2, a_2 \mapsto b_1, b_2 \mapsto a_1$ gives

$$\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q \rightsquigarrow \mathsf{x}^q + \mathsf{xy} + \mathsf{y}^p, \;\; \mathsf{x}^p + \mathsf{xy} \rightsquigarrow \mathsf{xy} + \mathsf{y}^p.$$

Then we prove the non-isomorphisms in the statement by using the following fact. If Type A potentials f and g on Q are isomorphic, then $\dim_{\mathbb{C}} \mathfrak{Jac}(f) = \dim_{\mathbb{C}} \mathfrak{Jac}(g)$, and further by 3.2.5 there is an equality of sets

$$\{\dim_{\mathbb{C}} \mathfrak{Jac}(f)/e_1, \; \dim_{\mathbb{C}} \mathfrak{Jac}(f)/e_3\} = \{\dim_{\mathbb{C}} \mathfrak{Jac}(g)/e_1, \; \dim_{\mathbb{C}} \mathfrak{Jac}(g)/e_3\}. \tag{6.1.E}$$

The following table lists $\dim_{\mathbb{C}} \mathfrak{Jac}(f)$, $\dim_{\mathbb{C}} \mathfrak{Jac}(f)/e_1$ and $\dim_{\mathbb{C}} \mathfrak{Jac}(f)/e_3$ for each f in each item, using Toda's formula 2.4.2.

	$\dim_{\mathbb{C}} \mathfrak{Jac}(f)$	$\dim_{\mathbb{C}} \mathfrak{Jac}(f)/e_1$	$\dim_{\mathbb{C}} \mathfrak{Jac}(f)/e_3$
(1)	20	6	6
(2)	$9s+2$	6	6
$\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q$	$4p+4q+4$	$4q-2$	$4p-2$
(4)	∞	6	6
$\mathsf{x}^p + \mathsf{xy}$	∞	∞	$4p-2$
(6)	∞	∞	∞

Now, all Jacobi algebras in (1) have dimension 20, but are mutually non-isomorphic by 6.1.16. All Jacobi algebras in (2) are mutually non-isomorphic since they all have different dimensions.

For (3), we only need to prove that $\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q \ncong \mathsf{x}^r + \mathsf{xy} + \mathsf{y}^s$ for any $(p, q) \neq (r, s)$ and $(p, q) \neq (s, r)$. From the above table,

$$\{\dim_{\mathbb{C}} \mathfrak{Jac}(\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q)/e_1, \; \dim_{\mathbb{C}} \mathfrak{Jac}(\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q)/e_3\} = \{4q-2, 4p-2\},$$
$$\{\dim_{\mathbb{C}} \mathfrak{Jac}(\mathsf{x}^r + \mathsf{xy} + \mathsf{y}^s)/e_1, \; \dim_{\mathbb{C}} \mathfrak{Jac}(\mathsf{x}^r + \mathsf{xy} + \mathsf{y}^s)/e_3\} = \{4r-2, 4s-2\}.$$

Since $(p, q) \neq (r, s)$ and $(p, q) \neq (s, r)$, then the above two sets are not equal, and so $\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q \ncong \mathsf{x}^r + \mathsf{xy} + \mathsf{y}^s$ by (6.1.E).

For (5), since $\mathsf{x}^p + \mathsf{xy} \cong \mathsf{xy} + \mathsf{y}^p$, we only need to prove that $\mathsf{x}^p + \mathsf{xy} \ncong \mathsf{x}^q + \mathsf{xy}$ for any $p \neq q$.

From the above table,

$$\{\dim_{\mathbb{C}} \mathfrak{Jac}(\mathsf{x}^p + \mathsf{xy})/e_1,\ \dim_{\mathbb{C}} \mathfrak{Jac}(\mathsf{x}^p + \mathsf{xy})/e_3\} = \{\infty, 4p - 2\},$$
$$\{\dim_{\mathbb{C}} \mathfrak{Jac}(\mathsf{x}^q + \mathsf{xy})/e_1,\ \dim_{\mathbb{C}} \mathfrak{Jac}(\mathsf{x}^q + \mathsf{xy})/e_3\} = \{\infty, 4q - 2\}.$$

Since $p \neq q$, the above two sets are not equal, so $\mathsf{x}^p + \mathsf{xy} \ncong \mathsf{x}^q + \mathsf{xy}$ by (6.1.E).

The above shows that potentials in the same item are mutually non-isomorphic. We finally prove that the potentials in different items are mutually non-isomorphic.

Since Jacobi algebras in (1), (2) and (3) have finite dimension, while those in (4), (5) and (6) have infinite dimension, we only need to prove that the potentials in (1), (2) and (3) are mutually non-isomorphic, and the potentials in (4), (5) and (6) are mutually non-isomorphic, respectively.

From the above table, the Jacobi algebras in (1), (2), and (3) have dimensions 20, $9s + 2$ and $4p + 4q + 4$, respectively. Since $s \geq 3$ and $(p, q) \neq (2, 2)$, then $9s + 2 > 20$ and $4p + 4q + 4 > 20$, and so the potentials in (1) are not isomorphic to those in (2) and (3). To compare the potentials in (2) and (3), since $(p, q) \neq (2, 2)$, then $\{4q - 2, 4p - 2\} \neq \{6, 6\}$, then the potentials in (2) are not isomorphic to those in (3), by (6.1.E) and the table.

To compare the potentials in (4), (5) and (6), since $\{6, 6\}$, $\{\infty, 4p - 2\}$ and $\{\infty, \infty\}$ are mutually not equal, then the potentials in (4), (5) and (6) are mutually non-isomorphic by (6.1.E) and the table.

By the geometric realizations in 6.1.15, the Jacobi algebras in (1), (2), (3) are realized by crepant resolutions of isolated cA_3 singularities, and the those in (4), (5), (6) are realized by crepant resolutions of non-isolated cA_3 singularities. □

Remark 6.1.18. In 6.1.17, (4) is the limit of (2) by $s \to \infty$ or (1) by $\lambda \to \frac{1}{4}$. Similarly, (5) and (6) are the limits of (3) by $p \to \infty$ and $q \to \infty$. This parallels the fact that divisor-to-curve contractions are usually the limit of flops; see also [BW2].

Remark 6.1.19. In this section, for a Type A potential f on the doubled A_3 quiver without loops, we normalise f using the matrix $A_{12}^2(f)$ introduced in 5.1.2. For a Type A potential f on the doubled A_3 quiver with loops, or more generally on the doubled A_n quiver Q_n with $n \geq 4$, one would instead need to use matrices of the form $A_{ij}^d(f)$ (with $j - i \geq 2$ and $d \geq 2$) to normalise f. At present, it is unclear how to extend the normalisation process developed here to Type A potentials on Q_n for arbitrary n.

§ 6.2 | Derived equivalence classes

The purpose of this section is to prove 6.2.6, which gives the derived equivalence classes of Type A potentials with finite-dimensional Jacobi algebra on Q.

Given a Type A potential f on Q, by 6.1.14 and 6.1.15 we can realize f by some cA_3

$$\mathcal{R} \cong \frac{\mathbb{C}[[u,v,x,y]]}{uv - h_0h_1h_2h_3}$$

and $\mathcal{R}$-module $M = \mathcal{R} \oplus (u,h_0) \oplus (u,h_0h_1) \oplus (u,h_0h_1h_2)$. Denote the corresponding crepant resolution as $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathcal{R}$, so that $\Lambda_{\mathrm{con}}(\pi) \cong \underline{\operatorname{End}}_{\mathcal{R}}(M) \cong \mathfrak{Jac}(f)$.

Notation 6.2.1. We adopt the following notation. We first recall π^i, $\mathcal{X}^i$, M^i and $\pi^{\mathbf{r}}$, $\mathcal{X}^{\mathbf{r}}$, $M^{\mathbf{r}}$ in 3.3.1. By 4.2.15, there is a Type A potential g on $Q_{3,I}$ such that $\Lambda_{\mathrm{con}}(\pi^{\mathbf{r}}) \cong \mathfrak{Jac}(Q_{3,I}, g)$ for some $I \subseteq \{1,2,3\}$, and we set $f^{\mathbf{r}} := g$, which is well defined up to the isomorphism of Jacobi algebras. For $1 \leq i \leq 3$, write f^i for $f^{(i)}$.

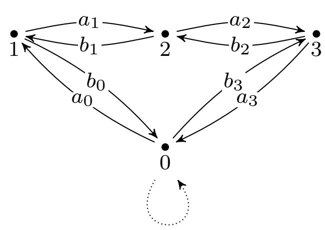
Since this section aims to classify the derived equivalence classes of Type A potentials, the definition of $f^{\mathbf{r}}$ and f^i up to the isomorphism of Jacobi algebras is harmless.

By 3.3.2, $\Lambda_{\mathrm{con}}(\pi^{\mathbf{r}}) \cong \underline{\operatorname{End}}_{\mathcal{R}}(M^{\mathbf{r}}) \cong \mathfrak{Jac}(f^{\mathbf{r}})$. If, in addition, $\mathcal{R}$ is isolated, by 2.3.8 $f^{\mathbf{r}} \simeq f$ (see the definition of $\simeq$ in 2.1.8). Moreover, by 2.3.6, $\mathcal{R}$ is isolated if and only if $\dim_{\mathbb{C}} \Lambda_{\mathrm{con}}(\pi) < \infty$ (equivalently, $\mathfrak{Jac}(f)$ is finite-dimensional).

Hence we transfer the question about the derived equivalence classes of Type A potentials on Q with finite-dimensional Jacobi algebra to that about the flops of crepant resolutions of isolated cA_3 singularities. The restriction to finite-dimensional Jacobi algebras is necessary because 2.3.8 requires $\mathcal{R}$ to be isolated.

