On quantum graph theory

–non-commutative graph theory–

M. Abu Omar

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Abstract

This research paper aims to introduce quantum graphs to newcomers. We adopt a clear and concise linear algebraic approach to simplify the concepts of quantum graphs. Our main perspective is viewing a quantum graph as a quantum adjacency matrix operator on a finite-dimensional C*-algebra (which can simply be thought of as a direct sum of matrix algebras), with the inner product defined by choosing a faithful positive linear functional on each matrix algebra summand of the direct sum. Our main result presents practical formulae for identifying isomorphic single-edged quantum graphs. An immediate corollary to this is that there is an infinite number of isomorphisms for a single-edged quantum graph on $M_n(\mathbb{C})$ for $n > 2$, which is surprising since all single-edged quantum graphs are isomorphic to one another for $n = 2$.

On quantum graph theory

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M.A.O.
University of Glasgow

B.Sc. Mathematics & Computer Science, 2019
M.Sc. Mathematics, 2021
M.Phil. Mathematics, 2025

Supervised by

Prof. Christian Voigt
University of Glasgow

This essay is dedicated to *Luma*, *Omar*, and my parents.

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Introduction

Quantum graphs, also known as *non-commutative graphs*, extend the notion of classical finite graphs into the realm of non-commutativity. What began as a subtopic in quantum information theory introduced in [5], is now a small but rich theory, drawing connections between operator algebras, quantum groups, non-commutative geometry, and quantum information theory.

The purpose of this thesis is threefold. The first is to give a detailed and deep introduction to the language that will be used in describing quantum graphs. Namely, we will look at things from string diagrams, to constructing inner products on finite-dimensional C^*-algebras, to studying automorphisms on these structures. The second is to give an introduction to quantum graphs. Finally, the third aim is to give some useful formulae and results pertaining to quantum graphs on M_n, with the end goal of proving that there are infinitely many non-isomorphisms on single-edged quantum graphs when $n > 2$.

This paper approaches the subject from a purely linear algebraic perspective, making it accessible to a broad range of mathematicians interested in linear algebra and its applications. This work has been formally verified in Lean [1]. While Lean itself is not discussed here, its presence guarantees precision and correctness.

The study of quantum graphs has progressed steadily over the past 15 years. Duan, et al. [5] introduced the concept of non-commutative graphs, drawing inspiration from quantum information theory. Shortly after, Weaver [17, 18] reformulated the notion of quantum graphs as quantum relations, and Musto, et al. [12] connected these perspectives via quantum adjacency matrix operators, showing their equivalence. Later, Daws [4] provided an accessible expository survey, aimed at introducing the topic to non-experts. They prove the equivalency between different notions in a purely algebraic method using operator algebras, while Matsuda [9] and, independently, Gromada [6], developed concrete examples and classification results on on 2×2 matrices M_2. In particular, they showed that the isomorphism of simple quantum graphs on M_2 is solely determined by the number of edges. This result prompted our investigation of higher-dimensional analogues.

Our analysis of M_3 (i.e., the set of 3×3 matrices) revealed to us that single-edged simple quantum graphs over M_3 are not always isomorphic. In our attempt to understand why this is not true for higher dimensions, we found that for *almost self-adjoint* matrices $x, y \in \mathrm{M}_n$ (where M_n is the set of $n \times n$ matrices) that have zero trace, we get $(\mathrm{M}_n, \mathrm{Tr}, A(x)) \cong (\mathrm{M}_n, \mathrm{Tr}, A(y))$ if and only if x is *almost similar* to y (see Theorem C.14). This means that, for $n > 2$, there are infinitely many non-isomorphisms of simple single-edged quantum graphs.

It is essential for readers to possess a strong background in linear algebra to fully engage with the content. Our intention is that this essay may serve as both an introduction to the topic of quantum graphs and as a foundation for further study and generalisation.

Notation

The inner product will be written as $\langle \cdot | \cdot \rangle$ and is linear in the right variable.

We write $\mathcal{L}(A, B)$ for the set of linear maps $A \to B$, and $\mathcal{L}(A) = \mathcal{L}(A, A)$. We write $\mathcal{B}(A, B)$ for the set of bounded linear maps $A \to B$ and, similarly, $\mathcal{B}(A) = \mathcal{B}(A, A)$.

Acknowledgements

First and foremost, I would like to thank my family for their love and support, without which none of this could have been possible. A special thanks also goes out to my friends *Nyla Chuku* and *Oghenetekevwe Kwakpovwe* for being there throughout all of this. Also, *Larissa Kroell* and *Junichiro Matsuda* for having useful mathematical discussions on quantum graphs with. Thanks also goes to my doctors, mentors, and advisors. I would also like to thank *Christian Voigt* for being patient with me. And thanks to everyone else who has made an impact in the last few years.

A Setting the scene

In this chapter, we set the scene by first looking at some string diagrams on $\mathbb{C}$-vector spaces and finite-dimensional Hilbert spaces. This will help us streamline calculations when dealing with algebraic and co-algebraic properties, such as the Frobenius equations (Theorem A.7).

We quickly go over some notation and recall some well-known results in linear algebra and Hilbert spaces in Sections A.III and A.IV, which can be quickly skimmed through.

We then define the Hilbert space induced by our finite-dimensional C^*-algebra B in Section A.V.

In Section A.II, we study the algebraic and co-algebraic structures of our C^*-algebra, where the co-algebraic structure is unravelled by the endowed Hilbert space. We also look at the modular automorphism σ on B in Section A.VI.

We then define and study four important maps that are used throughout, namely, the real map $\cdot^{\mathrm{r}}$ in Section A.VIII, the symmetry maps symm and symm′ in Section A.IX, and the Schur product map $\cdot \bullet \cdot$ in Section A.X. These maps are the cornerstones of the definition of a quantum graph taken as a quantum adjacency matrix (Chapter B).

In the penultimate section, we look at projections on $\mathrm{B} \otimes \mathrm{B}^{\mathrm{op}}$, which will be useful as we can define quantum graphs as such projections.

Finally, the last section looks at bimodules, which, again, will be useful as we can define quantum graphs as (B, B)-bimodule projections.

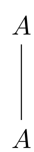
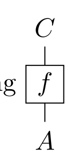
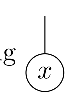
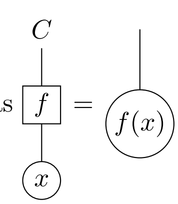
A.I String diagrams

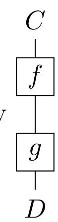
We begin with string diagrams, which are a powerful graphical calculus for simplifying complex algebraic calculations and will simplify later arguments.

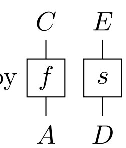
This section is based on the presentations given in [12] and [9].

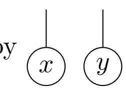
All string and planar diagrams are to be read from bottom to top.

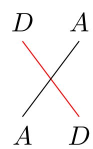
Let A, C, D, E be $\mathbb{C}$-vector spaces.

- The identity on A is denoted by the string  .
- A map $f \colon A \to C$ is denoted by the string  .
- Inputting an element $x \in A$ can be shown by the string  .
- So then $f(x)$ is shown as  .

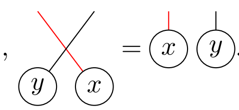
- Given $g\colon D \to A$, we have $f \circ g$ is denoted by .

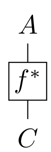
- Given $s\colon D \to E$, we have $f \otimes s$ is denoted by .

- Similarly, inputting an element $x \otimes y$ for $x \in A$ and $y \in D$, is denoted by .

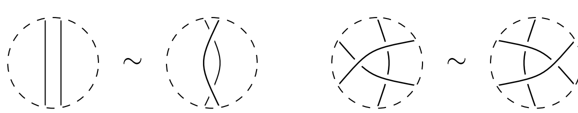
- Define $\varkappa_{A,D}$ as the identification $A \otimes D \cong D \otimes A$ given by $x \otimes y \mapsto y \otimes x$. The string diagram of $\varkappa_{A,D}$ is denoted by  .

The red and black strands are meant to highlight the fact that they are not intersecting, but overlapping (either way).

  So then it is clear that for any $x \in D$ and $y \in A$, we get, .

- When A and C are Hilbert spaces, then the adjoint of $f \in \mathcal{B}(A, C)$ is denoted by the string diagram , in other words, the string diagram of the adjoint of f is given by vertically reflecting the diagram of f.

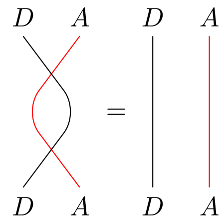
Deforming (i.e., stretching, compressing, and/or moving around the strings) the strings retain equality of the diagrams. This means we can perform planar isotopies (RO), Reidemeister moves (RII), and Reidemeister moves (RIII) [14] (see [2,8] for a concise introduction to knot theory). In knot theory, (RII) and (RIII) moves are shown as follows,



(RII) (RIII)

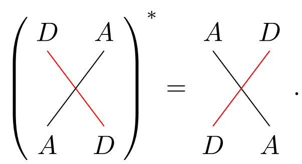
For our string diagrams, we alter the above definitions of (RII) and (RIII) so that there are no over and under crossings.

It is easy to see that string diagrams retain equality under regular isotopy. In particular, (RII) moves on string diagrams can be seen by the following,



The diagram on the left-hand side is exactly $\varkappa_{D,A}\varkappa_{A,D}$, which is the identity, and is exactly what the diagram on the right-hand side represents. This means $\varkappa_{A,D}^* = \varkappa_{A,D}^{-1}$:

Lemma A.1. $\varkappa_{A,D}^* = \varkappa_{A,D}^{-1} = \varkappa_{D,A}$. *In other words,*

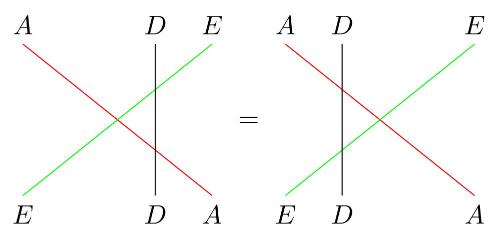


Proof. Let $x, w \in A$ and $y, z \in D$. Then we compute,

$$\begin{aligned}\left\langle x \otimes y \middle| \varkappa_{A,D}^*(z \otimes w)\right\rangle_{A\otimes D} &= \langle y \otimes x | z \otimes w\rangle_{D\otimes A} = \langle y|z\rangle_D \langle x|w\rangle_A \\ &= \langle x \otimes y | w \otimes z\rangle_{A\otimes D} = \left\langle x \otimes y \middle| \varkappa_{A,D}^{-1}(z \otimes w)\right\rangle_{A\otimes D}.\end{aligned}$$

Thus $\varkappa_{A,D}^* = \varkappa_{A,D}^{-1}$. ■

Similarly, (RIII) can be seen by the following,



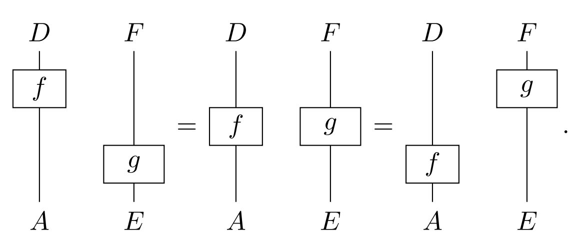
The diagram at the left-hand side is exactly $(\mathrm{id} \otimes \varkappa_{E,D})(\varkappa_{E,A} \otimes \mathrm{id})(\mathrm{id} \otimes \varkappa_{D,A})$, and the diagram at the right-hand side is exactly $(\varkappa_{D,A} \otimes \mathrm{id})(\mathrm{id} \otimes \varkappa_{E,A})(\varkappa_{E,D} \otimes \mathrm{id})$. An easy and quick computation shows that these two expressions are equal (they both flip the first and last tensor-element and keep the second in place).

The next result is about using heights as a graphical tool and is useful when manipulating string diagrams. The heights are shown to be algebraically irrelevant. It is simply regarded as planar isotopy when string diagrams are involved (i.e., we just move the boxes up/down).

Lemma A.2. *Given linear maps* $f\colon A \to D$ *and* $g\colon E \to F$, *we get*

$$(f \otimes \mathrm{id}_F)(\mathrm{id}_A \otimes g) = f \otimes g = (\mathrm{id}_D \otimes g)(f \otimes \mathrm{id}_E).$$

In other words,

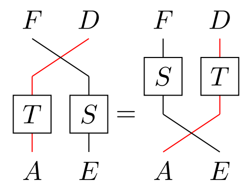


Proof. Obvious. ■

Lemma A.3. *Given linear maps* $T\colon A \to D$ *and* $S\colon E \to F$*, we get*

$$\varkappa_{D,F}(T \otimes S) = (S \otimes T)\varkappa_{A,E}.$$

In other words,



Proof. Straightforward computation. ■

A.II The algebraic and co-algebraic structures

With these diagrammatic conventions in place, we can now turn to algebras and co-algebras, which we will look at both diagrammatically and algebraically.

Notation. We work with a *strict* version (see more on this [13, p.37]) of the tensor product, i.e., for $\mathbb{C}$-vector spaces X, Y, Z, we have,

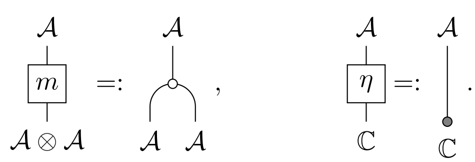
- $(x \otimes y) \otimes w = x \otimes (y \otimes w)$ for all $x \in X$, $y \in Y$, and $w \in W$,
- $\alpha \otimes x = \alpha x = x \otimes \alpha$ for all $\alpha \in \mathbb{C}$ and $x \in X$.

A.II.1 Algebras.

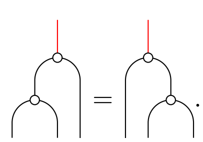
Definition A.4 (Algebra)**.** An *algebra* is a vector space together with a multiplication linear map $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a unit linear map $\eta\colon \mathbb{C} \to \mathcal{A}$, with a unit $1 \in \mathcal{A}$ such that $m(1 \otimes x) = m(x \otimes 1) = x$ for all $x \in \mathcal{A}$, and with the properties of associativity $m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m)$ and $m(\eta \otimes \mathrm{id}) = \mathrm{id} = m(\mathrm{id} \otimes \eta)$.

We call $(\mathcal{A}, m, \eta)$ an algebra if these properties hold.

In strings, m and η are denoted by:



Each relation in our algebra corresponds to equivalences on our diagrams. For instance, associativity of m [9, bottom of page 3] is essentially given by $m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m)$ (in other words, $(xy)z = x(yz)$); this is shown diagrammatically by the following,

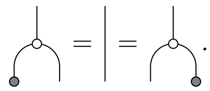


It is easy to *picture* the above equivalency by moving the vertical line (coloured in red) with the empty circle to the other empty circle.
To see this algebraically, we let $x, y, z \in B$, and compute,

$$m(m \otimes \mathrm{id})(x \otimes y \otimes z) = m(xy \otimes z) = (xy)z = x(yz) = m(x \otimes yz) = m(\mathrm{id} \otimes m)(x \otimes y \otimes z).$$

Thus $m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m)$.

Similarly, the relation $m(\eta \otimes \mathrm{id}) = \mathrm{id} = m(\mathrm{id} \otimes \eta)$ in strings is given by,

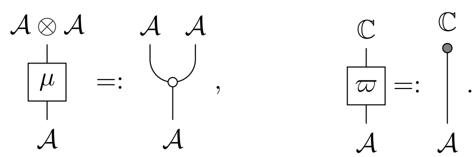


Algebraically, let $\alpha \in \mathbb{C}$ and $x \in \mathcal{A}$, then compute $m(\eta \otimes \mathrm{id})(\alpha \otimes x) = m(\alpha 1 \otimes x) = \alpha x = \alpha \otimes x$. Similarly, $m(\mathrm{id} \otimes \eta)(\alpha \otimes x) = m(\mathrm{id} \otimes \eta)(x \otimes \alpha) = \alpha x = \alpha \otimes x$.

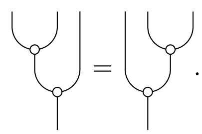
A.II.2 Co-algebras.

> **Definition A.5.** A vector space $\mathcal{A}$ is a *co-algebra* when it has a co-multiplication linear map $\mu\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and a co-unit linear map $\varpi\colon \mathcal{A} \to \mathbb{C}$ such that co-associativity is satisfied, i.e., $(\mu \otimes \mathrm{id})\mu = (\mathrm{id} \otimes \mu)\mu$, and the property $(\varpi \otimes \mathrm{id})\mu = \mathrm{id} = (\mathrm{id} \otimes \varpi)\mu$ is satisfied. We say $(\mathcal{A}, \mu, \varpi)$ is a co-algebra when those properties are satisfied.

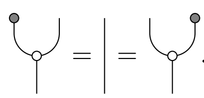
In strings, μ and ϖ are denoted by:



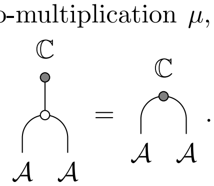
The co-associativity property [9, top of page 3] is then shown diagrammatically by,



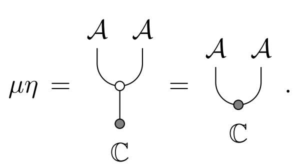
Finally, the property $(\varpi \otimes \mathrm{id})\mu = \mathrm{id} = (\mathrm{id} \otimes \varpi)\mu$ is shown diagrammatically by,



A.II.3 Algebras and co-algebras. Let $(\mathcal{A}, m, \eta, \mu, \varpi)$ be both an algebra and a co-algebra with multiplication map m, co-multiplication μ, unit map η, and co-unit ϖ. Then when m is composed with ϖ, we draw

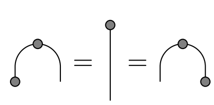


Similarly, we draw

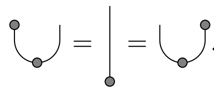


We summarise string diagrams and their connections to algebras and co-algebras in Appendix E.

Lemma A.6. *Given a vector space $\mathcal{A}$ which is both an algebra and a co-algebra with multiplication m, unit η, co-multiplication μ, and co-unit ϖ, we get $\varpi m(\eta \otimes \mathrm{id}) = \varpi = \varpi m(\mathrm{id} \otimes \eta)$:*



and $(\varpi \otimes \mathrm{id})\mu\eta = \eta = (\mathrm{id} \otimes \varpi)\mu\eta$*:*



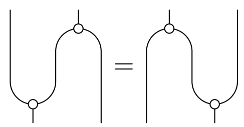
Proof. This is done by composing (unit_id) and (co_unit_id) with η and ϖ. Note that (unit_id) and (co_unit_id) are references to the properties highlighted in Appendix E. ∎

The following result highlights the power and usefulness of using string diagrams.

Theorem A.7 (the Frobenius equations)**.** *In general, given a vector space $\mathcal{A}$ which is both an algebra and a co-algebra with multiplication map m, unit map η, co-multiplication map μ, and co-unit map ϖ, then if*

$$(\mathrm{id} \otimes m)(\mu \otimes \mathrm{id}) = (m \otimes \mathrm{id})(\mathrm{id} \otimes \mu), \qquad (\diamond)$$

which in string diagrams is,

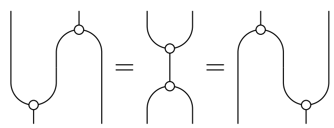


(*)

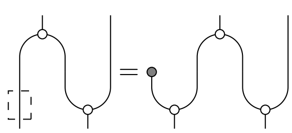
then we get the following equations (the Frobenius equations),

$$(\mathrm{id} \otimes m)(\mu \otimes \mathrm{id}) = \mu m = (m \otimes \mathrm{id})(\mathrm{id} \otimes \mu),$$

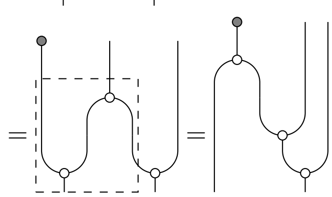
which in string diagrams is,



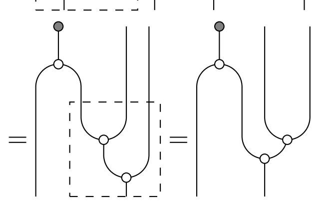
Proof. We show, using the algebraic and co-algebraic structure given in strings in Table 1 in Appendix E and the hypothesis (Equation (*)), that we get the Frobenius equations. Hence we compute,



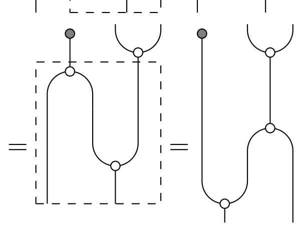
by (co_unit_id)



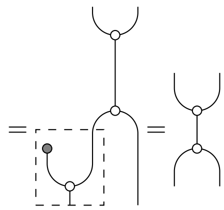
by (*)



by (co_mul_assoc)



by (*)



by (co_unit_id).

For the sake of completeness, we will show this algebraically to explore the usefulness and power of using string diagrams. We do this by demonstrating that it is not as easy to follow algebraically. "Planar isotopies" correspond to multiple algebraic identities, which can make the algebraic approach cumbersome.

In the following computation, we denote the identity operator on $\mathbb{C}$ by $\mathrm{id}_{\mathbb{C}}$ (and the identity operator on $\mathcal{A}$ remains as id). The idea is to use the fact that we have $\alpha \otimes x = x = x \otimes \alpha$ for all $\alpha \in \mathbb{C}$ and $a \in X$ for any algebra X.

We translate the calculations we made with the string diagrams above to compute,

$$
\begin{aligned}
(m \otimes \mathrm{id})(\mathrm{id} \otimes \mu) &= (m \otimes \mathrm{id})((\varpi \otimes \mathrm{id})\mu \otimes \mu) && \text{by (co\_unit\_id)}\\
&= (m \otimes \mathrm{id})(\varpi \otimes \mathrm{id}^{\otimes 3})\mu^{\otimes 2}\\
&= (m(\varpi \otimes \mathrm{id}^{\otimes 2}) \otimes \mathrm{id})\mu^{\otimes 2}\\
&= ((\mathrm{id}_{\mathbb{C}} \otimes m)(\varpi \otimes \mathrm{id}^{\otimes 2}) \otimes \mathrm{id})\mu^{\otimes 2}\\
&= ((\varpi \otimes \mathrm{id})(\mathrm{id} \otimes m) \otimes \mathrm{id})\mu^{\otimes 2} && \text{by A.2}\\
&= (\varpi \otimes \mathrm{id}^{\otimes 2})(\mathrm{id} \otimes m \otimes \mathrm{id})(\mu \otimes \mathrm{id}^{\otimes 2})(\mathrm{id} \otimes \mu)\\
&= (\varpi \otimes \mathrm{id}^{\otimes 2})((\mathrm{id} \otimes m)(\mu \otimes \mathrm{id}) \otimes \mathrm{id})(\mathrm{id} \otimes \mu)\\
&= (\varpi \otimes \mathrm{id}^{\otimes 2})((m \otimes \mathrm{id})(\mathrm{id} \otimes \mu) \otimes \mathrm{id})(\mathrm{id} \otimes \mu) && \text{by } (\diamond)\\
&= (\varpi \otimes \mathrm{id}^{\otimes 2})(m \otimes \mathrm{id}^{\otimes 2})(\mathrm{id} \otimes (\mu \otimes \mathrm{id})\mu)\\
&= (\varpi m \otimes \mathrm{id}^{\otimes 2})(\mathrm{id} \otimes (\mathrm{id} \otimes \mu)\mu) && \text{by (co\_mul\_assoc)}\\
&= (\varpi \otimes \mathrm{id}^{\otimes 2})(m \otimes \mathrm{id}^{\otimes 2})(\mathrm{id}^{\otimes 2} \otimes \mu)(\mathrm{id} \otimes \mu)\\
&= (\varpi \otimes \mathrm{id}^{\otimes 2})(\mathrm{id} \otimes \mu)(m \otimes \mathrm{id})(\mathrm{id} \otimes \mu) && \text{by A.2}\\
&= (\varpi \otimes \mathrm{id}^{\otimes 2})(\mathrm{id} \otimes \mu)(\mathrm{id} \otimes m)(\mu \otimes \mathrm{id}) && \text{by } (\diamond)\\
&= (\varpi \otimes \mathrm{id}^{\otimes 2})(\mathrm{id} \otimes \mu m)(\mu \otimes \mathrm{id})\\
&= (\mathrm{id}_{\mathbb{C}} \otimes \mu m)(\varpi \otimes \mathrm{id}^{\otimes 2})(\mu \otimes \mathrm{id}) && \text{by A.2}\\
&= \mu m((\varpi \otimes \mathrm{id})\mu \otimes \mathrm{id})\\
&= \mu m(\mathrm{id} \otimes \mathrm{id}) && \text{by (co\_unit\_id)}\\
&= \mu m.
\end{aligned}
$$

■

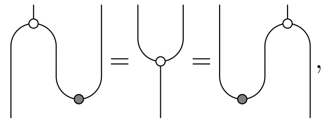
Using Theorem A.7, it thus suffices to show Equation (*) in order to show the Frobenius equation, when we have an algebraic and co-algebraic structure.

We get the following by composing η to the Frobenius equations (when the Frobenius equations are satisfied).

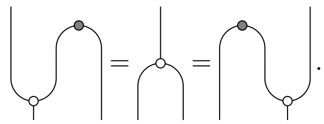
Proposition A.8 ([6, Corollary 1.6(2)]). *Let $(\mathcal{A}, m, \eta, \mu, \varpi)$ be an algebra and a co-algebra such that $(m \otimes \mathrm{id})(\mathrm{id} \otimes \mu) = (\mathrm{id} \otimes m)(\mu \otimes \mathrm{id})$. Then we have,*

(i) $(m \otimes \mathrm{id})(\mathrm{id} \otimes \mu\eta) = \mu = (\mathrm{id} \otimes m)(\mu\eta \otimes \mathrm{id})$,

in other words,



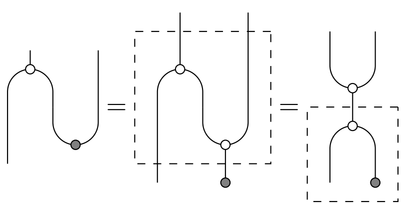
(ii) $(\mathrm{id} \otimes \varpi m)(\mu \otimes \mathrm{id}) = m = (\varpi m \otimes \mathrm{id})(\mathrm{id} \otimes \mu)$,
in other words,



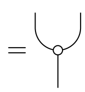
Proof. We only show Part (i), since Part (ii) is shown similarly. We compute,

$$\begin{aligned}(m \otimes \mathrm{id})(\mathrm{id} \otimes \mu\eta) &= (m \otimes \mathrm{id})(\mathrm{id} \otimes \mu)(\mathrm{id} \otimes \eta) \\ &= \mu m(\mathrm{id} \otimes \eta) && \text{by A.7} \\ &= \mu && \text{by (unit\_id).}\end{aligned}$$

Diagrammatically, the proof corresponds to:



by A.7

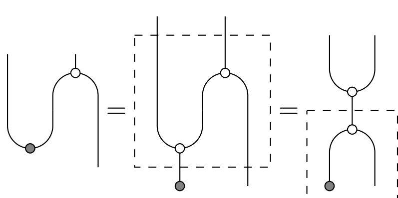


by (unit_id).

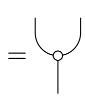
Similarly, we compute,

$$\begin{aligned}(\mathrm{id} \otimes m)(\mu\eta \otimes \mathrm{id}) &= (\mathrm{id} \otimes m)(\mu \otimes \mathrm{id})(\eta \otimes \mathrm{id}) \\ &= \mu m(\eta \otimes \mathrm{id}) && \text{by A.7} \\ &= \mu && \text{by (unit\_id).}\end{aligned}$$

In strings, this is:



by A.7



by (unit_id).

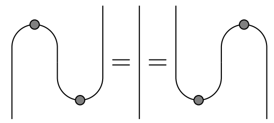
Thus $(m \otimes \mathrm{id})(\mathrm{id} \otimes \mu\eta) = \mu = (\mathrm{id} \otimes m)(\mu\eta \otimes \mathrm{id})$. ∎

The following result is sometimes referred to as the *snake equations*, and is given by composing η^* and η to the Frobenius equations (when the Frobenius equations are satisfied).

Proposition A.9 (Snake equations)**.** *Let* $(\mathcal{A}, m, \eta, \mu, \varpi)$ *be an algebra and a co-algebra such that* $(m \otimes \mathrm{id})(\mathrm{id} \otimes \mu) = (\mathrm{id} \otimes m)(\mu \otimes \mathrm{id})$*. Then we get*

$$(\varpi m \otimes \mathrm{id})(\mathrm{id} \otimes \mu\eta) = \mathrm{id} = (\mathrm{id} \otimes \varpi m)(\mu\eta \otimes \mathrm{id}),$$

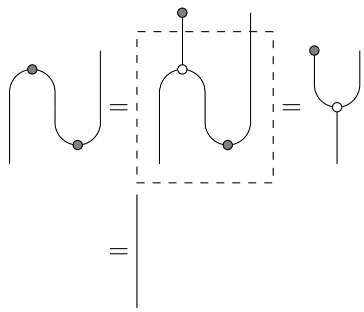
in other words,



Proof. We compute,

$$\begin{aligned}(\varpi m \otimes \mathrm{id})(\mathrm{id} \otimes \mu\eta) &= (\varpi \otimes \mathrm{id})(m \otimes \mathrm{id})(\mathrm{id} \otimes \mu\eta) \\ &= (\varpi \otimes \mathrm{id})\mu && \text{by A.8(i)} \\ &= \mathrm{id} && \text{by (co\_unit\_id).}\end{aligned}$$

In strings, this is:



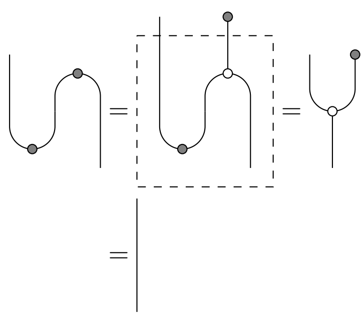
by A.8(i)

by (co_unit_id).

Similarly, we compute,

$$\begin{aligned}(\mathrm{id} \otimes \varpi m)(\mu\eta \otimes \mathrm{id}) &= (\mathrm{id} \otimes \varpi)(\mathrm{id} \otimes m)(\mu\eta \otimes \mathrm{id}) \\ &= (\mathrm{id} \otimes \varpi)\mu && \text{by A.8(i)} \\ &= \mathrm{id} && \text{by (co\_unit\_id).}\end{aligned}$$

In strings this is exactly:



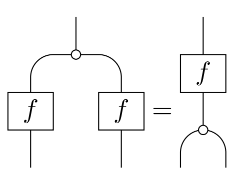
by A.8(i)

by (co_unit_id).

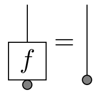
■

Proposition A.10. *Let $(\mathcal{A}_1, m_1, \eta_1)$ and $(\mathcal{A}_2, m_2, \eta_2)$ be algebras. Let $f\colon \mathcal{A}_1 \to \mathcal{A}_2$ be a linear map. Then we have,*

(i) $m_2 \circ (f \otimes f) = f \circ m_1 \Leftrightarrow \forall x, y \in \mathcal{A}_1 : f(xy) = f(x)f(y)$, *in other words,*



$\Leftrightarrow$ *f preserves multiplication,* *(ii)* $f \circ \eta_1 = \eta_2 \Leftrightarrow f(1) = 1$, *in other words,*



$\Leftrightarrow$ *f preserves the unit.*

Proof.

(i) For any $x, y \in \mathcal{A}_1$, we have,

$$m_2 \circ (f \otimes f)(x \otimes y) = f \circ m_1(x \otimes y) \Leftrightarrow m_2(f(x) \otimes f(y)) = f(xy) \\ \Leftrightarrow f(x)f(y) = f(xy).$$

(ii) We have, $f \circ \eta_1 = \eta_2 \Leftrightarrow \forall x \in \mathbb{C} : f(\eta_1(x)) = \eta_2(x) \Leftrightarrow \forall x \in \mathbb{C} : xf(1) = x1 \Leftrightarrow f(1) = 1.$

■

From the above proposition, we see that for an algebra homomorphism $f \colon \mathcal{A}_1 \to \mathcal{A}_2$, i.e., a linear map that preserves multiplication and preserves the unit, we get $m_2 \circ (f \otimes f) = f \circ m_1$ and $f \circ \eta_1 = \eta_2$.

A.II.4 Tensor products of co-algebras. Let $(\mathcal{A}_1, m_1, \eta_1)$ and $(\mathcal{A}_2, m_2, \eta_2)$ be algebras. Then $\mathcal{A}_1 \otimes \mathcal{A}_2$ is an algebra with multiplication linear map $(m_1 \otimes m_2)(\mathrm{id} \otimes \varkappa \otimes \mathrm{id})$ and unit linear map $\eta_1 \otimes \eta_2$. Similarly, when $(\mathcal{A}_1, \mu_1, \varpi_1)$ and $(\mathcal{A}_2, \mu_2, \varpi_2)$ are co-algebras, then $\mathcal{A}_1 \otimes \mathcal{A}_2$ is also a co-algebra with co-multiplication $(\mathrm{id} \otimes \varkappa \otimes \mathrm{id})(\mu_1 \otimes \mu_2)$ and co-unit $\varpi_1 \otimes \varpi_2$. It is left as an exercise to check that this indeed forms a co-algebra.

A.II.5 Finite-dimensional Hilbert space algebras induce a co-algebra. Let $(\mathcal{A}, m, \eta)$ be a finite-dimensional algebra and a Hilbert space. Then we can let m^* be the co-multiplication and η^* be the co-unit. Recall that by T^*, we mean the Hilbert space adjoint of T.

Proposition A.11. *Let $(\mathcal{A}, m, \eta)$ be a finite-dimensional algebra and Hilbert space. Then we can form a co-algebra $(\mathcal{A}, m^*, \eta^*)$ by letting m^* be the co-multiplication and η^* be the co-unit.*

There is also the opposite implication: if we start from a finite-dimensional co-algebra and Hilbert space $(\mathcal{A}, \mu, \varpi)$ then we can form an algebra $(\mathcal{A}, \mu^*, \varpi^*)$.

Proof.

($\Rightarrow$) **comul_assoc:** We have the following equivalences,

$$(m^* \otimes \mathrm{id})m^* = (\mathrm{id} \otimes m^*)m^* \Leftrightarrow (m(m \otimes \mathrm{id}))^* = (m(\mathrm{id} \otimes m))^* \\ \Leftrightarrow m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m),$$

which is true since $\mathcal{A}$ is an algebra.

counit_comul_id: Similarly, taking adjoints, we get

$$(\eta^* \otimes \mathrm{id})m^* = \mathrm{id} = (\mathrm{id} \otimes \eta^*)m^* \Leftrightarrow m(\eta \otimes \mathrm{id}) = \mathrm{id} = m(\mathrm{id} \otimes \eta),$$

which is true since $\mathcal{A}$ is an algebra.

($\Leftarrow$) We define multiplication on $\mathcal{A}$ by $(x, y) \mapsto \mu^*(x \otimes y)$, which is bilinear. So our multiplication linear map is μ^*. We let the unit linear map be ϖ^*.

We get associativity and $\mu^*(\varpi^* \otimes \mathrm{id}) = \mathrm{id} = \mu^*(\mathrm{id} \otimes \varpi^*)$ by following an analogous proof of the above.

We leave the proof of showing $\varpi^*(1)$ is indeed a unit in $\mathcal{A}$ as an exercise to the reader.

■

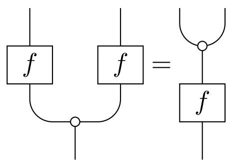
Lemma A.12. *Let $(\mathcal{A}, m, \eta)$ be a finite-dimensional algebra and Hilbert space. Then η^* is given by $y \mapsto \langle 1|y\rangle$.*

Proof. For $y \in \mathcal{A}$ and $x \in \mathbb{C}$, we have

$$\langle x|\eta^*(y)\rangle = \langle \eta(x)|y\rangle = \overline{x}\langle 1|y\rangle = \langle x|\langle 1|y\rangle\rangle.$$

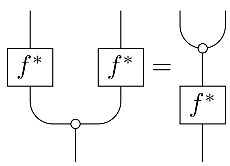
Thus $\eta^*(y) = \langle 1|y\rangle$. ■

Corollary A.13. *Let $(\mathcal{A}_1, m_1, \eta_1), (\mathcal{A}_2, m_2, \eta_2)$ be finite-dimensional algebras and Hilbert spaces. Then, for a linear map $f\colon \mathcal{A}_1 \to \mathcal{A}_2$, we get*
$(f \otimes f) \circ m_1^* = m_2^* \circ f \Leftrightarrow \forall x, y \in \mathcal{A}_2 : f^*(xy) = f^*(x)f^*(y)$, in other words,



$\Leftrightarrow$ f^* preserves multiplication,

moreover, $(f^* \otimes f^*) \circ m_2^* = m_1^* \circ f^* \Leftrightarrow \forall x, y \in \mathcal{A}_1 : f(xy) = f(x)f(y)$, i.e.,



$\Leftrightarrow$ f preserves multiplication.

Proof. Take adjoints of Proposition A.10. ■

Corollary A.14. *Let $(\mathcal{A}_1, m_1, \eta_1)$ and $(\mathcal{A}_2, m_2, \eta_2)$ be finite-dimensional algebras and Hilbert spaces. Then, given an algebra homomorphism $f\colon \mathcal{A}_1 \to \mathcal{A}_2$, we get*

1. *$m_2 \circ (f \otimes f) = f \circ m_1$,*
2. *$f \circ \eta_1 = \eta_2$,*
3. *$(f^* \otimes f^*) \circ m_2^* = m_1^* \circ f^*$.*

Proof. This is immediate from Proposition A.10 and Corollary A.13. ■

Proposition A.15. *Let $(\mathcal{A}_1, m_1, \eta_1), (\mathcal{A}_2, m_2, \eta_2)$ be finite-dimensional algebras and Hilbert spaces, and let $f\colon \mathcal{A}_1 \to \mathcal{A}_2$ be a linear map. Then f is an algebra homomorphism if and only if f^* is a co-algebra homomorphism (i.e., a linear map such that $(f^* \otimes f^*)m_2^* = m_1^* f^*$ and $\eta_1^* f^* = \eta_2^*$).*

Proof.

$$\begin{aligned} f^* \text{ is a co-algebra hom} &\Leftrightarrow (f^* \otimes f^*) \circ m_2^* = m_1^* \circ f^* \text{ and } \eta_1^* \circ f^* = \eta_2^* \\ &\Leftrightarrow m_2 \circ (f \otimes f) = f \circ m_1 \text{ and } f \circ \eta_1 = \eta_2 \\ &\Leftrightarrow f \text{ is an algebra hom.} \end{aligned}$$

■

A.III Ket-bra operators

In this short section, we go over some notation and well-known results in C*-algebras and Hilbert spaces.

Let E_1, E_2 be inner product spaces over $\mathbb{C}$.

> **Definition A.16** (ket, bra)**.** A *ket* operator $|\cdot\rangle$ is defined as the linear map $E_1 \to \mathcal{B}(\mathbb{C}, E_1)$ given by $x \mapsto (\alpha \mapsto \alpha x)$. A *bra* operator $\langle\cdot|$ is defined as the anti-linear map $E_1 \to \mathcal{B}(E_1, \mathbb{C})$ given by $x \mapsto (y \mapsto \langle x|y\rangle)$. A *ket-bra* operator $|\cdot\rangle\langle\cdot|$ is given by composing a $|\cdot\rangle$ with a $\langle\cdot|$, i.e., $|x\rangle\langle y| = |x\rangle \circ \langle y|$.

So a ket-bra is a linear map from E_2 to the anti-linear map $E_1 \to \mathcal{B}(E_1, E_2)$ and is given by

$$x \mapsto (y \mapsto (u \mapsto \langle y|u\rangle x)).$$

It is easy to see that we get $|x\rangle^* = \langle x|$ for any $x \in E_1$. Composing a bra with a ket gives us a scalar multiplication of the inner product, i.e., $\langle x||y\rangle = |\langle x|y\rangle\rangle$. So then $\langle x||y\rangle(1) = \langle x|y\rangle$.

Given an orthonormal basis (u_i) of a $\mathbb{C}$-inner product space E, we get $\sum_i |u_i\rangle\langle u_i| = \mathrm{id}$.

Lemma A.17. *Given Hilbert spaces E_1, E_2, E_3, linear maps $T_1 \in \mathcal{B}(E_2, E_3)$, $T_2 \in \mathcal{B}(E_3, E_1)$, and elements $x \in E_2$, $y \in E_1$, we get,*

(i) $T_1 \circ |x\rangle = |T_1(x)\rangle$,

(ii) $\langle y| \circ T_2 = \langle T_2^*(y)|$,

(iii) $|x\rangle\langle y|^* = |y\rangle\langle x|$.

Proof. This is a direct computation. ■

Remark A.18. For x in a Hilbert space $\mathcal{H}$, we have $|x\rangle = |y\rangle$ if and only if $x = y$ and similarly $\langle x| = \langle y|$ if and only if $x = y$ (straightforward computation). ♢

Proposition A.19. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and let $a, c \in \mathcal{H}_1 \setminus \{0\}$ and $b, d \in \mathcal{H}_2 \setminus \{0\}$. Then if $|a\rangle\langle b| = |c\rangle\langle d|$, then there exists some $0 \neq \alpha, \beta \in \mathbb{C}$ such that $a = \alpha c$ and $b = \alpha\beta d$.*

Proof. As $|a\rangle\langle b| = |c\rangle\langle d|$, we get $\langle b|b\rangle a = \langle d|b\rangle c$, which means $a = \dfrac{\langle d|b\rangle}{\|b\|^2} c$ (since $b \neq 0$). Taking adjoints of the hypothesis, we get $|b\rangle\langle a| = |d\rangle\langle c|$, and as $a \neq 0$, we also get

$$b = \frac{\langle c|a\rangle}{\|a\|^2} d = \frac{\langle d|b\rangle\, \|c\|^2}{\|a\|^2\, \|b\|^2} d.$$

Clearly, $\langle d|b\rangle \neq 0$ (otherwise, we get $a = 0$). Thus we can let $\alpha = \dfrac{\langle d|b\rangle}{\|b\|^2}$ and $\beta = \dfrac{\|c\|^2}{\|a\|^2}$. ■

Given a finite-dimensional $\mathbb{C}$-inner product space E, we say (u_i) is an *eigenbasis* of $x \in \mathcal{B}(E)$ in E with corresponding eigenvalues (λ_i), when (u_i) is an orthonormal basis of E such that each u_i is an eigenvector of x with a corresponding eigenvalue λ_i, in other words, $x(u_i) = \lambda_i u_i$ for each i.

Theorem A.20 (spectral theorem [3, Theorem 7.9])**.** *Given a finite-dimensional inner product space E over $\mathbb{C}$ and $T \in \mathcal{B}(E)$, then,*
$TT^* = T^*T \;\Leftrightarrow$ *there exists an eigenbasis of T in E.*

Proof. See [3, Theorem 7.9]. □

Thus any normal operator on a finite-dimensional Hilbert space has an orthonormal eigenbasis.

Corollary A.21. *Given a finite-dimensional inner product space E over $\mathbb{C}$ and $T \in \mathcal{B}(E)$, then, $TT^* = T^*T$ if and only if $T = \sum_i \lambda_i |u_i\rangle\langle u_i|$ for an eigenbasis (u_i) of T in E with corresponding eigenvalues (λ_i).*

Proof. Suppose $TT^* = T^*T$. Then from Theorem A.20, we know there exists an eigenbasis (u_i) of T in E. This means $\sum_i |u_i\rangle\langle u_i| = \text{id}$. Also $T(u_i) = \lambda_i u_i$ for each i, by definition of an eigenbasis.

So then, for any $a \in E$, we compute,

$$T(a) = \sum_i T(|u_i\rangle\langle u_i|(a)) = \sum_i \langle u_i|a\rangle T(u_i) = \sum_i \langle u_i|a\rangle \lambda_i u_i = \sum_i \lambda_i |u_i\rangle\langle u_i|(a).$$

Thus $T = \sum_i \lambda_i |u_i\rangle\langle u_i|$.

If, on the other hand, $T = \sum_i \lambda_i |u_i\rangle\langle u_i|$ for eigenbasis (u_i) of T with corresponding eigenvalues (λ_i). Then one can easily see that we get $TT^* = T^*T$. ■

Definition A.22 ([11, pg. 45, 46]). Given a C*-algebra $\mathcal{A}$ and elements $a, b \in \mathcal{A}$, we define the partial order $a \leq b$ as $b - a$ being self-adjoint and $\text{Spectrum}(b - a) \subseteq [0, \infty)$.

So, we can then say $x \in \mathcal{A}$ is *positive semi-definite* (non-negative) if it is self-adjoint and $\text{Spectrum}(x) \subseteq [0, \infty)$.

We say $x \in \mathcal{A}$ is *positive definite* if it is self-adjoint and $\text{Spectrum}(x) \subseteq (0, \infty)$. This is equivalent to positive semi-definiteness and invertibility. I.e., $x \in \mathcal{A}$ is positive definite if and only if $0 \leq x$ and x is invertible.

Theorem A.23 ([11, Theorem 2.2.1]). *Given $0 \leq a$ in a C*-algebra $\mathcal{A}$, there exists a unique element $0 \leq b \in \mathcal{A}$ such that $a = b^2$. This is known as the unique positive square root of the positive semi-definite element a, and can be denoted by $\sqrt{a}$.*

Proof. See [11, Theorem 2.2.1]. □

Theorem A.24 ([11, Theorem 2.2.5(1)]). *Given any element x in a C*-algebra $\mathcal{A}$, we get $0 \leq x$ if and only if $x = y^*y$ for some $y \in \mathcal{A}$.*

Proof. See [11, Theorem 2.2.5(1)]. □

Lemma A.25. *Given an algebra $\mathcal{A}$ over $\mathbb{C}$ and $x, y \in \mathcal{A}$, we have*
$\text{Spectrum}(xy) \setminus \{0\} = \text{Spectrum}(yx) \setminus \{0\}$.

Proof. Let $\lambda \in \mathbb{C} \setminus \{0\}$. Then we want to show that $\lambda 1 - xy$ is invertible if and only if $\lambda 1 - yx$ is invertible, which is equivalent to showing $1 - (\lambda^{-1}x)y$ is invertible if and only if $1 - y(\lambda^{-1}x)$ is invertible (since $\lambda \neq 0$).

So then it clearly suffices to show that for any $a, b \in \mathcal{A}$, we get $1 - ab$ is invertible implies $1 - ba$ is invertible. Suppose $a, b \in \mathcal{A}$, and $1 - ab$ is invertible. Then $1 - ba$ is also invertible with inverse $1 + b(1 - ab)^{-1}a$, since,

$$\begin{aligned}(1 - ba)(1 + b(1 - ab)^{-1}a) &= 1 - ba + b(1 - ab)^{-1}a - bab(1 - ab)^{-1}a \\ &= 1 - ba + b(1 - ab)(1 - ab)^{-1}a \\ &= 1 - ba + ba = 1,\end{aligned}$$

and, analogously,

$$\begin{aligned}(1 + b(1 - ab)^{-1}a)(1 - ba) &= 1 - ba + b(1 - ab)^{-1}a - b(1 - ab)^{-1}aba \\ &= 1 - ba + b(1 - ab)^{-1}(1 - ab)a \\ &= 1 - ba + ba = 1.\end{aligned}$$

■

Lemma A.26. *Given positive elements $x, y \in \mathcal{A}$, we have $xy = yx$ if and only if $0 \leq xy$.*

Proof. Firstly, xy is self-adjoint if and only if $yx = y^*x^* = (xy)^* = xy$, as both x and y are self-adjoint. So if $0 \leq xy$, then we know xy is self-adjoint, and so $xy = yx$. Suppose that we have $xy = yx$. We want to show Spectrum$(xy) \subseteq [0, \infty)$.

Now, using Theorem A.24, we let $a, b \in \mathcal{A}$ such that $x = a^*a$ and $y = b^*b$. So then we compute,

$$\begin{aligned} \text{Spectrum}(xy) \setminus \{0\} &= \text{Spectrum}(a^*ab^*b) \setminus \{0\} \\ &= \text{Spectrum}(ab^*ba^*)\{0\} && \text{by A.25} \\ &= \text{Spectrum}((ba^*)^*ba^*) \setminus \{0\} \subseteq [0, \infty) \setminus \{0\} && \text{by A.24.} \end{aligned}$$

Thus $0 \leq xy$. ■

Let E be a finite-dimensional inner product space over $\mathbb{C}$. For $x \in \mathcal{B}(E)$, we say it is *positive semi-definite* if $0 \leq x$ which means it is self-adjoint and has non-negative spectrum. It is easy to see that this is equivalent ([11, Theorem 2.3.5]) to $0 \leq \langle u|x(u)\rangle$ for all $u \in E$. We say $x \in \mathcal{L}(E)$ is *positive definite* if $0 < \langle u|x(u)\rangle$ for all $0 \neq u \in E$ (which is clearly equivalent to positive definite-ness from Definition A.22). Note that we drop the self-adjoint-edness requirement since we are working on $\mathbb{C}$ and is implied by $0 \leq \langle a|x(a)\rangle$ (more specifically, $\langle a|x(a)\rangle \in \mathbb{R}$ if and only if x is self-adjoint).

