Finite-dimensional DG-algebras and Reflexive DG-categories

by

Isambard Goodbody

A thesis submitted in fulfilment of the requirements
for the degree of

Doctor of Philosophy

at the

School of Mathematics & Statistics
College of Science & Engineering
University of Glasgow

July 2025

Summary

In this thesis, we study algebra and geometry from the point of view of the derived category. This is a gadget associated to an algebraic or geometric object which contains its homological information. In this way, many fundamental invariants and properties can be read off from the derived category. To make this precise, one should enhance the derived category by giving it the structure of a DG-category. In many cases of interest, such as for reasonable schemes, the derived category is determined by a smaller gadget: a DG-algebra. The objects of study in this thesis are DG-categories and the closely related DG-algebras.

In Chapter 3, we prove a series of foundational results about finite-dimensional DG-algebras as introduced by Orlov. We begin with a derived version of the Nakayama lemma, and then prove the main result which states that a certain functor reflects perfection. It generalises the fact that the simples over a finite-dimensional algebra can detect if a module has finite projective dimension. There are two dual versions of this reflecting-perfection result; the covariant dual version is used to give an equivalent characterisation of being Gorenstein for finite-dimensional DG-algebras. The contravariant dual version is used to study Koszul duality for finite-dimensional DG-algebras.

The remainder of this thesis concerns reflexive DG-categories. These were introduced by Kuznetsov and Shinder to generalise a duality in algebra and geometry between the perfect complexes and the bounded derived categories. The two categories abstracting these are the perfect and cohomologically finite modules over a DG-category. In Chapter 4, we take a new perspective on reflexive DG-categories by proving that they are the reflexive objects in the closed symmetric monoidal category of DG-categories localised at Morita equivalences. This provides a moral justification for the existence of some common information between these two categories.

In Chapter 5, we apply this monoidal characterisation to strengthen the duality between the perfect and cohomologically finite modules of a reflexive DG-category. The first consequence is that these two categories have the same Hochschild cohomol-

ogy and derived Picard groups. We go on to show that adjunctions between reflexive DG-categories are uniquely determined by adjunctions on the cohomologically finite modules. These results combine to show that the isomorphisms on derived Picard groups are compatible with spherical twists.

Chapter 6 gives three new ways of producing reflexive DG-categories. First, we show that proper coconnective DG-algebras with semisimple zeroth cohomology are reflexive. This includes new examples from representation theory and topology. Secondly, we make a connection between reflexive DG-categories and approximable triangulated categories. As applications, we show that Azumaya algebras over proper schemes are reflexive and that proper schemes and proper connective DG-algebras are reflexive over any field. Finally, we show that one can glue reflexive DG-categories using semiorthogonal decompositions. Motivating examples in this direction come from graded gentle algebras and Fukaya categories.

Contents

Acknowledgements

I'm grateful to a large number of people who've helped me throughout this PhD. I would like to thank my supervisor, Greg Stevenson, for his patient explanations and interesting discussions on a wide range of topics. His ability to maintain such a deep and broad understanding of mathematics has always been inspiring. I would like to thank Matt Booth and Sebastian Opper for taking an interest in this work and for many useful discussions and comments. In particular, for their encouragement to improve the results of Section 6.2 to their current state. I had many helpful conversations with Timmy de Deyn and Kabeer Manali Rahul regarding the results of Section 6.3. I would like to thank Dmitri Orlov for correspondence regarding Example 3.1.6 and Alexander Kuznetsov for comments on a previous version of Section 4. I'm grateful to anonymous reviewers for comments on the subjects of Chapter 3 and Section 6.3.

I've learnt a lot from the algebraic geometry research group in Glasgow. In particular, from Timmy de Deyn, Marina Godinho, Sam Lewis, Dave Murphy, Franco Rota and Parth Shimpi. I'm also extremely grateful to Andy Baker for generously sharing so much of his time and passion for mathematics. I'd like to thank Gwyn Bellamy and Simone Castellan for many interesting discussions and new perspectives. Outside of Glasgow, I have benefited from many conversations with people in the algebra, geometry and homotopy theory research communities. I would especially like to mention the groups in Aarhus, Antwerp and Prague, Ben Briggs, Julius Frank, Martin Gallauer, Bernhard Keller, Janina Letz and Clover May.

I'm grateful to the broader mathematical community in Glasgow who have made it such an enjoyable and stimulating place to work. I'd also like to thank Michael Wemyss for useful mathematical conversations and for the administrative role he played as my official supervisor in Glasgow. I'm grateful to the Aarhus mathematics department for their warm welcome during my stay there.

I'd also like to thank my friends outside of mathematics, my family, and Izzy for their support and their interest in this work.

Author's declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

This thesis is on the subject of derived noncommutative algebraic geometry. The fundamental idea is that homological algebra can be used to extract information about a geometric or algebraic object. This information can take the form of properties such as smoothness or invariants like dimension. The homological information of an algebraic or geometric object can be nicely wrapped up into its derived category. In this way, the derived category is thought of as a geometric object of study in its own right. The question of when two rings have the same derived category was answered in [Ric89] by Rickard's derived Morita theory. The analogous question in geometry was studied by Bondal and Orlov [BO01]. Derived noncommutative algebraic geometry crosses this divide by noting that some schemes have the same derived category as noncommutative algebras (beginning with [Bei78]). In [BVdB03], it was shown that every reasonable scheme has the same derived category as a DG-algebra: a more highly structured algebraic object. For this reason, DG-algebras are often thought of as noncommutative schemes (whether or not they come from an actual scheme). Generically, these DG-algebras appearing from geometry are not just ordinary algebras and have some higher structure.

The derived category of an algebraic or geometric object can be viewed with different kinds of structures. Originally, it was considered as a triangulated category, but it was soon realised that this structure alone is insufficient for many purposes. For example, one cannot extract homological invariants like Hochschild (co)homology and K-theory from the triangulated structure alone. Furthermore, there is no coherent theory of triangulated categories allowing for constructions such as tensor products or functor categories. For this reason, the derived category is often given an enhancement. In this thesis, we will view the derived category as a DG-category, which is the usual enhancement taken in algebra and geometry.

The first part of this thesis studies a class of noncommutative schemes called finite-dimensional DG-algebras as introduced by Orlov in [Orl20]. The cohomology of a DG-algebra is an ordinary algebra which has no higher structure. This process of passing to cohomology is analogous to passing from a topological space to its homotopy groups. Alongside this, there is another discrete algebra associated to a DG-algebra: its underlying algebra. This is not a homotopy invariant process, and yet, Orlov showed that this approach can be used to extract information about the higher structure of finite-dimensional DG-algebras. In Chapter 3, we generalise some results about finite-dimensional algebras to finite-dimensional DG-algebras including the Nakayama lemma and the ability of the simples to detect finite projective dimension. Dual versions are used to detect being Gorenstein and study Koszul duality for finite-dimensional DG-algebras. Along the way, we give some equivalent characterisations of Gorenstein DG-algebras.

The remainder of this work concerns reflexive DG-categories. Reflexivity is about a duality in algebra and geometry between the perfect complexes and the bounded derived category. The relationship between these two derived categories has been of interest in algebraic geometry since their conception. It is well known (see [Buc21]) that the difference between these categories is a measure of singularity. Results indicating that these two triangulated categories determine each other and satisfy a kind of duality have been in the air for a long time. This is implicit in [Ric89], [BVdB03], explicit in [Bal11], [Che21], and provided motivation for the development of two kinds of completions of triangulated categories in [Kra20] and [Nee21b]. Reflexive DG-categories were defined in [KS25] to abstract this duality in an enhanced setting. That is to say, the DG-structure is taken into account, instead of just the triangulated structure.

For a reflexive DG-category, there is a considerable amount of common information between its perfect derived category and its category of cohomologically finite modules; these are the generalisations of the perfect and bounded derived categories in the geometric and algebraic examples. It was shown in [KS25] that there is a bijection between the semiorthogonal decompositions of these two categories and an isomorphism between their triangulated autoequivalence groups. This information lives at the triangulated level and in this thesis, we show that there is also a significant amount of enhanced information shared between these two categories.

The main result of Chapter 4 is a new perspective on reflexive DG-categories as the reflexive objects in a closed symmetric monoidal category. The reflexive objects are those which can be recovered from their duals in a precise way. This provides a moral justification for the existence of the common information between the perfects and the cohomologically finite modules.

This monoidal viewpoint also has a practical use; it can be used to prove that these two categories have even more in common. Specifically, in Chapter 5 we show that the Hochschild cohomology of a reflexive DG-category coincides with that of its cohomologically finite modules. We emphasise that it is essential to work in the enhanced setting to study this invariant. Another consequence of the monoidal characterisation is that the derived Picard groups of these two categories are isomorphic. This result can be thought of as an enhanced version of Kuznetsov and Shinder's result that they have the same triangulated autoequivalence groups. We also show that an (enhanced) adjunction between the perfect derived categories of reflexive DG-categories is uniquely determined by an adjunction between their cohomologically finite modules. Originally introduced in [ST01], spherical twists are certain elements of the derived Picard group of a DG-category. Using this adjunction result, we show that the isomorphisms on derived Picard groups are compatible with spherical twists.

Chapter 6 focuses on producing new examples of reflexive DG-categories. In [KS25], projective schemes and proper connective DG-algebras (this includes finite-dimensional algebras) were shown to be reflexive over perfect fields. In fact, a more general version of the former example appeared in [BNP17]. In [LU22], it was shown that some special types of Fukaya categories are reflexive. We expand this list by giving three new ways of producing reflexive DG-categories which work over any field.

We show that proper coconnective DG-algebras with semisimple zeroth cohomology are reflexive. Examples of this form come from simple-minded collections and cochain algebras on topological spaces. The key tools used here are the weight structures developed in [KN13]. As part of the proof, we study the Koszul duality between connective and coconnective DG-algebras.

The second method compares reflexivity to the theory of approximable triangulated categories. This is a new approach to studying the relationship between the perfect complexes and the bounded derived category at the triangulated level. It has had many applications, such as in the study of strong generators and uniqueness of enhancements (see [Nee21b] for a survey). Using Neeman's representability theorems, we give a strong generation condition for an approximable DG-category to be reflexive. As consequences, we show that proper schemes and proper connective DG-algebras are reflexive over any field and that Azumaya algebras over proper schemes are reflexive.

The third approach studies how reflexivity behaves under gluing. We show that a proper DG-category is reflexive if it admits a semiorthogonal decomposition into reflexive pieces. This allows us to build new reflexive DG-categories out of old ones. Motivating examples in this direction are graded gentle algebras and Fukaya categories.

Chapter 2

Preliminaries

We fix a field k throughout and let $\mathcal{C}(k)$ denote the category of chain complexes over k. We will use cohomological grading and let H^i and Z^i denote the i-th cohomology and i-th cycle group of a complex. For a complex M, we will write $H^*(M) = \bigoplus_{i\in\mathbb{Z}} H^i(M)$. Modules will refer to left modules. If $\mathcal{C}$ is a category, $\mathrm{Hom}_{\mathcal{C}}(-,-)$ denotes the set of morphisms in $\mathcal{C}$.

If R is a ring, we let $\mathrm{Mod}\, R$ denote the category of left R-modules and we write $\mathrm{Hom}_R(-,-) \coloneqq \mathrm{Hom}_{\mathrm{Mod}\, R}(-,-)$. We let $\mathcal{D}(R)$ denote the derived category of R and $\mathcal{D}^{\mathrm{perf}}(R)$ the perfect complexes i.e. the replete category of compact objects in $\mathcal{D}(R)$. If R is left Noetherian, $\mathrm{mod}\, R$ will denote the finitely generated left R-modules and $\mathcal{D}^b(\mathrm{mod}\, R)$, the bounded derived category of finitely generated modules. We will write $\mathcal{D}^b(k)$ for $\mathcal{D}^b(\mathrm{mod}\, k) = \mathcal{D}^{\mathrm{perf}}(k)$. If X is a scheme, $\mathrm{QCoh}(X)$ will denote the category of quasicoherent sheaves on X, $\mathcal{D}_{\mathrm{QCoh}}(X)$ will denote the unbounded derived category of all $\mathcal{O}_X$-modules with quasicoherent cohomology and $\mathcal{D}^{\mathrm{perf}}(X)$ will denote the perfect complexes on X. If X is Noetherian, $\mathrm{coh}(X)$ will denote the coherent sheaves on X and $\mathcal{D}^b_{\mathrm{coh}}(X)$ will denote the category of $\mathcal{O}_X$-modules with coherent cohomology.

We will often use the language of triangulated categories. A standard reference is [Nee01]. We will use Σ for the shift functor of a triangulated category. Recall a subcategory of a triangulated category is thick if it is closed under shifts, summands and cones. If $\mathcal{S} \subseteq \mathcal{T}$ is a collection of objects, we let $\mathrm{thick}(\mathcal{S})$ denote the smallest thick subcategory containing $\mathcal{S}$. If $\mathcal{T}$ is a triangulated category with coproducts, we will let $\mathcal{T}^c \subseteq \mathcal{T}$ denote the compact objects in $\mathcal{T}$ i.e. the objects t such that $\mathrm{Hom}_{\mathcal{T}}(t,-)$ commutes with coproducts.

Throughout, we will also use the language of model categories, enriched categories, monoidal categories and 2-categories. Standard references for these subjects are [Hov99], [Hir02], [Kel82] and [EGNO15].

§ 2.1 | DG-categories

The main objects of study in this thesis are DG-categories. We will use them to enhance derived categories and we consider them as noncommutative schemes. Most of the results stated in this Chapter can be found in [Kel06], [Toë11] or [Jas21].

Definition 2.1.1. A DG-category is a category enriched over the symmetric monoidal category $\mathcal{C}(k)$.

We will write $\mathcal{A}(a,b)$ for the chain complex of morphisms from a to b in a DG-category $\mathcal{A}$. By DG-functors, DG-natural transformations, DG-adjoints, etc. we mean the $\mathcal{C}(k)$-enriched notions. Note this implies DG-natural transformations are objectwise closed. Indeed, by definition, a $\mathcal{C}(k)$-natural transformation $\alpha\colon F \to G$ between DG-functors $F, G\colon \mathcal{A} \to \mathcal{B}$ consists of a chain map $k \to \mathcal{B}(F(a), G(a))$ for every $a \in \mathcal{A}$ satisfying naturality conditions. As it is a chain map, the image of $k \to \mathcal{B}(F(a), G(a))$ must be contained in $Z^0\mathcal{B}(F(a), G(a))$.

Definition 2.1.2. If $\mathcal{A}$ is a DG-category, the underlying category $Z^0\mathcal{A}$ of $\mathcal{A}$ is the category with the same objects as $\mathcal{A}$ but with morphism sets from a to b given by $Z^0\mathcal{A}(a,b)$. The homotopy category $H^0\mathcal{A}$ of $\mathcal{A}$ is the category with the same objects as $\mathcal{A}$ but with morphism sets from a to b given by $H^0\mathcal{A}(a,b)$.

A DG-category $\mathcal{A}$ is said to be an enhancement of any category equivalent to $H^0\mathcal{A}$. Note that a DG-functor F induces ordinary functors on the underlying and homotopy categories which we will denote $Z^0(F)$ and $H^0(F)$ respectively.

Definition 2.1.3. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a DG-functor.

1. We say F is quasi-fully faithful if the induced maps $F\colon \mathcal{A}(x,y) \to \mathcal{B}(Fx, Fy)$ are quasi-isomorphisms for all $x, y \in \mathcal{A}$.
2. We say F is quasi-essentially surjective if $H^0(F)\colon H^0\mathcal{A} \to H^0\mathcal{B}$ is essentially surjective.
3. We say F is a quasi-equivalence if it is quasi-fully faithful and quasi-essentially surjective.

Remark 2.1.4. We recall some basic ways of producing DG-categories from others.

1. The opposite $\mathcal{A}^{op}$ of a DG-category $\mathcal{A}$ has the same objects as $\mathcal{A}$ with $\mathcal{A}^{op}(a,b) = \mathcal{A}(b,a)$ and composition induced by the symmetry isomorphism from the tensor product of chain complexes (which includes a sign rule).
2. The tensor product $\mathcal{A} \otimes \mathcal{B}$ of DG-categories $\mathcal{A}, \mathcal{B}$ has objects given by pairs of objects in $\mathcal{A}$ and $\mathcal{B}$ and morphisms $\mathcal{A}(a,a') \otimes_k \mathcal{B}(b,b')$.
3. The enveloping DG-category of $\mathcal{A}$ is $\mathcal{A}^e := \mathcal{A}^{op} \otimes \mathcal{A}$.

Remark 2.1.5. Almost all triangulated categories in algebra and geometry admit DG-enhancements. For example, the homotopy category of any k-linear additive category is enhanced by the DG-category of chain complexes. The Drinfeld quotient gives an enhancement of the (bounded or unbounded) derived category of any Abelian category (provided its morphisms form a set).

Remark 2.1.6. Most other ways of enhancing k-linear triangulated categories are equivalent to the DG-category approach. It was shown in [Kel94] that stable categories of Frobenius exact categories admit DG-enhancements. Furthermore, any A_∞-category over k is equivalent to a DG-category in a suitable sense (see [LH02]). Pretriangulated DG-categories are equivalent to k-linear stable infinity categories. See [Coh13] or [Don24].

Example 2.1.7. A k-algebra A can be viewed as a DG-category with a single object whose endomorphism complex is A concentrated in degree zero.

§ 2.2 | Derived categories of DG-categories

We turn now to looking at modules over DG-categories. Recall that $\mathcal{C}(k)$ admits an internal hom $\underline{\mathrm{Hom}}_k(-,-) \in \mathcal{C}(k)$ which for $M, N \in \mathcal{C}(k)$ is given by

$$\underline{\mathrm{Hom}}_k(M,N)^n = \prod_{i\in\mathbb{N}} \mathrm{Hom}_{\mathrm{Mod}\,k}(M^i, N^{i+n})$$

and with differential $(f^i)_{i\in\mathbb{Z}} \mapsto (d_N^{n+i} f^i - (-1)^n f^{i+1} d_M^i)_{i\in\mathbb{Z}}$. The self-enrichment means there is a DG-category whose objects are chain complexes over k and morphism complexes given by $\underline{\mathrm{Hom}}_k(-,-)$. As an abuse of notation, we will write $\mathcal{C}(k)$ for this DG-category as well for the ordinary Abelian category. See Remark 2.9.2 for a related abuse of notation.

Definition 2.2.1. For a small DG-category $\mathcal{A}$, we let $\mathcal{C}(\mathcal{A})$ denote the DG-category of DG-functors $\mathcal{A} \to \mathcal{C}(k)$ with chain complex of morphisms given by the enriched end

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(M,N) := \int_{a\in\mathcal{A}} \underline{\mathrm{Hom}}_k(M(a), N(a)).$$

We call objects of $\mathcal{C}(\mathcal{A})$ $\mathcal{A}$-left modules, objects of $\mathcal{C}(\mathcal{A}^{op})$ right $\mathcal{A}$-modules and objects of $\mathcal{C}(\mathcal{A}^{op} \otimes \mathcal{B})$ $\mathcal{B}$–$\mathcal{A}$-bimodules.

Remark 2.2.2. For any $a \in \mathcal{A}$, there is the associated corepresentable left module $\mathcal{A}(a,-) \in \mathcal{C}(\mathcal{A})$ and the representable right module $\mathcal{A}(-,a) \in \mathcal{C}(\mathcal{A}^{op})$. The diagonal bimodule $\mathcal{A} \in \mathcal{C}(\mathcal{A}^e)$ is given by $(a,a') \mapsto \mathcal{A}(a,a')$.

Remark 2.2.3. The elements of $\underline{\mathrm{Hom}}_{\mathcal{A}}(M,N)$ can be explicitly described as graded natural transformations with differential inherited from $\underline{\mathrm{Hom}}_k(M,N)$. For example, see [Jas21] or [AL17].

Remark 2.2.4. The morphisms in $Z^0\mathcal{C}(\mathcal{A})$ can be identified with DG-natural transformations between DG-functors $\mathcal{A} \to \mathcal{C}(k)$. Also $Z^0\mathcal{C}(\mathcal{A})$ is a complete and cocomplete Abelian category with limits and colimits defined objectwise. For example, a sequence $M \to N \to L$ in $Z^0\mathcal{C}(\mathcal{A})$ is short exact if for every $a \in \mathcal{A}$ the induced sequences

$$0 \to M(a) \to N(a) \to L(a) \to 0$$

are short exact sequences in the Abelian category $\mathcal{C}(k)$.

Definition 2.2.5. For a small DG-category $\mathcal{A}$, we say a morphism $f\colon M \to N \in Z^0\mathcal{C}(\mathcal{A})$ is null-homotopic if there is a morphism $s \in \underline{\mathrm{Hom}}_{\mathcal{A}}(M,N)^{-1}$ such that $d(s) = f$. We say morphisms $f, g\colon M \to N \in Z^0\mathcal{C}(\mathcal{A})$ are homotopic if $f-g$ is null-homotopic.

Remark 2.2.6. Using the explicit description of $\underline{\mathrm{Hom}}_{\mathcal{A}}(M,N)^n$ as graded natural transformations of degree n, the condition in Definition 2.2.5 that $d(s) = f$ states that for every $a \in \mathcal{A}$ and $i \in \mathbb{Z}$, there is an equality of maps $M(a)^i \to N(a)^i$

$$f_a^i = d_{N(a)}^{i-1} s_a^i + s_a^{i+1} d_{M(a)}^i.$$

Homotopy gives an equivalence relation on morphisms in $Z^0\mathcal{C}(\mathcal{A})$ and by definition, the morphisms in $H^0\mathcal{C}(\mathcal{A})$ are DG-natural transformations $f\colon M \to N$ up to homotopy. We will call a morphism $f\colon M \to N \in Z^0\mathcal{C}(\mathcal{A})$ a homotopy equivalence if it is an isomorphism in $H^0\mathcal{C}(\mathcal{A})$.

Example 2.2.7. If A is a k-algebra viewed as a DG-category, then $Z^0\mathcal{C}(A)$ is the Abelian category of chain complexes over A and $H^0\mathcal{C}(A)$ is the triangulated homotopy category of A.

Remark 2.2.8. If $\mathcal{A}$ is a small DG-category, then $H^0\mathcal{C}(\mathcal{A})$ admits a triangulated structure generalising that of Example 2.2.7. The shift of a module is defined pointwise, and one can define the cone of a morphism $f\colon M \to N \in H^0\mathcal{C}(\mathcal{A})$ by $\mathrm{cone}(f)(a) := \mathrm{cone}(f_a)$. For more details see [Kel94] or [Jas21].

We now recall the construction of the derived category of a DG-category which mirrors that of the derived category of a ring. The original reference is [Kel94]. It will be useful to have two quasi-equivalent approaches; one uses model category theory, and one uses triangulated category theory.

Definition 2.2.9. If $\mathcal{A}$ is a small DG-category, we say a morphism $f\colon M \to N \in Z^0\mathcal{C}(\mathcal{A})$ is a quasi-isomorphism if $f_a\colon M(a) \to N(a)$ is a quasi-isomorphism in $\mathcal{C}(k)$ for all $a \in \mathcal{A}$. A DG-module $M \in \mathcal{C}(\mathcal{A})$ is acyclic if $M(a) \in \mathcal{C}(k)$ is acyclic for all $a \in \mathcal{A}$.

Remark 2.2.10. Any homotopy equivalence in $Z^0\mathcal{C}(\mathcal{A})$ is a quasi-isomorphism and so the notion of quasi-isomorphism is well defined for morphisms in $H^0\mathcal{C}(\mathcal{A})$. A morphism in $H^0\mathcal{C}(\mathcal{A})$ is a quasi-isomorphism if and only if its cone (in the sense of Remark 2.2.8) is acyclic. Therefore, there are equivalences

$$Z^0\mathcal{C}(\mathcal{A})[\text{q-iso}^{-1}] \simeq H^0\mathcal{C}(\mathcal{A})[\text{q-iso}^{-1}] \simeq H^0\mathcal{C}(\mathcal{A})/\operatorname{Acyc}$$

between the Gabriel-Zisman localisation at quasi-isomorphisms and the Verdier localisation at acyclic modules.

The derived category of $\mathcal{A}$ will be a DG-category enhancing the equivalent categories in Remark 2.2.10. To do this we will identify $H^0\mathcal{C}(\mathcal{A})[\text{q-iso}^{-1}]$ with a subcategory of $H^0\mathcal{C}(\mathcal{A})$. The first approach uses model category theory and the following notion.

Definition 2.2.11. If $\mathcal{A}$ is a small DG-category, then $P \in \mathcal{C}(\mathcal{A})$ is semiprojective if it is a summand of an object $F = \operatorname{colim} F_i$ in $Z^0\mathcal{C}(\mathcal{A})$ where $F_n \hookrightarrow F_{n+1}$ and F_{n+1}/F_n is a summand of a corepresentable.

The following result is well known and can be proved using a variant of Theorem 11.3.2 in [Hir02].

Proposition 2.2.12. *If $\mathcal{A}$ is a small DG-category, then $Z^0\mathcal{C}(\mathcal{A})$ admits a model category structure in which weak equivalences are quasi-isomorphisms, the cofibrant objects are the semiprojective modules and all objects are fibrant.*

The above model structure is known as the projective model structure on $Z^0\mathcal{C}(\mathcal{A})$. There is also an injective model structure where all objects are cofibrant and the fibrant objects are the so called semi-injectives. See [Kel06].

Remark 2.2.13. The homotopy equivalence relations on morphisms between cofibrant and fibrant objects in $Z^0\mathcal{C}(\mathcal{A})$ coming from the projective and injective model structures are the same as the notion of homotopy in the sense of Definition 2.2.5. It follows that the Gabriel-Zisman localisation of $H^0\mathcal{C}(\mathcal{A})$ at quasi-isomorphisms admits fully faithful left and right adjoints

$$H^0\mathcal{C}(\mathcal{A}) \underset{R}{\overset{Q}{\leftrightarrows}} H^0\mathcal{C}(\mathcal{A})[\text{q-iso}^{-1}]$$

given by the cofibrant Q and fibrant R replacement functors in the two model structures. The image of Q consists of the semiprojective modules, and the image of R consists of the semi-injective modules.

Remark 2.2.14. By Corollary 13.2.4 in [Rie14] (see also Theorem 4.20 in [Sch18]) and since k is a field, we can lift the fibrant and cofibrant replacement functors to DG-functors. So there are DG-functors $Q\colon \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ and $R\colon \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ such that the image of Q is consists of semiprojectives and the image of R consists of semi-injectives. Furthermore, there are DG-natural transformations $\varepsilon\colon Q \to 1_{\mathcal{C}(\mathcal{A})}$ and $\eta\colon 1_{\mathcal{C}(\mathcal{A})} \to R$ which are objectwise quasi-isomorphisms.

The semiprojectives and semi-injectives give two ways of enhancing the derived category of a DG-category. We now consider a more algebraic approach. Instead of semiprojective modules, we consider the following notion which only depends on the homotopy equivalence class of a module.

Definition 2.2.15. If $\mathcal{A}$ is a small DG-category, we say that $P \in \mathcal{C}(\mathcal{A})$ is K-projective if the complex $\underline{\operatorname{Hom}}_{\mathcal{A}}(P, M)$ is acyclic for every acyclic $M \in \mathcal{C}(\mathcal{A})$.

Remark 2.2.16. In Section 3.1 of [Kel94], it was shown that any semiprojective DG-module is K-projective and any K-projective module is homotopy equivalent to a semiprojective module. It follows that the DG-subcategory of $\mathcal{C}(\mathcal{A})$ consisting of semiprojectives is quasi-equivalent to the DG-subcategory consisting of K-projectives.

Remark 2.2.17. One can show using triangulated category theory (as in [Jas21]) that the Verdier localisation of $H^0\mathcal{C}(\mathcal{A})$ at the acyclic modules, Acyc, admits fully faithful left and right adjoints.

$$H^0\mathcal{C}(\mathcal{A}) \underset{i}{\overset{p}{\leftrightarrows}} H^0\mathcal{C}(\mathcal{A})/\operatorname{Acyc}$$

The image of p consists of the K-projective modules and the image of i consists of the similarly defined K-injective modules. Remark 2.2.16 implies that these must coincide with the adjoints of Remark 2.2.13.

Definition 2.2.18. For a DG-category $\mathcal{A}$, the derived category of $\mathcal{A}$ is the full DG-subcategory $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{C}(\mathcal{A})$ consisting of K-projective $\mathcal{A}$-modules.

Remark 2.2.19. For a small DG-category $\mathcal{A}$, objects $M, N \in \mathcal{C}(\mathcal{A})$ represent objects in $H^0\mathcal{C}(\mathcal{A})/\operatorname{Acyc}$ and the chain complex of morphisms between the corresponding objects in $\mathcal{D}(\mathcal{A})$ will be denoted

$$\underline{\mathbb{R}\operatorname{Hom}}_{\mathcal{A}}(M, N) := \underline{\operatorname{Hom}}_{\mathcal{A}}(Q(M), Q(N))$$

where Q denotes the cofibrant replacement functor in the projective model structure.

Remark 2.2.20. By Remark 2.2.16, the inclusion of semiprojectives into $\mathcal{D}(\mathcal{A})$ is a quasi-equivalence. Similarly, the DG-category of semi-injectives is quasi-equivalent to the DG-category of K-injectives. Furthermore, $\mathcal{D}(\mathcal{A})$ is quasi-equivalent to the DG-category of semi-injectives. Indeed, the DG-functors are provided by Remark 2.2.14. So for $M, N \in \mathcal{C}(\mathcal{A})$, we have quasi-isomorphisms

$$\begin{aligned}\underline{\mathbb{R}\operatorname{Hom}}_{\mathcal{A}}(M, N) &\simeq \underline{\operatorname{Hom}}_{\mathcal{A}}(Q(M), N) \simeq \underline{\operatorname{Hom}}_{\mathcal{A}}(M, R(N)) \\ &\simeq \underline{\operatorname{Hom}}_{\mathcal{A}}(p(M), N) \simeq \underline{\operatorname{Hom}}_{\mathcal{A}}(M, i(N))\end{aligned}$$

using the notation of Remarks 2.2.13 and 2.2.17.

Remark 2.2.21. The K-projectives form the smallest triangulated subcategory of $H^0\mathcal{C}(\mathcal{A})$ closed under coproducts containing the corepresentable $\mathcal{A}$-modules. For example, see [Jas21]. This shows that $H^0\mathcal{D}(\mathcal{A})$ is a triangulated subcategory of $H^0\mathcal{C}(\mathcal{A})$.

Remark 2.2.22. By Remark 2.2.17, the localisation functor restricts to a triangle equivalence

$$H^0\mathcal{D}(\mathcal{A}) \xrightarrow{\sim} H^0\mathcal{C}(\mathcal{A})/\operatorname{Acyc}$$

and provides the required enhancement.

Remark 2.2.23. Another approach to enhancing the derived category of a DG-category directly as a localisation was introduced by Drinfeld in [Dri04].

§ 2.3 | Pretriangulated DG-categories

A pretriangulated DG-category is one whose homotopy category admits a canonical triangulated structure. This notion goes back to [BK90] using twisted complexes, although we take a different approach.

Remark 2.3.1. By Remark 2.2.21, $H^0\mathcal{D}(\mathcal{A})$ admits a triangulated structure. It was shown in [Kel94] that $H^0\mathcal{D}(\mathcal{A})$ is compactly generated by the corepresentables. We let $\mathcal{D}^{\mathrm{perf}}(\mathcal{A})$ denote the full DG-subcategory of $\mathcal{D}(\mathcal{A})$ whose objects are the compact objects in the triangulated category $H^0\mathcal{D}(\mathcal{A})$. It follows that $H^0\mathcal{D}^{\mathrm{perf}}(\mathcal{A})$ is the thick subcategory of $H^0\mathcal{D}(\mathcal{A})$ generated by the corepresentables. We say a module $M \in \mathcal{C}(\mathcal{A})$ is perfect if it is quasi-isomorphic to an object in $\mathcal{D}^{\mathrm{perf}}(\mathcal{A})$.

Remark 2.3.2. It will be useful to note the following source of triangles in $H^0\mathcal{D}(\mathcal{A})$. If $\mathcal{A}$ is a DG-category, and $0 \to M \to N \to L \to 0$ is a short exact sequence in $Z^0\mathcal{C}(\mathcal{A})$ then its image in $H^0\mathcal{D}(\mathcal{A}) \simeq H^0\mathcal{C}(\mathcal{A})/\operatorname{Acyc}$ is a triangle. For example, a similar proof is given in [Sta25, Tag 09KZ].

Remark 2.3.3. The enriched Yoneda embedding gives a DG-fully faithful functor from a small DG-category $\mathcal{A}$ into its module category, $\mathcal{A}^{op} \hookrightarrow \mathcal{C}(\mathcal{A}); a \mapsto \mathcal{A}(a,-)$. Clearly it factors through $\mathcal{D}^{\mathrm{perf}}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A}) \subseteq \mathcal{C}(\mathcal{A})$. Note that as it is DG-fully faithful it is also quasi-fully faithful.

Definition 2.3.4. A small DG-category $\mathcal{A}$ is pretriangulated if the image of $H^0\mathcal{A}$ in $H^0\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op}$ is a triangulated subcategory. We say that $\mathcal{A}$ is pretriangulated and idempotent complete if $H^0\mathcal{A}$ is also idempotent complete.

Remark 2.3.5. Note that a small DG-category $\mathcal{A}$ is pretriangulated and idempotent complete if and only if $\mathcal{A}^{op} \to \mathcal{D}^{\mathrm{perf}}(\mathcal{A})$ is a quasi-equivalence. Indeed, the corepresentables form a set of compact generators of $H^0\mathcal{D}(\mathcal{A})$ and so the thick subcategory they generate coincides with the compact objects in $\mathcal{D}(\mathcal{A})$ by Theorem 2.1 in [Nee96].

Remark 2.3.6. If $\mathcal{A}$ is a small DG-category and $H^0S \subseteq H^0\mathcal{D}(\mathcal{A})$ is a small thick subcategory, then the DG-subcategory S is pretriangulated and idempotent complete. Indeed, this follows since $\underline{\mathrm{Hom}}_{\mathcal{A}}(\Sigma M, N) \simeq \Sigma^{-1}\underline{\mathrm{Hom}}_{\mathcal{A}}(M, N)$ and the Yoneda embedding $S^{op} \hookrightarrow \mathcal{D}^{\mathrm{perf}}(S)$ sends the cone of a morphism $f\colon M \to N$ to the cocone of $\underline{\mathrm{Hom}}_{\mathcal{A}}(f, -)$ in $\mathcal{D}(S)$.

The following pretriangulated idempotent complete DG-category will be a key feature of this this thesis.