In order to present the NCCRs $\operatorname{End}_{\mathcal{R}}(M)$ and $\operatorname{End}_{\mathcal{R}}(M^{\mathbf{r}})$, we adopt the following.

Definition 6.2.2. *Define the quiver $\mathcal{Q}$ from Q by adding a new vertex 0, paths a_0, b_0, a_3, b_3 and a possible loop at vertex 0, as illustrated below.*



Quiver $\mathcal{Q}$

Since f^i might be a potential in $Q_{3,I}$ for some $I \neq \emptyset$, and we aim to classify the derived equivalence classes of Type A potentials on $Q := Q_{3,\emptyset}$, we need the following lemma.

Lemma 6.2.3. *Given a potential of Type A $f = \kappa_1 \mathsf{x}^p + \mathsf{xy} + \kappa_2 \mathsf{y}^q$ in Q, the following statements hold.*

(1) $\kappa_1 \neq 0$ *and* $p = 2 \iff f^1$ *is a potential on* Q.

(2) $\kappa_2 \neq 0$ *and* $q = 2 \iff f^3$ *is a potential on* Q.

(3) $\kappa_1, \kappa_2 \neq 0$ *and* $p = q = 2 \iff f^2$ *is a potential on* Q.

Proof. By 6.1.15, we can realize f by

$$\mathcal{R} \cong \frac{\mathbb{C}[[u, v, x, y]]}{uv - h_0 h_1 h_2 h_3}$$

and $\mathcal{R}$-module $M = \mathcal{R} \oplus (u, h_0) \oplus (u, h_0 h_1) \oplus (u, h_0 h_1 h_2)$ where $h_0 = \kappa_1 p \mathsf{x}^{p-1} + y$, $h_1 = x$, $h_2 = y$, $h_3 = x + \kappa_2 q \mathsf{y}^{q-1}$. For $1 \leq i \leq 3$, $\mathfrak{Jac}(Q_{3,I}, f^i) \cong \underline{\mathrm{End}}_{\mathcal{R}}(M^i)$ for some $I \subseteq \{1, 2, 3\}$.

(1) Since $M^1 = \mathcal{R} \oplus (u, h_1) \oplus (u, h_1 h_0) \oplus (u, h_1 h_0 h_2)$, by 3.3.3 $I = \emptyset \iff \kappa_1 \neq 0$ and $p = 2$.

(2) Since $M^2 = \mathcal{R} \oplus (u, h_0) \oplus (u, h_0 h_2) \oplus (u, h_0 h_2 h_1)$, by 3.3.3 $I = \emptyset \iff \kappa_2 \neq 0$ and $q = 2$.

(3) Since $M^3 = \mathcal{R} \oplus (u, h_0) \oplus (u, h_0 h_1) \oplus (u, h_0 h_1 h_3)$, by 3.3.3 $I = \emptyset \iff \kappa_1, \kappa_2 \neq 0$ and $p = q = 2$. □

Lemma 6.2.4. *Suppose that f is a Type A potential on Q. Then the following holds.*

(1) *If* $f = \mathsf{x}^2 + \mathsf{xy} + \lambda \mathsf{y}^2$ *with* $\lambda \neq 0$, *then* $f^1 \cong \mathsf{x}^2 + \mathsf{xy} + (\frac{1}{4} - \lambda)\mathsf{y}^2 \cong f^3$ *and* $f^2 \cong \mathsf{x}^2 + \mathsf{xy} + \frac{1}{16\lambda}\mathsf{y}^2$.

(2) *If* $f = \mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 + \mathsf{x}^p$ *with* $p \geq 3$, *then* $f^1 \cong \mathsf{x}^2 + \mathsf{xy} + \mathsf{y}^p$ *and* $f^3 \cong \mathsf{x}^p + \mathsf{xy} + \mathsf{y}^2$.

Proof. Suppose that $f = \mathsf{x}^2 + \mathsf{xy} + \lambda \mathsf{y}^p$. By 6.1.15, we can realize f by

$$\mathcal{R} \cong \frac{\mathbb{C}[[u, v, x, y]]}{uv - h_0 h_1 h_2 h_3}$$

and $\mathcal{R}$-module $M = \mathcal{R} \oplus (u, h_0) \oplus (u, h_0 h_1) \oplus (u, h_0 h_1 h_2)$ where $h_0 = 2\mathsf{x} + y$, $h_1 = x$, $h_2 = y$ and $h_3 = x + \lambda p \mathsf{y}^{p-1}$. Since $M^1 = \mathcal{R} \oplus (u, h_1) \oplus (u, h_1 h_0) \oplus (u, h_1 h_0 h_2)$, then by 3.3.3 $\mathrm{End}_{\mathcal{R}}(M^1)$ can be presented by $\mathcal{Q}$ with relations

$$\mathsf{x} b_1 - \mathsf{y} b_1 = 2 b_1 b_0 a_0, \quad b_2 \mathsf{x} - b_2 \mathsf{y} = 2(a_3 b_3 b_2 - \lambda p b_2 \mathsf{y}^{p-1}),$$
$$a_1 \mathsf{x} - a_1 \mathsf{y} = 2 b_0 a_0 a_1, \quad \mathsf{x} a_2 - \mathsf{y} a_2 = 2(a_2 a_3 b_3 - \lambda p \mathsf{y}^{p-1} a_2),$$

plus some other relations that factor through the vertex 0 (and so will not be relevant below). Hence $\underline{\mathrm{End}}_{\mathcal{R}}(M^1)$ can be presented by Q with relations

$$\mathsf{x} b_1 - \mathsf{y} b_1 = 0, \quad b_2 \mathsf{x} - b_2 \mathsf{y} = -2\lambda p b_2 \mathsf{y}^{p-1},$$
$$a_1 \mathsf{x} - a_1 \mathsf{y} = 0, \quad \mathsf{x} a_2 - \mathsf{y} a_2 = -2\lambda p \mathsf{y}^{p-1} a_2.$$

Thus $\underline{\mathrm{End}}_{\mathcal{R}}(M^1) \cong \mathfrak{Jac}(Q, f^1)$ where $f^1 \cong \frac{1}{2}\mathsf{x}^2 - \mathsf{xy} + \frac{1}{2}\mathsf{y}^2 - 2\lambda \mathsf{y}^p$. Normalizing by applying $a_1 \mapsto -\sqrt{2} a_1$ and $a_2 \mapsto \frac{1}{\sqrt{2}} a_2$ to f^1 gives

$$f^1 \mapsto \mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 - 2^{1-\frac{p}{2}} \lambda \mathsf{y}^p.$$

Setting $p = 2$ in the above potential proves the f^1 statement in (1). The proof of the f^3 statement in (1) is similar.

For $p \geq 3$ and $\lambda \neq 0$, applying $a_1 \mapsto \frac{1}{2}b_2, b_1 \mapsto a_2, a_2 \mapsto 2b_1, b_2 \mapsto a_1$ gives

$$\mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 - 2^{1-\frac{p}{2}}\lambda\mathsf{y}^p \rightsquigarrow \mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 - 2^{1+\frac{p}{2}}\lambda\mathsf{x}^p.$$

Then since $p \geq 3$ and $\lambda \neq 0$, by 6.1.14(2),

$$\mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 - 2^{1+\frac{p}{2}}\lambda\mathsf{x}^p \cong \mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 + \mathsf{x}^p.$$

Thus $(\mathsf{x}^2 + \mathsf{xy} + \mathsf{y}^p)^1 = \mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 + \mathsf{x}^p$. Since flopping is an involution, this proves the f^1 statement in (2). The proof of the f^3 statement in (2) is similar.

Then we finally prove the f^2 statement in (1). In this case, $g_0 = 2\mathsf{x} + y$, $g_1 = x$, $g_2 = y$ and $g_3 = x + 2\lambda\mathsf{y}$. Since $M^2 = \mathcal{R} \oplus (u, h_0) \oplus (u, h_0h_2) \oplus (u, h_0h_2h_1)$, then by 3.3.3 $\operatorname{End}_{\mathcal{R}}(M^2)$ can be presented by $\mathcal{Q}$ with relations

$$\begin{aligned} \mathsf{x}b_1 + 2\mathsf{y}b_1 &= b_1b_0a_0, \quad 2\lambda b_2\mathsf{x} + b_2\mathsf{y} = a_3b_3b_2, \\ a_1\mathsf{x} + 2a_1\mathsf{y} &= b_0a_0a_1, \quad 2\lambda\mathsf{x}a_2 + \mathsf{y}a_2 = a_2a_3b_3, \end{aligned}$$

plus some other relations that factor through the vertex 0 (and so will not be relevant below). Hence $\underline{\operatorname{End}}_{\mathcal{R}}(M^2)$ can be presented by Q with relations

$$\begin{aligned} \mathsf{x}b_1 + 2\mathsf{y}b_1 &= 0, \quad 2\lambda b_2\mathsf{x} + b_2\mathsf{y} = 0, \\ a_1\mathsf{x} + 2a_1\mathsf{y} &= 0, \quad 2\lambda\mathsf{x}a_2 + \mathsf{y}a_2 = 0. \end{aligned}$$

Thus $\underline{\operatorname{End}}_{\mathcal{R}}(M^2) \cong \mathcal{J}\mathrm{ac}(Q, f^2)$ where $f^2 \cong \mathsf{x}^2 + \mathsf{xy} + \frac{1}{16\lambda}\mathsf{y}^2$. ☐

Recall the definition of the generalised GV tuple $N(\pi)$ in 5.4.1.