Lemma A.27. *Given a finite-dimensional inner product space E over $\mathbb{C}$ and $T \in \mathcal{L}(E)$, we get*

T is positive semi-definite $\Leftrightarrow T = \sum_i |v_i\rangle\langle v_i|$ for some tuple (v_i) in E.

Proof.

($\Rightarrow$) Suppose $0 \leq T$. As positive semi-definiteness of T implies $TT^* = T^*T$, we can use Theorem A.20 and let (v_i) be the eigenbasis of T in E with corresponding eigenvalues (λ_i). Note that, by the above discussion, as $0 \leq T$, we also get each $0 \leq \lambda_i$. So then let each $x_i = \sqrt{\lambda_i} u_i$. Then we have $\sum_i |x_i\rangle\langle x_i| = \sum_i \sqrt{\lambda_i}\overline{\sqrt{\lambda_i}} |u_i\rangle\langle u_i| = \sum_i \lambda_i |u_i\rangle\langle u_i| = T$, where the last equality comes from Corollary A.21.

($\Leftarrow$) Suppose we have some tuple (v_i) in E such that $T = \sum_i |v_i\rangle\langle v_i|$. Then, for any $x \in E$, we get $\langle x|T(x)\rangle = \sum_i \langle x|v_i\rangle\langle v_i|x\rangle = \sum_i |\langle x|v_i\rangle|^2 \geq 0$. Thus T is positive semi-definite.

■

A.IV From linear maps to matrices

In this section, we go over our notation for matrices and then identify our linear operators as matrices as it is usually easier to work with.

Let $n, m \in \mathbb{N}$.

Notation.

- We denote $M_{n,m}$ to be the set of $n \times m$ matrices over $\mathbb{C}$.
- When $n = m$, we write M_n instead.
- For $x \in \mathbb{N}$, we write $[x]$ to mean $\{1, \ldots, x\}$.

Given any $a \in M_{n,m}$, we say a_{ij} for (i, j)-th entry of a. We write $e_{ij} \in M_n$ for the standard basis in matrix form (i.e., $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$, in other words, it has value 1 at (i, j) and 0 elsewhere). So a typical matrix $a \in M_n$ is given by $a = \sum_{i,j} a_{ij}e_{ij}$, and $\text{Tr}(e_{ij}) = \delta_{ij}$, where Tr is the trace of a matrix given by the sum of the diagonal, i.e., $\text{Tr}(a) = \sum_i a_{ii}$.

We have the following well-known isomorphism.

Definition A.28. Given orthonormal bases $b = (b_i), c = (c_j)$ of finite-dimensional Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, respectively, then we define the linear isomorphism

$$\mathcal{M}_{b,c} \colon \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \cong M_{\dim \mathcal{H}_2, \dim \mathcal{H}_1},$$

to be given by the map $T \mapsto ((k,p) \mapsto \langle c_k | T(b_p)\rangle)$.
Its inverse is given by $A \mapsto \sum_{i,j} A_{ij} |c_i\rangle\langle b_j|$.

When $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$ and $b = c$, then we write $\mathcal{M}_b \colon \mathcal{L}(\mathcal{H}) \cong M_{\dim \mathcal{H}}$ instead of $\mathcal{M}_{b,b}$. $\mathcal{M}_b$ is then a *-algebra isomorphism.

Example A.29. For example, for $\mathcal{M}_e \colon \mathcal{L}(\mathbb{C}^n) \cong M_n$ given by the standard orthonormal basis $e = (e_i)$ of $\mathbb{C}^n$. Then we have that our map is given by $\mathcal{M}_e(T)_{ij} = \langle e_i | T(e_j)\rangle$ for $T \in \mathcal{L}(\mathbb{C}^n)$, and its inverse is given by $A \mapsto (x \mapsto Ax)$ (see Corollary A.31).

Definition A.30. Given an orthonormal basis $b = (b_i)$ of a finite-dimensional Hilbert space $\mathcal{H}$, we define R_b to be the linear isomorphism $\mathcal{H} \cong \mathbb{C}^{\dim \mathcal{H}}$ given by $R_b(x)_i = \langle b_i | x\rangle$ with its inverse given by $x \mapsto \sum_i x_i b_i$.

Corollary A.31. *Let $e = (e_i), f = (f_j)$ be the standard orthonormal bases of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, where the inner products on $\mathbb{C}^n$ and $\mathbb{C}^m$ are given by the canonical inner product $\langle x|y\rangle = x^* y$. Then $R_e = \mathrm{id}$, $R_f = \mathrm{id}$, and $\mathcal{M}_{e,f}^{-1}$ is given by $A \mapsto (x \mapsto Ax)$.*
In other words, $\mathcal{M}_{e,f}^{-1}$ is is simply the matrix identified as a linear map (where the action is given by multiplying the matrix by column vectors).

Proof. This is clear and straightforward. ■

Remark A.32. Let $b = (b_i)$ and $c = (c_j)$ be orthonormal bases of finite-dimensional Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. Let R_b, R_c, respectively, be the identifications $\mathcal{H}_1 \cong \mathbb{C}^{\dim \mathcal{H}_1}$ and $\mathcal{H}_2 \cong \mathbb{C}^{\dim \mathcal{H}_2}$ given by b, c, as defined above. Then, a direct computation verifies that we get

$$\mathcal{M}_{b,c} \colon T \mapsto \mathcal{M}_{e,f}(R_c T R_b^{-1}),$$

where e and f are the standard orthonormal bases of $\mathbb{C}^{\dim \mathcal{H}_1}$ and $\mathbb{C}^{\dim \mathcal{H}_2}$, respectively, so that the identification $\mathcal{M}_{e,f}$ is exactly the one from Corollary A.31. Similarly, we get

$$\mathcal{M}_{b,c}^{-1} \colon A \mapsto R_c^{-1} \mathcal{M}_{e,f}^{-1}(A) R_b.$$

◊

Lemma A.33. *Let $e = (e_i)$ be an orthonormal basis of a finite-dimensional Hilbert space $\mathcal{H}$. Then $R_e^* = R_e^{-1}$. In other words, this is an isometry.*

Proof. Let $x \in \mathcal{H}$ and $y \in \mathbb{C}^{\dim \mathcal{H}}$. Then we compute,

$$\begin{aligned}\langle x | R_e^*(y)\rangle_{\mathcal{H}} &= \langle R_e(x) | y\rangle_{\mathbb{C}^{\dim \mathcal{H}}} = \sum_i \langle R_e(x)_i | y_i\rangle_{\mathbb{C}} = \sum_i \langle \langle e_i | x\rangle_{\mathcal{H}} | y_i\rangle_{\mathbb{C}} \\ &= \sum_i \langle x | e_i\rangle_{\mathcal{H}} y_i = \sum_i \langle x | y_i e_i\rangle_{\mathcal{H}} = \left\langle x \middle| R_e^{-1}(y)\right\rangle_{\mathcal{H}}.\end{aligned}$$

Thus $R_e^* = R_e^{-1}$. ■

Lemma A.34. *Given orthonormal bases $b = (b_i)$ and $c = (c_j)$ of finite-dimensional Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, and elements $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, we have $\mathcal{M}_{c,b}(|x\rangle\langle y|) = R_b(x) R_c(y)^*$.*

Proof. This is a direct computation. For any $i \in [\dim \mathcal{H}_1], j \in [\dim \mathcal{H}_2]$, we compute,

$$\mathcal{M}_{c,b}(|x\rangle\langle y|)_{ij} = \langle b_i||x\rangle\langle y|(c_j)\rangle = \langle b_i|x\rangle\langle y|c_j\rangle = R_b(x)_i\overline{R_c(y)_j} = \left(R_b(x)R_c(y)^*\right)_{ij}.$$

Thus $\mathcal{M}_{c,b}(|x\rangle\langle y|) = R_b(x)R_c(y)^*$. ∎

Similarly to linear maps, we say (u_i) is an *eigenbasis* of $x \in M_n$ in $\mathbb{C}^n$ with corresponding eigenvalues (λ_i), when (u_i) is an orthonormal basis of $\mathbb{C}^n$ such that each u_i is an eigenvector of $\mathcal{M}_e^{-1}(x)$ with corresponding eigenvalue λ_i, i.e., $xu_i = \mathcal{M}_e^{-1}(x)(u_i) = \lambda_i u_i$, where e is the standard orthonormal basis (e_i) of $\mathbb{C}^n$.

A matrix $U \in M_n$ is *unitary* if $U^*U = 1 = UU^*$.

Lemma A.35. *$U \in M_n$ is unitary if and only if its columns form an orthonormal basis of $\mathbb{C}^n$.*

Proof.

(⇒) Suppose U is unitary. Let (u_i) be an orthonormal basis of $\mathbb{C}^n$. Then for any $i, j \in [n]$ we get $\langle Uu_i|Uu_j\rangle = \langle u_i|U^*Uu_j\rangle = \langle u_i|u_j\rangle = \delta_{ij}$. So the tuple of the columns of U, $(Uu_1, \dots, Uu_n)$, forms an orthonormal basis of $\mathbb{C}^n$.

(⇐) Let (u_i) be an orthonormal basis of $\mathbb{C}^n$. Then by (⇒), we know (Uu_i) is also an orthonormal basis. Let $x \in \mathbb{C}^n$. Then

$$\|Ux\|^2 = \left\|\sum_i \langle u_i|x\rangle Uu_i\right\|^2 = \sum_i |\langle u_i|x\rangle|^2 = \left\|\sum_i \langle u_i|x\rangle u_i\right\|^2 = \|x\|^2.$$

∎

Using Lemma A.34, we can see that if we have $A \in M_n$ such that $AA^* = A^*A$, then Corollary A.21 tells us that we get $A = \sum_i \lambda_i u_i u_i^*$ for an eigenbasis (u_i) of A in $\mathbb{C}^n$ with corresponding eigenvalues (λ_i). And by Lemma A.35, we get a unitary matrix $U = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \in M_n$. So then we can write,

$$A = \sum_i \lambda_i u_i u_i^* = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{bmatrix} u_1^* \\ \vdots \\ u_n^* \end{bmatrix} = UDU^*,$$

where $D \in M_n$ is a diagonal matrix containing the eigenvalues (λ_i) of A. This is a specialized version of the *Schur decomposition theorem* [7, Theorem 2.3.1].

A.V The inner product

Any finite-dimensional C*-algebra B can be decomposed as $B = \bigoplus_i M_{n_i}$ for matrix algebras M_{n_i} over $\mathbb{C}$ [15, Theorem 11.2]. In this section, we endow B with an inner product from a chosen faithful and positive linear functional. We do this by first defining the inner product on a single summand M_{n_i}.

A.V.1 Constructing our inner product on M_n. In this sub-section we construct our inner product on M_n.

Definition A.36. By a *linear functional* on an algebra A, we mean a linear map $A \to \mathbb{C}$.

Definition A.37. Given a linear functional $\phi\colon M_n \to \mathbb{C}$, we define $\phi_Q = \sum_{i,j} \phi(e_{ij})e_{ji}$.

Lemma A.38. *Given a linear functional $\phi\colon M_n \to \mathbb{C}$, we get ϕ is given by $x \mapsto \operatorname{Tr}(\phi_Q\, x)$.*

Proof. For any $x \in M_n$, we compute,

$$\operatorname{Tr}(\phi_Q\, x) = \sum_{i,j} \phi(e_{ij}) \operatorname{Tr}(e_{ji} x) = \sum_{i,j} \phi(e_{ij}) x_{ij} = \sum_{i,j} \phi(x_{ij} e_{ij}) = \phi(x).$$

Thus we get ϕ is given by $x \mapsto \operatorname{Tr}(\phi_Q\, x)$. ■

We have already defined a partial order on a general C*-algebra in Definition A.22 and in terms of operators on a Hilbert space (see above Lemma A.27). So then, in matrix terms, $x \in M_n$ is positive semi-definite when $0 \leq \mathcal{M}_e^{-1}(x) \in \mathcal{B}(\mathbb{C}^n)$, where e is the standard orthonormal basis (e_i) of $\mathbb{C}^n$. In other words, $0 \leq x \in M_n$ when $0 \leq a^* x a$ for all $a \in \mathbb{C}^n$ [7, Definition 4.1.9].

Similarly, we say $x \in M_n$ is positive definite when $\mathcal{M}_e^{-1}(x) \in \mathcal{B}(\mathbb{C}^n)$ is positive definite, where e is the standard orthonormal basis (e_i) of $\mathbb{C}^n$ (analogue of Definition A.22) In other words, $x \in M_n$ being positive definite is defined by requiring $0 < a^* x a$ for any non-zero $a \in \mathbb{C}^n$ [7, Definition 4.1.9]. Similar to the reason on positive semi-definite-ness, the self-adjoint requirement can be dropped since we are working in $\mathbb{C}$.

Remark A.39. Given a vector $x \in \mathbb{C}^n$, the matrix $\mathcal{M}_e(|x\rangle\langle x|) = xx^*$ is positive semi-definite. And, interestingly, when $2 \leq n$, the matrix $\mathcal{M}_e(|x\rangle\langle x|) = xx^*$ is never positive definite. For a proof of the latter, it suffices to show that there exists a non-zero orthogonal vector $y \in \mathbb{C}^n$ (i.e., $\langle y|x\rangle = y^* x = 0$). ◊

Lemma A.40. *Let $U, x \in M_n$ such that U is invertible. Then*
UxU^* is positive-definite $\Leftrightarrow$ x is positive-definite.
Analogously, $0 \leq UxU^* \;\Leftrightarrow\; 0 \leq x$.

Proof. We get the following equivalences,

$$\begin{aligned} UxU^* \text{ is positive-definite} &\Leftrightarrow \forall 0 \neq v \in \mathbb{C}^n : 0 < \langle v|UxU^* v\rangle = \langle U^* v|xU^* v\rangle \\ &\Leftrightarrow x \text{ is positive-definite.} \end{aligned}$$

Where the second equivalency follows since U is invertible and so for any $v \in \mathbb{C}^n$, we get $U^* v \neq 0$ if and only if $v \neq 0$. ■

Given a self-adjoint matrix $x \in M_n$ and $r \in \mathbb{R}$ and an orthonormal basis (u_i) of $\mathbb{C}^n$ consisting of the eigenvectors of x with corresponding eigenvalues (α_i), then we can define x^r as the matrix $\sum_i \alpha_i^r u_i u_i^*$. Equivalently, via the Schur decomposition theorem (see end of Section A.IV), $x^r = UD^rU^*$ for unitary $U \in M_n$ and diagonal $D \in M_n$. Note that the eigenvalues of a self-adjoint matrix are real. Moreover, if x is positive semi-definite, then x^r is also positive semi-definite. And if x is positive definite, then so is x^r.

Let $0 \leq x$. Then, by the above argument, we have a unitary matrix $U \in M_n$ and diagonal matrix $D \in M_n$ (with non-negative entries as the eigenvalues of a positive semi-definite matrix are non-negative[1]) such that $x = x^1 = UD^1U^* = UD^{1/2}U^*UD^{1/2}U^* = x^{1/2}x^{1/2}$.

Definition A.41. Given C*-algebras A, C, we say $f\colon A \to C$ is a *positive map* when $0 \leq f(a)$ for all $0 \leq a$. In other words, f maps positive elements in A to positive elements in C.

So then, a linear functional f on M_n is a *positive map*[2] when $0 \leq f(x^* x)$ for any matrix $x \in M_n$.

Corollary A.42. *Given a linear functional ϕ on M_n, we have,*
ϕ is a positive map $\Leftrightarrow$ $\forall\, (x_i) \in \mathbb{C}^n : 0 \leq \sum_i \phi(x_i x_i^*)$.
In other words, ϕ is a positive map if and only if $0 \leq \sum_i \phi(x_i x_i^*)$ for all tuples (x_i) in $\mathbb{C}^n$.

[1] We can instead use Lemma A.40, to see that we also get $0 \leq D$; and as D is diagonal, we get that that all of its entries are non-negative.

[2] This is consistent with the current literature; see [11] for example.

Proof. Let e be the standard orthonormal basis (e_i) of $\mathbb{C}^n$. Then, since for any matrix x we have $0 \leq x$ if and only if $x = y^*y$ for some $y \in M_n$ by A.24, we have ϕ is positive if and only if $0 \leq \phi(x)$ for any positive semi-definite matrix $x \in M_n$. And so, we have the following equivalences,

$$\begin{aligned} \phi \text{ is positive} &\Leftrightarrow \forall\, 0 \leq x \in M_n : 0 \leq \phi(x) \\ &\Leftrightarrow \forall (x_i) \in \mathbb{C}^n : 0 \leq \sum_i \phi\left(\mathcal{M}_e(|x_i\rangle\langle x_i|)\right) && \text{by A.27} \\ &\Leftrightarrow \forall (x_i) \in \mathbb{C}^n : 0 \leq \sum_i \phi(x_i x_i^*) && \text{by A.34.} \end{aligned}$$

Thus ϕ being positive is equivalent to $0 \leq \sum_i \phi(x_i x_i^*)$ for any tuple (x_i) in $\mathbb{C}^n$. ■

Lemma A.43. *Given a linear functional ϕ on M_n, we have,*
ϕ is positive $\Leftrightarrow$ $0 \leq \phi_Q$.
Here, ϕ_Q is the matrix associated with ϕ defined in Definition A.37.

Proof. By Lemma A.38 we have $\phi(x) = \operatorname{Tr}(\phi_Q\, x)$ for all $x \in M_n$.

($\Rightarrow$) Suppose ϕ is positive. By Corollary A.42, we get $0 \leq \sum_i \phi(x_i x_i^*)$ for any tuple (x_i) in $\mathbb{C}^n$. So then for any $x \in \mathbb{C}^n$, we have $0 \leq \phi(xx^*) = \operatorname{Tr}(\phi_Q\, xx^*) = x^*\phi_Q\, x$, which means ϕ_Q is positive semi-definite.

($\Leftarrow$) Suppose Q is positive semi-definite. Then as mentioned above, since ϕ_Q is positive semi-definite, we have $\phi_Q = \phi_Q^{1/2}\phi_Q^{1/2}$, where $\phi_Q^{1/2}$ is also positive semi-definite (and so is self-adjoint). So for any $x \in M_n$, we have $\phi(x^*x) = \operatorname{Tr}(\phi_Q\, x^*x) = \operatorname{Tr}\left((x\phi_Q^{1/2})^*(x\phi_Q^{1/2})\right) \geq 0$.

Thus ϕ is positive if and only if our unique matrix ϕ_Q is positive semi-definite. ■

> **Definition A.44.** We say a linear functional ϕ on A is *tracial* if $\phi(xy) = \phi(yx)$ for all $x, y \in A$.

Proposition A.45. *Given a linear functional $\phi\colon M_n \to \mathbb{C}$, we have,*
ϕ is tracial $\Leftrightarrow$ $\exists!\, \alpha \in \mathbb{C} : \phi_Q = \alpha 1$.
Here, ϕ_Q is the matrix associated with ϕ defined in Definition A.37.

Proof.

($\Rightarrow$) Suppose ϕ is tracial. By Lemma A.38 we have $\phi(x) = \operatorname{Tr}(\phi_Q\, x)$ for any $x \in M_n$. So then for any $x, y \in M_n$, we have,

$$\sum_{i,j,k} (\phi_Q)_{ij} x_{jk} y_{ki} = \operatorname{Tr}(\phi_Q\, xy) = \phi(xy) = \phi(yx) = \operatorname{Tr}(\phi_Q\, yx) = \sum_{i,j,k} (\phi_Q)_{ij} y_{jk} x_{ki}.$$

Claim: $\forall p, q, r \in [n] : (\phi_Q)_{pq} = \delta_{pq}(\phi_Q)_{rr}$.

Let $p, q, r \in [n]$. Then by our hypothesis, we get,

$$(\phi_Q)_{pq} = \sum_{i,j,k} (\phi_Q)_{ij}(e_{qr})_{jk}(e_{rp})_{ki} = \sum_{i,j,k} (\phi_Q)_{ij}(e_{rp})_{jk}(e_{qr})_{ki} = \delta_{pq}(\phi_Q)_{rr}.$$

By the claim, we see that our matrix ϕ_Q is diagonal since $(\phi_Q)_{pq} = 0$ when $p \neq q$. And we also see that for any $p, q \in [n]$, we get $(\phi_Q)_{pp} = (\phi_Q)_{qq}$. So then let $i \in [n]$ and let $\alpha = (\phi_Q)_{ii}$. Then, by the above claim we get, $(\phi_Q)_{jk} = \delta_{jk}(\phi_Q)_{ii} = (\alpha 1)_{jk}$ for any $j, k \in [n]$. Thus $\phi_Q = \alpha 1$. Clearly this is unique, since for $\alpha, \beta \in \mathbb{C}$, if $\alpha \operatorname{Tr}(x) = \beta \operatorname{Tr}(x)$ for any $x \in M_n$, then $\alpha = \alpha \operatorname{Tr}(e_{ii}) = \beta \operatorname{Tr}(e_{ii}) = \beta$.

(⇐) Let $\alpha \in \mathbb{C}$ such that $\phi_Q = \alpha 1$. Then for any $x, y \in M_n$, we have

$$\phi(xy) = \alpha \operatorname{Tr}(xy) = \alpha \operatorname{Tr}(yx) = \phi(yx).$$

Thus ϕ is tracial if and only if there exists a unique complex number α such that $\phi_Q = \alpha 1$. ■

Remark A.46. Using the above Proposition A.45, we get ϕ is tracial and positive if and only if ϕ is given by $x \mapsto \alpha \operatorname{Tr}(x)$ for some unique non-negative $\alpha \in \mathbb{C}$. ◊

Corollary A.47. *If $A \in M_n$. Then*
$0 \leq A$ *and is invertible* ⇔ A *is positive-definite.*

Proof.

(⇒) Suppose $0 \leq A$ and $A = A^{1/2}A^{1/2}$ is invertible. Then $A^{1/2}$ is also invertible and positive semi-definite. Let $v \in \mathbb{C}^n$ be non-zero. Then, we compute,

$$\langle v|Av\rangle = \left\langle v \middle| A^{1/2}A^{1/2}v \right\rangle = \left\langle A^{1/2}v \middle| A^{1/2}v \right\rangle > 0,$$

since $A^{1/2}v \neq 0$ (as $A^{1/2}$ is invertible). Note that, in the second equality, we use the self-adjointedness of $A^{1/2}$ since it is positive semi-definite. So we are done.

(⇐) Suppose $0 < A$. Then obviously $0 \leq A$, so we only need to check if it is invertible. Suppose the contrary, i.e., A is not invertible. Then there exists a non-zero $v \in \mathbb{C}^n$ such that $Av = 0$. But then, by the hypothesis, we get $0 < \langle v|Av\rangle = 0$, which is a contradiction. Thus A is invertible.

■

Lemma A.48. *Given a positive definite matrix $Q \in M_n$, we have $\operatorname{Tr}(Qx^*x) = 0$ if and only if $x = 0$ for any $x \in M_n$.*

Proof. We have $\operatorname{Tr}\left(\left(xQ^{1/2}\right)^*\left(xQ^{1/2}\right)\right) = \operatorname{Tr}(Qx^*x) = 0$ if and only if $xQ^{1/2} = 0$. As $Q^{1/2}$ is positive definite, we get it is also invertible by Corollary A.47, and so $xQ^{1/2} = 0$ if and only if $x = 0$. And so we are done. ■

Definition A.49. A positive linear functional f on A is said to be *faithful*[3] if $f(x) = 0$ if and only if $x = 0$ for any non-negative element $0 \leq x \in A$.

Lemma A.50. *A positive linear functional ϕ on M_n being faithful is equivalent to*
$\phi(x^*x) = 0 \Leftrightarrow x = 0$ *for any* $x \in M_n$.

Proof. This is true since any matrix $x \in M_n$ is positive semi-definite if and only if $x = y^*y$ for some matrix $y \in M_n$ by Theorem A.24. ■

Proposition A.51. *Given a linear functional $\phi\colon M_n \to \mathbb{C}$, we have*
ϕ *is a positive and faithful map* ⇔ ϕ_Q *is positive-definite.*
Again, ϕ_Q is the matrix associated with ϕ defined in Definition A.37.

Proof. By Lemma A.38, we have ϕ is given by $x \mapsto \operatorname{Tr}(\phi_Q\, x)$, and by Lemma A.43, we know ϕ is positive if and only if $0 \leq \phi_Q$. So we need to show that faithfulness of a positive linear functional is equivalent to ϕ_Q being positive definite.

($\Rightarrow$) Suppose ϕ is faithful and positive. So we have $\phi(x) = 0$ if and only if $x = 0$ for any positive semi-definite matrix $x \in M_n$. Let $0 \neq x \in \mathbb{C}^n$. Then we have $xx^* \neq 0$ as $x \neq 0$. This means, by faithfulness and positivity of ϕ we get $\phi(xx^*) \neq 0$ as xx^* is a non-zero positive semi-definite matrix. And so $0 < \phi(xx^*) = \mathrm{Tr}(\phi_Q\, xx^*) = x^*\phi_Q\, x$, which means ϕ_Q is positive definite.

($\Leftarrow$) Suppose ϕ_Q is positive definite. Then for any $x \in M_n$, we get $\phi(x^*x) = \mathrm{Tr}(\phi_Q\, xx^*) = 0$ if and only if $x = 0$ using Lemma A.48.

Thus ϕ is a faithful and positive linear functional if and only if our unique matrix ϕ_Q is positive definite. ■

Proposition A.52. *Given a linear functional $\phi\colon M_n \to \mathbb{C}$, we get ϕ is star-preserving (i.e., $\phi(x^*) = \phi(x)^*$ for all $x \in M_n$) if and only if ϕ_Q is self-adjoint.*
Here, ϕ_Q is the matrix associated with ϕ defined in Definition A.37.

Proof.

($\Rightarrow$) Suppose $\phi(x^*) = \overline{\phi(x)}$ for all $x \in M_n$. We let $x \in M_n$, and compute using Lemma A.38,

$$\mathrm{Tr}(\phi_Q\, x^*) = \phi(x^*) = \overline{\phi(x)} = \overline{\mathrm{Tr}(\phi_Q\, x)} = \mathrm{Tr}(x^*\phi_Q^*),$$

And so $\phi_Q = \phi_Q^*$.

($\Leftarrow$) Suppose ϕ_Q is self-adjoint. Using Lemma A.38, ϕ is given by $x \mapsto \mathrm{Tr}(\phi_Q\, x)$. And so for any $x \in M_n$, we get $\phi(x^*) = \mathrm{Tr}(\phi_Q\, x^*) = \mathrm{Tr}((x\phi_Q)^*) = \overline{\mathrm{Tr}(x\phi_Q)} = \overline{\phi(x)}$. Thus, ϕ is real.

■

Corollary A.53. *Given a linear functional $\phi\colon M_n \to \mathbb{C}$, then,*
ϕ is positive and faithful $\Leftrightarrow$ $M_n \times M_n \to \mathbb{C}\colon (x, y) \mapsto \phi(x^*y)$ defines an inner product.

Proof. For any $x, y \in M_n$, let $\langle x|y\rangle_\phi = \phi(x^*y)$. Then, clearly,

$$\langle x|\alpha y + \beta z\rangle_\phi = \phi(x^*(\alpha y + \beta z)) = \alpha\phi(x^*y) + \beta\phi(x^*z) = \alpha\langle x|y\rangle_\phi + \beta\langle x|z\rangle_\phi,$$

by linearity of ϕ, for any $x, y, z \in M_n$ and $\alpha, \beta \in \mathbb{C}$.

If ϕ is faithful, we have $\langle x|x\rangle_\phi = \phi(x^*x) = 0$ if and only if $x = 0$ for any $x \in M_n$. And ϕ is positive if and only if $0 \leq \langle x|x\rangle_\phi$ for any $x \in M_n$. So if $\langle\cdot|\cdot\rangle_\phi$ defines an inner product on M_n, then we get ϕ is faithful and positive. So it remains to show that, given ϕ is a faithful and positive linear functional, we get $\overline{\langle x|y\rangle_\phi} = \langle y|x\rangle_\phi$ for any $x, y \in M_n$.

Suppose ϕ is faithful and positive. By Proposition A.51, we get our unique matrix ϕ_Q is positive definite (and so is self-adjoint). Using Proposition A.52, we get ϕ is star-preserving, and so, for any $x, y \in M_n$, we get, $\overline{\langle x|y\rangle_\phi} = \overline{\phi(x^*y)} = \phi(y^*x) = \langle y|x\rangle_\phi$.

Therefore, $\langle\cdot|\cdot\rangle_\phi\colon (x, y) \mapsto \phi(x^*y)$ is a well-defined inner product on M_n. ■

Combining Proposition A.51 and Corollary A.53, we get the following.

Theorem A.54. *Given a linear functional $\phi\colon M_n \to \mathbb{C}$, then the following are equivalent,*

(i) *ϕ is positive and faithful,*

(ii) *$\exists!\, Q \in M_n : Q$ is positive-definite and $\forall x \in M_n : \phi(x) = \mathrm{Tr}(Qx)$,*

(iii) *$M_n \times M_n \to \mathbb{C}\colon (x, y) \mapsto \phi(x^*y)$ defines an inner product on M_n.*

■

A.V.2 On finite-dimensional C*-algebras. We can now define the Hilbert space on our finite-dimensional C*-algebra $B = \bigoplus_i M_{n_i}$ by choosing linear functionals on each summand M_{n_i}.

Definition A.55. We let B be a multi-matrix algebra $\bigoplus_{i=1}^{\mathfrak{K}} M_{n_i}$. (So a general finite-dimensional C*-algebra [15, Theorem 11.2].)

A typical element $x \in B$ is written as $x = \bigoplus_i x_i$ for each $x_i \in M_{n_i}$. The standard basis of B is given by the tuple $\left[(e_{s,ij})_{i,j=1}^{n_s}\right]_{s=1}^{\mathfrak{K}}$ where each $e_{s,ij}$ is the matrix e_{ij} of the s-th summand of the direct sum $\bigoplus_i M_{n_i}$.

For each $i \in [\mathfrak{K}]$, we define the projection map $p_i \colon \bigoplus_j M_{n_j} \to M_{n_i}$ given by $x \mapsto x_i$. For each i, we also define the inclusion map $\iota_i \colon M_{n_i} \hookrightarrow \bigoplus_j M_{n_j}$ given by $x_i \mapsto (0, \ldots, 0, x_i, 0, \ldots, 0)$. Clearly $p_i \iota_j(x) = \delta_{i,j} x$ for $x \in M_{n_j}$. Using this notation, for $x \in B$, we write

$$x = \sum_i \iota_i(x_i) = \sum_{i,j,k} \iota_i(x_{i,jk} e_{jk}) = \sum_{i,j,k} x_{i,jk} e_{i,jk},$$

where each $x_{i,jk}$ is the input of x at j,k on the i-th summand. Elements of tensor products of direct sums, say $x \otimes y \in \bigoplus_{i,j} M_{n_i} \otimes M_{g_j}$, are given by $(x \otimes y)_{i,ab}^{j,cd} = x_{i,ac} y_{j,bd}$.

Definition A.56. For each i, we fix a faithful and positive linear functional ψ_i on M_{n_i}, and we let $Q_i \in M_{n_i}$ be our unique positive definite matrix such that $\psi_i \colon x \mapsto \operatorname{Tr}(Q_i x)$ (so each $Q_i = \sum_{j,k} \psi_i(e_{jk}) e_{kj}$) – see Proposition A.51.

We let ψ be our faithful positive linear functional on B given by $\psi = \sum_i \psi_i \circ p_i$, where each p_i is the projection map $\bigoplus_j M_{n_j} \to M_{n_i}$, and we let $Q = \bigoplus_i Q_i$. So then, given $x \in B$, we get $\psi(x) = \operatorname{Tr}(Qx)$, where Tr here is defined by the sum of the diagonals in each matrix block.

Using Theorem A.54, we can now define our Hilbert space.

Definition A.57. We define the inner product on each M_{n_i} by

$$\langle x|y\rangle_{\psi_i} = \psi_i(x^*y) = \operatorname{Tr}(Q_i x^* y),$$

for all $x, y \in M_{n_i}$. We denote (M_{n_i}, ψ_i) to be the Hilbert space given by this inner product. We define the inner product on B by

$$\langle x|y\rangle_{\psi} = \psi(x^*y) = \operatorname{Tr}(Q x^* y),$$

for all $x, y \in B$, where Tr here is defined by the sum of the diagonals in each matrix block. We denote (B, ψ) to be the Hilbert space given by this inner product. Note that, unless necessary, we usually leave out the subscript in the inner product.

Remark A.58. By definition, we obviously have $\langle x|y\rangle_{\psi} = \sum_i \langle x_i|y_i\rangle_{\psi_i}$ for $x, y \in B = \bigoplus_i M_{n_i}$. ◊

Let $\mathcal{H}_1, \mathcal{H}_2$ be finite-dimensional Hilbert spaces. Then, given some $p \in \mathbb{N}$, $x \in \mathcal{H}_1$, we write $x^{\otimes p}$ to mean x tensored with itself p times; this element is in $\mathcal{H}_1^{\otimes p}$ (i.e., $\mathcal{H}_1$ tensored with itself p times). Given a linear map $f \colon \mathcal{H}_1 \to \mathcal{H}_2$, we write $f^{\otimes p}$ to mean the linear map $\mathcal{H}_1^{\otimes p} \to \mathcal{H}_2^{\otimes p}$ given by f tensored with itself p times. The inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is given by $\langle x \otimes y|z \otimes w\rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \langle x|z\rangle_{\mathcal{H}_1} \langle y|w\rangle_{\mathcal{H}_2}$.

So then, the inner product on $B^{\otimes p}$ is given by $\langle x|y\rangle_{B^{\otimes p}} = \psi^{\otimes p}(x^*y) = \operatorname{Tr}(Q^{\otimes p} x^* y)$. Similarly, for positive and faithful linear functionals ψ, ϕ on B, H, respectively, the inner product on $(B, \psi) \otimes (H, \phi)$ is given by

$$\langle x \otimes y|z \otimes w\rangle_{\psi \otimes \phi} = (\psi \otimes \phi)(x^* z \otimes y^* w) = \operatorname{Tr}(Q_\psi x^* z) \operatorname{Tr}(Q_\phi y^* w).$$

Proposition A.59. *The adjoint of ψ on (B,ψ) and the canonical inner product space on $\mathbb{C}$ is given by $\mathbb{C} \to B\colon x \mapsto x1$. In other words, $\psi^* = |1\rangle$.*

Proof. For any $x \in \mathbb{C}$ and $y \in B$, we have,

$$\langle \psi^*(x)|y\rangle_\psi = \langle x|\psi(y)\rangle_{\mathbb{C}} = \overline{x}\psi(y) = \overline{x}\langle 1|y\rangle_\psi = \langle x1|y\rangle_\psi.$$

Thus $\psi^*(x) = x1$ for any $x \in \mathbb{C}$. ■

Proposition A.60 ([9, Proposition 2.5])**.** *We get $f = \left[\iota_s(e_{ij}Q_s^{-1/2})_{i,j=1}^{n_s}\right]_{s=1}^{\mathfrak{K}}$ is an orthonormal basis of (B,ψ).*
Moreover, we get R_f (see Definition A.30) is given by $R_f(x)_{s,ij} = (xQ^{1/2})_{s,ij}$.

Proof. This is linearly independent since for any $a \in B$, we get $\sum_{s,i,j} a_{s,ij}e_{s,ij}\iota_s(Q_s^{-1/2}) = 0$ if and only if $\sum_{s,i,j} a_{s,ij}e_{s,ij} = 0$. And since $(e_{s,ij})$ is a basis on B, we get that each $a_{s,ij} = 0$.
For any $r,s \in [\mathfrak{K}]$, $a,b \in [n_r]$, and $c,d \in [n_s]$, we get

$$\begin{aligned}\left\langle \iota_r(e_{ab}Q_r^{-1/2})\middle|\iota_s(e_{cd}Q_s^{-1/2})\right\rangle_\psi &= \delta_{r,s}\left\langle e_{ab}Q_r^{-1/2}\middle|e_{cd}Q_r^{-1/2}\right\rangle_{\psi_r} \\ &= \delta_{r,s}\operatorname{Tr}(Q_rQ_r^{-1/2}e_{ba}e_{cd}Q_r^{-1/2}) = \delta_{r,s}\delta_{a,c}\delta_{b,d}.\end{aligned}$$

And so this is orthonormal. Thus $(\iota_s(e_{ij}Q_s^{-1/2}))$ is an orthonormal basis of B. ■

Note that, using Proposition A.11, we have unravelled the co-algebraic structure of our finite-dimensional C*-algebra B.

Lemma A.61. *Let $i \in [\mathfrak{K}]$. Then,*

(i) $\iota_i^* = p_i$, where $\iota_i\colon M_{n_i} \hookrightarrow B$ and $p_i\colon B \to M_{n_i}$, in other words, $\iota_i^*(x) = x_i$ for $x \in B$,

(ii) $m^*\iota_i = (\iota_i \otimes \iota_i)m^*$, i.e., $m^*\iota_i(x_i) = (\iota_i \otimes \iota_i)m^*(x_i)$ for $x \in B$,

(iii) $\iota_i m = m(\iota_i \otimes \iota_i)$.

Proof.

(i) Let $y \in M_{n_i}$. Then we compute,

$$\langle \iota_i^*(x)|y\rangle_{\psi_i} = \langle x|\iota_i(y)\rangle_\psi = \langle x_i|y\rangle_{\psi_i}.$$

Where the last equality easily follows from $\operatorname{Tr}(Qx^*\iota_i(y)) = \operatorname{Tr}(Q_ix_i^*y)$.

(ii) Let $y,z \in B$. Then,

$$\begin{aligned}\langle m^*\iota_i(x_i)|y\otimes z\rangle_{\psi^{\otimes 2}} &= \langle \iota_i(x_i)|yz\rangle_\psi = \langle x_i|y_iz_i\rangle_{\psi_i} = \langle m^*(x_i)|y_i\otimes z_i\rangle_{\psi_i} \\ &= \langle m^*(x_i)|(\iota_i^*\otimes\iota_i^*)(y\otimes z)\rangle_{\psi_i} \qquad \text{by (i)} \\ &= \langle (\iota_i\otimes\iota_i)m^*(x_i)|y\otimes z\rangle_{\psi^{\otimes 2}}.\end{aligned}$$

(iii) $\iota_i m(x_i \otimes y_i) = \iota_i(x_iy_i) = \iota_i(x_i)\iota_i(y_i) = m(\iota_i\otimes\iota_i)(x_i\otimes y_i)$, for $x,y \in B$.

■

Proposition A.62. *The adjoint of the multiplication map m on the Hilbert spaces (M_{n_s},ψ_s) and $(M_{n_s}\otimes M_{n_s},\psi_s\otimes\psi_s)$ is given by*

$$x \mapsto \sum_{i,j,k,l} x_{il}Q_{s,kj}^{-1}(e_{ij}\otimes e_{kl}).$$

So then, for $x \in B$, we get $m^*(x) = \sum_{s,i,j,k,l} x_{s,il}Q_{s,kj}^{-1}(e_{s,ij}\otimes e_{s,kl})$, where $x = \bigoplus_s x_s$, for each $x_s \in M_{n_s}$.

Proof. Let $x, y, z \in M_{n_s}$, and then we compute,

$$\begin{aligned}
\langle x \otimes y | m^*(z) \rangle_{\psi_s^{\otimes 2}} &= \langle m(x \otimes y) | z \rangle_{\psi_s} = \langle xy | z \rangle_{\psi_s} = \operatorname{Tr}(Q_s (xy)^* z) = \operatorname{Tr}(x^* z Q_s y^*) \\
&= \sum_k (x^* z Q_s y^*)_{kk} = \sum_{q,k} \delta_{kq} (x^* z Q_s y^*)_{qk} = \sum_{q,k} (Q_s^{-1} Q_s)_{kq} (x^* z Q_s y^*)_{qk} \\
&= \sum_{i,j,k,l,q,a} z_{il} Q_{s,kj}^{-1} Q_{s,jq} x_{qi}^* Q_{s,la} y_{ak}^* \\
&= \sum_{i,j,k,l,r,p,o,q,a,b} z_{il} Q_{s,kj}^{-1} Q_{s,rq} x_{qo}^* \delta_{io} \delta_{jr} Q_{s,pa} y_{ab}^* \delta_{kb} \delta_{lp} \\
&= \sum_{i,j,k,l,r,p} z_{il} Q_{s,kj}^{-1} (Q_s x^* e_{ij})_{rr} (Q_s y^* e_{kl})_{pp} \\
&= \sum_{i,j,k,l} z_{il} Q_{s,kj}^{-1} \operatorname{Tr}(Q_s x^* e_{ij}) \operatorname{Tr}(Q_s y^* e_{kl}) \\
&= \sum_{i,j,k,l} z_{il} Q_{s,kj}^{-1} \langle x | e_{ij} \rangle_{\psi_s} \langle y | e_{kl} \rangle_{\psi_s} \\
&= \sum_{i,j,k,l} z_{il} Q_{s,kj}^{-1} \langle x \otimes y | e_{ij} \otimes e_{kl} \rangle_{\psi_s^{\otimes 2}}.
\end{aligned}$$

Thus $m^*(z) = \sum_{i,j,k,l} z_{il} Q_{s,kj}^{-1} (e_{ij} \otimes e_{kl})$ for any $z \in M_{n_s}$ and $s \in [\mathfrak{K}]$.

By extension, we thus get

$$\begin{aligned}
m^*(x) &= \sum_a m^* \iota_a (x_a) = \sum_a (\iota_a \otimes \iota_a) m^*(x_a) && \text{by A.61(ii)} \\
&= \sum_{a,i,j,k,l} x_{a,il} Q_{a,kj}^{-1} (\iota_a \otimes \iota_a)(e_{ij} \otimes e_{kl}) && \text{by above} \\
&= \sum_{a,i,j,k,l} x_{a,il} Q_{a,kj}^{-1} (e_{a,ij} \otimes e_{a,kl}).
\end{aligned}$$

■

Lemma A.63. *We have $mm^*(x) = \sum_i \operatorname{Tr}(Q_i^{-1}) \iota_i (x_i)$ for all $x \in B$.*

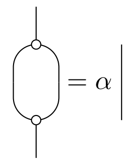
Proof. Let $x \in B$, then we compute,

$$\begin{aligned}
mm^*(x) &= \sum_a mm^* \iota_a (x_a) \\
&= \sum_a \iota_a mm^*(x_a) && \text{by A.61(ii),(iii)} \\
&= \sum_{a,i,j,k,l} x_{a,il} Q_{a,kj}^{-1} \iota_a m(e_{ij} \otimes e_{kl}) && \text{by A.62} \\
&= \sum_{a,i,j,k,l} x_{a,il} Q_{a,kj}^{-1} \delta_{jk} \iota_a (e_{il}) = \sum_{a,i,j,l} x_{a,il} Q_{a,jj}^{-1} e_{a,il} \\
&= \sum_{a,i,l} \operatorname{Tr}(Q_a^{-1}) x_{a,il} e_{a,il} = \sum_a \operatorname{Tr}(Q_a^{-1}) \iota_a (x_a).
\end{aligned}$$

Thus $mm^* = \sum_i \operatorname{Tr}(Q_i^{-1}) \iota_i p_i$. ■

Proposition A.64. *Given $\alpha \in \mathbb{C}$, we get $mm^* = \alpha \operatorname{id}$ if and only if $\operatorname{Tr}(Q_i^{-1}) = \alpha$ for all*

$i \in [\mathfrak{K}]$. *In other words,*



$$\Leftrightarrow \ \forall i \in [\mathfrak{K}] : \mathrm{Tr}(Q_i^{-1}) = \alpha.$$

Proof.

($\Leftarrow$) If $\mathrm{Tr}(Q_i^{-1}) = \alpha$ for all $i \in [\mathfrak{K}]$, then for all $x \in B$ we have,

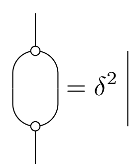
$$mm^*(x) = \sum_i \mathrm{Tr}(Q_i^{-1})\iota_i(x_i) = \alpha \sum_i \iota_i(x_i) = \alpha x.$$

($\Rightarrow$) If $mm^* = \alpha\,\mathrm{id}$. Then for $i \in [\mathfrak{K}]$, we get

$$\alpha 1_i = mm^*(1)_i = \sum_s \mathrm{Tr}(Q_s^{-1})\iota_s(1_s)_i = \mathrm{Tr}(Q_i^{-1})1_i.$$

∎

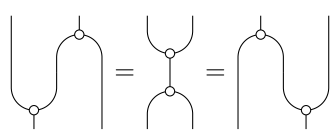
Definition A.65. We call our linear functional ψ a δ-form functional if $mm^* = \delta^2\,\mathrm{id}$ for some $0 < \delta$. In other words, ψ is a δ-form when



for some $0 < \delta$.

From this point forward, we assume our linear functional ψ is of δ-form.

Proposition A.66 (Frobenius equation [9, Equation 2.3] & [6, Proposition 1.5])**.**



Algebraically, we have $(m \otimes \mathrm{id})(\mathrm{id} \otimes m^*) = m^*m = (\mathrm{id} \otimes m)(m^* \otimes \mathrm{id})$.

Proof. It suffices to show $(m \otimes \mathrm{id})(\mathrm{id} \otimes m^*) = m^*m$, as we get the other equality by taking adjoints.

Let $x, y, z, w \in B$. Let $m^*(y) = \sum_i \alpha_i \otimes \beta_i$ for some tuples $(\alpha_i), (\beta_i)$ in B. Then we compute,

$$\begin{aligned}
\langle (m \otimes \mathrm{id})(\mathrm{id} \otimes m^*)(x \otimes y) | z \otimes w \rangle &= \sum_i \langle (m \otimes \mathrm{id})(x \otimes \alpha_i \otimes \beta_i) | z \otimes w \rangle \\
&= \sum_i \langle x\alpha_i \otimes \beta_i | z \otimes w \rangle = \sum_i \langle x\alpha_i | z \rangle \langle \beta_i | w \rangle \\
&= \sum_i \psi(\alpha_i^* x^* z) \langle \beta_i | w \rangle = \sum_i \langle \alpha_i | x^* z \rangle \langle \beta_i | w \rangle \\
&= \sum_i \langle \alpha_i \otimes \beta_i | x^* z \otimes w \rangle = \langle m^*(y) | x^* z \otimes w \rangle \\
&= \langle y | x^* z w \rangle = \psi(y^* x^* z w) = \langle xy | zw \rangle
\end{aligned}$$

$$= \langle m^* m(x \otimes y) | z \otimes w \rangle.$$

Thus $(m \otimes \mathrm{id})(\mathrm{id} \otimes m^*) = m^* m$. ■

Remark A.67. Thus, our Hilbert space (B, ψ) is a Frobenius algebra (i.e., a vector space that is both an algebra and a co-algebra which satisfies the Frobenius equations). ♢

A.VI The modular automorphism σ

In this section we define the modular automorphism.

Definition A.68 ([4, top of page 9]). Given $t \in \mathbb{R}$, we define the algebra automorphism $\sigma_t \colon B \cong B$ to be given by $a \mapsto Q^{-t} a Q^t$ with inverse $a \mapsto Q^t a Q^{-t}$ (so $\sigma_t^{-1} = \sigma_{-t}$), such that $\psi \circ \sigma_t = \psi$.

Lemma A.69. *For any $t, s \in \mathbb{R}$, we get,*

(i) $\sigma_t \sigma_s = \sigma_{t+s}$,

(ii) $\sigma_t(x)^* = \sigma_{-t}(x^*)$ *for any* $x \in B$,

(iii) $\sigma_t^* = \sigma_t$,

Proof.

(i) We get $\sigma_t \sigma_s(x) = \sigma_t(Q^{-s} x Q^s) = Q^{-t} Q^{-s} x Q^s Q^t = Q^{-(t+s)} x Q^{t+s} = \sigma_{t+s}(x)$ for any $x \in B$.

(ii) Let $x \in B$. Then $\sigma_t(x)^* = (Q^{-t} x Q^t)^* = Q^t x^* Q^{-t} = \sigma_{-t}(x^*)$.