Definition 2.3.7. For a small DG-category $\mathcal{A}$, we let $\mathcal{C}_{\mathrm{fd}}(\mathcal{A}) \subseteq \mathcal{C}(\mathcal{A})$ consist of those $M \in \mathcal{C}(\mathcal{A})$ such that for every $a \in \mathcal{A}$, $H^iM(a)$ is finite-dimensional for all $i \in \mathbb{Z}$ and $H^iM(a) = 0$ for $|i| >> 0$. We will refer to these as the cohomologically finite modules. Similarly, we let $\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$ consists of the cohomologically finite modules contained in $\mathcal{D}(\mathcal{A})$.

Remark 2.3.8. Note that the localisation functor restricts to an equivalence

$$H^0\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \xrightarrow{\sim} H^0\mathcal{C}_{\mathrm{fd}}(\mathcal{A})[\text{q-iso}^{-1}].$$

Example 2.3.9. If A is a finite-dimensional k-algebra viewed as a DG-category with one object, then $\mathcal{D}_{\mathrm{fd}}(A) \simeq \mathcal{D}^b(\operatorname{mod} A)$. If X is a projective scheme over a perfect field, then $\mathcal{D}_{\mathrm{fd}}(\mathcal{D}^{\mathrm{perf}}(X)) \simeq \mathcal{D}^b_{\mathrm{coh}}(X)$. See Proposition 6.1 of [KS25] or Theorem 3.0.2 of [BNP17] for a more general and relative version. We prove the result for proper schemes over any field using a different approach in Proposition 6.3.19. This shows that $\mathcal{D}_{\mathrm{fd}}$ can be thought of as a version of $\mathcal{D}^b_{\mathrm{coh}}$ for proper noncommutative schemes.

§ 2.4 | Tensor, Hom and Derived Functors

Many of the usual constructions of functors between derived categories of rings can be generalised to DG-categories. We recall some of these constructions here. Details can be found in [Gen15] or [AL17].

Remark 2.4.1. If $\mathcal{A}$ and $\mathcal{B}$ are DG-categories and $M \in \mathcal{C}(\mathcal{A}^{op} \otimes \mathcal{B})$ is a $\mathcal{B}$–$\mathcal{A}$-bimodule, then there is an adjoint pair of DG-functors

$$M \otimes_{\mathcal{A}} - \colon \mathcal{C}(\mathcal{A}) \rightleftarrows \mathcal{C}(\mathcal{B}) \colon \underline{\mathrm{Hom}}_{\mathcal{B}}(M, -)$$

defined as the enriched ends and coends

$$M \otimes_{\mathcal{A}} N = \int^{a \in \mathcal{A}} M(a, -) \otimes_k N(a) \quad \underline{\mathrm{Hom}}_{\mathcal{B}}(M, N) = \int_{b \in \mathcal{B}} \underline{\mathrm{Hom}}_k(M(-, b), N(b)).$$

Details can be found in [Gen15] or see [AL17] for an explicit construction.

Remark 2.4.2. The above DG-adjunction produces an ordinary adjunction on Z^0. If $Z^0\mathcal{C}(\mathcal{A})$ is considered with the projective model structure and $Z^0\mathcal{C}(\mathcal{B})$ is considered with the injective model structure, then $Z^0(M \otimes_{\mathcal{A}} -)$ preserves weak equivalences between cofibrant objects and $Z^0\,\underline{\mathrm{Hom}}_{\mathcal{B}}(M,-)$ preserves weak equivalences between fibrant objects. So the adjunction can be derived to an adjunction

$$M\otimes^{\mathbb{L}}_{\mathcal{A}}-\colon H^0\mathcal{C}(\mathcal{A})[\text{q-iso}^{-1}]\rightleftarrows H^0\mathcal{C}(\mathcal{B})[\text{q-iso}^{-1}]\colon \underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{B}}(M,-)$$

where the left derived functor is computed by taking semiprojective resolutions and the right derived functor is computed by taking semi-injective resolutions.

Remark 2.4.3. By Remark 2.2.14, the cofibrant and fibrant replacement functors can be lifted to DG-functors. This implies that there are DG-functors

$$M\otimes^{\mathbb{L}}_{\mathcal{A}}-\colon \mathcal{D}(\mathcal{A})\rightleftarrows \mathcal{D}(\mathcal{B})\colon \underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(M,-)$$

lifting the functors in Remark 2.4.2. Most of the time, it will be sufficient to work with the ordinary functors at the level of the homotopy category. A notable exception is given in Remark 2.4.6.

Remark 2.4.4. There is a more algebraic approach to the existence of the derived functors in Remark 2.4.2, which shows that they can be computed using K-projective and K-injective resolutions as opposed to the stronger notion of semiprojective and semi-injective resolutions. This is done, for example, in [Gen15].

Remark 2.4.5. Given a DG-functor $F\colon\mathcal{A}\to\mathcal{B}$, there are two bimodules ${}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}\in\mathcal{C}(\mathcal{A}^{op}\otimes\mathcal{B})$ and ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}\in\mathcal{C}(\mathcal{B}^{op}\otimes\mathcal{A})$ given by ${}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}(a,b)=\mathcal{B}(F(a),b)$ and ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}(b,a)=\mathcal{B}(b,F(a))$ for $a\in\mathcal{A}$, $b\in\mathcal{B}$. Then there are two associated adjoint pairs by Remark 2.4.1. By the Yoneda and coYoneda lemmas, there are isomorphisms of DG-functors

$$\mathrm{Res}(F)\simeq\underline{\mathrm{Hom}}_{\mathcal{B}}({}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}},-)\simeq{}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}\otimes_{\mathcal{B}}-\colon\mathcal{C}(\mathcal{B})\to\mathcal{C}(\mathcal{A})$$

where $\mathrm{Res}(F)$ sends $M\in\mathcal{C}(\mathcal{B})$ to the DG-functor $MF\in\mathcal{C}(\mathcal{A})$. Therefore, there is an adjoint triple

$$\begin{array}{c}\xrightarrow{\;\mathcal{B}\otimes^{\mathbb{L}}_{\mathcal{A}}-\;}\\ H^0\mathcal{D}(\mathcal{A})\xleftarrow{\;\mathrm{Res}(F)\;}H^0\mathcal{D}(\mathcal{B})\\ \xrightarrow[\;\underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(\mathcal{B},-)\;]{}\end{array}$$

where, as an abuse of notation, we have written $H^0\mathcal{D}(\mathcal{A})$ instead of the equivalent category $H^0\mathcal{C}(\mathcal{A})[\text{q-iso}^{-1}]$. Note that $\mathrm{Res}(F)$ preserves quasi-isomorphisms and so it descends to the homotopy categories without needing to be derived.

Remark 2.4.6. Given a DG-functor $F\colon\mathcal{A}\to\mathcal{B}$, the derived restriction functor $\mathrm{Res}(F)\colon H^0\mathcal{D}(\mathcal{B})\to H^0\mathcal{D}(\mathcal{A})$, constructed in Remark 2.4.5, is enhanced by the following composition of DG-functors

$$\mathcal{D}(\mathcal{B})\hookrightarrow\mathcal{C}(\mathcal{B})\xrightarrow{\mathrm{Res}(F)}\mathcal{C}(\mathcal{A})\xrightarrow{Q^{\mathcal{A}}}\mathcal{D}(\mathcal{A})$$

where $Q^{\mathcal{A}}$ denotes the DG-enhanced cofibrant replacement functor with respect to the projective model, structure as mentioned in Remark 2.2.14.

Remark 2.4.7. Given a DG-functor $F\colon \mathcal{A} \to \mathcal{B}$, the coYoneda lemma implies that $\mathcal{B} \otimes_{\mathcal{A}} -\colon \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ sends $\mathcal{A}(a,-) \mapsto \mathcal{B}(F(a),-)$. It follows that $\mathcal{B} \otimes_{\mathcal{A}} -$ preserves K-projectives and that the following diagram commutes up to isomorphism.

$$\begin{array}{ccc} H^0\mathcal{A} & \hookrightarrow & H^0\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op} \\ \downarrow{\scriptstyle H^0(F)} & & \downarrow{\scriptstyle \mathcal{B}\otimes_{\mathcal{A}}-} \\ H^0\mathcal{B} & \hookrightarrow & H^0\mathcal{D}^{\mathrm{perf}}(\mathcal{B})^{op} \end{array}$$

Remark 2.4.8. All of the induced functors mentioned in this section are triangulated. Specifically, the functors in Remark 2.4.2 are triangulated and any DG-functor between pretriangulated DG-categories induces a triangulated functor on H^0. Indeed, one can verify directly that if F is a DG-functor, then $\mathrm{Res}(F)$ is a triangulated functor. Then the above statements can be deduced from this fact.

Remark 2.4.9. A DG-functor $F\colon \mathcal{A} \to \mathcal{B}$ between pretriangulated DG-categories is a quasi-equivalence if and only if $H^0(F)$ is an equivalence of categories. Indeed, this follows from the existence of the triangulated shift functors on $H^0\mathcal{A}$ and $H^0\mathcal{B}$.

§ 2.5 | Dualities

In this section, we recall some dualities on the derived category of a DG-category.

Remark 2.5.1. Postcomposing by k-linear duality $(-)^{\vee} := \underline{\mathrm{Hom}}_k(-,k)\colon \mathcal{C}(k) \to \mathcal{C}(k)^{op}$ produces a DG-functor

$$(-)^{\vee}\colon \mathcal{C}(\mathcal{A})^{op} \to \mathcal{C}(\mathcal{A}^{op})$$

which preserves quasi-isomorphisms and so induces a functor $H^0\mathcal{D}(\mathcal{A})^{op} \to H^0\mathcal{D}(\mathcal{A}^{op})$ which can also be lifted to a DG-functor by Remark 2.2.14

Remark 2.5.2. Note that $(-)^{\vee}$ restricts to a quasi-equivalence $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{op} \simeq \mathcal{D}_{\mathrm{fd}}(\mathcal{A}^{op})$.

Remark 2.5.3. The derived tensor-hom adjunction reduces to the following quasi-isomorphism for any $M \in \mathcal{D}(\mathcal{A})$ and $N \in \mathcal{D}(\mathcal{A}^{op})$.

$$\underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(M, N^{\vee}) \simeq (N \otimes^{\mathbb{L}}_{\mathcal{A}} M)^{\vee}$$

Remark 2.5.4. For any $M \in \mathcal{C}(\mathcal{A} \otimes \mathcal{B})$, there is a contravariant DG-functor

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(-, M)\colon \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})^{op}$$

defined as

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(N, M) := \int_{a \in \mathcal{A}} \underline{\mathrm{Hom}}_k(N(a), M(a,-))$$

which sends $\mathcal{A}(a,-) \mapsto M(a,-)$. It can be derived to produce a functor

$$\underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(-,M)\colon H^0\mathcal{D}(\mathcal{A}) \to H^0\mathcal{D}(\mathcal{B})^{op}.$$

Remark 2.5.5. Consider the special case of Remark 2.5.4 with the diagonal bimodule $\mathcal{A} \in \mathcal{C}(\mathcal{A}^{op} \otimes \mathcal{A})$. This produces a DG-functor

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(-,\mathcal{A})\colon \mathcal{C}(\mathcal{A}^{op}) \to \mathcal{C}(\mathcal{A})^{op}$$

which restricts to a quasi-equivalence $\mathcal{D}^{\mathrm{perf}}(\mathcal{A}^{op}) \xrightarrow{\sim} \mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op}$. For example, see Proposition 3.3.3 in [Gen15].

§ 2.6 | DG-algebras

In many cases of interest, it is sufficient to work with DG-categories with a single object.

Definition 2.6.1. A DG-algebra A is a DG-category with one object or equivalently a chain complex with a multiplication satisfying $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for all homogenous $a, b \in A$.

Note that a module over a DG-algebra is a chain complex with an action compatible with the differential.

Example 2.6.2. DG-algebras are ubiquitous in algebra, geometry and topology.

1. Multiplicative cohomology theories such as singular cohomology of topological spaces or Hochschild cohomology of algebras are usually constructed as the cohomology of a DG-algebra.

2. In [BVdB03], it was shown that the derived category of a quasicompact, quasiseparated scheme is equivalent to the derived category of a DG-algebra. For this reason, DG-algebras are often thought of as noncommutative schemes.

We use the following grading convention.

Definition 2.6.3. A DG-algebra is connective if $H^i(A) = 0$ for $i > 0$ and coconnective if $H^i(A) = 0$ for $i < 0$.

Remark 2.6.4. The derived category of a connective DG-algebra A admits a t-structure whose heart is the category of modules over the ordinary algebra $H^0(A)$.

Remark 2.6.5. If $\mathcal{A}$ is a DG-category and $M \in \mathcal{D}(\mathcal{A})$, then set $E := \underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(M,M)$. Then $M \in \mathcal{C}(\mathcal{A} \otimes E)$ and there is a DG-functor

$$\underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(M,-)\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(E^{op})$$

which restricts to an equivalence $\mathrm{thick}(M) \simeq H^0\mathcal{D}^{\mathrm{perf}}(E^{op})$. Similarly, there is DG-functor

$$\underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(-, M) \colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(E)^{op}$$

which restricts to an equivalence $\mathrm{thick}(M) \simeq H^0\mathcal{D}^{\mathrm{perf}}(E)^{op}$

§ 2.7 | Homotopy Theory of DG-categories

We are interested in studying DG-categories up to homotopy. In this section, we detail how different notions of weak equivalence between DG-categories can be controlled using model structures. Let DGcat denote the category of small DG-categories with DG-functors between them.

Remark 2.7.1. In [Tab05], it is shown that DGcat admits a model category structure whose weak equivalences are quasi-equivalences, and every object is fibrant. We let Hqe denote the associated homotopy category.

The morphism sets in Hqe admit a description in terms of bimodules.

Definition 2.7.2. For DG-categories $\mathcal{A}, \mathcal{B}$, an $\mathcal{A}$–$\mathcal{B}$-bimodule $M \in \mathcal{C}(\mathcal{B}^{op} \otimes \mathcal{A})$ is right quasirepresentable if for every $a \in \mathcal{A}$, the right $\mathcal{B}$-module $M(-, a) \in \mathcal{C}(\mathcal{B}^{op})$ is quasi-isomorphic to a representable module $\mathcal{B}(-, b)$ for some $b \in \mathcal{B}$. We let $\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{rqr} \subseteq \mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})$ denote the DG-subcategory of K-projective right quasirepresentable bimodules.

A right quasirepresentable $\mathcal{A}$–$\mathcal{B}$ bimodule is also called a quasifunctor from $\mathcal{A}$ to $\mathcal{B}$.

Theorem 2.7.3 (Corollary 4.8, [Toë07])**.** *For* $\mathcal{A}, \mathcal{B} \in \mathrm{Hqe}$*, there is a bijection between* $\mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B})$ *and isomorphism classes of objects in* $H^0\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{rqr}$*.*

Another proof of Theorem 2.7.3 can be found in [CS15].

Remark 2.7.4. Given a quasifunctor M from $\mathcal{A}$ to $\mathcal{B}$, by Remark 2.4.2, there is a functor

$$- \otimes^{\mathbb{L}}_{\mathcal{A}} M \colon H^0\mathcal{D}(\mathcal{A}^{op}) \to H^0\mathcal{D}(\mathcal{B}^{op})$$

which sends representables to representables. Hence, by the Yoneda embedding, M induces a functor $H^0(\mathcal{A}) \to H^0(\mathcal{B})$.

Remark 2.7.5. Given a DG-functor $F \colon \mathcal{A} \to \mathcal{B}$ the functor DGcat $\to$ Hqe sends F to the quasifunctor $M_F \in \mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{rqr}$ given by $M_F(b, a) = \mathcal{B}(b, F(a))$. Indeed, this can be seen from the proof of Theorem 1.1 in [CS15].

Remark 2.7.6. By Theorem 2.7.3, any quasifunctor $M \in \mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{rqr}$ corresponds to a morphism in Hqe and so also to a DG-functor $\tilde{\mathcal{A}} \to \mathcal{B}$ where $\tilde{\mathcal{A}}$ denotes the cofibrant replacement of $\mathcal{A}$. We can extend Definition 2.1.3 by asking for this DG-functor to satisfy the corresponding conditions. We note that by Remark 2.4.9, these properties can usually be read off from the induced functor on H^0.

We will also be interested in the following weaker notion of equivalence.

Definition 2.7.7. A DG-functor $F\colon \mathcal{A} \to \mathcal{B}$ is a Morita equivalence if the functor $H^0\mathcal{D}(\mathcal{B}) \to H^0\mathcal{D}(\mathcal{A})$ described in Remark 2.4.5 is an equivalence.

Remark 2.7.8. Note that $F\colon \mathcal{A} \to \mathcal{B}$ is a Morita equivalence if and only if any of the three functors in Remark 2.4.5 are equivalences.

Remark 2.7.9. Any quasi-equivalence is a Morita equivalence. This follows from Remark 2.4.7 since the corepresentables form a set of compact generators.

Remark 2.7.10. The Yoneda embedding $\mathcal{A}^{op} \hookrightarrow \mathcal{D}^{\mathrm{perf}}(\mathcal{A})$ is a Morita equivalence. Indeed, this also follows from Remark 2.4.7.

Remark 2.7.11. In [Tab07], another model structure on DGcat is defined whose weak equivalences are Morita equivalences. We will denote the associated homotopy category Hmo. This model structure is a left Bousfield localisation of the model structure in Remark 2.7.1. The fibrant objects are pretriangulated idempotent complete DG-categories. It follows that $\mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{D}^{\mathrm{perf}}(\mathcal{B}))$.

Similarly, there is a description of the morphism sets as bimodules.

Definition 2.7.12. For DG-categories $\mathcal{A}, \mathcal{B}$, an $\mathcal{A}$–$\mathcal{B}$ bimodule $M \in \mathcal{C}(\mathcal{B}^{op} \otimes \mathcal{A})$ is right perfect if for every $a \in \mathcal{A}$, $M(-,a) \in \mathcal{C}(\mathcal{B}^{op})$ is a perfect $\mathcal{B}^{op}$-module. We let $\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{\mathcal{B}-\mathrm{perf}} \subseteq \mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})$ denote the DG-subcategory of K-projective right perfect bimodules.

The following result follows from Theorem 2.7.3. See Proposition 4.4.8 of [Toë11].

Theorem 2.7.13. *For $\mathcal{A}, \mathcal{B} \in \mathrm{Hmo}$, there is a bijection between $\mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B})$ and isomorphism classes of objects in $H^0\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{\mathcal{B}-\mathrm{perf}}$.*

Remark 2.7.14. We note that $\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{\mathcal{B}-\mathrm{perf}}$ is a pretriangulated DG-category. Indeed, this follows from Remark 2.3.6 as $H^0\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{\mathcal{B}-\mathrm{perf}}$ is a small thick subcategory of $H^0\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})$.

§ 2.8 | Smooth and Proper DG-categories

In this section, we recall some notions which generalise smoothness and properness to DG-categories. See [Lun10] and [Orl16].

Definition 2.8.1. Let $\mathcal{A}$ be a small DG-category.

1. We say $\mathcal{A}$ is proper if $\mathcal{A}$ is a cohomologically finite $\mathcal{A}$-bimodule.
2. We say $\mathcal{A}$ is smooth if $\mathcal{A}$ is a perfect $\mathcal{A}$-bimodule.

Note that the former is equivalent to $H^*\mathcal{A}(a,b)$ being finite-dimensional for all $a, b \in \mathcal{A}$ and to the containment $\mathcal{D}^{\mathrm{perf}}(\mathcal{A}) \subseteq \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$.

Remark 2.8.2. In [Orl16], Orlov showed any separated scheme of finite type over a field is proper if and only if $\mathcal{D}^{\text{perf}}(X)$ is proper. In [LS16], it was shown that any separated quasiprojective scheme over k is smooth if and only if $\mathcal{D}^{\text{perf}}(X)$ is smooth.

Remark 2.8.3. If A is an algebra viewed as a DG-category with a single object, then it is proper if and only if it is a finite-dimensional algebra. It is smooth if and only if A is homologically smooth. Recall that over a perfect field, homological smoothness of a finite-dimensional algebra is equivalent to having finite global dimension.

Remark 2.8.4. If $\mathcal{A}$ is smooth, then $\mathcal{D}_{\text{fd}}(\mathcal{A}) \subseteq \mathcal{D}^{\text{perf}}(\mathcal{A})$ (see for example, Lemma 3.8 of [KS25]). The converse is not true, e.g. the power series ring is not smooth as shown below.

Proposition 2.8.5. *Let $k = \mathbb{Q}$, then $k[[x]]$ is not smooth as a DG-category.*

Proof. Suppose $R \coloneqq k[[x]]$ is perfect as an $A \coloneqq k[[x]]^e$-module and let I be the kernel of the multiplication map $A \to R$. Then there is a triangle in $H^0\mathcal{D}(A)$

$$I \to A \to R \to^{+}$$

Since A and R are perfect as A-modules, I is too as the perfects form a triangulated category. In particular I is a finitely generated A-module. It follows that $I \otimes_A R$ is a finitely generated R-module. But there are isomorphisms

$$I \otimes_A R \simeq I \otimes_A A/I \simeq I/I^2 \simeq \Omega_{R/k}$$

where $\Omega_{R/k}$ are the Kahler differentials of R over k. For example by [Sta25, Tag 00RW]. Hence $\Omega_{R/k}$ is finitely generated as an R module. By Exercise 25.4 in [Mat86], $\Omega_{k((x))/k} = \Omega_{R/k} \otimes_R k((x))$. So we see that $\Omega_{k((x))/k}$ is finite dimensional over $k((x))$. By Theorem 26.5 in [Mat86], the dimension of $\Omega_{k((x))/k}$ over $k((x))$ is given by the transcendence degree of $k((x))$ over k. Since k is countable and $k((x))$ is uncountable, $k((x))$ cannot have finite transcendence degree over k and so we have a contradiction. □

Remark 2.8.6. Any smooth DG-category is Morita equivalent to a DG-algebra. See, for example, Lemma 2.5 in [Efi10]. Some authors, such as [TV07], require by definition that proper DG-categories are Morita equivalent to DG-algebras.

§ 2.9 | Notation

We will make the following abuses of notation

Remark 2.9.1. Recall that for a DG-category $\mathcal{A}$, there is a triangle equivalence $H^0\mathcal{D}(\mathcal{A}) \xrightarrow{\sim} H^0\mathcal{C}(\mathcal{A})/\operatorname{Acyc}$. As is consistent with how one usually thinks of the derived category, we will make this an identification. For example, when we write $M \in \mathcal{D}(\mathcal{A})$ we usually mean that M is an arbitrary $\mathcal{A}$-module and non-necessarily K-projective

Remark 2.9.2. We will often omit from the notation any additional structure a category is being considered with. For example, we have already used $\mathcal{C}(k)$ to denote an ordinary category, a symmetric monoidal category and a DG-category. Similarly for a DG-category $\mathcal{A}$, we will often use the notation $\mathcal{D}(\mathcal{A})$ for both the DG-category and the triangulated category $H^0\mathcal{D}(\mathcal{A})$. It should be clear from context what structure we are considering the derived category with.

Remark 2.9.3. By Remark 2.4.9, a DG-functor between pretriangulated DG-categories is a quasi-equivalence (or quasi-fully faithful) if and only if the ordinary induced functor on H^0 is an equivalence (or fully faithful). In line with Remark 2.9.2, we often say equivalence when we mean quasi-equivalence.

Chapter 3

Reflecting and Detecting for Finite-dimensional DG-algebras

Finite-dimensional DG-algebras are a class of proper noncommutative schemes introduced by Orlov in [Orl20]. A powerful feature of a finite-dimensional DG-algebra is that its higher structure can be controlled by its underlying algebra. In this spirit, we generalise several results about finite-dimensional algebras to finite-dimensional DG-algebras. We begin with a version of the Nakayama lemma and then prove a reflecting-perfection theorem which generalises the fact that the simples of a finite-dimensional algebra can detect modules of finite projective dimension. A dual version of this latter result allows one to detect when a finite-dimensional DG-algebra is Gorenstein. Along the way, we prove some equivalent characterisations of Gorenstein DG-algebras. Finally, we use another dual version of the reflecting-perfection theorem to study Koszul duality for finite-dimensional DG-algebras.

The results of this chapter first appeared in [Goo24b]. We have improved on the published version by removing the separability assumptions.

§ 3.1 | Finite-dimensional DG-algebras

In this section, we recall some preliminaries on finite-dimensional DG-algebras. Most of the results are contained in [Orl20].

Definition 3.1.1. A DG-algebra A is finite-dimensional if A^i is finite-dimensional for all i and $A^i = 0$ for $|i| >> 0$.

Remark 3.1.2. By Corollary 2.6 in [Orl20], the class of DG-algebras which are quasi-isomorphic to finite-dimensional DG-algebras is closed under Morita equivalence.

Example 3.1.3. There are many naturally occurring examples of finite-dimensional DG-algebras.

1. If A is a proper connective DG-algebra, Raedschelders and Stevenson showed that A is quasi-isomorphic to a finite-dimensional DG-algebra. See Corollary 3.12 of

[RS22] or Appendix A of [Goo24a] for another proof by the same authors with no assumptions on the base field.

2. Recall a DG-algebra is formal if it is quasi-isomorphic to its cohomology viewed as a DG-algebra with no differential. Any formal proper DG-algebra clearly admits a finite-dimensional model. This includes examples such as the graded gentle algebras appearing in the study of Fukaya categories.

3. The DG-algebras associated to various smooth projective surfaces are computed and shown to be finite-dimensional in [Bod17].

Remark 3.1.4. Not every proper DG-algebra admits a finite-dimensional model. It was shown in Corollary 2.21 of [Orl20], that the Grothendieck group of a smooth finite-dimensional DG-algebra is free of finite rank. For example, this means the proper DG-algebra associated to an elliptic curve does not admit a finite-dimensional model.

Definition 3.1.5. A module M over a finite-dimensional DG-algebra A is strictly finite-dimensional if M^i is finite-dimensional for all i and $M^i = 0$ for $|i| >> 0$. Let $\mathcal{D}_{\mathrm{sf}}(A)$ denote the smallest thick subcategory of $\mathcal{D}_{\mathrm{fd}}(A)$ containing all strictly finite-dimensional modules.

Example 3.1.6. As mentioned in Remark 1.10 of [Orl23], there are cohomologically finite modules over finite-dimensional DG-algebras which are not quasi-isomorphic to a strictly finite-dimensional module. Let $A = k[x, y]/(x^6, y^3)$ with $|x| = 0, |y| = 1$ and zero differential. Theorem 5.4 of [Efi20] gives an example of a $k[x]/x^6$–$k[y]/y^3$-bimodule V with finite-dimensional cohomology. The theorem in loc. cit. implies that the associated gluing is a DG-algebra which does not satisfy Theorem 3.1.15. It follows that the gluing is not quasi-isomorphic to a finite-dimensional DG-algebra. Hence, V is not quasi-isomorphic to a strictly finite-dimensional A-module.

Remark 3.1.7. The notion of a strictly finite-dimensional module comes with a series of problems; it is not invariant under quasi-isomorphism, and it is not clear that cones of morphisms between strictly finite-dimensional modules admit strictly finite-dimensional models (this is the reason we take $\mathcal{D}_{\mathrm{sf}}(A)$ to be the thick closure of the strictly finite-dimensional modules). It is also not clear that Morita equivalences preserve $\mathcal{D}_{\mathrm{sf}}(A)$. The more natural category to study is $\mathcal{D}_{\mathrm{fd}}(A)$. Example 3.1.6 shows that there is a finite-dimensional DG-algebra and an object in $\mathcal{D}_{\mathrm{fd}}(A)$ which does not admit a strict finite-dimensional model. However it is not clear that every object of $\mathcal{D}_{\mathrm{sf}}(A)$ admits a strict finite-dimensional model and so the question of when the inclusion $\mathcal{D}_{\mathrm{sf}}(A) \subseteq \mathcal{D}_{\mathrm{fd}}(A)$ is an equality remains open in general.

Remark 3.1.8. Despite the issues raised in Remark 3.1.7, $\mathcal{D}_{\mathrm{sf}}(A)$ is still worthy of study. In many cases of interest, we do have equality $\mathcal{D}_{\mathrm{sf}}(A) = \mathcal{D}_{\mathrm{fd}}(A)$. Proposition 3.9 of [Orl23] shows that $\mathcal{D}_{\mathrm{sf}}(A) = \mathcal{D}_{\mathrm{fd}}(A)$ under a connectivity assumption. By Proposition 4.6 in [KN13], this equality holds under certain coconnectivity assumptions. This is applied in Section 6.2. In Theorem 9.4.4 of [BGO25], we show that if A is a proper graded gentle algebra, then $\mathcal{D}_{\text{sf}}(A) = \mathcal{D}_{\text{fd}}(A)$.

Another reason to study $\mathcal{D}_{\text{sf}}(A)$ is that it can detect properties of finite-dimensional DG-algebras. The following is Corollary 3.12 in [Orl23].

Proposition 3.1.9. *Suppose A is a finite-dimensional DG-algebra over a perfect field. Then A is smooth if and only if $\mathcal{D}^{\text{perf}}(A) = \mathcal{D}_{\text{sf}}(A)$ and in this case $\mathcal{D}_{\text{sf}}(A) = \mathcal{D}_{\text{fd}}(A)$.*

The category $\mathcal{D}_{\text{sf}}(A)$ admits a generator which generalises the simple modules over a finite-dimensional algebra. The following definition is made in [Orl20].

Definition 3.1.10. If A is a finite-dimensional DG-algebra, let $J = J(A)$ denote the Jacobson radical of the underlying (ungraded) algebra of A. The external and internal DG-ideals associated to J are the A-bimodules

$$J_+ \coloneqq J + d(J) \qquad \text{and} \qquad J_- \coloneqq \{r \in J \mid d(r) \in J\}.$$

Remark 3.1.11. Since $J_- \subseteq J \subseteq J_+$, we deduce that J_- is a nilpotent DG-ideal and that the underlying algebra of the quotient A/J_+ is semisimple. Furthermore, the inclusion $J_- \hookrightarrow J_+$ is a quasi-isomorphism.

Theorem 3.1.12 (Proposition 2.16, [Orl20])**.** *For a finite-dimensional DG-algebra A, there is a quasi-equivalence*

$$\mathcal{D}(A/J_-) \simeq \mathcal{D}(A/J_+) \simeq \mathcal{D}(D_1 \times \cdots \times D_n)$$

where $D_1, \ldots, D_n$ are finite-dimensional division algebras.

Remark 3.1.13. The D_i are exactly those appearing in the Artin-Wedderburn Theorem applied to the underlying algebra of A/J_+.

The next result follows immediately from [Orl20]. See also Lemma 5.9 of [RS22].

Theorem 3.1.14. *(The Radical Filtration) If A is a finite-dimensional DG-algebra, then*

$$\mathcal{D}_{\text{sf}}(A) = \text{thick}(A/J_-).$$

Proof. Let M be a strictly finite-dimensional A-module. Then there is a filtration of M by strictly finite-dimensional A-modules

$$0 = J_-^N M \subseteq J_-^{N-1} M \subseteq \cdots \subseteq J_- M \subseteq M$$

and the factors are $MJ_-^i/MJ_-^{i+1} \in \mathcal{D}(A/J_-) \simeq \mathcal{D}(A/J_+)$. So by Theorem 3.1.12, MJ_-^i/MJ_-^{i+1} is a finite sum of shifts of summands of A/J_-. The filtration then produces triangles in $\mathcal{D}(A)$ which exhibit $M \in \text{thick}(A/J_-)$. Since $\mathcal{D}_{\text{sf}}(A)$ is the smallest thick subcategory containing all strictly finite-dimensional modules, it follows that $\mathcal{D}_{\text{sf}}(A) \subseteq \text{thick}(A/J_-)$. The converse follows since A/J_- is strictly finite-dimensional. □

The following theorem shows that finite-dimensional DG-algebras admit categorical resolutions. See loc. cit. for the notions of admissible subcategories and exceptional collection.

Theorem 3.1.15 (Corollary 2.20, [Orl20]). *A proper DG-algebra A is quasi-isomorphic to a finite-dimensional DG-algebra if and only if there is a proper DG-algebra E and a fully faithful functor*

$$\mathcal{D}^{\mathrm{perf}}(A) \hookrightarrow \mathcal{D}^{\mathrm{perf}}(E)$$

such that $\mathcal{D}^{\mathrm{perf}}(E)$ has a full exceptional collection. If A/J is k-separable, then E is smooth, and the embedding is admissible if and only if A is smooth.

We note a consequence for Hochschild homology of finite-dimensional DG-algebras. Recall the Hochschild homology of a DG-algebra A is $HH_*(A) \coloneqq H^{-*}(A \otimes^{\mathbb{L}}_{A^e} A)$.

Corollary 3.1.16. *Suppose A is a smooth finite-dimensional DG-algebra such that A/J is k-separable. Then the Hochschild homology $HH_*(A)$ of A is concentrated in degree zero.*

Proof. In this case, $\mathcal{D}^{\mathrm{perf}}(A)$ is admissible in $\mathcal{D}^{\mathrm{perf}}(E)$ where E is as in Theorem 3.1.15. Hochschild homology is an additive invariant (see, for example, Theorem 5.2 in [Kel06]) and so $HH_*(A)$ is a summand of $HH_*(E)$. By Theorem 2.19 of [Orl20], $\mathcal{D}^{\mathrm{perf}}(E)$ admits a semiorthogonal decomposition whose components are $\mathcal{D}^{\mathrm{perf}}(S_i)$ where S_i are separable k-algebras. So again by additivity, $HH_*(E) \simeq \bigoplus_i HH_*(S_i)$. Since each S_i is k-separable, S_i^e has global dimension 0. Hence, $HH_*(S_i)$ is concentrated in degree zero. □

Remark 3.1.17. Using the HKR Theorem of [Mar01], [Căl05], [Mar08], one can produce many examples of smooth and proper schemes over $\mathbb{C}$ whose Hochschild homology is not concentrated in degree zero. Examples include smooth projective curves of genus $g > 0$, K_3 surfaces, many Fano 3-folds and so on. Hence, none of these geometric objects can be derived equivalent to a finite-dimensional DG-algebra. This provides plenty of proper DG-algebras which do not admit finite-dimensional models.

Corollary 3.1.16 was already proved for smooth and proper connective DG-algebras in [Shk07].

§ 3.2 | Reflecting Perfection

We generalise some facts about finite-dimensional algebras to finite-dimensional DG-algebras. The following is a derived version of the Nakayama lemma. Note that by Remark 3.1.11, J_+ can be replaced by J_- in the statement of any of results in this Section.

Proposition 3.2.1. *Let A be a finite-dimensional DG-algebra and suppose $M \in \mathcal{D}(A)$ is such that $A/J_+ \otimes^{\mathbb{L}}_A M \simeq 0$. Then $M \simeq 0$.*

Proof. Let
$$\mathcal{C} = \{X \in \mathcal{D}(A^{op}) \mid X \otimes^{\mathbb{L}}_A M \simeq 0\} \subseteq \mathcal{D}(A^{op}).$$
Then $A/J_+ \in \mathcal{C}$ and $\mathcal{C}$ is thick as it is the kernel of a triangulated functor. By the radical filtration, we have that $A \in \operatorname{thick}_{A^{op}}(A/J_+)$ and so $A \in \mathcal{C}$. Hence, $M \simeq A \otimes^{\mathbb{L}}_A M \simeq 0$. □

Recall that $\mathcal{D}^b(k) = \mathcal{D}^b(\operatorname{mod} k)$ is the bounded derived category of finite-dimensional k-vector spaces or equivalently, the subcategory of $\mathcal{D}(k)$ consisting of complexes with finite-dimensional cohomology.

