Derived symmetries induced by relatively spherical contraction algebras

by

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Abstract

This thesis constructs homological symmetries (or, more precisely, derived autoequivalences) of algebras and varieties.

In more detail, given a surjective ring morphism $p\colon A \to B$, this thesis constructs endomorphisms $\mathcal{T}\colon \mathrm{D}(\mathrm{Mod}\, A) \to \mathrm{D}(\mathrm{Mod}\, A)$ and $\mathcal{C}\colon \mathrm{D}(\mathrm{Mod}\, B) \to \mathrm{D}(\mathrm{Mod}\, B)$ called the *twist* and *cotwist* around the extension of scalars functor induced by p. Moreover, we prove that these endomorphisms are equivalences in two settings: (1) twists for Gorenstein orders, and (2) twists induced by Frobenius exact categories.

(1) When A is a Gorenstein order and B is self-injective, then the twist $\mathcal{T}$ and cotwist $\mathcal{C}$ are equivalences provided that B is perfect as an A-module and satisfies a certain Tor-vanishing condition. In fact, under these assumptions, $\mathcal{C}$ is a shift of the Nakayama functor of B. If, moreover, A is an order over a three-dimensional ring, then we prove that the Tor-vanishing condition is equivalent to the ring-theoretic condition that $\ker p = (\ker p)^2$.

(2) Given a Frobenius exact category $\mathcal{E}$ and an object $x \in \mathcal{E}$, let $A = \mathrm{End}_{\mathcal{E}}(x)$, $B = \underline{\mathrm{End}}_{\mathcal{E}}(x)$ so that there is a natural surjection $p\colon A \to B$. In this setting, we show that the functors $\mathcal{T}$ and $\mathcal{C}$ are equivalences if B satisfies "hidden smoothness" and "spherical" criteria.

We furthermore apply the technology developed in (1) and (2) to construct derived autoequivalences of varieties. More specifically, given a crepant contraction $f\colon X \to Y$ between varieties satisfying mild conditions, there is an associated epimorphism of $\mathcal{O}_Y$-algebras $\pi\colon \mathcal{A} \to \mathcal{A}_{\mathrm{con}}$ which, affine locally, induces a surjection of algebras. Therefore, using techniques in non-commutative geometry, we apply the technology of (1) and (2) to construct autoequivalences of $\mathrm{D^b}(\mathrm{coh}\, X)$.

These results extend the construction of the *noncommutative twist*, introduced by Donovan and Wemyss, to more general settings. As a corollary, we also obtain that the noncommutative twist is in fact a spherical twist, and we discuss how our results extend previous works on spherical twists induced by crepant contractions.

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Author's declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Contents

Chapter 1

Introduction

This thesis studies *homological symmetries* of noncommutative algebras and of varieties which admit tilting bundles. More specifically, it studies the *noncommutative twist* introduced by Donovan and Wemyss [DW16, §5.3], and extends its construction to more general settings.

The modern notion of symmetry in science is phrased in terms of invariance under transformations: a symmetry of an object is a transformation applied to the object which leaves it invariant. For instance, a square is invariant under a rotation of $\frac{\pi}{2}$. In general, the $\frac{2\pi}{n}$ rotation of a regular polygon P with n sides can be interpreted as an automorphism of P, meaning that it can be viewed as an invertible transformation $r\colon P \to P$. The symmetries of P can be collected into a group $\operatorname{Aut}(P)$, and the study of such a group amounts to the study of the symmetries of P.

Symmetries play an important role in science and mathematics via the philosophy that the symmetries of an object constrain its properties. For example, a regular polygon which has a rotational symmetry of $\frac{2\pi}{q}$ must have $n = mq$ sides for some positive integer m. Hence, the group $\operatorname{Aut}(P)$ constrains the geometric properties of P.

This thesis concerns homological symmetries of algebras and varieties. Varieties are geometric spaces defined (locally) as solutions to polynomial equations. By homological symmetries, we mean symmetries of the objects $\mathrm{D^b}(\operatorname{mod} A)$ and $\mathrm{D^b}(\operatorname{coh} X)$, known, respectively, as the bounded derived category of finitely generated modules over an algebra A and the bounded derived category of coherent sheaves of a variety X. More specifically, this thesis constructs symmetries in $\operatorname{Aut}(\mathrm{D^b}(\operatorname{mod} A))$ and $\operatorname{Aut}(\mathrm{D^b}(\operatorname{coh} X))$.

The categories $\mathrm{D^b}(\operatorname{mod} A)$ and $\mathrm{D^b}(\operatorname{coh} X)$ package the homological properties of A and X. These are interesting objects to study since they are flexible, whilst also carrying much information about A and X. For instance, when X is a smooth projective variety with ample or anti-ample canonical bundle, then it is possible to reconstruct X from $\mathrm{D^b}(\operatorname{coh} X)$ [BO01].

In fact, the automorphism group $\operatorname{Aut}(\mathrm{D^b}(\operatorname{coh} X))$ has close connections with the stability manifold of X and with mirror symmetry. This, together with the fact that the symmetries in $\operatorname{Aut}(\mathrm{D^b}(\operatorname{coh} X))$ also constrain numerical invariants of X, have made the investigation of homological symmetries an important area of study in algebraic geometry and representation theory. The description of these automorphism groups in general is difficult, and many authors have contributed to constructing autoequivalences for certain classes of derived categories (see, for example, [Add16; BB17; BO01; Don24; BT11; DW19b; Orl03; ST01] in algebraic geometry and [BDL23; GM17; JY22; KS02; MY01; Qiu15] in representation theory).

The key ideas of this work are summarised as follows.

§ 1.1: Twists and cotwists. In order to construct automorphisms of the category $\mathrm{D^b}(\operatorname{mod} A)$, we first construct the *twist* $\mathcal{T}\colon \mathrm{D}(\operatorname{Mod} A) \to \mathrm{D}(\operatorname{Mod} A)$ and the *cotwist* $\mathcal{C}\colon \mathrm{D}(\operatorname{Mod} B) \to \mathrm{D}(\operatorname{Mod} B)$ around the extension of scalars functor induced by a ring morphism $p : A \to B$.

§ 1.2: Spherical twists for Gorenstein orders. When A is a Gorenstein order, we specify conditions on B such that $\mathcal{T}$ and $\mathcal{C}$ constructed in § 1.1 are autoequivalences. The central new idea in this section is that, when B is self-injective, then $\mathcal{C}$ is a shift of the Nakayama functor of B.

§ 1.3: Spherical twists induced by Frobenius categories. Given an exact Frobenius category $\mathcal{E}$ and an object $x \in \mathcal{E}$, let $A = \operatorname{End}_{\mathcal{E}}(x)$, $B = \underline{\operatorname{End}}_{\mathcal{E}}(x)$ so that there is a natural surjection $p\colon A \to B$. Then, we specify a "hidden smoothness" and a "spherical" criteria on A so that the functors $\mathcal{T}$ and $\mathcal{C}$ constructed in § 1.1 are autoequivalences. The key novelty in this section is that we construct and control what we call *partially minimal projective resolutions* in non Krull-Schmidt settings.

§ 1.4: Crepant contractions. The technology developed in § 1.3 is applied to construct twist automorphisms of $\mathrm{D^b}(\operatorname{coh} X)$ induced by crepant contractions $X \to Y$ between very singular varieties (see 2.7.1 for the precise setup). This result extends the noncommutative twist in [DW19b] to more singular settings, and, moreover, generalises [BB22, 5.18].

§ 1.1 Twists and cotwists

Twists around spherical objects were introduced by Seidel and Thomas [ST01] in order to obtain derived automorphisms of complex smooth projective varieties. This construction was generalised to twists around spherical functors by [AL17] [Ann13], as well as other authors. For a survey on the development of this technology, see, for example, [Add16] or [Seg18].

Twists around functors were introduced in order to construct automorphisms of triangulated categories. Given (enhanced) triangulated categories $\mathcal{A}$, $\mathcal{B}$ and a functor $F\colon \mathcal{A} \to \mathcal{B}$, then [AL17] constructs two closely related endofunctors $\mathcal{T}\colon \mathcal{B} \to \mathcal{B}$ and

$\mathcal{C}\colon \mathcal{A} \to \mathcal{A}$ called the twist and cotwist around F, respectively. When both $\mathcal{T}$ and $\mathcal{C}$ are equivalences, then F is called a *spherical functor* and $\mathcal{T}$ is called a *spherical twist*.

Properties of the cotwist afford control over properties of the twist. For instance, one can prove that $\mathcal{T}$ is an equivalence based on properties of $\mathcal{C}$. Therefore, given an autoequivalence T of a triangulated category $\mathcal{B}$, it is interesting to characterise T as a twist around a functor $S\colon \mathcal{A} \to \mathcal{B}$ whose domain category $\mathcal{A}$ is simpler (e.g. $\mathcal{A}$ is the derived category of a finite dimensional algebra) than the codomain category $\mathcal{B}$ (e.g. $\mathcal{B}$ is the category of an infinite dimensional algebra).

With this in mind, our first goal is to explicitly calculate the twist and cotwist around the restriction of scalars functor $F = (-) \otimes^{\mathbf{L}}_B B\colon \mathrm{D}(\mathrm{Mod}\, B) \to \mathrm{D}(\mathrm{Mod}\, A)$ induced by a ring morphism $p\colon A \to B$. Our strategy for this amounts to expressing the unit and counit, respectively, of the adjunction between F and its right adjoint F^{R} as the natural transformations

$$\begin{aligned}
&\mathbf{R}\mathrm{Hom}_A(p,-)\colon \mathbf{R}\mathrm{Hom}_A({}_AB,-) \to \mathbf{R}\mathrm{Hom}_A({}_AA,-),\\
&\mathbf{R}\mathrm{Hom}_A(s,-)\colon \mathbf{R}\mathrm{Hom}_B({}_BB,-) \to \mathbf{R}\mathrm{Hom}_B({}_BB \otimes^{\mathbf{L}}_A B_B,-),
\end{aligned}$$

induced by bimodule morphisms p and s. Here, s is the natural morphism

$$s\colon {}_BB \otimes^{\mathbf{L}}_A B_B \to {}_BB_B$$

in $\mathrm{D}(B \otimes_{\mathbb{Z}} B^{\mathrm{op}})$. From this observation, it is straightforward to deduce the first main result of this thesis.

Theorem A (3.1.3, 3.2.6)**.** *Let $p\colon A \to B$ be a ring morphism, and consider the restriction of scalars functor $F = (-) \otimes^{\mathbf{L}}_B B\colon \mathrm{D}(\mathrm{Mod}\, B) \to \mathrm{D}(\mathrm{Mod}\, A)$.*

(a) *The twist around F is*

$$\mathcal{T} = \mathbf{R}\mathrm{Hom}_A(\mathrm{cone}(p)[-1],-)\colon \mathrm{D}(\mathrm{Mod}\, A) \to \mathrm{D}(\mathrm{Mod}\, A).$$

(b) *Consider the morphism $s\colon {}_BB \otimes^{\mathbf{L}}_A B_B \to {}_BB_B$ in $\mathrm{D}(B \otimes_{\mathbb{Z}} B^{\mathrm{op}})$. Then, the cotwist around F is*

$$\mathcal{C} = \mathbf{R}\mathrm{Hom}_B(\mathrm{cone}(s),-)\colon \mathrm{D}(\mathrm{Mod}\, B) \to \mathrm{D}(\mathrm{Mod}\, B).$$

§ 1.2 Spherical twists for Gorenstein orders

Given Theorem A, it is natural to ask whether the restriction of scalars functor F is spherical. Although this is not known in general, we are able to specify this within certain contexts.

Let $p\colon A \to B$ be a surjection between R-algebras where R is local Cohen-Macaulay with $\dim R = d$ and dualizing module ω_R. Suppose that A is a *Gorenstein R-order* (i.e. there is an A-bimodule isomorphism $\mathbf{R}\mathrm{Hom}_R(A, \omega_{\mathcal{R}}) \cong A$).

Theorem B (4.2.1, 4.2.4)**.** *Suppose that B is perfect (i.e. has finite projective dimension) as an A-module. Then, the twist around the functor $F = (-) \otimes^{\mathbf{L}}_A B$ is $\mathcal{T} = \mathbf{R}\mathrm{Hom}_A(\ker p, -)$.*

Moreover, if B is self-injective (i.e. it is injective as a module over itself) and if $\mathrm{H}^k({}_B B \otimes^{\mathbf{L}}_A B_B) = 0$ for all $k \neq 0, -d$, then F is a spherical functor, and the cotwist is around F is $\mathcal{C} = [-d-1] \cdot \mathcal{N}$, where $\mathcal{N}$ is the Nakayama functor of B.

If A has finite global dimension, then B is automatically perfect. Moreover, when $d = 3$ in Theorem B, then the vanishing condition on $\mathrm{H}^*({}_B B \otimes^{\mathbf{L}}_A B_B)$ is equivalent to the ring-theoretic condition $(\ker p)^2 = \ker p$. Hence, for $d = 3$ and A of finite global dimension, Theorem B states that self-injective quotients of A satisfying $(\ker p)^2 = \ker p$ generate autoequivalences of $\mathrm{D^b}(\mathrm{mod}\, A)$.

§ 1.3 Spherical twists induced by Frobenius categories

Let $\mathcal{E}$ be an R-linear Frobenius exact category, then, as is recalled in § 2.2, its stable category $\underline{\mathcal{E}}$ is triangulated and admits a suspension functor Σ. Let $x \in \mathcal{E}$ and suppose that $\mathcal{E}$ and x satisfy the mild conditions of § 5.1.1. Then, $A = \mathrm{End}_{\mathcal{E}}(x)$ and $B = \underline{\mathrm{End}}_{\mathcal{E}}(x)$ are R-algebras and there is a natural quotient morphism $p\colon A \to B$. The key point is that the twist around $p\colon A \to B$ is spherical if the action of the suspension functor on the object x is periodic up to additive closure. The precise statement is the following.

Theorem C (5.2.13)**.** *Consider the functor $F = (-) \otimes^{\mathbf{L}}_A B$. If*

(C1) *B is perfect over A and*

(C2) *there exists $t \in \mathbb{Z}$ such that $\mathrm{Ext}^k_A(B, S) = 0$ for all $k \neq 0, t$ and all simple B-modules S*

then F is spherical. The twist around F is

$$\mathcal{T} = \mathbf{R}\mathrm{Hom}_A(\ker p, -)\colon\ \mathrm{D^b}(\mathrm{mod}\, A) \to \mathrm{D^b}(\mathrm{mod}\, A)$$

and the cotwist is

$$\mathcal{C} = \mathbf{R}\mathrm{Hom}_B(\underline{\mathrm{Hom}}_{\mathcal{E}}(x, \Sigma^{t-1}x), -)[1-t]\colon\ \mathrm{D^b}(\mathrm{mod}\, B) \to \mathrm{D^b}(\mathrm{mod}\, B).$$

We interpret (C1) in Theorem C as a "hidden smoothness criterion" and (C2) as a "spherical criterion". If B satisfies (C2), we say that B is *t-relatively spherical*. The key point in the proof of theorem C is that the following conditions are equivalent.

Theorem D (5.1.28)**.** *Fix $t \in \mathbb{Z}$ with $2 \leqslant t \leqslant \dim R$. Then, the following are equivalent*

(1) *B is perfect over A and, further, $\operatorname{Ext}^k_A(B, S) = 0$ for all $k \neq 0, t$ and all simple B-modules S.*

(2) *$\operatorname{Ext}^i_{\underline{\mathcal{E}}}(x, x) = 0$ for $0 < i < t-1$, and $\operatorname{add}_{\underline{\mathcal{E}}} \Sigma^{-t+1} x = \operatorname{add}_{\underline{\mathcal{E}}} x$.*

If the condition (2) holds, then we construct and control what we call *partially minimal projective resolutions* in non Krull-Schmidt settings. It follows from this resolution and through the technology of tilting bimodules that $\mathcal{T}$ and $\mathcal{C}$ are equivalences.

§ 1.4 Crepant contractions

Various authors have investigated derived symmetries of a variety X induced by *crepant contractions* [BB22; Don24; DW19b]. When X admits a relative tilting bundle and Y is complete locally a hypersurface, [DW19b] constructs a derived symmetry of X called the noncommutative twist. The results in chapter 6 extend this construction and, furthermore, prove that the noncommutative twist is in fact a twist. Consequently, we obtain new derived autoequivalences of schemes with at worst Gorenstein singularities.

Consider a crepant contraction $f\colon X \to Y$ satisfying mild conditions and admitting a relative tilting bundle $\mathcal{P}$ (see 2.7.1 for the precise setup).

Suppose first that $Y = \operatorname{Spec} R$ is affine. Then, there is a derived equivalence

$$\Psi_{\mathcal{P}}\colon \ \mathrm{D^b}(\operatorname{coh} X) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{mod} \operatorname{End}_R(f_*\mathcal{P})).$$

Write $\Lambda = \operatorname{End}_R(f_*\mathcal{P})$, and let $[\operatorname{add} R]$ be the ideal of $\operatorname{End}_R(f_*\mathcal{P})$ consisting of maps $f_*\mathcal{P} \to f_*\mathcal{P}$ which factor through a finite projective R-module. Consider the *contraction algebra* $\Lambda_{\mathrm{con}} := \operatorname{End}_R(f_*\mathcal{P})/[\operatorname{add} R]$. We may view the contraction algebra as a Λ_{con}-Λ-bimodule. When doing so, we will sometimes write ${}_{\Lambda_{\mathrm{con}}}\Lambda_{\mathrm{con}\Lambda}$ for clarity.

Theorem E (6.3.7)**.** *If Λ_{con} is perfect over Λ and t-relatively spherical, then the functor*

$$\Psi^{-1}_{\mathcal{P}} \cdot (-) \otimes^{\mathbf{L}}_{\Lambda_{\mathrm{con}}} \Lambda_{\mathrm{con}\Lambda}\colon \ \mathrm{D^b}(\operatorname{mod} \Lambda_{\mathrm{con}}) \to \mathrm{D^b}(\operatorname{coh} X)$$

is spherical. Moreover, the twist $\mathcal{T}_R$ and cotwist $\mathcal{C}_R$ around $\Psi^{-1}_{\mathcal{P}} \cdot (-) \otimes^{\mathbf{L}}_{\Lambda_{\mathrm{con}}} \Lambda_{\mathrm{con}\Lambda}$ are described in 6.3.7.

Example 6.3.10 explains that the above extends [BB22, 5.18] by dropping the one-dimensional fibre assumption and the hypersurface singularity assumption on Y. Moreover, this result extends [DW19b, 5.7] by dropping the hypersurface assumption.

When Y is not affine, we use the technology of noncommutative schemes to obtain a global analogue of Theorem E. Given f as above with Y not necessarily affine, then there is a noncommutative scheme $(Y, f_*\mathcal{E}nd_X(\mathcal{P}))$ and an equivalence

$$\Psi_{\mathcal{P}}\colon \ \mathrm{D^b}(\operatorname{coh} X) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{coh}(Y, f_*\mathcal{E}nd_X(\mathcal{P}))).$$

Consider the sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A} := f_*\mathcal{E}nd_X(\mathcal{P})$ and write $\mathcal{A}_{\mathrm{con}} = \mathcal{A}/\mathcal{I}$ where $\mathcal{I}$ is constructed in [DW19b, 2.8] and recalled in 2.7.5.

Theorem F (6.4.5)**.** *Suppose that $\mathcal{A}_{\mathrm{con}}$ is perfect in $\mathrm{D^b}(\mathrm{coh}(Y, \mathcal{A}))$. Then, the functor*

$$\mathcal{T}_Y := \Psi_{\mathcal{P}}^{-1} \cdot \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -) \cdot \Psi_{\mathcal{P}} \colon \mathrm{D^b}(\mathrm{coh}\, X) \to \mathrm{D^b}(\mathrm{coh}\, X)$$

is the twist around the functor $\Psi_{\mathcal{P}}^{-1} \cdot (-) \otimes^{\mathbf{L}}_{\mathcal{A}_{\mathrm{con}}} \mathcal{A}_{\mathrm{con}}$. Moreover, if $\mathcal{A}_{\mathrm{con}}$ is t-relatively spherical, then

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -) \colon \mathrm{D^b}(\mathrm{coh}(Y, \mathcal{A})) \to \mathrm{D^b}(\mathrm{coh}(Y, \mathcal{A}))$$

is an equivalence. Consequently,

$$\mathcal{T}_Y \colon \mathrm{D^b}(\mathrm{coh}\, X) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{coh}\, X)$$

is an autoequivalence of $\mathrm{D^b}(\mathrm{coh}\, X)$.

§ 1.5 Conventions

Modules will, by convention be right modules. We will write $\mathrm{Mod}\, R$ for the category of modules over the ring R, $\mathrm{mod}\, R$ for the category of finitely generated modules, $\mathrm{proj}\, R$ for the category of finitely generated projective modules, and $\mathrm{fl}\, R$ for the category of finite length modules. An R-S-bimodule L will be denoted ${}_RL_S$ and will by convention be a left R-module and a right S-module. We will often refer to an R-S-bimodule as an $S \otimes_{\mathbb{Z}} R^{\mathrm{op}}$-module. Consider bimodules ${}_RL_S$, ${}_RM_S$, and ${}_SN_R$. We will sometimes abbreviate the Hom-set $\mathrm{Hom}_S({}_RL_S, {}_RM_S)$ as $\mathrm{Hom}_S({}_RL, {}_RM)$. Similarly, the tensor product ${}_RL_S \otimes_S {}_SM_T$ will often be written as ${}_RL \otimes_S M_T$.

Let F, G be morphisms in a category, then we will denote the composition "G then F" as $F \cdot G$.

Given an additive category $\mathcal{A}$ and an object $X \in \mathcal{A}$, $\mathrm{add}_{\mathcal{A}}\, X$ denotes the full subcategory of $\mathcal{A}$ whose objects are finite direct sums of summands of X.

Finally, given a triangulated category $\mathcal{T}$ with suspension functor Σ, we will often abbreviate a distinguished triangle

$$a \to b \to c \to \Sigma a$$

as

$$a \to b \to c \to^{+}$$

Chapter 2

Background

§ 2.1 Semi-perfect and self-injective algebras

§ 2.1.1 Semi-perfect algebras

In the world of representation theory, finite dimensional algebras are much better understood than infinite dimensional ones. This is because there are many important structure theorems for the representation theory of finite dimensional algebras which do not readily generalise to infinite dimensional cases. However, many of these structure theorems do generalise to semi-perfect algebras so that, in some sense, semi-perfect algebras can be thought of as a generalisation.

Definition 2.1.1 ([Kra15, 4.1])**.** Recall that an algebra A is *semi-perfect* if any of the following equivalent conditions hold.

(1) The category of finitely generated projective A-modules is Krull-Schmidt.

(2) As a module over itself, A admits a direct sum decomposition $A = \bigoplus_{i=1}^{n} P_i$ where P_i have local endomorphism rings.

(3) Every simple A-module admits a projective cover.

(4) Every finitely generated A-module admits a projective cover.

Let A be a semi-perfect algebra and consider its direct sum decomposition $A = \bigoplus_{i \in I} P_i$ as in the definition above. Since P_i have local endomorphism rings, they are indecomposable projective A-modules. Moreover, note that $\{P_i\}_{i \in I}$ are all of the indecomposable projective A-modules up to isomorphism.

Lemma 2.1.2. *If A is a semi-perfect algebra, then it has finitely many simple modules.*

Proof. It follows from the definition of semi-perfect algebras and from [Kra15, 3.6] that given a simple A-module S, there is an i such that $\pi\colon P_i \to S$ is the projective cover of S. We claim that there is a bijection between isomorphism classes of indecomposable projectives and isomorphism classes of simples.

Suppose that S and S' are simple modules which admit projective covers $\pi\colon P_i \to S$ and $\pi'\colon P_i \to S'$. Then, $S \cong P_i/\ker\pi$ and $S' \cong P_i/\ker\pi'$ so that $\ker\pi$ and $\ker\pi'$ are maximal submodules of P_i. Therefore, either $\ker\pi = \ker\pi'$ or $\ker\pi + \ker\pi' = P_i$. In the first case, it follows that $S \cong S'$. In the latter case, since both $\ker\pi$ and $\ker\pi'$ are essential by definition of projective covers, it must be that $\ker\pi = \ker\pi' = 0$ so that $S \cong S'$. □

Notation 2.1.3. We let S_i be the simple A-module whose projective cover is P_i. When A is finite-dimensional, let I_i be the indecomposable injective whose socle is S_i.

§ 2.1.2 Self-injective algebras

An algebra A is said to be *self-injective* if it is injective as a module over itself. When A is Noetherian, there is a well-known characterisation of self-injective algebras.

Proposition 2.1.4 (See e.g. [Wei94, 4.2.4])**.** *The following are equivalent.*

(1) *A is Noetherian and self-injective,*

(2) *All projective A-modules are injective,*

(3) *All injective A-modules are projective.*

We will often be interested in self-injective R-algebras which are module finite over a noetherian ring R. For such algebras, the following result tells us we can restrict to studying artinian algebras.

Proposition 2.1.5 ([GN02, 3.4(4)])**.** *Let R be a noetherian ring, and consider an R-algebra A which is module finite over R. Then, for a non-zero finitely generated A-module M,*

$$\dim_R M \leqslant \operatorname{inj.dim}_A M.$$

It is a direct consequence of 2.1.5 above that a self-injective R-algebra over a noetherian ring which is module finite over R and is also an algebra over a field k must be finite dimensional.

Let A be a finite-dimensional algebra viewed as a path-algebra of a quiver Q with relations R. Then, self-injectivity of A can be thought of as a symmetry condition on (Q, R), since we want the class of indecomposable injective modules to match the class of indecomposable projective modules.

Example 2.1.6. Let Q_1 and Q_2 be the quivers

$$Q_1: \begin{tikzcd} & 0 \arrow[dr, "x_0"] & \\ 2 \arrow[ur, "x_2"] & & 1 \arrow[ll, "x_1"] \end{tikzcd} \qquad Q_2: \begin{tikzcd} & 0 \arrow[dr, shift left=2, "x_0"] \arrow[dr, "y_0"] \arrow[dr, shift right=2, "z_0"'] & \\ 2 \arrow[ur, shift left=2, "x_2"] \arrow[ur, "y_2"] \arrow[ur, shift right=2, "z_2"'] & & 1. \arrow[ll, shift right=2, "z_1"'] \arrow[ll, "y_1"] \arrow[ll, shift left=2, "x_1"] \end{tikzcd}$$

(1) Consider Q_1 with relations $R_1 = \langle x_0x_1x_2, x_1x_2x_0, x_2x_0x_1\rangle$. Then, the indecomposable projective and injective representations are, up to isomorphism,

$$I_2 \cong P_0 : \begin{tikzcd} & k \arrow[dr, "1"] & \\ k \arrow[ur, "0"] & & k \arrow[ll, "1"] \end{tikzcd} \qquad I_0 \cong P_1 : \begin{tikzcd} & k \arrow[dr, "0"] & \\ k \arrow[ur, "1"] & & k \arrow[ll, "1"] \end{tikzcd} \qquad I_1 \cong P_2 : \begin{tikzcd} & k \arrow[dr, "1"] & \\ k \arrow[ur, "1"] & & k. \arrow[ll, "0"] \end{tikzcd}$$

(2) Next, consider the quiver Q_2 with relations for $i \in \mathbb{Z}_3$

$$\begin{aligned} R_2 = \langle & x_iy_{i+1} - y_ix_{i+1}, x_iz_{i+1} - z_ix_{i+1}, y_iz_{i+1} - z_iy_{i+1}, \\ & x_ix_{i+1}, y_iy_{i+1}, z_iz_{i+1}\rangle_{i\in\mathbb{Z}_3}. \end{aligned}$$

Intuitively, R_2 means that the "variables x, y, z commute and they square to 0".

Then, the indecomposable projective representations P_0, P_1 and P_2 are

$$\begin{tikzcd} & k^2 \arrow[dr, shift left=2, "X_a"] \arrow[dr, "Y_a"] \arrow[dr, shift right=2, "Z_a"'] & \\ k^3 \arrow[ur, shift left=2, "X_c"] \arrow[ur, "Y_c"] \arrow[ur, shift right=2, "Z_c"'] & & k^3, \arrow[ll, shift right=2, "Z_b"'] \arrow[ll, "Y_b"] \arrow[ll, shift left=2, "X_b"] \end{tikzcd} \quad \begin{tikzcd} & k^3 \arrow[dr, shift left=2, "X_c"] \arrow[dr, "Y_c"] \arrow[dr, shift right=2, "Z_c"'] & \\ k^3 \arrow[ur, shift left=2, "X_b"] \arrow[ur, "Y_b"] \arrow[ur, shift right=2, "Z_b"'] & & k^2, \arrow[ll, shift right=2, "Z_a"'] \arrow[ll, "Y_a"] \arrow[ll, shift left=2, "X_a"] \end{tikzcd} \quad \begin{tikzcd} & k^3 \arrow[dr, shift left=2, "X_b"] \arrow[dr, "Y_b"] \arrow[dr, shift right=2, "Z_b"'] & \\ k^2 \arrow[ur, shift left=2, "X_a"] \arrow[ur, "Y_a"] \arrow[ur, shift right=2, "Z_a"'] & & k^3, \arrow[ll, shift right=2, "Z_c"'] \arrow[ll, "Y_c"] \arrow[ll, shift left=2, "X_c"] \end{tikzcd}$$

respectively. Here,

$$X_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Y_a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Z_a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$X_b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Z_b = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$X_c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y_c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z_c = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is easy to check that $I_i \cong P_i$ for all $i \in \mathbb{Z}_3$. Hence, the path algebra of (Q_2, R_2) is self-injective.

(3) Finally, we construct a non-example by breaking the symmetry in R_2 and adding the relation "$xyz = 0$". Consider the relations

$$\begin{aligned} R_3 = \langle & x_iy_{i+1} - y_ix_{i+1}, x_iz_{i+1} - z_ix_{i+1}, y_iz_{i+1} - z_iy_{i+1}, \\ & x_ix_{i+1}, y_iy_{i+1}, z_iz_{i+1}, x_iy_{i+1}z_{i+2}\rangle_{i\in\mathbb{Z}_3}. \end{aligned}$$

Then, the indecomposable projective representations are, up to isomorphism,

$$\begin{tikzcd} & k \arrow[dr, shift left=2, "(1,0,0)^{\mathrm{T}}"] \arrow[dr, "(0,1,0)^{\mathrm{T}}"] \arrow[dr, shift right=2, "(0,0,1)^{\mathrm{T}}"'] & \\ k^3 \arrow[ur, shift left=2, "0"] \arrow[ur, "0"] \arrow[ur, shift right=2, "0"'] & & k^3. \arrow[ll, shift right=2, "Z"'] \arrow[ll, "Y"] \arrow[ll, shift left=2, "X"] \end{tikzcd} \quad \begin{tikzcd} & k^3 \arrow[dr, shift left=2, "X"] \arrow[dr, "Y"] \arrow[dr, shift right=2, "Z"'] & \\ k \arrow[ur, shift left=2, "(1,0,0)^{\mathrm{T}}"] \arrow[ur, "(0,1,0)^{\mathrm{T}}"] \arrow[ur, shift right=2, "(0,0,1)^{\mathrm{T}}"'] & & k^3, \arrow[ll, shift right=2, "0"'] \arrow[ll, "0"] \arrow[ll, shift left=2, "0"] \end{tikzcd} \quad \begin{tikzcd} & k^3 \arrow[dr, shift left=2, "0"] \arrow[dr, "0"] \arrow[dr, shift right=2, "0"'] & \\ k^3 \arrow[ur, shift left=2, "X"] \arrow[ur, "Y"] \arrow[ur, shift right=2, "Z"'] & & k, \arrow[ll, shift right=2, "(0,0,1)^{\mathrm{T}}"'] \arrow[ll, "(0,1,0)^{\mathrm{T}}"] \arrow[ll, shift left=2, "(1,0,0)^{\mathrm{T}}"] \end{tikzcd}$$

Meanwhile, the indecomposable injective at vertex zero is

$$\begin{tikzcd} & k \arrow[dr, shift left=2, "0"] \arrow[dr, "0"] \arrow[dr, shift right=2, "0"'] & \\ k^3 \arrow[ur, shift left=2, "(1,0,0)"] \arrow[ur, "(0,1,0)"] \arrow[ur, shift right=2, "(0,0,1)"'] & & k^3 \arrow[ll, shift right=2, "Z"'] \arrow[ll, "Y"] \arrow[ll, shift left=2, "X"] \end{tikzcd}$$

which is not isomorphic to any of the indecomposable projectives.

Definition 2.1.7. Let A be a finite-dimensional self-injective algebra. Then, P_i is also an indecomposable injective A-module. It follows that its socle is a simple module which, by 2.1.2, must be S_j for some $j \in I$. The permutation of I defined by $\sigma\colon i \mapsto j$ is called the *Nakayama permutation* of A.

Example 2.1.8. (1) Consider the example 2.1.6 (1). Then, since $P_0 \cong I_2$, the socle of P_0 is S_2. That is, $\sigma(0) = 2$. Similarly, $\sigma(1) = 0$ and $\sigma(2) = 1$.

(2) Consider the algebra of 2.1.6 (2). Since $P_i \cong I_i$ for all $i \in \{0, 1, 2\}$, it follows that the socle of P_i is S_i and, thus, the Nakayama permutation of this algebra is the identity.

When A is is self-injective and finite dimensional, then the category $\operatorname{mod} A$ of finitely generated modules over A is equipped with an automorphism

$$\mathcal{N} := - \otimes_A DA\colon \operatorname{mod} A \to \operatorname{mod} A$$

called the *Nakayama functor*. Here, the dual $DA = \operatorname{Hom}_k(A, k)$ is equipped with its usual bimodule structure: for $f \in D(A)$ and $x \in A$, $(a \cdot f \cdot b)(x) = f(bxa)$. The Nakayama functor has the quasi-inverse [Iva11, 3.3]

$$\mathcal{N}^{-1} := - \otimes_A \operatorname{Hom}_A(DA, A)\colon \operatorname{mod} A \to \operatorname{mod} A$$

Self-injectivity for finite dimensional algebras is equivalent to the property that finitely generated indecomposable projective modules coincide with finitely generated indecomposable injectives [SY11, IV 3.7]. It thus follows that $\mathcal{N}$ and $\mathcal{N}^{-1}$ are exact and extend to automorphisms of $\mathrm{D^b}(\operatorname{mod} A)$.

§ 2.2 Frobenius exact categories

§ 2.2.1 Exact Categories

One of the main benefits of working in an abelian category is that we have access to nicely behaved exact sequences for which many diagram theorems hold (such as the snake lemma and the 3×3-lemma). In some sense, exact categories can be thought of a more general axiomatic setting in which we still have access to diagram lemmas and nicely behaved exact sequences.

Exact categories were introduced by Quillen in [Qui75], but there are multiple equivalent sets of axioms that can be used to define them. In this thesis, we will work with Keller's [Kel90] minimal axioms.

Definition 2.2.1. Let $\mathcal{E}$ be an additive category.

(1) A sequence of composable morphisms

$$0 \to a \xrightarrow{i} b \xrightarrow{p} c \to 0$$

is *exact* if i is the kernel of p and p is the cokernel of i. Such a sequence can be thought of as an exact pair (i, p).

(2) Fix a class S of exact pairs (i, p) in $\mathcal{E}$ that is closed under isomorphism. Then, the pairs $(i, p) \in S$ are called *admissible exact pairs*. The first morphism i in a pair $(i, p) \in S$ is called an *admissible monomorphism*(or admissible mono for short), and the second morphism p is called an *admissible epimorphism* (or admissible epi for short).

(3) The pair $(\mathcal{E}, S)$ is an *exact category* if the following axioms are satisfied.

Ex 0 The morphism $1_0 \colon 0 \to 0$ is an admissible epi,

Ex 1 The composition of two admissible epis is an admissible epi,

Ex 2 The pullback of an admissible epi $p \colon c \to d$ along an arbitrary morphism $g \colon b \to d$ exists and yields an admissible epi. That is, there is a cartesian square

$$\begin{tikzcd} a \arrow[r, "q"] \arrow[d, "f"] & b \arrow[d, "g"] \\ c \arrow[r, "p"] & d \end{tikzcd}$$

where q is an admissible epi.

Ex 2^{op} The pushout of an admissible mono $i \colon a \to b$ along an arbitrary morphism $f \colon a \to c$ exists and yields an admissible mono. That is, there is a cocartesian square

$$\begin{tikzcd} a \arrow[r, "i"] \arrow[d, "f"] & b \arrow[d, "g"] \\ c \arrow[r, "j"] & d \end{tikzcd}$$

where j is an admissible mono.

Example 2.2.2. (1) Any abelian category admits an exact structure by letting S be the class of all short exact sequences in the abelian category.

(2) Any additive category equipped with the set of split short exact sequences forms an exact category.

(3) Given a ring R that is Cohen-Macaulay (CM), the category of maximal Cohen-Macaulay modules

$$\operatorname{CM} R = \{M \in \operatorname{mod} R \mid \operatorname{Ext}^i_R(M, \omega_R) = 0 \,\forall\, i > 0\}$$

equipped with the usual exact structure in $\operatorname{mod} R$ forms an exact category.

(4) The category of locally free coherent $\mathcal{O}_X$-modules on a scheme X with the usual exact structure forms an exact category.

In what follows, for the sake of brevity, we will often refer to an exact category $(\mathcal{E}, S)$ as $\mathcal{E}$.

Definition 2.2.3. Let $\mathcal{E}$ be an exact category.

(1) A morphism $f\colon a \to b \in \mathcal{E}$ is *admissible* if it factors as

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & b \\ & c \arrow[ur, hook, "i"'] & \end{tikzcd}$$

where $p\colon a \to c$ is an admissible epi and $i\colon c \to b$ is an admissible mono.

(2) A sequence

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & b \arrow[rr, "g"] \arrow[dr, two heads, "q"'] & & c \\ & d \arrow[ur, hook, "i"'] & & e \arrow[ur, hook, "j"'] & \end{tikzcd}$$

of admissible morphisms $f\colon a \to b$ and $g\colon b \to c$ is *exact* if the sequence

$$0 \to d \xrightarrow{i} c \xrightarrow{q} e \to 0$$

is a short exact sequence.

It is possible to prove a version of the five lemma and the 3×3-lemma that holds in the context of exact categories. If, moreover, the exact category is idempotent complete, then a version of the snake lemma also holds. By idempotent complete we mean the following.

Definition 2.2.4. An additive category $\mathcal{E}$ is idempotent complete if every idempotent morphism $e\colon A \to A$ (i.e. $e^2 = e$) arises from a splitting

$$A = \ker e \oplus \operatorname{Im} e.$$

Finally, we note that there is an embedding theorem for exact categories [TT90], which states that for every exact category $\mathcal{E}$, there is an abelian category $\mathcal{A}$ and a fully faithful exact functor $i\colon \mathcal{E} \to \mathcal{A}$ that reflects exactness and, moreover, $\mathcal{E}$ is closed under extensions in $\mathcal{A}$. In other words, we can view any exact category as a full extension-closed subcategory of an abelian category.

§ 2.2.2 Stable category of an exact category

It is possible to construct two categories from an exact category $(\mathcal{E}, S)$: the projectively stable category and the injectively stable category.

Let $\mathcal{C}$ be an additive category. Recall that an *ideal* $\mathcal{I}$ of $\mathcal{C}$ consists of subgroups $\mathcal{I}(a,b) \subseteq \operatorname{Hom}_{\mathcal{C}}(a,b)$ such that for any morphism $f\colon a \to b \in \mathcal{I}(a,b)$ and any pair of morphisms $\alpha\colon a' \to a$ and $\beta\colon b \to b'$ in $\mathcal{C}$, then $\beta \cdot f \cdot \alpha \in \mathcal{I}(a',b')$.

Moreover, recall that a functor between exact categories is exact if it preserves admissible exact sequences.

Definition 2.2.5. (1) An object P in an exact category $(\mathcal{E}, S)$ is *S-projective* (or just projective for short) if the functor

$$\operatorname{Hom}_{\mathcal{E}}(P, -)\colon \mathcal{E} \to \mathrm{Ab}$$

is exact. Equivalently, an object P in $(\mathcal{E}, S)$ is projective if for every pair of morphisms $f\colon P \to X$ and $g\colon E \to X$ with g an admissible epi, then there exists a morphism $h\colon P \to E$ such that $g \cdot h = f$. Let $\operatorname{proj} \mathcal{E}$ denote the collection of S-projective objects in $\mathcal{E}$.

(2) Dually, an object $I \in \mathcal{E}$ is *S-injective* (or just injective for short) if the functor

$$\operatorname{Hom}_{\mathcal{E}}(-, I)\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Ab}$$

is exact. Let $\operatorname{inj} \mathcal{E}$ denote the collection of S-injective objects in $\mathcal{E}$.

Given an exact category $(\mathcal{E}, S)$ and $a, b \in \mathcal{E}$, let $[\operatorname{proj} \mathcal{E}]$ be the ideal of $\mathcal{E}$ defined by the subgroups $[\operatorname{proj} \mathcal{E}]_{a,b} \subseteq \operatorname{Hom}_{\mathcal{E}}(a,b)$ consisting of all maps which factor through a projective object. The *projectively stable category* of $(\mathcal{E}, S)$, denoted $\underline{\mathcal{E}}$, is the category whose objects are the same objects as $\mathcal{E}$ and whose morphisms between objects $a, b \in \underline{\mathcal{E}}$ are specified by

$$\operatorname{Hom}_{\underline{\mathcal{E}}}(a,b) = \underline{\operatorname{Hom}}_{\mathcal{E}}(a,b) := \operatorname{Hom}_{\mathcal{E}}(a,b)/[\operatorname{proj}\mathcal{E}]_{a,b}$$

The *injectively stable* category is defined dually.

Definition 2.2.6. Let $\mathcal{E}$ be an exact category. Recall that

(1) $\mathcal{E}$ has enough projectives if for each object $a \in \mathcal{E}$, there is an admissible exact sequence

$$0 \to k \to p \to a \to 0$$

with $p \in \operatorname{proj}\mathcal{E}$.

(2) $\mathcal{E}$ has enough injectives if for each object $a \in \mathcal{E}$, there is an admissible exact sequence

$$0 \to a \to i \to c \to 0$$

with $i \in \operatorname{inj}\mathcal{E}$.

Given an exact category with enough projectives, then it is possible to define a functor $\Omega\colon \mathcal{E} \to \underline{\mathcal{E}}$ in the following way. For each object $a \in \underline{\mathcal{E}}$ fix an admissible sequence $\mathcal{S}_a$

$$0 \to k(a) \xrightarrow{i_a} p(a) \xrightarrow{q_a} a \to 0$$

Then, define $\Omega(a) = k(a)$. Moreover, from the defining property of projective objects, it is straightforward to show that a morphism $f\colon a \to b$ induces a morphism $p(a) \to p(b)$ and a morphism $k(a) \xrightarrow{f'} k(b)$. Define $\Omega(f) = f'$. The induced morphism f' is not necessarily unique in $\mathcal{E}$, but it is unique up to stable equivalence. That is, if $f\colon a \to b$ induces two morphisms $f'\colon k(a) \to k(b)$ and $f''\colon k(a) \to k(b)$, then f' and f'' are in the same equivalence class in $\underline{\mathcal{E}}$. Hence, it is straightforward to show that this assignment descends to a well-defined functor $\Omega\colon \underline{\mathcal{E}} \to \underline{\mathcal{E}}$.

Moreover, it is not hard to check that for any two choices of admissible sequences $\{\mathcal{S}_a\}_{a\in\mathcal{E}}$ and $\{\mathcal{S}'_a\}_{a\in\mathcal{E}}$, then the resulting functors Ω and Ω' are naturally isomorphic.

§ 2.2.3 Frobenius exact categories and their stable categories

We are interested in a particular type of exact category, known as a Frobenius exact category, because its associated stable category is triangulated.

Definition 2.2.7. A *Frobenius exact category* $\mathcal{E}$ is an exact category that has enough projectives and injectives, and where, furthermore, $\operatorname{proj}\mathcal{E} = \operatorname{inj}\mathcal{E}$.

It turns out that when $\mathcal{E}$ is a Frobenius exact category, then Ω is an auto-equivalence of $\underline{\mathcal{E}}$ with inverse Ω^{-1}. Moreover, $(\underline{\mathcal{E}}, \Omega^{-1})$ has the structure of a triangulated category, with distinguished triangles having the form (up to isomorphism)

$$a \xrightarrow{f} b \xrightarrow{g} b \oplus^a p(\Omega^{-1}a) \xrightarrow{h} \Omega^{-1}a$$

where $f\colon a \to b$ is any morphism in $\underline{\mathcal{E}}$, and $p(\Omega^{-1}a)$ is a projective-injective object in $\mathcal{E}$ admitting an admissible epi $p(\Omega^{-1}a) \to \Omega^{-1}a$. Further, the object $b \oplus^a p(\Omega^{-1}a)$ and morphisms g and h are constructed by taking the pushout of f along the morphism $j_{\Omega^{-1}a}\colon a \to p(\Omega^{-1}a)$ which induces the diagram

$$\begin{tikzcd}
0 \arrow[r] & a \arrow[r, "j_{\Omega^{-1}a}"] \arrow[d, "f"] & p(\Omega^{-1}a) \arrow[r, "q_{\Omega^{-1}a}"] \arrow[d] & \Omega^{-1}a \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & b \arrow[r, "g"] & b \oplus^a p(\Omega^{-1}a) \arrow[r, "h"] & \Omega^{-1}a \arrow[r] & 0
\end{tikzcd}$$

Example 2.2.8 (See e.g. [Bue09, 9.8. 13.8])**.** Let $\mathcal{A}$ be an additive category. Consider the category of chain complexes $\mathrm{Ch}(\mathcal{A})$ equipped with exact structure S, where S is the class of component-wise split short exact sequences. Then, $(\mathrm{Ch}(\mathcal{A}), S)$ forms a Frobenius exact category. The projective-injective objects are the *contractible complexes* (i.e. chain complexes which are isomorphic to a direct sum of complexes of the form

$$\cdots \to 0 \to a \xrightarrow{1} a \to 0 \to \cdots$$

for $a \in \mathcal{A}$). The stable category of $(\mathrm{Ch}(\mathcal{A}), S)$ is triangle equivalent to the the homotopy category $\mathrm{K}(\mathcal{A})$.

Example 2.2.9 (See e.g. [Buc21, 4.8])**.** Given a Gorenstein ring R, the category of maximal Cohen-Macaulay modules $\mathrm{CM}\,R$ equipped with the usual exact structure in $\mathrm{mod}\,R$ forms a Frobenius exact category.

Example 2.2.10 (See e.g. [Kva20, 9,10])**.** More generally, let R be a ring and let $M \in \mathrm{Mod}\,R$. A *complete resolution* of M is an exact sequence P

$$P\colon\ \cdots \to P_{-1} \xrightarrow{d^0} P_0 \xrightarrow{d^1} P_1 \to \cdots$$

of projective R-modules P_i such that $\mathrm{Im}\, d^0 \cong M$ and $\mathrm{Hom}_R(P, Q)$ is exact for any projective R-module Q.

Consider the category

$$\mathrm{GP}\,R = \{N \in \mathrm{Mod}\,R \mid \text{N admits a complete resolution}\}$$

of Gorenstein projective R-modules. Then, $\mathrm{GP}\,R$ with the usual exact structure in $\mathrm{Mod}\,R$ is Frobenius exact.

The following lemmas about Frobenius exact categories will be useful throughout.

Lemma 2.2.11 ([Dug12, 2.2]). *For any map $f\colon x \to y \in \mathcal{E}$, there exists an object $z \in \mathcal{E}$ and a morphism $p\colon q \to y \in \mathcal{E}$ where $q \in \operatorname{proj} \mathcal{E}$ such that*

$$0 \to z \xrightarrow{g} x \oplus q \xrightarrow{(f,p)} y \to 0$$

is an admissible exact sequence in $\mathcal{E}$ that induces a triangle

$$z \xrightarrow{g} x \xrightarrow{f} y \to^{+}$$

in $\underline{\mathcal{E}}$.

This lemma has the following useful corollary.

Corollary 2.2.12. *Suppose that $x, y \in \underline{\mathcal{E}}$ are isomorphic. Then, there exists $p_0, p_1 \in \operatorname{proj} \mathcal{E}$ such that $p_0 \oplus x \cong p_1 \oplus y$ in $\mathcal{E}$.*

Lemma 2.2.13 ([Hap88, I.2.3,I.2.7], see also 2.2.11). *Let $\mathcal{E}$ be a Frobenius exact category. Then, an admissible sequence*

$$0 \to a \xrightarrow{u} b \xrightarrow{v} c \to 0$$

in $\mathcal{E}$ gives rise to a triangle

$$a \xrightarrow{u} b \xrightarrow{v} c \to^{+}$$

in $\underline{\mathcal{E}}$. Moreover, every triangle in $\underline{\mathcal{E}}$

$$x \xrightarrow{u} y \xrightarrow{v} z \to^{+}$$

lifts to an admissible sequence in $\mathcal{E}$

$$0 \to x \xrightarrow{u'} y \oplus p \xrightarrow{(u,\pi)} z \to 0$$

where $p \in \operatorname{proj} \mathcal{E}$ and u' is equivalent to u in $\underline{\mathcal{E}}$.