Lemma 6.2.5. *Let $\pi_k \colon \mathcal{X}_k \to \operatorname{Spec} \mathcal{R}_k$ be two crepant resolutions of isolated cA_n singularity $\mathcal{R}_k$ for $k = 1, 2$. If $\Lambda_{\mathrm{con}}(\pi_1)$ is derived equivalent to $\Lambda_{\mathrm{con}}(\pi_2)$, then $N(\pi_1) = N(\pi_2)$.*

Proof. Since $\Lambda_{\mathrm{con}}(\pi_1)$ is derived equivalent to $\Lambda_{\mathrm{con}}(\pi_2)$ and each $\mathcal{R}_i$ is isolated, then $\mathcal{R}_1 \cong \mathcal{R}_2$ by 2.3.7, and so π_1 and π_2 are two crepant resolutions of a same cA_n singularity and connected by a sequence of flops. Thus $N(\pi_1) = N(\pi_2)$ by [NW, 5.4] and 3.3.8. ☐

Theorem 6.2.6. *The following groups the Type A potentials on Q with finite-dimensional Jacobi algebra into sets, where all the Jacobi algebras in a given set are derived equivalent.*

(1) $\{\mathsf{x}^2 + \mathsf{xy} + \lambda'\mathsf{y}^2 \mid \lambda' = \lambda, \frac{1-4\lambda}{4}, \frac{1}{4(1-4\lambda)}, \frac{\lambda}{4\lambda-1}, \frac{4\lambda-1}{16\lambda}, \frac{1}{16\lambda}\}$ *for any* $\lambda \neq 0, \frac{1}{4}$.

(2) $\{\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^2, \mathsf{x}^2 + \mathsf{xy} + \mathsf{y}^p, \mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 + \mathsf{x}^p\}$ *for* $p \geq 3$.

(3) $\{\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q, \mathsf{x}^q + \mathsf{xy} + \mathsf{y}^p\}$ *for* $p \geq 3$ *and* $q \geq 3$.

Moreover, the Jacobi algebras of the sets in (1)–(3) are all mutually not derived equivalent, and in particular the Jacobi algebras of different sets in the same item are not derived equivalent. In (1) there are no further basic algebras in the derived equivalence class, whereas *in* (2)–(3) *there are an additional finite number of basic algebras in the derived equivalence class.*

Proof. By 6.1.17 and 2.3.6, the potentials in the statement are precisely the Type A potentials on Q with finite-dimensional Jacobi algebra, thus they exhaust all possibilities.

Firstly, we prove that the Jacobi algebras in each given set are derived equivalent. By 2.3.8, given a Type A potential f with finite-dimensional and $\mathfrak{J}\mathrm{ac}(f) \cong \Lambda_{\mathrm{con}}(\pi)$, if we want to obtain all the basic algebras that are derived equivalent to $\mathfrak{J}\mathrm{ac}(f)$, we only need to calculate all iterated flops from π. So we consider f^i for $1 \leq i \leq 3$ in the following.

(1) Suppose that $f = \mathsf{x}^2 + \mathsf{xy} + \lambda\mathsf{y}^2$ where $\lambda \neq 0, \frac{1}{4}$.

By 6.2.4, $f^1 \cong \mathsf{x}^2+\mathsf{xy}+(\frac{1}{4}-\lambda)\mathsf{y}^2 \cong f^3$ and $f^2 \cong \mathsf{x}^2+\mathsf{xy}+\frac{1}{16\lambda}\mathsf{y}^2$. Repeating the same argument, we have $f^{(12)} \cong \mathsf{x}^2 + \mathsf{xy} + \frac{1}{4(1-4\lambda)}\mathsf{y}^2$, $f^{(21)} \cong \mathsf{x}^2 + \mathsf{xy} + \frac{4\lambda-1}{16\lambda}\mathsf{y}^2$ and $f^{(121)} \cong \mathsf{x}^2 + \mathsf{xy} + \frac{\lambda}{4\lambda-1}\mathsf{y}^2$. Repeating this process, only six numbers appear, so by 2.3.8 there are no further basic algebras in this derived equivalence class.

(2) Suppose that $f = \mathsf{x}^2 + \mathsf{xy} + \frac{1}{4}\mathsf{y}^2 + \mathsf{x}^p$ where $p \geq 3$.

By 6.2.4, $f^1 \cong \mathsf{x}^2 + \mathsf{xy} + \mathsf{y}^p$ and $f^3 \cong \mathsf{x}^p + \mathsf{xy} + \mathsf{y}^2$, and thus the three potentials in the statement are derived equivalent. Since $p \geq 3$, then f^{12}, f^{13}, f^{31} and f^{32} are not on Q by 6.2.3, and so there are additional basic algebras in this derived equivalence class.

(3) By 6.1.17, $\mathsf{x}^p + \mathsf{xy} + \mathsf{y}^q \cong \mathsf{x}^q + \mathsf{xy} + \mathsf{y}^p$, and thus the two potentials in the statement are derived equivalent. Suppose that $f = \mathsf{x}^p+\mathsf{xy}+\mathsf{y}^q$. Since $p \geq 3$ and $q \geq 3$, then f^1, f^2 and f^3 are not on Q by 6.2.3, and so there are additional basic algebras in this derived equivalence class.

The wall–chamber decomposition of the movable cone for a cA_3 crepant resolution is governed by the type A_3 root system (see e.g. [W2, 5.24, §7]). These chambers are precisely the Weyl chambers, so their number equals the order of the Weyl group, namely $\#W(A_3) = |S_4| = 24$. Each chamber corresponds to a crepant resolution.

Moreover, the double A_3 quiver Q admits a natural involution that sends $e_1 \mapsto e_3$, $e_2 \mapsto e_2$, and $e_3 \mapsto e_1$ (equivalently, exchanging x and y in the potentials). This symmetry identifies certain Jacobi algebras, so that there are at most 12 distinct isomorphism classes. Consequently, the number of additional basic algebras appearing in the derived equivalence classes in cases (2) and (3) of 6.2.6 is finite.

Secondly, we prove that the Jacobi algebras in different sets in (1)–(3) are all mutually not derived equivalent.

Given any potential f in the statement, by 6.1.15 we can find a Type A_3 crepant resolution π such that $\Lambda_{\mathrm{con}}(\pi) \cong \mathfrak{J}\mathrm{ac}(f)$. By the definition of generalised GV invariants 3.1.1, the generalised GV tuple of each set in (1)–(3) is:

① $(1, 1, 1, 1, 1, 1)$,

② $(1, 1, 1, p-1, 1, 1)$,

③ $(1, 1, 1, p-1, q-1, 1)$.

Suppose that f_1 and f_2 are potentials in the statement with $f_1 \simeq f_2$ and each $\mathfrak{J}\mathrm{ac}(f_i) \cong \Lambda_{\mathrm{con}}(\pi_i)$, where $\pi_i \colon \mathcal{X}_i \to \operatorname{Spec} \mathcal{R}_i$ is a Type A_3 crepant resolution. Then $\Lambda_{\mathrm{con}}(\pi_1)$ is derived equivalent to $\Lambda_{\mathrm{con}}(\pi_2)$. Since each $\mathfrak{J}\mathrm{ac}(f_i)$ is finite-dimensional, then each $\mathcal{R}_i$ is isolated by 2.3.6, and so $N(\pi_1) = N(\pi_2)$ by 6.2.5. So if we want to prove that two potentials are not derived equivalent, we only need to prove that their corresponding generalised GV tuples are not equal.

Since $p \geq 3$, then any generalised GV tuple in ① is different from that in ②, and so any set of potentials in (1) is not derived equivalent to that in (2). Since $q \geq 3$, then any generalised GV tuple in ② is different from that in ③, and so any set of potentials in (2) is not derived equivalent to that in (3). Similar for (1) and (3).

Next, consider two sets of potentials in the same item. Given a potential f in (1), we have already exhausted all 6 potentials that are derived equivalent to f in the above proof. Thus by 2.3.8, different sets of potentials in (1) are not derived equivalent. Since different generalised GV tuples in ② are not equal, different sets of potentials in (2) are not derived equivalent. Similar for (3). $\square$

Remark 6.2.7. It is usually hard to give the derived equivalence class of an algebra A. But when A is $\mathfrak{J}\mathrm{ac}(f)$ for a Type A potential f on $Q_{n,I}$, there is a Type A_n crepant resolution $\pi \colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ such that $A \cong \Lambda_{\mathrm{con}}(\pi)$ by 4.2.12. If further A is finite-dimensional over $\mathbb{C}$, then $\mathcal{R}$ is isolated by 2.3.6. So we can apply 2.3.8 to get the *full* derived equivalence class of A by calculating all iterated flops from π.

This is why we restrict this section to the cases of Type A potential on Q with finite-dimensional Jacobi algebra. Furthermore, as indicated in §4.2.3, 2.3.8 does not extend directly to non-isolated cDV singularities, and thus 6.2.6 likewise does not extend directly to non-isolated cA_n singularities, although an appropriate generalisation may still be possible.

Remark 6.2.8. In cases (2) and (3) of 6.2.6, there are additional basic algebras in the derived equivalence class. These algebras are isomorphic to the Jacobi algebras of some potentials on $Q_{3,I}$ where $I \neq \emptyset$ (see the proof in 6.2.6).