Lemma 3.2.2. *Let A be a finite-dimensional DG-algebra and suppose $M \in \mathcal{D}(A)$ is such that $A/J_+ \otimes^{\mathbb{L}}_A M \in \mathcal{D}^b(k)$ or $\underline{\mathbb{R}\mathrm{Hom}}_A(M, A/J_+) \in \mathcal{D}^b(k)$. Then $M \in \mathcal{D}_{\mathrm{fd}}(A)$.*

Proof. The former case is similar to the proof of Proposition 3.2.1 but take $\mathcal{C} = \{X \in \mathcal{D}(A^{op}) \mid X \otimes^{\mathbb{L}}_A M \in \mathcal{D}^b(k)\}$. If $\underline{\mathbb{R}\mathrm{Hom}}_A(M, A/J_+) \in \mathcal{D}^b(k)$, then since $(A/J_+)^\vee \in \operatorname{thick}_A(A/J_+)$, we have that $(A/J_+ \otimes^{\mathbb{L}}_A M)^\vee \simeq \underline{\mathbb{R}\mathrm{Hom}}_A(M, (A/J_+)^\vee) \in \mathcal{D}^b(k)$. Therefore, $A/J_+ \otimes^{\mathbb{L}}_A M \in \mathcal{D}^b(k)$ so by above $M \in \mathcal{D}_{\mathrm{fd}}(A)$. □

The following result generalises the fact that the simples of a finite-dimensional algebra can detect modules with finite projective dimension.

Theorem 3.2.3. *Suppose A is a finite-dimensional DG-algebra and $M \in \mathcal{D}(A)$ is such that $\underline{\mathbb{R}\mathrm{Hom}}_A(M, A/J_+) \in \mathcal{D}^b(k)$ or $A/J_+ \otimes^{\mathbb{L}}_A M \in \mathcal{D}^b(k)$. Then $M \in \mathcal{D}^{\mathrm{perf}}(A)$.*

Proof. By Theorem 2.19 in [Orl20], there is a finite-dimensional DG-algebra E admitting an exceptional collection and a functor $\pi^* \colon \mathcal{D}(A) \hookrightarrow \mathcal{D}(E)$ which admits a right adjoint π_*. We claim that π_* sends $\mathcal{D}_{\mathrm{fd}}(E)$ to $\mathcal{D}_{\mathrm{sf}}(A)$. We have that,
$$\pi_*(M) \coloneqq \underline{\mathbb{R}\mathrm{Hom}}_E(P_n, -) \simeq \underline{\mathbb{R}\mathrm{Hom}}_E(P_n, E) \otimes^{\mathbb{L}}_E - \simeq \underline{\mathrm{Hom}}_E(P_n, E) \otimes^{\mathbb{L}}_E -$$
where P_n is a summand of E. So π_* can be computed using K-projective resolutions. By Theorem 2.19 in loc. cit., $\mathcal{D}^{\mathrm{perf}}(E) = \mathcal{D}_{\mathrm{fd}}(E)$. Then, by Proposition 2.5 in loc. cit., any object in $\mathcal{D}_{\mathrm{fd}}(E)$ admits a K-projective resolution which is strictly finite-dimensional. Therefore, $\pi_*(E/J_+) \in \operatorname{thick}_A(A/J_+)$. It follows that if $\underline{\mathbb{R}\mathrm{Hom}}_A(M, A/J_+) \in \mathcal{D}^b(k)$, then $\underline{\mathbb{R}\mathrm{Hom}}_A(M, \pi_*(E/J_+)) \in \mathcal{D}^b(k)$. Therefore,
$$\underline{\mathbb{R}\mathrm{Hom}}_E(\pi^*(M), E/J_+) \simeq \underline{\mathbb{R}\mathrm{Hom}}_A(M, \pi_*(E/J_+)) \in \mathcal{D}^b(k).$$
So by Lemma 3.2.2, $\pi^*(M) \in \mathcal{D}_{\mathrm{fd}}(E) = \mathcal{D}^{\mathrm{perf}}(E)$. As noted in the proof of Proposition 6.9 in [KS25], π^* reflects perfection and so $M \in \mathcal{D}^{\mathrm{perf}}(A)$. If $A/J_+ \otimes^{\mathbb{L}}_A M \in \mathcal{D}^b(k)$, then since $A/J_+^\vee \in \operatorname{thick}_{A^{op}}(A/J_+)$, we have that
$$\underline{\mathbb{R}\mathrm{Hom}}_A(M, A/J_+) \simeq ((A/J_+)^\vee \otimes^{\mathbb{L}}_A M)^\vee \in \mathcal{D}^b(k)$$
and so by above we have that $M \in \mathcal{D}^{\mathrm{perf}}(A)$. □

Corollary 3.2.4. *If A is a finite-dimensional DG-algebra, then*

$$\begin{aligned}\mathcal{D}^{\mathrm{perf}}(A) &= \{M \in \mathcal{D}(A) \mid A/J_+ \otimes^{\mathbb{L}}_A M \in \mathcal{D}^b(k)\} \\ &= \{M \in \mathcal{D}(A) \mid \underline{\mathbb{R}\mathrm{Hom}}_A(M, A/J_+) \in \mathcal{D}^b(k)\}.\end{aligned}$$

Proof. The result follows from Theorem 3.2.3 since both of the functors $A/J_+ \otimes^{\mathbb{L}}_A -$ and $\underline{\mathbb{R}\mathrm{Hom}}_A(-, A/J_+)$ send perfects to $\mathcal{D}^b(k)$. Indeed, they both send $A \mapsto A/J_+$. □

We note a consequence for the left homologically finite objects considered in [KS25].

Definition 3.2.5. Suppose $\mathcal{T}$ is a triangulated category. Define the following thick subcategories.

$$\begin{aligned}\mathcal{T}^{\mathrm{lhf}} &:= \{t \in \mathcal{T} \mid \bigoplus \mathrm{Hom}_{\mathcal{T}}(t, \Sigma^i t') \in \operatorname{mod} k \text{ for all } t' \in \mathcal{T}\} \\ \mathcal{T}^{\mathrm{rhf}} &:= \{t \in \mathcal{T} \mid \bigoplus \mathrm{Hom}_{\mathcal{T}}(t', \Sigma^i t) \in \operatorname{mod} k \text{ for all } t' \in \mathcal{T}\}\end{aligned}$$

Corollary 3.2.6. *Suppose A is a finite-dimensional DG-algebra. Then $\mathcal{D}_{\mathrm{fd}}(A)^{\mathrm{lhf}} = \mathcal{D}_{\mathrm{sf}}(A)^{\mathrm{lhf}} = \mathcal{D}^{\mathrm{perf}}(A)$.*

Proof. Since $\mathcal{D}_{\mathrm{fd}}(A)^{\mathrm{lhf}}$ and $\mathcal{D}_{\mathrm{sf}}(A)^{\mathrm{lhf}}$ are thick and contain A, they contain $\mathcal{D}^{\mathrm{perf}}(A)$. Conversely, if $M \in \mathcal{D}_{\mathrm{fd}}(A)^{\mathrm{lhf}}$ or $\mathcal{D}_{\mathrm{sf}}(A)^{\mathrm{lhf}}$ then $\underline{\mathbb{R}\mathrm{Hom}}_A(M, A/J_-) \in \mathcal{D}^b(k)$ and so $M \in \mathcal{D}^{\mathrm{perf}}(A)$ by Theorem 3.2.3. □

§ 3.3 | Detecting Gorenstein DG-algebras

We take our definition of a Gorenstein DG-algebra from [Jin20].

Definition 3.3.1. A proper DG-algebra A is Gorenstein if $\mathcal{D}^{\mathrm{perf}}(A) = \mathrm{thick}_A(A^\vee)$.

Remark 3.3.2. This condition is equivalent to requiring that both $A^\vee \in \mathcal{D}^{\mathrm{perf}}(A)$ and $A^\vee \in \mathcal{D}^{\mathrm{perf}}(A^{op})$. This follows since $A^\vee \in \mathcal{D}^{\mathrm{perf}}(A^{op}) = \mathrm{thick}_{A^{op}}(A)$ if and only if $A \in \mathrm{thick}_A(A^\vee)$ using the equivalence $(-)^\vee$. This implies that A is Gorenstein if and only if A^{op} is. One can see that for finite-dimensional algebras, Definition 3.3.1 is the Iwanga-Gorenstein condition.

Proposition 3.3.3. *Let A be a proper DG-algebra. The functor $(-)^\vee$ restricts to an equivalence $\mathcal{D}^{\mathrm{perf}}(A) \simeq \mathcal{D}^{\mathrm{perf}}(A^{op})^{op}$ if and only if A is Gorenstein.*

Proof. Since A is a proper DG-algebra, we can restrict $(-)^\vee$ to an equivalence.

$$(-)^\vee \colon \mathcal{D}^{\mathrm{perf}}(A) = \mathrm{thick}_A(A) \xrightarrow{\sim} \mathrm{thick}_{A^{op}}(A^\vee)^{op} \subseteq \mathcal{D}_{\mathrm{fd}}(A^{op})^{op} \tag{3.1}$$

By Equation (3.1), the functor $(-)^\vee$ restricts to an equivalence on $\mathcal{D}^{\mathrm{perf}}(A)$ if and only if $\mathcal{D}^{\mathrm{perf}}(A^{op}) = \mathrm{thick}_{A^{op}}(A^\vee)$. This is exactly the condition that A^{op} is Gorenstein. The result then follows since A is Gorenstein if and only if A^{op} is. □

Recall from Remark 2.5.5, one always has an equivalence $\mathcal{D}^{\mathrm{perf}}(A) \simeq \mathcal{D}^{\mathrm{perf}}(A^{op})^{op}$ but in general it is not given by $(-)^\vee$. One justification for this definition is the relationship to Serre functors in the sense of [BK89]. The next result generalises a well-known fact about finite-dimensional algebras.

Proposition 3.3.4. *A proper DG-algebra A is Gorenstein if and only if $\mathcal{D}^{\mathrm{perf}}(A)$ admits a Serre functor. In this case, the Serre functor is*

$$A^\vee \otimes_A^{\mathbb{L}} - \colon \mathcal{D}^{\mathrm{perf}}(A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{perf}}(A).$$

Proof. Suppose $\mathcal{D}^{\mathrm{perf}}(A)$ admits a Serre functor $S\colon \mathcal{D}^{\mathrm{perf}}(A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{perf}}(A)$. Then we have isomorphisms of functors $\mathcal{D}^{\mathrm{perf}}(A)^{op} \to \operatorname{mod} k$.

$$\begin{aligned} \operatorname{Hom}_{\mathcal{D}^{\mathrm{perf}}(A)}(-, S(A)) &\simeq \operatorname{Hom}_{\mathcal{D}^{\mathrm{perf}}(A)}(A, -)^\vee \\ &\simeq \operatorname{Hom}_{\mathcal{D}_{\mathrm{fd}}(A^{op})}(A, (-)^\vee) \\ &\simeq \operatorname{Hom}_{\mathcal{D}_{\mathrm{fd}}(A)}(-, A^\vee) \end{aligned}$$

The first follows from the Serre functor, the second since $(-)^\vee$ commutes with cohomology and the third holds by Remark 2.5.2. Therefore, there is a morphism $f : S(A) \to A^\vee$ corresponding to $1_{S(A)}$ and by the Yoneda lemma, the isomorphism is given by $\operatorname{Hom}_{\mathcal{D}(A)}(-, f)$. In particular, $H^i(f) = \operatorname{Hom}_{\mathcal{D}(A)}(\Sigma^{-i}A, f)$ is an isomorphism for all i. Therefore, f is a quasi-isomorphism and so $A^\vee \simeq S(A)$ is perfect. If $\mathcal{D}^{\mathrm{perf}}(A)$ admits a Serre functor, then so does $\mathcal{D}^{\mathrm{perf}}(A)^{op} \simeq \mathcal{D}^{\mathrm{perf}}(A^{op})$. So one also has $A^\vee \in \mathcal{D}^{\mathrm{perf}}(A^{op})$. Hence, A is Gorenstein.

Conversely, by Proposition 3.3.3 and Remark 2.5.5, $\underline{\mathbb{R}\mathrm{Hom}}_A(-, A)^\vee \simeq A^\vee \otimes_A^{\mathbb{L}} -$ is an equivalence. For $M, N \in \mathcal{D}^{\mathrm{perf}}(A)$ there are natural equivalences

$$\underline{\mathbb{R}\mathrm{Hom}}_A(M, \underline{\mathbb{R}\mathrm{Hom}}_A(N, A)^\vee)^\vee \simeq \underline{\mathbb{R}\mathrm{Hom}}_A(N, A) \otimes_A^{\mathbb{L}} M \simeq \underline{\mathbb{R}\mathrm{Hom}}_A(N, M)$$

where the last equivalence holds since it holds at $N = A$ and the class of objects for which it is an equivalence is thick. □

Proposition 3.3.4 generalises Theorem 4.3 of [Shk07]. The following is a dual version of Corollary 3.2.4.

Proposition 3.3.5. *For a finite-dimensional DG-algebra A*

$$\operatorname{thick}_A(A^\vee) = \{M \in \mathcal{D}_{\mathrm{fd}}(A) \mid \underline{\mathbb{R}\mathrm{Hom}}_A(A/J_-, M) \in \mathcal{D}^b(k)\}.$$

Proof. Corollary 3.2.4 applied to A^{op} implies

$$\mathcal{D}^{\mathrm{perf}}(A^{op}) = \{M \in \mathcal{D}_{\mathrm{fd}}(A^{op}) \mid M \otimes_A^{\mathbb{L}} A/J_- \in \mathcal{D}^b(k)\}.$$

Applying the equivalence $(-)^\vee$ gives

$$\operatorname{thick}_A(A^\vee) = \{M^\vee \in \mathcal{D}_{\mathrm{fd}}(A) \mid M \otimes_A^{\mathbb{L}} A/J_- \in \mathcal{D}^b(k)\}.$$

This can be rewritten using the equivalence $(-)^\vee$ as

$$\text{thick}_A(A^\vee) = \{N \in \mathcal{D}_{\text{fd}}(A) \mid N^\vee \otimes_A^{\mathbb{L}} A/J_- \in \mathcal{D}^b(k)\}.$$

Now, $N^\vee \otimes_A^{\mathbb{L}} A/J_- \in \mathcal{D}^b(k)$ if and only if $(N^\vee \otimes_A^{\mathbb{L}} A/J_-)^\vee \in \mathcal{D}^b(k)$. Hence, the result follows since

$$(N^\vee \otimes_A^{\mathbb{L}} A/J_-)^\vee \simeq \underline{\mathbb{R}\text{Hom}}_A(A/J_-, N^{\vee\vee}) \simeq \underline{\mathbb{R}\text{Hom}}_A(A/J_-, N). \qquad \square$$

We deduce that reflecting perfection for $\underline{\mathbb{R}\text{Hom}}_A(A/J_-, -)$ is related to being Gorenstein.

Theorem 3.3.6. *Let A be a finite-dimensional DG-algebra. Then A is Gorenstein if and only if both of the functors*

$$\underline{\mathbb{R}\text{Hom}}_A(A/J_-, -)\colon \mathcal{D}_{\text{fd}}(A) \to \mathcal{D}(A/J_-),$$
$$\underline{\mathbb{R}\text{Hom}}_{A^{op}}(A/J_-, -)\colon \mathcal{D}_{\text{fd}}(A^{op}) \to \mathcal{D}(A^{op}/J_-)$$

reflect perfection.

Proof. Suppose both functors reflect perfection. Proposition 3.3.5 applied to A and A^{op} implies that $A^\vee \in \mathcal{D}^{\text{perf}}(A)$ and $A^\vee \in \mathcal{D}^{\text{perf}}(A^{op})$. Hence, A is Gorenstein. The converse also follows from Proposition 3.3.5 applied to A and A^{op}. $\square$

Note that by Theorem 3.1.12, an object of $\mathcal{D}(A/J_-)$ is perfect if and only if it has finite-dimensional cohomology.

Corollary 3.3.7. *Let A be a finite-dimensional DG-algebra. Then A is Gorenstein if and only if both $\underline{\mathbb{R}\text{Hom}}_A(A/J_-, A)$ and $\underline{\mathbb{R}\text{Hom}}_{A^{op}}(A/J_-, A)$ have finite-dimensional cohomology.*

Proof. If A is Gorenstein, then so is A^{op}. By Proposition 3.3.5, $\underline{\mathbb{R}\text{Hom}}_A(A, A/J_-)$ and $\underline{\mathbb{R}\text{Hom}}_{A^{op}}(A, A/J_-)$ are in $\mathcal{D}^b(k)$. Conversely, it follows from Proposition 3.3.5 that $A \in \text{thick}_A(A^\vee)$ and $A \in \text{thick}_{A^{op}}(A^\vee)$. Hence, A is Gorenstein. $\square$

For not necessarily proper DG-algebras, another definition of Gorenstein was given in [FJ03]. We can compare these.

Proposition 3.3.8. *Let A be a proper DG-algebra. If A is Gorenstein, then there are inverse equivalences*

$$\underline{\mathbb{R}\text{Hom}}_A(-, A)\colon \mathcal{D}_{\text{fd}}(A)^{op} \underset{\longleftarrow}{\overset{\sim}{\longrightarrow}} \mathcal{D}_{\text{fd}}(A^{op})\colon \underline{\mathbb{R}\text{Hom}}_{A^{op}}(-, A).$$

Conversely, suppose that $\mathcal{D}^{\text{perf}}(A) = \mathcal{D}_{\text{fd}}(A)^{\text{lhf}}$ and $\mathcal{D}^{\text{perf}}(A^{op}) = \mathcal{D}_{\text{fd}}(A^{op})^{\text{lhf}}$. If both $\underline{\mathbb{R}\text{Hom}}_A(-, A)$ and $\underline{\mathbb{R}\text{Hom}}_{A^{op}}(-, A)$ preserve cohomologically finite modules, then A is Gorenstein.

Proof. If A is proper and Gorenstein, then $A^\vee \otimes^{\mathbb{L}}_A -$ is an equivalence on $\mathcal{D}(A)$ and so its adjoint $\underline{\mathbb{R}\mathrm{Hom}}_A(A^\vee, -)$ is too. It follows from the adjunction that $\underline{\mathbb{R}\mathrm{Hom}}_A(A^\vee, -)$ restricts to an equivalence on $\mathcal{D}_{\mathrm{fd}}(A)$. Then there is an equivalence

$$\underline{\mathbb{R}\mathrm{Hom}}_{A^{op}}(-, A) \simeq \underline{\mathbb{R}\mathrm{Hom}}_A(A^\vee, (-)^\vee) \colon \mathcal{D}_{\mathrm{fd}}(A^{op}) \xrightarrow{\sim} \mathcal{D}_{\mathrm{fd}}(A)^{op},$$

as required. Dually, since A^{op} is also Gorenstein, $\underline{\mathbb{R}\mathrm{Hom}}_A(-, A)$ restricts to the desired equivalence. They are inverse equivalences since there is an adjunction

$$\underline{\mathbb{R}\mathrm{Hom}}_A(-, A) \colon \mathcal{D}(A) \rightleftarrows \mathcal{D}(A^{op})^{op} \colon \underline{\mathbb{R}\mathrm{Hom}}_{A^{op}}(-, A).$$

Conversely, if $\underline{\mathbb{R}\mathrm{Hom}}_{A^{op}}(-, A)$ preserves cohomologically finite modules, then so does $\underline{\mathbb{R}\mathrm{Hom}}_A(A^\vee, -)$. It follows that $A^\vee \in \mathcal{D}_{\mathrm{fd}}(A)^{\mathrm{lhf}} = \mathcal{D}^{\mathrm{perf}}(A)$. Dually the other condition implies that $A^\vee \in \mathcal{D}^{\mathrm{perf}}(A^{op})$. Hence, A is Gorenstein. ☐

Remark 3.3.9. We note that the conditions $\mathcal{D}^{\mathrm{perf}}(A) = \mathcal{D}_{\mathrm{fd}}(A)^{\mathrm{lhf}}$ and $\mathcal{D}^{\mathrm{perf}}(A^{op}) = \mathcal{D}_{\mathrm{fd}}(A^{op})^{\mathrm{lhf}}$ hold for finite-dimensional DG-algebras by Corollary 3.2.6. They also hold for any proper reflexive (see Definition 4.3.1) DG-algebra by Proposition 6.1.1.

§ 3.4 | The Koszul Dual

We have seen that any finite-dimensional DG-algebra has a semisimple quotient $A \to A/J_-$ and A/J_- generates $\mathcal{D}_{\mathrm{sf}}(A)$. This sets up a version of Koszul duality for finite-dimensional DG-algebras.

Definition 3.4.1. The Koszul dual of a finite-dimensional DG-algebra A is the DG-algebra

$$A^! \coloneqq \underline{\mathbb{R}\mathrm{Hom}}_A(A/J_-, A/J_-).$$

The aim of this section is to record various derived equivalences between $A^!$ and A which swap the perfects and the finite-dimensional modules.

Definition 3.4.2. If A is a DG-algebra, let $\mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}(A) \coloneqq \mathcal{D}^{\mathrm{perf}}(A) \cap \mathcal{D}_{\mathrm{fd}}(A) \subseteq \mathcal{D}(A)$.

Remark 3.4.3.

1. If A is a proper DG-algebra, then $\mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}(A) = \mathcal{D}^{\mathrm{perf}}(A)$ and if A is smooth then $\mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}(A) = \mathcal{D}_{\mathrm{fd}}(A)$. Indeed, this follows from Lemma 3.8 in [KS25].

2. There are DG-algebras for which $\mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}(A)$ is properly contained in both $\mathcal{D}^{\mathrm{perf}}(A)$ and $\mathcal{D}_{\mathrm{fd}}(A)$. As noted in [Orl23], the infinite Kronecker algebra is an example.

3. In the language of [KS25], a DG-algebra A is hfd-closed if $\mathcal{D}_{\mathrm{fd}}(A) = \mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}(A)$ and $\mathcal{D}_{\mathrm{fd}}(A^{op}) = \mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}(A^{op})$.

4. If A is a connective finite-dimensional DG-algebra over a perfect field, then $A^!$ is smooth as noted in the proof of Proposition 6.9 in [KS25] and so $\mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}(A^!) = \mathcal{D}_{\mathrm{fd}}(A^!)$.

5. If $A = k[x]/x^2$ with $|x| = 1$, then $A^! = k[[t]]$ with $|t| = 0$ which is not smooth but still satisfies $\mathcal{D}_{\text{fd}}^{\text{perf}}(A^!) = \mathcal{D}_{\text{fd}}(A^!)$. Indeed, if R is any left Noetherian k-algebra of finite global dimension, then every finite-dimensional module is finitely generated and so admits a finite resolution by finitely generated projectives. Hence, $\mathcal{D}_{\text{fd}}(R) \subseteq \mathcal{D}^{\text{perf}}(R)$.

Theorem 3.4.4. *Let A be a finite-dimensional DG-algebra. Then there are equivalences*

$$\begin{array}{ccc} \mathcal{D}_{\text{sf}}(A)^{op} & \xrightarrow[\sim]{\underline{\mathbb{R}\text{Hom}}_A(-,A/J_-)} & \mathcal{D}^{\text{perf}}(A^!) \\ \uparrow & & \uparrow \\ \mathcal{D}^{\text{perf}}(A)^{op} & \xrightarrow{\sim} & \mathcal{D}_{\text{fd}}^{\text{perf}}(A^!) \end{array}$$

Proof. The top functor is an equivalence by Remark 2.6.5. Since $A \mapsto A/J_- \in \mathcal{D}_{\text{fd}}(A^!)$, the equivalence sends $\mathcal{D}^{\text{perf}}(A)^{op}$ to $\mathcal{D}_{\text{fd}}^{\text{perf}}(A^!)$ as shown. If $X \in \mathcal{D}_{\text{fd}}^{\text{perf}}(A^!)$, then $X \simeq \underline{\mathbb{R}\text{Hom}}_A(Y, A/J_-)$ for some $Y \in \mathcal{D}_{\text{sf}}(A)$. But then by Theorem 3.2.3, $Y \in \mathcal{D}^{\text{perf}}(A)$, as required. Therefore, the bottom functor is also essentially surjective. □

We note that $A^!$ is left hfd-closed if and only if it is right hfd-closed.

Proposition 3.4.5. *If A is a finite-dimensional DG-algebra, then $\mathcal{D}_{\text{fd}}^{\text{perf}}((A^!)^{op}) = \mathcal{D}_{\text{fd}}((A^!)^{op})$ if and only if $\mathcal{D}_{\text{fd}}^{\text{perf}}(A^!) = \mathcal{D}_{\text{fd}}(A^!)$.*

Proof. We claim that $(-)^\vee$ restricts to an equivalence as shown

$$\begin{array}{ccc} \mathcal{D}_{\text{fd}}(A^!) & \xrightarrow[\sim]{(-)^\vee} & \mathcal{D}_{\text{fd}}((A^!)^{op})^{op} \\ \uparrow & & \uparrow \\ \mathcal{D}_{\text{fd}}^{\text{perf}}(A^!) & \xrightarrow[\sim]{(-)^\vee} & \mathcal{D}_{\text{fd}}^{\text{perf}}((A^!)^{op})^{op} \end{array}$$

from which the result follows immediately. We need to show that for any $X \in \mathcal{D}_{\text{fd}}^{\text{perf}}(A^!)$ $X^\vee \in \mathcal{D}^{\text{perf}}((A^!)^{op})$. Since A generates $\mathcal{D}^{\text{perf}}(A)$, we have that $\underline{\mathbb{R}\text{Hom}}_A(A, A/J_-) \simeq A/J_-$ generates $\mathcal{D}_{\text{fd}}^{\text{perf}}(A^!)$. Therefore, it is enough to show that $A/J_-^\vee \in \mathcal{D}^{\text{perf}}((A^!)^{op})$. We have $A/J_-^\vee \simeq \underline{\mathbb{R}\text{Hom}}_A(A/J_-, A^\vee) \in \mathcal{D}((A^!)^{op})$. Since $A^\vee \in \mathcal{D}_{\text{sf}}(A) = \text{thick}_A(A/J_-)$, we have that $A/J_-^\vee \in \text{thick}_{A^!}(\underline{\mathbb{R}\text{Hom}}_A(A/J_-, A/J_-)) = \mathcal{D}^{\text{perf}}((A^!)^{op})$. □

We now deduce covariant versions of the two equivalences in Theorem 3.4.4.

Theorem 3.4.6. *Suppose A is a finite-dimensional DG-algebra. Then there are equivalences*

$$\underline{\mathbb{R}\text{Hom}}_A(A/J_-, -) \colon \mathcal{D}_{\text{sf}}(A) \xrightarrow{\sim} \mathcal{D}^{\text{perf}}((A^!)^{op})$$
$$(A/J_-)^\vee \otimes_A^{\mathbb{L}} - \colon \mathcal{D}^{\text{perf}}(A) \xrightarrow{\sim} \mathcal{D}_{\text{fd}}^{\text{perf}}((A^!)^{op})$$

Proof. The first equivalence follows from Remark 2.6.5. Composing the bottom equivalence from Theorem 3.4.4 with the equivalence $(-)^\vee$ in the proof of Proposition 3.4.5

states that

$$\underline{\mathbb{R}\mathrm{Hom}}_A(-, A/J_-)^\vee : \mathcal{D}^{\mathrm{perf}}(A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}(A^!)^{op} \xrightarrow{\sim} \mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}((A^!)^{op})$$

is an equivalence. Then we note that

$$\underline{\mathbb{R}\mathrm{Hom}}_A(-, A/J_-)^\vee \simeq \underline{\mathbb{R}\mathrm{Hom}}_A(-, A/J_-^{\vee\vee})^\vee \simeq A/J_-^\vee \otimes^{\mathbb{L}}_A -. \qquad \square$$

Remark 3.4.7. The Koszul dual $A^!$ has an augmentation from which we can recover A. Applying $\underline{\mathbb{R}\mathrm{Hom}}_A(-, A/J_-)$ to $A \to A/J_-$ gives a map of left $A^!$-modules $A/J_- \leftarrow A^!$. And $\underline{\mathbb{R}\mathrm{Hom}}_A(-, A/J_-)$ induces an equivalence of DG-algebras

$$A = \underline{\mathbb{R}\mathrm{Hom}}_A(A, A)^{op} \xrightarrow{\sim} \underline{\mathbb{R}\mathrm{Hom}}_{A^!}(A/J_-, A/J_-) =: A^{!!}.$$

This says that A is dc-complete with respect to A/J_- in the language of [DGI06].

Chapter 4

Reflexive DG-categories as Reflexive Objects

Reflexive DG-categories were introduced in [KS25] by Kuznetsov and Shinder to abstract the duality between the perfect and bounded derived categories of projective schemes. They showed that there is a significant amount of common information between $\mathcal{A}$ and $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ for a reflexive DG-category $\mathcal{A}$; they have the same triangulated autoequivalence groups and their semiorthogonal decompositions are in bijection.

The main result of this chapter is a new characterisation of reflexive DG-categories as the reflexive objects in the closed symmetric monoidal category Hmo. In a way that can be made precise, the reflexive objects are those for which information can be recovered from their duals. In Hmo, the dual of a DG-category $\mathcal{A}$ is $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. This provides a moral justification of why there is some common information between $\mathcal{A}$ and $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ when $\mathcal{A}$ is reflexive. In Chapter 5, this characterisation will be applied to produce more common information between these two categories.

We begin by recalling the abstract notion of a reflexive object in a closed symmetric monoidal category. In Section 4.2, we give the construction of the closed monoidal structure on Hmo due to [Toë07], and note that the dual of a DG-category $\mathcal{A}$ in Hmo is $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. In Section 4.3, we provide a detailed construction of the evaluation map appearing in the definition of a reflexive DG-category (which is skipped in [KS25]) as a natural transformation between functors on Hmo. In Section 4.4, we recall that Hmo can be viewed as a bicategory and that the dual functor $\mathcal{D}_{\mathrm{fd}}(-)$ lifts to this level. The results of this section appeared in the preprint [Goo24c].

§ 4.1 | Reflexivity in Monoidal Categories

We collect some well-known results on monoidal categories. Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a closed symmetric monoidal category whose internal hom we denote $\mathrm{hom}(-,-)$. The dual functor is

$$D := \mathrm{hom}(-, \mathbf{1}) \colon \mathcal{M} \to \mathcal{M}^{op}$$

and it is left adjoint to $D\colon \mathcal{M}^{op} \to \mathcal{M}$. Indeed, there are natural isomorphisms

$$\hom(x, Dy) \simeq \hom(x \otimes y, \mathbf{1}) \simeq \hom(y \otimes x, \mathbf{1}) \simeq \hom(y, Dx). \tag{4.1}$$

The unit and the counit coincide as a natural transformation

$$\mathrm{eval}\colon 1_{\mathcal{M}} \to DD.$$

The following definition appeared in [DP84].

Definition 4.1.1. An object $x \in \mathcal{M}$ is reflexive if eval_x is an isomorphism. Let $\mathrm{Refl}(\mathcal{M}) \subseteq \mathcal{M}$ denote the full subcategory of reflexive objects of $\mathcal{M}$.

Proposition 4.1.2. *If $\mathcal{M}$ is a closed symmetric monoidal category, then D restricts to a self-inverse equivalence*

$$D\colon \mathrm{Refl}(\mathcal{M}) \xrightarrow{\sim} \mathrm{Refl}(\mathcal{M})^{op}$$

and $\mathrm{Refl}(\mathcal{M})$ *is maximal with this property i.e. if D restricts to an equivalence on any subcategory $\mathcal{C} \subseteq \mathcal{M}$ then $\mathcal{C} \subseteq$* $\mathrm{Refl}(\mathcal{M})$.

Proof. The triangle identity of the adjunction implies that D restricts to a contravariant self-adjunction on $\mathrm{Refl}(\mathcal{M})$. Since the unit and counit are isomorphisms for all reflexive objects, it is an equivalence. If D restricts to an equivalence on $\mathcal{C}$, then the unit and counit must be isomorphisms and so every object of $\mathcal{C}$ is reflexive. □

Remark 4.1.3. We consider some canonical morphisms in $\mathcal{M}$.

1. For $x \in \mathcal{M}$, the counit of the adjunction $- \otimes x \dashv \hom(x, -)$ evaluated at the unit produces a map $\varepsilon^x_{\mathbf{1}}\colon Dx \otimes x \to \mathbf{1}$.
2. For $x, y \in \mathcal{M}$, the adjunct of $\varepsilon^x_{\mathbf{1}} \otimes \varepsilon^y_{\mathbf{1}}$ produces maps

$$\mu_{x,y}\colon Dx \otimes Dy \to D(x \otimes y)$$

   which make $D\colon \mathcal{M} \to \mathcal{M}^{op}$ a lax monoidal functor.
3. For $x, y \in \mathcal{M}$, there is a map

$$Dx \otimes y \to \hom(x, y)$$

   defined as the image of $\varepsilon^x_{\mathbf{1}}$ under

$$\mathrm{Hom}(Dx \otimes x, \mathbf{1}) \xrightarrow{-\otimes y} \mathrm{Hom}(Dx \otimes x \otimes y, y) \simeq \mathrm{Hom}(Dx \otimes y, \hom(x, y)).$$

Reflexivity should be compared to dualisability.

Definition 4.1.4. An object in $x \in \mathcal{M}$ is dualisable if the canonical map

$$Dx \otimes x \to \hom(x, x)$$

is an isomorphism. Let $\mathrm{Dual}(\mathcal{M})$ denote the full subcategory of dualisable objects in $\mathcal{M}$.

There are various equivalent definitions of dualisability. The following proposition can be extracted from [DP84] and [LMSM86].