Lemma 2.2.14. *Let $x, y \in \mathcal{E}$.*

(1) *If $\operatorname{Ext}^1_{\mathcal{E}}(x, y) = 0$, then*

$$\operatorname{Ext}^1_{\underline{\mathcal{E}}}(x, y) := \operatorname{Hom}_{\underline{\mathcal{E}}}(x, \Omega^{-1} y) = 0.$$

(2) *Given an admissible sequence*

$$0 \to y \xrightarrow{u} z \xrightarrow{v} x \to 0$$

in $\mathcal{E}$ and an object $a \in \mathcal{E}$ such that $\operatorname{Hom}_{\underline{\mathcal{E}}}(a, v)$ is epi, then $\operatorname{Hom}_{\mathcal{E}}(a, v)$ is epi.

Proof. To prove (1), suppose that $\mathrm{Ext}^1_{\underline{\mathcal{E}}}(x, y) = 0$ and let $\underline{w} \colon x \to \Omega^{-1} y$ be a morphism in $\underline{\mathcal{E}}$. Then, there is a triangle

$$y \xrightarrow{\underline{u}} z \xrightarrow{\underline{v}} x \xrightarrow{\underline{w}} \Omega^{-1} y$$

in $\underline{\mathcal{E}}$ which by 2.2.13 lifts to an admissible exact sequence

$$0 \to y \xrightarrow{u'} p \oplus z \xrightarrow{(f,v)} x \to 0$$

with $\underline{u'} = \underline{u}$. Since $\mathrm{Ext}^1_{\underline{\mathcal{E}}}(x, y) = 0$, this sequence splits. In particular, $\underline{u'} = \underline{u}$ splits in $\underline{\mathcal{E}}$. Whence, the triangle defined by the morphism $\underline{w} \colon x \to \Omega^{-1} y$ splits, forcing $\underline{w} \colon x \to \Omega^{-1} y$ to be zero. We may thus conclude that $\mathrm{Ext}^1_{\underline{\mathcal{E}}}(x, y) = 0$.

For statement (2), let a be an object in $\mathcal{E}$ and consider an admissible sequence in $\mathcal{E}$

$$0 \to y \xrightarrow{u} z \xrightarrow{v} x \to 0$$

such that $\mathrm{Hom}_{\underline{\mathcal{E}}}(a, v)$ is epi. We claim that $\mathrm{Hom}_{\mathcal{E}}(a, v) \colon \mathrm{Hom}_{\mathcal{E}}(a, z) \to \mathrm{Hom}_{\mathcal{E}}(a, x)$ is epi.

Let $f \in \mathrm{Hom}_{\mathcal{E}}(a, x)$. Since $\mathrm{Hom}_{\underline{\mathcal{E}}}(a, v)$ is an epimorphism, there is a $\underline{g} \in \mathrm{Hom}_{\underline{\mathcal{E}}}(a, z)$ such that $\underline{v} \cdot \underline{g} = \underline{f}$. Equivalently, $\alpha = v \cdot g - f \colon a \to x$ factors through a projective $p \in \operatorname{proj} \mathcal{E}$. Note that $f = v \cdot g - \alpha$. We will show that α factors through v.

Indeed, consider the following diagram

$$\begin{tikzcd} z \arrow[r, two heads, "v"] & x & a \arrow[l, "\alpha"'] \arrow[dl, "\lambda"] \\ & p \arrow[ul, dashed, "\gamma"] \arrow[u, "q"'] & \end{tikzcd}$$

where the arrow γ exists because v is an admissible epi and p is projective. Here, the inner triangles commute by construction. Hence,

$$(v \cdot \gamma) \cdot \lambda = q \cdot \lambda = \alpha$$

Thus, we may rewrite $f = v \cdot g - v \cdot \gamma \cdot \lambda = \mathrm{Hom}_{\mathcal{E}}(a, v)(g - \gamma \cdot \lambda)$ proving that $\mathrm{Hom}_{\mathcal{E}}(a, v)$ is epi. □

§ 2.3 Tilting theory

In a broad sense, tilting theory is a method of constructing equivalences between categories. In this section, we will recall the notions of tilting complexes of modules and of sheaves. As there are many parallels between the two concepts, we conclude the section by explaining their connection.

§ 2.3.1 Derived functors

The equivalences constructed via the technology of tilting theory often involve the derived Hom and derived tensor product functors. In this subsection, we briefly recall their construction.

Notation 2.3.1. To fix notation, let A and B be rings. By an A-B-bimodule, we mean a module over the ring $B \otimes_{\mathbb{Z}} A^{\mathrm{op}}$. Given any ring S, we will consider its associated homotopy category $\mathrm{K}(S) := \mathrm{K}(\operatorname{Mod} S)$, as well as its derived category $\mathrm{D}(S) := \mathrm{D}(\operatorname{Mod} S)$.

Definition 2.3.2. The Hom-complex functor

$$\operatorname{Hom}^*_A(-,-)\colon\ \mathrm{K}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{K}(A) \to \mathrm{K}(B)$$

sends complexes $L \in \mathrm{K}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$ and $M \in \mathrm{K}(A)$ to the complex $\operatorname{Hom}^*_A(L, M)$ which, at degree k, is given by the B-module

$$\operatorname{Hom}^*_A(L, M)^k = \prod_{p+q=k} \operatorname{Hom}_A(L^{-p}, M^q).$$

The differential is defined by the rule

$$d^k(f) = d_M \cdot f + (-1)^{k+1} f \cdot d_L$$

for all $f \in \operatorname{Hom}^*_A(L, M)^k$.

Definition 2.3.3. Similarly, there is a tensor product functor

$$\operatorname{Tot}(- \otimes_B -)\colon\ \mathrm{K}(B) \times \mathrm{K}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}}) \to \mathrm{K}(A)$$

sending complexes $L \in \mathrm{K}(B)$ and $M \in \mathrm{K}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$ to the complex $\operatorname{Tot}(L \otimes_B M)$ which, at degree k, is given by the A-module

$$\operatorname{Tot}(L \otimes_B M)^k = \bigoplus_{p+q=k} L^p \otimes_B M^q.$$

with differential at k defined as

$$d^k = \sum_{p+q=k} d^p_L \otimes 1_{M^q} + (-1)^p 1_{L^p} \otimes d^q_M$$

The functors in 2.3.2 and 2.3.3 are used to define the derived Hom and derived tensor products as follows.

First, recall that a chain complex I over an abelian category $\mathcal{A}$ is *K-injective* if for every acyclic complex $M \in \mathrm{K}(\mathcal{A})$, then $\operatorname{Hom}_{\mathrm{K}(\mathcal{A})}(M, I) = 0$. Let $\operatorname{k-inj} \mathcal{A}$ denote the full subcategory of $\operatorname{Ch}(\mathcal{A})$ consisting of K-injective complexes.

Given a complex $M \in \mathrm{D}(\mathcal{A})$, then a *$K$-injective resolution* of M is a chain map $M \to I$ which is a quasi-isomorphism and where I is a K-injective complex. Moreover, we will say that $\mathrm{D}(\mathcal{A})$ has *enough K-injectives* if every complex $M \in \mathrm{D}(\mathcal{A})$ admits a K-injective resolution.

Furthermore, a complex Q of A^{op}-modules is *K-flat* if for every acyclic complex M of right A-modules, the total complex $\mathrm{Tot}(M \otimes_A Q)$ is acyclic.

Given a complex M of A-modules, then a K-flat (resp. projective) resolution of M is a a chain map $P \to M$ and which is a quasi-isomorphism and where P is a K-flat (resp. projective) complex.

Definition 2.3.4. For complexes $M \in \mathrm{D}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$ and $N \in \mathrm{D}(\mathrm{Mod}\, A)$, let $N \to I$ be a K-injective resolution. Then

$$\mathbf{R}\mathrm{Hom}_A(M, N) := \mathrm{Hom}^*_A(M, I) \in \mathrm{D}(B).$$

Further, the complex $\mathbf{R}\mathrm{Hom}_A(M, N)$ does not depend (up to quasi-isomorphism) on the choice of K-injective resolution.

Let $P \to M$ be a K-projective resolution of M. Then, it is a classical fact that there is an quasi-isomorphism

$$\mathbf{R}\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}^*_A(Q, N)$$

which is natural in M and N.

Definition 2.3.5. For complexes $M \in \mathrm{D}(\mathrm{Mod}\, A)$ and $N \in \mathrm{D}(\mathrm{Mod}\, B \otimes_{\mathbb{Z}} A^{\mathrm{op}})$, let $P \to N$ be a resolution in $\mathrm{D}(\mathrm{Mod}\, B \otimes_{\mathbb{Z}} A^{\mathrm{op}})$ which is K-flat as a complex of B^{op}-modules. Then

$$M \otimes^{\mathbf{L}}_A N := \mathrm{Tot}(M \otimes_A P) \in \mathrm{D}(B).$$

§ 2.3.2 Tilting theory of algebras

In this section we are concerned with constructing equivalences between the derived categories of algebras. In this context, tilting theory can be viewed as an extension of the Morita theorem.

Recall that a *progenerator* in the category of modules $\mathrm{Mod}\, A$ over a ring A is a module $P \in \mathrm{Mod}\, A$ such that P is projective and P generates $\mathrm{Mod}\, A$ (i.e. every object $M \in \mathrm{Mod}\, A$ is a quotient of a direct sum of copies of P). Note that a projective P generates $\mathrm{Mod}\, A$ if and only if $\mathrm{Add}_A P = \mathrm{Add}_A A$.

Theorem 2.3.6 (See e.g. [Coh91, 4.5,5.4])**.** *Let A and B be k-algebras, where k is a field, then following are equivalent.*

(1) *There is a k-linear equivalence $F\colon \mathrm{Mod}\, A \to \mathrm{Mod}\, B$.*

(2) *There is an A-B-bimodule (with central k-action) M such that*

$$- \otimes_A M\colon \mathrm{Mod}\, A \to \mathrm{Mod}\, B$$

is an equivalence.

(3) *There exists a finitely generated progenerator* $P \in \operatorname{Mod} B$ *with* $A \cong \operatorname{End}_B(P)$.

The following is an extension of this theorem for derived categories.

Theorem 2.3.7 ([Ric89], [Ric91])**.** *Let A and B be k-algebras, where k is a field, then following are equivalent.*

(1) *There is a k-linear triangulated equivalence* $F\colon \mathrm{D}(\operatorname{Mod} A) \to \mathrm{D}(\operatorname{Mod} B)$.

(2) *There is a complex of A-B-bimodules M such that the functor*

$$- \otimes^{\mathbf{L}}_A M\colon \mathrm{D}(\operatorname{Mod} A) \to \mathrm{D}(\operatorname{Mod} B)$$

is an equivalence.

(3) *There is a complex T of B-modules such that*

(a) *T is perfect (i.e. $T \in \mathrm{K}^{\mathrm{b}}(\operatorname{proj} B)$).*

(b) *T generates $\mathrm{D}(\operatorname{Mod} B)$ as a triangulated category with infinite direct sums,*

(c) *T satisfies*

$$\operatorname{Hom}_{\mathrm{D}(\operatorname{Mod} S)}(T, T[n]) \cong \begin{cases} A & n = 0 \\ 0 & n \neq 0. \end{cases}$$

In addition to the original references, the reader is also referred to [Kel98] for a didactic exposition of 2.3.7. Roughly, given the bimodule ${}_AM_B$ as in (2), one may take $T := M_B$. Conversely, given a complex T as in (3), then one may construct an A-B-bimodule M such that T is quasi-isomorphic to M_B.

If the conditions of 2.3.7 hold, then the functor $- \otimes^{\mathbf{L}}_A M$ is called a *standard equivalence*, T is called a *tilting complex* and M is called a *two-sided tilting complex*. If M is concentrated in degree zero, we may refer to it as a *tilting bimodule*. It is worth noting that an example of an equivalence between derived categories of algebras which is not a standard equivalence is not known.

The following characterisation of two-sided tilting complexes will be useful to this thesis.

Proposition 2.3.8 ([Miy03, 1.8])**.** *Let k be a field and A and B be k-algebras. Then, for a complex M of A-B-bimodules, the following are equivalent.*

(1) *M is a two-sided tilting complex.*

(2) *M is such that*

(a) *M is biperfect. That is, M is perfect a complex of right B-modules and as a complex of left A-modules.*

(b) *The right multiplication morphism*

$$\rho_B \colon B \to \mathbf{R}\mathrm{Hom}_A(M, M)$$

is an isomorphism in $\mathrm{D}(\mathrm{Mod}\, B \otimes_k B^{\mathrm{op}})$.

(c) *The left multiplication morphism*

$$\lambda_A \colon A \to \mathbf{R}\mathrm{Hom}_B(M, M)$$

is an isomorphism in $\mathrm{D}(\mathrm{Mod}\, A \otimes_k A^{\mathrm{op}})$.

Moreover, it will be important that under the correct conditions, tilting equivalences restrict to the bounded derived category of finitely generated modules. This is a well-known fact, but we present a proof here for convenience.

Lemma 2.3.9. *Let R be a commutative noetherian ring, and let A, B be R-algebras which are module finite over R. Consider a complex M of finitely generated A-B-bimodules. Suppose that M is perfect as a complex of B-modules and is such that there is an equivalence*

$$- \otimes^{\mathbf{L}}_A M \colon \mathrm{D}(\mathrm{Mod}\, A) \to \mathrm{D}(\mathrm{Mod}\, B).$$

Then, $- \otimes^{\mathbf{L}}_A M$ restricts to an equivalence

$$- \otimes^{\mathbf{L}}_A M \colon \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, B).$$

Proof. We will prove that the functor

$$\mathbf{R}\mathrm{Hom}_B(M, -) \colon \mathrm{D}(\mathrm{Mod}\, B) \to \mathrm{D}(\mathrm{Mod}\, A)$$

restricts to the bounded derived categories of finitely generated modules. If this is the case, then its inverse $- \otimes^{\mathbf{L}}_A M$ also restricts.

Since M is a perfect complex of finitely generated B-modules, it admits a finite projective resolution $Q \to M$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, B)$. It follows that the complex of R-modules $\mathbf{R}\mathrm{Hom}_B(M, X) = \mathrm{Hom}^*_B(Q, X)$ has bounded cohomology for any $X \in \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, B)$. Necessarily, then, $\mathbf{R}\mathrm{Hom}_B(M, X)$ has bounded cohomology in $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ and is, thus, quasi-isomorphic to a bounded complex by taking the canonical truncation.

It remains to show that $\mathbf{R}\mathrm{Hom}_B(M, X)$ is a complex of finitely generated A-modules. Let $Q' \to M$ be a projective resolution of M in $\mathrm{D}(\mathrm{mod}\, B \otimes_R A^{\mathrm{op}})$. Then, $\mathbf{R}\mathrm{Hom}_B(M, X) = \mathrm{Hom}^*_B(Q', X)$. Since X is bounded, it suffices to show that the A-module $\mathrm{Hom}_B(U, N)$ is finitely generated if U is finitely generated as an $B \otimes_R A^{\mathrm{op}}$-module and N is finitely generated as an B-module.

Since A, B and $B \otimes_R A^{\mathrm{op}}$ are module finite over R, then finite generation of a module over A and over $B \otimes_R A^{\mathrm{op}}$ is equivalent to finite generation over R. Since R is noetherian, the standard argument in the commutative case implies that $\mathrm{Hom}_B(U, N)$ is finitely generated over R. Therefore, it is finite over A, which concludes the proof. □

§ 2.3.3 Tilting in geometry

Tilting also finds applications in geometry as a method of constructing derived equivalences between varieties and noncommutative rings. Tilting theory therefore reduces the homological algebra of varieties to homological algebra of rings.

Definition 2.3.10. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a ringed space. Recall that a complex of sheaves $\mathcal{F} \in \mathrm{D}(\mathrm{Qcoh}\,\mathcal{X})$ is *perfect* if there exists an open covering $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i}$ for each i is quasi-isomorphic to a bounded complex of sheaves which are summands of finite free $\mathcal{O}_X|_{U_i}$-modules.

Definition 2.3.11. Let $\mathcal{X}$ be a Noetherian scheme. A *tilting complex* on $\mathcal{X}$ is a complex $\mathcal{V} \in \mathrm{D}(\mathrm{Qcoh}\,\mathcal{X})$ such that

(1) $\mathcal{V}$ is perfect,

(2) $\mathcal{V}$ generates $\mathrm{D}(\mathrm{Qcoh}\,\mathcal{X})$ as a triangulated category with infinite direct sums,

(3) $\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\,\mathcal{X})}(\mathcal{V}, \mathcal{V}[n]) = 0$ for all $n \neq 0$.

If, moreover, $\mathcal{V}$ is a vector bundle, then we say that $\mathcal{V}$ is a *tilting bundle.*

Proposition 2.3.12. *Let $\mathcal{X}$ be a Noetherian scheme admitting a tilting complex $\mathcal{V}$. Write $\Lambda = \mathrm{End}_{\mathrm{D}(\mathrm{Qcoh}\,\mathcal{X})}(\mathcal{V})$. Then, the functor*

$$\mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\,\mathcal{X})}(\mathcal{V}, -)\colon\ \mathrm{D}(\mathrm{Qcoh}\,\mathcal{X}) \to \mathrm{D}(\mathrm{Mod}\,\Lambda)$$

is an equivalence. If furthermore $\mathcal{X}$ is quasi-projective, then the equivalence restricts to

$$\mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{coh}\,\mathcal{X})}(\mathcal{V}, -)\colon\ \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{X}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$$

Proof. The fact the $\mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\,\mathcal{X})}(\mathcal{V}, -)$ is an equivalence is [HB07, 7.6]. Essentially, the result follows from the definition of a tilting complex and a theorem of Keller which we recall in 2.3.14. The fact that the equivalence restrict when $\mathcal{X}$ is quasi-projective is proved in [Rou08, 6.4, 6.11, 6.12]. It is also a special case of 2.6.7(2) below. □

Example 2.3.13 ([Bei78])**.** In $\mathbb{P}^n$, the vector bundle $\mathcal{V} = \mathcal{O} \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(n)$ is a tilting complex, and thus there is derived equivalence

$$\mathrm{D}(\mathrm{Qcoh}\,\mathbb{P}^n) \xrightarrow{\sim} \mathrm{D}(\mathrm{Mod}\,\mathrm{End}_{\mathrm{D}(\mathrm{Qcoh}\,\mathbb{P}^n)}(\mathcal{V})).$$

For $n = 1$, the algebra $\Lambda = \operatorname{End}_{\mathrm{D}(\operatorname{Qcoh}\mathbb{P}^1)}(\mathcal{V})$ is the path algebra of the 2-Kronecker quiver

$$\bullet \rightrightarrows \bullet$$

§ 2.3.4 Tilting for algebraic categories

The definitions 2.3.7 and 2.3.11 of tilting complexes in $\mathrm{D}(\operatorname{Mod} R)$ and $\mathrm{D}(\operatorname{Qcoh}\mathcal{X})$ are clearly very similar. In this section we recall a theorem by Keller [Kel94] which explains the parallels between these two definitions.

Let k be a field and let $\mathcal{E}$ be a k-linear Frobenius exact category with infinite direct sums. Then, $\underline{\mathcal{E}}$ is a triangulated category with infinite direct sums. Recall that an object $X \in \underline{\mathcal{E}}$ is *compact* if the functor $\operatorname{Hom}_{\underline{\mathcal{E}}}(X, -)$ commutes with direct sums. Suppose that $\underline{\mathcal{E}}$ admits a compact object X which generates $\underline{\mathcal{E}}$ as a triangulated category with infinite direct sums.

Theorem 2.3.14 ([Kel94, §4.3]). *Let $\mathcal{E}$ be as above. Then, there is a DG-algebra A and a triangle equivalence $F\colon \underline{\mathcal{E}} \xrightarrow{\sim} \mathrm{D}(\operatorname{Mod} A)$ such that $F(X) \cong A$. If, moreover, $\operatorname{Hom}_{\underline{\mathcal{E}}}(X, \Sigma^n X) = 0$ for all $n \neq 0$, then we may assume that A is a k-algebra.*

For a didactic proof of the above result, we refer the reader to [Kel98].

In fact, if $\mathcal{A}$ is an abelian category and if $\mathrm{D}(\mathcal{A})$ has enough K-injectives, then $\mathrm{D}(\mathcal{A}) \cong \mathrm{K}(\text{k-inj}\,\mathcal{A})$, and so 2.2.8 shows that $\mathrm{D}(\mathcal{A})$ is the stable category of $\mathrm{Ch}(\text{k-inj}\,\mathcal{A})$ equipped with the split exact structure. In this case, it is possible to see from the proof of 2.3.14 presented in [Kel98] that we may take $A = \mathbf{R}\operatorname{Hom}_{\mathcal{A}}(X, X)$ and the functor $F = \mathbf{R}\operatorname{Hom}_{\mathcal{A}}(X, -)$.

Hence, in situations where $\mathrm{D}(\operatorname{Mod} R)$ and $\mathrm{D}(\operatorname{Qcoh}\mathcal{X})$ have enough K-injectives, then they can be identified with stable categories of Frobenius exact categories, and, moreover, since perfect complexes in $\mathrm{D}(\operatorname{Mod} R)$ and $\mathrm{D}(\operatorname{Qcoh}\mathcal{X})$ are precisely the compact objects, then the fact that tilting complexes 2.3.7 and 2.3.11 induce derived equivalences follows from 2.3.14.

§ 2.4 Twists and cotwists

Consider a functor $S\colon \mathcal{A} \to \mathcal{B}$ between triangulated categories which admits a right adjoint R and a left adjoint L. Then, naively, the twist around S is a functor T which sits in a functorial triangle

$$SR \xrightarrow{\varepsilon^{\mathrm{R}}} 1_{\mathcal{B}} \xrightarrow{\alpha} T \xrightarrow{\gamma} {}^{+} \tag{2.4.A}$$

where ε^{R} is the counit of the adjoint pair (S, R). By a functorial triangle, we mean that the functors and natural transformations in (2.4.A) evaluated on any $a \in A$ induce a triangle

$$SR(a) \xrightarrow{\varepsilon^{\mathrm{R}}_a} a \xrightarrow{\alpha_a} T(a) \xrightarrow{\gamma_a} {}^{+}$$

in $\mathcal{B}$. Furthermore, the cotwist around S is a functor C which sits in a functorial triangle

$$C \to 1_{\mathcal{A}} \xrightarrow{\eta^{\mathrm{R}}} RS \to^{+}$$

where $\eta^{\mathrm{R}} \colon 1_{\mathcal{A}} \to RS$ is the unit of the adjunction (S, R).

However,since cones in triangulated categories are not functorial, there are two main issues with this naive approach in defining twists and cotwists. First, it is not possible to guarantee the existence of functors T and C and, moreover, even if such existence were assumed, it is not possible to guarantee the uniqueness of T and C, not even up to isomorphism. More precisely, assume there are functors T, T' and natural transformations α, β such that for all objects $b \in \mathcal{B}$ there are triangles

$$\begin{tikzcd}
SR(b) \arrow[r, "\varepsilon_b^{\mathrm{R}}"] \arrow[d, equal] & b \arrow[r, "\alpha_b"] \arrow[d, equal] & T(b) \arrow[r] & {+} \\
SR(b) \arrow[r, "\varepsilon_b^{\mathrm{R}}"] & b \arrow[r, "\beta_b"] & T'(b) \arrow[r] & {+}
\end{tikzcd}$$

Then, the axioms of triangulated categories guarantee that there are isomorphisms $f_b \colon T(b) \to T'(b)$, but we cannot guarantee that the choice of morphisms $(f_b)_{b \in \mathcal{B}}$ is natural. Hence, although T and T' are isomorphic on objects, we cannot guarantee that they are isomorphic as functors.

One way to solve this issue, is to assume that $\mathcal{A}$ and $\mathcal{B}$ are enhanced triangulated categories and the functors T and C are exact functors which descend from DG-functors over the enhancements of $\mathcal{A}$ and $\mathcal{B}$. Then, it is possible to both guarantee the existence and uniqueness, up to isomorphism, of T and C. This approach will be made more precise in § 2.4.4.

§ 2.4.1 DG-categories

In this section, we briefly recall some basic generalities about DG-categories and DG-enhancements of triangulated categories. For a more complete overview of these topics, the reader is referred to, for example, [Toë11; Tab15; CS17].

Definition 2.4.1. Let R be a commutative ring.

(1) A *DG-category* is a category $\mathcal{C}$ such that, for all objects $a, b, c \in \mathcal{C}$, the Hom-sets $\mathcal{C}(a, b)$ are $\mathbb{Z}$-graded R-modules equipped with a differential $d \colon \mathcal{C}(a, b) \to \mathcal{C}(a, b)$ of degree 1, and the composition maps

$$\mathcal{C}(a, b) \otimes_R \mathcal{C}(b, c) \to \mathcal{C}(a, c)$$

are morphisms of chain complexes.

(2) Given a DG-category $\mathcal{C}$, its *homotopy category* $\mathrm{H}^0(\mathcal{C})$ is the category whose ob-

jects are the objects of $\mathcal{C}$ and whose morphisms are defined by

$$\mathrm{H}^0(\mathcal{C})(a,b) := \mathrm{H}^0(\mathcal{C}(a,b))$$

for all $a, b \in \mathcal{C}$.

(3) A *DG-functor* is a functor $F\colon \mathcal{C} \to \mathcal{D}$ between DG-categories such that, for all objects $a, b \in \mathcal{C}$, the map

$$F_{a,b}\colon \mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$$

is a morphism of chain complexes of R-modules. It is easy to see that a DG-functor F induces a functor $\mathrm{H}^0(F)\colon \mathrm{H}^0(\mathcal{C}) \to \mathrm{H}^0(\mathcal{D})$.

(4) Let **dgCat** denote the category whose objects are small DG-categories and whose morphisms are DG-functors.

Example 2.4.2. Consider the category $\mathrm{Ch}_{\mathrm{dg}}(R)$ whose objects are complexes of R-modules and whose morphisms are defined by

$$\mathrm{Ch}_{\mathrm{dg}}(R)(a,b) := \mathrm{Hom}^*_R(a,b)$$

for any $a, b \in \mathrm{Ch}(R)$.

Given a DG-category $\mathcal{C}$, then a *module over* $\mathcal{C}$ is a DG-functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ch}_{\mathrm{dg}}(R)$. A morphism between $\mathcal{C}$-modules M and N is a natural transformation $\phi\colon M \to N$ such that $\phi_c \in \mathrm{Ch}_{\mathrm{dg}}(R)(Mc, Nc)^0$ for all $c \in \mathcal{C}$. Let $\mathrm{dgMod}\,\mathcal{C}$ denote the category of modules over $\mathcal{C}$. A morphism $f\colon M \to N$ of $\mathcal{C}$-modules is a *quasi-isomorphism* if $f_c\colon M_c \to N_c$ is a quasi-isomorphism of chain complexes.

A module $M \in \mathrm{dgMod}\,\mathcal{C}$ is *perfect* if it is a compact object in $\mathrm{D}(\mathcal{C})$. Moreover, $M \in \mathrm{dgMod}\,\mathcal{C}$ is *acyclic* if, for all objects $a \in \mathcal{C}$, $M(a) \in \mathrm{Ch}_{dg}(R)$ is an acyclic complex. Let $\mathrm{Ac}(\mathcal{C})$ denote the full subcategory of $\mathrm{dgMod}\,\mathcal{C}$ consisting of acyclic complexes.

Definition 2.4.3. The *derived category* $\mathrm{D}(\mathcal{C})$ of a DG-category $\mathcal{C}$ is the Verdier quotient

$$\mathrm{D}(\mathcal{C}) := \mathrm{H}^0(\mathrm{dgMod}\,\mathcal{C})/\mathrm{H}^0(\mathrm{Ac}(\mathcal{C})),$$

which turns out to be triangulated (See, for example, [CS17, §1.1] or [Tab15, §1.1.2]).

We, furthermore, note that a DG-functor $F\colon \mathcal{C} \to \mathcal{D}$ induces functors

$$\mathrm{Ind}_F\colon\ \mathrm{dgMod}\,\mathcal{C} \to \mathrm{dgMod}\,\mathcal{D}$$
$$\mathrm{Res}_F\colon\ \mathrm{dgMod}\,\mathcal{D} \to \mathrm{dgMod}\,\mathcal{C}$$

such that the pair $(\mathrm{Ind}_F, \mathrm{Res}_F)$ forms an adjoint pair $\mathrm{Ind}_F \dashv \mathrm{Res}_F$. It turns out that the functor Res_F preserves acyclic modules, and so descends to a functor between the respective derived categories. For further details on these functors see [Tab15, §1.1.2] or [Dri04, §14].

Given two DG-categories $\mathcal{C}$ and $\mathcal{D}$, then one can define a DG-category $\mathcal{C} \otimes_R \mathcal{D}^{\mathrm{op}}$, whose objects are pairs (c, d) for c and d objects in $\mathcal{C}$ and $\mathcal{D}$, respectively. Given pairs (c, d) and (c', d'), morphisms $(c, d) \to (c', d')$ are $\mathcal{C} \otimes_R \mathcal{D}$ is defined by

$$\mathcal{C} \otimes_R \mathcal{D}((c, d), (c', d')) := \mathcal{C}(c, c') \otimes_R \mathcal{D}(d', d).$$

A $\mathcal{D}$-$\mathcal{C}$-*bimodule* is a functor $\mathcal{C}^{\mathrm{op}} \otimes_R \mathcal{D} \to \mathrm{Ch}_{\mathrm{dg}}(R)$. As with modules, one may define the category $\mathcal{D}$-dgMod-$\mathcal{C}$ of $\mathcal{D}$-$\mathcal{C}$-bimodules, as well as the derived category $\mathrm{D}(\mathcal{D}\text{-}\mathcal{C})$.

Example 2.4.4. Given a DG-category $\mathcal{A}$, we may always define the *diagonal bimodule* $\mathcal{A}$,

$$\mathcal{A} \colon \mathcal{A}^{\mathrm{op}} \otimes_R \mathcal{A} \to \mathrm{Ch}_{\mathrm{dg}}(R)$$

by letting, for all $a, b \in \mathcal{A}$, ${}_a\mathcal{A}_b := \mathrm{Hom}_{\mathcal{A}}(a, b)$.

A $\mathcal{D}$-$\mathcal{C}$-bimodule x is $\mathcal{C}$-*perfect* if, for all objects $d \in \mathcal{D}$, the $\mathcal{C}$-module

$$x(-, d) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ch}_{\mathrm{dg}}(R)$$

is perfect. A $\mathcal{D}$-*perfect* bimodule $x \in \mathcal{C}\text{-}\mathrm{dgMod}\text{-}\mathcal{D}$ is defined similarly.

§ 2.4.2 Tensor, Hom, and their derived functors

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be DG-categories, and let $x \in \mathcal{A}$-dgMod-$\mathcal{B}$. Then, as in [AL17, §2.1.4] or [Dri04, §14], one can define functors

$$- \otimes_{\mathcal{A}} x \colon \mathcal{C}\text{-}\mathrm{dgMod}\text{-}\mathcal{A} \to \mathcal{C}\text{-}\mathrm{dgMod}\text{-}\mathcal{B},$$
$$\mathrm{Hom}_{\mathcal{B}}(x, -) \colon \mathcal{C}\text{-}\mathrm{dgMod}\text{-}\mathcal{B} \to \mathcal{C}\text{-}\mathrm{dgMod}\text{-}\mathcal{A}.$$

A module $c \in \mathrm{dgMod}\,\mathcal{C}$ is *K-projective* if $\mathrm{H}^0(\mathrm{dgMod}\,\mathcal{C})(c, a) = 0$ for all $a \in \mathrm{Ac}\,\mathcal{C}$. The full subcategory of K-projective modules over $\mathcal{C}$ will be denoted $\text{k-proj}\,\mathcal{C}$. Given a module $b \in \mathrm{dgMod}\,\mathcal{C}$, then a *K-projective resolution* of b is a module $c \in \text{k-proj}\,\mathcal{C}$ together with a quasi-isomorphism $f \colon c \to b$.

It turns out that if x is K-projective, then functors $- \otimes_{\mathcal{A}} x$ and $\mathrm{Hom}_{\mathcal{B}}(x, -)$ preserve acyclic complexes, and so they descend to functors between the corresponding derived categories. Moreover, it turns out that K-projective resolutions exist. Hence, fixing a K-projective resolution $f \colon c \to x$, we may define the functors

$$- \otimes^{\mathbf{L}}_{\mathcal{A}} x \colon \mathrm{D}(\mathcal{C}\text{-}\mathcal{A}) \to \mathrm{D}(\mathcal{C}\text{-}\mathcal{B}),$$
$$\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(x, -) \colon \mathrm{D}(\mathcal{C}\text{-}\mathcal{B} \to \mathrm{D}(\mathcal{C}\text{-}\mathcal{A}).$$

by letting $- \otimes^{\mathbf{L}}_{\mathcal{A}} x := - \otimes_{\mathcal{A}} c$ and $\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(x, -) := \mathrm{Hom}_{\mathcal{B}}(c, -)$. For the precise details on these constructions, the reader is referred to [AL17, 2.1.6] or [Dri04, §14].

Proposition 2.4.5 ([AL17, 2.1,2.2])**.** *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be DG-categories, and consider the bimodule $x \in \mathcal{A}\text{-dgMod-}\mathcal{B}$.*

(1) *If x is $\mathcal{B}$-perfect, then there is an isomorphism of functors*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(x, -) \cong (-) \otimes^{\mathbf{L}}_{\mathcal{B}} \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(x, \mathcal{B}),$$

(2) *If x is $\mathcal{A}$-perfect, then the functor $(-) \otimes^{\mathbf{L}}_{\mathcal{A}} x \colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ admits left and right adjoints*

$$(-) \otimes^{\mathbf{L}}_{\mathcal{B}} \mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(x, \mathcal{A}) \colon \mathrm{D}(\mathcal{B}) \to \mathrm{D}(\mathcal{A}),$$
$$(-) \otimes^{\mathbf{L}}_{\mathcal{B}} \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(x, \mathcal{B}) \colon \mathrm{D}(\mathcal{B}) \to \mathrm{D}(\mathcal{A}),$$

respectively.

Definition 2.4.6 ([AL17])**.** Let $x \in \mathcal{A}\text{-}\mathrm{dgMod}\text{-}\mathcal{B}$, then the counit and unit of the adjoint pair $(- \otimes^{\mathbf{L}}_{\mathcal{A}} x, \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(x, -))$ evaluated at the diagonal bimodules

$$\mathrm{tr} \colon \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(x, \mathcal{B}) \otimes^{\mathbf{L}}_{\mathcal{A}} x \to \mathcal{B}$$
$$\mathrm{act} \colon \mathcal{A} \to \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(x, x)$$

are called the *derived trace* and *derived action* maps.

§ 2.4.3 DG enhancements of triangulated categories

We are interested in DG-categories which are enhancements of $\mathrm{D}(\mathrm{Mod}\, A)$, where A is a ring. There are many notions of enhancements, and, following [AL17], we will use that of *Morita enhancements*. As a result, we are interested in working with DG-categories up to *Morita equivalences*.

Definition 2.4.7. A *Morita equivalence* is a DG-functor $F \colon \mathcal{C} \to \mathcal{D}$ such that the induced functor $\mathrm{Res}_F \colon \mathrm{D}(\mathcal{D}) \to \mathrm{D}(\mathcal{C})$ is a triangulated equivalence.

Hence, we are interested in working in **Mrt(dgCat)**, which is the localisation of **dgCat** with respect to Morita equivalences. The morphisms of **Mrt(dgCat)** are called *quasi-functors*. It follows from [Toë11], that there is a one-to-one correspondence between *quasi-functors* $\mathcal{C} \to \mathcal{D}$ and objects in $\mathrm{D}^{\mathcal{D}\text{-perf}}(\mathcal{C}\text{-}\mathcal{D})$ (i.e. $\mathcal{C}$-$\mathcal{D}$ bimodules which are $\mathcal{D}$-perfect).

Definition 2.4.8. Let $\mathcal{T}$ be a triangulated category.

(1) Then, a *Morita enhancement* of $\mathcal{T}$ is a DG-category $\mathcal{A}$ together with a triangulated equivalence $E \colon \mathrm{D}(\mathcal{A}) \to \mathcal{T}$.

(2) A *small Morita enhancement* of $\mathcal{T}$ is a DG-category $\mathcal{A}$ together with a triangulated equivalence $\mathrm{D}_c(\mathcal{A}) \cong \mathcal{T}$. Here, $\mathrm{D}_c(\mathcal{A})$ denotes the full triangulated subcategory of $\mathrm{D}(\mathcal{A})$ consisting of compact objects.

Example 2.4.9. Let A be an R-algebra. Then, we may view A as a DG-algebra concentrated in degree zero, and we can consider the DG-category $\mathcal{A}$ with one object x and with morphisms defined by $\mathcal{A}(x,x) := A$. Then, a DG-module over $\mathcal{A}$ corresponds to a chain complex of A-modules, and so we can easily check that $\mathcal{A}$ is a Morita enhancement of $\mathrm{D}(\operatorname{Mod} A)$.

§ 2.4.4 Twists and cotwists more precisely

Following [AL17], let $\mathcal{A}$ and $\mathcal{B}$ be DG-categories and suppose that $S \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$ is both $\mathcal{A}$- and $\mathcal{B}$-perfect. Then, S induces the exact functor

$$s := (-) \otimes^{\mathbf{L}}_{\mathcal{A}} S\colon\ \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$$

which, by 2.4.5, admits left and right adjoints

$$\begin{aligned} l &:= (-) \otimes^{\mathbf{L}}_{\mathcal{B}} \mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(S, \mathcal{A})\colon\ \mathrm{D}(\mathcal{B}) \to \mathrm{D}(\mathcal{A}), \\ r &:= (-) \otimes^{\mathbf{L}}_{\mathcal{B}} \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(S, \mathcal{B})\colon\ \mathrm{D}(\mathcal{B}) \to \mathrm{D}(\mathcal{A}). \end{aligned}$$

Write $L := \mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(S, \mathcal{A}) \in \mathrm{D}(\mathcal{B}\text{-}\mathcal{A})$, $R := \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(S, \mathcal{B}) \in \mathrm{D}(\mathcal{B}\text{-}\mathcal{A})$, and, further, let $SR := R \otimes^{\mathbf{L}}_{\mathcal{A}} S \in \mathrm{D}(\mathcal{B}\text{-}\mathcal{B})$. Notice that the exact functor underlying the bimodule SR is exactly the composition $s \cdot r$.

Definition 2.4.10. With S, R and L as above,

(1) The *twist* around S is

$$T := \mathrm{cone}(\mathrm{tr}\colon SR \to \mathcal{B}) \in \mathrm{D}(\mathcal{B}\text{-}\mathcal{B}).$$

(2) The *cotwist* around S is the functor

$$C := \mathrm{cone}(\mathrm{act}\colon \mathcal{A} \to RS)[-1] \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{A}).$$

(3) The *twist* around s is

$$t := (-) \otimes^{\mathbf{L}}_{\mathcal{B}} T\colon\ \mathrm{D}(\mathcal{B}) \to \mathrm{D}(\mathcal{B}).$$

(4) The *cotwist* around s is

$$c := (-) \otimes^{\mathbf{L}}_{\mathcal{A}} C\colon\ \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A}).$$

As a straightforward consequence of the definition of twist and cotwist (see [AL17, §5] for details), there is a functorial triangle

$$sr \xrightarrow{\varepsilon^{\mathrm{R}}} 1_{\mathcal{B}} \xrightarrow{\tau} t \xrightarrow{\alpha} {}^{+} \tag{2.4.B}$$

in $\mathrm{D}(\mathcal{B})$, where $\varepsilon^{\mathrm{R}}\colon sr \to 1_{\mathcal{B}}$ is the counit of the adjunction (s, r). Similarly, there is a functorial triangle

$$c \xrightarrow{\gamma} 1_{\mathcal{A}} \xrightarrow{\eta^{\mathrm{R}}} rs \xrightarrow{\beta} ^{+} \tag{2.4.C}$$

in $\mathrm{D}(\mathcal{A})$ where $\eta^{\mathrm{R}}\colon 1_{\mathcal{A}} \to rs$ is the unit of the adjunction (s, r).

Henceforth, unless stated otherwise, when we refer to twist and cotwist functors, we mean the exact functors t and c underlying the bimodules T and C.

Remark 2.4.11. Notice that, strictly speaking, the twist t and cotwist c as defined in 2.4.12 are only defined on derived categories $\mathrm{D}(\mathcal{A})$ and $\mathrm{D}(\mathcal{B})$. However, it is straightforward to extend this definition to triangulated categories admitting Morita enhancements.

Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories admitting Morita enhancements $\mathcal{A}$ and $\mathcal{B}$, respectively. Then, as part of the data of a Morita enhancement, there are triangle equivalences $E\colon\ \mathrm{D}(\mathcal{A}) \to \mathcal{C}$ and $E'\colon\ \mathrm{D}(\mathcal{B}) \to \mathcal{D}$.

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which descends from a quasi-functor on the enhancements. By this, we mean that there exists an $S \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$ such that

$$F \cong E' \cdot (-) \otimes^{\mathbf{L}}_{\mathcal{A}} S \cdot E^{-1}.$$

Then, we say that the twist around F is $E' \cdot t \cdot (E')^{-1}$ and the cotwist is $E \cdot c \cdot E^{-1}$.

The twist and the cotwist around a functor are closely related, so that properties of C afford control over properties of T. Therefore, it is an interesting problem to characterise a functor F as a twist around a functor $s\colon\ \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ whose domain category $\mathrm{D}(\mathcal{A})$ is simpler (e.g. $\mathrm{D}(\mathcal{A})$ is the derived category of a finite dimensional algebra) than the codomain category $\mathrm{D}(\mathcal{B})$ (e.g. $\mathrm{D}(\mathcal{B})$ is the category of an infinite dimensional algebra). In particular, it is often possible to prove that the twist around a functor s is an equivalence, given knowledge of the cotwist. This intuition is made precise by the "2 out of 4" lemma.

Lemma 2.4.12 ([AL17, 5.1])**.** *Let $s\colon\ \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ be as above. If any two of the following conditions hold, then the other two also hold.*

(1) *The twist $t\colon\ \mathrm{D}(\mathcal{B}) \to \mathrm{D}(\mathcal{B})$ is an equivalence,*

(2) *The cotwist $c\colon\ \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ is an equivalence,*

(3) *With α as in (2.4.B), there is a natural isomorphism*

$$lt[-1] \xrightarrow{l(\alpha[-1])} lsr \xrightarrow{\varepsilon^{\mathrm{R}}_r} r$$

(4) *With* β *as in* (2.4.C) *and letting* η^{L} *denote the unit of the adjunction* (l, s)*, there is a natural isomorphism*
$$r \xrightarrow{\eta^{\mathrm{L}}_r} rsl \xrightarrow{\beta_l} cl[1]$$

Definition 2.4.13 ([AL17, 1.1])**.** A functor $s\colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ as above is *spherical* if any two of the conditions in 2.4.12 hold. When s is spherical, t is called a *spherical twist.*

In practice, it is hard to check whether the specific natural transformation in 2.4.12 (4) is an isomorphism, and so the following result will be key.

Theorem 2.4.14 ([Add16, 1], see also [Ann13] and [Rou08])**.** *With notation as above, if the cotwist* $c\colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ *is an equivalence, and if there exists an isomorphism of functors* $r \cong cl[1]$*, then* s *is spherical.*

Example 2.4.15. Let X be an n-dimensional smooth projective variety over a field k, and let ω_X denote the canonical bundle on X. Then, $\mathrm{D^b}(\operatorname{coh} X)$ admits a small Morita enhancement $\mathcal{A}$ (see e.g. [AL17, 4.3]).

As defined in [ST01], an object $\mathcal{E} \in \mathrm{D^b}(\operatorname{coh} X)$ is spherical if $\mathcal{E} \otimes^{\mathbf{L}} \omega_X \cong \mathcal{E}$ and, moreover,
$$\operatorname{Ext}^i_X(\mathcal{E}, \mathcal{E}) = \begin{cases} k & i = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

One may consider that $\mathcal{A}$-module $\bar{\mathcal{E}}\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ch_{dg}}(k)$ corresponding to the image of $\mathcal{E}$ across the equivalence $\mathrm{D^b}(\operatorname{coh} X) \cong \mathrm{D}_c(\mathcal{A})$. This module may be viewed as an object in $\mathrm{D}(\mathrm{Ch_{dg}}(k) - \mathcal{A})$, and the exact functor underlying $\bar{\mathcal{E}}$ is
$$s = - \otimes \mathcal{E}\colon\ \mathrm{D^b}(k) \to \mathrm{D^b}(\operatorname{coh} X) \xrightarrow{\sim} \mathrm{D}_c(\mathcal{A})$$

Since $\mathcal{E}$ is a spherical object, it follows from [ST01], that s is spherical. Moreover, the exact functor t sits in a functorial triangle
$$\mathcal{E} \otimes^{\mathbf{L}}_X \mathbf{R}\mathrm{Hom}_X(\mathcal{E}, -) \to 1_{\mathrm{D^b}(\operatorname{coh} X)} \to t \to^{+}.$$

This thesis will be particularly interested in the following class of examples.

Example 2.4.16. Let A, B be R-algebras and such that there is an R-algebra morphism $p\colon A \to B$. Notice that we may view B as an A-bimodule.

Consider the Morita enhancements $\mathcal{A}$ and $\mathcal{B}$ of $\mathrm{D}(\operatorname{Mod} A)$ and $\mathrm{D}(\operatorname{Mod} B)$, respectively, as in 2.4.9. Then, a DG-bimodule $S \in \mathrm{D}(\mathcal{B}\text{-}\mathcal{A})$ is just a chain complex of B-A-bimodules. Assuming that B is perfect as an A-module, and letting $S := {}_B B_A$, so that $R = \mathbf{R}\mathrm{Hom}_A({}_B B, {}_A A)$ and $L = {}_A B_B$, then the underlying exact functors induce the standard restriction-extension of scalars adjunction

$$\begin{tikzcd} \mathrm{D}(B) \arrow[rr, "s=(-)\otimes^{\mathbf{L}}{}_B B_A" description] & & \mathrm{D}(A) \arrow[ll, bend right, "l=(-)\otimes^{\mathbf{L}}{}_A B_B"'] \arrow[ll, bend left, "r=\mathbf{R}\mathrm{Hom}_A({}_B B_A,-)"] \end{tikzcd} \tag{2.4.D}$$

The twist and cotwist around S are the bimodules

$$T = \operatorname{cone}(\varepsilon^{\mathrm{R}}_A\colon \mathbf{R}\mathrm{Hom}_A({}_B B, {}_A B) \otimes^{\mathbf{L}}_B {}_B B_A \to {}_A A_A)$$
$$C = \operatorname{cone}(\eta^{\mathrm{R}}_B\colon {}_B B_B \to \mathbf{R}\mathrm{Hom}_A({}_B B, {}_B B \otimes^{\mathbf{L}}_B B_A))[-1]$$

where $\varepsilon^{\mathrm{R}}_A$ and η^{R}_B are the counit and unit of the tensor-hom adjunction $(-) \otimes^{\mathbf{L}}_B {}_B B_A \dashv \mathbf{R}\mathrm{Hom}_A({}_B B, -)$ evaluated at the bimodules ${}_A A_A$ and ${}_B B_B$, respectively.

It follows that the twist and cotwist around s are

$$t = (-) \otimes^{\mathbf{L}}_A T,$$
$$c = (-) \otimes^{\mathbf{L}}_B C.$$

§ 2.5 Crepant contractions to affine schemes

An important landmark in the study of derived symmetries is Bondal and Orlov's theorem [BO01, 3.1], which states that the derived autoequivalence group for smooth projective varieties with ample or anti-ample canonical bundle is as small as possible. Varieties with trivial canonical bundle, however, may have more complicated derived autoequivalence groups. Following the philosophy in [Don24], one may expect derived symmetries to arise from contractions against which the canonical bundle is trivial.

Recall that a *variety* over a field k is a reduced, separated scheme of finite type over k.

Definition 2.5.1. (1) As in [DW19b, 2.1], we say that a *contraction* is a projective birational morphism $f\colon X \to Y$ between normal varieties over a field k with Y quasi-projective and $\mathbf{R}f_*\mathcal{O}_X = \mathcal{O}_Y$.

(2) In view of the definition above of a contraction, we define a *complete local contraction* to be a projective birational morphism $f\colon X \to Y = \operatorname{Spec} R$ between normal schemes over a field k with R complete local and $\mathbf{R}f_*\mathcal{O}_X = \mathcal{O}_Y$.