Next, recall the definition of the quaternion type quiver algebra $A_{p,q}(\mu)$ in [E1, H2], which is the completion of the path algebra of the quiver Q

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left, "a_1"] & \underset{2}{\bullet} \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \underset{3}{\bullet} \arrow[l, bend left, "b_2"] \end{tikzcd}$$

modulo the relations

$$a_1a_2b_2 - (a_1b_1)^{p-1}a_1, b_2b_1a_1 - \mu(b_2a_2)^{q-1}b_2, a_2b_2b_1 - (b_1a_1)^{p-1}b_1, b_1a_1a_2 - \mu(a_2b_2)^{q-1}a_2,$$

where $\mu \in \mathbb{C}$ and $p, q \geq 2$. Note we have fewer relations than in Erdmann [E1] since we are working with the completion. In fact $A_{p,q}(\upmu) \cong \mathfrak{Jac}(Q, f)$, where

$$f = \frac{1}{p}\mathsf{x}^p - \mathsf{xy} + \frac{\upmu}{q}\mathsf{y}^q \cong \mathsf{x}^p + \mathsf{xy} + (-1)^q p^{-\frac{q}{p}} q^{-1} \upmu \mathsf{y}^q.$$

Denote $B_{p,q}(\uplambda) := \mathfrak{Jac}(Q, f)$ where $f = \mathsf{x}^p + \mathsf{xy} + \uplambda\mathsf{y}^q$. Thus $A_{p,q}(\upmu) \cong B_{p,q}((-1)^q p^{-\frac{q}{p}} q^{-1}\upmu)$.

The following improves various results of Erdmann and Holm [E1, H2].

Corollary 6.2.9. *The following groups those algebras $A_{p,q}(\upmu)$ which are finite-dimensional into sets, where all the algebras in a given set are derived equivalent.*

(1) $\{A_{2,2}(\upmu') \mid \upmu' = \upmu, 1-\upmu, \frac{1}{1-\upmu}, \frac{\upmu}{\upmu-1}, \frac{\upmu-1}{\upmu}, \frac{1}{\upmu}\}$ *for* $\upmu \neq 0, 1$.

(2) $\{A_{p,q}(1), A_{q,p}(1)\}$ *for* $(p, q) \neq (2, 2)$.

Moreover, the algebras of the sets in (1)–(2) *are all mutually not derived equivalent. In* (1) *there are no further basic algebras in the derived equivalence class, whereas in* (2) *there are an additional finite number of basic algebras in the derived equivalence class.*

Proof. Since $A_{p,q}(\upmu) \cong B_{p,q}((-1)^q p^{-\frac{q}{p}} q^{-1}\upmu)$, in particular $A_{2,2}(\upmu) \cong B(\frac{\upmu}{4})$, then by 6.1.17 the $A_{p,q}(\upmu)$ in the statement are precisely the finite-dimensional ones up to isomorphism.

Then we prove that the algebras in each set are derived equivalent.

(1) Since $A_{2,2}(\upmu) \cong B_{2,2}(\frac{\upmu}{4}) = \mathfrak{Jac}(\mathsf{x}^2 + \mathsf{xy} + \frac{\upmu}{4}\mathsf{y}^2)$, then by 6.2.6(1) the algebras in each set of (1) are derived equivalent. Moreover, again by 6.2.6(1) there are no further basic algebras in the derived equivalence class.

(2) When $(p, q) \neq (2, 2)$, $B_{p,q}((-1)^q p^{-\frac{q}{p}} q^{-1}) \cong B_{p,q}(1)$ by the proof of 6.1.14(3), thus $A_{p,q}(1) \cong B_{p,q}(1)$. Similarly, $A_{q,p}(1) \cong B_{q,p}(1)$. Thus by 6.2.6(2)(3) the algebras in each set of (2) are derived equivalent. Moreover, again by 6.2.6(2)(3) there are an additional finite number of basic algebras in the derived equivalence class.

By 6.2.6 the algebras of the sets in (1)–(2) are all mutually not derived equivalent. $\square$

Chapter 7

Appendix

The purpose of this appendix is to prove 7.0.18, which gives a quiver presentation (7.0.A) of $\mathrm{End}_{\mathbb{S}}(N)$. This is used to prove the geometric realization in §4.2.

We first introduce the reduction system and the Diamond Lemma. For a quiver Q, we denote the set of paths of degree i by Q_i where the degree is with respect to the path length, and write $Q_{\geq i} = \bigcup_{j\geq i} Q_j$ for the set of paths of degree $\geq i$.

Definition 7.0.1. [B3, §1] *Given a field k, a* reduction system *R for the path algebra kQ is a set of pairs*

$$R = \{(s, \varphi_s) \mid s \in S \text{ and } \varphi_s \in kQ\}$$

where

(1) *S is a subset of $Q_{\geq 2}$ such that s is not a sub-path of s' when $s \neq s' \in S$.*

(2) *For all $s \in S$, s and φ_s have the same head and tail.*

(3) *For each pair $(s, \varphi_s) \in R$, φ_s is* irreducible, *meaning we can write $\varphi_s = \sum_i \lambda_i p_i$ where each $0 \neq \lambda_i \in k$, and each p_i does not contain elements in S as a sub-path.*

Definition 7.0.2. *Let $(s, \varphi_s) \in R$ and let q, r be two paths such that $qsr \neq 0$ in kQ. Following [CS, §2] a* basic reduction *$\mathfrak{r}_{q,s,r} : kQ \to kQ$ is defined as the k-linear map uniquely determined by the following: for any path p*

$$\mathfrak{r}_{q,s,r}(p) = \begin{cases} q\varphi_s r & \text{if } p = qsr \\ p & \text{if } p \neq qsr \end{cases}$$

Sometimes we write $p \to q\varphi_s r$ instead of $\mathfrak{r}_{q,s,r}(p) = q\varphi_s r$ for simplicity.

Definition 7.0.3. *A* reduction *$\mathfrak{r}$ is defined as a composition $\mathfrak{r}_{q_n,s_n,r_n} \circ \cdots \circ \mathfrak{r}_{q_2,s_2,r_2} \circ \mathfrak{r}_{q_1,s_1,r_1}$ of basic reductions for some $n \geq 1$. We say a path p is* reduction-finite *if for any infinite sequence of reductions $(\mathfrak{r}_i)_{i\in\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $\mathfrak{r}_n \circ \cdots \circ \mathfrak{r}_2 \circ \mathfrak{r}_1(p) = \mathfrak{r}_{n_0} \circ \cdots \circ \mathfrak{r}_2 \circ \mathfrak{r}_1(p)$.*

A path may contain many sub-paths in S, so one may obtain different elements in kQ after performing different reductions.

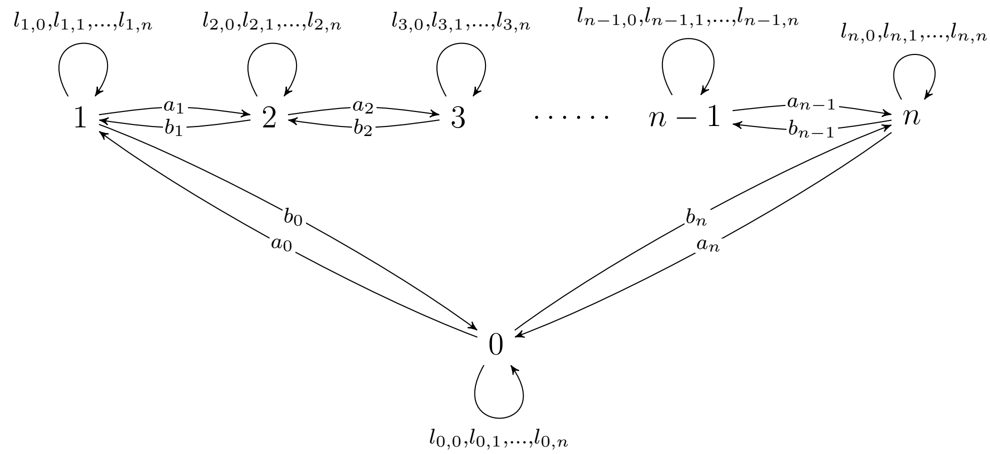
Definition 7.0.4. [B3, §1] *Let R be a reduction system for kQ. A path $pqr \in Q_{\geq 3}$ for $p, q, r \in Q_{\geq 1}$ is an* overlap ambiguity *of R if pq, $qr \in S$. We say that an overlap ambiguity pqr with $pq = s$ and $qr = s'$ is* resolvable *if $\varphi_s r$ and $p\varphi_{s'}$ are reduction-finite and $\mathfrak{r}(\varphi_s t) = \mathfrak{r}'(p\varphi_{s'})$ for some reductions $\mathfrak{r}$, $\mathfrak{r}'$.*

Theorem 7.0.5. (Diamond Lemma) [B3, 1.2] *Let $R = \{(s, \varphi_s)\}_{s\in S}$ be a reduction system for kQ. Let $I = \langle s - \varphi_s\rangle_{s\in S} \subset kQ$ be the corresponding two-sided ideal and write $A = kQ/I$ for the quotient algebra. If R is reduction-finite, then the following are equivalent:*

(1) *All overlap ambiguities of R are resolvable.*

(2) *The image of the set of irreducible paths under the projection $kQ \to A$ forms a k-basis of A.*

Consider the following quiver Q with relations I.



$$I := \begin{cases} l_{t,i}a_t = a_t l_{t+1,i}, \ l_{t+1,i}b_t = b_t l_{t,i}, \ l_{t,i}l_{t,j} = l_{t,j}l_{t,i}, \\ l_{t,t} = a_t b_t, \ l_{t+1,t} = b_t a_t \text{ for any } t \in \mathbb{Z}/(n+1) \text{ and } 0 \leq i, j \leq n. \end{cases} \tag{7.0.A}$$

Then define the reduction system R for the path algebra kQ to be

$$\begin{aligned} R := \{&(l_{t,i}a_t, a_t l_{t+1,i}), (l_{t+1,i}b_t, b_t l_{t,i}), (a_t b_t, l_{t,t}), (b_t a_t, l_{t+1,t}), (l_{tj}l_{ti}, l_{ti}l_{tj}) \mid \\ &\text{for any } 0 \leq i \leq n, \ t \in \mathbb{Z}/(n+1) \text{ and } j > i\}. \end{aligned} \tag{7.0.B}$$

We next prove that R is reduction-finite and all overlap ambiguities of R are resolvable.