Proposition 4.1.5. *Let $\mathcal{M}$ be a closed symmetric monoidal category and $x \in \mathcal{M}$. The following are equivalent.*

1. *x is dualisable.*
2. *There is a map $c\colon \mathbf{1} \to x \otimes Dx$ such that the composites*

$$x \xrightarrow{c \otimes x} x \otimes Dx \otimes x \xrightarrow{x \otimes \varepsilon_1} x$$
$$Dx \xrightarrow{Dx \otimes c} Dx \otimes x \otimes Dx \xrightarrow{\varepsilon_1 \otimes Dx} Dx$$

   are both the identity.
3. *The canonical map $Dx \otimes y \to \hom(x, y)$ is an isomorphism for all $y \in \mathcal{M}$.*
4. *x is reflexive and $\mu_{x,y}\colon Dx \otimes Dy \to D(x \otimes y)$ is an isomorphism for all $y \in \mathcal{M}$.*
5. *x is reflexive and $\mu_{x,Dx}\colon Dx \otimes DDx \to D(x \otimes Dx)$ is an isomorphism.*
6. *x is reflexive and Dx is dualisable.*

Proposition 4.1.6. *Suppose $\mathcal{M}$ is a closed symmetric monoidal category. Then* $\mathrm{Dual}(\mathcal{M})$ *is a symmetric monoidal subcategory of $\mathcal{M}$ and D restricts to a strong monoidal equivalence*

$$D\colon \mathrm{Dual}(\mathcal{M}) \xrightarrow{\sim} \mathrm{Dual}(\mathcal{M})^{op}.$$

and $\mathrm{Dual}(\mathcal{M})$ *is maximal with respect to this property.*

Proof. If x and y are dualisable, then by (3) and (4) of Proposition 4.1.5,

$$\begin{aligned} D(x \otimes y) \otimes x \otimes y &\simeq Dx \otimes Dy \otimes x \otimes y \\ &\simeq \hom(x, Dy \otimes x \otimes y) \\ &\simeq \hom(x, \hom(y, x \otimes y)) \\ &\simeq \hom(x \otimes y, x \otimes y). \end{aligned}$$

One can check this is the canonical map and so $x \otimes y$ is dualisable. By (4), D restricts to a strong monoidal equivalence. Conversely, if D restricts to an equivalence on a monoidal subcategory $\mathcal{C}$, then we must have $\mathcal{C} \subseteq \mathrm{Refl}(\mathcal{M})$. Since the equivalence is strong monoidal, we have that $\mu_{x,Dx}$ is an isomorphism for all $x \in \mathcal{C}$. Then by (5), $x \in \mathrm{Dual}(\mathcal{M})$. □

Remark 4.1.7. Dualisability is a notion of smallness and coincides with other notions of smallness in many cases. The dualisable objects correspond to the finite-dimensional representations over a finite group; finitely generated projectives in the module category of commutative ring; perfect complexes in the derived category of a commutative ring; and finite spectra in the stable homotopy category.

Example 4.1.8. In general, there are more reflexive objects than dualisable objects. In $\operatorname{Mod}\mathbb{Z}$, the Specker group $\prod_{\mathbb{N}}\mathbb{Z}$ is reflexive but not dualisable. In $\mathcal{D}(R)$, for a commutative ring R, the object $\bigoplus_{\mathbb{Z}}\Sigma^i R$ is reflexive but not dualisable.

Remark 4.1.9. The reflexive objects do not form a monoidal subcategory in general. Consider $M \coloneqq \bigoplus_{i\in\mathbb{Z}}\Sigma^i k \in \mathcal{D}(k)$. This is reflexive but $M \otimes_k M$ is not for cardinality reasons.

Proposition 4.1.10. *Reflexive objects and dualisable objects are closed under retracts.*

Proof. Suppose $x \xrightarrow{f} y \xrightarrow{g} x$ are such that $gf = 1_x$ and x is reflexive. Then, by naturality of eval, eval_x is a retract of eval_y. Then the result holds since isomorphisms are closed under retracts. The proof is similar for dualisable objects. □

Remark 4.1.11. We note that the dual functor $D\colon \mathcal{M} \to \mathcal{M}^{op}$ lifts to an $\mathcal{M}$-enriched functor (see Section 1.6 of [Kel82]). This means that for every pair of objects $x, y \in \mathcal{M}$ there is a morphism in $\mathcal{M}$

$$D_{x,y}\colon \hom(x,y) \to \hom(Dy, Dx)$$

subject to the usual associativity conditions and such that the following diagram commutes.

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{M}}(1, \hom(x,y)) & \xrightarrow{\mathrm{Hom}(1,D)} & \mathrm{Hom}_{\mathcal{M}}(1, \hom(Dy,Dx)) \\
\wr\downarrow & & \wr\downarrow \\
\mathrm{Hom}_{\mathcal{M}}(x,y) & \xrightarrow{\quad D \quad} & \mathrm{Hom}_{\mathcal{M}}(Dy,Dx)
\end{array}$$

where the bottom map is the action of the functor $D\colon \mathcal{M} \to \mathcal{M}^{op}$. Furthermore, Equation 4.1 shows that $D \dashv D$ is an $\mathcal{M}$-enriched adjunction (see Section 1.11 in [Kel82]) and so the following diagram commutes

$$\begin{array}{ccc}
\hom(x,y) & \xrightarrow{\quad D_{x,y} \quad} & \hom(Dy,Dx) \\
 & {\scriptstyle \hom(x,\mathrm{eval}_y)}\searrow \qquad \nearrow{\scriptstyle \sim} & \\
 & \hom(x, DDy) &
\end{array}$$

The following proposition follows immediately from Remark 4.1.11.

Proposition 4.1.12. *If $\mathcal{M}$ is a closed symmetric monoidal category and $y \in \mathcal{M}$ is reflexive, then*

$$D_{x,y}\colon \hom(x,y) \to \hom(Dy, Dx)$$

is an isomorphism for all $x \in \mathcal{M}$.

§ 4.2 | Duals in the Morita category

In this section, we recall Toën's description of the internal hom in the symmetric monoidal categories Hqe and Hmo. We include a proof that the dual in Hmo of a DG-category is its category of cohomologicaly finite modules.

Remark 4.2.1. The tensor product of DG-categories makes DGcat a closed symmetric monoidal category whose internal hom we denote $\hom_{\mathrm{DGcat}}(-,-)$. The symmetric monoidal and model structures are not compatible but as we are over a field, the tensor product preserves quasi-equivalences and so descends to

$$- \otimes - \colon \ \mathrm{Hqe} \times \mathrm{Hqe} \to \mathrm{Hqe}\,.$$

Over a commutative ring, the tensor product can also be derived to Hqe.

The next result follows from a more general statement about the simplicial enrichment associated to the model category. A direct proof is given in [CS15].

Theorem 4.2.2 (Theorem 1.3, [Toë07]). $(\mathrm{Hqe}, \otimes, k)$ *is a closed symmetric monoidal category with internal hom given by* $\hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B}) \simeq \mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{rqr}$.

Remark 4.2.3. Note that by Theorem 2.7.3, there is a bijection between $\mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B})$ and isomorphism classes of objects in $H^0 \hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B})$.

A similar version of the following result is mentioned at the beginning of Section 7 in [Toë07].

Proposition 4.2.4. *For a DG-category $\mathcal{A}$, there is an equivalence*

$$\hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{D}^b(k)) \simeq \mathcal{D}_{\mathrm{fd}}(\mathcal{A}).$$

Proof. Let Q denote the cofibrant replacement functor on DGcat for the model structure in Remark 2.7.1. By definition, the functor $\mathrm{Hom}_{\mathrm{Hqe}}(- \otimes \mathcal{A}, \mathcal{D}_{\mathrm{fd}}(k))$ is represented by $\hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{D}_{\mathrm{fd}}(k))$. We will show that $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ also represents this functor and deduce the equivalence. Let iso denote the isomorphisms classes of objects in a category and Ho the homotopy category of a model category. Consider the following chain of isomorphisms.

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Hqe}}(-, \mathcal{D}_{\mathrm{fd}}(\mathcal{A})) &\simeq \mathrm{Hom}_{\mathrm{Hqe}}(Q(-), \mathcal{D}_{\mathrm{fd}}(\mathcal{A})) \\
&\simeq \mathrm{iso}\,\mathrm{Ho}(\hom_{\mathrm{DGcat}}(Q(-), \mathcal{C}_{\mathrm{fd}}(\mathcal{A}))) \\
&\simeq \mathrm{iso}\,\mathrm{Ho}(\hom_{\mathrm{DGcat}}(Q(-), \hom_{\mathrm{DGcat}}(\mathcal{A}, \mathcal{C}_{\mathrm{fd}}(k)))) \\
&\simeq \mathrm{iso}\,\mathrm{Ho}(\hom_{\mathrm{DGcat}}(Q(-) \otimes \mathcal{A}, \mathcal{C}_{\mathrm{fd}}(k))) \\
&\simeq \mathrm{iso}\,\mathrm{Ho}(\hom_{\mathrm{DGcat}}(Q(Q(-) \otimes \mathcal{A}), \mathcal{C}_{\mathrm{fd}}(k))) \\
&\simeq \mathrm{Hom}_{\mathrm{Hqe}}(Q(Q(-) \otimes \mathcal{A}), \mathcal{D}^b(k)) \\
&\simeq \mathrm{Hom}_{\mathrm{Hqe}}(- \otimes \mathcal{A}, \mathcal{D}^b(k))
\end{aligned}$$

The first follows since there is a natural quasi-equivalence $Q(-) \to 1_{\mathrm{Hqe}}$. The second is Lemma 6.2 of [Toë07] applied to $M_0 = \mathcal{C}_{\mathrm{fd}}(\mathcal{A})$. The third is the definition of $\mathcal{C}_{\mathrm{fd}}(\mathcal{A})$ and the fourth is the underived tensor-hom adjunction. The fifth follows since $\mathrm{Ho}(\mathrm{hom}_{\mathrm{DGcat}}(\mathcal{B}, \mathcal{C}_{\mathrm{fd}}(k)) = \mathcal{D}_{\mathrm{fd}}(\mathcal{B})$ and the quasi-equivalence $Q(\mathcal{B}) \to \mathcal{B}$ induces an equivalence $\mathcal{D}_{\mathrm{fd}}(\mathcal{B}) \simeq \mathcal{D}_{\mathrm{fd}}(Q(\mathcal{B}))$ for any DG-category $\mathcal{B}$. The sixth is another application of Lemma 6.2. □

Remark 4.2.5. The symmetric monoidal structure can be derived to Hmo by setting $\mathcal{A} \otimes \mathcal{B} := \mathcal{D}^{\mathrm{perf}}(\mathcal{A} \otimes \mathcal{B})$. For more details see [Toë11].

The following well-known result follows from Theorem 4.2.2.

Theorem 4.2.6. *The symmetric monoidal category* Hmo *is closed with internal hom given by* $\mathrm{hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{hom}_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{D}^{\mathrm{perf}}(\mathcal{B})) \simeq \mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{\mathcal{B}-\mathrm{perf}}$

We deduce the following from Theorem 4.2.6 and Proposition 4.2.4.

Corollary 4.2.7. *The dual of a DG-category $\mathcal{A}$ in the closed symmetric monoidal category* Hmo *is* $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$.

Remark 4.2.8. Note that Proposition 4.2.4 could also be extracted directly from Theorem 4.2.6 by observing that $\mathcal{D}(k^{op} \otimes \mathcal{A})^{k-\mathrm{perf}} \simeq \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. However we include the direct proof of Proposition 4.2.4 as it will be used in the proof of Lemma 4.3.16.

§ 4.3 | Reflexive DG-categories

We will take the following as our definition of a reflexive DG-category.

Definition 4.3.1. A DG-category $\mathcal{A}$ is reflexive if it represents a reflexive object in Hmo.

Let us spell out what this means. By Proposition 4.2.4, the evaluation map gives a morphism in Hmo

$$\mathrm{eval}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{A}$$

which induces a triangulated functor

$$\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op} \to \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{A}$$

which is an equivalence if and only if $\mathcal{A}$ is reflexive. We will also be interested in the following weaker condition.

Definition 4.3.2. A DG-category $\mathcal{A}$ is semireflexive if $\mathrm{eval}_{\mathcal{A}}$ is quasi-fully faithful.

Remark 4.3.3. Note semireflexivity is equivalent to fully faithfulness of the triangulated functor $\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op} \to \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{A}$.

Remark 4.3.4. Quasi-fully faithful functors in Hqe can be encoded as the homotopy monomorphisms of the model category (see Lemma 2.4 of [Toë07]). So one could also encode semireflexivity in an abstract setting by considering monoidal categories with some higher structure.

Remark 4.3.5. In [KS25], reflexivity is defined for pretriangulated DG-categories by writing down an explicit evaluation functor $\mathcal{A} \to \mathcal{D}_{\text{fd}}\mathcal{D}_{\text{fd}}(\mathcal{A})$. In this section, we provide a detailed construction of Kuznetsov and Shinder's evaluation map as a natural transformation between endofunctors on Hmo and show that it coincides with the abstract evaluation map $\text{eval}_{\mathcal{A}}$ above.

Remark 4.3.6. Recall there are dualities $\mathcal{D}^{\text{perf}}(\mathcal{A}^{op}) \simeq \mathcal{D}^{\text{perf}}(\mathcal{A})^{op}$ and $\mathcal{D}_{\text{fd}}(\mathcal{A})^{op} \simeq \mathcal{D}_{\text{fd}}(\mathcal{A}^{op})$ which can be composed with $\text{eval}_{\mathcal{A}}$ to give equivalent formulations of reflexivity. These are studied in Lemma 3.10 of [KS25] and in particular, it is shown that $\mathcal{A}$ is reflexive if and only if $\mathcal{A}^{op}$ is.

Remark 4.3.7. By Lemma 3.13 in [KS25] or Proposition 4.1.2, if $\mathcal{A}$ is a reflexive DG-category, then so is $\mathcal{D}_{\text{fd}}(\mathcal{A})$.

Example 4.3.8. Many commonly studied DG-categories in algebra, geometry and topology turn out to be reflexive. For example, see [BNP17], [KS25], [LU22], [BGO25]. For a detailed survey of the currently known examples, see the beginning of Chapter 6.

Remark 4.3.9. By Remark 2.2.14, we can assume that that the projective model structure on $\mathcal{C}(\mathcal{A})$ admits a cofibrant replacement DG-functor. Therefore, we have DG-functors

$$\mathcal{D}(\mathcal{A}) \xhookrightarrow{i^{\mathcal{A}}} \mathcal{C}(\mathcal{A}) \xrightarrow{Q^{\mathcal{A}}} \mathcal{D}(\mathcal{A})$$

and a DG-natural transformation $i^{\mathcal{A}}\varepsilon^{\mathcal{A}} \colon Q^{\mathcal{A}} \to 1_{\mathcal{C}(\mathcal{A})}$ which is an objectwise quasi-isomorphism. Note these DG-functors restrict to cohomologically finite modules.

To construct $\mathcal{D}_{\text{fd}}$ as a functor on Hmo and to write down an explicit evaluation map we will use the following technical lemma.

Lemma 4.3.10. *Suppose $F, G \colon \mathcal{A} \to \mathcal{B}$ are DG-functors and $\alpha \colon F \to G$ is a DG-natural transformation such that for every $a \in \mathcal{A}$, the map $\alpha_a \in \text{Hom}_{H^0\mathcal{B}}(F(a), G(a))$ is an isomorphism. Then F and G represent the same morphism in* Hqe.

Proof. The quasifunctors representing F and G are M_F and $M_G \in \mathcal{C}(\mathcal{B}^{op} \otimes \mathcal{A})$ where $M_F(b, a) = \mathcal{B}(b, F(a))$ and $M_G(b, a) = \mathcal{B}(b, G(a))$. There are composition maps for any $b \in \mathcal{B}$ $a \in \mathcal{A}$

$$\mathcal{B}(b, F(a)) \otimes_k \mathcal{B}(F(a), G(a)) \to \mathcal{B}(b, G(a)).$$

Composing with $\alpha_a \in \mathcal{B}(F(a), G(a))$ produces chain maps $M_F(b, a) \to M_G(b, a)$ which are natural and give rise to map $M_F \to M_G \in \mathcal{C}(\mathcal{B}^{op} \otimes \mathcal{A})$. Taking cohomology produces maps for every i

$$H^i\mathcal{B}(b, F(a)) \otimes_k H^0\mathcal{B}(F(a), G(a)) \to H^i\mathcal{B}(b, G(a)).$$

Since α_a is an isomorphism in $H^0\mathcal{B}$, composing with the class α_a represents induces an isomorphism $H^*M_F(b,a) \to H^*M_G(b,a)$. Therefore, $M_F \simeq M_G \in \mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})$. Hence, the isomorphism class of M_F equals the isomorphism class of M_G and the corresponding elements of $\mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B})$ are equal. □

Definition 4.3.11. Define a functor $\mathcal{D}_{\mathrm{fd}}(-)\colon \mathrm{DGcat} \to \mathrm{Hqe}^{op}$ which sends $\mathcal{A} \mapsto \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ and $F\colon \mathcal{A} \to \mathcal{B}$ to the morphism represented by the DG-functor

$$\mathcal{D}_{\mathrm{fd}}(\mathcal{B}) \xhookrightarrow{i^{\mathcal{B}}} \mathcal{C}_{\mathrm{fd}}(\mathcal{B}) \xrightarrow{\mathrm{Res}(F)} \mathcal{C}_{\mathrm{fd}}(\mathcal{A}) \xrightarrow{Q^{\mathcal{A}}} \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$$

where $\mathrm{Res}(F)(M) = MF$.

Proposition 4.3.12. *The functor* $\mathcal{D}_{\mathrm{fd}}(-)\colon \mathrm{DGcat} \to \mathrm{Hqe}^{op}$ *in Definition 4.3.11 is well defined.*

Proof. We check $\mathcal{D}_{\mathrm{fd}}(-)$ is functorial. For a DG-category $\mathcal{A}$, $1_{\mathcal{A}}$ is sent to the quasi-functor representing the DG-functor $Q^{\mathcal{A}}i^{\mathcal{A}}\colon \mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \to \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. There is a DG-natural transformation $\varepsilon^{\mathcal{A}}\colon Q^{\mathcal{A}}i^{\mathcal{A}} \to 1_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}$ such that for any $M \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ the map $Q^{\mathcal{A}}(M) \to M$ is a quasi-isomorphism. Therefore, the morphisms $\varepsilon^{\mathcal{A}}_M \in \mathrm{Hom}_{H^0\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(Q^{\mathcal{A}}(M), M)$ are isomorphisms so by Lemma 4.3.10, $\mathcal{D}_{\mathrm{fd}}(1_{\mathcal{A}}) = 1_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})} \in \mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}), \mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$. Given DG-functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$, there is a DG-natural transformation

$$\alpha := Q^{\mathcal{A}}\,\mathrm{Res}(F)\varepsilon^{\mathcal{B}}_{\mathrm{Res}(G)i^{\mathcal{C}}} : \mathcal{D}_{\mathrm{fd}}(F)\mathcal{D}_{\mathrm{fd}}(G) \to \mathcal{D}_{\mathrm{fd}}(GF).$$

For any $M \in \mathcal{C}$, $\varepsilon^{\mathcal{B}}_{\mathrm{Res}(G)i^{\mathcal{C}}M}$ is a quasi-isomorphism in $\mathcal{C}_{\mathrm{fd}}(\mathcal{B})$. Restricting along F and applying $Q^{\mathcal{A}}$ preserves quasi-isomorphisms and so

$$\alpha_M \in \mathrm{Hom}_{H^0\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(Q^{\mathcal{A}}(Q^{\mathcal{B}}(MG)F), Q^{\mathcal{A}}(MGF))$$

is an isomorphism for every $M \in \mathcal{D}_{\mathrm{fd}}(\mathcal{C})$. Again by Lemma 4.3.10 we are done. □

Proposition 4.3.13. $\mathcal{D}_{\mathrm{fd}}(-)$ *descends to functors* $\mathrm{Hqe} \to \mathrm{Hqe}^{op}$ *and* $\mathrm{Hmo} \to \mathrm{Hmo}^{op}$.

Proof. If $\mathcal{D}_{\mathrm{fd}}$ descends to Hmo, then it certainly descends to Hqe. We must show that if $F\colon \mathcal{A} \to \mathcal{B}$ is a Morita equivalence, then $\mathcal{D}_{\mathrm{fd}}(F)$ is an isomorphism in Hmo. This is equivalent to showing that the DG-functor $\mathcal{D}_{\mathrm{fd}}(F)\colon \mathcal{D}_{\mathrm{fd}}(\mathcal{B}) \to \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ is a quasi-equivalence. If F is a Morita equivalence, then the unbounded restriction functor $\mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ is an equivalence by definition. This equivalence preserves the perfects as they are the compact objects. Note that $M \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ if and only if $\underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(P, M) \in \mathcal{D}^b(k)$ for every $P \in \mathcal{D}^{\mathrm{perf}}(\mathcal{A})$. Therefore, the equivalence restricts to an equivalence $\mathcal{D}_{\mathrm{fd}}(\mathcal{B}) \simeq \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. □

Proposition 4.3.14. *There are natural transformations* $\mathrm{ev}\colon 1_{\mathrm{Hqe}} \to \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(-)$ *and* $\mathrm{ev}\colon 1_{\mathrm{Hmo}} \to \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(-)$ *such that for each DG-category* $\mathcal{A}$ *and* $a \in \mathcal{A}$*,* $\mathrm{ev}_{\mathcal{A},a} \in \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ *is a* K*-projective resolution of the DG-functor* $\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \to \mathcal{D}^b(k)$ *which sends* $M \mapsto M(a)$*.*

Proof. For $\mathcal{A} \in \mathrm{DGcat}$, define

$$\overline{\mathrm{ev}}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{C}_{\mathrm{fd}}\mathcal{C}_{\mathrm{fd}}(\mathcal{A})$$

as follows. For $a \in \mathcal{A}$ set

$$\overline{\mathrm{ev}}_{\mathcal{A},a}\colon \mathcal{C}_{\mathrm{fd}}(\mathcal{A}) \to \mathcal{C}^b(k); M \mapsto M(a);\ \beta \mapsto \beta_a.$$

This defines a DG-functor and so $\overline{\mathrm{ev}}_{\mathcal{A},a} \in \mathcal{C}_{\mathrm{fd}}\mathcal{C}_{\mathrm{fd}}(\mathcal{A})$. Given some $f \in \mathcal{A}(a,b)^n$, set $\overline{\mathrm{ev}}_{\mathcal{A},f}\colon \overline{\mathrm{ev}}_{\mathcal{A},a} \to \overline{\mathrm{ev}}_{\mathcal{A},b}$ as the graded natural transformation whose value at $M \in \mathcal{C}_{\mathrm{fd}}(\mathcal{A})$ is

$$\overline{\mathrm{ev}}_{\mathcal{A},a}(M) = M(a) \xrightarrow{M(f)} M(b) = \overline{\mathrm{ev}}_{\mathcal{A},b}(M).$$

Then $f \mapsto \overline{\mathrm{ev}}_{\mathcal{A},f}$ is a chain map, and this constructs a DG-functor $\overline{\mathrm{ev}}_{\mathcal{A}}$. Define $\mathrm{ev}_{\mathcal{A}}$ as the composition

$$\mathrm{ev}_{\mathcal{A}}\colon \mathcal{A} \xrightarrow{\overline{\mathrm{ev}}_{\mathcal{A}}} \mathcal{C}_{\mathrm{fd}}\mathcal{C}_{\mathrm{fd}}(\mathcal{A}) \xrightarrow{\mathrm{Res}(i^{\mathcal{A}})} \mathcal{C}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \xrightarrow{Q^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}} \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$$

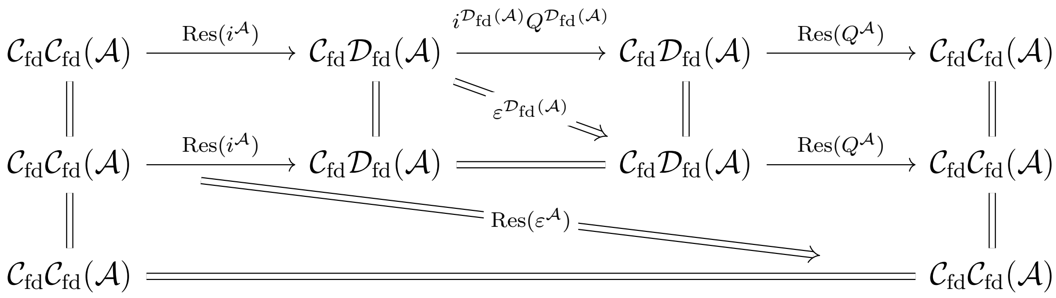
and view it as a morphism in Hqe and Hmo by taking its image along the functors $\mathrm{DGcat} \to \mathrm{Hqe} \to \mathrm{Hmo}$. We now check naturality. First, note that if $\alpha\colon F \to G$ is a DG-natural transformation between DG-functors $F, G\colon \mathcal{A} \to \mathcal{B}$, there is a DG-natural transformation $\mathrm{Res}(\alpha)\colon \mathrm{Res}(F) \to \mathrm{Res}(G)$ defined by

$$\mathrm{Res}(\alpha)_M(a)\colon \mathrm{Res}(F)(M)(a) = MF(a) \xrightarrow{M(\alpha_a)} MG(a) = \mathrm{Res}(G)(M)(a)$$

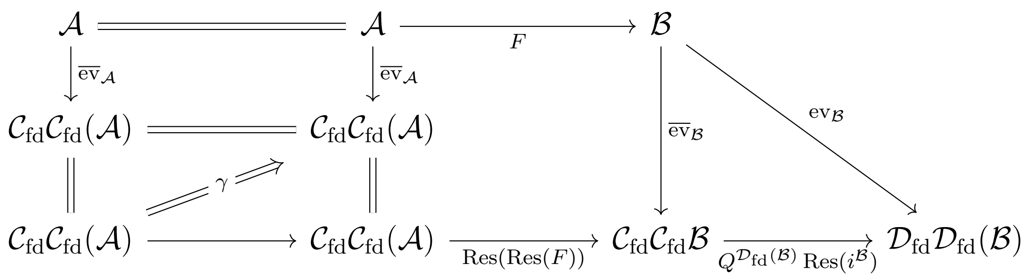
for $M \in \mathcal{C}(\mathcal{B})$ and $a \in \mathcal{A}$. Now, given $F\colon \mathcal{A} \to \mathcal{B} \in \mathrm{Hqe}$, note that since every object of Hqe is isomorphic to a cofibrant object, we can assume that $\mathcal{A}$ is cofibrant and so F is a DG-functor. We will define a DG-natural transformation ϕ of the following form.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad F \quad} & \mathcal{B} \\ \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} & \overset{\phi}{\Rightarrow} & \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{B}}} \\ \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{A} & \xrightarrow[\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(F)]{} & \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{B} \end{array}$$

First, let $\gamma^{\mathcal{A}} = \gamma\colon \mathrm{Res}(Q^{\mathcal{A}})i^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}Q^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}\,\mathrm{Res}(i^{\mathcal{A}}) \to 1_{\mathcal{C}_{\mathrm{fd}}\mathcal{C}_{\mathrm{fd}}(\mathcal{A})}$ denote the 2-cell defined as



Then let ϕ be the 2-cell



where the middle square commutes by naturality of $\overline{\text{ev}}$ and the right triangle commutes by definition. By Lemma 4.3.10, it is enough to show that ϕ_a is a quasi-isomorphism. For any $a \in \mathcal{A}$, we claim that $\operatorname{Res}\operatorname{Res}(F)\gamma_a$ is a quasi-isomorphism of $\mathcal{C}_{\text{fd}}(\mathcal{B})$ modules. If $M \in \mathcal{C}_{\text{fd}}(\mathcal{B})$, then $(\operatorname{Res}\operatorname{Res}(F)\gamma_a)_M$ is the composition

$$\begin{array}{c}
\left(\operatorname{Res}\operatorname{Res}(F)\operatorname{Res}(Q^{\mathcal{A}})i^{\mathcal{D}_{\text{fd}}(\mathcal{A})}Q^{\mathcal{D}_{\text{fd}}(\mathcal{A})}\operatorname{Res}(i^{\mathcal{A}})\overline{\text{ev}}_{\mathcal{A},a}\right)(M) \\
\Big\| \\
i^{\mathcal{D}_{\text{fd}}(\mathcal{A})}Q^{\mathcal{D}_{\text{fd}}(\mathcal{A})}(\overline{\text{ev}}_{\mathcal{A},a}i^{\mathcal{A}})(Q^{\mathcal{A}}(MF)) \\
\Big\downarrow {\scriptstyle \varepsilon^{\mathcal{D}_{\text{fd}}(\mathcal{A})}_{\overline{\text{ev}}_{\mathcal{A},a}i^{\mathcal{A}}}(Q^{\mathcal{A}}(MF))} \\
(\overline{\text{ev}}_{\mathcal{A},a}i^{\mathcal{A}})(Q^{\mathcal{A}}(MF)) = i^{\mathcal{A}}Q^{\mathcal{A}}(MF)(a) \\
\Big\downarrow {\scriptstyle \varepsilon^{\mathcal{A}}_{MF}(a)} \\
MF(a)
\end{array}$$

This is a quasi-isomorphism since $\varepsilon^{\mathcal{D}_{\text{fd}}(\mathcal{A})}_{\overline{\text{ev}}_{\mathcal{A},a}i^{\mathcal{A}}}$ is a quasi-isomorphism of $\mathcal{D}_{\text{fd}}(\mathcal{A})$-modules and $\varepsilon^{\mathcal{A}}_{MF}$ is a quasi-isomorphism of $\mathcal{A}$-modules. Hence, $\operatorname{Res}\operatorname{Res}(F)\gamma_a$ is a quasi-isomorphism for all a. Therefore, $\operatorname{Res}(i^{\mathcal{B}})\operatorname{Res}\operatorname{Res}(F)\gamma_a$ is a quasi-isomorphism for all $a \in \mathcal{A}$ as restriction preserves quasi-isomorphisms. Therefore,

$$\phi_a = Q^{\mathcal{D}_{\text{fd}}(\mathcal{B})}\operatorname{Res}(i^{\mathcal{B}})\operatorname{Res}\operatorname{Res}(F)\gamma_a$$

is a quasi-isomorphism since $Q^{\mathcal{D}_{\text{fd}}(\mathcal{B})}$ preserves quasi-isomorphisms. As they commute in Hqe, the naturality diagrams also commute in Hmo. □

Proposition 4.3.15. *There are adjunctions*

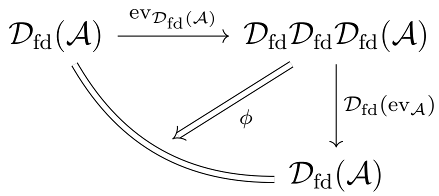
$$\mathcal{D}_{\mathrm{fd}}(-)\colon \mathrm{Hqe} \rightleftarrows \mathrm{Hqe}^{op} \colon \mathcal{D}_{\mathrm{fd}}(-)$$

and

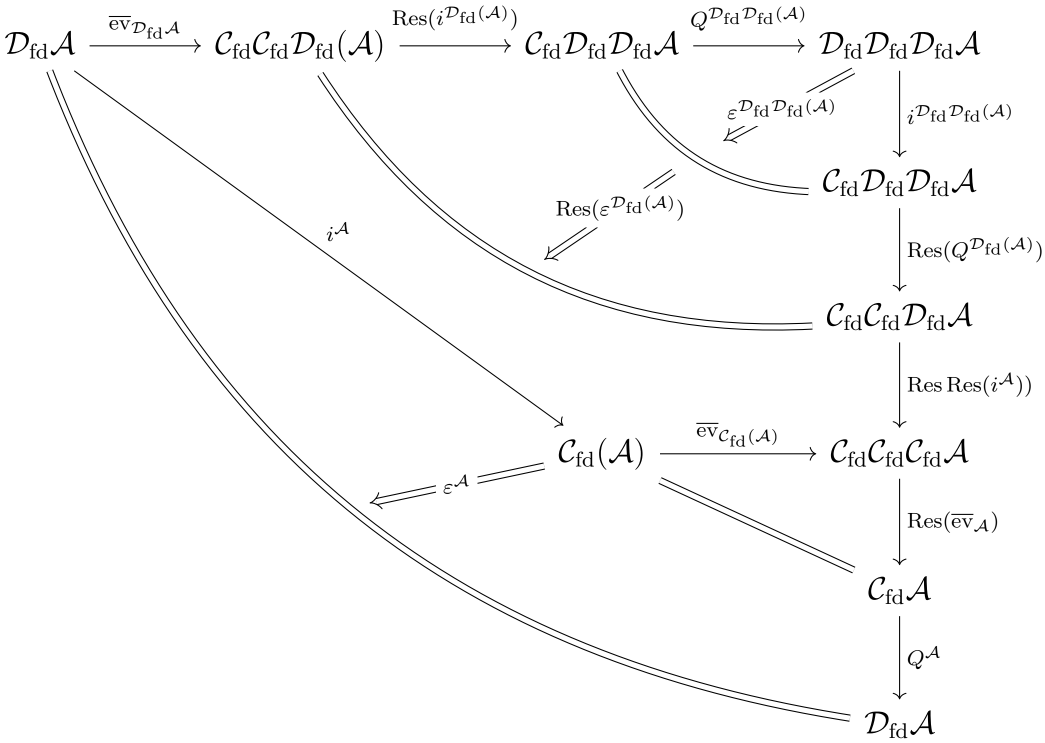
$$\mathcal{D}_{\mathrm{fd}}(-)\colon \mathrm{Hmo} \rightleftarrows \mathrm{Hmo}^{op} \colon \mathcal{D}_{\mathrm{fd}}(-)$$

whose unit and counit are both given by ev.

Proof. We will show that the triangle identity holds for $\mathcal{D}_{\mathrm{fd}}(-)$ on Hqe with candidate unit and counit given by $\mathrm{ev}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}$. To do this, we construct a DG-natural transformation of the form



Define ϕ as the following 2-cell.