(3) Suppose that X and Y are Gorenstein schemes with canonical sheaves ω_X and ω_Y, respectively. Recall that a (complete local) contraction $f\colon X \to Y$ is *crepant* if $f^*\omega_Y \cong \omega_X$.

In this section we will work within the following setup

Setup 2.5.2. Let $f\colon X \to \operatorname{Spec} R$ be a crepant (complete local) contraction and assume that X admits a tilting bundle $\mathcal{V}$ containing $\mathcal{O}_X$ as a summand.

Example 2.5.3. (1) A (complete local) crepant contraction $f\colon X \to \operatorname{Spec} R$ with fibres that are at most one-dimensional satisfies 2.5.2 [VdB04, 3.2.5, 3.2.8].

(2) Consider a (complete local) crepant contraction $g\colon X \to \operatorname{Spec} R$ with fibres that are at most two-dimensional. If X admits an ample globally generated line bundle $\mathcal{L}$ such that $\mathbf{R}^2 f_* \mathcal{L}^{-1} = 0$, then g satisfies 2.5.2 [TU10].

Lemma 2.5.4 ([DW19b, 2.5]). *If R is Gorenstein, and $f\colon X \to \operatorname{Spec} R$ satisfies 2.5.2, then the natural map $\operatorname{End}_X(\mathcal{V}) \to \operatorname{End}_R(f_*\mathcal{V})$ is an isomorphism. Moreover, $f_*\mathcal{V} \in \operatorname{CM} R$. Therefore, there is an equivalence $\Psi_{\mathcal{V}}\colon \operatorname{D}(\operatorname{Qcoh} X) \xrightarrow{\sim} \operatorname{D}(\operatorname{Mod} \Lambda)$ where $\Lambda = \operatorname{End}_R(f_*\mathcal{V})$ is the endomorphism algebra of an object $f_*\mathcal{V} \in \operatorname{CM} R$ in a Frobenius exact category.*

§ 2.5.1 The contraction algebra

To a (complete local) contraction $f\colon X \to \operatorname{Spec} R$ satisfying 2.5.2, Donovan and Wemyss [DW16, 2.9] associate an invariant called the *contraction algebra*.

Definition 2.5.5. In the setting of lemma 2.5.4, recall that

(1) The *contraction algebra* Λ_{con} is the quotient

$$\Lambda_{\mathrm{con}} := \Lambda/[\operatorname{proj} R]$$

where $[\operatorname{proj} R]$ is the ideal of maps $f_*\mathcal{V} \to f_*\mathcal{V}$ which factor through a finitely generated projective R-module.

(2) Alternatively, observe that since $\mathcal{O}_X$ is a summand of $\mathcal{V}$ by assumption, then $f_*\mathcal{O}_X = R$ is a summand of $f_*\mathcal{V}$. Let $e_0\colon f_*\mathcal{V} \to f_*\mathcal{V}$ be the idempotent in Λ corresponding to projection onto R. Then

$$\Lambda_{\mathrm{con}} := \Lambda/\Lambda e_0 \Lambda.$$

This invariant is known to package a lot of the information associated to a contraction. Indeed, it was shown in [DW16; HT18] that the contraction algebra recovers classical invariants associated to contractions, and furthermore Λ_{con} classifies smooth contractions when R has at worst isolated cDV singularities [JKM23, Appendix A] [JKM24] [KLW25]. In this thesis, we are interested in the contraction algebra because the properties of this invariant can track whether the noncommutative twist functor (recalled in 2.5.3) induced by the contraction is a derived autoequivalence of X.

We end this subsection with a lemma that will be used throughout.

Lemma 2.5.6. *For any $A, B \in \operatorname{mod} \Lambda_{\mathrm{con}}$,*

$$\operatorname{Ext}^1_{\Lambda_{\mathrm{con}}}(A, B) \cong \operatorname{Ext}^1_{\Lambda}(A, B)$$

Proof. This is standard. Since multiplying by idempotents is exact, we may view $\operatorname{mod}\Lambda_{\mathrm{con}}$ as an extension closed subcategory of $\operatorname{mod}\Lambda$, and so the result follows. $\square$

§ 2.5.2 Why the contraction algebra?

The contraction algebra can been seen in three different lights, each of which presents some intuition on why one should expect this algebra to remember many of properties of a contraction $f\colon X \to \operatorname{Spec} R$ and of $\mathrm{D}(\operatorname{Qcoh} X)$.

Stable endomorphism algebra.

In the setting of proposition 2.5.4, recall that the pushforward $f_*\mathcal{V}$ of the tilting bundle $\mathcal{V}$ on X is a maximal Cohen-Macaulay R-module. Since R is Gorenstein, $\operatorname{CM} R$ is a Frobenius exact category and so it stable category $\underline{\operatorname{CM}}\, R$ is triangulated. Observe that it follows directly from the definition of Λ_{con} that $\Lambda_{\mathrm{con}} = \underline{\operatorname{End}}_R(f_*\mathcal{V})$.

Since we can identify $\mathrm{D}(\operatorname{Qcoh} X)$ with $\mathrm{D}(\operatorname{Mod}\Lambda)$, where $\Lambda = \operatorname{End}_R(f_*\mathcal{V})$, then we can view a choice of contraction $f\colon X \to \operatorname{Spec} R$ as a way to associate a Frobenius exact category to this tilting bundle. Then, the contraction algebra is an algebra usually much smaller than Λ which remembers properties of Λ (and thus of X) up to stable equivalences.

Semi-orthogonal decomposition.

In the setting of proposition 2.5.4, the contraction f_* induces a recollement diagram (see e.g. [BBDG18, §1.4] for the definition of recollement)

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i_*"] & \mathrm{D}(\operatorname{Qcoh} X) \arrow[l, bend right, "i^*"'] \arrow[r, "\mathbf{R}f_*"] & \mathrm{D}(\operatorname{Mod} R) \arrow[l, bend right, "\mathbf{L}f^*"'] \arrow[l, bend left, "f^!"] \end{tikzcd}$$

where $\mathcal{C} := \ker \mathbf{R}f_*$ is the full subcategory of $\mathrm{D}(\operatorname{Qcoh} X)$ consisting of complexes which are mapped to 0 under $\mathbf{R}f_*$, and $i\colon \mathcal{C} \to \mathrm{D}(\operatorname{Qcoh} X)$ is the natural inclusion functor (see [Bri02] for further details on this recollement).

Further in this setting, there is an equivalence $\Psi_{\mathcal{V}}\colon \mathrm{D}(\operatorname{Qcoh} X) \xrightarrow{\sim} \mathrm{D}(\operatorname{Mod}\Lambda)$ and a functor $\mathbf{R}\operatorname{Hom}_\Lambda(e_0\Lambda, -)\colon \mathrm{D}(\operatorname{Mod}\Lambda) \to \mathrm{D}(\operatorname{Mod} R)$ which satisfy the equation $\mathbf{R}\operatorname{Hom}_\Lambda(e_0\Lambda, -) \cdot \Psi_{\mathcal{V}} = \mathbf{R}f_*$ [Wem18, 2.14]. Thus, composing the recollement above with the equivalence $\Psi_{\mathcal{V}}\colon \mathrm{D}(\operatorname{Qcoh} X) \xrightarrow{\sim} \mathrm{D}(\operatorname{Mod}\Lambda)$ induces the recollement diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "i_*"] & \mathrm{D}(\operatorname{Mod}\Lambda) \arrow[l, bend right, "i^*"'] \arrow[r, "\mathbf{R}\operatorname{Hom}_\Lambda(e_0\Lambda,-)"] & \mathrm{D}(\operatorname{Mod} R) \arrow[l, bend right, "-\otimes^{\mathbf{L}}_R e_0\Lambda"'] \arrow[l, bend left, "\mathbf{R}\operatorname{Hom}_R(\Lambda e_0,-)"] \end{tikzcd}$$

where $\mathcal{C}' = \ker \mathbf{R}\operatorname{Hom}_\Lambda(e_0\Lambda, -)$ is the full subcategory of $\mathrm{D}(\operatorname{Mod}\Lambda)$ consisting of complexes which are mapped to 0 under $\mathbf{R}\operatorname{Hom}_\Lambda(e_0\Lambda, -)$. It turns out that $\mathcal{C}'$ can be described as the full subcategory of $\mathrm{D}(\operatorname{Mod}\Lambda)$ consisting of objects with cohomology in $\operatorname{Mod}\Lambda_{\mathrm{con}}$.

Hence, we obtain a semi-orthogonal decomposition $\langle \mathcal{C}, \mathrm{D}(\operatorname{Mod} R)\rangle$ of $\mathrm{D}(\operatorname{Qcoh} X)$ where the objects of $\mathcal{C}$ have cohomology controlled by Λ_{con}.

Deformation algebra.

Suppose that $f\colon X \to \operatorname{Spec} R$ satisfies 2.5.4 and, in addition, has fibres of dimension at most 1. That is, f contracts n curves $\{C_i\}_{i=1}^n$ above the origin to a point. Then, [DW19c] shows that there is a non-commutative deformation functor $\mathrm{Def}\colon \mathrm{Art}_n \to \mathrm{Sets}$ of the sheaves $\mathcal{O}_{C_i}$ which is pro-represented (up to a Morita equivalence) by Λ_{con}. In other words, the noncommutative deformation theory of the structure sheaves $\mathcal{O}_{C_i}$ of the contracting curves turns out to be controlled by Λ_{con}, and so one would expect Λ_{con} to encode much of the geometric information about the contraction.

Viewing Λ_{con} as a deformation algebra also justifies why it is canonically defined from the contraction in the setting of contractions with at most 1-dimensional fibres. A crepant contraction may admit different choices of tilting bundles on X, but deformation theory is intrinsic.

§ 2.5.3 The noncommutative twist

Given a (complete local) contraction $f\colon X \to \operatorname{Spec} R$ satisfying 2.5.4, Donovan and Wemyss in [DW19b] introduced the *noncommutative twist functor*

$$\mathcal{T} = \Psi_{\mathcal{V}}^{-1} \cdot \mathbf{R}\mathrm{Hom}_{\Lambda}([\operatorname{proj} R], -) \cdot \Psi_{\mathcal{V}}\colon\ \mathrm{D^b}(\operatorname{coh} X) \to \mathrm{D^b}(\operatorname{coh} X)$$

where $[\operatorname{proj} R]$ is the ideal recalled in 2.5.5. It is proven in [DW19b] that $\mathcal{T}$ is an auto-equivalence of $\mathrm{D^b}(\operatorname{coh} X)$ if R is complete locally a hypersurface and if the contraction algebra Λ_{con} is 3-relatively spherical. By 3-relatively spherical, we mean that Λ_{con} satisfies $\mathrm{Ext}^i_{\Lambda}(\Lambda_{\mathrm{con}}, S) = 0$ for all simples $S \in \operatorname{mod}\Lambda_{\mathrm{con}}$ and for all $i \neq 0, 3$.

This thesis is concerned with proving that $\mathcal{T}$ is an auto-equivalence in more general settings. We, moreover, prove that $\mathcal{T}$ is a twist around a spherical functor and compute its associated cotwist.

§ 2.6 Noncommutative schemes and relative tilting

As discussed in § 2.3.3, a tilting complex induces a derived equivalence between schemes and noncommutative rings. As explained in § 2.5, given a contraction $f\colon X \to \operatorname{Spec} R$ satisfying certain assumptions, one can construct a tilting bundle. In the more general setting of a contraction $f\colon X \to Y$, where Y is not necessarily affine, it is harder to construct tilting bundles. In this more general setting, however, it is still possible to use noncommutative algebra to identify $\mathrm{D}(\operatorname{Qcoh} X)$ with another derived category parametrised by Y.

In this section, we recall the more general notion of a relative tilting bundle.

This construction induces a derived equivalence between varieties and noncommutative schemes. One can think of a relative tilting bundle as a "global version" of a tilting bundle.

§ 2.6.1 Noncommutative schemes and their modules

We begin by recalling some basic theory about noncommutative schemes. Mostly, the point of this subsection is to set notation and also to convince the reader that noncommutative schemes behave somewhat similarly to commutative schemes.

Definition 2.6.1. (1) A *noncommutative scheme* is a pair $(\mathcal{X}, \mathcal{A})$ where $\mathcal{X}$ is a scheme and $\mathcal{A}$ is a sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras which is quasi-coherent as an $\mathcal{O}_{\mathcal{X}}$-module. We will often abbreviate noncommutative scheme as NC scheme.

(2) A NC scheme $(\mathcal{X}, \mathcal{A})$ is said to be *quasi-projective* if the underlying scheme $\mathcal{X}$ is quasi-projective.

(3) Moreover, $(\mathcal{X}, \mathcal{A})$ is said to be *noetherian* if $\mathcal{X}$ is a noetherian scheme and $\mathcal{A}$ is a coherent $\mathcal{O}_{\mathcal{X}}$-module.

(4) Given a NC scheme $(\mathcal{X}, \mathcal{A})$, let $\mathrm{Mod}(\mathcal{X}, \mathcal{A})$ denote the category of $\mathcal{A}$-modules. Moreover, the category of quasi-coherent (respectively, coherent) $\mathcal{A}$-modules will be denoted as $\mathrm{Qcoh}(\mathcal{X}, \mathcal{A})$ (respectively, $\mathrm{coh}(\mathcal{X}, \mathcal{A})$).

(5) Given NC schemes $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{X}, \mathcal{B})$, an $\mathcal{A}$-$\mathcal{B}$-*bimodule* is a sheaf of $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}}$-modules on $\mathcal{X}$.

(6) A *morphism of noncommutative schemes* $f\colon (\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$ is a pair $(f_{\mathcal{X}}, f^{\#})$ where $f_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of schemes and $f^{\#}\colon f_{\mathcal{X}}^{-1}\mathcal{B} \to \mathcal{A}$ is a morphism of $f_{\mathcal{X}}^{-1}\mathcal{O}_Y$-algebras.

Remark 2.6.2. Observe that due to standard sheaf theory, since $f_{\mathcal{X}}^{-1}$ is left adjoint to $(f_{\mathcal{X}})_*$, the morphism $f^{\#}\colon f_{\mathcal{X}}^{-1}\mathcal{B} \to \mathcal{A}$ induces a morphism $f^{\flat}\colon \mathcal{B} \to (f_{\mathcal{X}})_*\mathcal{A}$.

Definition 2.6.3. Similarly to the commutative case, a morphism $f\colon (\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$ of NC schemes induces the pullback and pushforward adjoint pair.

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{X}, \mathcal{A}) \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Mod}(\mathcal{Y}, \mathcal{B}). \arrow[l, bend right, "f^*"'] \end{tikzcd}$$

(1) The functor

$$f_*\colon \mathrm{Mod}(\mathcal{X}, \mathcal{A}) \to \mathrm{Mod}(\mathcal{Y}, \mathcal{B})$$

sends a $\mathcal{B}$-module $\mathcal{F}$ to the sheaf $f_*\mathcal{F}$ whose sections on a open set $V \subset \mathcal{Y}$ are defined as

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V).$$

The $\mathcal{B}$-module structure on this sheaf is induced by the morphism $f^\flat\colon \mathcal{B} \to (f_{\mathcal{X}})_*\mathcal{A}$.

(2) The functor
$$f^*\colon \operatorname{Mod}(\mathcal{Y}, \mathcal{B}) \to \operatorname{Mod}(\mathcal{X}, \mathcal{A})$$
sends a $\mathcal{B}$-module $\mathcal{F}$ to the tensor product $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{B}} \mathcal{A}$.

Observe that we may restrict the functor f^* to $f^*\colon \operatorname{Qcoh}(\mathcal{X}, \mathcal{B}) \to \operatorname{Qcoh}(\mathcal{X}, \mathcal{A})$. If, further, $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ are noetherian, then f^* restricts to $f^*\colon \operatorname{coh}(\mathcal{X}, \mathcal{B}) \to \operatorname{coh}(\mathcal{X}, \mathcal{A})$.

Lemma 2.6.4. *Let $(\mathcal{X}, \mathcal{A})$ be a noetherian noncommutative scheme. Then, the categories* $\operatorname{Mod}(\mathcal{X}, \mathcal{A})$*,* $\operatorname{Qcoh}(\mathcal{X}, \mathcal{A})$ *and* $\operatorname{coh}(\mathcal{X}, \mathcal{A})$ *are abelian.*

Proof. The statement for $\operatorname{Mod}(\mathcal{X}, \mathcal{A})$ follows from [KS02, 18.1.6(v)] (they actually prove the stronger result that $\operatorname{Mod}(\mathcal{X}, \mathcal{A})$ is Grothendieck). The fact that $\operatorname{Qcoh}(\mathcal{X}, \mathcal{A})$ is abelian (and also Grothendieck) follows from [KL15, 5.6]. Finally, the category $\operatorname{coh}(\mathcal{X}, \mathcal{A})$ is abelian by [KS02, Exercise 8.23]. □

Since $\operatorname{Qcoh}(\mathcal{X}, \mathcal{A})$ is abelian, so is the category $\operatorname{Ch}(\operatorname{Qcoh}(\mathcal{X}, \mathcal{A}))$ of chain complexes of quasi-coherent $\mathcal{A}$-modules. Moreover, we may also construct the derived category $\operatorname{D}(\operatorname{Qcoh}(\mathcal{X}, \mathcal{A}))$.

§ 2.6.2 Resolution properties

We are interested in derived functors between derived categories of NC schemes, and so it will be important to establish the existence of *K-flat* and *K-injective* resolutions (recalled in § 2.3.1) in $\operatorname{Qcoh}(\mathcal{X}, \mathcal{A})$. The existence of *Lp-resolutions* under certain settings will also play an important role.

Definition 2.6.5. Let $(\mathcal{X}, \mathcal{A})$ be a NC scheme.

(1) An $\mathcal{A}$-module $\mathcal{F}$ is *locally projective* if $\mathcal{F}(U)$ is a projective $\mathcal{A}(U)$-module for all affine open $U \subset \mathcal{X}$.

(2) Let $\mathcal{F} \in \operatorname{D}(\operatorname{Qcoh}(\mathcal{X}, \mathcal{A}))$. Then, an *Lp-resolution* of $\mathcal{F}$ is a quasi isomorphism $\mathcal{P} \to \mathcal{F}$ where $\mathcal{P}$ is a K-flat complex of locally projective $\mathcal{A}$-modules.

Lemma 2.6.6. *Let $(\mathcal{X}, \mathcal{A})$ be a noetherian NC scheme. Then,*

(1) *Any complex in* $\operatorname{Ch}(\operatorname{Qcoh}(\mathcal{X}, \mathcal{A}))$ *admits a K-injective resolution.*

(2) *Any complex in* $\operatorname{Ch}(\operatorname{Qcoh}(\mathcal{X}, \mathcal{A}))$ *admits a K-flat resolution.*

(3) *If $(\mathcal{X}, \mathcal{A})$ is quasi-projective, then any complex in* $\operatorname{Ch}(\operatorname{Qcoh}(\mathcal{X}, \mathcal{A}))$ *admits an Lp-resolution.*

Proof. (1) follows from [ATJLSS00, 5.4], (2) is [BDG17, 3.9], and (3) is [BDG17, 3.7, 3.10]. □

§ 2.6.3 Derived functors

Given NC schemes $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{X}, \mathcal{B})$, let $\mathcal{F}$ be a complex of $\mathcal{B}$-$\mathcal{A}$-bimodules and $\mathcal{G}$ a complex of $\mathcal{A}$-$\mathcal{B}$-bimodules. In what follows, we present several technical lemmas, the point of which is to establish that the functors

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, -)\colon \ & \mathrm{D}(\mathrm{Mod}(\mathcal{X}, \mathcal{A})) \to \mathrm{D}(\mathrm{Mod}(\mathcal{B}(\mathcal{X})) \\ \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, -)\colon \ & \mathrm{D}(\mathrm{Mod}(\mathcal{X}, \mathcal{A})) \to \mathrm{D}(\mathrm{Mod}(\mathcal{X}, \mathcal{B})) \\ (-) \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{G}\colon \ & \mathrm{D}(\mathrm{Mod}(\mathcal{X}, \mathcal{A})) \to \mathrm{D}(\mathrm{Mod}(\mathcal{X}, \mathcal{B})) \end{aligned}$$

behave as expected.

Proposition 2.6.7 ([Spa88, section 6], [BDG17, 3.12])**.** *Given NC schemes $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{X}, \mathcal{B})$, let $\mathcal{F}$ be a complex of $\mathcal{B}$-$\mathcal{A}$-bimodules and $\mathcal{G}$ a complex of $\mathcal{A}$-$\mathcal{B}$-bimodules.*

(1) *The derived functors*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, -)\colon \mathrm{D}(\mathrm{Mod}(\mathcal{X}, \mathcal{A})) \to \mathrm{D}(\mathrm{Mod}(\mathcal{B}(\mathcal{X})) \tag{2.6.A}$$

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, -)\colon \mathrm{D}(\mathrm{Mod}(\mathcal{X}, \mathcal{A})) \to \mathrm{D}(\mathrm{Mod}(\mathcal{X}, \mathcal{B})) \tag{2.6.B}$$

$$(-) \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{G}\colon \mathrm{D}(\mathrm{Mod}(\mathcal{X}, \mathcal{A})) \to \mathrm{D}(\mathrm{Mod}(\mathcal{X}, \mathcal{B})) \tag{2.6.C}$$

exist.

(2) *If $\mathcal{F}$ is a complex of bimodules which are locally finitely presented as $\mathcal{A}$-modules and $\mathcal{G}$ is a complex of quasi-coherent bimodules, then the functors* (2.6.B) *and* (2.6.C) *restrict to the derived category of quasi-coherent modules.*

(3) *Given $\mathcal{M} \in \mathrm{D}(\mathrm{Qcoh}(\mathcal{X}, \mathcal{A}))$, the complexes $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{M})$ and $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{M})$ can be calculated by using a K-injective resolution of $\mathcal{M}$. Moreover, the complex $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{M})$ can also be calculated using an Lp-resolution of $\mathcal{F}$, if it exists.*

(4) *The complex $\mathcal{M} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{G}$ can be computed by using a K-flat resolution of $\mathcal{M}$.*

(5) *There are natural isomorphisms*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{G}, -) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathbf{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{G}, -)) \tag{2.6.D}$$

$$\mathbf{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{G}, -) \cong \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathbf{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{G}, -)). \tag{2.6.E}$$

which induce the adjunction $- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{G} \dashv \mathbf{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{G}, -)$.

Proof. (1), (3), (4) and (5) are [BDG17, 3.12]. The proof of the fact that the functors (2.6.A), (2.6.B) and (2.6.C) preserve quasi-coherence under the assumptions of (2) is the same as in the commutative case, which is standard. □

Definition 2.6.8. Recall that an object $\mathcal{F} \in \mathrm{D}(\mathrm{Qcoh}(\mathcal{X}, \mathcal{A}))$ is called *perfect* if there exists an affine open covering $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i}$ for each i is quasi-isomorphic to a bounded complex of sheaves which are summands of finite free $\mathcal{A}|_{U_i}$-modules.

Proposition 2.6.9. *Let $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{X}, \mathcal{B})$ be quasi-projective noetherian NC schemes. Suppose that $\mathcal{F}$ is a complex of coherent $\mathcal{B}$-$\mathcal{A}$-bimodules which is perfect as complex of $\mathcal{A}$-modules. Moreover, suppose that $\mathcal{G}$ is a complex of $\mathcal{A}$-$\mathcal{B}$-bimodules which is perfect as a complex of $\mathcal{A}^{\mathrm{op}}$-modules. Then, the functors* (2.6.A), (2.6.B) *and* (2.6.C) *restrict to*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, -)\colon \mathrm{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{X}, \mathcal{A})) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \mathcal{B}(\mathcal{X})) \tag{2.6.F}$$

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, -)\colon \mathrm{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{X}, \mathcal{A})) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{X}, \mathcal{B})) \tag{2.6.G}$$

$$(-) \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{G}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{X}, \mathcal{A})) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{X}, \mathcal{B})) \tag{2.6.H}$$

Proof. Since $\mathcal{X}$ is quasi-projective, it follows from 2.6.6 (3) that $\mathrm{coh}(\mathcal{X}, \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}})$ and $\mathrm{coh}(\mathcal{X}, \mathcal{A})$ have enough Lp-modules and so the bounded complex $\mathcal{F}$ admits an Lp-resolution $\mathcal{Q}' \to \mathcal{F}$ as a $\mathcal{A}$-module and an Lp-resolution $\mathcal{Q} \to \mathcal{F}$ as an $\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}$-module. Since $\mathcal{F}$ is perfect as an $\mathcal{A}$-module, we may assume that $\mathcal{Q}'$ is bounded. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{X}, \mathcal{A}))$.

We would first like to show that $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{M})$ is a bounded complex of coherent $\mathcal{B}$-modules. To see that it is bounded, note that, as a complex of vector spaces, $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{M}) \cong \mathcal{H}om^*_{\mathcal{A}}(\mathcal{Q}', \mathcal{M})$. Since both $\mathcal{Q}'$ and $\mathcal{M}$ are bounded, so is $\mathcal{H}om^*_{\mathcal{A}}(\mathcal{Q}', \mathcal{M})$. Hence, necessarily, $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{M})$ is bounded as a complex of $\mathcal{B}$-modules.

Next, we would like to show that $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{M})$ is a complex of coherent modules. The question is local, so we may assume that $\mathcal{X} = \operatorname{Spec} R$ where, by assumption, R is a noetherian ring, and $\mathcal{A} = A^\sim$, $\mathcal{B} = B^\sim$ with A, B finitely presented R-algebras. Note that A and B are noetherian, since they are module finite over a noetherian ring.

Since $\mathcal{Q}$ and $\mathcal{M}$ are coherent, $\mathcal{Q} = Q^\sim$ and $\mathcal{M} = M^\sim$ where $Q \in \mathrm{mod}\, A \otimes_R B^{\mathrm{op}}$ and $M \in \mathrm{mod}\, A$. So, coherence of $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{M}) = \mathcal{H}om^*_{\mathcal{A}}(\mathcal{Q}, \mathcal{M})$ as a complex $\mathcal{B}$-modules is equivalent to $\mathrm{Hom}^*_A(Q, M)$ being a complex of finitely presented B-modules. Since M is bounded and A and B are noetherian, it suffices to show that $\mathrm{Hom}_A(U, N)$ is a finitely generated B-module if $U \in \mathrm{mod}\, A \otimes_R B^{\mathrm{op}}$ and $N \in \mathrm{mod}\, A$. Note that this is exactly the proof of 2.3.9.

The proofs of (2.6.F) and (2.6.H) are analogous to the above proof, and so are omitted. $\square$

§ 2.6.4 Relative tilting bundle

Consider a morphism $f\colon X \to Y$ of noetherian schemes. Recall that a bundle $\mathcal{P}$ on X is *tilting relative to* Y in the sense of e.g. [DW19b, 2.3(2)] if there is an equivalence

$$\Psi_{\mathcal{P}} := \mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{P}, -)\colon \mathrm{D^b}(\mathrm{coh}\, X) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{coh}(Y, \mathcal{A})) \tag{2.6.I}$$

where $\mathcal{A} := \mathbf{R}f_*\mathcal{E}nd_X(\mathcal{P}) = f_*\mathcal{E}nd_X(\mathcal{P})$ is a sheaf of $\mathcal{O}_Y$-algebras and $\mathrm{D^b}(\mathrm{coh}(Y, \mathcal{A}))$ is the bounded derived category of modules over $\mathcal{A}$.

Remark 2.6.10. Observe that when $Y = \mathrm{Spec}\, R$ is affine, then a tilting bundle relative to $\mathrm{Spec}\, R$ is a vector bundle $\mathcal{P}$ such that there is an equivalence

$$\Psi_{\mathcal{P}} := \mathbf{R}\mathrm{Hom}_X(\mathcal{P}, -)\colon \mathrm{D^b}(\mathrm{coh}\, X) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\, \Lambda)$$

where $\Lambda := \mathbf{R}f_*\mathcal{E}nd_X(\mathcal{P}) = \mathrm{End}_X(\mathcal{P})$ is an R-algebra.

Example 2.6.11. Suppose that $f\colon X \to Y$ is a crepant contraction with fibres that are at most one-dimensional. Then, [VdB04, 3.3.2] shows that X admits a relative tilting bundle.

§ 2.7 Crepant contractions more generally

This section can be viewed as the global analogue of § 2.5. We will work within the following set-up

Setup 2.7.1. Let $f\colon X \to Y$ be a crepant (complete local) contraction. Assume further that Y is a Gorenstein d-fold with $d \geqslant 2$, and that X admits a relative tilting bundle $\mathcal{P}$ containing $\mathcal{O}_X$ as a summand.

Notation 2.7.2. With f as in 2.7.1, write Z for the locus of points of Y onto which f is not an isomorphism.

Example 2.7.3. Let Y be a Gorenstein d-fold with $d \geqslant 2$. Then, a crepant contraction $f\colon X \to Y$ with fibres that are at most one-dimensional satisfies 2.7.1 [VdB04, 3.3.2].

Lemma 2.7.4 ([DW19b, 2.5])**.** *Under the setup 2.7.1, the natural map $f_*\mathcal{E}nd_X(\mathcal{P}) \to \mathcal{E}nd_Y(f_*\mathcal{P})$ is an isomorphism, and so $\mathcal{P}$ induces a derived equivalence*

$$\Psi_{\mathcal{P}} := \mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{P}, -)\colon \mathrm{D^b}(\mathrm{coh}\, X) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{coh}(Y, \mathcal{A}))$$

where $\mathcal{A} = \mathcal{E}nd_Y(f_*\mathcal{P})$.

§ 2.7.1 The sheafy contraction algebra

Given a crepant contraction $f\colon X \to Y$ satisfying the general setup 2.7.1, this section recalls the construction of the noncommutative enhancement $(Y, \mathcal{A}_{\mathrm{con}})$. The NC scheme

$(Y, \mathcal{A}_{\mathrm{con}})$ was first constructed in [DW19b] and is an invariant associated to f which retains much information about the contraction f and the category $\mathrm{D}(\operatorname{Qcoh} X)$. We may view $(Y, \mathcal{A}_{\mathrm{con}})$ as the global analogue of the contraction algebra introduced by [DW16] and recalled in § 2.5.

Notation 2.7.5. Under the general setup 2.7.1,

(1) Let $\mathcal{Q} := f_*\mathcal{P}$.

(2) Write $\mathcal{A}$ for the sheaf of $\mathcal{O}_Y$-algebras

$$\mathcal{A} := \mathbf{R}f_*\mathcal{E}nd_X(\mathcal{P}) = f_*\mathcal{E}nd_X(\mathcal{P}) \cong \mathcal{E}nd_Y(\mathcal{Q})$$

where the last isomorphism is 2.7.4.

(3) As in [DW19b, 2.8], let $\mathcal{I}$ be the ideal sheaf of $\mathcal{A}$ specified by the rule: For every open subset $j \colon V \hookrightarrow Y$,

$$\mathcal{I}(V) = \{s \in \operatorname{End}_V(\mathcal{Q}|_V) \mid s_v \colon \mathcal{Q}_v \to \mathcal{Q}_v \in [\operatorname{proj} \mathcal{O}_{Y,v}]\, \forall v \in V\}$$

where $[\operatorname{proj} \mathcal{O}_{Y,v}]$ denotes the ideal of $\operatorname{End}_{\mathcal{O}_{Y,v}}(\mathcal{Q}_v)$ consisting of maps which factor through $\operatorname{proj} \mathcal{O}_{Y,v}$.

(4) The sheaf $\mathcal{A}_{\mathrm{con}} := \mathcal{A}/\mathcal{I}$ is called the sheaf of contraction algebras in [DW19b, 2.12].

(5) For an affine open subset $j \colon U = \operatorname{Spec} R \hookrightarrow Y$ of $(Y, \mathcal{O}_Y)$, write $\Lambda_U := \mathcal{A}(U)$. It follows that $\mathcal{I}(U) = [\operatorname{proj} R]$ [DW19b, 2.10] and $\mathcal{A}_{\mathrm{con}}(U) = \Lambda_U/[\operatorname{proj} R] := (\Lambda_{\mathrm{con}})_U$ [DW19b, 2.15].

We will be interested in the NC schemes $(Y, \mathcal{A})$ and $(Y, \mathcal{A}_{\mathrm{con}})$.

Lemma 2.7.6. *The NC schemes $(Y, \mathcal{A})$ and $(Y, \mathcal{A}_{\mathrm{con}})$ are noetherian.*

Proof. By assumption, Y is a noetherian scheme. To see that $\mathcal{A}$ and $\mathcal{A}_{\mathrm{con}}$ are coherent as $\mathcal{O}_Y$-modules, we may assume that $Y = \operatorname{Spec} R$ for a noetherian ring R. Then, we may view $f_*\mathcal{P}$ as an R-module which is finitely generated because $\mathcal{P}$ is coherent and f is proper.

Since $f_*\mathcal{P}$ is finitely generated over R and R is noetherian, $\mathcal{A} = \mathcal{E}nd_R(f_*\mathcal{P}) = \Lambda^\sim$ where $\Lambda = \operatorname{End}_R(f_*\mathcal{P})$. Thus, $\Lambda = \operatorname{End}_R(f_*\mathcal{P})$ is also finitely generated. Similarly, $\mathcal{A}_{\mathrm{con}} = \mathcal{A}/\mathcal{I} = \Lambda_{\mathrm{con}}^\sim$ where $\Lambda_{\mathrm{con}} = \Lambda/[\operatorname{proj} R]$ is finitely generated since Λ is.

Therefore, we conclude that $\mathcal{A}$ and $\mathcal{A}_{\mathrm{con}}$ are coherent as $\mathcal{O}_Y$-modules. □

An interesting property of the noncommutative scheme $(Y, \mathcal{A}_{\mathrm{con}})$ is that the contraction theorem [DW19b, 2.16] implies that $\mathcal{A}_{\mathrm{con}}$ is supported on the singular locus Z of f. As a consequence, we often reduce questions about $(Y, \mathcal{A}_{\mathrm{con}})$ to questions about stalks $(\mathcal{A}_{\mathrm{con}})_z$ for $z \in Z$.

§ 2.8 Skew group algebras

This section briefly recalls the construction of skew-group algebras, as they are a good source of examples for *Gorenstein orders*, which are the central objects in § 4.2.

Setup 2.8.1. Let $n \geqslant 2$ and consider a finite group $G \subset \mathrm{GL}(n, \mathbb{C})$ acting on the $\mathbb{C}$-algebra $R = \mathbb{C}[[x_1, x_2, \dots, x_n]]$. Let R^G denote the ring of invariants and set $\mathbb{C}^n/G := \operatorname{Spec} R^G$.

Definition 2.8.2. Recall that the *skew-group* algebra $R\#G$ is the vector space $R \otimes_{\mathbb{C}} \mathbb{C}G$ equipped with a "skewed" multiplication rule. That is, for $r_1 \otimes g_1, r_2 \otimes g_2 \in R \otimes_{\mathbb{C}} \mathbb{C}G$,

$$(r_1 \otimes g_1)(r_2 \otimes g_2) := (r_1 g_1(r_2)) \otimes g_1 g_2.$$

Remark 2.8.3. The definition of the skew-group algebra becomes more intuitive through a few observations.

(1) Observe that there is an algebra homomorphism $\alpha\colon R\#G \to \operatorname{End}_{R^G}(R)$ sending $r_1 \otimes g_1 \in R\#G$ to the endomorphism $\mu_{r_1 \otimes g_1}\colon R \to R$ defined by the rule $\mu_{r_1 \otimes g_1}(r) = r_1 g_1(r)$ for all $r \in R$. The "skew" multiplication in $R\#G$ is defined so that α is a homomorphism.

(2) Note that $R\#G$-modules are precisely the R-modules with a G-equivariant action. From this observation, it follows that

$$\mathrm{D^b}(\operatorname{mod} R\#G) \cong \mathrm{D}^{\mathrm{b}}_G(\operatorname{mod} R) \cong \mathrm{D}^{\mathrm{b}}_G(\operatorname{coh} \operatorname{Spec} R)$$

where $\mathrm{D}^{\mathrm{b}}_G(\operatorname{mod} R)$ and $\mathrm{D}^{\mathrm{b}}_G(\operatorname{coh} \operatorname{Spec} R)$ denote the bounded derived categories of G-equivariant finite R-modules and G-equivariant coherent sheaves on $\operatorname{Spec} R$, respectively.

(3) An $R\#G$-module is projective if and only if it is projective as an R-module. This follows from two observations. First, as an R-module, $R\#G = R \otimes_{\mathbb{C}} \mathbb{C}G$ is a finite direct sum of R. Hence, if $M \in \operatorname{add}_{R\#G} R\#G$, then $M \in \operatorname{add}_R R$.

For the converse, observe that since the fixed point functor is exact, then, as in [LW12, 5.6], it is not hard to show that for any $M, N \in \operatorname{Mod} R\#G$

$$\operatorname{Ext}^i_{R\#G}(M, N) \cong (\operatorname{Ext}^i_R(M, N))^G$$

for all i. Hence, any $M \in \operatorname{Mod} R\#G$ such that $M \in \operatorname{add}_R R$ is a projective $R\#G$-module.

(4) From (3), since $R = \mathbb{C}[[x_1, \dots, x_n]]$ has global dimension n, it follows that $R\#G$ also has global dimension n.

To compute examples, we are interested in presenting $R\#G$ as a quiver with relations. This can be done as follows.

Definition 2.8.4 ([McK80])**.** Consider a finite group $G \subset \mathrm{GL}(n, \mathbb{C})$ and let ρ_V be the natural n-dimensional representation of G. Moreover, let $\{\rho_0, \rho_1, \cdots, \rho_m\}$ be a complete set of irreducible $\mathbb{C}$-representations of G. Recall that the *McKay quiver* of (G, ρ_V) has vertices ρ_i for $0 \leqslant i \leqslant k$. There are m_{ij} arrows $\rho_i \to \rho_j$, where m_{ij} is the multiplicity of ρ_i in the decomposition of $\rho_V \otimes \rho_j$ as a sum of irreducible representations.

When G is a finite abelian group, then the McKay quiver can be described in an alternate way.

Proposition 2.8.5 (See e.g. [BSW10])**.** *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite abelian group and let ρ_V be an n-dimensional representation of G as a subgroup of $\mathrm{GL}(n, \mathbb{C})$. Moreover, let $\{\rho_0, \rho_1, \cdots, \rho_m\}$ be a complete set of irreducible $\mathbb{C}$-representations of G. Then, it is possible to choose a basis of V that diagonalises the action of G. Let σ_i be the one-dimensional representation of G defined by setting $\sigma_i(g)$ to be the i-th diagonal element of $g \in G$. Notice that $\rho_V = \bigoplus_{i=1}^n \sigma_i$. Hence, the McKay quiver can be described as the directed graph with vertices ρ_i and an arrow*

$$\sigma_i \otimes \rho_k \xrightarrow{x_{\sigma_i}^{\rho_k}} \rho_k.$$

for all $0 \leqslant i \leqslant n$ and $0 \leqslant k \leqslant m$.

In what follows, we will be interested in the case when G is a *small group*. That is, no element in G fixes a hyperplane in $\operatorname{Spec} R$. It is easy to check that any finite group $G \subset \mathrm{SL}(n, \mathbb{C})$ is small see e.g. [IT13, §3].

Theorem 2.8.6 ([IT13; BSW10; CMT07; LW12])**.** *Let G and R be as in 2.8.1. Assume further that G is small. Then, the following hold.*

(1) *The homomorphism $\alpha \colon R\#G \to \mathrm{End}_{R^G}(R)$ of 2.8.3 is an isomorphism.*

(2) *$R\#G$ can be presented, up to a Morita equivalence, as the completion of the path algebra of the McKay quiver, subject to relations given by a higher superpotential.*

(3) *Suppose further that G is abelian. Then, the relations in (2) can be written as*

$$\langle \{ x_{\sigma_j}^{\rho_k} x_{\sigma_i}^{\sigma_j \otimes \rho_k} = x_{\sigma_i}^{\rho_k} x_{\sigma_i}^{\sigma_i \otimes \rho_k} \mid \forall\, 1 \leqslant k \leqslant m, 1 \leqslant i, j \leqslant n \} \rangle.$$

Proof. Statement (1) is [LW12, 5.15] (see also [IT13, 4.2]). Statements (2) and (3) are [BSW10, 3.2] and [BSW10, 4.1], respectively. For statement (3), see also [CMT07]. □

Example 2.8.7. Let ε_3 be a primitive third root of unity, and let $G \subset \mathrm{SL}(3, \mathbb{C})$ be the group $G = \langle g \rangle$ where

$$g = \begin{pmatrix} \varepsilon_3 & 0 & 0 \\ 0 & \varepsilon_3 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}.$$

Then, G acts on $R = \mathbb{C}[[x, y, z]]$ as $g(f(x, y, z)) = f(\varepsilon_3 x, \varepsilon_3 y, \varepsilon_3 z)$.

The McKay quiver of G is

$$Q_2 : \quad \begin{tikzcd} & 0 \arrow[dr, shift left=2, "x_0"] \arrow[dr, "y_0"] \arrow[dr, shift right=2, "z_0"'] & \\ 2 \arrow[ur, shift left=2, "x_2"] \arrow[ur, "y_2"] \arrow[ur, shift right=2, "z_2"'] & & 1. \arrow[ll, "z_1"] \arrow[ll, shift left=2, "y_1"] \arrow[ll, shift left=4, "x_1"] \end{tikzcd}$$

The skew group algebra $R\#G \cong \mathrm{End}_{R^G}(R)$ can be presented as the complete path algebra A of Q_2 subject to the relations

$$R_1 = \langle\langle x_i y_{i+1} - y_i x_{i+1}, x_i z_{i+1} - z_i x_{i+1}, y_i z_{i+1} - z_i y_{i+1} \rangle\rangle_{i \in \mathbb{Z}_3}.$$

Chapter 3

Twist around ring morphisms

In this section, unless mentioned otherwise, let $p\colon A \to B$ be a ring morphism, and assume throughout that B is perfect as an A-module. Then, there is a triangle

$$\operatorname{cone}(p)[-1] \xrightarrow{i} A \xrightarrow{p} B \to^{+} \tag{3.0.A}$$

of A-bimodules.

The morphism $p\colon A \to B$ induces the restriction-extension of scalars adjoint pairs (F^{LA}, F), (F, F^{RA}) on the derived category, as highlighted below.

$$\begin{tikzcd} \mathrm{D}(B) \arrow[rr, "{F=-\otimes^{\mathbf{L}}{}_B B_A \cong \mathbf{R}\mathrm{Hom}_B({}_A B_B,-)}" description] & & \mathrm{D}(A) \arrow[ll, bend right, "{F^{\mathrm{LA}}=-\otimes^{\mathbf{L}}{}_A B_B}"'] \arrow[ll, bend left, "{F^{\mathrm{RA}}=\mathbf{R}\mathrm{Hom}_A({}_B B_A,-)}"] \end{tikzcd} \tag{3.0.B}$$

In what follows, we will characterise the twist and cotwist around the restriction of scalars functor F, motivated by applications to geometric settings induced by tilting bundles and crepant contractions.

§ 3.1 The twist

The goal of this section is to show that

$$\mathcal{T} := \mathbf{R}\mathrm{Hom}_A({}_A \operatorname{cone}(p)[-1]_A, -)\colon\ \mathrm{D}(A) \to \mathrm{D}(A)$$

is the twist around F in the sense of 2.4.10.

Recall there is a functor

$$\mathrm{Hom}^*_A(-,-)\colon\ \mathrm{K}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{K}(A) \to \mathrm{K}(B)$$

which sends complexes $L \in \mathrm{K}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$ and $M \in \mathrm{K}(A)$ to the complex $\mathrm{Hom}^*_A(L, M)$ which, at degree k, is given by the B-module

$$\mathrm{Hom}^*_A(L, M)^k = \prod_{p+q=k} \mathrm{Hom}_A(L^{-p}, M^q).$$

The differential is defined by the rule

$$d^k(f) = d_M \cdot f + (-1)^{k+1} f \cdot d_L$$

for all $f \in \mathrm{Hom}_A^*(L, M)^k$. Similarly, there is a functor

$$\mathrm{Tot}(- \otimes_B -)\colon \mathrm{K}(B) \times \mathrm{K}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}}) \to \mathrm{K}(A)$$

sending complexes $L \in \mathrm{K}(B)$ and $M \in \mathrm{K}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$ to the complex $\mathrm{Tot}(L \otimes_B M)$ which, at degree k, is given by the A-module

$$\mathrm{Tot}(L \otimes_B M)^k = \bigoplus_{p+q=k} L^p \otimes_B M^q.$$

with differential at k defined as

$$d^k = \sum_{p+q=k} d_L^p \otimes 1_{M^q} + (-1)^p 1_{L^p} \otimes d_M^q$$

Lemma 3.1.1. *The counit ε of the derived tensor-hom adjunction has component at $a \in \mathrm{D}(A)$*

$$\varepsilon_a\colon \mathbf{R}\mathrm{Hom}_A({}_B B_A, a) \otimes_B^{\mathbf{L}} B_A \to a$$

given as the composition $j_a^{-1} \cdot \gamma_a$, where $j_a\colon a \to I$ is a K-injective resolution of a and γ_a is the chain map

$$\gamma_a = (\gamma_a^j)\colon \mathrm{Tot}(\mathrm{Hom}_A^*({}_B B_A, I) \otimes_B B_A) \to I$$

which at degree j is specified by the evaluation map

$$\gamma_a^j\colon \mathrm{Hom}_A({}_B B_A, I^j) \otimes_B B_A \to I^j$$

sending $f \otimes x \mapsto f(x)$.

Proof. This lemma is a straightforward diagram chase of the identity along the isomorphism

$$\mathrm{Hom}_{\mathrm{D}(B)}(\mathbf{R}\mathrm{Hom}_A(B,-), \mathbf{R}\mathrm{Hom}_A(B,-)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(A)}(\mathbf{R}\mathrm{Hom}_A(B,-) \otimes_B^{\mathbf{L}} B, -)$$

which defines the derived tensor-hom adjunction. As we could not find a proof in the literature, we present one in appendix A. □

Lemma 3.1.2. *There is a commutative diagram*

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_A({}_A B, -) \arrow[r, "\mathbf{R}\mathrm{Hom}_A(p,-)"] \arrow[d, "\wr"] & \mathbf{R}\mathrm{Hom}_A({}_A A, -) \arrow[d, "\wr"] \\
\mathbf{R}\mathrm{Hom}_A({}_B B, -) \otimes_B^{\mathbf{L}} B_A \arrow[r, "\varepsilon"] & 1_{\mathrm{D}(A)}
\end{tikzcd}$$

of endofunctors on $\mathrm{D}(A)$.