(iii) Let $x, y \in B$. Then we compute

$$\begin{aligned} \langle x | \sigma_t^*(y) \rangle = \langle \sigma_t(x) | y \rangle = \psi(\sigma_t(x)^* y) &= \psi(\sigma_{-t}(x^*) y) && \text{by (ii)} \\ &= \mathrm{Tr}(Q \sigma_{-t}(x^*) y) = \mathrm{Tr}(Q Q^t x^* Q^{-t} y) \\ &= \mathrm{Tr}(Q x^* Q^{-t} y Q^t) = \langle x | \sigma_t(y) \rangle. \end{aligned}$$

Thus σ_t is self-adjoint.

■

Lemma A.70. *Let $x, y, z \in B$. Then we have,*

(i) $\langle x | yz \rangle = \langle y^* x | z \rangle$,

(ii) $\langle xy | z \rangle = \langle y | x^* z \rangle$,

(iii) $\langle xy | z \rangle = \langle x | z \sigma_{-1}(y^*) \rangle$,

(iv) $\langle x | yz \rangle = \langle x \sigma_{-1}(z^*) | y \rangle$,

(v) $\langle x | y \rangle = \langle y^* | \sigma_{-1}(x^*) \rangle$.

Proof. We quickly compute,

(i) $\langle x | yz \rangle = \mathrm{Tr}(Q x^* yz) = \mathrm{Tr}(Q (y^* x)^* z) = \langle y^* x | z \rangle$.

(ii) $\langle xy | z \rangle = \mathrm{Tr}(Q y^* x^* z) = \langle y | x^* z \rangle$.

(iii) $\langle xy | z \rangle = \mathrm{Tr}(Q y^* x^* z) = \mathrm{Tr}(Q x^* z (Q y^* Q^{-1})) = \langle x | z \sigma_{-1}(y^*) \rangle$.

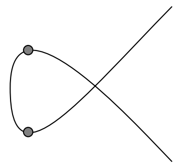
(iv) $\langle x | yz \rangle = \overline{\langle yz | x \rangle} = \overline{\langle y | x \sigma_{-1}(z^*) \rangle} = \langle x \sigma_{-1}(z^*) | y \rangle$, where we have used (iii) for the second equality.

(v) This is immediate from parts (i) and (iv).

■

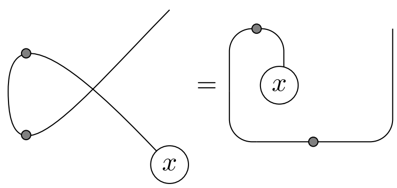
Remark A.71. For $x, y \in B$, combining Lemma A.70(v) with Lemma A.69(iii), we also get $\langle x|y\rangle = \langle \sigma_{-1/2}(y^*) | \sigma_{-1/2}(x^*)\rangle$. ◊

Proposition A.72. *Applying a left-handed twist corresponds to applying the automorphism* σ_1*, i.e.,*



$= \sigma_1.$

To make this clear, this is exactly saying that for any $x \in B$*, we have*



$= \sigma_1(x).$

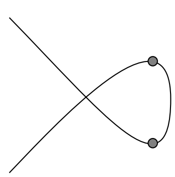
In other words, $(\eta^* m \otimes \mathrm{id})(\mathrm{id} \otimes \varkappa)(m^*\eta \otimes \mathrm{id}) = \sigma_1$.

Proof. Let $x, y \in B$, and let $m^*(1) = \sum_i \alpha_i \otimes \beta_i$ for some tuples $(\alpha_i), (\beta_i)$ in B. Then we compute,

$$
\begin{aligned}
&\langle (\eta^* m \otimes \mathrm{id})(\mathrm{id} \otimes \varkappa)(m^*\eta \otimes \mathrm{id})(x)|y\rangle \\
&= \sum_i \langle (\eta^* m \otimes \mathrm{id})(\mathrm{id} \otimes \varkappa)(\alpha_i \otimes \beta_i \otimes x)|y\rangle \\
&= \sum_i \langle (\eta^* m \otimes \mathrm{id})(\alpha_i \otimes x \otimes \beta_i)|y\rangle \\
&= \sum_i \langle \eta^*(\alpha_i x)\beta_i|y\rangle = \sum_i \langle \alpha_i x|1\rangle\langle \beta_i|y\rangle \\
&= \sum_i \langle \alpha_i|\sigma_{-1}(x^*)\rangle\langle \beta_i|y\rangle && \text{by A.70(iii)} \\
&= \sum_i \langle \alpha_i \otimes \beta_i|\sigma_{-1}(x^*) \otimes y\rangle = \langle m^*(1)|\sigma_{-1}(x^*) \otimes y\rangle \\
&= \langle 1|\sigma_{-1}(x^*)y\rangle = \langle \sigma_1(x)|y\rangle && \text{by A.70(i),A.69(ii).}
\end{aligned}
$$

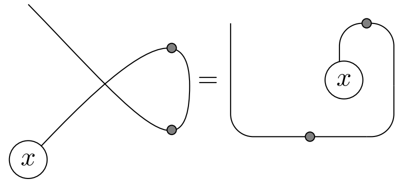
Thus the left-handed twist is exactly σ_1. ■

Proposition A.73. *Applying a right-handed twist corresponds to applying the automorphism* σ_{-1}*, i.e.,*



$= \sigma_{-1}.$

To make this clear, this is exactly saying that for any $x \in B$, we have



$= \sigma_{-1}(x).$

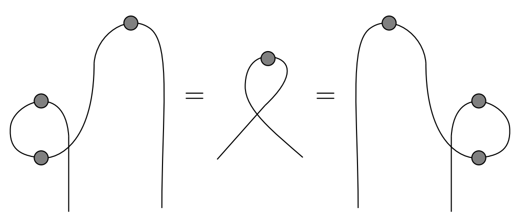
In other words, $(\mathrm{id} \otimes \eta^* m)(\varkappa \otimes \mathrm{id})(\mathrm{id} \otimes m^* \eta) = \sigma_{-1}$.

Proof. We let $x, y \in B$ and $m^*(1) = \sum_i \alpha_i \otimes \beta_i$ for some tuples $(\alpha_i), (\beta_i)$ in B. Then we compute,

$$\begin{aligned}
&\langle (\mathrm{id} \otimes \eta^* m)(\varkappa \otimes \mathrm{id})(\mathrm{id} \otimes m^* \eta)(x) | y \rangle \\
=&\sum_i \langle (\mathrm{id} \otimes \eta^* m)(\varkappa \otimes \mathrm{id})(x \otimes \alpha_i \otimes \beta_i) | y \rangle \\
=&\sum_i \langle (\mathrm{id} \otimes \eta^* m)(\alpha_i \otimes x \otimes \beta_i) | y \rangle \\
=&\sum_i \langle \langle 1 | x\beta_i \rangle \alpha_i | y \rangle = \sum_i \langle \alpha_i | y \rangle \langle x\beta_i | 1 \rangle \\
=&\sum_i \langle \alpha_i | y \rangle \langle \beta_i | x^* \rangle && \text{by A.70(ii)} \\
=&\langle m^*(1) | y \otimes x^* \rangle = \langle 1 | yx^* \rangle \\
=&\langle \sigma_{-1}(x) | y \rangle && \text{by A.70(iv).}
\end{aligned}$$

■

Proposition A.74 ([9, Equation 2.6])**.** *We have,*



In other words, $\eta^* m(\sigma_1 \otimes \mathrm{id}) = \eta^* \circ m \circ \varkappa = \eta^* m(\mathrm{id} \otimes \sigma_{-1})$.

Proof. For any $x, y \in B$, we compute,

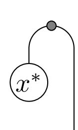
$$\eta^* m(\sigma_1 \otimes \mathrm{id})(x \otimes y) = \psi(\sigma_1(x) y) = \psi(x \sigma_{-1}(y)) = \eta^* m(\mathrm{id} \otimes \sigma_{-1})(x \otimes y).$$

And

$$\begin{aligned}
\eta^* \circ m \circ \varkappa(x \otimes y) &= \psi(yx) = \langle y^* | x \rangle = \langle x^* | \sigma_{-1}(y) \rangle \\
&= \psi(x \sigma_{-1}(y)) = \eta^* m(\mathrm{id} \otimes \sigma_{-1})(x \otimes y).
\end{aligned}$$

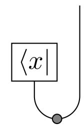
Thus $\eta^* m(\sigma_1 \otimes \mathrm{id}) = \eta^* \circ m \circ \varkappa = \eta^* m(\mathrm{id} \otimes \sigma_{-1})$. ■

Lemma A.75. *Given $x \in B$, we have,*

$\eta^* m(x^* \otimes \cdot) =$  $= \langle x |$.

Proof. This is an obvious and quick computation. ∎

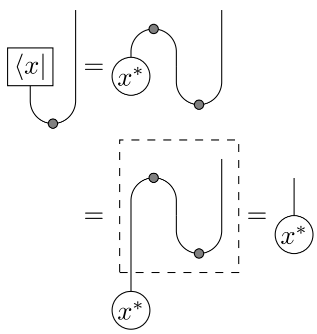
Lemma A.76. *Given $x \in B$, we have*

$$((\langle x| \otimes \mathrm{id})m^*\eta(1) =$$

$$= x^*.$$

Proof.

$$\begin{aligned}
\langle((\langle x| \otimes \mathrm{id})m^*(1)|y\rangle &= \sum_i \langle((\langle x| \otimes \mathrm{id})(\alpha_i \otimes \beta_i)|y\rangle \\
&= \sum_i \langle \alpha_i | x\rangle \langle \beta_i | y\rangle \\
&= \langle m^*(1) | x \otimes y\rangle = \langle 1 | xy\rangle \\
&= \langle x^* | y\rangle && \text{by A.70(i).}
\end{aligned}$$

Using strings:

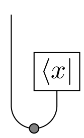


by A.76

by A.9.

∎

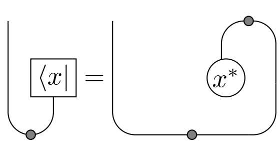
Lemma A.77. *Given $x \in B$, we have*


$$= \sigma_{-1}(x^*).$$

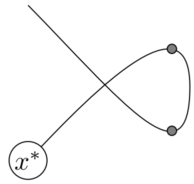
Proof.

$$\begin{aligned}
\langle(\mathrm{id} \otimes \langle x|)m^*\eta(1)|y\rangle &= \sum_i (\mathrm{id} \otimes \langle x|)(\alpha_i \otimes \beta_i)y \\
&= \sum_i \langle \beta_i | x\rangle \langle \alpha_i | y\rangle = \langle m^*(1) | y \otimes x\rangle \\
&= \langle 1 | yx\rangle = \langle \sigma_{-1}(x^*) | y\rangle && \text{by A.70(iv).}
\end{aligned}$$

With strings:



by A.76

$=$  $= \sigma_{-1}(x^*)$ by A.73.

The second equality can be seen by simply pulling the strand with x^* under/over the other strand. ■

Corollary A.78. *For any $t \in \mathbb{R}$, we have,*

(i) $m(\sigma_t \otimes \sigma_t) = \sigma_t m$,

(ii) $(\sigma_t \otimes \sigma_t)m^* = m^*\sigma_t$,

(iii) $\sigma_t \eta = \eta$,

(iv) $\eta^* \sigma_t = \eta^*$.

■

A.VII KMS inner product

Proposition A.79. *The following defines an inner product*

$$B \times B \to \mathbb{C}\colon (x, y) \mapsto \phi(x^* \sigma_{-1/2}(y)),$$

and is called the KMS-inner product. We denote this inner product by $\langle \cdot | \cdot \rangle_{KMS}$.

Proof. For any $x, y \in B$, let $\langle x|y\rangle_{\mathrm{KMS}} = \psi(x^*\sigma_{-1/2}(y))$. Now, for any $x, y \in B$ we get

$$\langle x|y\rangle_B = \big\langle \sigma_{1/4}(x) \big| \sigma_{1/4}(y) \big\rangle_{\mathrm{KMS}} \text{ and } \langle x|y\rangle_{\mathrm{KMS}} = \big\langle \sigma_{-1/4}(x) \big| \sigma_{-1/4}(y) \big\rangle_B.$$

So then we can easily see that $\langle \cdot|\cdot\rangle_{M_n}$ defines an inner product on B if and only if $\langle \cdot|\cdot\rangle_{\mathrm{KMS}}$ defines an inner product on B. ■

Given a linear map $A\colon (B_1, \psi_1) \to (B_2, \psi_2)$, we can define the canonical KMS map A_{KMS} to be be between the KMS-spaces.

Lemma A.80. *For $a \in (B_1, \psi_1)$ and $b \in (B_2, \psi_2)$, we have $|a\rangle\langle b|_{KMS} = \big|a\big\rangle\big\langle\sigma_{-1/2}(b)\big|$.*

Proof. Let $c \in B_2$. Then using Lemma A.69 we get

$$|a\rangle\langle b|_{\mathrm{KMS}}(c) = \langle b|c\rangle_{\mathrm{KMS}} a = \big\langle b \big| \sigma_{-1/2}(c) \big\rangle a = \big|a\big\rangle\big\langle \sigma_{-1/2}(b) \big|(c).$$

Thus $|a\rangle\langle b|_{\mathrm{KMS}} = \big|a\big\rangle\big\langle\sigma_{-1/2}(b)\big|$. ■

A.VIII Real (star-preserving) maps

In this section, we define what it means for a linear map to be *real* (also known as *star-preserving*) on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. Let $\mathcal{A}_1, \mathcal{A}_2$ be star-algebras.

> **Definition A.81.** We define the map $\cdot^r$ as the self-invertible anti-linear automorphism $\mathcal{L}(\mathcal{A}_1, \mathcal{A}_2) \cong \mathcal{L}(\mathcal{A}_1, \mathcal{A}_2)$ given by
>
> $$A \mapsto (a \mapsto A(a^*)^*).$$

Definition A.82 ([4, Definition 2.5])**.** We say $A \in \mathcal{L}(\mathcal{A}_1, \mathcal{A}_2)$ is *real* (or *star-preserving*) when $A(a^*) = A(a)^*$ for each $a \in \mathcal{A}_1$.

Lemma A.83. *Let $A \in \mathcal{L}(\mathcal{A}_1, \mathcal{A}_2)$. Then A is real if and only if $A^r = A$.*

Proof. Clearly $A^{\mathrm{r}}(x) = A(x^*)^* = A(x)$ if and only if $A(x^*) = A(x)^*$ for all $x \in \mathcal{A}$, which means A is real. ■

Given star-algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$, and linear maps $x\colon \mathcal{A}_1 \to \mathcal{A}_2$ and $y\colon \mathcal{A}_3 \to \mathcal{A}_4$, we clearly get $(x \otimes y)^{\mathrm{r}} = x^{\mathrm{r}} \otimes y^{\mathrm{r}}$.

Proposition A.84. *Given $A \in \mathcal{L}(\mathcal{A})$, we get* $\mathrm{Spectrum}(A^r) = \overline{\mathrm{Spectrum}(A)}$*.*
In fact, $x \in \ker(A - \lambda\,\mathrm{id})$ if and only if $x^* \in \ker(A^r - \bar{\lambda}\,\mathrm{id})$.

Proof. For any $x \in \mathcal{A}$, we have $A^{\mathrm{r}}(x^*) = A(x)^*$. So if x is an eigenvector of A with eigenvalue λ, then clearly $A^{\mathrm{r}}(x^*) = \bar{\lambda}x^*$, so x^* is an eigenvector of A^{r} with eigenvalue $\bar{\lambda}$. If, on the other hand, x^* is an eigenvector of A^{r} with eigenvalue $\bar{\lambda}$, then $A(x)^* = \bar{\lambda}x^*$, and so $A(x) = \lambda x$, which means x is an eigenvector of A with eigenvalue λ. ■

Remark A.85. Let $A \in \mathcal{L}(\mathcal{A})$ be real and let x be an eigenvector of A with corresponding eigenvalue λ. Then the eigenspaces of each eigenvalue of A are spanned by self-adjoint elements [10, Lemma 1.11]. ◊

Lemma A.86. *If $A\colon (B_1, \psi_1, \sigma_r) \to (B_2, \psi_2, \vartheta_r)$, then $A^{*r} = \sigma_1 A^{r*} \vartheta_{-1}$.*
Here, the ordered pairs (B_1, ψ_1, σ_r) and $(B_2, \psi_2, \vartheta_r)$ mean that we have finite-dimensional C^*-algebras B_1, B_2 with respective faithful and positive linear functionals ψ_1, ψ_2 and modular automorphisms σ_r, ϑ_r.

Proof. Let $x \in B_2$ and $y \in B_1$, and compute,

$$\begin{aligned}
\langle A^{*\mathrm{r}}(x)|y\rangle_{B_1} &= \left\langle A^*(x^*)^*\middle|y\right\rangle_{B_1} = \langle y^*|\sigma_{-1}(A^*(x^*))\rangle_{B_1} && \text{by A.70(ii),(iv)}\\
&= \langle A(\sigma_{-1}(y^*))|x^*\rangle_{B_2} && \text{by A.69(iii)}\\
&= \left\langle A(\sigma_1(y)^*)\middle|x^*\right\rangle_{B_2} = \langle x|\vartheta_{-1}A^{\mathrm{r}}\sigma_1(y)\rangle_{B_2} && \text{by A.70(ii),(iv)}\\
&= \langle \sigma_1 A^{\mathrm{r}*}\vartheta_{-1}(x)|y\rangle_{B_1} && \text{by A.69(iii).}
\end{aligned}$$

Thus, $A^{*\mathrm{r}} = \sigma_1 A^{\mathrm{r}*}\vartheta_{-1}$. ■

Proposition A.87 ([4, Lemma 5.7])**.** *Given elements $b \in (B_1, \psi_1, \sigma_r)$ and $a \in (B_2, \psi_2, \vartheta_r)$, we get $|a\rangle\langle b|^r = |a^*\rangle\langle\sigma_{-1}(b^*)|$.*
The ordered pairs here have the same meaning as in Lemma A.86.

Proof. For any $x \in B_1$, we compute,

$$\begin{aligned}
|a\rangle\langle b|^{\mathrm{r}}(x) &= (|a\rangle\langle b|(x^*))^* = \overline{\langle b|x^*\rangle}a^* = \langle x^*|b\rangle a^*\\
&= \langle 1|xb\rangle a^* && \text{by A.70(i)}\\
&= \langle \sigma_{-1}(b^*)|x\rangle a^* && \text{by A.70(iv)}\\
&= |a^*\rangle\langle\sigma_{-1}(b^*)|(x).
\end{aligned}$$

Thus $|a\rangle\langle b|^{\mathrm{r}} = |a^*\rangle\langle\sigma_{-1}(b^*)|$. ■

Proposition A.88 ([9, Proposition 2.15])**.** *If $A \in \mathcal{B}(B, \psi)$ is real, then A^* is real if and only if A commutes with σ_1.*

Proof. Suppose $A \in \mathcal{B}(B)$ is real. Firstly, taking adjoints and using Lemma A.69(iii), we get A commutes with σ_1 if and only if A^* commutes with σ_1, so it is enough to show that A^* is real if and only if A^* commutes with σ_1.

By Lemma A.86 we have $A^{*\mathrm{r}} = \sigma_1 A^{\mathrm{r}*}\sigma_{-1}$, which means $A^{*\mathrm{r}}\sigma_1 = \sigma_1 A^{\mathrm{r}*}$. Since we know A is real, we get $A^{\mathrm{r}} = A$ by Lemma A.83, and so $A^{*\mathrm{r}}\sigma_1 = \sigma_1 A^*$.

If A^* is real, then we have $A^{*\mathrm{r}} = A^*$ by Lemma A.83, and so $A^*\sigma_1 = \sigma_1 A^*$. If, on the other hand, A^* commutes with σ_1, then $A^{*\mathrm{r}}\sigma_1 = \sigma_1 A^* = A^*\sigma_1$, and so $A^{*\mathrm{r}} = A^*$, which by Lemma A.83 means A^* is real.

Therefore, given A is real, we get A^* is also real if and only if A commutes with σ_1. ■

Lemma A.89.

(i) $m^r = m\varkappa$,

(ii) $m^{*r} = \varkappa m^*$.

Recall that the map $\varkappa$ *is the identification* $B \otimes B \cong B \otimes B$ *given by* $x \otimes y \mapsto y \otimes x$.

Proof. (i) Let $x, y \in B$. Then

$$m^{\mathrm{r}}(x \otimes y) = m(x^* \otimes y^*)^* = (x^* y^*)^* = yx = m\varkappa(x \otimes y).$$

(ii)

$$\begin{aligned} m^{*\mathrm{r}} &= \sigma_1^{\otimes 2} m^{\mathrm{r}*}\sigma_{-1} && \text{by A.86} \\ &= \sigma_1^{\otimes 2}\varkappa m^*\sigma_{-1} && \text{by Part (i)} \\ &= \varkappa\sigma_1^{\otimes 2} m^*\sigma_{-1} && \text{by A.3} \\ &= \varkappa m^*\sigma_1\sigma_{-1} && \text{by A.78(ii)} \\ &= \varkappa m^* \end{aligned}$$

■

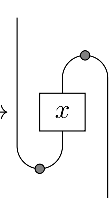
A.IX The symmetry maps

In this section we look at the symmetry maps which are found in [4, Equations 2 & 4 in Definition 2.4]. These maps turn out to correspond to the adjoint of the real map (i.e., $\mathrm{symm}(A) = A^{*r}$ and $\mathrm{symm}'(A) = A^{r*}$) – see Proposition A.92.

Definition A.90. We define symm as the linear map $\mathcal{B}(B_1, B_2) \to \mathcal{B}(B_2, B_1)$ given by

$$x \mapsto (\mathrm{id} \otimes \eta_2^* m_2)(\mathrm{id} \otimes x \otimes \mathrm{id})(m_1^*\eta_1 \otimes \mathrm{id}).$$

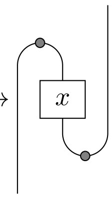
In other words,

symm: $x \mapsto$ 

We define symm′ as the linear map $\mathcal{B}(B_1, B_2) \to \mathcal{B}(B_2, B_1)$ given by

$$x \mapsto (\eta_2^* m_2 \otimes \mathrm{id})(\mathrm{id} \otimes x \otimes \mathrm{id})(\mathrm{id} \otimes m_1^*\eta_1).$$

In other words,

$\mathrm{symm}' \colon x \mapsto$ 

Proposition A.91 ([4, Proposition 5.3(ii,iii)]). *Let $a \in (B_1, \psi_1, \sigma_r), b \in (B_2, \psi_2, \vartheta_r)$. Then we have,*

(i) $\mathrm{symm}(|a\rangle\langle b|) = |\vartheta_{-1}(b^*)\rangle\langle a^*|$,

(ii) $\mathrm{symm}'(|a\rangle\langle b|) = |b^*\rangle\langle \sigma_{-1}(a^*)|$.

The ordered pairs here have the same meaning as in Lemma A.86.

Proof. (i) Let $m^*(1) = \sum_j \alpha_j \otimes \beta_j$ for tuples (α_j) and (β_j) in B_1, so that for any $x \in B_2$ and $y \in B_1$, we have,

$$\begin{aligned}
&\langle x|\mathrm{symm}(|a\rangle\langle b|)(y)\rangle = \langle x|(\mathrm{id} \otimes \eta_2^* m_2)(\mathrm{id} \otimes |a\rangle\langle b| \otimes \mathrm{id})(m_1^* \eta_1 \otimes \mathrm{id})(y)\rangle \\
&= \sum_j \langle x|(\mathrm{id} \otimes \eta_2^* m_2^*)(\mathrm{id} \otimes |a\rangle\langle b| \otimes \mathrm{id})(\alpha_j \otimes \beta_j \otimes y)\rangle \\
&= \sum_j \langle x|(\mathrm{id} \otimes \eta_2 m_2^*)(\alpha_j \otimes \langle b|\beta_j\rangle a \otimes y)\rangle \\
&= \sum_j \langle b|\beta_j\rangle \langle x|(\alpha_j \otimes \eta_2^*(ay))\rangle \\
&= \sum_j \langle b|\beta_j\rangle \langle 1|ay\rangle \langle x|\alpha_j\rangle \\
&= \sum_j \langle x \otimes b|\alpha_j \otimes \beta_j\rangle \langle 1|ay\rangle \\
&= \langle x \otimes b|m_2^*(1)\rangle \langle 1|ay\rangle = \langle xb|1\rangle \langle 1|ay\rangle \\
&= \langle x|\vartheta_{-1}(b^*)\rangle \langle a^*|y\rangle \qquad \text{by A.70(i),(iii)} \\
&= \langle x||\vartheta_{-1}(b^*)\rangle\langle a^*|(y)\rangle.
\end{aligned}$$

So $\mathrm{symm}(|a\rangle\langle b|) = |\vartheta_{-1}(b^*)\rangle\langle a^*|$ as claimed.

(ii) Analogously to the above, we let $m_1^*(1) = \sum_j \alpha_j \otimes \beta_j$ for each $\alpha_j, \beta_j \in B_1$, so that for any $x \in B_2$ and $y \in B_1$ we find,

$$\begin{aligned}
&\mathrm{symm}'(|a\rangle\langle b|) \\
&= \langle x|(\eta_2^* m_2 \otimes \mathrm{id})(\mathrm{id} \otimes |a\rangle\langle b| \otimes \mathrm{id})(\mathrm{id} \otimes m_1^* \eta_1)(y)\rangle \\
&= \sum_j \langle x|(\eta_2^* m_2 \otimes \mathrm{id})(\mathrm{id} \otimes |a\rangle\langle b| \otimes \mathrm{id})(y \otimes \alpha_j \otimes \beta_j)\rangle \\
&= \sum_j \langle x|(\eta_2^* m_2 \otimes \mathrm{id})(y \otimes \langle b|\alpha_j\rangle a \otimes \beta_j)\rangle \\
&= \sum_j \langle b|\alpha_j\rangle \langle x|(\eta_2^*(ya) \otimes \beta_j)\rangle \\
&= \sum_j \langle b|\alpha_j\rangle \langle 1|ya\rangle \langle x|\beta_j\rangle \\
&= \sum_j \langle b \otimes x|\alpha_j \otimes \beta_j\rangle \langle 1|ya\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle b \otimes x | m_1^*(1) \rangle \langle 1 | ya \rangle = \langle bx | 1 \rangle \langle 1 | ya \rangle \\
&= \langle x | b^* \rangle \langle \sigma_{-1}(a^*) | y \rangle && \text{by A.70(ii),(iv)} \\
&= \langle x | | b^* \rangle\!\langle \sigma_{-1}(a^*) | (y) \rangle.
\end{aligned}
$$

So then $\operatorname{symm}'(|a\rangle\!\langle b|) = |b^*\rangle\!\langle \sigma_{-1}(a^*)|$.

■

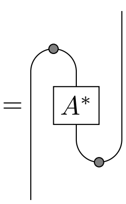
Proposition A.92. *For any operator* $A \in \mathcal{B}((B_1, \psi_1), (B_2, \psi_2))$*, we have* $\operatorname{symm}(A) = A^{r*}$ *and* $\operatorname{symm}'(A) = A^{*r}$.

Proof. Comparing part (ii) from Proposition A.95 with Proposition A.87, we see that for any $a \in B_1$ and $b \in B_2$, we have $\operatorname{symm}'(|a\rangle\!\langle b|) = |a\rangle\!\langle b|^{*\mathrm{r}}$. And so for any $A \colon B_1 \to B_2$, by writing $A = \sum_i |a_i\rangle\!\langle b_i|$ for some tuples $(a_i), (b_i)$ in B_2 and B_1 respectively, we can conclude $\operatorname{symm}'(A) = A^{*\mathrm{r}}$.

Analogously, comparing part (i) from Proposition A.91 with Proposition A.87, we see that for any $a \in B_1, b \in B_2$, we have $\operatorname{symm}(|a\rangle\!\langle b|) = |a\rangle\!\langle b|^{\mathrm{r}*}$. And so for any $A \colon B_1 \to B_2$, by writing $A = \sum_i |a_i\rangle\!\langle b_i|$ for some tuples $(a_i), (b_i)$ in B_2 and B_1 respectively, we can conclude $\operatorname{symm}(A) = A^{\mathrm{r}*}$. ■

Remark A.93. So by the above, we have $A \in \mathcal{B}(B)$ is real if and only if $\operatorname{symm}'(A^*) = A$; this is exactly [9, Lemma 2.13], but done via diagrams. We also get $A \in \mathcal{B}(B)$ is real if and only if $\operatorname{symm}(A) = A^*$; and this is exactly [6, Proposition 1.7(3)]. ◊

Corollary A.94. *Given* $A \in \mathcal{B}((B_1, \psi_1), (B_2, \psi_2))$*, we get*

$A^r =$ 

Proof. This is a direct consequence of Proposition A.92. ■

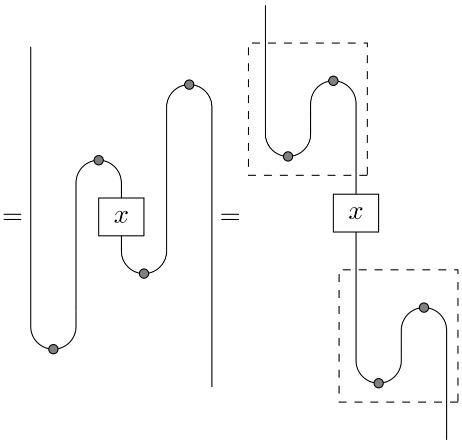
The next result tells us that our linear operators symm and symm′ are both invertible, where they are both inverses of each other. This means that for any operators $A_1 \colon B_1 \to B_2$ and $A_2 \colon B_2 \to B_1$, we get $\operatorname{symm}(A_1) = A_2$ if and only if $\operatorname{symm}'(A_2) = A_1$ ([4, Proposition 5.4]).

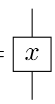
Corollary A.95. symm *is invertible with inverse* symm′.

Proof. For any $A \colon B_1 \to B_2$, we get $\operatorname{symm}(\operatorname{symm}'(A)) = \operatorname{symm}(A^{*\mathrm{r}}) = (A^{*\mathrm{r}})^{\mathrm{r}*} = A$ and $\operatorname{symm}'(\operatorname{symm}(A)) = \operatorname{symm}'(A^{\mathrm{r}*}) = (A^{\mathrm{r}*})^{*\mathrm{r}} = A$ using Proposition A.92. ■

Remark A.96. The above equivalency is mentioned, via string diagrams, in [9, Definition 2.19 (self-transpose)], which utilizes the snake-equations [12, II.B (5)] (see Proposition A.9). In particular, for any linear operator x, we have

$\mathrm{symm}(\mathrm{symm}'(x)) =$



$=$ 

by A.9.

Analogously, we can again use the snake-equations from Proposition A.9 to diagrammatically see that we get $\mathrm{symm}'(\mathrm{symm}(x)) = x$. ◊

Corollary A.97. $\mathrm{symm}^{\,r} = \mathrm{symm}^{-1}$.
Note that here, symm *is an operator in* $\mathcal{B}(\mathcal{B}(B,\psi),\mathcal{B}(B,\psi))$.

Proof. For any $A \in \mathcal{B}(B,\psi)$, we have $\mathrm{symm}^{\,\mathrm{r}}(A) = \mathrm{symm}(A^*)^* = A^{*\mathrm{r}} = \mathrm{symm}^{-1}(A)$, using Proposition A.92 and Corollary A.95. ■

Proposition A.98. *For* $A \in \mathcal{B}((B_1,\psi_1),(B_2,\psi_2))$ *and* $x \in B_1, y \in B_2$*, we get*

$$\psi_2(A(x)y) = \psi_1(x\,\mathrm{symm}(A)(y)).$$

Proof.

$$\begin{aligned}\psi_2(A(x)y) &= \langle A(x)^*|y\rangle_{B_2} = \langle A^{\mathrm{r}}(x^*)|y\rangle_{B_2}\\ &= \langle x^*|A^{\mathrm{r}*}(y)\rangle_{B_1} = \psi_1(x\,\mathrm{symm}(A)(y)) \qquad \text{by A.92.}\end{aligned}$$

■

The following result shows that a linear operator A satisfies $\mathrm{symm}(A) = A$ if and only if $\psi(A(x)y) = \psi(xA(y))$ for all x and y. So we can think of A satisfying $\mathrm{symm}(A) = A$ as being *symmetric* with respect to ψ.

Proposition A.99. *For* $A \in \mathcal{B}(B,\psi)$*, the following are equivalent,*

(i) $A^* = A^r$,

(ii) $\mathrm{symm}(A) = A$,

(iii) $\mathrm{symm}'(A) = A$,

(iv) $\forall x, y \in B : \psi(A(x)y) = \psi(xA(y))$.

Proof. The equivalence between (ii) and (iii) is already done in Corollary A.95. The equivalence between (i) and (iii) is easily seen using Proposition A.92. So it remains to show the equivalence between (iv) and the rest.

(ii) ⇒ *(iv)* Suppose $\operatorname{symm}(A) = A$. Then, by Proposition A.98, we get that for any $x, y \in B$,

$$\psi(A(x)y) = \psi(x \operatorname{symm}(A)(y)) = \psi(xA(y)).$$

(iv) ⇒ *(i)* Now suppose $\psi(A(x)y) = \psi(xA(y))$ for all $x, y \in B$. Then for any $x, y \in B$, we compute,

$$\langle A^*(x)|y\rangle = \langle x|A(y)\rangle = \psi(x^*A(y)) = \psi(A(x^*)y) = \left\langle A(x^*)^* \middle| y\right\rangle = \langle A^{\mathrm{r}}(x)|y\rangle.$$

Thus $A^* = A^{\mathrm{r}}$.

■

Corollary A.100. *For* $A\colon (B_1, \psi_1, \sigma_r) \to (B_2, \psi_2, \vartheta_r)$*, we have* $\operatorname{symm}'(A) \circ \vartheta_1 = \sigma_1 \circ \operatorname{symm}(A)$.

The ordered pairs here have the same meaning as in Lemma A.86.

Proof. Combine Lemma A.86 with Proposition A.92. ■

The following result slightly generalises [16, Lemma 2.1].

Corollary A.101. *For* $A \in \mathcal{B}(B, \psi)$*, we get* $\operatorname{symm}(A) = \operatorname{symm}'(A)$ *if and only if* $A\sigma_1 = \sigma_1 A$*. Moreover, if* $\operatorname{symm}(A) = A$*, then* A *commutes with* σ_1*.*

Proof. We only show the former, since if $\operatorname{symm}(A) = A$, then we get $\operatorname{symm}'(A) = A$ (Corollary A.95), and so $\operatorname{symm}(A) = \operatorname{symm}'(A)$.

We have the following equivalences,

$$\begin{aligned}
\operatorname{symm}(A) = \operatorname{symm}'(A) &\Leftrightarrow A^{\mathrm{r}*} = A^{*\mathrm{r}} && \text{by A.92}\\
&\Leftrightarrow \sigma_1 A^{\mathrm{r}*}\sigma_{-1} = A^{\mathrm{r}*} && \text{by A.86}\\
&\Leftrightarrow A^{\mathrm{r}*}\sigma_{-1} = \sigma_{-1}A^{\mathrm{r}*} &&\\
&\Leftrightarrow \sigma_{-1}A^{\mathrm{r}} = A^{\mathrm{r}}\sigma_{-1} && \text{by A.69(iii)}\\
&\Leftrightarrow \sigma_1 A = A\sigma_1 && \text{by A.69(iv).}
\end{aligned}$$

Thus $\operatorname{symm}(A) = \operatorname{symm}'(A)$ if and only if A commutes with σ_1. ■

Proposition A.102 ([9, Lemma 2.22])**.** *Let* $A \in \mathcal{B}(B, \psi)$*. Then,*

(i) if A *is self-adjoint and* $\operatorname{symm}(A) = A$*, then* A *is real,*

(ii) if A *is real and* $\operatorname{symm}(A) = A$*, then* A *is self-adjoint,*

(iii) if A *is real and self-adjoint, then* $\operatorname{symm}(A) = A$*.*

Proof. Note that A being real means $A^{\mathrm{r}} = A$ by Lemma A.83.

(i) Suppose $A^* = A$ and $\operatorname{symm}(A) = A$. Then we clearly get $A = A^{\mathrm{r}}$, as $\operatorname{symm}(A) = A$ is equivalent to $A^{\mathrm{r}} = A^*$ by Proposition A.99.

(ii) Suppose A is real and $\operatorname{symm}(A) = A$. Then, by Proposition A.99, we get $A^* = A^{\mathrm{r}} = A$.

(iii) Suppose A is real and self-adjoint. Then $A^* = A = A^{\mathrm{r}}$, and so by Proposition A.99, we get $\operatorname{symm}(A) = A$.

■

Lemma A.103. *Given any* $x \in \mathcal{B}((B_1, \psi_1), (B_2, \psi_2))$ *and* $y \in \mathcal{B}((B_3, \psi_3), (B_1, \psi_1))$*, we have*

$$\operatorname{symm}(x \circ y) = \operatorname{symm}(y) \circ \operatorname{symm}(x).$$

Proof. Using Proposition A.92, we compute,

$$\mathrm{symm}(x \circ y) = (x \circ y)^{\mathrm{r}*} = (x^{\mathrm{r}} \circ y^{\mathrm{r}})^* = y^{\mathrm{r}*} \circ x^{\mathrm{r}*} = \mathrm{symm}(y) \circ \mathrm{symm}(x).$$

■

Lemma A.104. *Given a linear map $A \in \mathcal{B}(B, \psi)$ and a *-homomorphism $f \colon B \to B_2$ such that $f^{*r} = f^*$, we get* $\mathrm{symm}(fAf^*) = f\,\mathrm{symm}(A)f^*$.

Proof. Using Proposition A.92, we get

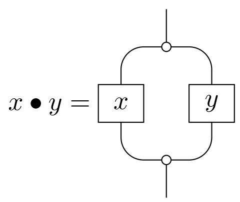
$$\mathrm{symm}(fAf^*) = fAf^* = (fAf^*)^{\mathrm{r}*} = (fA^{\mathrm{r}}f^*)^* = f\,\mathrm{symm}(A)f^*.$$

■

A.X The Schur product

We now define the Schur product map. Another name this sometimes goes by is the convolution product map.

Definition A.105. Let $\mathcal{A}_1$ be a co-algebra with co-multiplication μ_1 and let $\mathcal{A}_2$ be an algebra with multiplication map m_2. Then we define the *Schur product* $\cdot \bullet \cdot$ as the linear map $(\mathcal{A}_1 \to \mathcal{A}_2) \to \mathcal{L}(\mathcal{A}_1 \to \mathcal{A}_2)$ given by $x \mapsto (y \mapsto m_2(x \otimes y)\mu_1^*)$. In other words, for all linear maps $x, y \colon \mathcal{A}_1 \to \mathcal{A}_2$, we have

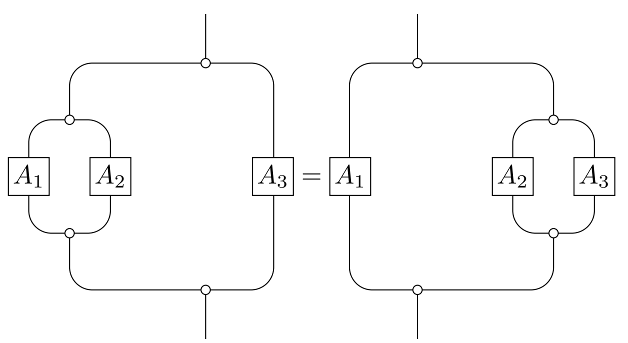


The following shows the associativity of the Schur product, which follows from the associativity and co-associativity of their respective multiplication maps.

Corollary A.106. *Given linear maps $A_1, A_2, A_3 \colon X_1 \to X_2$ on co-algebra X_1 and algebra X_2, then $\bullet$ is associative, i.e., we get*

$$(A_1 \bullet A_2) \bullet A_3 = A_1 \bullet (A_2 \bullet A_3)\,,$$

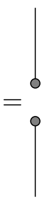
which in diagrams is,



Proof. This follows from the associativity (mul_assoc) and co-associativity (co_mul_assoc) of the multiplication maps. ■

So we already have a non-unital ring structure on $\mathcal{L}(A_1, A_2)$ for co-algebra A_1 and algebra A_2, where the product is given by the Schur product $\bullet$. Let us now define the unit.

Definition A.107. Let $\mathcal{A}_1$ be a co-algebra with co-unit ϖ_1 and let $\mathcal{A}_2$ be an algebra with unit η_2. Then we let the unit 1 on $(\mathcal{L}(\mathcal{A}_1, \mathcal{A}_2), \bullet)$ be given by $\eta_2\varpi_1$, which in diagrams is,

$$1 := \eta_2\varpi_1 =$$  .

Lemma A.108. $1 \bullet f = f = f \bullet 1$.

Thus $(\mathcal{L}(\mathcal{A}_1, \mathcal{A}_2), \bullet, 1 = \eta_2\varpi_1)$ *is a ring.*

Proof.

$$\begin{aligned} 1 \bullet f &= m_2(\eta_2\varpi_1 \otimes f)\mu_1 \\ &= m_2(\eta_2 \otimes \mathrm{id}_{\mathcal{A}_2})(\mathrm{id}_{\mathbb{C}} \otimes f)(\varpi_1 \otimes \mathrm{id}_{\mathcal{A}_1})\mu_1 \\ &= \mathrm{id}_{\mathbb{C}} \otimes f = f. \end{aligned}$$

The third equality follows from the algebraic and co-algebraic properties. The fourth follows from our notation (see under Section A.II).

Analogously,

$$\begin{aligned} f \bullet 1 &= m_2(f \otimes \eta_2\varpi_1)\mu_1 \\ &= m_2(\mathrm{id}_{\mathcal{A}_2} \otimes \eta_2)(f \otimes \mathrm{id}_{\mathbb{C}})(\mathrm{id}_{\mathcal{A}_1} \otimes \varpi_1)\mu_1 \\ &= f \otimes \mathrm{id}_{\mathbb{C}} = f. \end{aligned}$$

■

A 'quantum adjacency matrix' (see B.1) is essentially an operator that is an idempotent with respect to this ring structure. However, as our operator will be from B to B, saying 'idempotent' will be confusing (since that can mean idempotent with respect to the composition product), so we will say *Schur idempotent* to mean idempotent with respect to this ring structure.

The unit on this ring structure turns out to be our 'complete quantum adjacency operator' (see Definition B.32).

Remark A.109. While we are technically working over $\mathbb{C}$, this is all still generally true over any commutative semiring R. ♢

Proposition A.110. *Given* f, g, h, k, *we get:*

$$(f \otimes h) \bullet (g \otimes k) = (f \bullet g) \otimes (h \bullet k).$$

Proof. We leave the details of this as an exercise to the reader.
Hint: recall how the multiplication co-multiplication maps are defined on tensor products:

$$m_{E\otimes F} = (m_E \otimes m_F)(\mathrm{id} \otimes \varkappa_{F,E} \otimes \mathrm{id}),$$

$$\mu_{E\otimes F} = (\mathrm{id} \otimes \varkappa_{E,F} \otimes \mathrm{id})(\mu_E \otimes \mu_F).$$

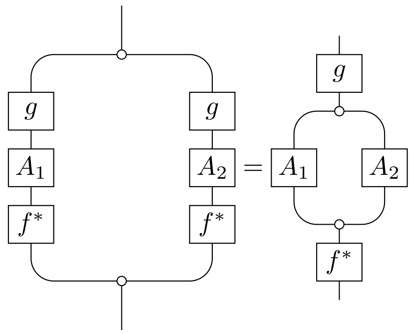
■

A.X.1 On Hilbert spaces.

Lemma A.111. *Let X_1, X_2, X_3, X_4 be finite-dimensional algebras with respective multiplication maps m_1, m_2, m_3, m_4 (with the induced co-algebras given by A.11). Then given algebra homomorphisms $f\colon X_2 \to X_1$ and $g\colon X_3 \to X_4$, and linear maps $A_1, A_2\colon X_2 \to X_3$, we have*

$$(gA_1f^*) \bullet (gA_2f^*) = g(A_1 \bullet A_2)f^*,$$

which in diagrams is,



Proof. We compute,

$$\begin{aligned}(gA_1f^*) \bullet (gA_2f^*) &= m_4(gA_1f^* \otimes gA_2f^*)m_1^* \\ &= m_4(g \otimes g)(A_1 \otimes A_2)(f^* \otimes f^*)m_1^* \\ &= g \circ m_3(A_1 \otimes A_2)m_2^* \circ f^* && \text{by A.14(i),(iii)} \\ &= g \circ (A_1 \bullet A_2) \circ f^*.\end{aligned}$$

■

The following shows that the Schur product on rank-one operators acts as a product on the left and right variables of the rank-one operators.

Proposition A.112 ([4, Proposition 5.3(iv)])**.** *For Hilbert spaces B_1 and B_2 which are also algebras (with the induced co-algebra given by A.11). Then given $a, c \in B_1$ and $b, d \in B_2$, we have*

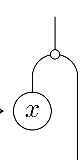
$$|a\rangle\langle b| \bullet |c\rangle\langle d| = |ac\rangle\langle bd|.$$

Proof.

$$\begin{aligned}|a\rangle\langle b| \bullet |c\rangle\langle d| &= m(|a\rangle\langle b| \otimes |c\rangle\langle d|)m^* = m|a \otimes c\rangle\langle b \otimes d|m^* \\ &= |m(a \otimes c)\rangle\langle m(b \otimes d)| = |ac\rangle\langle bd|.\end{aligned}$$

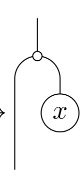
■

Definition A.113. We define the *-algebra homomorphism $\text{lmul}\colon B \to \mathcal{B}(B)$ to be the left multiplication map given by $y \mapsto (x \mapsto yx)$. In other words,

lmul: $x \mapsto$ 

,

We define the linear map $\text{rmul}\colon B \to \mathcal{B}(B)$ to be the right multiplication map given by $y \mapsto (x \mapsto xy)$. In other words,

rmul: $x \mapsto$ 

.

Remark A.114. For any $t \in \mathbb{R}$, we can write $\sigma_t = \mathrm{lmul}(Q^{-t})\,\mathrm{rmul}(Q^t)$. ◊

Lemma A.115. *For $x, y \in (B, \psi)$, we have,*

(i) $\mathrm{rmul}(xy) = \mathrm{rmul}(y)\,\mathrm{rmul}(x)$,

(ii) $\mathrm{rmul}(x)^* = \mathrm{rmul}(\sigma_{-1}(x^*))$,

(iii) $\mathrm{lmul}(x)\,\mathrm{rmul}(y) = \mathrm{rmul}(y)\,\mathrm{lmul}(x)$,

(iv) $f \circ \mathrm{lmul}(x) \circ f^{-1} = \mathrm{lmul}(f(x))$ *for invertible* $f \in B \to B_2$ *such that* $f(ab) = f(a)f(b)$ *for all* $a, b \in B$,

(v) $f \circ \mathrm{rmul}(x) \circ f^{-1} = \mathrm{rmul}(f(x))$ *for invertible* $f \in B \to B_2$ *such that* $f(ab) = f(a)f(b)$ *for all* $a, b \in B$,

(vi) $\mathrm{lmul}(x)^r = \mathrm{rmul}(x^*)$,

(vii) $\mathrm{rmul}(x)^r = \mathrm{lmul}(x^*)$.

Let e be the orthonormal basis $(\iota_s(e_{ij}Q_s^{-1}))_{s,ij}$ of (B, ψ) from Proposition A.60. Then,

(viii) $\mathcal{M}_e(\mathrm{lmul}(x)) = \bigoplus_i x_i \otimes 1_i$,

(ix) $\mathcal{M}_e(\mathrm{rmul}(x)) = \bigoplus_i 1_i \otimes \sigma_{1/2}(x_i)^T$.

Proof. Parts (i) and (iii) are straightforward computations and left as exercises.

(ii) Let $a, b \in B$. Then $\langle \mathrm{rmul}(x)^*(a) | b \rangle = \langle a | bx \rangle = \langle a\sigma_{-1}(x^*) | b \rangle = \langle \mathrm{rmul}(\sigma_{-1}(x^*))(a) | b \rangle$, where we have used Lemma A.70(iv) in the second equality.

(iv) Let $a \in B$. Then $f\,\mathrm{lmul}(x) f^{-1}(a) = f(xf^{-1}(a)) = f(x)a = \mathrm{lmul}(f(x))(a)$.

(v) Let $a \in B$. Then $f\,\mathrm{rmul}(x) f^{-1}(a) = f(f^{-1}(a)x) = af(x) = \mathrm{rmul}(f(x))(a)$.

(vi) Let $a \in B$. Then $\mathrm{lmul}(x)^{\mathrm{r}}(a) = \mathrm{lmul}(x)(a^*)^* = (xa^*)^* = ax^* = \mathrm{rmul}(x^*)(a)$. Thus $\mathrm{lmul}(x)^{\mathrm{r}} = \mathrm{rmul}(x^*)$.