Lemma 7.0.6. *The reduction system R (7.0.B) is reduction-finite.*

Proof. For any path p and any infinite sequence of reductions $(\mathfrak{r}_i)_{i\in\mathbb{N}}$, if there does not exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\mathfrak{r}_n \circ \cdots \circ \mathfrak{r}_1(p) = \mathfrak{r}_{n_0} \circ \cdots \circ \mathfrak{r}_1(p)$, then there must exist infinite basic reductions that can be applied to p consecutively. We prove that this is impossible. There are three types of path pairs in R:

(1) $(a_t b_t, l_{t,t}), (b_t a_t, l_{t+1,t})$.

(2) $(l_{t,i} a_t, a_t l_{t+1,i}), (l_{t+1,i} b_t, b_t l_{t,i})$.

(3) $(l_{t,j} l_{t,i}, l_{t,i} l_{t,j})$ for $j > i$.

The type (1) basic reduction decreases the path degree by one. The type (2) basic reduction moves a_t or b_t one step left, and $l_{t,i}$ or $l_{t+1,i}$ one step right in the path. Similarly, the type (3) basic reduction moves $l_{t,i}$ one step left, and $l_{t,j}$ one step right in the path for $j > i$.

Thus, any composition of these three types either decreases the path degree or moves a_t, b_t to the left, $l_{t,j}$ with the larger j to the right. Since the path degree of p is finite, we can only apply the basic reductions of these three types to p finitely many times. $\square$

Lemma 7.0.7. *All overlap ambiguities of the reduction system R (7.0.B) are resolvable.*

Proof. There are four types of overlap ambiguities in R (7.0.B): $l_{t,i}a_t b_t$, $l_{t+1,i} b_t a_t$, $l_{t,j} l_{t,i} a_k$, $l_{t+1,j} l_{t+1,i} b_t$ for $0 \leq i \leq n$, $t \in \mathbb{Z}/(n+1)$ and $j > i$. We next check that these overlap ambiguities are resolvable.

(1) When $t < i$, $(l_{t,i} a_t) b_t \to a_t (l_{t+1,i} b_t) \to (a_t b_t) l_{t,i} \to l_{t,t} l_{t,i}$, and $l_{t,i}(a_t b_t) \to l_{t,i} l_{t,t} \to l_{t,t} l_{t,i}$. The case of $t \geq i$ is similar.

(2) When $t < i$, $(l_{t+1,i} b_t) a_t \to b_t (l_{t,i} a_t) \to (b_t a_t) l_{t+1,i} \to l_{t+1,t} l_{t+1,i}$, and $l_{t+1,i}(b_t a_t) \to l_{t+1,i} l_{t+1,t} \to l_{t+1,t} l_{t+1,i}$. The case of $t \geq i$ is similar.

(3) $(l_{t,j} l_{t,i}) a_t \to l_{t,i}(l_{t,j} a_t) \to (l_{t,i} a_t) l_{t+1,j} \to a_t l_{t+1,i} l_{t+1,j}$,
$l_{tj}(l_{t,i} a_t) \to (l_{t,j} a_t) l_{t+1,i} \to a_t (l_{t+1,j} l_{t+1,i}) \to a_t l_{t+1,i} l_{t+1,j}$.

(4) $(l_{t+1,j} l_{t+1,i}) b_t \to l_{t+1,i}(l_{t+1,j} b_t) \to (l_{t+1,i} b_t) l_{t,j} \to b_t l_{t,i} l_{t,j}$,
$l_{t+1,j}(l_{t+1,i} b_t) \to (l_{t+1,j} b_t) l_{t,i} \to b_t (l_{t,j} l_{t,i}) \to b_t l_{t,i} l_{t,j}$. $\square$

Proposition 7.0.8. *Consider the quiver Q with relations I (7.0.A) and its reduction system R in (7.0.B). Then, the set of irreducible paths (with respect to R) of kQ under the projection $kQ \to kQ/I$ forms a k-basis of kQ/I.*

Proof. It is clear that the two-sided ideal generated by R (see 7.0.5) coincides with I (7.0.A). Since R is reduction-finite and all overlap ambiguities of R are resolvable by 7.0.6 and 7.0.7, the statement holds by 7.0.5. $\square$

Notation 7.0.9. For any $t \in \mathbb{Z}/(n+1)$, consider the following subsets of the set of paths on Q with head t.

(1) $\mathcal{A}_t := \{a_{t-i} \dots a_{t-2} a_{t-1} \mid \text{ all } i \in \mathbb{N}\}$.

(2) $\mathcal{B}_t := \{b_{t+i-1} \dots b_{t+1} b_t \mid \text{ all } i \in \mathbb{N}\}$.

(3) $\mathcal{L}_t := \{l_{t,0}^{i_1} l_{t,1}^{i_2} \dots l_{t,n}^{i_n} \mid \text{ all } i_1, i_2, \dots, i_n \in \mathbb{N} \cup \{0\}\}$.

(4) $\mathcal{A}_t\mathcal{L}_t := \{pq \mid \text{ all } p \in \mathcal{A}_t \text{ and } q \in \mathcal{L}_t\}$.

(5) $\mathcal{B}_t\mathcal{L}_t := \{pq \mid \text{ all } p \in \mathcal{B}_t \text{ and } q \in \mathcal{L}_t\}$.

(6) Then write $k\mathcal{A}_t$, $k\mathcal{B}_t$ and $k\mathcal{L}_t$ for the k-span of $\mathcal{A}_t$, $\mathcal{B}_t$ and $\mathcal{L}_t$ respectively.

(7) For any $A \in k\mathcal{A}_t$, write $(A)_{t-1}$ for the unique element in $k\mathcal{A}_{t-1}$ such that $A = (A)_{t-1}a_{t-1}$.

(8) For any $B \in k\mathcal{B}_t$, write $(B)_{t+1}$ for the unique element in $k\mathcal{B}_{t+1}$ such that $B = (B)_{t+1}b_t$.

(9) For any $L \in k\mathcal{L}_t$ and $0 \le s \le n$, write $(L)_s$ for the unique element in $k\mathcal{L}_s$, which is obtained by replacing $l_{t,0}, l_{t,1}, \dots, l_{t,n}$ in L by $l_{s,0}, l_{s,1}, \dots, l_{s,n}$.

We next describe all irreducible paths in Q, with respect to the reduction system R (7.0.B).

Proposition 7.0.10. *For any path p with head t in Q,*

$$p \text{ is irreducible } \iff p \in \mathcal{A}_t \cup \mathcal{B}_t \cup \mathcal{L}_t \cup \mathcal{A}_t\mathcal{L}_t \cup \mathcal{B}_t\mathcal{L}_t.$$

Proof. By the reduction system R (7.0.B), it is clear that each path in $\mathcal{A}_t, \mathcal{B}_t, \mathcal{L}_t, \mathcal{A}_t\mathcal{L}_t, \mathcal{B}_t\mathcal{L}_t$ is irreducible. We next prove the other direction. Since the head of p is t, p either ends with a_{t-1}, b_t or $l_{t,i}$ for some i. The proof splits into cases.

(1) p ends with a_{t-1}.

Write $p = qa_{t-1}$ for some q with head $t-1$. Then q either ends with a_{t-2}, b_{t-1} or $l_{t,i}$ for some i. However, if q either ends with b_{t-1} or $l_{t,i}$, then qa_{t-1} is reducible by R (7.0.B). Thus q can only end with a_{t-2}. Repeating the same process gives $p \in \mathcal{A}_t$.

(2) p ends with b_t.

Similar to (1), we can prove that $p \in \mathcal{B}_t$.

(3) p ends with $l_{t,i}$.

Write $p = ql_{t,i}$ for some q with head t. Then q either ends with a_{t-1}, b_t or $l_{t,j}$ for some j. If q ends with a_{t-1}, then $q \in \mathcal{A}_t$ by (1), and so $p \in \mathcal{A}_t\mathcal{L}_t$. Similarly, if q ends with b_t, then $p \in \mathcal{B}_t\mathcal{L}_t$. If q ends $l_{t,j}$, then $j \le i$; otherwise, it will contradict the irreducibility of $ql_{t,i}$. Repeating the same process gives $p \in \mathcal{L}_t$, $\mathcal{A}_t\mathcal{L}_t$ or $\mathcal{B}_t\mathcal{L}_t$. □

We next apply 7.0.8 and 7.0.10 to prove the exactness of a particular complex in 7.0.12. In the following, we write P_t for the k-span of the paths with head t in kQ/I (7.0.A).

Lemma 7.0.11. *The k-linear maps*

$$\begin{aligned} m_{l_{t,n}}\colon P_t &\to P_t, & m_{a_t}\colon P_t &\to P_{t+1} \\ f &\mapsto fl_{t,n} & f &\mapsto fa_t \end{aligned}$$

are injective for any $t \in \mathbb{Z}/(n+1)$.

Proof. We only prove $m_{l_{0,n}}$ and m_{a_0} and are injective, the other cases are similar. Since the reduction system R (7.0.B) is reduction-finite by 7.0.6, we can assume $f \in P_0$ is irreducible.

(1) $m_{l_{0,n}}$ is injective.