Proof. To prove the lemma, it suffices to construct natural isomorphisms

$$\begin{aligned} &m\colon \mathbf{R}\mathrm{Hom}_A({}_BB,-)\otimes^{\mathbf{L}}_B B_A \to \mathbf{R}\mathrm{Hom}_A({}_AB,-)\\ &n\colon \mathbf{R}\mathrm{Hom}_A({}_AA,-)\to 1_{\mathrm{D}(A)}\end{aligned}$$

such that the diagram

$$\begin{tikzcd} \mathbf{R}\mathrm{Hom}_A({}_AB,a) \arrow[r, "\mathbf{R}\mathrm{Hom}_A(p,a)"] \arrow[d, "m_a^{-1}"', "\wr"] & \mathbf{R}\mathrm{Hom}_A({}_AA,a) \arrow[d, "n_a"', "\wr"] \\ \mathbf{R}\mathrm{Hom}_A({}_BB,a)\otimes^{\mathbf{L}}_B B_A \arrow[r, "\varepsilon_a"] & a \end{tikzcd}$$

commutes for all $a\in\mathrm{D}(A)$. For each such a, fix a K-injective resolution $j_a\colon a\to I$ of a. Moreover, note that the functor $-\otimes_B B_A\colon B\to A$ is exact and so for any acyclic complex M in $\mathrm{D}(B)$, $\mathrm{Tot}(M\otimes_B B_A)$ is acyclic. Thus, B is K-flat in $\mathrm{D}(B^{\mathrm{op}})$. Hence, by the definition of the derived hom and tensor functors,

$$\begin{aligned} &(\mathbf{R}\mathrm{Hom}_A({}_BB,a)\otimes^{\mathbf{L}}_B B_A)^j = \mathrm{Hom}_A({}_BB,I^j)\otimes_B B_A,\\ &\mathbf{R}\mathrm{Hom}_A({}_AB,a)^j = \mathrm{Hom}_A({}_AB,I^j),\\ &\mathbf{R}\mathrm{Hom}_A({}_AA,a)^j = \mathrm{Hom}_A({}_AA,I^j),\end{aligned}$$

Let $m_a=(m_a^j)\colon \mathbf{R}\mathrm{Hom}_A({}_BB,a)\otimes^{\mathbf{L}}_B B_A\to\mathbf{R}\mathrm{Hom}_A({}_AB,a)$ be the chain map which, at degree j, is the natural multiplication isomorphism

$$m_a^j\colon\ \mathrm{Hom}_A({}_BB,I^j)\otimes_B B_A\to\mathrm{Hom}_A({}_BB,I^j).$$

Furthermore, set $n_a\colon\mathbf{R}\mathrm{Hom}_A({}_AA,a)\to a$ as the composition of the quasi-isomorphism j_a^{-1} with the chain isomorphism

$$\mathrm{ev}_1=(\mathrm{ev}_1^j)\colon\mathbf{R}\mathrm{Hom}_A({}_AA,a)\xrightarrow{\sim} I$$

where

$$\mathrm{ev}_1^j\colon\ \mathrm{Hom}_A({}_AA,I^j)\xrightarrow{\sim} I^j\colon f\mapsto f(1).$$

It is easy to check by diagram chasing that both $m=(m_a)_{a\in\mathrm{D}(A)}$ and $n=(n_a)_{a\in\mathrm{D}(A)}$ define natural transformations. We claim that for every $a\in\mathrm{D}(A)$, the diagram

$$\begin{tikzcd} \mathrm{Hom}^*_A({}_AB,I) \arrow[r, "\mathbf{R}\mathrm{Hom}_A(p,a)"] \arrow[d, "m_a^{-1}"', "\wr"] & \mathrm{Hom}^*_A({}_AA,I) \arrow[d, "n_a"', "\wr"] \\ \mathrm{Tot}(\mathrm{Hom}^*_A({}_BB,I)\otimes_B B_A) \arrow[r, "\varepsilon_a"] & a \end{tikzcd}$$

commutes. From 3.1.1, $\varepsilon_a = j_a^{-1} \cdot \gamma_a$. Hence, the above diagram commutes if and only the outer diagram in the following

$$\begin{tikzcd}
\mathrm{Hom}^*_A({}_AB, I) \arrow[rr, "\mathbf{R}\mathrm{Hom}_A(p,a)"] \arrow[d, "m_a^{-1}"', "\wr"] & & \mathrm{Hom}^*_A({}_AA, I) \arrow[d, "\mathrm{ev}_1"', "\wr"] \\
\mathrm{Tot}(\mathrm{Hom}^*_A({}_BB, I) \otimes_B B_A) \arrow[rr, dashed, "\gamma_a"] \arrow[d, "\gamma_a"] & & I \arrow[d, "j_a^{-1}"] \\
I \arrow[rr, "j_a^{-1}"] & & a
\end{tikzcd}$$

commutes. To see that this holds, consider first the composition $\gamma_a \cdot m_a^{-1}$. We may write this as a chain map which at degree j is

$$\gamma_a^j \cdot (m_a^{-1})^j \colon \mathrm{Hom}_A({}_AB, I^j) \to I^j$$

where, for $f \in \mathrm{Hom}_A({}_AB, I^j)$,

$$\gamma_a^j \cdot (m_a^{-1})^j(f) = \gamma_a^j(f \otimes 1) = f(1)$$

Next, consider the composition $\mathrm{ev}_1 \cdot \mathbf{R}\mathrm{Hom}_A(p, a)$. At degree j,

$$\mathrm{ev}_1^j \cdot \mathbf{R}\mathrm{Hom}_A(p, a)^j(f) = \mathrm{ev}_1^j(f \cdot p) = f(p(1)) = f(1).$$

Thus, since $(\gamma_a \cdot m_a^{-1})^j = (\mathrm{ev}_1 \cdot \mathbf{R}\mathrm{Hom}_A(p, a))^j$ for all j, it follows that the diagram commutes. $\square$

Corollary 3.1.3. *The functor*

$$\mathcal{T} \cong \mathbf{R}\mathrm{Hom}_A({}_A\,\mathrm{cone}(p)[-1]_A, -) \colon \mathrm{D}(A) \to \mathrm{D}(A)$$

is the twist around F.

Proof. Applying $\mathbf{R}\mathrm{Hom}_A(-, a)$ to the triangle (3.0.A) for each $a \in \mathrm{D}(A)$ induces the functorial triangle

$$\mathbf{R}\mathrm{Hom}_A({}_AB, -) \xrightarrow{f} \mathbf{R}\mathrm{Hom}_A({}_AA, -) \xrightarrow{g} \mathbf{R}\mathrm{Hom}_A(\mathrm{cone}(p)[-1], -) \to^+$$

where $f = \mathbf{R}\mathrm{Hom}_A(p, -)$ and $g = \mathbf{R}\mathrm{Hom}_A(i, -)$.

Given 3.1.2, there is commutative diagram

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_A(B, -) \arrow[r, "f"] \arrow[d, "\wr"] & \mathbf{R}\mathrm{Hom}_A(A, -) \arrow[r, "g"] \arrow[d, "\wr"] & \mathbf{R}\mathrm{Hom}_A(\mathrm{cone}(p)[-1], -) \arrow[r] \arrow[d, equal] & + \\
\mathbf{R}\mathrm{Hom}_A(B, -) \otimes_B^{\mathbf{L}} B \arrow[r, "\varepsilon"] & 1_{\mathrm{D}(A)} \arrow[r, "u"] & \mathbf{R}\mathrm{Hom}_A(\mathrm{cone}(p)[-1], -) \arrow[r] & +
\end{tikzcd}$$

where u is defined so that the diagram commutes. Hence, since the first row is exact, so is the bottom.

Recall from 2.4.16 that the twist around F is $\mathfrak{T} = - \otimes^{\mathbf{L}}_A T$ for a specific complex T of A-bimodules. Since ε is the counit, and the vertical morphisms in the diagram (3.1) are functorial, then $T \cong \mathbf{R}\mathrm{Hom}_A(\mathrm{cone}(p)[-1], A)$ as complexes of bimodules. Moreover, since $B \in \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\, A)$, then

$$\mathfrak{T} = - \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(\mathrm{cone}(p)[-1], A) \cong \mathbf{R}\mathrm{Hom}_A(\mathrm{cone}(p)[-1], -)$$

by [IR08, 2.10(1)]. □

Corollary 3.1.4. *Suppose further that $p\colon A \to B$ is a surjection. Then, the twist around F is*

$$\mathfrak{T} := \mathbf{R}\mathrm{Hom}_A({}_A \ker p_A, -)\colon \mathrm{D}(A) \to \mathrm{D}(A)$$

Proof. If p is a surjection, then the short exact sequence

$$0 \to \ker p \to A \to B \to 0$$

induces a triangle

$$\ker p \to A \to B \to^{+}$$

so that we may conclude that $\ker p = \mathrm{cone}(p)[-1]$. □

§ 3.2 The cotwist

The goal of this section is to compute the cotwist around F. This requires a handful of lemmas.

First, let $Q \to B$ be a K-projective resolution of $B \in \mathrm{D}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$. Then, it is straightforward to check that the quasi-isomorphism $Q \to B$ induces a quasi-isomorphism

$$s\colon {}_B B \otimes^{\mathbf{L}}_A B_B \to \mathrm{Tot}({}_B Q \otimes_A B_B).$$

Lemma 3.2.1. *There are functorial isomorphisms*

$$\nu\colon \mathbf{R}\mathrm{Hom}_B({}_B B_B, -) \to 1_{\mathrm{D}(B)}, \tag{3.2.A}$$

$$\mathbf{R}\mathrm{Hom}_B(s, -)\colon \mathbf{R}\mathrm{Hom}_B(\mathrm{Tot}({}_B Q \otimes_A B_B), -) \to \mathbf{R}\mathrm{Hom}_B({}_B B \otimes^{\mathbf{L}}_A B_B, -). \tag{3.2.B}$$

Proof. Let $q\colon Q \to B$ be a K-projective resolution of $B \in \mathrm{D}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$ and fix a complex $c \in \mathrm{D}(B)$.

The construction of ν in (3.2.A) is essentially the same as the construction of n in 3.1.2. To construct the quasi-isomorphism in (3.2.B), observe that the quasi-isomorphism

$$s\colon {}_B B \otimes^{\mathbf{L}}_A B_B \to \mathrm{Tot}({}_B Q \otimes_A B_B)$$

induces a quasi-isomorphism

$$\mathbf{R}\mathrm{Hom}_B(s, c)\colon \mathbf{R}\mathrm{Hom}_B(\mathrm{Tot}({}_B Q \otimes_A B_B), c) \to \mathbf{R}\mathrm{Hom}_B({}_B B \otimes^{\mathbf{L}}_A B_B, c)$$

which is natural in c. □

Lemma 3.2.2. *Let $\mathcal{I}$ denote the subcategory of* $\mathrm{D}(B)$ *consisting of K-injective complexes. Then, in $\mathcal{I}$, there are quasi-isomorphisms*

$$f\colon \operatorname{Hom}^*_B(\operatorname{Tot}({}_BQ \otimes_A B_B), -) \to \operatorname{Hom}^*_A({}_BQ_A, \operatorname{Tot}(- \otimes_B B_A)), \tag{3.2.C}$$

$$g\colon \operatorname{Hom}^*_A({}_BQ_A, \operatorname{Tot}(- \otimes_B Q_A)) \to \operatorname{Hom}^*_A({}_BQ_A, \operatorname{Tot}(- \otimes_B B_A)). \tag{3.2.D}$$

Proof. Let I be a K-injective complex in $\mathrm{D}(B)$ and let $q\colon Q \to B$ be a K-projective resolution of $B \in \mathrm{D}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$.

To prove (3.2.C), define the chain map

$$f_I = (f_I^j)\colon \operatorname{Hom}^*_B(\operatorname{Tot}({}_BQ \otimes_A B_B), I) \to \operatorname{Hom}^*_A(Q, \operatorname{Tot}(I \otimes_B B_A)).$$

where

$$f_I^j\colon \prod_{p+q=j} \operatorname{Hom}_B(Q^{-p} \otimes_A B_B, I^q) \to \prod_{p+q=j} \operatorname{Hom}_A(Q^{-p}, I^q \otimes_B B_A)$$

is such that, given a tuple $(\alpha^{p,q}) \in \prod_{p+q=j} \operatorname{Hom}_B(Q^{-p} \otimes_A B_B, I^q)$,

$$f_I^j(\alpha^{p,q})\colon x \mapsto \alpha^{p,q}(x \otimes_A 1) \otimes_B 1.$$

The morphism f_I^j is an isomorphism because it is a product of the composition of isomorphisms

$$v_I^{p,q}\colon \operatorname{Hom}_B(Q^{-p} \otimes_A B_B, I^q) \to \operatorname{Hom}_A(Q^{-p}, \operatorname{Hom}_B({}_AB_B, I^q))$$
$$w_I^{p,q}\colon (Q^{-p}, \operatorname{Hom}_B({}_AB_B, I^q)) \to (Q^{-p}, I^q \otimes_B B_A)$$

where $(v_I^{p,q}(\alpha^{p,q}))(x)(y) = \alpha^{p,q}(x \otimes y)$ and $(w + I^{p,q}(\beta^{p,q}))(x) = \beta^{p,q}(x)(1) \otimes_B 1$. It is not hard to check that f defined this way commutes with the differentials, so that it is indeed a chain map.

To prove (3.2.D), note that the quasi-isomorphism $q\colon Q \to {}_BB_A$ induces the quasi-isomorphism

$$g\colon \operatorname{Hom}^*_A({}_BQ_A, \operatorname{Tot}(- \otimes_B Q_A)) \to \operatorname{Hom}^*_A({}_BQ_A, \operatorname{Tot}(- \otimes_B B_A)).$$ □

To compute the cotwist, it will be helpful to know an explicit formulation of the unit of the derived tensor-hom adjunction.

Lemma 3.2.3. *Let $c \in \mathrm{D}(B)$ and $p_c\colon P \to c$ be a K-projective resolution of c. Moreover, let $q\colon Q \to B$ be a K-projective resolution of $B \in \mathrm{D}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$. Then, under the quasi-isomorphism*

$$h\colon \mathbf{R}\operatorname{Hom}_A({}_BB_A, c \otimes^{\mathbf{L}}_B B_A) \xrightarrow{\sim} \operatorname{Hom}^*_A({}_BQ_A, \operatorname{Tot}(P \otimes_B Q_A)),$$

the unit η of the derived tensor-hom adjunction has component $\eta_c = \lambda_c \cdot j_c$ where λ_c is *the chain map*

$$\lambda_c = (\lambda_c^j)\colon P \to \mathrm{Hom}_A^*({}_BQ_A, \mathrm{Tot}(P \otimes_B Q_A))$$

where

$$\lambda_c^j\colon P^j \to \prod_{p+q=j} \mathrm{Hom}_A(Q^{-p}, \bigoplus_{r+s=q} P^r \otimes_B Q^s)$$

sends $x \mapsto x \otimes - \in \mathrm{Hom}_A(Q^{-p}, P^j \otimes_B Q^{-p})$.

Proof. Similar to 3.1.1, this lemma is proved by diagram chasing the identity id $\in \mathrm{Hom}_{\mathrm{D}(A)}(c \otimes_B^{\mathbf{L}} B_A, c \otimes_B^{\mathbf{L}} B_A)$ across the standard isomorphism defining the tensor-hom adjunction. $\square$

Remark 3.2.4. Note that we may take the resolution $q\colon Q \to {}_BB_A$ to be a chain map (q^j) which vanishes in all degrees but zero. In which case, there is a chain map

$$\beta = (\beta^j)\colon \mathrm{Tot}({}_BQ \otimes_A B_B) \to {}_BB_B$$

where $\beta^j = 0$ for $j \neq 0$ and β^0 is induced by composing $q^0 \otimes_A 1_B\colon Q^0 \otimes_A B_B \to {}_BB \otimes_A B_B$ with the multiplication isomorphism $m\colon {}_BB \otimes_A B_B \to {}_BB_B$.

Since there is a quasi-isomorphism $s\colon {}_BB \otimes_A^{\mathbf{L}} B_B \to \mathrm{Tot}(Q \otimes_A B_A)$, it thus follows from 3.2.4 that there is a map

$$s \cdot \beta\colon {}_BB \otimes_A^{\mathbf{L}} B_B \to {}_BB_B.$$

Lemma 3.2.5. *There is a commutative diagram*

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_B({}_BB_B, -) \arrow[r, "\mathbf{R}\mathrm{Hom}_B(s\cdot\beta, -)"] \arrow[d, "\wr"] & \mathbf{R}\mathrm{Hom}_B({}_BB \otimes_A^{\mathbf{L}} B_B, -) \arrow[d, "\wr"] \\
1_{\mathrm{D}(B)} \arrow[r, "\eta"] & \mathbf{R}\mathrm{Hom}_A({}_BB_A, - \otimes_B^{\mathbf{L}} B_A)
\end{tikzcd}$$

Proof. Let $c \in \mathrm{D}(B)$. Fix a K-injective resolution $j_c\colon c \to I$ and a K-projective resolution $p_c\colon P \to c$. We claim that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_B({}_BB_B, c) \arrow[r, "\mathbf{R}\mathrm{Hom}_B(s\cdot\beta, c)"] \arrow[d, equal] & \mathbf{R}\mathrm{Hom}_B({}_BB \otimes^{\mathbf{L}}_A B_B, c) \arrow[d, "\mathbf{R}\mathrm{Hom}_B(s,c)^{-1}"] \\
\mathbf{R}\mathrm{Hom}_B({}_BB_B, c) \arrow[r, "\mathbf{R}\mathrm{Hom}_B(\beta, c)"] \arrow[d, equal] & \mathbf{R}\mathrm{Hom}_B(\mathrm{Tot}({}_BQ \otimes_A B_B), c) \arrow[d, equal] \\
\mathrm{Hom}^*_B({}_BB_B, I) \arrow[r, "\mathrm{Hom}^*_B(\beta, I)"] \arrow[d, "\mathrm{ev}_1"', "\wr"] & \mathrm{Hom}^*_B(\mathrm{Tot}({}_BQ \otimes_A B_B), I) \arrow[d, equal] \\
I \arrow[d, "p_c\cdot j_c"', "\wr"] & \mathrm{Hom}^*_B(\mathrm{Tot}({}_BQ \otimes_A B_B), I) \arrow[d, equal] \\
P \arrow[d, "\lambda_c"'] & \mathrm{Hom}^*_B(\mathrm{Tot}({}_BQ \otimes_A B_B), I) \arrow[d, "f"] \\
\mathrm{Hom}^*_A({}_BQ_A, \mathrm{Tot}(P \otimes_B Q_A)) \arrow[d, "\mathrm{Hom}^*_A({}_BQ_A, \mathrm{Tot}(p_c\cdot j_c \otimes_B Q_A)^{-1})"', "\wr"] & \mathrm{Hom}^*_A({}_BQ_A, \mathrm{Tot}(I \otimes_B B_A)) \arrow[d, equal] \\
\mathrm{Hom}^*_A({}_BQ_A, \mathrm{Tot}(I \otimes_B Q_A)) \arrow[d, "\mathrm{Hom}^*_A({}_BQ_A, \mathrm{Tot}(I \otimes_B q))"', "\wr"] & \mathrm{Hom}^*_A({}_BQ_A, \mathrm{Tot}(I \otimes_B B_A)) \arrow[d, equal] \\
\mathrm{Hom}^*_A({}_BQ_A, \mathrm{Tot}(I \otimes_B B_A)) \arrow[r, equal] & \mathrm{Hom}^*_A({}_BQ_A, \mathrm{Tot}(I \otimes_B B_A))
\end{tikzcd}$$

It is clear that the top two squares commute. Let $\alpha \in \mathrm{Hom}^*_A({}_BB_B, I)$ be a degree j map. Then, $\alpha = \alpha^j \in \mathrm{Hom}_A(B, I^j)$. Let us first chase α along the left-hand column. Well, $\mathrm{ev}_1(\alpha^j) = \alpha^j(1) \in I^j$. Hence, $p_c \cdot i_c(\alpha^j(1)) \in P^j$ so that

$$\lambda_c \cdot p_c \cdot i_c(\alpha^j(1)) = \lambda_c^j \cdot p_c \cdot i_c(\alpha^j(1)) = p_c \cdot i_c(\alpha^j(1)) \otimes (-) \in \prod_p \mathrm{Hom}_A(Q^{-p}, P^j \otimes_B Q^{-p}).$$

Note that we may rewrite this map as

$$\mathrm{Tot}(p_c \cdot i_c \otimes_B Q_A)(\alpha^j(1) \otimes (-))$$

Hence,

$$\Big(\mathrm{Hom}^*_A({}_BQ_A, \mathrm{Tot}(p_c \cdot i_c \otimes_B Q_A)^{-1}\Big)(p_c \cdot i_c(\alpha^j(1)) \otimes (-)) = \alpha^j(1) \otimes (-).$$

Finally,

$$\Big(\mathrm{Hom}^*_A({}_BQ_A, \mathrm{Tot}(I \otimes_B q))\Big)(\alpha^j(1) \otimes (-)) = \prod_p \alpha^j(1) \otimes_B q^{-p}(-) = \alpha^j(1) \otimes_B q^0(-)$$

because q^j vanish for all $j \neq 0$ by the choice of q in 3.2.4. Moreover, Since $q_0(x) \in B$ for all $x \in Q^0$ and α^j is a right B-module homomorphism,

$$\alpha^j(1) \otimes_B q^0(-) = \alpha^j(q^0(-)) \otimes_B 1.$$

Next, let us chase $\alpha = \alpha^j$ along the right-hand column. Well,

$$\mathrm{Hom}^*_B(\beta, I)(\alpha^j) \in \mathrm{Hom}_B(Q^0 \otimes B, I^j)$$

is such that

$$\mathrm{Hom}^*_B(\beta, I)(\alpha^j)(x \otimes y) = \alpha^j \cdot \beta(x \otimes y) = \alpha^j \cdot m \cdot (q^0 \otimes 1)(x \otimes y) = \alpha^j(q^0(x)y) = \alpha^j \cdot q^0(xy).$$

Moreover,

$$f(\alpha^j \cdot \beta)(x) = \alpha^j \cdot \beta(x \otimes_A 1) \otimes_B 1 = \alpha^j(q^0(x)) \otimes_B 1.$$

Thus, the diagram commutes. By composing each column with the functorial isomorphisms

$$\mathrm{Hom}^*_A({}_BQ_A, \mathrm{Tot}(I \otimes_B B_A)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_A({}_BB_A, I \otimes^{\mathbf{L}}_B B_A) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_A({}_BB_A, c \otimes^{\mathbf{L}}_B B_A)$$

It follows that the diagram in the statement commutes. $\square$

Corollary 3.2.6. *Let $C(s \cdot \beta)$ denote the cone of the morphism $s \cdot \beta\colon {}_BB \otimes^{\mathbf{L}}_A B_B \to {}_BB_B$ in $\mathrm{D}(B \otimes_{\mathbb{Z}} B^{\mathrm{op}})$. Then, the functor*

$$\mathcal{C} \cong \mathbf{R}\mathrm{Hom}_B({}_BC(s \cdot \beta)_B, -)$$

is the cotwist around F.

Proof. For each $c \in \mathrm{D}(B)$, applying $\mathbf{R}\mathrm{Hom}_B(-, c)$ to the triangle

$${}_BB \otimes^{\mathbf{L}}_A B_B \xrightarrow{s \cdot \beta} {}_BB_B \to C(s \cdot \beta) \to^+$$

induces first exact row in the diagram

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_B(C(s \cdot \beta, c), c) \arrow[r] \arrow[d, equal] & \mathbf{R}\mathrm{Hom}_B(B, c) \arrow[r, "\mathbf{R}\mathrm{Hom}_B(s \cdot \beta, c)"] \arrow[d, "\wr"] & \mathbf{R}\mathrm{Hom}_B(B \otimes^{\mathbf{L}}_A B, c) \arrow[r] \arrow[d, "\wr"] & {}^+ \\
\mathbf{R}\mathrm{Hom}_B(C(s \cdot \beta, c)_B, c) \arrow[r, "v"] & 1_{\mathrm{D}(B)} \arrow[r, "\eta"] & \mathbf{R}\mathrm{Hom}_A(B, - \otimes^{\mathbf{L}}_B B) \arrow[r] & {}^+
\end{tikzcd}$$

where the morphism v is defined so that the left-hand square commutes. The right-hand square commutes by 3.2.5. Hence, the bottom row is also exact.

Recall from 2.4.16 that the cotwist around F is $\mathcal{C} = - \otimes^{\mathbf{L}}_A C[-1]$ for a specific complex C of B-bimodules. Since η is the unit, and the vertical morphisms in the diagram (3.2) are functorial, then $C[-1] \cong \mathbf{R}\mathrm{Hom}_B({}_BC(s \cdot \beta)_B, B)$ as complexes of bimodules. Since $B \in \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\, A)$, then

$$\mathcal{C} = - \otimes^{\mathbf{L}}_B \mathbf{R}\mathrm{Hom}_B({}_BC(s \cdot \beta)_B, B) \cong \mathbf{R}\mathrm{Hom}_B({}_BC(s \cdot \beta)_B, -)$$

by [IR08, 2.10(1)]. $\square$

Proposition 3.2.7. *Fix $t \in \mathbb{Z}$ with $t \notin \{1, -1\}$. Suppose that $\mathrm{H}^k({}_B B \otimes^{\mathbf{L}}_A B_B) = 0$ for all $k \neq 0, t$. Then,*

$$\mathcal{C} = \mathbf{R}\mathrm{Hom}_B({}_B \operatorname{Tor}^A_{-t}(B, B)_B, -)[t-1]$$

Proof. The triangle of B-bimodules

$${}_B B \otimes^{\mathbf{L}}_A B_B \xrightarrow{s \cdot \beta} {}_B B_B \to {}_B C(s \cdot \beta)_B \to^{+}$$

induces the long exact sequence

$$\ldots \to \mathrm{H}^k(B \otimes^{\mathbf{L}}_A B) \xrightarrow{\mathrm{H}^k(s \cdot \beta)} \mathrm{H}^k(B) \to \mathrm{H}^k(C(s \cdot \beta)) \to \mathrm{H}^{k+1}(B \otimes^{\mathbf{L}}_A B) \to \ldots$$

on cohomology. Since B is concentrated in degree zero and $\mathrm{H}^k({}_B B \otimes^{\mathbf{L}}_A B_B)$ vanishes for all $k \neq 0, t$, it must be that $\mathrm{H}^k(C(s \cdot \beta)) = 0$ for all $k \neq -1, 0, t-1$. Moreover, note that $\mathrm{H}^0(s \cdot \beta)$ is an isomorphism. Hence,

$$\mathrm{H}^{-1}(C(s \cdot \beta)) = \mathrm{H}^0(C(s \cdot \beta)) = 0.$$

Therefore, $C(s \cdot \beta)$ is concentrated in degree $t-1$. It follows from this and the long exact cohomology sequence that

$$C(s \cdot \beta) \cong \mathrm{H}^{t-1}(C(s \cdot \beta))[t-1] \cong \mathrm{H}^t(B \otimes^{\mathbf{L}}_A B)[t-1].$$

Thus, the statement holds. $\square$

Chapter 4

Spherical twists for Gorenstein orders

In this chapter, we will consider a surjection $p\colon A \to B$ with additional natural assumptions on A and B under which we can prove that F is spherical.

§ 4.1 Setting

Motivated by geometric settings, we will work within the following setup.

Setup 4.1.1. Let $\mathcal{R}$ be a noetherian local Cohen-Macaulay (CM) ring of Krull dimension d with canonical module $\omega_{\mathcal{R}}$ and residue field k. Assume that A is a module finite $\mathcal{R}$-algebra, so the surjection $p\colon A \to B$ implies that B is also module finite over $\mathcal{R}$.

As the algebra A is module finite over $\mathcal{R}$, its derived category is equipped with the Nakayama functor defined in [IR08, §3], namely

$$- \otimes^{\mathbf{L}}_A \omega_A\colon \mathrm{D}^-(\operatorname{mod} A) \to \mathrm{D}^-(\operatorname{mod} A)$$

where $\omega_A := \mathbf{R}\mathrm{Hom}_{\mathcal{R}}(\Lambda, \omega_{\mathcal{R}})$ is a dualizing complex for $\mathrm{D}(\operatorname{mod} A)$ in the sense of [Yek92]. Moreover, [IR08, 3.5,3.7] proves a form of Serre duality by showing the existence of the following functorial isomorphisms.

(1) For any $a \in \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$ and $b \in \mathrm{K}^{\mathrm{b}}(\operatorname{proj} A)$

$$\mathbf{R}\mathrm{Hom}_A(a, b \otimes^{\mathbf{L}}_A \omega_A) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{R}}(\mathbf{R}\mathrm{Hom}_A(b, a), \omega_{\mathcal{R}}). \tag{4.1.A}$$

(2) For any $a \in \mathrm{D}^{\mathrm{b}}(\operatorname{fl} A)$ and $b \in \mathrm{K}^{\mathrm{b}}(\operatorname{proj} A)$

$$\mathrm{Hom}_{\mathrm{D}(\operatorname{mod} A)}(a, b \otimes^{\mathbf{L}}_A \omega_A[d]) \cong D\,\mathrm{Hom}_{\mathrm{D}(\operatorname{mod} A)}(b, a). \tag{4.1.B}$$

Remark 4.1.2. Although [IR08, §3] states the above results for Gorenstein rings, their proof applies word for word to any CM ring equipped with a canonical module once $\mathcal{R}$ is replaced by $\omega_{\mathcal{R}}$.

Recall that an algebra A is a *Gorenstein $\mathcal{R}$-order* if there is an A-bimodule isomorphism $\omega_A = \mathbf{R}\mathrm{Hom}_{\mathcal{R}}(A, \omega_{\mathcal{R}}) \cong A$. Moreover, an algebra is self-injective if it is injective as a module over itself (see § 2.1 for a brief overview on self-injective algebras, the Nakayama autoequivalence and the Nakayama permutation).

Setup 4.1.3. Under 4.1.1 assume in addition that A is a Gorenstein $\mathcal{R}$-order and that B is self-injective.

Remark 4.1.4. The key point of 4.1.3, is that since B is module finite over a noetherian ring and self-injective, 2.1.5 implies that B is finite dimensional.

Consider the functors

$$\begin{aligned}(-)^{\dagger} &:= \mathbf{R}\mathrm{Hom}_{\mathcal{R}}(-, \omega_{\mathcal{R}})\colon\ \mathrm{D}(\mathrm{mod}\,\mathcal{R}) \to \mathrm{D}(\mathrm{mod}\,\mathcal{R}),\\ D &:= \mathrm{Hom}_{\mathcal{R}}(-, k)\colon\ \mathrm{D}^{\mathrm{b}}(\mathrm{fl}\,\mathcal{R}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{fl}\,\mathcal{R}).\end{aligned}$$

Since $\omega_{\mathcal{R}}$ is a dualizing module, note that $(-)^{\dagger}$ induces a duality on $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\mathcal{R})$ so that $(-)^{\dagger\dagger} \cong 1_{\mathcal{R}}$.

Lemma 4.1.5. *Under setup 4.1.3, suppose that $B \in \mathrm{K}^{b}(\mathrm{proj}\,A)$. Then, there is a B-bimodule isomorphism*

$$_B \operatorname{Tor}^A_t(B, B)_B \cong {_B \operatorname{Ext}^{d-t}_A(DB, B)_B}$$

Proof. Consider the following isomorphisms,

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_A({_B B_A}^{\dagger}, {_B B_A}) &\cong \mathbf{R}\mathrm{Hom}_A({_B B_A}^{\dagger}, {_B B_A} \otimes^{\mathbf{L}}_A (\omega_A)_A) && \text{(by assumptions 4.1.3)}\\ &\cong \mathbf{R}\mathrm{Hom}_A({_B B_A}, {_B B_A}^{\dagger})^{\dagger} && \text{(by Serre duality (4.1.A))}\end{aligned}$$

Moreover, it follows from the derived tensor-hom adjunction that

$$\mathbf{R}\mathrm{Hom}_A({_B B_A}, {_B \mathbf{R}\mathrm{Hom}_{\mathcal{R}}(B, \omega_{\mathcal{R}})_A}) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{R}}({_B B_A} \otimes^{\mathbf{L}}_A B_B, \omega_{\mathcal{R}}).$$

Hence,

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_A({_B B_A}, {_B B_A}^{\dagger})^{\dagger} &\cong ({_B B_A} \otimes^{\mathbf{L}}_A B_B)^{\dagger\dagger}\\ &\cong {_B B_A} \otimes^{\mathbf{L}}_A B_B. && \text{(since } (-)^{\dagger} \text{ is a duality)}\end{aligned}$$

Thus, there is are bimodule isomorphisms

$$\begin{aligned}{_B \operatorname{Ext}^{-t}_A(B^{\dagger}, B)_B} &= \mathrm{H}^{-t}(\mathbf{R}\mathrm{Hom}_A({_B B^{\dagger}}_A, {_B B_A}))\\ &\cong \mathrm{H}^{-t}({_B B_A} \otimes^{\mathbf{L}}_A B_B)\\ &= {_B \operatorname{Tor}^A_t(B, B)_B}.\end{aligned}$$

Lemma [IR08, 3.6] states that there is a functorial isomorphism $(-)^\dagger \cong [-d] \cdot D$ on $\mathrm{D^b}(\mathrm{fl}\,\mathcal{R})$. Hence,

$$_B \operatorname{Tor}_t^A(B,B)_B \cong {}_B \operatorname{Ext}_A^{-t}(B^\dagger, B)_B \cong {}_B \operatorname{Ext}_A^{d-t}(DB, B)_B. \qquad \square$$

§ 4.2 Spherical twists for Gorenstein orders

Having introduced the setting of this chapter and its main properties in the previous section, we construct spherical twists for Gorenstein orders in this section.

Proposition 4.2.1. *Under setup 4.1.3, let $\mathcal{N}(-)$ denote the Nakayama autoequivalence of B. Suppose that $B \in \mathrm{K}^b(\operatorname{proj} A)$ and $\mathrm{H}^k({}_B B \otimes_A^{\mathbf{L}} B_B) = 0$ for all $k \neq 0, -d$. Then, the cotwist around F is $\mathcal{C} \cong [-d-1] \cdot \mathcal{N}$.*

Proof. Combining 3.2.7 with 4.1.5. necessarily

$$\mathcal{C} = \mathbf{R}\mathrm{Hom}_B(\mathrm{Hom}_A(DB, B), -).$$

Moreover, for self-injective algebras, the functor $\mathbf{R}\mathrm{Hom}_B(\mathrm{Hom}_A(DB, B), -)$ is naturally isomorphic to the Nakayama functor by [Iva11, 3.3]. We may thus conclude the statement. $\square$

Lemma 4.2.2. *Under setup 4.1.3, suppose that $\dim \mathcal{R} = 3$ and $B \in \mathrm{K}^b(\operatorname{proj} A)$. Then, $\mathrm{H}^k({}_B B \otimes_A^{\mathbf{L}} B_B) = 0$ for all $k \neq 0, -3$ if and only if $(\ker p)^2 \cong \ker p$ as ideals of A.*

Proof. Since $B \in \mathrm{K}^b(\operatorname{proj} A)$, we may apply the Auslander-Buchsbaum formula for Gorenstein orders [IW14a, 2.16]

$$\operatorname{p.dim}_A B \leqslant \operatorname{p.dim}_A B + \operatorname{depth}_{\mathcal{R}} B = 3.$$

Whence, $\mathrm{H}^k({}_B B \otimes_A^{\mathbf{L}} B_B) = 0$ for $k > 0$ and $k < -3$. Moreover,

$$\begin{aligned} &\operatorname{Tor}_1^A(B,B) \cong \operatorname{Ext}_A^2(DB,B) \cong D\operatorname{Ext}_A^1(B,DB), && \text{(by 4.1.5 and Serre duality (4.1.B))} \\ &\operatorname{Tor}_2^A(B,B) \cong \operatorname{Ext}_A^1(DB,B). && \text{(by 4.1.5)} \end{aligned}$$

Since B is self-injective, there an isomorphism $DB \cong B$ as right B- (and hence A-) modules. Therefore, $\operatorname{Ext}_A^1(B, DB) \cong \operatorname{Ext}_A^1(B,B) \cong \operatorname{Ext}_A^1(DB, B)$ as vector spaces, and so there are vector space isomorphisms

$$\mathrm{H}^{-1}({}_B B \otimes_A^{\mathbf{L}} B_B) = \operatorname{Tor}_1^A(B,B) \cong \operatorname{Ext}_A^1(B,B) \cong \operatorname{Tor}_2^A(B,B) = \mathrm{H}^{-2}({}_B B \otimes_A^{\mathbf{L}} B_B).$$

Consequently, to prove the statement, it suffices to prove that $\operatorname{Tor}_1^A(B,B)$ vanishes if and only if $(\ker p)^2 = \ker p$ as ideals of A.

It is a classical result that $\operatorname{Tor}_1^A(B,B) = \operatorname{Tor}_1^A(A/\ker p, A/\ker p) \cong \ker p/(\ker p)^2$.

Indeed, applying the functor $-\otimes_A A/\ker p$ to the exact sequence

$$0 \to \ker p \to A \to A/\ker p (\cong B) \to 0 \tag{4.2.A}$$

yields the exact sequence

$$0 \to \mathrm{Tor}_1^A(B,B) \to \ker p \otimes_A A/\ker p \xrightarrow{f} A \otimes_A A/\ker p \to A/\ker p \otimes_A A/\ker p \to 0.$$

It is easy to check that $f=0$ and, thus, $\mathrm{Tor}_1^A(B,B) \cong \ker p \otimes_A A/\ker p$. Moreover, applying the functor $-\otimes \ker p$ to (4.2.A) induces an exact sequence

$$\ker p \otimes_A \ker p \xrightarrow{g} A \otimes_A \ker p \to A/\ker p \otimes_A \ker p \to 0.$$

It is straightforward that $\mathrm{Im}\, g \cong (\ker p)^2$. We may conclude that

$$\mathrm{Tor}_1^A(B,B) \cong A/\ker p \otimes_A \ker p \cong \ker p/(\ker p)^2$$

and thus the statement follows. □

Corollary 4.2.3. *Under setup 4.1.3, suppose that* $\dim \mathcal{R} = 3$ *and* $B \in \mathrm{K}^b(\mathrm{proj}\, A)$. *If* $(\ker p)^2 \cong \ker p$, *then, the cotwist around F is* $\mathcal{C} \cong [-4]\cdot \mathcal{N}$, *where* $\mathcal{N}(-)$ *denotes the Nakayama autoequivalence of B.*

Proof. In light of 4.2.1, the key point to prove is that $\mathrm{H}^k({}_BB \otimes^{\mathbf{L}}_A B_B) = 0$ for all $k \neq 0, -3$ if $\dim \mathcal{R} = 3$. This follows from 4.2.2. □

Proposition 4.2.4. *Under setup 4.1.3, suppose that* $B \in \mathrm{K}^b(\mathrm{proj}\, A)$. *Then,*

$$F^{\mathrm{RA}} \cong [1]\cdot \mathcal{C} \cdot F^{\mathrm{LA}}.$$

on $\mathrm{D^b}(\mathrm{mod}\, A)$. *In particular, both* F^{RA} *and* F^{LA} *preserve bounded complexes.*

Hence, if $\mathrm{H}^k({}_BB \otimes^{\mathbf{L}}_A B_B) = 0$ *for all* $k \neq 0, d$, *then* $F\colon \mathrm{D^b}(\mathrm{mod}\, B) \to \mathrm{D^b}(\mathrm{mod}\, A)$ *is spherical.*

Proof. Let $c \in \mathrm{D^b}(A)$, Then, there are the following functorial isomorphisms

$$\begin{aligned}
F^{\mathrm{RA}}(c) &= \mathbf{R}\mathrm{Hom}_A({}_BB, c) \\
&\cong \mathbf{R}\mathrm{Hom}_A({}_BB, c \otimes^{\mathbf{L}}_A A_A) \\
&\cong c \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A({}_BB, A) && \text{(by [IR08, 2.10(2)], since B is perfect)} \\
&\cong c \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A({}_BB, A^\dagger) && \text{(since A is a Gorenstein $\mathcal{R}$-order)} \\
&\cong c \otimes^{\mathbf{L}}_A ({}_BB_A)^\dagger && \text{(by derived tensor-hom adjunction)} \\
&\cong c \otimes^{\mathbf{L}}_A DB_B[-d] && \text{(by [IR08, 3.6] since B is finite dimensional)} \\
&\cong c \otimes^{\mathbf{L}}_A B_B \otimes^{\mathbf{L}}_B DB_B[-d] \\
&\cong [1]\cdot \mathcal{C} \cdot F^{\mathrm{LA}}(c) && \text{(by 4.2.1 since $\mathcal{C} = (-) \otimes^{\mathbf{L}}_B DB[-d-1]$.)}
\end{aligned}$$

Since B is perfect as a right A-module, then F^{LA} preserves bounded complexes. Moreover, $\mathcal{C}$ preserves bounded complexes as well, and so necessarily F^{RA} preserves bounded complexes.

Next, suppose that $\mathrm{H}^k({}_B B \otimes^{\mathbf{L}}_A B_B) = 0$ for all $k \neq 0, d$. Then, the fact that F is spherical follows immediately from the above combined with 4.2.1 and the 2 out of 4 property 2.4.12. □

Since under setup 4.1.3, by 4.1.4, B is automatically finite dimensional, it has finitely many simples up to isomorphism. Let $\{S_j\}_{j=1}^k$ be a set containing one representative from each isomorphism class of simple modules and let σ be the Nakayama permutation of B.

Corollary 4.2.5. *Under setup 4.1.3, assume $B \in \mathrm{K}^b(\operatorname{proj} A)$ and $\mathrm{H}^k({}_B B \otimes^{\mathbf{L}}_A B_B) = 0$ for all $k \neq 0, d$. Then, the diagram*

$$\begin{tikzcd} \mathrm{D^b}(\operatorname{mod} A) \arrow[r, "\mathcal{T}"] & \mathrm{D^b}(\operatorname{mod} A) \\ \mathrm{D^b}(\operatorname{mod} B) \arrow[u, "F"] \arrow[r, "\mathcal{N}(-)[-d+1]"] & \mathrm{D^b}(\operatorname{mod} B) \arrow[u, "F"] \end{tikzcd}$$

commutes. In particular,

$$\mathbf{R}\mathrm{Hom}_A(\ker p, {}_B B) \cong {}_B DB_A[-d+1],$$
$$\mathbf{R}\mathrm{Hom}_A(\ker p, S_i) \cong S_{\sigma^{-1}(i)A}[-d+1].$$

Proof. By 4.2.4, F is spherical, so it follows from [Add16, section 1.3] and 4.2.1 that $\mathcal{T} \cdot F(-) \cong F \cdot \mathcal{C}(-)[2] \cong F \cdot \mathcal{N}(-)[-d+1]$. □

Corollary 4.2.6. *Under setup 4.1.3, suppose that $\dim \mathcal{R} = 3$ and $B \in \mathrm{K}^b(\operatorname{proj} A)$. If $(\ker p)^2 = \ker p$, then, $F\colon \mathrm{D^b}(\operatorname{mod} B) \to \mathrm{D^b}(\operatorname{mod} A)$ is spherical. Moreover, the diagram*

$$\begin{tikzcd} \mathrm{D^b}(\operatorname{mod} A) \arrow[r, "\mathcal{T}"] & \mathrm{D^b}(\operatorname{mod} A) \\ \mathrm{D^b}(\operatorname{mod} B) \arrow[u, "F"] \arrow[r, "\mathcal{N}(-)[-2]"] & \mathrm{D^b}(\operatorname{mod} B) \arrow[u, "F"] \end{tikzcd}$$

commutes. In particular,

$$\mathbf{R}\mathrm{Hom}_A(\ker p, {}_B B) \cong {}_B DB_A[-2]$$
$$\mathbf{R}\mathrm{Hom}_A(\ker p, S_i) \cong S_{\sigma^{-1}(i)A}[-2]$$

Proof. This directly follows from 4.2.3, 4.2.4, 4.2.5. □

Example 4.2.7. Let R be a commutative noetherian complete local Gorenstein ring with at worst isolated hypersurface singularities and $\dim R = 3$. Moreover, consider $M \in \operatorname{CM} R$ a maximal Cohen-Macaulay module with $\operatorname{Ext}^1_R(M, M) = 0$, so that $\operatorname{End}_R(M) \in \operatorname{CM} R$. Assume further that R is a summand of M, and write $\Lambda = \operatorname{End}_R(M)$. Let $[\operatorname{add} R]$ be the ideal of Λ consisting of maps $M \to M$ which factor through $\operatorname{add} R$, and set $\Lambda_{\text{con}} = \Lambda/[\operatorname{add} R] = \underline{\operatorname{End}}_R(M)$. We will consider the natural surjection $\pi \colon \Lambda \to \Lambda_{\text{con}}$, and argue that it satisfies the assumptions of 4.2.6.

Since CM modules are reflexive and $M \in \operatorname{CM} R$, observe that Λ is a Gorenstein R-order. To see this, we refer to the results [IW14a, 2.22(2)] and [IR08, 3.8 (1)⇒(3)] which state that if M is a reflexive R-module and, moreover, $\Lambda = \operatorname{End}_R(M) \in \operatorname{CM} R$, then Λ is a Gorenstein R-order. Since Λ is module finite, it satisfies 4.1.1.

Next, we claim that $(\ker \pi)^2 = \ker \pi$. Well, $\ker \pi = [\operatorname{add} R] = \Lambda e_0 \Lambda$, where $e_0 \colon M \to R \hookrightarrow M$ is the idempotent corresponding to the projection onto the summand R of M. It is easy to check that $(\Lambda e_0 \Lambda)^2 = \Lambda e_0 \Lambda$. Additionally, $\Lambda_{\text{con}} \in \mathrm{K}^{\mathrm{b}}(\operatorname{proj} \Lambda)$ by [Wem18, A.7(3)] (see also 5.2.17 later). It remains to show that Λ_{con} is self-injective.

Since R is a Gorenstein isolated hypersurface singularity, the category $\underline{\operatorname{CM}} R$ of stable maximal Cohen-Macaulay modules is Hom-finite and 2-Calabi-Yau (see e.g. [BIKR08, §1]). The property 2-CY means that there is a functorial isomorphism $\underline{\operatorname{Hom}}_R(N, \Omega^{-2} L) \cong D \underline{\operatorname{Hom}}_R(N, L)$ for all $L, M \in \operatorname{CM} R$. Moreover, R being a hypersurface means that there is an isomorphism $\Omega^2 \cong \operatorname{id}$ of endofunctors of $\underline{\operatorname{CM}} R$ [Eis80, 6.1]. Thus, there is an isomorphism of Λ_{con}-bimodules

$$\Lambda_{\text{con}} = \underline{\operatorname{Hom}}_R(M, M) \cong \underline{\operatorname{Hom}}_R(M, \Omega^{-2} M) \cong D \underline{\operatorname{Hom}}_R(M, M) = D\Lambda_{\text{con}}.$$

Therefore, Λ_{con} is self-injective (see also [BIKR08, 7.1]).

Whence, we may apply 4.2.6, and we may conclude that 4.2.6 extends [DW16, 5.11,5.12].

§ 4.2.1 Some non-examples

This section constructs some non-examples of 4.2.6, with the aim of illustrating that the conditions that B is self-injective and $(\ker p)^2 = \ker p$ in 4.2.6 are both strictly necessary for our proof.

Let $G \subset \mathrm{SL}(3, \mathbb{C})$ be a finite group acting on the $\mathbb{C}$-algebra $R = \mathbb{C}[[x, y, z]]$. Observe that R^G is a Gorenstein ring, since $G \subset \mathrm{SL}(3, \mathbb{C})$ (see e.g. [BH93, 6.4.9]).

Lemma 4.2.8. *The skew-group algebra $R\#G$ satisfies the assumptions of setup 4.1.1. Moreover, it is a Gorenstein R^G-order.*

Proof. To prove that $R\#G$ satisfies the assumptions of setup 4.1.1, it suffices to show that $R\#G$ is a module finite R^G-algebra. It is clear that $R\#G = R \otimes_{\mathbb{C}} \mathbb{C}G$ is a finite free R-module. Moreover, by [LW12, 5.4], R is a finite R^G-module, and so it follows that $R\#G$ is a finite R^G-module. It remains to show that $R\#G$ is a Gorenstein R^G-order.

Observe that by 2.8.6(1), $R\#G \cong \mathrm{End}_{R^G}(R)$. Given this fact, we will again use [IW14a, 2.22(2)] and [IR08, 3.8 (1)⇒(3)] which state that $\mathrm{End}_{R^G}(R)$ is a Gorenstein (R^G)-order if R is a reflexive R^G-module and, moreover, $\mathrm{End}_{R^G}(R) \in \mathrm{CM}\, R^G$. It is a well-known fact that $R \in \mathrm{CM}\, R^G$ (see e.g. [LW12, 5.16] or [Aus86, 2.2]). Since CM modules are reflexive, this concludes the proof. □

Example 4.2.9. As in 2.8.7, let ε_3 be a primitive third root of unity, and let $G \subset \mathrm{SL}(3, \mathbb{C})$ be the group $G = \langle g \rangle$ where

$$g = \begin{pmatrix} \varepsilon_3 & 0 & 0 \\ 0 & \varepsilon_3 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}.$$

Then, G acts on $R = \mathbb{C}[[x, y, z]]$ as $g(f(x, y, z)) = f(\varepsilon_3 x, \varepsilon_3 y, \varepsilon_3 z)$.

The skew group algebra $R\#G$ can be presented as the completion of the path algebra $\mathbb{C}Q_2$ where Q_2 is the quiver

$$Q_2 : \quad \begin{tikzcd} & 0 \arrow[dr, "x_0"] \arrow[dr, shift left=2, "y_0"] \arrow[dr, shift left=4, "z_0"] & \\ 2 \arrow[ur, "x_2"] \arrow[ur, shift right=2, "y_2"'] \arrow[ur, shift right=4, "z_2"'] & & 1 \arrow[ll, "z_1"] \arrow[ll, shift left=2, "y_1"] \arrow[ll, shift left=4, "x_1"] \end{tikzcd}$$

modulo the relations $R_0 = \langle\langle x_i y_{i+1} - y_i x_{i+1}, x_i z_{i+1} - z_i x_{i+1}, y_i z_{i+1} - z_i y_{i+1} \rangle\rangle_{i \in \mathbb{Z}_3}$.

This algebra admits the self-injective quotient $p \colon R\#G \to B$ constructed in 2.1.6(2). That is, B is the path algebra $\mathbb{C}Q_2$ subject to the relations

$$\begin{aligned} R_2 = \langle\langle & x_i y_{i+1} - y_i x_{i+1}, x_i z_{i+1} - z_i x_{i+1}, y_i z_{i+1} - z_i y_{i+1}, \\ & x_i x_{i+1}, y_i y_{i+1}, z_i z_{i+1} \rangle\rangle_{i \in \mathbb{Z}_3}. \end{aligned}$$

Intuitively, R_2 means that the "variables x, y, z commute and they square to 0".