(vii) Using the above, we get $\mathrm{rmul}(x)^{\mathrm{r}} = \mathrm{lmul}(x^*)^{\mathrm{rr}} = \mathrm{lmul}(x^*)$.

(viii) For any $r, s \in [\mathfrak{K}]$, and i, j, k, l, we compute,

$$\begin{aligned}
\mathcal{M}_e(\mathrm{lmul}(x))_{s,ij}^{r,kl} &= \sum_a \mathcal{M}_e(\mathrm{lmul}\,\iota_a(x_a))_{s,ij}^{r,kl} \\
&= \sum_a \left\langle \iota_s(e_{ij}Q_s^{-1/2}) \middle| \iota_a(x_a)\iota_r(e_{kl}Q_r^{-1/2}) \right\rangle \\
&= \delta_{r,s}(x_s e_{kl})_{ij} = \delta_{r,s} x_{s,ik} \delta_{l,j} \\
&= \bigoplus_a (x_a \otimes 1_a)_{s,ij}^{r,kl}.
\end{aligned}$$

Thus $\mathcal{M}_e(\mathrm{lmul}(x)) = \bigoplus_i x_i \otimes 1_i = \bigoplus_i \mathcal{M}_{e_i}(\mathrm{lmul}(x_i))$.

(ix) For any $r, s \in [\mathfrak{K}]$, and i, j, k, l, we compute,

$$\begin{aligned}
\mathcal{M}_e(\mathrm{rmul}(x))_{s,ij}^{r,kl} &= \sum_a \mathcal{M}_e(\mathrm{rmul}\,\iota_a(x_a))_{s,ij}^{r,kl} \\
&= \sum_a \left\langle \iota_s(e_{ij}Q_s^{-1/2}) \middle| \iota_r(e_{kl}Q_r^{-1/2})\iota_a(x_a) \right\rangle \\
&= \delta_{r,s}(e_{r,kl}\sigma_{1/2}(x))_{r,ij} \\
&= \delta_{r,s}\delta_{k,i}\sigma_{1/2}(x)_{r,lj} = \bigoplus_a (1_a \otimes \sigma_{1/2}(x_a)^T)_{s,ij}^{r,kl}.
\end{aligned}$$

Thus $\mathcal{M}_e(\mathrm{rmul}(x)) = \bigoplus_a 1_a \otimes \sigma_{1/2}(x_a)^T = \bigoplus_a \mathcal{M}_{e_a}(\mathrm{rmul}(x_a))$.

■

Proposition A.116 ([4, Lemma 5.24]). *Let $a, b \in (B, \psi)$. Then we have,*

(i) $|a\rangle\langle b| \bullet \mathrm{id} = \mathrm{lmul}(ab^*)$,

(ii) $\mathrm{id} \bullet |a\rangle\langle b| = \mathrm{rmul}(\sigma_{-1}(b^*)a)$.

Moreover, $|a\rangle\langle b| \bullet \mathrm{id} = \mathrm{id}$ *(resp.,* $|a\rangle\langle b| \bullet \mathrm{id} = 0$*) if and only if* $ab^* = 1$ *(resp.,* $ab^* = 0$*), and, similarly,* $\mathrm{id}\bullet|a\rangle\langle b| = \mathrm{id}$ *(resp.,* $\mathrm{id}\bullet|a\rangle\langle b| = 0$*) if and only if* $\sigma_{-1}(b^*)a = 1$ *(resp.,* $\sigma_{-1}(b^*)a = 0$*).*

Proof. Let $\mathrm{id} = \sum_t |f_t\rangle\langle f_t|$ for some orthonormal basis (f_t) in (B, ψ).

(i) For any $x, y \in B$, we compute,

$$\begin{aligned}
\langle |a\rangle\langle b| \bullet \mathrm{id}(x)|y\rangle &= \langle m(|a\rangle\langle b| \otimes \mathrm{id})m^*(x)|y\rangle \\
&= \sum_t \langle m(|a\rangle\langle b| \otimes |f_t\rangle\langle f_t|)m^*(x)|y\rangle \\
&= \sum_t \langle |af_t\rangle\langle bf_t|(x)|y\rangle && \text{by A.112} \\
&= \sum_t \langle x|bf_t\rangle\langle af_t|y\rangle \\
&= \sum_t \langle b^*x|f_t\rangle\langle f_t|a^*y\rangle && \text{by A.70(i),(ii)} \\
&= \langle b^*x|a^*y\rangle = \langle ab^*x|y\rangle = \langle \mathrm{lmul}(ab^*)(x)|y\rangle && \text{by A.70(ii).}
\end{aligned}$$

Thus $|a\rangle\langle b| \bullet \mathrm{id} = \mathrm{lmul}(ab^*)$. This equals id (resp., 0) if and only if ab^* equals 1 (resp., 0).

(ii) Now, for any $x, y \in B$, we compute,

$$\begin{aligned}
\langle \mathrm{id} \bullet |a\rangle\langle b|(x)|y\rangle &= \langle m(\mathrm{id} \otimes |a\rangle\langle b|)m^*(x)|y\rangle \\
&= \sum_t \langle m(|f_t\rangle\langle f_t| \otimes |a\rangle\langle b|)m^*(x)|y\rangle \\
&= \sum_t \langle |f_ta\rangle\langle f_tb|(x)|y\rangle && \text{by A.112} \\
&= \sum_t \langle x|f_tb\rangle\langle f_ta|y\rangle \\
&= \sum_t \langle x\sigma_{-1}(b^*)|f_t\rangle\langle f_t|y\sigma_{-1}(a^*)\rangle && \text{by A.70(iii),(iv)} \\
&= \langle x\sigma_{-1}(b^*)|y\sigma_{-1}(a^*)\rangle \\
&= \langle x\sigma_{-1}(b^*)a|y\rangle = \langle \mathrm{rmul}(\sigma_{-1}(b^*)a)(x)|y\rangle && \text{by A.70(iii).}
\end{aligned}$$

Thus $\mathrm{id} \bullet |a\rangle\langle b| = \mathrm{rmul}(\sigma_{-1}(b^*)a)$. This equals id (resp., 0) if and only if $\sigma_{-1}(b^*)a$ equals 1 (resp., 0).

■

Remark A.117. Let $a, b \in B$. Using Lemma A.115, we can rewrite the above to say exactly, $|a\rangle\langle b| \bullet \mathrm{id} = \mathrm{lmul}(a)\mathrm{lmul}(b)^*$ and $\mathrm{id} \bullet |a\rangle\langle b| = \mathrm{rmul}(a)\mathrm{rmul}(b)^*$. So we can see a very nice correlation between the two formulas, one uses left multiplication, while the other uses right. ◊

Lemma A.118. *Let B_1, B_2 be finite-dimensional C*-algebras with the respective multiplication maps m_1, m_2 and faithful and positive linear functionals ψ_1, ψ_2. Then for linear maps $x, y \in \mathcal{B}((B_1, \psi_1), (B_2, \psi_2))$, we have*

(i) $(x \bullet y)^r = y^r \bullet x^r$,

(ii) $(x \bullet y)^* = x^* \bullet y^*$.

Proof.

(i) Let $\varkappa_1 = \varkappa_{B_1, B_1}$ and $\varkappa_2 = \varkappa_{B_2, B_2}$. Then we compute,

$$\begin{aligned}(x \bullet y)^{\mathrm{r}} &= {m_2}^{\mathrm{r}} (x \otimes y)^{\mathrm{r}} m_1^{*\mathrm{r}} \\ &= m_2 \varkappa_2 (x^{\mathrm{r}} \otimes y^{\mathrm{r}}) \varkappa_1 m_1^* && \text{by A.89} \\ &= m_2 (y^{\mathrm{r}} \otimes x^{\mathrm{r}}) \varkappa_1 \varkappa_1 m_1^* && \text{by A.3} \\ &= m_2 (y^{\mathrm{r}} \otimes x^{\mathrm{r}}) m_1^* = y^{\mathrm{r}} \bullet x^{\mathrm{r}}.\end{aligned}$$

(ii) This is again clear from the definition:

$$(x \bullet y)^* = (m_2 (x \otimes y) m_1^*)^* = m_1 (x^* \otimes y^*) m_2^* = x^* \bullet y^*.$$

■

Remark A.119. From Lemma A.118(i), we can deduce that the set of bounded linear operators on our finite-dimensional C*-algebra with a faithful and positive linear functional ψ can be upgraded from a ring (multiplication given by $\cdot \bullet \cdot$ and 1 given by $\eta\eta^*$, see Lemma A.108) to a *-ring, where the star operation is given by $\cdot^{\mathrm{r}}$. ♢

The following result generalises parts of [4, Propositions 5.25 & 5.26], and shows this directly without needing to look at its corresponding projections. In particular, given a real linear map $A \in \mathcal{B}(B)$, we show that A is (ir)reflexive, i.e., if $A \bullet \mathrm{id} = \mathrm{id}$ (respectively, if $A \bullet \mathrm{id} = 0$), if and only if it is (ir)reflexive′, i.e., if $\mathrm{id} \bullet A = \mathrm{id}$ (respectively, if $\mathrm{id} \bullet A = 0$). In [4], they show this by requiring A to be a *quantum adjacency matrix*, but this is not needed. This result is passively shown via string diagrams within the proof of [9, Proposition 2.26].

Proposition A.120. *Let $A_1, A_2, A_3 \in \mathcal{B}(B_1, B_2)$ all be real linear maps. Then* $A_1 \bullet A_2 = A_3 \Leftrightarrow A_2 \bullet A_1 = A_3$. *This means that when $A \in \mathcal{B}(B)$ is real, then $A \bullet \mathrm{id} = \mathrm{id}$ if and only if $\mathrm{id} \bullet A = \mathrm{id}$, and, analogously, $A \bullet \mathrm{id} = 0$ if and only if $\mathrm{id} \bullet A = 0$.*

Proof. As A_1, A_2, A_3 are all real, we get the following equivalences by using Lemma A.118(i),

$$A_1 \bullet A_2 = A_3 \Leftrightarrow {A_1}^{\mathrm{r}} \bullet {A_2}^{\mathrm{r}} = {A_3}^{\mathrm{r}} \Leftrightarrow A_2 \bullet A_1 = A_3.$$

■

Corollary A.121. *Given $A_1, A_2 \in \mathcal{B}(B)$ and an algebra homomorphism $f \colon B \to B_2$, we get $(f A_1 f^*) \bullet (f A_2 f^*) = f(A_1 \bullet A_2) f^*$.*

Proof. Immediate from Lemma A.111. ■

Corollary A.122. *Given $A_1, A_2 \in \mathcal{B}(B_1, B_2)$, we get*

$$\mathrm{symm}(A_1 \bullet A_2) = \mathrm{symm}(A_2) \bullet \mathrm{symm}(A_1).$$

Proof. Use Proposition A.92 and Lemma A.118. ■

A.XI Automorphisms on B

A.XI.1 Automorphisms on $\mathbf{M_n}$. Here we show that any algebra automorphism on M_n is inner (i.e., given by $x \mapsto yxy^{-1}$ for some invertible $y \in M_n$) – see Proposition A.124. We then show that any star-algebra automorphism on M_n is unitarily inner (i.e., given by $x \mapsto UxU^*$ for some unitary matrix $U \in \mathrm{U_n}$).

Notation.

- $\mathrm{GL_n}$: general linear group of degree n (i.e., the set of invertible $n \times n$ matrices)
- $\mathrm{U_n}$: unitary group of degree n (i.e., the set of $n \times n$ unitary matrices)
- $\mathrm{SU_n}$: special unitary group of degree n (i.e., the set of unitary $n \times n$ matrices with determinant one)

So $\mathrm{SU_n} \subset \mathrm{U_n} \subset \mathrm{GL_n}$.

Definition A.123. An automorphism f on X is said to be *inner* if it is given by conjugation, i.e., $f\colon x \mapsto yxy^{-1}$ for some invertible $y \in X$.

Recall the identification $\mathcal{M}_u\colon \mathcal{L}(\mathcal{H}) \cong M_n$ given by an orthonormal basis u of Hilbert space $\mathcal{H}$, where $\mathcal{M}_u(x)_{ij} = \langle u_i | x(u_j)\rangle$ for any $x \in \mathcal{L}(\mathcal{H})$ and $i, j \in [n]$. (See Definition A.28.)

Theorem A.124. *Any algebra automorphism $f\colon M_n \cong M_n$ is inner.*

Proof. Let $f\colon M_n \cong_{\mathrm{a}} M_n$ be an algebra automorphism. We want to show that there exists a matrix $y \in M_n$ such that $f(x) = yxy^{-1}$ for any $x \in M_n$. It suffices to show that there exists a linear isomorphism $y \in \mathcal{L}(\mathbb{C}^n)$ such that $f(x)\mathcal{M}_e(y) = \mathcal{M}_e(y)x$ for all $x \in M_n$, where e is the standard orthonormal basis (e_i) of $\mathbb{C}^n$.

Assume $0 < n$, otherwise this is trivial. So we know there exists non-zero vectors in $\mathbb{C}^n$. Let $0 \neq \chi, \zeta \in \mathbb{C}^n$. This implies $\chi\zeta^* \neq 0$ (since $\chi, \zeta \neq 0$ implies $\chi_i, \zeta_j \neq 0$ for some i, j, which then implies $(\chi\zeta^*)_{ij} = \chi_i\zeta_j^* \neq 0$).
Now let $0 \neq \xi \in \mathbb{C}^n$ such that $f(\chi\zeta^*)\xi \neq 0$ (this exists since $f(\chi\zeta^*) \neq 0$ because f is an isomorphism - and so is injective).
Now we define our linear map $T \in \mathcal{L}(\mathbb{C}^n)$ by $x \mapsto f(x\zeta^*)\xi$.

This is clearly a well-defined linear map: for any $u, v \in \mathbb{C}^n$ and $\alpha, \beta \in \mathbb{C}$, we have,

$$\begin{aligned} T(\alpha u + \beta v) &= f((\alpha u + \beta v)\zeta^*)\xi = f(\alpha u\zeta^* + \beta v\zeta^*)\xi \\ &= \alpha f(u\zeta^*)\xi + \beta f(v\zeta^*)\xi = \alpha T(u) + \beta T(v). \end{aligned}$$

Note that for any $x \in \mathbb{C}^n$, we get $\mathcal{M}_e(T)x = \mathcal{M}_e^{-1}(\mathcal{M}_e(T))(x) = T(x)$ using Corollary A.31.

It remains to show: (i) this linear map is an isomorphism and; (ii) for any $x \in M_n$ we have $f(x)\mathcal{M}_e(T) = \mathcal{M}_e(T)x$. We begin with the latter.

(ii) For any $a \in M_n$ and $x \in \mathbb{C}^n$, we compute,

$$\begin{aligned} (\mathcal{M}_e(T)a)x = T(ax) &= f(ax\zeta^*)\xi = f(a(x\zeta^*))\xi = f(a)f(x\zeta^*)\xi \\ &= f(a)T(x) = (f(a)\mathcal{M}_e(T))x \end{aligned}$$

Thus, for any $a \in M_n$, we have $\mathcal{M}_e(T)a = f(a)\mathcal{M}_e(T)$.

(i) To show that T is an isomorphism, it suffices to show that it is surjective. So we need to show that for any $x \in \mathbb{C}^n$ we have $x \in \operatorname{im} T$ (i.e., there exists some $v \in \mathbb{C}^n$ such that $x = T(v)$).

Let $x \in \mathbb{C}^n$.

We know $T(\chi) = f(\chi\zeta^*)\xi \neq 0$. This means that there exists some $k \in [n]$ such that $T(\chi)_k \neq 0$ (so $T(\chi)_k \in \mathbb{C}$ is invertible). Then we can let $d \in \mathbb{C}^n$ be the vector given by

$$d_i = \begin{cases} T(\chi)_i^{-1} & \text{if } i = k, \\ 0 & \text{otherwise} \end{cases}$$

(i.e., has the inverse of $T(\chi)$ at k and 0 elsewhere). Then we clearly have $\langle d|T(\chi)\rangle = 1$.

Because f is an isomorphism, we know its surjective, and so there exists a matrix $b \in M_n$ such that $f(b) = xd^*$.

And so we get $f(b)T(\chi) = xd^*T(\chi) = x\langle d|T(\chi)\rangle = x$.

It now remains to show that we have $x = T(b\chi)$, which can be seen from the following computation,

$$\begin{aligned} x &= f(b)T(\chi) = (f(b)\mathcal{M}_e(T))\chi \\ &= (\mathcal{M}_e(T)b)\chi && \text{by (ii) above} \\ &= \mathcal{M}_e(T)(b\chi) = T(b\chi). \end{aligned}$$

Thus $x \in \operatorname{im} T$. And so T is surjective (which then implies T is bijective - and so it is an isomorphism).

Thus, our algebraic automorphism f is inner. ■

Remark A.125. The above is a purely linear algebraic proof of a specialized version of the *Skolem-Noether theorem* [19] which shows that for algebra homomorphisms f and g from a simple algebra B to a finite-dimensional simple and central algebra A, we get that they differ by a unit, i.e., $g(x) = uf(x)u^{-1}$ for some invertible element $u \in A$. ♢

Lemma A.126. *Given a Hilbert space $\mathcal{H}$, we have $\mathcal{B}(\mathcal{H})' = \{\alpha \operatorname{id} : \alpha \in \mathbb{C}\}$.*
In other words, $x \in \mathcal{B}(\mathcal{H})$ commutes with all operators $y \in \mathcal{B}(\mathcal{H})$ if and only if $x = \alpha \operatorname{id}$ for some $\alpha \in \mathbb{C}$.

Proof. Let $x \in \mathcal{B}(\mathcal{H})$. Obviously, if $x = \alpha \operatorname{id}$ for some $\alpha \in \mathbb{C}$, then it commutes with every other operator. Now suppose x commutes with every operator in $\mathcal{B}(\mathcal{H})$. So this means $|a\rangle\langle x^*(b)| = |a\rangle\langle b|x = x|a\rangle\langle b| = |x(a)\rangle\langle b|$ for all $a, b \in \mathcal{H}$. Suppose there exists some non-zero $a \in \mathcal{H}$, otherwise this is trivial. Then, for any $b \in \mathcal{H}$, we have

$$x(b) = \frac{\|a\|^2}{\|a\|^2}x(b) = \frac{1}{\|a\|^2}|x(b)\rangle\langle a|(a) = \frac{1}{\|a\|^2}|b\rangle\langle x^*(a)|(a) = \frac{\langle x^*(a)|a\rangle}{\|a\|^2}b.$$

Thus $x = \alpha \operatorname{id}$ where $\alpha = \langle x^*(a)|a\rangle / \|a\|^2$. ■

From an analogue of the above lemma, we can say that the center of M_n is exactly $\operatorname{Span}(1)$.

Remark A.127. A more general statement of the above lemma is that the center of the set of continuous linear operators on a topological vector space E with separating functional is trivial. The proof follows the fact that it has a separating functional (i.e., for $0 \neq x \in E$, there exists a functional f such that $f(x) \neq 0$). Any normed space has a separating functional by the Hahn-Banach theorem. ♢

Proposition A.128. *Any *-automorphism on M_n is given by $y \mapsto xyx^*$ for some $x \in \mathrm{U_n}$.*

Proof. We know any algebra automorphism on M_n is inner by Theorem A.124. Let f be a *-automorphism on M_n. Then as we know any *-automorphism is an algebra automorphism, we get f is given by $y \mapsto xyx^{-1}$ for some $x \in \mathrm{GL_n}$. Then for any $y \in M_n$, we get the following equivalences,

$$f(y^*) = f(y)^* \Leftrightarrow xy^*x^{-1} = \left(xyx^{-1}\right)^* = (x^*)^{-1}y^*x^* \Leftrightarrow (x^*x)y^* = y^*(x^*x).$$

Then by an analogue of Lemma A.126, we have that there exists a scalar $\alpha \in \mathbb{C}$ such that $x^*x = \alpha 1$. We have x^*x is positive semi-definite (since for any vector $a \in \mathbb{C}^n$, we get $a^*x^*xa = (xa)^* xa \geq 0$). And as x is invertible we get x^* is invertible, and so x^*x is invertible. So then by Lemma A.47 we get x^*x is positive definite, and so $0 < \alpha$. This means we have $1 = \alpha^{-1}(x^*x) = (\alpha^{-1/2}x)^*(\alpha^{-1/2}x)$, which means $\alpha^{-1/2}x$ is unitary. Thus f is given by $y \mapsto xyx^{-1} = (\alpha^{-1/2}x)y(\alpha^{-1/2}x)^{-1}$ for $\alpha^{-1/2}x \in \mathrm{U_n}$. ■

Remark A.129. An analogous proof to the above can show an analogue to the Skolem-Noether for *-homomorphisms f and g (where f is surjective) from a simple *-algebra B to a simple finite-dimensional ordered and central *-algebra A, i.e., they differ by a unitary element u such that $g(x) = uf(x)u^*$. ◊

A.XI.2 Automorphisms on B. Now for $B = \bigoplus_i M_{n_i}$, (*-)algebra automorphisms f on B are not always (unitarily) inner. In particular, if any two of the block matrices in the multi-matrix have equal dimension, then it would not necessarily be inner. For example, an algebra automorphism on $M_n \oplus M_n$ given by $x \oplus y \mapsto y \oplus x$ is not inner. In this section, we see that any (*-)algebra automorphism on $B = \bigoplus_i M_{n_i}$ will either be (unitarily) inner or a product of an (unitarily) inner automorphism with a permutation on B.

Lemma A.130. *An (*-)algebra automorphism f on B is (unitarily) inner if and only if there exists (*-)algebra automorphisms f_i on M_{n_i} for each i such that $f = \bigoplus_i f_i$.*

Proof. If f is inner, then there is some invertible element $U \in B$ such that f is given by $x \mapsto UxU^{-1}$. So, for any $x \in B$, we get $f(x) = \bigoplus_i U_i x_i U_i^{-1}$. This clearly means it is decomposable into a direct sum of inner automorphisms.

On the other hand, when we have $f = \bigoplus_i f_i$ for each f_i being an automorphism on M_{n_i}, then we can use Theorem A.124 to get that these are inner, and so obviously f is then also inner. ■

Lemma A.131. *Given an algebra automorphism f on B, we have $f(\mathrm{Z}(B)) = \mathrm{Z}(B)$, where $\mathrm{Z}(B)$ is the center of B.*

Proof. Let $x \in B$. Then we have the following equivalences.

$$\begin{aligned} x \in \mathrm{Z}(B) &\Leftrightarrow \forall y \in B : xy = yx \\ &\Leftrightarrow \forall y \in B : f^{-1}(xy) = f^{-1}(yx) \\ &\Leftrightarrow \forall y \in B : f^{-1}(x)f^{-1}(y) = f^{-1}(y)f^{-1}(x) \\ &\Leftrightarrow f^{-1}(x) \in \mathrm{Z}(B) \\ &\Leftrightarrow x \in f(\mathrm{Z}(B)). \end{aligned}$$

■

Lemma A.132. $\mathrm{Z}(B) = \{x \in B : \forall i : x_i \in \mathrm{Span}(1)\}$.
In other words, if we let $E_i \in B$ be the inclusion of the identities in each summand, i.e., $E_i = \iota_i(1)$ so that we have $(E_i)_j = \delta_{i,j} 1 \in M_{n_j}$, then the center of B is exactly the span of the inclusions E_i's.

Proof. Follows from the center of each M_{n_i} being equal to Span(1) (which follows from an analogue of Lemma A.126). ■

Recall that for $x \in \mathbb{N}$, we write $[x]$ to mean $\{1, \ldots, x\}$.

Notation. We permute the matrix blocks of the same size so that they are grouped together. More specifically, let $B = \bigoplus_{i=1}^{k} B_i$, where each $B_i = \bigoplus_{j=1}^{t_i} M_{n_i}$, such that if $n_i = n_j$, then $i = j$. So, for $x \in B$, we write $x = \sum_{i,j} \iota_i \iota_j(x_{i,j})$, where each $x_{i,j} \in M_{n_i}$.

As we have $n_i = n_j$ only when $i = j$, we get that there exists a linear isomorphism $M_{n_i} \cong M_{n_j}$ if and only if $i = j$.

Similarly, there exists an algebra isomorphism $B_i \cong B_j$ if and only if $i = j$ (otherwise, if $i \neq j$, then there would exist an algebra isomorphism $M_{n_i} \cong M_{n_j}$, which would then imply $i = j$ by the previous argument). This then means that for an algebra automorphism $f \colon B \cong B$, there exists algebra automorphisms $f_i \colon B_i \cong B_i$ such that $f = \bigoplus_i f_i$.

Lemma A.133. *Given an algebra automorphism f in $B = \bigoplus_a B_a$, there exists an inner algebra automorphism $g \colon B \to B$ and an outer automorphism $h \colon B \to B$ such that $f = gh$, if and only if, for each i, there exists a permutation $h_i \colon [t_i] \to [t_i]$ such that $f(\iota_i \iota_j(1)) = \iota_i \iota_{h_i(j)}(1)$.*
Notation: given a permutation $\sigma \colon [t_i] \to [t_i]$, the induced automorphism $h \colon B_i \to B_i$ will be given by $h(\iota_j(x)) = \iota_{\sigma(j)}(x)$ for $x \in M_{n_i}$.

Proof. Using the above argument, it suffices to show that for an algebra automorphism $f \colon B_a \to B_a$ for any $a \in [k]$, we get that there exists an inner automorphism $g \colon B_a \to B_a$ and a permutation h in B_a such that $f = gh$ if and only if $f(\iota_i(1)) = \iota_{h(j)}(1)$ for some permutation h in $[t_a]$.

(⇒) Suppose there exists an inner automorphism g and a permutation h such that $f = gh$. Then by Lemma A.130, we get that there exists automorphisms g_j in M_{n_a} such that $g = \bigoplus_j g_j$. And so we compute,

$$\begin{aligned} f(\iota_j(1)) &= gh(\iota_j(1)) = g(\iota_{h(j)}(1)) \\ &= \iota_{h(j)}(g_{h(j)}(1)) = \iota_{h(j)}(1). \end{aligned}$$

(⇐) Suppose $f(\iota_j(1)) = \iota_{h(j)}(1)$ for some permutation h in $[t_a]$. Then for any $x \in B_a$, we compute,

$$\begin{aligned} f(x) &= \sum_j f(\iota_j(x_j)) = \sum_j f(\iota_j(1)\iota_j(x_j)) \\ &= \sum_j f(\iota_j(1)) f(\iota_j(x_j)) \\ &= \sum_j \iota_{h(j)}(1) f(\iota_j(x_j)) \\ &= \sum_j \iota_{h(j)} \left([f(\iota_j(x_j))]_{h(j)} \right) \\ &= \left(\sum_j \iota_{h(j)} \circ p_{h(j)} \circ f \circ \iota_j \circ p_j \right)(x). \end{aligned}$$

Let each $g_j = p_j \circ f \circ \iota_{h^{-1}(j)}$. And let $g = \sum_j \iota_j \circ g_j \circ p_j$, so that we have $g \circ \iota_j = \iota_j \circ g_j$. Then let $h = \sum_j \iota_{h(j)} \circ p_j$. Then we have,

$$\begin{aligned} f &= \sum_j \iota_{h(j)} \circ p_{h(j)} \circ f \circ \iota_j \circ p_j \\ &= \sum_j \iota_{h(j)} \circ g_{h(j)} \circ p_j \\ &= g \circ \sum_j \iota_{h(j)} \circ p_j \end{aligned}$$

$$= g \circ h.$$

■

Theorem A.134. *Given an algebra automorphism $f : B \cong B$ where $B = \bigoplus_{i=1}^{k} B_i$ and each $B_i = \bigoplus_{j=1}^{t_i} M_{n_i}$, there exists an inner automorphism g in B and permutation $h = \bigoplus_i h_i$ in B, where each h_i is a permutation in B_i, such that $f = gh$.*

Proof. It suffices to show that, given an algebra automorphism f in B_a, there exists an inner automorphism g in B_a and a permutation h in B_a such that $f = gh$.

For each $i \in [t_a]$, let $E_i \in B_a$ such that $(E_i)_j = \delta_{i,j} 1 \in M_{n_a}$. In other words, each $E_i = \iota_i(1)$. It is clear that we have $E_i E_j = \delta_{i,j} E_i$.

Using Lemma A.133, it suffices to show that there exists a permutation h on $[t_a]$ such that each $f(E_i) = E_{h(i)}$.

As each $E_i \in \mathrm{Z}(B_a)$ (Lemma A.132), we also get each $f(E_i) \in \mathrm{Z}(B_a)$ (using Lemma A.131). So then, for each $i \in [t_a]$, we let $\alpha_i \in \mathbb{C}^{t_a}$ such that $f(E_i) = \sum_j (\alpha_i)_j E_j$.

We also clearly have $\sum_i E_i = 1$, and so we compute,

$$1 = f(1) = f\left(\sum_i E_i\right) = \sum_i f(E_i) = \sum_{i,j} (\alpha_i)_j E_j,$$

which is equivalent to saying $\sum_i (\alpha_i)_j = 1$ for all $j \in [t_a]$.

Now for any $p, q \in [\mathfrak{K}]$ such that $p \neq q$, we get

$$0 = f(E_p E_q) = f(E_p) f(E_q) = \sum_{i,j} \alpha_{p,i} \alpha_{q,j} E_i E_j = \sum_i \alpha_{p,i} \alpha_{q,i} E_i,$$

which is equivalent to saying $\alpha_{p,j} \alpha_{q,j} = 0$ for all $j \in [t_a]$.

As each $E_i \neq 0$, we have that for any $i \in [t_a]$, there exists $j \in [t_a]$ such that $\alpha_{i,j} \neq 0$.

If we take A to be the matrix given by $A_{ij} = \alpha_{i,j}$, then it suffices to show if $\alpha_{i,j} \neq 0$, then $\alpha_{i,j} = 1$ and it would be the only non-zero value in its row and column. This is because this would mean that $f(E_i) = E_{h(i)}$ for some permutation h.

Suppose $\alpha_{i,j} \neq 0$ for some i, j (we know that this exists as each $E_i \neq 0$). Then the condition $\alpha_{p,j} \alpha_{i,j} = 0$ for all $p \neq i$, implies that we have $\alpha_{i,j}$ being the only non-zero value in its column. And the condition $\sum_p \alpha_{p,j} = 1$ implies $\alpha_{i,j} = 1$.

All we now need to show is that it is the only non-zero value in its row. As for each row $p \neq i$ we have at least one non-zero value, then we know it cannot be in column j (as $\alpha_{p,j} = 0$). So this means $\alpha_{p,q} \neq 0$ for $q \neq j$, and so $\alpha_{k,q} = 0$ for all $k \neq p$, which means $\alpha_{i,q} = 0$ for all $q \neq j$, i.e., $\alpha_{i,j}$ is the only non-zero value. ■

Lemma A.135. *Given an inner automorphism f on B, we get*

$$f^* = \mathrm{rmul}(Q^{-1}) \circ \left(f^{-1}\right)^r \circ \mathrm{rmul}(Q).$$

In other words, we get $f^*(x) = f^{-1}(Qx^*)^* Q^{-1}$ for any $x \in B$.
Moreover, when f is a *-automorphism on B, we get $f^*(x) = f^{-1}(xQ)Q^{-1}$ for all $x \in B$.

Proof. Let $U \in B$ be the invertible element such that f is given by $x \mapsto UxU^{-1}$. Then, for any $x, y \in B$, we compute,

$$\langle f^*(x) | y \rangle = \langle x | f(y) \rangle = \mathrm{Tr}(Qx^* U y U^{-1}) = \mathrm{Tr}(QQ^{-1}U^{-1}Qx^*Uy)$$

$$= \operatorname{Tr}(QQ^{-1}f^{-1}(Qx^*)y) = \Big\langle \big(f^{-1}\big)^{\mathrm{r}}(xQ)Q^{-1}\Big|y\Big\rangle.$$

Thus f^* is given by $x \mapsto (f^{-1})^{\mathrm{r}}(xQ)Q^{-1}$.

If f is a *-automorphism, then $(f^{-1})^{\mathrm{r}} = f^{-1}$, and so the result then follows. ■

Corollary A.136. *More generally, given an algebra automorphism $f = gh$ on B, where g is an inner algebra automorphism on B and h is a permutation on the direct summands of equal sizes, we get*

$$f^* = h^{-1} \circ \operatorname{rmul}(Q^{-1}) \circ (g^{-1})^r \circ \operatorname{rmul}(Q).$$

In other words, we get $f^*(x) = h^{-1}(g^{-1}(Qx^*)^*Q^{-1})$ for any $x \in B$.
Moreover, when f is a *-automorphism on B, then we get $f^*(x) = f^{-1}(xQ)h^{-1}(Q^{-1})$ for all $x \in B$.

Proof. Let σ be the permutation on $[\mathfrak{K}]$ such that, for each i, $h(\iota_i(x_i)) = \iota_{\sigma(i)}(x_i)$ and $h^{-1}(\iota_i(x_i)) = \iota_{\sigma^{-1}(i)}(x_i)$. Equally, we have $h^{-1}(\iota_{\sigma(i)}(x_{\sigma(i)})) = \iota_i(x_{\sigma(i)})$ for each i. So then, for any $x, y \in B$, we compute,

$$\begin{aligned}\langle h^*(x)|y\rangle = \langle x|h(y)\rangle &= \sum_i \big\langle x\big|\iota_{\sigma(i)}(y_i)\big\rangle = \sum_i \big\langle x_{\sigma(i)}\big|y_i\big\rangle \\ &= \sum_i \big\langle \iota_i(x_{\sigma(i)})\big|y\big\rangle = \sum_i \big\langle h^{-1}(\iota_{\sigma(i)}(x_{\sigma(i)}))\big|y\big\rangle = \big\langle h^{-1}(x)\big|y\big\rangle.\end{aligned}$$

Thus $h^* = h^{-1}$. The result then follows from Lemma A.135. ■

The following result tells us when a *-algebra automorphism f in B is an *isometry* (i.e., $\|f(x)\| = \|x\|$ for all $x \in B$).

Lemma A.137. *Given a *-algebra automorphism f in B, the following are equivalent,*

(i) $f(Q) = Q$,

(ii) $f^* = f^{-1}$,

(iii) $\psi \circ f = \psi$,

(iv) $\forall x, y \in B : \langle f(x)|f(y)\rangle = \langle x|y\rangle$,

(v) $\forall x \in B : \|f(x)\| = \|x\|$.

Moreover, by letting $f = gh$ for some inner g and permutation h on the direct summands (Theorem A.134), then we also have equivalency with,

(vi) $g(Q) = Q$.

Proof.

(i) ⇔ *(iii)* First note that we get f is trace-preserving, since $f = gh$ for some inner g and permutation h on the direct summands (Theorem A.134), and clearly h is trace-preserving, and g is also clearly trace-preserving as $g\colon x \mapsto UxU^*$ for some unitary $U \in B$. Note that the trace here, is the usual trace on $B = \bigoplus_i M_{n_i}$, i.e., $\operatorname{Tr}(x) = \sum_{i,t} x_{i,tt}$ (sum of the diagonals in each matrix block) for all x.

(⇒) Suppose $f(Q) = Q$. Then we also have $f^{-1}(Q) = Q$. And so

$$\psi(f(x)) = \operatorname{Tr}(Qf(x)) = \operatorname{Tr}(f^{-1}(Q)x) = \operatorname{Tr}(Qx) = \psi(x),$$

for any $x \in B$.

(⇐) Suppose $\psi \circ f = \psi$. Then for any $x \in B$, we have

$$\psi(x) = \psi(f(x)) = \mathrm{Tr}(Qf(x)) = \mathrm{Tr}(f^{-1}(Qf(x))) = \mathrm{Tr}(f^{-1}(Q)x).$$

And so $f^{-1}(Q) = Q$ by uniqueness of Q (see Lemma A.38).

(iii) ⇔ *(iv)* For any $x, y \in B$, we have $\langle f(x)|f(y)\rangle = \psi(f(x)^* f(y)) = \psi \circ f(x^* y)$.

(⇒) Suppose $\psi \circ f = \psi$. Then $\langle f(x)|f(y)\rangle = \psi \circ f(x^* y) = \psi(x^* y) = \langle x|y\rangle$ for any $x, y \in B$.

(⇐) Suppose $\psi \circ f(x^* y) = \langle f(x)|f(y)\rangle = \langle x|y\rangle = \psi(x^* y)$ for any $x, y \in B$. Then for any $x \in B$, we have $\psi \circ f(x) = \psi \circ f(1^* x) = \psi(1^* x) = \psi(x)$.

(iv) ⇔ *(ii)*

$$\begin{aligned}\forall x, y \in B : \langle f(x)|f(y)\rangle = \langle x|y\rangle &\Leftrightarrow \forall x, y \in B : \langle f^* f(x)|y\rangle = \langle x|y\rangle \\ &\Leftrightarrow \forall x \in B : f^* f(x) = x \\ &\Leftrightarrow f^* f = \mathrm{id} \Leftrightarrow f^* = f^{-1}.\end{aligned}$$

(v) ⇔ *(ii)* We have the following equivalences,

$$\begin{aligned}\forall x \in B : \|f(x)\| = \|x\| &\Leftrightarrow \forall x \in B : \langle (f^* f - \mathrm{id})(x)|x\rangle = 0 \\ &\Leftrightarrow f^* f = \mathrm{id} \\ &\Leftrightarrow f^* = f^{-1}.\end{aligned}$$

(ii) ⇔ *(vi)*

$$\begin{aligned}f^* = f^{-1} &\Leftrightarrow \forall x \in B : f^{-1}(xQ)h^{-1}(Q^{-1}) = f^{-1}(x) && \text{by A.136} \\ &\Leftrightarrow \forall x \in B : f^{-1}(xQ) = f^{-1}(x)h^{-1}(Q) \\ &\Leftrightarrow \forall x \in B : xQ = xg(Q) \\ &\Leftrightarrow Q = g(Q)\end{aligned}$$

■

Remark A.138*.* Saying $f\colon B \to B$ is both a *-algebra and a co-algebra automorphism is equivalent to saying f is an isometric *-algebra automorphism. ◊

Corollary A.139. *For a *-algebra automorphism $f = gh$ in B, for some inner automorphism g in B and permutation h on the direct summands, then when f is an isometry, we get $h(Q) = Q$.*

Proof. Using Lemma A.137, we get $f(Q) = Q$ and $g(Q) = Q$. So then it is clear that we also get $h(Q) = Q$. ■

Corollary A.140. *Given a *-algebra automorphism $f = gh$ in B, where $g\colon x \mapsto UxU^*$ for some unitary $U \in B$, then f is an isometry if and only if $UQ = QU$.*

Proof. By Lemma A.137, we get f is an isometry if and only if $g(Q) = Q$, which is true if and only if $UQ = UQ$. ■

Corollary A.141. *Given an isometric *-algebra automorphism $f = gh$ in B, such that g is given by $x \mapsto UxU^*$ for some unitary $U \in B$, we have,*

(i) $m \circ (f \otimes f) = f \circ m$,

(ii) $(f \otimes f) \circ m^* = m^* \circ f$,

(iii) $f \circ \eta = \eta$,

(iv) $\eta^* \circ f = \eta^*$,

(v) $UQ = QU$.

Proof. As f is an isometry, Lemma A.137 tells us that we get $\psi \circ f = \psi$, which is exactly part (iv). Part (v) is given by Lemma A.140, and the rest is given by Lemma A.14. ∎

Proposition A.142. *Given C^*-algebras X_1, X_2, X_3, X_4, then for a linear map $A\colon X_2 \to X_3$, a surjective *-homomorphism $f\colon X_1 \to X_2$, and an injective *-homomorphism $g\colon X_3 \to X_4$, we get A is real if and only if gAf is real.*

Proof. As f and g are both *-homomorphisms, we get $f^{\mathrm{r}} = f$ and $g^{\mathrm{r}} = g$. So then we have the following equivalences,

$$\begin{aligned} gAf \text{ is real} &\Leftrightarrow (gAf)^{\mathrm{r}} = gAf && \text{by Lemma A.83} \\ &\Leftrightarrow g^{\mathrm{r}} A^{\mathrm{r}} f^{\mathrm{r}} = gAf \\ &\Leftrightarrow gA^{\mathrm{r}} f = gAf \\ &\Leftrightarrow A^{\mathrm{r}} = A \Leftrightarrow A \text{ is real} && \text{by Lemma A.83.} \end{aligned}$$

Note that, in the fourth equivalence, we used the fact that g is injective and f is surjective. ∎

Using the above, we get that for a *-isomorphism $f\colon X_1 \cong X_2$ and a linear operator $A \in \mathcal{B}(X_2)$, we get A is real if and only if $f^{-1}Af$ is.

Remark A.143. Given a unitary $x \in B$, if f_x is the automorphism on B given by $y \mapsto xyx^*$, then for $U, V \in B$, we have $f_U = f_V$ if and only if $xU^*V = U^*Vx$ for all $x \in B$, and this is true if and only if $U^*V = \alpha 1$ for some $\alpha \in \mathbb{C}$ (see Lemma A.126), which means $V = \beta U$ for some $\beta \in \mathbb{C}$. ♢

Lemma A.144. *Given a unitary $U \in B$, let f_U be the inner *-automorphism on B, given by $x \mapsto UxU^*$. Then we have $\mathcal{M}_e(f_U) = \bigoplus_i U_i \otimes \overline{\sigma_{-1/2}(U_i)}$, where e is the orthonormal basis $(\iota_s(e_{ij}Q_s^{-1/2}))_{s,ij}$ of B from Proposition A.60.*

Proof. Using Lemma A.115(viii),(ix) we get

$$\begin{aligned} \mathcal{M}_e(f_U) &= \mathcal{M}_e(\mathrm{lmul}(U)\,\mathrm{rmul}(U^*)) = \mathcal{M}_e(\mathrm{lmul}(U))\mathcal{M}_e(\mathrm{rmul}(U^*)) \\ &= \left(\bigoplus_i U_i \otimes 1_i\right)\left(\bigoplus_i 1_i \otimes \sigma_{1/2}(U_i^*)^T\right) \\ &= \bigoplus_i U_i \otimes \sigma_{1/2}(U_i^*)^T \\ &= \bigoplus_i U_i \otimes \overline{\sigma_{-1/2}(U_i)}. \end{aligned}$$

∎

Using the above lemma, we get the matrix of its inverse is $\mathcal{M}_e(f_{U^*}) = \bigoplus_i U_i^* \otimes \overline{\sigma_{-1/2}(U_i^*)}$, and the matrix of its adjoint is $\mathcal{M}_e(f_U^*) = \mathcal{M}_e(f_U)^* = \bigoplus_i U_i^* \otimes \overline{\sigma_{1/2}(U_i^*)}$.

Remark A.145. When our linear functional ψ is tracial (so $Q = \alpha 1$ for some $0 < \alpha$), then $\mathcal{M}_e(f_U) = \bigoplus_i U_i \otimes \overline{U_i}$ for some unitary $U \in B$, where e is the same orthonormal basis of B as above, i.e., $(\iota_s(e_{ij}Q_s^{-1/2}))_{s,ij}$ from Proposition A.60. ♢

Corollary A.146. *Let f be an algebra isomorphism between algebras A_1 and A_2, and let $x \in A_1$. Then*
$$\text{Spectrum}(f(x)) = \text{Spectrum}(x).$$

Proof. Let $\lambda \in \mathbb{C}$. Then we have the following equivalences,

$$\begin{aligned}\lambda \in \text{Spectrum}(f(x)) &\Leftrightarrow \lambda 1 - f(x) \text{ is not invertible}\\ &\Leftrightarrow \lambda f(1) - f(x) \text{ is not invertible}\\ &\Leftrightarrow f(\lambda 1 - x) \text{ is not invertible}\\ &\Leftrightarrow \lambda 1 - x \text{ is not invertible}\\ &\Leftrightarrow \lambda \in \text{Spectrum}(x).\end{aligned}$$

■

A.XII Projections

In this section, we go over some well-known results on idempotents and projections.

We assume that scalar multiplication by the natural numbers is always injective here.

A.XII.1 Idempotents.

Lemma A.147. *Given idempotent elements p and q in a ring R, we have $p+q$ is idempotent if and only if they anti-commute (i.e., $pq + qp = 0$).*

Proof. $(p+q)(p+q) = p+q$ if and only if $p+pq+qp+q = p+q$ if and only if $pq+qp = 0$. ■

Lemma A.148. *If idempotent a and element b in a ring R anti-commute, then $ab = 0$, which implies they commute. So anti-commutativity implies commutativity when one of them is idempotent.*

Proof. Suppose $ab + ba = 0$. Then $0 = a(ab + ba)a = aba + aba = 2aba$ which means $aba = 0$. And so $0 = a(ab + ba) = ab + aba = ab$. And since they anti-commute, ba is also equal to 0. Thus $ab = ba$. ■

Lemma A.149. *For an element a in a ring R, we get $1 - a$ is idempotent if and only if a is.*

Proof. If a is idempotent, then $(1-a)(1-a) = 1-2a+a^2 = 1-2a+a = 1-a$, which means $1 - a$ is idempotent.

If, on the other hand, $1 - a$ is idempotent, then we already know $1 - (1 - a)$ is idempotent by the above. And, of course, $1 - (1 - a) = a$, so we are done. ■

Lemma A.150. *Given idempotent elements p and q in a ring R where scalar multiplication by the natural numbers is injective, then*
$$pq = p = qp \Leftrightarrow q - p \text{ is idempotent.}$$

Proof.

(⇒) Suppose $pq = p = qp$. Then we have

$$\begin{aligned}(q-p)^2 &= q^2 - pq - qp + p^2 \\ &= q - pq - qp + q && \text{since } p, q \text{ are idempotent}\\ &= q - p - p + p && \text{by hypothesis, } pq = p = qp\\ &= q - p.\end{aligned}$$

Thus $p - q$ is idempotent.

($\Leftarrow$) Suppose $q-p$ is idempotent. We want to show $pq = p = qp$. Now $q-p$ is idempotent means that we have $q - p = (q-p)^2 = q^2 - qp - pq + p^2 = q - qp - pq + p$ and so

$$p + p = qp + pq. \tag{*}$$

We also have

$$\begin{aligned} & q(p+p) = q(qp+pq) && \text{and} && (p+p)q = (qp+pq)q \\ \Leftrightarrow\quad & qp + qp = qp + qpq && \text{and} && pq + pq = qpq + pq \\ \Leftrightarrow\quad & qp = qpq && \text{and} && pq = qpq. \end{aligned}$$

Hence $qp = pq$. And so it suffices to show $pq = p$. From Equation (*) we get $2p = 2pq$. And so the result then follows.

■

Let E be a vector space for the remainder of this section. Let U, V be subspaces of E such that $E = U \oplus V$. We say $p \in \mathcal{L}(E)$ is a *projection onto U along its complement V* if for all $x = u + v \in U \oplus V$, we have $p(x) = u$. This means $U = \operatorname{im} p$ and $V = \ker p$. We usually write $p_{U,V}$ to mean the projection onto U along its complement V. So if we have two subspaces U, V of E and write $p_{U,V}$, then it is assumed that we have $E = U \oplus V$. It is clear that we get $p_{U,V} + p_{V,U} = \mathrm{id}$.

Lemma A.151. *Given an idempotent operator $p \in \mathcal{L}(E)$, i.e., $p^2 = p$ and $x \in E$, we have $x \in \operatorname{im} p \Leftrightarrow p(x) = x$.*

Proof.

($\Rightarrow$) Suppose $x \in \operatorname{im} p$. Then we have that there exists some $y \in E$ such that $p(y) = x$. And so since $p^2 = p$, we clearly get $px = p(p(y)) = p(y) = x$.

($\Leftarrow$) Suppose $p(x) = x$. Then obviously $x = p(x) \in \operatorname{im} p$.

■

Lemma A.152. *Given $T \in \mathcal{L}(E)$, we have,*
$T^2 = T \;\Leftrightarrow\; p_{\operatorname{im} T, \ker T} = T.$
In other words, T is idempotent if and only if T is the projection onto its image along its kernel.

Proof.

($\Rightarrow$) Suppose $T^2 = T$.

Claim: $\operatorname{im} T \cap \ker T = \{0\}$.

We already know $\{0\} \subseteq \operatorname{im} T \cap \ker T$. So suppose $x \in \operatorname{im} T$ and $x \in \ker T$. Then there exists some $y \in E$ such that $T(y) = x$ and $T(x) = 0$. So then $x = T(y) = T(T(y)) = T(x) = 0$. And so $\operatorname{im} T \cap \ker T = \{0\}$.

Claim: $\operatorname{im} T + \ker T = E$.

We need to show that any element in E can be written as an element of $\operatorname{im} T + \ker T$. Let $x \in E$. Then $x - T(x) \in \ker T$ since

$$T(x - T(x)) = T(x) - T(T(x)) = T(x) - T(x) = 0.$$

And so we get $x = T(x) + (x - T(x)) \in \operatorname{im} T + \ker T$, as $T(x) \in \operatorname{im} T$.