We first write $f = \sum_i \lambda_i p_i$ as a linear combination of irreducible paths where each $\lambda_i \in k$. Since p_i is irreducible and there are no paths in S (7.0.B) that end with $l_{0,n}$, $p_i l_{0,n}$ is also irreducible. Thus if $f l_{0,n} = \sum_i \lambda_i p_i l_{0,n} = 0$, then each $\lambda_i = 0$ by 7.0.8, and so $f = 0$.

(2) m_{a_0} is injective.

Since $f \in P_0$, by 7.0.10 we can write f as a linear combination of irreducible terms

$$f = \lambda A + \mu B + \beta L + \sum_i \lambda_i A_i L_i + \sum_j \mu_j B_j L_j,$$

where each $\lambda, \mu, \beta, \lambda_i, \mu_j \in k$, and $A, A_i \in k\mathcal{A}_0$, and $B, B_j \in k\mathcal{B}_0$, and $L, L_i, L_j \in k\mathcal{L}_0$. Thus

$$\begin{aligned}
fa_0 &= \lambda A a_0 + \mu B a_0 + \beta L a_0 + \sum_i \lambda_i A_i L_i a_0 + \sum_j \mu_j B_j L_j a_0 \\
&= \lambda A a_0 + \mu (B)_1 b_0 a_0 + \beta L a_0 + \sum_i \lambda_i A_i L_i a_0 + \sum_j \mu_j (B_j)_1 b_0 L_j a_0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } B = (B)_1 b_0 \text{ and } B_j = (B_j)_1 b_0\text{)} \\
&\to \lambda A a_0 + \mu (B)_1 l_{1,0} + \beta a_0 (L)_1 + \sum_i \lambda_i A_i a_0 (L_i)_1 + \sum_j \mu_j (B_j)_1 l_{1,0} (L_j)_1. \qquad (7.0.C) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } b_0 L_j a_0 \to b_0 a_0 (L_j)_1 \to l_{1,0}(L_j)_1\text{)}
\end{aligned}$$

By 7.0.10, each term in (7.0.C) is irreducible. We next claim that each term in (7.0.C) differs from the others.

Since $A_i L_i$ are different for different i, $A_i a_0 (L_i)_1$ are different for different i. Similarly, $(B_j)_1 l_{1,0} (L_j)_1$ are different for different j. Since $\deg(A_i) \geq 1$, $A_i a_0 (L_i)_1$ is different from $a_0 (L)_1$ for each i. Similarly, $(B_j)_1 l_{1,0} (L_j)_1$ is different from $(B)_1 l_{1,0}$ for each j. Thus we proved the claim.

So by 7.0.8 the terms in (7.0.C) descend to give basis elements of kQ/I. Thus if $fa_0 = 0$, then each $\lambda, \mu, \beta, \lambda_i, \mu_j$ is zero, and so $f = 0$. Thus m_{a_0} is injective. $\square$

Proposition 7.0.12.

$$0 \to P_0 \xrightarrow[d_4]{(a_0, b_n)} P_1 \oplus P_n \xrightarrow[d_3]{\begin{pmatrix} l_{1,n} & -b_0 b_n \\ -a_n a_0 & l_{n,0} \end{pmatrix}} P_1 \oplus P_n \xrightarrow[d_2]{\binom{b_0}{a_n}} P_0 \xrightarrow[d_1]{} k[l_{0,1}, l_{0,2}, \ldots, l_{0,n-1}] \to 0$$

is an exact sequence of k-linear maps in $\mathrm{k}Q/I$ (7.0.A).

Proof. This sequence is a chain complex from the relations I (7.0.A). The exactness at the last three indexes is from [W1, §6]. By 7.0.11, we have d_4 is injective, and thus this complex is exact at the first index. So we only need to prove that $\ker d_3 \subseteq \operatorname{im} d_4$. It suffices to prove that, for any $(f, g) \in P_1 \oplus P_n$,

$$fl_{1,n} = ga_n a_0 \Rightarrow (f, g) = (ha_0, hb_n) \text{ for some } h \in P_0.$$

Since the reduction system R (7.0.B) is reduction-finite by 7.0.6, we can assume that f and g are irreducible. Since f is irreducible and there are no paths in S (7.0.B) that end with $l_{1,n}$, then $fl_{1,n}$ is also irreducible. Since $g \in P_n$, by 7.0.10 we can write g as a linear combination of irreducible terms

$$g = \lambda A + \mu B + \sum_{i=0}^{n} \beta_i L_i l_{n,i} + \sum_{i=0}^{n} \sum_j \lambda_{ij} A_{ij} K_{ij} l_{n,i} + \sum_k \mu_k B_k J_k,$$

where each $\lambda, \mu, \beta_i, \lambda_{ij}, \mu_k \in k$, and $A, A_{ij} \in k\mathcal{A}_n$, and $B, B_k \in k\mathcal{B}_n$, and $L_i, K_{ij}, J_k \in k\mathcal{L}_n$. Since $L_i l_{n,i}$ is irreducible, $L_i \in k\langle l_{n,0}, \dots, l_{n,i}\rangle$ for each i. Similarly, $K_{ij} \in k\langle l_{n,0}, l_{n,1}, \dots, l_{n,i}\rangle$ for each i and j.

Multiplying g on the right by $a_n a_0$, $ga_n a_0$ equals

$$\begin{aligned}
&\lambda A a_n a_0 + \mu B a_n a_0 + \sum_i \beta_i L_i l_{n,i} a_n a_0 + \sum_{i,j} \lambda_{ij} A_{ij} K_{ij} l_{n,i} a_n a_0 + \sum_k \mu_k B_k J_k a_n a_0 \\
&= \lambda A a_n a_0 + \mu (B)_{01} b_0 b_n a_n a_0 + \sum_i \beta_i L_i l_{n,i} a_n a_0 + \sum_{i,j} \lambda_{ij} A_{ij} K_{ij} l_{n,i} a_n a_0 \\
&+ \sum_k \mu_k (B_k)_{01} b_0 b_n J_k a_n a_0 \qquad (\text{since } B = (B)_0 b_n = (B)_{01} b_0 b_n,\ B_k = (B_k)_0 b_n = (B_k)_{01} b_0 b_n) \\
&\to \lambda A a_n a_0 + \mu (B)_{01} l_{1,0} l_{1,n} + \sum_i \beta_i a_n a_0 (L_i)_1 l_{1,i} + \sum_{i,j} \lambda_{ij} A_{ij} a_n a_0 (K_{ij})_1 l_{1,i} \\
&+ \sum_k \mu_k (B_k)_{01} l_{1,0} (J_k)_1 l_{1,n} \qquad (7.0.D) \\
&\qquad (\text{since } b_0 b_n J_k a_n a_0 \to b_0 b_n a_n a_0 (J_k)_1 \to b_0 l_{0,n} a_0 (J_k)_1 \to b_0 a_0 l_{1,n} (J_k)_1 \to l_{1,0} (J_k)_1 l_{1,n}) \\
&= \lambda A a_n a_0 + \mu (B)_{01} l_{1,0} l_{1,n} + \sum_{i=0}^{n-1} \beta_i a_n a_0 (L_i)_1 l_{1,i} + \beta_n a_n a_0 (L_n)_1 l_{1,n} + \\
&\sum_{i=0}^{n-1} \sum_j \lambda_{ij} A_{ij} a_n a_0 (K_{ij})_1 l_{1,i} + \sum_j \lambda_{nj} A_{nj} a_n a_0 (K_{nj})_1 l_{1,n} + \sum_k \mu_k (B_k)_{01} l_{1,0} (J_k)_1 l_{1,n} \\
&= \lambda A a_n a_0 + \sum_{i=0}^{n-1} \beta_i a_n a_0 (L_i)_1 l_{1,i} + \sum_{i=0}^{n-1} \sum_j \lambda_{ij} A_{ij} a_n a_0 (K_{ij})_1 l_{1,i} + \mathsf{f}_1 l_{1,n}. \qquad (7.0.E) \\
&\qquad (\text{set } \mathsf{f}_1 := \mu (B)_{01} l_{1,0} + \beta_n a_n a_0 (L_n)_1 + \textstyle\sum_j \lambda_{nj} A_{nj} a_n a_0 (K_{nj})_1 + \sum_k \mu_k (B_k)_{01} l_{1,0} (J_k)_1)
\end{aligned}$$

We claim that each term in (7.0.D) is irreducible. To see this, we consider the terms in (7.0.D) separately.

(1) By the reduction system R (7.0.B) $A a_n a_0$ is irreducible.

(2) Since $l_{1,0} l_{1,n} \in \mathcal{L}_1$ and $(B)_{01} \in \mathcal{B}_1$, $(B)_{01} l_{1,0} l_{1,n}$ is irreducible by 7.0.10.

(3) Since $L_i \in k\langle l_{n,0}, \dots, l_{n,i}\rangle$, $(L_i)_1 \in k\langle l_{1,0}, \dots, l_{1,i}\rangle$, so $(L_i)_1 l_{1,i} \in k\mathcal{L}_1$. Thus $a_n a_0 (L_i)_1 l_{1,i}$ is irreducible by 7.0.10.

(4) Since $K_{ij} \in k\langle l_{n,0}, \dots, l_{n,i}\rangle$, $(K_{ij})_1 \in k\langle l_{1,0}, \dots, l_{1,i}\rangle$, so $(K_{ij})_1 l_{1,i} \in k\mathcal{L}_1$. Thus we have $A_{ij} a_n a_0 (K_{ij})_1 l_{1,i}$ is irreducible by 7.0.10.