Note, however, that

$$\ker p = \langle\langle x_i x_{i+1}, y_i y_{i+1}, z_i z_{i+1} \rangle\rangle_{i \in \mathbb{Z}_3}.$$

is such that $(\ker p)^2 \neq \ker p$. This is straightforward to see since $\ker p$ contains paths of length 2 and the shortest non-zero path in $(\ker p)^2$ is of length 4. In fact, any quotient of $R\#G$ by an admissible ideal I (i.e. an ideal which only contains paths of length greater than or equal to 2) will not satisfy $I^2 = I$.

From this example, we conclude that the assumption $(\ker p)^2 = \ker p$ is not implied by the self-injectivity of B.

Example 4.2.10. With the notation of 4.2.9, observe that, when viewed as an R^G-module, R contains R^G as a summand [Aus86] [LW12, 5.20]. Hence, $R\#G \cong \mathrm{End}_{R^G}(R)$

has an idempotent $e_0 \colon R \to R^G \hookrightarrow R$ corresponding to projection onto R^G. Let $I = (R\#G)e_0(R\#G)$. Observe that there is a natural epi $q \colon R\#G \to R\#G/I$ and that $I^2 = I$. Moreover, The algebra $R\#G/I$ can be presented as the path algebra of the following quiver

$$Q_3 : \qquad 2 \overset{z_1}{\underset{x_1}{\overset{\longleftarrow\!\!-\!\!-\!\!-\!\!-}{\underset{\longleftarrow\!\!-\!\!-\!\!-\!\!-}{\xleftarrow{\quad y_1 \quad}}}}} 1.$$

A simple calculation reveals that indecomposable injectives do not match the indecomposable projectives, and so $R\#G/I$ is not self-injective. Therefore, it does not induce a derived equivalence as in 4.2.6. We may also conclude that the condition $(\ker q)^2 = \ker q$ does not guarantee self-injectivity.

§ 4.2.2 Why not the contraction algebra?

As recalled in § 2.5.1, to a crepant contraction $f \colon X \to \operatorname{Spec} \mathcal{R}$ satisfying 2.5.2, it is possible to associate a surjection $p \colon \Lambda \to \Lambda_{\mathrm{con}}$, where Λ is derived equivalent to X and Λ_{con} is the contraction algebra. In the setting of example 4.2.10, the quotient singularity $\mathbb{C}^3/G := \operatorname{Spec} R^G$ admits a particular crepant resolution $X \to \operatorname{Spec} R^G$ that turns out to satisfy 2.5.2 and, moreover, $\Lambda = R\#G$ [BKR01]. The associated contraction algebra is $\Lambda_{\mathrm{con}} = R\#G/I$. This epimorphism does not induce a derived equivalence as in 4.2.6 via the discussion in example 4.2.10, .

As recalled in § 2.5.1, the contraction algebra is an invariant which remembers much of the information associated to a contraction (for more details see § 2.5.1), and so it is an interesting object to study. However, from the point of view of constructing derived equivalences, example 4.2.10 and corollary 4.2.6 suggest that it might be more interesting to study other quotients of $R\#G$.

Chapter 5

Spherical twists induced by Frobenius categories

With geometric applications in mind, we would like to construct a spherical twist around the restriction of scalars functor when the algebra B is not finite dimensional. As a result, in this section, we introduce a different algebraic setting which allows us to construct a spherical twist in other geometric settings.

Given an object X in a Frobenius exact category $\mathcal{E}$ satisfying mild conditions, we will specify when the restriction of scalars functor F induced by quotient morphism $\pi\colon \operatorname{End}_{\mathcal{E}}(X) \to \underline{\operatorname{End}}_{\mathcal{E}}(X)$ is spherical. The key point is roughly that F is spherical if the action of the suspension functor on the object x is periodic up to additive closure. This is proved in § 5.2.

§ 5.1 Setting and fundamentals

In this section, we introduce the setup and some fundamental results necessary for § 5.2. Most notably, we show that the suspension functor acts periodically on X up to additive closure if and only if X has relatively spherical properties. This characterisation is important to our applications because relatively spherical properties are easier to control in the geometric setting. In the process, § 5.1.2 constructs what we call partially minimal projective resolutions.

§ 5.1.1 Setup

Recall that an additive category $\mathcal{C}$ is said to be $\mathcal{R}$-linear, where $\mathcal{R}$ is a ring, if the Hom-sets in the category are finitely generated $\mathcal{R}$-modules, and composition respects this structure. Assume further than $\mathcal{C}$ is an exact category. Then, the collection of projective objects in $\mathcal{C}$ will be denoted by $\operatorname{proj}\mathcal{C}$, and an object $P \in \mathcal{C}$ will be called an additive progenerator if $\operatorname{add}_{\mathcal{C}} P = \operatorname{proj}\mathcal{C}$.

With an eye towards applications, we will often work within the following set up.

Setup 5.1.1. Let $\mathcal{R}$ be a Cohen-Macaulay (CM) noetherian local ring of Krull dimension d with coefficient field k and canonical module ω_R. Let $\mathcal{E}$ be an $\mathcal{R}$-linear idempotent complete Frobenius exact category with an additive progenerator P. Assume that the stable category $\mathcal{D} := \mathcal{E}/\operatorname{proj}\mathcal{E}$ is Krull-Schmidt.

Notation 5.1.2. Since $\mathcal{D}$ is triangulated, we will denote its suspension functor by Σ.

We are particularly interested in examples where $\mathcal{E} = \operatorname{CM}\mathcal{R}$, the category of maximal Cohen-Macaulay modules over a Gorenstein ring $\mathcal{R}$, as these find natural applications in geometry.

Notation 5.1.3. Suppose that $\mathcal{E}$ is as in Setup 5.1.1. Then, given an object $X \in \mathcal{E}$, set $\Lambda = \operatorname{End}_{\mathcal{E}}(X)$ and let $[\operatorname{proj}\mathcal{E}]$ be the ideal in Λ of maps $X \to X$ which factor through a projective object in $\mathcal{E}$. Furthermore, let

$$\Lambda_{\text{con}} = \operatorname{End}_{\mathcal{D}}(X) = \underline{\operatorname{End}}_{\mathcal{E}}(X) := \Lambda/[\operatorname{proj}\mathcal{E}].$$

Remark 5.1.4. Note that Λ satisfies 4.1.1, so that $\omega_\Lambda := \mathbf{R}\operatorname{Hom}_{\mathcal{R}}(\Lambda, \omega_{\mathcal{R}})$ is a dualizing complex for Λ which induces a form of Serre duality as in (4.1.A) and (4.1.B).

We will often place additional assumptions on X, which are naturally satisfied in the applications we have in mind.

Setup 5.1.5. With the assumptions as in 5.1.1 and notation as above, let $X \in \mathcal{E}$ be such that $X \cong P \oplus \bigoplus_{i=1}^n X_i^{a_i}$, where $a_i \in \mathbb{N}$ with $a_i > 0$, and X_i are pairwise non-isomorphic, indecomposable objects in $\mathcal{D}$.

Remark 5.1.6. Since the suspension functor Σ is an automorphism of $\mathcal{D}$, it is clear that if $X \in \mathcal{E}$ satisfies 5.1.5, then so does $P \oplus \Sigma^k X$.

Note that for any $X \in \mathcal{E}$, there is a $Q \in \operatorname{proj}\mathcal{E}$ such that $Q \oplus X$ satisfies setup 5.1.5. To see this, note that, since $\mathcal{D}$ is Krull-Schmidt, any $X \in \mathcal{E}$ is stably isomorphic to $\bigoplus_{i=1}^n X_i^{a_i}$ with X_i and a_i as in setup 5.1.5. Consequently, by 2.2.12, there is an isomorphism $Q_1 \oplus X \cong Q_2 \oplus \bigoplus_{i=1}^n X_i^{a_i}$ in $\mathcal{E}$ with $Q_1, Q_2 \in \operatorname{proj}\mathcal{E}$. Thus, letting $Y_1 = Q_2 \oplus X_1$ and $Y_i = X_i$ for $i \neq 1$, we may write

$$(P \oplus Q_1 \oplus Q_2^{a_1-1}) \oplus X \cong P \oplus \bigoplus_{i=1}^k Y_i^{b_i}.$$

Notation 5.1.7. Given $X = P \oplus X'$ with $X' = \bigoplus_{i=1}^n X_i^{a_i}$ as in 5.1.5, let $I = \{1, 2, \ldots, n\}$ and consider the following idempotents in Λ

(1) $e_i \colon X \to X_i$, projection onto the summand X_i for $i \in I$,

(2) $e_0 \colon X \to P$, projection onto P,

(3) $e_{X'} \colon X \to X'$, projection onto X'.

With this notation,

$$\begin{aligned}
&\mathcal{E}(X, X_i) = e_i\Lambda \quad \text{and } \mathcal{D}(X, X_i) = e_i\Lambda_{\mathrm{con}},\\
&\mathcal{E}(X, X') = e_{X'}\Lambda \text{ and } \mathcal{D}(X, X') = e_{X'}\Lambda_{\mathrm{con}},\\
&\mathcal{E}(X, P) = e_0\Lambda.
\end{aligned}$$

Lemma 5.1.8. *Under setup 5.1.5, Λ_{con} is semi-perfect (see e.g. § 2.1 for some background on semi-perfect algebras) and, therefore, has finitely many simple modules up to isomorphism.*

Proof. Since $\mathrm{add}_{\mathcal{D}}(X)$ is Krull-Schmidt (and thus idempotent complete), there is an equivalence $\mathcal{D}(X, -)\colon \mathrm{add}_{\mathcal{D}}(X) \to \mathrm{proj}\,\Lambda_{\mathrm{con}}$ [Kra15, 2.3]. Therefore, $\mathrm{proj}\,\Lambda_{\mathrm{con}}$ is Krull-Schmidt. It follows from 2.1.1 that Λ_{con} is semi-perfect. The fact that semi-perfect rings have finitely many simple modules up to isomorphism is well-known (see, for example, 2.1.2). □

Since Λ_{con} is semi-perfect and $e_i\Lambda_{\mathrm{con}}$ is an indecomposable summand of Λ_{con}, it follows that $e_i\Lambda_{\mathrm{con}}/\,\mathrm{rad}(e_i\Lambda_{\mathrm{con}})$ is simple by [Kra15, 4.1 (3) ⇒ (1)]. In fact, these are all of the simple Λ_{con} modules up to isomorphism.

Notation 5.1.9. Write $S_i := e_i\Lambda_{\mathrm{con}}/\,\mathrm{rad}(e_i\Lambda_{\mathrm{con}})$.

We conclude this section by recalling a well-known result which will often be used throughout the thesis.

Lemma 5.1.10. *Let $\mathcal{A}$ be an additive idempotent complete category and consider objects $X, Y, Z \in \mathcal{A}$.*

(1) *If $Y \in \mathrm{add}_{\mathcal{A}} X$, then functor $\mathrm{Hom}_{\mathcal{A}}(X, -)$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathrm{End}_{\mathcal{A}}(X)}(\mathrm{Hom}_{\mathcal{A}}(X, Y), \mathrm{Hom}_{\mathcal{A}}(X, Z)).$$

which is natural in Y and Z.

(2) *The canonical composition map*

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \otimes_{\mathrm{End}_{\mathcal{A}}(Y)} \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

is an isomorphism if $Z \in \mathrm{add}_{\mathcal{A}}(Y)$ or $X \in \mathrm{add}_{\mathcal{A}}(Y)$.

§ 5.1.2 Partially minimal projective resolutions

The key tool for characterising when the action of the suspension functor on an object is periodic, up to additive closure, will be a (partially) minimal projective resolution of Λ_{con}. Ideally, we would like to construct the minimal projective resolution of Λ_{con} as a Λ-module, but since Λ is not necessarily semi-perfect, such a resolution need not exist. However, it is possible to construct a projective resolution which has sufficiently nice properties.

Notation 5.1.11. Under setup 5.1.5 and assuming the notation 5.1.9, write

$$\mathcal{S} := \bigoplus_{i \in I} S_i.$$

Definition 5.1.12. Under setup 5.1.5, we will say that a projective resolution

$$\dots \xrightarrow{f_3} Q_2 \xrightarrow{f_2} Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \to 0$$

of finitely generated Λ-modules is *partially minimal* if $\operatorname{Hom}_\Lambda(f_i, \mathcal{S}) = 0$ for all $i > 0$.

We will prove in 5.1.19 that under setup 5.1.5 every finitely generated Λ-module admits such a resolution. In order to construct it, we will need to establish some technical results. Our approach is to define an ideal which is somewhat similar to the radical and then to adapt the standard proof that minimal resolutions exist to our more general setting.

Definition 5.1.13. Under setup 5.1.5, let $M \in \operatorname{mod} \Lambda$. Set

$$I(M) = \left\{ Y \subseteq M \;\middle|\; \begin{array}{l} Y \text{ is a submodule with } Me_0\Lambda \subseteq Y, \text{ and} \\ \text{if there is a submodule } V \text{ with } Y \subseteq V \subseteq M, \text{ then} \\ V = M \end{array} \right\}$$

and

$$\operatorname{rad}_0 M = \bigcap_{Y \in I(M)} Y.$$

Remark 5.1.14. Note that in 5.1.13, Y is not necessarily proper. Consequently, $\operatorname{rad}_0 M$ is the intersection of M with all of the maximal submodules of M which contain the set Me_0. It is clear that $\operatorname{rad} M \subseteq \operatorname{rad}_0 M$, since $\operatorname{rad} M$ is the intersection of all maximal submodules of M. Moreover, if M does not have any proper submodules which contain the set Me_0, then $\operatorname{rad}_0 M = M$. For example, $\operatorname{rad}_0(e_0\Lambda) = e_0\Lambda$.

The following lemma is an adaptation of the usual characterisation of essential epimorphisms.

Lemma 5.1.15. *Under setup 5.1.5, let $\phi\colon M \to N$ be an epimorphism in $\operatorname{mod}\Lambda$. Then, the following are equivalent.*

(1) $\ker \phi \subseteq \operatorname{rad}_0 M$.

(2) *For any submodule $V \subseteq M$ with $Me_0 \subseteq V$ and $\ker\phi + V = M$, then $V = M$.*

(3) *Let $\alpha\colon L \to M$ be a morphism in $\operatorname{mod}\Lambda$ with $Me_0 \subseteq \operatorname{Im}\alpha$. Then, α is an epimorphism provided that the composition $\phi \cdot \alpha$ is an epimorphism.*

Proof. Suppose that (1) holds. and consider V as in (2). If $V = M$, then there is nothing to prove. If $V \subsetneq M$ with $Me_0 \subseteq V$ and $\ker \phi + V = M$, then there exists a maximal submodule $W \subsetneq M$ with $Me_0 \subseteq V \subseteq W$. It follows that $M = \ker \phi + W = W$, which is a contradiction. Hence, $V = M$, so that (2) holds.

Next, suppose that (2) holds. Then, given a morphism $\alpha \colon L \to M$ with $Me_0 \subseteq \operatorname{Im} \alpha$ and $\phi \cdot \alpha$ an epimorphism, it easy to see that $\operatorname{Im} \alpha + \ker \phi = M$, since for all $m \in M$, there exists an $\ell \in L$ such that $\phi(m) = \phi(\alpha(\ell))$. Hence, $m - \alpha(\ell) \in \ker \phi$ and we conclude that $\operatorname{Im} \alpha + \ker \phi = M$. Thus, by (2), $\operatorname{Im} \alpha = M$ and (3) holds.

Finally, suppose that (3) holds. If M does not have proper submodules $V \subseteq M$ with $Me_0 \subseteq M$, then by 5.1.14, $\operatorname{rad}_0 M = M$ and so (1) clearly holds. So, suppose that there is a maximal submodule $V \subset M$ with $Me_0 \subseteq V$. Suppose for a contradiction that $\ker \phi \nsubseteq V$. Note that the inclusion $\alpha \colon V \to M$ is such that $Me_0 \subseteq \operatorname{Im} \alpha$. Moreover, since V is maximal, $\ker \phi + V = M$. This implies that the composition $\phi \cdot \alpha$ is an epimorphism. It thus follows that α is an epimorphism. That is, $V = M$. This is a contradiction, and so $\ker \phi \subseteq V$. Since this holds for any $V \in I(M)$, then $\ker \phi \subseteq \bigcap_{V \in I(M)} V = \operatorname{rad}_0 M$ and so (1) holds. □

Lemma 5.1.16. *Under setup 5.1.5, let $f \colon X \to Y$ and $g \colon Y \to Z$ be epimorphisms which satisfy the equivalent assumptions of 5.1.15. Then, $g \cdot f$ satisfies 5.1.15.*

Proof. We test the third condition in 5.1.15. Let $\alpha \colon A \to X$ be a morphism with $Xe_0 \subseteq \operatorname{Im} \alpha$ such that $g \cdot f \cdot \alpha$ is an epimorphism. Since f is an epimorphism, it is clear that

$$Ye_0 = f(X)e_0 = f(Xe_0) \subseteq \operatorname{Im}(f \cdot \alpha).$$

Now since g satisfies 5.1.15, it follows that $f \cdot \alpha$ is an epimorphism. Whence α is an epimorphism since f satisfies 5.1.15. □

The strategy now is to show that each Λ-module admits an epimorphism as in 5.1.15, and then use this fact to show that any Λ-module admits a partially minimal projective resolution.

Lemma 5.1.17. *Under setup 5.1.5, let M be a semi-simple Λ-module. Then, there exists a Λ-module epimorphism $\phi \colon Q \to M$, where Q is a projective Λ-module and* $\ker \phi \subseteq \operatorname{rad}_0 Q$.

Proof. To first see that each simple Λ_{con}-module admits such an epimorphism, consider the quotient morphisms $\pi_i \colon e_i\Lambda \to e_i\Lambda_{\mathrm{con}}$ and $\alpha_i \colon e_i\Lambda_{\mathrm{con}} \to e_i\Lambda_{\mathrm{con}}/\operatorname{rad}(e_i\Lambda_{\mathrm{con}}) \cong S_i$ for $i \in I$. Note that

$$\ker \alpha_i = \operatorname{rad}(e_i\Lambda_{\mathrm{con}}) \subseteq \operatorname{rad}_0(e_i\Lambda_{\mathrm{con}}),$$
$$\ker \pi_i = e_i[\operatorname{proj} \mathcal{E}] = e_i\Lambda e_0\Lambda.$$

Since $\operatorname{rad}_0(e_i\Lambda)$ is the intersection of modules Y with $(e_i\Lambda)e_0\Lambda \subseteq Y \subseteq e_i\Lambda$, necessarily $\ker \pi_i = e_i\Lambda e_0\Lambda \subseteq \operatorname{rad}_0(e_i\Lambda)$.

It thus follows from 5.1.16 that the composition

$$\phi_i \colon e_i\Lambda \xrightarrow{\pi_i} e_i\Lambda_{\mathrm{con}} \xrightarrow{\alpha_i} S_i$$

is such that $\ker \phi_i \subseteq \operatorname{rad}_0(e_i\Lambda)$.

Next, let S be a simple Λ-module which is not a Λ_{con}-module. Then, S is not annihilated by e_0, so that

$$\operatorname{Hom}_\Lambda(e_0\Lambda, S) \cong \operatorname{Hom}_\Lambda(\Lambda, S)e_0 \cong Se_0 \cong S \neq 0.$$

Hence, we may choose a nonzero $\phi \colon e_0\Lambda \to S$ which, since S is simple, is surjective. By 5.1.14, $\ker \phi \subseteq e_0\Lambda = \operatorname{rad}_0(e_0\Lambda)$.

In summary, given a simple Λ-module T, write

$$P(T) := \begin{cases} e_i\Lambda & \text{if } T \cong S_i \text{ for } i \in I, \\ e_0\Lambda & \text{otherwise.} \end{cases}$$

Then, given an arbitrary semi-simple module $M = \bigoplus_{j=1}^k T_j$, by the above there exists an epimorphism $\phi = \bigoplus_{j=1}^k \phi_j$ with

$$\ker \phi \subseteq \bigoplus_i^k \operatorname{rad}_0 P(T_j) = \operatorname{rad}_0 \left(\bigoplus_i^k P(T_j) \right). \qquad \square$$

Proposition 5.1.18. *Under setup 5.1.5, let $M \in \operatorname{mod}\Lambda$. There exists projective Λ-module Q and an epimorphism $\psi \colon Q \to M$ such that $\ker\psi \subseteq \operatorname{rad}_0(Q)$.*

Proof. Since Λ is module finite over a local ring, it is semilocal by [Lam91, 20.6]. In particular, for any $M \in \operatorname{mod}\Lambda$, it is well-known that $M/\operatorname{rad} M$ is semi-simple. To see this, note that $M/\operatorname{rad} M = M/M\operatorname{rad}\Lambda$ is a semi-simple $\Lambda/\operatorname{rad}\Lambda$-module. Since simple Λ-modules are precisely the simple $\Lambda/\operatorname{rad}\Lambda$ modules, it follows that $M/\operatorname{rad} M$ is a semi-simple Λ-module.

By 5.1.17, there is an epimorphism $\phi \colon W \to M/\operatorname{rad} M$ with W projective and $\ker\phi \subseteq \operatorname{rad}_0 W$. Since W is projective, ϕ must factor as $\psi \colon W \to M$ followed by $q \colon M \to M/\operatorname{rad} M$. Since $\ker q \subseteq \operatorname{rad} M$ and ϕ is an epimorphism, so is ψ, by the classical version of 5.1.15 (see, for example, [Kra15, 3.3]). Finally, since $\phi = q \cdot \psi$, we have $\ker\psi \subseteq \ker\phi \subseteq \operatorname{rad}_0 W$, as required. $\square$

Theorem 5.1.19. *Under setup 5.1.5, let $M \in \operatorname{mod}\Lambda$. Then, M admits a partially minimal projective resolution.*

Proof. From 5.1.18, there is an epimorphism $f_0 \colon Q_0 \to M$ with $\ker f_0 \subseteq \operatorname{rad}_0 Q_0$ and Q_0 projective. We will define a projective resolution $f \colon Q \to X$ inductively. Suppose that $f_n \colon Q_n \to Q_{n-1}$ is such that $\ker f_n \subseteq \operatorname{rad}_0 Q_n$. Then, by 5.1.18, we may choose an epimorphism $\pi_{n+1} \colon Q_{n+1} \to \ker f_n$ with Q_{n+1} projective and $\ker \pi_{n+1} \subseteq \operatorname{rad}_0 Q_{n+1}$. Let $i_n \colon \ker f_n \to Q_n$ denote the natural inclusion and set $f_{n+1} = i_n \cdot \pi_{n+1}$.

We claim that this projective resolution is partially minimal. To see this, let S be a simple Λ_{con}-module and suppose that $g \colon Q_n \to S$ is nonzero. Then, $\ker g$ is maximal and, moreover, contains $Q_n e_0$ since S is annihilated by e_0. Therefore, $\operatorname{rad}_0 Q_n \subseteq \ker g$ and so

$$\operatorname{Im} f_{n+1} = \ker f_n \subseteq \operatorname{rad}_0 Q_n \subseteq \ker g.$$

It follows that $g \cdot f_{n+1} = 0$, and thus $\operatorname{Hom}_\Lambda(f_{n+1}, \mathcal{S}) = 0$. □

§ 5.1.3 Relatively spherical properties

The goal of this section is to prove that if Λ_{con} has certain Ext vanishing properties, then it admits a projective resolution as in (5.1.E) below. The existence of this partially minimal resolution is an essential step to understand the main theorem in § 5.1.4.

Definition 5.1.20. Under setup 5.1.5, fix $t \in \mathbb{Z}$ with $2 \leqslant t \leqslant d$. We will say that Λ_{con} is *t-relatively spherical* if $\operatorname{Ext}^k_\Lambda(\Lambda_{\mathrm{con}}, \mathcal{S}) = 0$ for $k \neq 0, t$.

The definition we use of t-relatively spherical is based on [DW19b, 4.3]. In the case where Λ_{con} is basic, the definitions are the same.

We require the following two technical lemmas.

Lemma 5.1.21. *Under setup 5.1.5, let S_i be the simple Λ_{con}-module as in 5.1.9. Then, the following hold.*

(1) $\operatorname{Hom}_\Lambda(\mathcal{E}(X, X_i), S_j) \cong \delta_{ij} S_j$ *as vector spaces for all $i \in \{0, 1, \dots, n\}$ and $j \in I$.*

(2) *If $U \in \operatorname{add} e_0\Lambda$, then $\operatorname{Hom}_\Lambda(U, S_j) = 0$ for all $j \in I$.*

(3) *If $U \in \operatorname{add} \Lambda$ and $\operatorname{Hom}_\Lambda(U, S_j) = 0$ for all $j \in I$, then $U \in \operatorname{add} e_0\Lambda$.*

Proof. (1) First, note that $\operatorname{Hom}_\Lambda(e_i\Lambda, S_i) \neq 0$ since the composition of quotient maps

$$e_i\Lambda \xrightarrow{\pi_i} e_i\Lambda_{\mathrm{con}} \xrightarrow{\alpha_i} S_i$$

is nonzero. In fact, since $\operatorname{Hom}_\Lambda(e_i\Lambda, S_i) \neq 0$, there is a vector space isomorphism $\operatorname{Hom}_\Lambda(e_i\Lambda, S_i) \cong S_i$ sending $\alpha \mapsto \alpha(e_i)$. Similarly, $\operatorname{Hom}_\Lambda(\Lambda, S_i) \cong S_i$ so that, by counting dimensions, it must be that $\operatorname{Hom}_\Lambda(e_i\Lambda, S_j) = 0$ for $i \neq j$.

(2) The statement is clear from the fact that $\operatorname{Hom}_\Lambda(e_0\Lambda, S_j) = 0$ for all $j \in I$.

(3) Suppose that $U \in \operatorname{add} \Lambda$. Then, by additive equivalence 5.1.10 (1), $U = \mathcal{E}(X, Y)$ for $Y \in \operatorname{add}_{\mathcal{E}} X$. It follows that Y is stably isomorphic to $\bigoplus_{i=1}^n X_i^{b_i}$ for $b_i \in \mathbb{N}$. Consequently, by 2.2.12, there is an isomorphism $Q \oplus Y \cong Q' \oplus \bigoplus_{i=1}^n X_i^{b_i}$ in $\mathcal{E}$ with $Q, Q' \in \operatorname{add} P$. Now,

$$\operatorname{Hom}_\Lambda(\mathcal{E}(X, Q \oplus Y), S_j) = \operatorname{Hom}_\Lambda(\mathcal{E}(X, Q), S_j) \oplus \operatorname{Hom}_\Lambda(\mathcal{E}(X, Y), S_j)$$

Since $\mathcal{E}(X,Q) \in \operatorname{add} e_0\Lambda$, (2) implies that $\operatorname{Hom}_\Lambda(\mathcal{E}(X,Q),S_j) = 0$. Moreover, since $U = \mathcal{E}(X,Y)$, it follows by assumption that $\operatorname{Hom}_\Lambda(\mathcal{E}(X,Y),S_j) = 0$. Thus,

$$0 = \operatorname{Hom}_\Lambda(\mathcal{E}(X,Q\oplus Y),S_j) = \operatorname{Hom}_\Lambda(\mathcal{E}(X,Q'\oplus \bigoplus_{i=1}^{n} X_i^{b_i}),S_j).$$

In particular, this implies that

$$\operatorname{Hom}_\Lambda(\mathcal{E}(X,X_j^{b_j}),S_j) = 0$$

for all $j \in \{1,\dots,n\}$. By (1), it must be that $b_j = 0$ for all j. Thus, Y vanishes in $\mathcal{D}$. That is, $Y \in \operatorname{add} P$ and $U = \mathcal{E}(X,Y) \in \operatorname{add}\mathcal{E}(X,P) = \operatorname{add} e_0\Lambda$. □

Lemma 5.1.22. *Under setup 5.1.5, fix $i \in I$. Then, there exists a partially minimal projective resolution of $e_i\Lambda_{\mathrm{con}}$*

$$\dots \xrightarrow{q_2} \mathcal{E}(X,W_1) \xrightarrow{q_1} \mathcal{E}(X,X_i) \to e_i\Lambda_{\mathrm{con}} \to 0 \tag{5.1.A}$$

such that

(1) $q_1 = \mathcal{E}(X,\alpha_1)$ *for some morphism* $\alpha_1\colon W_1 \to X_i \in \mathcal{E}$.

(2) $\alpha_1\colon W_1 \to X_i$ *is an admissible epi and, hence, admits a kernel* $\ker\alpha_1 \in \mathcal{E}$.

(3) $K_1 := \ker q_1 \cong \mathcal{E}(X,\ker\alpha_1)$

Further, for $j > 1$, the following hold.

(4) $q_j = \mathcal{E}(X,\alpha_j)$ *for some morphism* $\alpha_j\colon W_j \to W_{j-1} \in \mathcal{E}$.

(5) α_j *is an admissible epimorphism onto* $\ker\alpha_{j-1}$*, and so it admits a kernel* $\ker\alpha_j$.

(6) $K_j := \ker q_j \cong \mathcal{E}(X,\ker\alpha_j)$.

Proof. We will build the projective resolution inductively.

(a) Consider first the case where $j = 1$. By the proof of 5.1.19 and 5.1.17, there is an exact sequence

$$0 \to K_1 \xrightarrow{j_1} \mathcal{E}(X,U_1) \xrightarrow{f_1} \mathcal{E}(X,X_i) \xrightarrow{q_0} e_i\Lambda_{\mathrm{con}} \to 0 \tag{5.1.B}$$

where $\operatorname{Hom}_\Lambda(f_1,\mathcal{S}) = 0$.

Additive equivalence applied to $X \in \mathcal{E}$ implies that $f_1 = \mathcal{E}(X,\beta_1)$ for some morphism $\beta_1\colon U_1 \to X_i$. By 2.2.11, there is an exact sequence

$$0 \to \ker(p_1,\beta_1) \to Q_1 \oplus U_1 \xrightarrow{(p_1,\beta_1)} X_i \to 0$$

in $\mathcal{E}$ where $Q_1 \in \operatorname{add}_{\mathcal{E}} P$. Applying $\mathcal{E}(X,-)$ to this sequence induces

$$0 \longrightarrow \mathcal{E}(X, \ker(p_1, \beta_1)) \xrightarrow{\mathcal{E}(X,(p_1,\beta_1))} \mathcal{E}(X, X_i) \longrightarrow \operatorname{Coker} \mathcal{E}(X, (p_1, \beta_1)) \longrightarrow 0$$

We claim that there is an isomorphism $e_i\Lambda_{\mathrm{con}} \cong \operatorname{Coker} \mathcal{E}(X, (p_1, \beta_1))$, which holds if and only if $\operatorname{Im} \mathcal{E}(X, (p_1, \beta)) = e_i[\operatorname{proj} \mathcal{E}]$. Let $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{E}(X, Q_1 \oplus U_1)$.

First, observe that by exactness of (5.1.B), $\operatorname{Im} \mathcal{E}(X, \beta_1) = \ker q_0$ and by the proof of 5.1.17, $\ker q_0 = e_i[\operatorname{proj} \mathcal{E}]$. Consequently, $\beta_1 g_1 \colon X \to X_i$ factors through $\operatorname{proj} \mathcal{E}$. Since $Q_1 \in \operatorname{proj} \mathcal{E}$, it is clear that $p_1 g_1$ also factors through $\operatorname{proj} \mathcal{E}$. Hence, $(p_1, \beta_1) \cdot g = p_1 g_1 + \beta_1 g_1 \in e_i[\operatorname{proj} \mathcal{E}]$. That is, $\operatorname{Im} \mathcal{E}(X, (p_1, \beta_1)) \subseteq e_i[\operatorname{proj} \mathcal{E}]$.

Next, suppose that $f \colon X \to X_i$ factors through $Q \in \operatorname{proj} \mathcal{E}$, say via

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "a"'] & & X_i \\ & Q \arrow[ur, "b"'] & \end{tikzcd}$$

Then, since $(p_1, \beta_1) \colon Q_1 \oplus U_1 \to X_i$ is an epi, there is a map $c \colon Q \to Q_1 \oplus U_1$ such that $b = (p_1, \beta_1) \cdot c$. Thus, $f = (p_1, \beta_1) \cdot c \cdot a$. Hence, $f \in \operatorname{Im} \mathcal{E}(X, (p_1, \beta_1))$, and so $\operatorname{Im} \mathcal{E}(X, (p_1, \beta_1)) = e_i[\operatorname{proj} \mathcal{E}]$. Consequently, $e_i\Lambda_{\mathrm{con}} = \operatorname{Coker} \mathcal{E}(X, (p_1, \beta_1))$, as desired.

Let $\alpha_1 = (p_1, \beta_1)$, $q_1 = \mathcal{E}(X, \alpha_1)$ and $W_1 = Q_1 \oplus U_1$. Then, there is an exact sequence

$$0 \longrightarrow \mathcal{E}(X, \ker \alpha_1) \xrightarrow{i_1} \mathcal{E}(X, Q_1) \oplus \mathcal{E}(X, U_1) \xrightarrow{\mathcal{E}(X,\alpha_1)} \mathcal{E}(X, X_i) \longrightarrow e_i\Lambda_{\mathrm{con}} \longrightarrow 0 \tag{5.1.C}$$

where $\alpha_1 \colon U_1 \to X_i$ is an admissible epimorphism. This proves (1)-(3). Furthermore, to prove the partially minimal property, consider a map of Λ-modules $f \colon \mathcal{E}(X, X_i) \to \mathcal{S}$. Then,

$$f \cdot \mathcal{E}(X, \alpha_1) = f \cdot (\mathcal{E}(X, p_1), \mathcal{E}(X, \beta_1)) = (f \cdot \mathcal{E}(X, p_1), f \cdot \mathcal{E}(X, \beta_1)).$$

Since $\operatorname{Hom}_\Lambda(f_1, \mathcal{S}) = 0$ by choice of $f_1 = \mathcal{E}(X, \beta_1)$, necessarily $f \cdot \mathcal{E}(X, \beta_1) = 0$. In addition, $f \cdot \mathcal{E}(X, p_1) \in \operatorname{Hom}_\Lambda(\mathcal{E}(X, Q_1), \mathcal{S})$ where $\mathcal{E}(X, Q_1) \in \operatorname{add} e_0\Lambda$. Therefore, by 5.1.21(2), $f \cdot \mathcal{E}(X, p_1) = 0$. Consequently, $\operatorname{Hom}_\Lambda((\mathcal{E}(X, \alpha_1), \mathcal{S}) = 0$.

(b) Consider the case $j = 2$. Observe that the proof of 5.1.19 implies that there is an exact sequence

$$0 \to K_2 \to \mathcal{E}(X, U_2) \xrightarrow{f_2} \mathcal{E}(X, \ker \alpha_1) \to 0.$$

where $\mathrm{Hom}_\Lambda(i_1 \cdot f_2, \mathcal{S}) = 0$.

Additive equivalence applied to $X \in \mathcal{E}$ implies that $f_2 = \mathcal{E}(X, \beta_2)$ for some morphism $\beta_2 \colon U_2 \to \ker \alpha_1$. By 2.2.11, there is an exact sequence

$$0 \to \ker(p_2, \beta_2) \xrightarrow{i_2} Q_2 \oplus U_2 \xrightarrow{(p_2,\beta_2)} \ker \alpha_1 \to 0.$$

with $Q_2 \in \mathrm{add}_\mathcal{E} P$. Applying $\mathcal{E}(X, -)$ induces the exact sequence

$$0 \to \mathcal{E}(X, \ker(p_2, \beta_2)) \to \mathcal{E}(X, Q_2) \oplus \mathcal{E}(X, U_2) \xrightarrow{(\mathcal{E}(X,p_2), f_2)} \mathcal{E}(X, \ker \alpha_1)$$

Moreover, the last map must be an epi because component f_2 is an epi. Hence, by letting $W_2 = Q_2 \oplus U_2$, $\alpha_2 = (p_2, \beta_2)$ and $q_2 = \mathcal{E}(X, \alpha_2)$, there is an exact sequence

$$0 \to \mathcal{E}(X, \ker \alpha_2) \xrightarrow{i_2} \mathcal{E}(X, Q_2) \oplus \mathcal{E}(X, U_2) \xrightarrow{q_2} \mathcal{E}(X, \ker \alpha_1) \to 0$$

which satisfies (4)-(6). Additionally, given $h \colon \mathcal{E}(X, W_1) \to \mathcal{S}$, then

$$h \cdot i_1 \cdot q_2 = (h \cdot i_1 \cdot \mathcal{E}(X, p_2), h \cdot i_1 \cdot \mathcal{E}(X, \beta_2)).$$

Since $Q_2 \in \mathrm{add}\, P$, the map $h \cdot i_1 \cdot \mathcal{E}(X, p_2) \colon \mathcal{E}(X, Q_2) \to \mathcal{S}$ vanishes by 5.1.21(2). Moreover, by choice of $f_2 = \mathcal{E}(X, \beta_2)$, $\mathrm{Hom}_\Lambda(i_1 \cdot f_2, \mathcal{S}) = 0$, and so $h \cdot i_1 \cdot f_2 = 0$. We may conclude that $\mathrm{Hom}_\Lambda(i_1 \cdot q_2, \mathcal{S}) = 0$, proving the partially minimal property.

(c) The case for general $j \geqslant 2$ is similar and yields exact sequences

$$0 \to \mathcal{E}(X, \ker \alpha_j) \to \mathcal{E}(X, Q_j) \oplus \mathcal{E}(X, U_j) \xrightarrow{q_j} \mathcal{E}(X, \ker \alpha_j) \to 0 \tag{5.1.D}$$

which satisfy (4)-(6) and, moreover, $\mathrm{Hom}_\Lambda(i_{j-1} \cdot q_j, \mathcal{S}) = 0$.

By splicing (5.1.C) and (5.1.D), we obtain the required partially minimal projective resolution. □

Setup 5.1.23. Under setup 5.1.5, assume further that Λ_{con} is t-relatively spherical and is perfect over Λ.

Proposition 5.1.24. *Under setup 5.1.23, fix $i \in I$. Then, $e_i\Lambda_{\mathrm{con}}$ admits a partially minimal projective resolution*

$$\begin{aligned} 0 \longrightarrow \mathcal{E}(X, Q_t) \oplus \mathcal{E}(X, \Sigma^{-t+1} X_i) \longrightarrow \mathcal{E}(X, Q_{t-1}) \longrightarrow \dots \longrightarrow \mathcal{E}(X, Q_1) \\ \longrightarrow \mathcal{E}(X, X_i) \longrightarrow e_i\Lambda_{\mathrm{con}} \longrightarrow 0 \end{aligned} \tag{5.1.E}$$

where

(1) *Each* $Q_j \in \operatorname{add} P$*,*

(2) *There exists a permutation* τ *of the set* I *such that* $X_{\tau(i)} \cong \Sigma^{-t+1} X_i$ *in* $\mathcal{D}$*.*

Proof. If X is projective, then Λ_{con} vanishes and so it is clear (5.1.E) can be constructed. So, suppose that X is not projective.

By 5.1.19, Λ_{con} admits a partially minimal projective resolution

$$\dots \xrightarrow{q_2} \mathcal{E}(X, W_1) \xrightarrow{q_1} \mathcal{E}(X, X_i) \to e_i\Lambda_{\text{con}} \to 0 \tag{5.1.F}$$

and, since under setup 5.1.23 Λ_{con} is perfect as a Λ-module, we may assume that (5.1.F) has finite length s. We will break this proof into several steps, where each step further refines (5.1.F).

(a) We will first show that $W_j \in \operatorname{add} P$ for $0 < j \leqslant s$ with $j \neq t$.

Recall that by the definition of partially minimal projective resolution, applying the functor $\operatorname{Hom}_\Lambda(-, \mathcal{S})$ to (5.1.F) yields

$$\operatorname{Hom}_\Lambda(\mathcal{E}(X, W_j), \mathcal{S}) \cong \operatorname{Ext}^j_\Lambda(\Lambda_{\text{con}}, \mathcal{S})$$

which is zero since $j \neq 0, t$ and Λ_{con} is relatively spherical. Thus, by 5.1.21 (3), $\mathcal{E}(X, W_j) \in \operatorname{add} e_0\Lambda$. By additive equivalence 5.1.10 (1), $W_j \in \operatorname{add} P$.

(b) The next step is to show that we may assume $s = t$. This requires some technical results. Observe that by (a) and 5.1.22, (5.1.F) can be split into sequences

$$0 \to \mathcal{E}(X, \ker \alpha_1) \xrightarrow{i_1} \mathcal{E}(X, W_1) \xrightarrow{\mathcal{E}(X,\alpha_1)} \mathcal{E}(X, X_i) \xrightarrow{q_0} e_i\Lambda_{\text{con}} \to 0 \tag{5.1.G}$$

$$0 \to \mathcal{E}(X, \ker \alpha_j) \xrightarrow{i_j} \mathcal{E}(X, W_j) \xrightarrow{\mathcal{E}(X,\alpha_j)} \mathcal{E}(X, \ker \alpha_{j-1}) \to 0. \tag{5.1.H}$$

$$0 \to \mathcal{E}(X, W_s) \xrightarrow{i_j} \mathcal{E}(X, W_{s-1}) \xrightarrow{\mathcal{E}(X,\alpha_{s-1})} \mathcal{E}(X, \ker \alpha_{s-2}) \to 0. \tag{5.1.I}$$

for $1 < j < s$. Here, $\alpha_1 \colon W_1 \to X_i$ and $\alpha_j \colon W_k \to \ker \alpha_{j-1}$ are admissible epimorphisms in $\mathcal{E}$ such that there are exact sequences

$$0 \to \ker \alpha_1 \to W_1 \to X_i \to 0 \tag{5.1.J}$$

$$0 \to \ker \alpha_j \to W_j \to \ker \alpha_{j-1} \to 0. \tag{5.1.K}$$

in $\mathcal{E}$.

We will specify $\ker \alpha_j$ for $1 \leqslant j \leqslant s$.

(b)(i) When $1 \leqslant j < t$, then $W_j \in \operatorname{add} P$ by (a). Hence, (5.1.J) implies that $\ker \alpha_1 \cong P_1 \oplus \Sigma^{-1} X$ for some projective P_1. Thus, inductively, it must be that $\ker \alpha_j \cong P_j \oplus \Sigma^{-j} X$ with $P_j \in \operatorname{add} P$.

(b)(ii) For $j > t$, it follows from (a) that $W_j \in \operatorname{add} P$, so that (5.1.K) implies that $\ker \alpha_j = P_j \oplus \Sigma^{-j+t} \ker \alpha_t$.

(c) We now show that we may assume that $s = t$. If $s > t$, then, by (5.1.I) and (b)(ii),

$$\mathcal{E}(X, W_s) \cong \mathcal{E}(X, \ker \alpha_{s-1}) = \mathcal{E}(X, P_{s-1} \oplus \Sigma^{-s+1+t} \ker \alpha_t).$$

Since $\mathcal{E}(X, W_s)$ is projective Λ-module, so is $\mathcal{E}(X, P_{s-1} \oplus \Sigma^{-s+1+t} \ker \alpha_t)$. It follows that W_s, $P_{s-1} \oplus \Sigma^{-s+1+t} \ker \alpha_t \in \operatorname{add} X$. Hence, additive equivalence 5.1.10 (1) implies that $W_s \cong P_{s-1} \oplus \Sigma^{-s+1+t} \ker \alpha_t$ in $\mathcal{E}$.

Since $W_s \in \operatorname{add} P$ by (a), it must be that $\Sigma^{-s+1+t} \ker \alpha_t = 0$ in $\mathcal{D}$. Because Σ is an automorphism of $\mathcal{D}$, this implies that $\ker \alpha_t = 0$ in $\mathcal{D}$. Thus, the exact sequence (5.1.K) for $j = t$ induces an isomorphism $W_t \cong \ker \alpha_{t-1}$ in $\mathcal{D}$.

Lifting this to $\mathcal{E}$ using 2.2.12, $Q \oplus W_t \cong Q' \oplus \ker \alpha_{t-1}$ in $\mathcal{E}$ with $Q, Q' \in \operatorname{add} P$. Since $Q \oplus W_t \in \operatorname{add}_{\mathcal{E}} X$, it follows that $\ker \alpha_{t-1} \in \operatorname{add}_{\mathcal{E}} X$. Therefore, the kernel $K_{t-1} = \mathcal{E}(X, \ker \alpha_{t-1})$ is projective. Whence, we may truncate the projective resolution at K_{t-1} and, thus, may assume $s = t$.

(d) We claim that we may assume $W_t = P_t \oplus \Sigma^{-t+1} X_i$ in $\mathcal{E}$ for $P_t \in \operatorname{add} P$. To see this, note that it is a direct a consequence of (c) that we may take $W_t = \ker \alpha_{t-1}$ in $\mathcal{E}$. Moreover, (b)(i) implies that $\ker \alpha_{t-1} \cong P_t \oplus \Sigma^{-t+1} X_i$ in $\mathcal{E}$ for $P_t \in \operatorname{add} P$. Hence, the claim follows.

(e) Since $W_t \in \operatorname{add}_{\mathcal{D}} X$ by construction, $\Sigma^{-t+1} X_i \in \operatorname{add}_{\mathcal{D}} X$. Moreover, $\Sigma^{-t+1} X_i$ must be indecomposable in $\mathcal{D}$ because X_i is indecomposable and Σ is an automorphism. Thus, $\Sigma^{-t+1} X_i \cong X_j$ for some $j \in I$.

By combining (a), (c), (d) and (e), we may write (5.1.A) as

$$0 \to \mathcal{E}(X, Q_t) \oplus \mathcal{E}(X, \Sigma^{-t+1} X_i) \to \ldots \to \mathcal{E}(X, Q_1) \to \mathcal{E}(X, X_i) \to e_i \Lambda_{\text{con}} \to 0 \quad (5.1.L)$$

where $Q_t \in \operatorname{add} P$. Thus, (1) holds.

To prove (2), note that (e) implies that $\Sigma^{-t+1} X_i \cong X_j$ in $\mathcal{D}$ for some $j \in I$. Since Σ is an automorphism, the association $\tau \colon i \mapsto j$ is a permutation. $\square$

If we place additional assumptions on Λ_{con}, we obtain finer control over the resolution. The assumptions that we place on Λ_{con} will imply that it is self-injective and, as a result, the permutation that we get in 5.1.24 is the Nakayama permutation of Λ_{con} (see § 2.1 for a brief overview of self-injective algebras and the Nakayama permutation).

Setup 5.1.25. Under setup 5.1.23, assume $t = d := \dim \mathcal{R}$ with $d \geqslant 2$. Suppose further that $\dim_k \Lambda_{\text{con}} < \infty$, and that there is a Λ-bimodule isomorphism $\Lambda_{\text{con}} \otimes^{\mathbf{L}}_{\Lambda} \omega_{\Lambda} \cong \Lambda_{\text{con}}$

.

Corollary 5.1.26. *Under setup 5.1.25, the following hold.*

(1) Λ_{con} *is self-injective,*

(2) *for $i \in I$, $e_i\Lambda_{\mathrm{con}}$ admits the following partially minimal projective resolution*

$$0 \longrightarrow \mathcal{E}(X, Q_d) \oplus \mathcal{E}(X, X_{\sigma(i)}) \longrightarrow \mathcal{E}(X, Q_{d-1}) \longrightarrow \dots \longrightarrow \mathcal{E}(X, Q_1) \longrightarrow \mathcal{E}(X, X_i) \longrightarrow e_i\Lambda_{\mathrm{con}} \longrightarrow 0$$

where $Q_j \in \operatorname{add} P$ for all $1 \leqslant j \leqslant d$ and σ is the Nakayama permutation of Λ_{con}.

Proof. Observe that Λ_{con} is finite-dimensional, and so it is self-injective if and only if $\operatorname{Ext}^1_{\Lambda_{\mathrm{con}}}(\mathcal{S}, \Lambda_{\mathrm{con}}) = 0$. This latter condition holds by the following chain of vector space isomorphisms

$$\begin{aligned}
\operatorname{Ext}^1_{\Lambda_{\mathrm{con}}}(\mathcal{S}, \Lambda_{\mathrm{con}}) &\cong \operatorname{Ext}^1_{\Lambda}(\mathcal{S}, \Lambda_{\mathrm{con}}) && \text{(by 2.5.6)}\\
&\cong \operatorname{Ext}^1_{\Lambda}(\mathcal{S}, \Lambda_{\mathrm{con}} \otimes^{\mathbf{L}}_{\Lambda} \omega_\Lambda) && \text{(by assumptions in 5.1.25)}\\
&\cong D\operatorname{Ext}^{d-1}_{\Lambda}(\Lambda_{\mathrm{con}}, \mathcal{S}) && \text{(by Serre duality (4.1.B))}\\
&= 0 && \text{(Λ_{con} is d-relatively spherical with $d \geqslant 2$.)}
\end{aligned}$$

Thus, (1) holds. Let σ be the Nakayama permutatiom of Λ_{con}.