The pentagon commutes by naturality of $\overline{\mathrm{ev}}$ and the unlabelled triangle is the triangle identity for $\mathcal{C}_{\mathrm{fd}}(-) \dashv \mathcal{C}_{\mathrm{fd}}(-)$. Let β denote the composite of the first three non-trivial two-cells in ϕ, so that

$$\beta\colon\ \mathrm{Res}(\overline{\mathrm{ev}}_{\mathcal{A}})\,\mathrm{Res}\,\mathrm{Res}(i^{\mathcal{A}})\,\mathrm{Res}(Q^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})})i^{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{A}}Q^{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{A}}\,\mathrm{Res}(i^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})})\overline{\mathrm{ev}}_{\mathcal{D}_{\mathrm{fd}}\mathcal{A}} \to i^{\mathcal{A}}$$

and $\phi = \varepsilon^{\mathcal{A}}Q^{\mathcal{A}}\beta$. Then, given $M \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ and $a \in \mathcal{A}$, $\beta_M(a)$ is the morphism in $\mathcal{D}^b(k)$ given by

$$\begin{array}{c}
(\operatorname{Res}(\overline{\mathrm{ev}}_{\mathcal{A}}) \operatorname{Res}\operatorname{Res}(i^{\mathcal{A}}) \operatorname{Res}(Q^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}) i^{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{A}} Q^{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{A}} \operatorname{Res}(i^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})})\overline{\mathrm{ev}}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A}),M})(a) \\
\Big\| \\
i^{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})} Q^{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(\operatorname{Res}(i^{\mathcal{D}_{\mathrm{fd}}\mathcal{A}})\overline{\mathrm{ev}}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A}),M})(Q^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(\operatorname{Res}(i^{\mathcal{A}})\overline{\mathrm{ev}}_{\mathcal{A},a})) \\
\Big\downarrow \varepsilon^{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}_{\operatorname{Res}(i^{\mathcal{D}_{\mathrm{fd}}\mathcal{A}})\overline{\mathrm{ev}}_{\mathcal{D}_{\mathrm{fd}}\mathcal{A},M}}(Q^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(\operatorname{Res}(i^{\mathcal{A}})\overline{\mathrm{ev}}_{\mathcal{A},a})) \\
\operatorname{Res}(i^{\mathcal{D}_{\mathrm{fd}}\mathcal{A}})\overline{\mathrm{ev}}_{\mathcal{D}_{\mathrm{fd}}\mathcal{A},M}(Q^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(\operatorname{Res}(i^{\mathcal{A}})\overline{\mathrm{ev}}_{\mathcal{A},a})) = (i^{\mathcal{D}_{\mathrm{fd}}\mathcal{A}} Q^{\mathcal{D}_{\mathrm{fd}}\mathcal{A}})(\operatorname{Res}(i^{\mathcal{A}})\overline{\mathrm{ev}}_{\mathcal{A},a})(M) \\
\Big\downarrow \varepsilon^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}_{\operatorname{Res}(i^{\mathcal{A}})\overline{\mathrm{ev}}_{\mathcal{A},a}}(M) \\
\operatorname{Res}(i^{\mathcal{A}})\overline{\mathrm{ev}}_{\mathcal{A},a}(M) = i^{\mathcal{A}}(M)(a)
\end{array}$$

This is a quasi-isomorphism as both ε's are pointwise quasi-isomorphisms. Therefore, β_M is a quasi-isomorphism of $\mathcal{A}$-modules and so is $Q^{\mathcal{A}}(\beta_M)$. So the above 2-cell evaluated at M is

$$\mathcal{D}_{\mathrm{fd}}(\mathrm{ev}_{\mathcal{A}})\, \mathrm{ev}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(M) \xrightarrow[\sim]{Q^{\mathcal{A}}(\beta_M)} Q^{\mathcal{A}} i^{\mathcal{A}}(M) \xrightarrow[\sim]{\varepsilon^{\mathcal{A}}_M} M$$

which is a quasi-isomorphism. Therefore, the square commutes in Hqe by Lemma 4.3.10 and so also in Hmo and we are done. □

Lemma 4.3.16. *There is an isomorphism* $\mathcal{D}_{\mathrm{fd}}(-) \simeq \hom_{\mathrm{Hqe}}(-, \mathcal{D}^b(k))$ *of functors on* Hqe *and an isomorphism of functors* $\mathcal{D}_{\mathrm{fd}}(-) \simeq \hom_{\mathrm{Hmo}}(-, k)$ *on* Hmo.

Proof. It is enough to prove the first statement and as before we need only check naturality for DG-functors $F\colon \mathcal{A} \to \mathcal{B}$. Recall that for any cofibrant $X \in$ Hqe, there is an isomorphism by Lemma 6.2 in [Toë07]

$$\mathrm{Hom}_{\mathrm{Hqe}}(X, \mathcal{D}_{\mathrm{fd}}(\mathcal{A})) \simeq \mathrm{iso}\,\mathrm{Ho}(\hom_{\mathrm{DGcat}}(X, \mathcal{C}_{\mathrm{fd}}(\mathcal{A}))). \tag{6.7.1}$$

using the same notation as the proof of Proposition 4.2.4. We first show the following diagram commutes

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Hqe}}(X, \mathcal{D}_{\mathrm{fd}}(\mathcal{B})) & \xrightarrow{\sim} & \mathrm{iso}\,\mathrm{Ho}(\hom_{\mathrm{DGcat}}(X, \mathcal{C}_{\mathrm{fd}}(\mathcal{B}))) \\
\Big\downarrow {\scriptstyle \mathcal{D}_{\mathrm{fd}}(F)\circ} & & \Big\downarrow {\scriptstyle \hom_{\mathrm{DGcat}}(X, \mathrm{Res}(F))} \\
\mathrm{Hom}_{\mathrm{Hqe}}(X, \mathcal{D}_{\mathrm{fd}}(\mathcal{A})) & \xrightarrow{\sim} & \mathrm{iso}\,\mathrm{Ho}(\hom_{\mathrm{DGcat}}(X, \mathcal{C}_{\mathrm{fd}}(\mathcal{A})))
\end{array}$$

where the right vertical functor is well defined on the homotopy categories since $\mathrm{Res}(F)$ preserves weak equivalences in $\mathcal{C}_{\mathrm{fd}}(\mathcal{B})$ and the weak equivalences in $\hom_{\mathrm{DGcat}}(X, \mathcal{C}_{\mathrm{fd}}(\mathcal{B}))$ are defined pointwise. Any $f\colon X \to \mathcal{D}_{\mathrm{fd}}(\mathcal{B}) \in$ Hqe can be modelled as an actual DG-functor since X is cofibrant. The proof of Lemma 6.2 in loc. cit. shows that f is sent along the top right composition to

$$X \xrightarrow{f} \mathcal{D}_{\mathrm{fd}}(\mathcal{B}) \xrightarrow{i^{\mathcal{A}}} \mathcal{C}_{\mathrm{fd}}(\mathcal{B}) \xrightarrow{\mathrm{Res}(F)} \mathcal{C}_{\mathrm{fd}}(\mathcal{A}).$$

The image of f along the bottom left composition is

$$X \xrightarrow{f} \mathcal{D}_{\text{fd}}(\mathcal{B}) \xrightarrow{i^{\mathcal{A}}} \mathcal{C}_{\text{fd}}(\mathcal{B}) \xrightarrow{\text{Res}(F)} \mathcal{C}_{\text{fd}}(\mathcal{A}) \xrightarrow{Q^{\mathcal{A}}} \mathcal{D}_{\text{fd}}(\mathcal{A}) \xrightarrow{i^{\mathcal{A}}} \mathcal{C}_{\text{fd}}(\mathcal{A}).$$

Then, since $\varepsilon\colon i^{\mathcal{A}}Q^{\mathcal{A}} \to 1_{\mathcal{C}_{\text{fd}}(\mathcal{A})}$ is a pointwise quasi-isomorphism, these two objects are equal in $\operatorname{iso}\operatorname{Ho}(\hom_{\text{DGcat}}(X, \mathcal{C}_{\text{fd}}(\mathcal{A}))$, as required. Next, note that the following diagram commutes

$$\begin{array}{ccc} \operatorname{iso}\operatorname{Ho}(\hom_{\text{DGcat}}(X, \mathcal{C}_{\text{fd}}(\mathcal{B}))) & \xrightarrow{\sim} & \operatorname{iso}\operatorname{Ho}(\hom_{\text{DGcat}}(X \otimes \mathcal{B}, \mathcal{C}^b(k)) \\ \downarrow{\scriptstyle \hom_{\text{DGcat}}(X,\ \text{Res}(F))} & & \downarrow{\scriptstyle \hom_{\text{DGcat}}(X\otimes F,\ \mathcal{C}^b(k))} \\ \operatorname{iso}\operatorname{Ho}(\hom_{\text{DGcat}}(X, \mathcal{C}_{\text{fd}}(\mathcal{A}))) & \xrightarrow{\sim} & \operatorname{iso}\operatorname{Ho}(\hom_{\text{DGcat}}(X \otimes \mathcal{A}, \mathcal{C}^b(k)) \end{array}$$

as it commutes at the non-derived level by naturality of the tensor-hom adjunction. Lemma 6.2 of [Toë07] gives us naturality of the isomorphism in Equation (6.7.1) in X. Hence, the following diagram commutes.

$$\begin{array}{ccc} \operatorname{iso}\operatorname{Ho}(\hom_{\text{DGcat}}(X \otimes \mathcal{B}, \mathcal{C}^b(k))) & \xrightarrow{\sim} & \operatorname{Hom}_{\text{Hqe}}(X \otimes \mathcal{B}, \mathcal{D}^b(k)) \\ \downarrow{\scriptstyle \hom_{\text{DGcat}}(X\otimes F,\ \mathcal{C}^b(k))} & & \downarrow{\scriptstyle \circ(X\otimes F)} \\ \operatorname{iso}\operatorname{Ho}(\hom_{\text{DGcat}}(X \otimes \mathcal{A}, \mathcal{C}^b(k))) & \xrightarrow{\sim} & \operatorname{Hom}_{\text{Hqe}}(X \otimes \mathcal{A}, \mathcal{D}^b(k)) \end{array}$$

Finally, note that the square below commutes by definition of the functoriality of an internal hom.

$$\begin{array}{ccc} \operatorname{Hom}_{\text{Hqe}}(X \otimes \mathcal{B}, \mathcal{D}^b(k)) & \xrightarrow{\sim} & \operatorname{Hom}_{\text{Hqe}}(X, \hom_{\text{Hqe}}(\mathcal{B}, \mathcal{D}^b(k))) \\ \downarrow{\scriptstyle \circ(X\otimes F)} & & \downarrow{\scriptstyle \hom_{\text{Hqe}}(F,\mathcal{D}^b(k))\circ} \\ \operatorname{Hom}_{\text{Hqe}}(X \otimes \mathcal{A}, \mathcal{D}^b(k)) & \xrightarrow{\sim} & \operatorname{Hom}_{\text{Hqe}}(X, \hom_{\text{Hqe}}(\mathcal{A}, \mathcal{D}^b(k))) \end{array}$$

Pasting the previous four diagrams together states that the image under Yoneda embedding of the following diagram commutes.

$$\begin{array}{ccc} \mathcal{D}_{\text{fd}}(\mathcal{B}) & \xrightarrow{\sim} & \hom_{\text{Hqe}}(\mathcal{B}, \mathcal{D}_{\text{fd}}(k)) \\ \downarrow{\scriptstyle \mathcal{D}_{\text{fd}}(F)} & & \downarrow{\scriptstyle \hom_{\text{Hqe}}(F,\mathcal{D}^b(k))} \\ \mathcal{D}_{\text{fd}}(\mathcal{A}) & \xrightarrow{\sim} & \hom_{\text{Hqe}}(\mathcal{B}, \mathcal{D}^b(k)) \end{array}$$

Therefore, the diagram commutes in Hqe. □

Remark 4.3.17. Reflexive DG-categories were originally defined in Definition 3.11 of [KS25]. If $\mathcal{T}$ is a pretriangulated DG-category, then by the Yoneda Lemma, the composition

$$\mathcal{T}^{op} \simeq \mathcal{D}^{\text{perf}}(\mathcal{T}) \xrightarrow{\text{ev}_{\mathcal{T}}} \mathcal{D}_{\text{fd}}\mathcal{D}_{\text{fd}}(\mathcal{T})$$

sends $t \in \mathcal{T}$ to the functor which sends $M \in \mathcal{D}_{\text{fd}}(\mathcal{T})$ to $M(t)$. In [KS25], this functor is denoted $\mathbf{ev}_{\mathcal{T}}$ and so by Lemma 3.10 of loc. cit. we see that reflexivity of $\mathcal{T}$ in their sense is equivalent to the functor $\mathrm{ev}_{\mathcal{T}}$ being a quasi-equivalence. Therefore the following theorem shows that our definition of reflexivity coincides with theirs.

Theorem 4.3.18. *A DG-category $\mathcal{A}$ is reflexive (semireflexive) if and only if $\mathrm{ev}_{\mathcal{A}}$ is a quasi-equivalence (quasi-fully faithful).*

Proof. By uniqueness of adjunctions, the composition

$$\mathcal{A} \xrightarrow{\mathrm{ev}_{\mathcal{A}}} \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \simeq \hom_{\mathrm{Hqe}}(\hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{D}^b(k)), \mathcal{D}^b(k))$$

is $\mathrm{eval}_{\mathcal{A}}$ of Section 4.3 and so the result follows. □

The following consequence will be of great use. It is an enhanced version of Corollary 3.16 in [KS25]. Also note we only require $\mathcal{B}$ to be reflexive.

Corollary 4.3.19. *Suppose $\mathcal{B}$ is a reflexive (semireflexive) DG-category and $\mathcal{A}$ is any DG-category. Then the morphism in* Hmo

$$\hom_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) \to \hom_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}), \mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$$

is a quasi-equivalence (quasi-fully faithful) and the map

$$\mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) \to \mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}), \mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$$

is an isomorphism (monomorphism).

Proof. If $\mathcal{B}$ is reflexive, this follows immediately from Proposition 4.1.12. If $\mathcal{B}$ is semireflexive, $\mathrm{eval}_{\mathcal{B}}$ is quasi-fully faithful. Then by Corollary 6.6 in [Toë07], the morphism $\hom_{\mathrm{Hqe}}(\mathcal{A}, \mathrm{eval}_{\mathcal{B}})$ is also quasi-fully faithful. By Remark 4.1.11, it follows that the morphism in the statement is quasi-fully faithful. The second result follows by applying H^0 and restricting to isomorphism classes of objects. □

Remark 4.3.20. We note the connection to corepresentability. The Yoneda embedding is an isomorphism in Hmo and so for a DG-category $\mathcal{A}$, $\mathcal{A}$ is reflexive if and only if $\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op}$ is. So we're interested in the composition

$$\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op} \xrightarrow{\mathrm{ev}_{\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op}}} \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op}) \simeq \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{A}.$$

The equivalence is given by restricting along $\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}_{\mathrm{fd}}(\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op})$ which sends $N \mapsto \underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(-, N)$. Hence, the long composite is

$$\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op} \to \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{A};\ M \mapsto \underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(M, -).$$

Thus, $\mathcal{A}$ is reflexive if and only if every DG-functor $F \colon \mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \to \mathcal{D}^b(k)$ is corepresented by some $M \in \mathcal{D}^{\mathrm{perf}}(\mathcal{A})$ and every morphism in $\mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$ is determined by a morphism between the corepresenting objects.

§ 4.4 | The Dual as a Pseudofunctor

The categories Hmo and Hqe admit the higher structure of bicategories. In this section, we note that the functor $\mathcal{D}_{\mathrm{fd}}(-)$ lifts to this level. The following result is well known. For a detailed construction, see [Gen15] or [Ima25].

Proposition 4.4.1. *There are bicategories* $\underline{\mathrm{Hqe}}$ *and* $\underline{\mathrm{Hmo}}$ *with the same objects as* Hqe *and* Hmo *and the categories of morphisms are given by* $\underline{\mathrm{Hom}}_{\mathrm{Hqe}}(\mathcal{A},\mathcal{B}) \coloneqq \mathcal{D}(\mathcal{B}^{op}\otimes\mathcal{A})^{rqr}$ *and* $\underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A},\mathcal{B}) \coloneqq \mathcal{D}(\mathcal{B}^{op}\otimes\mathcal{A})^{\mathcal{B}-\mathrm{perf}}$.

Remark 4.4.2. To be precise, and recalling Remarks 2.9.1 and 2.9.2, $\underline{\mathrm{Hom}}_{\mathrm{Hqe}}(\mathcal{A},\mathcal{B})$ is the subcategory of $H^0\mathcal{C}(\mathcal{B}^{op}\otimes\mathcal{A})[\text{q-iso}^{-1}]$ consisting of right quasirepresentable modules and similarly for $\underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A},\mathcal{B})$.

Remark 4.4.3. The composition functors in $\underline{\mathrm{Hqe}}$ and $\underline{\mathrm{Hmo}}$ are both induced by the derived tensor products of bimodules

$$-\otimes^{\mathbb{L}}_{\mathcal{A}} -\colon \mathcal{D}(\mathcal{B}^{op}\otimes\mathcal{A})\otimes_k \mathcal{D}(\mathcal{C}^{op}\otimes\mathcal{B}) \to \mathcal{D}(\mathcal{C}^{op}\otimes\mathcal{A})$$

which are defined as the derived functors of coends as in Remark 2.4.1.

Remark 4.4.4. One can truncate any bicategory to produce an ordinary category with the same objects and morphism sets given by isomorphism classes of objects in the hom categories. By Theorems 2.7.3 and 2.7.13, Hqe and Hmo are the 1-categorical truncations of $\underline{\mathrm{Hqe}}$ and $\underline{\mathrm{Hmo}}$.

Lemma 4.4.5. *Suppose* $\mathcal{A},\mathcal{B}$ *are small DG-categories,* $X \in \mathcal{D}(\mathcal{B}^{op}\otimes\mathcal{A})^{\mathcal{B}-\mathrm{perf}}$ *and* $M \in \mathcal{D}_{\mathrm{fd}}(\mathcal{B})$. *Then* $X\otimes^{\mathbb{L}}_{\mathcal{B}} M \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$.

Proof. Note that the following diagram of functors commutes up to isomorphism, since it commutes at the underived level

$$\begin{array}{ccc} \mathcal{D}(\mathcal{B}^{op}\otimes\mathcal{A}) & \xrightarrow{-\otimes^{\mathbb{L}}_{\mathbb{B}} M} & \mathcal{D}(\mathcal{A}) \\ \downarrow{\scriptstyle e_a} & & \downarrow{\scriptstyle e_a} \\ \mathcal{D}(\mathcal{B}^{op}) & \xrightarrow{-\otimes^{\mathbb{L}}_{\mathcal{B}} M} & \mathcal{D}(k) \end{array}$$

where the vertical maps are given by evaluation at $a\in\mathcal{A}$. The coYoneda lemma implies that for any $b\in\mathcal{B}$, $\mathcal{B}(-,b)\otimes^{\mathbb{L}}_{\mathcal{B}} M \simeq M(b)\in\mathcal{D}^b(k)$. Since $\mathcal{D}^{\mathrm{perf}}(\mathcal{B}^{op})$ is the thick subcategory generated by representables, it follows that the bottom functor sends perfects to $\mathcal{D}^b(k)$. If $X\in\mathcal{D}(\mathcal{B}^{op}\otimes\mathcal{A})$ is right perfect, then $e_a(X)\in\mathcal{D}^{\mathrm{perf}}(\mathcal{B}^{op})$. So as the diagram commutes, $X\otimes^{\mathbb{L}}_{\mathcal{B}} M(a)\in\mathcal{D}^b(k)$ for all $a\in\mathcal{A}$. Therefore, $X\otimes^{\mathbb{L}}_{\mathcal{B}} M\in\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. ☐

Proposition 4.4.6. *There are pseudofunctors*

$$\mathcal{D}_{\mathrm{fd}}(-)\colon \underline{\mathrm{Hqe}}\to\underline{\mathrm{Hqe}}^{op} \quad \mathcal{D}_{\mathrm{fd}}(-)\colon \underline{\mathrm{Hmo}}\to\underline{\mathrm{Hmo}}^{op}$$

whose truncations agree with $\mathcal{D}_{\mathrm{fd}}(-)\colon \mathrm{Hqe} \to \mathrm{Hqe}^{op}$ *and* $\mathcal{D}_{\mathrm{fd}}(-)\colon \mathrm{Hmo} \to \mathrm{Hmo}^{op}$*.*

Proof. We will prove the statement for Hqe. Define the pseudofunctor on objects by sending $\mathcal{A}$ to $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. First, consider the DG-functor

$$\tilde{\phi}\colon \mathcal{C}(\mathcal{B}^{op} \otimes \mathcal{A}) \to \mathcal{C}(\mathcal{C}_{\mathrm{fd}}(\mathcal{A})^{op} \otimes \mathcal{C}_{\mathrm{fd}}(\mathcal{B}))$$

defined by sending X to the DG-functor

$$\mathcal{C}_{\mathrm{fd}}(\mathcal{A})^{op} \otimes \mathcal{C}_{\mathrm{fd}}(\mathcal{B}) \to \mathcal{C}(k); (M, N) \mapsto \underline{\mathrm{Hom}}_{\mathcal{A}}(M, X \otimes_{\mathcal{B}} N).$$

The universal property of the end and coend construction guarantee this is sufficiently functorial. Then define ϕ as the derived functor of the composition

$$\mathcal{C}(\mathcal{B}^{op} \otimes \mathcal{A}) \xrightarrow{\tilde{\phi}} \mathcal{C}(\mathcal{C}_{\mathrm{fd}}(\mathcal{A})^{op} \otimes \mathcal{C}_{\mathrm{fd}}(\mathcal{B})) \to \mathcal{C}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{op} \otimes \mathcal{D}_{\mathrm{fd}}(\mathcal{B}))$$

where the last map is induced from the inclusions $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{op} \subseteq \mathcal{C}_{\mathrm{fd}}(\mathcal{A})^{op}$ and $\mathcal{D}_{\mathrm{fd}}(\mathcal{B}) \subseteq \mathcal{C}_{\mathrm{fd}}(\mathcal{B})$. We claim ϕ sends right perfects to right quasirepresentables. Indeed, suppose $X \in \mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})$ is right perfect and K-projective and $N \in \mathcal{D}_{\mathrm{fd}}(\mathcal{B})$ so that

$$\phi(X)(-, N) = \underline{\mathrm{Hom}}_{\mathcal{A}}(-, X \otimes_{\mathcal{B}} N) \in \mathcal{D}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{op}).$$

Then $\phi(X)(-, N)$ is right quasirepresentable if $X \otimes_{\mathcal{B}} N \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. Since X and N are K-projective, so is $X \otimes_{\mathcal{B}} N$ by Lemma 3.4.1 of [Gen15]. Since X is right perfect and N has finite-dimensional cohomology, Lemma 4.4.5 implies $X \otimes_{\mathcal{B}} N$ does too. Therefore, $X \otimes_{\mathcal{B}} N \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ and we have functors

$$\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{rqr} \to \mathcal{D}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{op} \otimes \mathcal{D}_{\mathrm{fd}}(\mathcal{B}))^{rqr}.$$

Now, let $X \in \mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{rqr}$ and $Y \in \mathcal{D}(\mathcal{C}^{op} \otimes \mathcal{B})^{rqr}$ which we may assume to be K-projective. By the coYoneda lemma, there are natural isomorphisms in $\mathcal{D}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{op} \otimes \mathcal{D}_{\mathrm{fd}}(\mathcal{C}))$

$$\underline{\mathrm{Hom}}_{\mathcal{B}}(-, Y \otimes_{\mathcal{C}} -) \otimes^{\mathbb{L}}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{B})} \underline{\mathrm{Hom}}_{\mathcal{A}}(-, X \otimes_{\mathcal{B}} -) \simeq \underline{\mathrm{Hom}}_{\mathcal{A}}(-, X \otimes_{\mathcal{B}} Y \otimes_{\mathcal{C}} -).$$

This gives the required functoriality 2-cells for the pseudofunctor $\mathcal{D}_{\mathrm{fd}}(-)$. By taking X and Y to be K-projective resolutions of the diagonal bimodules, the above isomorphisms also give the unitality 2-cells. The above 2-cells satisfy the pseudofunctor axioms by the universal properties of ends and coends.

Now, we check that the above pseudofunctor agrees with the functor defined in Proposition 4.3.13. If $F\colon \mathcal{A} \to \mathcal{B}$ is a DG-functor, then recall $\mathcal{D}_{\mathrm{fd}}(F)$ is the DG-functor

$$\mathcal{D}_{\mathrm{fd}}(\mathcal{B}) \xhookrightarrow{i^{\mathcal{B}}} \mathcal{C}_{\mathrm{fd}}(\mathcal{B}) \xrightarrow{\mathrm{Res}(F)} \mathcal{C}_{\mathrm{fd}}(\mathcal{A}) \xrightarrow{Q^{\mathcal{A}}} \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$$

Now, F is represented in Hqe by $M_F \in \mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})$ where $M_F(b, a) = \mathcal{B}(b, F(a))$. Then

$$\phi(M_F) = \underline{\mathrm{Hom}}_{\mathcal{A}}(-, Q^{\mathcal{B}^{op}\otimes\mathcal{A}}(M_F) \otimes_{\mathcal{B}} -) \in \mathcal{D}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{op} \otimes_{\mathcal{B}} \mathcal{D}_{\mathrm{fd}}(\mathcal{B}))$$

For $N \in \mathcal{D}_{\mathrm{fd}}(\mathcal{B})$, there are natural quasi-isomorphisms

$$Q^{\mathcal{B}^{op}\otimes\mathcal{A}}(M_F) \otimes_{\mathcal{B}} N \simeq M_F \otimes_{\mathcal{B}} N \simeq NF \simeq Q^{\mathcal{A}}(NF) = \mathcal{D}_{\mathrm{fd}}(F)(N) \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$$

where the first follows since N is K-projective and the second by the coYoneda lemma. If $M \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$, then since it is K-projective $\underline{\mathrm{Hom}}_{\mathcal{A}}(M, -)$ preserves quasi-isomorphisms so

$$\phi(M_F)(M, N) = \underline{\mathrm{Hom}}_{\mathcal{A}}(M, Q(M_F) \otimes_{\mathcal{B}} N) \simeq \underline{\mathrm{Hom}}_{\mathcal{A}}(M, \mathcal{D}_{\mathrm{fd}}(F)(N)) = M_{\mathcal{D}_{\mathrm{fd}}(F)}(M, N)$$

where $M_{\mathcal{D}_{\mathrm{fd}}(F)} \in \mathcal{D}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{op} \otimes \mathcal{D}_{\mathrm{fd}}(\mathcal{B}))$ is the bimodule representing $\mathcal{D}_{\mathrm{fd}}(F)$. Then $\phi(M_F)$ and $M_{\mathcal{D}_{\mathrm{fd}}(F)}$ have the same isomorphism class in $\mathcal{D}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{op} \otimes \mathcal{D}_{\mathrm{fd}}(\mathcal{B}))^{rqr}$ and so represent the same morphism in Hqe. Any morphism $f\colon \mathcal{A} \to \mathcal{B}$ in Hqe factors as

$$\mathcal{A} \xrightarrow[\sim]{(M_G)^{-1}} \tilde{\mathcal{A}} \xrightarrow{M_F} \mathcal{B}$$

for DG-functors $G\colon \tilde{\mathcal{A}} \to \mathcal{A}$ and $F\colon \tilde{\mathcal{A}} \to \mathcal{B}$ such that G is a quasi-equivalence. And recall $\mathcal{D}_{\mathrm{fd}}(f)$ is defined by setting $\mathcal{D}_{\mathrm{fd}}(M_G^{-1})$ as the inverse of $\mathcal{D}_{\mathrm{fd}}(G)$ in Hqe. It then follows that $\phi(f) = \mathcal{D}_{\mathrm{fd}}(f) \in$ Hqe, as required. The proof of the statement for Hmo is similar. □

Remark 4.4.7. Recall that $\underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) = \mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{\mathcal{B}-\mathrm{perf}}$ is a triangulated category. We note the induced functor

$$\mathcal{D}_{\mathrm{fd}}(-)\colon \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) \to \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}), \mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$$

is triangulated. Indeed, one can see this directly from its construction as a derived functor or it follows as it can be viewed as H^0 of a DG-functor between pretriangulated DG-categories.

Remark 4.4.8. This 2-categorical language can be used to enhance the notion of an adjoint pair of triangulated functors. This will be considered in Section 5.3.

Chapter 5

Invariants of Reflexive DG-categories

In this chapter, we apply the monoidal characterisation of reflexive DG-categories developed in Chapter 4 to produce more common information between a reflexive DG-category and its dual.

First, we study Hochschild cohomology. We begin by recalling its interpretation in Hmo which appeared in [Toë07]. In Subsection 5.1.2, we show that for a semireflexive DG-category $\mathcal{A}$, there is a quasi-isomorphism of DG-algebras between the Hochschild cohomology of $\mathcal{A}$ and of $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. The inclusion of a proper DG-category into its cohomologically finite modules induces a map on Hochschild cohomology. We show in Subsection 5.1.3 that this too is an isomorphism. A fortiori, we show that any subcategories sitting between the cohomologically finite modules and the perfects have the same Hochschild cohomology. The results of Section 5.1 appeared in [Goo24b].

In Section 5.2, we study derived Picard groups. These are the enhanced symmetry groups of the derived category of a DG-category. We show that for a reflexive DG-category, there is an isomorphism on derived Picard groups between the perfect derived category and the cohomologically finite modules. This is an enhanced version of the isomorphism between triangulated autoequivalence groups which appears in [KS25].

One can enhance the notion of a triangulated adjoint pair between derived categories using the bicategory $\underline{\mathrm{Hmo}}$. In Section 5.3, we show that adjunctions between reflexive DG-categories are uniquely determined by adjunctions between their duals. Finally, in Section 5.4, we study spherical twists. These are certain elements of the derived Picard group originally considered in [ST01]. We use the results of Section 5.3 to show that the isomorphisms on derived Picard groups are compatible with spherical twists.

§ 5.1 | Hochschild Cohomology

Hochschild cohomology is a fundamental invariant appearing in algebra, geometry and topology. We take the following as our definition of the Hochschild cohomology of a DG-category.

Definition 5.1.1. The Hochschild cohomology of a DG-category $\mathcal{A}$ is the cohomology of the complex

$$HH(\mathcal{A}) := \underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A})$$

where $\mathcal{A}$ is viewed as the diagonal bimodule.

We will write $HH^*(\mathcal{A})$ for $H^*(HH(\mathcal{A}))$. Note that $HH(\mathcal{A})$ is an endomorphism object in a DG-category and so is a DG-algebra.

Example 5.1.2. Viewing a k-algebra as a DG-category with one object, this clearly generalises the usual notion of Hochschild cohomology of an algebra. The Hochschild cohomology of a quasicompact separated scheme X is defined as $\underline{\mathbb{R}\mathrm{Hom}}_{X\times X}(\Delta, \Delta)$ where Δ denotes the pushforward of the structure sheaf along the diagonal map. The HKR Theorem of [Mar01],[Căl05],[Mar08] gives a geometric description of the Hochschild cohomology of X. It follows from [Toë07] that the Hochschild cohomology of X coincides with the Hochschild cohomology of the DG-category $\mathcal{D}^{\mathrm{perf}}(X)$.

Remark 5.1.3. There is also a Hochschild chain complex for DG-categories. See [LVdB05].

§ 5.1.1 | Hochschild cohomology via the Morita category

We begin by recalling a result of Toën showing that Hochschild cohomology can be encoded in Hqe.

Theorem 5.1.4 (Corollary 8.1, [Toë07])**.** *If $\mathcal{A}$ is a DG-category, then*

$$HH(\mathcal{A}) \simeq \hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{A})(1_{\mathcal{A}}, 1_{\mathcal{A}})$$

the endomorphism DG-algebra of $1_{\mathcal{A}} \in \hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{A})$.

In Corollary 8.2 in [Toë07], it is shown that $HH(\mathcal{A}) \simeq HH(\mathcal{D}\mathcal{A})$ and so Hochschild cohomology is invariant under Morita equivalence. We will need the small version which is well known and can be proved in the same way.

Corollary 5.1.5. *If $\mathcal{A}$ is a DG-category, then*

$$HH(\mathcal{A}) \simeq HH(\mathcal{D}^{\mathrm{perf}}(\mathcal{A})).$$

Proof. By Corollary 6.6 in [Toë07], the Yoneda embedding $\mathcal{A} \hookrightarrow \mathcal{D}^{\mathrm{perf}}(\mathcal{A}^{op})$ induces a quasi-fully faithful functor

$$\hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{A}) \hookrightarrow \hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{D}^{\mathrm{perf}}(\mathcal{A}^{op})).$$

By Theorem 7.2 (2) in [Toë07], there is an equivalence

$$\hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{D}^{\mathrm{perf}}(\mathcal{A}^{op})) \simeq \hom_{\mathrm{Hqe}}(\mathcal{D}^{\mathrm{perf}}(\mathcal{A}^{op}), \mathcal{D}^{\mathrm{perf}}(\mathcal{A}^{op}))$$

also induced by the Yoneda embedding. Therefore, it induces a quasi-isomorphism of DG-algebras $HH(\mathcal{A}) \simeq HH(\mathcal{D}^{\mathrm{perf}}(\mathcal{A}^{op}))$. Finally, as $\mathcal{D}^{\mathrm{perf}}(\mathcal{A}^{op}) \simeq \mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op}$, we are done, since Hochschild cohomology is clearly invariant under taking opposites. □

Remark 5.1.6. It follows that Hochschild cohomology can be computed in Hmo as well as Hqe:

$$HH(\mathcal{A}) \simeq \hom_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{A})(1_{\mathcal{A}}, 1_{\mathcal{A}}).$$

§ 5.1.2 | Hochschild cohomology and Reflexivity

As in Remark 4.1.11, recall we have natural maps for any $\mathcal{A}, \mathcal{B} \in$ Hmo

$$\hom_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) \to \hom_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}), \mathcal{D}_{\mathrm{fd}}(\mathcal{A})). \tag{5.1}$$

Applying H^0 and restricting to isomorphism classes of objects gives the action of the functor $\mathcal{D}_{\mathrm{fd}}(-)$ on morphisms in Hmo.

Theorem 5.1.7. *If $\mathcal{A}$ is a semireflexive DG-category, then there is a quasi-isomorphism of DG-algebras*

$$HH(\mathcal{A}) \simeq HH(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})).$$

Proof. Since the morphism in Equation (5.1) lifts the action of the functor $\mathcal{D}_{\mathrm{fd}}(-)$, it sends $1_{\mathcal{A}}$ to $1_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}$. It is quasi-fully faithful by Corollary 4.3.19 and so it induces an isomorphism on HH^* by Theorem 5.1.4. □

Remark 5.1.8. We note that similar Hochschild cohomology isomorphisms are already known.

1. In Theorem 4.4.1 of [LVdB05], it is shown that $HH^*(\mathcal{A})$ is isomorphic to $HH^*(\mathcal{B})$ for any subcategory $\mathcal{A}^{op} \subseteq \mathcal{B} \subseteq \mathcal{D}(\mathcal{A})$. In the non-proper case this does not apply to our situation since $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ does not always meet this condition.
2. In [Kel19], it was shown that Koszul duality produces isomorphisms on Hochschild cohomology.

The main point of Theorem 5.1.7 is to demonstrate the power of the monoidal characterisation of reflexivity and not to increase the generality for which these isomorphisms hold.

Remark 5.1.9. We note that Hochschild cohomology of a DG-category admits the higher structure of an E_2 or B_∞ algebra which is exhibited as a Lie bracket on the cohomology. We give a heuristic argument for why the isomorphisms of Theorem 5.1.7 should preserve this structure. Morally, the E_2 structure is a result of the two compatible multiplications on $\hom_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{A})(1_{\mathcal{A}}, 1_{\mathcal{A}})$: one from vertical composition and one from horizontal composition. Since the pseudofunctor $\mathcal{D}_{\mathrm{fd}}(-)$ preserves both of these compositions, it should preserve the E_2-structure. Hence, the equivalence

$HH(\mathcal{A}) \simeq HH\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ should respect the higher structure. To make this precise one should use infinity-categorical foundations where this point of view on the E_2-structure can be more easily encoded.