Thus $\operatorname{im} T \oplus \ker T = E$ by definition. So we can talk about the projection $p_{\operatorname{im} T, \ker T}$. We now only need to show $p_{\operatorname{im} T, \ker T} = T$. Let $x \in E$, then there exists some $y \in \operatorname{im} T$ and $z \in \ker T$ such that $x = y + z$. Then $p_{\operatorname{im} T, \ker T}(x) = p_{\operatorname{im} T, \ker T}(y + z) = y$ and $T(x) = T(y + z) = T(y) + T(z) = y$ by Lemma A.151 and $T(z) = 0$ since $z \in \ker T$. So we are done.

(⇐) Let $p_{\operatorname{im} T, \ker T} = T$ (and so by definition we have $E = \operatorname{im} T \oplus \ker T$). Then for any $x \in E$, there exists some $y \in \operatorname{im} T$ and $z \in \ker T$ such that $x = y + z$. And so $p^2_{\operatorname{im} T, \ker T}(x) = p_{\operatorname{im} T, \ker T}(y) = y = p_{\operatorname{im} T, \ker T}(x)$.

■

Lemma A.153. *If $p, q \in \mathcal{L}(E)$ such that q is idempotent, then* $qp = p \Leftrightarrow \operatorname{im} p \subseteq \operatorname{im} q$.

Proof. Suppose $p, q \in \mathcal{L}(E)$ are idempotent. Then

$$\forall x \in E : q(p(x)) = p(x) \Leftrightarrow p(x) \in \operatorname{im} q \qquad \text{by Lemma A.151.}$$

And so the result then follows. ■

Lemma A.154. *Given an idempotent operator $T \in \mathcal{L}(E)$, we have* $\ker(T) = \operatorname{im}(\operatorname{id} - T)$.

Proof. Using Lemma A.152, we have $\operatorname{im}(\operatorname{id} - T) = \operatorname{im}(\operatorname{id} - p_{\operatorname{im} T, \ker T}) = \operatorname{im} p_{\ker T, \operatorname{im} T} = \ker T$. ■

Lemma A.155. *Given $p, q \in \mathcal{L}(E)$ such that q is idempotent, then $pq = p$ if and only if* $\ker q \subseteq \ker p$.

Proof. Using Lemma A.154, it suffices to show $\operatorname{im}(\operatorname{id} - q) \subseteq \ker p$. It is clear that this is true if and only if $p(\operatorname{id} - q) = 0$. And so we are done. ■

Lemma A.156. *Let $p, q \in \mathcal{L}(E)$ such that p is idempotent. Then $pqp = qp$ if and only if $\operatorname{im} p$ is invariant under q.*

Proof. We have the following equivalences,

$$\begin{aligned} pqp = qp &\Leftrightarrow \operatorname{im}(qp) \subseteq \operatorname{im} p && \text{by A.153} \\ &\Leftrightarrow q(\operatorname{im} p) \subseteq \operatorname{im} p \\ &\Leftrightarrow \operatorname{im} p \text{ is invariant under } q. \end{aligned}$$

■

Lemma A.157. *Let $p, q \in \mathcal{L}(E)$ such that p is idempotent. Then $pqp = pq$ if and only if $\ker p$ is invariant under q.*

Proof. Analogously to Lemma A.156, $pqp = pq$ if and only if $\ker p \subseteq \ker pq$ using Lemma A.155, which is true if and only if $q(\ker p) \subseteq \ker p$, in other words, $\ker p$ is invariant under q. ■

Corollary A.158. *Let $p, q \in \mathcal{L}(E)$ such that p is idempotent. Then $pq = qp$ if and only if both $\operatorname{im} p$ and $\ker p$ are invariant under q.*

Proof. This should be clear using both Lemmas A.156 and A.157. ■

Proposition A.159. *Two idempotent operators S and T on a vector space E are equal if and only if their images and kernels are equal, i.e., $\operatorname{im} S = \operatorname{im} T$ and $\ker S = \ker T$.*

Proof. Suppose $\operatorname{im} S = \operatorname{im} T$ and $\ker S = \ker T$. And let $x \in E$. We use Lemma A.152 to get S is the projection onto its image along its kernel, i.e., $S = p_{\operatorname{im} S, \ker S}$, which means $E = \operatorname{im} S \oplus \ker S$.

So then let $y \in \operatorname{im} S$ and $z \in \ker S$ such that $x = y+z$. Then $S(x) = S(y+z) = S(y)+S(z) = y$ using Lemma A.151 and the fact $z \in \ker S$. And as $\operatorname{im} S = \operatorname{im} T$ and $\ker S = \ker T$, we also get $T(x) = T(y+z) = T(y) + T(z) = y$. Thus $S(x) = T(x)$ for all x. ■

Now the following result is made easy by the above lemmas. Recall that the commutant of $M \subseteq \mathcal{B}(\mathcal{H})$ is defined by $M' = \{y \in \mathcal{B}(\mathcal{H}) : xy = yx, \forall x \in M\}$ and the bicommutant is defined by $M'' = (M')'$. Then we say a *von Neumann algebra* is a unital *-subalgebra $M \subseteq \mathcal{B}(\mathcal{H})$ such that $M = M''$.

Proposition A.160 ([4, Lemma 5.10])**.** *Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $e \in \mathcal{B}(\mathcal{H})$ be idempotent. Then $e \in M$ if and only if both $\operatorname{im} e$ and $\ker e$ are M'-invariant subspaces.*

Proof. This follows from Lemma A.158. ■

A.XII.2 Projections.

Definition A.161. A *projection* is a self-adjoint idempotent.

Lemma A.162. *Give projections e and f, we have $e+f$ is a projection if and only if $ef = 0$.*

Proof. Using Lemma A.147, we know $e + f$ is idempotent if and only if they anti-commute. And, of course, $e + f$ is self-adjoint since e and f are. So it suffices to show that elements e and f anti-commute if and only if $ef = 0$. If $ef = 0$, then $fe = 0$ using adjoints, and so they anti-commute. If, on the other hand, e and f anti-commute, then using Lemma A.148, we know $ef = 0$. ■

Lemma A.163. *Given projections e and f, we have $f-e$ is a projection if and only if $ef = e$ (or, equivalently, if and only if $fe = e$).*

Proof. Left as an exercise to the reader. Hint: use Lemma A.150. ■

Lemma A.164. *Given a continuous linear map $T \in \mathcal{B}(\mathcal{H})$, we get $(\operatorname{im} T)^\perp = \ker T^*$.*

Proof. Let $x \in \mathcal{H}$. Then,

$$\begin{aligned} x \in \ker T^* &\Leftrightarrow T^*(x) = 0 \\ &\Leftrightarrow \forall y, \langle y | T^*(x) \rangle = 0 \\ &\Leftrightarrow \forall y, \langle T(y) | x \rangle = 0 \\ &\Leftrightarrow \forall a \in \operatorname{im} T, \langle a | x \rangle = 0 \\ &\Leftrightarrow x \in (\operatorname{im} T)^\perp. \end{aligned}$$

■

Lemma A.165. *Projection operators e and f on a Hilbert space are equal if and only if their image is, i.e., $e = f$ if and only if $\operatorname{im} e = \operatorname{im} f$.*

Proof. Using Lemma A.159, we know $e = f$ if and only if their image and kernel are equal. And using Lemma A.164, we know $\ker e = \ker e^* = (\operatorname{im} e)^\perp$ and similarly $\ker f = (\operatorname{im} f)^\perp$. So then this is equivalent to only their images being equal. Thus we are done. ■

Proposition A.166. *Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $e \in \mathcal{B}(\mathcal{H})$ be a projection. Then $e \in M$ if and only if $\operatorname{im} e$ is a M'-invariant subspace.*

Proof. This is left as an exercise to the reader. Hint: use Proposition A.160 and the fact that a von Nuemann algebra is a *-subalgebra. ∎

Lemma A.167. *Let $T \in \mathcal{B}(E)$. Then $TT^* = T^*T$ if and only if $\|T^*(x)\| = \|T(x)\|$ for all $x \in E$.*

Proof. We quickly compute,

$$\begin{aligned} TT^* = T^*T &\Leftrightarrow \forall x \in E, \langle x|TT^*(x)\rangle = \langle x|T^*T(x)\rangle \\ &\Leftrightarrow \forall x, \langle T^*(x)|T^*(x)\rangle = \langle T(x)|T(x)\rangle \\ &\Leftrightarrow \forall x, \|T^*(x)\| = \|T(x)\|. \end{aligned}$$

∎

Corollary A.168. *Let $T \in \mathcal{B}(E)$ be normal (i.e., $TT^* = T^*T$). Then $(\operatorname{im} T)^\perp = \ker T$.*

Proof. It suffices to show $\ker T^* = \ker T$ using Lemma A.164. Let $x \in E$. By Lemma A.167, we know $\|T^*(x)\| = \|T(x)\|$. So then we compute,

$$\begin{aligned} x \in \ker T^* &\Leftrightarrow T^*(x) = 0 \\ &\Leftrightarrow \|T^*(x)\| = 0 \Leftrightarrow \|T(x)\| = 0 \\ &\Leftrightarrow T(x) = 0 \Leftrightarrow x \in \ker T. \end{aligned}$$

∎

Proposition A.169. *Let $e \in \mathcal{B}(E)$ be idempotent. Then the following are equivalent,*

(i) $(\operatorname{im} e)^\perp = \ker e$,

(ii) *e is normal,*

(iii) *e is self-adjoint,*

(iv) $0 \leq e$.

Proof. We already know $(iv) \Rightarrow (iii) \Rightarrow (ii)$. And $(ii) \Rightarrow (i)$ from Corollary A.168.

$(i) \Rightarrow (iii)$ Suppose $(\operatorname{im} e)^\perp = \ker e$. Let $x, y \in E$. We want to show $\langle x|e(y)\rangle = \langle e(x)|y\rangle$. Let $x = a + b$ and $y = c + d$ such that $a, c \in \operatorname{im} e$ and $b, d \in \ker e$ (since e is idempotent, see Proposition A.152). Using Lemma A.151, we have $e(a) = a$ and $e(c) = c$. Then $\langle x|e(y)\rangle = \langle a + b|e(c + d)\rangle = \langle a|c\rangle + \langle b|c\rangle = \langle a|c\rangle$, where the last equality follows from our hypothesis (i.e., $(\operatorname{im} e)^\perp = \ker e$). Similarly, $\langle e(x)|y\rangle = \langle a|c\rangle + \langle a|d\rangle = \langle a|c\rangle$. Thus $\langle e(x)|y\rangle = \langle x|e(y)\rangle$, as desired.

$(ii) \Rightarrow (iii)$ Suppose e is normal. Then it suffices to show that $e = e^*e$ (since e^*e is self-adjoint). Then note that we also get $\mathrm{id} - e$ is normal. So then we have the following equivalences,

$$\begin{aligned} e = e^*e &\Leftrightarrow \forall x, \|(e - e^*e)(x)\| = 0 \\ &\Leftrightarrow \forall x, \left\|(\mathrm{id} - e)^* e(x)\right\| = 0 \\ &\Leftrightarrow \forall x, \|(\mathrm{id} - e)e(x)\| = 0 && \text{by A.167} \\ &\Leftrightarrow (\mathrm{id} - e)e = 0 \Leftrightarrow e^2 = e. \end{aligned}$$

And this is true as e is idempotent. Thus $e = e^*e$, and so e is self-adjoint.

(iii) $\Rightarrow$ *(iv)* Suppose e is self-adjoint, so then $e^2 = e^* = e$. Then, for any $x \in E$, we get $0 \leq \langle e(x)|e(x)\rangle = \langle x|e^*e(x)\rangle = \langle x|e(x)\rangle$. Thus e is positive semi-definite.

■

Now let E be a finite-dimensional $\mathbb{C}$-inner product space. We know for any subspace $U \subseteq E$, we have the direct sum decomposition $E = U \oplus U^\perp$. We denote our *orthogonal projection* of E onto $U \subseteq E$ as the operator $P_U \in \mathcal{L}(E)$ given by $P_U(v) = u$ where $\operatorname{im} P_U = U$ is orthogonal to $\ker P_U = U^\perp$. So the orthogonal projection onto U is simply the projection onto U along its complement $U^\perp$.

Lemma A.170. *Given a subspace U of E, then if (u_i) is an orthonormal basis of U, then $P_U = \sum_i |u_i\rangle\langle u_i|$.*

Proof. Let (u_i) be an orthonormal basis of U and let $p = \sum_i |u_i\rangle\langle u_i|$. Then for any $y \in E$, we have $y = p(y) + (\mathrm{id} - p)(y)$. We compute,

$$p^2 = \sum_{i,j} |u_i\rangle\langle u_i||u_j\rangle\langle u_j| = \sum_{i,j} \langle u_i|u_j\rangle |u_i\rangle\langle u_j| = \sum_i |u_i\rangle\langle u_i| = p.$$

And so p is a projection. Let $x \in E$, then there exists some $y \in U$ and $z \in U^\perp$ such that $x = y + z \in U \oplus U^\perp$. Since (u_i) is an orthonormal basis of U, we get $p(y) = y$. So then we have $p(x) = y = P_U(x)$ for any $x \in E$. ■

A.XII.3 Projections on $B \otimes B^{\mathrm{op}}$.

Definition A.171. Let B^{op} be the *opposite algebra* to B, which is the same space as B, but with B's reversed product, i.e., $a^{\mathrm{op}} b^{\mathrm{op}} = (ba)^{\mathrm{op}}$ for $a^{\mathrm{op}}, b^{\mathrm{op}} \in B^{\mathrm{op}}$.

So, to be exact, we define $\mathrm{op}\colon B \cong_1 B^{\mathrm{op}}$ by $a \mapsto a^{\mathrm{op}}$ with inverse $a^{\mathrm{op}} \mapsto a$. Taking adjoints on B^{op} is given by $\mathrm{op} \circ {}^* \circ \mathrm{op}^{-1}$. So this means $(x^{\mathrm{op}})^* = (x^*)^{\mathrm{op}}$. Given an orthonormal basis (f_i) of B, we can let our orthonormal basis of B^{op} be given by applying our opposite map op to each element in our orthonormal basis of B, i.e., (f_i^{op}). We can then define the inner product on B^{op} to be given by

$$\langle a^{\mathrm{op}}|b^{\mathrm{op}}\rangle_{B^{\mathrm{op}}} = \mathrm{Tr}(Qa^*b) = \langle a|b\rangle_B.$$

We also define the opposite modular automorphism σ_t^{op} on B^{op} be given by $\mathrm{op} \circ \sigma_t \circ \mathrm{op}^{-1}$. So $\sigma_t^{\mathrm{op}}(x^{\mathrm{op}}) = \sigma_t(x)^{\mathrm{op}}$.

Remark A.172. A direct corollary to Proposition A.69 is that, for any $t, s \in \mathbb{R}$ and $x^{\mathrm{op}} \in B^{\mathrm{op}}$, we get $\sigma_t^{\mathrm{op}} \sigma_s^{\mathrm{op}} = \sigma_{t+s}^{\mathrm{op}}$, $\sigma_t^{\mathrm{op}}(x^{\mathrm{op}})^* = \sigma_{-t}^{\mathrm{op}}\left((x^{\mathrm{op}})^*\right)$, and $(\sigma_t^{\mathrm{op}})^* = \sigma_t^{\mathrm{op}}$. ♢

So then, given an orthonormal basis (f_i) of B, we get $(f_i \otimes f_j^{\mathrm{op}})$ is an orthonormal basis of $B \otimes B^{\mathrm{op}}$. The inner product on $B \otimes B^{\mathrm{op}}$ is given by

$$\langle a \otimes b^{\mathrm{op}}|c \otimes d^{\mathrm{op}}\rangle_{B \otimes B^{\mathrm{op}}} = \langle a|c\rangle_B \langle b^{\mathrm{op}}|d^{\mathrm{op}}\rangle_{B^{\mathrm{op}}} = \langle a|c\rangle_B \langle b|d\rangle_B = \langle a \otimes b|c \otimes d\rangle_{B \otimes B}.$$

Definition A.173 ([4, Definition 5.2])**.** We define the *tensor swap map* ς as the self-invertible and real (i.e., star-preserving) linear automorphism on $B \otimes B^{\mathrm{op}}$ which is given by $a \otimes b^{\mathrm{op}} \mapsto b \otimes a^{\mathrm{op}}$.

Remark A.174. So $\varsigma = (\mathrm{op}^{-1} \otimes \mathrm{op})\varkappa_{B,B^{\mathrm{op}}}$. ♢

Lemma A.175. *For any $x, y \in B \otimes B^{\mathrm{op}}$, we get $\varsigma(xy) = \varsigma(y)\varsigma(x)$ and $\varsigma(x^*) = \varsigma(x)^*$.*

Proof. Direct computation. ■

Lemma A.176. *For any $x, y \in \mathbb{R}$, we get $\varsigma \circ (\sigma_x \otimes \sigma_y^{\mathrm{op}}) = (\sigma_y \otimes \sigma_x^{\mathrm{op}}) \circ \varsigma$.*

Proof. Let $x, y \in \mathbb{R}$ and $\alpha, \beta \in B$. Then, we compute,

$$\begin{aligned}\varsigma(\sigma_x \otimes \sigma_y^{\mathrm{op}})(\alpha \otimes \beta^{\mathrm{op}}) &= \varsigma(\sigma_x(\alpha) \otimes \sigma_y(\beta)^{\mathrm{op}}) = \sigma_y(\beta) \otimes \sigma_x(\alpha)^{\mathrm{op}} \\ &= (\sigma_y \otimes \sigma_x^{\mathrm{op}})(\beta \otimes \alpha^{\mathrm{op}}) = (\sigma_y \otimes \sigma_x^{\mathrm{op}})\varsigma(\alpha \otimes \beta^{\mathrm{op}}).\end{aligned}$$

Thus $\varsigma(\sigma_x \otimes \sigma_y^{\mathrm{op}}) = (\sigma_y \otimes \sigma_x^{\mathrm{op}})\varsigma$. ∎

Proposition A.177. *Given $t, s \in \mathbb{R}$, there exists a linear isomorphism $\mathcal{B}(B, \psi) \cong B \otimes B^{\mathrm{op}}$.*

Proof. It is enough to define our maps on rank-one operators and simple tensors, as we can simply extend this. We define our linear map $\Psi_{t,s} \colon |a\rangle\langle b| \mapsto \sigma_t(a) \otimes (\sigma_s(b)^*)^{\mathrm{op}}$ and its inverse linear map $\Psi_{t,s}^{-1} \colon a \otimes b^{\mathrm{op}} \mapsto |\sigma_{-t}(a)\rangle\langle\sigma_{-s}(b^*)|$.

We now check that this is a well-defined inverse.

$$\begin{aligned}\Psi_{t,s}^{-1}\Psi_{t,s}(|a\rangle\langle b|) &= \Psi_{t,s}^{-1}\left(\sigma_t(a) \otimes (\sigma_s(b)^*)^{\mathrm{op}}\right) \\ &= |\sigma_{-t}(\sigma_t(a))\rangle\langle\sigma_{-s}(\sigma_s(b))| \\ &= |\sigma_0(a)\rangle\langle\sigma_0(b)| = |a\rangle\langle b| && \text{by A.69(i).}\end{aligned}$$

And, we also compute,

$$\begin{aligned}\Psi_{t,s}\Psi_{t,s}^{-1}(a \otimes b^{\mathrm{op}}) &= \Psi_{t,s}(|\sigma_{-t}(a)\rangle\langle\sigma_{-s}(b^*)|) \\ &= \sigma_t(\sigma_{-t}(a)) \otimes (\sigma_s(\sigma_{-s}(b^*))^*)^{\mathrm{op}} \\ &= \sigma_t\sigma_{-t}(a) \otimes \sigma_{-s}\sigma_s(b)^{\mathrm{op}} && \text{by A.69(ii)} \\ &= \sigma_0(a) \otimes \sigma_0(b)^{\mathrm{op}} = a \otimes b^{\mathrm{op}} && \text{by A.69(i).}\end{aligned}$$

Thus $\Psi_{t,s}$ is a linear isomorphism. ∎

> **Definition A.178** ([4, middle of page 9])**.** Given $t, s \in \mathbb{R}$, we define the linear isomorphism $\mathcal{B}(B, \psi) \cong B \otimes B^{\mathrm{op}}$ by $\Psi_{t,s} \colon |a\rangle\langle b| \mapsto \sigma_t(a) \otimes (\sigma_s(b)^*)^{\mathrm{op}}$ and its inverse by $\Psi_{t,s}^{-1} \colon a \otimes b^{\mathrm{op}} \mapsto |\sigma_{-t}(a)\rangle\langle\sigma_{-s}(b^*)|$. (See Proposition A.177.)

Lemma A.179. *For any $s, p, r, t \in \mathbb{R}$, we get $(\sigma_s \otimes \sigma_p^{\mathrm{op}}) \circ \Psi_{r,t} = \Psi_{s+r,-p+t}$.*

Proof. By linearity, it suffices to show this for $|a\rangle\langle b|$ where $a, b \in B$. So let $a, b \in B$ and compute,

$$\begin{aligned}(\sigma_s \otimes \sigma_p^{\mathrm{op}})\Psi_{r,t}(|a\rangle\langle b|) &= (\sigma_s \otimes \sigma_p^{\mathrm{op}})\left(\sigma_r(a) \otimes (\sigma_t(b)^*)^{\mathrm{op}}\right) \\ &= \sigma_{s+r}(a) \otimes \sigma_{-p+t}(b)^{\mathrm{op}} && \text{by A.69(i),(ii)} \\ &= \Psi_{s+r,-p+t}(|a\rangle\langle b|).\end{aligned}$$

∎

Proposition A.180 ([4, Proposition 5.3 & Lemma 5.7])**.** *Let $A, T \in \mathcal{B}(B, \psi)$ and $t, s \in \mathbb{R}$. Then,*

(i) $\Psi_{t,s}(A^*) = (\sigma_{t-s} \otimes \sigma_{t-s}^{\mathrm{op}})\varsigma(\Psi_{t,s}(A)^*)$,

(ii) $\Psi_{t,s}(\mathrm{symm}(A)) = (\sigma_{t+s-1} \otimes \sigma_{-(t+s)}^{\mathrm{op}})\varsigma(\Psi_{t,s}(A))$,

(iii) $\Psi_{t,s}(\mathrm{symm}'(A)) = (\sigma_{t+s} \otimes \sigma_{1-t-s}^{\mathrm{op}})\varsigma(\Psi_{t,s}(A))$,

(iv) $\Psi_{t,s}(A \bullet T) = \Psi_{t,s}(A)\Psi_{t,s}(T)$,

(v) $\Psi_{t,s}(A^r) = (\sigma_{2t} \otimes \sigma_{1-2s}^{\mathrm{op}})\Psi_{t,s}(A)^*$.

Proof. Since we can write any linear map $A \in \mathcal{B}(B,\psi)$ as $\sum_i |\alpha_i\rangle\langle\beta_i|$ for some tuples $(\alpha_i), (\beta_i)$ in B, it is enough to show that these results are true for $A = |a\rangle\langle b|$ and $T = |c\rangle\langle d|$ for $a, b, c, d \in B$.

(i) We compute,

$$\begin{aligned}
(\sigma_{t-s} \otimes \sigma_{t-s}^{\mathrm{op}})\varsigma(\Psi_{t,s}(|a\rangle\langle b|)^*) &= (\sigma_{t-s} \otimes \sigma_{t-s}^{\mathrm{op}})\varsigma(\sigma_{-t}(a^*) \otimes \sigma_s(b)^{\mathrm{op}}) \\
&= (\sigma_{t-s} \otimes \sigma_{t-s}^{\mathrm{op}})(\sigma_s(b) \otimes \sigma_{-t}(a^*)^{\mathrm{op}}) \\
&= \sigma_t(b) \otimes \sigma_{-s}(a^*)^{\mathrm{op}} && \text{by A.69(i)} \\
&= \sigma_t(b) \otimes (\sigma_s(a)^*)^{\mathrm{op}} && \text{by A.69(ii)} \\
&= \Psi_{t,s}(|b\rangle\langle a|) \\
&= \Psi_{t,s}(|a\rangle\langle b|^*) && \text{by A.17(iii).}
\end{aligned}$$

Thus $\Psi_{t,s}(A^*) = (\sigma_{t-s} \otimes \sigma_{t-s}^{\mathrm{op}})\varsigma(\Psi_{t,s}(A)^*)$.

(ii) We compute,

$$\begin{aligned}
&(\sigma_{t+s-1} \otimes \sigma_{-(t+s)}^{\mathrm{op}})\varsigma(\Psi_{t,s}(|a\rangle\langle b|)) \\
&= \varsigma(\sigma_{-(t+s)} \otimes \sigma_{t+s-1}^{\mathrm{op}})\Psi_{t,s}(|a\rangle\langle b|) && \text{by A.176} \\
&= \varsigma(\Psi_{-s,1-t}(|a\rangle\langle b|)) && \text{by A.179} \\
&= \varsigma(\sigma_{-s}(a) \otimes (\sigma_{1-t}(b)^*)^{\mathrm{op}} \\
&= \sigma_{1-t}(b)^* \otimes \sigma_{-s}(a)^{\mathrm{op}} \\
&= \sigma_t(\sigma_{-1}(b^*)) \otimes (\sigma_s(a^*)^*)^{\mathrm{op}} && \text{by A.69(i),(ii)} \\
&= \Psi_{t,s}(|\sigma_{-1}(b^*)\rangle\langle a^*|) \\
&= \Psi_{t,s}\left(\mathrm{symm}(|a\rangle\langle b|)\right) && \text{by A.91(i).}
\end{aligned}$$

Thus $\Psi_{t,s}(\mathrm{symm}(A)) = (\sigma_{t+s-1} \otimes \sigma_{-(t+s)}^{\mathrm{op}})\varsigma(\Psi_{t,s}(A))$.

(iii) We compute,

$$\begin{aligned}
&(\sigma_{t+s} \otimes \sigma_{1-t-s}^{\mathrm{op}})\varsigma(\Psi_{t,s}(|a\rangle\langle b|)) \\
&= \varsigma(\sigma_{1-t-s} \otimes \sigma_{t+s}^{\mathrm{op}})\Psi_{t,s}(|a\rangle\langle b|) && \text{by A.176} \\
&= \varsigma(\Psi_{1-s,-t}(|a\rangle\langle b|)) && \text{by A.179} \\
&= \varsigma(\sigma_{1-s}(a) \otimes (\sigma_{-t}(b)^*)^{\mathrm{op}} \\
&= \sigma_{-t}(b)^* \otimes \sigma_{1-s}(a)^{\mathrm{op}} \\
&= \sigma_t(b^*) \otimes (\sigma_s(\sigma_{-1}(a^*))^*)^{\mathrm{op}} && \text{by A.69(i),(ii)} \\
&= \Psi_{t,s}(|b^*\rangle\langle\sigma_{-1}(a^*)|) \\
&= \Psi_{t,s}\left(\mathrm{symm}'(|a\rangle\langle b|)\right) && \text{by A.91(ii).}
\end{aligned}$$

Therefore, $\Psi_{t,s}(\mathrm{symm}'(A)) = (\sigma_{t+s} \otimes \sigma_{1-t-s}^{\mathrm{op}})\varsigma(\Psi_{t,s}(A))$.

(iv) We have

$$\begin{aligned}
\Psi_{t,s}(|a\rangle\langle b|)\Psi_{t,s}(|c\rangle\langle d|) &= \left(\sigma_t(a) \otimes \left(\sigma_s(b)^*\right)^{\mathrm{op}}\right)\left(\sigma_t(c) \otimes \left(\sigma_s(d)^*\right)^{\mathrm{op}}\right) \\
&= \sigma_t(a)\sigma_t(c) \otimes \left((\sigma_s(b)\sigma_s(d))^*\right)^{\mathrm{op}} \\
&= \sigma_t(ac) \otimes (\sigma_s(bd)^*)^{\mathrm{op}} \\
&= \Psi_{t,s}(|ac\rangle\langle bd|) \\
&= \Psi_{t,s}\left(|a\rangle\langle b| \bullet |c\rangle\langle d|\right) && \text{by A.112.}
\end{aligned}$$

Thus $\Psi_{t,s}(A \bullet T) = \Psi_{t,s}(A)\Psi_{t,s}(T)$.

(v) Finally, we compute,

$$
\begin{aligned}
(\sigma_{2t} \otimes \sigma^{\mathrm{op}}_{1-2s})(\Psi_{t,s}(|a\rangle\langle b|))^* &= (\sigma_{2t} \otimes \sigma^{\mathrm{op}}_{1-2s})(\sigma_t(a)^* \otimes \sigma_s(b)^{\mathrm{op}}) \\
&= (\sigma_{2t} \otimes \sigma^{\mathrm{op}}_{1-2s})(\sigma_{-t}(a^*) \otimes \sigma_s(b)^{\mathrm{op}}) && \text{by A.69(ii)} \\
&= \sigma_t(a^*) \otimes \sigma_{1-s}(b)^{\mathrm{op}} && \text{by A.69(i)} \\
&= \sigma_t(a^*) \otimes \left(\sigma_s\left(\sigma_{-1}(b^*)\right)^*\right)^{\mathrm{op}} && \text{by A.69(i),(ii)} \\
&= \Psi_{t,s}(|a^*\rangle\langle\sigma_{-1}(b^*)|) \\
&= \Psi_{t,s}(|a\rangle\langle b|^{\mathrm{r}}) && \text{by A.87(iv).}
\end{aligned}
$$

Thus $\Psi_{t,s}(A^{\mathrm{r}}) = (\sigma_{2t} \otimes \sigma^{\mathrm{op}}_{1-2s})\Psi_{t,s}(A)^*$.

■

Corollary A.181. *Let $A \in \mathcal{B}(B,\psi)$ and $t,s \in \mathbb{R}$. Then,*

(i) $A^* = A \Leftrightarrow \Psi_{t,s}(A) = \varsigma(\Psi_{s,t}(A)^*)$,

(ii) $\mathrm{symm}(A) = A \Leftrightarrow \Psi_{t,s}(A) = \varsigma(\Psi_{-s,1-t}(A))$,

(iii) $\mathrm{symm}'(A) = A \Leftrightarrow \Psi_{t,s}(A) = \varsigma(\Psi_{1-s,-t}(A))$,

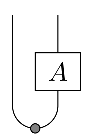
(iv) $A \bullet A = A \Leftrightarrow \Psi_{t,s}(A)^2 = \Psi_{t,s}(A)$,

(v) *A is real* $\Leftrightarrow \Psi_{t,s}(A)^* = \Psi_{-t,1-s}(A)$.

Proof. These follow directly from Proposition A.180, by combining them with Lemmas A.176 and A.179. ■

Remark A.182. Using the above Corollary A.181(v), we get A is real if and only if $\Psi_{0,1/2}(A)$ is self-adjoint. (See [4, Proposition 5.21].) ♢

Lemma A.183. *For $A \in \mathcal{B}(B,\psi)$, we have*



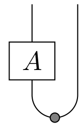
$= (\mathrm{id} \otimes \mathrm{op}^{-1})\varsigma\Psi_{0,1}(A).$

Proof. It suffices to show this for $A = |x\rangle\langle y|$ for $x,y \in B$. Let $a,b \in B$ and let $m^*(1) = \sum_i \alpha_i \otimes \beta_i$ for tuples $(\alpha_i),(\beta_i)$ in B. We compute,

$$
\begin{aligned}
\langle(\mathrm{id} \otimes |x^{\mathrm{op}}\rangle\langle y|)m^*(1)|a \otimes b^{\mathrm{op}}\rangle &= \sum_i \langle(\mathrm{id} \otimes |x^{\mathrm{op}}\rangle\langle y|)(\alpha_i \otimes \beta_i)|a \otimes b^{\mathrm{op}}\rangle \\
&= \sum_i \langle \alpha_i \otimes \langle y|\beta_i\rangle x^{\mathrm{op}}|a \otimes b^{\mathrm{op}}\rangle \\
&= \sum_i \langle \alpha_i|a\rangle\langle\beta_i|y\rangle\langle x^{\mathrm{op}}|b^{\mathrm{op}}\rangle \\
&= \langle m^*(1)|a \otimes y\rangle\langle x^{\mathrm{op}}|b^{\mathrm{op}}\rangle \\
&= \langle 1|ay\rangle\langle x^{\mathrm{op}}|b^{\mathrm{op}}\rangle \\
&= \langle\sigma_{-1}(y^*)|a\rangle\langle x^{\mathrm{op}}|b^{\mathrm{op}}\rangle \\
&= \langle\sigma_{-1}(y^*) \otimes x^{\mathrm{op}}|a \otimes b^{\mathrm{op}}\rangle \\
&= \left\langle \varsigma(x \otimes \sigma_1(y)^{*\mathrm{op}})\middle| a \otimes b^{\mathrm{op}}\right\rangle \\
&= \langle\varsigma\Psi_{0,1}(|x\rangle\langle y|)|a \otimes b^{\mathrm{op}}\rangle.
\end{aligned}
$$

Thus $\varsigma\Psi_{0,1}(A) = (\mathrm{id} \otimes \mathrm{op})(\mathrm{id} \otimes A)m^*(1)$. ■

Proposition A.184. *For $A \in \mathcal{B}(B,\psi)$, we have*



$$= (\mathrm{id} \otimes \mathrm{op}^{-1})\Psi_{0,0}(A).$$

Proof.

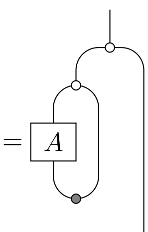
$$\begin{aligned}
\langle (|x\rangle\langle y| \otimes \mathrm{id}) m^*(1) | a \otimes b\rangle &= \sum_i \langle (|x\rangle\langle y| \otimes \mathrm{id})(\alpha_i \otimes \beta_i) | a \otimes b\rangle \\
&= \sum_i \langle \alpha_i | y\rangle \langle x|a\rangle\langle \beta_i | b\rangle \\
&= \langle m^*(1) | y \otimes b\rangle \langle x|a\rangle \\
&= \langle 1 | yb\rangle\langle x|a\rangle = \langle y^* | b\rangle \langle x|a\rangle \\
&= \langle x \otimes y^* | a \otimes b\rangle \\
&= \big\langle (\mathrm{id} \otimes \mathrm{op}^{-1})\Psi_{0,0}(|x\rangle\langle y|) \big| a \otimes b\big\rangle.
\end{aligned}$$

■

Proposition A.185. *Given $A \in \mathcal{B}(B,\psi)$, we have,*

(i) $A \bullet \mathrm{id} = \mathrm{lmul}\big(m(\mathrm{id} \otimes \mathrm{op}^{-1})\Psi_{0,0}(A)\big)$,
in other words,

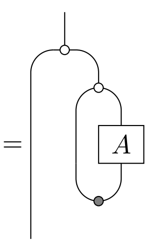
$$A \bullet \mathrm{id} =$$



,

(ii) $\mathrm{id} \bullet A = \mathrm{rmul}\big(m(\mathrm{id} \otimes \mathrm{op}^{-1})\varsigma(\Psi_{0,1}(A))\big)$,
in other words,

$$\mathrm{id} \bullet A =$$



.

Moreover, $A \bullet \mathrm{id}$ equals id (resp., 0) if and only if $m(\mathrm{id} \otimes \mathrm{op}^{-1})\Psi_{0,0}(A)$ equals 1 (resp., 0) and $\mathrm{id} \bullet A$ equals id (resp., 0) if and only if $m(\mathrm{id} \otimes \mathrm{op}^{-1})\varsigma(\Psi_{0,1}(A))$ equals 1 (resp., 0).

Proof. Let $A = \sum_i |a_i\rangle\langle b_i|$ for some tuples $(a_i), (b_i)$ in B.

(i) We compute,

$$\begin{aligned}
A \bullet \mathrm{id} = \sum_i |a_i\rangle\langle b_i| \bullet \mathrm{id} &= \sum_i \mathrm{lmul}(a_i b_i^*) && \text{by A.116(i)} \\
&= \sum_i \mathrm{lmul}\left(m(\mathrm{id} \otimes \mathrm{op}^{-1})(a_i \otimes (b_i^*)^{\mathrm{op}})\right) \\
&= \sum_i \mathrm{lmul}\left(m(\mathrm{id} \otimes \mathrm{op}^{-1})\Psi_{0,0}(|a_i\rangle\langle b_i|)\right)
\end{aligned}$$

$$= \mathrm{lmul}\left(m(\mathrm{id}\otimes \mathrm{op}^{-1})\Psi_{0,0}(A)\right).$$

This equals id (resp., 0) if and only if $m(\mathrm{id}\otimes\mathrm{op}^{-1})\Psi_{0,0}(A)$ equals 1 (resp., 0).

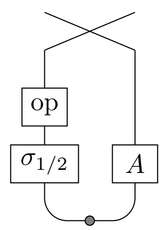
(*ii*) We compute,

$$\begin{aligned}
\mathrm{id}\bullet A &= \sum_i \mathrm{id}\bullet |a_i\rangle\langle b_i| = \sum_i \mathrm{rmul}(\sigma_{-1}(b_i^*)a_i) && \text{by A.116(ii)}\\
&= \sum_i \mathrm{rmul}(m(\mathrm{id}\otimes\mathrm{op}^{-1})(\sigma_{-1}(b_i^*)\otimes a_i^{\mathrm{op}}))\\
&= \sum_i \mathrm{rmul}\left(m(\mathrm{id}\otimes\mathrm{op}^{-1})\varsigma(a_i\otimes\sigma_1(b_i)^{*\mathrm{op}})\right)\\
&= \sum_i \mathrm{rmul}\left(m(\mathrm{id}\otimes\mathrm{op}^{-1})\varsigma\Psi_{0,1}(|a_i\rangle\langle b_i|)\right)\\
&= \mathrm{rmul}\left(m(\mathrm{id}\otimes\mathrm{op}^{-1})\varsigma\Psi_{0,1}(A)\right).
\end{aligned}$$

And this equals id (resp., 0) if and only if $m(\mathrm{id}\otimes\mathrm{op}^{-1})\varsigma(\Psi_{0,1}(A))$ equals 1 (resp., 0).

■

Proposition A.186 ([9, Definition 1.2])**.** *For $A\in\mathcal{B}(B,\psi)$, we have*



$= \Psi_{0,1/2}(A).$

Proof. We begin with the following claim.
Claim: $\varkappa_{B,B^{\mathrm{op}}} = (\mathrm{op}\otimes\mathrm{op}^{-1})\varsigma$. (1)

$$\begin{aligned}
(\mathrm{op}\otimes\mathrm{op}^{-1})\varsigma(a\otimes b^{\mathrm{op}}) &= (\mathrm{op}\otimes\mathrm{op}^{-1})(b\otimes a^{\mathrm{op}}) = b^{\mathrm{op}}\otimes a\\
&= \varkappa_{B,B^{\mathrm{op}}}(a\otimes b^{\mathrm{op}}).
\end{aligned}$$

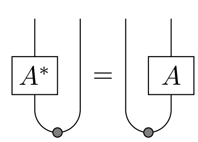
We then compute,

$$\begin{aligned}
&\varkappa_{B^{\mathrm{op}},B}(\mathrm{op}\otimes\mathrm{id})(\sigma_{1/2}\otimes A)m^*(1)\\
&= \varkappa_{B^{\mathrm{op}},B}(\mathrm{op}\circ\sigma_{1/2}\otimes\mathrm{op}^{-1})\varsigma\Psi_{0,1}(A) && \text{by A.183}\\
&= \varkappa_{B^{\mathrm{op}},B}(\mathrm{op}\otimes\mathrm{op}^{-1})\varsigma\Psi_{0,1/2}(A)\\
&= \varkappa_{B^{\mathrm{op}},B}\varkappa_{B,B^{\mathrm{op}}}\Psi_{0,1/2}(A) && \text{by Claim (1)}\\
&= \Psi_{0,1/2}(A).
\end{aligned}$$

■

Proposition A.187 ([9, Equation 1.1])**.** *For $A\in\mathcal{B}(B,\psi)$, we have,*

A is real ⇔



.

Proof.

$$(\mathrm{id}\otimes\mathrm{op}^{-1})\Psi_{0,0}(A^*) = (\mathrm{id}\otimes\mathrm{op}^{-1})\varsigma\Psi_{0,1}(A) \Leftrightarrow \Psi_{0,0}(A^*) = \varsigma\Psi_{0,1}(A)$$

$$\begin{aligned}&\Leftrightarrow \varsigma\Psi_{0,0}(A)^* = \varsigma\Psi_{0,1}(A)\\&\Leftrightarrow \Psi_{0,0}(A)^* = \Psi_{0,1}(A)\\&\Leftrightarrow A \text{ is real.}\end{aligned}$$

■

Proposition A.188. *Given $A \in \mathcal{B}(B,\psi)$ and $t,s,a,b \in \mathbb{R}$, then* $(\sigma_a \otimes \sigma_b^{\mathrm{op}})\Psi_{t,s}(A) = \Psi_{t,s}(\sigma_a A \sigma_{-b})$.

Proof. Let $A = \sum_i |x_i\rangle\langle y_i|$ for some tuples $(x_i),(y_i)$ in B. Then we compute,

$$\begin{aligned}(\sigma_a \otimes \sigma_b^{\mathrm{op}})\Psi_{t,s}(A) &= \sum_i \sigma_{a+t}(x_i) \otimes \sigma_{s-b}(y_i)^{*\mathrm{op}}\\&= \sum_i \Psi_{t,s}(|\sigma_a(x_i)\rangle\langle\sigma_{-b}(y_i)|)\\&= \sum_i \Psi_{t,s}(\sigma_a|x_i\rangle\langle y_i|\sigma_{-b}) && \text{by A.69(iii)}\\&= \Psi_{t,s}(\sigma_a A \sigma_{-b}).\end{aligned}$$

Thus $(\sigma_a \otimes \sigma_b^{\mathrm{op}})\Psi_{t,s}(A) = \Psi_{t,s}(\sigma_a A \sigma_{-b})$. ■

A.XII.4 $B = M_n$. This subsection contains some computations that will be useful for later on for when $B = M_n$ in Chapter C.

Lemma A.189. *Let e be the orthonormal basis $(e_{ij}Q^{-1/2})_{ij}$ on M_n (see Proposition A.60). Then for any $x,y \in M_n$, we have*

(i) $\mathcal{M}_e(|x\rangle\langle y|) = \varrho(xQ^{1/2})\varrho(yQ^{1/2})^*$,

(ii) $(\mathrm{id}\otimes\top^{-1})\Psi_{t,s}(\mathrm{id}) = \mathcal{M}_e\big(\big|Q^{-(t+s)-1/2}\big\rangle\big\langle Q^{t+s-3/2}\big|\big)$ *for any $t,s \in \mathbb{R}$.*

■

Proof. (i) For any i,j,k,l, we compute,

$$\begin{aligned}\mathcal{M}_e(|x\rangle\langle y|)_{ij}^{kl} &= \left\langle e_{ij}Q^{-1/2}\middle| |x\rangle\langle y|(e_{kl}Q^{-1/2}))\right\rangle\\&= \left\langle y\middle|e_{kl}Q^{-1/2}\right\rangle\left\langle e_{ij}Q^{-1/2}\middle|x\right\rangle\\&= \overline{(yQ^{1/2})_{kl}}(xQ^{1/2})_{ij} = \left[\varrho(xQ^{1/2})\varrho(yQ^{1/2})^*\right]_{ij}^{kl}.\end{aligned}$$

Thus $\mathcal{M}_e(|x\rangle\langle y|) = \varrho(xQ^{1/2})\varrho(yQ^{1/2})^*$.

(ii) For any $t,s \in \mathbb{R}$, and a,b,c,d, we compute,

$$\begin{aligned}\left[(\mathrm{id}\otimes\top^{-1})\Psi_{t,s}(\mathrm{id})\right]_{ab}^{cd} &= \sum_{i,j}\left[(\mathrm{id}\otimes\top^{-1})\Psi_{t,s}\left(\left|e_{ij}Q^{-1/2}\right\rangle\left\langle e_{ij}Q^{-1/2}\right|\right)\right]_{ab}^{cd}\\&= \sum_{i,j} Q^{-t}e_{ij}Q^{t-1/2})_{ac}\,\overline{(Q^{-s}e_{ij}Q^{s-1/2})_{bd}}\\&= \sum_{i,j} Q_{ai}^{-t}Q_{jc}^{t-1/2}\overline{Q_{bi}^{-s}}\,\overline{Q_{jd}^{s-1/2}}\\&= Q_{ab}^{-(t+s)}\overline{Q_{cd}^{t+s-1}} = \left[\varrho(Q^{-(t+s)})\varrho(Q^{t+s-1})^*\right]_{ab}^{cd}\\&= \mathcal{M}_e\left(\left|Q^{-(t+s)-1/2}\right\rangle\left\langle Q^{t+s-3/2}\right|\right)_{ab}^{cd}.\end{aligned}$$

Where the last equality comes from part (i) above. And so the result then follows.

■

Proposition A.190. *Let f be the orthonormal basis $(e_{ij}Q^{-1/2})_{ij}$ on M_n (see Proposition A.60). Then,*

(i) $(\mathrm{id}\otimes\top^{-1})\Psi_{0,1/2}(A) = (A\otimes\mathrm{id})\mathcal{M}_f\big(\big|Q^{-1}\big\rangle\big\langle Q^{-1}\big|\big)$ *for any* $A\in\mathcal{B}(M_n)$,

(ii) $\mathcal{M}_f^{-1}(\mathrm{id}\otimes\top^{-1})\Psi_{0,1/2}(|x\rangle\langle y|) = \mathrm{lmul}(x)\mathrm{rmul}(y)^*$ *for any* $x,y\in M_n$,

(iii) $\Psi_{0,1/2}^{-1}(\mathrm{id}\otimes\top)\mathcal{M}_f(|x\rangle\langle y|) = \mathrm{lmul}(xQ)\mathrm{rmul}(Qy)^*$ *for any* $x,y\in M_n$.

Proof.

(i)

$$\begin{aligned}(\mathrm{id}\otimes\top^{-1})\Psi_{0,1/2}(A) &= \sum_{i,j}(\mathrm{id}\otimes\top^{-1})\Psi_{0,1/2}(|A(f_{ij})\rangle\langle f_{ij}|)\\ &= \sum_{i,j}A(f_{ij})\otimes\overline{\sigma_{1/2}(f_{ij})}\\ &= (A\otimes\mathrm{id})(\mathrm{id}\otimes\top^{-1})\Psi_{0,1/2}(\mathrm{id})\\ &= (A\otimes\mathrm{id})\mathcal{M}_f\big(\big|Q^{-1}\big\rangle\big\langle Q^{-1}\big|\big) && \text{by A.189(ii).}\end{aligned}$$

(ii)

$$\begin{aligned}\mathcal{M}_f\big(\mathrm{lmul}(x)\mathrm{rmul}(y)^*\big) &= \mathcal{M}_f(\mathrm{lmul}(x))\mathcal{M}_f(\mathrm{rmul}(y))^*\\ &= x\otimes\overline{\sigma_{1/2}(y)} && \text{by A.115(viii),(ix)}\\ &= (\mathrm{id}\otimes\top^{-1})\Psi_{0,1/2}(|x\rangle\langle y|).\end{aligned}$$

(iii) We let $a\in M_n$ and compute,

$$\begin{aligned}&\Psi_{0,1/2}^{-1}((\mathrm{id}\otimes\top)\mathcal{M}_f(|x\rangle\langle y|))(a)\\ &= \sum_{i,j,k,l}\mathcal{M}_f(|x\rangle\langle y|)_{ij}^{kl}\Psi_{0,1/2}^{-1}((\mathrm{id}\otimes\top)(e_{ik}\otimes e_{jl}))(a)\\ &= \sum_{i,j,k,l}(xQ^{1/2})_{ij}\overline{(yQ^{1/2})_{kl}}\big|e_{ik}\big\rangle\big\langle\sigma_{-1/2}(e_{jl})\big|(a)\\ &= \sum_{i,j,k,l}(xQ^{1/2})_{ij}\overline{(yQ^{1/2})_{kl}}\Big\langle Q^{1/2}e_{jl}Q^{-1/2}\Big|a\Big\rangle e_{ik}\\ &= \sum_{i,j,k,l}(xQ^{1/2})_{ij}\overline{(yQ^{1/2})_{kl}}\Big\langle e_{jl}Q^{-1/2}\Big|Q^{1/2}a\Big\rangle e_{ik}\\ &= \sum_{i,j,k,l}(xQ^{1/2})_{ij}\overline{(yQ^{1/2})_{kl}}(Q^{1/2}aQ^{1/2})_{jl}e_{ik}\\ &= \sum_{i,j,k}(xQ^{1/2})_{ij}(Q^{1/2}aQ)_{jk}e_{ik}\\ &\quad= \sum_{i,k}\Big[xQ^{1/2}Q^{1/2}aQ^{1/2}Q^{1/2}y^*\Big]_{ik}e_{ik}\\ &\quad= xQaQy^* = \mathrm{lmul}(xQ)\,\mathrm{rmul}(Qy^*)(a)\\ &\quad= \mathrm{lmul}(xQ)\mathrm{rmul}(Qy)^*(a).\end{aligned}$$

Note that the last equality follows from Lemma A.115(ii), in particular, for any $y\in M_n$, we get $\mathrm{rmul}(Qy)^* = \mathrm{rmul}(\sigma_{-1}(y^*Q)) = \mathrm{rmul}(Qy^*)$.