To prove (2), by 5.1.24 and since $t = d$, $e_i\Lambda_{\mathrm{con}}$ admits the partially minimal projective resolution

$$0 \longrightarrow \mathcal{E}(X, Q_d) \oplus \mathcal{E}(X, \Sigma^{-d+1}X_i) \longrightarrow \mathcal{E}(X, Q_{d-1}) \longrightarrow \dots \longrightarrow \mathcal{E}(X, Q_1) \longrightarrow \mathcal{E}(X, X_i) \longrightarrow e_i\Lambda_{\mathrm{con}} \longrightarrow 0 \tag{5.1.M}$$

where $W_k \in \operatorname{add} P$ for all $1 \leqslant k \leqslant d$ and $\Sigma^{-d+1}X_i \cong X_j$ in $\mathcal{D}$ for some $j \in I$. Hence, it suffices to prove that $j = \sigma(i)$.

As in 5.1.9, let S_j be the simple Λ_{con}-module which is a quotient of $e_j\Lambda_{\mathrm{con}}$. By applying $\operatorname{Hom}_\Lambda(-, S_j)$ to the projective resolution (5.1.M), we have that

$$\begin{aligned}
\operatorname{Hom}_\Lambda(\mathcal{E}(X, Q_d) \oplus \mathcal{E}(X, \Sigma^{-d+1}X_i), S_j) &\cong \operatorname{Ext}^d_\Lambda(e_i\Lambda_{\mathrm{con}}, S_j) && \text{(by partial minimality)}\\
&\cong D\operatorname{Hom}_\Lambda(S_j, e_i\Lambda_{\mathrm{con}} \otimes^{\mathbf{L}} \omega_\Lambda) && \text{(by Serre duality (4.1.B))}\\
&\cong D\operatorname{Hom}_\Lambda(S_j, e_i\Lambda_{\mathrm{con}}) && \text{(by assumptions in 5.1.25)}\\
&\cong \delta_{j\sigma(i)}S_j && \text{(by 2.1.7)}
\end{aligned}$$

Consequently,

$$\mathrm{Hom}_\Lambda(\mathcal{E}(X, Q_d) \oplus \mathcal{E}(X, \Sigma^{-d+1}X_i), S_{\sigma(i)}) = \mathrm{Hom}_\Lambda(\mathcal{E}(X, \Sigma^{-d+1}X_i), S_{\sigma(i)}) \neq 0$$

Therefore, by 5.1.21 (1), $\Sigma^{-d+1}X_i$ cannot be isomorphic to X_k for $k \neq \sigma(i)$. Since $\Sigma^{-d+1}X_i \cong X_j$ for some j, it must be that $\Sigma^{-d+1}X_i \cong X_{\sigma(i)}$ in $\mathcal{D}$. □

§ 5.1.4 Periodic suspension

With the partially minimal projective resolution of Λ_{con} constructed, we are able to specify precisely when the action of the suspension functor is periodic up to additive closure.

Definition 5.1.27. Let $Y \in \mathcal{D}$ and $k \in \mathbb{Z}$.

(1) If $k > 0$, we say that Y is *k-rigid* if $\mathrm{Ext}^i_{\mathcal{D}}(Y, Y) = 0$ for all $1 \leqslant i \leqslant k$.

(2) If $k < 0$, we say that Y is *k-rigid* if $\mathrm{Ext}^i_{\mathcal{D}}(Y, Y) = 0$ for all $k \leqslant i \leqslant -1$.

Theorem 5.1.28. *Under setup 5.1.5, fix $t \in \mathbb{Z}$ with $2 \leqslant t \leqslant d$. Then, the following are equivalent.*

(1) *Λ_{con} is perfect over Λ, and is t-relatively spherical in the sense of 5.1.20.*

(2) *X is $(-t+2)$-rigid in $\mathcal{D}$ and there exists a permutation τ of I such that, for all $i \in I$, $\Sigma^{-t+1}X_i \cong X_{\tau(i)}$ in $\mathcal{D}$.*

Proof. We begin by proving (1) ⇒ (2). If Λ_{con} is t-relatively spherical and perfect over Λ, then it follows from 5.1.24(2) that there exists a permutation τ of I such that $X_{\tau(i)} \cong \Sigma^{-t+1}X_i$ for all $i \in I$.

To see that X is $(-t+2)$-rigid, fix $i \in I$ and note that via (5.1.K) and (b)(i) in the proof of 5.1.24, there are exact sequences

$$0 \to P_j \oplus \Sigma^{-j}X_i \to W_j \xrightarrow{\alpha_j} P_{j-1} \oplus \Sigma^{-j+1}X_i \to 0 \tag{5.1.N}$$

for $1 < j < t$. Applying $\mathcal{E}(X, -)$ to these induces following diagram where the first column is exact.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{E}(X, P_j \oplus \Sigma^{-j} X_i) \arrow[r, equal] \arrow[d, "i_j"] & \mathcal{E}(X, P_j \oplus \Sigma^{-j} X_i) \arrow[d, "i_j"] \\
\mathcal{E}(X, W_j) \arrow[r, equal] \arrow[d, "q_j"] & \mathcal{E}(X, W_j) \arrow[d, "q_j"] \\
\mathcal{E}(X, P_{j-1} \oplus \Sigma^{-j+1} X_i) \arrow[r, equal] \arrow[d] & \mathcal{E}(X, P_{j-1} \oplus \Sigma^{-j+1} X_i) \arrow[d] \\
\mathcal{E}^1(X, P_j \oplus \Sigma^{-j} X_i) \arrow[d] & 0 \\
\mathcal{E}^1(X, W_j) &
\end{tikzcd}$$

The second column is exact by (5.1.H) for $1 < j < t$. Since $W_j \in \operatorname{add} P$ and is, therefore, injective, $\operatorname{Ext}^1_{\mathcal{E}}(X, W_j) = 0$. It thus follows that $\operatorname{Ext}^1_{\mathcal{E}}(X, P_j \oplus \Sigma^{-j} X_i) = 0$ for $1 < j < t$. By 2.2.14, if Ext^1 in $\mathcal{E}$ vanishes then so does Ext^1 in $\mathcal{D}$. Thus,

$$0 = \operatorname{Ext}^1_{\mathcal{D}}(X, P_j \oplus \Sigma^{-j} X_i) = \operatorname{Ext}^1_{\mathcal{D}}(X, \Sigma^{-j} X_i) \cong \operatorname{Ext}^{-j+1}_{\mathcal{D}}(X, X_i)$$

and so $\operatorname{Ext}^k_{\mathcal{D}}(X, X_i) = 0$ for $-t + 2 \leqslant j \leqslant -1$. Since this is true for all $i \in I$, we may conclude that X is $(-t + 2)$-rigid.

For the implication (2) $\Rightarrow$ (1), fix $i \in I$. Since $\mathcal{E}$ has enough projectives, there are exact sequences

$$0 \to \Sigma^{-j-1} X_i \xrightarrow{\iota_j} Q_j \xrightarrow{p_j} \Sigma^{-j} X_i \to 0 \tag{5.1.O}$$

for $0 \leqslant j \leqslant t-2$ with $Q_j \in \operatorname{add} P$. Applying $\mathcal{E}(X, -)$ to (5.1.O) yields exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}(X, \Sigma^{-j-1} X_i) \arrow[r, "\mathcal{E}(X, \iota_j)"] & \mathcal{E}(X, Q_j) \arrow[r, "\mathcal{E}(X, p_j)"] & \mathcal{E}(X, \Sigma^{-j} X_i) \arrow[dll, rounded corners] \\
 & & \mathcal{E}^1(X, \Sigma^{-j-1} X_i) \arrow[r] & 0
\end{tikzcd} \tag{5.1.P}$$

because Q_j is projective/injective and so $\operatorname{Ext}^1_{\mathcal{E}}(X, Q_j) = 0$. By assumption, X is $(-t + 2)$-rigid so that

$$0 = \operatorname{Ext}^k_{\mathcal{D}}(X, X_i) = \operatorname{Ext}^1_{\mathcal{D}}(X, \Sigma^{k-1} X_i)$$

for $-t + 2 \leqslant k \leqslant -1$. Whence, again by 2.2.14, $\mathcal{E}(X, p_k)$ is an epi for all such k.

Therefore, (5.1.P) induces exact sequences

$$0 \to \mathcal{E}(X, \Sigma^{-j-1}X_i) \xrightarrow{\mathcal{E}(X,\iota_j)} \mathcal{E}(X, Q_j) \xrightarrow{\mathcal{E}(X,p_j)} \mathcal{E}(X, \Sigma^{-j}X_i) \to 0 \tag{5.1.Q}$$

$$0 \to \mathcal{E}(X, \Sigma^{-1}X_i) \xrightarrow{\mathcal{E}(X,\iota_0)} \mathcal{E}(X, Q_0) \xrightarrow{\mathcal{E}(X,p_0)} \mathcal{E}(X, X_i) \to C \to 0 \tag{5.1.R}$$

for $1 \leqslant j \leqslant t-2$. Here, C denotes the cokernel of $\mathcal{E}(X, p_0)$.

We claim that $C \cong e_i\Lambda_{\mathrm{con}}$ as Λ-modules. It is clear that $\operatorname{Im}\mathcal{E}(X, p_0) \subseteq e_i[\operatorname{proj}\mathcal{E}]$. The inclusion $e_i[\operatorname{proj}\mathcal{E}] \subseteq \operatorname{Im}\mathcal{E}(X, p_0)$ follows from the fact that p_0 is a deflation so that any morphism from a projective to X_i must factor through p_0. Hence,

$$C \cong \mathcal{E}(X, X_i)/e_i[\operatorname{proj}\mathcal{E}] \cong e_i\Lambda_{\mathrm{con}}. \tag{5.1.S}$$

Thus, by splicing (5.1.Q) for $q \leqslant j \leqslant t-2$ and (5.1.R), we obtain the exact sequence,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}(X, \Sigma^{-t+1}X_i) \arrow[r] & \mathcal{E}(X, Q_{t-2}) \arrow[r] & \dots \arrow[r] & \mathcal{E}(X, Q_0) \arrow[dll, rounded corners] \\ & & \mathcal{E}(X, X_i) \arrow[r] & e_i\Lambda_{\mathrm{con}} \arrow[r] & 0 \end{tikzcd} \tag{5.1.T}$$

where $Q_j \in \operatorname{add} P$ for $0 \leqslant j \leqslant t-2$. By assumption, $\Sigma^{-t+1}X_i \cong X_{\tau(i)}$ so that $\Sigma^{-t+1}X_i \in \operatorname{add} X$. It follows that (5.1.T) is a projective resolution of $e_i\Lambda_{\mathrm{con}}$. Hence, $e_i\Lambda_{\mathrm{con}}$ is perfect over Λ. Since this is true for all $i \in I$, Λ_{con} is perfect over Λ.

Due to 5.1.21(2), applying $\operatorname{Hom}_\Lambda(-, \mathcal{S})$ to (5.1.T), yields that $\operatorname{Ext}^k_\Lambda(e_i\Lambda_{\mathrm{con}}, \mathcal{S})$ vanishes for all $k \neq 0, t$. For $k = t$,

$$\begin{aligned} \operatorname{Ext}^t_\Lambda(e_i\Lambda_{\mathrm{con}}, S_j) &\cong \operatorname{Hom}_\Lambda(\mathcal{E}(X, X_{\tau(i)}), S_j) \\ &\cong \operatorname{Hom}_\Lambda(e_{\tau(i)}\Lambda, S_j) \\ &\cong \delta_{\tau(i)j}S_j & \text{(by 5.1.21(1).)} \end{aligned}$$

Thus, we may conclude that (1) holds. □

Corollary 5.1.29. *Assume setup 5.1.25 with $d \geqslant 3$. Then, the following are equivalent*

(1) *Λ_{con} is perfect over Λ and d-relatively spherical,*

(2) *X is $(-d+2)$-rigid in $\mathcal{D}$ and there exists a permutation τ of I such that, for all $i \in I$, $\Sigma^{-d+1}X_i \cong X_{\tau(i)}$ in $\mathcal{D}$.*

If these conditions are satisfied, then Λ_{con} is self-injective and $\tau = \sigma$, the Nakayama permutation of Λ_{con}.

Proof. The equivalence (1) ⇔ (2) is the $t = d$ case of 5.1.28. The fact Λ_{con} is self-injective follows from 5.1.26(1). The fact that $\tau = \sigma$ follows by combining the Ext calculations in the proof of 5.1.26(2) and the Ext calculations at the end of the proof of 5.1.28. □

Corollary 5.1.30. *Under setup 5.1.5, suppose that either of the equivalent assumptions in 5.1.28 hold. Then, Λ_{con} is perfect as both a right and a left Λ-module, and X is $(t-2)$-rigid.*

Proof. In order to construct a projective resolution of Λ_{con} as a left Λ-module, recall that for $j \in \mathbb{Z}$ we have exact sequences in $\mathcal{E}$,

$$0 \to \Sigma^{j-1}X_i \xrightarrow{\iota_j} W_j \xrightarrow{p_j} \Sigma^j X_i \to 0$$

with $W_j \in \operatorname{proj}\mathcal{E}$. We may apply $\mathcal{E}(-, X)$ to these exact sequences so that, since X is $(-t+2)$-rigid in $\mathcal{D}$, the sequences

$$0 \to \mathcal{E}(\Sigma^j X_i, X) \to \mathcal{E}(W_j, X) \to \mathcal{E}(\Sigma^{j-1}X_i, X) \to 0 \tag{5.1.U}$$

$$0 \to \mathcal{E}(\Sigma X_i, X) \xrightarrow{\mathcal{E}(p_1,X)} \mathcal{E}(W_1, X) \xrightarrow{\mathcal{E}(\iota_1,X)} \mathcal{E}(X_i, X) \to C \to 0. \tag{5.1.V}$$

are exact for $2 \leqslant j \leqslant t-1$. Here, C denotes the cokernel of $\mathcal{E}(\iota_i, X)$. Similarly to the proof of 5.1.28 (2) $\Rightarrow$ (1), $\operatorname{Im}\mathcal{E}(i_0, X) = [\operatorname{proj}\mathcal{E}]e_i$ so that $C \cong \Lambda_{\text{con}}e_i$. Hence, we may splice the sequences (5.1.U) for $2 \leqslant j \leqslant t-1$ and (5.1.V) together to obtain the exact sequence,

$$0 \to \mathcal{E}(\Sigma^{t-1}X_i, X) \to \mathcal{E}(W_{t-1}, X) \to \ldots \to \mathcal{E}(W_1, X) \to \mathcal{E}(X_i, X) \to \Lambda_{\text{con}}e_i \to 0.$$

Since $\Sigma^{t-1}X_i \in \operatorname{add}X$ by assumption, we obtain a projective resolution for $\Lambda_{\text{con}}e_i$ of length t. By taking the sum of $\Lambda_{\text{con}}e_i$ over all $i \in I$, we obtain a finite projective resolution for Λ_{con} as a left Λ-module.

To see that X is $(t-2)$-rigid, consider j such that $1 \leqslant j \leqslant t-2$. Because X is $(-t+2)$-rigid in $\mathcal{D}$ and $\Sigma^{t-1}X_i \cong X_{\tau(i)}$, then

$$\operatorname{Hom}_{\mathcal{D}}(X, \Sigma^j X_i) = \operatorname{Hom}_{\mathcal{D}}(X, \Sigma^j\Sigma^{-t+1}X_{\tau(i)}) = \operatorname{Hom}_{\mathcal{D}}(X, \Sigma^{-t+1+j}X_{\tau(i)}) = 0$$

since $-t+2 \leqslant -t+1+j \leqslant -1$. □

§ 5.2 Spherical twists induced by Frobenius categories

Assuming setup 5.1.5, this section specifies when the restriction of scalars functor F induced by quotient morphism $\pi\colon \operatorname{End}_{\mathcal{E}}(X) \to \underline{\operatorname{End}}_{\mathcal{E}}(X)$ is spherical.

§ 5.2.1 Tilting

We first extend [DW16, section 5.3] to the setting 5.1.5 and show that the noncommutative twist functor

$$\mathcal{T} := \mathbf{R}\operatorname{Hom}_{\Lambda}([\operatorname{proj}\mathcal{E}], -)\colon\ \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$$

is an autoequivalence, by proving that it is functorially isomorphic to the composition of standard equivalences induced by tilting modules.

Unless mentioned otherwise, we assume the following throughout.

Setup 5.2.1. Under setup 5.1.5, assume that either of the equivalent assumptions in 5.1.28 hold.

Some tilting bimodules

Notation 5.2.2. For $k \in \mathbb{Z}$ consider

(1) the algebra $\Lambda_k := \operatorname{End}_{\mathcal{E}}(P \oplus \Sigma^k X)$,

(2) the Λ_{k-1}-Λ_k bimodules $D_k := \mathcal{E}(P \oplus \Sigma^k X, P \oplus \Sigma^{k-1} X)$.

(3) the Λ_k-Λ_{k-1} bimodules $I_k := \mathcal{E}(P \oplus \Sigma^{k-1} X, P \oplus \Sigma^k X)$.

Proposition 5.2.3. *The Λ-Λ_{-1}-bimodule $I_0 = \mathcal{E}(P \oplus \Sigma^{-1} X, X)$ is tilting.*

Proof. This is essentially [JY22, 5.1]. It suffices to prove criteria (a)-(c) of [Miy03, 1.8]. Namely,

(a) The bimodule I_0 is perfect as a right Λ_{-1}-module and as a left Λ-module.

(b) The right multiplication map $\rho\colon \Lambda_{-1} \to \mathbf{R}\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_0, I_0)$ is a Λ_{-1}-bimodule isomorphism.

(c) The left multiplication map $\lambda\colon \Lambda \to \mathbf{R}\mathrm{Hom}_{\Lambda_{-1}}(I_0, I_0)$ is a Λ-bimodule isomorphism.

There are exact sequences

$$0 \to P \oplus \Sigma^{k-1} X \xrightarrow{\iota_k} Q_k \xrightarrow{p_k} P \oplus \Sigma^k X \to 0 \tag{5.2.A}$$

in $\mathcal{E}$ to which we may apply $\mathcal{E}(-, X)$ and $\mathcal{E}(P \oplus \Sigma^{-1} X, -)$. Since $\operatorname{Ext}^1_{\mathcal{D}}(X, X)$ vanishes by assumptions 5.2.1, then it follows from 2.2.14 and its dual that the morphisms $\mathcal{E}(\iota_1, X)$ and $\mathcal{E}(P \oplus \Sigma^{-1} X, p_0)$ are epi. Hence, the projective resolutions of I_0 as a Λ^{op}-module and as Λ_{-1}-module are

$$0 \to \mathcal{E}(X, X) \to \mathcal{E}(Q_0, X) \to I_0 \to 0 \tag{5.2.B}$$

$$0 \to \mathcal{E}(P \oplus \Sigma^{-1} X, P \oplus \Sigma^{-1} X) \to \mathcal{E}(P \oplus \Sigma^{-1} X, Q_0) \to I_0 \to 0, \tag{5.2.C}$$

respectively. Thus, I_0 is biperfect.

To prove that $\rho\colon \Lambda_{-1} \to \mathbf{R}\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_0, I_0)$ is a bimodule isomorphism, apply the functor $\mathrm{Hom}_{\Lambda}(-, I_0)$ to the exact sequence (5.2.B) and $\mathcal{E}(P \oplus \Sigma^{-1} X, -)$ to the exact sequence (5.2.A) for $k = 0$. Since $\mathcal{E}(P \oplus \Sigma^{-1} X, p_0)$ is an epimorphism, the following diagram with exact columns commutes.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{E}(P\oplus\Sigma^{-1}X, P\oplus\Sigma^{-1}X) \arrow[r, "f_0=\mathcal{E}(-,X)"] \arrow[d] & \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_0, I_0) \arrow[d] \\
\mathcal{E}(P\oplus\Sigma^{-1}X, Q_0) \arrow[r, "g_0=\mathcal{E}(-,X)"] \arrow[d] & \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathcal{E}(Q_0, X), I_0) \arrow[d] \\
\mathcal{E}(P\oplus\Sigma^{-1}X, X) \arrow[r, "f_{-1}=\mathcal{E}(-,X)"] \arrow[d] & \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathcal{E}(X, X), I_0) \arrow[d] \\
0 & \mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathcal{E}(P\oplus\Sigma^{-1}X, X), I_0) \arrow[d] \\
 & 0
\end{tikzcd}$$

By additive equivalence 5.1.10 (1), f_{-1} and g_0 are isomorphisms, so that f_0 is also an isomorphism. Hence, $\mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(I_0, I_0) = 0$ so that $\mathbf{R}\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(I_0, I_0)$ is concentrated at degree zero and $\rho = f_0$, which is an isomorphism.

For the proof that $\lambda\colon \Lambda \to \mathbf{R}\mathrm{Hom}_{\Lambda_{-1}}(I_0, I_0)$ is an isomorphism, we apply $\mathcal{E}(-, X)$ to the exact sequence (5.2.A) for $k = 0$ and $\mathrm{Hom}_{\Lambda_{-1}}(-, I_0)$ to the the exact sequence (5.2.C). We thus have the commutative diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{E}(X, X) \arrow[r, "h_0=\mathcal{E}(P\oplus\Sigma^{-1}X,-)"] \arrow[d] & \mathrm{Hom}_{\Lambda_{-1}}(I_0, I_0) \arrow[d] \\
\mathcal{E}(Q_0, X) \arrow[r, "g_0'=\mathcal{E}(P\oplus\Sigma^{-1}X,-)"] \arrow[d] & \mathrm{Hom}_{\Lambda_{-1}}(\mathcal{E}(P\oplus\Sigma^{-1}X, Q_0), I_0) \arrow[d] \\
\mathcal{E}(P\oplus\Sigma^{-1}X, X) \arrow[r, "h_{-1}=\mathcal{E}(P\oplus\Sigma^{-1}X,-)"] \arrow[d] & \mathrm{Hom}_{\Lambda_{-1}}(\mathcal{E}(P\oplus\Sigma^{-1}X, P\oplus\Sigma^{-1}X), I_0) \arrow[d] \\
0 & \mathrm{Ext}^1_{\Lambda_{-1}}(I_0, I_0) \arrow[d] \\
 & 0
\end{tikzcd}$$

Since h_{-1} and g_0' are isomorphisms by 5.1.10 (1), so is h_0. Hence, $\mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(I_0, I_0) = 0$ so that $\mathbf{R}\mathrm{Hom}_{\Lambda_{-1}}(I_0, I_0)$ is concentrated at degree zero and $\lambda = h_0$ is an isomorphism. □

Proposition 5.2.4. *The Λ_{-1}-Λ-bimodule $D_0 = \mathcal{E}(X, P \oplus \Sigma^{-1}X)$ is tilting.*

Proof. As in 5.2.3, it suffices to prove criteria (a)-(c) of [Miy03, 1.8] for D_0. The exact sequences (5.1.Q) and their dual show that D_0 is biperfect. For the statement that the corresponding left multiplication map λ is an isomorphism, note that we may apply $\mathrm{Hom}_{\Lambda}(-, D_0)$ to the exact sequences (5.1.Q) and $\mathcal{E}(-, P \oplus \Sigma^{-1}X)$ to the exact sequences (5.2.A) for $-t+2 \leqslant k \leqslant -1$. By 5.1.30, X is $(t-2)$-rigid in $\mathcal{D}$ so

$$\mathrm{Ext}^1_{\mathcal{D}}(P \oplus \Sigma^k X, P \oplus \Sigma^{-1} X) = \mathrm{Ext}^{-k-1}_{\mathcal{D}}(P \oplus X, P \oplus X) = 0.$$

Hence, $\mathcal{E}(\iota_k, P \oplus \Sigma^{-1} X)$ is an epimorphism by 2.2.14.

Therefore, the following diagram with exact columns commutes for $-t+2 \leqslant k \leqslant -1$.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{E}(P \oplus \Sigma^k X, P \oplus \Sigma^{-1} X) \arrow[r, "f_k=\mathcal{E}(X,-)"] \arrow[d] & \mathrm{Hom}_\Lambda(\mathcal{E}(X, P \oplus \Sigma^k X), D_0) \arrow[d] \\
\mathcal{E}(Q_k, P \oplus \Sigma^{-1} X) \arrow[r, "g_k=\mathcal{E}(X,-)"] \arrow[d] & \mathrm{Hom}_\Lambda(\mathcal{E}(X, Q_k), D_0) \arrow[d] \\
\mathcal{E}(P \oplus \Sigma^{k-1} X, P \oplus \Sigma^{-1} X) \arrow[r, "f_{k-1}=\mathcal{E}(X,-)"] \arrow[d] & \mathrm{Hom}_\Lambda(\mathcal{E}(X, P \oplus \Sigma^{k-1} X), D_0) \arrow[d] \\
0 & \mathrm{Ext}^1_\Lambda(\mathcal{E}(X, P \oplus \Sigma^k X), D_0) \arrow[d] \\
 & 0
\end{tikzcd}$$

For $k-1 = -t+1$, $P \oplus \Sigma^{-t+1} X \in \mathrm{add}_{\mathcal{E}}\, X$. Since $Q_k \in \mathrm{add}_{\mathcal{E}}\, P$, it follows from 5.1.10 (1) that f_{-t+1} and g_{-t+2} are isomorphisms. Thus, f_{-t+2} must be an isomorphism. Proceeding inductively, for each $-t+2 \leqslant k \leqslant -1$, we find that f_k is an isomorphism. In particular, f_{-1} is an isomorphism so that

$$\mathrm{Ext}^1_\Lambda(D_0, D_0) = \mathrm{Ext}^1_\Lambda(\mathcal{E}(X, P \oplus \Sigma^{-1} X), D_0) = 0.$$

It follows that $\mathbf{R}\mathrm{Hom}_\Lambda(D_0, D_0)$ is concentrated at degree zero and $\lambda = f_{-1}$ is an isomorphism.

For the proof that ρ is an isomorphism, we apply $\mathcal{E}(X, -)$ to the exact sequences (5.2.A) and also $\mathrm{Hom}_{\Lambda_{-1}^{\mathrm{op}}}(-, D_0)$ to the the exact sequences

$$0 \to \mathcal{E}(P \oplus \Sigma^k X, P \oplus \Sigma^{-1} X) \to \mathcal{E}(Q_k, P \oplus \Sigma^{-1} X) \to \mathcal{E}(P \oplus \Sigma^{k-1} X, P \oplus \Sigma^{-1} X) \to 0$$

for $1 \leqslant k \leqslant t-2$. Letting

$$h_k = \mathcal{E}(-, P \oplus \Sigma^{-1} X) \colon \mathcal{E}(X, P \oplus \Sigma^k X) \to \mathrm{Hom}_{\Lambda_{-1}^{\mathrm{op}}}(\mathcal{E}(P \oplus \Sigma^k X, P \oplus \Sigma^{-1} X), \mathcal{E}(X, P \oplus \Sigma^{-1} X)),$$

we inductively show that h_k are isomorphisms for all $1 \leqslant k \leqslant t-2$. Therefore, $\mathbf{R}\mathrm{Hom}_{\Lambda_{-1}^{\mathrm{op}}}(D_0, D_0)$ is concentrated in degree zero and $\rho = h_1$. □

Corollary 5.2.5. *For any $k \in \mathbb{Z}$, the bimodules $I_k = \mathcal{E}(P \oplus \Sigma^{k-1} X, P \oplus \Sigma^k X)$ and $D_k = \mathcal{E}(P \oplus \Sigma^k X, P \oplus \Sigma^{k-1} X)$ are tilting.*

Proof. By 5.1.6, if X satisfies setup 5.1.5, so does $P \oplus \Sigma^k X$. Moreover, it is clear that if X satisfies 5.1.28(2), so does $P \oplus \Sigma^k X$. Thus, $P \oplus \Sigma^k X$ satisfies setup 5.2.1, and so we may apply 5.2.3 and 5.2.4 to $P \oplus \Sigma^k X$. $\square$

Therefore, the functors

$$\begin{aligned} \Psi_k &:= \mathbf{R}\mathrm{Hom}_{\Lambda_{k-1}}(I_k, -)\colon \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_{k-1}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_k) \\ \Phi_k &:= \mathbf{R}\mathrm{Hom}_{\Lambda_k}(D_k, -)\colon \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_k) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_{k-1}) \end{aligned}$$

are equivalences and, thus, so is their composition

$$\mathbf{R}\mathrm{Hom}_{\Lambda}(I_0 \otimes^{\mathbf{L}}_{\Lambda_{-1}} D_0, -)\colon \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda). \tag{5.2.D}$$

If $t > 2$, then we further consider the composition

$$\mathbf{R}\mathrm{Hom}_{\Lambda}(I_{t-1} \otimes^{\mathbf{L}}_{\Lambda_{t-2}} I_{t-2} \otimes^{\mathbf{L}}_{\Lambda_{t-1}} \cdots \otimes^{\mathbf{L}}_{\Lambda_1} I_1, -)\colon \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_{t-1}), \tag{5.2.E}$$

which is also an equivalence. We will mildly abuse notation and write $\Psi_0 \cdot \Phi_0$ for (5.2.D) and Ψ_1^{t-1} for the (5.2.E). Since $\mathrm{add}_{\mathcal{E}}\, \Sigma^{-t+1} X = \mathrm{add}_{\mathcal{D}}\, X$, there is a Morita equivalence between Λ_{t-1} and Λ. Hence, $\Psi_0 \cdot \Phi_0$ and Ψ_1^{t-1} be viewed as autoequivalences of Λ.

For the remainder of this section we will prove two statements. First, we claim that $\Psi_0 \cdot \Phi_0$ is naturally isomorphic to the functor

$$\mathcal{T} := \mathbf{R}\mathrm{Hom}_{\Lambda}([\mathrm{proj}\,\mathcal{E}], -)\colon \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda).$$

Moreover, for $t > 2$, we construct a Morita equivalence $\mathbb{F}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_{t-1}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ such that $\mathbb{F} \cdot \Psi_1^{t-1} \cong \mathcal{T}$.

Tracking the composition $\Psi_0 \cdot \Phi_0$

To see that $\Psi_0 \cdot \Phi_0$ is naturally isomorphic to $\mathcal{T}$, it suffices to prove the following.

Proposition 5.2.6. *There is a Λ-bimodule isomorphism $I_0 \otimes^{\mathbf{L}}_{\Lambda_{-1}} D_0 \xrightarrow{\sim} [\mathrm{proj}\,\mathcal{E}]$.*

Proof. We first claim that the complex $I_0 \otimes^{\mathbf{L}}_{\Lambda_{-1}} D_0$ is concentrated in degree zero, so that $I_0 \otimes^{\mathbf{L}}_{\Lambda_{-1}} D_0 \cong I_0 \otimes_{\Lambda_{-1}} D_0$.

To see this, recall that the projective resolution of I_0 as Λ_{-1}-module is (5.2.C). Hence, $I_0 \otimes^{\mathbf{L}}_{\Lambda_{-1}} D_0$ is represented by the chain complex

$$0 \to \mathcal{E}(P \oplus \Sigma^{-1} X, P \oplus \Sigma^{-1} X) \otimes_{\Lambda_{-1}} D_0 \xrightarrow{f} \mathcal{E}(P \oplus \Sigma^{-1} X, Q_0) \otimes_{\Lambda_{-1}} D_0 \to 0$$

Thus, to show the claim that $I_0 \otimes^{\mathbf{L}}_{\Lambda_{-1}} D_0$ has cohomology in degree zero, it suffices to check that f is a monomorphism.

Now, by applying $- \otimes D_0$ to (5.2.C), we obtain first column of the following commutative diagram with exact columns.

$$\begin{tikzcd}
 & 0 \arrow[d] \\
\mathcal{E}(P \oplus \Sigma^{-1}X, P \oplus \Sigma^{-1}X) \otimes_{\Lambda_{-1}} D_0 \arrow[r, "m_{-1}"] \arrow[d, "f"] & \mathcal{E}(X, P \oplus \Sigma^{-1}X) \arrow[d] \\
\mathcal{E}(P \oplus \Sigma^{-1}X, Q_0) \otimes_{\Lambda_{-1}} D_0 \arrow[r, "n_0"] \arrow[d] & \mathcal{E}(X, Q_0) \arrow[d] \\
I_0 \otimes_{\Lambda_{-1}} D_0 \arrow[r, "m_0"] \arrow[d] & \mathcal{E}(X, X) \arrow[d, "\pi"] \\
0 & \Lambda_{\mathrm{con}} \arrow[d] \\
 & 0
\end{tikzcd} \tag{5.2.F}$$

The second column is obtained by applying $\mathcal{E}(X, -)$ to the exact sequence (5.2.A) for $k = 0$. The horizontal maps are the natural composition maps and commutativity can be checked by diagram chasing. Since m_{-1} and n_0 are isomorphisms by 5.1.10 (2), it follows that f is a monomorphism, as required.

It remains to show that there is a Λ-bimodule isomorphism $I_0 \otimes_{\Lambda_{-1}} D_0 \cong [\operatorname{proj} \mathcal{E}]$. This follows from the snake lemma applied to the diagram (5.2.F), since it implies that

$$I_0 \otimes_{\Lambda_{-1}} D_0 \cong \ker \pi = [\operatorname{proj} \mathcal{E}].$$

Moreover, since both m_0 and π are bimodule morphisms, this isomorphism is indeed a Λ-bimodule isomorphism. □

Tracking the composition Ψ_1^{t-1}

In order to show that, up to a Morita equivalence, $\mathcal{T}$ is naturally isomorphic to Ψ_1^{t-1}, we will establish some lemmas.

Lemma 5.2.7. *There exists a Morita equivalence* $\mathbb{F}\colon \mathrm{D^b}(\operatorname{mod} \Lambda_{t-1}) \to \mathrm{D^b}(\operatorname{mod} \Lambda)$

Proof. Let $\Lambda_{t-1,\mathrm{con}} := \underline{\operatorname{End}}_{\mathcal{E}}(\Sigma^{t-1}X)$, and write $[\operatorname{proj} \mathcal{E}]_{t-1}$ for the kernel of the quotient morphism $\Lambda_{t-1} \to \Lambda_{t-1,\mathrm{con}}$.

The existence of the Morita equivalence $\mathbb{F}$ essentially follows from the fact that $\operatorname{add}_{\mathcal{D}}(\Sigma^{t-1}X) = \operatorname{add}_{\mathcal{D}}(X)$ due to 5.1.28. That is, $\operatorname{add}_{\mathcal{E}}(P \oplus \Sigma^{t-1}X) = \operatorname{add}_{\mathcal{E}}(X)$ and, thus, $\mathcal{E}(P \oplus \Sigma^{t-1}X, X)$ is a projective generator in $\operatorname{mod} \Lambda_{t-1}$. Moreover, it follows from additive equivalence 5.1.10 (1) that

$$\operatorname{Hom}_{\Lambda_{t-1}}(\mathcal{E}(P \oplus \Sigma^{t-1}X, X), \mathcal{E}(P \oplus \Sigma^{t-1}X, X)) \cong \mathcal{E}(X, X)$$

as algebras. Thus, $\mathbb{F}\colon \operatorname{Hom}_{\Lambda_{t-1}}(\mathcal{E}(P \oplus \Sigma^{t-1}X, X), -)\colon \operatorname{mod}\Lambda_{t-1} \to \operatorname{mod}\Lambda$ is a Morita equivalence. $\square$

Lemma 5.2.8. *Suppose that $2 < t \leqslant d$. Then, there is a bimodule isomorphism*

$$I_{t-2} \otimes^{\mathbf{L}}_{\Lambda_{t-1}} I_{t-1} \otimes^{\mathbf{L}}_{\Lambda_{k-2}} \dots \otimes^{\mathbf{L}}_{\Lambda_1} I_1 \cong \mathcal{E}(X, P \oplus \Sigma^{t-2}X) \tag{5.2.G}$$

Proof. We begin by showing that for $1 \leqslant k \leqslant t-2$, there are bimodule isomorphisms

$$I_k \otimes^{\mathbf{L}}_{\Lambda_{k-1}} I_{k-1} \otimes^{\mathbf{L}}_{\Lambda_{k-2}} \dots \otimes^{\mathbf{L}}_{\Lambda_1} I_1 \cong \mathcal{E}(X, P \oplus \Sigma^{k}X)$$

by induction on k. When $k = 1$, this is $I_1 = \mathcal{E}(X, P \oplus \Sigma^1 X)$, which holds trivially. Suppose that it holds for $k = s$. Then,

$$I_{s+1} \otimes^{\mathbf{L}}_{\Lambda_s} I_s \otimes^{\mathbf{L}}_{\Lambda_{k-2}} \dots \otimes^{\mathbf{L}}_{\Lambda_1} I_1 \cong I_{s+1} \otimes^{\mathbf{L}}_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X).$$

We first prove that $I_{s+1} \otimes^{\mathbf{L}}_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X)$ is concentrated in degree zero. As a Λ-module, $I_{s+1} \otimes^{\mathbf{L}}_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X)$ is represented by the complex

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}(P \oplus \Sigma^s X, P \oplus \Sigma^s X) \otimes_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X) \arrow[d, "f"] & \\
& \mathcal{E}(P \oplus \Sigma^s X, Q_{s+1}) \otimes_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X) \arrow[r] & 0
\end{tikzcd}$$

which is obtained by applying $\mathcal{E}(P \oplus \Sigma^s X, -)$ to the exact sequences (5.2.A) for $k = s+1$ and then tensoring termwise with $\mathcal{E}(X, P \oplus \Sigma^s X)$. As in the proof of 5.2.6, we need to show that f is a monomorphism.

By applying $\mathcal{E}(X, -)$ to the exact sequences (5.2.A) for $k = s+1$, we obtain the following commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] \\
\mathcal{E}(P \oplus \Sigma^s X, P \oplus \Sigma^s X) \otimes_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X) \arrow[r, "m_s"] \arrow[d, "f"] & \mathcal{E}(X, P \oplus \Sigma^s X) \arrow[d] \\
\mathcal{E}(P \oplus \Sigma^s X, Q_{s+1}) \otimes_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X) \arrow[r, "n_{s+1}"] \arrow[d] & \mathcal{E}(X, Q_{s+1}) \arrow[d, "\mathcal{E}(X, p_{s+1})"] \\
I_{s+1} \otimes_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X) \arrow[r, "m_{s+1}"] \arrow[d] & \mathcal{E}(X, P \oplus \Sigma^{s+1} X) \arrow[d, "q_s"] \\
0 & \operatorname{Ext}^1_{\mathcal{E}}(X, P \oplus \Sigma^s X) \arrow[d] \\
& 0
\end{tikzcd} \tag{5.2.H}$$

where the second column is exact. Now, m_s and n_{s+1} are isomorphisms by 5.1.10 (2), hence f is injective, proving the claim. Namely, that $I_{s+1} \otimes^{\mathbf{L}}_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X)$ has cohomology only in degree zero. Truncating in the category of bimodules

$$I_{s+1} \otimes^{\mathbf{L}}_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X) \cong I_{s+1} \otimes_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X).$$

We next show that

$$I_{s+1} \otimes_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X) \cong \mathcal{E}(X, P \oplus \Sigma^{s+1} X).$$

Since X is $(-t+2)$-rigid in $\mathcal{D}$, $\mathrm{Ext}^1_{\mathcal{D}}(X, P \oplus \Sigma^s X)$ vanishes for $0 \leqslant s \leqslant t-3$, which implies that $\mathcal{E}(X, p_{s+1})$ is an epi by 2.2.14. Therefore, $\mathrm{Ext}^1_{\mathcal{E}}(X, \oplus \Sigma^s X)$ vanishes for $0 \leqslant s \leqslant t-3$. That is, m_{s+1} is epi. Since m_s and n_{s+1} are isomorphisms, the snake lemma implies that m_{s+1} is a mono, and, thus, an isomorphism. In fact, m_{s+1} is a bimodule morphism, and so

$$I_{s+1} \otimes^{\mathbf{L}}_{\Lambda_s} \mathcal{E}(X, P \oplus \Sigma^s X) \cong \mathcal{E}(X, P \oplus \Sigma^{s+1} X)$$

as bimodules whenever $1 \leqslant s+1 \leqslant t-2$.

We may thus conclude that

$$I_k \otimes^{\mathbf{L}}_{\Lambda_{k-1}} I_{k-1} \otimes^{\mathbf{L}}_{\Lambda_{k-2}} \cdots \otimes^{\mathbf{L}}_{\Lambda_1} I_1 \cong \mathcal{E}(X, P \oplus \Sigma^k X)$$

for $1 \leqslant k \leqslant t-2$. In particular,

$$I_{t-2} \otimes^{\mathbf{L}}_{\Lambda_{t-1}} I_{t-1} \otimes^{\mathbf{L}}_{\Lambda_{k-2}} \cdots \otimes^{\mathbf{L}}_{\Lambda_1} I_1 \cong \mathcal{E}(X, P \oplus \Sigma^{t-2} X) \qquad \square$$

Proposition 5.2.9. *Suppose that $2 < t \leqslant d$. Then,*

$$\mathbb{F} \cdot \Psi_1^{t-1} \cong \mathcal{T}$$

where $\mathbb{F}$ is the Morita equivalence of 5.2.7.

Proof. Since

$$\mathbb{F} \cdot \Psi_1^{t-1} \cong \mathbf{R}\mathrm{Hom}_{\Lambda}(\mathcal{E}(P \oplus \Sigma^{t-1} X, X) \otimes^{\mathbf{L}}_{\Lambda_{t-1}} I_{t-1} \otimes^{\mathbf{L}}_{\Lambda_{t-2}} I_{t-2} \otimes^{\mathbf{L}}_{\Lambda_{t-1}} \cdots \otimes^{\mathbf{L}}_{\Lambda_1} I_1, -),$$

it suffices to prove that

$$[\mathrm{proj}\, \mathcal{E}] \cong \mathcal{E}(P \oplus \Sigma^{t-1} X, X) \otimes^{\mathbf{L}}_{\Lambda_{t-1}} I_{t-1} \otimes^{\mathbf{L}}_{\Lambda_{t-2}} I_{t-2} \otimes^{\mathbf{L}}_{\Lambda_{t-1}} \cdots \otimes^{\mathbf{L}}_{\Lambda_1} I_1 \tag{5.2.I}$$

as Λ-bimodules.

By (5.2.8), this reduces to proving that

$$[\mathrm{proj}\, \mathcal{E}] \cong \mathcal{E}(P \oplus \Sigma^{t-1} X, X) \otimes^{\mathbf{L}}_{\Lambda_{t-1}} I_{t-1} \otimes^{\mathbf{L}}_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2} X) \tag{5.2.J}$$

Consider the diagram (5.2.H) for $s = t-2$. In this case, the group $\mathrm{Ext}^1_{\mathcal{E}}(X, P \oplus \Sigma^s X)$

does not vanish, but the second column is still exact and the morphisms m_s and n_{s+1} are still isomorphisms. So, it follows that $I_{t-1} \otimes^{\mathbf{L}}_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X)$ is still concentrated in degree zero. That is,

$$I_{t-1} \otimes^{\mathbf{L}}_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X) \cong I_{t-1} \otimes_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X).$$

Moreover, since $\mathrm{add}_{\mathcal{E}}(P \oplus \Sigma^{t-1}X) = \mathrm{add}_{\mathcal{E}}(X)$, the bimodule $\mathcal{E}(P \oplus \Sigma^{t-1}X, X)$ is projective on either side, and so the right hand side of (5.2.J) is isomorphic to

$$M := \mathcal{E}(P \oplus \Sigma^{t-1}X, X) \otimes_{\Lambda_{t-1}} I_{t-1} \otimes_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X)$$

Furthermore, since $\mathrm{add}_{\mathcal{E}}(P \oplus \Sigma^{t-1}X) = \mathrm{add}_{\mathcal{E}}(X)$, by 5.1.10 (2) the composition map induces an isomorphism

$$M \cong \mathcal{E}(P \oplus \Sigma^{t-2}X, X) \otimes_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X).$$

It thus remains to show that

$$[\mathrm{proj}\,\mathcal{E}] \cong \mathcal{E}(P \oplus \Sigma^{t-2}X, X) \otimes_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X)$$

as Λ-bimodules.

By applying $\mathcal{E}(P \oplus \Sigma^{t-2}X, -)$ then $(-) \otimes_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X)$ to (5.2.A) for $k = 0$, and applying $\mathcal{E}(P \oplus \Sigma^{t-2}X, -)$ to (5.2.A) for $k = 0$, we obtain the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
\mathcal{E}(P \oplus \Sigma^{t-2}X, P \oplus \Sigma^{-1}X) \otimes_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X) \arrow[r, "h_{-1}"] \arrow[d] & \mathcal{E}(X, P \oplus \Sigma^{-1}X) \arrow[d, "\mathcal{E}(X,\iota_0)"] \\
\mathcal{E}(P \oplus \Sigma^{t-2}X, Q_0) \otimes_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X) \arrow[r, "g_0"] \arrow[d] & \mathcal{E}(X, Q_0) \arrow[d, "\mathcal{E}(X,p_0)"] \\
\mathcal{E}(P \oplus \Sigma^{t-2}X, X) \otimes_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X) \arrow[r, "h_0"] \arrow[d] & \mathcal{E}(X, X) \arrow[d, "q_0"] \\
0 & \mathrm{Ext}^1_{\mathcal{E}}(X, P \oplus \Sigma^{-1}X) \arrow[d] \\
 & 0
\end{tikzcd}$$

of Λ-bimodules, where the columns are exact and the horizontal maps are the composition morphisms (Here, the first column is exact because X is $(t-2)$-rigid by 5.1.30).

Since $\mathrm{add}_{\mathcal{E}}(P \oplus \Sigma^{t-1}X) = \mathrm{add}_{\mathcal{E}}(X)$, then $\mathrm{add}_{\mathcal{E}}(P \oplus \Sigma^{t-2}X) = \mathrm{add}_{\mathcal{E}}(P \oplus \Sigma^{-1}X)$. Hence, 5.1.10 (2) implies that h_{-1} and g_0 are isomorphisms. It follows from the snake lemma that h_0 is a mono. By commutativity of the diagram and exactness of its columns,

$$\mathcal{E}(P \oplus \Sigma^{t-2}X, X) \otimes_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X) \cong \operatorname{Im} h_0 \cong \operatorname{Im} \mathcal{E}(X, p_0) \cong \ker q_0$$

By (5.1.R) and (5.1.S), q_0 is the quotient map $q_0 \colon \Lambda \to \Lambda_{\mathrm{con}}$. Hence,

$$\mathcal{E}(P \oplus \Sigma^{t-2}X, X) \otimes_{\Lambda_{t-2}} \mathcal{E}(X, P \oplus \Sigma^{t-2}X) \cong [\operatorname{proj} \mathcal{E}]$$

as Λ-bimodules, which concludes the proof. $\square$

Establishing that $\mathcal{T}$ is an equivalence

Theorem 5.2.10. *Under setup 5.2.1, the functor*

$$\mathcal{T} = \mathbf{R}\mathrm{Hom}_{\Lambda}([\operatorname{proj} \mathcal{E}], -) \colon \mathrm{D}(\Lambda) \to \mathrm{D}(\Lambda)$$

is an equivalence which preserves the bounded derived category of finitely generated modules.

Proof. The fact that $\mathcal{T}$ is an equivalence follows by applying 5.2.6 or 5.2.9 and the fact that the bimodules I_k and D_k are tilting by 5.2.5.

To see that $\mathcal{T}$ restricts to an equivalence

$$\mathcal{T} = \mathbf{R}\mathrm{Hom}_{\Lambda}([\operatorname{proj} \mathcal{E}], -) \colon \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda),$$

observe that Λ_{con} is biperfect as a Λ-module via 5.1.30. Therefore, the Λ-bimodule exact sequence

$$0 \to [\operatorname{proj} \mathcal{E}] \to \Lambda \to \Lambda_{\mathrm{con}} \to 0$$

implies that $[\operatorname{proj} \mathcal{E}]$ is also biperfect. Thus, $\mathcal{T}$ preserves bounded complexes. The fact that it preserves complexes of finitely generated modules is clear, since Λ is module finite over a noetherian ring. $\square$

§ 5.2.2 Spherical twist and cotwist

This section assumes setup 5.2.1 throughout. We will describe the cotwist around the restriction of scalars functor F associated to $\pi \colon \Lambda \to \Lambda_{\mathrm{con}}$. As a result, we will establish that F is spherical.

Applying the results in chapter 3 to the canonical quotient morphism $\pi \colon \Lambda \to \Lambda_{\mathrm{con}}$ induces the following diagram of adjoint pairs.

$$\begin{tikzcd}
\mathrm{D}(\Lambda_{\mathrm{con}}) \arrow[rr, "F = - \otimes^{\mathbf{L}}_{\Lambda_{\mathrm{con}}} {}_{\Lambda_{\mathrm{con}}}\Lambda_{\mathrm{con}\,\Lambda} \cong \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}({}_{\Lambda}\Lambda_{\mathrm{con}\,\Lambda_{\mathrm{con}}}, -)" description] & & \mathrm{D}(\Lambda) \arrow[ll, bend right, "F^{\mathrm{LA}} = - \otimes^{\mathbf{L}}_{\Lambda} \Lambda_{\mathrm{con}\,\Lambda_{\mathrm{con}}}"'] \arrow[ll, bend left, "F^{\mathrm{RA}} = \mathbf{R}\mathrm{Hom}_{\Lambda}({}_{\Lambda_{\mathrm{con}}}\Lambda_{\mathrm{con}\,\Lambda}, -)"] \arrow[ll, phantom, shift right=3, "\perp"] \arrow[ll, phantom, shift left=3, "\perp"]
\end{tikzcd} \tag{5.2.K}$$

Lemma 5.2.11. *Under setup 5.2.1, the functors F, F^{LA} and F^{RA} preserve bounded complexes of finitely generated modules.*

Proof. Since Λ_{con} is biperfect by 5.1.30, this is a consequence of [DW16, 6.7]. □

To describe the cotwist, we require the following lemma.