Example 5.1.10. The analogous result for Hochschild homology of reflexive DG-categories is not true. The algebra $k[x]/x^2$ is reflexive and $\mathcal{D}_{\mathrm{fd}}(k[x]/x^2) \simeq \mathcal{D}^{\mathrm{perf}}(A)$ where A is the DG-algebra $k[t]$ with $|t| = 1$. But one can compute that $HH_*(A)$ is non-zero in negative (homological) degrees. This cannot occur for the Hochschild homology of an algebra.

§ 5.1.3 | Hochschild cohomology for Proper DG-categories

If $\mathcal{A}$ is proper, then it is semireflexive (see Lemma 3.14 of [KS25] or Proposition 5.1.13) and so by Theorem 5.1.7, there is a quasi-isomorphism $HH(\mathcal{A}) \simeq HH(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$. There is also a map $HH(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})) \to HH(\mathcal{A})$ induced by the inclusion $\mathcal{A}^{op} \hookrightarrow \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. We show that this map is also a quasi-isomorphism. In fact, we work more generally with any intermediary category.

Remark 5.1.11. Suppose $\mathcal{A}$ is a proper DG-category and there is a DG-subcategory $\mathcal{A}^{op} \subseteq \mathcal{B} \subseteq \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. The Yoneda embedding of $\mathcal{B}$ restricted to $\mathcal{A}$ factors as

$$\begin{array}{ccc} \mathcal{B}^{op} & \xhookrightarrow{Y} & \mathcal{D}(\mathcal{B}) \\ \uparrow & & \uparrow \\ \mathcal{A} & \xhookrightarrow{Y|_{\mathcal{A}}} & \mathcal{D}_{\mathrm{fd}}(\mathcal{B}) \end{array}$$

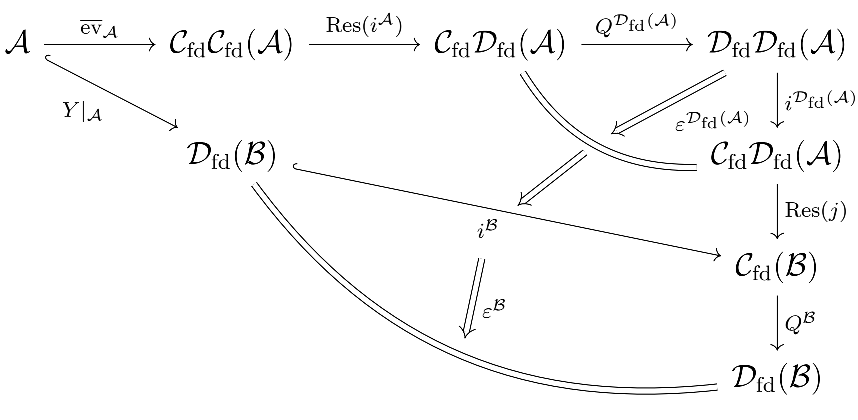
since $\underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(a,-), b) \in \mathcal{D}^b(k)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Lemma 5.1.12. *Suppose $\mathcal{A}$ is a proper DG-category and $\mathcal{A}^{op} \subseteq \mathcal{B} \subseteq \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$, and denote the inclusion by $j\colon \mathcal{B} \hookrightarrow \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. Then the composite*

$$\mathcal{A} \xrightarrow{\mathrm{ev}_{\mathcal{A}}} \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \xrightarrow{\mathcal{D}_{\mathrm{fd}}(j)} \mathcal{D}_{\mathrm{fd}}(\mathcal{B})$$

and the restricted Yoneda embedding $Y|_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{D}_{\mathrm{fd}}(\mathcal{B})$ represent the same morphism in Hqe.

Proof. Consider the diagram below.



There is nothing derived about the pentagon in this diagram. Both DG-functors $\mathcal{A} \to \mathcal{C}_{\mathrm{fd}}(\mathcal{B})$ can be explicitly described. For example, on objects the top route sends $a \in \mathcal{A}$ to the $\mathcal{B}$-module

$$\mathcal{B} \to \mathcal{D}^b(k); b \mapsto b(a); \beta \mapsto \beta_a$$

The bottom sends a to

$$\mathcal{B} \to \mathcal{D}^b(k); b \mapsto \mathcal{B}(\mathcal{A}(a,-),b); \beta \mapsto \mathcal{B}(\mathcal{A}(a,-),b);$$

By the $\mathcal{C}(k)$-enriched Yoneda lemma, there is a natural isomorphism between these two DG-functors $\mathcal{A} \to \mathcal{C}_{\mathrm{fd}}(\mathcal{B})$ as indicated. Since $\varepsilon^{\mathcal{B}}$ is a pointwise quasi-isomorphism, it remains to see that $Q^{\mathcal{B}} \operatorname{Res}(j)\varepsilon^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}$ is too. This holds since $\varepsilon^{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}$ is a pointwise quasi-isomorphism which is preserved by $\operatorname{Res}(j)$ and so is sent to a pointwise quasi-isomorphism by $Q^{\mathcal{B}}$. So by Lemma 4.3.10 we are done. □

We recover Lemma 3.14 of [KS25] by taking $\mathcal{B} = \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$.

Proposition 5.1.13. *Any proper DG-category is semireflexive.*

Theorem 5.1.14. *Suppose $\mathcal{A}$ is a proper DG-category and $\mathcal{A}^{op} \subseteq \mathcal{B} \subseteq \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. Then $HH^*(\mathcal{A}) \simeq HH^*(\mathcal{B})$.*

Proof. Denote the inclusion $j \colon \mathcal{B} \hookrightarrow \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. Consider the diagram

$$\begin{array}{ccccc}
\hom_{\mathrm{Hqe}}(\mathcal{A},\mathcal{A}) & \hookrightarrow & \hom_{\mathrm{Hqe}}(\mathcal{A},\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})) & \xrightarrow{\sim} & \hom_{\mathrm{Hqe}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}),\mathcal{D}_{\mathrm{fd}}(\mathcal{A})) \\
& \searrow & \downarrow{\scriptstyle \hom_{\mathrm{Hqe}}(\mathcal{A},\mathcal{D}_{\mathrm{fd}}(j))} & & \downarrow{\scriptstyle \hom_{\mathrm{Hqe}}(j,\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))=:j^*} \\
& & \hom_{\mathrm{Hqe}}(\mathcal{A},\mathcal{D}_{\mathrm{fd}}(\mathcal{B})) & \xrightarrow{\sim} & \hom_{\mathrm{Hqe}}(\mathcal{B},\mathcal{D}_{\mathrm{fd}}(\mathcal{A})) \\
& & & & \uparrow{\scriptstyle j_*:=\hom_{\mathrm{Hqe}}(\mathcal{B},j)} \\
& & & & \hom_{\mathrm{Hqe}}(\mathcal{B},\mathcal{B})
\end{array}$$

where the triangle is $\hom_{\mathrm{Hqe}}(\mathcal{A},-)$ applied to Lemma 5.1.12. The square commutes by naturality of the adjunction. From this we see that j^* is quasi-fully faithful when restricted to the image of $\hom_{\mathrm{Hqe}}(\mathcal{A},\mathcal{A})$. The long composite on the top is the map in Equation 5.1 and so $1_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}$ is in the image of $\hom_{\mathrm{Hqe}}(\mathcal{A},\mathcal{A})$. It follows that the map

$$j^* \colon HH(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})) \xrightarrow{\sim} \hom_{\mathrm{Hqe}}(\mathcal{B},\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))(j^*(1_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}), j^*(1_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}))$$

is a quasi-isomorphism. By Corollary 6.6 in [Toë07], the map

$$j_* : HH^*(\mathcal{B}) \xrightarrow{\sim} \hom_{\mathrm{Hqe}}(\mathcal{B},\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))(j_*(1_{\mathcal{B}}), j_*(1_{\mathcal{B}}))$$

is a quasi-isomorphism. Applying H^0 to $\hom_{\mathrm{Hqe}}(-,-)$ and taking isomorphism classes of objects gives $\operatorname{Hom}_{\mathrm{Hqe}}(-,-)$ and so $j^*(1_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}) \simeq j \simeq j_*(1_{\mathcal{B}}) \in H^0 \hom_{\mathrm{Hqe}}(\mathcal{B},\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$. Therefore, there is a quasi-isomorphism $HH(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})) \simeq HH^*(\mathcal{B})$. By Proposition

5.1.13, $\mathcal{A}$ is semireflexive so by Theorem 5.1.7, $HH(\mathcal{A}) \simeq HH(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$ and we are done. □

Corollary 5.1.15. *If A is a finite-dimensional DG-algebra, then*

$$HH(A) \simeq HH(\mathcal{D}_{\mathrm{sf}}(A)) \simeq HH(A^!).$$

Proof. This follows from Theorem 5.1.14 by taking $\mathcal{B} = \mathcal{D}_{\mathrm{sf}}(A) \simeq \mathcal{D}^{\mathrm{perf}}(A^!)$. □

Remark 5.1.16. Theorem 5.1.14 also follows from Theorem 4.4.1 in [LVdB05].

Remark 5.1.17. The ideas in the proof of Theorem 5.1.14 should be compared to the limited functoriality of Hochschild cohomology. Keller showed that Hochschild cohomology is functorial with respect to quasi-fully faithful DG-functors in [Kel03]. This can also be seen from Toën's description. If $F \colon \mathcal{A} \to \mathcal{B} \in \mathrm{Hqe}$, there is a cospan in Hqe

$$\hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{A}) \xrightarrow{F_*} \hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B}) \xleftarrow{F^*} \hom_{\mathrm{Hqe}}(\mathcal{B}, \mathcal{B})$$

If F is quasi-fully faithful, then by Corollary 6.6 in [Toë07], F_* is quasi-fully faithful. There are isomorphisms $F_*(1_{\mathcal{A}}) \simeq F \simeq F^*(1_{\mathcal{B}}) \in H^0 \hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B})$. Hence, there is a cospan of DG-algebras

$$\hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{A})(1_{\mathcal{A}}, 1_{\mathcal{A}}) \xrightarrow{\sim} \hom_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{B})(F, F) \leftarrow \hom_{\mathrm{Hqe}}(\mathcal{B}, \mathcal{B})(1_{\mathcal{B}}, 1_{\mathcal{B}})$$

which is a morphism from $HH(\mathcal{B}) \to HH(\mathcal{A})$ in Hqe. As it depends on a choice of isomorphism $F_*(1_{\mathcal{A}}) \simeq F$, it is not clear this construction is functorial. This can presumably be fixed by working with the additional structure provided by the bicategory $\underline{\mathrm{Hqe}}$ but we will not pursue this here. We note only that, whatever choices are made, it follows from the proof of Theorem 5.1.14, that for a proper DG-category $\mathcal{A}$, the induced map $HH(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})) \to HH(\mathcal{A})$ is a quasi-isomorphism.

§ 5.2 | Derived Picard Groups

The derived Picard group of a DG-category is the enhanced version of the group of triangulated autoequivalences of its derived category. In Corollary 3.16 of [KS25], it was shown that $\mathcal{D}^{\mathrm{perf}}(\mathcal{A})$ and $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ have isomorphic triangulated autoequivalence groups for reflexive DG-categories. An enhanced version of this theorem follows immediately from the monoidal characterisation of reflexive DG-categories.

Definition 5.2.1. The derived Picard group $\mathrm{DPic}(\mathcal{A})$ of a DG-category $\mathcal{A}$ is the group of automorphisms of $\mathcal{A}$ in Hmo.

Remark 5.2.2. Using Theorem 2.7.13, we see that $\mathrm{DPic}(\mathcal{A})$ is the group of invertible objects the monoidal category $\mathcal{D}(\mathcal{A}^e)$ up to isomorphism.

Remark 5.2.3. As it is functorial, $\mathcal{D}_{\mathrm{fd}}(-)$ restricts to $\mathrm{DPic}(\mathcal{A}) \to \mathrm{DPic}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))^{op}$ for any $\mathcal{A}, \mathcal{B} \in \mathrm{Hmo}$.

Theorem 5.2.4. *If $\mathcal{A}$ is a reflexive (semireflexive) DG-category, the group homomorphism*

$$\mathrm{DPic}(\mathcal{A}) \to \mathrm{DPic}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))^{op}$$

is an isomorphism (monomorphism).

Proof. If $\mathcal{A}$ is semireflexive, then by Corollary 4.3.19, the induced map is a monomorphism. If $\mathcal{A}$ is reflexive and $F \in \mathrm{DPic}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$, then by Corollary 4.3.19, $F = \mathcal{D}_{\mathrm{fd}}(G)$ for some $G \in \mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{A})$. Since functors preserve isomorphisms, $\mathcal{D}_{\mathrm{fd}}(F) \in \mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A}), \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$ is an automorphism. As $\mathcal{A}$ is reflexive and by Proposition 4.1.2, under the isomorphism $\mathcal{A} \simeq \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$, G corresponds to $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(F)$. Therefore, $G \in \mathrm{DPic}(\mathcal{A})$, as required. □

Remark 5.2.5. One can also consider a categorified version of the derived Picard group. If $\underline{\mathrm{DPic}}(\mathcal{A})$ denotes the full DG-subcategory of $\mathrm{hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{A})$ with objects corresponding to automorphisms, then $\mathrm{DPic}(\mathcal{A})$ can be identified with the isomorphism classes of objects in $H^0(\underline{\mathrm{DPic}}(\mathcal{A}))$. The proof of Theorem 5.2.4 shows that if $\mathcal{A}$ is reflexive then there is a quasi-equivalence of DG-categories $\underline{\mathrm{DPic}}(\mathcal{A}) \simeq \underline{\mathrm{DPic}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$ which induces an anti-monoidal equivalence on H^0.

§ 5.3 | Adjoint Functors

In this section, we study the behaviour of adjunctions after applying $\mathcal{D}_{\mathrm{fd}}$. We show that an adjunction between reflexive DG-categories is determined by an adjunction between their categories of cohomologically finite modules.

Definition 5.3.1. We say that $L \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B})$ is left adjoint to $R \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{B}, \mathcal{A})$ if L is left adjoint to R in the bicategory $\underline{\mathrm{Hmo}}$ i.e. there are 2-morphisms $\eta\colon 1_{\mathcal{A}} \to RL \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{A})$ and $\varepsilon\colon LR \to 1_{\mathcal{B}} \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{B}, \mathcal{B})$ such that the following compositions

$$L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon_L} L \quad \text{and} \quad R \xrightarrow{\eta_R} RLR \xrightarrow{R\varepsilon} R$$

are both the identity (up to the unitor and associator 2-cells).

Remark 5.3.2. We note that if f, f' are isomorphic 1-morphisms in a bicategory, then f admits a left or right adjoint if and only if f' does. Since the morphisms in Hmo are isomorphism classes of 1-morphisms in $\underline{\mathrm{Hmo}}$, it makes sense to say that a morphism in Hmo admits an adjoint.

Remark 5.3.3. It follows from Proposition 3.5.5 in [Gen15] that $M \in \mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B})$ admits a left adjoint if and only if its representative in $\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{\mathcal{B}\text{–perf}}$ is also perfect as a left $\mathcal{A}$-module. Furthermore, the left adjoint can be described as a dual over $\mathcal{A}$. There is also a characterisation for existence of a right adjoint.

Remark 5.3.4. Suppose that $L \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B})$ and $R \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{B}, \mathcal{A})$ are an adjoint pair. Then they induce an adjoint pair of triangulated functors between $\mathcal{D}^{\mathrm{perf}}(\mathcal{A})$

and $\mathcal{D}^{\mathrm{perf}}(\mathcal{B})$ and so they can be thought of as enhanced adjoints. See Corollary 2.2 (1) in [AL17].

We note that $\mathcal{D}_{\mathrm{fd}}(-)$ preserves adjoints.

Lemma 5.3.5. *If $L \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B})$ is left adjoint to $R \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{B}, \mathcal{A})$, then $\mathcal{D}_{\mathrm{fd}}(R) \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}), \mathcal{D}_{\mathrm{fd}}(\mathcal{B}))$ is left adjoint to $\mathcal{D}_{\mathrm{fd}}(L) \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}), \mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$. If η is the unit and ε the counit of $L \dashv R$, then the unit and counit of $\mathcal{D}_{\mathrm{fd}}(R) \dashv \mathcal{D}_{\mathrm{fd}}(L)$ are given by $\mathcal{D}_{\mathrm{fd}}(\varepsilon)$ and $\mathcal{D}_{\mathrm{fd}}(\eta)$ (up to unitor and associator 2-cells).*

Proof. Applying the pseudofunctor $\mathcal{D}_{\mathrm{fd}}(-)$ to the triangle identities in Definition 5.3.1 implies the required triangle identities commute. □

Theorem 5.3.6. *If $\mathcal{A}$ and $\mathcal{B}$ are reflexive DG-categories, then $M \in \mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B})$ admits a left (right) adjoint if and only if $\mathcal{D}_{\mathrm{fd}}(M)$ admits a right (left) adjoint.*

Proof. The forward direction is Lemma 5.3.5. If $\mathcal{D}_{\mathrm{fd}}(M)$ admits a right adjoint, then $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(M)$ admits a left adjoint by Lemma 5.3.5. Since $\mathcal{A}$ and $\mathcal{B}$ are reflexive, M corresponds to $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(M)$ under the isomorphism

$$\mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{A}, \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}\mathcal{B}).$$

It follows that M also admits a left adjoint. □

Definition 5.3.7. Let $\mathrm{Hom}^{L}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) \subseteq \mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B})$ denote the subset consisting of left adjoints and $\mathrm{Hom}^{R}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) \subseteq \mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B})$ denote the subset consisting of right adjoints.

Corollary 5.3.8. *If $\mathcal{A}$ and $\mathcal{B}$ are reflexive DG-categories, there are isomorphisms*

$$\mathrm{Hom}^{L}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}^{R}_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}), \mathcal{D}_{\mathrm{fd}}(\mathcal{A})),$$
$$\mathrm{Hom}^{R}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}^{L}_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}), \mathcal{D}_{\mathrm{fd}}(\mathcal{A})).$$

Proof. This follows from Theorem 5.3.6 and Corollary 4.3.19. □

Remark 5.3.9. Corollary 5.3.8 should be compared to Theorem 3.17 of [KS25]. There it is shown that, for reflexive DG-categories, there are bijections between left admissible subcategories of $\mathcal{A}$ and right admissible subcategories of $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. Left (right) admissible subcategories are triangulated fully faithful functors admitting left (right) adjoints.

Remark 5.3.10. Similarly to Remark 5.2.5, a categorified version of Corollary 5.3.8 holds. The isomorphisms in the statement are induced by quasi-equivalences between DG-subcategories of $\mathrm{hom}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B})$ and of $\mathrm{hom}_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}), \mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$ consisting of left and right adjoints. Note by Remark 5.3.3, these DG-subcategories admit explicit descriptions.

§ 5.4 | Spherical Twists

Spherical twists of derived categories were introduced in [ST01] in order to construct autoequivalences of the bounded derived category of coherent sheaves. A more general and DG-enhanced framework was given in [AL17] which produces elements of $\mathrm{DPic}(\mathcal{A})$ and $\mathrm{DPic}(\mathcal{B})$ from certain DG-functors $\mathcal{A} \to \mathcal{B}$. In this section, we show that the isomorphisms of Theorem 5.2.4 are compatible with spherical twists.

Remark 5.4.1. Recall from Section 4.4 that there is a bicategory $\underline{\mathrm{Hmo}}$ whose truncation is Hmo. The morphism categories $\underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A},\mathcal{B})$ are given by $\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{A})^{\mathcal{B}-\mathrm{perf}}$. As this is a triangulated category by Remark 2.7.14, it makes sense to speak of the cone of a 2-morphism in $\underline{\mathrm{Hmo}}$.

Definition 5.4.2. Suppose $\mathcal{A},\mathcal{B}$ are DG-categories and $S \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A},\mathcal{B})$ admits a right adjoint R and a left adjoint L (in the sense of Definition 5.3.1).

1. The twist of S is the 1-morphism $T \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{B},\mathcal{B})$ given by the cone of the counit $SR \to 1_{\mathcal{B}}$.
2. The dual twist of S is the 1-morphism $T' \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{B},\mathcal{B})$ given by the cocone of the unit $1_{\mathcal{B}} \to SL$.
3. The cotwist of S is the 1-morphism $F \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A},\mathcal{A})$ given by the cocone of the unit $1_{\mathcal{A}} \to RS$.
4. The dual cotwist of S is the 1-morphism $F \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A},\mathcal{A})$ given by the cone of the counit $LS \to 1_{\mathcal{A}}$.

We say that S is a spherical functor if T and F are both isomorphisms in Hmo.

Remark 5.4.3. The cone of a morphism in a triangulated category is defined up to non-unique isomorphism. In [AL17], explicit descriptions of the adjoints, units and counits are given which avoids this issue. However, since we are only interested in the spherical twists as elements of $\mathrm{DPic}(\mathcal{A})$, this definition suffices for us.

Remark 5.4.4. It was shown in Proposition 5.3 in [AL17] that (up to isomorphism of 1-cells), T is right adjoint to T' and F is right adjoint to F' so S is a spherical functor if and only if F' and T' are isomorphisms.

Remark 5.4.5. The original notion of a spherical object, introduced in [ST01], is the same information as a spherical functor out of $\mathcal{D}^b(k)$. Spherical functors with more general domains have also been studied. For example, see [HK25], [God24], [Add16] and the references therein.

Lemma 5.4.6. *Suppose $\mathcal{A}$, $\mathcal{B}$ are DG-categories and $S \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A},\mathcal{B})$ admits left and right adjoints denoted L and R. Then $\mathcal{D}_{\mathrm{fd}}(R)$ is left adjoint to $\mathcal{D}_{\mathrm{fd}}(S)$ and $\mathcal{D}_{\mathrm{fd}}(L)$ is right adjoint to $\mathcal{D}_{\mathrm{fd}}(S)$. If T and F are the twist and cotwist of S, then $\mathcal{D}_{\mathrm{fd}}(T)$ is the dual cotwist and $\mathcal{D}_{\mathrm{fd}}(F)$ is the dual twist of $\mathcal{D}_{\mathrm{fd}}(S)$.*

Proof. The existence of the adjunctions is Lemma 5.3.5 and furthermore if η and ε denote the unit and counit of the adjunction $S \dashv R$, then $\mathcal{D}_{\mathrm{fd}}(\varepsilon)$ is the unit and $\mathcal{D}_{\mathrm{fd}}(\eta)$ is the counit the adjunction of $\mathcal{D}_{\mathrm{fd}}(R) \dashv \mathcal{D}_{\mathrm{fd}}(S)$. By Remark 4.4.7, there is a triangle

$$\mathcal{D}_{\mathrm{fd}}(R)\mathcal{D}_{\mathrm{fd}}(S) \simeq \mathcal{D}_{\mathrm{fd}}(SR) \xrightarrow{\mathcal{D}_{\mathrm{fd}}(\eta)} 1_{\mathcal{D}_{\mathrm{fd}}(\mathcal{B})} \to \mathcal{D}_{\mathrm{fd}}(T)$$

in $\underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}), \mathcal{D}_{\mathrm{fd}}(\mathcal{B}))$. Therefore, $\mathcal{D}_{\mathrm{fd}}(T)$ is the dual cotwist of $\mathcal{D}_{\mathrm{fd}}(S)$. The argument that $\mathcal{D}_{\mathrm{fd}}(F)$ is the dual twist of $\mathcal{D}_{\mathrm{fd}}(S)$ is similar. □

Theorem 5.4.7. *Suppose $\mathcal{A}$, $\mathcal{B}$ are DG-categories and $S \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B})$ admits left and right adjoints. If S is a spherical functor, then $\mathcal{D}_{\mathrm{fd}}(S)$ is a spherical functor. If $\mathcal{A}$ and $\mathcal{B}$ are reflexive, then the converse is true.*

Proof. If S is a spherical functor, then its twist T and cotwist F are isomorphisms in Hmo and so $\mathcal{D}_{\mathrm{fd}}(T)$ and $\mathcal{D}_{\mathrm{fd}}(F)$ are too. By Lemma 5.4.6, these are the dual twist and dual cotwist of $\mathcal{D}_{\mathrm{fd}}(S)$. So by Remark 5.4.4 we are done. If $\mathcal{A}$ and $\mathcal{B}$ are reflexive, then $\mathcal{D}_{\mathrm{fd}}(F)$ and $\mathcal{D}_{\mathrm{fd}}(T)$ are isomorphisms if and only if F and T are isomorphisms by Theorem 5.2.4. So the converse holds. □

Definition 5.4.8. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be DG-categories.

1. We say that $T \in \mathrm{DPic}(\mathcal{B})$ is an $\mathcal{A}$-spherical (dual) twist if there is a spherical functor $S \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{A}, \mathcal{B})$ such that T is the (dual) twist of S.

2. We say that $F \in \mathrm{DPic}(\mathcal{B})$ is a $\mathcal{C}$-spherical (dual) cotwist if there is a spherical functor $S \in \underline{\mathrm{Hom}}_{\mathrm{Hmo}}(\mathcal{B}, \mathcal{C})$ such that F is the (dual) cotwist of S.

3. Let $\mathrm{STwi}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathrm{DPic}(\mathcal{B})$ denote the subgroup of $\mathrm{DPic}(\mathcal{B})$ generated by $\mathcal{A}$-spherical twists (or equivalently by their inverses: the $\mathcal{A}$-spherical dual twists).

4. Let $\mathrm{SCoTwi}_{\mathcal{C}}(\mathcal{B}) \subseteq \mathrm{DPic}(\mathcal{B})$ denote the subgroup of $\mathrm{DPic}(\mathcal{B})$ generated by the $\mathcal{C}$-spherical cotwists (or equivalently by their inverses: the $\mathcal{C}$-spherical dual cotwists).

Remark 5.4.9. It was shown in [Seg18], that any element of $\mathrm{DPic}(\mathcal{B})$ can be written as the spherical twist associated to a spherical functor. Therefore, $\mathrm{DPic}(\mathcal{B})$ can be written as a union of subgroups of the form $\mathrm{STwi}_{\mathcal{A}}(\mathcal{B})$.

Lemma 5.4.10. *If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are DG-categories, then the group homomorphism*

$$\mathrm{DPic}(\mathcal{B}) \to \mathrm{DPic}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}))^{op}$$

sends $\mathcal{A}$-spherical twists to $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$-spherical dual cotwists and it sends $\mathcal{C}$-spherical cotwists to $\mathcal{D}_{\mathrm{fd}}(\mathcal{C})$-spherical dual twists. Hence, it restricts to group homomorphisms

$$\mathrm{STwi}_{\mathcal{A}}(\mathcal{B}) \to \mathrm{SCoTwi}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}))^{op},$$
$$\mathrm{SCoTwi}_{\mathcal{C}}(\mathcal{B}) \to \mathrm{STwi}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{C})}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}))^{op}.$$

Proof. This follows immediately from Lemma 5.4.6. □

Theorem 5.4.11. *There are group isomorphisms*

$$\mathrm{STwi}_{\mathcal{A}}(\mathcal{B}) \xrightarrow{\sim} \mathrm{SCoTwi}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}))^{op},$$
$$\mathrm{SCoTwi}_{\mathcal{C}}(\mathcal{B}) \xrightarrow{\sim} \mathrm{STwi}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{C})}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}))^{op}.$$

for any reflexive DG-categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

Proof. There are isomorphisms

$$\mathrm{DPic}(\mathcal{B}) \xrightarrow{\sim} \mathrm{DPic}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}))^{op} \xrightarrow{\sim} \mathrm{DPic}(\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})) \simeq \mathrm{DPic}(\mathcal{B})$$

where the first two follow from applications of Theorem 5.2.4 to $\mathcal{B}$ and to $\mathcal{D}_{\mathrm{fd}}(\mathcal{B})$ and the last follows since there is an isomorphism $\mathcal{B} \simeq \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})$ in Hmo. The long composite is the identity. Indeed, the composite of the first two maps sends an automorphism T to $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(T)$. Then naturality of the evaluation map $\mathcal{B} \to \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})$ implies the last map sends $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(T)$ to T. By Lemma 5.4.10 and since Morita equivalences preserve spherical twists (see [AL17]), the above composite restricts as follows.

$$\begin{aligned}\mathrm{STwi}_{\mathcal{A}}(\mathcal{B}) \hookrightarrow \mathrm{SCoTwi}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}))^{op} &\hookrightarrow \mathrm{STwi}_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})) \\ &\simeq \mathrm{STwi}_{\mathcal{A}}(\mathcal{B})\end{aligned}$$

However, since the composite is the identity, it follows that these maps are all isomorphisms, as required. The proof for cotwists is similar. □

Remark 5.4.12. As in Remarks 5.2.5 and 5.3.10, we note the group isomorphisms in Theorem 5.4.11 are induced by quasi-equivalences between DG-subcategories of $\underline{\mathrm{DPic}}(\mathcal{B})$ and $\underline{\mathrm{DPic}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{B}))^{op}$.

Chapter 6

Producing Reflexive DG-categories

This chapter is about examples of reflexive DG-categories in algebra, geometry and topology. We begin by recalling what is already known. In [KS25], two families of examples are shown to be reflexive over perfect fields: proper connective DG-algebras (Proposition 6.9) and projective schemes (Proposition 6.1). A more general version of the latter example, with a stricter field assumption, appeared in Remark 1.2.6 of [BNP17]. They showed that proper algebraic spaces over characteristic zero fields are reflexive. They use an infinity categorical framework and work relative to a perfect derived stack. An example of a different flavour was considered in [LU22]. Theorem 6.11 of loc. cit. shows that Fukaya categories of Milnor fibres of weighted homogeneous isolated hypersurface singularities satisfying some additional conditions are reflexive. In this example, the cohomologically finite modules correspond to the wrapped Fukaya category. To the best of the author's knowledge, these were all of the known examples prior to the work of this thesis. There are also the less interesting examples: opposites and duals of reflexive DG-categories, and smooth and proper DG-categories (see Remark 6.1.4).

We begin with some general criteria for a proper DG-category to be reflexive. One is related to the left homologically finite objects, and another is based on the Koszul dual. In Section 6.2, we show that proper coconnective DG-algebras with semisimple zeroth cohomology are reflexive. Along the way, we prove some results of independent interest about Koszul duality between connective and coconnective DG-algebra. We deduce reflexivity for cochain algebras on finite CW-complexes and certain subcategories of derived categories of Abelian categories.

In Section 6.3, we compare the theory of reflexive DG-categories to that of approximable triangulated categories. This theory was developed by Neeman to abstract the duality between the perfect and bounded derived categories of schemes at the triangulated level. In this theory, the bounded derived category is described as the completion of the perfects in an analogous way to the completion of a metric space. For a proper DG-category whose derived category is approximable, we identify its completion with the cohomologically finite modules. Combined with a representability theorem of Neeman, this gives a strong generation condition for such a DG-category to be reflexive. As applications, we show that proper connective DG-algebras and proper schemes are reflexive over any field and show that Azumaya algebras over proper schemes are reflexive.

Finally in Section 6.4, we study how reflexivity behaves under gluing via semiorthogonal decompositions. Semiorthogonal decompositions are a fundamental way of producing new noncommutative schemes and it is natural to consider what properties gluing preserves. It is clear from the monoidal characterisation of reflexivity that if a reflexive DG-category admits a semi-orthogonal decomposition, then the pieces must also be reflexive. We prove a partial converse to this. We show that if a semireflexive DG-category admits a semiorthogonal decomposition into reflexive pieces, then it too is reflexive. We give a small example which demonstrates how this can be used to build new reflexive DG-categories. However, the main application of this gluing result appears in [BGO25] where we show that all proper graded gentle algebras are reflexive. This in turn means that almost all Fukaya categories associated to a marked surface are reflexive.

We note that there are plenty of non-examples: any k-algebra which has no finite-dimensional modules cannot be reflexive or by [BGO25] any commutative Noetherian ring local ring which is not complete. However, we are unaware of any proper DG-category which is not reflexive.

The results of Section 6.3 appeared in the preprint [Goo24a], where the assumption of working over a field is relaxed to a commutative Noetherian ring. In the joint project [BGO25], we have made a connection between reflexivity and *dc*-completion in the sense of [DGI06]. This allows us to generalise the properness assumptions appearing in this chapter and show that many other familiar examples of DG-categories appearing in algebra and topology are reflexive. The corresponding generalised versions of the results in Section 6.2 (using the same arguments) appear in [BGO25]. As mentioned, the results of Section 6.4 also appear in [BGO25].

§ 6.1 | Reflexivity for Proper DG-categories

We begin by giving some reflexivity criteria for proper DG-categories. Some of these arguments first appeared for finite-dimensional DG-algebras in [Goo24b]. First, we note a relationship between reflexivity and homologically finite objects as in Definition 3.2.5.

Proposition 6.1.1. *If $\mathcal{A}$ is a proper reflexive DG-category, then $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{\mathrm{lhf}} = \mathcal{D}^{\mathrm{perf}}(\mathcal{A})$.*

Proof. Since $\underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}(a,-), M) \simeq M(a)$, we have that $\mathcal{D}^{\mathrm{perf}}(\mathcal{A}) \subseteq \mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{\mathrm{lhf}}$. If $M \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{\mathrm{lhf}}$, then $\underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(M, -) \in \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. By reflexivity, there is an $M' \in$

$\mathcal{D}^{\mathrm{perf}}(\mathcal{A})$ and equivalences $\underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(M,-) \simeq \underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(M',-) \in \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. Taking cohomology implies there is an isomorphism $\mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(M',-) \simeq \mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}(M,-)$ of functors on $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. Since $M, M' \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$, the ordinary Yoneda lemma implies that $M \simeq M' \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. Therefore, M is perfect. □

We give a condition for the converse to hold.

Proposition 6.1.2. *Suppose $\mathcal{A}$ is proper and $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{\mathrm{lhf}} = \mathcal{D}^{\mathrm{perf}}(\mathcal{A})$. Then $\mathcal{A}$ is reflexive if and only if $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \subseteq \mathcal{D}^{\mathrm{perf}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ as subcategories of $\mathcal{D}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} & & \mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{op} & \xrightarrow{\sim} & \mathcal{D}^{\mathrm{perf}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})) & & \\ & & \uparrow & & \uparrow & & \\ \mathcal{A} & \hookrightarrow & \mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op} & \xrightarrow{\sim} & \mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})) & \hookrightarrow & \mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})) \end{array}$$

where the top map is the Yoneda embedding and $\mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}$ indicates the intersection of the perfects and the cohomologically finite modules, as in Definition 3.4.2. We claim that the image of $\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op}$ under the equivalence is $\mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$ as shown in the diagram. As the top map is a contravariant equivalence, it restricts to a contravariant equivalence between $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{\mathrm{lhf}}$ and $\mathcal{D}^{\mathrm{perf}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))^{\mathrm{rhf}}$. Note that for any DG-category $\mathcal{B}$, $\mathcal{D}^{\mathrm{perf}}(\mathcal{B})^{\mathrm{rhf}} = \mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}(\mathcal{B})$. By Lemma 5.1.12, the long composite on the bottom coincides with the evaluation map for $\mathcal{A}$. Therefore, $\mathcal{A}$ is reflexive if and only if $\mathcal{D}^{\mathrm{perf}}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) = \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. This occurs exactly when $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \subseteq \mathcal{D}^{\mathrm{perf}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. □

Remark 6.1.3. Recall the condition that $\mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}\mathcal{A}) \subseteq \mathcal{D}^{\mathrm{perf}}\mathcal{D}_{\mathrm{fd}}\mathcal{A}$ appeared in Section 3.4 for DG-algebras and is one half of the condition that $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ is hfd-closed in the sense of [KS25].