(5) Since $l_{1,0}(J_k)_1 l_{1,n} \in k\mathcal{L}_1$, $(B_k)_{01} l_{1,0}(J_k)_1 l_{1,n}$ is irreducible by 7.0.10.

We next claim that each term in (7.0.D) differs from the others.

Since each $a_n a_0 (L_i)_1 l_{1,i}$ ends with $l_{1,i}$, $a_n a_0 (L_i)_1 l_{1,i}$ are different for different i. Since $A_{ij} K_{ij} l_{n,i}$ are different for different i and j, $A_{ij} a_n a_0 (K_{ij})_1 l_{1,i}$ are also different for different i and j. Similarly, $(B_k)_{01} l_{1,0}(J_k)_1 l_{1,n}$ are different for different k. Since $\deg(A_{ij}) \geq 1$, $A_{ij} a_n a_0$ is different from $a_n a_0$, so $A_{ij} a_n a_0 (K_{ij})_1 l_{1,i}$ is different from $a_n a_0 (L_i)_1 l_{1,i}$. Similarly, $(B_k)_{01} l_{1,0}(J_k)_1 l_{1,n}$ is different from $(B)_{01} l_{1,0} l_{1,n}$. So we proved the claim.

Since (7.0.E) is obtained by combining the terms in (7.0.D) that end with $l_{1,n}$, each term in (7.0.E) is also irreducible and differs from the others. So the terms in (7.0.E) descend to give different basis elements of kQ/I by 7.0.8.

Recall that $f l_{1,n} = g a_n a_0$ and $f l_{1,n}$ is irreducible. Since only $\mathsf{f}_1 l_{1,n}$ ends with $l_{1,n}$ in $g a_n a_0$ (7.0.E), then all terms in $g a_n a_0$ except $\mathsf{f}_1 l_{1,n}$ are zero, namely $\lambda = 0$, $\beta_i = 0$ and $\lambda_{ij} = 0$ for any j and $0 \leq i \leq n-1$. So

$$\begin{aligned}
g &= \mu B + \beta_n L_n l_{n,n} + \sum_j \lambda_{nj} A_{nj} K_{nj} l_{n,n} + \sum_k \mu_k B_k J_k \\
&= \mu (B)_0 b_n + \beta_n L_n a_n b_n + \sum_j \lambda_{nj} A_{nj} K_{nj} a_n b_n + \sum_k \mu_k (B_k)_0 (J_k)_0 b_n \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{since } B_k = (B_k)_0 b_n \text{ and } b_n J_k = (J_k)_0 b_n) \\
&= h b_n. \qquad (\text{set } h := \mu (B)_0 + \beta_n L_n a_n + \textstyle\sum_j \lambda_{nj} A_{nj} K_{nj} a_n + \sum_k \mu_k (B_k)_0 (J_k)_0)
\end{aligned}$$

Thus $g a_n a_0 = h b_n a_n a_0 = h a_0 l_{1,n}$. Together with $f l_{1,n} = g a_n a_0$ gives $f l_{1,n} = h a_0 l_{1,n}$, and so $f = h a_0$ by 7.0.11. Thus $(f, g) = (h a_0, h b_n)$, proving the claim. $\square$

With the exact sequence in 7.0.12, we can calculate the vector space dimension of each graded degree piece of P_t in (7.0.F), which will be used to prove the isomorphism in 7.0.18.

Notation 7.0.13. In the following, we adopt a new definition of degree of Q (7.0.A), which differs from path length in 2.1.1(4).

(1) Define $\deg(a_i) = \deg(b_i) = 1$ and $\deg(l_{t,i}) = 2$ for each i and t.

(2) With respect to this degree, write $P_{t,d}$ for the graded piece of degree d of P_t.

(3) With notation in 7.0.9, write $\mathcal{A}_{t,d}, \mathcal{B}_{t,d}, \mathcal{L}_{t,d}$, $(\mathcal{A}_t\mathcal{L}_t)_d$ and $(\mathcal{B}_t\mathcal{L}_t)_d$ for the subset of degree d paths in $\mathcal{A}_t$, $\mathcal{B}_t$, $\mathcal{L}_t$, $\mathcal{A}_t\mathcal{L}_t$ and $\mathcal{B}_t\mathcal{L}_t$ respectively.

(4) Write D_d for the vector space dimension of $P_{0,d}$.

By the symmetry of the quiver Q and relations I (7.0.A), D_d is also the vector space dimension of $P_{t,d}$ for $1 \leq t \leq n$. By 7.0.12, for any integer d, there is an exact sequence

$$0 \to P_{0,d} \to P_{1,d+1} \oplus P_{n,d+1} \to P_{1,d+3} \oplus P_{n,d+3} \to P_{0,d+4} \to T_{d+4} \to 0,$$

where T_{d+4} denotes the degree $d+4$ piece of $k[l_{0,1}, l_{0,2}, \ldots, l_{0,n-1}]$. Thus

$$D_d - 2D_{d+1} + 2D_{d+3} - D_{d+4} + E_{d+4} = 0 \tag{7.0.F}$$

where $E_{d+4} = \dim_k T_{d+4}$.

Since 7.0.10 describes all irreducible paths in Q, we can calculate D_d for each d in 7.0.14. Moreover, we can verify that the D_d in 7.0.14 satisfies (7.0.F) with some basic calculations. We write $|S|$ for the number of elements in a set S, and denote $C(m, n)$ as the number of n-combinations from a given set T of m elements.

Proposition 7.0.14. *With notation as above, the following holds.*

$$D_d = \begin{cases} 2\sum_{i=0}^{(d-1)/2} C(n+i, n), & \text{if } d \text{ odd} \\ C(n+d/2, n) + 2\sum_{i=0}^{d/2-1} C(n+i, n), & \text{if } d \text{ even.} \end{cases}$$

In particular, we have the vector space dimension of the graded degree d piece of kQ/I, which is $(n+1)D_d$.

Proof. Since D_d is the vector space dimension of $P_{t,d}$ for any t, by 7.0.8 D_d is equal to the number of the irreducible paths with head t and degree d. Recall the notation in 7.0.9 and 7.0.13. By 7.0.10, for any path p with head t and degree d,

$$p \text{ is irreducible} \iff p \in \mathcal{A}_{t,d} \cup \mathcal{B}_{t,d} \cup \mathcal{L}_{t,d} \cup (\mathcal{A}_t\mathcal{L}_t)_d \cup (\mathcal{B}_t\mathcal{L}_t)_d.$$

Thus $D_d = |\mathcal{A}_{t,d} \cup \mathcal{B}_{t,d} \cup \mathcal{L}_{t,d} \cup (\mathcal{A}_t\mathcal{L}_t)_d \cup (\mathcal{B}_t\mathcal{L}_t)_d|$. Since the intersection of any two different sets above is empty,

$$D_d = |\mathcal{A}_{t,d}| + |\mathcal{B}_{t,d}| + |\mathcal{L}_{t,d}| + |(\mathcal{A}_t\mathcal{L}_t)_d| + |(\mathcal{B}_t\mathcal{L}_t)_d|.$$

We first claim that $|\mathcal{A}_{t,d}| = 1$ and $|\mathcal{B}_{t,d}| = 1$ for each d, and

$$|\mathcal{L}_{t,d}| = \begin{cases} 0, & \text{if } d \text{ odd} \\ C(n+d/2, n), & \text{if } d \text{ even.} \end{cases}$$

Since $\mathcal{A}_{t,d} := \{a_{t-d} \ldots a_{t-2}a_{t-1}\}$, $|\mathcal{A}_{t,d}| = 1$. Similarly, $|\mathcal{B}_{t,d}| = 1$. Since the degree of each loop is two (see 7.0.13), if d is odd, then $\mathcal{L}_{t,d} = \emptyset$, and so $|\mathcal{L}_{t,d}| = 0$. Now we consider the case of d is even. Since any $p \in \mathcal{L}_{t,d}$ has the form of $l_{t,0}^{i_0} l_{t,1}^{i_1} \ldots l_{t,n}^{i_n}$ where each i_j is a positive integer and $2i_0 + 2i_1 + \cdots + 2i_n = d$, $|\mathcal{L}_{t,d}| = C(n+d/2, n)$. Thus we proved the claim.

By the definition of $(\mathcal{A}_t\mathcal{L}_t)_d$ in 7.0.13, $|(\mathcal{A}_t\mathcal{L}_t)_d| = \sum_{0<i<d} |\mathcal{A}_{t,i}||\mathcal{L}_{t,d-i}|$. Then we split into

two cases. When d is odd,

$$
\begin{aligned}
|(\mathcal{A}_t\mathcal{L}_t)_d| &= \sum_{0<i<d} |\mathcal{A}_{t,i}||\mathcal{L}_{t,d-i}| \\
&= |\mathcal{A}_{t,1}||\mathcal{L}_{t,d-1}| + |\mathcal{A}_{t,3}||\mathcal{L}_{t,d-3}| + \cdots + |\mathcal{A}_{t,d-2}||\mathcal{L}_{t,2}| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } |\mathcal{L}_{t,d}| = 0 \text{ when } d \text{ is odd)} \\
&= C(n + (d-1)/2, n) + C(n + (d-3)/2, n) + \cdots + C(n+1, n) \\
&= \sum_{i=1}^{(d-1)/2} C(n+i, n).
\end{aligned}
$$