Lemma 5.2.12. *Under setup 5.2.1, $K := F^{\mathrm{LA}} \cdot F(\Lambda_{\mathrm{con}}) = {}_{\Lambda_{\mathrm{con}}}\Lambda_{\mathrm{con}\Lambda} \otimes^{\mathbf{L}}{}_{\Lambda}\Lambda_{\mathrm{con}\Lambda_{\mathrm{con}}}$ has cohomology concentrated in degrees 0 and $-t$.*

Proof. Let $B = \Lambda_{\mathrm{con}\Lambda} \otimes^{\mathbf{L}}{}_{\Lambda}\Lambda_{\mathrm{con}\Lambda_{\mathrm{con}}}$, which is just K with the left module structure forgotten. Then, B can be calculated by applying $- \otimes {}_{\Lambda}\Lambda_{\mathrm{con}\Lambda_{\mathrm{con}}}$ term-wise to the projective resolution obtained by summing (5.1.E) over all $i \in I$ with the correct multiplicities a_i:

$$0 \to \mathcal{E}(X, Q_t) \oplus \mathcal{E}(X, Y_i) \to \mathcal{E}(X, Q_{t-1}) \to \ldots \to \mathcal{E}(X, Q_1) \to \mathcal{E}(X, X') \to \Lambda_{\mathrm{con}} \to 0.$$

Since each $Q_j \in \operatorname{add} P$ for $1 \leqslant j \leqslant t$, then $\mathcal{E}(X, Q_j) \in \operatorname{add} e_0\Lambda$. That is, the tensor product $\mathcal{E}(X, Q_j) \otimes {}_{\Lambda}\Lambda_{\mathrm{con}\Lambda_{\mathrm{con}}}$ is a summand of $(e_0\Lambda)^{k_j} \otimes {}_{\Lambda}\Lambda_{\mathrm{con}\Lambda_{\mathrm{con}}}$ for some $k_j \in \mathbb{N}$. However, e_0 annihilates Λ_{con} so that $(e_0\Lambda \otimes {}_{\Lambda}\Lambda_{\mathrm{con}\Lambda_{\mathrm{con}}})^{k_j} = 0$. Hence, $B^j = 0$ for $i \neq 0, -t$. Since B and K are quasi-isomorphic as Λ_{con}-modules, it follows that

$$\mathrm{H}^j(K) = \mathrm{H}^j(B) = 0$$

for $i \notin \{-t, 0\}$. □

Theorem 5.2.13. *Under setup 5.2.1, the following hold.*

(1) *The functor*

$$\mathcal{T} := \mathbf{R}\mathrm{Hom}_{\Lambda}([\operatorname{proj} \mathcal{E}], -) \colon \mathrm{D}(\Lambda) \to \mathrm{D}(\Lambda)$$

is the twist around F. Moreover, the restriction

$$\mathcal{T} := \mathbf{R}\mathrm{Hom}_{\Lambda}([\operatorname{proj} \mathcal{E}], -) \colon \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$$

is the twist around F restricted to $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_{\mathrm{con}})$.

(2) *The functor*

$$\begin{aligned} \mathcal{C} &= \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}(\mathrm{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}}), -)[-t-1] \\ &\cong \mathrm{Hom}_{\Lambda_{\mathrm{con}}}(\mathcal{D}(X, \Sigma^{-t+1} X), -)[-t-1], \end{aligned}$$

is the cotwist around F. Hence, it is an equivalence. Moreover, the restriction

$$\mathcal{C} = \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}(\mathrm{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}}), -)[-t-1] \colon \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_{\mathrm{con}}) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_{\mathrm{con}}).$$

is the twist around F restricted to $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_{\mathrm{con}})$.

(3) *The functor F is spherical.*

Proof. (1) The fact that $\mathcal{T}$ is the twist is immediate from 3.1.3. The restriction statement follows from 5.2.11.

(2) The fact that

$$\mathcal{C} = \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}({}_{\Lambda_{\mathrm{con}}}\mathrm{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}})_{\Lambda_{\mathrm{con}}}, -)[-t-1]$$

is an immediate consequence of 5.2.12 and 3.2.7. The isomorphism

$$\mathcal{C} \cong \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}({}_{\Lambda_{\mathrm{con}}}\mathcal{D}(X, \Sigma^{-t+1}X)_{\Lambda_{\mathrm{con}}}, -)[-t-1]$$

follows from the Λ_{con}-bimodule isomorphism

$$\mathrm{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}}) \cong \mathcal{D}(X, \Sigma^{-t+1}X)$$

due to [Dug12, 3.1(3)].

Since $\mathrm{add}_{\mathcal{D}}\, X = \mathrm{add}_{\mathcal{D}}\, \Sigma^{-t+1}X$, it follows that $\mathcal{D}(X, \Sigma^{t-1}X)$ is projective on either side and that $\mathrm{add}\, \mathcal{D}(X, \Sigma^{t-1}X) = \mathrm{add}\, \Lambda_{\mathrm{con}}$. Hence, $\mathcal{D}(X, \Sigma^{t-1}X)$ is a projective generator in $\mathrm{mod}\, \Lambda_{\mathrm{con}}$, so that it immediately follows from Morita theory that $\mathcal{C}$ is an equivalence.

(3) The cotwist is an equivalence by (2), and the twist is an equivalence by 5.2.10. Hence, by the the 2 out of 4 property 2.4.12, F is spherical. □

Remark 5.2.14. The theorem [Dug12, 3.1(3)] quoted above is stated for the case where $\Lambda_{\mathrm{con}} = \underline{\mathrm{End}}_{\mathcal{E}}(X)$ is the endomorphism algebra of a cluster tilting object in $\mathcal{E}$. However, the proof works word for word using only the rigidity assumption.

Corollary 5.2.15. *Under setup 5.2.1, assume that X is basic. Then,*

$$\mathcal{C} \cong \mathrm{Hom}_{\Lambda_{\mathrm{con}}}({}_{\sigma}(\Lambda_{\mathrm{con}})_1, -)[-t-1]$$

where $\sigma\colon \Lambda_{\mathrm{con}} \to \Lambda_{\mathrm{con}}$ is the automorphism of algebras induced by an isomorphism $\Sigma^{-t+1}X \to X$.

Proof. If X is basic, then 5.1.28 implies that there is an isomorphism $\Sigma^{-t+1}X \to X$ in $\mathcal{D}$ which induces an isomorphism $\mathcal{D}(X, \Sigma^{-t+1}X) \cong {}_{\sigma}(\Lambda_{\mathrm{con}})_1$ of Λ_{con}-bimodules. Hence, it follows from 5.2.13 (2) that

$$\mathcal{C} \cong \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}({}_{\sigma}(\Lambda_{\mathrm{con}})_1, -)[-t-1].$$

which, since ${}_{\sigma}(\Lambda_{\mathrm{con}})_1$ is projective on either side, is equivalent to the statement. □

Corollary 5.2.16. *Under setup 5.2.1, the Λ_{con}-bimodule ${}_{\Lambda_{\mathrm{con}}}\mathrm{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}})_{\Lambda_{\mathrm{con}}}$ is projective as a right and left bimodule.*

Proof. By the proof of 5.2.13 (2),

$$\operatorname{Tor}_t^{\Lambda}(\Lambda_{\text{con}}, \Lambda_{\text{con}}) \cong \mathcal{D}(X, \Sigma^{-t+1} X)$$

Our assumptions imply that $\operatorname{add}_{\mathcal{D}} X = \operatorname{add}_{\mathcal{D}} \Sigma^{-t+1} X$, so that the statement follows. □

Example 5.2.17. Let R be a commutative noetherian complete local Gorenstein ring with at worst isolated hypersurface singularities and with $\dim R = 3$.

Let $M \in R$ be as in 4.2.7 with $\Lambda = \operatorname{End}_R(M)$ and $\Lambda_{\text{con}} = \Lambda/[\operatorname{add} R]$. We claim that the epimorphism $\pi\colon \Lambda \to \Lambda_{\text{con}}$ satisfies the assumptions of 4.2.6. Recall that by 4.2.7, it suffices to show that Λ_{con} is perfect. Well, since R is hypersurface, $\Omega^{-2} = \Sigma^2 \cong \text{id}$ on $\underline{\text{CM}}\, R$ [Eis80, 6.1]. Since by assumption $\operatorname{Ext}_R^1(M, M) = 0$, it follows from 5.1.28 that Λ_{con} is perfect.

Chapter 6

Spherical twists induced by crepant contractions

This section applies the theory developed in chapters 4 and 5 to obtain derived autoequivalences of schemes with at worst Gorenstein singularities.

§ 6.1 Setting

Throughout, we will work within setup 2.7.1. Namely,

Setup. *Let $f\colon X \to Y$ be a crepant (complete local) contraction. Assume further that Y is a Gorenstein d-fold with $d \geqslant 2$, and that X admits a relative tilting bundle $\mathcal{P}$ containing $\mathcal{O}_X$ as a summand.*

Given a morphism $f\colon X \to Y$ satisfying the setup above, we write write Z for the locus of points of Y onto which f is not an isomorphism.

The following lemma will be useful to construct spherical twists on $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$.

Lemma 6.1.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories which admit Morita enhancements $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $S \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$ be biperfect, and suppose that there is an equivalence $E\colon \mathcal{B} \to \mathcal{B}'$ of DG-categories. Observe that one may view E as a $\mathcal{B}$-$\mathcal{B}'$-bimodule and E^{-1} as a $\mathcal{B}'$-$\mathcal{B}$-bimodule.*

If T is the twist around S, then $E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} T \otimes^{\mathbf{L}}_{\mathcal{B}} E$ is the twist around ES. Moreover, if C is the cotwist around S, then it is also the cotwist around ES. Hence, if S is spherical, so is ES.

Proof. Recall from §2.4 that there is an adjunction

$$(-) \otimes^{\mathbf{L}}_{\mathcal{A}} S\colon \mathrm{D}(\mathcal{A}\text{-}\mathcal{A}) \to \mathrm{D}(\mathcal{A}\text{-}\mathcal{B}) \dashv (-) \otimes^{\mathbf{L}}_{\mathcal{B}} R\colon \mathrm{D}(\mathcal{A}\text{-}\mathcal{B}) \to \mathrm{D}(\mathcal{A}\text{-}\mathcal{A})$$

which induces the Hom-space isomorphisms

$$\Phi_{(A,B)}\colon \operatorname{Hom}_{\mathrm{D}(\mathcal{A}\text{-}\mathcal{A})}(A, B \otimes^{\mathbf{L}}_{\mathcal{B}} R) \to \operatorname{Hom}_{\mathrm{D}(\mathcal{A}\text{-}\mathcal{B})}(A \otimes^{\mathbf{L}}_{\mathcal{A}} S, B).$$

that are natural for all $A \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{A})$ and $B \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$. Using these isomorphisms, it is easy to check that there are adjoint pairs

$$(-) \otimes^{\mathbf{L}}_{\mathcal{A}} ES\colon \mathrm{D}(\mathcal{A}\text{-}\mathcal{A}) \to \mathrm{D}(\mathcal{A}\text{-}\mathcal{B}') \dashv (-) \otimes^{\mathbf{L}}_{\mathcal{B}'} E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} R\colon \mathrm{D}(\mathcal{A}\text{-}\mathcal{B}') \to \mathrm{D}(\mathcal{A}\text{-}\mathcal{A}).$$

To see this, observe that for each $A \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{A})$ and $C \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{B}')$, there are Hom-set isomorphisms

$$\Psi_{A,C}\colon \mathrm{Hom}_{\mathrm{D}(\mathcal{A}\text{-}\mathcal{A})}(A, C \otimes^{\mathbf{L}}_{\mathcal{B}'} E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} R) \xrightarrow{\Phi_{(A, C\otimes^{\mathbf{L}}_{\mathcal{B}'}, E^{-1})} \otimes^{\mathbf{L}}_{\mathcal{B}} E} \mathrm{Hom}_{\mathrm{D}(\mathcal{A}\text{-}\mathcal{B}')}(A \otimes^{\mathbf{L}}_{\mathcal{A}} S \otimes^{\mathbf{L}}_{\mathcal{B}} E, C)$$

which are natural in A and C. Let $RE^{-1} := E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} R$.

Hence, as in §2.4, the twist T and T' around S and ES are defined (up to isomorphism) as

$$\begin{aligned} T &:= \mathrm{cone}(\mathrm{tr}\colon SR \to \mathcal{B}) \in \mathrm{D}(\mathcal{B}\text{-}\mathcal{B}), \\ T' &:= \mathrm{cone}(\mathrm{tr}'\colon E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} SR \otimes^{\mathbf{L}}_{\mathcal{B}} E \to \mathcal{B}') \in \mathrm{D}(\mathcal{B}'\text{-}\mathcal{B}'), \end{aligned}$$

respectively. Here,

$$\mathrm{tr} := \varepsilon_{\mathcal{B}}, \text{ and } \mathrm{tr}' := \varepsilon'_{\mathcal{B}'}$$

where ε and ε' are the counits of the adjunctions $(- \otimes^{\mathbf{L}}_{\mathcal{A}} S, - \otimes^{\mathbf{L}}_{\mathcal{A}} R)$ and $(- \otimes^{\mathbf{L}}_{\mathcal{A}} ES, - \otimes^{\mathbf{L}}_{\mathcal{A}} RE^{-1})$, respectively.

Observe, then, that if $\mathrm{tr}' = E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathrm{tr} \otimes^{\mathbf{L}}_{\mathcal{B}} E$, then the triangle

$$SR \xrightarrow{\mathrm{tr}} \mathcal{B} \to T \to^{+}$$

in $\mathrm{D}(\mathcal{B}\text{-}\mathcal{B})$ induces the triangle

$$E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} SR \otimes^{\mathbf{L}}_{\mathcal{B}} E \xrightarrow{\mathrm{tr}'} \mathcal{B}' \to E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} T \otimes^{\mathbf{L}}_{\mathcal{B}} E \to^{+}$$

in $\mathrm{D}(\mathcal{B}\text{-}\mathcal{B})$, from which we may conclude that $T' \cong E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} T \otimes^{\mathbf{L}}_{\mathcal{B}} E$, as required. Thus, it suffices to show that $\varepsilon'_{\mathcal{B}'} = E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} \varepsilon_{\mathcal{B}} \otimes^{\mathbf{L}}_{\mathcal{B}} E$.

The counit at $\mathcal{B}'$ $\varepsilon'_{\mathcal{B}'}$ is specified by

$$\begin{aligned} \varepsilon'_{\mathcal{B}'} &= \Psi_{(\mathcal{B}' \otimes^{\mathbf{L}}_{\mathcal{B}'} E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} R,\ \mathcal{B}')}(1_{\mathcal{B}' \otimes^{\mathbf{L}}_{\mathcal{B}'} E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} R}) \\ &= \Psi_{(E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} R,\ \mathcal{B}')}(1_{E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} R}) \\ &= (\Phi_{(E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} R,\ E^{-1})}(1_{E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} R}) \otimes^{\mathbf{L}}_{\mathcal{B}} E \\ &= \varepsilon_{E^{-1}} \otimes^{\mathbf{L}}_{\mathcal{B}} E. \end{aligned}$$

However, it follows from [AL17, p.12] that $\varepsilon_{E^{-1}} = E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} \varepsilon_{\mathcal{B}}$. Hence,

$$\varepsilon'_{\mathcal{B}'} = E^{-1} \otimes^{\mathbf{L}}_{\mathcal{B}} \varepsilon_{\mathcal{B}} \otimes^{\mathbf{L}}_{\mathcal{B}} E$$

as required.

The fact that the cotwist is the same follows from $\mathrm{act} = \mathrm{act}'$. $\square$

§ 6.2 Complete local setting

Assume now that $f\colon X \to Y = \operatorname{Spec} \mathcal{R}$ is a complete local contraction as in 2.7.1 with $(\mathcal{R}, \mathfrak{m})$ a complete local ring. In this context, let $\Lambda = \operatorname{End}_{\mathcal{R}}(f_*\mathcal{P})$ and write $\Lambda_{\mathrm{con}} = \underline{\operatorname{End}}_{\mathcal{R}}(f_*\mathcal{P})$.

Remark 6.2.1. Since Y is affine and f is crepant, it follows from [DW19b, 2.5(1)] that $f_*\mathcal{P} \in \operatorname{CM} \mathcal{R}$.

Lemma 6.2.2. *The category $\mathcal{E} = \operatorname{CM} \mathcal{R}$ satisfies setup 5.1.1 and the module $f_*\mathcal{P}$ satisfies setup 5.1.5. Moreover, Λ is a Gorenstein $\mathcal{R}$-order (i.e. there exists an isomorphism $\Lambda_{\mathrm{con}} \otimes \omega_\Lambda \cong \Lambda_{\mathrm{con}}$ of Λ-bimodules).*

Proof. Since $\mathcal{R}$ is a complete local Gorenstein ring, $\operatorname{CM} \mathcal{R}$ is a Krull-Schmidt Frobenius category with $\operatorname{proj} \mathcal{E} = \operatorname{add} \mathcal{R}$. It follows that $\operatorname{CM} \mathcal{R}$ satisfies 5.1.1.

By the definition of a relative tilting bundle, $\mathcal{P} = \mathcal{O}_X \oplus \mathcal{P}_0$. Since $\operatorname{CM} \mathcal{R}$ is Krull-Schmidt, $f_*\mathcal{P}$ isomorphic to $R \oplus Q \oplus \bigoplus_{i=1}^n M_i^{a_i}$ where $Q \in \operatorname{add} R$, $a_i \in \mathbb{N}$ with $a_i > 0$, and M_i are pairwise non-isomorphic indecomposable modules in $\operatorname{CM} \mathcal{R}$ which are not projective. It follows that M_i are also indecomposable in $\underline{\operatorname{CM}} \mathcal{R}$. Hence, $f_*\mathcal{P}$ satisfies 5.1.5.

Finally, we will argue that $\omega_\Lambda \cong \Lambda$ as Λ-bimodules. Well, since f is crepant, $\Lambda = \operatorname{End}_{\mathcal{R}}(f_*\mathcal{P}) \in \operatorname{CM} \mathcal{R}$ by [IW14b, 4.8]. Because of this and because Cohen-Macaulay modules are reflexive, [IW14a, 2.22(2)] and [IR08, 3.8 (1)⇒(3)] imply that $\omega_\Lambda \cong \Lambda$ as Λ-bimodules. $\square$

Proposition 6.2.3. *Suppose that Λ_{con} is perfect over Λ and t-relatively spherical for some $t \in \mathbb{Z}$ with $2 \leqslant t \leqslant d$. Then, $[\operatorname{add} \mathcal{R}]$ is a tilting Λ-bimodule. Whence, X admits the derived autoequivalence*

$$\Phi_{\mathcal{R}} := \Psi_{\mathcal{P}}^{-1} \cdot \mathbf{R}\mathrm{Hom}_\Lambda([\operatorname{add} \mathcal{R}], -) \cdot \Psi_{\mathcal{P}}\colon\ \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \tag{6.2.A}$$

Proof. As Λ_{con} is perfect and relatively spherical, the pair $\mathcal{E} = \operatorname{CM} \mathcal{R}$ and $X = f_*\mathcal{P}$ satisfies setup 5.2.1, by 6.2.2. Thus, 5.2.10 implies that $[\operatorname{add} \mathcal{R}]$ is a tilting bimodule and $\Phi_{\mathcal{R}}$ is an equivalence. $\square$

Proposition 6.2.4. *Suppose that Λ_{con} is perfect over Λ and t-relatively spherical for some $t \in \mathbb{Z}$ with $2 \leqslant t \leqslant d$. Then,*

$$\Phi_{\mathcal{R}}\colon\ \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$$

is the spherical twist around the functor

$$\Psi_{\mathcal{P}}^{-1} \cdot F\colon\ \mathrm{D^b}(\operatorname{mod} \Lambda_{\mathrm{con}}) \to \mathrm{D^b}(\operatorname{coh} X).$$

The cotwist is

$$\mathcal{C} = \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}({}_{\Lambda_{\mathrm{con}}}\operatorname{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}})_{\Lambda_{\mathrm{con}}}, -)[-t-1]. \tag{6.2.B}$$

If $\dim_k \Lambda_{\mathrm{con}} < \infty$ *and* $t = d$*, then the cotwist is naturally isomorphic to* $[-d-1]\cdot\mathcal{N}$*, where* $\mathcal{N}$ *is the Nakayama functor of* Λ_{con}*.*

Proof. The fact that $\mathcal{T} = \mathbf{R}\mathrm{Hom}_{\Lambda}([\operatorname{add} R], -)\colon\ \mathrm{D^b}(\operatorname{mod}\Lambda) \to \mathrm{D^b}(\operatorname{mod}\Lambda)$ is the twist around the functor $F = - \otimes^{\mathbf{L}}_{\Lambda_{\mathrm{con}}} \Lambda_{\mathrm{con}}\colon\ \mathrm{D^b}(\operatorname{mod}\Lambda_{\mathrm{con}}) \to \mathrm{D^b}(\operatorname{mod}\Lambda)$ follows from 5.2.13(1). Moreover, since the pair $\mathcal{E} = \mathrm{CM}\,\mathcal{R}$ and $X = f_*\mathcal{P}$ satisfies setup 5.2.1 by 6.2.2, we may apply 5.2.13(2) to conclude that the cotwist around F is given by the functor

$$\mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}({}_{\Lambda_{\mathrm{con}}}\operatorname{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}})_{\Lambda_{\mathrm{con}}}, -)[-t-1].$$

Hence, either 2.4.11 or 6.1.1 imply that $\Phi_{\mathcal{R}}$ is the twist around $\Psi_{\mathcal{P}}^{-1} \cdot F$ and that the cotwist around this functor is given by (6.2.B). Since both the twist and cotwist are equivalences by 5.2.13, it follows that $\Psi_{\mathcal{P}}^{-1} \cdot F$ is spherical.

Finally, we note that if $\dim_k \Lambda_{\mathrm{con}} < \infty$ and $t = d$, then 5.1.26(1) implies that Λ_{con} is self-injective. Moreover, $\mathrm{H}^k(\Lambda_{\mathrm{con}} \otimes^{\mathbf{L}}_{\Lambda} \Lambda_{\mathrm{con}}) = 0$ for all $k \neq 0, -d$ by 5.2.12.

Hence, $\pi\colon \Lambda \to \Lambda_{\mathrm{con}}$ satisfies the assumptions of 4.2.1. Whence, the cotwist is naturally isomorphic to $[-d-1]\cdot\mathcal{N}$. □

We end this section by remarking that checking whether Λ_{con} is relatively spherical can be simplified in this setting.

Proposition 6.2.5. *The contraction algebra* Λ_{con} *is* t*-relatively spherical if and only if* $\operatorname{Ext}^j_{\Lambda}(\Lambda_{\mathrm{con}}, \mathcal{S}) = 0$ *for* $0 < j < t$*,* $t < j < d$*.*

Proof. Suppose that $\operatorname{Ext}^j_{\Lambda}(\Lambda_{\mathrm{con}}, \mathcal{S}) = 0$ for $0 < j < t$ and $t < j < d$. Then, Λ_{con} is t-relatively spherical if $\operatorname{Ext}^j_{\Lambda}(\Lambda_{\mathrm{con}}, \mathcal{S}) = 0$ for $j > d$. Well, the proof of 6.2.2 shows that $\Lambda \in \mathrm{CM}\,\mathcal{R}$ and that ω_{Λ} is a projective Λ-module. Thus, Λ is a Gorenstein $\mathcal{R}$-order in the sense of [IW14a, 1.6, 2.4]. Whence, we may apply the Auslander-Buchsbaum formula for Gorenstein orders [IW14a, 2.16]

$$\operatorname{p.dim}_{\Lambda}(\Lambda_{\mathrm{con}}) \leqslant \operatorname{p.dim}_{\Lambda}(\Lambda_{\mathrm{con}}) + \operatorname{depth}_{\mathcal{R}}(\Lambda_{\mathrm{con}}) = d.$$

That is, necessarily, $\operatorname{Ext}^j_{\Lambda}(\Lambda_{\mathrm{con}}, \mathcal{S}) = 0$ for $j > d$. □

Example 6.2.6. Assume that $Y = \operatorname{Spec}\mathcal{R}$ has at worst hypersurface singularities. Moreover, suppose that the exceptional locus of f has codimension greater than one and that f has fibres which are at most one-dimensional. Then, [VdB04, 3.2.11] implies that X has a tilting bundle $\mathcal{V} = \mathcal{O}_X \oplus \mathcal{P}$. Let $\Lambda = \operatorname{End}_{\mathcal{R}}(f_*\mathcal{V})$ and $\Lambda_{\mathrm{con}} = \underline{\operatorname{End}}_{\mathcal{R}}(f_*\mathcal{V})$.

It is not hard to check that $\operatorname{Ext}^1_{\mathcal{R}}(f_*\mathcal{P}, f_*\mathcal{P}) = 0$. Moreover, [Eis80, 6.1] implies that $\Omega^2 \cong \mathrm{id}$ on $\mathrm{CM}\,\mathcal{R}$. It thus follows from 5.1.28 that Λ_{con} is 3-relatively spherical and perfect over Λ. Hence, we may apply 6.2.4.

Additionally, since the isomorphism is $\Omega^2 \cong \mathrm{id}$ is functorial and by the proof of 5.2.13(2), it follows that

$$\operatorname{Tor}^{\Lambda}_t(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}}) \cong \underline{\operatorname{Hom}}_{\mathcal{R}}(f_*\mathcal{V}, \Sigma^2 f_*\mathcal{V}) \cong \underline{\operatorname{Hom}}_{\mathcal{R}}(f_*\mathcal{P}, f_*\mathcal{P}) = \Lambda_{\mathrm{con}}.$$

as Λ_{con}-bimodules. Hence, the cotwist is $1_{\mathrm{D^b}(\operatorname{mod}\Lambda_{\mathrm{con}})}[-4]$.

§ 6.3 Zariski local setting

In this section assume that $f\colon X \to Y = \operatorname{Spec} R$ is as in 2.7.1 where R need not be a complete local ring. In this context, let $\Lambda = \operatorname{End}_{\mathcal{R}}(f_*\mathcal{P})$ and write $\Lambda_{\mathrm{con}} = \underline{\operatorname{End}}_R(f_*\mathcal{P})$.

Notation 6.3.1. For each $x \in \operatorname{MaxSpec} R$, let R_x denote the localisation of R at x and let $\mathcal{R}_x$ denote the completion of R_x at the unique maximal ideal. Moreover, given an R-module, let M_x denote its localisation at x and write $\widehat{M}$ for its completion. Recall that if M is a finitely generated R-module, $\widehat{M} = M \otimes_R R_x \otimes_{R_x} \mathcal{R}_x$.

§ 6.3.1 Constructing the equivalence

The first goal of this section is to prove that the functor

$$\Phi_R = \Psi_{\mathcal{P}}^{-1} \cdot \mathbf{R}\mathrm{Hom}_{\Lambda}([\operatorname{add} R], -) \cdot \Psi_{\mathcal{P}}\colon\ \mathrm{D^b}(\operatorname{coh} X) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{coh} X) \tag{6.3.A}$$

is an equivalence if Λ_{con} is perfect over Λ and relatively spherical. This amounts to proving that the ideal $[\operatorname{add} R] \subseteq \Lambda$ is tilting. It is possible to reduce this fact to a statement in the complete local setting.

Lemma 6.3.2. *Let A be a module finite R-algebra. Let M be a finitely generated A-bimodule. If there is an n such that for all $x \in \operatorname{MaxSpec} R$, $\operatorname{p.dim}_{\widehat{A}}(\widehat{M}) \leqslant n$, $\operatorname{p.dim}_{\widehat{A}^{\mathrm{op}}}(\widehat{M}) \leqslant n$, and $\widehat{M}$ is tilting, then M is tilting.*

Proof. We will show that M is tilting because it satisfies (a)-(c) of [Miy03, 1.8]. For (a), to see that M is perfect as a right A-module, let

$$\ldots \to P_n \xrightarrow{p_n} P_{n-1} \xrightarrow{p_{n-1}} \ldots \to P_0 \to M \to 0$$

be a projective resolution $p\colon P \to M \to 0$ in $\operatorname{mod} A$. For finitely generated modules over noetherian rings, localisation and completion are exact functors, and so the complex $\widehat{p}\colon \widehat{P} \to \widehat{M} \to 0$ is exact for all $x \in \operatorname{MaxSpec} R$. Moreover, it is clear that if P_i is a summand of A^k for some $k \in \mathbb{Z}$, then $\widehat{P}_i = P_i \otimes_R \mathcal{R}_x$ is a summand of $A^k \otimes_R \mathcal{R}_x = \widehat{A}^k$.

Hence, $\widehat{P}_i$ is a projective $\widehat{A}$-module, and we may conclude that $\widehat{p}\colon \widehat{P} \to \widehat{M} \to 0$ is a projective resolution of $\widehat{M}$.

Because $\operatorname{p.dim}_{\widehat{A}}(\widehat{M}) \leqslant n$, necessarily $\ker \widehat{p}_n$ is a projective $\widehat{A}$-module. Equivalently (because localisation and completion are exact), $\widehat{\ker p_n}$ is a projective $\widehat{A}$-module for all $x \in \operatorname{MaxSpec} R$. That is,

$$\operatorname{Ext}^1_A(\ker p_n, N) \otimes_R \mathcal{R}_x = \operatorname{Ext}^1_{\widehat{A}}(\widehat{\ker p_n}, \widehat{N}) = 0$$

for all $N \in \operatorname{mod} A$ by [DW16, 2.15]. Since $\operatorname{Ext}^1_A(\ker p_n, N) \otimes_R \mathcal{R}_x$ vanishes for all $x \in \operatorname{MaxSpec} R$, it must be that $\operatorname{Ext}^1_A(\ker p_n, N) = 0$. Consequently, $\ker p_n$ is a finitely generated projective A-module, and we may truncate $p\colon P \to M \to 0$ to a projective resolution of length n. That is, $\operatorname{p.dim}_A(M) \leqslant n$. The proof that $\operatorname{p.dim}_{A^{\mathrm{op}}}(M) \leqslant n$ is similar, and so it is omitted.

To prove (b), we would like to show that the natural right multiplication map $\lambda\colon A \to \mathbf{R}\mathrm{Hom}_A(M, M)$ is a bimodule isomorphism in the derived category. First, since $\widehat{M}$ is tilting for all $x \in \operatorname{MaxSpec} R$, then the natural right multiplication map $\widehat{A} \to \mathbf{R}\mathrm{Hom}_{\widehat{A}}(\widehat{M}, \widehat{M})$ is a quasi-isomorphism. Thus, for $i \neq 0$,

$$0 = \operatorname{Ext}^i_{\widehat{A}}(\widehat{M}, \widehat{M}) = \operatorname{Ext}^i_A(M, M) \otimes_R \mathcal{R}_x.$$

Since $\operatorname{Ext}^i_A(M, M) \otimes_R \mathcal{R}_x$ vanishes for all $x \in \operatorname{MaxSpec} R$, necessarily $\operatorname{Ext}^i_A(M, M) = 0$ for $i \neq 0$. Therefore, it suffices to show that the degree zero multiplication map $\mathrm{H}^0(\lambda)\colon A \to \operatorname{Hom}_A(M, M)$ is an isomorphism.

Well, $\widehat{M}$ is tilting so that the natural multiplication map $\gamma_x\colon \widehat{A} \to \operatorname{Hom}_{\widehat{A}}(\widehat{M}, \widehat{M})$ is an isomorphism. It easy to check that $\mathrm{H}^0(\lambda) \otimes_R \mathcal{R}_x = \gamma_x$, and so $\mathrm{H}^0(\lambda) \otimes_R \mathcal{R}_x$ is an isomorphism for all $x \in \operatorname{MaxSpec} R$. Thus, $\mathrm{H}^0(\lambda)$ must be an isomorphism, as required. The dual argument works to show that (c) holds, that is, the natural left multiplication map is an isomorphism. □

Since $[\operatorname{add} R] \otimes_R R_x \otimes_{R_x} \mathcal{R}_x \cong [\operatorname{add} \mathcal{R}_x]$ as $\widehat{\Lambda}$-bimodules by [DW16, 2.16], we obtain the following corollary.

Corollary 6.3.3. *If* $[\operatorname{add} \mathcal{R}_x]$ *is tilting for all closed points* $x \in Z$*, then* $[\operatorname{add} R]$ *is tilting.*

Proof. The goal is to apply 6.3.2 to $[\operatorname{add} R]$ since $[\operatorname{add} R] \otimes_R R_x \otimes_{R_x} \mathcal{R}_x \cong [\operatorname{add} \mathcal{R}_x]$. Therefore, we need to check that the assumptions of 6.3.2 are satisfied.

If $x \in Z$ and $[\operatorname{add} \mathcal{R}_x]$ is a tilting module, then both $\operatorname{p.dim}_{\widehat{\Lambda}}[\operatorname{add} \mathcal{R}_x] < \infty$ and $\operatorname{p.dim}_{\widehat{\Lambda}^{\mathrm{op}}}[\operatorname{add} \mathcal{R}_x] < \infty$. We claim that $\operatorname{p.dim}_{\widehat{\Lambda}}[\operatorname{add} \mathcal{R}_x] \leqslant d$ and $\operatorname{p.dim}_{\widehat{\Lambda}^{\mathrm{op}}}[\operatorname{add} \mathcal{R}_x] \leqslant d$ for all $x \in Z$.

First, note that the pair $\mathcal{E} = \operatorname{CM} \mathcal{R}_x$ and $f_*\mathcal{P} \otimes_R \mathcal{R}_x$ are in the situation of 6.2.2, so that $\widehat{\Lambda} = \operatorname{End}_{\mathcal{R}_x}(\widehat{f_*\mathcal{P}})$ is a Gorenstein $\mathcal{R}$-order. Thus, we may apply the Auslander-

Buchsbaum formula for Gorenstein orders [IW14a, 2.16] to conclude that

$$\mathrm{p.\,dim}_{\widehat{\Lambda}}[\mathrm{add}\,\mathcal{R}_x] \leqslant \mathrm{p.\,dim}_{\widehat{\Lambda}}[\mathrm{add}\,\mathcal{R}_x] + \mathrm{depth}_{\mathcal{R}_x}([\mathrm{add}\,\mathcal{R}_x]) = d$$

(Here, we use the fact that R is equicodimensional). The dual argument works to prove that $\mathrm{p.\,dim}_{\widehat{\Lambda}^{\mathrm{op}}}[\mathrm{add}\,\mathcal{R}_x] \leqslant d$.

Therefore, it suffices to prove that for all $x \notin Z$, then $\mathrm{p.\,dim}_{\widehat{\Lambda}}([\mathrm{add}\,\mathcal{R}_x]) \leqslant d$, $\mathrm{p.\,dim}_{\widehat{\Lambda}^{\mathrm{op}}}([\mathrm{add}\,\mathcal{R}_x]) \leqslant d$, and $[\mathrm{add}\,\mathcal{R}_x]$ is tilting. This is an immediate consequence of [DW19b, 4.4] since this result implies that for $x \notin Z$, $\widehat{\Lambda}_{\mathrm{con}} = 0$. Therefore, because $[\mathrm{add}\,\mathcal{R}_x]$ is the kernel of the quotient $\widehat{\Lambda} \to \widehat{\Lambda}_{z,\mathrm{con}}$, it follows that $[\mathrm{add}\,\mathcal{R}_x] \cong \widehat{\Lambda}$, and clearly this satisfies the desired properties. □

The following shows that the relatively spherical property can also be checked complete locally.

Lemma 6.3.4. *The following are equivalent*

(1) Λ_{con} *is t-relatively spherical and perfect over* Λ*,*

(2) $\widehat{\Lambda}_{z,\mathrm{con}}$ *is t-relatively spherical and perfect over* $\widehat{\Lambda}_z$ *for all closed points* $z \in Z$*.*

Proof. The key to this proof is that $\widehat{\Lambda}_{\mathrm{con}} \cong \widehat{\Lambda}/[\mathrm{add}\,\mathcal{R}_x]$ for all $x \in \mathrm{MaxSpec}\,R$, due to [DW16, 2.16].

The implication (1) $\Rightarrow$ (2) is clear. Suppose that (2) holds. Then, for $j \neq 0, t$ and $z \in Z$

$$0 = \mathrm{Ext}^j_{\widehat{\Lambda}}(\widehat{\Lambda}_{z,\mathrm{con}}, \mathcal{S}_z) \cong \mathrm{Ext}^j_{\Lambda}(\Lambda_{\mathrm{con}}, \mathcal{S}) \otimes \mathcal{R}_z.$$

Moreover, $\widehat{\Lambda}_{x,\mathrm{con}} = 0$ for $x \notin Z$ [DW19b, 4.4]. Hence, $\mathrm{Ext}^j_{\Lambda}(\Lambda_{\mathrm{con}}, \mathcal{S})$ vanishes on all closed points, and so it must be zero.

Moreover, since (2) holds, $\mathrm{p.\,dim}_{\widehat{\Lambda}_z}(\widehat{\Lambda}_{z,\mathrm{con}}) = t$ for all $z \in Z$. Hence, the same argument to prove (a) in the proof of 6.3.2 works to show that $\mathrm{p.\,dim}_{\Lambda}(\Lambda_{\mathrm{con}}) \leqslant t$. □

Proposition 6.3.5. *Suppose that* Λ_{con} *perfect over* Λ *and t-relatively spherical for some* $t \in \mathbb{Z}$ *with* $2 \leqslant t \leqslant d$*. Then there is a derived autoequivalence*

$$\Phi_R = \Psi_{\mathcal{P}}^{-1} \cdot \mathbf{R}\mathrm{Hom}_{\Lambda}([\mathrm{add}\,R], -) \cdot \Psi_{\mathcal{P}} \colon \mathrm{D^b}(\mathrm{coh}\,X) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{coh}\,X) \tag{6.3.B}$$

Proof. To prove the statement, it suffices to show that the ideal $[\mathrm{add}\,R] \subseteq \Lambda$ is tilting which, by 6.3.3, follows from the fact that $[\mathrm{add}_{\mathcal{R}_x}]$ is tilting for all $x \in Z$. By 6.3.4 and the fact that completion and localisation are exact functors, it must be that $\widehat{\Lambda}_{\mathrm{con}}$ is perfect over $\widehat{\Lambda}$ and t-relatively spherical. Thus, 6.2.3 implies that $[\mathrm{add}_{\mathcal{R}_x}]$ is tilting for all $x \in Z$. □

§ 6.3.2 Proving that the twist is spherical

Consider the functor $F := - \otimes^{\mathbf{L}}_{\Lambda_{\mathrm{con}}} \Lambda_{\mathrm{con}\Lambda} \colon \mathrm{D^b}(\operatorname{mod}\Lambda_{\mathrm{con}}) \to \mathrm{D^b}(\operatorname{mod}\Lambda)$. The goal of the remainder of this section is to show that if Λ_{con} is perfect over Λ and t-relatively spherical, then $\Phi_R \colon \mathrm{D^b}(\operatorname{coh} X) \to \mathrm{D^b}(\operatorname{coh} X)$ is a spherical twist around the functor $\Psi_{\mathcal{P}}^{-1} \cdot F \colon \mathrm{D^b}(\operatorname{mod}\Lambda_{\mathrm{con}}) \to \mathrm{D^b}(\operatorname{coh} X)$.

Lemma 6.3.6. *Fix $x \in \operatorname{MaxSpec} R$. There is a $\widehat{\Lambda}_{\mathrm{con}}$-bimodule isomorphism*

$$\operatorname{Tor}_i^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}}) \otimes \mathcal{R}_x \cong \operatorname{Tor}_i^{\widehat{\Lambda}}(\widehat{\Lambda}_{\mathrm{con}}, \widehat{\Lambda}_{\mathrm{con}})$$

Proof. The proof follows as in e.g. [Wei94, 3.2.10]. □

Proposition 6.3.7. *If Λ_{con} is perfect over Λ and t-relatively spherical then*

$$\Phi_R \colon \mathrm{D^b}(\operatorname{coh} X) \to \mathrm{D^b}(\operatorname{coh} X)$$

is the twist around the functor

$$\Psi_{\mathcal{P}}^{-1} \cdot F \colon \mathrm{D^b}(\operatorname{mod}\Lambda_{\mathrm{con}}) \to \mathrm{D^b}(\operatorname{coh} X).$$

The cotwist is

$$\mathcal{C}_R = \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}({}_{\Lambda_{\mathrm{con}}} \operatorname{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}})_{\Lambda_{\mathrm{con}}}, -)[-t-1]. \tag{6.3.C}$$

Proof. In light of 3.1.3, the functor $\mathfrak{T} = \mathbf{R}\mathrm{Hom}_{\Lambda}([\operatorname{add} R], -) \colon \mathrm{D}(\operatorname{Mod}\Lambda) \to \mathrm{D}(\operatorname{Mod}\Lambda)$ is the twist around $F = - \otimes^{\mathbf{L}}_{\Lambda_{\mathrm{con}}} \Lambda_{\mathrm{con}} \colon \mathrm{D}(\operatorname{Mod}\Lambda_{\mathrm{con}}) \to \mathrm{D}(\operatorname{Mod}\Lambda)$.

Moreover, we claim that F, as well as its right and left adjoints, preserve bounded complexes of finitely generated modules. By [DW16, 6.7], it suffices to prove that Λ_{con} is biperfect. By assumption, $\operatorname{p.dim}_{\Lambda} \Lambda_{\mathrm{con}} = n < \infty$. It follows from 6.3.4 that $\widehat{\Lambda}_{x,\mathrm{con}}$ is t-relatively spherical and perfect. By 5.1.30, this means that $\operatorname{p.dim}_{\widehat{\Lambda}^{\mathrm{op}}} \widehat{\Lambda}_{\mathrm{con}} \leqslant t$ for all $x \in \operatorname{MaxSpec} R$. By the dual of proof of 6.3.2, it must be that $\operatorname{p.dim}_{\Lambda^{\mathrm{op}}} \Lambda_{\mathrm{con}} \leqslant t$. Consequently, Λ_{con} is biperfect.

Thus, we conclude that the restriction of $\mathfrak{T}$ to $\mathrm{D^b}(\operatorname{mod}\Lambda)$ is the twist around the restriction of F to $\mathrm{D^b}(\operatorname{mod}\Lambda_{\mathrm{con}})$. As a result, the fact that Φ_R is the twist around $\Psi_{\mathcal{P}}^{-1} \cdot F$ follows immediately from either 2.4.11 or 6.1.1.

Observe, moreover, that either 2.4.11 or 6.1.1 imply that the cotwist around $\Psi_{\mathcal{P}}^{-1} \cdot F$ can be identified with the twist around F. Therefore, by 3.2.7, the cotwist is as claimed if the complex $K = {}_{\Lambda_{\mathrm{con}}}\Lambda_{\mathrm{con}\Lambda} \otimes^{\mathbf{L}} {}_{\Lambda}\Lambda_{\mathrm{con}\Lambda_{\mathrm{con}}}$ has cohomology concentrated in degrees 0 and $-t$. We will show that these properties hold in this setup because they hold complete locally.

Fix $x \in \operatorname{MaxSpec} R$. By 6.3.6,

$$\mathrm{H}^{-i}(K) \otimes \mathcal{R}_x = \operatorname{Tor}_i^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}}) \otimes \mathcal{R}_x = \operatorname{Tor}_i^{\widehat{\Lambda}}(\widehat{\Lambda}_{\mathrm{con}}, \widehat{\Lambda}_{\mathrm{con}})$$

Hence, 5.2.12 implies that $\mathrm{H}^i(K) \otimes \mathcal{R}_x = 0$ for $i \neq 0, -t$. Since this is true for all $x \in \operatorname{MaxSpec} R$, it must be that $\mathrm{H}^i(K) = 0$ for $i \neq 0, -t$. Thus, even in the Zariski local setting, we may specify the cotwist and dual cotwist as in (6.3.C). □

We next show that ${}_{\Lambda_{\mathrm{con}}} \operatorname{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}})_{\Lambda_{\mathrm{con}}}$ is a tilting bimodule so that, by 6.3.7, the cotwist is an equivalence. To do this, we will use 6.3.2.

Lemma 6.3.8. *Suppose that Λ_{con} is perfect and t-relatively spherical. Then, the Λ_{con}-bimodule $\operatorname{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}})$ is tilting.*

Proof. Set $T = {}_{\Lambda_{\mathrm{con}}} \operatorname{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}})_{\Lambda_{\mathrm{con}}}$. Then, $\widehat{T} = \operatorname{Tor}_t^{\widehat{\Lambda}}(\widehat{\Lambda}_{\mathrm{con}}, \widehat{\Lambda}_{\mathrm{con}})$ by 6.3.6. It follows from 6.3.4 that $\widehat{\Lambda}_{\mathrm{con}}$ is t-relatively spherical and perfect, and so 6.2.4 implies that $\widehat{T}$ is tilting for all $x \in \operatorname{MaxSpec} R$. Note, moreover that $\widehat{T}$ is projective on either side by 5.2.16. Hence, 6.3.2 implies that T is tilting. □

In summary, we have proven the following theorem in this section.

Theorem 6.3.9. *If Λ_{con} is t-relatively spherical and perfect over Λ, then*

$$\Phi_R \colon \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$$

is a spherical twist around the functor

$$\Psi_{\mathcal{P}}^{-1} \cdot F \colon \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_{\mathrm{con}}) \to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X).$$

The cotwist is

$$\mathcal{C} = \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}({}_{\Lambda_{\mathrm{con}}} \operatorname{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}})_{\Lambda_{\mathrm{con}}}, -)[-t-1]. \tag{6.3.D}$$

If $\dim_k \Lambda_{\mathrm{con}} < \infty$ and $t = d$, then the cotwist is naturally isomorphic to $[-d-1] \cdot \mathcal{N}$, where $\mathcal{N}$ is the Nakayama functor of Λ_{con}.

Proof. The fact that Φ_R is a spherical twist follows from 6.3.5, 6.3.7 and 6.3.8.

If $\dim_k \Lambda_{\mathrm{con}} < \infty$, then it follows 5.1.26(1) that it is self-injective. Hence, Λ and Λ_{con} satisfy the assumptions of 4.2.1. Whence, the cotwist is naturally isomorphic to $[-d-1] \cdot \mathcal{N}$. □

Example 6.3.10. In this example we will show that 6.3.9 extends [BB22, 5.18] by dropping the one-dimensional fibre assumption on f and the hypersurface singularity assumption on Y.

Assume that $Y = \operatorname{Spec} \mathcal{R}$ has at worst hypersurface singularities. Moreover, suppose that the exceptional locus of f has codimension greater than one and that f has fibres which are at most one-dimensional. Similarly to 6.2.6, it follows from [VdB04, 3.2.11] that X has a tilting bundle $\mathcal{V} = \mathcal{O}_X \oplus \mathcal{P}$. Let $\Lambda = \operatorname{End}_{\mathcal{R}}(f_* \mathcal{V})$ and $\Lambda_{\mathrm{con}} = \underline{\operatorname{End}}_{\mathcal{R}}(f_* \mathcal{V})$.

By 6.2.6, $\widehat{\Lambda}_{\mathrm{con}}$ is 3-relatively spherical and perfect over $\widehat{\Lambda}$ for all $x \in \operatorname{MaxSpec} R$. Therefore, by 6.3.4, Λ_{con} is 3-relatively spherical and perfect over Λ, and so we may apply 6.3.9.

Furthermore, 6.3.9 implies that the cotwist is $\mathcal{C} = \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathrm{con}}}(T, -)[-4]$ where the bimodule $T = {}_{\Lambda_{\mathrm{con}}}\operatorname{Tor}_t^{\Lambda}(\Lambda_{\mathrm{con}}, \Lambda_{\mathrm{con}})_{\Lambda_{\mathrm{con}}}$. It follows from 6.2.6 and 6.3.6 that $\widehat{T} = \widehat{\Lambda}_{\mathrm{con}}$. Hence, we may conclude that T is a projective Λ_{con}-module, whence

$$\mathcal{C} = \mathrm{Hom}_{\Lambda_{\mathrm{con}}}(T, -)[-4].$$

§ 6.4 Global setting

Assuming the general setup 2.7.1, this section constructs a twist $\mathcal{T}$ in the category $\mathrm{D^b}(\operatorname{coh} X)$. Furthermore, we will establish that $\mathcal{T}$ is an equivalence if the noncommutative scheme $(Y, \mathcal{A}_{\mathrm{con}})$ (introduced in [DW19b, section 2] and recalled in § 2.7.1) satisfies certain relatively spherical properties. In what follows, we assume the notation 2.7.5.