Remark 6.1.4. If $\mathcal{D}^{\mathrm{perf}}(\mathcal{A}) = \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$, then $\mathcal{A}$ is reflexive. Indeed, this follows from Proposition 6.1.2 or one can check it directly. In particular, any smooth (or more generally regular) and proper DG-category is reflexive (using e.g. Lemma 3.8 of [KS25]). Alternatively, the smooth and proper DG-categories are exactly the dualisable objects in Hmo (see Theorem 1.43 in [Tab15]) and so they are reflexive by Proposition 4.1.5.

Using a similar argument, we see that reflexivity is related to generation of the cohomologically finite modules over the Koszul dual. Also, compare to Theorem 3.4.4.

Proposition 6.1.5. *Suppose A is a proper DG-algebra and $\mathcal{D}_{\mathrm{fd}}(A) = \mathrm{thick}_A(K)$ for some $K \in \mathcal{D}_{\mathrm{fd}}(A)$ and let $A^! := \underline{\mathbb{R}\mathrm{Hom}}_A(K,K)$. Then A is reflexive if and only if $\mathrm{thick}_{A^!}(K) = \mathcal{D}_{\mathrm{fd}}(A^!)$.*

Proof. Consider the following diagram.

$$\begin{array}{ccccc}
\mathcal{D}_{\mathrm{fd}}(A)^{op} & \xrightarrow[\sim]{\underline{\mathbb{R}\mathrm{Hom}}_A(-,K)} & \mathcal{D}^{\mathrm{perf}}(A^!) & & \\
\uparrow & & \uparrow & & \\
\mathcal{D}^{\mathrm{perf}}(A)^{op} & \xrightarrow{\sim} & \mathrm{thick}_{A^!}(K) & \hookrightarrow & \mathcal{D}_{\mathrm{fd}}(A^!)
\end{array}$$

The top map is an equivalence by Remark 2.6.5. The equivalence restricts as shown since it sends $A \mapsto K$. The equivalence $\mathcal{D}_{\mathrm{fd}}(A)^{op} \simeq \mathcal{D}^{\mathrm{perf}}(A^!)$ induces an equivalence $\mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(A)) \simeq \mathcal{D}_{\mathrm{fd}}(A^!); F \mapsto F(K)$. Therefore, A is reflexive if and only if the following composite is essentially surjective.

$$\mathcal{D}^{\mathrm{perf}}(A)^{op} \to \mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(A)) \simeq \mathcal{D}_{\mathrm{fd}}(A^!); M \mapsto \underline{\mathbb{R}\mathrm{Hom}}_A(M, K)$$

Therefore, A is reflexive if and only if the bottom map in the above diagram is essentially surjective. This occurs exactly when $\mathrm{thick}_{A^!}(K) = \mathcal{D}_{\mathrm{fd}}(A^!)$. □

§ 6.1.1 | Reflexivity Conditions for Finite-dimensional DG-algebras

Recall from Section 3.4 that if A is a finite-dimensional DG-algebra, then we let

$$A^! = \underline{\mathbb{R}\mathrm{Hom}}_A(A/J_-, A/J_-)$$

where J_- is Orlov's DG-radical as in Definition 3.1.10. We have the following condition for a finite-dimensional DG-algebra to be reflexive.

Proposition 6.1.6. *Let A be a finite-dimensional DG-algebra satisfying the condition* $\mathrm{thick}(A/J_-) = \mathcal{D}_{\mathrm{fd}}(A)$. *Then A is reflexive if and only if $\mathcal{D}_{\mathrm{fd}}(A^!) \subseteq \mathcal{D}^{\mathrm{perf}}(A^!)$ if and only if* $\mathrm{thick}_{A^!}(A/J_-) = \mathcal{D}_{\mathrm{fd}}(A^!)$.

Proof. By Corollary 3.2.6, $\mathcal{D}_{\mathrm{fd}}(A)^{\mathrm{lhf}} = \mathcal{D}^{\mathrm{perf}}(A)$. By Proposition 6.1.2, A is reflexive if and only if $\mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(A)) \subseteq \mathcal{D}^{\mathrm{perf}}(\mathcal{D}_{\mathrm{fd}}(A))$. Since A/J_- generates $\mathcal{D}_{\mathrm{fd}}(A)$, we have $\mathcal{D}_{\mathrm{fd}}(A) \simeq \mathcal{D}^{\mathrm{perf}}(A^!)^{op}$, as in the proof of Proposition 6.1.5. Therefore, A is reflexive if and only if $\mathcal{D}_{\mathrm{fd}}(\mathcal{D}^{\mathrm{perf}}(A^!)^{op}) \subseteq \mathcal{D}^{\mathrm{perf}}(\mathcal{D}^{\mathrm{perf}}(A^!)^{op})$. As there is a Morita equivalence $A^! \hookrightarrow \mathcal{D}^{\mathrm{perf}}(A^!)^{op}$, this occurs exactly when $\mathcal{D}_{\mathrm{fd}}(A^!) \subseteq \mathcal{D}^{\mathrm{perf}}(A^!)$. The second equivalent statement for reflexivity follows by taking $K = A/J_-$ in Proposition 6.1.5. □

Remark 6.1.7. Recall from Remark 3.1.7, the question of when A/J_- generates $\mathcal{D}_{\mathrm{fd}}(A)$ is subtle. See Remark 3.1.8 for some situations where this is satisfied.

Remark 6.1.8. See Remark 3.4.3 and Proposition 3.4.5 for some discussion around the condition $\mathcal{D}_{\mathrm{fd}}(A^!) \subseteq \mathcal{D}^{\mathrm{perf}}(A^!)$.

Remark 6.1.9. By the first part of Remark 3.1.8 and Remark 3.4.3 (4), the conditions of Theorem 6.1.6 hold for connective finite-dimensional DG-algebras over a perfect field. Combined with Example 3.1.3 (1), this recovers Proposition 6.9 of [KS25] which states that proper connective DG-algebras over perfect fields are reflexive. In Proposition 6.3.22, we give a different proof with no base field requirements.

Example 6.1.10. Take $A = k[x]/x^2$ with $|x| = 1$ so $A^! = k[[t]]$ with $|t| = 0$. As in Section 8 of [Goo24c], one can use Neeman's classification of thick subcategories to show that $\mathcal{D}_{\mathrm{fd}}(A) = \mathrm{thick}_A(k)$ (alternatively this follows Proposition 5.6 of [KN13]). Then by Remark 3.4.3 (5) and Proposition 6.1.6, A is reflexive. This example will be generalised in Section 6.2.

§ 6.2 | Coconnective DG-algebras

We study reflexivity for coconnective DG-algebras A such that $H^0(A)$ is a semisimple k-algebra (that is the global dimension of $H^0(A)$ is zero). We will use the following concept which was introduced in [Bon10] and [Pau08].

Definition 6.2.1. A weight structure $(\mathcal{T}^{w>0}, \mathcal{T}^{w\leq 0})$ on a triangulated category $\mathcal{T}$ is a pair of additive subcategories, closed under summands, that satisfy the following conditions.

1. $\mathcal{T}^{w>0}$ is closed under Σ^{-1} and $\mathcal{T}^{w\leq 0}$ is closed under Σ.
2. $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{w>0}, \mathcal{T}^{w\leq 0}) = 0$.
3. For every object $X \in \mathcal{T}$, there is a triangle
$$\sigma_{>0}X \to X \to \sigma_{\leq 0}X$$
with $\sigma_{>0}X \in \mathcal{T}^{w>0}$ and $\sigma_{\leq 0}X \in \mathcal{T}^{w\leq 0}$.

The key examples are given by the brutal truncations on the homotopy category of an additive category. We will make use of the following result.

Theorem 6.2.2 (Corollary 4.1, [KN13])**.** *Let A be a coconnective DG-algebra such that $H^0(A)$ is semisimple.*

1. *There is a weight structure $(\mathcal{D}(A)^{w>0}, \mathcal{D}(A)^{w\leq 0})$ on $\mathcal{D}(A)$ given by*
$$\mathcal{D}(A)^{w>0} = \{X \in \mathcal{D}(A) \mid H^i(X) = 0 \textit{ for } i \leq 0\},$$
$$\mathcal{D}(A)^{w\leq 0} = \{X \in \mathcal{D}(A) \mid H^i(X) = 0 \textit{ for } i > 0\}.$$
2. *For every $X \in \mathcal{D}(A)$, there is a triangle $\sigma_{>0}X \to X \to \sigma_{\leq 0}X$ such that the map $\sigma_{>0}X \to X$ induces an isomorphism on H^i for $i > 0$ and the map $X \to \sigma_{\leq 0}X$ induces an isomorphism on H^i for $i \leq 0$.*

Remark 6.2.3. If A is a proper coconnective DG-algebra such that $H^0(A)$ is semisimple, then $\sigma_{\leq 0}A$ will play the role of the simples. By Proposition 4.6 in [KN13], $\mathrm{thick}_A(\sigma_{\leq 0}A) = \mathcal{D}_{\mathrm{fd}}(A)$, and so we are in the situation of Proposition 6.1.5. Throughout this section, we let $A^! := \underline{\mathbb{R}\mathrm{Hom}}_A(\sigma_{\leq 0}A, \sigma_{\leq 0}A)$. Note that $A^!$ is connective by Lemma 5.2 of [KN13].

Remark 6.2.4. We note that the truncation triangles in a weight structure are not as well behaved as in a t-structure. They are not unique and they are not functorial. The following example also demonstrates that even for coconnective DG-algebras with $A^i = 0$ for $i < 0$, one can't always use the brutal truncations for the "special" triangles appearing in Theorem 6.2.2.

Example 6.2.5. Let A be the path algebra of the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

with $|\alpha| = 0$ and $|\beta| = 1$ and $d(\alpha) = \beta$. Then A is a coconnective DG-algebra with semisimple H^0. Indeed $A \simeq H^*(A) = k \times k$. The brutal truncation triangle of A is

$$(\beta) = A^1 \to A \to A^0 = (\alpha, e_1, e_2) \to^+$$

but this does not induce a quasiisomorphism $k^2 = H^0(A) \to H^0(A^0) = k^3$. The special truncation triangle of Theorem 6.2.2 is given by

$$(\alpha, \beta) = J \to A \to A/J = (e_0, e_1) \to^+$$

where the second map is in fact a quasi-isomorphism.

Remark 6.2.6. If A is a finite-dimensional DG-algebra which is also coconnective with semisimple H^0 then we have potentially two different candidates for the "simple" modules: $\sigma_{\leq 0}A$ and A/J_+. In general these two notions do not coincide as the following example shows.

Example 6.2.7. Let A be the path algebra of the quiver

$$1 \xrightarrow{\alpha} 2$$

Let α have degree 1 and define a differential by $d(e_1) = \alpha = -d(e_2)$. Then A is a DG-algebra and $A \simeq H^*(A) \simeq k$ but $H^0(A/J) = k \oplus k$. Therefore the triangle

$$J \to A \to A/J \to^+$$

does not induce isomorphisms $H^0(A) \xrightarrow{\sim} H^0(A/J)$. Hence A/J does not coincide with any $\sigma_{\leq 0}A$ shown to exist by Theorem 6.2.2. In this case one may take $\sigma_{\leq 0}A$ to be given by A itself.

To prove reflexivity, it will be useful to relate the simple $H^0(A)$-modules to the simple $H^0(A^!)$-modules. The first step is the following.

Theorem 6.2.8. *Suppose A is a coconnective DG-algebra such that $H^0(A)$ is semisimple. There is a bijection between the isomorphism classes of indecomposable summands of $H^0(A^!)$ (as $H^0(A^!)$-modules) and the isomorphism classes of simple $H^0(A)$-modules.*

Proof. Let $S := \sigma_{\leq 0}A$. Any indecomposable summand of $H^0(A^!)$ is of the form $P = H^0(A^!)e_P$ for some primitive idempotent e_P of $H^0(A^!) = \mathrm{Hom}_{\mathcal{D}(A)}(S, S)$. This idempotent splits in $\mathcal{D}(A)$ and produces an indecomposable summand $\overline{\phi(P)}$ of S, and so a summand $\phi(P)$ of $H^0(S) \simeq H^0(A)$. Suppose $\phi(P)$ is decomposable so that $\phi(P) \simeq \bigoplus S_i$ for some simple summands S_i of $H^0(A)$. By Lemma 4.5 of [KN13], there are indecomposable summands $\tilde{A}_i$ of A in $\mathcal{D}(A)$ lifting each of the S_i. By Lemma 4.4 in loc. cit., the isomorphism $\bigoplus S_i \to \phi(P)$ can be lifted to a map $f\colon \bigoplus A_i \to \overline{\phi(P)}$ which is an isomorphism on H^0. Since $\overline{\phi(P)} \in \mathcal{D}(A)^{w\leq 0}$, f factors as $\tilde{f}\colon \sigma_{\leq 0}(\bigoplus A_i) \to \overline{\phi(P)}$ and $H^0(\tilde{f})$ is an isomorphism. But then $\tilde{f}$ is a quasi-isomorphism and so $\overline{\phi(P)} \simeq \bigoplus \sigma_{\leq 0}A_i$, using Lemma 3.2 in loc. cit. This contradicts the indecomposablility of $\overline{\phi(P)}$. Therefore, $\phi(P)$ is an indecomposable summand of $H^0(A)$ and so a simple $H^0(A)$-module.

Given a simple $H^0(A)$-module, Corollary 4.7 of loc. cit. and its proof show that it can be lifted to a summand of S. The same argument as above shows that the lift is indecomposable in $\mathcal{D}(A)$ and so it corresponds to a primitive idempotent in $H^0(A^!)$.

Suppose that $\phi(P') \simeq \phi(P)$. Then by the uniqueness of Corollary 4.7, we have that $\overline{\phi(P)} \simeq \overline{\phi(P')}$. Therefore, there are isomorphisms

$$H^0(A^!)e_P \simeq \mathrm{Hom}_{\mathcal{D}(A)}(\overline{\phi(P)}, \sigma_{\leq 0}A,) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(\overline{\phi(P')}, \sigma_{\leq 0}A) \simeq H^0(A^!)e_{P'}$$

using the bijection between idempotents and summands. □

Remark 6.2.9. For an arbitrary ring, there can be indecomposable projectives which are not summands of the regular module. If $H^0(A)$ is a finite product of division algebras, we will show in Proposition 6.2.12 that $H^0(A^!)$ is semiperfect, and so the indecomposable projective $H^0(A^!)$-modules are exactly the indecomposable summands of $H^0(A^!)$ (for example, by Corollary 24.14 in [Lam01]).

We note that we can assume that $H^0(A)$ is a finite product of division algebras.

Lemma 6.2.10. *If A is a coconnective DG-algebra such that $H^0(A)$ is semisimple, then there is a coconnective DG-algebra A', which is Morita equivalent to A and such that $H^0(A')$ is a finite product of division algebras.*

Proof. Suppose $H^0(A) \simeq \bigoplus_{i=1,\dots,n, j=1,\dots,k_i} S_{i,j}$ is a decomposition of $H^0(A)$ into simple left modules such that $S_{i,j} \simeq S_{i,j'}$ for $1 \leq j, j' \leq k_i$ for all i. Then $H^0(A) \simeq M_{k_1}(D_1) \times \cdots \times M_{k_n}(D_n)$ for division algebras D_i. By Lemma 4.5 in [KN13], $A \simeq \bigoplus A_{i,j}$ where $H^0(A_{i,j}) \simeq S_{i,j}$. For each $i, 1 \leq j, j' \leq k_i$, there is an isomorphism $f\colon S_{i,j} \to S_{i,j'}$ which lifts to a map $A_{i,j} \to A_{i,j'}$ by Lemma 4.4 in loc. cit. By naturality of the isomorphism in Lemma 4.4, it follows that the lift is also an isomorphism. So if $A' := \underline{\mathbb{R}\mathrm{Hom}}_A(\bigoplus_i A_i, \bigoplus_i A_i)^{op}$ where $A_1, \dots A_n$ are a choice of isomorphism classes of the $A_{i,j}$, then we have $\mathcal{D}^{\mathrm{perf}}(A) \simeq \mathcal{D}^{\mathrm{perf}}(A')$. Furthermore, we have that $H^0(A')$ is a finite product of division algebras. □

Lemma 6.2.11. *If A is a coconnective DG-algebra such that $H^0(A)$ is semisimple, then $H^0(\sigma_{\leq 0}A)$ is a semisimple $H^0(A^!)$-module.*

Proof. Set $S = \sigma_{\leq 0}A$. Note there is a map of k-algebras induced by the functor H^0

$$\begin{aligned} H^0(A^!) = \mathrm{Hom}_{\mathcal{D}(A)}(S,S) &\twoheadrightarrow \mathrm{Hom}_{H^0(A)}(H^0(S),H^0(S)) \\ &\simeq \mathrm{Hom}_{H^0(A)}(H^0(A),H^0(A)) \\ &\simeq H^0(A)^{op} \end{aligned}$$

using the isomorphism $H^0(A) \simeq H^0(S)$. Since $H^0(A)$ is a semisimple k-algebra, it is a semisimple left $H^0(A^!)$-module. There is an isomorphism of left $H^0(A)$-modules $H^0(A) \simeq H^0(S)$. It remains to check that this is an isomorphism of left $H^0(A^!)$-modules, where $H^0(A)$ is viewed as a left $H^0(A^!)$-module via restriction along this ring map, and $H^0(S)$ is viewed as a left $H^0(A^!)$-module using the fact that $H^0(A^!)$ is the endomorphism ring of S in $\mathcal{D}(A)$. To check this, note that the map $H^0(A^!) \to H^0(A)^{op}$ can be identified with the map

$$H^0(A^!) = \mathrm{Hom}_{\mathcal{D}(A)}(S,S) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{D}(A)}(A,S) \xrightarrow[\sim]{f_*^{-1}} \mathrm{Hom}_{\mathcal{D}(A)}(A,A) \simeq H^0(A)^{op}$$

where $f\colon A \to S$ is the truncation map. One can then check directly that the isomorphism $H^0(A) \simeq H^0(S)$ is of left $H^0(A^!)$-modules. □

Proposition 6.2.12. *If A is a coconnective DG-algebra such that $H^0(A)$ is a finite product of division algebras, then $H^0(A^!)$ is a semiperfect ring.*

Proof. Suppose $H^0(A) = D_1 \times \cdots \times D_n$ is a product of division algebras. By Lemma 4.5 in [KN13], A splits into a direct sum of indecomposables $A \simeq \bigoplus A_i$ such that $H^0(A_i) \simeq D_i$. It follows that $S \coloneqq \sigma_{\leq 0}A \simeq \bigoplus \sigma_{\leq 0}A_i \coloneqq \bigoplus S_i$. Therefore, $H^0(A^!) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(\bigoplus S_i, \bigoplus S_i)$. The idempotents $e_i\colon S \to S_i \to S$ clearly form a complete set of orthogonal idempotents, so it remains to show that each $e_iH^0(A)e_i = \mathrm{Hom}_{\mathcal{D}(A)}(S_i,S_i)$ is local. Consider the map of algebras from the proof of Lemma 6.2.11

$$H^0(A^!) \to H^0(A)^{op}$$

induced by taking H^0. Clearly, it restricts to the subalgebras

$$\mathrm{Hom}_{\mathcal{D}(A)}(S_i,S_i) \to \mathrm{Hom}_{H^0(A)}(H^0(S_i),H^0(S_i)) \simeq \mathrm{Hom}_{H^0(A)}(H^0(A_i),H^0(A_i)) \simeq D_i^{op}$$

We note that this map reflects units. Indeed, if $f\colon S_i \to S_i$ is such that $H^0(f)$ is an isomorphism, then since the cohomology of S_i only lives in degree zero, f is a quasi-isomorphism. Since D_i is a division algebra, this implies that the kernel of the above map is exactly the set of non-units. Therefore, the non-units form an ideal and $\mathrm{Hom}_{\mathcal{D}(A)}(S_i,S_i)$ is local. □

The proof of Proposition 6.2.12 also shows the following.

Corollary 6.2.13. *If A is a coconnective DG-algebra such that $H^0(A)$ is a division algebra, then $H^0(A^!)$ is local.*

Lemma 6.2.14. *Suppose A is a coconnective DG-algebra such that $H^0(A)$ is a finite product of division algebras. Then there is an isomorphism $H^0(A^!)/\operatorname{rad} H^0(A^!) \simeq H^0(A)^{op}$. Furthermore, the simple $H^0(A^!)$-modules are exactly the indecomposable summands of $H^0(\sigma_{\leq 0}A)$.*

Proof. Let $J = \operatorname{rad} H^0(A^!)$. Since $H^0(A^!)$ is semiperfect, $H^0(A^!)/J$ is the maximal semisimple quotient of $H^0(A^!)$ and so the map $H^0(A^!) \twoheadrightarrow H^0(A)^{op}$ in the proof of Lemma 6.2.11 factors as a map $H^0(A^!)/J \twoheadrightarrow H^0(A)^{op}$. Since $H^0(A^!)$ is semiperfect, $H^0(A^!)/J$ is isomorphic to a product of N matrix rings over division algebras. Here N is the number of isomorphism classes of simples which equals the number of isomorphism classes of indecomposable projectives. The only quotient rings of $H^0(A^!)/J$ are products of its connected components and so $H^0(A)^{op}$ must be some product of connected components of $H^0(A^!)/J$. However by Theorem 6.2.8, the number of connected components of $H^0(A)^{op}$ and of $H^0(A)/J$ are equal. It follows that the map must be an isomorphism. Since $H^0(\sigma_{\leq 0}A) \simeq H^0(A)$ as $H^0(A^!)$-modules, the second statement follows. ☐

Theorem 6.2.15. *If A is a proper coconnective DG-algebra such that $H^0(A)$ is semisimple, then A is reflexive.*

Proof. By Lemma 6.2.10, we can assume that $H^0(A)$ is a product of division algebras. Set $S := \sigma_{\leq 0}A$. Proposition 4.6 (1) in [KN13] states that $\mathcal{D}_{\mathrm{fd}}(A) = \operatorname{thick}_A(S)$ so by Proposition 6.1.5, it is enough to show that $\operatorname{thick}_{A^!}(S) = \mathcal{D}_{\mathrm{fd}}(A^!)$. Suppose $M \in \mathcal{D}_{\mathrm{fd}}(A^!)$. Then, since $A^!$ is connective, $\mathcal{D}_{\mathrm{fd}}(A^!)$ admits the standard t-structure and so, by induction on length, we may assume that $M \simeq H^0(M)$. Since $H^0(M)$ is a finite-dimensional $H^0(A^!)$-module, it has finite length and admits a finite filtration whose factors are finite-dimensional simple $H^0(A^!)$-modules. By Lemma 6.2.14, these are all finite sums of summands of $H^0(S)$. The associated short exact sequences become triangles in $\mathcal{D}(A^!)$ and so it follows that $M \in \operatorname{thick}_{A^!}(S)$. ☐

Remark 6.2.16. The results of this section should be compared to the extensive literature relating simple-minded collections, silting objects, t-structures, and weight structures. See, for example, [AN09],[KN13], [KY14], [Bon24] [Fus24]. A simple-minded collection in an algebraic triangulated category is the same information as a coconnective DG-algebra A such that $H^0(A)$ is a finite product of division algebras, and a silting object is the same as a connective DG-algebra.

Example 6.2.17. If $\mathcal{T}$ is a proper pretriangulated DG-category admitting a simple-minded collection, then $\mathcal{T} \simeq \mathcal{D}^{\mathrm{perf}}(A)$ for a proper coconnective DG-algebra A such that $H^0(A)$ is semisimple. Therefore, $\mathcal{T}$ is reflexive.

Example 6.2.18. If $\mathcal{A}$ is a Hom-finite k-linear Abelian category (i.e. $\operatorname{Hom}_{\mathcal{A}}(M, N)$ is finite-dimensional for all $M, N \in \mathcal{A}$) and $S_1, \ldots, S_n$ are a collection of simples (not necessarily all of them) such that $\underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(\bigoplus S_i, \bigoplus S_i)$ is bounded, then the thick subcategory they generate is reflexive. This is satisfied if the S_i are all of finite projective dimension or all of finite injective dimension.

Example 6.2.19. Let A be a finite-dimensional algebra which admits a grading with $A/\operatorname{rad}(A)$ in degree zero. Then A can be viewed as a formal coconnective DG-algebra with semisimple zeroth cohomology which is then reflexive. This includes a large number of commonly studied graded algebras such as exterior algebras and truncated polynomial rings. Combined with Proposition 6.3.22, we see that $k[x]/x^2$ is a reflexive DG-algebra for any choice of grading $|x| \in \mathbb{Z}$.

Example 6.2.20. Let X be a topological space with finite singular cohomology such as a finite CW-complex, then $A = C^*(X; k)$ is reflexive. We investigate examples of this form more closely in [BGO25].

§ 6.3 | Approximable DG-categories

In this section, we produce new examples of reflexive DG-categories using the theory of approximable triangulated categories. This technology was invented to abstract the relationship between the bounded derived category and perfect complexes over a scheme. As applications, we show that proper schemes and proper connective DG-algebras are reflexive over any field. Furthermore, we show that Azumaya algebras over proper schemes are reflexive. The results of this section appear in [Goo24a], where we work more generally over a commutative Noetherian ring.

§ 6.3.1 | Approximable Triangulated Categories

Developed by Neeman, the theory of approximable triangulated categories has found many applications which are surveyed in [Nee21a] and [CNS24]. The main objects of study here are triangulated categories with coproducts and t-structures. In order to talk about approximable triangulated categories, we need some notation on how to build objects.

Definition 6.3.1. Let $\mathcal{T}$ be a triangulated category with coproducts, $S \subseteq \mathcal{T}$ a subcategory, and let $[a, b] \subseteq \mathbb{Z}$ be an interval.

1. Let $\langle S \rangle_1^{[a,b]} \subseteq \mathcal{T}$ consist of finite sums of i-th shifts of summands of objects of S where $i \in [a, b]$.

2. Let $\langle S \rangle_n^{[a,b]} \subseteq \mathcal{T}$ consist of all summands of objects X which fit into a triangle

$$X' \to X \to X''$$

   with $X' \in \langle S \rangle_1^{[a,b]}$ and $X'' \in \langle S \rangle_{n-1}^{[a,b]}$.

3. Let $\langle S \rangle^{[a,b]}$ be the union over n of all $\langle S \rangle_n^{[a,b]}$.

4. Let $\overline{\langle S\rangle_1}^{[a,b]}$ consist of all sums of i-th shifts of summands of objects of S where $i \in [a,b]$.

5. Let $\overline{\langle S\rangle}_n^{[a,b]}$ consist of all summands of objects X which fit into a triangle

$$X' \to X \to X''$$

with $X' \in \overline{\langle S\rangle}_1^{[a,b]}$ and $X'' \in \overline{\langle S\rangle}_{n-1}^{[a,b]}$.

If $[a,b] = \mathbb{Z}$, so that all shifts are allowed, we will omit the associated superscript.

Definition 6.3.2. A triangulated category is approximable if it has a compact generator G and a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and there is an integer $A > 0$ satisfying the following conditions.

1. $\Sigma^A G \in \mathcal{T}^{\leq 0}$.

2. $\mathrm{Hom}_{\mathcal{T}}(\Sigma^{-A}G, E) = 0$ for all $E \in \mathcal{T}^{\leq 0}$.

3. For every $F \in \mathcal{T}^{\leq 0}$, there is a triangle

$$E \to F \to D$$

with $D \in \mathcal{T}^{\leq -1}$ and $E \in \overline{\langle G\rangle}_A^{[-A,A]}$.

Remark 6.3.3. By Proposition 2.6 in [Nee21c], approximability is a property only of the triangulated category and doesn't depend on a choice of t-structure or compact generator.

Definition 6.3.4. Given a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on a triangulated category $\mathcal{T}$, we fix the following notation.

$$\mathcal{T}^{\leq n} := \Sigma^{-n}\mathcal{T}^{\leq 0}, \quad \mathcal{T}^{\geq n} := \Sigma^{-n}\mathcal{T}^{\geq 0}$$

$$\mathcal{T}^{-} := \bigcup_n \mathcal{T}^{\leq n}, \quad \mathcal{T}^{+} := \bigcup_n \mathcal{T}^{\geq n}, \quad \mathcal{T}^{b} := \mathcal{T}^{-} \cap \mathcal{T}^{+}$$

Two t-structures $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ on $\mathcal{T}$ are equivalent if there is some $N \geq 0$ such that $\mathcal{T}_1^{\leq -N} \subseteq \mathcal{T}_2^{\leq 0} \subseteq \mathcal{T}_1^{\leq N}$.

Remark 6.3.5. By Theorem A.1 in [ATJLSS03], any compact object in a triangulated category $\mathcal{T}$ with coproducts generates a t-structure. One can check that if G and G' are two compact generators of $\mathcal{T}$, then they generate equivalent t-structures. A t-structure is said to be in the preferred equivalence class if it is equivalent to one generated by a compact generator of $\mathcal{T}$. By Proposition 2.4 in [Nee21c], any t-structure on an approximable triangulated category $\mathcal{T}$ satisfying the conditions (1), (2) and (3) in Definition 6.3.2 is in the preferred equivalence class.

Example 6.3.6. We note the following examples of approximable triangulated categories.

1. If X is a quasicompact separated scheme, then $\mathcal{D}_{\mathrm{QCoh}}(X)$ is approximable, and the standard t-structure is in the preferred equivalence class (see Example 4.6 in [Nee21c]). More generally, the derived category of a quasicompact quasiseparated scheme is weakly approximable.

2. If $\mathcal{T}$ is a triangulated category with a compact generator G which satisfies $\mathrm{Hom}_{\mathcal{T}}(G, \Sigma^i G) = 0$ for $i > 0$, then $\mathcal{T}$ is approximable (see Remark 4.3 of [Nee21c]). For example, the derived category of a connective DG-algebra is approximable, and the standard t-structure is in the preferred equivalence class.

The following definition gives a notion of $\mathcal{D}^b_{\mathrm{coh}}$ for approximable triangulated categories.

Definition 6.3.7. Let $\mathcal{T}$ be an approximable triangulated category and $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ a t-structure in the preferred equivalence class.

1. Let $\mathcal{T}^-_c$ consist of all objects $F \in \mathcal{T}$ such that for any $n > 0$ there is a triangle

$$E \to F \to D$$

with $E \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\leq -n}$.

2. Let $\mathcal{T}^b_c = \mathcal{T}^-_c \cap \mathcal{T}^b$.

Remark 6.3.8. The subcategories $\mathcal{T}^-_c$ and $\mathcal{T}^b_c$ are thick and only depend on $\mathcal{T}$ and not on a choice of t-structure in the preferred equivalence class. See Lemma 2.10 in [Nee21c] and Remark 8.7 in [Nee21a].

Example 6.3.9. We can identify $\mathcal{T}^b_c$ in various examples.

1. For a quasicompact quasiseparated scheme X, $\mathcal{D}_{\mathrm{QCoh}}(X)^b_c$ is the category of pseudocoherent complexes with bounded cohomology. So if X is Noetherian, $\mathcal{D}_{\mathrm{QCoh}}(X)^b_c = \mathcal{D}^b_{\mathrm{coh}}(X)$. See the discussion after Proposition 8.10 in [Nee21a].

2. Suppose A is a connective DG-algebra such that $H^0(A)$ is a coherent ring and $H^n(A)$ is a finitely presented $H^0(A)$-module for all $n \in \mathbb{Z}$. Theorem A.5 in [BCMR$^+$24] shows that $\mathcal{D}(A)^b_c$ consists of all modules M such that $H^n(M)$ is finitely presented over $H^0(A)$ for all $n \in \mathbb{Z}$, and only finitely many are non-zero.

We will use the following representability theorems which hold for approximable triangulated categories.

Definition 6.3.10. For a k-linear triangulated category $\mathcal{T}$, a finite homological functor is a k-linear homological functor $F \colon \mathcal{T} \to \mathrm{Mod}\, k$ such that $\bigoplus_i F(\Sigma^i t) \in \mathrm{mod}\, k$ for every $t \in \mathcal{T}$.

Theorem 6.3.11 (Theorem 0.4, [Nee21c]). *Let $\mathcal{T}$ be an approximable triangulated category such that $\operatorname{Hom}_{\mathcal{T}}(X, Y)$ is finite-dimensional for all $X, Y \in \mathcal{T}^c$.*

1. *A k-linear homological functor $F\colon (\mathcal{T}^c)^{op} \to \operatorname{Mod} k$ is finite if and only if it is isomorphic to $\operatorname{Hom}_{\mathcal{T}}(-, M)$ for some $M \in \mathcal{T}^b_c$.*

2. *Suppose there is some $G \in \mathcal{T}^b_c$ and $N > 0$ such that $\mathcal{T} = \overline{\langle G \rangle}_N$. Then a k-linear homological functor $F\colon \mathcal{T}^b_c \to \operatorname{Mod} k$ is finite if and only if it is isomorphic to $\operatorname{Hom}_{\mathcal{T}}(M, -)$ for some $M \in \mathcal{T}^c$.*

Remark 6.3.12. Theorem 0.4 in [Nee21c] is significantly more powerful. It also makes statements about representability of (unenhanced) morphisms between finite functors on $\mathcal{T}^b_c$ and $\mathcal{T}^c$ and it works over a commutative Noetherian ring.

§ 6.3.2 | Reflexivity and Approximability

Definition 6.3.13. We say a small DG-category $\mathcal{A}$ is approximable if $\mathcal{D}(\mathcal{A})$ is an approximable triangulated category.

Remark 6.3.14. Since an approximable triangulated category admits a single compact generator, any approximable DG-category is Morita equivalent to a DG-algebra.

The following lemma will be used to relate reflexivity to approximability.

Lemma 6.3.15. *Suppose $\mathcal{T}$ is a k-linear approximable triangulated category and that $\operatorname{Hom}_{\mathcal{T}}(X, Y)$ is finite-dimensional for all $X, Y \in \mathcal{T}^c$. Then*

$$\mathcal{T}^b_c = \left\{ t \in \mathcal{T} \mid \bigoplus_i \operatorname{Hom}_{\mathcal{T}}(c, \Sigma^i t) \in \operatorname{mod} k \textit{ for all } c \in \mathcal{T}^c \right\}.$$

Proof. Let $\mathcal{S}$ denote the right-hand side. By Theorem 6.3.11 (1) (or directly), we see that $\mathcal{T}^b_c \subseteq \mathcal{S}$.