When d is even,

$$
\begin{aligned}
|(\mathcal{A}_t\mathcal{L}_t)_d| &= \sum_{0<i<d} |\mathcal{A}_{t,i}||\mathcal{L}_{t,d-i}| \\
&= |\mathcal{A}_{t,2}||\mathcal{L}_{t,d-2}| + |\mathcal{A}_{t,4}||\mathcal{L}_{t,d-4}| + \cdots + |\mathcal{A}_{t,d-2}||\mathcal{L}_{t,2}| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } |\mathcal{L}_{t,d}| = 0 \text{ when } d \text{ is odd)} \\
&= C(n + (d-2)/2, n) + C(n + (d-4)/2, n) + \cdots + C(n+1, n) \\
&= \sum_{i=1}^{d/2-1} C(n+i, n).
\end{aligned}
$$

Similarly, we also have

$$
|(\mathcal{B}_t\mathcal{L}_t)_d| = |(\mathcal{A}_t\mathcal{L}_t)_d| = \begin{cases} \sum_{i=1}^{(d-1)/2} C(n+i, n), & \text{if } d \text{ odd} \\ \sum_{i=1}^{d/2-1} C(n+i, n), & \text{if } d \text{ even.} \end{cases}
$$

Then we calculate D_d into two cases. When d is odd,

$$
\begin{aligned}
D_d &= |\mathcal{A}_{t,d}| + |\mathcal{B}_{t,d}| + |\mathcal{L}_{t,d}| + |(\mathcal{A}_t\mathcal{L}_t)_d| + |(\mathcal{B}_t\mathcal{L}_t)_d| \\
&= 1 + 1 + 0 + 2\sum_{i=1}^{(d-1)/2} C(n+i, n) \\
&= C(n, n) + C(n, n) + 2\sum_{i=1}^{(d-1)/2} C(n+i, n) \qquad \text{(since } C(n,n) = 1) \\
&= 2\sum_{i=0}^{(d-1)/2} C(n+i, n).
\end{aligned}
$$

When d is even,

$$\begin{aligned}
D_d &= |\mathcal{A}_{t,d}| + |\mathcal{B}_{t,d}| + |\mathcal{L}_{t,d}| + |(\mathcal{A}_t\mathcal{L}_t)_d| + |(\mathcal{B}_t\mathcal{L}_t)_d| \\
&= 1 + 1 + C(n+d/2,n) + 2\sum_{i=1}^{d/2-1} C(n+i,n) \\
&= C(n,n) + C(n,n) + C(n+d/2,n) + 2\sum_{i=1}^{d/2-1} C(n+i,n) \qquad \text{(since } C(n,n)=1\text{)} \\
&= C(n+d/2,n) + 2\sum_{i=0}^{d/2-1} C(n+i,n). \qquad \square
\end{aligned}$$

Notation 7.0.15. We next define

$$\mathcal{S} := \frac{k[u,v,x_0,x_1,\dots,x_n]}{uv - x_0x_1\dots x_n}, \tag{7.0.G}$$

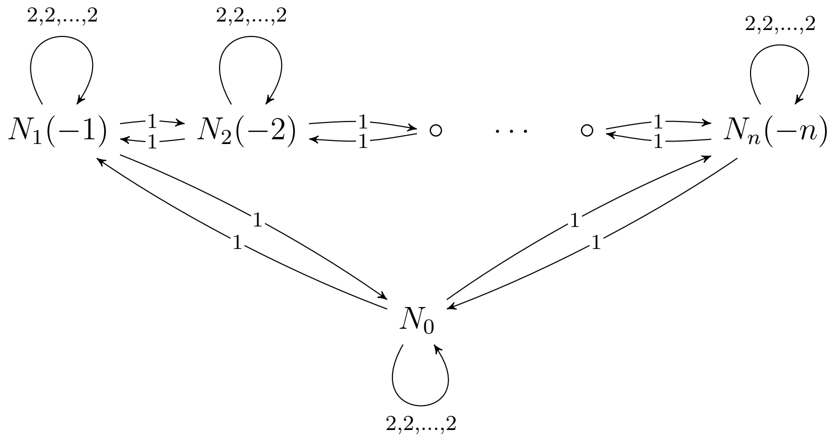
and consider the $\mathcal{S}$-module $N := \bigoplus_{i=0}^{n} N_i$ where $N_0 := \mathcal{S}$ and $N_i := (u, \prod_{j=0}^{i-1} x_j)$ for $1 \le i \le n$.

We will show that kQ/I (7.0.A) presents $\operatorname{End}_{\mathcal{S}}(N)$. By [IW3], every morphism in $\operatorname{End}_{\mathcal{S}}(N)$ can be obtained as a linear combination of compositions of the following maps.

$$\begin{tikzcd}
N_1 \arrow[loop above, "{x_0,x_1,\dots,x_n}"] \arrow[r, shift left, "x_1"] \arrow[drr, shift left, "inc"] & N_2 \arrow[loop above, "{x_0,x_1,\dots,x_n}"] \arrow[l, shift left, "inc"] \arrow[r, shift left, "x_2"] & \circ \arrow[l, shift left, "inc"] & \cdots & \circ \arrow[r, shift left, "x_{n-1}"] & N_n \arrow[loop above, "{x_0,x_1,\dots,x_n}"] \arrow[l, shift left, "inc"] \arrow[dlll, shift left, "\frac{x_n}{u}"] \\
 & & N_0 \arrow[loop below, "{x_0,x_1,\dots,x_n}"] \arrow[ull, shift left, "x_0"] \arrow[urrr, shift left, "u"] & & &
\end{tikzcd} \tag{7.0.H}$$

Thus there exists an obvious surjective homomorphism $kQ \twoheadrightarrow \operatorname{End}_{\mathcal{S}}(N)$. Since I gets sent to zero by inspection, this induces a surjective homomorphism $\psi\colon kQ/I \twoheadrightarrow \operatorname{End}_{\mathcal{S}}(N)$. We will show that ψ is an isomorphism, by counting graded pieces.

Notation 7.0.16. Grade $\mathcal{S}$ via $\deg(u) = \deg(v) = n+1$ and $\deg(x_0) = \cdots = \deg(x_n) = 2$. The particular choice of the graded shift of N given by $N := \bigoplus_{i=0}^{n} N_i(-i)$ induces a grading in $\operatorname{End}_{\mathcal{S}}(N)$, which explicitly grades each arrow in (7.0.H) as follows.



(7.0.I)

Notation 7.0.17. Parallel to the notation 7.0.13, we adopt the following notation.

(1) Set $Q_t := \operatorname{Hom}_{\mathbb{S}}(N, N_t(-t))$ for $0 \le t \le n$.

(2) With respect to (7.0.I), write $Q_{t,d}$ for the degree d graded piece of Q_t.

(3) Write D'_d for the vector space dimension of $Q_{0,d}$.

By the symmetry of (7.0.H), D'_d is also the vector space dimension of $Q_{t,d}$ for $1 \le t \le n$.

By [W3], we have the following exact sequence,

$$0 \to N_0 \xrightarrow{(x_0,u)} N_1 \oplus N_n \xrightarrow{\begin{pmatrix} x_n & -u \\ -\frac{x_0 x_n}{u} & x_0 \end{pmatrix}} N_1 \oplus N_n \xrightarrow{\begin{pmatrix} inc \\ \frac{x_n}{u} \end{pmatrix}} N_0 \to 0$$

Using the grading in 7.0.17, the above exact sequence becomes

$$0 \to N_0 \xrightarrow{d_4} N_1(-1) \oplus N_n(-n) \xrightarrow{d_3} N_1(-1) \oplus N_n(-n) \xrightarrow{d_2} N_0 \to 0. \tag{7.0.J}$$

where each d_i is homogeneous, and further $\deg(d_4) = 1 = \deg(d_2)$ and $\deg(d_3) = 2$. Applying $\operatorname{Hom}_{\mathbb{S}}(N, -)$ to (7.0.J) induces the following exact sequence,

$$0 \to Q_0 \xrightarrow{d_4} Q_1 \oplus Q_n \xrightarrow{d_3} Q_1 \oplus Q_n \xrightarrow{d_2} Q_0 \xrightarrow{d_1} \Lambda_{\mathrm{con}} \cong k[x_1, x_2, \dots, x_{n-1}] \to 0,$$

which is parallel to the one in 7.0.12. Thus for any integer d, there is an exact sequence

$$0 \to Q_{0,d} \to Q_{1,d+1} \oplus Q_{n,d+1} \to Q_{1,d+3} \oplus Q_{n,d+3} \to Q_{0,d+4} \to T'_{d+4} \to 0,$$

where T'_{d+4} denotes the degree $d+4$ piece of $k[x_1, x_2, \dots, x_{n-1}]$. Thus

$$D'_d - 2D'_{d+1} + 2D'_{d+3} - D'_{d+4} + E'_{d+4} = 0 \tag{7.0.K}$$

where $E'_{d+4} = \dim_k T'_{d+4}$.

Proposition 7.0.18. *With notation as above, ψ induces an isomorphism $kQ/I \xrightarrow{\sim} \operatorname{End}_{\mathbb{S}}(N)$.*

Proof. With the notation above, ψ is a graded surjective homomorphism, so it suffices to show that $D_d = D'_d$ for all d. Using (7.0.F), (7.0.K) and $E_d = E'_d$ for each d, we have $D_d = D'_d$ for each d by induction. $\square$

Corollary 7.0.19. *With respect to the degree in (7.0.I), for any d, the vector space dimension of the degree d graded piece of* $\mathrm{End}_{\mathfrak{S}}(N)$ *is equal to*

$$\begin{cases} 2(n+1)\sum_{i=0}^{(d-1)/2} C(n+i,n), & \textit{if } d \textit{ odd} \\ (n+1)C(n+d/2,n) + 2(n+1)\sum_{i=0}^{d/2-1} C(n+i,n), & \textit{if } d \textit{ even.} \end{cases}$$

Proof. This is immediate from 7.0.18 and 7.0.14. $\square$

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