■

A.XIII Bimodules

Bimodules provide another useful perspective for interpreting quantum graphs, as we will see in Section B.III.

Given rings A, B, we say $\mathcal{A}$ is an (A,B)-*bimodule* when it is both a left A-module given by the left scalar multiplication $\cdot_{\mathrm{l}}$ and a right B-module given by the right scalar multiplication $\cdot_{\mathrm{r}}$, such that $a \cdot_{\mathrm{l}} (b \cdot_{\mathrm{r}} c) = (a \cdot_{\mathrm{l}} b) \cdot_{\mathrm{r}} c$ for any $b \in \mathcal{A}$, $a \in A$, and $c \in B$ (in other words, it is a vector space equipped with these left and right actions). Given (A,B)-bimodules $\mathcal{A}, \mathcal{D}$, we call a linear map $P\colon \mathcal{A} \to \mathcal{D}$ an (A,B)-*bimodule map* if $P(a \cdot_{\mathrm{l}} b \cdot_{\mathrm{r}} c) = a \cdot_{\mathrm{l}} P(b) \cdot_{\mathrm{r}} c$, for $a \in A$, $c \in B$, and $b \in \mathcal{A}$.

For the purposes of this essay, we will focus on the bimodules that are inferred from our C*-algebra B. This algebra naturally gives rise to a (B,B)-bimodule $B \otimes B$ such that the left scalar multiplication $\cdot_{\mathrm{l}}\colon B \times B \otimes B \to B \otimes B$ is given by $(a, x \otimes y) \mapsto ax \otimes y$ and the right scalar multiplication $\cdot_{\mathrm{r}}\colon B \otimes B \times B \to B \otimes B$ is given by $(x \otimes y, a) \mapsto x \otimes ya$. In other words, $a \cdot_{\mathrm{l}} x = (\mathrm{lmul}(a) \otimes \mathrm{id})(x)$ and $x \cdot_{\mathrm{r}} a = (\mathrm{id} \otimes \mathrm{rmul}(a))(x)$. So then (B,B)-bimodule maps $P \in \mathcal{B}(B \otimes B)$ satisfy the property $P((\mathrm{lmul}(x) \otimes \mathrm{rmul}(y))(a)) = (\mathrm{lmul}(x) \otimes \mathrm{rmul}(y))P(a)$.

Given elements in $B \otimes B$, we can form (B,B)-bimodule maps in $\mathcal{B}(B \otimes B)$ in the obvious way. In particular, let $x, y \in B$, then $(\mathrm{rmul} \otimes \mathrm{lmul})(x \otimes y)$ is an (B,B)-bimodule map:

$$\begin{aligned}(\mathrm{rmul}(x) \otimes \mathrm{lmul}(y))(g \cdot_{\mathrm{l}} a \otimes b \cdot_{\mathrm{r}} h) &= gax \otimes ybh = g \cdot_{\mathrm{l}} (ax \otimes yb) \cdot_{\mathrm{r}} h \\ &= g \cdot_{\mathrm{l}} (\mathrm{rmul}(x) \otimes \mathrm{lmul}(y))(a \otimes b) \cdot_{\mathrm{r}} h.\end{aligned}$$

Lemma A.191. *Let $f \in \mathcal{B}(B \otimes B)$ and $P_1, P_2 \in \mathcal{B}(B)$. Then,*

(i) $(\mathrm{rmul} \otimes \mathrm{lmul})(p)(a \otimes b) = a \cdot_{\mathrm{l}} p \cdot_{\mathrm{r}} b$ *for all* $p \in B \otimes B$ *and* $a, b \in B$,

(ii) f *is an* (B,B)*-bimodule map* $\Leftrightarrow$ $(\mathrm{rmul} \otimes \mathrm{lmul})(f(1)) = f$,

(iii) $P_1(ab) = aP_1(b)$ *for all* $a, b \in B$ $\Leftrightarrow$ $P_1 = \mathrm{rmul}(P_1(1))$,

(iv) $P_1(ab) = P_1(a)b$ *for all* $a, b \in B$ $\Leftrightarrow$ $P_1 = \mathrm{lmul}(P_1(1))$,

(v) Let $P_1, P_2 \neq 0$. *Then* $P_1 \otimes P_2$ *is an* (B,B)*-bimodule map* $\Leftrightarrow$ $P_1 = \mathrm{rmul}(P_1(1))$ *and* $P_2 = \mathrm{lmul}(P_2(1))$.

Proof.

(i) $(\mathrm{rmul} \otimes \mathrm{lmul})(x \otimes y)(a \otimes b) = ax \otimes yb = a \cdot_{\mathrm{l}} (x \otimes y) \cdot_{\mathrm{r}} b$.

(ii) By the above, we know $(\mathrm{rmul} \otimes \mathrm{lmul})(f(1))(x \otimes y) = x \cdot_{\mathrm{l}} f(1) \cdot_{\mathrm{r}} y$ for all $x, y \in B$.

($\Rightarrow$) This is obvious.

($\Leftarrow$) Suppose $(\mathrm{rmul} \otimes \mathrm{lmul})(f(1)) = f$, which is simply $x \cdot_{\mathrm{l}} f(1) \cdot_{\mathrm{r}} y = f(x \otimes y)$ for all $x, y \in B$. Let $a, b, c, d \in B$. Then $f(ab \otimes cd) = ab \cdot_{\mathrm{l}} f(1) \cdot_{\mathrm{r}} cd = a \cdot_{\mathrm{l}} f(b \otimes c) \cdot_{\mathrm{r}} d$. Thus f is an (B,B)-bimodule map.

(iii) If $P_1(ab) = aP_1(b)$ for all $a, b \in B$, then obviously $P_1 = \mathrm{rmul}(P_1(1))$. So suppose $P_1 = \mathrm{rmul}(P_1(1))$ and let $a, b \in B$. Then $P_1(ab) = abP_1(1) = aP_1(b)$.

(iv) Analogously to the above, if $P_1(ab) = P_1(a)b$ for all $a, b \in B$, then we obviously get $P_1 = \mathrm{lmul}(P_1(1))$. So suppose $P_1 = \mathrm{lmul}(P_1(1))$ and let $a, b \in B$. Then $P_1(ab) = P_1(1)ab = P_1(a)b$.

(v) Suppose $P_1, P_2 \neq 0$. If $P_1 = \mathrm{rmul}(P_1(1))$ and $P_2 = \mathrm{lmul}(P_2(1))$, then we immediately get $P_1 \otimes P_2$ is an (B,B)-bimodule map. Let $a, b \in B$ such that $P_1(a) \neq 0$ and $P_2(b) \neq 0$. Now suppose $P_1 \otimes P_2$ is an (B,B)-bimodule map. Then for all $x \in B$, we get $xP_1(1) \otimes P_2(b) = P_1(x) \otimes P_2(b)$ by the hypothesis. And since $P_2(b) \neq 0$, we get $xP_1(1) = P_1(x)$ for all $x \in B$, which is what we needed.

Similarly, for all $x \in B$, we get $P_1(a) \otimes P_2(1)x = P_1(a) \otimes P_2(x)$ by the hypothesis. And since $P_1(a) \neq 0$, we get $P_2(1)x = P_2(x)$ for all $x \in B$, which is what we needed.

■

B Quantum graphs

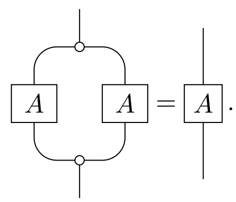
In this chapter, we define a quantum graph. There are multiple ways that one can go about defining a quantum graph, such as via quantum adjacency matrices, projections, bimodule maps, and via positive maps.

B.I Quantum adjacency matrices

The following definition is a slightly adapted version of [10, Definition 1.4].

Definition B.1 (quantum adjacency matrix)**.** We say an operator $A \in \mathcal{B}(B,\psi)$ is a *quantum adjacency matrix* if it satisfies *Schur idempotence* (Definition A.105): $A \bullet A = A$.

In other words, Schur idempotence is,



We say (B,ψ,A) is a *quantum graph* on B given by the quantum adjacency matrix operator $A \in \mathcal{B}(B,\psi)$.

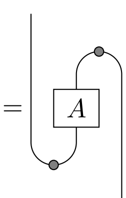
Furthermore, we say,

- (B,ψ,A) is real when A is real (also known as star-preserving) (Definition A.82),

  Recall A is real if and only if $A^{\mathrm{r}} = A$ (Proposition A.83), where A^{r} is given by $x \mapsto A(x^*)^*$.

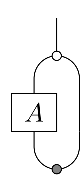
- (B,ψ,A) is self-adjoint when A is self-adjoint,
- (B,ψ,A) is *symmetric* if $\operatorname{symm}(A) = A$ (Definition A.90),

  Recall the definition of symm:

  $$\operatorname{symm}(A) := (\mathrm{id} \otimes \eta^* m)(\mathrm{id} \otimes A \otimes \mathrm{id})(m^*\eta \otimes \mathrm{id}) =$$ 

  Note that $\operatorname{symm}(A) = A$ is equivalent to $A^{\mathrm{r}} = A^*$ (see Proposition A.99).

- (B,ψ,A) is *(ir)reflexive* if $A \bullet \mathrm{id} = \mathrm{id}$ (respectively, if $A \bullet \mathrm{id} = 0$).

 $= 1$ (resp., $= 0$).

Remark B.2. We later show that (see Proposition B.39) a reflexive quantum adjacency matrix corresponds to an irreflexive quantum adjacency matrix. ◊

Remark B.3. We also know that given a real quantum graph (B,ψ,A), we get $A \bullet \mathrm{id} = \mathrm{id}$

(resp., $A \bullet \mathrm{id} = 0$) if and only if $\mathrm{id} \bullet A = \mathrm{id}$ (resp., $\mathrm{id} \bullet A = 0$). See Proposition A.120. ◊

B.II Quantum graphs as projections on $B \otimes B^{\mathrm{op}}$

In this section we see the bijection between real quantum graphs on B and projections on $B \otimes B^{\mathrm{op}}$. We then provide our first formula for a real quantum graph.

Lemma B.4. *Given $A \in \mathcal{B}(B, \psi)$, we have (B, ψ, A) is a real quantum graph if and only if $\Psi_{0,1/2}(A)$ is a projection.*

Proof. Let $e = \Psi_{0,1/2}(A)$. We have A is a quantum adjacency matrix if and only if $e^2 = e$ by Corollary A.181(iv). We also have A is real if and only if e is self-adjoint by Corollary A.181(v). So it is clear that by Proposition A.169, we get A is idempotent and real if and only if e is a projection. ■

Let $A \in \mathcal{B}(\mathrm{B}, \psi)$ be real and Schur idempotent (i.e., $A \bullet A = A$), then by Lemma B.4, we have $\Psi_{0,1/2}(A)$ is an orthogonal projection element in $B \otimes B^{\mathrm{op}}$. Transposing the second tensor and applying $\mathcal{M}^{-1} \otimes \mathcal{M}^{-1}$, we get $(\mathcal{M}^{-1} \otimes \mathcal{M}^{-1})(\mathrm{id} \otimes \top^{-1})\Psi_{0,1/2}(A) \in \bigoplus_{i,j=1}^{k} \mathcal{B}(\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j})$ is a direct sum of orthogonal projection operators. Thus we can choose subspaces $U_{i,j} \subseteq \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ for each $i, j \in [k]$ as the projected subspaces of the orthogonal projection, and we can let $(u_{i,j,s})_s$ be an orthonormal basis of each $U_{i,j}$, such that each $u_{i,j,s} = \sum_t x_{i,j,s,t} \otimes y_{i,j,s,t} \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ so that we can write

$$\begin{aligned}(\mathcal{M}^{-1} \otimes \mathcal{M}^{-1})(\mathrm{id} \otimes \top^{-1})\Psi_{0,1/2}(A) &= \bigoplus_{i,j} \sum_{s} |u_{i,j,s}\rangle\langle u_{i,j,s}| \\ &= \bigoplus_{i,j} \sum_{s,t,p} |x_{i,j,s,t}\rangle\langle x_{i,j,s,p}| \otimes |y_{i,j,s,t}\rangle\langle y_{i,j,s,p}|\end{aligned}$$

and so

$$(\mathrm{id} \otimes \top^{-1})\Psi_{0,1/2}(A) = \sum_{i,j=1}^{k} \sum_{s=1}^{\dim U_{i,j}} \sum_{t,p} \iota_i(x_{i,j,s,t}x_{i,j,s,p}^*) \otimes \iota_j(y_{i,j,s,t}y_{i,j,s,p}^*) \in B \otimes B.$$

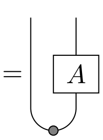
So then

$$\begin{aligned} A &= \sum_{i,j,s,t,p} \Psi_{0,1/2}^{-1}(\mathrm{id} \otimes \top)(\iota_i(x_{i,j,s,t}x_{i,j,s,p}^*) \otimes \iota_j(y_{i,j,s,t}y_{i,j,s,p}^*)) \\ &= \sum_{i,j,s,t,p} \Psi_{0,1/2}^{-1}\left(\iota_i(x_{i,j,s,t}x_{i,j,s,p}^*) \otimes \iota_j\left((y_{i,j,s,t}y_{i,j,s,p}^*)^{\top}\right)^{\mathrm{op}}\right) \\ &= \sum_{i,j,s,t,p} \left|\iota_i(x_{i,j,s,t}x_{i,j,s,p}^*)\right\rangle\!\left\langle\sigma_{-1/2}(\iota_j(\overline{y_{i,j,s,t}y_{i,j,s,p}^*}))\right|. \end{aligned}$$

B.III Quantum graphs as bimodule projections

In this section, we see

Definition B.5 ([10, Diagram (1.3)])**.** Let Υ be the linear isomorphism from $\mathcal{B}(B, \psi)$ to $B \otimes B$ given by

$$A \mapsto (\mathrm{id} \otimes A)m^*(1) =$$  .

Using Proposition A.183, we get $\Upsilon = (\mathrm{id} \otimes \mathrm{op}^{-1})\varsigma\Psi_{0,1}$; so $\Upsilon^{-1} = \Psi_{0,1}^{-1}\varsigma(\mathrm{id} \otimes \mathrm{op})$. So for $x, y \in B$, we get $\Upsilon(|x\rangle\langle y|) = \sigma_{-1}(y^*) \otimes x$ and $\Upsilon^{-1}(x \otimes y) = |y\rangle\langle \sigma_{-1}(x^*)|$.

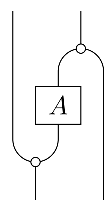
Lemma B.6. $\Upsilon^{-1}m^* = \mathrm{rmul}$.

Proof. Let $x, y, z \in B$, and let $m^*(x) = \sum_i \alpha_i \otimes \beta_i$ for tuples $(\alpha_i), (\beta_i)$ in B. Then we compute,

$$\begin{aligned}\big\langle \Upsilon^{-1}m^*(x)(y)\big|z\big\rangle &= \sum_i \big\langle \Upsilon^{-1}(\alpha_i \otimes \beta_i)(y)\big|z\big\rangle = \sum_i \langle |\beta_i\rangle\langle \sigma_{-1}(\alpha_i^*)|(y)|z\rangle \\ &= \sum_i \langle y|\sigma_{-1}(\alpha_i^*)\rangle\langle \beta_i|z\rangle = \sum_i \langle \alpha_i|y^*\rangle\langle\beta_i|z\rangle && \text{by A.70(v)} \\ &= \langle m^*(x)|y^*\otimes z\rangle = \langle x|y^*z\rangle = \langle yx|z\rangle && \text{by A.70(ii)} \\ &= \langle \mathrm{rmul}(x)(y)|z\rangle.\end{aligned}$$

Thus $\Upsilon \circ m^* = \mathrm{rmul}$. ∎

Lemma B.7. *Let* $A \in \mathcal{B}(\mathrm{B}, \psi)$*. We have*



$= (\mathrm{rmul} \otimes \mathrm{lmul})\Upsilon(A).$

In other words, $(\mathrm{id} \otimes m)(\mathrm{id} \otimes A \otimes \mathrm{id})(m^* \otimes \mathrm{id}) = (\mathrm{rmul} \otimes \mathrm{lmul})\Upsilon(A)$.

Proof. It suffices to show this for $A = |x\rangle\langle y|$ for $x, y \in B$. We compute,

$$\begin{aligned} &\langle (\mathrm{id}\otimes m)(\mathrm{id}\otimes |x\rangle\langle y| \otimes \mathrm{id})(m^*\otimes \mathrm{id})(a\otimes b)|c\otimes d\rangle \\ &= \sum_t \langle (\mathrm{id}\otimes m)(\mathrm{id}\otimes |x\rangle\langle y|\otimes \mathrm{id})(\alpha_t\otimes\beta_t\otimes b)|c\otimes d\rangle \\ &= \sum_t \langle \alpha_t \otimes \langle y|\beta_t\rangle xb|c\otimes d\rangle = \sum_t \langle \beta_t|y\rangle\langle\alpha_t|c\rangle\langle xb|d\rangle \\ &= \langle m^*(a)|c\otimes y\rangle\langle xb|d\rangle = \langle a|cy\rangle\langle xb|d\rangle \\ &= \langle a\sigma_{-1}(y^*)|c\rangle\langle xb|d\rangle = \langle a\sigma_{-1}(y^*)\otimes xb|c\otimes d\rangle \\ &= \langle (\mathrm{rmul}(\sigma_{-1}(y^*))\otimes \mathrm{lmul}(x))(a\otimes b)|c\otimes d\rangle \\ &= \langle (\mathrm{rmul}\otimes\mathrm{lmul})\Upsilon(|x\rangle\langle y|)(a\otimes b)|c\otimes d\rangle.\end{aligned}$$

Thus $(\mathrm{id}\otimes m)(\mathrm{id}\otimes A\otimes \mathrm{id})(m^*\otimes\mathrm{id}) = (\mathrm{rmul}\otimes\mathrm{lmul})\Upsilon(A)$ for any $A \in \mathcal{B}(B,\psi)$. ∎

Lemma B.8. $((\mathrm{rmul}\otimes\mathrm{lmul})\Upsilon(A))(a) = \Upsilon\circ(A\bullet\Upsilon^{-1}(a))$ *for all* $A\in\mathcal{B}(B,\psi)$ *and* $a \in B\otimes B$.

Proof. It suffices to show this for when $A = |x\rangle\langle y|$ and $a = \alpha\otimes\beta$ for $x, y, \alpha, \beta \in B$. We compute,

$$\begin{aligned}(\mathrm{rmul}\otimes\mathrm{lmul})\Upsilon(|x\rangle\langle y|)(\alpha\otimes\beta) &= (\mathrm{rmul}(y)^*\otimes\mathrm{lmul}(x))(\alpha\otimes\beta) = \alpha\sigma_{-1}(y^*)\otimes x\beta \\ &= \Upsilon\left(|x\beta\rangle\langle y\sigma_{-1}(\alpha^*)|\right) = \Upsilon\left(|x\rangle\langle y|\bullet|\beta\rangle\langle\sigma_{-1}(\alpha^*)|\right) \\ &= \Upsilon\left(|x\rangle\langle y|\bullet\Upsilon^{-1}(\alpha\otimes\beta)\right).\end{aligned}$$

∎

Proposition B.9. *There exists a linear isomorphism* Φ *from* $\mathcal{B}(B,\psi)$ *to* (B,B)*-bimodule maps* $\mathcal{B}(B\otimes B)$ *given by* $x\mapsto(\mathrm{rmul}\otimes\mathrm{lmul})(\Upsilon(x))$*, with inverse given by* $x\mapsto\Upsilon^{-1}(x(1))$*.*

Proof. Let $\Phi = (\mathrm{rmul} \otimes \mathrm{lmul})\Upsilon$. Then, clearly, Φ maps onto the set of (B, B)-bimodule maps $\mathcal{B}(B \otimes B)$. And let $\Phi^{-1} \colon x \mapsto \Upsilon^{-1}(x(1))$.

This is clearly a right-inverse: let $x \in \mathcal{B}(B \otimes B)$ be a (B, B)-bimodule map, and compute,

$$\Phi\Phi^{-1}(x) = (\mathrm{rmul} \otimes \mathrm{lmul})\Upsilon\Upsilon^{-1}(x(1)) = (\mathrm{rmul} \otimes \mathrm{lmul})(x(1)),$$

and using Lemma A.191(i), we see that this is exactly equal to x.

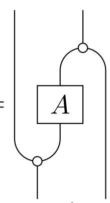
Now let $A \in \mathcal{B}(B)$. We then compute,

$$\Phi^{-1}\Phi(A) = \Upsilon^{-1}\left[\Upsilon\left[A \bullet \Upsilon^{-1}(\mathrm{id})\right]\right] = A \bullet \Upsilon^{-1}(\mathrm{id}) = A \bullet |1\rangle\langle 1| = A.$$

Thus Φ is also a left-inverse, and so is a linear isomorphism. ∎

Definition B.10 ([10, Diagram (1.5) - left diagram])**.** We define Φ as the linear isomorphism

$$\mathcal{B}(B, \psi) \cong \{x \in \mathcal{B}(B \otimes B) : x \text{ is a } (B, B)\text{-bimodule map}\},$$

given by $A \mapsto (\mathrm{rmul} \otimes \mathrm{lmul})\Upsilon(A) =$ 

, and its inverse given by $x \mapsto \Upsilon^{-1}(x(1))$. So for $x, y \in B$, we have $\Phi(|x\rangle\langle y|) = \mathrm{rmul}(y)^* \otimes \mathrm{lmul}(x)$.

Lemma B.11. *Let $A_1, A_2 \in \mathcal{B}(B, \psi)$. Then $\Phi(A_1)^* = \Phi(A_1{}^{\mathrm{r}})$ and $\Phi(A_1)\Phi(A_2) = \Phi(A_1 \bullet A_2)$. Moreover, (B, ψ, A) is a real quantum graph if and only if $\Phi(A)$ is a projection.*

Proof. It suffices to show these for when $A_1 = |a\rangle\langle b|$ and $A_2 = |c\rangle\langle d|$ for $a, b, c, d \in B$.

(i)

$$\begin{aligned}\Phi(|a\rangle\langle b|)^* &= \big(\mathrm{rmul}(b)^* \otimes \mathrm{lmul}(a)\big)^* = \mathrm{rmul}(\sigma_{-1}(b^*))^* \otimes \mathrm{lmul}(a^*)\\ &= \Phi(|a^*\rangle\langle\sigma_{-1}(b^*)|) = \Phi(|a\rangle\langle b|^{\mathrm{r}}).\end{aligned}$$

(ii)

$$\begin{aligned}\Phi(|a\rangle\langle b|)\Phi(|c\rangle\langle d|) &= (\mathrm{rmul}(b)^* \otimes \mathrm{lmul}(a))(\mathrm{rmul}(d)^* \otimes \mathrm{lmul}(c))\\ &= \mathrm{rmul}(d)\,\mathrm{rmul}(b)^* \otimes \mathrm{lmul}(a)\,\mathrm{lmul}(c)\\ &= \mathrm{rmul}(bd)^* \otimes \mathrm{lmul}(ac) = \Phi(|ac\rangle\langle bd|) = \Phi(|a\rangle\langle b| \bullet |c\rangle\langle d|).\end{aligned}$$

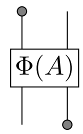
So, if $A \in \mathcal{B}(B, \psi)$ is a real Schur idempotent (i.e., (B, ψ, A) is a real quantum graph), then we get $\Phi(A)^* = \Phi(A^{\mathrm{r}}) = \Phi(A)$ and $\Phi(A)\Phi(A) = \Phi(A \bullet A) = \Phi(A)$. And so $\Phi(A)$ is a projection using Proposition A.169. ∎

In Matsuda [10], the inverse of Φ from (B, B)-bimodule maps on $\mathcal{B}(B \otimes B)$ to $\mathcal{B}(B)$ is given by $(\eta^* \otimes \mathrm{id})x(\mathrm{id} \otimes \eta)$. Let us call this map ϖ. We let $a \in B$ and compute,

$$\begin{aligned}\varpi(P \otimes Q)(a) &= (\eta^* \otimes \mathrm{id})(P \otimes Q)(a \otimes 1)\\ &= (\eta^* \otimes \mathrm{id})(P(a) \otimes Q(1))\\ &= \langle 1|P(a)\rangle Q(1)\\ &= |Q(1)\rangle\langle P^*(1)|(a) = Q|1\rangle\langle 1|P(a).\end{aligned}$$

The map ϖ is clearly a left-inverse of Φ as $\varpi\Phi(|x\rangle\langle y|) = \mathrm{lmul}(x)|1\rangle\langle 1|\mathrm{rmul}(y)^* = |x\rangle\langle y|$ for all $x, y \in B$. So, by the uniqueness of the inverse of Φ, we get $\Phi^{-1} = \varpi$.

Proposition B.12. *Given* $A \in \mathcal{B}(B, \psi)$*, we get* $(\eta^* \otimes \mathrm{id})\Phi(A)(\mathrm{id} \otimes \eta) = A$*, in other words,*

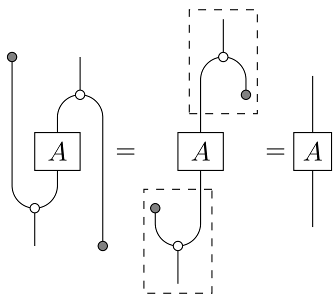


$$= A.$$

Proof. It suffices to show this for when $A = |x\rangle\langle y|$ for $x, y \in B$. We let $a \in B$ and compute,

$$\begin{aligned}
(\eta^* \otimes \mathrm{id})\Phi(|x\rangle\langle y|)(\mathrm{id} \otimes \eta)(a) &= (\eta^* \otimes \mathrm{id})(\mathrm{rmul}(y)^* \otimes \mathrm{lmul}(x))(a \otimes 1) \\
&= (\eta^*(\mathrm{rmul}(y)^*(a)) \otimes \mathrm{lmul}(x)(1)) \\
&= \eta^*(a\sigma_{-1}(y^*))x \\
&= \psi(y^* a)x = |x\rangle\langle y|(a).
\end{aligned}$$

With strings:



by (unit_id),(co_unit_id).

∎

The above tells us that there is another way (a pictorial way) to define the inverse of Φ. This is the definition that Matsuda [10, Diagram (1.5) - right diagram] chose.

However, if we did not know Φ is a linear isomorphism, then how do we go about showing that ϖ is a right-inverse of Φ? We begin with the following lemma.

Lemma B.13. *Let* $P \in \mathcal{B}(B, \psi)$*. Then* $P = \mathrm{rmul}(P(1))$ *if and only if* $\mathrm{symm}(P) = \mathrm{lmul}(P(1))$*.*

Proof. The key idea in this proof is that the inverse of symm is symm′.

$$\begin{aligned}
\mathrm{symm}(P) = \mathrm{lmul}(P(1)) &\Leftrightarrow P = \mathrm{symm}'(\mathrm{lmul}(P(1))) \\
&\Leftrightarrow P = \mathrm{lmul}(P(1))^{*\mathrm{r}} \\
&\Leftrightarrow P = \mathrm{lmul}(P(1)^*)^{\mathrm{r}} \\
&\Leftrightarrow P = \mathrm{rmul}(P(1)).
\end{aligned}$$

∎

Now it is easy to show that ϖ is a right-inverse of Φ. Let $P, Q \in \mathcal{B}(B, \psi)$ such that $P \otimes Q$ is an (B, B)-bimodule map. We assume P, Q are both non-zero, otherwise this is trivial. Then we get $P = \mathrm{rmul}(P(1))$ and $Q = \mathrm{lmul}(Q(1))$. By the above Lemma, we then get $\mathrm{symm}(P) = \mathrm{lmul}(P(1))$. And so,

$$\begin{aligned}
\Phi\varpi(P \otimes Q) = \Phi(|Q(1)\rangle\langle P^*(1)|) &= \mathrm{rmul}(\sigma_{-1}{P^*}^{\mathrm{r}}(1)) \otimes \mathrm{lmul}(Q(1)) \\
&= \mathrm{rmul}({P^{\mathrm{r}}}^*(1)) \otimes Q = \mathrm{rmul}(\mathrm{symm}(P)(1)) \otimes Q \\
&= \mathrm{rmul}(P(1)) \otimes Q = P \otimes Q.
\end{aligned}$$

B.III.1 Another formula for the adjacency matrix. Given a real quantum graph (B, A), we know $\Phi(A) \in \mathcal{B}(B \otimes B)$ is a projection, so then we let $U \subseteq B \otimes B$ be the projected subspace and let $(u_i)_i$ be an orthonormal basis of U.

$$\Phi(A) = \sum_i |u_i\rangle\langle u_i|.$$

And so,

$$A = \sum_{i,t} \langle u_i|1\rangle |b_{it}\rangle\langle \sigma_{-1}(a_{it}^*)|,$$

where we let each $u_i = \sum_{i,t} a_{it} \otimes b_{it}$.

B.IV Quantum graphs as positive maps

In this short section, we see how one can also define a quantum graph as a positive linear operator satisfying Schur idempotence.

Let $\mathcal{A}$ be a C*-algebra in this section and $\mathcal{H}$ be a Hilbert space.

We denote the orthogonal projection of $\mathcal{H}$ onto $U \subseteq \mathcal{H}$ by P_U.

Corollary B.14. *For all operators $T, S \in \mathcal{B}(\mathcal{H})$, if $TS = 0$, then $P_{\ker T} S = S$.*

Proof. Suppose $TS = 0$. Then we get $\operatorname{im} S \subseteq \ker T = \operatorname{im} P_{\ker T}$, and so Lemma A.153 tells us that we get $P_{\ker T} S = S$. ■

Lemma B.15. *A self-adjoint operator $x \in \mathcal{B}(\mathcal{H})$ can be written as $x = x_+ - x_-$ for some positive semi-definite operators $x_+, x_- \in \mathcal{B}(\mathcal{H})$ such that x_+, x_- both commute with x and $x_+ x_- = x_- x_+ = 0$.*

Proof. We let $\sqrt{x^2}$ be the unique positive square-root of x^2 (see Lemma A.23). We then let $x_+ = \frac{1}{2}(\sqrt{x^2} + x)$ and $x_- = \frac{1}{2}(\sqrt{x^2} - x)$. Then we clearly get $x = x_+ - x_-$.

It is clear that both x_+ and x_- are self-adjoint. We have both x_+ and x_- commute with x since $\sqrt{x^2}$ commutes with x. So then we get $x_+ x_- = 0$, and so $x_- x_+ = 0$. And by Corollary B.14, we get $P_{\ker x_+} x_- = x_-$.

Now $2P_{\ker x_+}\sqrt{x^2} = 2P_{\ker x_+} x_+ + 2P_{\ker x_+} x_- = 2P_{\ker x_+} x_- = 2x_- = \sqrt{x^2} - x$. So then $x = (1 - 2P_{\ker x_+})\sqrt{x^2}$ and $x_- = P_{\ker x_+}\sqrt{x^2}$. And so

$$x_+ = \frac{1}{2}(\sqrt{x^2} + x) = \frac{1}{2}(\sqrt{x^2} + (1 - 2P_{\ker x_+})\sqrt{x^2}) = (1 - P_{\ker x_+})\sqrt{x^2} = P_{(\ker x_+)^\perp}\sqrt{x^2}.$$

As x_+ and x_- are self-adjoint, we also get $x_- = \sqrt{x^2} P_{\ker x_+}$ and $x_+ = \sqrt{x^2} P_{(\ker x_+)^\perp}$. So then x_+ and x_- are products of commuting positive elements, and by Lemma A.26, this means both x_+ and x_- are positive. ■

Now we can state the needed result.

Proposition B.16. *Given $A \in \mathcal{B}(\mathcal{H})$, we get A is real if A is a positive map.*

Proof. Suppose A is a positive map.

Claim: if $x \in M_n$ is self-adjoint, then $A^{\mathrm{r}}(x) = A(x)$. $(*)$

Using Lemma B.15, we get positive semi-definite operators $x_+, x_- \in \mathcal{H}$ such that $x = x_+ - x_-$. So then we compute,

$$A^{\mathrm{r}}(x) = A^{\mathrm{r}}(x_+ - x_-) = A(x_+)^* - A(x_-)^* = A(x_+) - A(x_-).$$

The last equality follows from A being a positive map, so $A(x_+)$ is positive semi-definite.

Let $x \in \mathcal{H}$. Then we can write $x = a + ib$, where $a = \frac{1}{2}(x + x^*)$ and $b = \frac{1}{2i}(x - x^*)$. Clearly, both a and b are self-adjoint. So then by Claim (*), we get

$$A^{\mathrm{r}}(x) = A^{\mathrm{r}}(a) + iA^{\mathrm{r}}(b) = A(a) + iA(b) = A(x).$$

Thus A is real. ■

Lemma B.17. *Given a real Schur idempotent $A \in \mathcal{B}(B, \psi)$ (i.e., A is real and $A \bullet A = A$), we get A is a positive map (i.e., $0 \leq A(x^*x)$ for all $x \in B$).*

Proof. First, we have,

$$mm^* = \Phi(\mathrm{id}) = \Phi(\mathrm{id}^{\mathrm{r}}) = \Phi(\mathrm{id})^* = \sum_i \mathrm{rmul}(u_i) \otimes \mathrm{lmul}(u_i)^* = \sum_i \mathrm{rmul}(u_i) \otimes \mathrm{lmul}(u_i^*),$$

where $(u_i)_i$ is an orthonormal basis of B.

Let $x \in B$. Then using the above, we compute,

$$\begin{aligned} A(x^*x) &= (A \bullet A)m(x^* \otimes x) \\ &= \sum_i m(A \otimes A)(x^*u_i \otimes u_i^*x) \\ &= \sum_i A(x^*u_i)A(u_i^*x) = \sum_i A(u_i^*x)^*A(u_i^*x) \geq 0. \end{aligned}$$

Thus A is a positive map. ■

Theorem B.18 ([9, Proposition 2.23])**.** *Given $A \in \mathcal{B}(B, \psi)$ such that $A \bullet A = A$, we get A is real if and only if A is a positive map.*

Proof. This is done by combining Lemma B.17 and Proposition B.16. ■

The above result tells us that there is, yet, another equivalent definition for a quantum graph via positivity.

Theorem B.19. *In summary, for $A \in \mathcal{B}(B, \psi)$, we have the following are equivalent,*

1. *$A \bullet A = A$ and $A^r = A$,*
2. *$A \bullet A = A$ and A is a positive map,*
3. *$\Psi_{0,1/2}(A)$ is an orthogonal projection element in $B \otimes B^{\mathrm{op}}$,*
4. *$\Phi(A) = (\mathrm{rmul} \otimes \mathrm{lmul})(\mathrm{id} \otimes \mathrm{op}^{-1})\varsigma\Psi_{0,1}$ is an orthogonal projection (B, B)-bimodule map in $\mathcal{B}(B \otimes B)$,*
5. *there exists an orthonormal basis $((u_{i,j,p})_p)_{i,j}$ of $U \subseteq \bigoplus_{i,j} \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ such that*

$$(\mathrm{id} \otimes \top^{-1})\Psi_{0,1/2}(A) = \bigoplus_{i,j} \sum_p u_{ij,p} u_{ij,p}^*$$

6. *there exists an orthonormal basis $(u_i)_i$ of $U \subseteq B \otimes B$ such that*

$$\Phi(A) = \sum_i |u_i\rangle\langle u_i|$$

7. *For $B = M_n$, there exists an orthonormal basis $(u_i)_i$ of $U \subseteq M_n$ such that*

$$A = \sum_i \mathrm{lmul}(u_i Q)\mathrm{rmul}(Q u_i)^*.$$

■

B.V Number of edges

In this section, we discuss the number of edges and degree of a quantum graph.

Lemma B.20. *If $A \in \mathcal{B}(B, \psi)$ is real (i.e., star-preserving), then $\langle 1|A(1)\rangle \in \mathbb{R}$.*

Proof. $\langle 1|A(1)\rangle = \psi(A(1)) = \psi(A(1^*)^*) = \overline{\psi(A(1))} = \overline{\langle 1|A(1)\rangle}$. Thus $\langle 1|A(1)\rangle$ is real. ■

Lemma B.21. *If $A \in \mathcal{B}(B, \psi)$, then $\langle 1|A(1)\rangle = \langle 1|\Psi_{t,s}(A)\rangle$ for any $t, s \in \mathbb{R}$.*

Proof. Let $t, s \in \mathbb{R}$. Let $(\alpha_i), (\beta_i)$ be tuples in B such that $A = \sum_i |\alpha_i\rangle\langle\beta_i|$. Then,

$$\begin{aligned}
\langle 1|A(1)\rangle &= \sum_i \langle 1|\alpha_i\rangle\langle\beta_i|1\rangle \\
&= \sum_i \langle 1|\sigma_t(\alpha_i)\rangle\langle 1|\sigma_{-s}(\beta_i^*)\rangle \\
&= \sum_i \langle 1|\sigma_t(\alpha_i)\rangle\big\langle 1\big|\sigma_s(\beta_i)^{*\mathrm{op}}\big\rangle && \text{by A.69(ii)} \\
&= \langle 1|\Psi_{t,s}(A)\rangle.
\end{aligned}$$

Note that, in the second equality, we use the fact that the modular automorphism is an algebra automorphism (so $\sigma_t(1) = 1$) and that it is self-adjoint by Lemma A.69(iii). ■

Lemma B.22. *If $A \in \mathcal{B}(B, \psi)$, then $\langle 1|A(1)\rangle = \langle 1|\Phi(A)(1)\rangle$.*

Proof. Let $(\alpha_i), (\beta_i)$ be tuples in B such that $A = \sum_i |\alpha_i\rangle\langle\beta_i|$. Then,

$$\begin{aligned}
\langle 1|A(1)\rangle &= \sum_i \langle 1||\alpha_i\rangle\langle\beta_i|(1)\rangle \\
&= \sum_i \langle 1|\alpha_i\rangle\langle\beta_i|1\rangle \\
&= \sum_i \langle 1|\alpha_i\rangle\langle 1|\beta_i^*\rangle \\
&= \sum_i \langle 1|\mathrm{lmul}(\alpha_i)(1)\rangle\big\langle 1\big|\mathrm{rmul}(\beta_i)^*(1)\big\rangle \\
&= \sum_i \big\langle 1\big|(\mathrm{rmul}(\beta_i)^* \otimes \mathrm{lmul}(\alpha_i))(1)\big\rangle \\
&= \sum_i \langle 1|\Phi(|\alpha_i\rangle\langle\beta_i|)(1)\rangle = \langle 1|\Phi(A)(1)\rangle.
\end{aligned}$$

■

Proposition B.23. *Given a real quantum graph (B, ψ, A), we have $0 \leq \langle 1|A(1)\rangle \leq \|1\|^4$. This is the Hilbert space norm.*

Proof. By Theorem B.18, we know (B, ψ, A) is real if and only if A is a positive map. So then $0 \leq A(1)$ and so we let $A(1) = x^*x$ for some $x \in B$. Using the above lemma, we compute,

$$\begin{aligned} 0 \leq \langle x|x\rangle &= \langle 1|x^*x\rangle \\ &= \langle 1|A(1)\rangle = \langle 1|\Phi(A)(1)\rangle \\ &\leq \langle 1|(\mathrm{id} \otimes \mathrm{id})(1)\rangle \\ &= \|1\|^4 . \end{aligned}$$

■

Definition B.24. [9] A quantum graph (B, ψ, A) is said to be d-regular, for $d \in \mathbb{C}$, if it satisfies $A(1) = d1 = A^*(1)$. Here, d is defined as the *degree* of (B, ψ, A).

Lemma B.25. [9] *If (B, A) is a d-regular real quantum graph, then $0 \leq d \leq \|1\|^2$. Similarly to above, this is the Hilbert space norm.*

Proof. We can see that we have $\langle 1|A(1)\rangle / \|1\|^2 = d$. And so this follows from the above proposition. ■

B.V.1 Number of edges of real quantum graphs.

For classical graphs: Let G be the classical graph with a vertex set $V = \{v_1, \ldots, v_n\}$ and an edge set $E \subseteq V \times V$. Then the classical adjacency matrix $A \in M_n$ of G is given by $A_{ij} = 1$ if $(v_i, v_j) \in E$, and 0 otherwise. The number of classical edges in G is given by $\sum_{i,j} A_{ij}$. Identifying the classical adjacency matrix as $\mathcal{M}^{-1}(A) = \sum_{i,j} A_{ij} |e_i\rangle\langle e_j| \in \mathcal{B}(\mathbb{C}^n)$ (so that it is a real quantum adjacency matrix operator), then the trace of $\Psi_{0,1/2}(\mathcal{M}^{-1}(A)) = \sum_{i,j} A_{ij} e_i \otimes {e_j}^{*\mathrm{op}}$ is going to be the number of classical edges.

So then, in the tracial case, it makes sense to talk about the *number of edges* of real quantum graphs $(\mathrm{B}, \mathrm{Tr}, A)$, by letting it be the trace of the orthogonal projection $\Psi_{0,1/2}(A)$. (This corresponds to our definition of the number of edges of a quantum graph, see Proposition B.28.)

Definition B.26 ([6])**.** The *number of edges* of a quantum graph (B, ψ, A) is given by $\langle 1|A(1)\rangle$.

Corollary B.27. *Given any real quantum graph (B, ψ, A), the number of edges of (B, ψ, A) is 0 if and only if $A = 0$. Similarly, the number of edges of (B, ψ, A) is $\|1\|^4$ if and only if $A = |1\rangle\langle 1|$. Again, this is the Hilbert space norm.*

Proof. For $x, y \in B$, recall the KMS construction $\langle x|y\rangle_{\mathrm{KMS}} = \left\langle x\middle|\sigma_{-1/2}(y)\right\rangle$ from Section A.VII. And let A_{KMS} and $\Psi_{\mathrm{KMS},t,s}$ be the respective linear map and identification on this Hilbert space.

As (B, ψ, A) is a real quantum graph, we have $\Psi_{t,s}(A)$ is a projection (similarly, $\Psi_{\mathrm{KMS},t,s}(A_{\mathrm{KMS}})$ is a projection). So then,

$$\begin{aligned} \langle 1|A(1)\rangle = 0 &\Leftrightarrow \langle 1|A_{\mathrm{KMS}}(1)\rangle_{\mathrm{KMS}} = 0 \\ &\Leftrightarrow \langle 1|\Psi_{\mathrm{KMS},t,s}(A_{\mathrm{KMS}})\rangle_{\mathrm{KMS}} = 0 && \text{by B.21} \\ &\Leftrightarrow \eta^*(\Psi_{\mathrm{KMS},t,s}(A_{\mathrm{KMS}})^* \Psi_{\mathrm{KMS},t,s}(A_{\mathrm{KMS}})) = 0 \\ &\Leftrightarrow \Psi_{\mathrm{KMS},t,s}(A_{\mathrm{KMS}}) = 0 \Leftrightarrow A_{\mathrm{KMS}} = 0 \Leftrightarrow A = 0. \end{aligned}$$

Note that the reason we used the "KMS" construction is because of how we defined things. With our definition, the counit of $A \otimes A^{\mathrm{op}}$ is faithful when the inner product is constructed via KMS.

Now,

$$\begin{aligned}\langle 1|A(1)\rangle = \|1\|^4 &\Leftrightarrow \langle 1|(|1\rangle\langle 1| - A)(1)\rangle = 0 \\ &\Leftrightarrow A = |1\rangle\langle 1| && \text{by the above.}\end{aligned}$$

Note that the second equivalence follows from the fact that the operator $|1\rangle\langle 1| - A$ is a real quantum graph (Proposition B.39). ■

Proposition B.28. *For $\psi = \mathrm{Tr}$ (i.e., $\mathrm{Tr}\colon x \mapsto \sum_i \mathrm{Tr}(x_i)$), we have that the number of edges of a quantum graph $(\mathrm{B}, \mathrm{Tr}, A)$ is exactly the sum of the dimensions of the projected subspaces of the direct sum of orthogonal projections $(\mathcal{M}^{-1} \otimes \mathcal{M}^{-1}\top^{-1})\Psi_{0,1/2}(A) \in \bigoplus_{i,j} \mathcal{B}(\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j})$.*

Proof. Let $U_{i,j} \subseteq \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ for each $i, j \in [k]$ be the projected subspaces of the direct sum of the orthogonal projection. Then

$$\begin{aligned}\sum_{i,j} \dim(U_{i,j}) &= \mathrm{Tr}((\mathcal{M}^{-1} \otimes \mathcal{M}^{-1}\top^{-1})\Psi_{0,1/2}(A)) \\ &= (\mathrm{Tr} \otimes \mathrm{Tr} \circ \top^{-1})\Psi_{0,1/2}(A) \\ &= (\mathrm{Tr} \otimes \mathrm{Tr} \circ \mathrm{unop})\Psi_{0,1/2}(A) \\ &= \left\langle 1 \middle| \Psi_{0,1/2}(A) \right\rangle = \langle 1|A(1)\rangle.\end{aligned}$$

■

Proposition B.29. *Let our positive and faithful functional ψ on $B = \bigoplus_{i=1}^{\mathfrak{K}} M_{n_i}$ be a δ-form trace (i.e., $mm^* = \delta\,\mathrm{id}$ for some positive real number $\delta \in \mathbb{R}$ and $\psi = \mathrm{Tr}$). Then the trivial graph $(mm^*)^{-1}$ has $\mathfrak{K}$ edges.*

Proof. From the above proposition we know that the number of edges is exactly the sum of the dimensions of the projected subspaces. So then we compute,

$$\begin{aligned}&(\mathrm{Tr} \otimes \mathrm{Tr} \circ \mathrm{op}^{-1})\left(\Psi_{0,1/2}\left((mm^*)^{-1}\right)\right) \\ &= \delta^{-2}(\mathrm{Tr} \otimes \mathrm{Tr} \circ \mathrm{op}^{-1})(\Psi_{0,1/2}(\mathrm{id})) \\ &= \delta^{-2}\sum_{i,j,k}(\mathrm{Tr} \otimes \mathrm{Tr} \circ \mathrm{op}^{-1})\left(\Psi_{0,1/2}\left(\left|e_{k,ij}Q^{-1/2}\right\rangle\!\left\langle e_{k,ij}Q^{-1/2}\right|\right)\right) \\ &= \delta^{-2}\sum_{i,j,k}\mathrm{Tr}(e_{k,ij}Q^{-1/2})\,\mathrm{Tr}(e_{k,ji}Q^{-1/2}) \\ &= \delta^{-2}\sum_{i,j,k}Q_{k,ji}^{-1/2}Q_{k,ij}^{-1/2} \\ &= \delta^{-2}\sum_{k=1}^{\mathfrak{K}}\mathrm{Tr}(Q_k^{-1}) = \mathfrak{K}\end{aligned}$$

■

The number of classical edges of a trivial graph (i.e., a graph with all loops, so the adjacency matrix is the identity matrix) is equal to n. So each matrix block can be thought of as a "vertex" in the classical sense.

Proposition B.30. *Let* $(B = \bigoplus_{i=1}^{k} M_{n_i}, \psi, A)$ *be a real quantum graph. For each* $i, j \in [k]$*, let* $U_{ij} \subseteq \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ *be the projected subspaces of the direct sum of orthogonal projection operators (ref definition). Then, for any* $i, j \in [k]$*, if the graph is undirected (i.e.,* $A^* = A$*), then* $\dim U_{ij} = \dim U_{ji}$*.*

Proof. Let $\Psi_{0,1/2}(A) = \sum_t \alpha_t \otimes \beta_t^{\mathrm{op}}$. Then we compute,

$$\begin{aligned}
\dim U_{ij} &= (\mathrm{Tr} \otimes \mathrm{Tr} \circ \mathrm{unop})(\Psi_{0,1/2}(A)_{ij}) \\
&= (\mathrm{Tr} \otimes \mathrm{Tr} \circ \mathrm{unop})((\sigma_{-1/2} \otimes \sigma_{-1/2}^{\mathrm{op}})\varsigma(\Psi_{0,1/2}(A))_{ij}^*) \\
&= (\mathrm{Tr} \otimes \mathrm{Tr} \circ \mathrm{unop})(\varsigma(\Psi_{0,1/2}(A))_{ij}) \\
&= \sum_t \mathrm{Tr}(\beta_{t,i}) \,\mathrm{Tr}(\alpha_{t,j}) \\
&= (\mathrm{Tr} \otimes \mathrm{Tr} \circ \mathrm{unop})(\Psi_{0,1/2}(A)_{ji}) = \dim U_{ji}.
\end{aligned}$$

■

B.VI Some examples

B.VI.1 Complete and trivial quantum graph. We now consider two examples of a quantum adjacency matrix, namely, the complete quantum graph and the trivial quantum graph. We also see that subtracting any quantum adjacency matrix from the complete quantum graph will give us a nice correspondence between reflexive quantum graphs and irreflexive quantum graphs.