Conversely, suppose $X \in \mathcal{S}$. By assumption, $\operatorname{Hom}_{\mathcal{T}}(-, X)$ is a finite functor on $\mathcal{T}^c$ and so, by Theorem 6.3.11, we have a natural isomorphism $\alpha\colon \operatorname{Hom}_{\mathcal{T}}(-, X') \simeq \operatorname{Hom}_{\mathcal{T}}(-, X)$ of functors on $\mathcal{T}^c$ for some $X' \in \mathcal{T}^b_c$. By Lemma 8.5 (ii) in [Nee21c], X' admits a (strong) $\mathcal{T}^c$-approximating system in the terminology of loc. cit. and so Lemma 6.8 of loc. cit. applies and states that α lifts to a morphism $\hat{\alpha}\colon X' \to X$. Since $\alpha = \operatorname{Hom}_{\mathcal{T}}(-, \hat{\alpha})$ is an isomorphism restricted to compacts, and $\mathcal{T}$ is compactly generated, it follows that $\operatorname{Hom}_{\mathcal{T}}(-, \hat{\alpha})$ is an isomorphism on all $\mathcal{T}$. By the Yoneda lemma, $\hat{\alpha}$ is an isomorphism. So, $X \simeq X' \in \mathcal{T}^b_c$.

□

Remark 6.3.16. In the description of $\mathcal{T}^b_c$ in Lemma 6.3.15, it is equivalent to ask for $\bigoplus_i \operatorname{Hom}(c, \Sigma^i t)$ to be finite-dimensional for all c contained in a compact generating set.

In the presence of enhancements we get the following description.

Theorem 6.3.17. *Suppose $\mathcal{A}$ is an approximable DG-category such that $H^i\mathcal{A}(a,b) \in \operatorname{mod} k$ for all $a, b \in \mathcal{A}$ and $i \in \mathbb{Z}$ (e.g. $\mathcal{A}$ is proper). Then $\mathcal{D}(\mathcal{A})^b_c = \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$.*

Proof. The corepresentables $\mathcal{A}(a,-)$ for $a \in \mathcal{A}$ form a set of compact generators for $\mathcal{D}(\mathcal{A})$. So the assumption guarantees that $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(M,N)$ is finite-dimensional for any two $M, N \in \mathcal{D}^{\mathrm{perf}}(\mathcal{A})$. Since $\operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{A}(a,-), \Sigma^i M) \simeq H^i(M(a))$, the right-hand side of the equality in Lemma 6.3.15 can be identified with $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. □

Theorem 6.3.18. *Suppose $\mathcal{A}$ is a proper approximable DG-category. If there is an object $G \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ such that $\mathcal{D}(\mathcal{A}) = \overline{\langle G \rangle}_n$ for some $n \geq 1$, then $\mathcal{A}$ is reflexive.*

Proof. By Theorem 6.3.17, we have that $\mathcal{D}(\mathcal{A})^b_c = \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. As $\mathcal{A}$ is proper, it is enough to show the evaluation map is essentially surjective. Suppose that $F \in \mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})) = \mathcal{D}_{\mathrm{fd}}(\mathcal{D}(\mathcal{A})^b_c)$. Then $H^0(F) \colon \mathcal{D}(\mathcal{A})^b_c \to \operatorname{Mod} k$ is a finite homological functor. Theorem 6.3.11 gives that $H^0(F) \simeq \operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(M,-)$ for some $M \in \mathcal{D}^{\mathrm{perf}}(\mathcal{A})$. It follows (by e.g. Lemma 2.2 in [KS25]), that $F \simeq \underline{\mathbb{R}\mathrm{Hom}}_{\mathcal{A}}(M,-) \in \mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$. Therefore, F is in the image of the evaluation map $\mathcal{D}^{\mathrm{perf}}(\mathcal{A})^{op} \to \mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$, as required. □

§ 6.3.3 | Applications

We give a different proof of reflexivity for the geometric examples in [KS25], [BNP17] which requires no base field assumptions.

Proposition 6.3.19. *If X is a proper scheme over k, then $\mathcal{D}^b_{\mathrm{coh}}(X) \simeq \mathcal{D}_{\mathrm{fd}}(\mathcal{D}^{\mathrm{perf}}(X))$ and $\mathcal{D}^{\mathrm{perf}}(X)$ is reflexive.*

Proof. By Example 6.3.6 (1), $\mathcal{D}_{\mathrm{QCoh}}(X)$ is approximable and there are equivalences $\mathcal{D}(\mathcal{D}^{\mathrm{perf}}(X))^b_c \simeq \mathcal{D}_{\mathrm{QCoh}}(X)^b_c = \mathcal{D}^b_{\mathrm{coh}}(X)$. By Proposition 3.26 in [Orl16], $\mathcal{D}^{\mathrm{perf}}(X)$ is a proper DG-category. So Theorem 6.3.17 applies and $\mathcal{D}^b_{\mathrm{coh}}(X) \simeq \mathcal{D}_{\mathrm{fd}}(\mathcal{D}^{\mathrm{perf}}(X))$. By Theorem 2.3 of [Nee21b], there is an object $G \in \mathcal{D}^b_{\mathrm{coh}}(X)$ such that $\overline{\langle G \rangle}_n = \mathcal{D}(\mathcal{D}^{\mathrm{perf}}(X))$. We are done by Theorem 6.3.18. □

We will now verify the hypothesis of Theorem 6.3.18 for proper connective DG-algebras.

Lemma 6.3.20. *Suppose A is a proper connective DG-algebra. Then there is an $N \geq 1$ such that $\mathcal{D}(A) = \overline{\langle H^0(A)/\operatorname{rad} H^0(A) \rangle}_N$.*

Proof. Since A is connective, so is A^e and since A is bounded, the standard t-structure implies there is some N such that $A \in \langle H^*(A) \rangle_N \subseteq \mathcal{D}(A^e)$. There is a filtration by $H^0(A)$-bimodules

$$0 = J^{N'} H^*(A) \subseteq \cdots \subseteq JH^*(A) \subseteq H^*(A)$$

where $J = \operatorname{rad} H^0(A)$. Let $F_i = J^i H^*(A)/J^{i+1}H^*(A)$ for $i = 0, \ldots, N'-1$ which are $H^0(A)^{op} \otimes_k H^0(A)/J$-modules. The filtration implies that

$$H^*(A) \in \langle F_i \mid i = 0, \ldots, N'-1 \rangle_{N'} \subseteq \mathcal{D}(H^0(A)^e).$$

Combining with the above and restricting along the map $A^e \to H^0(A^e) \simeq H^0(A)^e$, then gives that $A \in \langle F_i \rangle_{N''} \subseteq \mathcal{D}(A^e)$ for some $N'' \geq 1$. So there is a sequence in $\mathcal{D}(A^e)$

$$0 \to G_0 \to G_1 \to \cdots \to G_{N''} = A$$

with the cone of $G_j \to G_{j+1}$ lying in $\langle F_i \rangle_1$. Now, if $M \in \mathcal{D}(A)$, tensoring the above diagram with M gives a diagram

$$0 \to G_0 \otimes^{\mathbb{L}}_A M \to G_1 \otimes^{\mathbb{L}}_A M \to \cdots \to G_{N''} \otimes^{\mathbb{L}}_A M = M$$

with the cone of each map lying in $\langle F_i \otimes^{\mathbb{L}}_A M \rangle_1$. Since each F_i is a semisimple left $H^0(A)$-module, we have that $F_i \otimes^{\mathbb{L}}_A M \in \mathcal{D}(H^0(A)/J)$. Therefore, $F_i \otimes^{\mathbb{L}}_A M \in \overline{\langle H^0(A)/J \rangle}_1 \subseteq \mathcal{D}(A^e)$ and so $M \in \overline{\langle H^0(A)/J \rangle}_{N''}$. □

Remark 6.3.21. Lemma 6.3.20 can also be proved using finite-dimensional DG-algebras. Suppose A is a finite-dimensional DG-algebra and J_- is as in Definition 3.1.10. Then, by a similar argument to Theorem 3.1.14, $\mathcal{D}(A) = \overline{\langle A/J_- \rangle}_N$ for a finite N. By Example 3.1.3 (1), any proper connective DG-algebra is quasi-isomorphic to a finite-dimensional DG-algebra.

Proposition 6.3.22. *If A is a proper connective DG-algebra, then A is reflexive.*

Proof. This follows from Example 6.3.6 (2), Theorem 6.3.18 and Lemma 6.3.20. □

Proposition 6.3.22 removes the perfect field assumption from Proposition 6.9 of [KS25]. We now turn to a new kind of example.

Definition 6.3.23. A mild noncommutative scheme $(\mathcal{A}, X)$ is a scheme X with a sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras such that $\mathcal{A}$ is quasicoherent when viewed as an $\mathcal{O}_X$-module. We say $(\mathcal{A}, X)$ is quasicompact or quasiseparated if the same holds for X.

For some background in this context see [DDLMR24a]. For a quasicompact quasiseparated mild noncommutative scheme, let $\mathcal{D}_{\mathrm{QCoh}}(\mathcal{A}, X)$ denote the derived category of $\mathcal{A}$-modules with quasicoherent cohomology. We recall that $\mathcal{D}_{\mathrm{QCoh}}(\mathcal{A}, X)$ is compactly generated, and the compact objects are the perfect complexes which we denote $\mathcal{D}^{\mathrm{perf}}(\mathcal{A}, X)$. If X is Noetherian and $\mathcal{A} \in \mathrm{coh}(X)$, then $\mathrm{coh}(\mathcal{A}, X)$ will denote the coherent $\mathcal{A}$-modules and $\mathcal{D}^b_{\mathrm{coh}}(\mathcal{A}, X)$ will denote the subcategory of $\mathcal{D}_{\mathrm{QCoh}}(\mathcal{A}, X)$ consisting bounded complexes with coherent cohomology.

Proposition 6.3.24. *Suppose $(\mathcal{A}, X)$ is a mild noncommutative scheme such that X is proper over k and $\mathcal{A}$ is coherent over X. Then $\mathcal{D}^b_{\mathrm{coh}}(\mathcal{A}, X) \simeq \mathcal{D}_{\mathrm{fd}}(\mathcal{D}^{\mathrm{perf}}(\mathcal{A}, X))$.*

Proof. A similar proof to that of Proposition 3.26 in [Orl16] shows that $\mathcal{D}^{\mathrm{perf}}(\mathcal{A}, X)$ is proper (see Proposition 5.8 of [Goo24a]). By Proposition 4.1 of [DDLMR24a], $\mathcal{D}_{\mathrm{QCoh}}(\mathcal{A}, X)$ is approximable and by Proposition 4.2 of loc. cit., $\mathcal{D}_{\mathrm{QCoh}}(\mathcal{A}, X)^b_c = \mathcal{D}^b_{\mathrm{coh}}(\mathcal{A}, X)$. Therefore, the result follows from Theorem 6.3.17. □

Definition 6.3.25. An Azumaya algebra over a scheme X is a mild noncommutative scheme $(\mathcal{A}, X)$ such that $\mathcal{A}$ is coherent over X and $\mathcal{A}_x$ is a central separable algebra over $\mathcal{O}_{X,x}$ for every $x \in X$.

Proposition 6.3.26. *Suppose $(\mathcal{A}, X)$ is an Azumaya algebra over a scheme X which is proper over k. Then $\mathcal{D}^{\mathrm{perf}}(\mathcal{A}, X)$ is reflexive.*

Proof. By Proposition 6.3.24 and the results quoted from [DDLMR24a] in its proof, the conditions of Theorem 6.3.18 will be satisfied if there is a $G \in \mathcal{D}^b_{\mathrm{coh}}(\mathcal{A}, X)$ such that $\mathcal{D}_{\mathrm{QCoh}}(\mathcal{A}, X) = \overline{\langle G \rangle}_n$. Such a G is constructed in the proof of Corollary 6.2 in [DDLMR24b][1]. □

§ 6.4 | Gluing Reflexive DG-categories

In this section, we show that one can produce new examples of reflexive DG-categories using semiorthogonal decompositions. Semiorthogonal decompositions are fundamental tools in the study of noncommutative algebraic geometry. See [Kuz14] for a survey.

Definition 6.4.1. A semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ of a triangulated category $\mathcal{T}$ is a pair of thick subcategories $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$ such that $\mathrm{thick}(\mathcal{A}, \mathcal{B}) = \mathcal{T}$ and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{B}, \mathcal{A}) = 0$.

For simplicity, we have considered semiorthogonal decompositions with two pieces. For many pieces, the definition can be iterated.

Remark 6.4.2. If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition, then the inclusion $\mathcal{A} \hookrightarrow \mathcal{T}$ admits a left adjoint and the inclusion $\mathcal{B} \hookrightarrow \mathcal{T}$ admits a right adjoint. This exhibits $\mathcal{A}$ and $\mathcal{B}$ as Bousfield localisations of $\mathcal{T}$. For any $t \in \mathcal{T}$, there are triangles

$$b \to t \to a$$

with $b \in \mathcal{B}$ and $a \in \mathcal{A}$.

Remark 6.4.3. Suppose $\mathcal{T}$ is a pretriangulated DG-category and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$ are DG-subcategories such that $H^0(\mathcal{T}) = \langle H^0(\mathcal{A}), H^0(\mathcal{B}) \rangle$ is a semiorthogonal decomposition. By Proposition 4.10 and Lemma 4.4 in [KL15], $\mathcal{T}$ is quasi-equivalent to a DG-category $\widetilde{\mathcal{T}}$ and there are DG-fully faithful functors $i_{\mathcal{A}} \colon \mathcal{A} \hookrightarrow \widetilde{\mathcal{T}}$, $i_{\mathcal{B}} \colon \mathcal{B} \hookrightarrow \widetilde{\mathcal{T}}$ such that $i_{\mathcal{A}}$ admits a DG-left adjoint and $i_{\mathcal{B}}$ admits a DG-right adjoint.

Proposition 6.4.4. *Suppose $\mathcal{T}$ is a reflexive pretriangulated DG-category and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$ are DG-subcategories such that $H^0(\mathcal{T}) = \langle H^0(\mathcal{A}), H^0(\mathcal{B}) \rangle$ is a semiorthogonal decomposition. Then $\mathcal{A}$ and $\mathcal{B}$ are reflexive DG-categories.*

[1] See also Theorem 3.15 in version 1 of the arXiv preprint [DDLMR24b].

Proof. Let $i_{\mathcal{A}} \colon \mathcal{A} \hookrightarrow \mathcal{T}$ and $i_{\mathcal{B}} \colon \mathcal{B} \hookrightarrow \mathcal{T}$ denote the DG-inclusion functors. By Remark 6.4.3, we can assume without loss of generality that $i_{\mathcal{A}}$ admits a DG-left adjoint $\pi_{\mathcal{A}}$ and $i_{\mathcal{B}}$ admits a DG-left right adjoint $\pi_{\mathcal{B}}$. Therefore, there are natural isomorphisms of DG-functors $\pi_{\mathcal{A}} i_{\mathcal{A}} \simeq 1_{\mathcal{A}}$ and $\pi_{\mathcal{B}} i_{\mathcal{B}} \simeq 1_{\mathcal{B}}$. In particular, $\mathcal{A}$ and $\mathcal{B}$ are retracts of $\mathcal{T}$ in Hmo. Therefore, by Remark 4.1.10, $\mathcal{A}$ and $\mathcal{B}$ are reflexive. □

We now prove a partial converse.

Theorem 6.4.5. *Suppose $\mathcal{T}$ is a semireflexive pretriangulated DG-category and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$ are DG-subcategories such that $H^0(\mathcal{T}) = \langle H^0(\mathcal{A}), H^0(\mathcal{B}) \rangle$ is a semiorthogonal decomposition. If $\mathcal{A}$ and $\mathcal{B}$ are reflexive DG-categories, then so is $\mathcal{T}$.*

Proof. Without loss of generality, we can assume that the semiorthogonal decomposition holds at the DG-level by Remark 6.4.3. Applying Lemma 3.7 in [KS25] twice gives a semiorthogonal decomposition

$$\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{T}) = \langle \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A}), \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B}) \rangle.$$

Let $i_{\mathcal{A}}$ and $i_{\mathcal{B}}$ denote the inclusions of $\mathcal{A}$ and $\mathcal{B}$ into $\mathcal{T}$ and $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ their adjoints. The proof of Lemma 3.7 in loc. cit. shows that the inclusions of $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ and $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})$ into $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{T})$ are given by $i_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})} := \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(i_{\mathcal{A}})$ and $i_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})} := \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(i_{\mathcal{B}})$ and their adjoints are $\pi_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})} := \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\pi_{\mathcal{A}})$ and $\pi_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})} := \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\pi_{\mathcal{B}})$. So naturality of ev implies that each of the squares below commute in Hmo.

$$\begin{array}{ccccc}
\mathcal{A} & \overset{\pi_{\mathcal{A}}}{\underset{i_{\mathcal{A}}}{\rightleftarrows}} & \mathcal{T} & \overset{i_{\mathcal{B}}}{\underset{\pi_{\mathcal{B}}}{\rightleftarrows}} & \mathcal{B} \\
\downarrow{\scriptstyle \mathrm{ev}_{\mathcal{A}}} & & \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{T}}} & & \downarrow{\scriptstyle \mathrm{ev}_{\mathcal{B}}} \\
\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) & \overset{\pi_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}}{\underset{i_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}}{\rightleftarrows}} & \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{T}) & \overset{i_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})}}{\underset{\pi_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})}}{\rightleftarrows}} & \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})
\end{array}$$

We need only check that $\mathrm{ev}_{\mathcal{T}}$ is essentially surjective. Suppose $M \in \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{T})$. Then there is a triangle

$$i_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})}\pi_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})}M \to M \to i_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}\pi_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}M.$$

Since $\mathcal{A}$ is reflexive, we have that $\pi_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}M \simeq \mathrm{ev}_{\mathcal{A}}(a)$ for some $a \in \mathcal{A}$. So we have

$$i_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}\pi_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}M \simeq i_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}\,\mathrm{ev}_{\mathcal{A}}(a) \simeq \mathrm{ev}_{\mathcal{T}}\, i_{\mathcal{A}}(a).$$

Similarly,

$$i_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})}\pi_{\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B})}M \simeq \mathrm{ev}_{\mathcal{T}}\, i_{\mathcal{B}}(b)$$

for some $b \in \mathcal{B}$. Since $\mathcal{T}$ is semireflexive, the morphism

$$\Sigma^{-1}\,\mathrm{ev}_{\mathcal{T}}\, i_{\mathcal{A}}(a) \to \mathrm{ev}_{\mathcal{T}}\, i_{\mathcal{B}}(b)$$

in the triangle above can be lifted to some $f \colon \Sigma^{-1} i_{\mathcal{A}}(a) \to i_{\mathcal{B}}(b) \in \mathcal{T}$. Therefore, it follows that $\mathrm{ev}_{\mathcal{T}}(\mathrm{cone}(f)) \simeq M$, as required. □

Example 6.4.6. Let A be the path algebra of the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

with relations $\alpha\beta$ and $\beta\alpha$. We can view A as a DG-algebra with zero differential and with grading given by some $|\alpha| \in \mathbb{Z}$ and $|\beta| \in \mathbb{Z}$. For general $|\alpha|$ and $|\beta|$, note that A does not satisfy the conditions of either Proposition 6.3.22 or Theorem 6.2.15. One can check that $\mathcal{D}^{\mathrm{perf}}(A) = \langle \mathcal{A}, \mathcal{B} \rangle$ where $\mathcal{A} = \mathrm{thick}(P_2)$ and $\mathcal{B} = \mathrm{thick}(\mathrm{cone}(\beta))$. One can compute that $\underline{\mathbb{R}\mathrm{Hom}}_A(P_2) = k$ and $\underline{\mathbb{R}\mathrm{Hom}}_A(\mathrm{cone}(\beta)) = k[x]/x^2$ with $|x| = |\alpha| + |\beta| - 1$. Since $k[x]/x^2$ is reflexive with any grading, Theorem 6.4.5 implies that A is reflexive. This is an example of a graded gentle algebra. In [BGO25], we use similar ideas to show that all graded gentle algebras are reflexive.

Example 6.4.7. In Theorem 5.4 of [Efi20], Efimov constructed an example of a proper DG-algebra B which does not admit a categorical resolution. The category $\mathcal{D}^{\mathrm{perf}}(B)$ admits a semiorthogonal decomposition with pieces given by the perfect derived categories of $k[x]/x^6$ and $k[y]/y^3$ with $|x| = 0$ and $|y| = 1$. These are both reflexive and so B is reflexive.

Remark 6.4.8. We emphasise that Example 6.4.7 is related to, but different from, Example 3.1.6. Reflexivity of the DG-algebra A, defined in Example 3.1.6, remains open due to the subtle issue of generation of the category of cohomologically finite modules.

Example 6.4.9. Let A be the upper triangular k-algebra

$$\begin{pmatrix} k & \bigoplus_{\mathbb{N}} k \\ 0 & k \end{pmatrix}$$

then $\mathcal{D}^{\mathrm{perf}}(A)$ admits a semiorthogonal decomposition with $\mathcal{D}^{\mathrm{perf}}(A) = \langle \mathcal{D}^b(k), \mathcal{D}^b(k) \rangle$. However one can compute that $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(A) \simeq \mathcal{D}^{\mathrm{perf}}(A^{!!})$ where $A^{!!}$ is the upper triangular k-algebra

$$\begin{pmatrix} k & \prod_{2^{\mathbb{N}}} k \\ 0 & k \end{pmatrix}$$

and that if A were reflexive then the map $A \to A^{!!}$ would be an isomorphism. Therefore, A is not semireflexive (and so also not reflexive).

Bibliography

[Add16] N. Addington. New derived symmetries of some hyperkähler varieties. *Algebr. Geom.*, 3(2):223–260, 2016.

[AL17] R. Anno and T. Logvinenko. Spherical DG-functors. *J. Eur. Math. Soc.*, 19(9):2577–2656, 2017.

[AN09] S. Al-Nofayee. Simple objects in the heart of a t-structure. *J. Pure Appl. Algebra*, 213(1):54–59, 2009.

[ATJLSS03] L. Alonso Tarrío, A. Jeremías López, and Ma. José Souto Salorio. Construction of t-structures and equivalences of derived categories. *Trans. Amer. Math. Soc.*, 355(6):2523–2543, 2003.

[Bal11] M. Ballard. Derived categories of sheaves on singular schemes with an application to reconstruction. *Adv. Math.*, 227(2):895–919, 2011.

[BCMR$^+$24] R. Biswas, H. Chen, K. Manali Rahul, C. J. Parker, and J. Zheng. Bounded t-structures, finitistic dimensions, and singularity categories of triangulated categories. *arXiv preprint: 2401.00130*, 2024.

[Bei78] A. A. Beilinson. Coherent sheaves on P^n and problems of linear algebra. *Funct. Anal. Its. Appl.*, 12(3):214–216, 1978.

[BGO25] M. Booth, I. Goodbody, and S. Opper. Reflexive dg categories in algebra and topology. *arXiv preprint: 2506.11213*, 2025.

[BK89] A. Bondal and M. Kapranov. Representable functors, Serre functors, and mutations. *Math. USSR Izv.*, 35(3):519–541, 1989.

[BK90] A. Bondal and M. Kapranov. Enhanced triangulated categories. *Mat. Sb. (Russian)*, 181:5:669–683, 1990.

[BNP17] D. Ben-Zvi, D. Nadler, and A. Preygel. Integral transforms for coherent sheaves. *J. Eur. Math. Soc.*, 19(12):3763–3812, 2017.

[BO01] A. Bondal and D. Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compos. Math.*, 125:327–344, 2001.

[Bod17] A. Bodzenta. Differential graded quivers of smooth rational surfaces. *Proc. Edinb. Math. Soc.*, 60(4):859–876, 2017.

[Bon10] M. V. Bondarko. Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general). *J. K-theory*, 6(3):387–504, 2010.

[Bon24] L. Bonfert. Derived projective covers and Koszul duality of simple-minded and silting collections. *arXiv preprint: 2309.00554*, 2024.

[Buc21] R. O. Buchweitz. *Maximal Cohen-Macaulay modules and Tate cohomology*, volume 262 of *Math. Surveys Monogr.* Amer. Math. Soc., 2021. With appendices by L.L. Avramov, B. Briggs, S.B. Iyengar and J.C. Letz.

[BVdB03] A. Bondal and M. Van den Bergh. Generators and representability of functors in commutative and noncommutative geometry. *Mosc. Math. J.*, 3(1):1–36, 2003.

[Căl05] A. Căldăraru. The Mukai pairing–II: the Hochschild–Kostant–Rosenberg isomorphism. *Adv. Math.*, 194:34–66, 2005.

[Che21] X. W. Chen. Representability and autoequivalence groups. *Math. Proc. Cambridge Philos. Soc.*, 171(3):657–668, 2021.

[CNS24] A. Canonaco, A. Neeman, and P. Stellari. Weakly approximable triangulated categories and enhancements: a survey. *Boll. Unione. Mat. Ital.*, 18:109–134, 2024.

[Coh13] L. Cohn. Differential graded categories are k-linear stable infinity categories. *arXiv preprint: 1308.2587*, 2013.

[CS15] A. Canonaco and P. Stellari. Internal homs via extensions of dg functors. *Adv. Math.*, 277:100–123, 2015.

[DDLMR24a] T. De Deyn, P. Lank, and K. Manali Rahul. Approximability and Rouquier dimension for noncommutative algebras over schemes. *arXiv preprint: 2408.04561*, 2024.

[DDLMR24b] T. De Deyn, P. Lank, and K. Manali Rahul. Descent and generation for noncommutative coherent algebras over schemes. *arXiv preprint: 2410.01785*, 2024.

[DGI06] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar. Duality in algebra and topology. *Adv. Math.*, 200(2):357–402, 2006.

[Don24] M. Doni. k-linear Morita theory. *arXiv preprint: 2406.15895*, 2024.

[DP84] A. Dold and D. Puppe. Duality, trace and transfer. *Proc. Steklov Inst. Math*, 154:85–103, 1984.

[Dri04] V. Drinfeld. DG quotients of DG categories. *J. Algebra*, 272(2):643–691, 2004.

[Efi10] A. I. Efimov. Formal completion of a category along a subcategory. *arXiv preprint: 1006.4721*, 2010.

[Efi20] A. I. Efimov. Categorical smooth compactifications and generalized Hodge-to-de Rham degeneration. *Invent. Math.*, 222:667–694, 2020.

[EGNO15] P. Etingov, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor categories*, volume 205 of *Math. Surveys Monogr.* Amer. Math. Soc., 2015.

[FJ03] A. Frankild and P. Jørgensen. Gorenstein differential graded algebras. *Israel Journal of Mathematics*, 135:327–353, 2003.

[Fus24] R. Fushimi. The correspondence between silting objects and t-structures for non-positive dg algebras. *arXiv preprint: 2312.17597*, 2024.

[Gen15] F. Genovese. Quasi-functors as lifts of Fourier-Mukai functors: the uniqueness problem. *PhD thesis, University of Pavia*, 2015.

[God24] M. Godinho. A twist on ring morphisms and crepant contractions. *arXiv preprint: 2410.24030*, 2024.

[Goo24a] I. Goodbody. Approximable triangulated categories and reflexive DG-categories. *arXiv preprint: 2411.09461*, 2024. With an appendix by T. Raedschelders and G. Stevenson.

[Goo24b] I. Goodbody. Reflecting perfection for finite-dimensional differential graded algebras. *Bull. Lond. Math. Soc.*, 56(12):3689–3707, 2024.

[Goo24c] I. Goodbody. Reflexivity and Hochschild cohomology. *arXiv preprint: 2403.09299*, 2024.

[Hir02] P. S. Hirschhorn. *Model categories and their localisations*, volume 99 of *Math. Surveys Monogr.* Amer. Math. Soc., 2002.

[HK25] A. Hochenegger and A. Krug. Relations among $\mathbb{P}$-twists. *Int. Math. Res. Not.*, 2025(7), 2025.

[Hov99] M. Hovey. *Model categories*, volume 63 of *Math. Surveys Monogr.* , Amer. Math. Soc., 1999.

[Ima25] Y. Imamura. A formal category theoretic approach to the homotopy theory of dg categories. *arXiv preprint: 2405.07873*, 2025.

[Jas21] G. Jasso. DG-categories. *Lecture notes, University of Bonn*, 2021.

[Jin20] H. Jin. Cohen-Macaulay differential graded modules and negative Calabi-Yau configurations. *Adv. Math.*, 374(18):107338, 2020.

[Kel82] M. Kelly. *Basic concepts of enriched category theory*, volume 64 of *Lecture Notes in Math.* Cambridge University Press, 1982.

[Kel94] B. Keller. Deriving DG categories. *Ann. Sci. Ec. Norm. Supér.*, 27(1):63–102, 1994.

[Kel03] B. Keller. Derived invariance of higher structures on the Hochschild complex. *Preprint*, 2003. Available at `https://webusers.imj-prg.fr/~bernhard.keller/publ/dih.pdf`.

[Kel06] B. Keller. On differential graded categories. *Proceedings of the International Congress of Mathematicians, Madrid, Eur. Math. Soc.*, II:151–190, 2006.

[Kel19] B. Keller. A remark on Hochschild cohomology and Koszul duality. *arXiv preprint: 1911.00965*, 2019.

[KL15] A. Kuznetsov and V.A. Lunts. Categorical resolutions of irrational singularities. *Int. Math. Res. Not.*, 2015(13):4536–4625, 2015.

[KN13] B. Keller and P. Nicolás. Weight structures and simple dg modules for positive dg algebras. *Int. Math. Res. Not.*, 2013(5):1028–1078, 2013.

[Kra20] H. Krause. Completing perfect complexes. *Math. Z.*, 296:1387–1427, 2020. With appendices by T. Barthel and B. Keller.

[KS25] A. Kuznetsov and E. Shinder. Homologically finite-dimensional objects in triangulated categories. *Selecta Math.*, 31(27):31–27, 2025.

[Kuz14] A. Kuznetsov. Semiorthogonal decompositions in algebraic geometry. *Proceedings of the International Congress of Mathematicians, Seoul*, II, 2014.

[KY14] S. Koenig and D. Yang. Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras. *Doc. Math.*, 19:403–438, 2014.

[Lam01] T. Y. Lam. *A first course in noncommutative rings.* Grad. Texts in Math. Springer, 2001.

[LH02] K. Lefèvre-Hasegawa. Sur les A_∞-catégories. *PhD thesis, Université Paris Diderot - Paris 7*, 2002.

[LMSM86] L. G. Lewis, J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Math.* Springer-Verlag, 1986.

[LS16] V. A. Lunts and O. M. Schnürer. New enhancements of derived categories of coherent sheaves and applications. *J. Algebra*, 446:203–274, 2016.

[LU22] Y. Lekili and K. Ueda. Homological mirror symmetry for Milnor fibers via moduli of A_∞-structures. *J. Topology*, 15(3):1058–1106, 2022.

[Lun10] V. A. Lunts. Categorical resolution of singularities. *J. Algebra*, 323(10):2977–3003, 2010.

[LVdB05] W. Lowen and M. Van den Bergh. Hochschild cohomology of abelian categories and ringed spaces. *Adv. Math.*, 198(1):172–221, 2005.

[Mar01] N. Markarian. Poincaré-Birkoff-Witt isomorphism, Hochshild homology and Riemann-Roch Theorem. *Preprint*, 2001. Available at `https://cds.cern.ch/record/735777`.

[Mar08] N. Markarian. The Atiyah class, Hochschild cohomology and the Riemann-Roch theorem. *Journal of the London Mathematical Society*, 79(1):129–143, October 2008.

[Mat86] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge studies in advanced mathematics.* Cambridge University Press, 1986.

[Nee96] A. Neeman. The Grothendieck duality theorem via Bousfield's techniques and Brown representability. *Journal of the American Mathematical Society*, 9(1):205–236, 1996.

[Nee01] A. Neeman. *Triangulated categories*, volume 148 of *Ann. of Math. Stud.* Princeton University Press, Princeton, 2001.

[Nee21a] A. Neeman. Approximable triangulated categories. *Representations of Algebras, Geometry and Physics, Contemp. Math.*, 769(4):111–155, 2021.

[Nee21b] A. Neeman. Strong generators in $D^{\mathrm{perf}}(X)$ and $D^b_{\mathrm{coh}}(X)$. *Ann. of Math.*, 193:689–732, 2021.

[Nee21c] A. Neeman. Triangulated categories with a single compact generator and two Brown representability theorems. *arXiv preprint: 1804.02240*, 2021.

[Orl16] D. Orlov. Smooth and proper noncommutative schemes and gluing of DG categories. *Adv. Math.*, 302:59–105, 2016.

[Orl20] D. Orlov. Finite-dimensional differential graded algebras and their geometric realizations. *Adv. Math.*, 366:107096, 2020.

[Orl23] D. Orlov. Smooth DG algebras and twisted tensor product. *Russian Math. Surveys*, 78(5):853–880, 2023.

[Pau08] D. Pauksztello. Compact corigid objects in triangulated categories and co-t-structures. *Centr. Eur. J. Math.*, 6:25–42, 2008.

[Ric89] J. Rickard. Morita theory for derived categories. *J. Lond. Math. Soc.*, s2-39(3):436–456, 1989.

[Rie14] E. Riehl. *Categorical homotopy theory.* New Math. Monogr. Cambridge University Press, 2014.

[RS22] T. Raedschelders and G. Stevenson. Proper connective differential graded algebras and their geometric realizations. *Eur. J. Math.*, 8:574–598, 2022.

[Sch18] O. M Schnürer. Six operations on dg enhancements of derived categories of sheaves. *Selelecta Math.*, 24:1805–1911, 2018.

[Seg18] E. Segal. All autoequivalences are spherical twists. *Int. Math. Res. Not.*, 2018(10):3137–3154, 2018.

[Shk07] D. Shklyarov. On Serre duality for compact homologically smooth DG algebras. *arXiv preprint: 0702590*, 2007.

[ST01] P. Seidel and R. Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Math. J.*, 108(1):37–108, 2001.

[Sta25] The Stacks Project Authors. The Stacks Project. `https://stacks.math.columbia.edu`, 2025.

[Tab05] G. Tabuada. Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. *C. R. Math. Acad. Sci. Paris*, 340(1):15–19, 2005.

[Tab07] G. Tabuada. Theorie homotopique des DG-categories. *PhD thesis, Université Paris Diderot - Paris 7*, 2007.

[Tab15] G. Tabuada. *Noncommutative motives*, volume 63 of *Univ. Lecture Ser.* Amer. Math. Soc., 2015.

[Toë07] B. Toën. The homotopy theory of dg-categories and derived Morita theory. *Invent. Math.*, 167(3):615–667, 2007.

[Toë11] B. Toën. *Lectures on DG-categories, Topics in Algebraic and Topological K-Theory*, volume 2008 of *Lecture Notes in Math.* Springer, 2011. 243–302.

[TV07] B. Toën and M. Vaquié. Moduli of objects in dg-categories. *Ann. Sci. Ec. Norm. Supér.*, 40(3):387–444, 2007.