§ 6.4.1 The twist

Proposition 6.4.1. *There are adjoint pairs (G^{LA}, G), (G, G^{RA}) as described below in (6.4.A).*

$$\begin{tikzcd}[column sep=huge]
\mathrm{D}(\mathrm{Qcoh}(Y, \mathcal{A}_{\mathrm{con}})) \arrow[rr, "G = - \otimes^{\mathbf{L}}{}_{\mathcal{A}_{\mathrm{con}}} \mathcal{A}_{\mathrm{con}} \cong \mathbf{R}\mathrm{Hom}_{\mathcal{A}_{\mathrm{con}}}(\mathcal{A}_{\mathrm{con}}, -)" description] & & \mathrm{D}(\mathrm{Qcoh}(Y, \mathcal{A})) \arrow[ll, bend right, "G^{\mathrm{LA}} = - \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{A}_{\mathrm{con}}"'] \arrow[ll, bend left, "G^{\mathrm{RA}} = \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}_{\mathrm{con}}, -)"]
\end{tikzcd} \tag{6.4.A}$$

Moreover, if $\mathcal{A}_{\mathrm{con}}$ is perfect as a $\mathcal{A}$-module, then G^{LA}, G and G^{RA} restrict to the bounded derived categories of coherent sheaves.

Proof. This follows from 2.6.7, 2.6.9 and 2.7.6. □

Our goal is to describe the twist around the functor G.

Recall that by the construction of $\mathcal{A}$ and $\mathcal{A}_{\mathrm{con}}$, there is an exact sequence

$$0 \to \mathcal{I} \xrightarrow{\iota} \mathcal{A} \xrightarrow{\pi} \mathcal{A}_{\mathrm{con}} \to 0 \tag{6.4.B}$$

of $\mathcal{A}$-bimodules.

Lemma 6.4.2. *If $\mathcal{A}_{\mathrm{con}}$ is perfect as an $\mathcal{A}$-module, so is $\mathcal{I}$.*

Proof. We may assume $Y = \operatorname{Spec} R$ with $\mathcal{A} = \Lambda^{\sim}$, $\mathcal{A}_{\mathrm{con}} = \Lambda_{\mathrm{con}}^{\sim}$ and $\mathcal{I} = [\operatorname{add} R]^{\sim}$. Then, $\mathcal{I}$ is perfect if and only if it has finite projective dimension. By assumption, Λ_{con} is quasi-isomorphic to a bounded complex P of projective Λ-modules and, thus, has finite projective dimension as a Λ-module.

The equation (6.4.B) reduces to

$$0 \to [\mathrm{add}\, R] \to \Lambda \to \Lambda_{\mathrm{con}} \to 0,$$

so that finite projective dimension of Λ_{con} implies finite projective dimension of $[\mathrm{add}\, R]$. ☐

Lemma 6.4.3. *There is a commutative diagram*

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{A}_{\mathrm{con}}, -) \arrow[r, "\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\pi, -)"] \arrow[d, "\wr"] & \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{A}, -) \arrow[d, "\wr"] \\
\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{A}_{\mathrm{con}}, -) \otimes^{\mathbf{L}}_{\mathcal{A}_{\mathrm{con}}} \mathcal{A}_{\mathrm{con}} \arrow[r, "\varepsilon"] & 1_{\mathrm{D}((Y,\mathcal{A}))}
\end{tikzcd}$$

of endofunctors on $\mathrm{D}(\mathrm{Qcoh}(Y, \mathcal{A}))$. *Here,* ε *is the counit of the adjunction induced by* 2.6.7(5).

Proof. Let $\mathcal{M} \in \mathrm{D}(\mathrm{Qcoh}(Y, \mathcal{A}))$ and let $j_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{J}$ be a K-injective resolution of $\mathcal{M}$. Then,

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{A}_{\mathrm{con}}, \mathcal{M}) = \mathcal{H}om^*_{\mathcal{A}}(\mathcal{A}_{\mathrm{con}}, \mathcal{J}),$$
$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{A}_{\mathrm{con}}, -) \otimes^{\mathbf{L}}_{\mathcal{A}_{\mathrm{con}}} \mathcal{A}_{\mathrm{con}} = \mathrm{Tot}(\mathcal{H}om^*_{\mathcal{A}}(\mathcal{A}_{\mathrm{con}}, \mathcal{J}) \otimes_{\mathcal{A}_{\mathrm{con}}} \mathcal{A}_{\mathrm{con}}).$$

Consider the isomorphisms of chain complexes

$$m \colon \mathrm{Tot}(\mathcal{H}om^*_{\mathcal{A}}(\mathcal{A}_{\mathrm{con}}, \mathcal{J}) \otimes_{\mathcal{A}_{\mathrm{con}}} \mathcal{A}_{\mathrm{con}}) \to \mathcal{H}om^*_{\mathcal{A}}(\mathcal{A}_{\mathrm{con}}, \mathcal{J})$$
$$\mathrm{ev}_1 \colon \mathcal{H}om^*_{\mathcal{A}}(\mathcal{A}, \mathcal{J}) \to \mathcal{J}$$

which are natural in $\mathcal{J}$. We may view m as the sheafification of the morphism which at every open set U acts as the natural multiplication morphism

$$\mathrm{Tot}(\mathrm{Hom}^*_{\mathcal{A}|_U}(\mathcal{A}_{\mathrm{con}}|_U, \mathcal{J}|_U) \otimes_{\mathcal{A}_{\mathrm{con}}(U)} \mathcal{A}_{\mathrm{con}}(U)) \to \mathrm{Hom}^*_{\mathcal{A}|_U}(\mathcal{A}_{\mathrm{con}}|_U, \mathcal{J}|_U)$$

Moreover, ev_1 is the morphism which at every open set U acts as the natural evaluation morphism at the identity section of $\mathcal{A}(U)$.

$$\mathrm{Hom}^*_{\mathcal{A}|U}(\mathcal{A}|_U, \mathcal{J}|_U) \to \mathcal{J}(U)$$

Note, moreover, that $\varepsilon_{\mathcal{M}} = j_{\mathcal{M}} \cdot \gamma_{\mathcal{M}}$ where $\gamma_{\mathcal{M}}$ is the natural evaluation map which defines the tensor-hom adjunction in the homotopy category of chain complexes of $\mathcal{A}$-modules. Hence, the commutativity of the diagram in the statement of the lemma is equivalent to commutativity for all $\mathcal{J}$ of the following diagram.

$$\begin{tikzcd}
\mathcal{H}om^*_{\mathcal{A}}(\mathcal{A}_{\mathrm{con}}, \mathcal{J}) \arrow[r, "\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\pi,-)"] \arrow[d, "m", "\wr"'] & \mathcal{H}om^*_{\mathcal{A}}(\mathcal{A}, \mathcal{J}) \arrow[d, "\mathrm{ev}_1", "\wr"'] \\
\mathrm{Tot}(\mathcal{H}om^*_{\mathcal{A}}(\mathcal{A}_{\mathrm{con}}, \mathcal{J}) \otimes_{\mathcal{A}_{\mathrm{con}}} \mathcal{A}_{\mathrm{con}}) \arrow[r, dashed, "\gamma_{\mathcal{M}}"] \arrow[d, "\gamma_{\mathcal{M}}"] & \mathcal{J} \arrow[d, "j_{\mathcal{M}}"] \\
\mathcal{J} \arrow[r, "j_{\mathcal{M}}^{-1}"] & \mathcal{M}
\end{tikzcd}$$

Whence, it suffices to check commutativity of the top square, which only involves morphisms of chain complexes. So, it is enough to check that the diagram commutes affine locally, which is just 3.1.2. □

Corollary 6.4.4. *The functor*

$$\mathcal{T} := \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -)\colon \mathrm{D}(\mathrm{Qcoh}(Y, \mathcal{A})) \to \mathrm{D}(\mathrm{Qcoh}(Y, \mathcal{A}))$$

is the twist around G. If, moreover, $\mathcal{A}_{\mathrm{con}}$ is perfect as an $\mathcal{A}$-module, then $\mathcal{T}$ restricts to the bounded derived category of coherent $\mathcal{A}$-modules.

Proof. The fact that $\mathcal{T}$ is the twist around G is just the global analogue of 3.1.3.

If $\mathcal{A}$ is perfect, then so is $\mathcal{I}$ by 6.4.2. Thus, $\mathcal{T}$ restricts to the bounded derived category of coherent $\mathcal{A}$-modules by 2.6.9. □

Corollary 6.4.5. *Suppose that $\mathcal{A}_{\mathrm{con}}$ is perfect as an $\mathcal{A}$-module. Then, the functor*

$$\Psi_{\mathcal{P}}^{-1} \cdot \mathcal{T} \cdot \Psi_{\mathcal{P}}\colon \mathrm{D^b}(\mathrm{coh}\, X) \to \mathrm{D^b}(\mathrm{coh}\, X)$$

is the twist around $\Psi_{\mathcal{P}}^{-1} \cdot G$.

Proof. This is immediate from either 2.4.11 or 6.1.1 and 6.4.4. □

§ 6.4.2 Equivalence criterion

The goal of this subsection is to prove that the twist $\mathcal{T}$ is an equivalence when the NC scheme $(Y, \mathcal{A}_{\mathrm{con}})$ satisfies certain spherical assumptions. In order to do so, the following technical lemma will be essential.

Lemma 6.4.6. *Let $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{X}, \mathcal{B})$ be noncommutative schemes. Let $i\colon U \hookrightarrow \mathcal{X}$ an open subset. Let $\mathcal{F} \in \mathrm{D}(\mathrm{Qcoh}(\mathcal{X}, \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}}))$ be a complex of bimodules which are locally finitely presented as $\mathcal{A}$-modules. Consider the adjunctions induced by 2.6.7(5)*

$$- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{F} \dashv \mathbf{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, -), \tag{6.4.C}$$

$$- \otimes^{\mathbf{L}}_{i^*\mathcal{A}} i^*\mathcal{F} \dashv \mathbf{R}\mathcal{H}om_{i^*\mathcal{B}}(i^*\mathcal{F}, -) \tag{6.4.D}$$

and let $\varepsilon^{\mathcal{X}}$ *and* $\eta^{\mathcal{X}}$ *denote the counit and unit of the adjunction* (6.4.C)*, respectively. Similarly, let* ε^{U} *and* η^{U} *denote the counit and unit of the adjunction* (6.4.D)*. Then,* $i^*(\varepsilon^{\mathcal{X}}_{\mathcal{F}}) = \varepsilon^{U}_{i^*\mathcal{F}}$ *and* $i^*(\eta^{\mathcal{X}}_{\mathcal{F}}) = \eta^{U}_{i^*\mathcal{F}}$*.*

Proof. We prove the statement $i^*(\varepsilon^{\mathcal{X}}_{\mathcal{F}}) = \varepsilon^{U}_{i^*\mathcal{F}}$, and omit the analogous proof of $i^*(\eta^{\mathcal{X}}_{\mathcal{F}}) = \eta^{U}_{i^*\mathcal{F}}$.

Write $\mathcal{O}(\mathcal{X})$ for the collection of open sets of $\mathcal{X}$. Let $\mathcal{M}, \mathcal{N} \in \mathrm{D}(\mathrm{Qcoh}(\mathcal{X}, \mathcal{B}))$, and fix a K-injective resolution $j\colon \mathcal{M} \to \mathcal{J}$ of $\mathcal{M}$. Moreover, let $p\colon \mathcal{Q} \to \mathcal{F}$ be a K-flat resolution of $\mathcal{F}$ as a $\mathcal{A}$-$\mathcal{B}$-module. From 2.6.7(5), there is an isomorphism

$$\begin{array}{ccc}
\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{N}, \mathbf{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{M})) & \xrightarrow[\sim]{} & \mathbf{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{F}, \mathcal{M}) \\
\| & & \| \\
\mathcal{H}om^*_{\mathcal{A}}(\mathcal{N}, \mathcal{H}om^*_{\mathcal{B}}(\mathcal{Q}, \mathcal{J})) & \xrightarrow[\sim]{\Phi^{(\mathcal{N},\mathcal{M})}} & \mathcal{H}om^*_{\mathcal{B}}(\mathrm{Tot}(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{Q}), \mathcal{J})
\end{array}$$

where $\Phi^{(\mathcal{N},\mathcal{M})}$ is an isomorphism of complexes of sheaves. Therefore,

$$\Phi^{(\mathcal{N},\mathcal{M})} = (\Phi^{(\mathcal{N},\mathcal{M})}_V)_{V \in \mathcal{O}(\mathcal{X})}$$

where each $\Phi^{(\mathcal{N},\mathcal{M})}_V$ is a map of complexes

$$\Phi^{(\mathcal{N},\mathcal{M})}_V\colon \mathrm{Hom}^*_{\mathcal{A}|_V}(\mathcal{N}|_V, \mathcal{H}om^*_{\mathcal{B}}(\mathcal{Q}|_V, \mathcal{J}|_V)) \to \mathrm{Hom}^*_{\mathcal{B}|_V}(\mathrm{Tot}(\mathcal{N}|_V \otimes^{\mathbf{L}}_{\mathcal{A}|_V} \mathcal{Q}|_V), \mathcal{J}|_V),$$

and these maps are compatible with restriction. That is, the diagram

$$\begin{array}{ccc}
\mathrm{Hom}^*_{\mathcal{A}}(\mathcal{N}, \mathcal{H}om^*_{\mathcal{B}}(\mathcal{Q}, \mathcal{J})) & \xrightarrow{\Phi^{(\mathcal{N},\mathcal{M})}_{\mathcal{X}}} & \mathrm{Hom}^*_{\mathcal{B}}(\mathrm{Tot}(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{Q}), \mathcal{J}) \\
\downarrow{\scriptstyle (-)|_U} & & \downarrow{\scriptstyle (-)|_U} \\
\mathrm{Hom}^*_{\mathcal{A}|_U}(\mathcal{N}|_U, \mathcal{H}om^*_{\mathcal{B}}(\mathcal{Q}|_U, \mathcal{J}|_U)) & \xrightarrow{\Phi^{(\mathcal{N},\mathcal{M})}_U} & \mathrm{Hom}^*_{\mathcal{B}|_U}(\mathrm{Tot}(\mathcal{N}|_U \otimes^{\mathbf{L}}_{\mathcal{A}|_U} \mathcal{Q}|_U), \mathcal{J}|_U)
\end{array} \tag{6.4.E}$$

commutes. Moreover,

$$\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{M})|_U = \mathbf{R}\mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{F}|_U, \mathcal{M}|_U), \qquad \text{(by [KS05, 18.4.6])}$$
$$(\mathcal{N} \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{F})|_U = \mathcal{N}|_U \otimes^{\mathbf{L}}_{\mathcal{A}|_U} \mathcal{F}|_U. \qquad \text{(by [KS05, 18.5.2])}$$

With these two equations in mind, note that the morphisms $\mathrm{H}^0(\Phi^{(\mathcal{N},\mathcal{M})}_X)$ and $\mathrm{H}^0(\Phi^{(\mathcal{N},\mathcal{M})}_U)$ induce the Hom-set isomorphisms defining the adjunctions (6.4.C) and (6.4.D), respectively. Therefore, when the complex $\mathcal{N} = \mathbf{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{M})$, then $\varepsilon^{\mathcal{X}}_{\mathcal{F}} = \mathrm{H}^0(\Phi^{(\mathcal{N},\mathcal{M})}_X)(\mathrm{id})$ and $\varepsilon^{U}_{i^*\mathcal{F}} = \mathrm{H}^0(\Phi^{(\mathcal{N},\mathcal{M})}_U)(\mathrm{id})$.

For $\mathcal{N} = \mathbf{R}\mathcal{H}om_{\mathcal{B}}(\mathcal{F}, \mathcal{M})$, the diagram (6.4.E) is

$$\begin{array}{ccc}
\mathrm{Hom}^*_{\mathcal{A}}(\mathcal{H}om^*_{\mathcal{B}}(\mathcal{Q}, \mathcal{J}), \mathcal{H}om^*_{\mathcal{B}}(\mathcal{Q}, \mathcal{J})) & \xrightarrow{\Phi^{(\mathcal{N},\mathcal{M})}_{\mathcal{X}}} & \mathrm{Hom}^*_{\mathcal{B}}(\mathrm{Tot}(\mathcal{H}om^*_{\mathcal{B}}(\mathcal{Q}, \mathcal{J}) \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{Q}), \mathcal{J}) \\
\downarrow{\scriptstyle (-)|_U} & & \downarrow{\scriptstyle (-)|_U} \\
\mathrm{Hom}^*_{\mathcal{A}|_U}(\mathcal{H}om^*_{\mathcal{B}|_U}(\mathcal{Q}|_U, \mathcal{J}|_U), \mathcal{H}om^*_{\mathcal{B}|_U}(\mathcal{Q}|_U, \mathcal{J}|_U)) & \xrightarrow{\Phi^{(\mathcal{N},\mathcal{M})}_U} & \mathrm{Hom}^*_{\mathcal{B}|_U}(\mathrm{Tot}(\mathcal{H}om^*_{\mathcal{B}|_U}(\mathcal{Q}|_U, \mathcal{J}|_U) \otimes^{\mathbf{L}}_{\mathcal{A}|_U} \mathcal{Q}|_U), \mathcal{J}|_U)
\end{array}$$

Thus,

$$\varepsilon^{\mathcal{X}}_{\mathcal{F}}|_U = \mathrm{H}^0(\Phi_X^{(\mathcal{N},\mathcal{M})})(\mathrm{id})|_U = \mathrm{H}^0(\Phi_U^{(\mathcal{N},\mathcal{M})})(\mathrm{id}\,|_U) = \mathrm{H}^0(\Phi_U^{(\mathcal{N},\mathcal{M})})(\mathrm{id}) = \varepsilon^U_{i^*\mathcal{F}}$$

as required. □

Definition 6.4.7. Similarly to [DW19b, 5.6], we will say that $\mathcal{A}_{\mathrm{con}}$ is t-relatively spherical if for all closed points $z \in Z$,

$$\mathrm{Ext}^i_{\widehat{\mathcal{A}}_z}(\widehat{\mathcal{A}}_{\mathrm{con},z}, \mathcal{S}) = 0$$

for all $i \neq 0, t$. Here, $\mathcal{S}$ is the direct sum of all simple $\widehat{\mathcal{A}}_{\mathrm{con},z}$ modules.

Theorem 6.4.8. *If $\mathcal{A}_{\mathrm{con}}$ is perfect in $\mathrm{D^b}(\mathrm{coh}(Y, \mathcal{A}))$ and t-relatively spherical, then*

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -)\colon\ \mathrm{D^b}(\mathrm{mod}\,\mathcal{A}) \to \mathrm{D^b}(\mathrm{mod}\,\mathcal{A})$$

is an equivalence. Consequently,

$$\Phi_X = \Psi_{\mathcal{P}}^{-1} \cdot \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{I}, -) \cdot \Psi_{\mathcal{P}}\colon\ \mathrm{D^b}(\mathrm{coh}\,X) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{coh}\,X) \tag{6.4.F}$$

is an autoequivalence of $\mathrm{D^b}(\mathrm{coh}\,X)$.

Proof. First, we show that $(Y, \mathcal{A})$ admits an affine open cover $\{U_i = \mathrm{Spec}\,R_i \to Y\}$ such that $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -)|_{U_i}$ is an equivalence. Observe that, by 6.3.5, the functor $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -)|_{U_i} \cong \mathbf{R}\mathrm{Hom}_{\Lambda_{U_i}}([\mathrm{add}\,R_i], -)$ is an equivalence if $(\Lambda_{\mathrm{con}})_{U_i}$ is perfect as a Λ_{U_i}-module and t-relatively spherical.

Well, since $\mathcal{A}_{\mathrm{con}}$ is assumed to be perfect in $\mathrm{D^b}(\mathrm{coh}(Y, \mathcal{A}))$, there exists an open affine cover $\{j_i\colon U_i = \mathrm{Spec}\,R_i \to Y\}$ such that $\mathcal{A}_{\mathrm{con}}|_{U_i}$ is perfect as a $\mathcal{A}|_{U_i}$-module for all i. Equivalently, since each U_i is affine, $(\Lambda_{\mathrm{con}})_{U_i}$ is perfect as a Λ_{U_i}-module.

Moreover, since the definition of t-relatively spherical is stalk-local, the condition that $\mathcal{A}_{\mathrm{con}}$ is t-relatively spherical implies $\mathcal{A}_{\mathrm{con}}|_{U_i}$ is t-relatively spherical. That is, for $z \in U_i$,

$$(\mathcal{A}_{\mathrm{con}}|_{U_i})_z = ((\widehat{\Lambda}_{\mathrm{con}})_{U_i})_z.$$

Hence, $(\Lambda_{\mathrm{con}})_{U_i}$ is t-relatively spherical by 6.3.4. Thus, by the proof of 6.3.5, we may conclude that the functor $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -)|_{U_i} \cong \mathbf{R}\mathrm{Hom}_{\Lambda_{U_i}}([\mathrm{add}\,R_i], -)$ is an equivalence for all U_i in the affine open cover.

We next show that this implies that the functor $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -)$ is an equivalence. Notice that it is an equivalence if and only if the unit $\eta\colon 1_{\mathcal{A}} \to \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, - \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I})$ and the counit $\varepsilon\colon \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -) \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I} \to 1_{\mathcal{A}}$ are an isomorphisms. Since the proofs that η and ε are isomorphisms are very similar, we prove the statement for η and omit the proof for ε.

Observe that η is an isomorphism if and only if η_a is an isomorphism for all $a \in$

$\mathrm{D^b}(\mathrm{mod}\,\mathcal{A})$. To show that this latter condition holds, consider the triangle

$$a \xrightarrow{\eta_a} \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, a \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I}) \to K_a \to^{+}$$

in $\mathrm{D^b}(\mathrm{coh}(Y, \mathcal{A}))$. We will show that $K_a = 0$.

Since U_i is affine, the inclusion $j_i \colon U_i \to Y$ is flat so that j_i^* is an exact functor. We thus have the following triangle

$$j_i^* a \xrightarrow{j_i^* \eta_a} j_i^* \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, a \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I}) \to j_i^*(K_a) \to^{+}$$

Now, by 6.4.6, this triangle is the same as

$$j_i^* a \xrightarrow{\eta_{j_i^* a}} \mathbf{R}\mathcal{H}om_{\mathcal{A}}(j_i^* \mathcal{I}, j_i^* a \otimes^{\mathbf{L}}_{\mathcal{A}} j_i^*(\mathcal{I})) \to j_i^*(K_a) \to^{+}$$

Where $\eta_{j_i^* a}$ is the unit of the Zariski local adjunction

$$- \otimes^{\mathbf{L}}_{\mathcal{A}} j_i^* \mathcal{I} \dashv \mathbf{R}\mathcal{H}om_{\mathcal{A}|_{U_i}}(j_i^* \mathcal{I}, -).$$

However, the functor $\mathbf{R}\mathcal{H}om_{\mathcal{A}|_U}(j^* \mathcal{I}, -) = \mathbf{R}\mathcal{H}om_{\mathcal{A}|_{U_i}}([\mathrm{add}\, R_i], -)$ is an equivalence by the above. It follows that $j_i^*(K_a) = 0$. Since K_a is zero on every U_i and $\{U_i\}_i$ is an open cover of Y, necessarily $K_a = 0$. Thus, η is an isomorphism. $\square$

Chapter 7

Concluding Remarks

We end this thesis with a short reflection on the technology that has been developed, as well as with an outlook towards possible extensions.

To recapitulate, this thesis computes the twist and cotwist around the restriction of scalars functor F associated to a ring morphism $p\colon A \to B$. By restricting our setting to surjective morphisms, Gorenstein orders, or stable endomorphism algebras, we are able to specify that F is spherical. In all of these settings, our motivating examples are surjections $\pi\colon \Lambda \to \Lambda_{\mathrm{con}}$ induced by crepant contractions. In order to simplify the discussion, let us assume throughout that A and also Λ have finite global dimension.

In the setting of Gorenstein orders (chapter 4) over 3-dimensional rings, given an surjective morphism $p\colon A \to B$, there are two tensions one needs to grapple with in order to assert that F is spherical. We need B to be self-injective and $\ker p = (\ker p)^2$. Remarkably, when $\mathcal{R}$ is a Gorenstein isolated cDV singularity, then the epimorphism $\pi\colon \Lambda \to \Lambda_{\mathrm{con}}$ satisfies these conditions. However, this might not be the case in more singular settings, as illustrated in example 4.2.10. Therefore, from the point of view of constructing derived equivalences, example 4.2.10 and corollary 4.2.6 suggest that it might be more interesting to study other self-injective quotients of Λ.

Furthermore, as soon as we move to non-isolated settings or higher dimensional settings, Λ_{con} is unlikely to be finite dimensional (see [DW19a, 4.8]) and, therefore, it is unlikely to be self-injective (see remark 4.1.4). So, with the goal of tackling such cases, we move to the setting of chapter 5.

In chapter 5, we prove that F is spherical given that B is t-relatively spherical. In particular, we recover [DW19b, 6.3] when Y is complete locally a hypersurface. As an example, we may consider $Y = \operatorname{Spec} \mathbb{C}[[u, v, x, y]]/(uv - x^2y)$. Then, by [IW18], Y admits 3 crepant resolutions which are derived equivalent to algebras $\Lambda_1 = \operatorname{End}_R(M_1)$, $\Lambda_2 = \operatorname{End}_R(M_2)$ and $\Lambda_3 = \operatorname{End}_R(M_3)$ where

$$\begin{aligned} M_1 &= R \oplus \langle u, x\rangle \oplus \langle u, xy\rangle \\ M_2 &= R \oplus \langle u, y\rangle \oplus \langle u, xy\rangle \\ M_3 &= R \oplus \langle u, x\rangle \oplus \langle u, x^2\rangle. \end{aligned}$$

Each resolution has an associated surjection $p_i\colon \Lambda_i \to \Lambda_{\mathrm{con}}^{(i)}$, and thus has associated functors $F_i := -\otimes_{\Lambda_i} \Lambda_{\mathrm{con}}^{(i)}$ and $\mathcal{T}_i := \mathbf{R}\mathrm{Hom}_{\Lambda_i}(\ker p_i, -)$.

It turns out that $R \oplus \Omega M_1 \cong M_1$, and so $\Lambda_{\mathrm{con}}^{(1)}$ is 2-relatively spherical. Whence, by either [DW19b, 6.3] or 5.2.13, $\mathcal{T}_1$ is a spherical twist around F_1. On the other hand, although $R \oplus \Omega^2 M_2 = M_2$, it does not hold that $\mathrm{Ext}^1_R(M_2, M_2) = 0$. Hence, $\Lambda_{\mathrm{con}}^{(2)}$ is not 3-relatively spherical and nothing can be said about $\mathcal{T}_2$.

However, there is a derived equivalence $\Phi\colon \mathrm{D^b}(\operatorname{mod}\Lambda_1) \to \mathrm{D^b}(\operatorname{mod}\Lambda_2)$ and so, by either 2.4.11 or 6.1.1, the composition $\Phi \cdot \mathcal{T}_2 \cdot \Phi_1^{-1}$ is a spherical twist around the functor $\Phi \cdot F_1$. The question we pose is whether $\Phi \cdot \mathcal{T}_2 \cdot \Phi^{-1}$ can be intrinsically characterised by an algebra B and a surjection $q\colon \Lambda_2 \to \Lambda_{\mathrm{con}}^{(2)} \to B$.

In fact, one can show that there is a tilting complex $\mathcal{P} \in \mathrm{D^b}(\operatorname{mod}\Lambda_{\mathrm{con}}^{(1)})$ such that $B := \mathrm{End}_{\Lambda_{\mathrm{con}}^{(1)}}(\mathcal{P})$ is indeed a quotient $q\colon \Lambda_2 \to \Lambda_{\mathrm{con}}^{(2)} \to B$. Therefore, there is a diagram

$$
\begin{tikzcd}
\mathrm{D^b}(\operatorname{mod}\Lambda_1) \arrow[rr, "\Phi", "\sim"'] & & \mathrm{D^b}(\operatorname{mod}\Lambda_2) \\
\mathrm{D^b}(\operatorname{mod}\Lambda_{\mathrm{con}}^{(1)}) \arrow[u, "F_1"] \arrow[rr, "-\otimes^{\mathbf{L}}_{\Lambda_{\mathrm{con}}^{(1)}} \underline{\mathrm{Hom}}(M_2, M_1)"] \arrow[dr, "\sim", "-\otimes^{\mathbf{L}}_{\Lambda_{\mathrm{con}}^{(1)}} \mathcal{P}"'] & & \mathrm{D^b}(\operatorname{mod}\Lambda_{\mathrm{con}}^{(2)}) \arrow[u, "F_2"'] \\
& \mathrm{D^b}(\operatorname{mod}B) \arrow[ur, "-\otimes_B B"'] &
\end{tikzcd}
$$

with respect to which there are interesting questions to ask:

(1) Does the diagram commute?

(2) Is the functor $\Phi \cdot \mathcal{T}_2 \cdot \Phi^{-1}$ a spherical twist around $-\otimes_B B_{\Lambda_2}$. That is, is there a natural isomorphism $\Phi \cdot F_1 \cong -\otimes_B B_{\Lambda_2}$?

(3) Can B be intrinsically characterised through properties of Λ_2?

(4) Can we deduce that $\Phi \cdot \mathcal{T}_2 \cdot \Phi^{-1}$ is a spherical twist based on properties of B?

In some sense, all of these questions fall under the umbrella question: from the point of view of constructing equivalences of $\mathrm{D^b}(\operatorname{mod}\Lambda_1)$, are other quotients of Λ (not just the contraction algebra) more interesting to study?

Appendix A

Chasing the evaluation map

Let A, B be rings such that there is a ring morphism $p\colon A \to B$. In this section, we compute the counit of the derived tensor-Hom adjunction by diagram chasing the identity along the isomorphism

$$\mathrm{Hom}_{\mathrm{D}(B)}(\mathbf{R}\mathrm{Hom}_A(B,-), \mathbf{R}\mathrm{Hom}_A(B,-)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(A)}(\mathbf{R}\mathrm{Hom}_A(B,-) \otimes_B^{\mathbf{L}} B, -)$$

which defines the derived tensor-hom adjunction.

Lemma A.0.1. *Let* $L \in \mathrm{K}(B)$, $M \in \mathrm{K}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$ *and* $N \in \mathrm{K}(A)$. *There is a canonical isomorphism*

$$\phi\colon \mathrm{Hom}_B^*(L, \mathrm{Hom}_A^*(M, N)) \xrightarrow{\sim} \mathrm{Hom}_A^*(\mathrm{Tot}(L \otimes_B M), N).$$

Proof. Let α be a degree j element in $\mathrm{Hom}_B^*(L, \mathrm{Hom}_A^*(M, N))$. Then,

$$\alpha \in \prod_{p+q=j} \mathrm{Hom}_B(L^{-p}, \mathrm{Hom}_A^q(M, N)) = \prod_{p+r+s=j} \mathrm{Hom}_B(L^{-p}, \mathrm{Hom}_A(M^{-r}, N^s)).$$

Hence, we may write $\alpha = (\alpha^{p,r,s})$. By the standard tensor-hom adjunction there are isomorphisms

$$\phi^{p,r,s}\colon \mathrm{Hom}_B(L^{-p}, \mathrm{Hom}_A(M^{-r}, N^s)) \xrightarrow{\sim} \mathrm{Hom}_A(L^{-p} \otimes_B M^{-r}, N^s).$$

Hence, there is an isomorphism $\phi^j = \prod_{p+r+s=j} \phi^{p,r,s}$:

$$\begin{aligned}
\mathrm{Hom}_B^*(L, \mathrm{Hom}_A^*(M, N))^j &= \prod_{p+r+s=j} \mathrm{Hom}_B(L^{-p}, \mathrm{Hom}_A(M^{-r}, N^s)) \\
&\xrightarrow[\phi^j]{\sim} \prod_{p+r+s=j} \mathrm{Hom}_A(L^{-p} \otimes_B M^{-r}, N^s) \\
&= \prod_{k+s=j} \mathrm{Hom}_A\Big(\bigoplus_{p+r=k} L^{-p} \otimes_B M^{-r}, N^s\Big) \\
&= \mathrm{Hom}_A^*(\mathrm{Tot}(L \otimes_B M), N)^j
\end{aligned}$$

It is straightforward to check that $\phi = (\phi^j)$ respects the differentials so that it is indeed a chain complex isomorphism. $\square$

Lemma A.0.2. *Let $L, M \in \mathrm{K}(B)$, then there is an isomorphism*

$$\mathrm{H}^j(\mathrm{Hom}^*(L, M)) \xrightarrow{f} \mathrm{Hom}_{\mathrm{K}(B)}(L, M[j]).$$

Proof. This isomorphism is well known. Explicitly, given a cocycle $\beta = (\beta^{p,q})$ in $\prod_{p+q=j} \mathrm{Hom}_R(L^{-p}, M^q)$, the isomorphism maps the equivalence class $[\beta]$ to the chain map $L \to M[j]$ in $\mathrm{K}(B)$ defined by the data $(\beta^{p,q})$. □

Lemma A.0.3. *Let $I \in \mathrm{K}(A)$ be a K-injective complex, and let P be a complex of A-B-bimodules which is K-flat as a complex of B^{op}-modules. Then, $\mathrm{Hom}^*_A(P, I)$ is a K-injective complex in $\mathrm{K}(B)$.*

Proof. Let M be an acyclic complex. Then,

$$\begin{aligned} \mathrm{Hom}_{\mathrm{K}(B)}(M, \mathrm{Hom}^*_A(P, I)) &\cong \mathrm{H}^0(\mathrm{Hom}^*_B(M, \mathrm{Hom}^*_A(P, I)) && \text{(by A.0.2)} \\ &\cong \mathrm{H}^0(\mathrm{Hom}^*_A(\mathrm{Tot}(M \otimes P), I)) && \text{(by A.0.1)} \\ &\cong \mathrm{Hom}_{\mathrm{K}(A)}(\mathrm{Tot}(M \otimes P), I). && \text{(by A.0.2)} \end{aligned}$$

Now, since M is acyclic and P is K-flat as a B^{op}-module, it must be that $\mathrm{Tot}(M \otimes P)$ is acyclic. Since I is K-injective, then $\mathrm{Hom}_{K(A)}(\mathrm{Tot}(M \otimes P), I) = 0$ by definition. Hence, by the definition of K-injective, $\mathrm{Hom}^*_A(P, I)$ is a K-injective complex. □

Lemma A.0.4. *Let $L, J \in \mathrm{D}(B)$ such that J is $K-$injective. Then, there is an isomorphism*

$$\mathrm{Hom}_{\mathrm{K}(B)}(L, J) \xrightarrow{g} \mathrm{Hom}_{\mathrm{D}(B)}(L, J).$$

Proof. The isomorphism $g\colon \mathrm{Hom}_{\mathrm{K}(R)}(L, J) \to \mathrm{Hom}_{\mathrm{D}(R)}(L, J)$ is well-known. It maps a morphism $L \xrightarrow{\alpha} J$ in $\mathrm{Hom}_{\mathrm{K}(R)}(L, J)$ to the roof $L = L \xrightarrow{\alpha} J$ in $\mathrm{Hom}_{\mathrm{D}(R)}(L, J)$. □

Proposition A.0.5. *Let $L \in \mathrm{K}(B)$, $M \in \mathrm{K}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$ and $N \in \mathrm{K}(A)$. Then, there is an isomorphism*

$$\phi^0_{L,M,N}\colon \mathrm{Hom}_{\mathrm{D}(B)}(L, \mathbf{R}\mathrm{Hom}_A(M, N)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(A)}(L \otimes^{\mathbf{L}}_B M, N)$$

which is natural in L, M, N.

Proof. Let $q_M\colon P \to M$ be a bimodule resolution of M which is K-flat as a complex of B^{op}-modules. Then, the statement follows from the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{D}(B)}(L, \mathbf{R}\mathrm{Hom}_A(M,N)) \arrow[r, "\psi^0_{L,M,N}"] \arrow[d, equal] & \mathrm{Hom}_{\mathrm{D}(A)}(L \otimes^{\mathbf{L}}_B M, I) \arrow[d, equal] \\
\mathrm{Hom}_{\mathrm{D}(B)}(L, \mathrm{Hom}^*_A(M,I)) \arrow[d, "\mathrm{Hom}_{\mathrm{D}(B)}(L,\mathrm{Hom}^*_A(q_M,I))", "\wr"'] & \mathrm{Hom}_{\mathrm{D}(A)}(\mathrm{Tot}(L \otimes P), I) \arrow[d, equal] \\
\mathrm{Hom}_{\mathrm{D}(B)}(L, \mathrm{Hom}^*_A(P,I)) \arrow[d, "g", "\wr"', "\text{by A.0.4 and A.0.3}"' near start] & \mathrm{Hom}_{\mathrm{D}(A)}(\mathrm{Tot}(L \otimes P), I) \arrow[d, "g"', "\wr"] \arrow[d, "\text{by A.0.4}"] \\
\mathrm{Hom}_{\mathrm{K}(B)}(L, \mathrm{Hom}^*_A(P,I)) \arrow[d, "f^{-1}", "\wr"', "\text{by A.0.2}"' near start] & \mathrm{Hom}_{\mathrm{K}(A)}(\mathrm{Tot}(L \otimes P), I) \arrow[d, "f^{-1}"', "\wr"] \arrow[d, "\text{by A.0.2}"] \\
\mathrm{H}^0(\mathrm{Hom}^*_B(L, \mathrm{Hom}^*_A(P,I)) \arrow[r, "\mathrm{H}^0(\phi)", "\sim\ \text{by A.0.1}"'] & \mathrm{H}^0(\mathrm{Hom}^*_A(\mathrm{Tot}(L \otimes P), I)
\end{tikzcd}$$

where $\psi^0_{L,M,N}$ is defined so that the diagram commutes. □

As in A.0.5, let $M \in \mathrm{K}(A \otimes_{\mathbb{Z}} B^{\mathrm{op}})$ and choose a bimodule resolution $P \to M$ of M that is K-flat as a B^{op}-module. Then, there is a quasi-isomorphism

$$s\colon \mathbf{R}\mathrm{Hom}_A(M,N) \otimes^{\mathbf{L}}_B M \to \mathrm{Hom}^*_A(P,I) \otimes^{\mathbf{L}}_B M.$$

Lemma A.0.6. *The counit ε of the adjunction $(-) \otimes^{\mathbf{L}}_B M \dashv \mathbf{R}\mathrm{Hom}_A(M,-)$ has component at $N \in \mathrm{D}(A)$*

$$\varepsilon_N\colon \mathbf{R}\mathrm{Hom}_A(M,N) \otimes^{\mathbf{L}}_B M \to N$$

given as the composition $j_N^{-1} \cdot \gamma_N \cdot s$. Here, $j_N\colon N \to I$ is a K-injective resolution of N and $\gamma_N = (\gamma^j_N)$ is a chain map

$$\gamma^j_N\colon \bigoplus_{p+q=j} \Big(\prod_{r+s=p} \mathrm{Hom}_A(P^{-r}, I^s)\Big) \otimes_B P^q \to I^j$$

sending $\alpha \otimes a \in \mathrm{Hom}_A(P^{-r}, I^s) \otimes P^q$ to zero if $-r \neq q$ and sending an element $\alpha \otimes a \in \mathrm{Hom}_A(P^q, I^j) \otimes P^q$ to $\alpha(a) \in I^j$.

Proof. The component ε_N of the counit is the image of the identity under the isomorphism $\phi^0_{\mathbf{R}\mathrm{Hom}_A(M,N),M,N}$ defined in A.0.5. That is, $\varepsilon_N = j_N^{-1} \cdot \psi^0_{\mathbf{R}\mathrm{Hom}_A(M,N),M,N}(\mathrm{id})$. Moreover, observe that since $\psi^0_{L,M,N}$ is natural in L, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{D}(B)}(\mathrm{Hom}^*_A(P,I), \mathbf{R}\mathrm{Hom}_A(M,N)) \arrow[r, "\psi^0_{\mathrm{Hom}^*_A(P,I),M,N}"] \arrow[d, "\mathrm{Hom}_{\mathrm{D}(B)}(\mathrm{Hom}^*_A(q_M,I),\mathbf{R}\mathrm{Hom}_A(M,N))"] & \mathrm{Hom}_{\mathrm{D}(A)}(\mathrm{Hom}^*_A(P,I) \otimes^{\mathbf{L}}_B M, I) \arrow[d, "\mathrm{Hom}_{\mathrm{D}(A)}(s,I)"] \\
\mathrm{Hom}_{\mathrm{D}(B)}(\mathbf{R}\mathrm{Hom}_A(M,N), \mathbf{R}\mathrm{Hom}_A(M,N)) \arrow[r, "\psi^0_{\mathbf{R}\mathrm{Hom}_A(M,N),M,N}"] & \mathrm{Hom}_{\mathrm{D}(A)}(\mathbf{R}\mathrm{Hom}_A(M,N) \otimes^{\mathbf{L}}_B M, I)
\end{tikzcd}$$

where the vertical arrows arrows are quasi-isomorphisms. Hence, $\psi^0_{\mathbf{R}\mathrm{Hom}_A(M,N),M,N}(\mathrm{id})$

equals

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{D}(A)}(s,I)\cdot\psi^0_{\mathrm{Hom}^*_A(P,I),M,N}\cdot\mathrm{Hom}_{\mathrm{D}(B)}(\mathrm{Hom}^*_A(q_M,I)^{-1},\mathbf{R}\mathrm{Hom}_A(M,N))(\mathrm{id})\\
&=\mathrm{Hom}_{\mathrm{D}(A)}(s,I)\cdot\psi^0_{\mathrm{Hom}^*_A(P,I),M,N}(\mathrm{Hom}^*_A(q_M,I)^{-1})\\
&=\psi^0_{\mathrm{Hom}^*_A(P,I),M,N}(\mathrm{Hom}^*_A(q_M,I)^{-1})\cdot s.
\end{aligned}$$

That is, $\varepsilon_N = j_N^{-1}\cdot\psi^0_{\mathrm{Hom}^*_A(P,I),M,N}(\mathrm{Hom}^*_A(q_M,I)^{-1})\cdot s$. Hence it suffices to show that

$$\psi^0_{\mathrm{Hom}^*_A(P,I),M,N}(\mathrm{Hom}^*_A(q_M,I)^{-1})=\gamma_N.$$

Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{D}(B)}(\mathrm{Hom}^*_A(P,I),\mathbf{R}\mathrm{Hom}_A(M,N)) \arrow[r, "\psi^0_{\mathrm{Hom}^*_A(P,I),M,N}"] \arrow[d, equal] & \mathrm{Hom}_{\mathrm{D}(A)}(\mathrm{Hom}^*_A(P,I)\otimes^{\mathbf{L}}_B M,I) \arrow[d, equal] \\
\mathrm{Hom}_{\mathrm{D}(B)}(\mathrm{Hom}^*_A(P,I),\mathrm{Hom}^*_A(M,I)) \arrow[d, "\mathrm{Hom}_{\mathrm{D}(B)}(\mathrm{Hom}^*_A(P,I),\mathrm{Hom}^*_A(q_M,I))", "\wr"'] & \mathrm{Hom}_{\mathrm{D}(A)}(\mathrm{Tot}(\mathrm{Hom}^*_A(P,I)\otimes P),I) \arrow[d, equal] \\
\mathrm{Hom}_{\mathrm{D}(B)}(\mathrm{Hom}^*_A(P,I),\mathrm{Hom}^*_A(P,I)) \arrow[d, "g", "\text{by A.0.4 and A.0.3}\ \wr"'] & \mathrm{Hom}_{\mathrm{D}(A)}(\mathrm{Tot}(\mathrm{Hom}^*_A(P,I)\otimes P),I) \arrow[d, "g"', "\wr\ \text{by A.0.4}"] \\
\mathrm{Hom}_{\mathrm{K}(B)}(\mathrm{Hom}^*_A(P,I),\mathrm{Hom}^*_A(P,I)) \arrow[d, "f^{-1}", "\text{by A.0.2}\ \wr"'] & \mathrm{Hom}_{\mathrm{K}(A)}(\mathrm{Tot}(\mathrm{Hom}^*_A(P,I)\otimes P),I) \arrow[d, "f^{-1}"', "\wr\ \text{by A.0.2}"] \\
\mathrm{H}^0(\mathrm{Hom}^*_B(\mathrm{Hom}^*_A(P,I),\mathrm{Hom}^*_A(P,I)) \arrow[r, "\mathrm{H}^0(\phi)", "\sim\ \text{by A.0.1}"'] & \mathrm{H}^0(\mathrm{Hom}^*_A(\mathrm{Tot}(\mathrm{Hom}^*_A(P,I)\otimes P),I)
\end{tikzcd}$$

Then, it is clear that

$$\mathrm{Hom}_{\mathrm{D}(B)}(\mathrm{Hom}^*_A(M,I),\mathrm{Hom}^*_A(q_M,I))(\mathrm{Hom}^*_A(q_M,I)^{-1})=\mathrm{id}\,.$$

So that $f^{-1}\cdot g(\mathrm{id})=[\beta]$, where

$$\beta=(\beta^j)\in\mathrm{Hom}^*_B(\mathrm{Hom}^*_A(P,I),\mathrm{Hom}^*_A(P,I))^0=\prod_j\mathrm{Hom}_B(\mathrm{Hom}^*_A(P,I)^j,\mathrm{Hom}^*_A(P,I)^j)$$

is the product over j of identity maps $\beta_j\colon \mathrm{Hom}^*_A(P,I)^j\to\mathrm{Hom}^*_A(P,I)^j$. That is,

$$\beta^j\colon\prod_{r+s=j}\mathrm{Hom}_A(P^{-r},I^s)\to\prod_{k+n=j}\mathrm{Hom}_A(P^{-k},I^n)$$

is the product of identity maps $\beta^j=(\beta^{r,s}\colon \mathrm{Hom}_A(P^{-r},I^s)\to\mathrm{Hom}_A(P^{-r},I^s))$. We may thus rewrite $\beta=(\beta^{j,r,s})$.

It follows from A.0.1 that

$$\mathrm{H}^0(\phi)([\beta])=[\phi(\beta)]=[\phi((\beta^{j,r,s}))]=[(\phi^{-j,r,s}(\beta^{j,r,s})]$$

where

$$\phi^{-j,r,s}(\beta^{j,r,s}) \in \mathrm{Hom}_A(\mathrm{Hom}^*_A(P^{-r}, I^s) \otimes P^{-r}, I^s)$$

is the map sending $\alpha \otimes a \mapsto \beta^{j,r,s}(\alpha)(a) = \alpha(a)$. Thus,

$$\phi(\beta)^j \in \mathrm{Hom}_A(\bigoplus_{q+n=j} (\prod_{r+s=q} \mathrm{Hom}_A(P^{-r}, I^s)) \otimes P^q, I^j)$$

sends $\alpha \otimes a \in \mathrm{Hom}_A(P^{-r}, I^s) \otimes P^q$ to zero if $-r \neq q$ and sends $\alpha \otimes a \in \mathrm{Hom}_A(P^q, I^j) \otimes P^q$ to $\alpha(a) \in I^j$.

Mapping $\mathrm{H}^0(\phi([\beta]))$ across f, we find that it corresponds to the chain map γ_N. For $g^{-1}(\gamma_N)$, we just view this map in the derived category.

In summary, the counit at N is

$$\varepsilon_N = j_N^{-1} \cdot \psi^0_{\mathrm{Hom}^*_A(P,I),M,N}(\mathrm{Hom}^*_A(q_M, I)^{-1}) \cdot s$$

where $\psi^0_{\mathrm{Hom}^*_A(P,I),M,N}(\mathrm{Hom}^*_A(q_M, I)^{-1})$ is represented in the derived category by the chain map γ_N, which concludes the proof. $\square$

Corollary A.0.7. *The counit ε of the adjunction $(-) \otimes^{\mathbf{L}}_B B \dashv \mathbf{R}\mathrm{Hom}_A(B, -)$ has component at $a \in \mathrm{D}(A)$*

$$\varepsilon_a \colon \mathbf{R}\mathrm{Hom}_A(B, a) \otimes^{\mathbf{L}}_B B \to a$$

given as the composition $j_a^{-1} \cdot \gamma_a$, where $j_a \colon a \to I$ is a K-injective resolution of a and γ_a is the chain map

$$\gamma_a = (\gamma_a^j) \colon \mathrm{Tot}(\mathrm{Hom}^*_A(B, I) \otimes_B B) \to I$$

which at degree j is specified by the evaluation map

$$\gamma_a^j \colon \mathrm{Hom}_A(B, I^j) \otimes_B B \to I^j$$

sending $f \otimes x \mapsto f(x)$.

Proof. This follows from A.0.6 by noting that B is flat as B^{op}-module, hence we need not take a K-flat resolution of B. $\square$

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