Proposition B.31. *The operator* $|1\rangle\langle 1|$ *is a reflexive, symmetric, real and self-adjoint quantum adjacency matrix.*

Proof. We check the axioms from Definition B.1. Using Proposition A.102, it suffices to check that it is a Schur idempotent, reflexive, symmetric, and self-adjoint operator.

self-adjoint We get $|1\rangle\langle 1|^* = |1\rangle\langle 1|$ by Proposition A.17(iii).

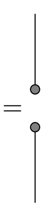
Schur idempotent We get $|1\rangle\langle 1| \bullet |1\rangle\langle 1| = |1\rangle\langle 1|$ by Proposition A.112.

symmetric We get $\mathrm{symm}(|1\rangle\langle 1|) = |\sigma_{-1}(1^*)\rangle\langle 1^*| = |1\rangle\langle 1|$ by Proposition A.91(i).

reflexive Lastly, we get $|1\rangle\langle 1| \bullet \mathrm{id} = \mathrm{lmul}(1) = \mathrm{id}$ by Proposition A.116(i).

■

Definition B.32 (Complete quantum graph [4, Definition 2.8])**.** The *complete quantum graph* is given by the reflexive, symmetric, real, and self-adjoint quantum graph $(B, \psi, |1\rangle\langle 1|)$. In strings, this is:

$$|1\rangle\langle 1| = \eta\eta^* =$$

.

Corollary B.33. *We have* $\mathrm{symm}(\mathrm{id}) = \mathrm{id}$.

Proof. This is exactly Lemma A.9. ■

We know $mm^* = \delta^2 \,\mathrm{id}$ from Proposition A.63. And so we get mm^* is invertible with inverse $\delta^{-2}\,\mathrm{id}$.

Proposition B.34. *We have that the operator* $(mm^*)^{-1} = \delta^{-2}\,\mathrm{id}$ *is a real, self-adjoint, symmetric and reflexive quantum adjacency matrix.*

Proof. We obviously have $\delta^{-2}\,\mathrm{id}$ is self-adjoint since $\delta^{-2} \in \mathbb{R}$. Idempotence and reflexivity is also clear as,

$$(mm^*)^{-1} \bullet (mm^*)^{-1} = \delta^{-4}\mathrm{id} \bullet \mathrm{id} = \delta^{-4} mm^* = (mm^*)^{-1},$$
$$(mm^*)^{-1} \bullet \mathrm{id} = \delta^{-2}\mathrm{id} \bullet \mathrm{id} = \delta^{-2} mm^* = \mathrm{id}\,.$$

Finally, by Corollary B.33, we have symmetry,

$$\mathrm{symm}\left((mm^*)^{-1}\right) = \delta^{-2}\,\mathrm{symm}(\mathrm{id}) = \delta^{-2}\,\mathrm{id} = (mm^*)^{-1}.$$

■

Definition B.35 (Trivial quantum graph [4, Definition 2.10])**.** The *trivial quantum graph* is given by the reflexive quantum graph $(B, \psi, (mm^*)^{-1})$.

B.VI.2 Classical finite graphs. We briefly discuss classical finite graphs at the start of Section B.V.1. A classical finite (un)directed graph is, of course, a quantum graph. Schur multiplication on $\mathcal{B}(\mathbb{C}^n)$ corresponds to entry-wise matrix multiplication (naïve matrix multiplication, also known as the Hadamard matrix product).

Lemma B.36. *Given* $x \in \mathbb{C}^n$*, we have* $m^*(x) = \sum_i x_i(e_i \otimes e_i)$*, where* $(e_i)_i$ *is the standard basis on* $\mathbb{C}^n$*.*

Proof. Left as an exercise. ■

Using the above lemma, it should be easy to see that for $x, y \in \mathcal{B}(\mathbb{C}^n)$ and the standard orthonormal basis $e = (e_i)_i$ on $\mathbb{C}^n$, we get $\mathcal{M}_e(x \bullet y)_{ij} = \mathcal{M}_e(x)_{ij}\mathcal{M}_e(y)_{ij}$.

This means for $x \in \mathcal{B}(\mathbb{C}^n)$, $x \bullet x = x$ if and only if $\mathcal{M}(x)_{ij}\mathcal{M}(x)_{ij} = \mathcal{M}(x)_{ij}$, which just means that all entries are either 0 or 1. So a quantum graph on $\mathbb{C}^n$ corresponds to a classical finite (un)directed graph.

All quantum graphs on $\mathcal{B}(\mathbb{C}^n)$ are real as $\mathcal{M}_e(x^{\mathrm{r}})_{ij} = \langle e_i | x^{\mathrm{r}}(e_j)\rangle = \overline{\mathcal{M}_e(x)_{ij}}$ for a quantum graph $x \in \mathcal{B}(\mathbb{C}^n)$, but $\mathcal{M}_e(x)_{ij}$ is either 0 or 1, so $\mathcal{M}_e(x^{\mathrm{r}})_{ij} = \mathcal{M}_e(x)_{ij}$.

A quantum graph on $\mathbb{C}^n$ being symmetric or self-adjoint both correspond to the classical finite graph being directed.

B.VI.3 Real quantum graphs. For a real quantum graph (B, ψ, A), we get

$$(\mathcal{M}^{-1} \otimes \mathcal{M}^{-1}\top^{-1})\Psi_{0,1/2}(A) \in \bigoplus_{i,j=1}^{k} \mathcal{B}(\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j})$$

is a direct sum of projection operators. Thus we can choose subspaces $U_{i,j} \subseteq \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ for each $i, j \in [k]$ as the projected subspaces of the orthogonal projection, and we can let $(u_{i,j,s})_s$ be an orthonormal basis of each $U_{i,j}$, such that each $u_{i,j,s} = \sum_t x_{i,j,s,t} \otimes y_{i,j,s,t} \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ so that we can write

$$A = \sum_{i,j,s,t,p} \left| \iota_i(x_{i,j,s,t} x^*_{i,j,s,p}) \right\rangle\!\!\left\langle \sigma_{-1/2}(\iota_j(\overline{y_{i,j,s,t} y^*_{i,j,s,p}})) \right|.$$

Example B.37. Let ψ be tracial with $Q = 1$.

Let $U_{1,1} = \operatorname{Span}(x \otimes y)$, $U_{1,2} = U_{2,1} = U_{2,2} = 0$, where $x, y \in \mathbb{C}^2$ such that $x^*x = 1 = y^*y$. Then $(M_2 \oplus M_2, A)$ is a real quantum graph such that

$$A = \left| \begin{pmatrix} xx^* & \\ & 0 \end{pmatrix} \right\rangle\!\!\left\langle \begin{pmatrix} \overline{yy^*} & \\ & 0 \end{pmatrix} \right|.$$

This is a single-edged graph as $\langle 1 | A(1) \rangle = 1$.

Now let $U_{1,1} = \operatorname{Span}(x \otimes y)$, $U_{1,2} = \operatorname{Span}(a \otimes b)$, $U_{2,1} = U_{2,2} = 0$, where $x, y, a, b \in \mathbb{C}^2$ such that $x^*x = y^*y = a^*a = b^*b = 1$. Then $(M_2 \oplus M_2, A)$ is a real quantum graph such that

$$A = \left| \begin{pmatrix} xx^* & \\ & 0 \end{pmatrix} \right\rangle\!\!\left\langle \begin{pmatrix} \overline{yy^*} & \\ & 0 \end{pmatrix} \right| + \left| \begin{pmatrix} aa^* & \\ & 0 \end{pmatrix} \right\rangle\!\!\left\langle \begin{pmatrix} 0 & \\ & \overline{bb^*} \end{pmatrix} \right|.$$

The number of edges is 2, as $\langle 1 | A(1) \rangle = 2$.

Another way to describe a real quantum graph (B, A) is via $\Phi(A)$, a (B, B)-bimodule orthogonal projection map in $\mathcal{B}(B \otimes B)$. So then let $U \subseteq B \otimes B$ be the projected subspace and (u_i) be an orthonormal basis of U. Then

$$A = \sum_{i,t} \langle 1 | u_i \rangle |b_{it}^* \rangle\!\langle a_{it}|,$$

where we let each $u_i = \sum_t a_{it} \otimes b_{it}$.

Example B.38. Let ψ be tracial with $Q = 1$.

Let $U = \operatorname{Span}(x \otimes y)$ where $x, y \in M_2 \oplus M_2$ such that $\langle x | x \rangle = \langle y | y \rangle = 1$. Then $(M_2 \oplus M_2, A, \operatorname{Tr})$ is a real quantum graph such that

$$A = \langle 1 | x \otimes y \rangle |y^* \rangle\!\langle x|.$$

Since $\Phi(A)$ is a (B, B)-bimodule map, using Lemma A.191(ii) the elements x and y need to also satisfy

$$\langle x | a \rangle \langle y | b \rangle x \otimes y = \langle x | 1 \rangle \langle y | 1 \rangle ax \otimes yb,$$

for all a, b.

B.VII (Ir)reflexive complements

Proposition B.39 ([4, Proposition 6.8])**.** *Let $A \in \mathcal{B}(B, \psi)$. Then, (B, ψ, A) is a (symmetric) ((ir)reflexive) quantum graph if and only if $(B, \psi, |1\rangle\!\langle 1| - A)$ is a (symmetric) (irreflexive, (resp. reflexive)) quantum graph.*

Proof. We check A is self-adjoint, idempotent and symmetric if and only if $|1\rangle\!\langle 1| - A$ is. We then check A satisfies reflexivity (respectively, irreflexivity) if and only if $|1\rangle\!\langle 1| - A$ satisfies irreflexivity (respectively, reflexivity). Note that by Definition B.32, we get $|1\rangle\!\langle 1|$ is a reflexive quantum adjacency matrix.

self-adjoint $|1\rangle\!\langle 1| - A^* = |1\rangle\!\langle 1|^* - A^* = (|1\rangle\!\langle 1| - A)^* = |1\rangle\!\langle 1| - A$ if and only if $A^* = A$.

idempotence From Lemma A.108, we know $(B, \bullet, 1 = \eta\eta^* = |1\rangle\!\langle 1|)$ is a ring, and so $|1\rangle\!\langle 1| - A$ is Schur idempotent by Lemma A.149.

symmetry We have the following equivalences,

$$|1\rangle\!\langle 1| - A \text{ satisfies symmetry} \Leftrightarrow \operatorname{symm}(|1\rangle\!\langle 1| - A) = |1\rangle\!\langle 1| - A$$

$$\begin{aligned}&\Leftrightarrow \operatorname{symm}(|1\rangle\langle 1|) - \operatorname{symm}(A) = |1\rangle\langle 1| - A\\ &\Leftrightarrow \operatorname{symm}(A) = A\\ &\Leftrightarrow A \text{ satisfies symmetry.}\end{aligned}$$

Note that the second last equivalence follows since $\operatorname{symm}(|1\rangle\langle 1|) = |1\rangle\langle 1|$ since it is a quantum adjacency matrix (Definition B.32).

(ir)reflexive We compute,

$$(|1\rangle\langle 1| - A) \bullet \mathrm{id} = |1\rangle\langle 1| \bullet \mathrm{id} - A \bullet \mathrm{id} = \mathrm{id} - A \bullet \mathrm{id}.$$

So then we have the following equivalences,

$$\begin{aligned}|1\rangle\langle 1| - A \text{ satisfies (ir)reflexivity} &\Leftrightarrow \mathrm{id} - A \bullet \mathrm{id} = \mathrm{id} \;\; (\text{resp., } \mathrm{id} - A \bullet \mathrm{id} = 0)\\ &\Leftrightarrow A \bullet \mathrm{id} = 0 \;(\text{resp., } A \bullet \mathrm{id} = \mathrm{id})\\ &\Leftrightarrow A \text{ satisfies irreflexivity (resp., reflexivity).}\end{aligned}$$

■

Proposition B.40 ([4, Proposition 6.7])**.** *Let $A \in \mathcal{B}(B, \psi)$. Denote E as the trivial graph, i.e., $E = (mm^*)^{-1}$. Then A is a reflexive quantum adjacency matrix if and only if $A - E$ is an irreflexive quantum adjacency matrix.*

Proof. Firstly, we know E is a reflexive quantum adjacency matrix, so $E^* = E$ and satisfies Schur idempotence (i.e., $E \bullet E = E$), symmetry (i.e., $\operatorname{symm}(E) = E$), and reflexivity (i.e., $E \bullet \mathrm{id} = \mathrm{id}$).

- $(A - E)^* = A - E \Leftrightarrow A^* - E^* = A - E \Leftrightarrow A^* - E = A - E \Leftrightarrow A^* = A$
- By linearity, we get $(A - E) \bullet \mathrm{id} = A \bullet \mathrm{id} - E \bullet \mathrm{id} = A \bullet \mathrm{id} - \mathrm{id}$. Thus, we have $(A - E) \bullet \mathrm{id} = 0$ if and only if $A \bullet \mathrm{id} = \mathrm{id}$.
- Again, by linearity, we get $\operatorname{symm}(A - E) = \operatorname{symm}(A) - \operatorname{symm}(E) = \operatorname{symm}(A) - E$. This means we have $\operatorname{symm}(A - E) = A - E$ if and only if we have $\operatorname{symm}(A) = A$.
- Finally, we only need to check idempotence.

  (⇒) Suppose A is a reflexive quantum adjacency matrix. Then, using Proposition A.120, we get,

$$\begin{aligned}(A - E) \bullet (A - E) &= A \bullet A - E \bullet A - A \bullet E + E \bullet E\\ &= A \bullet A - \delta^{-2}(A \bullet \mathrm{id} + \mathrm{id} \bullet A) + E\\ &= A - 2\delta^{-2}\,\mathrm{id} + E = A - E.\end{aligned}$$

  (⇐) Suppose $A - E$ is an irreflexive quantum adjacency matrix. Then, using Proposition A.120 again, we get

$$\begin{aligned}A \bullet A &= ((A - E) + E) \bullet ((A - E) + E)\\ &= (A - E) \bullet (A - E) + (A - E) \bullet E + E \bullet (A - E) + E \bullet E\\ &= A - E + \delta^{-2}((A - E) \bullet \mathrm{id} + \mathrm{id} \bullet (A - E)) + E = A - E + E = A.\end{aligned}$$

Thus A is a reflexive quantum adjacency matrix if and only if $A - (mm^*)^{-1}$ is an irreflexive quantum adjacency matrix. ■

Corollary B.41 ([4, Corollaries 6.9 & 6.10])**.** *Let $A \in \mathcal{B}(B,\psi)$. Then $|1\rangle\langle 1| - (mm^*)^{-1} - A$ is an irreflexive quantum adjacency matrix if and only if A is also an irreflexive quantum adjacency matrix.*
Analogously, $|1\rangle\langle 1| + (mm^*)^{-1} - A$ is a reflexive quantum adjacency matrix if and only if A is also a reflexive quantum adjacency matrix.

Proof. By Proposition B.39, we get $|1\rangle\langle 1| - ((mm^*)^{-1} + A)$ is an irreflexive quantum adjacency matrix if and only if $(mm^*)^{-1} + A$ is a reflexive quantum adjacency matrix. And by Proposition B.40, we get $(mm^*)^{-1} + A$ is a reflexive quantum adjacency matrix if and only if $A = (mm^*)^{-1} + A - (mm^*)^{-1}$ is an irreflexive quantum adjacency matrix. Thus $|1\rangle\langle 1| - (mm^*)^{-1} - A$ is an irreflexive quantum adjacency matrix if and only if A is.

Analogously, by Proposition B.39 we get $|1\rangle\langle 1| - (A - (mm^*)^{-1})$ is reflexive if and only if $A - (mm^*)^{-1}$ is irreflexive, which is true if and only if A is reflexive by Proposition B.40. ■

Definition B.42. We define the *irreflexive complement* of $A \in \mathcal{B}(B,\psi)$ to be given by $|1\rangle\langle 1| - (mm^*)^{-1} - A$.
And we define the *reflexive complement* of $A \in \mathcal{B}(B,\psi)$ to be given by $|1\rangle\langle 1| + (mm^*)^{-1} - A$.

We denote the irreflexive complement of $A \in \mathcal{B}(B,\psi)$ by $A^{\complement_i}$. Clearly, taking the irreflexive complement of A twice gives us exactly A as seen by the following,

$$A^{\complement_i\complement_i} = |1\rangle\langle 1| - (mm^*)^{-1} - (|1\rangle\langle 1| - (mm^*)^{-1} - A) = A.$$

And for any $A_1, A_2 \in \mathcal{B}(B,\psi)$, we get $A_1^{\complement_i} = A_2^{\complement_i}$ if and only if $A_1 = A_2$.

We denote the reflexive complement of $A \in \mathcal{B}(B,\psi)$ by $A^{\complement_r}$. Analogously to the above, taking the reflexive complement of A twice gives us exactly A. And for any $A_1, A_2 \in \mathcal{B}(B,\psi)$, we get $A_1^{\complement_r} = A_2^{\complement_r}$ if and only if $A_1 = A_2$.

One can then easily notice that the reflexive complement of the complete graph is the trivial graph, i.e., $|1\rangle\langle 1|^{\complement_r} = (mm^*)^{-1}$; and similarly $\left((mm^*)^{-1}\right)^{\complement_r} = |1\rangle\langle 1|$.

B.VIII Isomorphisms

Definition B.43 ([10, Definition 3.1])**.** Let B_1, B_2 be finite-dimensional C*-algebras. Let $A_1 \in \mathcal{B}(B_1)$ and $A_2 \in \mathcal{B}(B_2)$ such that (B_1, A_1) and (B_2, A_2) are quantum graphs. Then a *graph homomorphism* from (B_1, A_1) to (B_2, A_2) is defined by a *-algebra homomorphism $f\colon B_1 \to B_2$ such that $A_2 \bullet (fA_1f^*) = fA_1f^*$.

Remark B.44. If f is an isometric *-isomorphism $B_1 \cong B_2$ such that $fA_1 = A_2f$, then it is a bijective graph homomorphism since $A_2 \bullet (fA_1f^*) = A_2 \bullet A_2 = A_2$. ♢

Proposition B.45. *If $A \in \mathcal{B}(B,\psi)$ and f is an isometric *-automorphism on B, then, (B,ψ,A) is a(n) (self-adjoint) (real) (symmetric) ((ir)-reflexive) quantum graph if and only if $(B,\psi,f^{-1}Af)$ is. Moreover, the number of edges are the same.*

Proof. We have $\left(f^{-1}Af\right) \bullet \left(f^{-1}Af\right) = f^{-1}(A \bullet A)f$ from Corollary A.121. And so we get $f^{-1}(A \bullet A)f = f^{-1}Af$ if and only if $A \bullet A = A$. This means (B,ψ,A) is a quantum graph if and only if $(B,\psi,f^{-1}Af)$ is.

real We already know $f^{-1}Af$ is real if and only if A is from Lemma A.142.

symmetric We use Lemma A.104 to get A is symmetric if and only if $f^{-1}Af$ is.

self-adjoint We get $f^{-1}Af$ is self-adjoint if and only if A is self-adjoint, since,

$$f^{-1}A^*f = f^*A^*(f^{-1})^* = (f^{-1}Af)^* = f^{-1}Af \Leftrightarrow A^* = A.$$

The first equality follows from f being isometric (Lemma A.137).

(ir)reflexive We have $(f^{-1}Af) \bullet \mathrm{id} = (f^{-1}Af) \bullet (f^{-1}\,\mathrm{id}\, f) = f^{-1}(A \bullet \mathrm{id})f$ from Corollary A.121. So then $f^{-1}(A \bullet \mathrm{id})f = \mathrm{id}$ if and only if $A \bullet \mathrm{id} = ff^{-1} = \mathrm{id}$. Similarly, we get $f^{-1}(A \bullet \mathrm{id})f = 0$ if and only if $A \bullet \mathrm{id} = 0$.

Thus it is clear that we get $f^{-1}Af$ is a(n) (self-adjoint) (real) (symmetric) ((ir)reflexive) quantum adjacency matrix if and only if A is.

Finally, the number of edges are clearly equal: $\langle 1 | f^{-1}Af(1) \rangle = \langle 1 | f^{-1}A(1) \rangle = \langle 1 | A(1) \rangle$, where we used the fact that our automorphism is unital in the first equality, and that our automorphism is isometric in the second. ■

We can now define what it means for two quantum graphs on B to be *isomorphic* to each other. In particular, $(B, \psi, A_1) \cong (B, \psi, A_2)$ when there exists an isometric *-automorphism f on B such that $fA_2 = A_1 f$.

Definition B.46. We say a quantum graph (B, ψ, A_1) is *isomorphic to* a quantum graph (B, ψ, A_2) if there exists an isometric *-automorphism f on B such that $fA_1 = A_2 f$, in other words, the following diagram commutes,

$$\begin{array}{ccc} B & \xrightarrow{f} & B \\ {\scriptstyle A_1}\downarrow & & \downarrow{\scriptstyle A_2} \\ B & \xrightarrow[f]{} & B \end{array}$$

We denote this by $(B, \psi, A_1) \cong (B, \psi, A_2)$.

B.IX Quantum isomorphisms

Everything in this section is only used in this section. Quantum isomorphisms are not mentioned or used outside of this section.

Let B_1, B_2 be finite-dimensional C*-algebras with faithful and positive linear functionals ψ_1, ψ_2 such that they are, respectively, of δ_1 and δ_2 forms, i.e., $m_1 m_1^* = \delta_1^2\, \mathrm{id}$ and $m_2 m_2^* = \delta_2^2\, \mathrm{id}$, for some $0 < \delta_1, \delta_2$.

Definition B.47 (quantum function [9, Def 2.29], [12, Def 3.11, 4.3])**.** We say (H, P) is a *quantum function* for a finite-dimensional Hilbert space H and a linear map $P\colon (B_1 \otimes H) \to (H \otimes B_2)$ when it satisfies the following:

1. $P(\eta \otimes \mathrm{id}) = (\mathrm{id} \otimes \eta)$,
2. $(\mathrm{id} \otimes m)(P \otimes \mathrm{id})(\mathrm{id} \otimes P) = P(m \otimes \mathrm{id})$,
3. $(\eta^* m \otimes \mathrm{id})(\mathrm{id} \otimes P^* \otimes \mathrm{id})(\mathrm{id} \otimes m^* \eta) = P$,
   equivalently, $P^* = (\mathrm{id} \otimes \eta^* m)(\mathrm{id} \otimes P \otimes \mathrm{id})(m^* \eta \otimes \mathrm{id})$.

We say a quantum function (H, P) is a *quantum bijection* if

1. $(\mathrm{id} \otimes \eta^*)P = (\eta^* \otimes \mathrm{id})$,
2. $(P \otimes \mathrm{id})(\mathrm{id} \otimes P)(m^* \otimes \mathrm{id}) = (\mathrm{id} \otimes m^*)P$.

Lemma B.48. *A quantum function* (H, P) *satisfies* $(\mathrm{id} \otimes \eta^*)P = (\eta^* \otimes \mathrm{id})$ *if and only if* $P^*P = \mathrm{id}$.

Proof.

(⇒) Suppose $(\mathrm{id}\otimes\eta^*)P = (\eta^*\otimes\mathrm{id})$. Then

$$\begin{aligned}
P^*P &= (\mathrm{id}\otimes\eta^* m)(\mathrm{id}\otimes P\otimes\mathrm{id})(m^*\eta\otimes\mathrm{id})P\\
&= (\mathrm{id}\otimes\eta^* m)(\mathrm{id}\otimes P\otimes\mathrm{id})(\mathrm{id}\otimes P)(m^*\eta\otimes\mathrm{id})\\
&= (\mathrm{id}\otimes(\mathrm{id}\otimes\eta^*)(\mathrm{id}\otimes m)(P\otimes\mathrm{id})(\mathrm{id}\otimes P))(m^*\eta\otimes\mathrm{id})\\
&= (\mathrm{id}\otimes(\mathrm{id}\otimes\eta^*)P(m\otimes\mathrm{id}))(m^*\eta\otimes\mathrm{id})\\
&= (\mathrm{id}\otimes(\eta^*\otimes\mathrm{id})(m\otimes\mathrm{id}))(m^*\eta\otimes\mathrm{id})\\
&= ((\mathrm{id}\otimes\eta^* m)(m^*\eta\otimes\mathrm{id})\otimes\mathrm{id})\\
&= (\mathrm{id}\otimes\mathrm{id}) && \text{by A.9}\\
&= \mathrm{id}
\end{aligned}$$

(⇐) Suppose $P^*P = \mathrm{id}$. Then we have,

$$(\mathrm{id}\otimes\eta^*)P = (P^*(\mathrm{id}\otimes\eta))^* = (P^*P(\eta\otimes\mathrm{id}))^* = (\eta\otimes\mathrm{id})^* = (\eta^*\otimes\mathrm{id}).$$

■

Lemma B.49. *A quantum function (H,P) such that $(P\otimes\mathrm{id})(\mathrm{id}\otimes P)(m^*\otimes\mathrm{id}) = (\mathrm{id}\otimes m^*)P$ has its adjoint as its right-inverse, i.e., $PP^* = \mathrm{id}$.*

Proof.

$$\begin{aligned}
PP^* &= P(\mathrm{id}\otimes\eta^* m)(\mathrm{id}\otimes P\otimes\mathrm{id})(m^*\eta\otimes\mathrm{id})\\
&= (\mathrm{id}\otimes\eta^* m)(P\otimes\mathrm{id})(\mathrm{id}\otimes P\otimes\mathrm{id})(m^*\eta\otimes\mathrm{id})\\
&= (\mathrm{id}\otimes\eta^* m)((P\otimes\mathrm{id})(\mathrm{id}\otimes P)(m^*\otimes\mathrm{id})(\eta\otimes\mathrm{id})\otimes\mathrm{id})\\
&= (\mathrm{id}\otimes\eta^* m)((\mathrm{id}\otimes m^*)P(\eta\otimes\mathrm{id})\otimes\mathrm{id})\\
&= (\mathrm{id}\otimes\eta^* m)((\mathrm{id}\otimes m^*)(\mathrm{id}\otimes\eta)\otimes\mathrm{id})\\
&= (\mathrm{id}\otimes(\mathrm{id}\otimes\eta^* m)(m^*\eta\otimes\mathrm{id}))\\
&= (\mathrm{id}\otimes\mathrm{id}) = \mathrm{id}\,.
\end{aligned}$$

■

Proposition B.50 ([9, Lemma 2.34])**.** *A quantum function (H,P) is quantum bijective if and only if its adjoint is its inverse (i.e., $PP^* = \mathrm{id}$ and $P^*P = \mathrm{id}$).*

Proof. Using the above two lemmas, it remains to show that if we have P^* is the inverse of P, then it satisfies $(P\otimes\mathrm{id})(\mathrm{id}\otimes P)(m^*\otimes\mathrm{id}) = (\mathrm{id}\otimes m^*)P$.

Suppose P^* is the inverse of P. Then we compute,

$$\begin{aligned}
(P\otimes\mathrm{id})(\mathrm{id}\otimes P)(m^*\otimes\mathrm{id}) &= ((m\otimes\mathrm{id})(\mathrm{id}\otimes P^*)(P^*\otimes\mathrm{id}))^*\\
&= (P^*P(m\otimes\mathrm{id})(\mathrm{id}\otimes P^*)(P^*\otimes\mathrm{id}))^*\\
&= (P^*(\mathrm{id}\otimes m)(P\otimes\mathrm{id})(\mathrm{id}\otimes P)(\mathrm{id}\otimes P^*)(P^*\otimes\mathrm{id}))^*\\
&= (P^*(\mathrm{id}\otimes m))^* = (\mathrm{id}\otimes m^*)P.
\end{aligned}$$

Thus (H,P) is quantum bijective if and only if P^* is its inverse. ■

Definition B.51 (quantum isomorphism of quantum graphs [9, Def 2.35], [12, Def 5.11])**.** We say quantum graphs (B_1, ψ_1, A_1) and (B_2, ψ_2, A_2) are *quantum isomorphic* if there exists a quantum bijective function (H, P) with $P\colon (B_1 \otimes H) \to (H \otimes B_2)$ and with H being a finite-dimensional Hilbert space, such that $P(A_1 \otimes \mathrm{id}) = (\mathrm{id} \otimes A_2)P$. In other words, $P(A_1 \otimes \mathrm{id})P^* = (\mathrm{id} \otimes A_2)$. Using Proposition B.50, P being quantum bijective means it is an isometry (in other words, $PP^* = \mathrm{id}$ and $P^*P = \mathrm{id}$).

Proposition B.52. *For any finite-dimensional Hilbert space H and quantum bijective function (H, P) with $P\colon (B_1 \otimes H) \to (H \otimes B_2)$, we get*

1. $P((mm^*)^{-1} \otimes \mathrm{id})P^* = \dfrac{\delta_2^2}{\delta_1^2}(\mathrm{id} \otimes (mm^*)^{-1})$,
   this says that the trivial quantum graph in B_1 is quantum isomorphic to a scalar factor of the trivial quantum graph in B_2.
2. $P(|1\rangle\langle 1| \otimes \mathrm{id})P^* = (\mathrm{id} \otimes |1\rangle\langle 1|)$,
   this says that the complete quantum graph in B_1 is quantum isomorphic to the complete quantum graph in B_2.

Proof. 1.

$$P((mm^*)^{-1} \otimes \mathrm{id})P^* = \delta_1^{-2} PP^* = \delta_1^{-2}\,\mathrm{id} = \delta_1^{-2}\delta_2^2(\mathrm{id} \otimes (mm^*)^{-1}).$$

2.

$$\begin{aligned} P(|1\rangle\langle 1| \otimes \mathrm{id})P^* &= P(\eta\eta^* \otimes \mathrm{id})P^* \\ &= (P(\eta \otimes \mathrm{id}))(P(\eta \otimes \mathrm{id}))^* \\ &= (\mathrm{id} \otimes \eta)(\mathrm{id} \otimes \eta)^* \\ &= (\mathrm{id} \otimes \eta\eta^*) = (\mathrm{id} \otimes |1\rangle\langle 1|). \end{aligned}$$

■

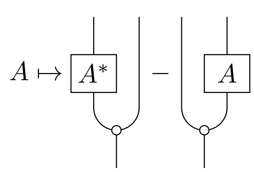
B.X Graph gradient and degrees

In this section we summarize the notion of graph gradients and degrees that appear in [10, Section 2].

Definition B.53 ([10, Definition 2.1])**.** Let $\nabla\colon \mathcal{B}(B, \psi) \to (B \to B \otimes B)$ be the *graph gradient*, given by

$$A \mapsto (A^* \otimes \mathrm{id} - \mathrm{id} \otimes A)m^*.$$

In other words,



Lemma B.54 ([10, Proposition 2.4(1)])**.** *Let $A \in \mathcal{B}(B, \psi)$. Then*

$$\nabla(A) = \Phi(A^r)(\eta \otimes \mathrm{id}) - \Phi(A)(\mathrm{id} \otimes \eta).$$

Proof. It suffices to show this for $A = |x\rangle\langle y|$ for $x, y \in B$.

Let $a, b, c \in B$ and $m^*(a) = \sum_i \alpha_i \otimes \beta_i$ for some tuples $(\alpha_i), (\beta_i)$ in B. So then we compute,

$$\langle \nabla(|x\rangle\langle y|)(a) | b \otimes c\rangle = \big\langle (|x\rangle\langle y|^* \otimes \mathrm{id} - \mathrm{id} \otimes |x\rangle\langle y|)m^*(a) \big| b \otimes c \big\rangle$$

$$
\begin{aligned}
&= \sum_i \langle (|y\rangle\langle x| \otimes \mathrm{id} - \mathrm{id} \otimes |x\rangle\langle y|)(\alpha_i \otimes \beta_i) | b \otimes c\rangle \\
&= \sum_i \langle |y\rangle\langle x|(\alpha_i) \otimes \beta_i | b \otimes c\rangle - \langle \alpha_i \otimes |x\rangle\langle y|(\beta_i) | b \otimes c\rangle \\
&= \sum_i \langle \alpha_i | x\rangle \langle y | b\rangle \langle \beta_i | c\rangle - \langle \beta_i | y\rangle \langle \alpha_i | b\rangle \langle x | c\rangle \\
&= \sum_i \langle \alpha_i \otimes \beta_i | x \otimes c\rangle \langle y | b\rangle - \langle \alpha_i \otimes \beta_i | b \otimes y\rangle \langle x | c\rangle \\
&= \langle m^*(a) | x \otimes c\rangle \langle y | b\rangle - \langle m^*(a) | b \otimes y\rangle \langle x | c\rangle \\
&= \langle a | xc\rangle \langle y | b\rangle - \langle a | by\rangle \langle x | c\rangle \\
&= \langle x^* a | c\rangle \langle y | b\rangle - \langle a\sigma_{-1}(y^*) | b\rangle \langle x | c\rangle \\
&= \langle y \otimes x^* a - a\sigma_{-1}(y^*) \otimes x | b \otimes c\rangle \\
&= \big\langle (\mathrm{rmul}(y) \otimes \mathrm{lmul}(x)^*)(1 \otimes a) - (\mathrm{rmul}(y)^* \otimes \mathrm{lmul}(x))(a \otimes 1) \big| b \otimes c \big\rangle \\
&= \big\langle (\Phi(|x\rangle\langle y|)^*(\eta \otimes \mathrm{id})\tau^{-1} - \Phi(|x\rangle\langle y|)(\mathrm{id} \otimes \eta)\varkappa^{-1}\tau^{-1})(a) \big| b \otimes c \big\rangle.
\end{aligned}
$$

Thus $\nabla(A) = \Phi(A^{\mathrm{r}})(\eta \otimes \mathrm{id}) - \Phi(A)(\mathrm{id} \otimes \eta)$ for any $A \in \mathcal{B}(B, \psi)$. ■

If A is a real Schur idempotent, then we know $\Phi(A)$ is an orthogonal projection. And so, using the above, we get $\Phi(A)\nabla(A) = \nabla(A)$, in other words, the range of $\nabla(A)$ is contained in the subspace $\Phi(A)$ projects onto.

Lemma B.55 ([10, Proposition 2.3])**.** *Let $A \in \mathcal{B}(B, \psi)$. Then*

$$\Upsilon^{-1}(\nabla(A)(x)) = \mathrm{rmul}(x) A^r - A\,\mathrm{rmul}(x).$$

Proof. It suffices to show this for $A = |a\rangle\langle b|$ for $a, b \in B$. We compute,

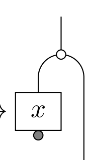
$$
\begin{aligned}
\Upsilon^{-1}(\nabla(|a\rangle\langle b|)(x)) &= \Upsilon^{-1}\left(\Phi(|a\rangle\langle b|^{\mathrm{r}})(1 \otimes x) - \Phi(|a\rangle\langle b|)(x \otimes 1)\right) \\
&= \Upsilon^{-1}\left(\Phi(|a^*\rangle\langle \sigma_{-1}(b^*)|)(1 \otimes x) - \Phi(|a\rangle\langle b|)(x \otimes 1)\right) \\
&= \Upsilon^{-1}(\mathrm{rmul}(\sigma_{-1}(b^*))^*(1) \otimes \mathrm{lmul}(a^*)(x) - \mathrm{rmul}(b)^*(x) \otimes \mathrm{lmul}(a)(1)) \\
&= \Upsilon^{-1}(b \otimes a^* x - x\sigma_{-1}(b^*) \otimes a) \\
&= |a^* x\rangle\langle \sigma_{-1}(b^*)| - |a\rangle\langle b\sigma_{-1}(x^*)| \\
&= |\mathrm{rmul}(x)(a^*)\rangle\langle \sigma_{-1}(b^*)| - |a\rangle\langle \mathrm{rmul}(\sigma_{-1}(x^*))(b)| \\
&= |\mathrm{rmul}(x)(a^*)\rangle\langle \sigma_{-1}(b^*)| - \big|a\big\rangle\big\langle \mathrm{rmul}(x)^*(b)\big| \\
&= \mathrm{rmul}(x)|a\rangle\langle b|^{\mathrm{r}} - |a\rangle\langle b|\,\mathrm{rmul}(x).
\end{aligned}
$$

Thus $\Upsilon^{-1}(\nabla(A)(x)) = \mathrm{rmul}(x)A^{\mathrm{r}} - A\,\mathrm{rmul}(x)$ for any $A \in \mathcal{B}(B, \psi)$. ■

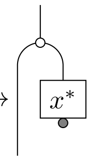
If A is real, then, by the above, we get $\Upsilon^{-1}(\nabla(A)(x)) = \mathrm{rmul}(x)A - A\,\mathrm{rmul}(x)$.

Definition B.56 ([10, Definition 2.5])**.**

- Let D_{in} be the linear map $\mathcal{B}(\mathcal{B}(B, \psi))$ given by $x \mapsto m(x \otimes \mathrm{id})(\eta \otimes \mathrm{id})$. In other words,

$x \mapsto$ 

- Let D_{out} be the anti-linear map $\mathcal{B}(\mathcal{B}(B,\psi))$ given by $x \mapsto m(\text{id} \otimes x^*)(\text{id} \otimes \eta)$. In other words,

$x \mapsto$ 

Corollary B.57. *Given $x \in \mathcal{B}(B,\psi)$, we get $D_{\text{in}}(x) = \text{lmul}(x(1))$ and $D_{\text{out}}(x) = \text{rmul}(x^*(1))$.*

Proof. Let $a \in B$ and compute,

$$D_{\text{in}}(x)(a) = m(x \otimes \text{id})(\eta \otimes \text{id})(a) = m(x \otimes \text{id})(1 \otimes a) = x(1)a = \text{lmul}(x(1))(a).$$

Thus $D_{\text{in}}(x) = \text{lmul}(x(1))$. We let $a \in B$ again and compute,

$$D_{\text{out}}(x)(a) = m(\text{id} \otimes x^*)(\text{id} \otimes \eta)(a) = m(\text{id} \otimes x^*)(a \otimes 1) = ax^*(1) = \text{rmul}(x^*(1))(a).$$

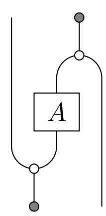
Thus $D_{\text{out}}(x) = \text{rmul}(x^*(1))$, as desired. ∎

Proposition B.58. *Given $x \in \mathcal{B}(B,\psi)$, if $\text{symm}(x) = x$, then $D_{\text{in}}(x)^r = D_{\text{out}}(x)$.*

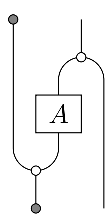
Proof. Firstly, $D_{\text{in}}(x)^{\text{r}} = \text{lmul}(x(1))^{\text{r}} = \text{rmul}(x(1)^*) = \text{rmul}(x^{\text{r}}(1))$. So then using Proposition A.99, we have $x^* = x^{\text{r}}$, and so the result then follows. ∎

Lemma B.59. *Let $A \in \mathcal{B}(B,\psi)$. Then,*

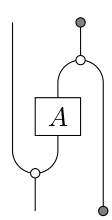
(i) $(\text{id} \otimes \eta^*)\Phi(A)(\eta \otimes \text{id}) = A^{r*}$,

 $= A^{r*}$.

(ii) $(\eta^* \otimes \text{id})\Phi(A)(\eta \otimes \text{id}) = D_{\text{in}}(A)$,

 $= D_{\text{in}}(A)$.

(iii) $(\text{id} \otimes \eta^*)\Phi(A)(\text{id} \otimes \eta) = D_{\text{out}}(A^r)$.

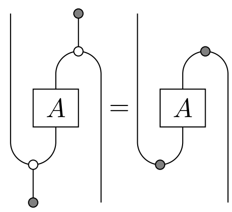
 $= D_{\text{out}}(A^r)$.

Proof. It suffices to show this for when $A = |x\rangle\langle y|$ for $x, y \in B$.

(i) Let $a \in B$ and compute,

$$\begin{aligned}(\mathrm{id}\otimes\eta^*)\Phi(|x\rangle\!\langle y|)(\eta\otimes\mathrm{id})(a) &= (\mathrm{id}\otimes\eta^*)(\mathrm{rmul}(y)^*\otimes\mathrm{lmul}(x))(1\otimes a)\\ &= \big(\mathrm{rmul}(y)^*(1)\otimes\eta^*(\mathrm{lmul}(x)(a))\big)\\ &= \eta^*(xa)\sigma_{-1}(y^*) = \psi(x^{**}a)\sigma_{-1}(y^*)\\ &= |\sigma_{-1}(y^*)\rangle\!\langle x^*|(a) = |x^*\rangle\!\langle\sigma_{-1}(y^*)|^*(a) = |x\rangle\!\langle y|^{\mathrm{r}*}(a).\end{aligned}$$

With strings:

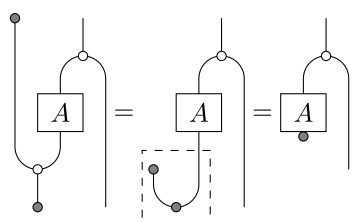


$= \mathrm{symm}(A) = A^{\mathrm{r}*}$ by A.92.

(ii) Let $a \in B$ and compute,

$$\begin{aligned}(\eta^*\otimes\mathrm{id})\Phi(|x\rangle\!\langle y|)(\eta\otimes\mathrm{id})(a) &= (\eta^*\otimes\mathrm{id})(\mathrm{rmul}(y)^*\otimes\mathrm{lmul}(x))(1\otimes a)\\ &= \eta^*(\mathrm{rmul}(y)^*(1))\,\mathrm{lmul}(x)(a)\\ &= \psi(\sigma_{-1}(y^*))xa\\ &= \psi(y^*)xa = \mathrm{lmul}(|x\rangle\!\langle y|(1))(a)\\ &= D_{\mathrm{in}}(|x\rangle\!\langle y|)(a).\end{aligned}$$

With strings:

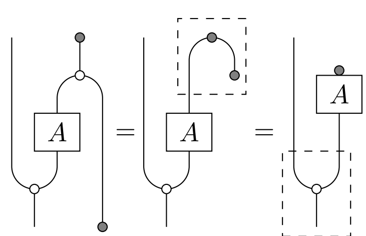


by A.6.

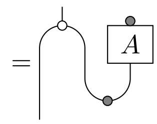
(iii) Let $a \in B$ and compute,

$$\begin{aligned}(\mathrm{id}\otimes\eta^*)\Phi(|x\rangle\!\langle y|)(\mathrm{id}\otimes\eta)(a) &= (\mathrm{id}\otimes\eta^*)(\mathrm{rmul}(y)^*\otimes\mathrm{lmul}(x))(a\otimes 1)\\ &= \eta^*(\mathrm{lmul}(x)(1))\mathrm{rmul}(y)^*(a) = \psi(x)a\sigma_{-1}(y^*)\\ &= \mathrm{rmul}(\psi(x^{**})\sigma_{-1}(y^*))(a)\\ &= \mathrm{rmul}(|\sigma_{-1}(y^*)\rangle\!\langle x^*|(1))(a)\\ &= \mathrm{rmul}(|x^*\rangle\!\langle\sigma_{-1}(y^*)|^*(1))(a)\\ &= \mathrm{rmul}(|x\rangle\!\langle y|^{\mathrm{r}*}(1))(a) = D_{\mathrm{out}}(|x\rangle\!\langle y|^{\mathrm{r}})(a).\end{aligned}$$

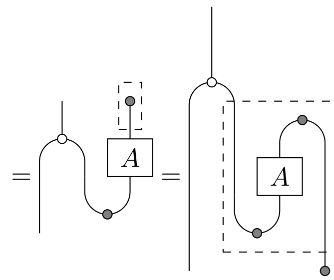
With strings:



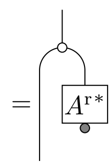
by A.6



by A.8(i)



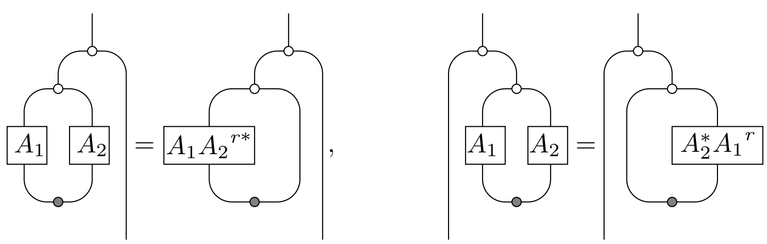
by A.6



by A.92.

■

Lemma B.60. *$D_{\text{in}}(A_1 \bullet A_2) = \left(A_1 {A_2}^{r*}\right) \bullet \mathrm{id}$ and $D_{\text{out}}(A_1 \bullet A_2) = \mathrm{id} \bullet (A_2^* {A_1}^r)$, in other words,*



Proof. It suffices to show these for when $A_1 = |a\rangle\langle b|$ and $A_2 = |c\rangle\langle d|$ for $a, b, c, d \in B$. So then we compute,

$$\begin{aligned} D_{\text{in}}(|a\rangle\langle b| \bullet |c\rangle\langle d|) &= \mathrm{lmul}(|ac\rangle\langle bd|(1)) = \langle bd|1\rangle \,\mathrm{lmul}(ac) \\ &= (\langle bd|1\rangle |a\rangle\langle c^*|) \bullet \mathrm{id} = (\langle b|\sigma_{-1}(d^*)\rangle |a\rangle\langle c^*|) \bullet \mathrm{id} \\ &= (|a\rangle\langle b||\sigma_{-1}(d^*)\rangle\langle c^*|) \bullet \mathrm{id} = \left(|a\rangle\langle b||c\rangle\langle d|^{\mathrm{r}*}\right) \bullet \mathrm{id}. \end{aligned}$$

Thus $D_{\text{in}}(A_1 \bullet A_2) = \left(A_1 {A_2}^{\mathrm{r}*}\right) \bullet \mathrm{id}$ for any $A_1, A_2 \in \mathcal{B}(B, \psi)$.

Analogously, we compute,

$$\begin{aligned} D_{\text{out}}(|a\rangle\langle b| \bullet |c\rangle\langle d|) &= \mathrm{rmul}(|bd\rangle\langle ac|(1)) = \langle ac|1\rangle \,\mathrm{rmul}(bd) \\ &= \mathrm{id} \bullet (\langle ac|1\rangle |d\rangle\langle \sigma_{-1}(b^*)|) \\ &= \mathrm{id} \bullet (\langle c|a^*\rangle |d\rangle\langle \sigma_{-1}(b^*)|) \\ &= \mathrm{id} \bullet (|d\rangle\langle c||a^*\rangle\langle \sigma_{-1}(b^*)|) = \mathrm{id} \bullet \left(|c\rangle\langle d|^* |a\rangle\langle b|^{\mathrm{r}}\right). \end{aligned}$$

Thus $D_{\text{out}}(A_1 \bullet A_2) = \mathrm{id} \bullet (A_2^* {A_1}^{\mathrm{r}})$ for any $A_1, A_2 \in \mathcal{B}(B, \psi)$. ■

Proposition B.61 ([10, Lemma 2.6])**.** *Given $A \in \mathcal{B}(B, \psi)$, we get,*

$$\nabla(A)^* \nabla(A) = D_{\text{in}}(A \bullet A^r) - A \bullet A - A^* \bullet A^* + D_{\text{out}}(A^r \bullet A).$$

Proof. We compute,

$$\nabla(A)^* \nabla(A) = m(A \otimes \mathrm{id} - \mathrm{id} \otimes A^*)(A^* \otimes \mathrm{id} - \mathrm{id} \otimes A) m^*$$

$$
\begin{aligned}
&= (AA^*) \bullet \mathrm{id} - A \bullet A - A^* \bullet A^* + \mathrm{id} \bullet (A^* A) \\
&= D_{\mathrm{in}}(A \bullet A^{\mathrm{r}}) - A \bullet A - A^* \bullet A^* + D_{\mathrm{out}}(A^{\mathrm{r}} \bullet A) \qquad \text{by B.60.}
\end{aligned}
$$

■

Lemma B.62 ([10, Proposition 2.4(2)])**.** *Given a real Schur-idempotent $A \in \mathcal{B}(B, \psi)$, we have $\nabla(A)(xy) = \nabla(A)(x) \cdot_{\mathrm{r}} y + x \cdot_{\mathrm{l}} \nabla(A)(y)$.*

Proof.

$$
\begin{aligned}
\nabla(A)(xy) &= \Phi(A^{\mathrm{r}})(1 \otimes xy) - \Phi(A)(xy \otimes 1) = \Phi(A)(1 \otimes xy - xy \otimes 1) \\
&= \Phi(A)(1 \otimes xy - x \otimes y) + \Phi(A)(x \otimes y - xy \otimes 1) \\
&= \Phi(A)((1 \otimes x - x \otimes 1) \cdot_{\mathrm{r}} y) + \Phi(A)(x \cdot_{\mathrm{l}} (1 \otimes y - y \otimes 1)) \\
&= \Phi(A)(1 \otimes x - x \otimes 1) \cdot_{\mathrm{r}} y + x \cdot_{\mathrm{l}} \Phi(A)(1 \otimes y - y \otimes 1) \\
&= \nabla(A)(x) \cdot_{\mathrm{r}} y + x \cdot_{\mathrm{l}} \nabla(A)(y).
\end{aligned}
$$

■

C Single-edged real quantum graphs on $B = M_n$

In this chapter, we study the possible isomorphisms for single-edged real quantum graphs over a faithful and positive linear functional ψ on M_n. By single-edged, here, we mean a quantum graph such that its projection projects onto a one-dimensional subspace (so is given by a single element). For tracial functionals, the dimension of the projected subspace is equal to its number of edges, so in that case, 'single-edged' does in fact mean a quantum graph with one edge (see Proposition B.28).

Let us first summarise what we did before.

Firstly, we fix a faithful and positive linear functional ψ on B, and let $Q \in B$ be the positive-definite matrix such that ψ is given by $x \mapsto \operatorname{Tr}(Qx)$. A linear map $A \in \mathcal{B}(B, \psi)$ is a quantum adjacency matrix when $A \bullet A = A$. We say (B, ψ, A) is a quantum graph when A is a quantum adjacency matrix operator on B. We let Ψ be the linear isomorphism from $\mathcal{B}(B, \psi)$ to $B \otimes B^{\mathrm{op}}$ given by $|x\rangle\langle y| \mapsto x \otimes \sigma_{1/2}(y)^{*\mathrm{op}}$. We have (B, ψ, A) is a real quantum graph if and only if $\Psi(A)$ is an orthogonal projection. Applying $(\mathrm{id} \otimes \top^{-1})$ makes it an orthogonal projection on $B \otimes B$ instead of on $B \otimes B^{\mathrm{op}}$.

C.I Minimal projections

Definition C.1. We say a projection $x \in \mathcal{B}(B)$ is *minimal* if there exists a one-dimensional subspace $V \subseteq B$ such that x projects onto V.

If $x \in \mathcal{B}(B)$ is a minimal projection, then there exists a one-dimensional subspace V such that x projects onto V, i.e., $x = P_V$. So then let (v) be an orthonormal basis of V. Then we can write $x = |v\rangle\langle v|$ by Lemma A.170. So then for any $0 \neq y \in U$, we get $x = \frac{1}{\|y\|^2}|y\rangle\langle y|$. Then we can define a surjective map $B \setminus \{0\} \to \{\text{minimal projections in } \mathcal{B}(B)\}$, given by $y \mapsto \frac{1}{\|y\|^2}|y\rangle\langle y|$. Two minimal projections given by $0 \neq x, y \in B$ are equal if and only if x and y are co-linear, i.e., $\frac{1}{\|y\|^2}|y\rangle\langle y| = \frac{1}{\|x\|^2}|x\rangle\langle x|$ if and only if $\exists \beta \in \mathbb{C} \setminus \{0\} : y = \beta x$ by Proposition A.19 (in other words, this map is "almost injective" – discussed more in the next section). So then this map is bijective up to a scalar multiple (i.e., it is surjective and "almost injective").

Using Proposition A.190(iii), we get that all single-edged real linear operators on B that satisfy Schur idempotence are given by,

$$\Psi_{0,1/2}^{-1}(\mathrm{id} \otimes \top)\mathcal{M}\left(\frac{1}{\|x\|^2}|x\rangle\langle x|\right) = \frac{1}{\|x\|^2}\operatorname{lmul}(xQ)\operatorname{rmul}(Qx)^*,$$

for some $x \in B \setminus \{0\}$. Recall $\mathcal{M}$ is the identification $\mathcal{B}(M_n) \cong M_{n \times n} \cong M_n \otimes M_n$ (Section A.IV).

C.II Single-edged real quantum graphs

In this section we define a surjective function A from the set of non-zero elements in M_n to the set of single-edged real quantum graphs on M_n. Again, by a single-edged real quantum graph (M_n, x), we mean a real quantum graph (M_n, x) such that its projection is given by a single element.

Definition C.2. Let

$$A \colon M_n \setminus \{0\} \to \{y \in \mathcal{B}(M_n, \psi) : (M_n, \psi, y) \text{ is a single-edged real quantum graph}\}$$

be given by $x \mapsto \frac{1}{\|x\|^2} \operatorname{lmul}(xQ)\operatorname{rmul}(Qx)^*$.

In other words, $A(x) = \Psi_{0,1/2}^{-1}(\mathrm{id} \otimes \top)\mathcal{M}\left(\frac{1}{\|x\|^2}|x\rangle\langle x|\right)$, where $\top\colon M_n \cong_{\mathrm{a}} M_n^{\mathrm{op}}$ is given by $x \mapsto (x^T)^{\mathrm{op}}$ with its inverse given by $x^{\mathrm{op}} \mapsto x^T$ (see Proposition A.190(iii)). Recall, from Section A.IV, $\mathcal{M}$ is the identification $\mathcal{B}(M_n) \cong M_{n\times n} \cong M_n \otimes M_n$.

Proposition C.3. *Given any $0 \neq \alpha \in \mathbb{C}$ and $0 \neq x \in M_n$, we have $A(\alpha x) = A(x)$.*

Proof. We quickly compute,

$$A(\alpha x) = \frac{1}{\|\alpha x\|^2} \operatorname{lmul}(\alpha x Q)\operatorname{rmul}(\alpha Q x)^* = \frac{|\alpha|^2}{|\alpha|^2 \|x\|^2} \operatorname{lmul}(xQ)\operatorname{rmul}(Qx)^* = A(x).$$

■

The above proposition tells us that our map A is not injective. However, it is *almost injective* (defined below).

Definition C.4. Given $\mathbb{C}$-vector spaces V_1, V_2, we say that a function $T\colon V_1 \setminus \{0\} \to V_2$ is *almost injective* if for all $0 \neq v, w \in V_1$, if $T(v) = T(w)$, then there exists some $0 \neq \beta \in \mathbb{C}$ such that $v = \beta w$.

Lemma C.5. *$A\colon M_n \setminus \{0\} \to \mathcal{B}(M_n, \psi)$ is an almost injective function.*

Proof. Let x, y be non-zero elements in M_n and suppose $A(x) = A(y)$. Then we have

$$\Psi_{0,1/2}^{-1}(\mathrm{id} \otimes \top)\mathcal{M}\left(\left|\frac{x}{\|x\|}\right\rangle\!\left\langle\frac{x}{\|x\|}\right|\right) = \Psi_{0,1/2}^{-1}(\mathrm{id} \otimes \top)\mathcal{M}\left(\left|\frac{y}{\|y\|}\right\rangle\!\left\langle\frac{y}{\|y\|}\right|\right).$$

As $\Psi_{0,1/2}^{-1}$, $(\mathrm{id} \otimes \top)$ and $\mathcal{M}$ are isomorphisms, we have the ket-bras are equal, so using Proposition A.19, we get $\|y\| x = \alpha \|x\| y$ for some non-zero $\alpha \in \mathbb{C}$. Thus $x = \frac{\alpha \|x\|}{\|y\|} y$, and so the function A is almost injective. ■

We have $A(Q^{-1})$ is the trivial graph since,

$$A(Q^{-1}) = \frac{1}{\|Q^{-1}\|^2} \operatorname{lmul}(Q^{-1}Q)\operatorname{rmul}(QQ^{-1})^* = \frac{1}{\|Q^{-1}\|^2} \mathrm{id} = \operatorname{Tr}(Q^{-1})^{-1} \mathrm{id} = (mm^*)^{-1}.$$

Lemma C.6. *If $x \in M_n \setminus \{0\}$, then $A(x)$ is self-adjoint if and only if* $\operatorname{symm}(A(x)) = A(x)$.

Proof. Since A maps non-zero elements to real single-edged quantum graphs, we can use Proposition A.102(ii),(iii) to get $A(x)$ is self-adjoint if and only if $\operatorname{symm}(A(x)) = A(x)$. ■

C.II.1 Conditions for self-adjoint-edness and (ir)reflexivity. So far, we know that for any non-zero element $x \in M_n$, we get $A(x)$ is both real and satisfies Schur idempotence (i.e., $A(x) \bullet A(x) = A(x)$). In this section we find the conditions we need to put on $x \in M_n \setminus \{0\}$ to get $A(x)$ is self-adjoint/symmetric and (ir)reflexive.

It turns out that we get $A(x)$ is self-adjoint if and only if x is co-linear to some self-adjoint element and commutes with Q. We define this property (i.e., being co-linear to some self-adjoint element and commuting with Q) as being *almost self-adjoint via restricted Q* (see below).

Definition C.7 (almost self-adjoint)**.** We say an element $x \in M_n$ is *almost self-adjoint via restricted* Q if there exists some $\alpha \in \mathbb{C}$ and a self-adjoint element $y \in M_n$ such that $x = \alpha y$ and $xQ = Qx$.

Remark C.8. Let $x \in M_n \setminus \{0\}$ be almost self-adjoint via restricted Q, then x commutes with Q and $x = \alpha y$ for some $\alpha \in \mathbb{C}$ and self-adjoint element $y \in B$. Obviously, we also get y commutes with Q. ♢

Lemma C.9. *Let $x \in M_n \setminus \{0\}$. Then $A(x)$ is self-adjoint if and only if x is almost self-adjoint via restricted Q.*

Proof. We have the following equivalences,

$$\begin{aligned}
&A(x) \text{ is self-adjoint} \\
\Leftrightarrow\ & \operatorname{rmul}(Qx)\operatorname{lmul}(xQ)^* = \operatorname{lmul}(xQ)\operatorname{rmul}(Qx)^* \\
\Leftrightarrow\ & \operatorname{lmul}(Qx^*)\operatorname{rmul}(Qx^*)^* = \operatorname{lmul}(xQ)\operatorname{rmul}(Qx)^* && \text{by A.115(ii),(iii)} \\
\Leftrightarrow\ & |Qx^*\rangle\langle Qx^*| = |xQ\rangle\langle Qx| && \text{by A.190(ii)} \\
\Leftrightarrow\ & \operatorname{lmul}(Q)|x^*\rangle\langle x^*|\operatorname{lmul}(Q) = \operatorname{rmul}(Q)|x\rangle\langle x|\operatorname{lmul}(Q) \\
\Leftrightarrow\ & \sigma_{-1}|x^*\rangle\langle x^*| = |x\rangle\langle x|.
\end{aligned}$$

Note that the last equivalence follows as $\operatorname{lmul}(Q)$ is invertible and $\sigma_{-1} = \operatorname{lmul}(Q)\operatorname{rmul}(Q^{-1})$.

(⇒) Suppose $A(x)$ is self-adjoint.
Then, by the above equivalences, we have

$$\sigma_{-1}|x^*\rangle\langle x^*| = |x\rangle\langle x|, \tag{1}$$

which, by Proposition A.91(i), is exactly $\operatorname{symm}(|x\rangle\langle x|) = |x\rangle\langle x|$. We then also get $\operatorname{symm}'(|x\rangle\langle x|) = |x\rangle\langle x|$ by Corollary A.95. So then using Proposition A.91(ii), we get $|x^*\rangle\langle x^*| = |x\rangle\langle \sigma_1(x)|$. By Equation (1), we also have $|x^*\rangle\langle x^*| = |\sigma_1(x)\rangle\langle x|$. So then we get $|x\rangle\langle \sigma_1(x)| = |\sigma_1(x)\rangle\langle x|$. And so $\sigma_1(x) = \alpha x$ where $\alpha = \dfrac{\|\sigma_{1/2}(x)\|^2}{\|x\|^2}$. Clearly, $0 < \alpha \in \mathbb{R}$. Then $|x^*\rangle\langle x^*| = |\sqrt{\alpha}x\rangle\langle\sqrt{\alpha}x|$, and so, by Proposition A.19, we get a non-zero $\beta \in \mathbb{C}$ such that $x^* = \beta\sqrt{\alpha}x$. So then we have $x = \overline{\beta}\sqrt{\alpha}x^*$ and $x = \beta^{-1}\sqrt{\alpha}^{-1}x^*$, which means $0 = x - x = (\overline{\beta}\sqrt{\alpha} - \beta^{-1}\sqrt{\alpha}^{-1})x^*$ which is true if and only if $\|\beta\|^2\alpha = 1$. So there exists some non-zero $\gamma \in \mathbb{C}$ such that $\gamma^2 = \beta\sqrt{\alpha}$. Then $\|\gamma\|^2 = 1$ as this is true if and only if $\|\gamma^2\|^2 = \|\beta\sqrt{\alpha}\|^2 = \|\beta\|^2\alpha = 1$.

We have γx is self-adjoint since this is true if and only if $\overline{\gamma}x^* = \gamma x$, which is true if and only if $x^* = \gamma^2 x = \beta\sqrt{\alpha}x$. And we know this is true from before, so γx is self-adjoint. Now let $y = \gamma x$, then we have $x = \|\gamma\|^2 x = \overline{\gamma}y$. So x is co-linear to a self-adjoint element.

Now from $|x^*\rangle\langle x^*| = |x\rangle\langle\sigma_1(x)|$, we get $|y\rangle\langle y| = \alpha|y\rangle\langle y|$, and so $(1-\alpha)|y\rangle\langle y| = 0$ which is true if and only if $\alpha = 1$. Thus $\sigma_1(x) = \alpha x = x$, which means x commutes with Q. Thus x is almost self-adjoint via restricted Q.

(⇐) Suppose x is almost self-adjoint via restricted Q. So we have x commutes with Q, and we let $\alpha \in \mathbb{C}$ and $y \in M_n$ such that $y^* = y$ and $x = \alpha y$. Then we compute,

$$|\sigma_{-1}(x^*)\rangle\langle x^*| = |x^*\rangle\langle x^*| = \alpha\bar{\alpha}|y\rangle\langle y| = |x\rangle\langle x|.$$

And by the above equivalences, this means $A(x)$ is self-adjoint, so we are done. ∎

For $A(x)$ to be irreflexive, we need x to have trace zero. For it to be reflexive, x will need to be co-linear to Q^{-1}. So this means that there is only one single-edged reflexive real quantum graph and that is exactly the trivial graph $(M_n, \psi, A(Q^{-1}))$.

Lemma C.10. *Let $x \in M_n \setminus \{0\}$. Then*

(i) $A(x) \bullet \mathrm{id} = 0 \Leftrightarrow \operatorname{Tr}(x) = 0$,

(ii) $A(x) \bullet \mathrm{id} = \mathrm{id} \Leftrightarrow \exists\, \alpha \in \mathbb{C} \setminus \{0\} : x = \alpha Q^{-1}$.

Proof. By applying our linear equivalence $\Psi_{0,1/2}$ and Propositions A.180(iv) and A.189(iv), we get

$$\begin{aligned}\Psi_{0,1/2}(A(x) \bullet \mathrm{id}) &= \Psi_{0,1/2}(A(x))\Psi_{0,1/2}(\mathrm{id}) \\ &= \frac{1}{\|x\|^2}(\mathrm{id} \otimes \top)\mathcal{M}(|x\rangle\langle x|)(\mathrm{id} \otimes \top)\mathcal{M}\left(\left|Q^{-1}\right\rangle\!\left\langle Q^{-1}\right|\right) \\ &= \frac{1}{\|x\|^2}(\mathrm{id} \otimes \top)\mathcal{M}\left(|x\rangle\langle x|\left|Q^{-1}\right\rangle\!\left\langle Q^{-1}\right|\right) \\ &= \frac{\left\langle x\middle|Q^{-1}\right\rangle}{\|x\|^2}(\mathrm{id} \otimes \top)\mathcal{M}\left(\left|x\right\rangle\!\left\langle Q^{-1}\right|\right). \end{aligned} \tag{1}$$

(i) Equation (1) equals 0 if and only if $\left\langle x\middle|Q^{-1}\right\rangle = 0$ or $\left|x\right\rangle\!\left\langle Q^{-1}\right| = 0$. And we know $\left|x\right\rangle\!\left\langle Q^{-1}\right|$ is non-zero since both x and Q^{-1} are non-zero. So this is true if and only if $\left\langle x\middle|Q^{-1}\right\rangle = 0$. Obviously, expanding this, we get $0 = \left\langle x\middle|Q^{-1}\right\rangle = \operatorname{Tr}(Qx^*Q^{-1}) = \operatorname{Tr}(x^*) = \overline{\operatorname{Tr}(x)}$, which is true if and only if $\operatorname{Tr}(x) = 0$, so we are done.

(ii) Note that we have $\left\langle x\middle|Q^{-1}\right\rangle \neq 0$, otherwise we get $\operatorname{Tr}(x) = 0$ and so $A(x) \bullet \mathrm{id} = 0$ by Part (i). Using Proposition A.189(iv), we get Equation (1) equals $\Psi_{0,1/2}(\mathrm{id})$ if and only if $\left\langle x\middle|Q^{-1}\right\rangle\left|x\right\rangle\!\left\langle Q^{-1}\right| = \|x\|^2\left|Q^{-1}\right\rangle\!\left\langle Q^{-1}\right|$.
This is true if and only if $\left|\left\langle x\middle|Q^{-1}\right\rangle x - \|x\|^2 Q^{-1}\right\rangle\!\left\langle Q^{-1}\right| = 0$, which is then true if and only if $\left\langle x\middle|Q^{-1}\right\rangle x = \|x\|^2 Q^{-1}$. The result then follows as $\left\langle x\middle|Q^{-1}\right\rangle \neq 0$.

■

C.III Describing isomorphisms on single-edges

In this section, we study when we get $(M_n, A(x)) \cong (M_n, A(y))$ for non-zero elements $x, y \in M_n$. Theorem C.14 is one of the main results in this thesis, and is a classification for single-edged real quantum graphs.

Corollary C.11. *Let f be an isometric *-automorphism on M_n, and let $x, y \in M_n$. Then* $f^{-1} \circ |x\rangle\langle y| \circ f = \left|f^{-1}(x)\right\rangle\!\left\langle f^{-1}(y)\right|$.
Analogously, $f \circ |x\rangle\langle y| \circ f^{-1} = |f(x)\rangle\langle f(y)|$.

Proof. This is done using Lemmas A.17(i),(ii) and A.135. ■

Lemma C.12. *Let $x \in M_n$ and f be an isometric *-automorphism on M_n. Then*
$f^{-1} \circ A(x) \circ f = A(f^{-1}(x))$.
Analogously, $f \circ A(x) \circ f^{-1} = A(f(x))$.

Proof. Let $U \in M_n$ be the unitary such that f is given by $x \mapsto UxU^*$ (see Proposition A.128). Then, as f is an isometry, we use Lemma A.137 to get $UQ = QU$, and also $U^*Q = QU^*$.

By Lemma A.144, we know $\mathcal{M}(f) = U \otimes \overline{\sigma_{-1/2}(U)}$. Also note $\mathcal{M}(f^{-1}) = \mathcal{M}(f)^*$.

Using Lemma A.137 again, we get $\|x\| = \left\|f^{-1}(x)\right\|$.

Then we compute,

$$\begin{aligned}
\mathcal{M}(f^{-1}A(x)f) &= \mathcal{M}(f^{-1})\mathcal{M}(A(x))\mathcal{M}(f)\\
&= \frac{1}{\|x\|^2}\left(U\otimes\overline{\sigma_{-1/2}(U)}\right)^*\left(xQ\otimes\overline{\sigma_{1/2}(Qx)}\right)\left(U\otimes\overline{\sigma_{-1/2}(U)}\right)\\
&= \frac{1}{\|x\|^2}\left(U^*xQU\otimes\overline{Q^{-1/2}U^*Q^{1/2}Q^{-1/2}QxQ^{1/2}Q^{1/2}UQ^{-1/2}}\right)\\
&= \frac{1}{\|x\|^2}\left(U^*xUQ\otimes\overline{Q^{-1/2}U^*QxQUQ^{-1/2}}\right)\\
&= \frac{1}{\|x\|^2}\left(f^{-1}(x)Q\otimes\overline{Q^{-1/2}QU^*xUQQ^{-1/2}}\right)\\
&= \frac{1}{\|x\|^2}\left(f^{-1}(x)Q\otimes\overline{Q^{1/2}f^{-1}(x)Q^{1/2}}\right)\\
&= \frac{1}{\|f^{-1}(x)\|^2}\left(f^{-1}(x)Q\otimes\overline{\sigma_{1/2}(Qf^{-1}(x))}\right) = \mathcal{M}(A(f^{-1}(x))).
\end{aligned}$$

Thus $f^{-1}A(x)f = A(f^{-1}(x))$. ∎

We finally come to our main result. The following tells us that any two single-edged real quantum graphs $(M_n,\psi,A(x))$ and $(M_n,\psi,A(y))$ given by non-zero elements $x,y\in M_n$ are isomorphic if and only if there exists a non-zero $\beta\in\mathbb{C}$ and a unitary $U\in M_n$ such that $x=U(\beta y)U^*$ and $UQ=QU$.

> **Definition C.13.** We say an element $x\in M_n$ is *almost similar via restricted* Q to $y\in M_n$ if there exists a unitary $U\in M_n$ and $\beta\in\mathbb{C}\setminus\{0\}$ such that $x=\beta UyU^*$ and $UQ=QU$. Equivalently (see Lemma A.140), x is almost similar via restricted Q to y if there exists a non-zero $\beta\in\mathbb{C}$ and an isometric *-automorphism f on M_n such that $x=f(\beta y)$.

Theorem C.14. *Let ψ be a positive and faithful linear functional on M_n, where we endow M_n with the inner product $\langle a|b\rangle=\psi(a^*b)=\mathrm{Tr}(Qa^*b)$ for all $a,b\in M_n$, where $Q\in M_n$ is the unique positive definite element such that $\psi(a)=\mathrm{Tr}(Qa)$ for all $a\in M_n$. Let $x,y\in M_n\setminus\{0\}$. Then*

x and y are almost similar via restricted Q
$\Leftrightarrow\ (M_n,\psi,A(x))\cong(M_n,\psi,A(y))$.

Proof.

($\Rightarrow$) Suppose there exists some non-zero $\beta\in\mathbb{C}$ and an isometric *-automorphism f on M_n such that $x=f(\beta y)$. Then by Propositions C.12 and C.3 we get

$$f^{-1}A(x)f = A(f^{-1}(x)) = A(\beta y) = A(y).$$

Thus $(M_n,\psi,A(x))\cong(M_n,\psi,A(y))$.

($\Leftarrow$) Suppose we have an isometric *-automorphism f on M_n such that $A(x)f=fA(y)$. Then by Proposition C.12 we get $A(f^{-1}(x))=A(y)$. And as A is an almost injective function (see Lemma C.5), we get that there exists some non-zero complex number α such that $f^{-1}(x)=\alpha y$. This means we get $x=f(\alpha y)$. So then we are done.

∎

C.IV Describing isomorphisms of single-edges on tracial functionals

A nice corollary to Theorem C.14 is that when we have ψ is tracial, then $(M_n,\psi,A(x))$ is isomorphic to $(M_n,\psi,A(y))$ if and only if x and βy, for some $\beta\in\mathbb{C}$, have equal spectra (see Corollary C.22).

C.IV.1 Some properties for almost self-adjoint elements. We first quickly recover some easy well-known results for normal matrices, but instead apply it to almost self-adjoint matrices.

Lemma C.15. *Let $x \in M_n$ be an almost self-adjoint matrix. Then*
x *is upper-triangular* $\Leftrightarrow$ x *is diagonal.*

Proof. As x is almost self-adjoint, we let $\alpha \in \mathbb{C}$ and $y \in M_n$ such that y is self-adjoint and $x = \alpha y$. Now αy being upper-triangular means that for any $i, j \in [n]$, if $j < i$, then $\alpha y_{ij} = 0$. And αy being diagonal means that for any $i, j \in [n]$, if $i \neq j$, then $\alpha y_{ij} = 0$.

($\Rightarrow$) Suppose that we have αy is upper-triangular, i.e., for any $i, j \in [n]$, if $j < i$, then $\alpha y_{ij} = 0$. Let $i, j \in [n]$ such that $i \neq j$. We assume $i < j$, otherwise this is exactly our hypothesis when $j < i$. Then by our hypothesis we know $\alpha y_{ji} = 0$. We assume $\alpha \neq 0$, otherwise this is trivial. So then $y_{ji} = 0$. And since $y = y^*$, we get $y_{ij} = y^*_{ij} = \overline{y_{ji}} = 0$. Thus $\alpha y_{ij} = 0$.

($\Leftarrow$) If it is diagonal, then it is already upper-triangular.

■

Definition C.16. We say that two elements $x, y \in M_n$ are *similar* if there exists some unitary $U \in M_n$ such that $UxU^* = y$.

Proposition C.17. *Given two diagonal matrices $D_1, D_2 \in M_n$ such that they have the same diagonal entries with the same multiplicities, then there exists a permutation matrix that transforms one into the other, i.e.,*
$\exists P \in U_n : D_2 = PD_1P^*$. ■

Lemma C.18 (Schur decomposition [7, Theorem 2.3.1(a)]). *Let $A \in M_n$. Then*
$\exists\, (U \in U_n)\, (D \in M_n) : A = UDU^*$ *and D is upper-triangular.* ■

Lemma C.19. *Let $A_1, A_2 \in M_n$ be almost self-adjoint. Then*
A_1, A_2 have equal eigenvalues with the same multiplicities $\Leftrightarrow$ A_1, A_2 are similar.

Proof.

($\Rightarrow$) Suppose A_1, A_2 have equal eigenvalues with the same multiplicities. Since they are almost self-adjoint, we let $\alpha_1, \alpha_2 \in \mathbb{C}$ and $y_1, y_2 \in M_n$ such that y_1, y_2 are self-adjoint and $A_1 = \alpha_1 y_1$ and $A_2 = \alpha_2 y_2$.

Using the Schur decomposition Lemma C.18, we write $A_1 = \mathcal{U}_1 D_1 \mathcal{U}_1^*$ and $A_2 = \mathcal{U}_2 D_2 \mathcal{U}_2^*$ where D_1, D_2 are upper-triangular matrices and $\mathcal{U}_1, \mathcal{U}_2$ are unitary matrices.

Claim: We have $D_1 = \mathcal{U}_1^* A_1 \mathcal{U}_1$ and $D_2 = \mathcal{U}_2^* A_2 \mathcal{U}_2$ are almost self-adjoint.

We have $D_1 = \mathcal{U}_1^* A_1 \mathcal{U}_1 = \alpha_1 \mathcal{U}_1^* y_1 \mathcal{U}_1$, and similarly $D_2 = \alpha_2 \mathcal{U}_2^* y_2 \mathcal{U}_2$. As this is a *-inner automorphism, we get

$$(\mathcal{U}_1^* y_1 \mathcal{U}_1)^* = \mathcal{U}_1^* y_1^* \mathcal{U}_1 = \mathcal{U}_1^* y_1 \mathcal{U}_1$$

Analogously, $(\mathcal{U}_2^* y_2 \mathcal{U}_2)^* = \mathcal{U}_2^* y_2 \mathcal{U}_2$. Thus D_1 and D_2 are almost self-adjoint.

Now, by Lemma C.15, we know an almost self-adjoint matrix is upper triangular if and only if it is diagonal. So D_1 and D_2 are diagonal.

Since A_1, A_2 have equal eigenvalues with the same multiplicities, we know $D_2 = PD_1P^*$, where P is a permutation matrix (so is unitary) by Proposition C.17. We then compute,

$$\begin{aligned} A_2 &= \mathcal{U}_2 D_2 \mathcal{U}_2^* = \mathcal{U}_2 (PD_1P^*) \mathcal{U}_2^* \\ &= (\mathcal{U}_2 P) D_1 (\mathcal{U}_2 P)^* \end{aligned}$$

$$= (\mathcal{U}_2 P)(\mathcal{U}_1^* A_1 \mathcal{U}_1)(\mathcal{U}_2 P)^*$$
$$= (\mathcal{U}_2 P \mathcal{U}_1^*) A_1 (\mathcal{U}_2 P \mathcal{U}_1^*)^*.$$

Obviously, $\mathcal{U}_2 P \mathcal{U}_1^*$ is unitary since $(\mathcal{U}_2 P \mathcal{U}_1^*)(\mathcal{U}_1 P^* \mathcal{U}_2^*) = 1 = (\mathcal{U}_1 P^* \mathcal{U}_2^*)(\mathcal{U}_2 P \mathcal{U}_1^*)$. Thus, we have shown that there exists a unitary matrix $U = \mathcal{U}_2 P \mathcal{U}_1^*$ such that $U A_1 U^* = A_2$.

($\Leftarrow$) Suppose $U A_1 U^* = A_2$ for some $U \in U_n$. Then the result follows from Proposition A.25.

■

Definition C.20.

- We say that two matrices x, y have *almost equal spectra* if there exists some $0 \neq \beta \in \mathbb{C}$ such that x and βy have equal eigenvalues with the same multiplicities.
- We say that two matrices x, y are *almost similar* if there exists some $0 \neq \beta \in \mathbb{C}$ such that x and βy are similar.

By definition, we have that Lemma C.19 corresponds to the following.

Corollary C.21. *Let $x, y \in M_n$ be almost self-adjoint. Then*
x and y have almost equal spectra $\Leftrightarrow$ x and y are almost similar. ■

Now we are ready to state the corollary to Theorem C.14 for when $B = M_n$ and ψ is tracial.

Corollary C.22. *Let $x, y \in M_n \setminus \{0\}$ such that they are both almost self-adjoint. Then $(M_n, \mathrm{Tr}, A(x))$ is isomorphic to $(M_n, \mathrm{Tr}, A(y))$ if and only if x is almost similar to y (equivalently, by Corollary C.21, if x and y have almost equal spectra).*

Proof. By Proposition A.45 we get $Q = \alpha 1$ for some $0 < \alpha \in \mathbb{R}$ (this is positive and real since Q is positive definite). This means any matrix will obviously commute with Q. So the result then follows from Theorem C.14. ■

C.V Isomorphisms on M_n for tracial functionals

In this section, we continue working on tracial functionals, so by Proposition A.45, we get $Q = \alpha 1$ for some $0 < \alpha \in \mathbb{R}$ (positivity and realness of α follows from Q being positive definite). Note that this means we get $(M_n, \mathrm{Tr}, A(1))$ is the trivial graph. Section C.V.2 contains the remaining main results of this thesis, and are classifications for certain types of quantum graphs.

C.V.1 Isomorphisms on M_2. The following result shows that all single-edged irreflexive quantum graphs on M_2 are isomorphic. This has already been done in [6, Theorem 3.11] and [9, Theorem 3.1], but we give an easier proof.

Proposition C.23. *All single-edged self-adjoint (ir)reflexive real quantum graphs on M_2 are isomorphic.*
In other words, for any almost self-adjoint matrices $x, y \in M_2 \setminus \{0\}$ that have zero trace, we get $(M_2, \mathrm{Tr}, A(x)) \cong (M_2, \mathrm{Tr}, A(y))$.

Proof. By Theorem C.14 it is enough to say that the matrix x is almost similar to y (or, equivalently, by Corollary C.21, that they have almost equal spectra).

For a two-by-two matrix to be have zero trace, it would need some $\alpha \in \mathbb{C}$ and $-\alpha$ to be its eigenvalues. This means that any almost self-adjoint matrix that has zero trace has almost equal spectra to any other almost self-adjoint matrix that has zero trace. Thus we are done. ■

C.V.2 Isomorphisms on M_n for $n > 2$. Now recall that for any non-zero, almost self-adjoint matrices $x, y \in M_n$ that have zero trace, we get $(M_n, \mathrm{Tr}, A(x))$ is isomorphic to $(M_n, \mathrm{Tr}, A(y))$ if and only if there exists some non-zero $\beta \in \mathbb{C}$ such that x and βy have equal eigenvalues with the same multiplicities (Corollary C.22). When $n > 2$, there are infinite possibilities of non-similar matrices x, y, which means there are infinite possibilities of non-isomorphisms for single-edged self-adjoint, irreflexive, and real quantum graphs on M_n for tracial functionals.

Theorem C.24. *For $n > 2$, we get infinitely many non-isomorphisms for single-edged self-adjoint, irreflexive, and real quantum graphs on M_n for tracial $\psi = \mathrm{Tr}$.*

Proof. Let $x, y \in M_n$. Propositions C.9 and C.10(i) tell us that we get $A(x)$ and $A(y)$ are self-adjoint, irreflexive, and real quantum adjacency matrices if and only if x and y are almost self-adjoint and have zero trace.

Then by Theorem C.14 we have $(M_n, \mathrm{Tr}, A(x)) \ncong (M_n, \mathrm{Tr}, A(y))$ if and only if x and y are not similar (or, equivalently, by Corollary C.21, do not have almost equal spectra).

As the trace equals 0 for both x and y, this means that their eigenvalues add up to 0. Since $n > 2$, we do not necessarily get that they are almost equal.

For example, over M_3, if we have the spectrum of x equals $(-2, -1, 3)$ and the spectrum of y equals $(-2, 1, 1)$. Then there does not exist a β such that $(-2, -1, 3) = (-2\beta, \beta, \beta)$. ■

Corollary C.25. *For $n > 2$, we also have infinitely many non-isomorphisms between self-adjoint reflexive and real quantum graphs on M_n with 2-quantum edges.*

Proof. Given $A_1 \in \mathcal{B}(M_n, \mathrm{Tr})$ is a self-adjoint irreflexive and real quantum adjacency matrix, we get $A(1) + A_1$ is a self-adjoint reflexive and real quantum adjacency matrix by Proposition B.40. Clearly, if A_1 is single-edged, then $A(1) + A_1$ has exactly 2-edges. Given another single-edged self-adjoint irreflexive and real quantum adjacency matrix $A_2 \in \mathcal{B}(M_n, \mathrm{Tr})$, it is then clear that we get $(M_n, \mathrm{Tr}, A(1) + A_1)$ is isomorphic to $(M_n, \mathrm{Tr}, A(1) + A_2)$ if and only if (M_n, Tr, A_1) is isomorphic to (M_n, Tr, A_2). So, it then follows from Theorem C.24 since we know there are infinitely many non-isomorphisms for single-edged self-adjoint irreflexive and real quantum graphs. ■

Corollary C.26. *For $n > 2$, there are infinitely many non-isomorphisms between self-adjoint irreflexive and real quantum graphs on M_n with $n^2 - 2$ edges.*

Proof. Similarly to the previous proof, this follows directly from the fact that two self-adjoint irreflexive and real quantum graphs are isomorphic if and only if their irreflexive complements are isomorphic; in particular, see Proposition B.41 and Theorem C.24. ■

Corollary C.27. *For $n > 2$, there are infinitely many non-isomorphisms between self-adjoint, reflexive, and real quantum graphs on M_n with $n^2 - 1$ edges.*

Proof. Similarly to the previous proofs, this follows directly from Proposition B.39 and Theorem C.24. ■

This means, for $n = 3$, so far, we know that there are infinitely many non-isomorphisms for quantum graphs with 1, 2, 7, and 8 edges.

C.VI Adding single-edges

In this section, we discuss adding single-edged self-adjoint and real quantum graphs $(M_n, \psi, A(x))$ and $(M_n, \psi, A(y))$. As expected, this only works when x and y are orthogonal.

Proposition C.28. *Given self-adjoint real quantum graphs $(M_n, \psi, A(x))$ and $(M_n, \psi, A(y))$ for non-zero $x, y \in M_n$, we get $(M_n, \psi, A(x) + A(y))$ is a self-adjoint and real quantum graph if and only if $\langle x|y\rangle = 0$.*
In other words, given almost self-adjoint via restricted Q elements $0 \neq x, y \in M_n$, we get $(M_n, \psi, A(x) + A(y))$ is a self-adjoint and real quantum graph if and only if x is orthogonal to y, i.e., $\langle x|y\rangle = 0$.

Proof. Let $x, y \in M_n \setminus \{0\}$ be almost self-adjoint via restricted Q. So let $\alpha, \beta \in \mathbb{C}$ and $a, b \in M_n$ such that a and b are self-adjoint and $x = \alpha a$ and $y = \beta b$. So then we also have a and b both commute with Q. This means we get,

$$\langle a|b\rangle = \operatorname{Tr}(Qa^*b) = \operatorname{Tr}(Qab) = \operatorname{Tr}(aQb) = \operatorname{Tr}(Qb^*a) = \langle b|a\rangle.$$

Obviously, $\langle x|y\rangle = 0$ if and only if $\langle a|b\rangle = 0$ as α, β are non-zero. So it suffices to show that Equation (1) is equivalent to $\langle a|b\rangle = 0$.

We have $A(x) + A(y)$ is Schur idempotent if and only if

$$A(x) \bullet A(y) + A(y) \bullet A(x) = 0. \tag{1}$$

Now, applying our linear isomorphism $\Psi_{0,1/2}$ to Equation (1) and using A.180(iv), we get,

$$\begin{aligned}
&\Psi_{0,1/2}\left[A(x) \bullet A(y) + A(y) \bullet A(x)\right] \\
&= \Psi_{0,1/2}(A(x))\Psi_{0,1/2}(A(y)) + \Psi_{0,1/2}(A(y))\Psi_{0,1/2}(A(x)) \\
&= \frac{1}{\|x\|^2\|y\|^2}(\mathrm{id} \otimes \top)\mathcal{M}(|x\rangle\langle x||y\rangle\langle y| + |y\rangle\langle y||x\rangle\langle x|) \\
&= \frac{1}{\|x\|^2\|y\|^2}(\mathrm{id} \otimes \top)\mathcal{M}(\langle x|y\rangle|x\rangle\langle y| + \langle y|x\rangle|y\rangle\langle x|) \\
&= \frac{\|\alpha\|^2\|\beta\|^2\langle a|b\rangle}{\|x\|^2\|y\|^2}(\mathrm{id} \otimes \top)\mathcal{M}(|a\rangle\langle b| + |b\rangle\langle a|).
\end{aligned}$$

So Equation (1) becomes,

$$\langle a|b\rangle(|a\rangle\langle b| + |b\rangle\langle a|) = 0. \tag{2}$$

So we need to show that Equation (2) is equivalent to $\langle a|b\rangle = 0$.

($\Rightarrow$) Suppose we have Equation (2). Then we get $\langle a|b\rangle = 0$ or $|a\rangle\langle b| + |b\rangle\langle a| = 0$. If $\langle a|b\rangle = 0$, then we are done. If, on the other hand, $|a\rangle\langle b| + |b\rangle\langle a| = 0$, then we have $\langle b|c\rangle a + \langle a|c\rangle b = 0$ for all $c \in B$. This means we get $\|b\|^2 a + \langle a|b\rangle b = 0$, and so $a = \frac{\langle a|b\rangle}{\|b\|^2} b$. So then we compute,

$$0 = |a\rangle\langle b| + |b\rangle\langle a| = \frac{2\langle a|b\rangle}{\|b\|^2}|b\rangle\langle b|.$$

Which is true if and only if $\langle a|b\rangle = 0$, so we are done.

($\Leftarrow$) Suppose $\langle a|b\rangle = 0$. Then we obviously get Equation (2).

■

Corollary C.29. *Given a tuple (x_i) in M_n such that each $(M_n, \psi, A(x_i))$ is a self-adjoint and real quantum graph, then*
$(M_n, \psi, \sum_i A(x_i))$ *is a self-adjoint and real quantum graph*
$\Leftrightarrow\ \forall i, j \in [n] : i \neq j \Rightarrow \langle x_i|x_j\rangle = 0.$ ■

Remark C.30. It is clear that we can only add up to n self-adjoint and real quantum graphs which satisfy the orthogonality requirement. Otherwise, we would not get orthogonality (and so it would not be a self-adjoint and real quantum graph by the above corollary). ◊

Definition C.31.

- We say a tuple of elements (x_i) in M_n is *simultaneously similar* to a tuple of elements (y_i) in M_n if there exists an isometric *-automorphism f on M_n such that each $x_i = f(y_i)$. We denote this by $(x_i) \sim_Q (y_i)$.
- We say a tuple of elements (x_i) in M_n is *simultaneously almost similar* to a tuple of elements (y_i) in M_n if there exists a tuple of non-zero complex numbers (β_i) such that $(x_i) \sim_Q (\beta_i y_i)$.

Lemma C.32. *Given tuples (x_i) and (y_i) in M_n, if* $\mathrm{Spectrum}(\sum_i x_i) \neq \mathrm{Spectrum}(\sum_i y_i)$, *then $(x_i) \nsim_Q (y_i)$.*

Proof. We show its contrapositive statement. So suppose $(x_i) \sim_Q (y_i)$. Then there exists an isometric *-automorphism f on M_n such that for each $x_i = f(y_i)$. And so we get $\mathrm{Spectrum}(\sum_i x_i) = \mathrm{Spectrum}(f(\sum_i y_i)) = \mathrm{Spectrum}(\sum_i y_i)$. ■

Corollary C.33. *Let $x, y, z, w \in M_n \setminus \{0\}$ such that $(M_n, \psi, A(x)+A(y))$ and $(M_n, \psi, A(z)+A(w))$ are quantum graphs. Then*
(x, y) is simultaneously almost similar to (z, w)
$\Rightarrow$ $(M_n, \psi, A(x) + A(y)) \cong (M_n, \psi, A(z) + A(w))$.

Proof. Suppose (x, y) is simultaneously almost similar to (z, w). Then we let f be an isometric *-automorphism on M_n and we let $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that $x = f^{-1}(\alpha z)$ and $y = f^{-1}(\beta w)$. Then we have

$$\begin{aligned} A(x) + A(y) &= A(f^{-1}(\alpha z) + A(f^{-1}(\beta w)) \\ &= A(\alpha f^{-1}(z)) + A(\beta f^{-1}(w)) \\ &= A(f^{-1}(z)) + A(f^{-1}(w)) && \text{by C.3} \\ &= f^{-1}(A(z) + A(w))f && \text{by C.12} \end{aligned}$$

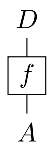
Which is what we needed. ■

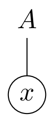
E String diagrams

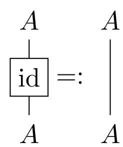
In this chapter, we list a summary of the language of our string diagrams, and operations one can use to manipulate said diagrams.

All diagrams are to be read from bottom to top. All diagrams have outputs (denoted by labelling the top of the diagram) in a space.

Let A, D, E, F be $\mathbb{C}$-vector spaces. The alphabet of our string diagrams consists of variables $x, y, \ldots$ in our spaces, and functions $f, g, \ldots$. We can concatenate *appropriate* diagrams to create new diagrams. We can also tensor the diagrams to create new diagrams. In particular:

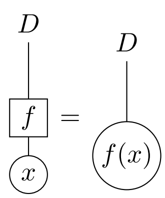
- a linear map $f\colon A \to D$ is given by 

- inputting $x \in A$ is given by 

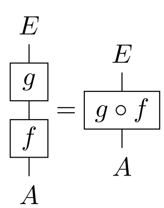
- We let the identity map $\mathrm{id}\colon A \to A$ be given by 

.

- We can concatenate a linear map $f\colon A \to D$ with an element $x \in A$, i.e., we can draw $f(x)$. This is done by placing the string diagram representing f on top of the diagram representing x:

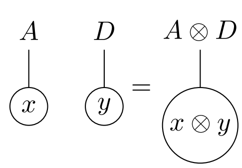


- When we concatenate two diagrams that represent linear maps, then this is simply the composition of the maps. In particular, for $f\colon A \to D$ and $g\colon D \to E$, we have,

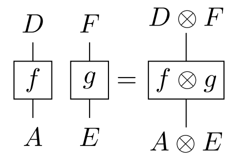


- We can perform planar isotopies on our diagrams. So we can stretch, compress, and move the strings around.

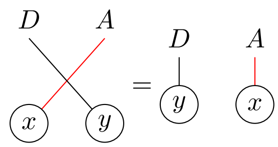
- To apply a tensor product of $x \in A$ with $y \in D$, i.e., $x \otimes y$, we place the diagram of y to the right of the diagram of x:



- Similarly, to apply a tensor product of linear map $f\colon A \to D$ with $g\colon E \to F$, we place the diagram of g to the right of f:

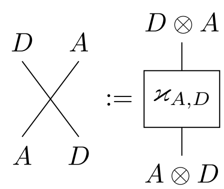


- We can overlap the strings. For $x \in A$ and $y \in D$, we get,



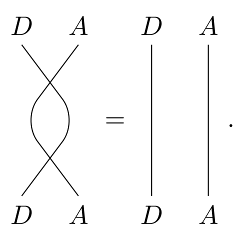
Note that the different colours here are meant to only highlight the fact that they are not intersecting, but overlapping (either way). Generally, we neither need nor use colours.

- Recall that we defined $\varkappa_{A,D}$ to be the tensor swap-map on $A \otimes D$, i.e., $\varkappa\colon A \otimes D \cong D \otimes A$ and is given by $x \otimes y \mapsto y \otimes x$. In strings, this is given by,

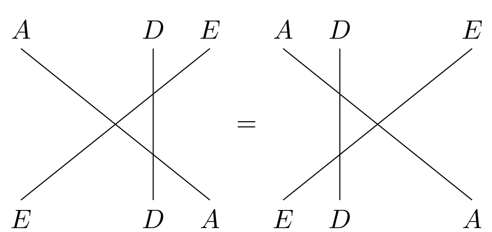


Obviously, $\varkappa_{A,D}(x \otimes y)$ is exactly the preceding point.

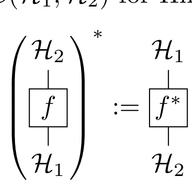
- We can move strings under/over other strings (also known as the Reidemeister II move (RII)):



- We can also perform Reidemeister III moves (RIII):



- We can apply the adjoint to a diagram representing a bounded linear map by vertically reflecting the diagram. So for $f \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ for Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, we have



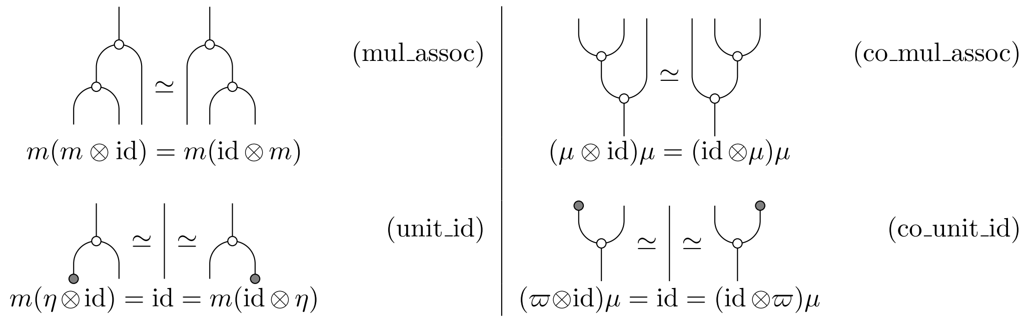
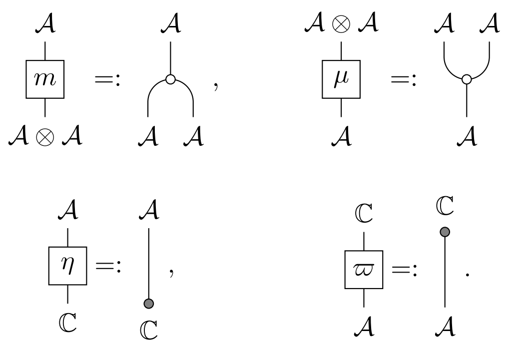
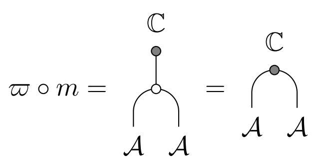


Table 1: algebraic and co-algebraic structures in string diagrams

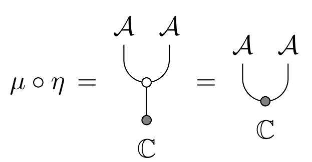
Now let $\mathcal{A}$ be an algebra and a co-algebra, with multiplication map $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, co-multiplication map $\mu\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, unit map $\eta\colon \mathbb{C} \to \mathcal{A}$, and co-unit map $\varpi\colon \mathcal{A} \to \mathbb{C}$. In strings, we define these as:



When m is composed with ϖ, then we draw this as



Similarly, when composing η with μ, we have



The algebraic and co-algebraic properties of $\mathcal{A}$ are given by the following,

The above table gives us more operations to use when manipulating diagrams.

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